COMPOSITIO MATHEMATICA

JAKOB LEVITZKI

On the equivalence of the nilpotent elements of a semi simple ring

On the equivalence of the nilpotent elements of a semi simple ring

by

Jakob Levitzki

Jerusalem

Introduction.

In the present note it is shown that each class of equivalent nilpotent elements of a semi simple ring [1]) can be characterised by certain characteristic numbers; two nilpotent elements belong to the same class if and only if their characteristic numbers coincide. This is achieved by reducing each nilpotent element to a certain normal form. As a consequence we find that the number of classes is finite. Applying our results to square matrices in a commutative or *non commutative* field we find that each nilpotent matrix can be transformed into a Jordan form [2]). Essential use is made of the notion of the „rank" (see I, 2) which yields an interpretation of the rank of a matrix based on the theory of the ideals in a simple ring. The convenience of this interpretation is also shown by a few examples in the appendix to this paper.

I.   Notations and preliminary remarks.

1.   If $\mathfrak{W}_1, \mathfrak{W}_2, \ldots, \mathfrak{W}_m$ are r.h. (read: right hand) ideals of a semi simple ring S, then we denote by $(\mathfrak{W}_1, \mathfrak{W}_2, \ldots, \mathfrak{W}_m)$ the r.h. ideal $\mathfrak{W}$ which consists of all the elements of the form $\sum_{i=1}^m r_i, r_i \epsilon \mathfrak{W}_i$. If from $\sum_{i=1}^m r_i = 0$, $r_i \epsilon \mathfrak{W}_i$ follows always $r_i = 0$, $i = 1, \ldots, m$ then $\mathfrak{W}$ is called the direct sum of the $\mathfrak{W}_i$ and we

---

[1])   Two elements a and b of S are called equivalent if for a regular element r of S the relation $r^{-1}ar = b$ holds.

[2])   The Jordan Form of a nilpotent matrix is $\begin{pmatrix} a_1 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & a_m \end{pmatrix}$ where each a_λ is a matrix of the form $\begin{pmatrix} 0 & \varepsilon & 0 \ldots 0 \\ 0 & 0 & \varepsilon \ldots 0 \\ \ldots \ldots \ldots \\ 0 & 0 & 0 \ldots \varepsilon \\ 0 & 0 & 0 \ldots 0 \end{pmatrix}$, ε being the unit of the field.

write $\mathfrak{W} = \mathfrak{W}_1 + \mathfrak{W}_2 + \ldots + \mathfrak{W}_m$ or $\mathfrak{W} = \Sigma_{i=1}^{m}\mathfrak{W}_i$. If in $\Sigma_{i=1}^{m}\mathfrak{W}_i$ the $\mathfrak{W}_i$ are primitive (i.e. the $\mathfrak{W}_i$ are different from zero and do not contain any subideals except $\mathfrak{W}_i$ and the zero ideal) then the number m is called the length of $\mathfrak{W}$; notation: $|\mathfrak{W}| =$ length of $\mathfrak{W}$. Similar notations are used for l.h. (read: left hand) ideals. If $\mathfrak{W}$ is the zero ideal, then $|\mathfrak{W}| = 0$.

2. LEMMA 1. *If $a \epsilon S$, $b \epsilon S$ then $|abS| \leq |aS|$; $|Sab| \leq |Sb|$.*

Proof. This is an immediate consequence of $abS \subseteq aS$ and $Sab \subseteq Sb$.

LEMMA 2. *If $a \epsilon S$, $b \epsilon S$ then $|abS| \leq |bS|$ and $|Sab| \leq |Sa|$.*

Proof. Suppose $bS = \Sigma_{i=1}^{m}\mathfrak{W}_i$, where the $\mathfrak{W}_i$ are primitive r.h. ideals, i.e. $|bS| = m$. Multiplying by a we obtain $abS = (a\mathfrak{W}_1, a\mathfrak{W}_2, \ldots, a\mathfrak{W}_m)$. Each $a\mathfrak{W}_i$ is (as easily seen) either a primitive r.h. ideal or zero, i.e. $|abS| \leq |bS|$. Similarly follows $|Sab| \leq |Sa|$.

LEMMA 3. *If $a \epsilon S$ and $a^2 = a$ then $|Sa| = |aS|$.*

Proof. For $a = 0$ the lemma is trivial. For $a \neq 0$ let $aS = \Sigma_{i=1}^{m}\mathfrak{W}_i$, where the $\mathfrak{W}_i$ are primitive r.h. ideals, i.e. $|aS| = m$. Let $a = \Sigma_{i=1}^{m}a_i$, $a_i \epsilon \mathfrak{W}_i$, then from $a^2 = a$ we obtain (by known argument) the relations $a_i a_k \Big\} = \begin{matrix} 0 \\ a_i \end{matrix}$ if $\begin{matrix} i \neq k \\ i = k \end{matrix}$, and hence $Sa = \Sigma_{i=1}^{m} Sa_i$, which implies $|Sa| \geq |aS|$ since $a_i \neq 0$, $i = 1, \ldots, m$. Similarly follows the relation $|aS| \geq |Sa|$, and hence $|aS| = |Sa|$.

THEOREM 1. *If $a \epsilon S$ then $|aS| = |Sa|$.*

Proof. Suppose $S = aS + \mathfrak{W}$, where $\mathfrak{W}$ is a r.h. ideal. The unit e of S has a representation of the form $e = ab + e'$, where $e' \epsilon \mathfrak{W}$ and (as easily seen) $(ab)^2 = ab$; $|abS| = |aS|$. By lemma 3 we have therefore $|abS| = |Sab| = |aS|$ and hence (by lemma 2) $|Sa| \geq |aS|$. Similarly follows $|aS| \geq |Sa|$ and hence $|Sa| = |aS|$, q.e.d.

NOTATION: If $a \epsilon S$, then the common length of aS and Sa is called the rank of a and is denoted by $|a|$.

3. From 2 follows easily: If $a_i \epsilon S$ $(i = 1, \ldots, m)$ and if $(\Sigma_{i=1}^{m} a_i)S = \Sigma_{i=1}^{m}a_i S$ then also $S\Sigma_{i=1}^{m}a_i = \Sigma_{i=1}^{m}Sa_i$. In this case we have the relation $|\Sigma_{i=1}^{m}a_i| = \Sigma_{i=1}^{m}|a_i|$ and we say that the a_i form a direct sum. Conversely from $|\Sigma_{i=1}^{m}a_i| = \Sigma_{i=1}^{m}|a_i|$ it follows that the r.h. ideals $a_i S$ (resp. the l.h. ideals Sa_i) form a direct sum. Thus the sum of idempotent elements e_i $(i = 1, \ldots, m)$ which form an orthogonal system $\left(\text{i.e. } e_i e_k \Big\} = \begin{matrix} 0 \\ e_i \end{matrix} \text{ if } \begin{matrix} i \neq k \\ i = k \end{matrix}\right)$ is direct.

4. From 2 and 3 it follows easily, that if $a = \sum_{i=1}^{m} a_i$, $|a| = \sum_{i=1}^{m} |a_i|$ and $ab = 0$ (or $ba = 0$) then $a_i b = 0$ (or resp. $ba_i = 0$) for each $i = 1, \ldots, m$.

5. LEMMA 1. *Let S be a semi simple ring and $\mathfrak{W}$, $\mathfrak{l}$ primitive r.h. resp. l.h. ideals. If $\mathfrak{l} \cdot \mathfrak{W} \neq 0$ then the subring $\mathfrak{W}\mathfrak{l}$ of S is a field.*

Proof. Suppose $r_1 l_1 \cdot r_2 l_2 = 0$, where $r_i \in \mathfrak{W}$, $l_i \in \mathfrak{l}$, $i = 1, 2$. Then either $r_1 l_1$ or $r_2 l_2$ is zero, since otherwise it would follow from $\mathfrak{l}\mathfrak{W} \neq 0$ and from the primitivity of $\mathfrak{W}$ and $\mathfrak{l}$ that $\mathfrak{l}\mathfrak{W} = (Sr_1 l_1)(r_2 l_2 S) = S(r_1 l_1 r_2 l_2)S = 0$. The ring aSa has, therefore, no divisors of zero. From $\mathfrak{l}\mathfrak{W} \neq 0$ follows also by similar argument for $r \in \mathfrak{W}$, $l \in \mathfrak{l}$, $rl \neq 0$ the relation $rl \cdot \mathfrak{W}\mathfrak{l} = \mathfrak{W}\mathfrak{l}$, i.e. each equation $rl \cdot x = r'l'$, $r' \in \mathfrak{W}$, $l' \in \mathfrak{l}$ in $\mathfrak{W}\mathfrak{l}$ has a solution which is unique, since $\mathfrak{W}\mathfrak{l}$ has no divisors of zero. Similarily, each equation $x \cdot rl = r'l'$ can be uniquely solved, i.e. $\mathfrak{W}\mathfrak{l}$ is a field.

LEMMA 2. *If A is a finite or infinite set of elements of a semi simple ring S then the subring ASA is either nilpotent or it contains a principal idempotent element e and has a representation $ASA = eSe + N$ where eSe is a semi simple ring and N is the radical of ASA.*

Proof. Let $\mathfrak{W}$ and $\mathfrak{l}$ be a r.h. and a l.h. primitive subideal of AS and SA respectively. If the ring $\mathfrak{W}\mathfrak{l}$ is nilpotent for each $\mathfrak{W}$ and $\mathfrak{l}$, then also ASA is nilpotent. If now for certain $\mathfrak{W}$ and $\mathfrak{l}$ the ring $\mathfrak{W}\mathfrak{l}$ is not nilpotent, then as in lemma 1 it follows that $\mathfrak{W}\mathfrak{l}$ is a field since in this case $\mathfrak{l}\mathfrak{W} \neq 0$; hence ASA contains idempotent elements. Let now e be an idempotent element of highest rank which lies in ASA, and suppose $AS = eS + \overline{\mathfrak{W}}$, $SA = Se + \bar{\mathfrak{l}}$ where $\overline{\mathfrak{W}}$ and $\bar{\mathfrak{l}}$ are chosen such that $e\overline{\mathfrak{W}} = \bar{\mathfrak{l}}e = 0$. Then $\bar{\mathfrak{l}}\overline{\mathfrak{W}} = 0$; in fact, suppose $\bar{\mathfrak{l}}\overline{\mathfrak{W}} \neq 0$ then there exists a primitive l.h. ideal $\mathfrak{l}$ and a primitive r.h. ideal $\mathfrak{W}$ such that $\mathfrak{l}\mathfrak{W} \neq 0$; $\mathfrak{l} \subseteq \bar{\mathfrak{l}}$; $\mathfrak{W} \subseteq \overline{\mathfrak{W}}$. Hence (Lemma 1) $\mathfrak{W}\mathfrak{l}$ is a field and has a unit e'. Since $e\mathfrak{l}\mathfrak{W} = \mathfrak{l}\mathfrak{W}e = 0$ we have $ee' = e'e = 0$ and therefore $|e + e'| = |e| + |e'|$ which contradicts the assumption that e is an idempotent of highest rank as $e + e'$ is an idempotent which lies in ASA and has the rank $|e| + 1$. From $\bar{\mathfrak{l}}\overline{\mathfrak{W}} = 0$ follows that the ring $N = eS\bar{\mathfrak{l}} + \overline{\mathfrak{W}}Se + \overline{\mathfrak{W}}\bar{\mathfrak{l}}$ is a nilpotent ideal in ASA which together with $ASA = eSe + N$ completes the proof.

REMARK. If $\overline{\mathfrak{W}}$ and $\bar{\mathfrak{l}}$ are different from zero and if e_1 and e^* are units of $\overline{\mathfrak{W}}$ and $\bar{\mathfrak{l}}$ respectively, then $e^* e_1 = 0$. We set $e_2 = e^* - e_1 e^*$, then e_2 is also a unit of $\bar{\mathfrak{l}}$ and $ASA = eSe + e_1 SA + ASe_2$, where e, e_1, e_2 are orthogonal.

6. LEMMA 3. *If* $|a+b| = |a| + |b|$ *and* $ab = ba = 0$, $a \neq 0$, $b \neq 0$ *then two orthogonal elements* α, β *can be found such that* $a = \alpha a \alpha$, $b = \beta b \beta$.

Proof. Since aS and bS form a direct sum we may write $S = aS + bS + \mathfrak{W}$ where $\mathfrak{W}$ is a r.h. ideal in S. If E is the unit of S we have $E = e_1 + e_2 + e_3$ where $e_1 a = a$; $e_2 b = b$; $e_3 \mathfrak{W} = \mathfrak{W}$ and $e_i e_k \big\} = \begin{smallmatrix} 0 \\ e_i \end{smallmatrix}$ if $\begin{smallmatrix} i \neq k \\ i=k \end{smallmatrix}$. Similarly considering the l.h. ideals Sa and Sb, we find $\bar{e}_1$ and $\bar{e}_2$ such that $a\bar{e}_1 = a$, $b\bar{e}_2 = b$ and $\bar{e}_i \bar{e}_k \big\} = \begin{smallmatrix} 0 \\ e_i \end{smallmatrix}$ if $\begin{smallmatrix} i \neq k \\ i=k \end{smallmatrix}$. From $ab = ba = 0$ follows further $\bar{e}_1 e_2 = \bar{e}_2 e_1 = 0$. If now e is a principal idempotent in the ring aSa (in case aSa is nilpotent we set $e = 0$), then from $aSab = baSa = 0$ and $eS \subseteqq aS$ $Se \subseteqq Sa$ follows $eb = be = 0$ and $e_1 e = e$. Setting $e' = e_1 - ee_1$; $e_1' = e' + e$ we have $e'e = ee' = 0$, $e_1' e_1 = e_1$, $e_1' a = a$ where e', e_1' are idempotent. Similarly we find $\bar{e}_1'$, $\bar{e}'$ such that $e\bar{e}' = \bar{e}'e = 0$, $\bar{e}_1 e_1' = \bar{e}_1$, $a\bar{e}_1' = a$. If we set $aSa = eSe' + eS\bar{e}' + e'Se + e'S\bar{e}'$ then (see 5) $\bar{e}'e' = 0$, hence the element $\alpha = e + e' + (\bar{e}' - e'\bar{e}')$ is idempotent and $\alpha a = a\alpha = \alpha a \alpha$. Since now $xb = bx = 0$ if we substitute for x any of the elements e, e', $\bar{e}'$, it follows that $\alpha b = b\alpha = 0$, i.e. the element b lies in the ring $(E - \alpha)S(E - \alpha)$, which completes the proof as we may set $\beta = E - \alpha$.

LEMMA 4. *If* $a = \sum_{i=1}^{m} a_i$, $|a| = \sum_{i=1}^{m} |a_i|$ *and* $a_i a_k = 0$ *for* $i \neq k$, *then there exists an orthogonal system of elements* α_i $(i=1, \ldots, m)$ *such that* $a_i = \alpha_i a_i \alpha_i$.

Proof. We prove by induction. For $m = 2$ the lemma is true according to lemma 3. For $m > 2$ we set $b = \sum_{i=2}^{m} a_i$, then a_1 and b satisfy the condition of lemma 3, hence, there exist two idempotents α_1 and β such that $a_1 = \alpha_1 a_1 \alpha_1$, $b = \beta b \beta$, $\alpha_1 \beta = \beta \alpha_1 = 0$. For the elements a_i $(i = 2, \ldots, m)$ of the semi simple ring $\beta S \beta$ we may assume the validity of the lemma (since the number of the elements in this system is less than m), hence there exists a system of orthogonal elements $\alpha_2, \ldots, \alpha_m$ satisfying together with α_1 the relations in question.

II. Reduction to the normal form.

LEMMA 5. *Let* S *be a simple ring and* $c_{i,k}$ $(i, k = 1, \ldots, n)$ *an arbitrary matric-basis of* S. *Suppose* $c \epsilon S$; $c = \sum_{i=1}^{m} c_i$; $|c_i| = 1$; $c_i c_k \big\} \begin{smallmatrix} =0 \\ \neq 0 \end{smallmatrix}$ if $\begin{smallmatrix} k \neq i+1 \\ k = i+1 \end{smallmatrix}$. *Then the element* c *is equivalent to* $\sum_{\lambda=1}^{m} c_{\lambda, \lambda+1}$.

Proof. The r.h. ideal $c_{i+1}S$ and the l.h. ideal Sc_i satisfy the conditions of lemma 1, hence, there exists an idempotent $\bar{v}_{i+1, i+1}$

which is the unit of the field $c_{i+1}Sc_i$ and therefore $c_i\bar{c}_{i+1,i+1} = c_i$; $\bar{c}_{i+1,i+1}c_{i+1} = c_{i+1}$, $i = 1, \ldots, m-1$. From $c_ic_k = 0$, $k \neq i+1$ follows further $\bar{c}_{i,i}\bar{c}_{k,k} = 0$, $i \neq k$.

If $c^*_{1,1}$ resp. $c^*_{m+1,m+1}$ is an arbitrary l.h. unit resp. r.h. unit of c_1S resp. Sc_m then we set $\bar{c}_{1,1} = c^*_{1,1} - c^*_{1,1}\sum_{i=2}^{m}\bar{c}_{i,i}$, $\bar{c}_{m+1,m+1} = c^*_{m+1,m+1} - \sum_{i=2}^{m}\bar{c}_{i,i}c^*_{m+1,m+1}$, and thus obtain an orthogonal system $\bar{c}_{i,i}$, $i = 1, \ldots, m+1$. Setting further $\bar{c}_{i,i+1} = c_i$, $\bar{c}_{i,i+\lambda} = \prod_{\varrho=0}^{\lambda-1}c_{i+\varrho}$, then the $\bar{c}_{i,i+\lambda}$; $(i = 1, \ldots, m, \lambda = 1, \ldots, m-i+1)$ are different from zero as a consequence of $|c_i| = 1$, and $c_ic_{i+1} \neq 0$ as in this case we have $c_ic_{i+1}S = c_iS$ and therefore in general $\prod_{\varrho=0}^{\lambda-1}c_{i+\varrho}S = c_iS \neq 0$; we complete now the $\bar{c}_{i,k}$ $(i \leq k, i, k = 1, \ldots, m+1)$ (by known argument) to a matric basis $\bar{c}_{i,k}$ $(i, k = 1, \ldots, n)$ of S. According to a theorem of Artin [3]) there exists a regular element r which satisfies the relations $r^{-1}c_{i,k}r = \bar{c}_{i,k}$, $i, k = 1, \ldots, n$. Since $a = \sum_{i=1}^{m}\bar{c}_{i,i+1}$ we obtain $r^{-1}\left(\sum_{i=1}^{m}c_{i,i+1}\right)r = a$, q.e.d.

THEOREM 2. *Let* S *be a simple ring and* $c_{i,k}$ $(i, k = 1, \ldots, n)$ *an arbitrary matric basis of* S. *Suppose further* $a \in S$, $a = \sum_{i=1}^{q}a_i$, $|a| = \sum_{i=1}^{q}|a_i|$, $a_i = \sum_{\lambda=1}^{n_i}t_{i,\lambda}$, $|t_{i,\lambda}| = 1$, *and* $t_{i,\lambda}t_{i,\sigma} \begin{Bmatrix} = \\ \neq \end{Bmatrix} 0$ *if* $\begin{matrix} \sigma \neq \lambda+1 \\ \sigma = \lambda+1 \end{matrix}$. *Then* a *is equivalent to* $\sum_{\lambda=1}^{n_1}c_{\lambda,\lambda+1} + \sum_{\lambda=n_1+2}^{n_1+n_2+1}c_{\lambda,\lambda+1} + \cdots + \sum_{\lambda=n_1+\ldots+n_{q-1}+q}^{n_1+\ldots+n_q+q-1}c_{\lambda,\lambda+1}$.

Proof. The proof follows immediately by combining lemma 4 and lemma 5.

THEOREM 3. *If* S *is a semi simple ring and* a *is a nilpotent element of index* $m + 1$ [4]) *then there exists an orthogonal system* d_i $(i = 1, \ldots, q)$, *each* d_i *being of rank* 1 *such that*

$$(1) \qquad aS = \sum_{i=1}^{n_1}a^id_1S + \sum_{i=1}^{n_2}a^id_2S + \ldots + \sum_{i=1}^{n_q}a^id_qS,$$

$$a^\lambda d_\sigma \begin{Bmatrix} \neq \\ = \end{Bmatrix} 0 \text{ if } \begin{matrix} \lambda \leq n_\sigma \\ \lambda > n_\sigma \end{matrix} \quad (\sigma = 1, \ldots, q, \text{ and } m = n_1 \geq n_2 \geq \ldots \geq n_q).$$

Proof. From $Sa^{\lambda+1} \subseteq Sa^\lambda$ follows easily $Sa^{\lambda+1} \subset Sa^\lambda$ if $\lambda \leq m$ since $Sa^{\lambda+1} = Sa^\lambda$ implies $Sa^{\lambda+\varrho} = Sa^\lambda$ for each ϱ, hence $a^\sigma \neq 0$ for each σ which contradicts $a^{m+1} = 0$. If $l'_{\lambda-1}$ denotes the set of all elements $l'_{\lambda-1}$ such that $l'_{\lambda-1} \in Sa^{\lambda-1}$ and $l'_{\lambda-1}a = 0$, then $l'_{\lambda-1}$ is evidently a l.h. ideal which is different from zero in case

[3]) E. ARTIN, Zur Theorie der hyperkomplexen Zahlen [Hamb. Abh. **5** (1927), 251—260].

[4]) The index is $m + 1$ if $a^{m+1} = 0$ and $a^k \neq 0$ for $k \leq m$.

$\lambda \leqq m+1$ since $a^m \epsilon S a^{\lambda-1}$, $a^m \epsilon \mathfrak{l}'_{\lambda-1}$, and $a^m \neq 0$. From $S a^{\lambda-1} \supset S a^{\lambda}$ follows $\mathfrak{l}'_{\lambda-1} \supseteq \mathfrak{l}'_{\lambda}$, $\lambda=1, \ldots, m+1$. Setting $S a^m = \mathfrak{l}_m + \mathfrak{l}'_m$ then obviously $\mathfrak{l}_m = 0$ (since $a^{m+1}=0$). We further set $S a^{m-1} = \mathfrak{l}_{m-1} + \mathfrak{l}'_{m-1}$; then obviously $\mathfrak{l}_{m-1} \neq 0$, and if $l_{m-1} a = 0$, $l_{m-1} \epsilon \mathfrak{l}_{m-1}$ then $l_{m-1} = 0$. From $S a^{m-2} \supset S a^{m-1}$ follows $S a^{m-2} \supset \mathfrak{l}_{m-1}$. From $l'_{m-2} + l_{m-1} = 0$; $l'_{m-2} \epsilon \mathfrak{l}_{m-2}$; $l_{m-1} \epsilon \mathfrak{l}_{m-1}$ follows $l'_{m-2} a + l_{m-1} a = 0$, and since $l'_{m-2} a = 0$ we have $l_{m-1} a = 0$ which implies $l_{m-1} = 0$, and hence also $l'_{m-2} = 0$, in other words, the ideals $\mathfrak{l}'_{m-2}$ and $\mathfrak{l}_{m-1}$ form a direct sum.

We set $S a^{m-2} = \mathfrak{l}'_{m-2} + \mathfrak{l}_{m-1} + \bar{\mathfrak{l}}_{m-1}$ and $\mathfrak{l}_{m-2} = \mathfrak{l}_{m-1} + \bar{\mathfrak{l}}_{m-1}$, then $S a^{m-2} = \mathfrak{l}'_{m-2} + \mathfrak{l}_{m-2}$, and it is easily seen that from $l_{m-2} a = 0$, $l_{m-2} \epsilon \mathfrak{l}_{m-2}$ follows $l_{m-2} = 0$, further, $\mathfrak{l}_{m-2} \supseteq \mathfrak{l}_{m-1}$. Thus by induction we obtain the decompositions

$$(2) \qquad S a^{\lambda} = \mathfrak{l}_{\lambda} + \mathfrak{l}'_{\lambda}, \quad \mathfrak{l}_{\lambda+1} \subseteq \mathfrak{l}_{\lambda}, \quad \mathfrak{l}'_{\lambda+1} \subseteq \mathfrak{l}'_{\lambda} \quad (\lambda = 1, \ldots, m)$$

where each relation $l_{\lambda} a = 0$, $l_{\lambda} \epsilon \mathfrak{l}_{\lambda}$ implies $l_{\lambda} = 0$. We now set $\mathfrak{l}'_{\lambda} = \mathfrak{l}'_{\lambda+1} + \mathfrak{l}^*_{\lambda}$ $(\lambda = 1, \ldots, m)$; then $\mathfrak{l}'_m = \mathfrak{l}^*_m$, and we have $S a^{\varrho} = \mathfrak{l}_{\varrho} + \sum_{i=\varrho}^m \mathfrak{l}^*_i$; in particular $S a = \mathfrak{l}_1 + \sum_{i=1}^m \mathfrak{l}^*_i$. If we set $S = S a + \mathfrak{l}$, and $E = e_1 + \sum_{i=1}^m e^*_i + e$ where E is the unit of S, and $e_1 \epsilon \mathfrak{l}_1$, $e^*_i \epsilon \mathfrak{l}^*_i$, $e \epsilon \mathfrak{l}$, then (as it is well known) we have $e_1^2 = e_1$, $e^{*2}_i = e^*_i$ and $e_1 e^*_i = e^*_i e_1 = e^*_i e^*_j = 0$ for $i \neq j$.

The element

$$(3) \qquad e'_{\lambda} = \sum_{i=\lambda}^m e^*_i$$

is a unit of $\mathfrak{l}'_{\lambda}$, and we have $e_1 e'_{\lambda} = e'_{\lambda} e_1 = 0$, and for $\lambda \geq \tau$, the relations $e'_{\lambda} e'_{\tau} = e'_{\tau} e'_{\lambda} = e'_{\lambda}$. If further e_{λ} denotes any l.h. unit of $\mathfrak{l}_{\lambda}$, then from $e_1 e'_{\lambda} = 0$ follows also $e_{\varrho} e'_{\lambda} = 0$ since $e_{\varrho} \epsilon \mathfrak{l}_1$ and $\mathfrak{l}_1 = S e_1$. Suppose now

$$(4) \qquad a^{\lambda} = l_{\lambda} + l'_{\lambda}$$

where $l_{\lambda} \epsilon \mathfrak{l}_{\lambda}$, $l'_{\lambda} \epsilon \mathfrak{l}'_{\lambda}$. Then $a^{\lambda+\varrho} e'_{\lambda} = (l_{\lambda+\varrho} + l'_{\lambda+\varrho}) e'_{\lambda} = (l_{\lambda+\varrho} e_{\lambda+\varrho} + l'_{\lambda+\varrho} e'_{\lambda+\varrho}) e'_{\lambda} = l'_{\lambda+\varrho} e'_{\lambda+\varrho} e'_{\lambda} = l'_{\lambda+\varrho} e'_{\lambda+\varrho} = (l_{\lambda+\varrho} + l'_{\lambda+\varrho}) e'_{\lambda+\varrho} = a^{\lambda+\varrho} e'_{\lambda+\varrho}$, hence

$$(5) \qquad a^{\lambda+\varrho} e'_{\lambda} = a^{\lambda+\varrho} e'_{\lambda+\varrho}.$$

From $a^{\lambda} e^*_{\varrho} = (l_{\lambda} + l'_{\lambda}) e^*_{\varrho}$ and $l_{\lambda} \epsilon S e$, follows $l_{\lambda} e^*_{\varrho} = 0$, hence $a^{\lambda} e^*_{\varrho} = l'_{\lambda} e^*_{\varrho} = l'_{\lambda} e'_{\lambda} e^*_{\varrho} = l'_{\lambda} \sum_{i=\lambda}^m e^*_i e^*_{\varrho}$ i.e.:

$$(6) \qquad a^{\lambda} e^*_{\varrho} = 0 \quad \text{if} \quad \varrho < \lambda.$$

Since $|a^{\lambda}| = |l_{\lambda}| + |l'_{\lambda}|$ it follows from I, 3 that

$$(7) \qquad a^{\lambda} S = l_{\lambda} S + l'_{\lambda} S.$$

Multiplying (4) by a we obtain $a^{\lambda+1} = l_\lambda a$ since $l'_\lambda a = 0$; therefore $a^{\lambda+1}S = l_\lambda a S = l_\lambda S$ [5]). From (4) we obtain (in view of (3)) $a^\lambda e'_\lambda = l'_\lambda e'_\lambda = l'_\lambda$ and hence

$$(8) \qquad a^\lambda S = a^{\lambda+1}S + a^\lambda e'_\lambda S, \quad |a^\lambda e'_\lambda| = |e'_\lambda|.$$

From (8) follows easily

$$(9) \qquad a^\lambda S = a^\lambda e'_\lambda S + a^{\lambda+1}e'_{\lambda+1}S + \ldots + a^m e'_m S \quad (\lambda = 1, \ldots, m).$$

Using (9) for $\lambda = 1$ and (3) we further obtain

$$(10) \qquad aS = \Sigma_{i=1}^m a^i e_m^* S + \Sigma_{i=1}^{m-1} a^i e_{m-1}^* S + \ldots + \Sigma_{i=1}^2 a^i e_2^* S + a e_1^* S.$$

Not all the r.h. ideals which appear in the right side of (10) are necessarily different from zero, since some of the e_j^* may vanish. We denote by $c_1, \ldots, c_t$ those e_j^* which are different from zero, and may assume that $c_1 = e_m^*$ since $e_m^* = e'_m \neq 0$. Then $m_1 = m$, and we have

$$(11) \qquad aS = \Sigma_{i=1}^{m_1} a^i c_1 S + \Sigma_{i=1}^{m_2} a^i c_2 S + \ldots + \Sigma_{i=1}^{m_t} a^i c_t S$$

where we may assume $m = m_1 > m_2 > \ldots > m_t$. The c_i satisfy the relations

$$(12) \qquad \left. c_i c_k \right\} = \begin{array}{l} 0 \\ c_i \end{array} \text{ if } \begin{array}{l} i \neq k \\ i = k \end{array}, \quad \left. a^\lambda c_i \right\} \begin{array}{l} = \\ \neq \end{array} 0 \text{ if } \begin{array}{l} \lambda > m_i \\ \lambda \leq m_i \end{array}.$$

We now represent each $c_\lambda S$ as a direct sum of primitive r.h. ideals: $c_\lambda S = \Sigma_{\sigma=1}^{p_\lambda} c_\lambda^{(\sigma)} S$ where the $c_\lambda^{(\sigma)}$ are chosen such that $c_\lambda = \Sigma_{\sigma=1}^{p_\lambda} c_\lambda^{(\sigma)}$, $\left. c_\lambda^{(\sigma)} c_\lambda^{(\sigma')} \right\} = \begin{array}{l} 0 \\ c_\lambda^{(\sigma)} \end{array}$ if $\begin{array}{l} \sigma \neq \sigma' \\ \sigma = \sigma' \end{array}$.

From $c_\lambda c_{\lambda'} = 0$ for $\lambda \neq \lambda'$ follows further $c_\lambda^{(\sigma)} c_{\lambda'}^{(\sigma')} = 0$ if $\lambda \neq \lambda'$, and we obtain from (12):

$$(13) \qquad \left. c_\lambda^{(\sigma')} c_{\lambda'}^{(\sigma)} \right\} = \begin{array}{l} c_\lambda^{(\sigma)} \\ 0 \end{array} \text{ if } \begin{array}{l} \lambda = \lambda', \sigma = \sigma' \\ \text{either } \lambda \neq \lambda' \text{ or } \sigma \neq \sigma' \end{array}, \text{ and } \left. a^\tau c_\lambda^{(\sigma)} \right\} \begin{array}{l} = \\ \neq \end{array} 0 \text{ if } \begin{array}{l} \tau > n_\lambda \\ \tau \leq n_\lambda \end{array}.$$

From (11) and the second part of (8) we now obtain

$$(14) \qquad aS = \Sigma_{\sigma=1}^{p_1} \Sigma_{\lambda=1}^{m_1} a^\lambda c_1^{(\sigma)} S + \Sigma_{\sigma=1}^{p_2} \Sigma_{\lambda=1}^{m_2} a^\lambda c_2^{(\sigma)} S + \ldots + \\ + \Sigma_{\sigma=1}^{p_t} \Sigma_{\lambda=1}^{m_t} a^\lambda c_t^{(\sigma)} S.$$

Setting for the sake of simplification $c_1^{(\sigma)} = d_\sigma$, $c_i^{(\sigma)} = d_{p_1+\ldots+p_{i-1}+\sigma}$ we obtain

$$(15) \qquad aS = \Sigma_{i=1}^{n_1} a^i d_1 S + \Sigma_{i=1}^{n_2} a^i d_2 S + \ldots + \Sigma_{i=1}^{n_q} a^i d_q S$$

---

[5]) The last equality is a consequence of $l_\lambda a S \subseteq l_\lambda S$ and $|l_\lambda a| = |l_\lambda|$ (see the definition of l_λ).

where $m_i = n_{p_1+p_2+\ldots p_{i-1}+\lambda}$, $\lambda = 1,\ldots, p_i$, $p_0 = 0$. Since (according to (13)) the d_i form an orthogonal system, and since from the primitivity of the $c_\lambda^{(\sigma)}S$ follows $|d_i| = 1$, the relations (15) and (13) complete the proof of the theorem.

THEOREM 4. *If S is a semi simple ring and a is a nilpotent element of index $m+1$, then a can be represented in the form $a = \Sigma_{i=1}^q a_i$ where $|a| = \Sigma_{i=1}^q |a_i|$ and $a_i a_{i'} = 0$ if $i \neq i'$. Further, each a_i has the form $a_i = \Sigma_{k=1}^{n_i} r_{i,k}$ where*

$$(16) \quad |a_i| = \Sigma_{k=1}^{n_i} |r_{i,k}|, \; |r_{i,k}| = 1, \; r_{i,k} r_{i,k'} \left.\begin{matrix}= \\ \neq\end{matrix}\right\} 0 \text{ if } \begin{matrix}k' \neq k+1 \\ k' = k+1\end{matrix};$$

$$r_{i,k} r_{i'k'} = 0 \text{ if } i \neq i', \text{ and } m = n_1 \geq n_2 \geq \ldots \geq n_q.$$

Proof. From (15) it follows that a can be represented in the form

$$(17) \qquad a = \Sigma_{i=1}^{n_1} a^i r_1^{(i)} + \Sigma_{i=1}^{n_2} a^i r_2^{(i)} + \ldots + \Sigma_{i=1}^{n_q} a^i r_q^{(i)}$$

where $r_\lambda^{(i)} \epsilon \, d_\lambda S$, $|a^i r_\lambda^{(i)}| = 1$, and hence $|a| = \Sigma_{i=1}^q n_i$.

We multiply both sides of (17) right hand by $a^j r_\lambda^{(j)}$, then: $a^{j+1} r_\lambda^{(j)} = \Sigma_{i=1}^{n_1} a^i r_1^{(i)} \, a^j r_\lambda^{(j)} + \ldots + \Sigma_{i=1}^{n_q} a^i r_q^{(i)} \, a^j r_\lambda^{(j)}$. Since $a^{j+1} r_\lambda^{(j)} \epsilon \, a^{j+1} d_\lambda S$; $a^i r_\sigma^{(i)} \, a^j r_\lambda^{(j)} \varepsilon \, a^i d_\sigma S$ and since the ideals $a^k d_\sigma S$ form a direct sum it follows in case $j = n_\lambda$ that $a^{j+1} r_\lambda^{(j)} = 0$ (see (13)) and therefore $a^i r_\sigma^{(i)} \, a^j r_\sigma^{(j)} = 0$ for each i, σ, λ; in case $j < n_\lambda$ we obtain $a^{j+1} r_\lambda^{(j)} = a^{j+1} r_\sigma^{(j+1)} \, a^j r_\sigma^{(j)} \neq 0$ while $a^i r_\sigma^{(i)} a^j r_\lambda^{(j)} = 0$ for each $i \neq j+1$. Thus by setting $r_{i,k+1} = a^{n_i-k} r_i^{(n_i-k)}$, $a_i = \Sigma_{k=1}^{n_i} r_{i,k}$, we have in addition to $|r_{i,k}| = 1$ the relations

$$(18) \qquad a = \Sigma_{i=1}^q a_i, \; a_i = \Sigma_{\lambda=1}^{n_i} r_{i,\lambda}, \; r_{i,k} r_{i'k'} = 0 \text{ if } i \neq i',$$

i.e. $a_i a_{i'} = 0$ if $i \neq i'$ and $r_{i,k} r_{i,k'} \left.\begin{matrix}= \\ \neq\end{matrix}\right\} 0$ if $\begin{matrix}k' \neq k+1 \\ k' = k+1\end{matrix}$ q.e.d.

DEFINITION. If S is a simple ring, then the representation (18) is called the normal representation of the nilpotent element a of S, and the numbers $n_1, n_2, \ldots, n_q$ the characteristic of a.

Above definition is justified by the following

THEOREM 5. *Two nilpotent elements of a simple ring are equivalent if and only if their characteristic numbers coincide.*

Proof. The proof follows easily from theorems 2 and 4.

COROLLARY. *The number of the different classes of equivalent nilpotent elements of a semi simple ring is finite.*

REMARK. It is easily seen how lemma 5 in II, theorem 2, the above definition and theorem 5 should be modified in case S is semi simple.

III.   Application to matrices.

Let $\begin{pmatrix} a_1^1 \cdots a_n^1 \\ \cdots\cdots \\ a_1^m \cdots a_n^m \end{pmatrix}$ be a matrix in an arbitrary field F. If for $1 \leqq r \leqq m,\ 1 \leqq s_1 < s_2 < \ldots < s_r \leqq m$, and $fs_1, fs_2, \ldots, fs_r, fs_i \epsilon F$ where $fs_i \neq 0$ at least for one i, the relations $\Sigma_{i=1}^r fs_i a_k^{s_i}$ $(k = 1, \ldots, n)$ hold, then the r rows $a_1^{s_i}, \ldots, a_n^{s_i}$ $(i = 1, \ldots, r)$ are called left hand (in short: l.h.) dependent. Otherwise these rows are called l.h. independent. Similarly the right hand (in short: r.h.) dependence of the rows, as well as the l.h. and the r.h. dependence and independence of the columns is defined.

DEFINITION. The maximal number of l.h. independent rows of a matrix is called the l.h. rank of the rows. Similarly the r.h. rank of the rows as well as the l.h. rank and the r.h. rank of the columns is defined.

In general, the r.h. rank of the rows (columns) is different from the l.h. rank of the rows (columns). Moreover, the following theorem can be easily proved: *If the r.h. rank of the rows (columns) of each matrix of second order is equal to the l.h. rank of the rows (columns), then the field F is commutative.*

As to the l.h. rank of the rows (columns) and the r.h. rank of the columns (rows) it will be shown in present paragraph that they are always equal to each other.

We now consider the set S of all square matrices $a = \begin{pmatrix} \alpha_1^1 \cdots \alpha_n^1 \\ \cdots\cdots \\ \alpha_1^n \cdots \alpha_n^n \end{pmatrix}$ of order n (a shorter notation: $a = (\alpha_k^i)$). Defining equality, sum and product as follows

$$(\alpha_k^i) = (\beta_k^i)\ \text{if}\ \alpha_k^i = \beta_k^i;\ (\alpha_k^i) + (\beta_k^i) = (\gamma_k^i)\ \text{if}\ \alpha_k^i + \beta_k^i = \gamma_k^i;$$
$$(\alpha_k^i)(\beta_k^i) = (\delta_k^i)\ \text{if}\ \delta_k^i = \Sigma_{l=1}^n \alpha_l^i \beta_k^l \quad (i, k = 1, \ldots, n),$$

we obtain, as it is well known, a simple ring of length n.

THEOREM 6. *If $a \epsilon S$, then the length of Sa is equal to the l.h. rank of the rows of a and the length of aS is equal to the r.h. rank of the columns of a.*

Proof. We denote by a_λ $(\lambda = 1, \ldots, n)$ the matrix, the λth row of which coincides with the λth row of a while all the other elements are zeros. We assume (for the convenience of notation) that the r l.h. independent rows are the r first ones. From the l.h. dependence of any $r + 1$ rows follows for each $r + \lambda$, $\lambda = 1, \ldots,$ $n - r$ the existence of r elements $f_i^{r+\lambda}$ $(i = 1, \ldots, r)$, such that

$a_k^{r+\lambda} = \Sigma_{t=1}^{r} f_t^{r+\lambda} a_k^t$, $k = 1, \ldots, n$. If further $b_{r+\lambda}$ $(\lambda = 1, \ldots, n)$ denotes the matrix the $(r+\lambda)^{\text{th}}$ row of which is equal to $f_1^{r+\lambda}, \ldots, f_r^{r+\lambda}, 0, \ldots, 0$ then $a_{r+\lambda} = b_{r+\lambda} \Sigma_{i=1}^{r} a_i$ $(\lambda = 1, \ldots, n-r)$, and hence, the l.h. ideal Sa is a subset of $(Sa_1, \ldots, Sa_r)$ (since $a = \Sigma_{i=1}^{n} a_i$).

On the other hand, if e_k denotes the matrix all the elements of which are zeros, except the element in the kth row and k^{th} column which is equal to the unit of the field F, then $e_k a = a_k$ and hence $(Sa_1, \ldots, Sa_r) \subseteq Sa$; therefore $(Sa_1, \ldots, Sa_r) = Sa$. If further $\Sigma_{i=1}^{r} s_i a_i = 0$; $s_i \epsilon S$, it follows from the l.h. independence of the r first rows that all the elements of the λth column of s_λ are zeros, and hence $s_i a_i = 0$ $(i = 1, \ldots r)$; in other words, the Sa_i $(i = 1, \ldots, r)$ form a direct sum: $Sa = Sa_1 + \ldots + Sa_r$. Since finally (as easily verified) the l.h. ideals Sa_i $(i = 1, \ldots, r)$ are primitive, the first part of the theorem is proved. The second part follows similarly.

By I, 2 and theorem 6 we now obtain:

THEOREM 7. *The l.h. rank of the rows of a matrix is equal to the r.h. rank of the columns.*

As to the r.h. rank of the rows and the l.h. rank of the columns of a matrix a, their equality follows by considering the transpose of a and applying the theorem just proved.

It is clear how the results of **II** should be applied to the matrices of the ring S: *Each nilpotent matrix can be transformed to a Jordan form, the structure of which is interpreted by theorems 4 and 2.*

IV. Appendix.

It might be of some interest to show the usefulness of the interpretation of the rank stated in theorem 6 also by the following examples. As in section III we denote by S the simple ring of all matrices of nth order in an arbitrary field F.

1. If $a \epsilon S$ and $|a| = |S|$, then a is called a regular matrix. In this case we have $Sa = aS = S$. This implies for each $b \epsilon S$ the relations $Sab = Sb$, $baS = bS$, and hence: $|ab| = |ba| = |b|$.

2. If $a \epsilon S, b \epsilon S$ then $\alpha)$ $|a+b| \leqq |a|+|b|$; $\beta)$ $|a+b| \geqq |a|-|b|$;

$\gamma)$ $|ab| \Big\} \leqq \dfrac{|a|}{|b|}$.

In fact, α is an immediate consequence of $(aS, bS) \supseteq (a+b)S$. To prove β we write $a = a + b + (-e)b$, where e is the unit

26

of S; then by α and 1 we have $|a| \leqq |a+b| + |b|$. Relations γ follow from $abS \subseteq aS$, $Sab \subseteq Sb$.

3.   If $a \epsilon S$, $b \epsilon S$, $c \epsilon S$ then $|ab| + |bc| \leqq |b| + |abc|$. [6])

In fact, since $bS \supseteq bcS$, we may write $bS = bcS + dS$, where $d \epsilon S$, $|d| = |b| - |bc|$. Multiplying by a we obtain $abS = (abcS, adS)$ and therefore $|ab| \leqq |abc| + |ad| \leqq |abc| + |d| = |abc| + |b| - |bc|$, or $|ab| + |bc| \leqq |abc| + |b|$, q.e.d.

Remark. Specialising the above inequality we obtain the *"law of nullity"* of Sylvester: $|ab| \leqq |a|$, $|ab| \leqq |b|$, $|ab| \geqq |a| + |b| - |S|$.

4.   If $a \epsilon S$, $a^s \neq 0$ and $s > |a|$, then $a^t \neq 0$ for each natural number t. Since $aS \supseteq a^2 S \supseteq \ldots \supseteq a^s S$ and since $a^i S \supset a^{i+1} S$ $(i = 1, \ldots, s-1)$ would involve $|a| \geqq s$ contradictory to the assumption, it follows that for a certain i $(1 \leqq i < s)$ the relation $a^i S = a^{i+1} S$ must hold, which by successive multiplication gives (by induction) $a^i S = a^{i+k} S$ for each k, and hence $a^t \neq 0$ for each t.

5.   The theorem in 4 may be considered as a special case of the following: If $a \epsilon S$, $a_i \epsilon S$, $b_i \epsilon S$, $a_i = ab_i$ $(i = 1, \ldots, s)$, $a_1 \cdot a_2 \cdots a_s \neq 0$ and $s > |a|$ then there exists an element c of S and a naturel number t such that $c = c_1 \cdot c_2 \cdots c_t$ where the c_i are elements of the set $a_1, a_2, \ldots, a_s$ and $c^k \neq 0$ for each k [7]).

---

[6])   For matrices in a commutative field this was proved by FROBENIUS, Über den Rang einer Matrix [Sitzungsber. d. Akad. Berlin **1911**, 20—29].

[7])   This follows easily from § 2 of the paper: J. LEVITZKI, Über nilpotente Unterringe [Math. Ann. **105** (1931), 620—627].