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Non-linear difference equations

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Non-linear difference equations

by

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1. Introduction.

In the following pages an investigation is given of the analytic character of solutions (in the neighborhood of the singular point $x = \infty$) of *the non-linear n-th order difference equation*

$$\text{(A)} \qquad y(x+n) = x^{\frac{w}{\alpha}}\, a(x, y(x)) \quad (\textit{integers } w,\ \alpha;\ \alpha \geqq 1)\,[1],$$

where

$$\text{(1)} \qquad a(x, y(x)) = a(x, y(x), y(x+1), \ldots, y(x+n-1))$$

and the function $a(x, y_0, y_1, \ldots, y_{n-1})$ *is analytic in* $z\left(=x^{\frac{1}{\alpha}}\right)$, $y_0, y_1, \ldots, y_{n-1}$ *at* $(z = \infty,\ y_0 = y_1 = \ldots = y_{n-1} = 0)$ *while*

$$\text{(2)} \qquad a(x, 0, 0, \ldots, 0) = 0.$$

That is,

$$\text{(3)} \quad a(x, y_0, y_1, \ldots, y_{n-1}) = a_1(x, y_0, \ldots, y_{n-1}) + a_2(x, y_0, \ldots, y_{n-1}),$$

$$\text{(3a)} \quad a_1(x, y_0, \ldots, y_{n-1}) = b_0(x)\, y_0 + b_1(x)\, y_1 + \ldots + b_{n-1}(x)\, y_{n-1},$$

$$\text{(3b)} \quad a_2(x, y_0, \ldots, y_{n-1}) = \sum_{m=2}^{\infty} \sum_{i_0+\ldots+i_{n-1}=m} a_{i_0, i_1, \ldots, i_{n-1}}(x) y_0^{i_0} y_1^{i_1} \cdots y_{n-1}^{i_{n-1}}$$

$$(i_0, i_1, \ldots, i_{n-1} \geqq 0).$$

[1]) The fraction $\frac{w}{\alpha}$ is in its lowest terms.

where

$$b_i(x) = \sum_{\nu=0}^{\infty} b_{i,\nu}\, x^{-\frac{\nu}{\alpha}} \tag{4}$$

$$a_{i_0,\ldots,i_{n-1}}(x) = \sum_{\nu=0}^{\infty} a_{i_0,\ldots,i_{n-1};\nu}\, x^{-\frac{\nu}{\alpha}}, \tag{4a}$$

the series (4), (4a) being convergent for $|x| \geqq r(>0)$ [2]); moreover, the series in the second member of (3b) converges for

$$|x| \geqq r;\ |y_i| \leqq \varrho \quad (\varrho > 0;\ i = 0, 1, \ldots, n-1) \text{ [3])}. \tag{5}$$

The problem at hand has a significance in the theory of non-linear difference equations somewhat analogous to that which certain recent investigations due to Trjitzinsky [4]) have in the field of non-linear differential equations.

In a certain sense a non-linear difference system

$$y_i(x+1) = a_i(x, y_1(x), \ldots, y_n(x)) \quad (i = 1, 2, \ldots, n) \tag{6}$$

is equivalent to a single difference equation of finite order. This can be inferred with the aid of the following heuristic considerations. Let $A_0 = A_0(x, y_1, \ldots, y_n)$ be a function, for a moment arbitrary, of the displayed variables. On letting

$$y(x) = A_0(x, y_1(x), \ldots, y_n(x)) \tag{7}$$

we have

$$\begin{aligned} y(x+1 &= A_0(x+1, y_1(x+1), \ldots, y_n(x+1)) \\ &= A_0(x+1, a_1, \ldots, a_n) = A_1(x, y_1(x), \ldots, y_n(x)). \end{aligned}$$

Suppose

$$y(x+\nu) = A_\nu(x, y_1(x), \ldots, y_n(x)) \quad (\text{fixed } \nu \geqq 1). \tag{8}$$

With the aid of (6) we then obtain the relation

$$y(x+\nu+1) = A_{\nu+1}(x, y_1(x), \ldots, y_n(x)). \tag{8a}$$

In order to be able to obtain from (7) a succession of relations (8) ($\nu = 1, 2, \ldots$), the function A_0 must be chosen so that a transition from (8) to (8a) should be possible for $\nu = 0, 1, \ldots$ If, for

[2]) r is taken sufficiently great so that the circumference of the circle $|x| = r$ is interior the domain of analyticity of the corresponding functions.

[3]) ϱ is taken sufficiently small so that the function (3b) is analytic in $z\left(=x^{\frac{1}{\alpha}}\right)$, $y_0, \ldots, y_{n-1}$ in the closed region defined by (5).

[4]) Analytic Theory of Non-linear Singular Differential Equations. This work will appear in the Mémorial des Sciences Mathématiques, Paris; in the sequel it will be referred to as (T_1).

instance, A_0 is also so chosen that the Jacobian of $A_0, A_1, \ldots, A_{n-1}$ with respect to $y_1, y_2, \ldots, y_n$ does not vanish in a suitable domain W of the complex variables $x, y_1, \ldots, y_n$, from (8; $\nu=0, 1, \ldots, n-1$) it would be possible to obtain a set of relations

$$(9) \qquad y_j(x) = b_j(x, y(x), y(x+1), \ldots, y(x+n-1)) \qquad (j=1, 2, \ldots, n).$$

Substituting these in (8; $\nu=n$) we obtain an equation

$$(10) \qquad y(x+n) = A(x, y(x), y(x+1), \ldots, y(x+n-1))\ [5].$$

Given a system (6) it is always possible to obtain in a manner outlined above, or by a slightly modified method, a single difference equation of finite order so that, whenever solutions of the latter are known, those of the system can be constructed. In view of the facts outlined above the following can be observed. *From the results obtained for* (A) *conclusions of similar character can be inferred regarding every system* (6), *which at* $x=\infty$ *has a singular point of the same type as that of* (A) *and for which* $a_i(x, 0, \ldots, 0) = 0$ $(i=1, \ldots, n)$.

Of importance in the sequel will be the linear equation

$$(\mathrm{B}) \quad L(x, y(x)) \equiv y(x+n) - x^{\frac{w}{\alpha}} a_1(x, y(x), y(x+1), \ldots, y(x+n-1)) = 0$$

(cf. (3), (3a), (4)), related to the problem (A). This is the equation to which (A) reduces when $a_2(x, y_0, \ldots, y_{n-1}) \equiv 0$.

In the theory of linear difference equations of outstanding importance are the fundamental developments of N. E. Nörlund[6]). These depend largely on the use of Laplace integrals and convergent factorial series. Of other contributors we shall mention R. D. Carmichael, J. Horn, G. D. Birkhoff and the present author [7]).

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[5]) In practice one would of course choose A_0 as simple as possible.

[6]) Cf., for instance, N. E. Nörlund, Lecons sur les équations linéaires aux différences finies [Paris, 1929].

[7]) G. D. Birkhoff and Trjitzinsky, Analytic Theory of Singular Difference Equations [Acta Mathematica **60** (1932), 1—89]; in the sequel referred to as (BT).

Trjitzinsky, Analytic Theory of Linear q-difference Equations [Acta Mathematica **61** (1933), 1—38].

Trjitzinsky, Laplace Integrals and Factorial Series in the Theory of Linear Differential and Linear Difference Equations [Trans. Amer. Math. Soc. **37** (1935), 80—146]; in the sequel referred to as (T_2).

Trjitzinsky, Linear Difference Equations Containing a Parameter [Annali di Matematica **14** (1935/36), 181—214].

The above papers contain numerous references to the literature in the field of linear difference equations.

The equation (B) has been treated in (BT) from the point of view of the asymptotic properties of solutions (in the neighborhood of $x=\infty$). In consequence of known investigations, due to G. D. Birkhoff of the formal aspects of the theory of linear difference equations [8]) it can be said that, provided $b_0(x)$ (cf. (3a)) is not identically zero, the equation (B) has a set of n (formally linearly independent) formal solutions, each of the type

$$s(x) = e^{Q(x)} x^r \{x\}_\nu . \tag{11}$$

Here

$$Q(x) = \mu\, x \log x + q_0 x + q_1 x^{\frac{p-1}{p}} + \dots + q_{p-1} x^{\frac{1}{p}} \tag{11a}$$

$$\left(\mu = \frac{l}{p};\ \text{integers } l,\ p;\ p \geqq 1\right)$$

and

$$\{x\}_\nu = {}_0\varrho(x) + {}_1\varrho(x) \log x + \dots + {}_\nu\varrho(x) \log^\nu x \tag{12}$$

where the ${}_j\varrho(x)$ are series, in general divergent, of the form

$$ {}_j\varrho(x) = \sum_{\gamma=0}^{\infty} {}_j\varrho_\gamma x^{-\frac{\gamma}{p}} \qquad (j = 0, 1, \dots, \nu). \tag{12a}$$

The integer p can be chosen the same for all series (11).

Throughout this work it will be understood that the following assumption has been made.

Hypothesis A. *The linear equation* (B), *associated with the problem* (A), *is effectively of order* n; *that is,* $b_0(x) \not\equiv 0$ (cf. (3a)). *Moreover, amongst the functions* $Q(x)$ (cf. (11a)), *involved in the full set of formal solutions* (11) *of the equation* (B), *there is at least one for which*

$$\Re Q^{(1)}(x) \not\equiv 0 \tag{13}$$ [9]).

This hypothesis excludes precisely the case when every $Q(x)$ is of the form

$$Q(x) = q_0 x \tag{14}$$

where q_0 is a purely imaginary number.

The condition that the equation (B) should be effectively of order n is made mainly for the sake of simplicity. In fact, from

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[8]) G. D. Birkhoff, Formal Theory of Irregular Linear Difference Equations [Acta Mathematica **54** (1930), 205—246].

[9]) $\Re a$ denotes the real part of a.

the developments given in the sequel one could infer without difficulty that results of essentially the same type, as those given in the text, would hold if the linear equation (B) were allowed to be of order less than the n-th. On the other hand, the condition that at least one of the $Q(x)$ should satisfy (13) is more essential. This condition will enable construction of certain solutions of (A), using a solution of (B) as a first approximation, when the variable x is in a suitable region extending to infinity to the left or to the right [10]). Whenever all of the $Q(x)$ are of the form $q_0 x$, with q_0 a purely imaginary number, substantially different methods would have to be used.

The problem of n-th ($n \geqq 2$) order non-linear difference equations has never been considered before from the point of view of the present work [11]). On the other hand, the first order problem (under various assumptions) has been treated by a number of writers. Of the developments of the latter kind most relevant, in so far as our present point of view is concerned, is a succession of contributions due to J. Horn [12]). This author obtains formal solutions $s(x; p(x))$ as series in positive integral powers of an arbitrary periodic function $p(x)$ (of period unity). The coefficients of the various powers of the periodic function are functions which he expresses with the aid of convergent Laplace integrals, leading to expressions involving convergent factorial series (exponential summability of corresponding formal power series [13])). On the other hand, the series $s(x; p(x))$, itself, is shown to be convergent.

Now, with the problem formulated as it is in the present work, results of the type of those obtained by Horn in general will not hold. While we shall obtain formal solutions $s(x; p_1(x), p_2(x), \ldots, p_m(x))$ ($p_1(x), \ldots, p_m(x)$ arbitrary functions of period unity) as series in positive integral powers of $p_1(x), \ldots, p_m(x)$, *in general it will be impossible to express the coefficients of this series with the aid of convergent factorial series.* In fact, the linear problem (B)

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[10]) Regions of this type will be defined more precisely in the sequel.

[11]) We might mention some developments regarding questions of stability (a problem analogous to that in the theory of a type of non-linear differential systems) in connection with certain difference systems: O. Perron [Journ. reine ang. Math. **161** (1929), 41—64]; Ta Li [Acta Mathematica **63** (1934), 99—141].

[12]) For instance, Über nichtlineare Differenzengleichungen [Archiv der Math. und Physik **25** (1916), 137—148]; Über eine nichtlineare Differenzengleichung [Jahresber. D. Math. Ver. **26** (1917), 230—251]; Zur Theorie der nichtlinearen Differenzengleichungen [Math. Zeitschr. **1** (1918), 80—114].

[13]) These power series are in general divergent.

is a special case of the problem (A); on the other hand, it has been shown by Trjitzinsky (cf. (T_2)) that the formal solutions of an equation (B) are not always expressible with the aid of convergent factorial series. Accordingly, asymptotic methods will be used to investigate the character of the coefficients of the various monomials

$$p_1^{j_1}(x)p_2^{j_2}(x)\cdots p_m^{j_m}(x)$$

involved in $s(x;\, p_1(x), \ldots, p_m(x))$. Moreover, convergence of the series $s(x;\, p_1(x), \ldots, p_m(x))$ in general is not to be expected. However, we shall construct "actual solutions" (analytic, for $x \neq \infty$, in certain regions extending to infinity) which in a certain sense, to be specified precisely in the sequel, are asymptotic to the corresponding formal series.

Finally, it is to be noted that for $n = 1$ the results of the present work will continue to relate to a problem heretofore not treated — a problem to which the methods of Horn would continue to be inapplicable, unless certain additional hypotheses were made.

2. Formal solutions. (Case I.)

Let $p_1(x)$ be an arbitrary function of period unity and consider the series

(1) $s(x) = y_1(x)p_1(x) + y_2(x)p_2(x) + \ldots + y_\nu(x)p_1^\nu(x) + \ldots$

Formally

$$\text{(1a)} \qquad s^{i_\nu}(x+\nu) = p_1^{i_\nu}(x)\left(\sum_{j=1}^{\infty} y_j(x+\nu)p_1^{j-1}(x)\right)^{i_\nu}.$$

In (3b; § 1) substitute $y_i = s(x+i)$ $(i=0, 1, \ldots, n-1)$. It will follow that

$$\text{(2)} \qquad a_2(x, s(x), s(x+1), \ldots, s(x+n-1)) =$$

$$\sum_{m=2}^{\infty} p_1^m(x) \sum_{i_0+\ldots+i_{n-1}=m} a_{i_0,\ldots,i_{n-1}}(x) \prod_{\nu=0}^{n-1}\left(\sum_{j=1}^{\infty} y_j(x+\nu)p_1^{j-1}(x)\right)^{i_\nu}$$

$$= \Psi_2(x)p_1^2(x) + \Psi_3(x)p_1^3(x) + \ldots \quad (i_0, \ldots, i_{n-1} \geqq 0).$$

In the second member of (2) the expression following $p_1^m(x)$ can be arranged as a series

(3) $F_m = \Psi_{m,0}(x) + \Psi_{m,1}(x)p_1(x) + \Psi_{m,2}(x)p_1^2(x) + \ldots$

From (2) one can then obtain

$$\Psi_j(x) = \sum_{k=2}^{j} \Psi_{k,\,j-k}(x) \qquad (j=2,3,\ldots). \tag{4}$$

We now proceed to compute the $\Psi_{m,\gamma}(x)$ $(m \geqq 2; \gamma \geqq 0)$. There is a development

$$\left(\sum_{j=0}^{\infty} y_{j+1}(x+\nu)p_1^j(x)\right)^{i_\nu} = \sum_{\beta=0}^{\infty} \varphi_{i_\nu;\beta}(x+\nu)p_1^\beta(x) \tag{5}$$

where

$$\varphi_{i_\nu;\beta}(x+\nu) = \sum y_{j_1+1}(x+\nu)y_{j_2+1}(x+\nu)\cdots y_{j_{i_\nu}+1}(x+\nu) \tag{5a}$$
$$(j_1,\ldots,j_{i_\nu} \geqq 0; j_1+j_2+\ldots+j_{i_\nu}=\beta).$$

It is possible then to write

$$\prod_{\nu=0}^{n-1}\left(\sum_{j=1}^{\infty} y_j(x+\nu)p_1^{j-1}(x)\right)^{i_\nu} = \sum_{\gamma=0}^{\infty} {}_\gamma\varphi_{i_0,\,i_1,\ldots,\,i_{n-1}}(x)p_1^\gamma(x), \tag{6}$$

$${}_\gamma\varphi_{i_0,\,i_1,\ldots,\,i_{n-1}}(x) = \sum \varphi_{i_0;\beta_0}(x)\varphi_{i_1;\beta_1}(x+1)\cdots\varphi_{i_{n-1};\beta_{n-1}}(x+n-1) \tag{6a}$$
$$(\beta_0,\beta_1,\ldots,\beta_{n-1} \geqq 0;\ \beta_0+\beta_1+\ldots+\beta_{n-1}=\gamma).$$

Thus, in consequence of the definition of F_m, by (3) and by (6a) it is inferred that

$$\Psi_{m,\gamma}(x) = \sum a_{i_0,\,i_1,\ldots,\,i_{n-1}}(x)\,{}_\gamma\varphi_{i_0,\ldots,\,i_{n-1}}(x) \tag{7}$$
$$(i_0+i_1+\ldots+i_{n-1}=m;\ i_0,i_1,\ldots,i_{n-1} \geqq 0).$$

Substitution of (7) in (4) will yield

$$\Psi_j(x) = \sum_{k=2}^{j} \sum a_{i_0,\,i_1,\ldots,\,i_{n-1}}(x)\,{}_{j-k}\varphi_{i_0,\ldots,\,i_{n-1}}(x) \tag{8}$$
$$(i_0+\ldots+i_{n-1}=k;\ i_0,\ldots,i_{n-1} \geqq 0).$$

From (8), by virtue of (6a) and (5a), it is finally concluded that, for $j = 2, 3, \ldots,$

$$\Psi_j(x) = \sum_{k=2}^{j} \sum{}''' a_{i_0,\ldots,\,i_{n-1}}(x) \cdot \tag{9}$$
$$\sum{}'' \prod_{\nu=0}^{n-1} \sum{}' y_{j_1+1}(x+\nu)\,y_{j_2+1}(x+\nu)\cdots y_{j_{i_\nu}+1}(x+\nu)$$

where

$$\sum{}' = \sum \qquad (j_1, j_2, \ldots, j_{i_\nu} \geqq 0;\ j_1+j_2+\ldots+j_{i_\nu}=\beta_\nu), \tag{10}$$

$$\sum{}'' = \sum \qquad (\beta_0, \beta_1, \ldots, \beta_{n-1} \geqq 0;\ \beta_0+\beta_1+\ldots+\beta_{n-1}=j-k), \tag{11}$$

$$\sum{}''' = \sum \qquad (i_0, i_1, \ldots, i_{n-1} \geqq 0;\ i_0+i_1+\ldots+i_{n-1}=k). \tag{12}$$

This is analogous to 2 formula given in (T_1).
In particular

$$\text{(13)}\qquad \begin{aligned} \Psi_2(x) = \Psi_{2,0}(x) = \\ = \Sigma\, a_{i_0,\ldots,i_{n-1}}(x) y_1^{i_0}(x) y_1^{i_1}(x+1)\cdots y_1^{i_{n-1}}(x+n-1) \\ (i_0, i_1, \ldots, i_{n-1} \geqq 0;\ i_0 + \ldots + i_{n-1} = 2). \end{aligned}$$

On taking account of (B; § 1), in view of (3; § 1), equation (A) can be written in the form

$$\text{(14)}\qquad L(x, y(x)) = x^{\frac{w}{\alpha}} a_2(x, y(x), y(x+1), \ldots, y(x+n-1)).$$

Now L is a linear difference operator, while $p_1(x)$ in (1) is of period one. Hence substitution of (1) into (14) will yield, in consequence of (2),

$$\text{(15)}\qquad \sum_{j=1}^{\infty} p_1^j(x) L(x, y_j(x)) = x^{\frac{w}{\alpha}} \sum_{j=2}^{\infty} p_1^j(x) \Psi_j(x).$$

This leads to a succession of recursion difference equations

$$\text{(16)}\qquad L(x, y_1(x)) = 0,$$

$$\text{(17)}\qquad L(x, y_j(x)) = x^{\frac{w}{\alpha}} \Psi_j(x) \qquad (j = 2, 3, \ldots).$$

Examination of $\Psi_j(x)$, as given by (9), enables one to conclude that $\Psi_j(x)$ is independent of $y_j(x)$, $y_{j+1}(x)$, ... This fact will be signified by writing

$$\text{(18)}\qquad \Psi_j(x) = \Psi_j(x; y_1(x), y_2(x), \ldots, y_{j-1}(x) \quad (j = 2, 3, \ldots).$$

Accordingly, one may expect that with a proper choice of $y_1(x)$ a sequence of functions $y_j(x)$ $(j = 1, 2, \ldots)$ can be found so that the equations (16) and (17) are all satisfied. Corresponding to such a sequence there would be on hand a formal solution (1) of the problem (A). We shall now proceed to find such a sequence.

A linear n-th order non-homogeneous difference equation

$$\text{(19)}\qquad z(x+n) + a_1(x) z(x+n-1) + \ldots + a_n(x) z(x) = a(x) \qquad (a_n(x) \neq 0)$$

can be solved as follows. Let

$$z_1(x), z_2(x), \ldots, z_n(x)$$

constitute a full set of solutions of the homogeneous problem obtained by replacing $a(x)$ by zero. Let $(a_{i,j})$

denote a matrix of n^2 elements $a_{i,j}$ $(i, j=1, \ldots, n)$, with $a_{i,j}$ in the i-th row and j-th column. Moreover, let

$$(a_{i,j})^{-1}$$

be the inverse of the matrix $(a_{i,j})$. Now the $z_i(x)$ $(i=1, \ldots, n)$ form a full set of solutions. Accordingly, the determinant

$$|(z_i(x+j))|$$

is not identically zero. Thus, functions $\bar{z}_{i,j}(x)$ $(i, j=1, \ldots, n)$ can be defined by the matrix relation

$$(\bar{z}_{i,j}(x)) = (z_i(x+j))^{-1}.$$

Then

$$z(x) = \sum_{\lambda=1}^{n} z_\lambda(x) \underset{u=x}{\mathbf{S}} a(u)\bar{z}_{n,\lambda}(u), \tag{19a}$$

where

$$\underset{u=x+1}{\mathbf{S}} h(u) - \underset{u=x}{\mathbf{S}} h(u) = h(x) \tag{20}$$

will constitute a solution of (19) provided that the operation of summation, designated by $\underset{u=x}{\mathbf{S}}$, can be carried out.

Thus, with

$$y_{1:1}(x),\ y_{1:2}(x), \ldots, y_{1:n}(x) \tag{21}$$

denoting a full set of solutions of (16) [14]) and with the functions $\bar{y}_{i,j}(u)$ $(i, j=1, \ldots, n)$ defined by the matrix relation

$$(\bar{y}_{i,j}(u)) = (y_{1:i}(u+j))^{-1},$$

(17) can be written in the form

$$y_j(x) = \sum_{\lambda=1}^{n} y_{1:\lambda}(x) \underset{u=x}{\mathbf{S}} u^{\frac{w}{\alpha}} \Psi_j(u)\bar{y}_{n,\lambda}(u) \qquad (j=2, 3, \ldots). \tag{22}$$

Definition 1. *Let* $\{x\}_\nu$ *denote any expression* (12; § 1), *where the* ${}_j\varrho(x)$ $(j=0, 1, \ldots, \nu)$ *are series, possibly divergent, of the form* (12a; § 1). *Let* K *denote a region extending to infinity. The symbol* $[x]_\nu$ *will denote generically a function, defined in* K $(x \neq \infty)$, *such that*

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[14]) Such a set exists since by Hypothesis A (§ 1) (16) is effectively of order n.

(23) $$[x]_\nu \sim \{x\}_\nu \qquad (x \text{ in } K)$$ [15].

In consequence of the developments given in (BT) *there exists a region K satisfying conditions* (α), (β), (γ), (δ).

(α) When x is in K we have $|x| \geqq r_0$ (>0) and $x-1$ is also in K; K contains the part of the negative axis of reals for which $|x| \geqq r_0$ (a portion of this axis may enter as a part of the boundary of K).

(β) Part of the boundary of K consists of an arc of the circle $|x| = r_0$, while the rest of the boundary consists of two-non-intersecting curves, B_u, B_l, extending from the extremities of this arc and possessing limiting directions at infinity; B_u in the second quadrant and B_l in the third. Either, one of these curves is coincident (for $|x| \geqq r_0$) with the negative axis of reals — in which case the other curve recedes indefinitely from the axis — or both curves B_u, B_l recede indefinitely [16]) from this axis.

(γ) Consider the functions $Q(x)$ (cf. (11a; § 1)), associated with a full set of formal solutions (11; § 1) of the linear problem (B; § 1). The functions $\Re Q^{(1)}(x)$ (= real part of $Q^{(1)}(x)$) maintain the same ordering in K. As a matter of notation write

(24) $$\Re Q_1^{(1)} = \Re Q_2^{(1)}(x) = \ldots = \\ = \Re Q_\delta^{(1)}(x) > \Re Q_{\delta+1}^{(1)}(x) \geqq \Re Q_{\delta+2}^{(1)}(x) \geqq \ldots \geqq \Re Q_n^{(1)}(x) \\ (x \text{ in } K)$$ [17]).

(δ) The problem (B; § 1) has a full set of solutions $y_{1:i}(x)$ ($i=1, \ldots, n$), analytic in K ($x \neq \infty$), such that (with the notation of Def. 1) we have

(25) $$y_{1:i}(x) = e^{Q_i(x)} x^{r_i} [x]_{\nu(i)} \qquad (i=1, \ldots, n;\ x \text{ in } K)$$ [18]).

In the sequel, unless stated otherwise, K will denote a region satisfying the above conditions (α), ..., (δ). Henceforth, unless stated to the contrary, the set of functions (21) involved in the relations (22) will be the one referred to in these conditions.

DEFINITON 2. *Let K be a region extending to infinity, but not*

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[15]) That is, $[x]_\nu = {}_0\bar{\varrho}(x) + {}_1\bar{\varrho}(x)\log x + \ldots + {}_\nu\bar{\varrho}(x)\log^\nu x$ where the ${}_j\bar{\varrho}(x)$ are functions correspondingly asymptotic (in K) to the series ${}_j\varrho(x)$ ($j = 0, 1, \ldots, \nu$).

[16]) That is, if B_u for instance is such a curve, we have $\Im x$ (= imaginary part of x) $\to \infty$, as x recedes to infinity along B_u.

[17]) The $Q_i(x)$ correspond to the formal solutions $s_i(x)$ $\left(=\{x\}_{\nu(i)} x^r \exp Q(x);\ i = 1, 2, \ldots, n\right)$. Here $>$ may become $=$ along B_u or B_l.

[18]) That is, $y_{1:i}(x) \backsim s_i(x)$ ($i = 1, \ldots, n$) in K.

necessarily satisfying $(\alpha), \ldots, (\delta)$. *A real function* $f(x)$ *will be said to be monotone in* K *to the left, provided*

$$(26) \qquad f(x_1) \leqq f(x_2)$$

whenever x_1 *and* x_2 *are in* K, *while*

$$(26a) \qquad \Im x_1 = \Im x_2, \quad \Re x_1 < \Re x_2.$$

When in (26) $\leqq$ *is replaced by* $\geqq$ *the function* $f(x)$ *will be termed monotone in* K *to the right.*

With K denoting a region satisfying $(\alpha), \ldots, (\delta)$, it is concluded that *the functions*

$$(27) \qquad f_{1,\lambda}(x) = |e^{Q_{1,\lambda}(x)}| \qquad (Q_{1,\lambda}(x) = Q_1(x) - Q_\lambda(x))$$

are all monotone in K *to the left.* In fact, whenever x_1 and x_2 satisfy the conditions of Def. 2, in view of (24) we shall have

$$\Re Q^{(1)}_{1,\lambda}(u)du \geqq 0$$

for u on the rectilinear segment (x_1, x_2). Hence

$$\Re[Q_{1,\lambda}(x_2) - Q_{1,\lambda}(x_1)] = \int_{x_1}^{x_2} \Re Q^{(1)}_{1,\lambda}(u)du \geqq 0.$$

Accordingly $\Re Q_{1,\lambda}(x)$ is monotone in K to the left. Whence the same will be true for the function $f_{1,\lambda}(x)$.

On writing

$$(28) \qquad Q_i(x) = \mu_i x \log x + q_{i,0}x + q_{i,1}x^{\frac{p-1}{p}} + \ldots + q_{i,p-1}x^{\frac{1}{p}} \qquad (i = 1, 2, \ldots, n)$$

(cf. (11a; § 1)) *it is noted that* (24) *implies*

$$(29) \qquad \mu_1 = \mu_2 = \ldots = \mu_\delta \geqq \mu_{\delta+1} \geqq \mu_{\delta+2} \geqq \ldots \geqq \mu_n.$$

This is a consequence of the following considerations. From (28) we obtain

$$\Re Q^{(1)}_i(x) = \mu_i(1 + \log|x|) + g_i(x),$$

where $g_i(x)$ is real and $|g_i(x)|$ is bounded in K. Thus,

$$(29a) \qquad \Re Q^{(1)}_i(x) - \Re Q^{(1)}_j(x) = (\mu_i - \mu_j)(1 + \log|x|) + g_{i,j}(x)$$
$$(g_{i,j}(x) \text{ real; } |g_{i,j}(x)| \text{ bounded in } K).$$

If $\Re Q^{(1)}_i(x) = \Re Q^{(1)}_j(x)$ (in K) then necessarily $\mu_i = \mu_j$; in fact, it can be shown that we would then have

$$Q_i(x) - Q_j(x) = q_{i,j}x \quad (\Re q_{i,j} = 0).$$

If $\Re Q_i^{(1)}(x) > \Re Q_j^{(1)}(x)$ (in K) necessarily $\mu_i \geqq \mu_j$. In fact, if we had $\mu_i - \mu_j < 0$ it would follow that the function defined by (29a) would approach $-\infty$, as $|x| \to \infty$ (in K). This constitutes a contradiction to our assumption

$$\Re Q_i^{(1)}(x) - \Re Q_j^{(1)}(x) > 0 \quad (\text{in } K).$$

CASE I. *There exists a region K satisfying not only conditions* (α), (β), (γ), (δ) (*cf. the text preceding Def.* 2) *but also the following. The function*

(30) $$\left|e^{Q_1(x)}\right|$$

is monotone in K to the left (*terminology of Def.* 2) *and*

(30a) $$e^{Q_1(x)} \sim 0 \qquad (\text{in } K)$$ [19]).

Now

(31) $$\Re x \log x = |x| \left[\cos \bar{x} \log |x| - \bar{x} \sin \bar{x}\right] \quad (\bar{x}=\text{angle of } x).$$

Hence, whenever $\mu_1 \neq 0$, the dominant component of $\Re Q_1(x)$ is

(31a) $$-\mu_1 |x| \log |x|,$$

provided x is in a sufficiently close neighborhood of the negative axis of reals. Accordingly, case I cannot take place when $\mu_1 < 0$, as (30a) would not then hold. If $\mu_1 = 0$ and if, furthermore, we have

(31b) $$Q_1(x) = q_{1,0}\, x \qquad (\Re q_{1,0} = 0)$$

condition (30a) could not be satisfied [20]) in K. If $\mu_1 > 0$ consideration of (31) leads one to the conclusion that (30a) will certainly hold in every region Γ defined by the inequalities

(31c) $$\frac{\pi}{2} + \varepsilon \leqq \bar{x} \leqq \frac{3\pi}{2} - \varepsilon \qquad (|x| \geqq r_0)$$

where ε is a fixed positive number, however small. *Now* $|\exp Q_1(x)|$ *is monotone to the left in every region* (*extending to infinity to the left*) *in which*

(32) $$\Re Q_1^{(1)}(x) \geqq 0 .$$

This follows by a reasoning analogous to that previously employed

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[19]) That is, $\exp Q_1(x) \sim 0 + 0x^{-1} + 0x^{-2} + \ldots$ (in K). Here and throughout, unless the contrary is stated, asymptotic relations are in the ordinary sense (i.e., to ∞ of terms).

[20]) (30a) would fail along every line (in K) parallel to the axis of reals.

in connection with (27). On writing

$$\Re Q_1^{(1)}(x) = \mu_1(1 + \log|x|) + g_1(x)$$

it is noted that $|g_1(x)| \leqq g_1 \quad (|x| \geqq r_0)$. Hence, when $\mu_1 > 0$, (32) will be satisfied for $|x| \geqq r_0$, provided r_0 has been chosen suitably great [21]). Thus (with $\mu_1 > 0$) take a region satisfying (α), . . ., (δ). Consider the part common to such a region and to Γ (cf. (31c)) [22]). Call this common part K. In the region K the conditions of case I will hold. When $\mu_1 = 0$ and $Q_1(x)$ is not of the form (31b) it may happen that for no region (of stated type and extending to the left) are the conditions of case I all satisfied.

The following essential facts have been established.

If case I holds in a region K (satisfying (α), . . ., (δ)) necessarily $\mu_1 \geqq 0$ (also cf. (29)) *and $Q_1(x)$ cannot be of the form* (31b).

If $\mu_1 > 0$ case I will certainly hold in some region K.

We shall now proceed to investigate, for the case I, the character of the coefficients $y_j(x)$, involved in the formal solution (1). With the $y_{1:i}(x)$ $(i = 1, \ldots, n)$ denoting a set of solutions of (B; § 1), referred to in connection with (25), the function $y_1(x)$ (a solution of (16)) will be taken as

$$(33) \qquad y_1(x) = y_{1:1}(x) + c_2(x)y_{1:2}(x) + \ldots + c_\delta(x)y_{1:\delta}(x).$$

Here $c_2(x), \ldots, c_\delta(x)$ are arbitrary functions of period unity, analytic [23]) in K.

In consequence of (24)

$$(33a) \qquad Q_i(x) = Q_1(x) + \sqrt{-1}\, q_i x \quad (\text{real } q_i;\ i = 1, 2, \ldots, \delta).$$

Hence by (33) and (25), generically,

$$(33b) \quad y_1(x) = e^{Q_1(x)}\left(g_1(x)[x]_{\nu(1)} + g_2(x)[x]_{\nu(2)} + \cdots + g_\delta(x)[x]_{\nu(\delta)}\right)$$

(x in K; cf. Def. 1) where

$$(34) \quad g_i(x) = c_i(x)e^{\sqrt{-1}q_i x}x^{r_i} \quad (i = 1, \ldots, \delta;\ c_1(x) \equiv 1;\ q_1 = 0).$$

At this point it will be convenient to introduce the following definition.

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21) We make such a choice.

22) Γ can be replaced by any more extensive region, confined to the second and third quadrants, in which $\exp Q_1(x) \sim 0$.

23) Since, as stated in (β), at least one of the curves B_u, B_l recedes indefinitely from the axis of reals, the $c_i(x)$ will be analytic at least in a half plane bounded by a line parallel to the axis of reals.

DEFINITION 3. *Let q_i $(i = 1, 2, \ldots, \delta;\ q_1 = 0)$ be real numbers and let $c_2(x), \ldots, c_\delta(x)$ be arbitrary functions of period unity, analytic (for $x \neq \infty$) in a region K extending to infinity. Let functions $g_i(x)$ $(i=1, \ldots, \delta)$ be defined by* (34). *We then shall write generically*

$$(35)\qquad [x]_N^p = \Sigma\, g_1^{i_1}(x) g_2^{i_2}(x) \cdots g_\delta^{i_\delta}(x) f_{i_1, i_2, \ldots, i_n}(x)$$
$$(i_1, i_2, \ldots, i_\delta \geqq 0;\ i_1 + i_2 + \ldots + i_\delta = p;\ x \text{ in } K),$$

where p is a positive integer and

$$(35a)\qquad f_{i_1, i_2, \ldots, i_n}(x) = [x]_N \qquad (x \text{ in } K;\ \textit{cf. Def. } 1).$$

The corresponding formal expression will be designated as $\{x\}_N^p$. *Thus*

$$(35b)\qquad [x]_N^p \sim \{x\}_N^p = \Sigma\, g_1^{i_1}(x) \cdots g_\delta^{i_\delta}(x) \{x\}_N$$
$$(i_1, \ldots, i_\delta \geqq 0;\ i_1 + \ldots + i_\delta = p;\ x \text{ in } K;\ \textit{cf. Def. } 1).$$

In accordance with this notation

$$(36)\qquad y_1(x) = e^{Q_1(x)} [x]_{N(1)}^1 \qquad (x \text{ in } K)\ ^{24)}.$$

Before proceeding further the following Lemma will be stated.

LEMMA 1. *Let K denote a region extending to infinity* [25]). *Consider a function*

$$(37)\qquad f(x) = e^{Q(x)} [x]_N^p \qquad (x \text{ in } K;\ \textit{cf. Def. } 3),$$

where

$$(37a)\qquad Q(x) = \mu x \log x + {}_0q x + {}_1q x^{1-\frac{1}{p}} + \ldots + {}_{p-1}q x^{\frac{1}{p}}$$
$$(\mu \text{ rational}).$$

Then

$$(38)\qquad f(x+\nu) = e^{Q(x)} x^{\nu\mu} [x]_N^p \qquad (\nu = 0, 1, \ldots, n-1)$$

provided $x + n - 1$ is in K. Moreover, formally

$$(38a)\qquad \underset{t=x}{\mathrm{S}}\, e^{Q(t)} t^r \{t\}_N = e^{Q(x)} x^{r-\mu} \{x\}_N$$

when $\mu > 0$. On the other hand, when $\mu < 0$ we shall have

$$(38b)\qquad \underset{t=x}{\mathrm{S}}\, e^{Q(t)} t^r \{t\}_N = e^{Q(x)} x^r \{x\}_N .$$

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[24]) $N(1)$ is the greatest of the numbers $\nu(1), \nu(2), \ldots, \nu(\delta)$.

[25]) This region is to be exterior a circle $|x| = r_0$ (> 0).

Whenever $\mu = 0$

$$\text{(39)}\qquad \mathop{\mathbf{S}}_{t=x} e^{Q(t)} t^r \{t\}_N = e^{Q(x)} x^{r+h} \{x\}_N = e^{Q(x)} x^{r+1} \{x\}_N ,$$

where h *is a rational number,* $0 \leqq h \leqq 1$. *When* $Q(x) \equiv 0$ *in the last two members of* (39) $\{x\}_N$ *is sometimes replaced by* $\{x\}_{N+1}$.

A determination of h more exact than that given above is possible, but is not essential for our purposes. On the other hand, employment of a more precise value of h would considerably complicate the subsequent developments. The above Lemma can be proved with the aid of certain considerations of a formal character; it is essentially a consequence of Birkhoff's work in the theory of linear difference equations (formal aspects).

We shall now determine the character of the function $\Psi_2(x)$ (cf. (13)). By (36), in consequence of Lemma 1, it follows that

$$\text{(40)}\qquad y_1(x+\nu) = e^{Q_1(x)} x^{\nu\mu_1} [x]^1_{N(1)}$$

$$(\nu = 0, 1, \ldots, n-1; x+n-1 \text{ in } K).$$

Now

$$\text{(41)}\qquad [x]^u_l [x]^{u_1}_{l_1} = [x]^{u+u_1}_{l+l_1} .$$

Consequently

$$y_1^{i_\nu}(x+\nu) = e^{i_\nu Q_1(x)} x^{\nu i_\nu \mu_1} [x]^{i_\nu}_{i_\nu N(1)} .$$

Thus, since in (13) $i_0 + \ldots + i_{n-1} = 2$ $(i_0, \ldots, i_{n-1} \geqq 0)$, it is inferred that

$$\text{(42)}\qquad \prod_{\nu=0}^{n-1} y_1^{i_\nu}(x+\nu) = e^{2Q_1(x)} x^{\mu_1 i^1} [x]^2_{2N(1)} \qquad (x+n-1 \text{ in } K)$$

where

$$\text{(42a)}\qquad i^1 = i_1 + 2i_2 + \ldots + (n-1)i_{n-1} \leqq 2(n-1).$$

Now $\mu_1 \geqq 0$ (cf. italics preceding (33)). Hence by (42a) i^1 in (42) can be replaced by $2(n-1)$. Since

$$a_{i_0, \ldots, i_{n-1}}(x) = [x]_0 \qquad \text{(in } K\text{)},$$

from (13) we then can obtain the relation

$$\text{(43)}\qquad \Psi_2(x) = e^{2Q_1(x)} x^{2(n-1)\mu_1} [x]^2_{2N(1)} \quad (x+n-1 \text{ in } K).$$

The following notation will be introduced. K^1 *will denote the region formed by translating the boundary of* K *to the left parallel to the axis of reals through the distance* $n-1$.

It is observed that (43) holds for x in K'.

Suppose now that K is a region with respect to which the conditions (α), ..., (δ) hold. We shall substitute (25) in (22) *without, for a moment, assuming that the conditions of case* I *are satisfied in* K. With the aid of Lemma 1 a direct computation will yield the result

$$(44) \qquad y_j(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda} [x]_{\nu(\lambda)} S_\lambda^x(\Psi_j(u)) \qquad (j=2, 3, \ldots)$$

where

$$(44a) \qquad S_\lambda^x(\Psi_j(u)) = \mathop{\mathbf{S}}_{u=x} \Psi_j(u) e^{-Q_\lambda(u)} u^{-r_\lambda+\overline{m}} [u]_{q(\lambda)}.$$

Here $\overline{m}$ is a rational number ($\overline{m} \geqq 0$) [26].

We now assume case I and proceed to calculate $y_2(x)$ with the aid of (44; $j=2$) and (43). The summand involved in (44a) will be of the form

$$(45) \quad {}_2h_\lambda(u) = e^{2Q_1(u)-Q_\lambda(u)} u^{2(n-1)\mu_1 - r_\lambda + \overline{m}} [u]^2_{2N(1)+q(\lambda)} \qquad (u \text{ in } K^1).$$

In view of Def. 3 (cf. (33a), (34)) from (45) we obtain

$$(45a) \quad {}_2h_\lambda(u) = \Sigma B_{i_1, \ldots, i_\delta}(u) \qquad (i_1 + \ldots + i_\delta = 2;\ i_1, \ldots, i_\delta \geqq 0).$$

Here

$$(45b) \qquad B_{i_1,\ldots,i_\delta}(u) = c_1^{i_1}(u) \cdots c_\delta^{i_\delta}(u)\, e^{2Q_1(u)-Q_\lambda(u)+\sqrt{-1}uq(i_1,\ldots,i_\delta)} \cdot$$
$$\cdot\, u^{2(n-1)\mu_1 - r_\lambda + \overline{m} + r(i_1,\ldots,i_\delta)} \cdot [u]_{2N(1)+q(\lambda)} \qquad (u \text{ in } K^1)$$

where

$$(45c) \quad q(i_1, \ldots, i_\delta) = i_1 q_1 + i_2 q_2 + \ldots + i_\delta q_\delta;$$
$$r(i_1, \ldots, i_\delta) = i_1 r_1 + i_2 r_2 + \ldots + i_\delta r_\delta.$$

In the function

$$Q(u) = 2Q_1(u) - Q_\lambda(u) + \sqrt{-1}uq(i_1, \ldots, i_\delta) = \mu\, u \log u + \ldots$$

we have

$$(45d) \qquad \mu = 2\mu_1 - \mu_\lambda = \mu_1 + (\mu_1 - \mu_\lambda) \geqq \mu_1 (\geqq 0) \qquad (\lambda = 1, 2, \ldots, n)$$

in consequence of (29) and in view of the italicized statement preceding (33). We shall write

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[26]) Neither the value of $\overline{m}$ nor those of the integers $q(\lambda)$ ($\lambda = 1, \ldots, n$) need to be known precisely; $\overline{m}$ depends essentially on $\frac{w}{\alpha}$ (cf. (22)) and on the μ_i ($i = 1, \ldots, n$).

(45e) $$\underset{u=x}{\mathbf{S}}\, {}_2h_\lambda(u) = \Sigma \underset{u=x}{\mathbf{S}}\, B_{i_1, \ldots, i_\delta}(u) \quad (x \text{ in } K').$$

In evaluating the "sums", here involved, the following method will be used:

(46) $$\underset{u=x}{\mathbf{S}}\, B(u) = B(x-1) + B(x-2) + \ldots \qquad (x \text{ in } K').$$

The applicability of this method is due to the fact that

$$\left|e^{2Q_1(u)-Q_\lambda(u)}\right| = \left|e^{Q_1(u)}\right|\left|e^{Q_1(u)-Q_\lambda(u)}\right|$$

is a product of two functions monotone in K' to the left (cf. Def. 2 and the statements in connection with (27) and (30)), while in K' the asymptotic relation (30a) is satisfied. The function

(46a) $$\underset{u=x}{\mathbf{S}}\, B_{i_1, \ldots, i_\delta}(x)$$

so defined in K' will be analytic in K' ($x \neq \infty$). Moreover, the asymptotic form of this function will be that indicated in Lemma 1, namely

(46b) $$\begin{aligned} &c_1^{i_1}(x) \cdots c_\delta^{i_\delta}(x) e^{2Q_1(x)-Q_\lambda(x)+\sqrt{-1}xq(i_1, \ldots, i_\delta)} \cdot \\ &\cdot x^{2(n-1)\mu_1 - r_\lambda + \bar{m} + r(i_1, \ldots, i_\delta)} x^{h_2}[x]_{2N(1)+q(\lambda)} \end{aligned} \qquad (x \text{ in } K').$$

When $\mu_1 = 0$ *we may take* h_2 *as a rational number* h *independent of* λ *and such that* $0 \leqq h \leqq 1$ (cf. Lemma 1). On the other hand, *when* $\mu_1 > 0$ *we let* $h_2 = -\mu_1$ [27]). Thus, in view of (45e), (45b), (45a), (45) and (44a), it is inferred that

(47) $$S_\lambda^x(\Psi_2(u)) = e^{2Q_1(x)-Q_\lambda(x)} x^{2(n-1)\mu_1 - r_\lambda + \bar{m} + h_2}[x]^2_{2N(1)+q(\lambda)}$$

(x in K'; $\lambda = 1, \ldots, n$). Whence by (44)

(48) $$y_2(x) = e^{2Q_1(x)} x^{\nu_2}[x]^2_{N(2)} \qquad (x \text{ in } K'),$$

(48a) $$\nu_2 = 2(n-1)\mu_1 + \bar{m} + h_2 .$$

Now by (17) $L(x, y_2(x)) = x^{\frac{w}{\alpha}} \Psi_2(x)$; that is, by (B; § 1) and by (3a; § 1),

(49) $$\begin{aligned} y_2(x+n) &= \\ &= x^{\frac{w}{\alpha}}[b_0(x)y_2(x) + \ldots + b_{n-1}(x)\, y_2(x+n-1)] + x^{\frac{w}{\alpha}}\Psi_2(x) \end{aligned}$$

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[27]) According to (38a) we could take $h_2 = -(2\mu_1 - \mu_2)$. One then could write $x^{h_2} = x^{-\mu_1} x^{-(\mu_1-\mu_\lambda)} = x^{-\mu_1}[x]_0$, since the rational number $\mu_1 - \mu_\lambda \geqq 0$ ($\lambda = 1, 2, \ldots, n$); h_2 is independent of $i_1, \ldots, i_\delta$.

where the $b_i(x)$ are given by (4; § 1). Whence

(49a) $$y_2(x+\nu) =$$
$$= (x-n+\nu)^{\overline{\alpha}^{w}}[b_0(x-n+\nu)\, y_2(x-n+\nu) + \\ + \dots + b_{n-1}(x-n+\nu)\, y_2(x+\nu-1)] \\ + (x-n+\nu)^{\overline{\alpha}^{w}}\Psi_2(x-n+\nu) \qquad (\nu = 0, 1, \dots, n-1).$$

The leading term in $x^{\overline{\alpha}^{w}} b_i(x)$ is x^{c_i} $(c_i \lesseqgtr (n-i)\mu_1)$. From (49a; $\nu=1$) we find the form of $y_2(x+1)$ in K'. Using (48) and substituting the known form of $y_2(x+1)$ in the second member of (49a; $\nu=2$) the character of $y_2(x+2)$ throughout K' can be determined. We then substitute the known forms of $y_2(x+1)$, $y_2(x+2)$ in (49a; $\nu=3$) and with the aid of (48) establish the asymptotic character of $y_2(x+3)$ (x in K'). After a finite number of steps the asymptotic forms of the functions

$$y_2(x+\nu) \qquad (\nu = 0, 1, \dots, n-1)$$

are determined for x in K'. To carry this out at each step the asymptotic form of $\Psi_2(x-n+\nu)$ ($\nu=0, 1, \dots, n-1$; x in K') is employed. This form is known by (43) [28]. Following these lines it is shown that necessarily (48) holds for x in K [29]. That is,

(50) $$y_2(x+\nu) = e^{2Q_1(x)} x^{y_2+2\nu\mu_1} [x]^2_{N(2)}$$
$$(\nu = 0, 1, \dots, n-1;\ x \text{ in } K').$$

Assume now that, for x in K' and a value of $j(\geqq 3)$, we have

(51) $$y_r(x+\nu) = e^{rQ_1(x)} x^{y_r+r\nu\mu_1} [x]^r_{N(r)}$$
$$(\nu = 0, 1, \dots, n-1;\ r = 1, 2, \dots, j-1),$$

(51a) $$\Psi_r(x) = e^{rQ_1(x)} x^{\Psi_r} [x]^r_{k(r)} \qquad (r = 1, 2, \dots, j-1),$$

where

(51b) $$y_r = \Psi_r + \bar{m} + h_r = \alpha_0 r^2 + \alpha_1 r - \alpha_2,$$

(51c) $h_r = h$ (*for* $\mu_1=0$), [30] $h_r = -(r-1)\mu_1$ (*for* $\mu_1>0$)

and, for $\mu_1 > 0$,

(52) $$\alpha_0 = (n-2)\frac{\mu_1}{2},\ \alpha_1 = n\frac{\mu_1}{2} + \bar{m},\quad \alpha_2 = (n-1)\mu_1 + \bar{m}$$

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[28]) (43) holds for x in K'; on the other hand, when x is in K' the points $x - n + \nu$ ($\nu = 0, 1, \dots, n-1$) will also be in K'.

[29]) Cf. the statement subsequent to (43).

[30]) This is the same number h as involved in the text subsequent to (46b).

while, for $\mu_1 = 0$,

(52a) $$\alpha_0 = 0, \quad \alpha_1 = \bar{m} + h, \quad \alpha_2 = \bar{m} + h.$$

The relations (51), . . ., (52a) have been established for $j = 3$; that is, for $r = 1,2$. We shall now carry out the induction. With the aid of (51) the asymptotic form of $\Psi_j(x)$ can be inferred by virtue of (9), (10), (11) and (12) (cf. (18)). On writing

$$y_{j_r+1}(x+\nu) = e^{(j_r+1)Q_1(x)} x^{y_{j_r+1}+(j_r+1)\nu\mu_1} [x]^{j_r+1}_{N(j_r+1)}$$

it follows that

(53) $$\prod_{r=1}^{i_\nu} y_{j_r+1}(x+\nu) = \\ = e^{(\beta_\nu+i_\nu)Q_1(x)} x^{\Sigma y_{j_r+1}+(\beta_\nu+i_\nu)\nu\mu_1} [x]^{\beta_\nu+i_\nu}_{N(j_1,j_2,\ldots,j_{i_\nu})}$$

since, by (10), $j_1 + j_2 + \ldots + j_{i_\nu} = \beta_\nu$. Now, in view of (51b) and since by (52) and (52a) we have $\alpha_0 + \alpha_1 - \alpha_2 = 0$, it follows that

(53a) $$y_{j_r+1} = \alpha_0 j_r^2 + (2\alpha_0 + \alpha_1) j_r .$$

It is noted that, *subject to the condition that the* x_i $(i = 1, 2, \ldots, m)$ *be non-negative integers such that* $x_1 + x_2 + \ldots + x_m = K$ $(\geqq 0)$, *the maximum value of* $x_1^2 + x_2^2 + \ldots + x_m^2$ *will be* K^2. Accordingly, in view of (10) and (53a) and provided $\alpha_0 \geqq 0$, it is immediately inferred that

(53b) $$\Sigma\, y_{j_r+1} \leqq \alpha_0 \beta_\nu^2 + (2\alpha_0 + \alpha_1)\beta_\nu .$$

The two members in (53b) are rational numbers. Hence the second member of (53) is of the form

(54) $$e^{(\beta_\nu+i_\nu)Q_1(x)} x^{\alpha_0\beta_\nu^2 + (2\alpha_0+\alpha_1)\beta_\nu + \nu(\beta_\nu+i_\nu)\mu_1} [x]^{\beta_\nu+i_\nu}_{N'_{\beta_1,\ldots,\beta_{n-1}}},$$

where $N'_{\beta_1,\ldots,\beta_{n-1}}$ is the greatest of the numbers $N(j_1, \ldots, j_{i_\nu})$ [31]) The summation (with respect to $j_1, \ldots, j_{i_\nu}$) of (10), extended over (53), will accordingly yield a function F_ν of the form (54) (x in K'; $\nu = 0, 1, \ldots, n-1$).

Now, by (11) and (12) and in consequence of the italicized statement subsequent to (53a),

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[31]) with $j_1, \ldots, j_{i_\nu}$ subject to the conditions of (10).

$$(54a)\qquad \sum_0^{n-1} \beta_\nu = j - k, \quad \sum_0^{n-1} i_\nu = k, \quad \sum_0^{n-1} \beta_\nu^2 \leqq (j-k)^2,$$
$$\sum_0^{n-1} \nu(\beta_\nu + i_\nu) \leqq (n-1)j.$$

By (54) and (54a)

$$(55)\qquad \prod_{\nu=0}^{n-1} F_\nu = e^{jQ_1(x)} x^{\alpha_0(j-k)^2 + (2\alpha_0+\alpha_1)(j-k)+(n-1)j\mu_1} \cdot$$
$$\cdot [x]^j_{N''_{\beta_1, \ldots, \beta_{n-1}}} = F_{\beta_1, \ldots, \beta_{n-1}}.$$

Thus, with $N'''_{i_0, \ldots, i_{n-1}}$ denoting the greatest of the numbers $N''_{\beta_1, \ldots, \beta_{n-1}}$, it is observed that

$$\Sigma'' F_{\beta_1, \ldots, \beta_{n-1}} = F^{i_0, \ldots, i_{n-1}}$$

(cf. (11)) is a function of the same form as the second member of (55), except that $N''_{\beta_1, \ldots, \beta_{n-1}}$ is replaced by $N'''_{i_0, \ldots, i_{n-1}}$. Let ${}_kN$ denote the greatest of the numbers $N'''_{i_0, \ldots, i_{\nu-1}}$ $(i_1, \ldots, i_{n-1}$ subject to the conditions of (12)). It is noted that

$${}_kF = \Sigma''' a_{i_0, \ldots, i_{n-1}}(x)\, F^{i_0, \ldots, i_{n-1}} \qquad \text{(cf. (12))}$$

is a function whose asymptotic form for x in K' is given by the second member of (55) with $N''_{\beta_1, \ldots, \beta_{n-1}}$ replaced by ${}_kN$. Let $k(j)$ be the greatest of the numbers ${}_kN$ $(k = 2, \ldots, j)$. It is then inferred that [32])

$$(56)\qquad \Psi_j(x) = \sum_{k=2}^{j} {}_kF = e^{jQ_1(x)} x^{\Psi_j} [x]^j_{k(j)} \qquad (x \text{ in } K'),$$

where Ψ_j is the maximum of the numbers

$$(56a)\qquad l_{j,k} = \alpha_0(j-k)^2 + (2\alpha_0+\alpha_1)(j-k) + (n-1)j\mu_1 \qquad (k = 2, 3, \ldots, j).$$

When $\mu_1 = 0$, by (52a), $l_{j,k} = \alpha_1(j-k)$ $(k = 2, \ldots, j)$. Thus,

$$(57)\qquad \Psi_j = (\bar{m} + h)(j-2) \qquad (\text{for } \mu_1 = 0).$$

When $\mu_1 > 0$ the maximum of $l_{j,k}$ $(k = 2, 3, \ldots, j)$ is attained for $k = 2$. Whence in consequence of (52) one may write

---

[32]) The precise nature of the dependence of $k(j)$ (of $N(j)$, as well) on is not essential for our purposes. In general, $\lim k(j) = \infty$ and $\lim N(j) = \infty$, as $j \to \infty$.

(57a)
$$\Psi_j = \alpha_0 j^2 + \left(\mu_1 + n\frac{\mu_1}{2} + \overline{m}\right)j - 2\alpha_1$$
$$= \alpha_0 j^2 + (\alpha_1 + \mu_1)j - 2\alpha_1 \qquad (\text{for } \mu_1 > 0).$$

The results (57), (57a) are precisely in agreement with what becomes of the relations (51), . . ., (52a) when j is replaced by $j+1$. To complete the induction it remains to prove that (51) holds when j is replaced by $j+1$; that is, for $r=j$. For the present we continue to assume that $\alpha_0 \geqq 0$. To establish the form of $y_j(x)$ use will be made of (44) where $\Psi_j(u)$ is given by (56). On designating the summand involved in (44a) by ${}_jh_\lambda(u)$, it will follow that

(58)
$$ {}_jh_\lambda(u) = e^{jQ_1(u)-Q_\lambda(u)} u^{\Psi_j - r_\lambda + \overline{m}} [u]^j_{k(j)+q(\lambda)} \qquad (u \text{ in } K').$$

Furthermore

(58a)
$$\mathop{\mathbf{S}}_{u=x} {}_jh_\lambda(u) = S^x_\lambda(\Psi_j(u)) = \Sigma \mathop{\mathbf{S}}_{u=x} B_{i_1,\ldots,i_\delta}(u)$$
$$(i_1, \ldots, i_\delta \geqq 0;\ i_1 + \ldots + i_\delta = j)$$

where

(58b)
$$B_{i_1,\ldots,i_\delta}(u) =$$
$$= c_1^{i_1}(u) \cdots c_\delta^{i_\delta}(u) e^{Q(u)} u^{\Psi_j - r_\lambda + \overline{m} + r(i_1,\ldots,i_\delta)} [u]_{k(j)+q(\lambda)},$$

(58c)
$$Q(u) = jQ_1(u) - Q_\lambda(u) + \sqrt{-1}uq(i_1, \ldots, i_\delta) =$$
$$= (j\mu_1 - \mu_\lambda)u \log u + \ldots$$

and the $q(i_1, \ldots, i_\delta)$, $r(i_1, \ldots, i_\delta)$ are given by (45c), subject to the conditions

$$i_1, \ldots, i_\delta \geqq 0; \quad i_1 + \ldots + i_\delta = j.$$

Now

(59)
$$\mu = j\mu_1 - \mu_\lambda = (j-1)\mu_1 + (\mu_1 - \mu_\lambda) \geqq (j-1)\mu_1 (\geqq 0) \qquad (\lambda = 1, \ldots, n)$$

for the same reasons in consequence of which (45d) holds. Moreover,

(59a)
$$\left| e^{jQ_1(u)-Q_\lambda(u)} \right| = \left| e^{Q_1(u)} \right|^{j-1} \left| e^{Q_1(u)-Q_\lambda(u)} \right|$$

is monotone in K' to the left (Def. 2). Thus, in view of (30a), the various ,,sums'' of the last member of (58a) can be evaluated according to the method (46). The asymptotic form of the function

(59b) $$\underset{u=x}{\mathbf{S}}\, B_{i_1,\ldots,i_\delta}(u) \qquad (x \text{ in } K'),$$

so defined, would be that obtained by the corresponding formal "summation" according to Lemma 1. Whence this form will be

(59c) $$c_1^{i_1}(x)\cdots c_\delta^{i_\delta}(x)e^{Q(x)}x^{\psi_j-r_\lambda+\bar{m}+r(i_1,\ldots,i_\delta)}x^{h_j}[x]_{k(j)+q(\lambda)}$$

(cf. (58b), (58c)), where

(59d) $$h_j = h \text{ (for } \mu_1 = 0); \; h_j = -(j-1)\mu_1 \quad (\text{for } \mu_1 > 0).$$

Here h is independent of j and λ and is identical with the number so denoted in the italics subsequent to (46b). When $\mu_1 > 0$ it is possible to take $h_j = -(j\mu_1 - \mu_\lambda)$ (cf. Lemma 1). Now

$$x^{-(j\mu_1-\mu_\lambda)} = x^{-(j-1)\mu_1}x^{-(\mu_1-\mu_\lambda)} = x^{-(j-1)\mu_1}[x]_0$$

since, by (59), $\mu_1 - \mu_\lambda$ is a non-negative number. Also it is noted that for some λ ($\lambda = 1$, certainly) $j\mu_1 - \mu_\lambda = (j-1)\mu_1$. Hence we can take $h_j = -(j-1)\mu_1$ (for $\mu_1 > 0$) — and this will entail no loss of précision. Since (59b) has in K' the form (59c), in consequence of (58a) it is inferred that

(59e) $$S_\lambda^x(\Psi_j(u)) = e^{jQ_1(x)-Q_\lambda(x)}x^{\psi_j-r_\lambda+\bar{m}+h_j}[x]^j_{k(j)+q(\lambda)}$$

(x in K'; $\lambda = 1, \ldots, n$). Whence, by (44),

(60) $$y_j(x) = e^{jQ_1(x)}x^{y_j}[x]^j_{N(j)} \qquad (x \text{ in } K'),$$

where

(60a) $$y_j = \Psi_j + \bar{m} + h_j\,.$$

When $\mu_1 > 0$, by (60a), (59d) and (57a) we obtain

(60b) $$y_j = \alpha_0 j^2 + \alpha_1 j - \alpha_2$$

where the numbers α_0, α_1, α_2 are defined by (52). When $\mu_1 = 0$ from (60a), (59d) and (57) it will follow that (60b) holds with α_0, α_1, α_2 defined by (52a). The relations

(60c) $$y_j(x+\nu) = e^{jQ_1(x)}x^{y_j+j\nu\mu_1}[x]^j_{N(j)}$$
$$(\nu = 0, 1, \ldots, n-1;\ x \text{ in } K')$$

can be proved with the aid of the equalities

(61) $$y_j(x+\nu) = (x-n+\nu)^{\frac{w}{\alpha}}[b_0(x-n+\nu)y_j(x-n+\nu) + \ldots$$
$$+ b_{n-1}(x-n+\nu)y_j(x+\nu-1)] + (x-n+\nu)^{\frac{w}{\alpha}}\Psi_j(x-n+\nu)$$
$$(\nu = 0, 1, \ldots, n-1)$$

following the lines employed in the derivation of (50) with the aid of (49a).

Thus, for the case when $\alpha_0 \geqq 0$ (cf. (52)), the induction is complete. *The relations* (51), . . ., (52a) *hold* (*in this case*) *for all values of* j ($j = 3, 4, \ldots$).

The inequality $\alpha_0 \geqq 0$ fails (with $\mu_1 \geqq 0$) if and only if

(62) $$\mu_1 > 0, \quad n = 1.$$

Under (62) we obtain

(62a) $$\Psi_j(x) = \sum_{k=2}^{j} a_k(x) \Sigma\, y_{j_1+1}(x) y_{j_2+1}(x) \cdots y_{j_k+1}(x)$$
$$(j_1, \ldots j_k \geqq 0; j_1 + \ldots + j_k = j - k)$$

for $j = 2, 3, \ldots$ Moreover [33]),

(62b) $$y_j(x) = e^{Q_1(x)} x^{r_1} [x]_0 \underset{u=x}{\mathrm{S}}\, e^{-Q_1(u)} u^{-r_1} [u]_0 \Psi_j(u) \quad (j = 2, 3, \ldots).$$

With $y_1(x) = [x]_0^1 \cdot \exp Q_1(x)$, it follows that $\Psi_2(x) = a_2(x) y_1^2(x)$ is of the form $[x]_0^2 \cdot \exp 2Q_1(x)$. For $j = 2$ the summand in (62b) will be

(63) $$e^{Q_1(u)} u^{-r_1} [u]_0^2 .$$

Thus with the aid of Lemma 1 one can infer that the „sum" involved in (62b) is

$$e^{Q_1(x)} x^{-r_1-\mu_1} [x]_0^2 .$$

Whence by (62b)

(63a) $$y_2(x) = e^{2Q_1(x)} x^{-\mu_1} [x]_0^2 .$$

Thus the constants y_1, y_2, Ψ_2 have the values

(64) $$y_1 = 0, \quad y_2 = -\mu_1; \quad \Psi_2 = 0 .$$

Assume that for x *in* K *and a value of* $j (\geqq 3)$ *we have*

(65) $$y_r(x) = e^{rQ_1(x)} x^{y_r} [x]_0^r \qquad (r = 1, 2, \ldots, j-1),$$

(65a) $$\Psi_r(x) = e^{rQ_1(x)} x^{\Psi_r} [x]_0^r \qquad (r = 2, \ldots, j-1),$$

where

(66) $$y_r = -\mu_1 r + \mu_1, \quad \Psi_r = 0.$$

For $j = 3$ this has been established. The form of the function

---

[33]) When (62) holds $\dfrac{w}{\alpha} = \mu_1$ and $\overline{m} = 0$.

$\Psi_j(x)$ can be derived with the aid of (65; $r=1, 2, \ldots, j-1$) and (62a). We have

$$\prod_{r=1}^{k} y_{j_r+1}(x) = \prod_{r=1}^{k} e^{(j_r+1)Q_1(x)} x^{-\mu_1 j_r} [x]_0^{j_r} \tag{67}$$

$$= e^{jQ_1(x)} x^{-\mu_1(j-k)} [x]_0^{j-k} \qquad (x \text{ in } K)$$

since $j_1 + \ldots + j_k = j - k$ (by (62a)). The function

$$a_k(x) \sum \prod_{r=1}^{k} y_{j_r+1}(x) \tag{67a}$$

$$(j_1, \ldots, j_r \geqq 0;\ j_1 + \ldots + j_k = j - k)$$

will possess the asymptotic form of the last member of (67). Since the rational numbers $-\mu_1(j-k)$ ($k=2, \ldots, j$) satisfy the inequalities

$$-\mu_1(j-k) \leqq 0,$$

with the equality sign taking place for $k = j$, in view of the form of (67a) from (62a) one can infer that

$$\Psi_j(x) = e^{jQ_1(x)} [x]_0^j \qquad (x \text{ in } K). \tag{67b}$$

In consequence of (67b) the summand involved in (62b) is seen to have the form

$$e^{(j-1)Q_1(u)} u^{-r_1} [u]_0^j \qquad (u \text{ in } K).$$

As $(j-1)Q_1(u) = (j-1)\mu u \log \mu + \ldots$, it is observed that the "sum" of (62b) is

$$e^{(j-1)Q_1(x)} x^{-r_1-(j-1)\mu_1} [x]_0^j \qquad (x \text{ in } K).$$

Whence

$$y_j(x) = e^{jQ_1(x)} x^{-j\mu_1+\mu_1} [x]_0^j \qquad (x \text{ in } K).$$

Accordingly, for x in K, the relations (65), (65a), (66) hold for all j ($j=3, 4, \ldots$); that is, they are valid for $r = 1, 2, \ldots$

LEMMA 2. *Consider the problem* (A; § 1) *under the Hypothesis* A (§ 1). *There exists then a region K* (*extending to the left*) *satisfying the conditions* (α), (β), (γ), (δ) (*cf. text in connection with and preceding to* (24), (25)).

When case I *is on hand* (*cf. italics in connection with* (30), (30a) *and K denotes a region referred to in the formulation of this case, the equation* (A) *will have a formal solution*

$$s(x) = y_1(x)p_1(x) + y_2(x)p_1^2(x) + \ldots + y_j(x)p_1^j(x) + \ldots, \tag{68}$$

where $y_1(x)$ *is a solution of the linear problem* (B; § 1) *and is given by* (33), *and the* $y_j(x)$ *are of the form*

$$y_j(x) = e^{jQ_1(x)}\, x^{\gamma_j}\, [x]^j_{N(j)} \tag{68a}$$
$$(j = 1, 2, \ldots; \quad x \text{ in } K; \text{ cf. Def. } 3)\,^{34}).$$

In (68) *and* (68a) δ *arbitrary functions*

$$p_1(x),\ c_2(x),\ c_3(x), \ldots, c_\delta(x),$$

each of period unity, are involved. Moreover, the numbers γ_j *are of the form*

$$\gamma_j = \alpha_0 j^2 + \alpha_1 j - \alpha_2 . \tag{69}$$

In (69), *when* $n = 1$ *and* $\mu_1 \neq 0$ (*i.e.* $\mu_1 > 0$),

$$\alpha_0 = 0, \quad \alpha_1 = -\mu_1, \quad \alpha_2 = -\mu_1 . \tag{69a}$$

When $n \geqq 2$ *and* $\mu_1 \neq 0$ (*i.e.* $\mu_1 > 0$)

$$\alpha_0 = (n-2)\frac{\mu_1}{2}, \ \alpha_1 = n\frac{\mu_1}{2} + \bar{m}, \ \alpha_2 = (n-1)\mu_1 + \bar{m} \tag{69b}$$

On the other hand, when $\mu_1 = 0$ *and* $n \geqq 1$,

$$\alpha_0 = 0, \quad \alpha_1 = \bar{m} + h, \quad \alpha_2 = \bar{m} + h. \tag{69c}$$

Here the non-negative rational number $\bar{m}$ *is the one involved in* (44a), *and* h *is a rational number* (*independent of* j) *such that* $0 \leqq h \leqq 1$.

3. Formal solutions. (Case II.)

In the last section formal solutions (generally divergent) were obtained in a region K extending „to the left". In a linear problem the corresponding results „on the right" could be inferred almost immediately. This is not the case for the non-linear problem on hand. In this section an analogue of Lemma 2 (§ 2) will be obtained with K denoting a certain region extending to the right.

In view of the developments given in (BT) *there exists a region satisfying conditions* $(\bar{\alpha})$, $(\bar{\beta})$, $(\bar{\gamma})$, $(\bar{\delta})$.

$(\bar{\alpha})$ When x is in K we have $|x| \geqq r_0 (> 0)$ and $x + 1$ is also in K, K contains a part of the positive axis of reals [35]).

---

[34]) In applying Def. (3) the numbers q_i therein involved are those from (33a); moreover, the region K is identical with the one so designated in our Lemma.

[35]) This part might be a component of the boundary of K.

($\bar{\beta}$) The wording of this condition is the same as that of (β) (§ 2), except that B_l is to be in the fourth quadrant and B_u in the first and reference is made to the positive axis of reals.

($\bar{\gamma}$) Consider the functions $Q(x)$ (cf. (11a: § 1)), associated with a full set of formal solutions (11; § 1) of the linear problem (B; § 1). The functions $\Re Q^{(1)}(x)$ maintain the same ordering in K. As a matter of notation write

$$
\begin{aligned}
(1) \qquad & \Re Q_1^{(1)}(x) = \Re Q_2^{(1)}(x) = \ldots = \\
& = \Re Q_\delta^{(1)}(x) < \Re Q_{\delta+1}^{(1)}(x) \leqq \Re Q_{\delta+2}^{(1)}(x) \leqq \ldots \leqq \Re Q_n^{(1)}(x)
\end{aligned}
$$

(x in K; cf. foot-note in connection with (24; § 2)).

($\bar{\delta}$) The problem (B; § 1) has a full set of solutions $y_{1:i}(x)$ $(i = 1, \ldots, n)$ analytic in K $(x \neq \infty)$, such that

$$
(2) \qquad y_{1:i}(x) = e^{Q_i(x)}\, x^{r_i}[x]_{\nu(i)} \quad (i = 1, \ldots, n;\ x \text{ in } K).
$$

In consequence of (1) *the functions*

$$
(3) \qquad f_{1,\lambda}(x) = \left|e^{Q_1(x) - Q_\lambda(x)}\right| \qquad (\lambda = 1, \ldots, n)
$$

are all monotone in K to the right (Def. 2 (§ 2)). If we write

$$
(4) \qquad Q_i(x) = \mu_i x \log x + q_{i,0}x + q_{i,1}x^{\frac{p-1}{p}} + \ldots + q_{i,p-1}x^{\frac{1}{p}}
$$

it is observed that necessarily

$$
(5) \qquad \mu_1 = \mu_2 = \ldots = \mu_\delta \leqq \mu_{\delta+1} \leqq \mu_{\delta+2} \leqq \ldots \leqq \mu_n.
$$

CASE II. *There exists a region K satisfying the above conditions* ($\bar{\alpha}$), ..., ($\bar{\delta}$) *as well as the following. The function*

$$
(6) \qquad \left|e^{Q_1(x)}\right|
$$

is monotone in K to the right (Def. 2 (§ 2)) *and*

$$
(6a) \qquad e^{Q_1(x)} \sim 0 \qquad (\textit{in } K).
$$

By a reasoning analogous to that employed for a similar purpose in § 2 the following is inferred.

If case II *holds in a region K (satisfying* ($\bar{\alpha}$), ..., ($\bar{\delta}$)) *necessarily* $\mu_1 \leqq 0$ *and* $Q_1(x)$ *is not of the form* $q_{1,0}x$ $(\Re q_{1,0} = 0)$.

If $\mu_1 < 0$ *case* II *will certainly hold in some region K.*

We shall now give developments for the case II. Thus $\mu_1 \leqq 0$.

The function $y_1(x)$ will be defined by (33; § 2), (34; § 2) except, of course, K will be the region of case II. With

$$
(7) \qquad y_1(x) = e^{Q_1(x)}\, x^{y_1}[x]^1_{N(1)} \qquad (x \text{ in } K;\ y_1 = 0),
$$

in consequence of Lemma 1 (§ 2) it will follow that

(7a) $$y_1(x+\nu) = e^{Q_1(x)} x^{\nu\mu_1} [x]^1_{N(1)}$$
$$(x \text{ in } K;\ \nu = 0, 1, \ldots, n-1).$$

On making use of (13; § 2) and on noting that

$$i_0, \ldots, i_{n-1} \geqq 0;\ i_0 + \ldots + i_{n-1} = 2$$

it is inferred that

(7b) $$\prod_{\nu=0}^{n-1} y_1^{i_\nu}(x+\nu) = e^{2Q_1(x)} x^{\mu_1 i'} [x]^2_{2N(1)} \qquad (x \text{ in } K),$$

(7c) $$i' = i_1 + 2i_2 + \ldots + (n-1)i_{n-1}.$$

When $n = 1$, $i' = 0$. When $n > 1$

(7d) $$i' \geqq 0$$

where the equality sign is attained for $i_0 = 2, i_1 = i_2 = \ldots = i_{n-1} = 0$. Whence the second member has the form

(8) $$e^{2Q_1(x)} [x]^2_{2N(1)} \qquad (x \text{ in } K).$$

Accordingly, by (13; § 2) it follows that

(9) $$\Psi_2(x) = e^{2Q_1(x)} x^{\Psi_2} [x]^2_{k(2)} \qquad (x \text{ in } K)$$

where

(9a) $$\Psi_2 = 0.$$

On recalling that formulas (44; § 2), (44a; § 2) serve to determine the $y_j(x)$ $(j = 2, 3, \ldots)$ not only for the case I, it is found possible to compute $y_2(x)$ with the aid of (44; § 2) and (9). We have

(10) $$S^x_\lambda(\Psi_2(u)) = \underset{u=x}{\mathrm{S}}\, e^{2Q_1(u) - Q_\lambda(u)} u^{\Psi_2 - r_\lambda + \bar{m}} [u]^2_{k(2)+q(\lambda)}$$

where

(11) $\bar{m} = 0$ (for $n=1$); rational $\bar{m} \geqq 0$ (for $n>1$).

Now the function

$$\left|e^{2Q_1(u) - Q_\lambda(u)}\right| = \left|e^{Q_1(u)}\right| \left|e^{Q_1(u) - Q_\lambda(u)}\right|$$

is monotone in K to the right since it is a product of two functions possessing such a property (cf. (6) and the statement in connection with (3)). Moreover, by (6a) this function is asymptotic in K to zero. Hence the „sum" involved in (10) can be evaluated according to the formula

$$\underset{u=x}{\mathbf{S}}\ B(u) = -B(u) - B(u+1) - B(u+2) - \dots .$$

This method of „summation" will be used throughout this section. With the aid of Lemma 1 (§ 2) from (10) one can obtain

$$(12)\qquad S_{\lambda}^{x}(\Psi_2(u)) = e^{2Q_1(x) - Q_{\lambda}(x)}\, x^{\Psi_2 - r_{\lambda} + \bar{m} + h_2}\, [x]^2_{K(2)+q(\lambda)} \qquad (x \text{ in } K).$$

On writing $2Q_1(x) - Q_{\lambda}(x) = \mu x \log x + \dots$ $(\mu = 2\mu_1 - \mu_{\lambda})$ it is inferred that in (12)

$$(12\text{a})\qquad 0 \leqq -\mu_1 \leqq -\mu = \mu_{\lambda} - 2\mu_1 .$$

The inequalities here involved follow from (5). Thus, we may take

$$(13)\qquad h_2 = 0 \qquad (\text{if } \mu_{\lambda} - 2\mu_1 \neq 0).$$

This certainly will be the case when $\mu_1 \neq 0$ and also when $\mu_{\lambda} \neq \mu_1$. The alternative to (13) is

$$(13\text{a})\qquad \mu_1 = \mu_{\lambda} = 0.$$

If for at least one value of λ (13a) holds the number h_2, independent of λ, could be given as a certain rational number h',

$$(13\text{b})\qquad 0 \leqq h_2 = h' \leqq 1$$ [36]).

If for no value of λ (13a) *takes place, necessarily $\mu_1 \neq 0$ and h_2 is defined, independently of λ, by* (13)[37]). In other words, when $\mu_1 = 0$ h_2 will be given by (13b) and when $\mu_1 \neq 0$ it will be given by (13). Whence, by (12) and (44; § 2),

$$(14)\qquad y_2(x) = e^{2Q_1(x)}\, x^{y_2}\, [x]^2_{N(2)} \qquad (x \text{ in } K),$$

$$(14\text{a})\qquad y_2 = \Psi_2 + \bar{m} + h_2.$$

In view of (9a) and (14a) and in consequence of the above definition of h_2 it will follow that

$$(14\text{b})\qquad \begin{aligned} y_2 &= \bar{m} + h' \ (\geqq 0) \qquad (\text{for } \mu_1 = 0)^{38)}, \\ y_2 &= \bar{m} \qquad\quad (\geqq 0) \qquad (\text{for } \mu_1 \neq 0). \end{aligned}$$

Here, for $n = 1$, $\bar{m}$ may be replaced by zero. On using (9; $j = 3$; § 2), for $n > 1$, and (62a; § 2), for $n = 1$, in view of the established forms of $y_1(x)$ and $y_2(x)$ it is concluded that $\Psi_3 = y_2$. It also can be shown that $y_3 = \Psi_3 + \bar{m} + h_3$ where

---

[36]) h' is the greatest one of the several numbers h ($\lambda = 1, \dots$) of Lemma 1 (§ 2).

[37]) Conversely, when $\mu_1 \neq 0$ (i.e. $\mu_1 < 0$) h_2 is so defined.

[38]) For $n = 1$ the first line of (14b) reduces to $y_2 = h' = h$ $(\geqq 0)$.

$$h_3 = 0 \qquad (\text{if } \mu_1 \neq 0),\ h_3 = h' \ (0 \leqq h' \leqq 1;\ \text{if } \mu_1 = 0)$$ [39].

Assume now that for x in K and a value of j ($\geqq 3$) we have

(15) $$y_r(x) = e^{rQ_1(x)}\, x^{y_r} [x]^r_{N(r)} \qquad (r = 1, \ldots, j-1),$$

(15a) $$\Psi_r(x) = e^{rQ_1(x)}\, x^{\Psi_r} [x]^r_{K(r)} \qquad (r = 2, \ldots, j-1)$$

where

(16) $$\Psi_r = y_{r-1} \qquad (r = 2, 3, \ldots, j-1),$$

(16a) $$y_r = b_1 r + b_2 \qquad (r = 1, \ldots, j-1),$$

with

(16b) $$b_1 = \bar{m},\ b_2 = -\bar{m} \qquad (\textit{if } \mu_1 \neq 0),$$

and

(16c) $$b_1 = \bar{m} + h',\ b_2 = -\bar{m} - h' \qquad (\textit{if } \mu_1 = 0).$$

The above has been established previously for $j = 3$. Perform now the substitution of (15) into (9; § 2). We have

$$y_{j_r+1}(x+\nu) = e^{(j_r+1)Q_1(x)}\, x^{y_{j_r+1} + (j_r+1)\nu\mu_1} [x]^{j_r+1}_{N(j_r+1)}$$

for x in K and $\nu = 0, 1, \ldots, n-1$. Thus, for x in K

(17) $$\prod_{r=1}^{i_\nu} y_{j_r+1}(x+\nu) = e^{(\beta_\nu + i_\nu) Q_1(x)}\, x^{y' + (\beta_\nu + i_\nu)\nu\mu_1} [x]^{\beta_\nu + i_\nu}_{N(j_1, j_2, \ldots, j_{i_\nu})}$$

since $j_1 + j_2 + \ldots + j_{i_\nu} = \beta_\nu$ (10; § 2). Here

(17a) $$y' = \Sigma\, y_{j_r+1} = b_1 \Sigma\, (j_r+1) + b_2 = b_1\beta_\nu.$$

Thus the second member of (17) is of the form

(18) $$e^{(\beta_\nu + i_\nu)Q_1(x)}\, x^{b_1\beta_\nu + \nu(\beta_\nu + i_\nu)\mu_1} [x]^{\beta_\nu + i_\nu}_{N'_{\beta_1, \ldots, \beta_{n-1}}}$$

where $N'_{\beta_1, \ldots, \beta_{n-1}}$ is the greatest of the numbers $N(j_1, \ldots, j_{i_\nu})$. The function F_ν, resulting from the extension of the summation (10; § 2) over the left member of (17), will have the form (18). Whence, by (11; § 2) and (12; § 2),

(18a) $$F_{\beta_1, \ldots, \beta_{n-1}} = \prod_{\nu=0}^{n-1} F_\nu = e^{jQ_1(x)}\, x^{b_1(j-k) + f'\mu_1} [x]^j_{N''_{\beta_1, \ldots, \beta_{n-1}}}$$

$(x \text{ in } K)$

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[39]) This can be established in a way analogous to that employed in proving (14a), (14b).

where

(18b) $$f' = \sum_{\nu=0}^{n-1} \nu(\beta_\nu + i_\nu) \geqq 0$$

(under (11; § 2), (12; § 2)) [40]).

Thus, since $\mu_1 \leqq 0$, in (18a) f' may be replaced by zero. Accordingly, it is inferred that

$$\Sigma'' F_{\beta_1,\ldots,\beta_{n-1}} = F^{i_0,\ldots,i_{n-1}} \qquad \text{(cf. (11; § 2))}$$

is a function of the form

(19) $$e^{jQ_1(x)} x^{b_1(j-k)} [x]^j_{N'''_{i_0,\ldots,i_{n-1}}} \qquad (x \text{ in } K).$$

Furthermore, it is observed that

$${}_kF = \Sigma''' a_{i_0,\ldots,i_{n-1}}(x) F^{i_0,\ldots,i_{n-1}} \quad \text{(cf. (12; § 2))}$$

has, in K, the asymptotic form (19) with $N''_{\beta_1,\ldots,\beta_{n-1}}$ replaced by ${}_kN$. Finally, it is inferred that

(20) $$\Psi_j(x) = \sum_{k=2}^{j} {}_kF = e^{jQ_1(x)} x^{\Psi_j} [x]^j_{k(j)} \qquad (x \text{ in } K)$$

where Ψ_j is the greatest of the numbers

$$b_1(j-k) \qquad (k=2,\ldots,j).$$

Now, by (16b) and (16c), $b_1 \geqq 0$. Thus

(20a) $$\Psi_j = b_1(j-2) = b_1(j-1) + b_2 = y_{j-1}$$

(cf. (16a)). This result implies validity of (16) for $r=j$.

With the aid of (20) the form of $y_j(x)$ can be found. By (44; § 2)

(21) $$S^x_\lambda(\Psi_j(u)) = \underset{u=x}{\mathbf{S}}\, e^{jQ_1(u)-Q_\lambda(u)} u^{\Psi_j - r_\lambda + \overline{m}} [u]^j_{k(j)+q(\lambda)}\,.$$

Since, by (6) and in view of the statement in connection with (3), the functions

$$\left|e^{Q_1(u)}\right|, \ \left|e^{Q_1(u)-Q_\lambda(u)}\right|$$

are monotone in K to the right, it is noted that

(21a) $$\left|e^{jQ_1(u)-Q_\lambda(u)}\right| = \left|e^{Q_1(u)}\right|^{j-1} \left|e^{Q_1(u)-Q_\lambda(u)}\right|$$

possesses the same property. By (6a) the function (21a) is asymptotic in K to zero. On taking account of the relationship between the „actual" summation method of this section with the corresponding formal situation and on noting that

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[40]) For some terms the equality sign of (18b) actually takes place.

$$jQ_1(u) - Q_\lambda(u) = \mu u \log u + \ldots \quad (\mu = j\mu_1 - \mu_\lambda),$$

application of Lemma 1 (§ 2) enables one to infer that

$$(22) \qquad S_\lambda^x(\Psi_j(u)) = e^{jQ_1(x)-Q_\lambda(x)} x^{\Psi_j - r_\lambda + \bar{m} + h_j} [x]^j_{k(j)+q(\lambda)}$$

(x in K) where h_j can be selected independent of λ as follows:

$$(22a) \qquad h_j = 0 \qquad (\text{if } \mu_1 \neq 0),$$

$$(22b) \qquad h_j = h' \qquad (0 \leqq h' \leqq 1;\ \text{if } \mu_1 = 0).$$

Here h' is independent of j. The reasoning in this connection is precisely of the same type as that employed in the determination of h_2 (cf. (13), (13b)). By (22) and (44; § 2) we accordingly have

$$(23) \qquad y_j(x) = e^{jQ_1(x)} x^{y_j} [x]^j_{N(j)} \qquad (x \text{ in } K)$$

where, by (20a),

$$(24) \qquad y_j = \Psi_j + \bar{m} + h_j = y_{j-1} + \bar{m} + h_j .$$

Whence it is concluded that

$$y_j = b_1 j + b_2$$

where b_1, b_2 are given by (16b), if $\mu_1 \neq 0$, and by (16c), if $\mu_1 = 0$. This completes the induction. The following Lemma can be now formulated.

Lemma 3. *Consider the equation* (A; § 1) *under the Hypothesis* A (§ 1). *There exists then a region K (extending to the right) satisfying the conditions* ($\bar{\alpha}$), ($\bar{\beta}$), ($\bar{\gamma}$), ($\bar{\delta}$) *(cf. the beginning of this section).*

When case II *(italics in connection with* (6) *and* (6a)) *is on hand and K denotes a region referred to in the statement of this case, the problem* (A) *will possess a formal solution*

$$(25) \quad s(x) = y_1(x)p_1(x) + y_2(x)p_1^2(x) + \ldots + y_j(x)p_1^j(x) + \ldots$$

where $y_1(x)$ is a solution of the linear equation (B; § 1) *and is given by* (7), *and the $y_1(x)$ are of the form*

$$(25a) \qquad y_j(x) = e^{jQ_1(x)} x^{y_j} [x]^j_{N(j)}$$

$$(j = 1, 2, \ldots;\ x \text{ in } K;\ \text{cf. Def. 3 (§ 2)})\ ^{41}).$$

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[41]) In applying Def. 3 (§ 2) it is noted that the involved constants are those from the relations $Q_i(x) = Q_1(x) + \sqrt{-1}q_i x$ ($i=1, 2, \ldots, \delta$; cf. (1)).

In (25) *and* (25a) *there enter* δ *arbitrary functions of period unity,*

$$p_1(x), c_2(x), \ldots, c_\delta(x).$$

The numbers y_j *involved in* (25a) *are of the form*

$$y_j = b_1 j + b_2 \qquad (j = 1, 2, \ldots) \tag{26}$$

where

$$b_1 = \bar{m}, \qquad b_2 = -\bar{m} \qquad (\text{if } \mu_1 \neq 0), \tag{26a}$$

$$b_1 = \bar{m} + h', \quad b_2 = -(\bar{m}+h') \quad (\text{if } \mu_1 = 0). \tag{26b}$$

Here h' *is rational and* $0 \leqq h' \leqq 1$. *The non-negative rational number* $\bar{m}$ *is the one involved in* (44a; § 2)[42]).

Note. In Lemma 2 (§ 2) and in the above Lemma the expression $[x]^j_{N(j)}$ involves functions (analytic in K for $x \neq \infty$) asymptotic in K to series of the form $d_0 + d_1 x^{-\frac{1}{p_j}} + d_2 x^{-\frac{2}{p_j}} + \ldots$ (p_j integer). It is possible to have $p_j \to \infty$ as $j \to \infty$.

4. Transformations.

Consider the formal solution (68; § 2) corresponding to the Case I (§ 2). Let N be a positive integer, however large. Apply the transformation

$$y(x) = Y(x) + \varrho(x) \tag{1}$$

to the equation (A; § 1). Here $\varrho(x)$ is to be the new variable and

$$Y(x) = \sum_{j=1}^{N-1} y_j(x) p_1^j(x). \tag{2}$$

In (2) the $y_j(x)$ are the functions so denoted in Lemma 2 (§ 2). They possess in K the asymptotic forms (68a; § 2).

On taking account of the statement in connection with (5; § 1) it becomes manifest that the inequalities

$$|Y(x)| \leqq \varrho' < \varrho \tag{2a}$$

should be satisfied for x in K at least sufficiently far from the origin.

Now the q_i, involved in $[x]^j_{N(j)}$, are real. Hence by (35; § 2) and (34; § 2), on writing $x = u + \sqrt{-1}\, v$, it is inferred that

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[42]) The formula (44a; § 2) referring, of course, to the region K now under consideration.

$$(3)\qquad \left|[x]^j_{N(j)}\right| \leqq \Sigma \left|c_1^{i_1}(x)\cdots c_\delta^{i_\delta}(x)\right| e^{-(i_1 q_1+\cdots+i_\delta q_\delta)v}\cdot$$
$$\cdot\left|x^{i_1 r_1+\cdots+i_\delta r_\delta}\right| \left|[x]_{N(j)}\right|$$
$$(i_1,\ldots,i_\delta \geqq 0;\ i_1+\ldots+i_\delta = j).$$

By Def. 1 (§ 2)

$$(3a)\qquad \left|[x]_{N(j)}\right| < b_{N(j)}\left|x\right|^{\varepsilon}\qquad (\varepsilon>0;\ x \text{ in } K)$$

where ε can be taken as small as desired. Let

$$(3b)\qquad r' = \max(\Re r_1, \Re r_2, \ldots, \Re r_\delta).$$

From (3) in view of (3a) it will follow that

$$(4)\qquad \left|[x]^j_{N(j)}\right| < b'(j)\left|x\right|^{jr'+\varepsilon}\Sigma\left|e^{-q_1 v}c_1(x)\right|^{i_1}\cdots\left|e^{-q_\delta v}c_\delta(x)\right|^{i_\delta}.$$

Assume that the periodic functions

$$(5)\qquad p_1(x),\ c_2(x),\ldots,c_\delta(x)$$

are analytic in K $(x\neq\infty)$. *Designate by* $f(v)$ *a function for which the inequalities*

$$(5a)\qquad \left|p_1(x)\right| \leqq f(v);\ \left|p_1(x)c_i(x)\right| e^{-q_i v} \leqq f(v)\qquad (i=2,\ldots,\delta).$$

all hold for x *in* K $(v=\Im x)$.

In the sequel it will be assumed that the above function $f(v)$ *is such that*

$$(6)\qquad e^{Q_1(x)}f(v)\sim 0\qquad (x \text{ in } K),$$

where the asymptotic relationship is with respect to x. *That is,* (6) *is equivalent to the set of inequalities, valid for* x *in* K,

$$(6a)\qquad \left|e^{Q_1(x)}f(v)\right| \leqq A_m\left|x\right|^{-m}\qquad (m=1,2,\ldots).$$

It is clear that (6) will certainly hold in every subregion of K for which $a\leqq v\leqq b$; this being so no matter what the function $f(v)$ is. It is to be noted that $f(v)$ specifies the rate with which the absolute values of the periodic functions (5) may vary away from the axis of reals. It is not difficult to show that the variety of analytic functions (5) for which (6) holds in a region K, of the type stated before, is quite extensive.

In consequence of (4) and (5a)

$$(7)\qquad \begin{aligned}\left|y_j(x)p_1^j(x)\right| &= \left|e^{jQ_1(x)}x^{\nu j}[x]^j_{N(j)}\,p_1^j(x)\right|\\ &\leqq b''(j)\left|e^{Q_1(x)}f(v)\right|^j\left|x\right|^{\nu_j+jr'+\varepsilon}\qquad (x \text{ in } K).\end{aligned}$$

Whence, by (6a),

$$y_j(x)p_1^j(x) \sim 0 \qquad (x \text{ in } K) \tag{7a}$$

and, in view of (2),

$$Y(x) \sim 0 \qquad (x \text{ in } K). \tag{8}$$

Let K_λ denote the part of K in which $|x| \geqq \lambda$.

The relationship (8) implies that (2a) *holds in a region $K_{\lambda'}$, provided λ' (depending on N) is chosen sufficiently great. The inequality* (2a) *can be secured in the original region $K = K_{r_0}$ if the function $f(v)$, involved in* (5a), *is sufficiently small (depending on N).*

If the transformation (1) is now carried out it is observed that the equation satisfied by $\varrho(x)$ in general would have a meaning only when

$$|\varrho(x)| \leqq \varrho'' \qquad (\varrho' + \varrho'' = \varrho;\ x \text{ in } K). \tag{9}$$

We have, for $x + n - 1$ in $K_{\lambda'}$,

$$\begin{aligned} &a_2(x, Y(x) + \varrho(x), \ldots, Y(x+n-1) + \varrho(x+n-1)) \\ &= a_2(x, Y(x), \ldots, Y(x+n-1)) + \\ &\qquad + \Sigma\, \alpha_{i_0, i_1, \ldots, i_{n-1}}(x) \varrho^{i_0}(x) \cdots \varrho^{i_{n-1}}(x+n-1) \\ &(i_0 + \ldots + i_{n-1} \geqq 1;\ i_0, \ldots, i_{n-1} \geqq 0;\ \text{cf. } (3b;\ \S\ 1)), \end{aligned} \tag{10}$$

where

$$\alpha_{i_0, \ldots, i_{n-1}}(x) = \frac{1}{i_0! \cdots i_{n-1}!} \frac{\partial^{i_0 + \cdots + i_{n-1}} a_2}{\partial y_0^{i_0} \cdots \partial y_{n-1}^{i_{n-1}}} \tag{10a}$$
$$(y_0 = Y(x),\ y_1 = Y(x+1), \ldots, y_{n-1} = Y(x+n-1)).$$

With (2a) satisfied for x in $K (= K_{\lambda'}$ or $K_{r_0})$ it follows that

$$|Y(x+i)| \leqq \varrho' \quad (i = 0, 1, \ldots, n-1;\ x + n - 1 \text{ in } K). \tag{10b}$$

Thus, whenever

$$|\varrho(x+i)| \leqq \varrho'' \quad (i = 0, 1, \ldots, n-1;\ x + n - 1 \text{ in } K), \tag{10c}$$

that is whenever (9) holds in K, it is concluded that the series in the second member of (10) is absolutely convergent for $x + n - 1$ in K. In view of (10a)

$$\begin{aligned} &\alpha_{i_0, \ldots, i_{n-1}}(x) = \\ &= \Sigma\, a_{i_0 + j_0, \ldots, i_{n-1} + j_{n-1}}(x) C_{i_0}^{i_0 + j_0} \cdots C_{i_{n-1}}^{i_{n-1} + j_{n-1}} Y^{j_0}(x) \cdots Y^{j_{n-1}}(x+n- \\ &\qquad (j_0 + \ldots + j_{n-1} \geqq 2 - (i_0 + \ldots + i_{n-1});\ j_0, \ldots, j_{n-1} \geqq 0), \end{aligned} \tag{11}$$

the involved series being convergent for $x+n-1$ in K. Consequently with the aid of (8) one can infer that

$$\alpha_{i_0,\ldots,i_{n-1}}(x)=a_{i_0,\ldots,i_{n-1}}(x)+\beta_{i_0,\ldots,i_{n-1}}(x) \tag{12}$$
$$(i_0,\ldots,i_{n-1}\geqq 0;\ i_0+\ldots+i_{n-1}\geqq 2),$$

while

$$\alpha_{i_0,\ldots,i_{n-1}}(x)=\beta_{i_0,\ldots,i_{n-1}}(x)\quad (\text{when } i_0+\ldots+i_{n-1}=1). \tag{12a}$$

Here the $\beta_{i_0,\ldots,i_{n-1}}(x)$ are analytic for $x+n-1$ in $K(x\neq\infty)$ and, moreover,

$$\beta_{i_0,\ldots,i_{n-1}}(x)\sim 0\quad (x+n-1 \text{ in } K;\ i_0+\ldots+i_{n-1}\geqq 1). \tag{12b}$$

With L denoting the linear operator of (B; § 1) consider the function

$$-F_N(x)=L(Y(x))-x^{\frac{w}{\alpha}}a_2(x,Y(x),\ldots,Y(x+1-n)), \tag{13}$$

which certainly is defined for $x+n-1$ in $K(=K_{\lambda'}$ or $K_{r_0})$. If $Y(x)$ were replaced by the formal (generally divergent) series $s(x)$ of Lemma 2 (§ 2) and the resulting expression were formally expanded in powers of $p_1(x)$ we would obtain the series

$$\sum_{j=1}^{\infty}\Gamma_j(x;y_1(x),\ldots,y_{j-1}(x))\,p_1^j(x), \tag{14}$$

where

$$\Gamma_1(x;y_1(x),\ldots,y_{j-1}(x))\equiv L(x,y_1(x))=0, \tag{14a}$$

$$\Gamma_j(x,y_1(x),\ldots,y_{j-1}(x))\equiv L(x,y_j(x))-x^{\frac{w}{\alpha}}\Psi_j(x)=0 \tag{14b}$$
$$(j=2,3,\ldots)$$

(cf. (2; § 2), (9; § 2), (16; § 2), (17; § 2)). Now $Y(x)$ is $s(x)$ with the $y_j(x)$ $(j=N,N+1,\ldots)$ all replaced by zero. Thus

$$-F(x)=\sum_{j\geqq 1}\bar{\Gamma}_j(x)p_1^j(x), \tag{15}$$

$$\bar{\Gamma}_1(x)=L(x,y_1(x))=0, \tag{15a}$$

$$\bar{\Gamma}_j(x)=L(x,y_j(x))-x^{\frac{w}{\alpha}}\bar{\Psi}_j(x) \tag{15b}$$
$$(j=2,3,\ldots;\ y_j(x)\equiv 0\ (j\geqq N)).$$

It is observed that $\bar{\Psi}_j(x)$ is $\Psi_j(x)$ with $y_j(x)$ $(j\geqq N)$ replaced by zero. Hence, since $\Psi_j(x)$ depends only on $y_1(x),\ldots,y_{j-1}(x)$,

$$\bar{\Psi}_j(x) = \Psi_j(x) \qquad (j=2,3,\ldots,N). \tag{16}$$

For $j > N$ the $\bar{\Psi}_j(x)$ have the asymptotic form of the corresponding $\Psi_j(x)$. By (15b), (16) and (14b)

$$\bar{\Gamma}_j(x) = 0 \qquad (j=2,3,\ldots,N-1), \tag{17}$$

$$\bar{\Gamma}_N(x) = -x^{\frac{w}{\alpha}}\Psi_N(x), \tag{17a}$$

$$\bar{\Gamma}_j(x) = -x^{\frac{w}{\alpha}}\bar{\Psi}_j(x) \qquad (j=N+1,N+2,\ldots). \tag{17b}$$

Whence, in view of (15a), (17), (17a), (17b), from (15) one can infer that the function $F_N(x)$ is of the form

$$\begin{aligned} F_N(x) &= x^{\frac{w}{\alpha}} \sum_{j\geqq N} \bar{\Psi}_j(x) p_1^j(x) \\ &= x^{\frac{w}{\alpha}} \sum_{j\geqq N} e^{jQ_1(x)} x^{\Psi_j} [x]^j_{k(j)}\, p_1^j(x) \qquad (x+n-1 \text{ in } K). \end{aligned} \tag{18}$$

Now, in consequence of (5a),

$$|p_1^j(x)|\,|[x]^j_{k(j)}| < h_j |x|^{jr'+\varepsilon} f^j(v). \tag{19}$$

Hence, on taking account of (6a) and of the satisfied conditions of convergence of the series (18), it is concluded that

$$F_N(x) = e^{NQ_1(x)} f^N(v)\, x^{\Psi_N + Nr' + \frac{w}{\alpha} + \varepsilon}\, G_N(x) \tag{20}$$

where

$$|G_N(x)| \leqq G_N \qquad (x+n-1 \text{ in } K). \tag{20a}$$

Of course, $F_N(x)$ is defined in x for $x+n-1$ in K $(x \neq \infty)$. It is to be noted that, as can be seen from the developments of § 2,

$$\Psi_j = 0 \qquad (\text{if } n=1 \text{ and } \mu_1 \neq 0 \text{ (i.e. } \mu_1 > 0)) \tag{21}$$

and

$$\Psi_j = \alpha_0 j^2 + j(\alpha_1 + \mu_1) - 2\alpha_1 \quad (\text{cf. } (52;\ \S\ 2),\ (52a;\ \S\ 2)), \tag{21a}$$

when $n=1$ and $\mu_1 = 0$ or when $n > 1$.

Substitution of (1) into (A; § 1) will yield, by (10),

$$\begin{aligned} L(Y(x)) + L(\varrho(x)) &= \\ x^{\frac{w}{\alpha}} a_2(x, Y(x)) &+ x^{\frac{w}{\alpha}} \sum \alpha_{i_0,\ldots,i_{n-1}}(x)\, \varrho^{i_0}(x) \cdots \varrho^{i_{n-1}}(x+n-1) \\ &(i_0 + \ldots + i_{n-1} \geqq 1;\ i_0, \ldots, i_{n-1} \geqq 0). \end{aligned}$$

Thus, in consequence of (13)

$$(22)\quad L(\varrho(x)) = F_N(x) + x^{\frac{w}{\alpha}} \sum \alpha_{i_0,\dots,i_{n-1}}(x)\varrho^{i_0}(x)\cdots\varrho^{i_{n-1}}(x+n-1)$$
$$(i_0 + \dots + i_{n-1} \geqq 1).$$

Transposing the terms of the second member of (22), linear in $\varrho(x), \dots, \varrho(x+n-1)$, to the left we obtain

$$(23)\quad L_1(\varrho(x)) = F_N(x) + x^{\frac{w}{\alpha}}\, {}_1H(x, \varrho(x), \dots, \varrho(x+n-1))$$

where

$$(23a)\quad {}_1H(x, \varrho(x), \dots, \varrho(x+n-1)) =$$
$$\sum \alpha_{i_0,\dots,i_{n-1}}(x)\varrho^{i_0}(x)\cdots\varrho^{i_{n-1}}(x+n-1)$$
$$(i_0 + \dots + i_{n-1} \geqq 2;\ i_0, \dots, i_{n-1} \geqq 0;\ \text{cf. (12), (12b)})$$

and

$$(23b)\quad L_1(\varrho(x)) =$$
$$\varrho(x+n) - x^{\frac{w}{\alpha}}[{}_1b_{n-1}(x)\varrho(x+n-1) + \dots + {}_1b_0(x)\varrho(x)].$$

Here, by (12a) *and* (12b),

$$(23c)\quad {}_1b_i(x) - b_i(x) \sim 0$$
$$(i=0, 1, \dots, n-1;\ \text{cf. (3a; § 1), (4; § 1)})$$

when $x + n - 1$ *is in* K. Also it is to be recalled that $F_N(x)$ *is a function satisfying* (20) *and* (20a). *Moreover*, ${}_1H(x, \varrho_0, \varrho_1, \dots, \varrho_{n-1})$ *is absolutely convergent when* $x + n - 1$ *is in* K ($= K_{\lambda'}$ *or* K_{r_0}), *provided*

$$(23d)\quad |\varrho_i| \leqq \varrho'' \quad (i=0, 1, \dots, n-1;\ \text{cf. (9)}).$$

Let t_N denote a positive number, for the present not specified. Consider

$$(24)\quad H_N(x) = e^{NQ_1(x)} f^N(v) x^{t_N} \qquad (v = \mathfrak{J}x).$$

We have

$$(24a)\quad H_N(x+i) = H_N(x) x^{iN\mu_1} h_i(x) \qquad (i=0, 1, \dots, n)$$

where

$$(24b)\quad h_i(x) = [x]_0, \quad \frac{1}{h_n(x)} = [x]_0.$$

The further transformation

$$(25) \qquad \varrho(x) = H_N(x)\zeta(x)$$

will be now applied to (23). By (24a) and (24b), in view of (23b), it is inferred that

$$(26) \qquad L_1(\varrho(x)) = H_N(x)x^{nN\mu_1}h_n(x)\, L'(\zeta(x)),$$

$$(26a) \qquad L'(\zeta(x)) = \zeta(x+n) - x^{w'}\sum_{i=0}^{n-1} b'_i(x)\zeta(x+i),$$

where

$$(26b) \qquad w' = \frac{w}{\alpha} - N\mu_1, \quad b'_i(x) = [x]_0 \quad (x+n-1 \text{ in } K)\text{[43]}.$$

Since, with $i_0, \ldots, i_{n-1} \geqq 0$ and $i_0 + \ldots + i_{n-1} = m \quad (m \geqq 0)$,

$$x^{(i_1+2i_2+\ldots+(n-1)i_{n-1})N\mu_1} = x^{m(n-1)N\mu_1}[x]_0$$

and since

$$\varrho^{i\nu}(x+\nu) = H_N^{i\nu}(x)x^{\nu i_\nu N\mu_1}h_\nu^{i\nu}(x)\zeta^{i\nu}(x+\nu)$$

from (23a) it follows that

$$(27) \quad {}_1H(x, \varrho(x), \ldots, \varrho(x+n-1)) = H_N^2(x)x^{2(n-1)N\mu_1}H'(x, \zeta(x)),$$

$$(27a) \quad H'(x, \zeta(x)) = \sum_{m=2}^{\infty} H_N^{m-2}(x)x^{(m-2)(n-1)N\mu_1} \cdot \sum_{i_0+\ldots+i_{n-1}=m} {}_1\alpha_{i_0,\ldots,i_{n-1}}(x)\,\zeta^{i_0}(x)\cdots\zeta^{i_{n-1}}(x+n-1)$$

where

$$(27b) \qquad {}_1\alpha_{i_0,\ldots,i_{n-1}}(x) = [x]_0 \qquad (x+n-1 \text{ in } K).$$

On taking account of the convergence properties of ${}_1H(x, \varrho(x), \ldots \varrho(x+n-1)$ stated in connection with (23a), (23d), in view of (6) and of the way the series (27a) was derived, it is concluded that $H'(x, \zeta_0, \ldots, \zeta_{n-1})$ converges absolutely and uniformly for $x+n-1$ in $K_\lambda(\lambda \geqq \lambda'$ or $\lambda \geqq r_0$, as the case may be)[44]), provided

$$(28) \qquad |\zeta_i| \leqq \zeta' = \zeta'(\lambda) \qquad (i = 0, 1, \ldots, n-1).$$

The essential fact is to be noted that

$$(28a) \qquad \lim_{\lambda \to \infty} \zeta'(\lambda) = \infty.$$

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[43]) In this connection use has been made of the fact that, for the case now under consideration, μ_1 is rational and non-negative so that $x^{-jN\mu_1} = [x]_0$ $(j = 0, 1, \ldots, n-1)$.

[44]) K_λ, if not coincident with K (which is $K_{\lambda'}$ or K_{r_0}), is to be a subregion of K.

That is, given a number ${}_1\zeta$, however large, there exists a value $\lambda = \lambda(N, {}_1\zeta)$ so that $H'(x, \zeta_0, \ldots, \zeta_{n-1})$ converges, when $x + n - 1$ is in K_λ, for $|\zeta_i| \leqq {}_1\zeta$ $(i = 0, \ldots, n-1)$.

It is also possible to secure convergence of $H'(x, \zeta_0, \ldots, \zeta_{n-1})$ *for* $x+n-1$ *in the original region K and for* $|\zeta_i| \leqq {}_1\zeta$ $(i=0, \ldots, n-1)$; ${}_1\zeta$ *as great as desired), provided the function* $f(v)$, *involved in the inequalities satisfied by the periodic functions, be sufficiently small (depending on* ${}_1\zeta$).

By (20) and (24)

$$F_N(x) = H_N(x)\, x^{\Psi_N + Nr' + \frac{w}{\alpha} + \varepsilon - t_N}\, G_N(x).$$

Hence, in view of (26) and (27), application of the transformation (25) to (23) will yield

$$(29) \qquad L'(\zeta(x)) = x^{\tau'} \frac{1}{h_n(x)} G_N(x) + x^{\tau''} \frac{1}{h_n(x)} H_N(x)\, H'(x, \zeta(x)),$$

$$(29a) \quad \tau' = \Psi_N + Nr' + \frac{w}{\alpha} + \varepsilon - t_N - nN\mu_1, \quad \tau'' = \frac{w}{\alpha} + (n-2)N\mu_1$$

(cf. (26), (26a), (26b), (24b), (20a), (24), (27a).

Lemma 4. *Consider case* I (§ 2) *and the formal solution of* (A; § 1), *relating to this case and specified in Lemma* 2 (§ 2). *Let N be an integer, however large. Define the function* $Y(x)$ *by* (2). *Let the periodic functions* (5), *involved in the formal solution, be subject to the conditions stated in italics in connection with* (5), ..., (6a). $Y(x)$ *will satisfy* (2a) *and* (8) *for x in the region K* (*which is* $K_{\lambda'}$ *or* K_{r_0} *of the italicized statement subsequent to* (8)). *The transformation*

$$(30) \qquad y(x) = Y(x) + H_N(x)\zeta(x) \quad (\text{cf. } (24), (5a))$$

will yield the equation (29), (29a). *The series* (27a) *representing* $H'(x, \zeta(x))$ *satisfies the convergence conditions stated in connection with* (28), (28a) (*also cf. the subsequent statement in italics*).

An analogous Lemma, corresponding to the case II (§ 3), will be now established. The transformation will be

$$(31) \qquad y(x) = Y(x) + \varrho(x), \quad Y(x) = \sum_{j=1}^{N-1} y_j(x) p_1^j(x)$$

(cf. (25; § 3)). We again make the assumption stated in connection with (5), ..., (6a), with K having the new significance. Then (8) and (2a) will be satisfied for x in $K(=K_{\lambda'}$ or K_{r_0}; cf. the statement subsequent to (8)). The formulas (9), ..., (20a) will continue to hold provided that, throughout, the statement $x+n-1$ in K be replaced by x in K. On taking account of section 3 it is observed that the relations (21) and (21a) are to be replaced by

(32) $\quad \Psi_j = y_j \quad (j=2,3,\ldots;$ cf. (26; § 3); (26a; § 3); (26b; § 3)).

The relations (22), ...,(25) (x in K) will hold. Since in the case II (§ 3) $\mu_1 \leqq 0$, the result of the transformation (25) will be different. The relation (26) will hold with

$$(33) \qquad L'(\zeta(x)) = \zeta(x+n) - x^{\frac{w}{\alpha}} \sum_{i=0}^{n-1} {}_1b_i(x) x^{-(n-i)N\mu_1} \frac{h_i(x)}{h_n(x)} \zeta(x+i)$$
$$= \zeta(x+n) - x^{w'} \sum_{i=0}^{n-1} b_i'(x)\zeta(x+i)$$

where

$$(33a) \qquad w' = \frac{w}{\alpha} - nN\mu_1, \quad b_i'(x) = [x]_0 \qquad (x \text{ in } K).$$

Since

$$\varrho(x+\nu) = H_N(x) x^{\nu N\mu_1} h_\nu(x)\zeta(x+\nu) \quad (\nu = 0, \ldots, n-1),$$

by (23a)

$${}_1H(x, \varrho(x)) = \sum_{m=2}^{\infty} H_N^m(x) \sum_{i_0+\ldots+i_{n-1}=m} \alpha_{i_0,\ldots,i_{n-1}}(x) h_0^{i_0}(x) \cdots h_{n-1}^{i_{n-1}}(x)$$
$$\cdot x^{i'N\mu_1} \zeta^{i_0}(x)\zeta^{i_1}(x+1) \cdots \zeta^{i_{n-1}}(x+n-1)$$

where $i' = i_1 + 2i_2 + \ldots + (n-1)i_{n-1}$. It is to be noted that $i' \geqq 0$ [45]). Since $\mu_1 (\leqq 0)$ is rational, by (24b),

$$h_0^{i_0}(x) \cdots h_{n-1}^{i_{n-1}}(x) x^{i'N\mu_1} = [x]_0.$$

Hence

$$(34) \qquad {}_1H(x, \varrho(x), \ldots, \varrho(x+n-1)) = H_N^2(x) H'(x, \zeta(x)),$$

$$(34a) \qquad H'(x, \zeta(x)) =$$
$$= \sum_{m=2}^{\infty} H_N^{m-2}(x) \sum_{i_0+\ldots+i_{n-1}=m} {}_1\alpha_{i_0,\ldots,i_{n-1}}(x) \zeta^{i_0}(x) \cdots \zeta^{i_{n-1}}(x+n-1)$$

where

$$(34b) \qquad {}_1\alpha_{i_0,\ldots,i_{n-1}}(x) = [x]_0 \qquad (x \text{ in } K).$$

For x in K, the convergence properties of the series (34a) will be of the same description as those stated for the series representing $H'(x, \zeta(x))$ in the case I. Since

$$F_N(x) = H_N(x) x^{\Psi_N + Nr' + \frac{w}{\alpha} + \varepsilon - t_N} G_N(x)$$

---

[45]) The equality sign is attained for $i_0 = m$, $i_1 = i_2 = \ldots = i_{n-1} = 0$.

(Ψ_N defined by (32)), in view of (26) (with $L'(\zeta(x))$ given by (33), (33a) and (34)), from (23) it is inferred that *the transformed equation is of the form* (29), *where* $L'(\zeta(x))$ *is given by* (33) and (33a), τ' *is defined as in* (29a) (*with* Ψ_N *given by* (32)), *the function* $G_N(x)$ *is subject to the inequality* (20a) (x *in* K), *and* $H'(x, \zeta(x))$ *is defined by* (34a), (34b); *moreover,*

$$\tau'' = \frac{w}{\alpha} - nN\mu_1.$$

LEMMA 5. *Consider case* II (§ 3) *and the formal solution of* (A; § 1), *relating to this case and specified in Lemma* 3 (§ 3). *Let* N *be an integer, however large. Let* $Y(x)$ *be the function of* (31). *Let the periodic functions* (5), *involved in the formal solution, be subject to the same conditions as stated in Lemma* 4. *For* x *in* K $Y(x)$ *will satisfy conditions analogous to those which were stated with respect to the function* $Y(x)$ *of Lemma* 4. *The transformation*

(35) $$y(x) = Y(x) + H_N(x)\zeta(x) \qquad \text{(cf. (31), (24))}$$

will yield the equation (29), *the expressions therein involved being specified by the italicized statement preceding this Lemma. Moreover, the series* (34*a*) *representing* $H'(x, \zeta(x))$ *will have, for* x *in* K, *the same convergence properties as the corresponding series* (27a) *of Lemma* 4.

The function $H_N(x)$ involved in Lemmas 4 and 5 contains a positive number t_N whose value will be specified more precisely in the sequel.

5. Solutions for the case I.

The transformation (30; § 4) applied to (A; § 1) will yield, when the case I (§ 2) is on hand, the equation (29; § 4), (29a; § 4) referred to in Lemma 4 (§ 4). A solution of this equation will be found in the form of a convergent series

(1) $$\zeta(x) = \zeta_0(x) + \zeta_1(x) + \dots$$

On writing

(1a) $$z_j(x) = \zeta_0(x) + \zeta_1(x) + \dots + \zeta_j(x) \qquad (j = 0, 1, \dots)$$

the terms of the series (1) will be determined in succession with the aid of the equations

(2) $$L'(\zeta_0(x)) = t_0(x) = x^{\tau'} h_n^{-1}(x) G_N(x),$$

(2a) $$L'(\zeta_1(x)) = t_1(x) = x^{\tau''} h_n^{-1}(x) H_N(x) H'(x, z_0(x)),$$

(2b) $$L'(\zeta_j(x)) = t_j(x) = \\ = x^{\tau''} h_n^{-1}(x) H_N(x) \left(H'(x, z_{j-1}(x)) - H'(x, z_{j-2}(x))\right) \\ (j = 2, 3, \ldots).$$

Under appropriate convergence conditions, which will be proved in the sequel, the series (1) will represent a solution of (29; § 4). This can be inferred by adding the corresponding members of the equations (2), (2a), (2b) ($j=2, 3, \ldots$).

The equation $L'(\zeta(x)) = 0$ (cf. (26a; § 4)) has solutions which, in view of (26; § 4) and (25; § 4) are of the form

(3) $$\zeta(x) = H_N^{-1}(x) \varrho(x)$$

where $\varrho(x)$ satisfies $L_1(\varrho(x)) = 0$ (cf. (23b; § 4), (23c; § 4)). The solutions of $L_1(\varrho(x)) = 0$ are asymptotically the same as those which one would obtain solving $L(\varrho(x)) = 0$ [46]). This is a consequence of the relations (23c; § 4).

Consider the equation

(4) $$L'(\zeta(x)) = t(x).$$

On multiplying the both members of (4) by $H_N(x) x^{nN\mu_1} h_n(x)$ in view of (26; § 4) it is observed that (4) is equivalent to

(4a) $$L_1(\varrho(x)) = H_N(x) x^{nN\mu_1} h_n(x) t(x) \quad (\text{cf. } (3)).$$

It is to be noted that in section 2 the equation $L(y_j(x)) = x^{\frac{w}{\alpha}} \Psi_j(x)$ was solved with the aid of formulas (44; § 2), (44a; § 2). Thus, on replacing $y_j(x)$ by $\varrho(x)$ and $x^{\frac{w}{\alpha}} \Psi_j(x)$ by the second member of (4a), as well as recalling the statement subsequent to (3), it is concluded that a solution of (4a) can be given by

(5) $$\varrho(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda} \varrho_\lambda(x)\, \Gamma_\lambda^x(t(u)),$$

(5a) $$\Gamma_\lambda^x(t(u)) = \mathop{\mathrm{S}}_{u=x} H_N(u) e^{-Q_\lambda(x)} u^{nN\mu_1 - r_\lambda + \overline{m} - \frac{w}{\alpha}} \bar{\varrho}_\lambda(u) t(u)$$

where

(5b) $$\varrho_\lambda(x) = [x]_{\nu(\lambda)}, \quad \bar{\varrho}_\lambda(x) = [x]_{q(\lambda)}, \\ |\varrho_\lambda(x)|, \; |\bar{\varrho}_\lambda(x)| < h_1 |x|^\varepsilon \quad (\varepsilon > 0;\ \lambda = 1, \ldots, n;\ x \text{ in } K).$$

Substitution of (24; § 4) into (5a) will yield, by (5b), the inequalities

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[46]) $L(y(x))$ is the part of $y(x+n) - x^{\frac{w}{\alpha}} a(x, y(x))$ (cf. (A; § 1)) linear in $y(x), y(x+1), \ldots, y(x+n-1)$ (cf. (16; § 2)).

$$(6) \qquad |\Gamma_\lambda^x(t(u))| < h_1 f^N(v) \underset{u=x}{\mathbf{S}} |e^{NQ_1(u)-Q_\lambda(u)} u^{-r_\lambda}|\, |u|^{t'_N} |t(u)|$$

$$\left(t'_N = t_N + nN\mu_1 + \bar{m} - \frac{w}{\alpha} + \varepsilon;\ \lambda = 1, \ldots, n;\ x \text{ in } K\right)$$

provided that, with $B_\lambda(u)$ denoting the "summand" displayed in (6), the series

$$(6a) \qquad \underset{u=x}{\mathbf{S}} B_\lambda(u) = B(x-1) + B(x-2) + \ldots$$

converges for x in K $(\lambda = 1, \ldots, n)$ [47].

If both members of (5) are multiplied by $H_N^{-1}(x)$, in view of (3) it is concluded that

$$(7) \qquad \zeta(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x) - NQ_1(x)} x^{r_\lambda} f^{-N}(v) x^{-t_N} \varrho_\lambda(x) \Gamma_\lambda^x(t(u))$$

constitutes a solution of (4), provided the series (6a) $(\lambda = 1, 2, \ldots)$ converge. In consequence of (6), (5b) and (7) it is inferred that the solutions of the equations (2), (2a), (2b) $(j = 2, 3, \ldots)$ satisfy the inequalities

$$(8) \qquad |\zeta_j(x)| < h_1^2 \sum_{\lambda=1}^{n} |e^{Q_\lambda(x) - NQ_1(x)} x^{r_\lambda}|\, |x|^{-t_N + \varepsilon} \underset{u=x}{\mathbf{S}} B_{\lambda,j}(u)$$

$$(j = 0, 1, \ldots)$$

in any region in which the series

$$(8a) \qquad \underset{u=x}{\mathbf{S}} B_{\lambda,j}(u) = \sum_{\nu=1}^{\infty} B_{\lambda,j}(x-\nu) \qquad (\lambda = 1, \ldots, n)$$

converge. Here

$$(8b) \qquad B_{\lambda,j}(u) = |e^{NQ_1(u) - Q_\lambda(u)} u^{-r_\lambda}|\, |u|^{t'_N} |t_j(u)|$$

where t'_N is defined by (6) and the $t_j(u)$ are determined in succession with the aid of (2), (2a), (2b) $(j = 2, 3, \ldots)$.

The number t_N involved in τ' (cf. (29; § 4)) can be chosen suitably great so that equation (2) has a solution $\zeta_0(x)$ such that

$$(9) \qquad |\zeta_0(x+\nu)| < l_0 \quad (\nu = 0, 1, \ldots, n-1;\ x \text{ in } K_{\lambda'}).$$

The details of the proof of this fact will be omitted. However, we shall note that for $\nu = 0$ (9) can be secured with

$$(10) \qquad t_N \geqq \Psi_N + Nr' + \bar{m} + 3\varepsilon + 1.$$

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[47]) The „sums" are throughout evaluated by means of series.

Here, as seen from the developments of section 2,

$$\Psi_N = \alpha_0 N^2 + (\alpha_1+\mu_1)N - 2\alpha_1, \quad \alpha_0 = (n-2)\frac{\mu}{2} \tag{11}$$

$$\text{(when } \mu_1 \neq 0 \text{ and } n > 1\text{)},$$

$$\Psi_N = N(\bar{m}+h) - 2(\bar{m}+h) \quad \text{(when } \mu_1 = 0\text{)}, \tag{11a}$$

$$\Psi_N = 0 \quad \text{(when } \mu_1 \neq 0 \text{ and } n = 1\text{)}. \tag{11b}$$

The inequalities (9) can be established in succession for $\nu = 1, 2, \ldots, n-1$ with the aid of the equation (2) itself, provided we take

$$t_N = \alpha_0 N^2 + \alpha_1' N + \alpha_2' \tag{12}$$

where α_1' (> 0), α_2' are suitable numbers. Moreover, to satisfy (9) it is noted that *λ' can be selected independent of N when $\mu_1 = 0$ and also when $n \leqq 2$; λ' may depend on N when $\mu_1 \neq 0$ and $n > 2$.*

Now the series $H'(x, {}_0\zeta, \ldots, {}_{n-1}\zeta)$ related to the transformation of Lemma 4 (§ 4) converges, as stated in section 4, provided (28; § 4) holds. It is to be noted that *$\zeta'(=\zeta'(\lambda'))$ can be made as great as desired by taking λ' sufficiently great or by taking $f(v)$ sufficiently small. Thus,* by whichever device, *the inequality*

$$\zeta' > 2l_0 \tag{13}$$

can certainly be secured.

We shall now proceed to show that the functions $\zeta_j(x)$ can be determined in succession subject to the inequalities

$$|\zeta_j(x+\nu)| < l_0 2^{-j} \tag{14}$$

$$(\nu = 0, 1, \ldots, n-1;\ j = 1, 2, \ldots;\ x+n-1 \text{ in } K_{\lambda'}).$$

If (14) were demonstrated *the series* (1) *would be absolutely and uniformly convergent and it would represent a solution* $\zeta(x)$ of (29; § 4); *moreover, we would have*

$$|\zeta(x+\nu)| < 2l_0 \ (< \zeta') \tag{15}$$

$$(\nu = 0, 1, \ldots, n-1;\ x+n-1 \text{ in } K_{\lambda'}).$$

In view of the character of the series $H'(x, {}_0z, {}_1z, \ldots, {}_{n-1}z)$ it is concluded that *there exists a constant M such that, provided*

$$|{}_\nu z_j| \leqq 2l_0, \ |{}_\nu z_{j-1}| \leqq 2l_0 \qquad (\nu = 0, \ldots, n-1), \tag{16}$$

necessarily

$$\begin{aligned} &|H'(x, {}_0z_j, \ldots, {}_{n-1}z_j) - H'(x, {}_0z_{j-1}, \ldots, {}_{n-1}z_{j-1})| \\ &< M[|{}_0\zeta_j| + |{}_1\zeta_j| + \ldots + |{}_{n-1}\zeta_j|] \qquad (x+n-1 \text{ in } K_{\lambda'}) \end{aligned} \tag{16a}$$

where ${}_\nu\zeta_j = {}_\nu z_j - {}_\nu z_{j-1}$.

Write $z_{-1}(x) \equiv 0$. Then $H'(x, z_{-1}(x)) = 0$. Whence, in view of (16), (16a) and (9) (where $\zeta_0(z) = z_0(x)$) it is concluded that

$$(16b)\quad |H'(x, z_0(x))| < M[|\zeta_0(x)| + |\zeta_0(x+1)| + \ldots + |\zeta_0(x+n-1)|] < nMl_0 \qquad (x+n-1 \text{ in } K_{\lambda'}).$$

Write

$$(17)\qquad W(x) = x^{\tau''} h_n^{-1}(x) H_N(x).$$

Let w'' be a positive number, to be specified more precisely in the sequel. By (29a; § 4) and (24; § 4) we have, with $C(>0)$ assigned however small,

$$(17a)\quad |W(x)| = |e^{Q_1(x)} f(v)|^N |x|^{t_N + \tau''} |h_n^{-1}(x)| < |x|^{-w''-2} C \qquad (x \text{ in } K_{\lambda'}),$$

provided λ' (depending on C) is sufficiently great; here λ' can be selected independent of N whenever Ψ_N and w'' are linear in N [48]). This follows by (6; § 4). By (2a), (17), (17a), (16b)

$$(18)\qquad |t_1(x)| < nMl_0 C\, |x|^{-w''-2} \quad (x+n-1 \text{ in } K_{\lambda'}).$$

By (8b; $j=1$) and (18)

$$(19)\quad B_{\lambda,1}(x) < nMl_0 C\, |e^{NQ_1(x) - Q_\lambda(x)} x^{-r_\lambda}|\, |x|^{t'_N - w''} |x|^{-2} = B^*_{\lambda,1}(x) |x|^{-2} \quad (x+n-1 \text{ in } K_{\lambda'}).$$

Choose λ' sufficiently great so that, for $x+n-1$ in $K_{\lambda'}$, the functions $B^*_{\lambda,1}(x)$ ($\lambda = 1, \ldots, n$; cf. (19)) are monotone to the left [49]). Such a choice is possible since the functions $|\exp Q_1(x)|$, $f_{1,\lambda}(x)$ (cf. (27; § 2)) possess this property. It is again noted that λ' is independent of N when Ψ_N is linear in N (provided w' is linear in N) [50]). In consequence of the above italicized statement and by (19)

$$(19a)\quad \mathop{\mathrm{S}}_{u=x} B_{\lambda,1}(u) = \sum_{\nu=1}^{\infty} B_{\lambda,1}(x-\nu) < \sum_{\nu=1}^{\infty} B^*_{\lambda,1}(x-\nu) |x-\nu|^{-2} < B^*_{\lambda,1}(x) \sum_{\nu=1}^{\infty} |x-\nu|^{-2} < \omega B^*_{\lambda,1}(x) |x|^{-1} \quad (x+n-1 \text{ in } K_{\lambda'})\ [51]).$$

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[48]) By (12) and since $\tau'' = \frac{w}{\alpha} + (n-2)N\mu_1$ we then will have $t_N + \tau'' + w''$ linear in N. If w'' is linear in N but Ψ_N is quadratic then t_N is given by (12), with $\alpha_0 \neq 0$, and $t_N + \tau'' + w''$ will contain the quadratic term $\alpha_0 N^2$.

[49]) It will be shown in the sequel that w'' can be taken as the greater one of the numbers 0 and H (cf. (20a)).

[50]) The quadratic term in Ψ_N, t_N (12), t'_N (6) is present or not simultaneously. If present, it is $\alpha_0 N^2$.

[51]) Here ω is from a special case of the inequality of Birkhoff, $\sum_1^{\infty} |x-\nu|^{-2} < \omega x^{-1}$.

Thus, by (8; $j=1$), (19a) and (19)

$$|\zeta_1(x)| < h_1^2 n^2 \omega M l_0 C |x|^{-w''+H} \quad (x+n-1 \text{ in } K_{\lambda'}), \tag{20}$$

$$H = nN\mu_1 + \bar{m} - \frac{w}{\alpha} + 2\varepsilon - 1. \tag{20a}$$

At this stage it will be convenient to prove the following Lemma.

Lemma 6. *Consider the equation*

$$L'(\zeta(x)) = t(x), \tag{21}$$

where

$$L'(\zeta(x)) = \zeta(x+n) - x^{w'} \sum_{i=0}^{n-1} b'_i(x)\zeta(x+i) \tag{21a}$$

$$\left(w' = \frac{w}{\alpha} - N\mu_1;\ b'_i(x) = [x]_0 \quad (x+n-1 \textit{ in } K);\ \text{cf. } (26a;\ \S\, 4)\right)$$

is the difference polynomial involved in (2). *Let* $t(x)$ *be a function such that*

$$|t(x)| < t\,|x|^{-w''-2} \qquad (x+n-1 \textit{ in } K_{\lambda'}) \tag{21b}$$

and let λ' *be a number satisfying the italicized statements subsequent to* (17a) *and* (19). *Then, provided the number* w'' (*independent of* t) *is sufficiently great, equation* (21) *will possess a solution* $\zeta(x)$ *for which*

$$|\zeta(x+\nu)| < tE \quad (\nu = 0, 1, \ldots, n-1;\ x+n-1 \textit{ in } K_{\lambda'}) \tag{22}$$

where E *is a constant depending only on the operator* L'.

On taking account of how (20) was established as a consequence of (18), when solving the equation $L(\zeta_1(x)) = t_1(x)$, it is concluded that (21), (21a), (21b) imply that

$$|\zeta(x)| < tE_0 |x|^{-w''+H},\ E_0 = h_1^2 n\omega \tag{23}$$

when $x+n-1$ is in $K_{\lambda'}$. Here H is given by (20a). Take w'' so that

$$w'' \geqq 0,\ w'' \geqq H. \tag{23a}$$

The equation (21) can be written in the form

$$\zeta(x+\nu) = (x-n+\nu)^{w'} \sum_{r=\nu-n}^{\nu-1} b'_{r+n-\nu}(x-n+\nu)\zeta(x+r) + t(x-n+\nu). \tag{24}$$

When $\mu_1 = 0$ we have $\frac{w}{\alpha} = 0$ so that $w' = 0$. If $\mu_1 \neq 0$ (i.e.

$\mu' > 0$) with N sufficiently great it will follow that $w' \leqq 0$. Thus, in any case $w' < 0$. By (21a)

(24a) $$|b'_{r+n-u}(x-n+\nu)(x-n+\nu)^{w'}| < b'|x|^{w'}$$
$$(x+n-1 \text{ in } K;\ \nu = 0, 1, \ldots, n-1).$$

Also, in view of (21b),

(24b) $$|t(x-n+\nu)| < t|x|^{-w''-2}$$
$$(x+n-1 \text{ in } K_{\lambda'};\ \nu = 0, \ldots, n-1).$$

Thus, in consequence of (24)

(25) $$|\zeta(x+\nu)| < b'|x|^{w^1} \sum_{r=\nu-n}^{\nu-1} |\zeta(x+r)| + t|x|^{-w''-2}$$
$$(x+n-1 \text{ in } K_{\lambda'};\ \nu = 0, \ldots, n-1).$$

By (23) and since $-w'' + H \leqq 0$

(26) $$|\zeta(x+r)| < tE_0|x|^{-w''+H}$$
$$(x+n-1 \text{ in } K_{\lambda'};\ n = 1-n, 2-n, \ldots, 0).$$

Thus, by (25; $\nu = 1$) and (26) we have, for $x+n-1$ in $K_{\lambda'}$,

(27) $$|\zeta(x+1)| < t[nb'E_0|x|^{w'-w''+H} + |x|^{-w''-2}].$$

It can be always supposed that $|x| \geqq 1$. In view of (20a) and (21a) it is observed that $-2 - w' - H < 0$. Thus by (27) we have, when $x+n-1$ is in $K_{\lambda'}$,

(27a) $$|\zeta(x+1)| < tE_1|x|^{w'-w''+H},\quad E_1 = nb'E_0 + 1.$$

This implies that (23) can be replaced by a more exact inequality, whenever $w' < 0$. For the purpose on hand the additional precision is not necessary. Accordingly, in view of (27a) and since $w' \leqq 0$, we shall write

(27b) $$|\zeta(x+1)| < tE_1|x|^{-w''+H} \quad (x+n-1 \text{ in } K_{\lambda'}).$$

When $n = 1$ the Lemma is established in consequence of (23) and since $-w'' + H \leqq 0$. When $n = 2$ the truth of the Lemma is inferred with the aid of (27b).

Consider the remaining case when $n > 2$. *Suppose that for some* i $(2 \leqq i \leqq n-1)$

(28) $$|\zeta(x+r)| < tE_r|x|^{-w''+H}$$
$$(r = 0, 1, \ldots, i-1;\ x+n-1 \text{ in } K_{\lambda'})$$

where

(28a) $$E_r = nb'E_{r-1} + 1 \qquad (r = 1, \ldots, i-1).$$

The relations (28), (28a) have been previously established for $i=2$. Since $-w''+H \leqq 0$ and $|x| \geqq 1$ it follows that (with $\Re x \leqq 0$) $|x-\lambda|^{-w''-H} \leqq |x|^{-w''+H}$ whenever $\lambda \geqq 0$. Hence the particular inequality (28; $r=i-1$) will imply

(28b) $$|\zeta(x+r)| < tE_{i-1}|x|^{-w''+H}$$
$$r=i-1, i-2, \ldots, 0, -1, \ldots; \quad (x+n-1 \text{ in } K_{\lambda'}).$$

In consequence of (28b) and (25; $\nu=i$)

(29) $$|\zeta(x+i)| < t[nb'E_{i-1}|x|^{w'-w''+H} + |x|^{-w''-2}]$$
$$(x+n-1 \text{ in } K_{\lambda'}).$$

Since $-2-w'-H<0$ and $w' \leqq 0$ it follows that

(29a) $$|\zeta(x+i)| < tE_i|x|^{w'-w''+H} \leqq tE_i|x|^{-w''+H}$$
$$(E_i = nb'\,E_{i-1}+1; \; x+n-1 \text{ in } K_{\lambda'}).$$

Accordingly, by induction it is inferred that

(30) $$|\zeta(x+\nu)| < tE_{n-1}|x|^{-w''+H}$$
$$(\nu=0, 1, \ldots, n-1; \; x+n-1 \text{ in } K_{\lambda'})$$

where E_{n-1} is a number, depending only on L' and determined by means of the relations (28a; $r=1, 2, \ldots, n-1$) (cf. (23)).

Thus it is observed that *the above Lemma holds with* $E=E_{n-1}$ (*cf.* (30)) *and* w'' *equal to the greater one of the numbers* 0 *and* H (*cf.* (20a)). Consequently it is noted that w'' can be taken independent of N, when $\mu_1=0$, and linear in N in the contrary case.

In consequence of Lemma 6 and of (18), (20) it is seen that the equation (2a) has a solution $\zeta_1(x)$ such that

(31) $$|\zeta_1(x+\nu)| < t_1E \quad (\nu=0, \ldots, n-1; \; x+n-1 \text{ in } K_{\lambda'})$$

where $t_1=nMl_0C$. *Now* E *is independent of* C. As stated before, $C(>0)$ can be chosen at will [52]). To achieve conformity with (14; $j=1$) take C so that $t_1E \leqq l_0 2^{-1}$; that is,

(32) $$C \leqq \frac{1}{2nME}.$$

We thus have (14) satisfied for $j=0$ (cf. (9)) and for $j=1$, so that

$$|z_0(x+\nu)| < 2l_0, \quad |z_1(x+\nu)| < 2l_0$$
$$(\nu=0, \ldots, n-1; \; x+n-1 \text{ in } K_{\lambda'}; \text{ cf. (1a)}).$$

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[52]) In general λ' increases as C is diminished (cf. (17a)).

Whence, in view of the statement in connection with (16), (16a),

$$(33)\quad |H'(x, z_1(x)) - H'(x, z_0(x))| < M[|\zeta_1(x)| + \ldots + |\zeta_1(x+n-1)|] < nMl_0\, 2^{-1} \qquad (x+n-1 \text{ in } K_{\lambda'}).$$

By (17), (2b; $j=2$) and in consequence of (33)

$$(34)\qquad |t_2(x)| < |W(x)|\, nMl_0\, 2^{-1} \quad (x+n-1 \text{ in } K_{\lambda'}).$$

Thus, by (17a) and (32),

$$(34a)\qquad |t_2(x)| < t_2|x|^{-w''-2} \qquad (t_2 = l_0 E^{-1} 2^{-2})$$

when $x+n-1$ is in $K_{\lambda'}$. Lemma 6 (with $t = t_2$) is applicable to the equation (2b; $j=2$). Hence this equation possesses a solution $\zeta_2(x)$ for which

$$(35)\qquad \zeta_2(x+\nu) < t_2 E = l_0 2^{-2} \qquad (\nu = 0, 1, \ldots, n-1;\ x+n-1 \text{ in } K_{\lambda'})\,..$$

This establishes (14; $j=2$).

Suppose now that (14) holds for $j = 0, 1, \ldots, r-1$ $(r \geqq 3)$. By (1a) it is then inferred that, for $x+n-1$ in $K_{\lambda'}$,

$$|z_j(x+\nu)| \leqq |\zeta_0(x+\nu)| + \ldots + |\zeta_j(x+\nu)| < l_0(1+2^{-1} + \ldots + 2^{-j}) < 2l_0 \qquad (\nu = 0, \ldots, n-1;\ j = 0, \ldots, r-1).$$

Thus, with the aid of the statement in connection with (16), (16a) it is concluded that

$$(36)\quad |H'(x, z_{r-1}(x)) - H'(x, z_{r-2}(x))| < M[|\zeta_{r-1}(x)| + \ldots + |\zeta_{r-1}(x+n-1)|] < nMl_0 2^{-(r-1)} \qquad (x+n-1 \text{ in } K_{\lambda'}).$$

Furthermore it will follow that

$$|t_r(x)| < |W(x)|\, nMl_0 2^{-(r-1)} \quad (x+n-1 \text{ in } K_{\lambda'})$$

(cf. (2b; $j=r$)).

Whence, by virtue of (17a) and (32),

$$(37)\qquad |t_r(x)| < t_r|x|^{-w''-2} \qquad (t_r = l_0 E^{-1} 2^{-r})$$

for $x+n-1$ in $K_{\lambda'}$. Application of Lemma 6 to the equation (2b; $j=r$) enables us to assert that

$$(38)\qquad |\zeta_r(x+\nu)| < t_r E = l_0 2^{-r} \qquad (\nu = 0, \ldots, n-1;\ x+n-1 \text{ in } K_{\lambda'}).$$

Thus by induction (14) has been established for $j=0, 1, \ldots$ ($x+n-1$ in $K_{\lambda'}$). The series (1) will accordingly possess the properties stated in connection with (15). It will represent a solution of the equation (29; § 4).

LEMMA 7. *Consider case* I (§ 2) *and let K be a corresponding region. Let*

$$(39) \qquad s(x) = \sum_{j=1}^{\infty} y_j(x) p_1^j(x) \qquad \left(y_j(x) = e^{jQ_1(x)} x^{y_j} [x]_{N(j)}^j;\ x \text{ in } K\right)$$

be the formal solution of (A; § 1) *relating to this case* (*cf.* *Lemma* 2 (§ 2)). *The constants* $y_j = \alpha_0 j^2 + \alpha_1 j - \alpha_2$ $(j=1, 2, \ldots)$ *are given by* (69; § 2), (69a; § 2), (69b; § 2), (69c; § 2). *Let K_r denote the subset of K for which $|x| \geqq r$.*

Given a positive integer N, however large, there exists a number λ' which may depend on N when y_j is quadratic in j and which is independent of N when y_j is linear in j so that, provided the arbitrary periodic functons

$$p_1(x),\ c_2(x), \ldots, c_\delta(x)$$

involved in the formal solution are subject to the conditions stated in connection with (5; § 4), (5a; § 4), (6; § 4), *we have a solution $y(x)$ of* (A; § 1) *such that*

$$(39\text{a}) \qquad y(x) \sim s(x) \qquad (x \text{ in } K_{\lambda'}).$$

The asymptotic relation (39a) *is in the following sense:*

$$(40) \qquad y(x) = \sum_{j=1}^{N-1} y_j(x) p_1^j(x) + e^{NQ_1(x)} f^N(v) x^{t_N} \zeta(x) \qquad (cf.\ (12)).$$

Here $f(v)$ ($v = \Im(x)$; cf. (5a; § 4)) *is to be chosen sufficiently small* (*depending on N*), *t_N is given by* (12) *and $\zeta(x)$ is a function such that*

$$(41) \qquad |\zeta(x+\nu)| < 2\, l_0$$
$$(\text{cf. } (15);\ \nu = 0, 1, \ldots, n-1;\ x+n-1 \text{ in } K_{\lambda'}).$$

The solution $y(x)$ is analytic in every finite part of $K_{\lambda'}$. *At $x=\infty$ it has a singular point.*

NOTE. The function $f(v)$ of (5a; § 4) can be taken independent of N in which case λ', in general, would have to be chosen depending on N even if y_j is linear in j. To take $f(v)$ "sufficiently small" merely means to take $f(v) = c f_1(v)$ where c is a sufficiently small constant and $f_1(v)$ is a function such that

$$e^{Q_1(x)} f_1(v) \sim 0 \qquad (x \text{ in } K)\ ^{53}).$$

53) The asymptotic relation here is with respect to x.

The choice of λ' is conditioned by the several italicized statements subsequent to (8; § 4), (12; § 5), in connection with (13; § 5) and subsequent to (17a), (19).

6. The existence theorem.

We shall now proceed to establish an analogue to Lemma 7 corresponding to case II (§ 3). We now have $\mu_1 \leqq 0$.

It will be convenient to state briefly certain previously established, but scattered in various places, facts relating to case II. The constants y_j involved in the coefficients of the formal solution of (A; § 1) are

$$(1) \qquad y_j = b_1 j + b_2 \qquad (j=1,2,\ldots),$$

$$(1a) \qquad b_1 = \bar{m}, \qquad b_2 = -m \qquad (\mu_1 < 0),$$

$$(1b) \qquad b_1 = \bar{m} + h', \qquad b_2 = -(\bar{m}+h') \qquad (\mu_1 = 0).$$

The transformation of Lemma 5 (§ 4) is

$$(2) \qquad y(x) = Y(x) + H_N(x)\,\zeta(x), \quad Y(x) = \sum_{j=1}^{N-1} y_j(x) p_1^j(x),$$

$$(2a) \qquad H_N(x) = e^{NQ_1(x)} f^N(v) x^{t_N}$$

where $t_N(\geqq 0)$ is for the present undefined. The result of this transformation is

$$(3) \qquad L'(\zeta(x)) = x^{\tau'} h_n^{-1}(x) G_N(x) + W(x) H'(x, \zeta(x)),$$

$$(3a) \qquad \tau' = y_{N-1} + Nr' + \frac{w}{\alpha} + \varepsilon - t_N - nN\mu_1,$$

$$(3b) \qquad |G_N(x)| < G_N \qquad (x \text{ in } K),$$

$$(3c) \qquad W(x) = x^{\tau''} h_n^{-1}(x) H_N(x) \qquad \left(\tau'' = \frac{w}{\alpha} - nN\mu_1\right).$$

The series (34a; § 4) representing $H'(x, \zeta_0, \ldots, \zeta_{n-1})$ converges for x in $K_{\lambda'}$, provided

$$(4) \qquad |\zeta_i| \leqq \zeta' \qquad (i=0,\ldots,n-1).$$

The number ζ' can be made as great as desired by suitable choice of λ' or $f(v)$.

With the symbols involved possessing the meaning just indicated we proceed as in case I up to and including formula (5; § 5). The „sums" are now to be evaluated according to the formula

$$(5) \qquad \underset{u=x}{\mathbf{S}}\, \Psi(u) = -\Psi(x) - \Psi(x+1) - \ldots \qquad (x \text{ in } K).$$

Finally, it can be shown that, for the case under consideration, inequalities (8; § 5), (8b; § 5) hold wherever the series

$$\sum_{\nu=0}^{\infty} B_{\lambda,j}(x+\nu) \qquad (\lambda=1,\ldots,n) \tag{6}$$

converge.

In consequence of (2; § 5) and of (3b) and (3a) it is concluded that, provided

$$t_N = b_1'N + b_2' \tag{7}$$

where b_1', b_2' are suitable numbers (independent of N) [54]), we shall have

$$|\zeta_0(x)| < l_0 \qquad (x \text{ in } K_{\lambda'})\ ^{55}). \tag{8}$$

As in case I (cf. (13; § 5)), it is now arranged that $\zeta' > 2l_0$ (ζ' the number involved in (4)).

In order to obtain a bounded solution of (3) it will be sufficient to secure the inequalities

$$|\zeta_j(x)| < l_0 2^{-j} \qquad (j=1,2,\ldots;\ x \text{ in } K_{\lambda'}). \tag{9}$$

Following the lines of the corresponding developments presented in section 5, these inequalities are proved. In carrying out the details of this proof *no Lemma of the type of Lemma* 6 (§ 5) *is necessary*; moreover, all the statements of the demonstration are to be made for x in $K_{\lambda'}$.

LEMMA 8. *Consider case* II (§ 3) *and let K be a corresponding region. Let*

$$s(x) = \sum_{j=1}^{\infty} y_j(x)p_1^j(x) \quad (y_j(x) = e^{jQ_1(x)}x^{y_j}[x]^j_{N(j)};\ x \text{ in } K) \tag{10}$$

be the formal solution of (A; § 1) (*cf. Lemma* 3 (§ 3)). *The constants* $y_j = b_1 j + b_2$, *here involved, are given by* (1a), (1b).

Given a positive integer N, however large, a number λ′ can be found, independent of N, so that, with the δ arbitrary periodic functions involved in (10) *subject to the conditions of the type imposed in Lemma* 7, *there is on hand a solution y(x) of the equation* (A; § 1) *such that*

$$y(x) \sim s(x) \qquad (x \text{ in } K_{\lambda'}). \tag{11}$$

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[54]) Such a choice can be made.

[55]) (6; § 5) implies (9; § 5) as relating to case II.

The asymptotic relation (11) *implies that* $y(x)$ *is of the form*

$$(12) \qquad y(x) = \sum_{j=1}^{N-1} y_j(x) p_1^j(x) + e^{NQ_1(x)} f^N(v) x^{tN} \zeta(x) \qquad (cf.\ (7))$$

where

$$(12a) \qquad |\zeta(x)| < 2l_0 \qquad (x \text{ in } K_{\chi'}).$$

Here $f(v)$ $(v = \Im x)$ *is the function involved in the inequalities satisfied by the periodic functions and* (*depending on* N) *is to be chosen sufficiently small. The solution* $y(x)$ *is analytic in* $K_{\chi'}$ $(x \neq \infty)$.

It is possible to take $f(x)$ independent of N. But then λ' may have to depend on N.

It will be now proved that, under Hypothesis A (§ 1), either case I (§ 2) or case II (§ 3) is certain to be on hand. For this purpose the following Lemma will be essential.

LEMMA 9. *Write* $x = \varrho e^{\sqrt{-1}\,\Theta}$. *Suppose that no curve extending to infinity and satisfying the equation*

$$(13) \qquad \Re\left[q_0 x^{\frac{r}{p}} + q_1 x^{\frac{r-1}{p}} + \ldots + q_{r-1} x^{\frac{1}{p}}\right] = 0$$

$$(\textit{integer } p \geqq 1;\ 1 \leqq r \leqq p;\ q_0 \neq 0;\ q_i = |q_i| e^{\sqrt{-1}\,\bar{q}_i})$$

is coincident with, say, the ray $\Theta = \pi$. *If, then, there exists an infinite branch* B *satisfying* (13) *and having at infinity the limiting direction* π, *necessarily* B *will recede indefinitely from this ray* [56]).

It is observed that (13) can be written in the form

$$(14) \qquad 0 = H(\varrho, \Theta) \equiv \sum_{i=0}^{r-i} |q_i| \varrho^{\frac{r-i}{p}} \cos\left(\bar{q}_i + \frac{r-i}{p}\,\Theta\right).$$

Since there exists a curve B satisfying the conditions stated in the Lemma necessarily

$$(15) \quad \cos\left(\bar{q}_0 + \frac{r}{p}\,\pi\right) = \cos\left(\bar{q}_1 + \frac{r-1}{p}\pi\right) = \ldots =$$

$$= \cos\left(\bar{q}_{\gamma-1} + \frac{r-\gamma+1}{p}\,\pi\right) = 0,$$

$$\cos\left(\bar{q}_\gamma + \frac{r-\gamma}{p}\,\pi\right) \neq 0 \quad (1 \leqq \gamma \leqq r-1;\ q_\gamma \neq 0)\,^{57}).$$

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[56]) That is, along B, $\varrho \sin(\Theta - \pi)$ (or just $\varrho(\Theta - \pi)$) will approach $\pm\infty$ as $\varrho \to \infty$.

[57]) In (15) the expressions $\cos\left(\bar{q}_i + \dfrac{r-i}{p}\pi\right)$ which correspond to $q_i = 0$ are to be deleted.

The equation of B is

(16) $$\Theta = \pi + h(\varrho) \qquad (h(\varrho) \to 0 \text{ as } \varrho \to \infty).$$

To obtain more information regarding $h(\varrho)$ substitute (16) in (14). Thus, in view of (15),

$$0 = H(\varrho, \Theta) = \sum_{i=0}^{\gamma-1} \lambda_i |q_i| \varrho^{\frac{r-i}{p}} \sin\left(\frac{r-i}{p} h(\varrho)\right)$$
$$+ \sum_{i=\lambda}^{r-1} |q_i| \varrho^{\frac{r-i}{p}} \cos\left[\bar{q}_i + \frac{r-i}{p}(\pi + h(\varrho))\right] \qquad (\lambda_i = \pm 1).$$

Whence on writing

$$\sin\left[\frac{r-i}{p} h(\varrho)\right] = \omega_i(\varrho) \frac{r-i}{p} h(\varrho) \qquad (i = 0, \ldots, \gamma-1),$$

where necessarily $\omega_i(\varrho) \to 1$ as $\varrho \to \infty$, it follows that

$$\varrho^{-\frac{r}{p}} H(\varrho, \Theta) = \sum_{i=0}^{\gamma-1} \lambda_i \omega_i(\varrho) |q_i| \varrho^{-\frac{i}{p}} \left(\frac{r-i}{p}\right) h(\varrho)$$
$$+ \varrho^{-\frac{\gamma}{p}} \left[|q_\gamma| \cos\left(\bar{q}_\gamma + \frac{r-\gamma}{p}\pi\right) + \omega'(\varrho)\right] = 0$$

where $\omega'(\varrho) \to 0$ as $\varrho \to \infty$. Hence

(17) $$h(\varrho)\left[\lambda_0 |q_0| \frac{r}{p} + \omega''(\varrho)\right] +$$
$$+ \varrho^{-\frac{\gamma}{p}} \left[|q_\gamma| \cos\left(\bar{q}_\gamma + \frac{r-\gamma}{p}\pi\right) + \omega'(\varrho)\right] = 0$$
$$(\omega''(\varrho) \to 0 \text{ as } \varrho \to \infty).$$

Accordingly it is concluded that

(17a) $$h(\varrho) = \varrho^{-\frac{\gamma}{p}} (k_0 + \bar{\omega}(\varrho))$$

where $\bar{\omega}(\varrho) \to 0$, as $\varrho \to \infty$, and

(17b) $$k_0 = \frac{-|q_\gamma| \cos\left(\bar{q}_\gamma + \frac{r-\gamma}{p}\pi\right)}{\lambda_0 |q_0| \frac{r}{p}} \neq 0, \ \neq \infty.$$

Therefore

$$\sin(\theta - \pi) = \sin h(\varrho) = \varrho^{-\frac{\gamma}{p}} (k_0 + \omega_1(\varrho))$$

$(\omega_1(\varrho) \to 0$, as $\varrho \to \infty)$ and

$$\varrho \sin(\Theta - \pi) = \varrho^{1-\frac{\gamma}{p}} (k_0 + \omega_1(\varrho)) \to \infty$$

as $\varrho \to \infty$, since $1 - \frac{\gamma}{p} > 0$. This establishes the Lemma. *Following similar lines it is possible to extend this Lemma to include cases when the ray $\theta = \pi$ is replaced by any other ray* [58]).

If some of the numbers μ are positive the greatest μ can be denoted as μ_1. We then have case I (§ 2). If some of the numbers μ are negative the least μ can be denoted as μ_1. There is then case II (§ 3) on hand.

When all the μ's are zero the $Q(x)$ are of the form

$$(18) \qquad Q(x) = qx + P(x) \quad (q = q' + \sqrt{-1}\, q''),$$

$$(18a) \qquad P(x) = ax^{\alpha} + bx^{\beta} + \dots + lx^{\lambda}$$

$$(1 > \alpha > \beta > \dots > \lambda > 0; \ \angle a = \bar{a}, \ \angle b = \bar{b}, \dots, \angle l = \bar{l}).$$

Here $\alpha, \beta, \dots, \lambda$ are rational, and $a \neq 0$, if and only if $P(x) \neq 0$. On writing $Q_i(x) = q_i x + P_i(x)$ the subscript i will be attached to the symbols

$$q', \ q''; \ a, \ b, \dots, l; \ \alpha, \ \beta, \dots, \lambda.$$

Whenever in some region, extending to infinity,

$$(19) \qquad \Re Q_i^{(1)}(x) \geqq \Re Q_j^{(1)}(x)$$

it necessarily follows that

$$(19a) \qquad q_i' \geqq q_j'.$$

In fact, (19) implies

$$q_i' + \alpha_i |a_i| \, |x|^{\alpha_i - 1} \cos(\bar{a}_i + (\alpha_i - 1)\bar{x}) + \dots \geqq q_j' +$$
$$+ \alpha_j |a_j| |x|^{\alpha_j - 1} \cos(\bar{a}_j + (\alpha_j - 1)\bar{x}) + \dots \qquad (\bar{x} = \angle x);$$

that is,

$$q_i' - q_j' \geqq f_{i,j}(x) \qquad (f_{i,j}(x) \to 0 \text{ as } |x| \to \infty).$$

This establishes the above italicized statement. Let, now, K denote a region satisfying the conditions $(\alpha), \dots, (\delta)$ of § 2. Interior K

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[58]) The Lemma will in general break down when $r > p$. A treatment of some questions of this type is included in

W. J. Trjitzinsky, Analytic Theory of Linear Differential Equations [Acta Mathematica 62 (1934), 167—226].

(20) $\Re Q_1^{(1)}(x) = \ldots = \Re Q_\delta^{(1)}(x) > \Re Q_{\delta+1}^{(1)}(x) \geqq \ldots \geqq \Re Q_n^{(1)}(x)$.

Whence, with all the μ_j assumed zero, we have

$$q_1' = \ldots = q_\delta' \geqq q_{\delta+1}' \geqq \ldots \geqq q_n'.$$

If there exist some positive numbers q', necessarily $q_1' > 0$. Hence, on noting that

$$\Re Q_1^{(1)}|x| = q_1^1 + \alpha_1|a_1|\,|x|^{\alpha_1-1} \cos\,(\bar{a}_1+(\alpha_1-1)\bar{x}) + \ldots,$$

it is observed that

(21) $$\Re Q_1^{(1)}(x) > 0$$

for $|x| \geqq r_0$ (r_0 sufficiently great). As a consequence of (21) it is concluded that $|\exp\, Q_1(x)|$ *is monotone in K to the left.* Let $\Re'$ denote any one of the set of regions (extending to infinity) in which $\Re Q_1(x) \leqq 0$. The boundaries of $\Re'$ (extending to infinity) will be denoted by B_r and B_l. These curves satisfy the equation

$$\Re Q_1(x) \equiv |q_1|\,|x| \;\cos\;(\bar{q}_1+\bar{x}) + |a_1|\,|x|^{\alpha_1} \cos\;(\bar{a}_1+\alpha_1\bar{x}) + \ldots = 0.$$

Their limiting directions, $\bar{x}_r$, $\bar{x}_l$, are distinct and are found amongst the values $\bar{x}$ satisfying

(22) $$\cos\,(\bar{q}_1+\bar{x}) = 0,$$

that is, amongst the values

(22a) $$\bar{x}^{(m)} = (\tfrac{1}{2}+m)\pi - \bar{q}_1 \quad (m = 0, \pm 1, \pm 2, \ldots).$$

As a matter of notation, let B_r denote the boundary of $\Re'$ with the smaller limiting direction at infinity. It can be shown that

(22b) $$\bar{x}_r = (\tfrac{1}{2}+2k)\,\pi - \bar{q}_1,\; \bar{x}_l = \bar{x}_r + \pi$$

where K is an integer. Now, since $\Re q_1 = q_1' > 0$, there exists an integer ν so that

(23) $$(2\nu - \tfrac{1}{2})\pi < \angle\, q_1 = \bar{q}_1 < (2\nu+\tfrac{1}{2})\pi.$$

By (22b) and (23)

(23b) $$[2(K-\nu) + 1]\pi > \bar{x}_r > 2(K-\nu)\pi\,.$$

Thus the following is inferred. The curves B_r and B_l have either direction at infinity not coincident with those of either extremity of the axis of reals. B_r extends into the first or second quadrant, while B_l extends into the third or fourth. The region $\Re'$ extends to the left of the simple curve consisting of B_r, B_l and of an arc

γ of a circle $|x| = r_0$. In $\mathfrak{R}'$ we have $|\exp Q_1(x)| \leqq 1$. A subregion $\overline{\mathfrak{R}}$ of $\mathfrak{R}'$ can be formed so that

$$e^{Q_1(x)} \sim 0 \quad (x \text{ in } \overline{\mathfrak{R}}), \tag{24}$$

while the boundary of $\overline{\mathfrak{R}}$ consists of a simple curve whose constituent parts are: the arc γ and curves $\overline{B}_r$, $\overline{B}_l$, extending from the extremities of this arc and possessing at infinity the limiting directions of B_r and B_l, respectively. Thus, it is observed that *when all the μ's are zero, while there exist some positive $q'(= \Re q)$, there is case I on hand in a region $\overline{K}$.* The latter region can be selected as the part common to the regions K (satisfying the conditions $(\alpha), \ldots, (\delta)$ of § 2) and $\overline{\mathfrak{R}}$. Similarly, it is proved that *when all the μ's are zero, while there exist some negative q', we have case* II (§ 3) *in a certain region extending to the right.*

It remains to consider the situation when all the μ's and the q''s are zero. All the $Q(x)$ are then of the form

$$Q(x) = \sqrt{-1}\, q''x + P(x), \; P(x) = ax^{\alpha} + \ldots + lx^{\lambda} \tag{25}$$
$$(\text{real } q''; \; 1 > \alpha > \ldots > \lambda > 0).$$

As stated before, $a \neq 0$ if and only if $P(x) \not\equiv 0$. In consequence of the Hypothesis A (§ 1) not all the $P(x)$ are identically zero.

In the sequel it will be convenient to make use of the following definition.

Definition 4. *Generically $\Gamma = \Gamma(\theta', \theta'')$ is to denote a region extending to infinity and bounded by a simple curve T. This curve is to consist of an arc γ of a circle $|x| = r_0$ (r_0 sufficiently great) and of two infinite branches,*

$$B' = B(\theta'), \; B'' = B(\theta''),$$

extending from the extremities of γ and possessing at infinity the limiting directions θ' and θ'', respectively. Here $\theta' < \theta''$. The interior of $\Gamma(\theta', \theta'')$ is in the counter clockwise direction from $B(\theta')$ to $B(\theta'')$. The number $\theta'' - \theta'$ is defined as the angle of $\Gamma(\theta', \theta'')$.

Consider a particular $P(x)$ for which $a \neq 0$. The relation $\Re P(x) \leqq 0$ is satisfied in a finite number of regions Γ'_i of the form

$$\Gamma'_i = \Gamma(\theta'_i, \theta''_i). \tag{26}$$

Here θ'_i, θ''_i are the directions of the rays extending from the origin and bounding a sector in which

$$\cos(\bar{a} + \alpha\bar{x}) \leqq 0 \quad (\bar{a} = \angle a; \; \bar{x} = \angle x).$$

Thus one may take

$$\theta_i' = \frac{1}{\alpha}\left[\pi\left(\frac{1}{2}+2i\right)-\bar{a}\right];\ \theta_i'' = \frac{1}{\alpha}\left[\pi\left(\frac{3}{2}+2i\right)-\bar{a}\right] \tag{26a}$$

$$(i = 0,\ \pm 1,\ \pm 2, \ldots).$$

The angle of Γ_i' is

$$\theta_i'' - \theta_i' = \frac{\pi}{\alpha} > \pi. \tag{26b}$$

Corresponding to every region Γ_i' a suitable subregion Γ_i can be found of the form

$$\Gamma_i = \Gamma(\theta_i', \theta_i''), \tag{27}$$

with boundaries $B_i (= B(\theta_i'))$ and $B_i'' (= B(\theta_i''))$, such that

$$e^{P(x)} \sim 0 \qquad (x \text{ in } \Gamma_i). \tag{28}$$

The angle of Γ_i is of course given by (26b). Whenever

$$2\nu\pi \leqq \theta_i' < \pi + 2\nu\pi \qquad (\nu \text{ an integer}) \tag{29}$$

it follows that

$$2\nu\pi + \frac{\pi}{\alpha} \leqq \theta_i''.$$

Now $\pi < \frac{\pi}{\alpha}$. Thus (29) implies that

$$\pi + 2\nu\pi < \theta_i''$$

and, in view of (29),

$$\theta_i' < \pi + 2\nu\pi < \theta_i''. \tag{29a}$$

Hence, whenever (29) holds, *the negative axis of reals (with the direction $\pi + 2\nu\pi$) is interior the region Γ_i, the limiting directions at infinity of the boundaries B_i', B_i'' of Γ_i being distinct from that of the axis just referred to.* The alternative to (29) is

$$2k\pi - \pi \leqq \theta_i' < 2k\pi \qquad (k \text{ an integer}). \tag{30}$$

Adding $\frac{\pi}{\alpha}$ to the first two members of (30) we obtain

$$2k\pi + \frac{\pi}{\alpha} - \pi \leqq \theta_i''.$$

Whence, since $-\pi + \frac{\pi}{\alpha} > 0$, $2k\pi < \theta_i''$. In conjunction with (30) this signifies that

$$\theta_i' < 2k\pi < \theta_i''. \tag{30a}$$

Thus, (30) implies that *the positive axis of reals (with the direction*

$2k\pi$) *is interior the region* Γ_i. Moreover, *the limiting directions at infinity of the boundaries of* Γ_i *will be distinct from that of this axis.* The following has been established.

If $P(x)$, *a function involved in* (25), *is not identically zero then exp* $P(x) \sim 0$ *in a region* Γ_i *which either has the properties stated subsequent to* (29a) *or has the properties described following* (30a).

Consequently it is concluded that the case when all the μ's and all the q''s are zero falls into two subcases.

Subcase A_1. *Amongst the functions* $P(x)$ *involved in* (25) *there is one for which* (28) *holds in a region* Γ_i, *satisfying the conditions of the italicized statement subsequent to* (29a).

Subcase A_2. *There exists a function* $P(x)$ *for which* (28) *holds in a region* Γ_i *subject to the conditions of the italics following* (30a).

Consider subcase A_1. As stated in § 2 there exists a region K satisfying the conditions $(\alpha), \ldots, (\delta)$ of § 2. This region can be selected, for instance, so that its lower boundary consists of the negative axis (sufficiently far out) and so that its upper boundary is in the second quadrant. Moreover, K can be so selected that the negative axis referred to in the preceding statement has the direction $\pi + 2\nu\pi$, where ν is the integer involved in (29a). Let K_1 denote the part common to K and Γ_i, where Γ_i is the region mentioned in the formulation of the subcase A_1. The upper boundary of K_1 will certainly recede indefinitely from the negative axis. Interior K, and hence interior K_1,

$$(31)\qquad \Re P_1^{(1)}(x) = \ldots = \Re P_\delta^{(1)}(x) > \Re P_{\delta+1}^{(1)}(x) \geqq \ldots \geqq \Re P_n^{(1)}(x)$$

since $\Re Q_i^{(1)}(x) = \Re P_i^{(1)}(x)$. If $P_1(x) \not\equiv 0$ consider the curves P_1' extending to infinity and satisfying the equation

$$(32)\qquad \Re P_1^{(1)}(x) = 0.$$

If there are any curves P_1' extending into the closed region K_1, necessarily there could be only one such curve. This is a consequence of the fact that the limiting directions of the curves P_1' are roots of the equation $\cos\left(\bar{a}_1 + (\alpha_1 - 1)\bar{x}\right) = 0$ and thus differ from each other by at least $\left(\frac{\pi}{1-\alpha_1}\right)$ $(>\pi)$. If a curve P_1' extends into the closed region K_1 either this curve is coincident with the lower boundary of K_1 (i.e. the ray $\bar{x} = \pi + 2\nu\pi$) or it is in the second quadrant and necessarily indefinitely recedes from this boundary (cf. (BT)). In any case there exists a subregion K_2 of K_1, which sometimes is coincident with K_1, such that

$$(33)\qquad \Re P_1^{(1)}(x) \geqq 0 \text{ or } \Re P_1^{(1)}(x) \leqq 0 \quad (x \text{ in } K_2),$$

the lower boundary of K_2 being coincident with that of K_1 and the upper boundary receding indefinitely from the lower boundary [59]).

If the function $P(x)$, referred to in the formulation of subcase A_1, is coincident with $P_1(x)$ the following facts can be observed. Since in K_2 we have exp $P_1(x) \sim 0$, while (33) holds, necessarily

$$\Re P_1^{(1)}(x) = \Re Q_1^{(1)}(x) \geqq 0 \qquad (x \text{ in } K_2). \tag{34}$$

Now the number q_1'', occurring in the expression

$$Q_1(x) = \sqrt{-1}\, q_1'' x + P_1(x), \tag{34a}$$

is not defined for an additive term of the form $2\pi\omega$ where ω is any integer [60]). Choose ω so that $q_1'' \geqq 0$. Then

$$\left|e^{Q_1(x)}\right| = e^{-q_1'' v}\left|e^{P_1(x)}\right| \leqq \left|e^{P_1(x)}\right| \qquad (x \text{ in } K_2;\ v = \Im x) \tag{35}$$

so that

$$e^{Q_1(x)} \sim 0 \qquad (x \text{ in } K_2). \tag{35a}$$

Thus, in view of (34) and (35a), it is observed that, when $P(x) \equiv P_1(x)$, case I (§ 2) will take place in K_2.

Suppose now that $P(x) \not\equiv P_1(x)$, all of the $Q(x)$ being of the form (25). Then $P(x) = P_q(x)$ $(\delta < q \leqq n)$. Consider curves $P_{1,q}$ satisfying the equation

$$\Re\big(P_1(x) - P(x)\big) \equiv H(|x|, \bar{x}) = 0 \qquad (\bar{x} = \angle x).$$

Since, by (31), $\Re\big(P_1^{(1)}(x) - P_q^{(1)}(x)\big) > 0$ (interior K_1), it is possible to write

$$P_1(x) - P_q(x) = \sum_{i=0}^{r-1} q_i x^{\frac{r-i}{p}} \qquad (\text{integer } p \geqq 2;\ 1 \leqq r < p) \tag{36a}$$

where $q_0 \neq 0$. Then

$$H(|x|, \bar{x}) = \sum_{i=0}^{r-1} |q_i|\, |x|^{\frac{r-i}{p}} \cos\left(\bar{q}_i + \frac{r-i}{p}\bar{x}\right) \qquad (\bar{q}_i = \angle q_i). \tag{36b}$$

The limiting directions at infinity of the curves $P_{1,q}$ are roots of the equation

$$\cos\left(\bar{q}_0 + \frac{r}{p}\bar{x}\right) = 0 \qquad \left(0 < \frac{r}{p} < 1\right).$$

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59) The upper boundary is in the second quadrant; more precisely, in the sector $\frac{\pi}{2} + 2\nu\pi \leqq \bar{x} \leqq \pi + 2\nu\pi$.

60) In fact, if exp $Q(x) x^r \{x\}_N$ is a formal solution of the linear problem, exp $[Q(x) + 2\pi\omega\sqrt{-1}\, x] x^r \{x\}_N$ will also be a solution, because exp $(2\pi\omega\sqrt{-1} x)$ is a function of period unity.

Consequently one can infer that only one curve $P_{1,q}$ may extend into the closed region K_2. If the latter actually takes place, either the curve in question is the ray $\bar{x} = \pi + 2\nu\pi$ or it is in the second quadrant and is receding indefinitely from this ray. Whence there exists a subregion K_3 of K_2 (sometimes coincident with K_2) so that

$$(37) \qquad \Re(P_1(x) - P_q(x)) \geqq 0 \qquad (x \text{ in } K_3)$$

or

$$(37a) \qquad \Re(P_1(x) - P_q(x)) \leqq 0 \qquad (x \text{ in } K_3).$$

Moreover, K_3 can be so selected that its lower boundary is coincident with the ray $\bar{x} = \pi + 2\nu\pi$ (sufficiently far from the origin, of course) while its upper boundary recedes indefinitely from this ray. Since K_3 is a subset of K_2 and of K_1, in view of (31) and (33) we have

$$(38) \qquad \Re P_1^{(1)}(x) \geqq 0 \quad \text{or} \quad \Re P_1^{(1)}(x) \leqq 0 \qquad (x \text{ in } K_3),$$

$$(38a) \qquad \Re(P_1^{(1)}(x) - P_q^{(1)}(x)) > 0 \qquad (x \text{ interior } K_3).$$

Moreover,

$$(38b) \qquad e^{P_q(x)} \sim 0 \qquad (x \text{ in } K_3).$$

In examining the function $H(|x|, \bar{x})$ the following situations are seen to be possible:

(1^0) $w_0 = \cos\left(\bar{q}_0 + \frac{r}{p}(\pi + 2\nu\pi)\right) \neq 0;$

(2^0) $w_i = \cos\left(\bar{q}_i + \frac{r-i}{p}(\pi + 2\nu\pi)\right) = 0$

$(i = 0, \ldots, \gamma - 1;\ 1 \leqq \gamma \leqq r - 1)$ [61]),

$w_\gamma = \cos\left(\bar{q}_\gamma + \frac{r-\gamma}{p}(\pi + 2\nu\pi)\right) \neq 0 \quad (q_\gamma \neq 0);$

(3^0) $\cos\left(\bar{q}_i + \frac{r-i}{p}(\pi + 2\nu\pi)\right) = 0 \quad (i = 0, 1, \ldots, r-1).$

Write

$$(39) \qquad x = u + \sqrt{-1}\, v,\ x' = u' + \sqrt{-1}\, v,\ (v > 0;\ u \leqq u')$$

where x' is on the boundary of K_3. It is noted that x is then in K_3 and v can be made to approach $+\infty$ [62]); moreover, (39) implies that

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[61]) These equalities are considered only corresponding to the non zero q_i.

[62]) This is because the upper boundary of K_3 certainly recedes indefinitely away from the ray $x = \pi + 2v\pi$ into the second quadrant.

(39a) $\bar{x} = \pi + 2\nu\pi - h(v, |x|),\ h(v, |x|) = \sin^{-1}\left(\frac{v}{|x|}\right) \quad (\bar{x} = \angle x)$

where

$$0 < h(v, |x|) \leqq \frac{\pi}{2}.$$

When (1^0) takes place substitution of (39a) into (36b) will yield

$$(40) \qquad H(|x|, \bar{x}) = x^{\frac{r}{p}} \left[|q_0| \cos\left(\bar{q}_0 + \frac{r}{p}(\pi + 2\nu\pi)\right) + o_1(|x|, v)\right]$$

$$(o_1(|x|, v) \to 0 \text{ as } |x| \to \infty).$$

Hence with v positive and fixed the limit of $H(|x|, \bar{x})$, when $x \to \infty$ (in K_3), will be $+\infty$ or $-\infty$, depending on whether the left member in (1^0) is positive or negative. Now, by (37), (37a), $H(|x|, \bar{x})$ does not change sign in K_3. Thus, if w_0 (cf. (1^0)) is positive, (37) necessarily takes place. When $w_0 < 0$ we have (37a). In view of (38a) it is observed that $H(|x|, \bar{x})$ diminishes monotonically to the left along the line $v =$ constant. Hence w_0 must be negative. Along the line $v =$ constant, sufficiently far to the left, we shall have $H(|x|, \bar{x}) \leqq 0$. Thus, when (1^0) takes place, necessarily (37a) will hold throughout K_3. One then has

$$(41) \qquad |e^{P_1(x)}| \leqq |e^{P_q(x)}| \quad (x \text{ in } K_3)$$

so that, in view of (38b), it can be asserted that

$$(41a) \qquad e^{P_1(x)} \sim 0 \qquad (x \text{ in } K_3).$$

Choosing the number q_1'', involved in (34a), so that $q_1'' \geqq 0$ the relation

$$(41b) \qquad e^{Q_1(x)} \sim 0 \qquad (x \text{ in } K_3)$$

will be secured. In view of (41a) it is observed that (38) can hold only with the symbols $\geqq 0$. That is,

$$(42) \qquad \Re P_1^{(1)}(x) = \Re Q_1^{(1)}(x) \geqq 0 \quad (x \text{ in } K_3).$$

Whenever (1^0) *is on hand case* I (§ 2) *will certainly take place in a region* K_3 *specified above.*

When (2^0) is considered it is observed that $H(|x|, \bar{x})$ (cf. (36b)) is of the form

$$(43) \qquad H(|x|, \bar{x}) = \pm \frac{vr}{p} |q_0|\, |x|^{\frac{r-p}{p}} (1 + o_2(|x|, v))$$

$$+ |x|^{\frac{r-\gamma}{p}} \left[|q_\gamma| \cos\left(\bar{q}_\gamma + \frac{r-p}{p}(2\nu+1)\pi\right) + o_3(|x|, v)\right]$$

where

$$o_2(|x|, v) \to 0, \quad o_3(|x|, v) \to 0,$$

as $|x| \to \infty$ $(v > 0;\ x$ in $K_3)$. Since $\gamma < r < p$ we have $\gamma - r < 0$ and $\gamma - p < 0$. Whence, by (43),

$$\text{(43a)} \qquad H(|x|, \bar{x}) = |x|^{\frac{r-p}{\gamma}} [w_\gamma + o_4(|x|, v)]$$

$$\big(o_4(|x|, v) \to 0 \text{ as } |x| \to \infty \text{ in } K_3;\ v > 0;\ w_\gamma \neq 0\big).$$

Applying a reasoning of the same type as previously employed in proving the italicized statement subsequent to (42), it is now established that *case* I (§ 2) *is certain to take place in* K_3 *whenever* (2^0) *is on hand.*

To complete the treatment of subcase A_1 it remains to consider (3^0). The ray $\bar{x} = \pi + 2\nu\pi$ is then a $P_{1,q}$ curve (cf. (36)) [63]. If (37a) holds, (41) is obtained and, in view of (38b), it will follow that (41a) holds. We can then secure (41b). On the other hand, (41a) would imply that (38) can hold (in K_3) only with the symbols $\geqq 0$. Thus (42), and hence case I (§ 2), will be certain to be on hand in the region K_3.

We are thus brought to the consideration of (3^0), when the inequality (37) holds. Let x, u, v have the significance indicated in the statement in connection with (39). In view of (39a) and (3^0)

$$\cos\left(\bar{q}_i + \frac{r-i}{p}\bar{x}\right) = \lambda_i h_i,\ \lambda_i = \sin\left(\bar{q}_i + \frac{r-i}{p}(\pi + 2\nu\pi)\right) = \pm 1$$

where

$$0 < h_i = \sin\left(\frac{r-i}{p} h(v, |x|)\right) < \frac{r-i}{p} h(v, |x|) < \frac{r-i}{p} \operatorname{tg}\left|\frac{v}{x}\right|$$

$$(i = 0, \ldots, r-1).$$

Thus, by (36b),

$$\text{(44)} \quad H(|x|, \bar{x}) \leqq \sum_{i=0}^{r-1} |q_i|\, |x|^{\frac{r-i}{p}} h_i < \sum_{i=0}^{r-1} |q_i|\, |x|^{\frac{r-i}{p}} \left(\frac{r-i}{p}\right) \operatorname{tg}\left|\frac{v}{x}\right|$$

$$= |x|^{\frac{r}{p}} \operatorname{tg}\left|\frac{v}{x}\right| \left(\frac{r}{p}|q_0| + h'(|x|)\right) \qquad \left(\frac{r}{p}|q_0| > 0\right)$$

where $h'(|x|)$ *depends only on* $|x|$ and $h'(|x|) \to 0$, as $|x| \to \infty$. Whence there exists a constant b, independent of v, so that

$$\text{(44a)} \qquad 0 \leqq H(|x|, \bar{x}) < b|x|^{\frac{r}{p}} \operatorname{tg}\left|\frac{v}{x}\right| \qquad (x \text{ in } K_3).$$

Consider a curve T_c extending into the second quadrant and of

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[63]) There are no other curves extending into the closed region K_3.

the form

$$(45) \qquad v = c|x|^{\frac{p-r}{p}} \qquad (c > 0).$$

It recedes indefinitely from the negative axis of reals. Let K' denote the region lying in the second quadrant and bounded by the negative axis and by T_c. Since in K'

$$v \leqq c|x|^{\frac{p-r}{p}}$$

it will follow that

$$(45a) \qquad \operatorname{tg}\left|\frac{v}{x}\right| \leqq \operatorname{tg}\left(c|x|^{-\frac{r}{p}}\right) < B_c|x|^{-\frac{r}{p}} \qquad (x \text{ in } K')\ ^{64}).$$

Let K_4 denote the region common to K_3 and K'. In view of (44a) and (45a)

$$(46) \qquad H(|x|, \bar{x}) < b\, B_c = b' \qquad (x \text{ in } K_4).$$

The upper boundary of K_4 has at infinity the limiting direction of the ray $\bar{x} = \pi = 2\nu\pi$. This boundary recedes indefinitely from the ray. By (46)

$$\left|e^{P_1(x) - P_q(x)}\right| = e^{H(|x|, x)} < e^{b'} = c' \qquad (x \text{ in } K_4)$$

so that

$$(47) \qquad \left|e^{P_1(x)}\right| < c'\left|e^{P_q(x)}\right| \qquad (x \text{ in } K_4).$$

Hence, by virtue of (38b),

$$(47a) \qquad e^{P_1(x)} \sim 0 \qquad (x \text{ in } K_4).$$

By a suitable choice of q_1'' *the relation* (41b) *is now secured for x in K_4*. In view of (47a) it is again concluded that, inasmuch as (38) must hold throughout K_4 either with the symbols $\geqq$ or $\leqq$, the first is necessarily the case. *That is,* (42) *will hold in K_4*. The preceding two italicized statements enable us to assert that case I (§ 2) will take place in a region K_4 whenever (3^0) and (37) hold.

Using lines of reasoning analogous to those employed subsequent to the statements formulating subcases A_1, A_2 it can be proved that, whenever subcase A_1 is on hand, case I (§ 2) is certain to take place in a region bounded by the negative axis and extending into the third quadrant.

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[64]) Throughout we keep $|x| \geqq r_0$. Choose c sufficiently small or r_0 sufficiently great so that $cr_0^{-\frac{r}{p}} < \frac{\pi}{2}$. By a suitable choice of c and r_0 B_c can be made as close as desired to c.

Similarly, it is shown that case II (§ 3) will occur in regions extending into the second and in regions extending into the fourth quadrants, whenever subcase A_2 holds.

LEMMA 10. *Under Hypothesis* A (§ 1), *at least one of the cases* I (§ 2), II (§ 3) *is certain to take place. Moreover, whenever one of these cases occurs, either* (α') *the case will take place in a region containing in its interior one of the extremities of the axis of reals or* (α'') *it will hold in two distinct regions — one extending from the real axis upwards, the other downwards.*

When (α') *holds, both boundaries of the region recede indefinitely from that extremity of the axis of reals which is contained in the region.*

When (α'') *holds, the boundary of the region in which case* I (*or* II) *holds contains an extremity of the axis of reals; moreover, the boundary will contain a part receding indefinitely from this axis.*

The following Existence Theorem has been established.

EXISTENCE THEOREM. *Consider the non-linear difference equation* (A; § 1) *under the Hypothesis* A (§ 1). *Of the cases* I (§ 2) *and* II (§ 3) *at least one is certain to occur* (*cf. Lemma* 10). *In the case* I *the equation* (A) *has formal solutions as specified in Lemma* 2 (§ 2). *In the case* II *there is a formal solution as stated in Lemma* 3 (§ 3). *There exist „actual" solutions analytic in every finite part of the stated regions and involving a number of arbitrary periodic functions. At infinity the „actual" solutions in general have a singular point. They are asymptotically related in a certain sense, to the formal solutions. Their analytic character is specified by the Lemma* 7 (§ 5), *in the case* I, *and by the Lemma* 8 (§ 6), *in the case* II.

For the involved regions at least one boundary recedes indefinitely from an extremity of the axis of reals. Whence it is observed that, with the aid of successive applications of the equations (A; § 1), the analytic (asymptotic) character of the analytic continuations of the „actual" solutions can be always inferred — at least in a half plane of the form $\mathfrak{J}x \geqq c$ and in another half plane of the form $\mathfrak{J}x \leqq -c$ (c sufficiently great). It is also to be noted that under some hypotheses, more restrictive than Hypothesis A (§ 1), the coefficients in the formal solutions can be investigated with greater precision. Of significance is the problem, for the present put aside, to determine

(1) *Under what conditions are the coefficients of the formal series representable along the lines of Nörlund's methods* (*Laplace integrals, convergent factorial series*)*?*

(2) *Under what conditions is the formal series $s(x)$ convergent?* When $s(x)$ denotes a formal solution (cf. Lemmas 2 (§ 2), 3 (§ 3)) and an „actual" solution $y(x)$ is said to be asymptotic to $s(x)$, in the sense indicated in Lemmas 7 (§ 5) and 8 (§ 6), the term „asymptotic" is justified for the following reasons. The function $y(x)$, on one hand, is well defined by a succession of definite analytic processes. On the other hand, $y(x)$ can be represented by the first $N-1$ terms of the series $s(x)$ (in general divergent) with an error whose absolute value can be made as small as desired either by subjecting the involved periodic functions to suitable conditions (as stated in the text) or by excluding the interior of a circle $|x| = r_0$ where r_0 is sufficiently great.

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