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Fields of parallel vectors in non-analytic manifolds in the large

by

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Let $\mathfrak{M}$ be a coordinate manifold of class C^r. We assume $\mathfrak{M}$ to be connected in the topological sense and to admit an affine connection L with components $L^{\alpha}_{\beta\gamma}$ which are of class C^s as functions of the coordinates of $\mathfrak{M}$. Evidently we must have $s \leqq r - 2$ from the equations of transformation of the components of the connection L.

In the following paper we shall consider the question of the existence of fields of parallel vectors defined over open point sets in $\mathfrak{M}$. This problem has been previously discussed under the analytic hypothesis by T. Y. Thomas [1]). Indeed one of our objects will be to exhibit the essential differences between the analytic and non-analytic cases and to indicate why the algebraic characterization obtained under the analytic hypothesis can probably not be extended to the case of coordinate manifolds of class C^r.

1. It will be sufficient to assume for our purpose that the above integer s is $\geqq n + 1$ where n is the dimensionality of $\mathfrak{M}$.

Analytically we are concerned with the existence of solutions of the system of linear partial differential equations which define the field of parallel vectors in $\mathfrak{M}$. But our discussion will apply likewise to any invariantive system of linear equations in $\mathfrak{M}$ with the x coordinates of $\mathfrak{M}$ as independent variables and with unknowns which may be scalars, the components of tensors, etc. In other words our methods are representative of the treatment to be applied to this general category of differential equations defined over the manifold $\mathfrak{M}$. We shall define regular points in the following manner. Consider the set of equations

[1]) Fields of Parallel Vectors in the Large [Compositio Mathematica 3 (1936), 453—468].

$$(E_0) \qquad \xi^\mu B^\alpha_{\mu\beta\gamma} = 0,$$
$$(E_1) \qquad \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1} = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$(E_n) \qquad \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_n} = 0,$$

where the B's denote the components of the curvature tensor and its successive covariant derivatives. A point P of $\mathfrak{M}$ will be said to be *regular with respect to the system* $E_0 + \ldots + E_t$ where $t \leqq n-1$ if there exists a neighborhood $U(P)$ in which the rank of the matrix M of the B's of this system is constant. All other points of $\mathfrak{M}$ will be said to be *singular with respect to this system.* By this definition it is obvious that the regular points form an open point set and it is easily seen that the singular points are nowhere dense. To prove this last statement we observe that if P is a singular point any neighborhood $U(P)$ contains a point Q such that the rank of M at Q is greater than the rank of M at P. In fact there exists a neighborhood $U'(P) \subset U(P)$ in which the rank of M is at least equal to its rank at P. If $U'(P)$ did not contain a point Q_1 at which the rank of M is greater than the rank of M at P then P by definition would be a regular point. If Q_1 is a regular point the proof is complete. If Q_1 is a singular point then by the above argument $U(P)$ considered as a neighborhood of Q_1 will contain a point Q_2 such that the rank of M at Q_2 is greater than the rank of M at Q_1. Continuing we obtain a finite sequence of points $Q_1, Q_2, Q_3, \ldots$ in $U(P)$ such that the rank of M at Q_α is greater than the rank of this matrix at $Q_{\alpha-1}$. This sequence must contain a first point Q_m which is regular since a point at which the rank of M is n is necessarily regular. Since Q_m is regular there exists a neighborhood $U(Q_m) \subset U(P)$ which consists entirely of regular points. Hence the singular points with respect to the above system are nowhere dense.

Let us denote by R_t the set of points regular with respect to the system $E_0 + \ldots + E_t$ for $t = 0, \ldots, n-1$ and by S_t the corresponding sets of singular points. The points of the intersection

$$R = R_0 \cap R_1 \cap \ldots \cap R_{n-1}$$

will merely be said to be *regular*. A point of $\mathfrak{M}$ not in the set R will be said to be *singular*. Denoting by S the set of singular points in $\mathfrak{M}$ this set is the logical sum

$$S = S_0 + S_1 + \ldots + S_{n-1}.$$

Since the intersection of a finite number of open point sets is open it follows that R is an open point set. Also S is nowhere dense since it is the sum of a finite number of sets each of which is nowhere dense. To prove this let P be any point of S and $U(P)$ any neighborhood of P. Then there is a neighborhood $U_0 \subset U(P)$ composed entirely of points of R_0 since S_0 is nowhere dense. Again there is a neighborhood $U_1 \subset U_0$ such that U_1 contains only points of R_1 and so only points of R_0. Finally we get a neighborhood U_{n-1} containing only points of the intersection of $R_0, R_1, \ldots, R_{n-1}$, i.e. of R.

2. We shall now show that if P is a regular point in $\mathfrak{M}$ the set of equations E_n is linearly dependent on the set $E_0 + \ldots + E_{n-1}$ at P and in fact that there exists a neighborhood $U(P)$ of regular points in $\mathfrak{M}$ in which one can find equations with continuous coefficients expressing this dependence.

Case I. If all the coefficients B of the set of equations E_0 are zero at the point P then these coefficients vanish identically in some neighborhood $U(P)$ since P is a regular point in $\mathfrak{M}$. Hence in $U(P)$ the coefficients B of all the equations $E_1, \ldots, E_n$ will vanish and the above statement is therefore valid.

Case II. Let E_t for $t = 1, \ldots, n-1$ be the first set of equations possessing the property that it is linearly dependent on the set $E_0 + \ldots + E_{t-1}$ at P. Then the ranks of the matrices of the systems $E_0 + \ldots + E_{t-1}$ and $E_0 + \ldots + E_t$ will be equal at P. Call this rank r. Since P is a regular point there will be a neighborhood $U_1(P)$ in which the matrices of the above systems have the constant rank r. Any r independent equations of the system $E_0 + \ldots + E_{t-1}$ at P will be independent in some neighborhood $U(P) \subset U_1(P)$ and in this neighborhood $U(P)$ we can express the set E_t linearly in terms of the above independent equations with coefficients which have the same properties of continuity and differentiability as the coefficients of the set E_t. Hence we can find tensor relations of the form [2])

$$B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_t} = \sum_{s=0}^{s=t-1} A^{pqr_1\ldots r_s}_{h\beta\gamma\delta_1\ldots\delta_t} B^{h}_{\mu pq,r_1\ldots,r_s} \tag{1}$$

[2]) These equations may be taken to represent the dependence of the set E_t on the above r independent equations of the system $E_0 + \ldots + E_{t-1}$ in the coordinate system under consideration, those A's which do not correspond to these r independent equations having the value zero. To obtain these relations in any coordinate system we have merely to transform the A's as the components of tensors as indicated by their indices.

valid in $U(P)$ and having coefficients A continuous and with continuous partial derivatives to the order $n-t$ inclusive. By covariant differentiation of these relations we see that each of the sets of equations $E_{t+1}, \ldots, E_n$ can be expressed linearly in $U(P)$ in terms of the set $E_0 + \ldots + E_{t-1}$ with continuous coefficients.

Case III. If neither Case I nor Case II apply the set $E_0 + \ldots + E_{n-1}$ contains n independent equations at P. Then the set E_n is evidently dependent on the set $E_0 + \ldots + E_{n-1}$ at P. Since P is a regular point the rank of the matrix of the set $E_0 + \ldots + E_{n-1}$ will be n in some neighborhood $U_1(P)$. As in case II the set E_n will be linearly dependent on the set $E_0 + \ldots + E_{n-1}$ in some neighborhood $U(P) \subset U_1(P)$ and the coefficients of the equations expressing this dependence will be continuous functions of the coordinates in this neighborhood.

Since one of the above three cases must occur we see that *for any regular point P in $\mathfrak{M}$ there exists a neighborhood $U(P)$ in which the equations* (1) *for $t = n$ are valid with continuous coefficients A*. This neighborhood $U(P)$ can of course be taken to be a neighborhood composed entirely of regular points in $\mathfrak{M}$.

In the following the above italicized property of regular points in $\mathfrak{M}$ is the only one of which use will be made. A singular point P in $\mathfrak{M}$ which also possesses this property, i.e. for which a neighborhood $U(P)$ exists such that in $U(P)$ any equation of the set E_n is linearly expressable with continuous coefficients by means of the equations of the set $E_0 + \ldots + E_{n-1}$ or in other words such that (1) with $t = n$ is satisfied with continuous coefficients A in $U(P)$, will be called a *non-essential singular* point in $\mathfrak{M}$. All other singular points will be called *essential singular points*. Obviously the set composed of all regular and non-essential singular points in $\mathfrak{M}$ is open and its complement, i.e. the essential singular points in $\mathfrak{M}$, is nowhere dense.

3. Consider the open set R^* of regular and non-essential singular points in $\mathfrak{M}$. By the component of a point P of R^* we mean the greatest open connected point set in R^* which contains the point P. We denote such a component by $K(P)$. If $Q \subset K(P)$ then obviously $K(Q) = K(P)$. Thus the set R^* is divided into a finite or infinite number of components $K(P)$ with boundaries composed of essential singular points.

Let $C(t)$ for $0 \leqq t \leqq 1$ be a continuous arc (continuous map of the unit interval) in a particular component $K(P)$. Along this arc the set of equations E_n can be represented linearly in terms of the set $E_0 + \ldots + E_{n-1}$ with coefficients (components of tensors A) which are continuous functions of t in the interval $0 \leqq t \leqq 1$ (irrespective of coordinate transformations). To prove this take any value of $t = t'$ which will then correspond to a point $C(t')$ of the arc. Since $C(t')$ is a regular or non-essential singular point in $\mathfrak{M}$ there exists a neighborhood $U \subset R^*$ in which the equations

$$(2) \qquad B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_n} = \sum_{s=0}^{n-1} A^{pqr_1\ldots r_s}_{h\beta\gamma\delta_1\ldots\delta_n} B^{h}_{\mu pq,r_1,\ldots,r_s}$$

are valid with A's which are continuous functions of the coordinates. From the fact that the arc is a continuous map of the unit interval there will be some t-interval containing t' whose map lies entirely in the neigborhood U and for this t-interval equations (2) will hold with A's which are continuous functions of t. Corresponding to any value of $t = t'$ such that $0 \leqq t' \leqq 1$ there exists an interval containing t' in which the above statement is true. The whole interval $0 \leqq t \leqq 1$ can now be covered by a finite number of the above t-intervals $N_1, \ldots, N_m$ corresponding to increasing values of the variable t. We shall now construct from these m representations (2) of the equations E_n a single continuous representation valid for the entire interval $0 \leqq t \leqq 1$ as above stated.

Consider two successive t-intervals N_p and N_{p+1} and let t_p and t_{p+1} where $t_{p+1} > t_p$ be two values of the variable t lying in the intersection $N_p \cap N_{p+1}$. Obviously the entire interval $t_p \leqq t \leqq t_{p+1}$ lies in the intersection. We shall define a representation (2) of E_n in this intersection which will continue in a continuous manner the representation of E_n in N_p from the right and the representation of E_n in N_{p+1} from the left. Denote briefly by $B_n = \Sigma AB$ and $B_n = \Sigma \overline{A}B$ the above representations in the intersection $N_p \cap N_{p+1}$ with respect to a single coordinate system covering this intersection. If $F(t)$ is any continuous function of t in the interval $t_p \leqq t \leqq t_{p+1}$ then $B_n = \Sigma[(1-F)A + F\overline{A}]B$ will give a (continuous) representation of E_n in the interval. We have now merely to choose $F(t_p) = 0$ and $F(t_{p+1}) = 1$ to obtain the above continuation of the representation of E_n. By proceeding along the intervals $N_1, \ldots, N_m$ in succession we obtain the desired representation for the arc $C(t)$.

4. *We shall now show that any solution vector ξ of the system $E_0 + \ldots + E_{n+1}$ at any point Q of a particular component $K(P)$*

will result in a solution vector ξ' at any other point Q' of this component by parallel displacement of the vector ξ at Q along an arc of class C^1 lying in $K(P)$. Let $C(t)$ where $0 \leqq t \leqq 1$ be the arc joining Q to Q' so that $C(0) = Q$ and $C(1) = Q'$. By this parallel displacement we will obtain a vector $\xi(t)$ on the arc C with components $\xi^\alpha(t)$ of class C^1. On $C(t)$ put

$$
(3) \qquad
\begin{aligned}
&S^\alpha_{\beta\gamma} = \xi^\mu B^\alpha_{\mu\beta\gamma},\\
&S^\alpha_{\beta\gamma\delta_1} = \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&S^\alpha_{\beta\gamma\delta_1\ldots\delta_{n-1}} = \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_{n-1}}.
\end{aligned}
$$

By invariant differentiation of (3) with respect to t we obtain

$$
\begin{aligned}
&\frac{\vartheta S^\alpha_{\beta\gamma}}{\vartheta t} = S^\alpha_{\beta\gamma\delta_1}\frac{dx^{\delta_1}}{dt},\\
&\frac{\vartheta S^\alpha_{\beta\gamma\delta_1}}{\vartheta t} = S^\alpha_{\beta\gamma\delta_1\delta_2}\frac{dx^{\delta_2}}{dt},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\vartheta S^\alpha_{\beta\gamma\delta_1\ldots\delta_{n-1}}}{\vartheta t} = \sum_{s=0}^{n-1} A^{\alpha pqr_1\ldots r_s}_{h\beta\gamma\delta_1\ldots\delta_n} S^h_{pqr_1\ldots r_s}\frac{dx^{\delta_n}}{dt},
\end{aligned}
$$

where use has been made of the continuous representation (2) of E_n along the entire arc $C(t)$ in writing the last set of these equations. Since the left members of (3) and $S \equiv 0$ are solutions of the above system having the same initial values it follows that these two solutions are identical (uniqueness theorem).

Due to the fact the property of dependence or independence of vectors is invariant under parallel displacement it follows from the above result that the rank of the matrix of the system $E_0 + \ldots + E_{n-1}$ is constant in each component $K(P)$. As a consequence a non-essential singular point is a regular point with respect to the system $E_0 + \ldots + E_{n-1}$. We shall now prove conversely that if P is a regular point with respect to the system $E_0 + \ldots + E_{n-1}$ then P is either a regular point or a non-essential singular point in $\mathfrak{M}$. By hypothesis the rank of the matrix M of the system $E_0 + \ldots + E_{n-1}$ is constant in some neighborhood $U_1(P)$. Let the rank of M be r in $U_1(P)$. Then there exist r independent equations $E_{\alpha_1}, \ldots, E_{\alpha_r}$ in the system $E_0 + \ldots + E_{n-1}$ such that an rth ordered determinant D formed from the coefficients of these r equations will not vanish in some neighborhood $U(P) \subset U_1(P)$. At any regular point Q in $U(P)$ the system E_n can be expressed in terms of these r independent equations by means of a definite (i.e. the same for all points Q) set of equations

$$E_n(Q) = \sum_{i=1}^{r} \frac{D^{\alpha_i}(Q)}{D(Q)} E_{\alpha_i}(Q)$$

with coefficients which are rational functions of the coefficients of the equations E_{α_i} and E_n and having denominators depending only on the above determinant D. Since any point in $U(P)$ is a limit of regular points Q it follows that the above equations hold for all points in $U(P)$. This proves the above statement.

To sum up we now have the following result: *Any regular or non-essential singular point is a regular point with respect to the system* $E_0 + \ldots + E_{n-1}$ *and conversely any point which is regular with respect to this system is either a regular or non-essential singular point. In other words the set of regular and non-essential singular points is identical with the set of regular points with respect to the system* $E_0 + \ldots + E_{n-1}$.

As a by-product of the above we obtain the further result that the vector spaces defined at the points of $K(P)$ by the solutions of the system $E_0 + \ldots + E_{n-1}$ are parallel in the sense that the vector space at any point of $K(P)$ is carried into the vector space at any other point by parallel displacement along any arc $C(t)$ joining these points. In particular if the rank of the matrix of the system $E_0 + \ldots + E_{n-1}$ is $n-1$ at any point of $K(P)$ parallel displacement of the solution vector ξ of $E_0 + \ldots + E_{n-1}$ at a point $Q \subset K(P)$ to any other point of this component will result in a solution vector of this system which is determined to within a factor depending on the arc of displacement. Under this latter condition a single field of parallel vectors will exist in the component $K(P)$ for the case of a Riemann space since length of a vector is then invariant under parallel displacement.

5. Let $P(t, \varepsilon)$ be the continuous map in a component $K(P)$ of the unit square $0 \leqq t \leqq 1$, $0 \leqq \varepsilon \leqq 1$ such that $P(0, \varepsilon)$ and $P(1, \varepsilon)$ are fixed points Q and Q' for all values of ε, i.e. each ε-arc joins the points Q and Q'. We assume that the local representations $x^\alpha(t, \varepsilon)$ of this map with respect to any coordinate system have the following continuous derivatives:

$$\frac{\partial x^\alpha}{\partial t}, \; \frac{\partial x^\alpha}{\partial \varepsilon}, \; \frac{\partial^2 x^\alpha}{\partial t \partial \varepsilon} \left(= \frac{\partial^2 x^\alpha}{\partial \varepsilon \partial t} \right).$$

Hence $\frac{\partial x^\alpha}{\partial \varepsilon} = 0$ at Q and Q'. By parallel displacement of an arbitrary (but fixed) vector ξ at Q along ε-arcs of the above map $P(t, \varepsilon)$ we obtain a vector distribution $\xi(t, \varepsilon)$ defined in the unit square such that the components $\xi^\alpha(t, \varepsilon)$ with respect to any local x coordinate system are continuous and have the following continuous derivatives

$$\frac{\partial \xi^\alpha}{\partial t},\ \frac{\partial \xi^\alpha}{\partial \varepsilon},\ \frac{\partial^2 \xi^\alpha}{\partial t \partial \varepsilon}\ \left(= \frac{\partial^2 \xi^\alpha}{\partial \varepsilon \partial t}\right)$$

as follows from the existence theorem for differential equations. By using only the above derivatives we can deduce the following invariant relations

$$\frac{\vartheta}{\vartheta t}\left(\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon}\right) - \frac{\vartheta}{\vartheta \varepsilon}\left(\frac{\vartheta \xi^\alpha}{\vartheta t}\right) = B^\alpha_{\beta\gamma\delta}\, \xi^\beta \frac{\partial x^\gamma}{\partial \varepsilon} \frac{\partial x^\delta}{\partial t}.$$

We now observe that the second term in the left member vanishes since $\frac{\vartheta \xi^\alpha}{\vartheta t}$ is equal to zero by the parallel displacement. A necessary condition for the existence of a field of parallel vectors $\xi(x)$ in the component $K(P)$ is that the rank of the system $E_0 + \ldots + E_{n-1}$ be less than n in this component. Assuming such a rank for this system let us choose the initial values of the components of the above vector ξ at Q to be a non-trivial solution of the system $E_0 + \ldots + E_{n-1}$. Then by the result of § 4 the right members of the above relations will vanish along all ε-arcs. Hence these relations reduce to

$$\text{(4)} \qquad \frac{\vartheta}{\vartheta t}\left(\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon}\right) \equiv \frac{\partial}{\partial t}\left(\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon}\right) + L^\alpha_{\beta\gamma} \frac{\vartheta \xi^\beta}{\vartheta \varepsilon} \frac{\partial x^\gamma}{\partial t} = 0.$$

Since

$$\text{(5)} \qquad \frac{\vartheta \xi^\alpha}{\vartheta \varepsilon} = \frac{\partial \xi^\alpha}{\partial \varepsilon} + L^\alpha_{\beta\gamma}\, \xi^\beta \frac{\partial x^\gamma}{\partial \varepsilon}$$

and since $\frac{\partial \xi^\alpha}{\partial \varepsilon}$ and $\frac{\partial x^\gamma}{\partial \varepsilon}$ both vanish at Q we have that $\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon}$ is equal to zero at the point Q. Since $\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon} \equiv 0$ is a solution of (4) having the same initial values it follows from the existence theorem for ordinary differential equations that $\frac{\vartheta \xi^\alpha}{\vartheta \varepsilon}$ vanishes along any ε-arc. Hence in particular these derivatives vanish at the point Q'. Then from (5) it follows that $\frac{\partial \xi^\alpha}{\partial \varepsilon}$ vanishes at Q' because $\frac{\partial x^\gamma}{\partial \varepsilon}$ vanishes at this point. *We have now proved that we arrive at the same vector ξ' at the point Q' by parallel displacement of any solution vector ξ of the system $E_0 + \ldots + E_{n-1}$ at Q along any ε-arc of the map $P(t, \varepsilon)$.*

6. As a consequence of the above result it will follow that in any connected and simply connected open point set O contained in any component $K(P)$ the parallel displacement of any solution vector ξ of the system $E_0 + \ldots + E_{n-1}$ at a point Q to any other point Q' of O will be independent of the path of the displacement and hence will give rise to a field of parallel vectors $\xi(x)$ in O. Obviously the class of the components of the vectors $\xi(x)$ is one greater than that of the components of the connection L.

A necessary condition for the existence of a field of parallel vectors $\xi(x)$ over $\mathfrak{M}$ is that the system $E_0 + \ldots + E_{n-1}$ shall possess a non-trivial solution at any point of $\mathfrak{M}$ and this condition can be expressed by the vanishing of the resultant system R_1 of the equations $E_0 + \ldots + E_{n-1}$ over $\mathfrak{M}$. If now conversely $R_1 = 0$ over $\mathfrak{M}$ a field of parallel vectors exist in the open point sets O in any component $K(P)$. In particular if all the components $K(P)$ in $\mathfrak{M}$ are simply connected, a field of parallel vectors will exist in each of these components, but it may not be possible to choose these fields so that discontinuities will not arise at the essential singular points in $\mathfrak{M}$, i.e. at the boundaries of the various components $K(P)$. Whether or not the space $\mathfrak{M}$ is itself simply connected appears to be without especial significance in this connection. Here arises one of the essential differences in the problem of characterizing spaces admitting a field of parallel vectors under the non-analytic and analytic hypotheses. For in the analytic case the condition $R_1 = 0$ is both necessary and sufficient for the existence of a (continuous) field of parallel vectors over a simply connected space (Thomas, *loc. cit.*). Thus it appears that in a space $\mathfrak{M}$ of class C^r the various components $K(P)$ in $\mathfrak{M}$ play the same role as that of the entire space in the analytic case.

An investigation of the problem of characterizing spaces of class C^r admitting one or more continuous fields of parallel vectors which thus involves the construction of necessary and sufficient conditions for the removal of possible boundary discontinuities would be of interest but will not be considered here. In this connection it may be observed that if R_2 represents the set of all minors of order $n-1$ which can be formed from the matrix of the coefficients of the system $E_0 + \ldots + E_{n-1}$ then $R_1 = 0$, $R_2 \neq 0$, i.e. at least one of the minors of the set R_2 does not vanish at any point, over $\mathfrak{M}$ is a set of invariant conditions which are sufficient to insure the non-existence of essential singular points in $\mathfrak{M}$. Under these conditions there will exist only one component $K(P)$ and if $\mathfrak{M}$ is furthermore simply connected there will exist a field of parallel vectors in $\mathfrak{M}$. Analogous conditions can of course be given for the existence of more than one field of parallel vectors in $\mathfrak{M}$.

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