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Two theorems on product complexes

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Let A be a finite n-dimensional complex and A^s the ns-dimensional product complex $A \times A \times \cdots \times A$ of s factors A; then we have the theorems.

I. The Betti numbers of A are uniquely determined by those of A^s.

Let P_i and p_i represent the i-dimensional Betti numbers of A^s and A respectively, then we have [1])

$$P_i = \sum_{i_1+i_2+\ldots+i_s=i} p_{i_1} p_{i_2} \cdots p_{i_s} \quad (i=0, 1, 2, \ldots, ns).$$

Assume that when all the P_i $(i=0, 1, 2, \ldots, ns)$ are assigned, the p_i are uniquely determined for $i = 0, 1, \ldots, m$; then Künneth's formula gives

$$P_{m+1} = s p_{m+1} p_0^{s-1} + \sum_{\substack{i_1+i_2+\ldots+i_s=m+1 \\ i_j \leqq m \ (j=1,2,\ldots,s)}} p_{i_1} p_{i_2} \cdots p_{i_s}$$

thus uniquely determining p_{m+1}; again Künneth's formula gives $P_0 = p_0^s$ so that p_0 is uniquely determined by P_0, hence I follows by induction.

II. If A and B are finite complexes the Betti numbers of B are uniquely determined by those of A and $A \times B$.

Assume that when all the Betti numbers of A and $A\times B$ are assigned, the i-th Betti numbers of B for $i = 0, 1, \ldots, m$ are determined, then Künneth's formula gives

$$p_{m+1}(A\times B) = p_0(A) p_{m+1}(B) + \sum_{j=0,1,\ldots,m} p_{m-j+1}(A)\, p_j(B),$$

thus uniquely determining $p_{m+1}(B)$; hence II follows by induction.

These theorems are related to the problem of Ulam [2]):

If the squares A^2 and B^2 of the topological spaces A and B

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[1]) H. Künneth [Math. Ann. 90 (1923), 65—85 (18)].

respectively are homeomorphic, are A and B homeomorphic? It might be noted that in theorems I and II we may replace „Betti numbers" by „Betti numbers and coefficients of torsion" since Künneth's formula is valid for the modular Betti numbers and the modular and absolute Betti numbers determine [3]) the coefficients of torsion.

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[2]) S. Ulam [Fundamenta Math. **20** (1933), 285].

[3]) J. W. Alexander [Trans. Am. Math. Soc. **28** (1926), 301—329; 12.14].