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Summation Formulae and their Relation to Dirichlet's Series

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1. 1. Some thirty years ago Voronoï [1]) announced that, after several years of study devoted to a particular set of problems, he had arrived at the following result:

Let $\tau(n)$ be a numerical function, determined only for positive integer values of n; let $f(x)$ be continuous and have only a finite number of maxima and minima in an interval $a < x < b$. Then analytic functions $\delta(x)$ and $\alpha(x)$, dependent only on the numerical function $\tau(n)$, can be determined such that

$$\frac{1}{2}\sum_{n>a}^{n\leqq b} \tau(n)\, f(n) + \frac{1}{2}\sum_{n\geqq a}^{n<b} \tau(n)\, f(n)$$

$$= \int_a^b f(x)\,\delta(x)dx + \sum_{n=1}^{\infty} \tau(n) \int_a^b f(x)\,\alpha(nx)\,dx. \tag{1}$$

He then goes on to prove this statement when $\tau(n)$ is $d(n)$, the number of divisors of n. At one point of his lengthy memoir (at p. 462) he returns to the consideration of a general $\tau(n)$, finds $\delta(x)$, but does not attempt to find $\alpha(x)$.

That the formula (1) is true for a great variety of $\tau(n)$ is a conjecture that is forced upon anyone who works through a proof of a particular instance of it. Some striking results have been obtained by Kochliakov [2]) in a paper which starts from a generalisation of the identity

$$\varrho^{\frac{1}{2}} \sum_{n=0}^{\infty} r(n) e^{-n\pi\varrho} = \varrho^{-\frac{1}{2}} \sum_{n=0}^{\infty} r(n) e^{-n\pi/\varrho},$$

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[1]) Annales de l'Ecole Normale (3) 21 (1904); 207—267, 459—533. The conjecture is given on page 209.

[2]) Messenger of Math. 58 (1929), 1—23.

where $\varrho > 0$ and $r(n)$ is the number of ways of expressing n as the sum of two squares.

Two questions have led me to investigate the formula (1); they are closely related. The first question is

"Why, in the proved examples of (1), is the function $\alpha(x)$ the kernel of a transform of Fourier type?"

The best known examples are

$$\tau(n)=1, \qquad \alpha(x)=2\cos 2\pi x;$$
$$\tau(n)=r(n), \qquad \alpha(x)=\pi \mathrm{J}_0(2\pi\sqrt{x});$$
$$\tau(n)=d(n), \qquad \alpha(x)=4\mathrm{K}_0(4\pi\sqrt{x})-2\pi \mathrm{Y}_0(4\pi\sqrt{x}).$$

Save when $\tau(n)=1$, the proof of (1) is so beset with analytical difficulties that it is impossible to see why the function $\alpha(x)$ that occurs is a transform kernel; but in each case it is.

The answer to the first question is included in the answer to the second question.

"How can one arrive at the form for $\alpha(x)$ appropriate to a given $\tau(n)$, and is it in all cases the kernel of a transform?"

The analysis which follows is an attempt to answer these questions. A complete proof of (1) as it stands is nowhere given, though a modified form of it is proved. In the first place, (1) presents itself in the form

$$\tfrac{1}{2}\sum_{n>a}^{n\leqq b}\tau(n)f(n)+\tfrac{1}{2}\sum_{n\geqq a}^{n<b}\tau(n)f(n)$$
$$=\int_a^b f(x)\,dR_0(x)+\sum_{n=1}^{\infty}\frac{\tau(n)}{n}\int_a^b f(x)d\{\alpha_1(nx)\} \qquad (1,\text{a})$$

and, for general $\tau(n)$ as distinct from particular examples, we cannot change from Stieltjes integrals to Riemann or Lebesgue integrals by means of such equations as

$$dR_0(x)=\delta(x)\,dx, \qquad d\{\alpha_1(nx)\}=n\,\alpha(nx)\,dx,$$

unless we include the equivalents of these equations among our hypotheses. In the second place, restrictive conditions must govern the function defined by the Dirichlet's series

$$\Sigma\,\tau(n)n^{-s}$$

in order that our transformations may be carried out, and *additional* conditions must be imposed if the resulting function $\alpha_1(x)$ is to be a Fourier kernel (cf. § 4).

1.2. References to particular examples of the formula (1) are given at the end of the paper.

Section 2 sets out, as simply as possible, the transformations which underly the formulae (1) and (1, a): it makes no attempt to provide a rigorous proof of them. The remaining sections examine the extent to which the formal work of section 2 can be justified.

2. *Formal work.*

2. 1. Let $s = \sigma + it$, and let

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1}$$

where the series converges absolutely for $\sigma > 1$. Suppose that, for some positive b, the function $\psi(s)$ is defined over the strip $\sigma \geqq -b$ and that[3]), for large values of $|t|$,

$$\psi(\sigma + it) = o(|t|) \tag{2}$$

uniformly in $-b \leqq \sigma \leqq c$, where $c > 1$.

Then, by a well-known result in the theory of Dirichlet's series[4]),

$$\sum_{n=1}^{n<x} a_n + \tfrac{1}{2} a_x = \frac{1}{2\pi i} \mathrm{P} \int_{c-i\infty}^{c+i\infty} \psi(s) \frac{x^s}{s}\, ds \qquad (c > 1). \tag{3}$$

Write

$$D_0(x) = \sum_{n=1}^{n<x} a_n + \tfrac{1}{2} a_x \qquad (x \geqq 1),$$

$$D_0(x) = 0 \qquad (0 < x < 1).$$

Then

$$D_0(\theta) = \frac{1}{2\pi i} \mathrm{P} \int_{c-i\infty}^{c+i\infty} \psi(s) \frac{\theta^s}{s}\, ds$$

$$= \psi(0) + R_0(\theta) + \frac{1}{2\pi i} \mathrm{P} \int_{-b-i\infty}^{-b+i\infty} \psi(s) \frac{\theta^s}{s}\, ds, \tag{4}$$

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[3]) This particular assumption is easily avoided if we use a less direct route to our results.

[4]) (I) We adopt the convention that $a_x = 0$ when x is not an integer;

(II) Cauchy's principal value of the integral, denoted by P, is necessary when x is an integer, not otherwise; cf. Hardy and Riesz [General Theory of Dirichlet's Series (Cambridge, 1915), 12].

where [5]) $R_0(\theta)$ denotes the sum of the residues of

$$s^{-1}\psi(s)\theta^s$$

at the poles of $\psi(s)$ between $\sigma = -b$ and $\sigma = c$.

Now introduce the notation

$$A(s) = \frac{\psi(s)}{\psi(1-s)}, \tag{5}$$

so that, when $\sigma < 0$,

$$\psi(s) = A(s) \sum_{n=1}^{\infty} a_n n^{s-1}.$$

Hence, from (4), on reversing the order of summation and integration,

$$D_0(\theta) = \psi(0) + R_0(\theta) + \sum_{n=1}^{\infty} \frac{a_n}{n} \frac{1}{2\pi i} \mathrm{P} \int_{-b-i\infty}^{-b+i\infty} \frac{A(s)}{s} (n\theta)^s ds.$$

Write

$$\mathscr{A}_1(y) = \frac{1}{2\pi i} \mathrm{P} \int_{-b-i\infty}^{-b+i\infty} \frac{A(s)}{s} y^s ds. \tag{6}$$

We now have

$$D_0(\theta) = \psi(0) + R_0(\theta) + \sum_{n=1}^{\infty} \frac{a_n}{n} \mathscr{A}_1(n\theta). \tag{7}$$

Finally, we have, when $0 < \alpha < 1 < \beta$,

$$\sum_{n=1}^{n<\beta} a_n f(n) + \tfrac{1}{2} a_\beta f(\beta)$$
$$= \int_\alpha^\beta f(\theta) dD_0(\theta)$$
$$= \int_\alpha^\beta f(\theta) dR_0(\theta) + \int_\alpha^\beta f(\theta) d\left[\sum_{n=1}^{\infty} \frac{a_n}{n} \mathscr{A}_1(n\theta)\right],$$

or, on making a formal change [6]),

$$\int_\alpha^\beta f(\theta) dR_0(\theta) + \sum_{n=1}^{\infty} \frac{a_n}{n} \int_\alpha^\beta f(\theta) d[\mathscr{A}_1(n\theta)]. \tag{8}$$

[5]) The form assumes that $s = 0$ is not a pole of $\psi(s)$.

[6]) Formal changes of this sort, in which Stieltjes integrals occur, have been considered recently by L. C. Young [Journal London Math. Soc. 9 (1934), 119—126]; he gives references to other papers on the same topic.

If we assume that $A(s)$ is such that the path of integration in (6), which defines $\mathscr{A}_1(y)$, may be taken to be $(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, then $\mathscr{A}_1(y)$ has the form which characterises the kernel of a transform [7]), since

$$A(s)A(1-s) \equiv 1.$$

2.2. In the sections which follow we find a set of hypotheses which enable us to arrive at the functions $\mathscr{A}_1(y)$ when the condition 2.1 (2) is relaxed.

3. *The series* $\sum_{n=1}^{n<x} a_n(x-n)^k$.

3.1. HYPOTHESIS 1. *The function $\psi(s) = \psi(\sigma+it)$ is defined for $\sigma > 1$ by the series*

$$\sum_{n=1}^{\infty} a_n n^{-s} \qquad (a_n \text{ real}),$$

whose abscissa of absolute convergence is not greater than unity.

HYPOTHESIS 2. *There is a positive number b such that some process of analytic continuation defines $\psi(s)$ over a strip $-b \leqq \sigma \leqq 1$, and the only singularities of $\psi(s)$ in this strip are poles, finite in number, none of which lie in the strip $-b \leqq \sigma \leqq 0$.*

By a well known theorem [8]), if the abscissa of convergence is unity,

$$\psi(s) \text{ has a pole at } s = 1 \tag{1}$$

when all the a_n are positive or zero.

HYPOTHESIS 3. *The function $\psi(s)$ is of finite order in $\sigma \geqq -b$. That is, for some t_0 and some K,*

$$|\psi(s)| = O(|t|^K) \quad \textit{when } |t| > t_0 \textit{ and } \sigma \geqq -b. \tag{2}$$

By hypothesis 1, we may write [9]), when $\alpha > 1$ and $c > 1$,

$$D_{\alpha-1}(x) = \frac{\sum_{n=1}^{n<x} a_n(x-n)^\alpha}{\Gamma(\alpha)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(s)\frac{\Gamma(s)x^{s+\alpha}}{\Gamma(s+\alpha+1)}\,ds. \tag{3}$$

By hypothesis 3, we may write this, when $\alpha > K+1$, as

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[7]) HARDY and TITCHMARSH [Proc. London Math. Soc. (2) 35 (1933), 116—155]. WATSON [ibid. 156—199].

[8]) See, for example, TITCHMARSH [The Theory of Functions (Oxford, 1932), 294].

[9]) HARDY and RIESZ [loc. cit. [4]), 51; Theorem 40].

$$D_{\alpha-1}(x)=\frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)}+R_{\alpha-1}(x)$$

$$+\frac{x^{\alpha-1}}{2\pi i}\int_{-b-i\infty}^{-b+i\infty}\psi(s)\frac{\Gamma(s)}{\Gamma(s+\alpha)}x^s\,ds, \qquad (4)$$

where $R_{\alpha-1}(x)$ denotes the sum of the residues of

$$\frac{x^{s+\alpha-1}\Gamma(s)\psi(s)}{\Gamma(s+\alpha)}$$

at such poles [10]) of $\psi(s)$ as lie in the strip $-b\leqq\sigma\leqq 1$.

When we define $A(s)$ by means of the equation

$$A(s)=\frac{\psi(s)}{\psi(1-s)}, \qquad (5)$$

we get at once

$$D_{\alpha-1}(x)-\frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)}-R_{\alpha-1}(x)$$

$$=\frac{x^{\alpha-1}}{2\pi i}\int_{-b-i\infty}^{-b+i\infty}A(s)\frac{x^s\Gamma(s)}{\Gamma(s+\alpha)}\sum_{n=1}^{\infty}\frac{a_n}{n^{1-s}}ds, \qquad (6)$$

but before we can invert the order of integration and summation we must make some hypothesis about $A(s)$. We choose

HYPOTHESIS 4a. *For some B and some t_0,*

$$|A(-b+it)|=O(|t|^B) \text{ when } |t|>t_0. \qquad (7)$$

We now introduce the notation

$$\mathscr{A}_\alpha(y)=\frac{y^\alpha}{2\pi i}\int_{-b-i\infty}^{-b+i\infty}A(s)\frac{y^{s-1}\Gamma(s)}{\Gamma(s+\alpha)}ds, \qquad (8)$$

and then, when $\alpha>\max.(K+1,\, B+1)$ we may write equation (6) as

$$D_{\alpha-1}(x)-\frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)}-R_{\alpha-1}(x)=\sum_{n=1}^{\infty}a_n n^{-\alpha}\mathscr{A}_\alpha(nx). \qquad (9)$$

Moreover, it is readily seen, from (6), that the series (9) converges uniformly (and absolutely) to its sum in a range $x\geqq\delta>0$.

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[10]) (I) The form must be modified if $s=0$ is a pole of $\psi(s)$;

(II) By (1), there is always one such pole, namely, $s=1$, when $a_n\geqq 0$ and unity is the convergence abscissa.

4. *The transform associated with* $\psi(s)$.

4. 1. Having found an expression for $D_{\alpha-1}(x)$ [a fractional integral of $D_0(x)$] in terms of $\mathscr{A}_\alpha(y)$, we now examine what conditions will ensure that $\mathscr{A}_\alpha(y)$ is the fractional integral of a Fourier kernel.

Instead of the hypothesis (4a) we make the more stringent, but natural, extension of it

HYPOTHESIS 4b. *For* B *and some* t_0,

$$|A(\sigma+it)| = O(|t|^B)$$

when $|t| > t_0$ *and* $-b \leqq \sigma \leqq \frac{1}{2}$.

If we write

$$A_\alpha(y) = \frac{y^\alpha}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} A(s)\frac{\Gamma(s)y^{s-1}}{\Gamma(s+\alpha)}\,ds, \tag{1}$$

then, for $\alpha > B+1$,

$$\mathscr{A}_\alpha(y) = A_\alpha(y) + S_\alpha(y), \tag{2}$$

where $S_\alpha(y)$ is the sum of the residues of

$$\frac{\psi(s)\,\Gamma(s)\,y^{\alpha+s+1}}{\psi(1-s)\,\Gamma(s+\alpha)} \tag{3}$$

at its poles in $-b < \sigma < \frac{1}{2}$.

Before we can prove that the functions $\mathscr{A}_\alpha(y)$ which occur in 3. 1 (9) are the integrals of a Fourier kernel we must make yet another hypothesis.

HYPOTHESIS 5. *The function*

$$\frac{A(s)\,\Gamma(s)}{\Gamma(s+\alpha)} = \frac{\psi(s)\,\Gamma(s)}{\psi(1-s)\,\Gamma(s+\alpha)}$$

has no pole in the strip $-b \leqq \sigma \leqq \frac{1}{2}$.

It may be noted here that, in many cases, the pole of $\Gamma(s)$ at $s=0$ is neutralised by the pole of $\psi(1-s)$ at $s=0$ [cf. 3. 1 (1)].

With this final hypothesis, we may write

$$\mathscr{A}_\alpha(y) = A_\alpha(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\tfrac{1}{2}-it)\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}-it+\alpha)}\,y^{\alpha-\frac{1}{2}-it}dt. \tag{4}$$

4. 2. Since a_n is real, $\psi(\sigma+it)$ and $\psi(\sigma-it)$ are conjugate complex numbers; also, since $A(s)$ is defined as $\dfrac{\psi(s)}{\psi(1-s)}$, $A(\frac{1}{2}+it)$ and $A(\frac{1}{2}-it)$ are conjugate complex numbers. Further,

$$\left|A(\tfrac{1}{2}-it)\right| = +\sqrt{\{A(\tfrac{1}{2}-it)A(\tfrac{1}{2}+it)\}}$$
$$= +\sqrt{\left\{\frac{\psi(\frac{1}{2}-it)}{\psi(\frac{1}{2}+it)}\cdot\frac{\psi(\frac{1}{2}+it)}{\psi(\frac{1}{2}-it)}\right\}},$$

and so

$$\left|A(\tfrac{1}{2}-it)\right| = 1. \tag{1}$$

Thus the integral in 4. 1 (4) is absolutely convergent when $\alpha > 1$. Moreover, we can apply a general theorem, given by Watson in his paper on General Transforms [11]), to deduce the following results: —

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-\frac{1}{2}-it}\frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,dt \tag{2}$$

exists as a mean-square integral, and if its value is denoted by $\frac{\chi(y)}{y}$, then $\frac{\chi(y)}{y}$ is a function of class $L^2(0, \infty)$, $\frac{\chi(xy)}{y}$ is a kernel of a transform (in the sense of Watson's Theorem IA), and

$$\int_0^\infty \frac{\chi(xy)\chi(yz)}{y^2}\,dy = \text{Min}\,(x, z)$$

for all values of x, z in $(0, \infty)$.

The relation between this function $\chi(y)$ and the functions $A_\alpha(y)$, given by 4. 1 (4) when $\alpha > 1$, is readily deduced from standard theorems concerning mean-square integrals. For the mean-square convergence of $f_n(x)$ to $f(x)$ implies [12])

$$\lim_{n\to\infty}\int_0^x f_n(\theta)g(\theta)\,d\theta = \int_0^x f(\theta)g(\theta)\,d\theta \tag{3}$$

for all functions $g(\theta)$ of integrable square in $(0, x)$. In particular, when $\alpha > \frac{1}{2}$, the mean-square convergence of $f_n(x)$ to $f(x)$ implies

$$\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^x (x-\theta)^{\alpha-1}\theta f_n(\theta)d\theta = \frac{1}{\Gamma(a)}\int_0^x (x-\theta)^{\alpha-1}\theta f(\theta)\,d\theta.$$

So when we put $f(y) = \frac{\chi(y)}{y}$, defined by (2), we get

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[11]) Proc. London Math. Soc. (2) 35 (1933), 156—199 (162, 3).

[12]) See, for example, Titchmarsh [The Theory of Functions (Oxford, 1932), 389]

$$\lim_{q\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1} y\,dy \int_{-q}^{q} y^{-\frac{1}{2}-it}\frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,dt$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\chi(y)\,dy \qquad (4)$$

That is,

$$\lim_{q\to\infty}\frac{1}{2\pi}\int_{-q}^{q} x^{\alpha+\frac{1}{2}-it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{3}{2}-it+\alpha)}A(\tfrac{1}{2}-it)\,dt$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\chi(y)\,dy.$$

Hence, on using our previous notation, $A_{\alpha+1}(y)$ is, when $\alpha > \frac{1}{2}$, the αth integral of $\chi(y)$, where $\dfrac{\chi(xy)}{y}$ is a kernel of a transform.

5. *Convergence factors.*

5. 1. We have already (§ 3) observed that

$$D_0(x)=\sum_{n=1}^{n<x} a_n+\tfrac{1}{2}a_x=\frac{1}{2\pi i}\,\mathrm{P}\int_{c-i\infty}^{c+i\infty}\frac{x^s\psi(s)}{s}\,ds \qquad (c>1),$$

where the principal-value sign is required *only when x is an integer*, and that, when $\alpha > 1$,

$$D_{\alpha-1}(x)=\frac{1}{\Gamma(\alpha)}\sum_{n=1}^{n<x} a_n(x-n)^{\alpha-1}$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+\alpha-1}\psi(s)\Gamma(s)}{\Gamma(s+\alpha)}\,ds.$$

If now *we exclude integer values of x when $\alpha=1$*, we may write

$$D_{\alpha-1}(x)=\frac{x^{\alpha-1}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s\psi(s)\Gamma(s)}{\Gamma(s+\alpha)}\,ds \qquad (\alpha\geqq 1,\ c>1),$$

or, on introducing the notation

$$F(s)=x^s\psi(s)\frac{\Gamma(s)}{\Gamma(s+\alpha)},$$

$$D_{\alpha-1}(x)=\lim_{\delta\to 0}\frac{x^{\alpha-1}}{2\pi i}\left[\int_c^{c+i\infty} e^{\delta i(s-k)}F(s)\,ds+\int_{c-i\infty}^{c} e^{-\delta i(s-k)}F(s)\,ds\right], \quad (1)$$

where k is any real finite number and δ tends to zero through positive values.

Now suppose hypotheses 1, 2, 3 and 4b to be satisfied. Then

$$\frac{x^{\alpha-1}}{2\pi i}\left[\int_c^{c+i\infty} e^{\delta i(s-k)}F(s)\,ds+\int_{c-i\infty}^{c} e^{-\delta i(s-k)}F(s)\,ds\right]$$

$$=\frac{x^{\alpha-1}}{2\pi i}\left[\int_{-b}^{-b+i\infty} e^{\delta i(s-k)}F(s)\,ds+\int_{-b-i\infty}^{-b} e^{-\delta i(s-k)}F(s)\,ds\right]$$

$$+x^{\alpha-1}\left[\text{sum of residues of } e^{\delta i(s-k)}F(s) \text{ at its poles in } t>0,\ -b<\sigma\leqq 1\right]$$

$$+x^{\alpha-1}\left[\text{sum of residues of } e^{-\delta i(s-k)}F(s) \text{ at its poles in } t<0,\ -b<\sigma\leqq 1\right]$$

$$-\frac{x^{\alpha-1}}{2\pi i}\int_{-b}^{c} e^{\delta i(s-k)}F(s)\,ds+\frac{x^{\alpha-1}}{2\pi i}\int_{-b}^{c} e^{-\delta i(s-k)}F(s)\,ds,$$

where the path of integration in the last integral has indentations below the real axis at each real pole of $F(s)$ in $-b<\sigma\leqq 1$, and the path of integration in the last integral but one has corresponding indentations above the real axis.

Accordingly, with the notation of section 3,

$$D_{\alpha-1}(x)-\frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)}-R_{\alpha-1}(x)$$

$$=\lim_{\delta\to 0}\frac{x^{\alpha-1}}{2\pi i}\left[\int_{-b}^{-b+i\infty} e^{\delta i(s-k)}F(s)\,ds+\int_{-b-i\infty}^{-b} e^{-\delta i(s-k)}F(s)\,ds\right]. \qquad (2)$$

In this, we may write

$$F(s)=\frac{\psi(s)}{\psi(1-s)}\cdot\frac{x^s\Gamma(s)}{\Gamma(s+\alpha)}\sum_{n=1}^{\infty}a_n n^{s-1}$$

and then integrate term-by-term. Thus, for $\alpha\geqq 1$,

$$D_{\alpha-1}(x)-\frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)}-R_{\alpha-1}(x)=\lim_{\delta\to 0}\sum_{n=1}^{\infty}a_n n^{-\alpha}\mathscr{A}_\alpha(nx,\delta), \qquad (3)$$

where

$$\mathscr{A}_\alpha(y,\delta)=\frac{1}{2\pi i}\int_{-b}^{-b+i\infty} e^{\delta i(s-k)}\frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)}y^{s+\alpha-1}\,ds$$

$$+\frac{1}{2\pi i}\int_{-b-i\infty}^{-b} e^{-\delta i(s-k)}\frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)}y^{s+\alpha-1}\,ds. \qquad (4)$$

If now we introduce hypothesis 5, so that the integrands in (4) have no poles between $\sigma = -b$ and $\sigma = \frac{1}{2}$, we may identify $\mathscr{A}_\alpha(y, \delta)$ with $A_\alpha(y, \delta)$, where

$$A_\alpha(y, \delta) = \frac{y^\alpha}{2\pi i}\int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} e^{\delta i(s-k)} \frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)} y^{s-1} ds$$
$$+ \frac{y^\alpha}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}} e^{-\delta i(s-k)} \frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)} y^{s-1} ds.$$

In these integrals put $s = \frac{1}{2} - it$; we get, on taking $k = \frac{1}{2}$,

$$A_\alpha(y, \delta) = \frac{1}{2\pi}\int_0^\infty e^{-\delta t} A(\tfrac{1}{2} - it) \frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}-it+\alpha)} y^{\alpha-\frac{1}{2}-it} dt$$
$$+ \frac{1}{2\pi}\int_{-\infty}^0 e^{\delta t} A(\tfrac{1}{2} - it) \frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}-it+\alpha)} y^{\alpha-\frac{1}{2}-it} dt. \tag{5}$$

Moreover, from (3), when hypotheses 1—5 are satisfied,

$$D_{\alpha-1}(x) - \frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)} - R_{\alpha-1}(x) = \lim_{\delta\to 0} \sum_{n=1}^\infty n^{-\alpha} a_n A_\alpha(nx, \delta) \tag{6}$$

for $\alpha > 1$ and positive values of x, and for $\alpha = 1$ and positive non-integer values of x.

5. 2. It is clear from the proceeding sub-section that the convergence factor exp $(-\delta|t|)$ in 5. 1 (5) is the factor appropriate to the „order" hypotheses 3, 4a and 4b. If these hypotheses are not satisfied, but are satisfied when

$$|t|^K \text{ is replaced by } \exp\left(|t|^K\right),$$

then a suitable convergence factor is $\exp\{-\delta \exp(|t|)\}$. Appropriate modifications of 5. 1 (1) are easily made to ensure such a factor in the corresponding modification of 5. 1 (5). That is to say, the order hypotheses 3, 4a and 4b, govern the detail rather than the general form of 5. 1 (3), (5) and (6).

This is not the case when we come to consider the summation formulae of § 6. There, it is necessary to our mode of proof to have an ordinary convergent sum at some stage of the work, and the order hypotheses 3, 4a, and 4b are required to ensure this. On the other hand, such a limitation may be due solely to the method of proof.

6. *The summation formula.*

6. 1. Let

$$r_0(x) = D_0(x) - \psi(0) - R_0(x),$$

$$r_1(x) = \int_0^x r_0(t)\,dt = D_1(x) - x\,\psi(0) - R_1(x),$$

and so on, the functions $D_\alpha(x)$, $R_\alpha(x)$ being those defined in § 3. Then, on hypotheses 1—3, 4b and 5 [13]), when α is sufficiently large

$$r_{\alpha-1}(x) = \sum_{n=1}^{\infty} n^{-\alpha} a_n A_\alpha(nx), \tag{1}$$

and the series is absolutely convergent, uniformly in any finite interval $a \leqq x \leqq b$.

We now take $0 < a < 1$ and, as in § 2, we write

$$\sum_{n=1}^{n<x} f(n)\,a_n + \tfrac{1}{2} f(x)\,a_x = \int_a^x f(\theta)\,d\,D_0(\theta).$$

We then replace the right hand side by

$$\int_a^x f(\theta)\,dr_0(\theta) + \int_a^x f(\theta)\,dR_0(\theta). \tag{2}$$

Suppose [14]) now that $f(\theta)$ and all its differential coefficients down to $f^{(\alpha)}(\theta)$ are bounded in (a, x) and that $f^{(\alpha-1)}(\theta)$ is the (R) integral of $f^{(\alpha)}(\theta)$. Integration by parts in the first integral of (2) gives

$$\Big[r_0(t)f(t) - r_1(t)f'(t) + \ldots + (-1)^{\alpha-1} r_{\alpha-1}(t) f^{(\alpha-1)}(t)\Big]_a^x$$
$$+ (-1)^\alpha \int_a^x r_{\alpha-1}(t) f^{(\alpha)}(t)\,dt + \int_a^x f(t)\,d\,R_0(t).$$

In this we can substitute for $r_{\alpha-1}(t)$ from (1) and so obtain

$$\sum_{n=1}^{n<x} f(n)\,a_n + \tfrac{1}{2} f(x) a_x = \int_a^x f(t)\,d\,R_0(t)$$
$$+ \Big[r_0(t)f(t) - r_1(t)f'(t) + \ldots + (-1)^{\alpha-1} r_{\alpha-1}(t) f^{(\alpha-1)}(t)\Big]_a^x$$
$$+ (-1)^\alpha \sum_{n=1}^{\infty} n^{-\alpha} a_n \int_a^x f^{(\alpha)}(t) A_\alpha(nt)\,dt. \tag{3}$$

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[13]) If hypothesis 5 is omitted, the only change necessary is the replacement of the functions $A_\alpha(x)$ which follow by the functions $\mathscr{A}_\alpha(x)$ of 3.1; cf. 4. 1(2) and (3).

[14]) It is more convenient to work with the Riemann-Stieltjes integral than with the Lebesgue-Stieltjes integral.

6. 2. We now show that, with the hypotheses of 6. 1, the last equation may be written as

$$\sum_{n=1}^{n<x} f(n)a_n = \int_a^x f(t)\,dR_0(t) + \lim_{\delta\to 0}\sum_{n=1}^{\infty} n^{-1}a_n \int_a^x f(t)\,d\{A_1(nt,\delta)\}, \quad (1)$$

whenever x is not an integer.

If K and B are the constants in the hypotheses 3 and 4b respectively and $\alpha > \mathrm{Max}\,(K+1, B+1)$, then α is sufficiently large to ensure the truth of 6. 1 (3). Also, for such an α, [cf. 3. 1 (8) and 4. 1 (4)]

$$A_\alpha(nx) = \mathscr{A}_\alpha(nx) = \frac{(nx)^{\alpha-b-1}}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} (nx)^{s+b}\frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)}\,ds,$$

the integral being absolutely convergent, and

$$A_\alpha(nx,\delta) = \mathscr{A}_\alpha(nx,\delta)$$

$$= \frac{(nx)^{\alpha-b-1}}{2\pi i}\int_{-b}^{-b+i\infty} e^{\delta i(s-\frac{1}{2})}(nx)^{s+b}\frac{A(s)\Gamma(s)}{\Gamma(s+\alpha)}\,ds$$

$$+ \frac{(nx)^{\alpha-b-1}}{2\pi i}\int_{-b-i\infty}^{-b} e^{-\delta i(s-\frac{1}{2})}(nx)^{s+b}\frac{A(s)\Gamma(s)}{\Gamma(s+a)}\,ds.$$

From these forms it is not difficult (though somewhat tedious) to prove that, given $\varepsilon > 0$, we can find δ_1 such that $\delta < \delta_1$ implies

$$\left|\sum_{n=1}^{N} n^{-\alpha}a_n\{A_\alpha(nt) - A_\alpha(nt,\delta)\}\right| < \varepsilon$$

for all t in a finite range $a \leqq t \leqq t_1$ and for all N, and to prove further that

$$\sum_{n=1}^{\infty} n^{-\alpha}a_n \int_a^x f^{(\alpha)}(t)A_\alpha(nt,\delta)\,dt$$

is convergent and tends to

$$\sum_{n=1}^{\infty} n^{-\alpha}a_n \int_a^x f^{(\alpha)}(t)A_\alpha(nt)\,dt$$

as δ tends to zero.

Hence 6. 1 (3) may be written as

$$\sum_{n=1}^{n<x} f(n)a_n + \tfrac{1}{2}f(x)a_x = \int_a^x f(t)\,dR_0(t)$$
$$+\Big[r_0(t)f(t) - r_1(t)f'(t) + \ldots + (-1)^{\alpha-1}r_{\alpha-1}(t)f^{(\alpha-1)}(t)\Big]_a^x$$
$$+ \lim_{\delta\to 0}(-1)^{\alpha}\sum_{n=1}^{\infty} n^{-\alpha}a_n \int_a^x f^{(\alpha)}(t)\,A_\alpha(nt,\delta)\,dt. \qquad (2)$$

On integrating once by parts, the last term in this becomes, when $\alpha \geqq 2$,

$$\Big[(-1)^{\alpha}f^{(\alpha-1)}(t)\lim_{\delta\to 0}\sum_{n=1}^{\infty} n^{-\alpha}a_n A_\alpha(nt,\delta)\Big]_a^x$$
$$+ \lim_{\delta\to 0}(-1)^{\alpha-1}\sum_{n=1}^{\infty} n^{1-\alpha}a_n \int_a^x f^{(\alpha-1)}(t)A_{\alpha-1}(nt,\delta)\,dt;$$

for, by 5. 1 (6), the limit in the square bracket is known to exist and the terms in that bracket are known to be equal to

$$(-1)^{\alpha}f^{(\alpha-1)}(t)\,r_{\alpha-1}(t).$$

Accordingly, when $\alpha \geqq 2$ one integration by parts in (2) above will reproduce that formula with α reduced by unity.

It is now clear that successive integration by parts will reduce (2) to the form

$$\sum_{n=1}^{n<x} f(n)a_n + \tfrac{1}{2}f(x)a_x$$
$$= \int_a^x f(t)\,dR_0(t) + r_0(x)f(x) - r_0(a)f(a)$$
$$- \lim_{\delta\to 0}\sum_{n=1}^{\infty} n^{-1}a_n \int_a^x f'(t)\,A_1(nt,\delta)\,dt. \qquad (3)$$

Since 5. 1 (6) is not necessarily true when x is an integer and $\alpha = 1$, we must exclude integer values of x before we can integrate by parts again. A final integration by parts gives the result (1).

6. 3. Finally, we note that the formula 6. 1 (3) may be reduced to a simpler form if, in addition to the hypotheses of 6. 1, we know that

$$r_{\alpha-1}(x) = \sum_{n=1}^{\infty} n^{-\alpha}a_n A_\alpha(nx) \qquad (\alpha \geqq 1). \qquad (1)$$

With the type of function $\psi(s)$ connected with the theory of lattice-points, one is often able to prove, independently of the transformations given in the present paper, that (1) is true provided the Riesz (R, n, k) sum of the infinite series is taken as the interpretation of the right hand side.

Let us suppose then, that for $\alpha \geqq 1$ and for some fixed k,

$$r_{\alpha-1}(x) = \lim_{N\to\infty} N^{-k} \sum_{n=1}^{N} n^{-\alpha} a_n (N-n)^k A_\alpha(nx). \tag{2}$$

The direct route to the appropriate summation formula is then, as in the beginning of 6. 1,

$$\sum_{n=1}^{n<x} f(n) a_n + \tfrac{1}{2} f(x) a_x$$

$$= \int_a^x f(t)\, dR_0(t) + \int_a^x f(t)\, dr_0(t)$$

$$= \int_a^x f(t)\, dR_0(t) + \int_a^x f(t)\, d\left\{ \lim_{N\to\infty} N^{-k} \sum_{n=1}^{N} n^{-1} a_n (N-n)^k A_1(nt) \right\},$$

followed by an inversion which, when justified, gives

$$\int_a^x f(t)\, dR_0(t) + \lim_{N\to\infty} N^{-k} \sum_{n=1}^{N} n^{-1} a_n (N-n)^k \int_a^x f(t)\, d\{A_1(nt)\}.$$

It is, of course, the justification of the last step which leads to the analytical difficulties. On the other hand, by introducing the hypotheses of 6. 1 [including those which relate to the differential coefficients of $f(\theta)$], we can establish 6. 1 (3) and then replace the last term of it by

$$(-1)^\alpha \lim_{N\to\infty} N^{-k} \sum_{n=1}^{N} n^{-\alpha} a_n (N-n)^k \int_a^x f^{(\alpha)}(t)\, A_\alpha(nt)\, dt.$$

Then, as in 6. 2 (but with a different type of limit), we can reduce the formula by integration by parts until we get to the form

$$\sum_{n=1}^{n<x} f(n) a_n + \tfrac{1}{2} f(x) a_x$$

$$= \int_a^x f(t)\, dR_0(t) + \sum_{n=1}^{\infty} a_n \int_a^x f(t)\, d\{A_1(nt)\}, \tag{3}$$

where the „sum" of the infinite series is its (R, n, k) sum [15]).

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[15]) I am indebted to Prof. J. R. Wilton for the idea that Riesz sums are applicable to the formula (3) as well as the formula (1).

There is the manifest proviso that, if (2) is false for certain values of x, e. g. integer values, then (3) is also false for such values of x.

7. *Extension of previous results.*

7. 1. Let us return now to the work of § 3. At equation (4) we there proved that, subject to the hypotheses 1, 2 and 3,

$$D_{\alpha-1}(x) = \frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)} + R_{\alpha-1}(x) + \frac{x^{\alpha-1}}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} \psi(s)\,\frac{\Gamma(s)}{\Gamma(s+\alpha)}\,x^s\,ds, \tag{1}$$

whenever $\alpha > K+1$.

Now let

$$\sum_{n=1}^{\infty} b_n n^{-s} \qquad (b_n \ \textit{real})$$

be any Dirichlet's series whose abscissa of absolute convergence does not exceed unity; let the series, and its analytic continuations, define a function $\varphi(s)$. Then the last term in (1), above, can be written as

$$\frac{x^{\alpha-1}}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} \frac{\psi(s)}{\varphi(1-s)}\cdot\frac{x^s\,\Gamma(s)}{\Gamma(s+\alpha)}\sum_{n=1}^{\infty} b_n n^{s-1}\,ds. \tag{2}$$

When we write

$$B(s) = \frac{\psi(s)}{\varphi(1-s)}$$

and make the hypothesis,

HYPOTHESIS 4c. *For some M and some t_0,*

$$|B(-b+it)| = O(|t|^M) \ \textit{when} \ |t| > t_0, \tag{3}$$

we can, when $\alpha > \operatorname{Max}(K+1, M+1)$, write (1) as

$$D_{\alpha-1}(x) - \frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)} - R_{\alpha-1}(x) = \sum_{n=1}^{\infty} b_n n^{-\alpha}\,\mathscr{B}_\alpha(nx), \tag{4}$$

where

$$\mathscr{B}_\alpha(y) = \frac{y^\alpha}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} B(s)\,\frac{y^{s-1}\,\Gamma(s)}{\Gamma(s+\alpha)}\,ds. \tag{5}$$

7. 2. When we write

$$C(s) = \frac{\varphi(s)}{\psi(1-s)}$$

it is clear that

$$B(s)\,C(1-s) \equiv 1$$

so that $B(s)$ and $C(s)$ have the forms appropriate to non-symmetrical Fourier formulae [16]) or what may be called cross-transforms.

It may be pointed out that a theory of such cross-transforms along the lines of Professor Watson's theory of General Transforms has not yet been developed. Some, at any rate, of its results should prove to be interesting.

8. *Particular examples.*

8. 1. The formula 1. 1 (1) when $\tau(n)=1$ is well known: it has, in various forms, been frequently investigated. It may be considered as Poisson's summation formula [17]) applied to an even function.

When $\tau(n)=r(n)$, the formula is that given by Landau in Vorlesungen über Zahlentheorie II, 274, Satz 559. Some applications of this formula have recently been published [18]).

The formula with $\tau(n) = d(n)$ was investigated at length by Voronoï (loc. cit.): more recent proofs [19]) of it appeared a few years ago.

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[16]) Hardy and Titchmarsh, loc. cit., § 1. 3 and the examples in § 1. 5. Dixon and Ferrar [Quart. J. of Math. (Oxford) 3 (1932), 55].

[17]) For example, Wilton [Journal London Math. Soc. 5 (1930), 276—279].

[18]) Dixon and Ferrar [Quart. J. of Math. (Oxford) 5 (1934), 48—63].

[19]) Dixon and Ferrar [Quart. J. of Math. (Oxford) 2 (1931), 31—54]. Wilton [ibid. 3 (1932), 26—32]. The transformations of the present paper are essentially those of the paper Dixon and Ferrar (1931). Although Professor A. L. Dixon has taken no active part in the preparation of the present paper, I am deeply indebted to him for that assistance which comes from our continued collaboration in this particular field of study.

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