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Chapter 3. The First Steps in Surface Theory

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5. Intersection Theory on Surfaces

I introduce the intersection product for divisors following [Beau]. This is the same as the topological intersection product as given in the Appendix. Riemann-Roch for divisors is proven and as a consequence, the genus formula for an irreducible curve. This will be used a lot in what follows.

In this section let S be an algebraic surface. For any line bundle $\mathcal{L}$ on S set

$$h^i(\mathcal{L}) = \dim H^i(S, \mathcal{L})$$

and

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) \quad \text{(Euler-Poincaré characteristic of } \mathcal{L}).$$

For a divisor D set

$$h^i(D) = h^i(\mathcal{O}(D)).$$

The topological Euler-Poincaré characteristic is the alternating sum

$$e(S) = \sum_j (-1)^j \dim H^j(\mathbb{C}_S).$$

Recall (see Appendix A2.7), that $H^i(S, \mathbb{C}_S)$ is finite-dimensional and zero for $i > 4$.

Let $C \subset S$ be an irreducible curve. Recall the notion of *normalisation* or *desingularisation*

$$\nu : \tilde{C} \longrightarrow C.$$

Here $\tilde{C}$ is a non-singular curve (projective manifold of dimension 1) and ν is a finite map which is an isomorphism outside the singularities of C. For existence and uniqueness let me refer to [G-H, p. 498] or [G, Chapter II]. One only needs to know that the construction is done locally (in the ordinary topology) and one can form the normalisation of any part of a curve in an open subset of S. Suppose now that two distinct irreducible curves C and C' meet in a point x. Suppose that at x one has local equations $f = 0$ for C and $g = 0$ for C', where $f, g \in \mathcal{O}_x$ the local ring of germs of holomorphic functions at x. Of course f nor g need to be irreducible. Let me recall (see e.g. [G, p.83]) how one may define the *intersection multiplicity of C and C' at x* denoted $(C, C')_x$. One first assumes that f is irreducible in the coordinate neighourhood U of x where it is defined. Choose coordinates in U such that $x = (0,0)$. Let $\Delta \subset \mathbb{C}$ be a suitable small disk centered at 0 in the t-plane and let $\nu : \Delta \to U$ be a local normalisation for $U \cap C$ and suppose that

$$g \circ \nu(t) = a_l t^l + \text{higher order terms}$$

with $a_l \neq 0$. This number l by definition is $(C, C')_x$. In general one can write $f = \prod_j f_j^{m_j}$ with f_j irreducible and defining locally an irreducible curve C_j. Let me extend the definition by linearity: $(C, C')_x = \sum_j m_j (C_j, C)_x$.

Remark 1. The intersection multiplicity of C and C' at x can also be defined as the dimension of the $\mathbb{C}$-vector-space $\mathcal{O}_{S,x}/(f,g)$. It is not difficult to show that this is the same as the preceding definition. See Problem 1.

From the preceding remark it is clear that $(C,C')_x = (C',C)_x$ so, defining the intersection number

$$(C,C') = \sum_x (C,C')_x$$

one obtains a symmetric pairing on the set of irreducible curves. I want to extend this definition to all divisors. This is not at all obvious, since one doesn't know what for instance (C,C) should be. The idea now is that in forming the intersection product one should be allowed to move a curve in its linear equivalence class. Since in particular, any very ample divisor can be moved at will in its linear equivalence class, one can define (C,C) for these divisors. Then, remembering that any divisor is the difference of two very ample divisors, one can define the intersection product for any two divisors. But of course, there are many ways to write a divisor as a difference of two very ample divisors and it is not clear that this yields a well-defined intersection product. Although one in principle can carry out this program, it is a little faster to follow Beauville's route [Beau, Chapt. 1].

Theorem 2. *For any two divisors D, D' one poses*

$$(D,D') = \chi(\mathcal{O}_S) - \chi(\mathcal{O}(-D)) - \chi(\mathcal{O}(-D')) + \chi(\mathcal{O}(-D-D')).$$

This defines a symmetric bilinear product on Pic S *extending the intersection product on irreducible curves. This product is the unique product on* Pic S *which satisfies the following properties;*

1. *If C and C' are two smooth curves which intersect transversally, (C,C') is exactly the number of intersection points,*
2. *it is symmetric: $(D,D') = (D',D)$,*
3. *it is linear: $(D+D',D'') = (D,D'') + (D',D'')$.*
4. *it depends on the linear equivalence classes of the divisors only.*

Note that this theorem implies that the intersection product only depends on the sheaves defined by the divisors, i.e. one can move a divisor in its linear equivalence class without changing intersection products.

Proof:

Step 1. Uniqueness. Given two divisors C and D, one can write them as differences of very ample divisors (Serre's Theorem), say $C = C_1 - C_2$ and $D = D_1 - D_2$. One can choose smooth curves C'_k in $|C_k|$, $D'_k \in |D_k|$ $(k = 1,2)$ such that C'_k meets D'_j transversally $j,k = 1,2$. This follows from Bertini. The four properties then totally determine (C,D), since $(C,D) = (C'_1,D'_1) - (C'_1,D'_2) - (C'_2,D'_1) + (C'_2,D'_2)$ and each of the four terms is equal to the number of intersection points of the curves involved.

Step 2. Let me prove that the definition for distinct irreducible curves C and C' with coincides with (C,C') as defined by means of local intersection numbers.

Choose a non-trivial section s, resp s' of the line bundle $\mathcal{O}(C)$, resp. $\mathcal{O}(C')$ vanishing on C, resp. C'. The sequence

$$0 \to \mathcal{O}_S(-C-C') \xrightarrow{(s',-s)} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \xrightarrow{\begin{pmatrix} s \\ s' \end{pmatrix}} \mathcal{O}_S \to \mathcal{O}_{C\cap C'} \to 0$$

then is exact. This simply is a translation of the exactness of the *Koszul sequence* at any point x of S:

$$0 \to \mathcal{O}_x \xrightarrow{(g,-f)} \mathcal{O}_x^{\oplus 2} \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} \mathcal{O}_x \to \mathcal{O}_x/(f,g) \to 0.$$

I leave it to the reader that this sequence is indeed exact. See Problem 2. From this exact sequence one immediately verifies the desired equality.

Step 3. Let me show that for any irreducible smooth curve C on S and every divisor D on S one has

$$(C,D) = \deg(D|C).$$

To show this you may employ the exact sequences

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

and

$$0 \to \mathcal{O}_S(-D-C) \to \mathcal{O}_S(-D) \to \mathcal{O}_C(-D) \to 0.$$

One finds that

$$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) = \chi(\mathcal{O}_C)$$
$$-\chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}(-C-D)) = -\chi(\mathcal{O}_C(-D))$$

and so $(C,D) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-D)) = -\deg(-D|C) = \deg(D|C)$ by the usual Riemann-Roch theorem on the smooth curve C.

Step 4. Completion of proof

Let me introduce for any three divisors D_1, D_2, D_3 the expression

$$s(D_1,D_2,D_3) = (D_1,D_2+D_3) - (D_1,D_2) - (D_1,D_3).$$

This is symmetric in the arguments as one readily verifies. It vanishes if D_1 is a non-singular curve (by Step 3.) and hence it also vanishes if D_2 or D_3 is a non-singular curve. One needs to see that it always vanishes. But as remarked before, one can write any divisor and hence also D_2 as a difference $D_2 = A - B$ with A and B very ample. Moreover, by Remark 4.22 one can suppose that A and B are smooth and connected. Now you write out $0 = s(D_1,D_2,B) = (D_1,A) - (D_1,D_2) - (D_1,B)$. From this it follows immediately that the product is bilinear. ■

Remark 3. Note that the latter coincides with the topological intersection product since this product also enjoys the preceding properties. This one sees as follows. Any divisor

$D = \sum n_i C_i$ yields a cohomology class $c(D) = \sum n_i c(C_i)$ where the class $c(C_i) \in H^2(S, \mathbb{Z})$ is defined as the Poincaré-dual of the fundamental class $h(C_i) \in H_2(S, \mathbb{Z})$ of C_i. Likewise one has $h(D) = \sum_i n_i h(C_i) \in H_2(M, \mathbb{Z})$, a class which is Poincaré dual to $c(D)$. For any two homology classes $c, c' \in H_2(S, \mathbb{Z})$ the intersection product (c, c') is defined in Appendix A.2 where it is shown that this product is the same as evaluating the cup product of the Poincaré dual classes on the orientation class $o_S \in H_4(S, \mathbb{Z})$. Clearly, one gets a pairing on divisors by setting $(D, D') = (h(D), h(D'))$. This pairing only depends on the linear equivalence class. Indeed, by Proposition A3.8 the first Chern class of a divisor $c_1(D)$ coincides with the fundamental cohomology class $c(D)$ and the first Chern class depends only on the isomorphism class $\mathcal{O}_M(D)$ of the divisor D. This shows that the fourth property holds. The first property is Claim A2.17. The remaining ones are trivial.

Examples

1. $S = \mathbb{P}^2$. Any divisor D is linearly equivalent to dL, with L a line. So, if $D \equiv dL$ and $D' \equiv d'L$ one finds that $(D, D') = dd'$ which is Bezout's theorem.

2. $S = \mathbb{P}^1 \times \mathbb{P}^1$. One has the two fibres $F = \mathbb{P}^1 \times a$ and $F' = b \times \mathbb{P}^1$ and leaving these away one gets $\mathbb{C}^2$ on which every divisor is the divisor of some rational function. It follows that all divisors on S are linearly equivalent to $mF + m'F'$. Clearly $(F, F') = 1$ and since on $\mathbb{P}^1$ any two points are linearly equivalent any two fibres of the form $\mathbb{P}^1 \times a$ are linearly equivalent and so $(F, F) = 0 = (F', F')$. This completely determines the intersection pairing.

Let me now show that one can derive a weak form of the Riemann-Roch theorem, using Serre-duality A3.3.

Theorem 4. (Riemann-Roch) *For any divisor D on S one has*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

Proof: By definition $(-D, D - K_S) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}(D)) - \chi(\mathcal{O}_S(K_S - D)) + \chi(\mathcal{O}(K_S))$. By Serre duality $\chi(\mathcal{O}(K_S)) = \chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S(K_S - D)) = \chi(\mathcal{O}_S(D))$. So $-(D, D) + (D, K_S) = (-D, D - K_S) = 2(\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(D))$. ∎

Remark 5. Using Serre duality once more one can rewrite the Riemann-Roch theorem as follows.

$$h^0(D) - h^1(D) + h^0(K_S - D) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

From this way of writing Riemann-Roch one derives an inequality which will be used a lot in the sequel

$$h^0(D) + h^0(K_S - D) \geq \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

Remark 6. The strong form of the Riemann-Roch theorem, also called Hirzebruch-Riemann-Roch theorem, expresses $\chi(\mathcal{O}_S)$ in topological invariants of S. For algebraic surfaces this goes back to Noether and therefore is called the Noether formula. It reads as follows

$$\chi(\mathcal{O}_S) = \frac{1}{12}((K_S, K_S) + e(S)).$$

That indeed the self intersection of K_S is a topological invariant follows since it equals the self-intersection of the first Chern class of the surface, which is a topological invariant. Let me refer to [G-H, Chapter 4.6] for a geometric proof of the Noether formula. For the general Riemann-Roch theorem for projective manifolds I refer to [Hir].

The Riemann-Roch theorem is also valid for line bundles $\mathcal{L}$ on any surface, even if these are not of the form $\mathcal{O}(D)$. Clearly, even to make sense of the Riemann-Roch formula, one needs to use here the topological definition of the intersection product.

Next, let me give a formula for the genus of an irreducible curve on a surface.

Lemma 7.

1. *For any effective divisor D on a surface S one has*

$$-\chi(\mathcal{O}_D) = \frac{1}{2}((D, D) + (D, K_S)).$$

2. (Genus formula) *For an irreducible curve C with genus $g = \dim H^1(\mathcal{O}_C)$ one has*

$$2g - 2 = (K_S, C) + (C, C).$$

Proof: There is an exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

which implies $-\chi(\mathcal{O}_D) = -\chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-D)) = \frac{1}{2}((D, D) + (D, K_S))$ by Riemann-Roch. If D is irreducible, $h^0(\mathcal{O}_D) = 1$ and hence $2g(D) - 2 = -2\chi(\mathcal{O}_D)$. ∎

Remark 8. The genus g as defined above for a singular curve C is related to the genus $\tilde{g}$ of its normalisation $\nu : \tilde{C} \to C$ as follows. There is an exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\tilde{C}} \to \Delta \to 0,$$

where $\Delta = \oplus \Delta_x$ is a sky-skraper sheaf concentrated in the singular points x of C. Taking the Euler-characteristics one gets

$$g(C) = g(\tilde{C}) + \sum_x \dim \Delta_x.$$

The important consequence is that

$$g(C) \geq g(\tilde{C}) \text{ with equality if and only if } C \text{ is smooth.}$$

Let me finish this section by saying a few words about the *dualising sheaf* of an effective divisor D on a surface. One defines it by

$$\omega_D = \mathcal{O}_D(K_S) \otimes \mathcal{O}_D(D)$$

which entails an exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S) \otimes \mathcal{O}_S(D) \xrightarrow{res} \omega_D \to 0.$$

For a divisor $D = \sum C_i$ where D is a curve with ordinary double points, this sequence leads to an explicit description of the dualising sheaf.

Proposition 9. *Let C be a curve with only ordinary double points $\{ p_1, \ldots, p_d \}$, let $\nu : \tilde{C} \to C$ be its normalisation and $\nu^{-1}(p_i) = \{ p_i', p_i'' \}$. The sheaf $\nu^{-1}\omega_C$ consists of the germs of meromorphic differential forms α on $\tilde{C}$ having at most poles of order 1 at the points $\{ p_i', p_i'' \}$, $i = 1, \ldots, d$ and such that $res_{p_i'}\alpha + res_{p_i''}\alpha = 0$.*

Proof: The residue homomorphism res in the preceding exact sequence can be computed as follows. Let $\{ u, v \}$ be local coordinates around a point of C and let f be a local equation for C such that $\dfrac{\partial f}{\partial v}$ is not identically zero on C. If

$$\varphi = \frac{h}{f} du \wedge dv$$

is a local holomorphic section of $\mathcal{O}_S(K_S) \otimes \mathcal{O}_S(C)$ (so h is holomorphic). Then $res(h) = (hdu/\dfrac{\partial f}{\partial v})|C$ as is easily verified. Now around an ordinary double point one can take $f = uv$ so that $res(h) = h\dfrac{du}{u}|\{ v = 0 \} = -h\dfrac{dv}{v}|\{ u = 0 \}$. Taking the normalisation separates the two branches and $\nu^*(h)$ becomes meromorphic on each branch with pole of order at most one and with opposite residues. ∎

Problems.

5.1. Prove the equivalence of the two definitions of intersection index. (See Remark 1). Hint: Interpret the number l as the dimension of the vector space $W := \mathcal{O}(\Delta)_0/(g \circ \nu(t))$ and construct a surjection $\mathcal{O}_x \to W$ whose kernel is (f, g).

5.2. Prove that the Koszul sequence is exact.

5.3. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Using the adjunction formula, show that the genus of C is equal to $\frac{1}{2}(d-1)(d-2)$.

5.4. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (a, b), i.e. $(C, F') = a$ and $(C, F) = b$. Derive a formula for the genus of C in terms of a and b.

6. Birational geometry of surfaces

Birational maps between surfaces can be described totally by blowing ups or sigma-processes. The resulting exceptional curves can be recognised by Castelnuovo's contraction criterion. As a result, minimal surfaces are the same as surfaces without -1-curves. I finish this section with a reformulation of some of the results in 'Mori-terminology'.

I start by recalling the notion of the σ-process. Let S be a surface and $x \in S$ which serves as the origin in a coordinate chart (z, w) in a neighbourhood U of x. Now define

$$\tilde{U} = \{ \ ((z,w),(Z,W)) \in U \times \mathbb{P}^1 \mid zW = wZ \ \}$$

where (Z, W) are the homogeneous coordinates of $\mathbb{P}^1$. Projection onto the first factor defines a holomorphic map

$$p : \tilde{U} \to U$$

which maps the curve $E = p^{-1}(x) = x \times \mathbb{P}^1$ onto x and $\tilde{U} \setminus E$ biholomorphically onto $U \setminus x$. Note that on $\tilde{U} \cap U \times \{ \ Z \neq 0 \ \}$ the coordinates $u = z, v = W/Z$ can be used and the map p is given by $p(u,v) = (u, uv)$. A similar remark applies to $\tilde{U} \cap U \times \{ \ W \neq 0 \ \}$.

Now glue S and $\tilde{U}$ over the set $\tilde{U} \setminus E = U \setminus x$. This yields a new smooth surface $\tilde{S}$ and a holomorphic map $p : \tilde{S} \to S$ which maps $E \cong \mathbb{P}^1$ to x and $p : \tilde{S} \setminus E \to S \setminus x$ is biholomorphic. This is *the σ-process* at x. The curve E is called the *exceptional curve*. The surface $\tilde{S}$ is called the *blow up* of S at x.

If S is algebraic then so is $\tilde{S}$. See Problem 1.

If C is any curve through x one can form the closure $\tilde{C}$ of the set $p^{-1}(C \setminus x) \cong C \setminus x$ in $\tilde{S}$. This curve is called the *strict transform* of the curve C.

Lemma 1. *If C is an irreducible curve passing with multiplicity m through x the total transform $p^*(C)$ is related to the strict transform by*

$$p^*(C) = \tilde{C} + mE.$$

Proof: Clearly $p^*(C) = \tilde{C} + kE$ with some integer k which one can determine by a local computation around x. Use coordinates (z, w) around x and (u, v) on part of $\tilde{U}$ as before. The local equation at x for C can be written as

$$f = f_m(z,w) + \text{higher order terms}$$

with f_m homogeneous of degree m. So in (u, v)-coordinates one has

$$f \circ p = f(u, uv) = u^m(f_m(1,v) + \ldots)$$

and so one must have $k = m$. ∎

Proposition 2. *Let S be an algebraic surface, $p : \tilde{S} \to S$ the blow up at x and let E be the exceptional curve.*

i. The homomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \to \operatorname{Pic} \tilde{S}$ *defined by* $(D, n) \mapsto p^*D + nE$ *is an isomorphism. A similar assertion holds for the Néron-Severi groups.*

ii. For any two divisors D, D' *on* S *one has* $(p^*D, p^*D') = (D, D')$, $(E, p^*D) = 0$ *and* $(E, E) = -1$

iii. One has $K_{\tilde{S}} = p^* + E$.

Proof: One can replace divisors by linearly equivalent divisors for which the components do not pass through x. Then the first two formulas in (ii). follow. Now choose a curve C passing through x with multiplicity 1 so that the strict transform $\tilde{C}$ meets the exceptional curve transversally. So $1 = (\tilde{C}, E) = (p^*C - E, E) = -(E, E)$. This completes the proof of (ii).

To show (i), note that the given map is clearly surjective (every irreducible curve distinct from E on $\tilde{S}$ is the strict transform of its image on S). To show injectivity, suppose that $p^*D + nE$ is linearly equivalent to 0. Intersecting with E and applying (ii) one finds that $n = 0$, but then D also must be linearly equivalent to zero. Replacing 'linearly equivalent' by 'homologically equivalent' yields an isomorphism on the level of the Néron-Severi groups.

To prove (iii), observe that p^*K_S and $K_{\tilde{S}}$ coincide outside E. So $K_{\tilde{S}} = K_S + mE$ for some integer m. The adjunction formula shows that $-2 = (K_{\tilde{S}}, E) + (E, E) = -m - 1$ and hence $m = 1$. ∎

Let me now show how to 'eliminate the points of indeterminacy' of a rational map $f : S \dashrightarrow \mathbb{P}^n$. One has $f = (f_0, \ldots, f_n)$ with f_i polynomials, the map f is not defined at the set where all the f_i simultaneously vanish on S. Since one can assume that the f_i have no common factor this set F must be finite. Let $f(S)$ be the Zariski-closure of $f(S \setminus F)$ in $\mathbb{P}^n$.

Proposition 3. *Let* $f : S \dashrightarrow \mathbb{P}^n$ *be a rational map. There is a sequence of blowings up* $S_m \xrightarrow{\sigma_m} S_{m-1} \xrightarrow{\sigma_{m-1}} \ldots \xrightarrow{\sigma_1} S_0 = S$ *such that the rational map* $f \circ \sigma_1 \circ \ldots \circ \sigma_m$ *is everywhere defined.*

Proof: One may assume that $f(S)$ is not contained in any hyperplane. But then the system of hyperplanes yields a linear system $|D|$ of divisors which have the points in F as base points. If $F = \emptyset$ you are ready. Otherwise, you blow up S at a point of F. Say $\sigma_1 : S_1 \to S$. Then you can write $\sigma_1^* D = D_1 + m_1 E$ with a certain multiplicity m_1 which can be chosen in such a way that $|D_1|$ does not have E in its base locus. If $|D_1|$ does not have a base locus you are ready. Otherwise you can blow up in one of the base points of the new linear system. One must see that this process stops and it is here that one makes essential use of the intersection theory. Indeed, at the k-th step one finds

$$(D_k, D_k) = (D_{k-1}, D_{k-1}) - m_k^2 < (D_{k-1}, D_{k-1})$$

but since $|D_k|$ has no curve in its base locus one has $(D_k, D_k) \geq 0$. So the self intersection numbers of the divisors in the linear systems constructed in this way must stabilise for some $k \leq (D, D)$ and then there are no more base points left. ∎

Examples 1. Let $S \subset \mathbb{P}^n$ be a surface and p a point of S. Projection from p is a rational map $S \dashrightarrow \mathbb{P}^{n-1}$ which is defined everywhere except at p. By blowing up S at p one obtains a morphism $\tilde{S} \to \mathbb{P}^{n-1}$, where $\tilde{S}$ is the blow up of S at p.

2. Let $Q \subset \mathbb{P}^3$ be a smooth quadric and let $Q \dashrightarrow \mathbb{P}^2$ be projection from a point on the quadric maps. The two lines on Q passing through p are mapped to two distinct points, say $\bar{p}$ and $\bar{q}$. Blowing up Q at p gives a surface $\tilde{Q}$ and a morphism $\tilde{Q} \to \mathbb{P}^2$ which is the blow up of $\mathbb{P}^2$ at the points $\bar{p}$ and $\bar{q}$.

In the proof of the preceding proposition I used that the points at which a birational map is not defined is finite. This implies that for a birational morphism the number of curves which map to points must be finite since at these points the inverse birational map is not defined. Let me prove that these are the only points at which the inverse is not defined.

Lemma 4. *Let S be a projective variety of dimension two and S' a projective surface. If a birational morphism $f : S \to S'$ has the property that f^{-1} is not defined at a certain point p' , the inverse image $f^{-1}(p')$ is a curve.*

Proof: One may replace S by an affine variety, say $j : S \subset \mathbb{C}^n$ (so that still $f^{-1}(p') \neq \emptyset$). The rational map $j \circ f^{-1} : S' \dashrightarrow \mathbb{C}^n$ is given by n rational functions of which the first may be assumed to be not defined at p', is of the form $\frac{P}{Q}$ with Q a non-constant polynomial with $Q(p') = 0$. One may assume that P and Q have no common factors. Let D be the curve defined by $f^*Q = 0$. On S one has

$$f^*P = x_1 \cdot f^*Q,$$

with x_1 the first coordinate function on $\mathbb{C}^n$. So on D both functions f^*P and f^*Q are zero and $D = f^{-1}D'$ with $D' = \{ P = Q = 0 \}$. Since P and Q have no common factor this must be a finite set containing p'. Restricting to a smaller Zariski-open neighbourhood of p' one may therefore assume that $D' = \{ p' \}$ and so $f^{-1}(p') = D$, a curve. ∎

Corollary 5. *If a birational map of surfaces is not defined at a certain point, the inverse map contracts a curve onto that point.*

Proof: Let $f : S \dashrightarrow S'$ be a birational map which is defined on the Zariski-open $U \subset S$ and let $\Gamma \subset U \times S'$ be its graph and let S_1 be its closure in $S \times S'$. This is a variety (of dimension 2) possibly with singularities. Let $q : S_1 \to S$ and $q' : S_1 \to S'$ be the two projections. Suppose that f is not defined at p. Then q^{-1} is likewise not defined at p. By the Lemma, q contracts a curve $\tilde{C}$ onto p. But, since $S_1 \subset S \times S'$, the image of $\tilde{C}$ in S by assumption is a curve which maps to p by f^{-1}. ∎

Proposition 6. (Universal property of the blowing up) *Let $f : S' \to S$ be a birational morphism between surfaces. Suppose that f^{-1} is not defined at $p \in S$. Then f factors as $f = \sigma \circ g$ with g a birational morphism and σ the blow up at p.*

Proof: Let $g = \sigma^{-1} \circ f$ and $h = g^{-1}$. One hase to show that g is a morphism. Let me assume that this is not the case. Then the previous lemma shows that $h(C) = p'$ for some curve $C \subset \tilde{S}$ and some point $p' \in S'$. Since then $\sigma(C) = f(p')$ the curve C must be the exceptional curve E and $f(p') = p$. Let u be any local coordinate at p (this means that the curve $\{ u = 0 \}$ passes simply through p. I claim that f^*u must be a local coordinate at p'. If not, it would be in $\mathfrak{m}_{p'}^2$ and hence also $h^*f^*u = \sigma^*u$ would be in $\mathfrak{m}_e^2$ for any $e \in E$ where h is defined, which is at all but finitely many points. But the blowing up has the property that any coordinate function on S at p lifts to a coordinate function at all points of E but one. On the other hand, there does exist u with $f^*u \in \mathfrak{m}_{p'}^2$. To see this one chooses local coordinates x, y at p and considers f^*y. If this is in $\mathfrak{m}_{p'}^2$ one sets $u = y$. Otherwise f^*y is a coordinate near p' and it vanishes with multiplicity one along $f^{-1}p$ in a neighbourhood of p'. So f^*y gives a local equation for $f^{-1}p$ and then $f^*x = v \cdot f^*y$ for some $v \in \mathcal{O}_{p'}$. So then $u = x - v(p') \cdot y$ has the property that $f^*u \in \mathfrak{m}_{p'}^2$ which completes the contradiction. ■

Remark 7. There is also a complementary universal property for the blowing up: if $h : \tilde{S} \to \mathbb{P}^n$ is a morphism wich contracts E, it factors as $g \circ \sigma$ with a morphism $g : S \to \mathbb{P}^n$.

This is much easier to see. One may write $h = (h_0, \ldots, h_n)$ and consider h_i/h_j which is a regular function on $S_j = \tilde{S} \cap \{ h_j \neq 0 \}$ and, by assumption, on $\sigma(S_j)$. So h also defines a morphism $g : S \to \mathbb{P}^n$ with $h = g \circ \sigma$.

Theorem 8. *Any birational morphism between surfaces is the composition of a sequence of blowings up and isomorphisms.*

Proof: Let $f : S \to S'$ be a birational morphism which is not an isomorphism. Then f^{-1} is not defined at some point p_1 and by the previous proposition, $f = \sigma_1 \circ f_1$ with σ_1 the blowing up at p_1 and f_1 a birational morphism. Observe that f contracts the curves which f_1 contracts but also at least one more curve, namely any curve which by f_1 is mapped to the exceptional curve for σ. So the number of curves contracted by f_1 is strictly less than the number of curves contracted by f. If no curves are contracted by f_1 the inverse map is a morphism and so f_1 is an isomorphism. Otherwise one can continue and write $f_1 = \sigma_2 \circ f_2$ with σ_2 a blowing up and f_2 a birational morphism contracting fewer curves than f_1. After a finite number of steps this process terminates. ■

Corollary 9. *Every birational map $S \dashrightarrow S'$ between surfaces fits into a commutative diagram*

$$\begin{tikzcd} & S'' \arrow[dl, "h"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed] & & S' \end{tikzcd}$$

with h and g a composition of blowings up and isomorphisms.

Proof: This follows immediately from Proposition 3 and Theorem 8. ∎

For the purpose of reducing the birational classification to a biregular classification, the previous theorem is important. One introduces the following basic definition, which underlines this.

Definition 10. *A surface S is minimal if every birational morphism $S \to S'$ is an isomorphism*

The previous theorem then shows that every surface can be mapped to a miminal surface by a birational morphism. Indeed, if S is not minimal, there is some surface S' and a birational morphism $S \to S'$ which, by the previous theorem is a sequence of blowings up and isomorphisms. Since under a blowing up the rank of the Néron-Severi-group increases by one, this process must terminate. It follows moreover, that on a non-minimal surface there must be exceptional curves for some σ-process. These are smooth rational curves with self intersection (-1). Let me call such curves (-1)-curves. These are always exceptional curves for a blowing up by the following theorem.

Theorem 11. (Castelnuovo's contraction criterion). *A smooth rational curve E on a surface S with $(E,E) = -1$ is the exceptional curve for a σ-process $S \to S'$.*

Proof: Choose a very ample divisor H on S such that $H^1(S, \mathcal{O}(H)) = 0$, which is possible by Serre's Theorem (Theorem 4.13). Now E has a certain degree $d = (H,E)$ with respect to the embedding given by $|H|$ and $H' = H + dE$ now has the property that $(H',E) = 0$. I want to show that in fact $|H'|$ gives a morphism $\sigma : S \to S'$ of S onto a smooth surface S'. The fact that $(H',E) = 0$ then implies that H' is trivial on E and so this morphism contracts E to a point p. I shall show that σ is an isomorphism from $S \setminus E$ to $S' \setminus p$. It then follows from Remark 7 that this morphism is the σ-process at p.

Let me construct a special basis for the sections of $\mathcal{O}(H')$. The long exact sequence associated to the sequences

$$0 \to \mathcal{O}(H + (i-1)E) \to \mathcal{O}(H + iE) \to \mathcal{O}_E(d-i) \to 0, \quad i = 1, \dots, d,$$

successively shows that $H^1(\mathcal{O}_S(H + iE)) = 0$ and that $H^0(\mathcal{O}_S(H + iE))$ surjects onto $H^0(\mathcal{O}_E(d-i))$. So I can take a basis for $H^0(\mathcal{O}(H+dE))$ by first taking a basis $\{ s_0, \dots, s_n \}$ for the sections of $\mathcal{O}(H)$ and adding sections which come from sections $\{ s_{i,0}, \dots, s_{i,d-i} \}$ of $\mathcal{O}(H + iE)$ which restrict to a basis for $H^0(\mathcal{O}_E(d-i))$. So, if s is a section of $\mathcal{O}_S(E)$ defining E I get the following basis for $H^0(\mathcal{O}(H'))$:

$$\{ s^d s_0, \dots, s^d s_n, s^{d-1} s_{1,0}, \dots, s^{d-1} s_{1,d-1}, \dots, s s_{d-1,0}, s s_{d-1,1}, s_{d,0}. \}$$

The rational map given by this basis is everywhere defined outside E and E is mapped entirely onto the point $p = (0, \dots, 1)$. Clearly it gives a biregular morphism of $S \setminus E$ onto $S' \setminus p$. One only needs to show that p is smooth on S'.

Let $U \subset S$ be an open neighbourhood of E where $s_{d,0} \neq 0$. Define the following sections of $\mathcal{O}_U(-E)$:

$$x = \frac{s_{d-1,0}}{s_{d,0}}, \quad y = \frac{s_{d-1,1}}{s_{d,0}}.$$

These restrict to a basis for $H^0(\mathcal{O}_E(1))$ and so I may assume that U is small enough so that x and y are not simultaneously zero on U and so one gets a morphism $z_2 : U \to \mathbb{P}^1$. The functions sx, sy define a morphism $z_1 : U \to \mathbb{C}^2$ and together with z_2 even a morphism $(z_1, z_2) : U \to \mathbb{C}^2 \times \mathbb{P}^1$ which in fact maps to the blow up $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ considered as submanifold of $\mathbb{C}^2 \times \mathbb{P}^1$. Summarising, one gets a morphism

$$z : U \longrightarrow \widetilde{\mathbb{C}^2},$$

which by construction, fits into the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "z"] \arrow[d, "\sigma"] & \widetilde{\mathbb{C}^2} \arrow[d, "\tau"] \\ \sigma(U) \arrow[r, "\tilde{z}"] & \mathbb{C}^2 \end{tikzcd}$$

where τ is the blowing up of $\mathbb{C}^2$ in the origin. Note that $\sigma(U)$ is open in S'.

Let me show that, replacing U by a smaller open neighbourhood of E (in the complex topology) the map $\tilde{z}$ is an isomorphism from $\sigma(U)$ onto its image which is an open neighbourhood of the origin in $\mathbb{C}^2$ and thus p is a smooth point.

First of all, by construction $z|E$ maps E isomorphically on E. Furthermore, z is a local isomorphism around each point of E (in the complex topology). Indeed, let (u, v) be coordinates on $\mathbb{C}^2$ and let (U, V) be homogeneous coordinates on $\mathbb{P}^1$. Then $\widetilde{\mathbb{C}^2}$ is the submanifold given by $uV = vU$. If $q \in E$, one may assume that $z(q) = ((0,1),(0,1)) \in \mathbb{C}^2 \times \mathbb{P}^1$. Now at $z(q)$ the functions $v, U/V$ are local coordinates on $\widetilde{\mathbb{C}}^2$. One has

$$z^*v = sy, \quad z^*(U/V) = x/y.$$

The first function vanishes with multiplicity 1 along E while the second function gives a local coordinate at q. This implies that in a neighbourhood of q where both functions are defined the map z is a local isomorphism.

Now one invokes an easy lemma from topology.

Lemma *Let $f : X \to Y$ be a continuous map between Hausdorff spaces which restricts to a homeomorphism $f|K \to f(K)$ on a compact set K and which is a local homeomorphism in the neighbourhood of each point of K. Then there exists an open neighbourhood of K which is mapped homeomorphically by f onto an open subset of Y.*

The proof of this lemma is left to the reader.

From the lemma one finds an open neigbourhood U of E which maps isomorphically to $W \subset \widetilde{\mathbb{C}^n}$. Consider the morphism $\sigma \circ z^{-1}|W$. It contracts the exceptional curve in W and hence, by Remark 7 it factors as $g \circ \tau$, where g is a morphism which must be the inverse of $\tilde{z}$. So $\sigma(U)$ is isomorphic to the open neighbourhood $\tau(W)$ of the origin in $\mathbb{C}^2$.

■

Corollary 12. *A surface is minimal if and only if it does not contain (-1)-curves.*

I end this section with a few remarks which are intended to illustrate the point of view of birational geometry since Mori theory came into existence.

As demonstrated previously, for surfaces there always exists some minimal model in the birational equivalence class of a given surface. In principle there could be many minimal models. It turns out that, with the exception of the ruled surfaces there is a unique minimal model up to isomorphism. By definition a *ruled surface* is any surface which admits a birational map onto $C \times \mathbb{P}^1$ with C a curve, so these are known from a birational point of view. However one still needs to know the distinct minimal ruled surfaces. This shall be done in the next sections.

In higher dimensions there need not exist a smooth minimal model. When the concept of minimal model is suitably modified, in order to have such a model one necessarily has to allow singularities in codimension ≥ 3. It turns out that you can only expect a minimal model if K, the canonical divisor is *nef* which means that K intersects non-negatively with any curve. Mori theory also shows that there is a basic distinction between the case K nef and K not nef. I shall illustrate this for surfaces.

Proposition 13. *If there exists a curve C on S with $(K_S, C) < 0$ and $(C, C) \geq 0$, all plurigenera of S are zero. If S is a surface with at least one non-vanishing plurigenus and C is a curve on S with $(K_S, C) < 0$, the curve C is an exceptional curve of the first kind, i.e. C is a smooth rational curve with $(C, C) = -1$.*

Proof: Let D be an effective pluricanonical divisor. Write it like $D = aC + R$. Since $(D, C) < 0$ the divisor D actually contains C, i.e. $a > 0$. Then $0 > m(K_S, C) = (D, C) = a(C, C) + (R, C) \geq a(C, C)$. Since this is ≥ 0 in the first case, one arrives at a contradiction: the plurigenera must all vanish. In the second case, if $(K_S, C) \leq -2$ the adjunction formula gives $(C, C) \geq 0$ and we again have a contradiction. So $(K_S, C) = -1$ and the adjunction formula shows that C is an exceptional curve of the first kind. ∎

Recall that the Kodaira-dimension $\kappa(S)$ of S is equal to $-\infty$ means that all plurigenera of S vanish. This is for instance the case for rational and ruled surfaces as will be shown in the next section. So using the notion of nef-ness and Kodaira-dimension there is a reformulation à la Mori for the previous Proposition.

Reformulation 14. *Suppose S is a surface whose canonical bundle is not nef. Then either S is not minimal or $\kappa(S) = -\infty$.*

Let me give a second illustration of the Mori-point of view with regards to the question of uniqueness of the minimal model.

Proposition 15. *Let S and S' be two surfaces and let $f : S \dashrightarrow S'$ be a birational map. If $K_{S'}$ is nef, f is a morphism. If moreover K_S is nef, f is an isomorphism.*

Proof: Let $\sigma : \tilde{X} \to X$ be the blow up of any surface with exceptional curve E and let $\tilde{C} \subset \tilde{X}$ an irreducible curve such that $C := \sigma(\tilde{C})$ is again a curve, one has

$$(K_{\tilde{X}}, \tilde{C}) = (\sigma^* K_X + E, \sigma^* C - mE) = (K_X, C) + m \geq (K_X, C).$$

So if K_X is nef there can be no curve $\tilde{C}$ on $\tilde{X}$ mapping to a curve on X and for which $(K_{\tilde{X}}, \tilde{C}) \leq -1$. Since any morphism is composed of blowings up this then also holds for an arbitrary morphism $X' \to X$.

Let me apply this in the present situation with $X = S'$. Resolve the points of indeterminacy of f. Choose a resolution where you need the minimal number of blowings up. One may suppose that one needs at least one blow up. Then the image $C = f(E)$ of the exceptional curve E of the last blow up must be a curve, which contradicts the preceding since $(K, E) = -1$ on the last blown up surface. So f is a morphism. Similarly, if K_S is nef, f^{-1} is a morphism and so f is an isomorphism. ■

Problems.

6.1. Prove that the blow up of an algebraic surface is again algebraic. Hint: define the blow-up $\widetilde{\mathbb{P}^n}$ of the point $p = (0, \ldots, 1)$ in $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ given by the bihomogeneous equations $X_iY_j - Y_jX_i = 0$, $i, j = 0, \ldots, n-1$. Next one shows that the Segre-embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ identifies $\mathbb{P}^n \times \mathbb{P}^m$ with a projective submanifold of $\mathbb{P}^{nm+n+m}$. So $\mathbb{P}^n \times \mathbb{P}^m$ is a projective manifold. Next, every subset of $\mathbb{P}^n \times \mathbb{P}^m$ given by bihomogeneous equations can be seen to correspond to a projective subvariety of $\mathbb{P}^{nm+n+m}$. Show that the blow-up of a surface $S \subset \mathbb{P}^n$ in p is the Zariski-closure of $S \setminus \{ p \}$ in $\widetilde{\mathbb{P}^n}$.

7. Ruled and rational surfaces

The minimal models of non-rational ruled surfaces are geometrically ruled. These are always projective bundles associated to rank 2 vector bundles. The minimal models for rational surfaces are the Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$.

Definition 1.

1. *A surface S is called a ruled surface if it is birationally isomorphic to $C \times \mathbb{P}^1$ where C is a smooth curve. If $C = \mathbb{P}^1$ one calls S rational.*

2. *A surface S is called geometrically ruled if there is a morphism $p : S \to C$ of maximal rank onto a smooth curve with fibres $\mathbb{P}^1$.*

Two remarks are in order. First, a surface is rational if and only if it is isomorphic to $\mathbb{P}^2$ since, as shown in section 7, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally isomorphic. Secondly, it is by no means clear that a geometrically ruled surface is actually ruled. This however is the case. In fact a little more is true.

Proposition 2. *If $f : S \to C$ is a surjective morphism of a surface S onto a curve C for which $c \in C$ is a regular value and whose fibre at c is isomorphic to $\mathbb{P}^1$, then there is a Zariski-open neighbourhood U of c in C such that $f^{-1}U$ is isomorphic to $U \times \mathbb{P}^1$ in a fibre preserving manner.*

Proof:
Step 1. $H^2(S, \mathcal{O}) = 0$.

Serre-duality implies that $H^2(S, \mathcal{O})$ is dual to $H^0(S, K_S)$. Suppose that this would be non-zero, i.e. that there would be an effective canonical divisor K. Let F be the fibre over c. Since $(F, F) = 0$ (a nearby fibre F' is linearly equivalent to F and so $(F, F) = (F, F') = 0$) the genus formula yields

$$-2 = (K, F) + (F, F) = (K, F).$$

If however K is effective you can write it as $K = nF + G$ where $n \geq 0$ and G is disjoint from F and so $(K, F) = (G, F) \geq 0$.

Step 2. Construction of a divisor H with $(H, F) = 1$.

Now observe that the exponential sequence and Step 1. yields a surjection

$$\mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbb{Z})$$

and therefore it is sufficient to produce a cohomology class h with $h \cdot f = 1$. Here $f = c_1(F)$ and the product on $H^2(S, \mathbb{Z})$ is the one coming from cup-product. (Remark 5.3). Now by Poincaré-duality the cup-product pairing is a perfect pairing on $\mathbb{H} := H^2(S, \mathbb{Z})/\mathrm{Torsion}$. This means that the natural map

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathrm{Hom}(\mathbb{H}, \mathbb{Z}), \\ x &\longmapsto \{y \mapsto y \cdot x\} \end{aligned}$$

is an isomorphism. The numbers $x \cdot f$ form an ideal in $\mathbb{Z}$, say (d) and so the linear functional $x \mapsto 1/d(x \cdot f)$ must be of the form $x \mapsto (x \cdot f')$ for some $f' \in \mathbb{H}$ for which one then has $f = df'$. (The element f' is called *primitive* and this shows that if one writes $x = nx'$ with $x' \in \mathbb{H}$ and n as large as possible, the resulting x' is primitive).

I claim that in our case $d = 1$. Look at $k = c_1(K)$. Since $f \cdot k = -2$ as shown before, one must have $f' \cdot k = -2/d$. Now by the genus formula $f(x) := x \cdot x + x \cdot k \pmod 2 = 0$ for x the class of an irreducible curve and hence, since $f(x)$ is linear, this is true for all of $\mathbb{H}$. In particular $-2/d$ must be even and so $d = 1$. But now f is primitive and so there exists some $h \in \mathbb{H}$ with $h \cdot f = 1$.

Step 3. End of proof.

Consider the exact sequence

$$0 \to \mathcal{O}_S(H + (r-1)F) \to \mathcal{O}_S(H + rF) \to \mathcal{O}_F(1) \to 0,$$

which in cohomology gives

$$H^0(\mathcal{O}_S(H + rF)) \xrightarrow{a_r} H^0(\mathcal{O}_F(1)) \to H^1(\mathcal{O}_S(H + (r-1)F)) \xrightarrow{b_r} H^1(\mathcal{O}_S(H + rF)) \to 0.$$

The map b_r being surjective for all r means that the dimensions of $H^1(\mathcal{O}_S(H + rF))$ form a decreasing sequence which eventually must become stable and then b_r becomes an isomorphism and hence a_r will be a surjection. For such an r pick a plane inside $H^0(\mathcal{O}_S(H+rF))$ which is mapped isomorphically onto $H^0(F,\mathcal{O}(1))$ by a_r and let P be the corresponding pencil. This pencil will separate points on F and so the possible fixed locus of P consists of points in fibres distinct from F or of curves in fibres disjoint from F. Let me take away all these fibres as well as the reducible fibres and look at the restriction P' of the pencil on this part of the surface, which is of the form $f^{-1}U$ with $U \subset C$ Zariski open. Now a generic member C_t of the moving part P' of P meets F in exactly one point, so if it would be reducible, it would contain some fibers. This however is not possible since then C_s with $s \neq t$ would meet C_t in the intersection points of these fibres and so they would be base points. It follows that P' is a pencil entirely consisting of sections of the restricted fibration. The pencil P' defines a morphism $g : f^{-1}U \to \mathbb{P}^1$ with fibres C_t meeting the fibres of $f^{-1}U \to U$ in exactly one point and so $(f,g) : f^{-1}U \to U \times \mathbb{P}^1$ is the desired fibre preserving isomorphism. ■

Let me next relate the geometrically ruled surfaces $S \to C$ to rank two vector bundles on C. If E is such a bundle you can replace every fibre E_x over $x \in C$ by the corresponding projective line. In this way you obtain $\mathbb{P}(E)$, a $\mathbb{P}^1$-bundle over C.

Proposition 3. *Every geometrically ruled surface $S \to C$ is isomorphic to the $\mathbb{P}^1$-bundle associated to some rank two vector bundle E on C. Two bundles $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are isomorphic if and only if $E' \cong E \otimes L$ for some line bundle L on C.*

Proof: Recall that isomorphism classes of line bundles on C are classified by the set $H^1(\mathcal{O}^*)$. One can introduce the sheaf $Gl(2,\mathcal{O}_C)$ of invertible 2×2-matrices with coefficients in $\mathcal{O}_C$ and the quotient sheaf $PGl(2,\mathcal{O}_C)$. The set $H^1(Gl(2,\mathcal{O}_C))$ classifies the rank 2-bundles on C and $H^1(PGl(2,\mathcal{O}_C))$ the $\mathbb{P}^1$-bundles. The defining exact sequence

$$1 \to \mathcal{O}_C^* \to Gl(2,\mathcal{O}_C) \to PGl(2,\mathcal{O}_C) \to 1$$

in cohomology gives an 'exact sequence of sets'.

$$H^1(\mathcal{O}_C^*) \xrightarrow{a} H^1(Gl(2,\mathcal{O}_C)) \xrightarrow{p} H^1(PGl(2,\mathcal{O}_C)) \to H^2(\mathcal{O}_C^*).$$

This means that $p(e) = p(e')$ if and only if $a(l)\cdot e = e'$ for some $l \in H^1(\mathcal{O}_C^*)$, where $H^1(\mathcal{O}_C^*)$ acts on the set of rank two bundles by the tensor product. The result follows if one can show that $H^2(\mathcal{O}^*) = 0$.This however follows immediately from the exponential sequence on C. ■

Let me now show that the minimal models of the non-rational ruled surfaces are exactly the geometrically ruled surfaces. First note a simple lemma from topology whose proof is left to the reader.

Lemma 4. *Let $f : S \to C$ be a surjective morphism of a surface onto a smooth curve. Assume that the smooth fibres are all connected. Then all fibres are connected.*

Proposition 5. *Let S be without (-1)-curves and let $f : S \to C$ be a morphism onto a smooth curve C such that the generic curve of f is $\mathbb{P}^1$. Then $f : S \to C$ gives S the structure of a geometrically ruled surface.*

Proof: As in previous arguments, for every fibre F one has $(F, K) = -2$ and hence no fibre can be multiple, i.e. $c_1(F)$ is primitive. So all irreducible fibres have genus 0 which, by Lemma 5.7 implies that they are all $\mathbb{P}^1$. All you have to do now is to to rule out the possibility of reducible fibres, since then Proposition 2 can be applied. So suppose that $F = \sum_i n_i C_i$ is a reducible fibre. Now compute $n_i C_i^2 = (C_i, F - \sum_{j \neq i} n_j C_j) = -\sum_{j \neq i} n_j (C_j, C_i) < 0$ since C_i meets at least one C_j (the fibre F is connected by the previous lemma). But then C_i has negative self intersection and since $(K, C_i) + (C_i, C_i) = 2g(C_i) - 2$ one concludes that $(K, C_i) \geq -1$ with equality if and only if C_i is a smooth rational curve with self intersection -1, i.e. a (-1)-curve. But these don't exist on S and so $(K, C_i) \geq 0$ and $(K, F) \geq 0$ whereas $(K, F) = -2$. This contradiction shows that there are no reducible fibres present and therefore the proof is complete. ∎

Corollary 6. *A minimal model of a non-rational ruled surface is geometrically ruled.*

Proof: Let S be minimal and let $S \dashrightarrow C \times \mathbb{P}^1$ be birational and consider the resulting rational map $S \to C$. It necessarily is a morphism, because otherwise one would have to blow up at least once to eliminate points of indeterminacy and such an exceptional curve would have to be mapped to a point on C (since C is not rational). But by the 'easy' universal property for blowing up (Remark 6.7) one can 'factor out' the σ-process for the exceptional curve without creating points of indeterminacy.

Since the generic fibre of $S \to C$ is $\mathbb{P}^1$ and since S does not contain (-1)-curves, the result follows from the preceding Proposition.

∎

Next topic: the rational geometrically ruled surfaces. For this, one needs Grothendieck's result on the splitting of vector bundles on $\mathbb{P}^1$.

Lemma 7. *Every vector bundle on $\mathbb{P}^1$ is the direct sum of line bundles.*

Proof: Let me first consider the question: 'when does an exact sequence of vector bundles (on any manifold) split?'. So let

$$0 \to V' \xrightarrow{a} V \xrightarrow{b} V'' \to 0$$

be an exact sequence of vector bundles. It splits by definition, if there is a subbundle of V which by b is mapped isomorphically onto V''. Equivalently, there should exist a homomorphism $c : V'' \to V$ such that $b \circ c = \mathrm{Id}_{V''}$. To put this into the language of exact sequences, note that applying $\mathrm{Hom}(V'', -)$ to the preceding exact sequence yields an exact sequence of vector bundles

$$0 \to \mathrm{Hom}(V'', V') \xrightarrow{a^*} \mathrm{Hom}(V'', V) \xrightarrow{b^*} \mathrm{Hom}(V'', V'') \to 0,$$

with e.g. $b^*(c) = b \cdot c$. Now $\mathrm{Id}_{V''}$ is a global section of $\mathrm{Hom}(V'', V'')$ and the splitting is equivalent to the existence of a global section c of $\mathrm{Hom}(V'', V)$ with $b^*(c) = \mathrm{Id}_{V''}$. Looking at the exact sequence in cohomology, one sees that it suffices that $H^1(\mathrm{Hom}(V'', V')) = 0$.

In this case one applies this to the following situation. Consider a vector bundle E and fix $k \in \mathbb{Z}$ such that the bundle $E(k')$ has no sections for $k' < k$ but does have a section for $k' = k$. This section has no zeroes, otherwise some $E(k')$ with $k' < k$ would have had a section. But then the section defines a trivial sub line bundle of $E(k)$ and hence an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E(k) \to F \to 0.$$

Now you twist this sequence by $\mathcal{O}(-1)$ and consider the resulting exact sequence. Since $H^0(E(k-1)) = 0$, the space $H^0(F(-1))$ goes injectively in $H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. So $H^0(F(-1))$ and hence also $H^0(F(-2))$ must vanish. Serre duality then gives that $0 = H^1(F^\vee) = H^1(\mathrm{Hom}(F, \mathcal{O}))$. By the previous considerations, the preceding sequence splits and so by induction E splits as a direct sum of line bundles.

■

By twisting by $\mathcal{O}_{\mathbb{P}^1}(k)$ one can always normalize a $\mathbb{P}^1$-bundle on $\mathbb{P}^1$, say $\mathbb{P}(F' \oplus F'')$ in such a way that F' becomes trivial. Then, upon writing $F'' = \mathcal{O}_{\mathbb{P}^1}(n)$ you arrive at the definition of the Hirzebruch surfaces

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

(Miles Reid suggests that the 'F' might stand for 'Fritz', Hirzebruch's first name).

Let me summarise the discussion so far in the form of a proposition.

Proposition 8. *A geometrically ruled surface over $\mathbb{P}^1$ is a Hirzebruch surface.*

Let me conclude this chapter by computing the invariants for the geometrically ruled surfaces. Let me first recall that $\mathbb{P}(E)$ always admits a divisor H with $(H, F) = 1$ (cf. proof of Proposition 2). Let us set $h = c_1(H)$.

Proposition 9. *Let $\varphi : S \to C$ be a geometrically ruled surface and let g be the genus of C. Then*

1. *$H^2(S, \mathbb{Z})$ is generated by the class f of a fibre and the class h. In fact* $\mathrm{Pic}(S) = f^* \mathrm{Pic}(S) \oplus \mathbb{Z} \cdot H$.
2. *$c_1(K_S) = -2h + (2g - 2 + d)f$, where $d = h \cdot h$. In particular $(K, K) = 8(1 - g)$.*

Proof:
1. First look at the Picard group. It suffices to show that divisors D on S with $(D, F) = 0$ are of the form $\varphi^* d$ with $d \in \mathrm{Pic}\, C$. If D would be effective this follows immediately, so it suffices to show that $D_n := D + nF$ is effective for n sufficiently large. Now $(D_n, D_n) = (D, D)$ while $(D_n, K_S) = (D, K) - 2n$, so $h^0(\mathcal{O}_S(D_n)) + h^0(\mathcal{O}_S(K_S - D_n)) \geq n + \text{constant}$ by the Riemann-Roch inequality. Since $h^0(\mathcal{O}_S(K_S - D_n)) = 0$ for large n (look at the degree of the divisor $K_S - D_n$ with respect to some very ample divisor on S) one indeed gets a section in $h^0(\mathcal{O}_S(D_n))$ for n large enough.

For the cohomology group, simply look at the exponential sequence by which $H^2(S, \mathbb{Z})$ is a quotient of $\mathrm{Pic}(S)$ and so is generated by f (all fibres are cohomologically equivalent) and h which are independent since $h \cdot f = 1$.

2. Write $c_1(K_S) = af + bh$ and intersect with f to get $b = -2$. Then one finds $(K_S, H) = a - 2d$ while the genus formula gives $2g - 2 = ((K_S, H)) + (H, H) = a - d$. ■

To compute the other invariants one needs

Proposition 10. *q, p_g and $P_n := \dim H^0(K_S^{\otimes n})$ are birational invariants.*

Proof: Let me give the proof for holomorphic 2-forms. The other cases are similar. So let $f : S \dashrightarrow S'$ be a birational map. f is a morphism outside a finite set F. So, if α is a holomorphic 2-form, $f^*\alpha$ is a rational 2-form and regular on $S \setminus F$. But then it is regular on S, since $f^*\alpha$ has poles in divisors at most. It follows that pulling back gives an injection

$$H^0(S', K_{S'}) \hookrightarrow H^0(S, K_S).$$

The inverse of f then yields an inverse to this map and so this is an isomorphism. ■

It follows that one can use the model $C \times \mathbb{P}^1$ to compute the invariants q, p_g and P_n. One has $\Omega^1_{C\times\mathbb{P}^1} = p^*\Omega^1_C \oplus q^*\Omega^1_{\mathbb{P}^1}$ (here p and q denote the projections onto the factors) and hence $K_{C\times\mathbb{P}^1} \cong p^*K_C \otimes q^*K_{\mathbb{P}^1}$. It then follows (see Problem 3.7) that $h^0(\Omega^1_{C\times\mathbb{P}^1}) = h^0(\Omega^1_C) + h^0(\Omega^1_{\mathbb{P}^1}) = g$, the genus of C and that $h^0(K_{C\times\mathbb{P}^1}) = h^0(K_C)\cdot h^0(K_{\mathbb{P}^1}) = 0$ and similarly one finds that $P_n = 0$. Summarising, you get

Lemma 11. *For a ruled surface S birationally isomorphic to $C \times \mathbb{P}^1$ one has $p_g = P_n = 0$ and $q = g(C)$.*

Next, let me turn to the invariants of the Hirzebruch surfaces. Observe that there are two natural types of sections of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, the unique section $\hat{S}$ which in each fibre assumes the value $(0,1)$ and the sections $S = S_p$ of the form $s(x) = (1, p(x))$ where p is a homogeneous polynomial of degree n. Clearly $(S, S) = n$ and one would like to compute $\hat{S}$ in terms of S and F. One hase $\hat{S} = S + aF$ since $\hat{S}$ is a section. Moreover $(S, \hat{S}) = 0$ since the two sections never meet. So $0 = (S, \hat{S}) = n + a$ and hence $a = -n$. It follows that $(\hat{S}, \hat{S}) = -n$. I claim that $\hat{S}$ is the unique curve with strictly negative self intersection. Indeed, if $cF + dS \equiv D \neq \hat{S}$ one hase $c = (D, \hat{S}) \geq 0$ and also $d = (D, F) \geq 0$ and so $(D, D) = d^2n + 2cd \geq 0$. It follows that the $\mathbb{F}_n$ with $n > 0$ are mutually non-isomorphic; they are distinguished by the self intersection number of the unique curve on them with negative self intersection. Since $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ any curve on this surface has non-negative self intersection and so $\mathbb{F}_0$ is not isomorphic to $\mathbb{F}_n$ with $n > 0$. Finally, $\mathbb{F}_1$ is non-minimal, since the section with negative self intersection is an exceptional curve. The other $\mathbb{F}_n$ are minimal since they do not contain an exceptional curve. Summarizing one has

Proposition 12. *The Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$ are minimal and pairwise non-isomorphic.*

Using Proposition 7.8 one finds

Corollary 13. *The Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$ are precisely the minimal geometrically ruled surfaces with $q = 0$.*

Problems.

7.1. Prove that $\mathbb{F}_1$ is the projective plane blown up in a point.

7.2. Give a direct proof for the fact that every algebraic line bundle on an affine curve is trivial.

7.3. The Hirzebruch surface $\mathbb{F}_n$ is a non-singular model of the cone on the rational normal curve in $\mathbb{P}^{n+1}$. Prove this by considering the linear system $|s| = |f^*\mathcal{O}_{\mathbb{P}^1}(n)|$ on $\mathbb{F}_n$.

7.4. With the linear system $|S + kF|$ with $k \geq 1$ one embeds $\mathbb{F}_n$ in $\mathbb{P}^{n+2k+1}$ as a surface of degree $n + 2k$. The fibres map to straight lines, the unique section with negative self intersection to a rational normal curve in $\mathbb{P}^k \subset \mathbb{P}^{n+2k+1}$ and the sections S_p are mapped to rational normal curves in a linear subspace (depending on p) of dimension $n + k$.
Conversely, if one starts with two disjunct subspaces of $\mathbb{P}^{n+2k+1}$ of dimensions k and $n + k$, take rational normal curves C and C' in these spaces, choose an isomorphism $u : C \to C'$ and joint $u \in C$ and $u' \in C$ by a straight line. The resulting surface is isomorphic to $\mathbb{F}_n$ embedded by means of $|S + kF|$.