Cahiers du Buro

T. V. Narayana

J. Zidek

Contributions to the Theory of Tournaments Part I The Combinatorics of Knock-Out Tournaments

Cahiers du Bureau universitaire de recherche opérationnelle. Série Recherche, tome 13 (1969), p. 3-18

<http://www.numdam.org/item?id=BURO_1969__13__3_0>

CONTRIBUTIONS TO THE THEORY OF TOURNAMENTS

PART I

THE COMBINATORICS OF KNOCK-OUT TOURNAMENTS

par

T. V. NARAYANA et J. ZIDEK

SUMMARY

The combinatorics of repeated round-robin tournaments has been studied in detail by statisticians interested in paired comparisons, as well as by mathematicians as a branch of graph theory. Although it has long been surmised that repeated knock-out tournaments could represent an alternative to repeated round-robin tournaments, very few results on the combinatorics of knock-out tournaments are known. In this paper, we introduce a class of random knock-out tournaments and study their elementary combinatorial properties. No effort is made in Part I to study the statistical or combinatorial aspects of *repeated* knock-out tournaments. Indeed, as will be apparent early in this paper, the systematic study of knock-out tournaments without repetition is best done with a high-speed computer to yield explicit numerical results. The statistical aspects are covered in Part II(*), while Part III(**) deals with the numerical and computational aspects e.g. results of simulation comparing repeated knock-out and round-robin tournaments.

1. INTRODUCTION

Random knock-out tournaments - apart from the classical case of $n = 2^t$ $(t = 1, 2, \ldots)$ players - have not been studied very often in the literature. A convenient summary, including bibliographies of known results, is provided by the recent books of David [2] and Moon [5]. One widely known elementary result is sometimes posed as an exercise in introductory textbooks on probability (cf. Tucker [6] p. 18) :

(*) à paraître dans Revue Roumaine de mathématiques pures et appliquées, 1969.

(**) en préparation.

$n = 2^t$ players of equal strenght play a knock-out tournament, being randomly paired off in each round. What is the probability that two given players A and B meet in some round of the tournament ? The answer $2/n$ could be obtained by noting that $(n-1)$ matches are played and any one of the $\binom{n}{2}$ pairs formed from n players is likely to play in a match.

Variants of the idea that a knock-out tournament has $(n-1)$ matches i.e. $(n-1)$ losers and a winner, each loser requiring one match to be "knocked-out", have appeared early in the literature. An example is the identity attributed to Mendelsohn by Moon ([5] p. 49, where an amusing exercise of Chisholm is also presented) ; namely, prove

$$n-1 = \sum_{t=1}^{\infty}\left[\frac{n+2^{t-1}-1}{2^t}\right]$$

where [x] denotes the greatest integer $\leqslant$ x.

Starting from this well-known idea, we shall define in Section 3, a class of random knock-out tournaments and study their combinatorics systematically. As a preliminary, a detailed discussion of the classical case is undertaken in Section 2. The two final sections are devoted to a brief comparison of tournaments (including an investigation of certain important special cases) and some concluding remarks.

Before proceeding to the study of the classical case i.e. $n = 2^t$ players of equal strength, we formulate our basic model. The model chosen is one where just one of the players A (say) has a probability p of winning against each of the other $(n-1)$ players B_i $(i = 1, \ldots, n-1)$, who are of equal strength. We shall assume, implicitly, that $p \geqslant 1/2$, although most, if not all our formulae, are valid for all p, $0 \leqslant p \leqslant 1$. This assumption expresses in simplified form the idea that a superior player is present. It is relevant to point out that, apart from the mathematical tractability of this model, it represents the important case of a single outlier. In comparing selection procedures for choosing the best object, this model provides a suitable basis for comparison ; we shall investigate in Part III whether this model represents a realistic approximation, at least in the qualitative sense, to more general models satisfying strong stochastic transitivity.

2. THE CLASSICAL CASE

Owing to its importance, we derive in detail the results for the classical case, where $n = 2^t$ players of equal strength (i.e .

$p = 1/2$) are matched off randomly against each other in each round. After the first round, 2^{t-1} losers are discarded and the process is continued until a single winner emerges.

Let R_n^i denote the probability that a given player A plays exactly i rounds in this tournament. Clearly $R_n^i = 0$ for $i > t$. As a preliminary to applying the inclusion - exclusion principle, we denote by $P_n^{[i]}$, P_n^i respectively, the probabilities that A meets exactly $B_1, \ldots, B_i$ and that A meets $B_1, \ldots, B_i$ in the tournament. Of course P_n^i includes the case where A meets $B_1, \ldots B_i$ and perhaps others during the tournament. Clearly, to the geometric distribution

$$R_n^i = \frac{1}{2^i} \quad (i < t) ;$$
$$R_n^t = \frac{1}{2^{t-1}}. \tag{1}$$

$P_n^{[i]}$ is immediately obtained from the R_n^i ; indeed, as the i opponents of A are equally likely to be any one of the $\binom{n-1}{i}$ combinations, we have

$$P_n^{[i]} = \frac{1}{2^i \binom{n-1}{i}} \quad (i < t) ;$$
$$P_n^{[t]} = \frac{1}{2^{t-1} \binom{n-1}{t}} . \tag{2}$$

The P_n^i are now obtained from the $P_n^{[i]}$ by analogy with the inclusion and exclusion principle. Alternatively, if A meets $B_1, \ldots, B_i$ and ν additional players, these ν additional players can be chosen in $\binom{n-i-1}{\nu}$ ways from the remaining players ($\nu = 0, 1, \ldots, t-i$).

Noting that these cases are mutually exclusive, an appropriate application of the theorem of total probabilities yields,

$$P_n^i = \sum_{\nu=0}^{t-1} P_n^{[i+\nu]} \binom{n-i-1}{\nu} \tag{3}$$

Substituting from (2) into (3), we have after an elementary simplification the relation connecting P_n^i and R_n^i, namely

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=0}^{t-i} R_n^{i+\nu} \binom{i+\nu}{i}, \tag{4}$$

where by definition, $R_n^i = P_n^i = 0$ for $i > t$ in the classical case.

Clearly the equations (4) may be considered as n simultaneous equations for the P_n^i in terms of the R_n^i. Inversely, given the P_n^i, if the R_n^i were unknown, we could solve for the R_n^i. Indeed the inverse of the matrix of coefficients of R_n^i in (4), is well known [cf. 3, p. 100 for a similar proof], and we obtain

$$R_n^i = \sum_{\nu=0}^{n-i-1} (-1) \; P_n^{i+\nu} \binom{n-1}{i+\nu} \binom{i+\nu}{i} \tag{5}$$

Of course $R_n^i = P_n^i = 0$ for $i > t$ in the classical case.

We have stated equations (4), (5) generally, since they may be validated for all random tournaments of our basic model, as defined in the next section.

We return now to the classical case where A has probability $p \neq 1/2$ of defeating any B_i. Since the B_i's are of equal strength, it is evident that equations (4), (5) continue to hold by the same "symmetry" arguments used as before.

However, now

$$\begin{aligned} R_n^i &= p^{i-1}q \qquad \text{for } i = 1, \ldots, t-1 \\ R_n^t &= p^{t-1} = p^{t-1}q + p^t. \end{aligned} \tag{6}$$

(Note : If A wins $t - 1$ rounds, he is assured of playing in the final round). Substituting these values in (4), we have, in the classical case with general p,

$$\binom{n-1}{i} P_n^i = x_i + p^t \binom{t}{i}, \tag{7}$$

where

$$x_i = \sum_{\nu=0}^{t-i} p^{i+\nu-1} q \binom{i+\nu}{i} \qquad i = 1, \ldots, t. \tag{8}$$

Now

$$px_i = \sum_{\nu=0}^{t-i} p^{i+\nu} q \binom{i+\nu}{i} = \sum_{\nu=1}^{t-1} p^{i+\nu-1} q \binom{i+\nu-1}{i} + p^t q \binom{t}{i}$$

Using

$$\binom{i+\nu}{i} - \binom{i+\nu-1}{i} = \binom{i+\nu-1}{i-1}$$

we obtain, for $i \geqslant 2$, from the last two equations

$$qx_i = \sum_{\nu=0}^{t-i} \left\{ p^{i+\nu-1} q\binom{i+\nu-1}{i-1} \right\} - p^t q\binom{t}{i}. \tag{9}$$

Comparing the quantity in brackets with (8) we obtain, for $i \geqslant 2$,

$$x_i = \frac{p}{q} x_{i-1} - \binom{t}{i-1} p^t - p^t \binom{t}{i}. \tag{10}$$

Since $x_1 = \dfrac{1-p^t}{q} - tp^t$ we are able to obtain x_i recursively and hence P_n^i.

Indeed

$$x_2 = \frac{p}{q} \left\{ \frac{1-p^t}{q} - tp^t \right\} - \binom{t}{1} p^t - \binom{t}{2} p^t$$

$$= \frac{1}{q^2} \left\{ p(1-p^t) - \binom{t}{1} p^t q \right\} - \binom{t}{2} p^t.$$

and by induction,

$$x_i = \frac{1}{q^i} \left\{ p^{i-1}(1-p^t) - p^t \left[\binom{t}{1} qp^{i-2} + \binom{t}{2} q^2 p^{i-3} + \ldots + \binom{t}{i-1} q^{i-1} \right] \right\} - \binom{t}{i} p^t. \tag{11}$$

Thus, from (7) and (11), we have explicit expressions for P_n^i :

$$\binom{n-1}{i} P_n^i = \left(\frac{p}{q}\right)^{i-1} \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{o} - \binom{t}{1} \frac{q}{p} - \ldots - \binom{t}{i-1} \left(\frac{q}{p}\right)^{i-1} \right]$$

$$(i = 1, \ldots, t). \tag{12}$$

When $p = q$, the expression for P_n^i simplifies, using $n = 2^t$. We then have,

$$\binom{n-1}{i} P_n^i = \frac{2}{n} \left[n - \binom{t}{o} - \binom{t}{1} - \ldots - \binom{t}{i-1} \right]$$

In particular, we obtain from (12),

$$P_n^1 = \frac{(1-p^t)}{q(n-1)}, \tag{13}$$

which naturally reduces to $\frac{2}{2^t} = \frac{2}{n}$, when $p = \frac{1}{2}$.

As a conclusion to this section, we make one further remark about P_n^1 which will be valid for random tournaments in general. Clearly $E(R)$, the expected number of rounds that A plays in the tournament, equals $(n-1)P_n^1$. Letting $\Pi = p^t$ be the probability that A wins the tournament, we obtain as a reformulation of (13) :

$$\Pi + q\ E(R) = 1.$$

3. RANDOM KNOCK-OUT TOURNAMENTS

When n is not a power of 2, say $n = 2^t + K$, $0 < K < 2^t$, it is usual in the literature [2, 5] to reduce the number of players to 2^t by matching 2K players randomly in a preliminary round. Motivated by this idea, we define for every integer $n \geqslant 2$, a random knock-out tournament with n players as a vector of positive integers $(m_1, \ldots, m_k)$ satisfying :

$$\begin{gathered} m_1 + \ldots + m_k = n - 1,\ m_k = 1. \\ 2m_1 \leqslant n. \\ 2m_i \leqslant n - m_1 - \ldots - m_{i-1}\ (i \geqslant 2). \end{gathered} \tag{14}$$

A tournament defined by the vector $(m_1, \ldots, m_k)$ is played as follows. On the first round $2m_1$ players, chosen at random from n, are paired off randomly. The remaining $n - m_1$ players have a bye for this round. The m_1 pairs yield m_1 losers who are eliminated from the tournament. We are then left with a tournament of $n - m_1$ players, with vector $(m_2, \ldots, m_k)$. This inductive rule is well defined for $n > 2$, since in the case $n = 2$ there is a unique tournament of one round.

We indicate here a few examples of tournaments, which have been studied in the literature and are of special importance in the applications. For any $n = 2^t + K$, $(0 \leqslant K < 2^t)$ we designate by

T_1 : the tournament with vector $m_i = \left[\frac{n + 2^{i-1} - 1}{2^i}\right]$

T_2 : the tournament with vector $(K, 2^{t-1}, 2^{t-2}, \ldots, 1)$

T_3 : the tournament with vector $(1, \ldots, 1)$.

When $K = 0$, both T_1, T_2 reduce to the classical case ; while if $K \neq 0$, T_1, T_2 consist of $(t+1)$ rounds. Of course, T_3 always consists of $(n-1)$ rounds. It is convenient to set

$$n_1 = n, \qquad n_i = n_{i-1} - m_{i-1} \quad (i \geqslant 2) \; ; \tag{15}$$

so that n_i is the number of players at the start of round i $(i = 1, \ldots, k)$.

Further, let

$$p_i = \frac{2m_i}{n_i}, \; q_i = 1 - p_i \quad (i = 1, \ldots, k) \; ; \tag{16}$$

so that p_i is the probability that a specified player, among the n_i qualified for round i, does not get a bye in the round. Finally from (14), $n_k = 2$ so that $p_k = 1$.

We emphasize once again that we are studying the model where one player A has the probability p of winning in a match against each of the other equally strong $B_1, \ldots, B_{n-1}$. Let $\Pi = \Pi\ (n, m_1, \ldots, m_k)$ denote the probability that A wins the tournament $(m_1, \ldots, m_k)$ where the m_i satisfy (14) Also, let us suppose we are given the vector

$$R_n = (R_n^1, \ldots, R_n^k) = R(n, m_1, \ldots, m_k)$$

where R_n^i represents the probability that A plays exactly i rounds. As $\sum_i R_n^i = 1$, $E(R_n) = \sum_i iR_n^i$ represents the expected number of rounds played by A in the tournament. The above definitions lead us to the following theorem.

Theorem 1. The probability $\Pi = \Pi_n$, that A wins the tournament $(m_1, \ldots, m_k)$ is given by

$$\Pi = (p_1 p + q_1)\ (p_2 p + q_2) \ldots (p_{k-1}\, p + q_{k-1})p. \tag{17}$$

Further

$$\Pi + q\ E(R_n) = 1, \qquad \text{where} \qquad q = 1 - p. \tag{18}$$

Proof. We prove both results by induction. Let us assume that (17) is true for *all* tournaments with $(n-1)$ or less players, since it is clearly true for all tournaments with $n = 2, 3, 4$ players. The inductive hypothesis assures us that

$$\Pi_{n_2} = \Pi(n_2, m_2, \ldots, m_k) = (p_2 p + q_2) \ldots (p_{k-1} p + q_{k-1})p \tag{19}$$

where $p_2, q_2, \ldots, p_{k-1}, q_{k-1}$ are defined by (16). Now player A can survive the first round of $(m_1, \ldots, m_k)$ in two exclusive and exhaustive ways, namely

a) playing and winning round 1 with probability pp_1

b) having a bye with probability q_1.

An elementary theorem of probability gives

$$\Pi_n = (p_1\ p + q_1)\,\Pi_{n_2} \tag{20}$$

and using (19) the proof is complete.

A similar inductive proof establishes (18) with the help of the recursion similar to (20) :

$$E(R_{n_1}) = p_1 q + p_1 p\ [\,E(R_{n_2}) + 1\,] + q_1 E\,(R_{n_2}).$$

Of course $E(R_{n_2})$ clearly refers to the tournament $(m_2, \ldots, m_k)$.

We restate (17) informally as follows : Let us suppose we multiply out the R. H. S. of (17), yielding

$$\Pi = b_{k-1}\ p + b_{k-2}p^2 + \ldots + b_o p^k = \underline{b}'\underline{w}, \tag{21}$$

where

$$\begin{aligned} \underline{b} &= (b_{k-1}, \ \ldots, \ b_o)' \\ \underline{w} &= (p, p^2, \ldots, p^k)' \end{aligned} \tag{22}$$

Now, given that A wins the tournament, let B^*_{k-i} be the probability that A receives exactly k-i byes in the course of the tournament $(i = 1, \ldots, k)$. Since the conditional probability that A wins the tournament given that he has received $k - i$ byes is p^i, we have

$$\Pi = B^*_{k-1}\ p + \ \ldots\ + B^*_o\ p^k. \tag{23}$$

Comparing (21), (23) $B^*_{k-i} \equiv b_{k-i}$, $(i = 1, \ldots, k)$; so that b_{k-i}, as defined by (21), is indeed the conditional probability that A receives exactly $k - i$ byes, given that A won the tournament. By an exactly analogous argument, R^i_n may be split up into the exclusive probabilities that A plays i rounds and wins the tournament or that A plays i rounds and loses. Since the details are obvious, we state

Theorem 2. Let $\underline{c} = (c_{k-1}, \ldots, c_o)$ where

$$c_{k-1} = 1, \quad c_{k-i} = 1 - \sum_{j=1}^{i-1} b_{k-j} \quad (i \geqslant 2). \tag{24}$$

Then

$$R_n^i = (p^i b_{k-i} + p^{i-1} q \, c_{k-i}). \tag{25}$$

Remarks : 1. Theorem 2 enables us, with the help of a computer, to calculate R_n^i explicity for any tournament.

2. Summing over all $i = 1, .., k$ on both sides of (25) we have

$$1 = \Pi + q/p \; \underline{c}'\underline{w}. \tag{26}$$

Comparing (26) and (18)

$$E(R_n) = \frac{1}{p} \underline{c}'\underline{w}.$$

From the obvious relations

$$\Pi + \Pi^* (n - 1) = 1, \qquad E(R_n) + (n - 1) \, E(R_n^*) = 2(n - 1)$$

where Π^*, $E(R_n^*)$ are, respectively, the probability of any B_i winning the tournament, and the expected number of rounds played by B_i, we have

$$E(R_n^*) = \frac{2 - \underline{c}'\underline{w}}{p(n - 1)} \, ; \; \Pi^* = \frac{1 - \underline{b}'\underline{w}}{(n - 1)} \, .$$

Clearly

$$\Pi^* + q \; E(R_n^*) = 2q.$$

Finally, let P_n^i, Q_n^i denote respectively the probabilities that A meets $B_1, B_2, \ldots,$ and B_i and (B_1 or B_2 or ... B_i) in the tournament. By the usual arguments involving inclusion and exclusion, the following relations hold :

Theorem 3.

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=0}^{n-i-1} R_n^{i+\nu} \binom{i+\nu}{i} \tag{28}$$

$$Q_n^i = \sum_{\nu=1}^{n-1} (-1)^{\nu+1} \, P_n^{i+\nu} \binom{i}{\nu}; \tag{29}$$

the reciprocal relations are also valid :

$$R_n^i = \sum_{\nu=0}^{n-i-1} (-1)^\nu P_n^{i+\nu} \binom{n-1}{i+\nu} \binom{i+\nu}{i}, \tag{30}$$

$$P_n^i = \sum_{\nu=1}^{n-1} (-1)^{\nu-1} Q_n^\nu \binom{i}{\nu}. \tag{31}$$

Remembering R_n^i, P_n^i are zero for $i > k$ for any tournament $(m_1, \ldots, m_k)$, the above equations yield explicit numerical results with the help of a computer.

4. A COMPARISON OF TOURNAMENTS.

In this section we give explicit results or recurrence relations for the R_n^i, P_n^i for some special tournamants, of interest in the applications, as a preliminary to a brief comparison of knock-out tournaments in general (as defined by (14)). Apart from the tournaments T_1, T_2, T_3, introduced in the last section, we consider one further example, which will prove to be of particular importance in the applications, labelled T_4. A study of T_4 also permits us to further generalize our definition of random knock-out tournaments, although we shall not do so explicitly.

For any $n > 2$, the tournament T_4 is played according to the vector $(1, 1, \ldots, 1)$, with the further restriction that the winner of any round automatically plays in the succeeding round. Thus , after a pair, drawn at random, plays in the first round, one of the remaining $(n - 2)$ bye players of round 1 meets the winner in round 2, and so on. While this tournament does not satisfy the définition (12), clearly Theorems 2, 3 apply. We have only to set $b_0 = 2/n$, $b_1 = \ldots = b_{n-2} = 1/n$, and Theorem 1 holds for T_4 if restated as (21) ; of course equation (17) is wrong in the context of T_4 and the b_{k-i} cannot be interpreted as the coefficients of p^i in the expansion of (17).

We consider for illustrative purposes T_1, as the details are similar for the other tournaments. For $n > 4$, let $n = 2^t + K$ with $0 \leqslant K < 2^t$. It is easily seen that

$$R_{2n}^{i+1} = pR_n^i \quad \text{for} \quad i = 1, 2, \ldots, t+1,$$

and

$$R_{2n}^1 = q.$$

From (28),

$$P_{2n}^{i} = \frac{1}{\binom{2n-1}{i}} \sum_{\nu=0}^{t+2-i} R_{2n}^{i+\nu} \binom{i+\nu}{\nu},$$

Using $\binom{i+\nu}{\nu} = \binom{i+\nu-1}{\nu-1} + \binom{i+\nu-1}{\nu}$, we have for $i > 2$

$$P_{2n}^{i} = \frac{p}{\binom{2n-1}{i}} \left[\sum_{\nu=0}^{t+2-i} R_n^{i+\nu-1} \binom{i+\nu-1}{\nu-1} + \sum_{\nu=0}^{t+2-i} R^{i+\nu-1} \binom{i+\nu-1}{\nu} \right] \quad (32)$$

Now each sum within the brackets can be simplified using (28) once again. By an easy simplification of (32), we have the following recurrence relation for $i > 2$:

$$P_{2n}^{i} = \frac{p\binom{n-1}{i}}{\binom{2n-1}{i}} \left[P_n^i + \frac{i}{n-i} P^{i-1} \right]. \quad (33)$$

A similar argument yields for $i > 2$

$$P_{2n-1}^{i} = \frac{\binom{n-1}{i}}{\binom{2n-1}{i}} \left[\frac{pi}{n-i} \frac{n-1}{n} P_n^{i-1} + \frac{\frac{1}{2} + p(n-1)}{n} P_n^i \right]. \quad (34)$$

These recurrence relations were obtained by Capell and Narayana [1] by a direct but longer combinatorial argument. Theorem 3 provides a simple derivation of many similar results ; if n is a power of two in (33), i.e. in the classical case, the explicit values (12) of P_n^i are immediately deduced.

A short summary of explicit values or recurrences for R_n^i, P_n^i are provided for the tournaments T_2, T_3, T_4 below. The results can be verified by combinatorial arguments directly, but the use of Theorems 2, 3 simplifies the details considerably.

Tournament T_2

$$\Pi = \frac{2K}{2^t + K} p^{t+1} + \frac{2^t - K}{2^t + K} p^t .$$

$$R_n^i = p^{i-1} q \quad (i = 1, \dots, t-1 ;$$

$$R_n^t = \frac{2K}{n} p^{t-1} q + \frac{2^t - K}{n} p^{t-1} ;$$

$$R_n^{t+1} = \frac{2K}{n} p^t .$$

$$\binom{n-1}{i} P_n^i = \left(\frac{p}{q}\right)^{i-1} \frac{p^t}{q} \left\{ \frac{1}{p^t} - \binom{t}{o} - \frac{q}{p}\binom{t}{1} - \ \dots \ - \left(\frac{q}{p}\right)^{i-1} \binom{t}{i-1} \left[\frac{1-2Kq}{n}\right] \right\} .$$

$$\Pi = \frac{(2p+n-2)\,n-1}{n!}$$

$$R_n^{i+1} = \frac{2}{n}\, p\; R_{n-1}^i + \left(1 - \frac{2}{n}\right) R_{n-1}^{i+1} \qquad (i = 1, \dots, n-2) ;$$

$$R_n^1 = q .$$

Explicit results for P_n^i appear to be cumbersome (if not impossible), but numerical results are easily obtained with a computer.

The interested reader is referred to [4] for further details.

Explicit results are possible in this case.

$$\Pi = \frac{1}{n} p^2 + \frac{1}{n} p \ + \dots + \frac{1}{n} p^{n-2} + \frac{2}{n} p^{n-1} =$$

$$= \frac{1}{n} p \left(\frac{1-p^{n-1}}{1-p}\right) + \frac{1}{n} p^{n-1} .$$

$$R_n^i = p^{i-1} q \, \frac{n-i}{n} + \frac{p^{i-1}}{n} (i < n-1), \quad R_n^{n-1} = p^{n-2} \frac{2}{n} .$$

$$P_n^i = \frac{p^{i-1}}{n\binom{n-1}{i}} \sum_{\nu=0}^{n-2-i} (q\,\overline{n-i-\nu} + 1) \binom{i+\nu}{i} p^{\nu} + p^{n-2} \frac{2}{n} .$$

In the rest of this section, we shall assume $1/2 \leqslant p \leqslant 1$, so that a stronger player A is present. We let $\Pi_i(p)$ $(i = 1,2,3,4)$ be the probability that A wins the corresponding tournament T_i, and state our results for all $n \geqslant 4$, since for $n = 2, 3$ only one tournament is possible.

Theorem 4. For any tournament with $n \geqslant 4$ players, and any (fixed) p, lying in the open interval $(1/2, 1)$,

$$\Pi_2(p) \geqslant \Pi_1(p) > \Pi_3(p) > \Pi_4(p).$$

Strict inequality holds between $\Pi_2(p)$, $\Pi_1(p)$ for all n for which the tournaments T_1, T_2 are different i.e. n is not of the form 2^t or $2^M(2^t - 1)$, M being a positive integer.

As the details of the proof are quite elementary though tedious, we restrict ourselves to stating the definition of descendant of a tournament, which is a useful tool in comparing tournaments.

Let $T = (m_1, \ldots, m_k)$ be a tournament vector for n players and for some $i < k$, let $m_i = m_i' + m_i''$, where m_i', m_i'' are positive integers. Then the tournament with vector $T' = (m_1, \ldots, m_{i-1}, m_i', m_i'', m_{i+1}, \ldots, m_k)$ will be called a first-stage descendant of T.

From Theorem 1, it is immediately verified that $\Pi(T) > \Pi(T')$ for fixed p in $(1/2, 1)$. Since it is fairly obvious how to define second-stage (and further) descendants of T, and that $(1, 1, \ldots, 1)$ is a descendant of every random knock-out tournament defined by (12), one part of our theorem follows easily. We refer to [4] for a complete proof of our theorem, and for a comparison of random knock-out tournaments with a stronger player present.

5. CONCLUSION.

We mention briefly some combinatorial problems arising from our work on random knock-out tournaments as defined by (12).

1/ For any tournament $(m_1, \ldots, m_k)$, let us consider Q_n^i, where Q_n^i is the probability that A meets B_1 or B_2 or ... B_i during the course of the tournament and is given by (29). Clearly, when $p = 1/2$ i.e. for a random knock-out tournament with all players of equal strength.

$$Q_n^1 = \frac{2}{n} \quad \text{and} \quad Q_n^{n-1} = 1.$$

Further from the definition of Q_n^i

$$Q_n^i < Q_n^{i+1} \qquad (i = 1, \ldots, n-2).$$

Given any real number x in $(0,1)$, and any tournament vector $(m_1,\ldots,m_k)$ of n players, there exists clearly a unique integer $i = i\,(x, m_1,\ldots,m_k)$ say, such that

$$Q_n^{i-1} < x \leqslant Q_n^i \quad (i = 1,\ldots,n-1).$$

Surely it is natural to define $Q_n^0 = 0$. When $p = 1/2$, and $x = 1/2$ so that $i(1/2, m_1,\ldots,m_k)$ corresponds to the "median", we have found by computer for all $n \leqslant 256$, the remarkable result that

$$i\left(\frac{1}{2}, m_1,\ldots,m_k\right) = \left\{\frac{n-1}{3}\right\} \qquad (35)$$

where $\{\frac{n-1}{3}\}$ represents the smallest integer $\geqslant \frac{n-1}{3}$. Although presumably, (35) may be established by cumbersome arguments i. e . for large n for example, and the results for small n verified by computer, no proof has been found by us. We thus prefer to treat (35) as a conjecture, a special case of the more general conjecture :

$$i\left(\frac{1}{2^I}\ ;\ m_1,\ldots,m_k\right) = \frac{n-1}{2^{I+1}-1} \qquad (36)$$

independently of $m_1,\ldots,m_k$.

2/ Our development, enables us by computer at least, to propose principles for solution to the following payoff problem, a special case of which was suggested to us by Moon.

Let us suppose that n players participate in a random knock-out tournament, and it is decided that every player who plays in a round receives one dollar, and there is a bonus B, for the winner of the tournament. The n players decide before hand that one amongst them has probability $p > 1/2$ of beating each of the others who are supposed to be of equal strength. What should be a fair value of the bonus B ?

One principle, which might be proposed for the value of B is that it should be such that the strongest player receives approximately the same *expected* amount, whatever random tournament (12) is chosen for the play-off. In the simple example chosen , Theorem 1 implies that $B = 1/q$, and irrespective of the tournament chosen, A's expected gain is

$$\Pi B + E(R) = \frac{\Pi}{q} + E(R) = \frac{1}{q}.$$

However, for different types of payoffs (not to speak of different principles of bonus) this problem appears to be of some practical interest and is being studied by H. Morin.

3/ We finally remark that if $T(n, k)$ represents the number of tournament vectors $(m_1, \ldots, m_k)$ with $n = 2^t + K$ players $0 < K < 2^t$, then clearly.

$$T(n, k) = 0 \quad \text{for} \quad k \leqslant t$$

$$\text{except for} \quad T(n, t) = 1 \quad \text{for} \quad n = 2^t.$$

Numerical values for $T(n, k)$ can be obtained for $k \geqslant t + 1$, using

$$T(n, k) = T(n-1, k-1) + T(n-2, k-1) + \ldots + T\left(\left[\frac{n+1}{2}\right], k-1\right)$$

A short table of values of $T(n, k)$ is presented.

n \ k	2	3	4	5	6	7	8	9
3	1							
4	1	1						
5		2	1					
6		2	3	1				
7		1	5	4	1			
8		1	6	9	5	1		
9			6	15	14	6	1	
10			6	21	29	20	7	1

ACKNOWLEDGEMENT

We are grateful to the Summer Research Institute, Vancouver, of the Canadian Mathematical Congress where this research was completed.

REFERENCES

[1] CAPELL, P. and NARAYANA, T.V. - "Contribution to the Theory of Tournaments", submitted for publication.

[2] DAVID, H.A. - "The Method of Paired Comparisons". London, Charles Griffin and Company Ltd. 1963.

[3] FELLER, W. - "An Introduction to Probability Theory and its Applications". Vol. I, 2nd Edition, New York, John Wiley and Sons, Inc. 1957.

[4] HOPKINS, M.J.D. - Unpublished M.Sc. Thesis, University of Alberta, 1969.

[5] MOON, J.W. - "Topics on Tournaments", New York, Holt, Rinehart and Winston, 1968.

[6] TUCKER, H.G. - "Introduction to Probability and Mathematical Statistics", New York, Academic Press.