T. V. NARAYANA
E. GOODMAN

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par

T. V. NARAYANA

(assisted by E. GOODMAN)

1. INTRODUCTION

The numbers

$$C_{r,s} = \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}$$

for r, s positive integers appear to have been introduced in [2]. In an expanded version of this paper [3], the relationship of the $C_{r,s}$ to ballot problems and the numbers $\frac{1}{n+1}\binom{2n}{n}$ $(n \geqslant 0)$ which are sometimes called the "Catalan" numbers was fully discussed. Considering the many and varied combinatorial interpretations of the so-called Catalan numbers, it is not surprising that the $C_{r,s}$ - which represent a "partition" of the Catalan numbers - have arisen independently on other occasions. G. Kreweras [1] obtained them as a special case in his elegant simultaneous treatment of the problems of Young and Simon Newcomb ; and again, certain results in the theory of statistical estimation [4, 5] suggest that

$$\sqrt{1 - 2u - 2v - 2uv + u^2 + v^2} = 1 - u - v - 2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} C_{r,s}\, u^r v^s \quad (1)$$

As Kreweras remarks [1, p. 31], direct analytic proofs of (1) are of interest, and in this note we present two proofs of the above expansion where

$$C_{r,s} = \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} \quad (r, s > 0). \qquad (2)$$

FIRST PROOF OF THE EXPANSION

This proof was suggested to us by A.P. Guinand and consists of verifying that the series on the R.H.S. of (2) satisfies a certain differential equation by essentially a simple though tedious bit of algebra. Consider

$$f(u, v) = \frac{1}{2}\{1 - u - v - \sqrt{1 - 2u - 2v - 2uv + u^2 + v^2}\}.$$

Clearly f satisfies the first order differential equation

$$(1 - 2u - 2v - 2uv + u^2 + v^2)\frac{\partial f}{\partial u} + (1 - u + v)f = v + uv - v^2 \qquad (3)$$

with the boundary condition $f(0, v) = 0$ for all v. It is required to show that

$$f(u, v) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}u^r v^s \qquad (4)$$

for sufficiently small u, v. As this series obviously satisfies the boundary condition $f(0, v) = 0$ for all v, we need only show that it satisfies the differential equation (3). Noting from (4) that

$$f = uv + u^2v + uv^2 + \text{terms of higher order},$$

and

$$\frac{\partial f}{\partial u} = v + 2uv + v^2 + \text{terms of high order}$$

we have by substitution of these values and simplification (as far as second order terms in u, v),

$$(1 - 2u - 2v + u^2 - 2uv + v^2)\frac{\partial f}{\partial u} + (1 - u + v)f = (v + uv - v^2)$$
$$+ \text{ higher order terms.} \qquad (5)$$

Comparing (3) and (5), it remains to show that all terms of third and higher order vanish in (5) on the R.H.S. We shall assume (or as will easily be verified in our second proof) that the series (4)

$$f(u, v) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}u^r v^s$$

converges aboslutely in a small neighbourhood around the origin. Term by term differentiation of (4) with respect to u yields

$$\frac{\partial f}{\partial u} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} ru^{r-1}v^{s}. \tag{6}$$

Consequently the coefficient of $u^r v^s$ in (5) is

$$\begin{aligned}&(r+1)C_{r+1,s} - 2rC_{r,s} - 2(r+1)C_{r+1,s-1} + (r-1)C_{r-1,s}\\ &-2rC_{r,s-1} + (r+1)C_{r+1,s-2} + C_{r,s} - C_{r-1,s} + C_{r,s-1}.\end{aligned} \tag{7}$$

Substituting for $C_{r,s}$ from (2) and simplifying by taking out a fac- $\frac{[(r+s-2)!]^2}{[(r-1)!\quad(s-1)!]^2}$ we can easily prove that (7) equals zero for all third and higher order terms. From the remark after (5), we have completed the proof.

PROOF AS A BINOMIAL EXPANSION

In our second proof we expand the left hand side of (1) and we then proceed to show that this expansion is identical to the right hand side. We write the left hand side of (1) as

$$\begin{aligned}T &= \sqrt{(1-u-v)^2 - 4uv}\\ &= (1-u-v)y\\ &= y - (u+v)y\end{aligned}$$

where

$$y = \sqrt{1 - \frac{4uv}{(1-u-v)^2}}. \tag{8}$$

Now for $-1 < \frac{4uv}{(1-u-v)^2} < 1$ we have

$$y = \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} \left(\frac{-4uv}{1-u-v}\right)^{r}$$

$$= 1 - 2\sum_{r=1}^{\infty} \frac{(2r-2)!}{r!\,(r-1)!} \frac{u^r v^r}{(1-u-v)^{2r}}. \tag{9}$$

Again for $-1 < u + v < 1$ we find that

$$[1 - (u+v)]^{-2r} = \sum_{s=0}^{\infty} \frac{(2r+s-1)!}{(2r-1)!\, s!} (u+v)^s \tag{10}$$

Substituting (10) into (9) we have

$$y = 1 - 2 \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \frac{u^r v^r (2r+s-1)!}{(2r-1)\, r!\, (r-1)!\, s!} (u+v)^s$$

so that

$$T = y - y(u+v)$$

$$= 1 - u - v - 2 \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \frac{u^r v^r (2r+s-2)!}{r!\,(r-1)!\,s!} (u+v)^s \tag{11}$$

$$= 1 - u - v - 2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} u^r v^r \binom{2r+s-3}{r-1} \binom{r+s-2}{r-1} \frac{1}{r} (u+v)^{s-1}.$$

We now show that the coefficient of $u^R v^S$ in (11) for $R \geqslant S$ is

$$\frac{\binom{R+S-1}{R} \binom{R+S-1}{S}}{R+S-1}.$$

We note that by considerations of symmetry, the requirement $R \geqslant S$ is no real restriction. Terms involving $u^R v^S$ will be obtained only when $r \leqslant S$, for if $r > S$ the powers of v are immediately too high. Now when $r = S$ and $s - 1 = R - S$ the term involving $u^R v^S$ is given by

$$u^S v^S \binom{R+S-2}{S-1} \binom{R-1}{S-1} \frac{1}{S} \binom{R-S}{o} u^{R-S} v^o .$$

Similarly when $r = S - 1$ and $S - 1 = R - S + 2$ the term involving $u^R v^S$ is given by

$$u^{S-1} v^{S-1} \binom{R+S-2}{S-2} \binom{R}{S-2} \frac{1}{S-1} \binom{R-S+2}{1} u^{R-S+1} v.$$

Continuing the procedure, we find that when $r = 1$ and $s - 1 = R + S - 2$, the term involving $u^R v^S$ is given by

$$uv \binom{R+S-2}{o} \binom{R+S-2}{o} \frac{1}{1} \binom{R+S-2}{S-1} u^{R-1} v^{S-1}.$$

Thus, upon collecting terms, we can write the coefficient of $u^R v^S$ as

$$\sum_{k=1}^{S} \binom{R+S-2}{S-k}\binom{R+k-2}{S-k}\binom{R-S+2k-2}{k-1}\frac{1}{S-k+1} \quad (12)$$

$$= \frac{1}{R}\binom{R+S-2}{S-1}\sum_{k=1}^{S}\binom{S-1}{k-1}\binom{R}{S-k+1}$$

It remains to show that

$$\frac{1}{R}\binom{R+S-2}{S-1}\sum_{k=1}^{S}\binom{S-1}{k-1}\binom{R}{S-k+1} = \frac{\binom{R+S-1}{R}\binom{R+S-1}{S}}{R+S-1} \quad (13)$$

or equivalently, upon dividing both sides of (13) by $\frac{1}{R}\binom{R+S-2}{S-1}$

and making the substitution $K = k-1$ that

$$\sum_{k=0}^{S-1}\binom{S-1}{K}\binom{R}{S-K} = \binom{R+S-1}{S}. \quad (14)$$

But (14) is a well-known identity ; one proof is to note $(1+x)^R (1+x)^{S-1} = (1+x)^{R+S-1}$, and collect the coefficient of x^S on both sides. Hence our proof is complete.

CONCLUDING REMARKS

Although it is easy enough, through elementary algebra to show the validity of the expansion [1] in a small neighbourhood of the origin, we do not claim to have a sure method of making such double series expansions in general ; nor have we investigated in detail the region of absolute convergence of the series. As Kreweras points out [1] such expansions are tedious (if not difficult) to establish directly. Indeed the value of

$$C_{r,s} = \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}$$

was first obtained by simple considerations in the theory of estimation [5] ; of course, once the form of $C_{r,s}$ is guessed, the expansion can easily be validated. Since the connections between series expansions and UMV estimates has been adequately discussed, the interested reader can consult the references in [4].

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