VIANNEY COMBET

Multi-solitons pour des équations non-linéaires dispersives surcritiques

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Multi-solitons pour des équations dispersives non-linéaires surcritiques

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Résumé

Cette thèse a pour objet le comportement asymptotique de solutions d'équations aux dérivées partielles dispersives non-linéaires surcritiques. À travers deux exemples-type de telles équations, l'équation de Korteweg-de Vries généralisée (gKdV) et l'équation de Schrödinger non-linéaire (NLS), on traite de la convergence en temps grand des solutions vers des solitons (solutions particulières globales de l'équation), ou des sommes de solitons.

Dans un premier temps, par une méthode de compacité, on obtient pour l'équation (gKdV) l'existence d'une solution convergeant vers un soliton mais n'étant pas un soliton, ce qui est une différence notable avec les cas sous-critique et critique. Puis, en utilisant une description du spectre de l'opérateur linéarisé autour d'un soliton et une méthode de point fixe, nous obtenons l'existence d'une famille à un paramètre caractérisant complètement de telles solutions.

En revenant à une méthode de compacité, nous arrivons dans un deuxième temps à obtenir un résultat similaire pour les multi-solitons de (gKdV), c'est-à-dire des solutions qui convergent vers une somme de solitons. Nous montrons que, étant donnés N solitons, il existe d'une part une famille à N paramètres de N-solitons, et que d'autre part cette famille caractérise tous les multi-solitons de (gKdV) surcritique. Ce résultat est à nouveau original par rapport aux cas sous-critique et critique, pour lesquels il y a existence et unicité des multi-solitons.

Enfin, en adaptant les techniques précédentes à l'équation (NLS) surcritique, nous sommes en mesure de prouver un résultat similaire de multi-existence des multi-solitons, mais sans pour autant obtenir de classification. On rappelle cependant que, même pour les cas sous-critique et critique, aucun résultat général de classification n'a encore été obtenu pour (NLS).

Mots-clefs : Multi-solitons, gKdV, NLS, surcritique, stabilité, comportement asymptotique.

MULTI-SOLITONS FOR SOME SUPERCRITICAL NONLINEAR DISPERSIVE EQUATIONS

Abstract

This thesis deals with the asymptotic behavior of solutions of supercritical nonlinear dispersive partial differential equations. Through two typical examples of such equations, the generalized Korteweg-de Vries equation (gKdV) and the nonlinear Schrödinger equation (NLS), we study the convergence of solutions, when time goes to infinity, towards solitons (particular global solutions of the equation), or sums of solitons.

First, by a compactness method, we obtain for the (gKdV) equation the existence of a solution converging to a soliton but not being a soliton, which is a notable difference with the subcritical and critical cases. Then, using a description of the spectrum of the linearized operator around a soliton and a fixed point method, we obtain the existence of a one-parameter family which completely characterizes all such solutions.

Second, returning to a compactness method, we can obtain a similar result for the multi-solitons of (gKdV), *i.e.* solutions which converge towards a sum of solitons. We show that, N solitons being given, there exists on the one hand an N-parameter family of N-solitons, and on the other hand that this family characterizes all multi-solitons of supercritical (gKdV). This result is also a new feature by comparison with the subcritical and critical cases, for which multi-solitons exist and are unique.

Finally, adapting previous techniques to supercritical (NLS), we prove a similar result of multi-existence of multi-solitons, but without classification. Nevertheless, we recall that, even for the subcritical and critical cases, no general result of classification has been proved for (NLS) yet.

Keywords : Multi-solitons, gKdV, NLS, supercritical, stability, asymptotic behavior.

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Première partie

Introduction

Sommaire

Cette thèse a pour objet les *multi-solitons* de quelques *équations dispersives non-linéaires surcritiques*. Dans une première section de ce chapitre introductif, nous nous proposons de détailler le contexte de cette étude dans le cas de l'équation de Korteweg-de Vries généralisée (gKdV) – exemple important d'équation dispersive non-linéaire – pour laquelle les résultats les plus complets ont été obtenus.

Ces résultats, ainsi que leur cheminement à partir de la problématique de départ, sont exposés dans une seconde section, à la fin de laquelle nous nous intéresserons plus en détail à une autre équation dispersive non-linéaire, abondamment étudiée pour ses applications physiques, afin d'illustrer la souplesse des méthodes employées : l'équation de Schrödinger non-linéaire (NLS).

On notera que cette seconde section est divisée en trois parties, faisant écho aux parties II, III et IV de la thèse, chacune de celles-ci étant un article à l'origine. Voici leurs références respectives :

II. Construction and characterization of solutions converging to solitons for supercritical gKdV equations. *Differential and Integral Equations*, 23(5–6) : 513–568.

III. Multi-soliton solutions for the supercritical gKdV equations. Accepté pour publication dans *Communications in Partial Differential Equations.*

IV. Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension. *Arxiv preprint arXiv:1008.4613.*

1 Contexte

1.1 L'équation de Korteweg-de Vries généralisée

Dans cette première section, on considère l'équation aux dérivées partielles suivante, appelée équation de Korteweg-de Vries généralisée :

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases} \tag{gKdV}$$

où $p \geqslant 2$ est un entier, $(t,x) \in \mathbb{R}^2$ et $u(t,x) \in \mathbb{R}$.

Avant toute chose, il convient de s'assurer que cette équation admet des solutions, au moins locales (on parle de caractère *localement bien posé* de l'équation). Si le problème de Cauchy a été abondamment étudié pour cette équation, ce sont Kenig, Ponce et Vega [11] qui ont apporté la contribution la plus fondamentale et la plus définitive à cette question, en donnant les espaces de Sobolev optimaux dans lesquels l'équation est bien posée, pour chaque valeur de p. En particulier, ils prouvent que (gKdV) est localement bien posée dans $H^1(\mathbb{R})$: pour $u_0 \in H^1(\mathbb{R})$, il existe $T > 0$ et une solution $u \in C^0([0,T), H^1(\mathbb{R}))$ de (gKdV) vérifiant $u(0) = u_0$, unique dans une certaine classe $Y_T \subset C^0([0,T), H^1(\mathbb{R}))$. De plus, si $T^* \geqslant T$ est le temps maximal d'existence pour u, alors soit $T^* = +\infty$ et la solution $u(t)$ est dite *globale en temps* ; soit $T^* < +\infty$, et alors $\|\partial_x u(t)\|_{L^2} \to +\infty$ quand $t \to T^*$ (on dit que la solution *explose en temps fini*). Cette dichotomie est appelée *critère d'explosion*. Ils montrent enfin qu'il y a propagation de la régularité : si $u_0 \in H^s(\mathbb{R})$ avec $s \geqslant 1$, alors $u(t) \in H^s(\mathbb{R})$ pour tout $t \in [0, T^*)$.

À partir de la forme même de l'équation, on peut ensuite noter les invariances suivantes :

- Invariance par translation : si $u(t,x)$ est solution de (gKdV), alors $w(t,x) = u(t - t_0, x - x_0)$ est aussi solution, pour tous $t_0, x_0 \in \mathbb{R}$.
- Invariance par *scaling* : si $u(t,x)$ est solution de (gKdV), alors $w(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$ est aussi solution, pour tout $\lambda > 0$.
- Symétrie en temps : si $u(t,x)$ est solution de (gKdV), alors $w(t,x) = u(-t,-x)$ est aussi solution.

Une autre propriété fondamentale de cette équation est la conservation pour des solutions H^1 des deux quantités suivantes : pour tout temps $t \in [0, T^*)$,

$$M(u(t)) = \int u^2(t) = M(u_0) \quad \text{(masse)},$$
$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u_0) \quad \text{(énergie)}.$$

Au vu de ces définitions, il apparaît clairement que le choix de $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 < +\infty\}$ comme espace fonctionnel est justifié par le fait que ce soit l'espace d'énergie associé à (gKdV).

Enfin, une dernière propriété générale de l'équation (gKdV) intéressante pour notre étude est la possibilité de *globalisation* des solutions *a priori* locales obtenues par [11]. Celle-ci peut être obtenue, dans le cas où $2 \leqslant p < 5$, par l'inégalité de Gagliardo-Nirenberg suivante :

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leqslant C_{\mathrm{GN}}(p) \Big(\int v_x^2\Big)^{\frac{p-1}{4}} \Big(\int v^2\Big)^{\frac{p+3}{4}}, \tag{1.1}$$

où $C_{\mathrm{GN}}(p) > 0$ est la constante optimale. En effet, pour de telles valeurs de p et par conservation de la masse et de l'énergie, il est clair que $\|\partial_x u(t)\|_{L^2}$ est bornée pour $t \in [0, T^*)$, ce qui impose $T^* = +\infty$ d'après le critère d'explosion énoncé plus haut.

À l'inverse, pour $p = 5$, Merle [20] et Martel et Merle [15] ont prouvé qu'il existe une grande classe de solutions de (gKdV) qui explosent en temps fini. Ainsi, on peut affirmer que $p = 5$ est l'exposant critique pour le comportement en temps long des solutions de (gKdV). On appellera donc *cas sous-critique* le cas où $p < 5$, *cas critique* lorsque $p = 5$, et enfin *cas surcritique* lorsque $p > 5$. Comme l'indique le titre de la thèse, c'est uniquement ce dernier cas qui nous intéressera pour notre étude, pour lequel aucun résultat d'explosion en temps fini n'a pour le moment été obtenu (même si l'explosion en temps fini est probable pour une large classe de solutions aussi).

1.2 La famille des solitons

Comme rappelé dans la section précédente, on considère dans cette thèse l'équation (gKdV) *surcritique*, pour laquelle il peut y avoir explosion de ses solutions. Ainsi, pour étudier le comportement asymptotique de solutions, qui nous intéresse ici, il ne paraît guère possible d'étudier des solutions autres que près d'un voisinage de solutions dont on sait qu'elles sont définies pour tout temps. Il se trouve que de telles solutions existent pour (gKdV), et possèdent en outre d'autres propriétés remarquables : on les appelle *ondes solitaires* ou *solitons.* Elles ont la particularité de garder un profil constant pour tout temps, et doivent leur existence à un équilibre subtil entre la dispersion induite par le terme dérivatif de l'équation ($\partial_x^3 u$) et la non-linéarité ($\partial_x(u^p)$).

Plus précisément, notons Q l'unique solution, aux translations près, de

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q'' + Q^p = Q, \quad \text{i.e. } Q(x) = \left(\frac{p+1}{2\mathrm{ch}^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}. \tag{1.2}$$

On rappelle que Q est l'unique minimiseur de l'inégalité de Gagliardo-Nirenberg (1.1). Autrement dit, pour $v \in H^1(\mathbb{R})$,

$$\|v\|_{L^{p+1}}^{p+1} = C_{\mathrm{GN}}(p)\|v_x\|_{L^2}^{\frac{p-1}{2}}\|v\|_{L^2}^{\frac{p+3}{2}} \iff \exists(\lambda_0, a_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} : v(x) = a_0 Q(\lambda_0 x + b_0). \tag{1.3}$$

Alors, pour tout $c_0 > 0$ et tout $x_0 \in \mathbb{R}$,

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0)$$

est une onde solitaire de (gKdV), où on a noté

$$Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x).$$

On peut remarquer que les ondes solitaires de (gKdV) se déplacent toujours *vers la droite*, à la vitesse $c_0 > 0$, et que Q_{c_0} n'est que la version « rescalée » de Q.

Une fois acquise l'existence de la famille à deux paramètres (R_{c_0,x_0}) des solitons de (gKdV), une question naturelle est de s'interroger sur la *stabilité* – dans un sens à préciser – d'une telle famille. La définition naturelle venant à l'esprit est la suivante : si u_0 est une donnée initiale assez proche d'un profil Q_{c_0}, la solution $u(t)$ associée à u_0 sera-t-elle proche de ce même profil pour tout temps ? Étant donnée l'invariance par translation que vérifie (gKdV), on est amené à définir le « tube », pour $\varepsilon > 0$,

$$U_\varepsilon = \{u \in H^1 \mid \inf_{y\in\mathbb{R}} \|u - Q_{c_0}(\cdot - y)\|_{H^1} \leqslant \varepsilon\}.$$

On dit alors que Q_{c_0} est *orbitalement stable* si, pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que, si $u_0 \in U_\delta$, alors la solution associée $u(t) \in U_\varepsilon$ pour tout $t \in \mathbb{R}$. On dira que Q_{c_0} est *instable* si Q_{c_0} n'est pas stable. La réponse à la question précédente est donnée par le théorème suivant.

Théorème. *Le profil Q_{c_0} est stable si et seulement si $p < 5$.*

Ce théorème est le fruit de plusieurs travaux, parmi lesquels on peut citer Cazenave et Lions [2], Weinstein [23], ou Grillakis, Shatah et Strauss [10] pour le cas $p < 5$; Martel et Merle [14] pour le cas $p = 5$; et Bona, Souganidis et Strauss [1] pour le cas $p > 5$. Cependant, on trouvera aussi une preuve de l'instabilité de Q_{c_0} pour $p > 5$

dans la partie II, car elle sera utile pour exhiber une solution limite qui répondra à la problématique posée dans cette partie.

Un autre aspect de la stabilité est le concept de *stabilité asymptotique*, qui tend à décrire qualitativement le comportement d'une solution qui reste proche d'un profil Q_{c_0} pour tout temps. Martel et Merle [17] ont démontré le résultat suivant, fruit de plusieurs travaux antérieurs sur le même thème.

Théorème. *Soient $p \geqslant 2$ et $c_0 > 0$. Il existe $\alpha_0 > 0$ tel que, si $u(t)$ est une solution globale vérifiant $u(t) \in U_{\alpha_0}$ pour tout $t \geqslant 0$, alors il existe $t \mapsto \rho(t) \in \mathbb{R}$ et $t \mapsto c(t) \in \mathbb{R}$ telles que*

$$u(t) - Q_{c(t)}(\cdot - \rho(t)) \to 0 \quad \textit{dans } H^1(x > \tfrac{c_0}{10}t) \textit{ lorsque } t \to +\infty.$$

De plus, si $p \neq 5$, alors il existe $c_+ > 0$ tel que $c(t) \to c_+$ lorsque $t \to +\infty$.

Deux remarques s'imposent à la lecture de ce théorème. Tout d'abord, il faut noter que, contrairement à la stabilité orbitale, ce résultat de stabilité asymptotique est vrai pour toute valeur de p. D'autre part, il énonce une convergence dans l'espace $H^1(x > \frac{c_0}{10}t)$ avec $t \geqslant 0$, et non dans tout l'espace $H^1(\mathbb{R})$.

C'est tout de même un résultat optimal dans le sens où, dans le cas sous-critique $p < 5$, une convergence dans tout l'espace $H^1(\mathbb{R})$ impliquerait que u *est* un soliton (voir Section 2.1), ce qui n'est bien sûr pas forcément le cas en toute généralité. Mais, pour $p > 5$, une telle convergence dans tout l'espace $H^1(\mathbb{R})$ vers un ou plusieurs solitons n'implique pas une conclusion similaire *a priori*. Cette problématique des multi-solitons, centrale dans cette thèse, est exposée dans la section suivante.

1.3 Les multi-solitons

On considère maintenant $N \geqslant 1$ solitons de vitesses différentes, définis par $2N$ paramètres

$$0 < c_1 < \cdots < c_N, \quad x_1, \ldots, x_N \in \mathbb{R}.$$

On définit le $j^{\text{ième}}$ soliton par $R_j(t) = R_{c_j,x_j}(t)$, et on appelle R la somme de ces N solitons, *i.e.* on pose $R(t) = \sum_{j=1}^N R_j(t)$. Il faut noter qu'à cause de la non-linéarité dans l'équation, R n'est jamais une solution de (gKdV) dès que $N \geqslant 2$.

Par contre, comme les solitons ont des vitesses différentes, ils ont des interactions exponentiellement décroissantes en temps. Une définition possible de N-soliton (qu'on appellera *multi-soliton* lorsque $N \geqslant 2$) est de considérer que c'est une solution de (gKdV) qui se comporte *asymptotiquement* comme R. Plus précisément, on appellera un N-soliton une solution u de (gKdV) telle que

$$\|u(t) - R(t)\|_{H^1(\mathbb{R})} \longrightarrow 0 \quad \text{lorsque} \quad t \to +\infty. \tag{1.4}$$

Plusieurs résultats d'existence et d'unicité de multi-solitons ont été prouvés, suivant les valeurs de p. Tout d'abord, dans les cas complètement intégrables $p = 2$ (équation originelle de Korteweg-de Vries) et $p = 3$ (équation dite de Korteweg-de Vries modifiée), on sait qu'il existe des multi-solitons pour (gKdV), dans un sens plus fort, comme conséquence de la méthode de *scattering* inverse (voir par exemple Miura [21]).

Plus généralement, pour tous les cas sous-critique et critique $p \leqslant 5$, Martel [13] a prouvé qu'étant donnés N solitons, il existe une *unique* solution φ de (gKdV) satisfaisant (1.4). De plus, φ converge vers R uniformément exponentiellement en temps dans toutes les normes $H^s(\mathbb{R})$, pour tout $s \geqslant 0$.

Un résultat similaire a été plus récemment obtenu par Côte, Martel et Merle [6] pour le cas surcritique $p > 5$. En effet, ils obtiennent aussi l'existence d'un N-soliton φ pour tout choix de N solitons, convergeant exponentiellement vers R en temps dans toutes les normes $H^s(\mathbb{R})$, en utilisant un argument topologique pour contrôler la nature instable des solitons dans le régime surcritique. En revanche, ils n'obtiennent pas l'unicité de la solution φ ainsi construite, et pour cause : on montrera dans cette thèse qu'une telle solution n'est en fait pas unique.

Pour comprendre cette différence qualitative, il convient tout d'abord d'appréhender le cas du 1-soliton, ce qui a été le point de départ de cette thèse.

2 Principaux résultats

2.1 Cas du 1-soliton pour (gKdV)

Comme suggéré en fin de section précédente, on s'intéresse au cas du 1-soliton pour l'équation de (gKdV) surcritique, c'est-à-dire aux solutions convergeant asymptotiquement dans tout $H^1(\mathbb{R})$ vers un soliton. En effet, si la situation est claire pour les cas sous-critique et critique comme nous allons le voir, puisque les 1-solitons sont *exactement* les solitons, il n'en est pas de même pour le cas surcritique.

Pour comprendre le cas sous-critique $p < 5$ (le cas critique se traitant similairement), il suffit de se baser sur la caractérisation variationnelle suivante de Q_c, que l'on peut trouver par exemple dans [1] sous forme locale.

Théorème. *Soient $p < 5$ et $c > 0$. Alors la fonctionnelle $v \in H^1(\mathbb{R}) \mapsto E(v) \in \mathbb{R}$ admet un minimum global $E_{\min} = E(Q_c)$ sous la contrainte $M(v) = M(Q_c)$. De plus, $\{v \in H^1(\mathbb{R}) \mid E(v) = E_{\min}\} = \{Q_c(\cdot - x_0)\ ;\ x_0 \in \mathbb{R}\}$.*

En particulier, si une solution converge vers un soliton, alors par conservation de la masse et de l'énergie au cours du temps, elle possède la même masse et la même énergie que ce soliton, et est donc un soliton d'après le théorème précédent. Concernant la démonstration de ce théorème, elle s'obtient facilement à partir de l'inégalité de Gagliardo-Nirenberg (1.1), son cas d'égalité (1.3) appliqué à $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$, et des identités de Pohozaev. En effet, pour $v \in H^1(\mathbb{R})$ tel que $M(v) = M(Q_c)$, on a

$$E(v) = \frac{1}{2}\int v_x^2 - \frac{1}{p+1}\int v^{p+1} \geqslant \frac{1}{2}\int v_x^2 - \frac{C_{\mathrm{GN}}(p)}{p+1}\left(\int v_x^2\right)^{\frac{p-1}{4}}\left(\int Q_c^2\right)^{\frac{p+3}{4}}$$

d'après (1.1). De plus, en posant $\theta \geqslant 0$ tel que $\theta^4 = \frac{\int v_x^2}{\int Q_c'^2}$, on obtient, par (1.3),

$$E(v) \geqslant \frac{1}{2}\int v_x^2 - \frac{1}{p+1}\left(\int v_x^2\right)^{\frac{p-1}{4}}\left(\int Q_c'^2\right)^{-\frac{p-1}{4}}\int Q_c^{p+1} = \frac{1}{2}\theta^4\int Q_c'^2 - \frac{1}{p+1}\theta^{p-1}\int Q_c^{p+1}.$$

Enfin, d'après l'identité $\int Q_c'^2 = \frac{p-1}{2(p+1)} \int Q_c^{p+1}$, on a $E(Q_c) = \frac{p-5}{4(p+1)} \int Q_c^{p+1}$, et donc aussi

$$\begin{aligned} E(v) - E(Q_c) &\geqslant \frac{p-1}{4(p+1)} \theta^4 \int Q_c^{p+1} - \frac{1}{p+1} \theta^{p-1} \int Q_c^{p+1} - \frac{p-5}{4(p+1)} \int Q_c^{p+1} \\ &\geqslant \frac{\int Q_c^{p+1}}{4(p+1)} \left[(p-1)\theta^4 - 4\theta^{p-1} + (5-p) \right]. \end{aligned}$$

Or une simple étude sur $\mathbb{R}_+$ de la fonction f définie par $f(\theta) = (p-1)\theta^4 - 4\theta^{p-1} + (5-p)$ indique qu'elle est décroissante sur $[0,1]$ et croissante sur $[1, +\infty[$, et donc, puisque $f(0) = 5 - p > 0$ et $f(1) = 0$, on en déduit que $f(\theta) \geqslant 0$ pour tout $\theta \geqslant 0$. Ainsi, $E(Q_c)$ est bien un minimum de l'énergie. De plus, si $E(v) = E(Q_c)$, alors il y a égalité dans l'inégalité de Gagliardo-Nirenberg, et il existe donc $x_0 \in \mathbb{R}$ tel que $v = Q_c(\cdot - x_0)$ d'après (1.3).

Par contre, lorsque $p > 5$, la caractérisation variationnelle précédente n'est plus valable. Pour le constater simplement, il suffit de calculer, pour $a > 0$ donné et la fonction $f_a(x) = \sqrt{a} Q_c(ax)$,

$$M(f_a) = M(Q_c), \quad E(f_a) = a^2 \int Q_c'^2 - \frac{1}{p+1} a^{\frac{p-1}{2}} \int Q_c^{p+1}.$$

Comme $\frac{p-1}{2} > 2$ dans ce cas, on en déduit que $E(f_a) \to -\infty$ lorsque $a \to +\infty$, et donc que $\inf\{E(v) \mid M(v) = M(Q_c)\} = -\infty$. Cette remarque faite, une question peut donc venir naturellement à l'esprit : existe-t-il des solutions de (gKdV) qui convergent vers des solitons, mais qui ne sont pas pour autant des solitons ?

La réponse, positive, a été apportée dans la première partie de la thèse, sous la forme du résultat suivant (voir Corollary 3.14 dans la partie II).

Théorème 1 ([3]). *Soit $p > 5$. Il existe une solution $w(t)$ de* (gKdV) *définie pour tout $t \geqslant 0$ et une fonction $t \mapsto \rho(t)$ telles que :*

(i) $\|w(t) - Q(\cdot - \rho(t))\|_{H^1(\mathbb{R})} \to 0$ *lorsque* $t \to +\infty$,

(ii) $\forall c > 0, \forall x_0 \in \mathbb{R}, w(0) \neq Q_c(\cdot + x_0)$.

La démonstration de ce théorème a été l'occasion de se familiariser avec les techniques et résultats standards de l'étude du comportement asymptotique des solitons : modulation, stabilité orbitale et asymptotique, propriétés de « presque monotonie » de la masse et de l'énergie, méthode de compacité, ou encore continuité faible du flot.

L'idée-clé de la démonstration, pour construire une solution w proche du profil Q pour tout temps, a été assez paradoxalement d'utiliser l'instabilité de Q, ce qui a permis de construire une suite de données initiales convergeant (faiblement) vers une donnée initiale limite répondant à notre problème. Plus précisément, à partir d'une suite explicite $(u_{0,n})$ vérifiant $\|u_{0,n} - Q\|_{H^1} \to 0$ pour $n \to +\infty$, nous avons (re)démontré l'instabilité de Q, ce qui nous a permis d'exhiber l'existence d'un $\delta > 0$ tel que

$$\forall n \geqslant 1, \exists T_n \in \mathbb{R}_+ \text{ tel que } \begin{cases} \inf_{y \in \mathbb{R}} \|u_n(T_n) - Q(\cdot - y)\|_{H^1} = \delta, \\ \forall t \in [0, T_n],\ \inf_{y \in \mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leqslant \delta, \end{cases}$$

où u_n est la solution associée à $u_{0,n}$.

Ensuite, de la suite $(u_n(T_n))$, bornée dans $H^1(\mathbb{R})$ et contenant un défaut persistant $\delta > 0$ en plus du profil Q, on a pu extraire une sous-suite convergeant vers un certain v_0 dans $H^1(\mathbb{R})$, et on a pu montrer qu'effectivement v_0 ne pouvait pas être un profil Q_c. Une fois cette donnée limite construite, nous l'avons « transformée » en donnée initiale, en utilisant la symétrie en temps-espace de l'équation de (gKdV), en posant $w_0(x) = v_0(-x)$. Puis, par un théorème de continuité faible du flot (dont la démonstration a été complètement rédigée en annexe la partie III, la partie II étant suffisamment longue), nous avons pu en déduire que la solution $w(t)$ associée à la donnée initiale w_0 était définie pour tout temps $t \geqslant 0$, et que $u_n(T_n - t)$ convergeait vers $w(t)$ faiblement dans H^1 (à une translation en espace près, due à la modulation). Enfin, étant données les estimées de $\|u_n(t) - Q\|_{H^1}$ pour $t \in [0, T_n]$, nous avons pu conclure que $w(t)$ convergeait dans tout $H^1(\mathbb{R})$ vers un profil Q_{c_+}, en utilisant conjointement la stabilité asymptotique de Q pour estimer la décroissance « à droite » de $w(t) - Q_{c_+}$, et les propriétés de monotonie de la masse et de l'énergie pour estimer la décroissance « à gauche » de la solution $w(t)$. Il est à noter que la décroissance « à gauche » de $w(t)$ a d'abord été obtenue classiquement « à droite » de $u_n(T_n - t)$, ce qui s'est traduit par une décroissance « à gauche » après le renversement en temps et en espace utilisé plus haut.

Ainsi, il a été construit ce que l'on a appelé une *solution spéciale* pour (gKdV) surcritique, c'est-à-dire un 1-soliton n'étant pas un soliton. Cependant, pendant la finalisation du théorème précédent, un article de Duyckaerts et Roudenko [8] concernant la même problématique, mais à propos de l'équation de Schrödinger cubique focalisante tridimensionnelle (cNLS-3d), est paru. Entre autres résultats, en adaptant un travail précédent de Duyckaerts et Merle [7], ils apportaient une réponse plus complète à la problématique des 1-solitons, en utilisant une information supplémentaire sur le spectre de l'opérateur linéarisé autour d'un soliton. En effet, ils ont obtenu toute une famille à un paramètre de solutions spéciales, qui en outre les caractérise toutes, sous hypothèse de convergence exponentielle (voir [8, Proposition 7.1] pour plus de précisions).

Or, il est apparu que le résultat spectral utilisé dans [8] pour (cNLS-3d) était encore vrai pour (gKdV) surcritique, d'après un papier de Pego et Weinstein [22] : le spectre de l'opérateur $\mathcal{L}$ linéarisé autour de Q, défini par $\mathcal{L}f = \partial_x^3 f - \partial_x f + p\partial_x(Q^{p-1}f)$, vérifie $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}$ (avec $e_0 > 0$), et $-e_0$ et e_0 sont des valeurs propres simples, associées à des fonctions propres notées $\mathcal{Y}_-$ et $\mathcal{Y}_+$ respectivement. En s'affranchissant de l'hypothèse de convergence exponentielle, le théorème suivant d'existence et de caractérisation des solutions spéciales a pu être prouvé pour (gKdV), ce qui constitue le résultat principal de la partie II (voir Theorem 1.1).

Théorème 2 ([3]). *Soit $p > 5$.*

1. *Il existe une famille à un paramètre $(U^A)_{A\in\mathbb{R}}$ de solutions de* (gKdV) *telle que, pour tout $A \in \mathbb{R}$,*
$$\lim_{t\to+\infty} \|U^A(t, \cdot + t) - Q\|_{H^1} = 0.$$
De plus, pour tout $A \in \mathbb{R}$, il existe $t_0 = t_0(A) \in \mathbb{R}$ tel que, pour tout $s \in \mathbb{R}$, il existe $C > 0$ telle que
$$\forall t \geqslant t_0, \quad \|U^A(t, \cdot + t) - Q - Ae^{-e_0 t}\mathcal{Y}_+\|_{H^s} \leqslant Ce^{-2e_0 t}.$$
2. *Si u est une solution de* (gKdV) *telle que $\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0$, alors il existe $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ et $x_0 \in \mathbb{R}$ tels que $u(t) = U^A(t, \cdot - x_0)$ pour $t \geqslant t_0$.*

La démonstration de la première partie de ce théorème a suivi le schéma de la démonstration de [8], en construisant le reliquat h^A défini par $h^A(t,x) = U^A(t,x+t) - Q(x)$ par une méthode de point fixe, autour d'une famille de solutions approchées de l'équation linéarisée autour de Q, approchant cette équation à tout ordre exponentiel dans toutes les normes $H^s(\mathbb{R})$. La principale difficulté de cette construction par rapport à [8] est venue du fait qu'il fallait utiliser des normes adaptées à (gKdV) pour boucler le point fixe, c'est-à-dire *a priori* les normes utilisées dans [11] pour le problème de Cauchy. En effet, par rapport à l'équation de Schrödinger, l'apparition d'une dérivée supplémentaire dans l'équation (gKdV) oblige à utiliser l'effet régularisant de Kato, et donc l'utilisation de normes *espace-temps* pour le point fixe (contrairement à des normes *temps-espace* pour l'équation de Schrödinger). Puisque nous n'étions pas à un niveau de régularité critique en s pour l'espace H^s, nous avons pu simplifier le point fixe en ne considérant que deux normes : la norme $L_t^\infty H_x^s$ et la norme $L_x^5 L_t^{10}$, cette dernière étant issue du problème de Cauchy pour (gKdV) critique [12].

Concernant la démonstration de la caractérisation des solutions spéciales, nous nous sommes aussi inspiré de la démarche de [8], avec quelques différences cependant. Tout d'abord, pour remplacer le théorème de coercivité de l'énergie linéarisée autour d'un soliton prouvé dans [8], nous avons pu nous appuyer sur un théorème équivalent prouvé dans [6], qui se base sur les fonctions propres *de l'adjoint* de l'opérateur linéarisé $\mathcal{L}$, celui-ci étant plus simple à appréhender que l'opérateur linéarisé lui-même pour (gKdV). À partir de ce résultat fondamental, nous avons pu transformer l'hypothèse de départ de convergence de u vers Q en une convergence exponentielle, grâce à un argument de systèmes dynamiques qui n'était pas présent dans [8]. Nous avons ensuite démontré que cette convergence pouvait être rendue exponentiellement décroissante à tout ordre, pourvu que l'on considère la différence $u - U^A$ et non $u - Q$, où $A \in \mathbb{R}$ est apparu naturellement à l'étape précédente. Si cette amélioration de la décroissance exponentielle est similaire à celle de [8], elle a été cependant obtenue d'une manière plus intrinsèque. Enfin, l'argument final d'unicité a aussi été obtenu par contraction, en utilisant les normes adéquates qui avaient servi à la construction des solutions spéciales.

Enfin, comme conséquence du théorème précédent, nous avons d'abord remarqué qu'à translations en espace et en temps près, il n'y avait « que » deux solutions spéciales, à savoir U^{-1} et U^1. Nous avons de plus établi que, suivant les normalisations des fonctions propres choisies, $U^{-1}(t)$ était globale en temps. Il pourrait être intéressant de s'intéresser à son comportement lorsque $t \to -\infty$. On pourrait aussi remarquer que le Théorème 2 est plus complet que le Théorème 1, mais il faut bien noter que la construction de la solution spéciale $w(t)$ du Théorème 1 n'a nécessité aucune information sur le spectre de l'opérateur linéarisé, et pourrait donc être adaptée à des équations dont le spectre de l'opérateur linéarisé est mal connu. En conclusion de la partie II, nous avons d'ailleurs établi un lien entre les deux théorèmes de cette partie, en identifiant $w(t)$ parmi la famille de solutions spéciales (U^A) : à translations en espace et en temps près, et à changement de *scaling* près, on a prouvé que $w = U^{-1}$.

Après avoir compris le cas du 1-soliton pour (gKdV) surcritique, qui présentait donc des différences qualitatives importantes avec les cas sous-critique et critique, il nous a semblé intéressant de s'attaquer au problème général des multi-solitons, c'est-à-dire au cas des N-solitons avec $N \geqslant 2$.

2.2 Cas des multi-solitons pour (gKdV)

Comme annoncé à la fin de la section précédente, on s'est ensuite intéressé au cas des multi-solitons pour (gKdV) surcritique. On rappelle que le contexte et les résultats déjà existants ont été présentés dans la Section 1.3. En adaptant les résultats et les techniques du cas du 1-soliton (partie II), de la construction d'*un* multi-soliton par Côte, Martel et Merle [6], et de la construction et l'unicité des multi-solitons pour les cas sous-critique et critique par Martel [13], nous avons pu obtenir le résultat optimal suivant. Il s'agit bien sûr du résultat central de cette thèse (voir Theorem 1.3 de la partie III).

Théorème 3 ([4]). *Soient* $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ *et* $x_1, \ldots, x_N \in \mathbb{R}$. *On note* $R = \sum_{j=1}^{N} R_j$ *avec* $R_j = R_{c_j,x_j}$.

1. *Il existe une famille à* N *paramètres* $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ *de solutions de* (gKdV) *telle que, pour tout* $(A_1,\ldots,A_N) \in \mathbb{R}^N$,

$$\lim_{t\to+\infty} \|\varphi_{A_1,\ldots,A_N}(t) - R(t)\|_{H^1} = 0,$$

et si $(A'_1,\ldots,A'_N) \neq (A_1,\ldots,A_N)$, *alors* $\varphi_{A'_1,\ldots,A'_N} \neq \varphi_{A_1,\ldots,A_N}$.

2. *Réciproquement, si* u *est une solution de* (gKdV) *vérifiant*

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0,$$

alors il existe $(A_1,\ldots,A_N) \in \mathbb{R}^N$ *tel que* $u = \varphi_{A_1,\ldots,A_N}$.

Ce résultat est optimal dans le sens où il clôt les problèmes d'existence et d'unicité des multi-solitons pour (gKdV) surcritique, pour N solitons donnés absolument quelconques (sachant qu'il ne peut exister de multi-soliton contenant deux solitons ayant la même vitesse), comme [13] l'avait fait pour les cas sous-critique et critique.

On peut aussi remarquer qu'il s'inscrit dans la continuité logique du cas des 1-solitons, puisque pour ceux-ci une famille à 1 paramètre les caractérisait tous, et que pour les N-solitons, une famille à N paramètres les caractérise tous similairement. La cohérence entre les deux théorèmes ne s'arrête pas ici, puisque la première partie du Théorème 3 est en fait basée sur la proposition cruciale suivante (voir Proposition 3.1 de la partie III). Par un simple argument de *scaling*, on note tout d'abord que l'opérateur $\mathcal{L}_{c_j}$ linéarisé autour de Q_{c_j}, défini par

$$\mathcal{L}_{c_j} f = \partial_x^3 f - c_j \partial_x f + p\partial_x(Q_{c_j}^{p-1} f),$$

admet $\pm e_j = \pm c_j^{3/2} e_0$ comme valeurs propres réelles, associées à des fonctions propres $Y_j^{\pm}$ (qui suivent cette fois le soliton R_j, puisqu'on ne peut plus recentrer le problème à vitesse nulle comme dans le cas du 1-soliton).

Proposition. *Soient* φ *un multi-soliton (donné par [6] par exemple),* $j \in [\![1, N]\!]$ *et* $A_j \in \mathbb{R}$. *Alors il existe* $t_0 > 0$ *et* $u \in C([t_0, +\infty), H^1)$ *solution de* (gKdV) *tels que*

$$\forall t \geqslant t_0, \quad \|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$

où $\gamma > 0$ *est un petit paramètre ne dépendant que des vitesses* c_k.

Quelques remarques s'imposent à la lecture de cette proposition. Tout d'abord, il faut noter que, lorsque $A_j \neq 0$, on perturbe effectivement le multi-soliton φ en obtenant un nouveau multi-soliton différent. Cela permet la construction de la famille $(\varphi_{A_1,\dots,A_N})$, pourvu que celle-ci soit faite dans l'ordre adéquat, c'est-à-dire de $j = 1$ à $j = N$, puisqu'il ne serait pas significatif de perturber le multi-soliton à l'ordre e_j avant l'ordre e_{j-1}, sachant que $e_j > e_{j-1} + \gamma$. D'autre part, on peut aussi remarquer que le membre de droite est en $e^{-(e_j+\gamma)t}$ et non e^{-2e_jt} comme pour le 1-soliton : cela s'explique simplement par les interactions entre les différents solitons, que mesure ce paramètre γ. Enfin, il faut noter que, puisque la perturbation se fait à un ordre élevé e_j, il était capital de pouvoir disposer d'un multi-soliton φ déjà construit au départ. En effet, si on remplaçait φ par la somme des solitons R, des termes source d'ordre γ d'interactions entre les solitons apparaîtraient dans les estimations, rendant vaines celles à l'ordre $e_j \gg \gamma$.

Par contre, si les résultats sont similaires, les idées des démonstrations du Théorème 2 et de la proposition précédente diffèrent sensiblement. En effet, comme expliqué dans la section précédente, l'idée-clé de la démonstration du Théorème 2 était un argument de point fixe, réalisé autour d'une famille de solutions approchées de l'équation linéarisée *autour d'un soliton*. Cependant, du fait des interactions entre solitons, il ne nous est pas apparu clair que la construction d'une famille de solutions approchées de l'équation linéarisée *autour d'un multi-soliton* était possible. Ainsi a été privilégiée une démonstration se basant uniquement sur des arguments de compacité et d'énergie, dans la lignée de [13, 6], l'utilisation d'un théorème de continuité faible du flot (démontré en annexe de la partie III) permettant de conclure facilement.

Suivant la stratégie de [6], on a donc considéré une suite croissant vers l'infini de temps S_n, une suite de paramètres $\mathfrak{b}_n = (b_{n,k})_{j<k\leqslant N} \in \mathbb{R}^{N-j}$ à déterminer, et u_n solution de

$$\begin{cases} \partial_t u_n + \partial_x[\partial_x^2 u_n + u_n^p] = 0, \\ u_n(S_n) = \varphi(S_n) + A_j e^{-e_j S_n} Y_j^+(S_n) + \sum_{k>j} b_{n,k} Y_k^+(S_n). \end{cases} \tag{2.1}$$

Cependant, on peut constater que, à la différence de [6], les paramètres $b_{n,k}$ destinés à contrôler la nature instable des solitons par un argument topologique ne sont choisis que pour $k > j$, les directions instables étant régies par l'équation elle-même pour $k \leqslant j$, puisque l'on se place à l'ordre e_j. Nous avons alors pu montrer la proposition suivante d'estimations uniformes, en utilisant certains arguments de la partie II, ainsi que des arguments de monotonie de l'énergie développés dans [13] afin de pouvoir appliquer le théorème de coercivité de l'énergie autour de chaque soliton, qui était déjà l'argument-clé dans la partie II.

Proposition. *Il existe $n_0 \geqslant 0$ et $t_0 > 0$ (indépendant de n) tels que ce qui suit soit vérifié. Pour chaque $n \geqslant n_0$, il existe $\mathfrak{b}_n \in \mathbb{R}^{N-j}$, vérifiant $\|\mathfrak{b}_n\| \leqslant 2e^{-(e_j+2\gamma)S_n}$, tel que la solution u_n de* (2.1) *soit définie sur l'intervalle $[t_0, S_n]$, et satisfasse*

$$\forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Il est à noter que la propriété de monotonie de l'énergie utilisée pour la construction n'était pas absolument nécessaire. En effet, une estimation « brute » en valeur absolue d'une fonctionnelle légèrement différente pouvait aussi donner le résultat souhaité, comme cela a été fait dans la partie IV (puisque les propriétés de monotonie ne sont plus vérifiées pour (NLS)). Elle a par contre été nécessaire dans la partie de classification des multi-solitons, notamment pour montrer qu'une simple convergence comme (1.4) pouvait s'améliorer en convergence exponentielle à un petit ordre $\gamma > 0$.

Justement, concernant la preuve de la classification des multi-solitons, celle-ci s'est basée sur le fait d'essayer d'améliorer à tout ordre exponentiel la convergence de départ (1.4). Comme dans le cas du 1-soliton, on constate qu'une fois atteint l'ordre e_1, on ne peut pas améliorer la convergence si on ne raffine pas le comportement asymptotique de u. En procédant étape par étape, de $j = 1$ à $j = N$ comme pour la construction, on arrive finalement à identifier $(A_1, \dots, A_N) \in \mathbb{R}^N$ et à démontrer que $\|u - \varphi_{A_1,\dots,A_N}\|_{H^1}$ converge vers 0 à tout ordre exponentiel. Finalement, un argument de *bootstrap* comme celui utilisé dans [13] pour prouver l'unicité permet de conclure que $u = \varphi_{A_1,\dots,A_N}$.

Enfin, une fois le Théorème 3 établi, nous nous sommes intéressé à l'équation de Schrödinger non-linéaire surcritique, afin de constater si la méthode employée dans la partie III était suffisamment souple pour être adaptée à une autre équation importante, tout en sachant qu'il était pour le moment vain d'essayer d'obtenir aussi la caractérisation des multi-solitons pour (NLS) surcritique, à cause de l'absence de propriété de monotonie de l'énergie mentionnée plus haut.

2.3 Cas des multi-solitons pour (NLS)

Comme annoncé en fin de section précédente, nous nous sommes intéressé dans la dernière partie de la thèse à la construction (et seulement à la construction) d'une famille à N paramètres de N-solitons pour (NLS) surcritique, le cas du 1-soliton ayant déjà été traité dans [8]. On rappelle que, même pour les cas sous-critique et critique, aucun résultat général de classification n'a été obtenu pour le moment. Enfin, comme des problèmes techniques sont survenus pour les dimensions supérieures à 2 (à cause du manque de régularité de l'équation à l'origine), nous avons décidé de traiter uniquement le cas unidimensionnel, les calculs étant déjà suffisamment techniques dans ce cas. Ainsi, nous avons considéré l'équation

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases} \tag{NLS}$$

où $p > 5$ est réel, $(t,x) \in \mathbb{R}^2$, et $u(t,x) \in \mathbb{C}$.

Depuis les travaux de Ginibre et Velo [9], on sait que (NLS) est bien posée dans $H^1(\mathbb{R})$. Autrement dit, pour tout $u_0 \in H^1(\mathbb{R})$, il existe $T > 0$ et une unique solution maximale $u \in C([0,T), H^1(\mathbb{R}))$ de (NLS). De plus, soit $T = +\infty$, soit $T < +\infty$ et alors $\lim_{t\to T} \|\partial_x u(t)\|_{L^2} = +\infty$. En outre, comme pour (gKdV), la masse et l'énergie d'une solution H^1 de (NLS) sont conservées, ainsi qu'une troisième quantité, appelée *moment*. Plus précisément, pour tout $t \in [0,T)$, on a

$$M(u(t)) = \int |u(t)|^2 = M(u_0) \quad \text{(masse)},$$
$$E(u(t)) = \frac{1}{2}\int |\partial_x u(t)|^2 - \frac{1}{p+1}\int |u(t)|^{p+1} = E(u_0) \quad \text{(énergie)},$$
$$P(u(t)) = \operatorname{Im}\int \partial_x u(t)\bar{u}(t) = P(u_0) \quad \text{(moment)}.$$

Enfin, on remarque que (NLS) admet les mêmes invariances que (gKdV), ainsi que deux autres additionnelles. En effet, si $u(t,x)$ est solution de (NLS), alors, pour tous $t_0, x_0, \gamma_0, v_0 \in \mathbb{R}$ et $\lambda > 0$, sont aussi solutions de (NLS) :

- par invariance par translation, $w(t,x) = u(t - t_0, x - x_0)$;
- par invariance par *scaling*, $w(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$;
- par invariance par rotation, $w(t,x) = u(t,x)e^{i\gamma_0}$;
- par symétrie en temps, $w(t,x) = \overline{u(-t,x)}$;
- par invariance galiléenne, $w(t,x) = u(t, x - v_0 t)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t)}$.

Au vu de ces multiples invariances, il est clair que les solitons de (NLS) pourront dépendre de plus de paramètres que ceux de (gKdV). Cependant, ils sont construits à partir du même profil Q déjà défini en (1.2), puisqu'avec cette définition $e^{it}Q(x)$ est solution (stationnaire) de (NLS). Ainsi, si on se donne des paramètres $c_0 > 0$ de *scaling*, $\gamma_0 \in \mathbb{R}$ de phase, $v_0 \in \mathbb{R}$ de vitesse, et $x_0 \in \mathbb{R}$ de translation en espace, alors

$$R_{c_0,\gamma_0,v_0,x_0}(t,x) = Q_{c_0}(x - v_0 t - x_0)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t + c_0 t + \gamma_0)},$$

avec toujours

$$Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x),$$

est un soliton de (NLS), se déplaçant sur la droite $x = v_0 t + x_0$. On peut noter que, contrairement à (gKdV), vitesse et *scaling* sont décorrélés, ce qui induira une légère différence dans la construction de la famille $(\varphi_{A_1,\ldots,A_N})$. Enfin on rappelle que, comme pour (gKdV) surcritique, les solitons de (NLS) surcritique sont instables (voir [10]).

On peut maintenant définir les multi-solitons de (NLS). On considère pour cela $N \geqslant 1$ solitons ayant des vitesses différentes, définis par $4N$ paramètres

$$v_1 < \cdots < v_N, \quad c_1, \ldots, c_N \in \mathbb{R}_+^*, \quad \gamma_1, \ldots, \gamma_N \in \mathbb{R}, \quad x_1, \ldots, x_N \in \mathbb{R},$$

et on pose $R_j(t) = R_{c_j,\gamma_j,v_j,x_j}(t)$. Comme pour (gKdV), on définit R comme la somme des solitons R_j, et on appelle N-soliton une solution u de (NLS) qui satisfait (1.4).

La question de l'existence des multi-solitons a fait l'objet de plusieurs travaux pour (NLS) sous-critique et critique. Tout d'abord, Merle [19] a établi un résultat d'existence pour le cas critique, comme conséquence d'un résultat d'explosion et de l'invariance conforme. Ce résultat a été étendu par Martel et Merle [16] au cas sous-critique, en utilisant des arguments déjà développés par Martel, Merle et Tsai [18] pour montrer la stabilité des multi-solitons dans H^1.

Concernant (NLS) surcritique, qui nous intéresse ici, on rappelle que le cas des 1-solitons a été complètement traité dans [8], pour une équation similaire à (NLS) (car aussi L^2-surcritique et H^1-sous-critique). Pour les multi-solitons, il se trouve que le résultat de Côte, Martel et Merle [6], établissant l'existence d'*au moins* un N-soliton et utilisé dans la partie III, est aussi valable pour (NLS) d'après ce même papier. Ainsi, puisque ce point de départ fondamental était encore valable pour (NLS), nous avons pu montrer le dernier résultat suivant (Theorem 1.3 de la partie IV).

Théorème 4 ([5]). *Soient* $p > 5$, $N \geqslant 2$, $v_1 < \cdots < v_N$, $(c_1, \ldots, c_N) \in (\mathbb{R}_+^*)^N$, $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ *et* $(x_1, \ldots, x_N) \in \mathbb{R}^N$. *On note* $R = \sum_{j=1}^N R_{c_j,\gamma_j,v_j,x_j}$.

Alors il existe $\gamma > 0$ *et une famille à* N *paramètres* $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ *de solutions de* (NLS) *tels que, pour tout* $(A_1, \ldots, A_N) \in \mathbb{R}^N$, *il existe* $C > 0$ *et* $t_0 > 0$ *tels que*

$$\forall t \geqslant t_0, \ \|\varphi_{A_1,\ldots,A_N}(t) - R(t)\|_{H^1} \leqslant Ce^{-\gamma t},$$

et si $(A'_1, \ldots, A'_N) \neq (A_1, \ldots, A_N)$, *alors* $\varphi_{A'_1,\ldots,A'_N} \neq \varphi_{A_1,\ldots,A_N}$.

Ce résultat est bien sûr l'équivalent pour (NLS) de la première partie du Théorème 3 concernant (gKdV). Sa démonstration est aussi très semblable, la première principale différence étant la technicité des calculs, bien plus importante pour (NLS) en raison de la plus grande complexité de l'équation linéarisée, et donc aussi de la fonctionnelle d'énergie à considérer. La seconde différence, plus profonde, vient du fait qu'on ait pu obtenir une estimée de la dérivée de la fonctionnelle d'énergie *en valeur absolue*, contrairement à (gKdV) où on utilisait une propriété de *monotonie* de celle-ci, qui est fausse pour (NLS). On s'est pour cela inspiré des techniques de localisation utilisées dans [6], ce qui nous a permis d'estimer la fonctionnelle d'énergie, et donc d'achever la démonstration du Théorème 4 de manière similaire à celle du Théorème 3.

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Part II

Construction and characterization of solutions converging to solitons for supercritical gKdV equations

Summary

Abstract

We consider the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^p) = 0, \quad (t,x) \in \mathbb{R}^2,$$

in the supercritical case $p > 5$, and we are interested in solutions which converge to a soliton in large time in H^1. In the subcritical case ($p < 5$), such solutions are forced to be exactly solitons by variational characterization [3, 22], but no such result exists in the supercritical case. In this paper, we first construct a "special solution" in this case by a compactness argument, *i.e.* a solution which converges to a soliton without being a soliton. Secondly, using a description of the spectrum of the linearized operator around a soliton [20], we construct a one parameter family of special solutions which characterizes all such special solutions. In the case of the nonlinear Schrödinger equation, a similar result was proved in [7, 8].

1 Introduction

1.1 The generalized Korteweg-de Vries equation

We consider the generalized Korteweg-de Vries equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases} \tag{gKdV}$$

where $(t,x) \in \mathbb{R}^2$ and $p \geqslant 2$ is an integer. The following quantities are formally conserved for solutions of (gKdV):

$$\int u^2(t) = \int u^2(0) \quad \text{(mass)}, \tag{1.1}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \quad \text{(energy)}. \tag{1.2}$$

Kenig, Ponce and Vega [12] have shown that the local Cauchy problem for (gKdV) is well-posed in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of (gKdV) satisfying $u(0) = u_0$ which is unique in some class $Y_T \subset C^0([0,T], H^1(\mathbb{R}))$. Moreover, if $T^* \geqslant T$ is the maximal time of existence of u, then either $T^* = +\infty$ which means that $u(t)$ is a global solution, or $T^* < +\infty$ and then $\|u(t)\|_{H^1} \to +\infty$ as $t \uparrow T^*$ ($u(t)$ is a finite time blow up solution). Throughout this paper, when referring to an H^1 solution of (gKdV), we mean a solution in the above sense. Finally, if $u_0 \in H^s(\mathbb{R})$ for some $s \geqslant 1$, then $u(t) \in H^s(\mathbb{R})$ for all $t \in [0, T^*)$.

In the case where $2 \leqslant p < 5$, it is standard that all solutions in H^1 are global and uniformly bounded by the energy and mass conservations and the following Gagliardo-Nirenberg inequality:

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leqslant C_{\text{GN}}(p)\Big(\int v_x^2\Big)^{\frac{p-1}{4}}\Big(\int v^2\Big)^{\frac{p+3}{4}} \tag{1.3}$$

with optimal constant $C_{\mathrm{GN}}(p) > 0$. In the case $p = 5$, the existence of finite time blow up solutions was proved by Merle [19] and Martel and Merle [16]. Therefore $p = 5$ is the critical exponent for the long time behavior of solutions of (gKdV). For $p > 5$, the existence of blow up solutions is an open problem.

We recall that a fundamental property of (gKdV) equations is the existence of a family of explicit traveling wave solutions. Let Q be the only solution (up to translations) of

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q'' + Q^p = Q, \quad \text{i.e. } Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}$$

Note that Q is the unique minimizer of the Gagliardo-Nirenberg inequality (1.3) (see [4] for the case $p = 5$ for example), *i.e.* for $v \in H^1(\mathbb{R})$:

$$\|v\|_{L^{p+1}}^{p+1} = C_{\mathrm{GN}}(p)\|v_x\|_{L^2}^{\frac{p-1}{2}}\|v\|_{L^2}^{\frac{p+3}{2}} \iff \exists(\lambda_0, a_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} : v(x) = a_0 Q(\lambda_0 x + b_0). \quad (1.4)$$

For all $c_0 > 0$ and $x_0 \in \mathbb{R}$, $R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0)$ is a solution of (gKdV), where

$$Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}\, x).$$

We call solitons these solutions though they are known to be solitons only for $p = 2, 3$ (in the sense that they are stable by interaction).

It is well-known that solitons are orbitally stable (see Definition 2.7) for $p < 5$ and unstable for $p \geqslant 5$. An important fact used by Weinstein [22] to prove their orbital stability when $p < 5$ is the following variational characterization of Q_{c_0}: if u is a solution of (gKdV) such that $E(u) = E(Q_{c_0})$ and $\int u^2 = \int Q_{c_0}^2$ for some $c_0 > 0$, then there exists $x_0 \in \mathbb{R}$ such that $u = Q_{c_0}(\cdot - x_0)$. As a direct consequence, if now $u(t)$ is a solution such that

$$\lim_{t \to +\infty} \inf_{y \in \mathbb{R}} \|u(t) - Q_{c_0}(\cdot - y)\|_{H^1(\mathbb{R})} = 0 \quad (1.5)$$

(*i.e.* u converges to Q_{c_0} in the suitable sense), then $u = R_{c_0,x_0}$. For $p = 5$, the same is true for similar reasons (see [23]).

In the present paper, we focus on the supercritical case $p > 5$. Some asymptotic results around solitons have been proved: orbital instability of solitons by Bona *et al.* [3] (see also [10]) and asymptotic stability (in some sense) by Martel and Merle [18] for example. But available variational arguments do not allow to classify all solutions of (gKdV) satisfying (1.5). In fact, in Section 3, we construct a solution of (gKdV) satisfying (1.5) which is not a soliton (we call *special solution* such a solution). In Section 4, by another method, we construct a whole family of such solutions, and we completely characterize solutions satisfying (1.5). This method is strongly inspired of arguments developed by Duyckaerts and Roudenko [8], themselves an adaptation of arguments developed by Duyckaerts and Merle [7]. For reader's convenience, we recall in the next section the results in [8] related to our paper.

1.2 The nonlinear Schrödinger equation case

We recall Duyckaerts and Roudenko's results for the nonlinear Schrödinger equation. They consider in [8] the 3d focusing cubic nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}^3). \end{cases} \tag{cNLS-3d}$$

This equation is $\dot{H}^{1/2}$-critical, and so L^2-supercritical like (gKdV) for $p > 5$, while [7] is devoted to the $\dot{H}^1$-critical equation. Similarly to (gKdV), (cNLS-3d) is locally well-posed in H^1, and solutions of (cNLS-3d) satisfy the following conservation laws:

$$E_{\mathrm{NLS}}[u](t) = \frac{1}{2}\int |\nabla u(t,x)|^2\,dx - \frac{1}{4}\int |u(t,x)|^4\,dx = E_{\mathrm{NLS}}[u](0),$$
$$M_{\mathrm{NLS}}[u](t) = \int |u(t,x)|^2\,dx = M_{\mathrm{NLS}}[u](0).$$

Moreover, if Q is the unique (in a suitable sense) solution of the nonlinear elliptic equation $-Q + \Delta Q + |Q|^2 Q = 0$, then $e^{it}Q(x)$ is a soliton solution of (cNLS-3d).

Theorem 2 in [8] states the existence of two radial solutions $Q^+(t)$ and $Q^-(t)$ of (cNLS-3d) such that $M_{\mathrm{NLS}}[Q^+] = M_{\mathrm{NLS}}[Q^-] = M_{\mathrm{NLS}}[Q]$, $E_{\mathrm{NLS}}[Q^+] = E_{\mathrm{NLS}}[Q^-] = E_{\mathrm{NLS}}[Q]$, $[0,+\infty)$ is in the time domain of definition of $Q^\pm(t)$, and there exists $e_0 > 0$ such that $\|Q^\pm(t) - e^{it}Q\|_{H^1} \leqslant Ce^{-e_0 t}$ for all $t \geqslant 0$. Moreover, $Q^-(t)$ is globally defined and scatters for negative time, and the negative time of existence of $Q^+(t)$ is finite.

They also prove the following classification theorem [8, Theorem 3].

Theorem ([8]). *Let u be a solution of* (cNLS-3d) *satisfying*

$$E_{\mathrm{NLS}}[u]M_{\mathrm{NLS}}[u] = E_{\mathrm{NLS}}[Q]M_{\mathrm{NLS}}[Q].$$

(a) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}$, then either u scatters or $u = Q^-$ up to the symmetries.

(b) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} = \|\nabla Q\|_{L^2}\|Q\|_{L^2}$, then $u = e^{it}Q$ up to the symmetries.

(c) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} > \|\nabla Q\|_{L^2}\|Q\|_{L^2}$ and u_0 is radial or of finite variance, then either the interval of existence of u is of finite length or $u = Q^+$ up to the symmetries.

In particular, if $\lim_{t\to+\infty} \|u(t) - e^{it}Q\|_{H^1} = 0$, then $u = e^{it}Q$, Q^+ or Q^- up to the symmetries.

Among the various ingredients used to prove the results above, one of the most important is a sharp analysis of the spectrum $\sigma(\mathcal{L}_{\mathrm{NLS}})$ of the linearized Schrödinger operator around the ground state solution $e^{it}Q$, due to Grillakis [9] and Weinstein [21]. They prove that $\sigma(\mathcal{L}_{\mathrm{NLS}}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover that e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}_{\mathrm{NLS}}$ with eigenfunctions $\mathcal{Y}_+^{\mathrm{NLS}}$ and $\mathcal{Y}_-^{\mathrm{NLS}} = \overline{\mathcal{Y}_+^{\mathrm{NLS}}}$. This structure, which is similar for (gKdV) according to Pego and Weinstein [20], will also be crucial to prove our main result (exposed in the next section).

1.3 Main result and outline of the paper

In this paper, we consider similar questions for the (gKdV) equation in the supercritical case $p > 5$. Recall that, similarly to the (cNLS-3d) case, Pego and Weinstein [20] have determined the spectrum of the linearized operator $\mathcal{L}$ around the soliton $Q(x-t)$: $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions $\mathcal{Y}_+$ and $\mathcal{Y}_-$ which are exponentially decaying (see Proposition 4.2 and Corollary 4.4). We now state precisely our main result.

Theorem 1.1. *Let $p > 5$.*

1. (Existence of a family of special solutions). *There exists a one-parameter family $(U^A)_{A\in\mathbb{R}}$ of solutions of* (gKdV) *such that*
$$\lim_{t\to+\infty} \|U^A(t, \cdot + t) - Q\|_{H^1} = 0.$$
Moreover, for all $A \in \mathbb{R}$, there exists $t_0 = t_0(A) \in \mathbb{R}$ such that, for all $s \in \mathbb{R}$, there exists $C > 0$ such that
$$\forall t \geqslant t_0, \quad \|U^A(t, \cdot + t) - Q - Ae^{-e_0 t}\mathcal{Y}_+\|_{H^s} \leqslant Ce^{-2e_0 t}.$$

2. (Classification of special solutions). *If u is a solution of* (gKdV) *such that*
$$\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0,$$
then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for $t \geqslant t_0$.

Remark 1.2. From Theorem 1.1, there are actually only three different special solutions U^A up to translations in time and in space: U^1, U^{-1} and $Q(\cdot - t)$ (see Proposition 4.12). This is of course related to the three solutions of (cNLS-3d) constructed in [8]: $Q^+(t)$, $Q^-(t)$ and $e^{it}Q$.

Remark 1.3. From Section 4.5, we can choose a normalization of $\mathcal{Y}_\pm$ so that, for $A < 0$, $\|\partial_x U^A\|_{L^2} < \|Q'\|_{L^2}$. Then $U^{-1}(t)$ is global, *i.e.* defined for all $t \in \mathbb{R}$. It would be interesting to investigate in more details its behavior as $t \to -\infty$. On the other hand, the behavior of $U^1(t)$ is not known for $t < t_0$.

Remark 1.4. By scaling, Theorem 1.1 extends to Q_c for all $c > 0$ (see Corollary 4.11 at the end of the paper).

The paper is organized as follows. In the next section, we recall some properties of the solitons, and in particular we recall the proof of their orbital instability when $p > 5$. This result is well-known [3], but our proof with an explicit initial data is useful to introduce some suitable tools to the study of solitons of (gKdV) (as modulation, Weinstein's functional, monotonicity, linearized equation, etc.). Moreover, it is the first step to construct *one* special solution in Section 3 by compactness, similarly as Martel and Merle [18]. This proof does not use the precise analysis of the spectrum of $\mathcal{L}$ due to Pego and Weinstein [20], and so can be hopefully adapted to equations for which the spectrum of the linearized operator is not well-known. To fully prove Theorem 1.1 (existence and uniqueness of a family of special solutions, Section 4), we rely on the method introduced in [7] and [8].

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2 Preliminary results

We recall here some well-known properties of the solitons and some results of stability around the solitons. We begin by recalling notation and simple facts on the functions $Q(x)$ and $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$ defined in Section 1.1.

Notation. They are available in the whole paper.

(a) $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R})$ scalar product, and $\perp$ the orthogonality with respect to $(\cdot,\cdot)$.

(b) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1+\xi^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R})\}$, and in particular $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.

(c) We denote $\frac{\partial}{\partial x}v = \partial_x v = v_x$ the partial derivative of v with respect to x, and $\partial_x^s = \partial^s$ the s-order partial derivative with respect to x when no confusion is possible.

(d) All numbers C, K appearing in inequalities are real constants (with respect to the context) strictly positive, which may change in each step of an inequality.

Claim 2.1. *For all $c > 0$, one has:*

(i) $Q_c > 0$, Q_c is even, Q_c is C^∞, and $Q_c'(x) < 0$ for all $x > 0$.

(ii) There exist $K_1, K_2 > 0$ such that: $\forall x \in \mathbb{R}$, $K_1 e^{-\sqrt{c}|x|} \leqslant Q_c(x) \leqslant K_2 e^{-\sqrt{c}|x|}$.

(iii) For all $j \geqslant 0$, there exists $C_j > 0$ such that $Q_c^{(j)}(x) \sim C_j e^{-\sqrt{c}|x|}$ as $|x| \to +\infty$. For all $j \geqslant 1$, there exists $C_j' > 0$ such that: $\forall x \in \mathbb{R}$, $|Q_c^{(j)}(x)| \leqslant C_j' e^{-\sqrt{c}|x|}$.

(iv) The following identities hold:

$$\int Q_c^2 = c^{\frac{5-p}{2(p-1)}} \int Q^2 \quad , \quad \int (Q_c')^2 = c^{\frac{p+3}{2(p-1)}} \int Q'^2. \tag{2.1}$$

2.1 Weinstein's functional linearized around Q

We introduce here the Weinstein's functional F and give an expression of $F(Q+a)$ for a small which will be very useful in the rest of the paper. We recall first that the energy of a function $\varphi \in H^1$ is defined by $E(\varphi) = \frac{1}{2}\int(\partial_x\varphi)^2 - \frac{1}{p+1}\int \varphi^{p+1}$.

Definition 2.2. Weinstein's functional is defined for $\varphi \in H^1$ by $F(\varphi) = E(\varphi) + \frac{1}{2}\int \varphi^2$.

Claim 2.3. *If $u_0 \in H^1$ and $u(t)$ solves* (gKdV) *with $u(0) = u_0$, then for all $t \in [0, T^*)$, $F(u(t)) = F(u_0)$. It is an immediate consequence of* (1.1) *and* (1.2).

Lemma 2.4 (Weinstein's functional linearized around Q). *For all $C > 0$, there exists $C' > 0$ such that, for all $a \in H^1$ satisfying $\|a\|_{H^1} \leqslant C$,*

$$F(Q+a) = F(Q) + \frac{1}{2}(La, a) + K(a) \tag{2.2}$$

where $La = -\partial_x^2 a + a - pQ^{p-1}a$, and $K : H^1 \to \mathbb{R}$ satisfies $|K(a)| \leqslant C''\|a\|_{H^1}^3$.

Proof. Let $a \in H^1$ be such that $\|a\|_{H^1} \leqslant C$. Then we have

$$\begin{aligned} E(Q+a) &= \frac{1}{2}\int (Q' + \partial_x a)^2 - \frac{1}{p+1}\int (Q+a)^{p+1} \\ &= E(Q) + \frac{1}{2}\int (\partial_x a)^2 + \int Q' \cdot \partial_x a \\ &\quad - \frac{1}{p+1}\int \left[(p+1)Q^p a + \frac{(p+1)p}{2} Q^{p-1}a^2 + R(a)\right] \\ &= E(Q) + \frac{1}{2}\int (\partial_x a)^2 - \int Qa - \frac{p}{2}\int Q^{p-1}a^2 - \frac{1}{p+1}\int R(a) \end{aligned}$$

since $Q'' + Q^p = Q$, and where $R(a) = \sum_{k=3}^{p+1} \binom{p+1}{k} Q^{p+1-k}a^k$. Since $\|a\|_{L^\infty} \leqslant C\|a\|_{H^1} \leqslant C$, then $|R(a)| \leqslant C|a|^3 \leqslant C\|a\|_{L^\infty}|a|^2$, and so $K(a) = -\frac{1}{p+1}\int R(a)$ satisfies $|K(a)| \leqslant C'\|a\|_{H^1}^3$. Moreover, we have more simply: $\int (Q+a)^2 = \int Q^2 + \int a^2 + 2\int Qa$. Finally we have

$$F(Q+a) = F(Q) + \frac{1}{2}\int a^2 + \frac{1}{2}\int (\partial_x a)^2 - \frac{p}{2}\int Q^{p-1}a^2 + K(a). \qquad \square$$

Claim 2.5 (Properties of L). *The operator L defined in Lemma 2.4 is self-adjoint and satisfies the following properties.*

(i) First eigenfunction: $LQ^{\frac{p+1}{2}} = -\lambda_0 Q^{\frac{p+1}{2}}$ *where* $\lambda_0 = \frac{1}{4}(p-1)(p+3) > 0$.

(ii) Second eigenfunction: $LQ' = 0$, *and* $\ker L = \{\lambda Q' \; ; \; \lambda \in \mathbb{R}\}$.

(iii) Scaling: If we denote $S = \frac{dQ_c}{dc}\Big|_{c=1}$, *then* $S(x) = \frac{1}{p-1}Q(x) + \frac{1}{2}xQ'(x)$ *and* $LS = -Q$.

(iv) Coercivity: There exists $\sigma_0 > 0$ *such that, for all* $u \in H^1(\mathbb{R})$ *satisfying* $(u, Q') = (u, Q^{\frac{p+1}{2}}) = 0$, *one has* $(Lu, u) \geqslant \sigma_0 \|u\|_{L^2}^2$.

Proof. The first three properties follow from straightforward computation, except for $\ker L$ which can be determined by ODE techniques, see [21, Proposition 2.8]. The property of coercivity follows easily from (i), (ii) and classical results on self-adjoint operators and Sturm-Liouville theory. $\square$

Lemma 2.6. *There exist $K_1, K_2 > 0$ such that, for all $\varepsilon \in H^1$ satisfying $\varepsilon \perp Q'$,*

$$(L\varepsilon, \varepsilon) = \int \varepsilon_x^2 + \int \varepsilon^2 - p\int Q^{p-1}\varepsilon^2 \geqslant K_1\|\varepsilon\|_{H^1}^2 - K_2\Big(\int \varepsilon Q^{\frac{p+1}{2}}\Big)^2.$$

Proof. By Claim 2.5, we already know that there exists $\sigma_0 > 0$ such that, for all ε satisfying $\varepsilon \perp Q^{\frac{p+1}{2}}$ and $\varepsilon \perp Q'$, we have $(L\varepsilon, \varepsilon) \geqslant \sigma_0\|\varepsilon\|_{L^2}^2$. The first step is to replace the L^2 norm by the H^1 one in this last inequality, which is easy if we choose σ_0 small enough. If we do not suppose $\varepsilon \perp Q^{\frac{p+1}{2}}$, we write $\varepsilon = \varepsilon_1 + aQ^{\frac{p+1}{2}}$ with $a = (\int \varepsilon Q^{\frac{p+1}{2}})(\int Q^{p+1})^{-1}$ such that $\varepsilon_1 \perp Q^{\frac{p+1}{2}}$ for the L^2 scalar product, but also for the bilinear form $(L\cdot, \cdot)$ since $Q^{\frac{p+1}{2}}$ is an eigenvector for L. Since $Q^{\frac{p+1}{2}} \perp Q'$, we obtain easily the desired inequality from the previous step. $\square$

2.2 Orbital stability and decomposition of a solution around Q

In this paper, we consider only solutions which stay close to a soliton. So it is important to define properly this notion, and the invariance by translation leads us to consider for $\varepsilon > 0$ the "tube"

$$U_\varepsilon = \{u \in H^1 \mid \inf_{y \in \mathbb{R}} \|u - Q_c(\cdot - y)\|_{H^1} \leqslant \varepsilon\}.$$

Definition 2.7. The solitary wave Q_c is (orbitally) *stable* if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u_0 \in U_\delta$, then the associated solution $u(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$. The solitary wave Q_c is *unstable* if Q_c is not stable.

Theorem 2.8. *Q_c is stable if and only if $p < 5$.*

Remark 2.9. 1. The stability part of this theorem is due to Benjamin [1], Bona [2], Cazenave and Lions [5], and Weinstein [21, 23]. Recall also that in general, the stability theory in H^1 requires the H^1 Cauchy theory due to Kenig, Ponce and Vega [12] for the (gKdV) equation.

2. The instability part of the theorem has been proved by Bona *et al.* [3] (see also [10]) for $p > 5$ and by Martel and Merle [15] for $p = 5$. Nevertheless, we give an explicit proof of the instability of Q when $p > 5$ (*i.e.* we exhibit an explicit sequence of initial data which contradicts the stability) which will be useful to construct the special solution by the compactness method (Section 3).

3. An important ingredient to prove this theorem is the following lemma of modulation close to Q. Its proof is based on the implicit function theorem (see for example [3, Lemma 4.1] for details). The orthogonality to Q' obtained by this lemma will be of course useful to exploit the coercivity of the bilinear form $(L\cdot,\cdot)$. Finally, we conclude this section by a simple but useful lemma which describes the effect of small translations on Q.

Lemma 2.10 (Modulation close to Q). *There exist $\varepsilon_0 > 0$, $C > 0$ and a unique C^1 map $\alpha : U_{\varepsilon_0} \longrightarrow \mathbb{R}$ such that, for every $u \in U_{\varepsilon_0}$, $\varepsilon = u(\cdot + \alpha(u)) - Q$ satisfies*

$$(\varepsilon, Q') = 0 \text{ and } \|\varepsilon\|_{H^1} \leqslant C \inf_{y \in \mathbb{R}} \|u - Q(\cdot - y)\|_{H^1} \leqslant C\varepsilon_0.$$

Lemma 2.11. *There exist $h_0 > 0$, $A_0 > 0$ and $\beta > 0$ such that:*

(i) if $|h| \leqslant h_0$ then $\beta h^2 \leqslant \|Q - Q(\cdot + h)\|_{H^1}^2 \leqslant 4\beta h^2$,

(ii) if $|h| > h_0$ then $\|Q - Q(\cdot + h)\|_{H^1}^2 > A_0$.

Proof. It is a simple application of Taylor's theorem to f defined by $f(a) = \|Q - Q(\cdot + a)\|_{H^1}^2$. □

2.3 Instability of Q for $p > 5$

In this section, we construct an explicit sequence $(u_{0,n})_{n \geqslant 1}$ of initial data which contradicts the stability of Q.

Proposition 2.12. *Let* $u_{0,n}(x) = \lambda_n Q(\lambda_n^2 x)$ *with* $\lambda_n = 1 + \frac{1}{n}$ *for* $n \geqslant 1$. *Then*

$$\int u_{0,n}^2 = \int Q^2 \quad , \quad E(u_{0,n}) < E(Q) \quad \text{and} \quad \|u_{0,n} - Q\|_{H^1} \xrightarrow[n\to\infty]{} 0. \qquad (2.3)$$

Proof. The first and the last facts are obvious thanks to substitutions and the dominated convergence theorem. For the energy inequality, we compute $E(u_{0,n}) = \frac{\lambda_n^4}{2}\int Q'^2 - \frac{\lambda_n^{p-1}}{p+1}\int Q^{p+1}$. But $2\int Q'^2 = \frac{p-1}{p+1}\int Q^{p+1}$ by Pohozaev identities, and so

$$\begin{aligned} E(u_{0,n}) - E(Q) &= \left[\frac{p-1}{4}(\lambda_n^4 - 1) - (\lambda_n^{p-1} - 1)\right] \cdot \frac{1}{p+1}\int Q^{p+1} \\ &= \left[\sum_{k=2}^{4}\left\{\frac{p-1}{4}\binom{4}{k} - \binom{p-1}{k}\right\}\frac{1}{n^k} - \sum_{k=5}^{p-1}\binom{p-1}{k}\frac{1}{n^k}\right] \cdot \frac{1}{p+1}\int Q^{p+1}. \end{aligned}$$

To conclude, it is enough to show that $\binom{p-1}{k} > \frac{p-1}{4}\binom{4}{k}$ for $k \in \{2,3,4\}$, which is equivalent to show that $\binom{p-2}{k-1} = \frac{k}{p-1}\binom{p-1}{k} > \frac{k}{4}\binom{4}{k} = \binom{3}{k-1}$, which is right since $p > 5$ and $k > 1$. □

Remark 2.13. We do not really need to know the explicit expression of $u_{0,n}$ to prove the instability of Q: initial data satisfying conditions (2.3) and decay in space would fit. For example, we could have chosen $\lambda_n = 1 - \frac{1}{n}$, so that conditions (2.3) hold for n large (in fact $E(u_{0,n}) - E(Q) \sim \frac{(p-1)(5-p)}{2(p+1)}\int Q^{p+1} \cdot \frac{1}{n^2} < 0$ as $n \to +\infty$ in this case).

Theorem 2.14. *Let* u_n *be the solution associated to* $u_{0,n}$ *defined in Proposition 2.12. Then*

$$\exists \delta > 0, \forall n \geqslant 1, \exists T_n \in \mathbb{R}_+ \text{ such that } \inf_{y\in\mathbb{R}} \|u_n(T_n) - Q(\cdot - y)\|_{H^1} > \delta. \qquad (2.4)$$

• We prove this theorem by contradiction, *i.e.* we suppose:

$$\forall \varepsilon > 0, \exists n_0 \geqslant 1, \forall t \in \mathbb{R}_+, \inf_{y\in\mathbb{R}} \|u_{n_0}(t) - Q(\cdot - y)\|_{H^1} \leqslant \varepsilon,$$

and we apply this assumption to ε_0 given by Lemma 2.10. Dropping n_0 for a while, the situation amounts in

$$\int u_0^2 = \int Q^2 \ , \ E(u_0) < E(Q) \ \text{ and } \ \forall t \in \mathbb{R}_+, \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \leqslant \varepsilon_0.$$

The last fact implies that $u(t) \in U_{\varepsilon_0}$ for all $t \in \mathbb{R}_+$, so Lemma 2.10 applies and we can define $x(t) = \alpha(u(t))$ which is C^1 by standard arguments (see [15] for example), and $\varepsilon(t,x) = u(t, x + x(t)) - Q(x)$ which satisfies $(\varepsilon(t), Q') = 0$ and $\|\varepsilon(t)\|_{H^1} \leqslant C\varepsilon_0$ for all $t \in \mathbb{R}_+$. Note that $x(t)$ is usually called the *center of mass* of $u(t)$. Before continuing the proof, we give the equation satisfied by ε and an interesting consequence on x'.

Proposition 2.15. *There exists* $C > 0$ *such that*

$$\varepsilon_t - (L\varepsilon)_x = (x'(t) - 1)(Q + \varepsilon)_x + R(\varepsilon),$$

where $\|R(\varepsilon(t))\|_{L^1} \leqslant C\|\varepsilon(t)\|_{H^1}^2$. *As a consequence, one has* $|x'(t) - 1| \leqslant C\|\varepsilon(t)\|_{H^1}$.

Proof. Since $u(t,x) = Q(x - x(t)) + \varepsilon(t, x - x(t))$ by definition of ε and $-\partial_t u = \partial_x^3 u + \partial_x(u^p)$, we obtain

$$x'(t)(Q+\varepsilon)_x - \varepsilon_t = Q_{xxx} + \varepsilon_{xxx} + (Q^p)_x + p(Q^{p-1}\varepsilon)_x + R(\varepsilon)$$

where

$$R(\varepsilon) = \frac{\partial}{\partial x}\left(\sum_{k=2}^{p}\binom{p}{k}Q^{p-k}\varepsilon^k\right) = \sum_{k=2}^{p}\binom{p}{k}\left[(p-k)Q'Q^{p-k-1}\varepsilon^k + kQ^{p-k}\varepsilon_x\varepsilon^{k-1}\right].$$

As $\|\varepsilon\|_{L^\infty} \leqslant C\|\varepsilon\|_{H^1} \leqslant C\varepsilon_0$, we have $|R(\varepsilon)| \leqslant C|\varepsilon|^2 + C'|\varepsilon_x\varepsilon|$, and so $R(\varepsilon)$ is such as expected. Moreover, since $La = -a_{xx} + a - pQ^{p-1}a$ and $Q'' + Q^p = Q$, we get

$$-\varepsilon_t - \varepsilon_{xxx} - p(Q^{p-1}\varepsilon)_x = Q_{xxx} + (Q^p)_x - x'(t)(Q+\varepsilon)_x + R(\varepsilon)$$

and so $-\varepsilon_t + (L\varepsilon)_x = Q_x - x'(t)(Q+\varepsilon)_x + \varepsilon_x + R(\varepsilon)$.

To obtain the estimate on x', we multiply the equation previously found by Q' and integrate. Since $(\varepsilon_t, Q') = (\varepsilon, Q')_t = 0$, it gives, with an integration by parts,

$$\int (L\varepsilon)Q'' = (x'-1)\int (Q'^2 + \varepsilon_x Q') + \int R(\varepsilon)Q'.$$

Since L is self-adjoint, we can write $(x'-1)\int(Q'^2 + \varepsilon_x Q') = \int(LQ'')\varepsilon - \int R(\varepsilon)Q'$. Now, from $|\int \varepsilon_x Q'| \leqslant \|\varepsilon_x\|_{L^2}\|Q'\|_{L^2} \leqslant \|\varepsilon\|_{H^1}\|Q'\|_{L^2} \leqslant C\varepsilon_0\|Q'\|_{L^2}$, we choose ε_0 small enough so that the last quantity is smaller than $\frac{1}{2}\int Q'^2$; and so we have

$$|x'-1| \leqslant \frac{2}{\int Q'^2}\left(\left|\int (LQ'')\varepsilon\right| + \left|\int R(\varepsilon)Q'\right|\right).$$

As $LQ'' \in L^2(\mathbb{R})$ and $Q' \in L^\infty(\mathbb{R})$, then following the estimate on $R(\varepsilon)$, we obtain the desired inequality by the Cauchy-Schwarz inequality. □

• Return to the proof of Theorem 2.14 and now consider

$$\zeta(x) = \int_{-\infty}^{x}\left(S(y) + \beta Q^{\frac{p+1}{2}}(y)\right)dy$$

for $x \in \mathbb{R}$, where S is defined in Claim 2.5 and β will be chosen later. We recall that $S(x) = \frac{1}{p-1}Q(x) + \frac{1}{2}xQ'(x)$ satisfies $LS = -Q$, and in particular $S(x) = o(e^{-|x|/2})$ when $|x| \to +\infty$, since $Q(x), Q'(x) \sim Ce^{-|x|}$ (see Claim 2.1). By integration, we have $\zeta(x) = o(e^{x/2})$ when $x \to -\infty$, and ζ is bounded on $\mathbb{R}$.

Now, the main idea of the proof is to consider the functional, defined for $t \in \mathbb{R}_+$,

$$J(t) = \int \varepsilon(t,x)\zeta(x)\,dx.$$

The first step is to show that J is defined and bounded in time thanks to the following proposition of decay properties of the solutions, and the second one is to show that $|J'|$ has a strictly positive lower bound, which will reach the desired contradiction. Firstly, choosing ε_0 small enough, we obtain the following proposition.

Proposition 2.16. *There exists $C > 0$ such that, for all $t \geqslant 0$ and all $x_0 > 0$,*

$$\int_{x>x_0} (u^2 + u_x^2)(t, x + x(t))\, dx \leqslant Ce^{-x_0/4}. \tag{2.5}$$

Remark 2.17. Inequality (2.5) holds for all solution u_n of (gKdV) associated to the initial data $u_{0,n}$ defined in Proposition 2.12, with $C > 0$ independent of n. Indeed, we have $u = u_{n_0}$ for some $n_0 \geqslant 1$, but the following proof shows that the final constant C does not depend of n_0.

Proof. It is based on the exponential decay of the initial data, and on monotonicity results that the reader can find in [17, Lemma 3]. We recall here their notation and their lemma of monotonicity.

◇ Let $\psi(x) = \frac{2}{\pi} \arctan(\exp(x/4))$, so that ψ is increasing, $\lim_{-\infty} \psi = 0$, $\psi(0) = \frac{1}{2}$, $\lim_{+\infty} \psi = 1$, $\psi(-x) = 1 - \psi(x)$ for all $x \in \mathbb{R}$, and $\psi(x) \sim Ce^{x/4}$ when $x \to -\infty$. Now let $x_0 > 0$, $t_0 > 0$, and define $\psi_0(t, x) = \psi(x - x(t_0) + \frac{1}{2}(t_0 - t) - x_0)$ for $0 \leqslant t \leqslant t_0$, and

$$\begin{cases} I_{x_0,t_0}(t) = \int u^2(t, x)\psi_0(t, x)\, dx, \\ J_{x_0,t_0}(t) = \int \left(u_x^2 + u^2 - \frac{2}{p+1} u^{p+1}\right)(t, x)\psi_0(t, x)\, dx. \end{cases}$$

Then, if we choose ε_0 small enough, there exists $K > 0$ such that, for all $t \in [0, t_0]$, we have

$$\begin{cases} I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \leqslant K \exp\left(-\frac{x_0}{4}\right), \\ J_{x_0,t_0}(t_0) - J_{x_0,t_0}(t) \leqslant K \exp\left(-\frac{x_0}{4}\right). \end{cases}$$

◇ Now, let us prove how this result can preserve the decay of the initial data to the solution for all time, *on the right* (which means for $x > x_0$ for all $x_0 > 0$). If we apply it to $t = 0$ and replace t_0 by t, we obtain, for all $t > 0$,

$$\int (u_x^2 + u^2)(t, x + x(t))\psi(x - x_0)\, dx \\ \leqslant C' \int (u_{0x}^2 + u_0^2)(x)\psi(x - x(t) + \frac{1}{2}t - x_0)\, dx + K'e^{-x_0/4}.$$

But by Proposition 2.15, we have $|x' - 1| \leqslant C\|\varepsilon\|_{H^1} \leqslant C\varepsilon_0$, thus if we choose ε_0 small enough, we have $|x' - 1| \leqslant \frac{1}{2}$, and so we obtain by the mean value inequality (notice that $x(0) = \alpha(u_{0,n_0}) = 0$) that $|x(t) - t| \leqslant \frac{1}{2}t$. We deduce that $-x(t) + \frac{1}{2}t \leqslant 0$, and since ψ is increasing, we obtain

$$\int (u_x^2 + u^2)(t, x + x(t))\psi(x - x_0)\, dx \leqslant C \int (u_{0x}^2 + u_0^2)(x)\psi(x - x_0)\, dx + Ke^{-x_0/4}.$$

◇ Now we explicit exponential decay of u_0. In fact, we have clearly $(u_{0x}^2 + u_0^2)(x) \sim Ce^{-2\lambda^2|x|} \leqslant Ce^{-2|x|}$ when $x \to \pm\infty$. Moreover, since $\psi(x) \leqslant Ce^{x/4}$ for all $x \in \mathbb{R}$, we have

$$\int (u_{0x}^2 + u_0^2)(x)\psi(x - x_0)\, dx \leqslant C \int (u_{0x}^2 + u_0^2)(x)e^{\frac{x-x_0}{4}}\, dx \\ \leqslant Ce^{-x_0/4} \int (u_{0x}^2 + u_0^2)(x)e^{x/4}\, dx \leqslant C'e^{-x_0/4}.$$

◇ Finally, we have more simply

$$\int (u_x^2 + u^2)(t, x + x(t))\psi(x - x_0)\,dx \geqslant \frac{1}{2}\int_{x>x_0} (u_x^2 + u^2)(t, x + x(t))\,dx,$$

and so the desired inequality. □

• Now that this proposition is proved, we can easily show the first step of the proof of Theorem 2.14.

1st step. We bound $|J(t)|$ independently of time by writing

$$J(t) = \int \varepsilon(t, x)\zeta(x)\,dx = \int_{x>0} \varepsilon(t, x)\zeta(x)\,dx + \int_{x<0} \varepsilon(t, x)\zeta(x)\,dx,$$

so that

$$\begin{aligned}|J(t)| &\leqslant \|\zeta\|_{L^\infty} \int_{x>0} (Q(x) + |u(t, x + x(t))|)\,dx + \sqrt{\int_{x<0} \varepsilon^2(t, x)\,dx}\sqrt{\int_{x<0} \zeta^2(x)\,dx} \\ &\leqslant \|\zeta\|_{L^\infty}\|Q\|_{L^1} + \|\zeta\|_{L^\infty} U + \|\varepsilon(t)\|_{L^2} V,\end{aligned}$$

where:

i) $\|\varepsilon(t)\|_{L^2} \leqslant \|\varepsilon\|_{H^1} \leqslant C\varepsilon_0 < +\infty$,

ii) $V^2 = \int_{x<0} \zeta^2(x)\,dx < +\infty$ since $\zeta^2(x) = o(e^x)$ when $x \to -\infty$,

iii) thanks to (2.5), we finally conclude the first step with

$$\begin{aligned}U = \int_{x>0} |u(t, x + x(t))|\,dx &= \sum_{n=0}^{+\infty} \int_n^{n+1} |u(t, x + x(t))|\,dx \\ &\leqslant \sum_{n=0}^{+\infty} \left(\int_{x>n} u^2(t, x + x(t))\,dx\right)^{1/2} \\ &\leqslant \|u(t, \cdot + x(t))\|_{L^2} + \sum_{n=1}^{+\infty} \left(\int_{x>n} u^2(t, x + x(t))\,dx\right)^{1/2} \\ &\leqslant C\varepsilon_0 + \|Q\|_{L^2} + C\sum_{n=1}^{+\infty} e^{-n/8} < +\infty.\end{aligned}$$

2nd step. We evaluate J' by using Proposition 2.15 and by integrating by parts:

$$\begin{aligned}J' &= \int \varepsilon_t\zeta = \int (L\varepsilon)_x\zeta + (x' - 1)\int Q_x\zeta + (x' - 1)\int \varepsilon_x\zeta + \int R(\varepsilon)\zeta \\ &= -\int \varepsilon L(\zeta') - (x' - 1)\int Q\zeta' - (x' - 1)\int \varepsilon\zeta' + \int R(\varepsilon)\zeta \\ &= -\int \varepsilon(LS + \beta LQ^{\frac{p+1}{2}}) - (x' - 1)\int Q(S + \beta Q^{\frac{p+1}{2}}) - (x' - 1)\int \varepsilon\zeta' + \int R(\varepsilon)\zeta.\end{aligned}$$

Now we take $\beta = -\left(\int QS\right)\left(\int Q^{\frac{p+3}{2}}\right)^{-1}$, so that the second integral is null. But, by (iv) of Claim 2.1,

$$\frac{d}{dc}\int Q_c^2 = 2\int Q_c \frac{dQ_c}{dc} = \left(\frac{5 - p}{2(p - 1)}\right) c^{\frac{5-p}{2(p-1)} - 1} \int Q^2 < 0$$

since $p > 5$, and so by taking $c = 1$, we remark that $\beta > 0$.

Moreover, since $Q^{\frac{p+1}{2}}$ is an eigenvector for L for an eigenvalue $-\lambda_0$ with $\lambda_0 > 0$ (see Claim 2.5), we deduce

$$\begin{aligned} J' &= -\int \varepsilon(-Q - \beta\lambda_0 Q^{\frac{p+1}{2}}) - (x'-1)\int \varepsilon\zeta' + \int R(\varepsilon)\zeta \\ &= \beta\lambda_0 \int \varepsilon Q^{\frac{p+1}{2}} + \int Q\varepsilon - (x'-1)\int \varepsilon\zeta' + \int R(\varepsilon)\zeta. \end{aligned}$$

But for the last three terms, we remark that:

a) the mass conservation $\int u^2(t) = \int u_0^2$ implies that $\int Q^2 + 2\int \varepsilon Q + \int \varepsilon^2 = \int Q^2$ and so $|\int Q\varepsilon| \leqslant \frac{1}{2}\int \varepsilon^2 \leqslant \frac{1}{2}\|\varepsilon\|^2_{H^1}$,

b) thanks to Proposition 2.15, we have $|-(x'-1)\int \varepsilon\zeta'| \leqslant |x'-1|\|\varepsilon\|_{L^2}\|\zeta'\|_{L^2} \leqslant C\|\varepsilon\|^2_{H^1}$,

c) still thanks to this proposition, we have $|\int R(\varepsilon)\zeta| \leqslant \|\zeta\|_{L^\infty}\|R(\varepsilon)\|_{L^1} \leqslant C\|\varepsilon\|^2_{H^1}$.

We have finally

$$J' = \beta\lambda_0 \int \varepsilon Q^{\frac{p+1}{2}} + K(\varepsilon) \tag{2.6}$$

where $K(\varepsilon)$ satisfies $|K(\varepsilon)| \leqslant C\|\varepsilon\|^2_{H^1}$. We now use identity (2.2) which claims

$$F(u(t)) = F(u_0) = F(Q) + \frac{1}{2}(L\varepsilon, \varepsilon) + K'(\varepsilon)$$

with $|K'(\varepsilon)| \leqslant C\|\varepsilon\|^3_{H^1}$. In other words, we have $(L\varepsilon,\varepsilon) + 2K'(\varepsilon) = 2[F(u_0) - F(Q)] = 2[F(u_{0,n_0}) - F(Q)] = -\gamma_{n_0}$ with $\gamma_{n_0} > 0$, since $\|u_{0,n_0}\|_{L^2} = \|Q\|_{L^2}$ and $E(u_{0,n_0}) < E(Q)$ by construction of u_{0,n_0}. To estimate the term $(L\varepsilon,\varepsilon)$, we use Lemma 2.6, so that if we denote $a(t) = \int \varepsilon Q^{\frac{p+1}{2}}$, we obtain

$$a^2(t) \geqslant \frac{K_1}{K_2}\|\varepsilon\|^2_{H^1} - \frac{1}{K_2}(L\varepsilon,\varepsilon) = \frac{\gamma_{n_0}}{K_2} + \frac{K_1}{K_2}\|\varepsilon\|^2_{H^1} + \frac{2}{K_2}K'(\varepsilon).$$

But $|K'(\varepsilon)| \leqslant C\|\varepsilon\|^3_{H^1}$ and $\|\varepsilon\|_{H^1} \leqslant C\varepsilon_0$, so if we take ε_0 small enough, we have

$$a^2(t) \geqslant K\|\varepsilon\|^2_{H^1} + \kappa_{n_0}$$

with $K, \kappa_{n_0} > 0$. In particular, $a^2(t) \geqslant \kappa_{n_0} > 0$, thus a keeps a constant sign, say positive. Then we have

$$a(t) \geqslant \sqrt{K\|\varepsilon\|^2_{H^1} + \kappa_{n_0}} \geqslant \sqrt{\frac{K}{2}}\|\varepsilon\|_{H^1} + \sqrt{\frac{\kappa_{n_0}}{2}} = K'\|\varepsilon\|_{H^1} + \kappa'_{n_0}.$$

But from (2.6), we also have $J'(t) = \beta\lambda_0 a(t) + K(\varepsilon)$ with $|K(\varepsilon)| \leqslant C\|\varepsilon\|^2_{H^1}$, and so

$$J'(t) \geqslant \beta\lambda_0 K'\|\varepsilon\|_{H^1} + \beta\lambda_0\kappa'_{n_0} - C\|\varepsilon\|^2_{H^1} \geqslant \beta\lambda_0\kappa'_{n_0} = \theta_{n_0} > 0,$$

if we choose as previously ε_0 small enough. But it implies that $J(t) \geqslant \theta_{n_0}t + J(0) \longrightarrow +\infty$ as $t \to +\infty$, which contradicts the first step and concludes the proof of the theorem. Note that if $a(t) < 0$, it is easy to show by the same arguments that $J'(t) \leqslant \theta'_{n_0} < 0$, so $\lim_{t\to+\infty} J(t) = -\infty$ and then the same conclusion.

3 Construction of a special solution by compactness

In this section, we prove the existence of a special solution by a compactness method. This result is of course weaker than Theorem 1.1, but it does not require the existence of $\mathcal{Y}_\pm$ proved in [20].

3.1 Construction of the initial data

Now Theorem 2.14 is proved, we can change T_n obtained in (2.4) in the *first* time which realizes this. In other words:

$$\exists\delta > 0, \forall n \geqslant 1, \exists T_n \in \mathbb{R}_+ \text{ such that } \begin{cases} \inf_{y\in\mathbb{R}} \|u_n(T_n) - Q(\cdot - y)\|_{H^1} = \delta, \\ \forall t \in [0, T_n],\ \inf_{y\in\mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leqslant \delta. \end{cases}$$

Remark 3.1. We have $T_n \longrightarrow +\infty$. Indeed, we would have $T_n < T_0$ for all n otherwise (after passing to a subsequence). But by Lipschitz continuous dependence on the initial data (see [12, Corollary 2.18]), we would have for n large enough

$$\sup_{t\in[0,T_0]} \|u_n(t) - Q(\cdot - t)\|_{H^1} \leqslant K\|u_{0,n} - Q\|_{H^1}.$$

But since $\|u_{0,n} - Q\|_{H^1} \xrightarrow[n\to\infty]{} 0$ by (2.3), we would have $\inf_{y\in\mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leqslant \frac{\delta}{2}$ for n large enough and for all $t \in [0, T_0]$, which is wrong for $t = T_n \in [0, T_0]$.

Now we can take δ smaller than ε_0, so that $u_n(t) \in U_{\varepsilon_0}$ for all $t \in [0, T_n]$ and so Lemma 2.10 applies. Thus, we can define $x_n(t) = \alpha(u_n(t))$ (notice that $x_n(0) = \alpha(u_{0,n}) = 0$) such that $\varepsilon_n(t) = u_n(t, \cdot + x_n(t)) - Q$ satisfies

$$\forall t \in [0, T_n], \begin{cases} (\varepsilon_n(t), Q') = 0, \\ \|\varepsilon_n(t)\|_{H^1} \leqslant C \inf_{y\in\mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leqslant C\delta. \end{cases}$$

Moreover, for $t = T_n$, we have more precisely

$$\delta \leqslant \|\varepsilon_n(T_n)\|_{H^1} \leqslant C\delta. \tag{3.1}$$

In particular, $\{\varepsilon_n(T_n)\}$ is bounded in H^1, and so by passing to a subsequence, we can define ε_∞ such that

$$\varepsilon_n(T_n) \rightharpoonup \varepsilon_\infty \text{ in } H^1 \text{ (weakly) and } v_0 = \varepsilon_\infty + Q.$$

Remark 3.2.

1. As announced in the introduction, one of the most important points in this section is to prove that we have constructed a non trivial object, *i.e.* v_0 is not a soliton (Proposition 3.4). This fact is quite natural since v_0 is the weak limit of $u_n(T_n, \cdot + x_n(T_n))$ which contains a persisting defect $\varepsilon_n(T_n)$.
2. Since the proof of Proposition 3.4 is mainly based on evaluating L^2 norms, the following lemma will be useful.

Lemma 3.3. *There exists $C_0 > 0$ such that, for n large enough, $\|\varepsilon_n(T_n)\|_{L^2} \geqslant C_0\delta$.*

Proof. It comes from the conservation of the Weinstein's functional F in time. In fact, we can write $F(Q+\varepsilon_n(T_n)) = F(Q+\varepsilon_n(0))$ where $\varepsilon_n(0) = u_{0,n} - Q$ satisfies $\|\varepsilon_n(0)\|_{H^1} \xrightarrow[n\to\infty]{} 0$ by (2.3). Then, by (2.2),

$$F(Q) + \frac{1}{2}(L\varepsilon_n(T_n), \varepsilon_n(T_n)) + K(\varepsilon_n(T_n)) = F(Q) + \frac{1}{2}(L\varepsilon_n(0), \varepsilon_n(0)) + K(\varepsilon_n(0)),$$

where $|K(a)| \leqslant C_1\|a\|_{H^1}^3$. It comes

$$\int \left[(\partial_x\varepsilon_n(T_n))^2 + \varepsilon_n^2(T_n) - pQ^{p-1}\varepsilon_n^2(T_n)\right] \leqslant C\|\varepsilon_n(0)\|_{H^1}^2 + K(\varepsilon_n(0)) - K(\varepsilon_n(T_n))$$

and so

$$\|\varepsilon_n(T_n)\|_{H^1}^2 \leqslant C\int \varepsilon_n^2(T_n) + C\|\varepsilon_n(0)\|_{H^1}^2 + C_1\|\varepsilon_n(0)\|_{H^1}^3 + C_1\|\varepsilon_n(T_n)\|_{H^1}^3.$$

Since $\|\varepsilon_n(0)\|_{H^1} \longrightarrow 0$, then by (3.1) we have, for n large enough,

$$\|\varepsilon_n(T_n)\|_{H^1}^2 \leqslant C\int \varepsilon_n^2(T_n) + C_1C\delta\|\varepsilon_n(T_n)\|_{H^1}^2 + \frac{\delta^2}{4}.$$

But if we choose δ small enough so that $C_1C\delta \leqslant \frac{1}{2}$, we obtain

$$\frac{\delta^2}{2} \leqslant \frac{1}{2}\|\varepsilon_n(T_n)\|_{H^1}^2 \leqslant C\int \varepsilon_n^2(T_n) + \frac{\delta^2}{4}$$

and finally $\int \varepsilon_n^2(T_n) \geqslant \frac{\delta^2}{4C}$. □

Proposition 3.4. *For all $c > 0$, $v_0 \neq Q_c$.*

Proof. We proceed by contradiction. In other words, we suppose that $v_n := u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup v_0 = \varepsilon_\infty + Q = Q_c$ weakly in H^1 for some $c > 0$. We recall that it implies in particular that $v_n \longrightarrow Q_c$ strongly in L^2 on compacts as $n \to +\infty$.

- *Decomposition of v_n.* Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ equals to 0 on $(-\infty, -1]$ and 1 on $[0, +\infty)$. Now let $A \gg 1$ to fix later and define $\varphi_A(x) = \varphi(x + A)$, so that $\varphi_A(x) = 0$ if $x \leqslant -A-1$ and 1 if $x \geqslant -A$. We also define $h_n = (1-\varphi_A)v_n$, $Q_c^A = Q_c\varphi_A$ and $z_n = \varphi_A v_n - \varphi_A Q_c = \varphi_A(v_n - Q_c)$, so that

$$v_n = (1-\varphi_A)v_n + \varphi_A v_n = h_n + z_n + Q_c^A.$$

- *Estimate of $\|z_n\|_{L^2}$.*

$$\begin{aligned}\int z_n^2 = \int (v_n - Q_c)^2\varphi_A^2 &\leqslant \int_{-A-1}^{A+1} (v_n - Q_c)^2 + \int_{x>A+1} (v_n - Q_c)^2 \\ &\leqslant \int_{-A-1}^{A+1} (v_n - Q_c)^2 + 2\int_{x>A+1} v_n^2 + 2\int_{x>A+1} Q_c^2 = I + J + K.\end{aligned}$$

Notice that $I \xrightarrow[n\to\infty]{} 0$ since $v_n \xrightarrow[n\to\infty]{} Q_c$ in L^2 on compacts. Moreover, thanks to the exponential decay of Q_c, we have $K \leqslant Ce^{-2\sqrt{c}A}$. Finally, we have $J \leqslant Ce^{-A/4}$ with C independent of n by Remark 2.17. In summary, there exists $\rho > 0$ such that $\int z_n^2 \leqslant Ce^{-\rho A}$ if $n \geqslant n(A)$.

- *Mass balance.* On one hand, we have by (2.3) and by the mass conservation $\int v_n^2 = \int u_{0,n}^2 = \int Q^2$. On the other hand, we can develop

$$\int v_n^2 = \int h_n^2 + \int (Q_c^A + z_n)^2 + 2\int_{-A-1}^{-A} v_n^2 \varphi_A (1-\varphi_A).$$

But note that $2\int_{-A-1}^{-A} v_n^2\varphi_A(1-\varphi_A) \underset{n\to\infty}{\longrightarrow} 2\int_{-A-1}^{-A} Q_c^2\varphi_A(1-\varphi_A) \leqslant Ce^{-\rho A}$, since $v_n \longrightarrow Q_c$ on compacts. Consequently,

$$\int Q^2 = \int h_n^2 + \int (Q_c^A)^2 + 2\int Q_c^A z_n + \int z_n^2 + a_n^A$$

where $a_n^A \geqslant 0$ satisfies $a_n^A \leqslant Ce^{-\rho A}$ for $n \geqslant n(A)$. Thanks to the previous estimate of $\|z_n\|_{L^2}$ and the Cauchy-Schwarz inequality, we deduce that

$$\int Q^2 = \int h_n^2 + \int (Q_c^A)^2 + a_n'^A$$

where $a_n'^A$ satisfies $|a_n'^A| \leqslant Ce^{-\rho A}$ for $n \geqslant n(A)$. But

$$\int (Q_c^A)^2 = \int Q_c^2\varphi_A^2 = \int Q_c^2 + \int Q_c^2(\varphi_A^2 - 1) \leqslant \int Q_c^2 + \int_{x<-A} Q_c^2 \leqslant \int Q_c^2 + Ce^{-\rho A}$$

and $\int Q_c^2 = c^{-\beta}\int Q^2$ with $\beta > 0$ since $p > 5$ (see Claim 2.1). In conclusion, we have the mass balance

$$(1 - c^{-\beta})\|Q\|_{L^2}^2 = \|h_n\|_{L^2}^2 + a_n''^A \tag{3.2}$$

where $a_n''^A$ still satisfies $|a_n''^A| \leqslant Ce^{-\rho A}$ for $n \geqslant n(A)$.

- *Upper bound of* $\|h_n\|_{L^2}$. We remark that, for $n \geqslant n(A)$, $\|h_n\|_{L^2} \leqslant C_1\delta$. Indeed, thanks to (3.1), we have

$$\|h_n\|_{L^2} \leqslant \|(1-\varphi_A)Q\|_{L^2} + \|\varepsilon_n(T_n)\|_{L^2} \leqslant Ce^{-\rho A} + C\delta \leqslant C_1\delta,$$

if we permanently fix A large enough so that $e^{-\rho A} \leqslant \delta^3$ (the power 3 will be useful later in the proof).

- *Upper bound of* $|c-1|$. Thanks to the previous point and the mass balance (3.2), we have $|1 - c^{-\beta}| \leqslant C\delta^2$. We deduce that c is close to 1, and so by Taylor's theorem that $|c-1| \leqslant K|1-c^{-\beta}| \leqslant C\delta^2$.

- *Lower bound of* $\|h_n\|_{L^2}$. We now prove that, for $n \geqslant n(A)$, $\|h_n\|_{L^2} \geqslant C_2\delta$. Firstly, we have, by Lemma 3.3,

$$\begin{aligned} C_0\delta &\leqslant \|\varepsilon_n(T_n)\|_{L^2} = \|v_n - Q\|_{L^2} = \|h_n + Q_c^A + z_n - Q\|_{L^2} \\ &\leqslant \|h_n\|_{L^2} + \|z_n\|_{L^2} + \|Q_c^A - Q_c\|_{L^2} + \|Q_c - Q\|_{L^2} = \|h_n\|_{L^2} + \|Q_c - Q\|_{L^2} + b_n^A, \end{aligned}$$

where $b_n^A = \|z_n\|_{L^2} + \|Q_c^A - Q_c\|_{L^2} \geqslant 0$ satisfies $b_n^A \leqslant Ce^{-\rho A}$ for $n \geqslant n(A)$. Moreover, if we denote $f(c) = \|Q_c - Q\|_{L^2}^2$ for $c > 0$, then f is C^∞ and $f(c) \geqslant 0 = f(1)$, hence 1 is a minimum of f, $f'(1) = 0$ and so, by Taylor's theorem, $f(c) \leqslant C(c-1)^2$, *i.e.* $\|Q_c - Q\|_{L^2} \leqslant C|c-1|$. Thanks to the previous point, we deduce that

$$C_0\delta \leqslant \|h_n\|_{L^2} + K\delta^2 + b_n^A \leqslant \|h_n\|_{L^2} + C\delta^2.$$

Finally, if we choose δ small enough so that $C\delta \leqslant \frac{C_0}{2}$, we reach the desired inequality.

- *Energy balance.* We now use the conservation of Weinstein's functional and (2.2) to write

$$F(u_0) = F(v_n) = F(Q + \varepsilon_n(T_n)) = F(Q) + \frac{1}{2}(L\varepsilon_n(T_n), \varepsilon_n(T_n)) + K(\varepsilon_n(T_n))$$

where $|K(\varepsilon_n(T_n))| \leqslant C\|\varepsilon_n(T_n)\|_{H^1}^3 \leqslant C\delta^3$ by (3.1). Now we decompose $\varepsilon_n(T_n)$ in

$$\varepsilon_n(T_n) = v_n - Q = h_n + z_n + Q_c^A - Q = (Q_c - Q) + (Q_c^A - Q_c) + (z_n + h_n)$$

in order to expand

$$\begin{aligned}(L\varepsilon_n(T_n), \varepsilon_n(T_n)) = &\,(L(Q_c - Q), Q_c - Q) + (L(z_n + h_n), z_n + h_n) \\ &+ (L(Q_c^A - Q_c), Q_c^A - Q_c) + 2(L(Q_c - Q), z_n + h_n) \\ &+ 2(L(Q_c - Q), Q_c^A - Q_c) + 2(L(Q_c^A - Q_c), z_n + h_n).\end{aligned}$$

We recall that $(La, b) = -\int a''b + \int ab - p\int Q^{p-1}ab$, and so, by the Cauchy-Schwarz inequality, $|(La, b)| \leqslant (\|a''\|_{L^2} + C\|a\|_{L^2})\|b\|_{L^2}$. Since we have $\|z_n + h_n\|_{L^2} \leqslant \|z_n\|_{L^2} + \|h_n\|_{L^2} \leqslant Ce^{-\rho A} + C_1\delta \leqslant C\delta$, we can estimate

$$\begin{aligned}|(L(Q_c - Q), z_n + h_n)| &\leqslant (\|Q_c'' - Q''\|_{L^2} + C\|Q_c - Q\|_{L^2})\|z_n + h_n\|_{L^2} \\ &\leqslant C|c - 1| \cdot C\delta \leqslant C\delta^3.\end{aligned}$$

Similarly, we have

$$\begin{aligned}|(L(Q_c^A - Q_c), z_n + h_n)| &\leqslant (\|\varphi_A''Q_c\|_{L^2} + 2\|\varphi_A'Q_c'\|_{L^2} + \|(\varphi_A - 1)Q_c''\|_{L^2} \\ &\quad + C\|Q_c^A - Q_c\|_{L^2})\|z_n + h_n\|_{L^2} \\ &\leqslant Ce^{-\rho A} \cdot C\delta \leqslant C\delta^3.\end{aligned}$$

Moreover, we have by integrating by parts $(La, b) = \int a'b' + \int ab - p\int Q^{p-1}ab$, and so $|(La, b)| \leqslant C\|a\|_{H^1}\|b\|_{H^1}$. It implies that

$$\begin{cases} |(L(Q_c - Q), Q_c - Q)| \leqslant C\|Q_c - Q\|_{H^1}^2 \leqslant C(c-1)^2 \leqslant C\delta^3, \\ |(L(Q_c^A - Q_c), Q_c^A - Q_c)| \leqslant C\|Q_c^A - Q_c\|_{H^1}^2 \leqslant Ce^{-2\rho A} \leqslant C\delta^3, \\ |(L(Q_c - Q), Q_c^A - Q_c)| \leqslant C\|Q_c - Q\|_{H^1}\|Q_c^A - Q_c\|_{H^1} \leqslant C|c-1|e^{-\rho A} \leqslant C\delta^3, \end{cases}$$

thanks to the estimate on $|c - 1|$ previously found. For the last term, we have

$$(L(h_n + z_n), h_n + z_n) = \|h_n + z_n\|_{H^1}^2 - p\int Q^{p-1}(h_n + z_n)^2$$

and

$$\begin{aligned}\int Q^{p-1}(h_n + z_n)^2 &\leqslant 2\int Q^{p-1}h_n^2 + 2\int Q^{p-1}z_n^2 \\ &\leqslant 2\int (1 - \varphi_A)^2 Q^{p-1}v_n^2 + 2\|Q\|_{L^\infty}^{p-1}\int z_n^2 \\ &\leqslant 2\int_{x<-A} Q^{p-1}v_n^2 + 2\|Q\|_{L^\infty}^{p-1}\int z_n^2.\end{aligned}$$

But $\|v_n\|_{L^\infty} \leqslant C\|v_n\|_{H^1} \leqslant C(\|\varepsilon_n(T_n)\|_{H^1} + \|Q\|_{H^1}) \leqslant C(K\delta + \|Q\|_{H^1}) = K'$, and so $\int_{x<-A} Q^{p-1}v_n^2 \leqslant C\int_{x<-A} Q^{p-1} \leqslant Ce^{-\rho A}$. As $\int z_n^2 \leqslant Ce^{-\rho A}$, we have

$$F(u_0) = F(Q) + \frac{1}{2}\|h_n + z_n\|_{H^1}^2 + d_n^A \geqslant F(Q) + \frac{1}{2}\|h_n + z_n\|_{L^2}^2 + d_n^A$$

where $|d_n^A| \leqslant C\delta^3$ for $n \geqslant n(A)$.

Moreover, we have

$\|h_n + z_n\|_{L^2}^2 - \|h_n\|_{L^2}^2 \leqslant \|z_n\|_{L^2}^2 + 2\|z_n\|_{L^2}\|h_n\|_{L^2} \leqslant Ce^{-2\rho A} + 2Ce^{-\rho A} \cdot C_1\delta \leqslant C\delta^3.$

Finally, energy balance provides us, for some N large enough,

$$F(u_0) \geqslant F(Q) + \frac{1}{2}\|h_N\|_{L^2}^2 + d'$$

with $|d'| \leqslant C\delta^3$.

- *Conclusion.* Since $F(u_0) < F(Q)$ by hypothesis, we obtain $\|h_N\|_{L^2}^2 \leqslant C\delta^3$. But we also have, by the lower bound of $\|h_n\|_{L^2}$, $\|h_N\|_{L^2}^2 \geqslant C_2^2\delta^2$. Gathering both information, we obtain $\frac{C_2^2}{C} \leqslant \delta$, which is clearly a contradiction if we choose δ small enough, and so concludes the proof of Proposition 3.4. □

3.2 Weak continuity of the flow

The main idea to obtain the special solution is to reverse the weak convergence of v_n to v_0 in time and in space, using the fact that $u(t,x)$ is a solution of (gKdV) if and only if $u(-t,-x)$ is also a solution. More precisely, we define $w_0 = \check{v}_0 \in H^1(\mathbb{R})$, *i.e.* for all $x \in \mathbb{R}$, $w_0(x) = v_0(-x)$.

Remark 3.5. For all $c > 0$ and all $x_0 \in \mathbb{R}$, one has

$$w_0 \neq Q_c(\cdot + x_0).$$

In fact, otherwise and since Q_c is even, we would have $v_0(x) = Q_c(x - x_0)$. But $v_n - Q = \varepsilon_n(T_n)$ and $(\varepsilon_n(T_n), Q') = (v_n, Q') = 0$, so by weak convergence in H^1, $(v_0, Q') = 0$. Thus, we would have $\int Q_c(x - x_0)Q'(x)\,dx = 0$, and if we show that $x_0 = 0$, we shall reach the desired contradiction since we have $v_0 \neq Q_c$ for all $c > 0$ by Proposition 3.4. To show this, consider $f(a) = \int Q_c(x-a)Q'(x)\,dx$ for $a \in \mathbb{R}$, which is odd since Q_c is even and Q' odd. In particular, $f(0) = 0$, and it is enough to show that $f(a) < 0$ for $a > 0$ to conclude (because we shall have $f(a) > 0$ for $a < 0$ by parity). But using again the parity of Q_c and Q', we have

$$f(a) = \int_0^a [Q_c(a-x) - Q_c(a+x)]Q'(x)\,dx + \int_a^{+\infty} [Q_c(x-a) - Q_c(x+a)]Q'(x)\,dx.$$

Since Q' is negative and Q_c is strictly decreasing on $\mathbb{R}_+$, both integrals are negative, and so $f(a) < 0$ for $a > 0$, as we desired.

Remark 3.6.

1. Now, w_0 being constructed, we show that the associated solution $w(t)$ is defined for all t positive, and can be seen as a weak limit (Proposition 3.8) in order to prove the convergence of $w(t)$ to a soliton.

2. The main ingredient of the proof of Proposition 3.8 is the following lemma of weak continuity of the flow, whose proof is inspired by [11, Theorem 5]. This proof is long and technical, and thus is not completely written in this paper.

Lemma 3.7. *Suppose that $z_{0,n} \rightharpoonup z_0$ in H^1, and that there exist $T > 0$ and $K > 0$ such that the solution $z_n(t)$ corresponding to initial data $z_{0,n}$ exists for $t \in [0,T]$ and $\sup_{t\in[0,T]} \|z_n(t)\|_{H^1} \leqslant K$. Then, for all $t \in [0,T]$, the solution $z(t)$ such that $z(0) = z_0$ exists, and $z_n(T) \rightharpoonup z(T)$ in H^1.*

Sketch of the proof. Let $T^* = T^*(\|z_0\|_{H^{\frac{3}{4}}}) > 0$ be the maximum time of existence of the solution $z(t)$, well defined by [12, Corollary 2.18] since $s = \frac{3}{4} > \frac{p-5}{2(p-1)} = s_c(p)$. We distinguish two cases, whether $T < T^*$ or not, and we show that this last case is in fact impossible.

1st case. Suppose that $T < T^*$. As $z(t)$ exists for $t \in [0, T]$ by hypothesis, it is enough to show that $z_n(T) \rightharpoonup z(T)$ in H^1. But since C_0^∞ is dense in H^{-1} and $\|z_n(T) - z(T)\|_{H^1} \leqslant \|z_n(T)\|_{H^1} + \|z(T)\|_{H^1} \leqslant K'$, it is enough to show that $z_n(T) \longrightarrow z(T)$ in $\mathcal{D}'(\mathbb{R})$. It is the end of this case, very similar to the proof in [11] (but using an H^3 regularization and so using some arguments like in [14, Section 3.4]), which is technical and not written in this paper consequently.

2nd case. Suppose that $T^* \leqslant T$ and let us show that it implies a contradiction. Indeed, there would exist $T' < T^*$ such that $\|z(T')\|_{H^{\frac{3}{4}}} \geqslant 2K$ (where K is the same constant as in the hypothesis of the lemma). But we can apply the first case with T' instead of T, so that $z_n(T') \rightharpoonup z(T')$ in H^1, and since $\|z_n(T')\|_{H^1} \leqslant K$, we obtain by weak convergence $\|z(T')\|_{H^{\frac{3}{4}}} \leqslant \|z(T')\|_{H^1} \leqslant K$, and so the desired contradiction. □

Proposition 3.8. *The solution $w(t)$ of* (gKdV) *such that $w(0) = w_0$ is defined for all $t \geqslant 0$, and $u_n(T_n - t, x_n(T_n) - \cdot) \rightharpoonup w(t)$ in H^1.*

Proof. As the assumption is clear for $t = 0$, we fix $T > 0$ and we show it for this T. As $\lim_{n\to+\infty} T_n = +\infty$ by Remark 3.1, we have $T_n \geqslant T$ for $n \geqslant n_0$. As a consequence, for $n \geqslant n_0$ and for $t \in [0, T]$, $z_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$ is well defined, solves (gKdV), and has for initial data

$$z_n(0) = u_n(T_n, x_n(T_n) - \cdot) = \check{v}_n \rightharpoonup \check{v}_0 = w_0 \quad \text{in } H^1.$$

Moreover, we have

$$\begin{aligned}\|z_n(t)\|_{H^1} &= \|u_n(T_n - t, x_n(T_n) - \cdot)\|_{H^1} \\ &\leqslant \|\varepsilon_n(T_n - t, x_n(T_n) - x_n(T_n - t) - \cdot)\|_{H^1} + \|Q(x_n(T_n) - x_n(T_n - t) - \cdot)\|_{H^1} \\ &\leqslant \|\varepsilon_n(T_n - t)\|_{H^1} + \|Q\|_{H^1} \leqslant C\delta + \|Q\|_{H^1} = K.\end{aligned}$$

By Lemma 3.7, we deduce that w exists on $[0, T]$, and $z_n(T) \rightharpoonup w(T)$ in H^1. □

3.3 Exponential decay on the left of w

The goal of this section is to prove an exponential decay on the "left" of w, using the exponential decay of u_n on the right. Since $\varepsilon_n(T_n - t) = u_n(T_n - t, \cdot + x_n(T_n - t)) - Q$ satisfies $(\varepsilon_n(T_n - t), Q') = 0$ and $\|\varepsilon_n(T_n - t)\|_{H^1} \leqslant C\delta$ for all $t \in [0, T_n]$, $u_n(T_n - t)$ is in the same situation as the situation of u summed up just before Proposition 2.15, with δ instead of ε_0 for the small parameter. In particular, by Remark 2.17, inequality (2.5) holds for $u_n(T_n - t)$ with C independent of n if we choose δ small enough. In other words, we have, for all $t \geqslant 0$ and $x_0 > 0$ (and n large enough),

$$\int_{x > x_0} (u_{nx}^2 + u_n^2)(T_n - t, x + x_n(T_n - t))\, dx \leqslant C e^{-x_0/4}. \tag{3.3}$$

But before passing to the limit, we have to define the "left" of w, *i.e.* the center of mass $x_w(t)$ of $w(t)$.

Lemma 3.9. *There exists $C > 0$ such that, for all $t \geqslant 0$,*

$$\inf_{y \in \mathbb{R}} \|w(t) - Q(\cdot - y)\|_{H^1} \leqslant C\delta.$$

Proof. Fix $t \geqslant 0$ and $n_0 \geqslant 0$ such that, for $n \geqslant n_0$, $T_n \geqslant t$. Since Q is even, we have

$$\varepsilon_n(T_n - t, x_n(T_n) - x_n(T_n - t) - \cdot) = u_n(T_n - t, x_n(T_n) - \cdot) - Q(\cdot - x_n(T_n) + x_n(T_n - t)).$$

Now, if we denote $w_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$ and $y_n(t) = x_n(T_n) - x_n(T_n - t)$, we have

$$\|w_n(t) - Q(\cdot - y_n(t))\|_{H^1} = \|\varepsilon_n(T_n - t)\|_{H^1} \leqslant C\delta.$$

But following the remark done at the beginning of this section, Proposition 2.15 is still valid, and so $|x_n'(t) - 1| \leqslant C\delta$ for $t \in [0, T_n]$. We deduce that $y_n(t) = \int_{T_n - t}^{T_n} x_n'(s)\,ds = \int_{T_n - t}^{T_n} (x_n'(s) - 1)\,ds + t$ satisfies $|y_n(t)| \leqslant C\delta t + t = Ct$. By passing to a subsequence, we can suppose that $\lim_{n\to\infty} y_n(t) = y(t)$. But now, we can write

$$\|w_n(t) - Q(\cdot - y(t))\|_{H^1} \leqslant C\delta + \|Q - Q(\cdot + (y_n(t) - y(t)))\|_{H^1} \leqslant C'\delta$$

for $n \geqslant N(t, \delta)$ by Lemma 2.11. Finally, since $w_n(t) \rightharpoonup w(t)$ in H^1 by Proposition 3.8, we obtain by weak convergence $\|w(t) - Q(\cdot - y(t))\|_{H^1} \leqslant C'\delta$, and the result follows. □

We can now choose δ small enough so that $C\delta \leqslant \varepsilon_0$, and so define $x_w(t) = \alpha(w(t))$ by Lemma 2.10, with notably $\|w(t, \cdot + x_w(t)) - Q\|_{H^1} \leqslant C\delta$. But to exploit (3.3), we have to show first that $y_n(t) = x_n(T_n) - x_n(T_n - t)$ is close to $x_w(t)$ for all t.

Lemma 3.10. *There exists $C > 0$ such that*

$$\forall t \geqslant 0, \exists n_0 \geqslant 0, \forall n \geqslant n_0,\ |x_w(t) - y_n(t)| \leqslant C\delta.$$

Proof. Let $t \geqslant 0$ and n be large enough such that $T_n \geqslant t$. We keep notation $w_n(t)$ and $y_n(t)$ of the previous proof, where we have already remarked that $|y_n(t)| \leqslant Ct$. For the same reason, we have $|x_w(t) - y_n(t)| \leqslant \Omega t$. Now choose $A(t) \gg 1$ such that $\|Q\|_{L^2(|x| \geqslant A(t) - \Omega t)} \leqslant \delta$. Since $w_n(t) \rightharpoonup w(t)$ in H^1, we have $\|w_n(t) - w(t)\|_{L^2(|x| \leqslant A(t))} \leqslant \delta$ for $n \geqslant n_0$. Moreover,

$$\|w(t) - Q(\cdot - x_w(t))\|_{H^1} \leqslant C\delta \quad \text{and} \quad \|w_n(t) - Q(\cdot - y_n(t))\|_{H^1} \leqslant C\delta,$$

and so, by the triangle inequality, $\|Q(\cdot - x_w(t)) - Q(\cdot - y_n(t))\|_{L^2(|x| \leqslant A(t))} \leqslant C\delta$. We deduce that, for $n \geqslant n_0$,

$$\begin{aligned}
\|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2} &\leqslant \sqrt{2}\|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \leqslant A(t))} \\
&\quad + \sqrt{2}\|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \geqslant A(t))} \\
&\leqslant C\delta + \sqrt{2}\|Q\|_{L^2(|x| \geqslant A(t))} \\
&\quad + \sqrt{2}\|Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \geqslant A(t))} \\
&\leqslant C\delta + 2\sqrt{2}\|Q\|_{L^2(|x| \geqslant A(t) - \Omega t)} \leqslant C\delta.
\end{aligned}$$

We conclude by choosing δ small enough so that $C\delta \leqslant A_0$, where A_0 is defined in Lemma 2.11, and we apply this lemma to reach the desired inequality (note that the lemma holds of course with the L^2 norm instead of the H^1 one). □

If we choose δ small enough so that $C\delta \leqslant 1$ (for example) in Lemma 3.10, we can now prove the following proposition.

Proposition 3.11. *There exists $C > 0$ such that, for all $t \geqslant 0$ and all $x_0 > 0$,*

$$\int_{x<-x_0-1} (w_x^2 + w^2)(t, x + x_w(t))\, dx \leqslant Ce^{-x_0/4}.$$

Proof. Let $t \geqslant 0$, $x_0 > 0$ and $n \geqslant n_0$ where n_0 is defined in Lemma 3.10. From (3.3) and the substitution $y = x_n(T_n) - x_n(T_n - t) - x = y_n(t) - x$, we obtain

$$\int_{x<y_n(t)-x_0} (u_{nx}^2 + u_n)(T_n - t, x_n(T_n) - x)\, dx \leqslant Ce^{-x_0/4}.$$

If we still denote $w_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$, we deduce by Lemma 3.10 that

$$\int_{x<-x_0-1+x_w(t)} (w_{nx}^2 + w_n^2)(t, x)\, dx \leqslant Ce^{-x_0/4}.$$

But $w_n(t) \rightharpoonup w(t)$ in H^1, so $w_n(t) \rightharpoonup w(t)$ and $w_{nx}(t) \rightharpoonup w_x(t)$ in L^2. Moreover, $\psi = \mathbb{1}_{(-\infty,-x_0-1+x_w(t))} \in L^\infty$ implies $w_n(t)\psi \rightharpoonup w(t)\psi$ and $w_{nx}(t)\psi \rightharpoonup w_x(t)\psi$ in L^2, thus by weak convergence $\int_{x<-x_0-1+x_w(t)} w^2(t,x)\, dx \leqslant Ce^{-x_0/4}$ and the same inequality for w_x, so the result follows by sum. □

3.4 Asymptotic stability and conclusion

The final ingredient to prove that $w(t)$ is a special solution is the theorem of asymptotic stability proved by Martel and Merle [18]. Indeed, thanks to Lemma 3.9, we can apply this theorem with $c_0 = 1$ if we choose δ small enough such that $C\delta < \alpha_0$. We obtain $c_+ > 0$ and $t \mapsto \rho(t) \in \mathbb{R}$ such that

$$\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(x>t/10)} \xrightarrow[t\to+\infty]{} 0. \tag{3.4}$$

Remark 3.12. As usual, $\rho(t)$ and c_+ are defined in [18] by a lemma of modulation close to Q, which gives the estimates $\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1} \leqslant C\delta$, $|\rho'(t) - 1| \leqslant C\delta$ and $|c_+ - 1| \leqslant C\delta$. We deduce that

$$\begin{aligned}
\|Q - Q(\cdot + \rho(t) - x_w(t))\|_{H^1} &= \|Q(\cdot - \rho(t)) - Q(\cdot - x_w(t))\|_{H^1} \\
&\leqslant \|Q - Q_{c_+}\|_{H^1} + \|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1} \\
&\quad + \|w(t) - Q(\cdot - x_w(t))\|_{H^1} \\
&\leqslant K|c_+ - 1| + C\delta + C'\delta \leqslant C''\delta.
\end{aligned}$$

Now, choosing δ small enough, $C''\delta \leqslant A_0$ and Lemma 2.11 gives $|x_w(t) - \rho(t)| \leqslant C\delta \leqslant 1$. Finally, Proposition 3.11 becomes

$$\forall t \geqslant 0, \forall x_0 > 2, \quad \int_{x<-x_0} (w_x^2 + w^2)(t, x + \rho(t))\, dx \leqslant C'e^{-x_0/4}. \tag{3.5}$$

We are now able to prove the main result of this section.

Theorem 3.13 (Existence of one special solution). *There exist $w(t)$ solution of* (gKdV) *defined for all $t \geqslant 0$, $c_+ > 0$ and $t \mapsto \rho(t)$ such that:*

(i) $\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(\mathbb{R})} \xrightarrow[t\to+\infty]{} 0,$

(ii) $\forall c > 0, \forall x_0 \in \mathbb{R}, w(0) \neq Q_c(\cdot + x_0)$.

Proof. By Remark 3.5, it is enough to prove (i). We have by the triangle inequality

$$\begin{aligned}\|w(t) - Q_{c_+}(\cdot - \rho(t))\|^2_{H^1(\mathbb{R})} &\leqslant \|w(t) - Q_{c_+}(\cdot - \rho(t))\|^2_{H^1(x>t/10)} + 2\|w(t)\|^2_{H^1(x<t/10)} \\ &\quad + 2\|Q_{c_+}(\cdot - \rho(t))\|^2_{H^1(x<t/10)} = \mathbf{I} + \mathbf{II} + \mathbf{III}.\end{aligned}$$

Since $|\rho'(t) - 1| \leqslant C\delta \leqslant \frac{1}{10}$ if we choose δ small enough, $|\rho(t) - t - \rho(0)| \leqslant \frac{1}{10}t$ and so, if we denote $\rho_0 = \rho(0) \in \mathbb{R}$, we have $\frac{t}{10} - \rho(t) \leqslant -\frac{4}{5}t - \rho_0$. We can now estimate:

- $\mathbf{I} \xrightarrow[t\to+\infty]{} 0$ by (3.4).
- For t large enough, we have $\frac{4t}{5} + \rho_0 > 2$, and so (3.5) gives

$$\begin{aligned}\frac{1}{2}\mathbf{II} &= \int_{x<t/10} (w_x^2 + w^2)(t,x)\,dx = \int_{x<t/10-\rho(t)} (w_x^2 + w^2)(t, x + \rho(t))\,dx \\ &\leqslant \int_{x<-4t/5-\rho_0} (w_x^2 + w^2)(t, x+\rho(t))\,dx \leqslant Ce^{-t/5} \xrightarrow[t\to+\infty]{} 0.\end{aligned}$$

- Finally, since $(Q'^2_{c_+} + Q^2_{c_+})(x) \leqslant Ce^{2\sqrt{c_+}x}$ for all $x \in \mathbb{R}$ (see Claim 2.1), we have

$$\begin{aligned}\frac{1}{2}\mathbf{III} &= \int_{x<t/10} (Q'^2_{c_+} + Q^2_{c_+})(x - \rho(t))\,dx = \int_{x<t/10-\rho(t)} (Q'^2_{c_+} + Q^2_{c_+})(x)\,dx \\ &\leqslant \int_{x<-4t/5-\rho_0} (Q'^2_{c_+} + Q^2_{c_+})(x)\,dx \leqslant C\int_{x<-4t/5-\rho_0} e^{2\sqrt{c_+}x}\,dx \leqslant Ce^{-\frac{8t}{5}\sqrt{c_+}}\end{aligned}$$

which also tends to 0 when $t \to +\infty$. This achieves the proof of Theorem 3.13. □

Corollary 3.14. *For all $c > 0$, there exist $w_c(t)$ solution of* (gKdV) *defined for all $t \geqslant 0$ and $t \mapsto \rho_c(t)$ such that:*

(i) $\|w_c(t, \cdot + \rho_c(t)) - Q_c\|_{H^1(\mathbb{R})} \xrightarrow[t\to+\infty]{} 0,$

(ii) $\forall c' > 0, \forall x_0 \in \mathbb{R}, w_c(0, \cdot + \rho_c(0)) \neq Q_{c'}(\cdot + x_0)$.

Proof. It is based on the scaling invariance of the (gKdV) equation: if $u(t,x)$ is a solution, then for all $\lambda > 0$, $\lambda^{\frac{2}{p-1}}u(\lambda^3 t, \lambda x)$ is also a solution. For $c > 0$ given, we thus define w_c by $w_c(t) = \lambda_c^{\frac{2}{p-1}} w(\lambda_c^3 t, \lambda_c x)$ with $\lambda_c = \sqrt{\frac{c}{c_+}}$, where w and c_+ are defined above. Since $Q_c(x) = \lambda_c^{\frac{2}{p-1}} Q_{c_+}(\lambda_c x)$, we have by substitution

$$\begin{aligned}\|w(t) - Q_{c_+}(\cdot - \rho(t))\|^2_{H^1} = \lambda_c^{\frac{p-5}{p-1}}\Big(&\|w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c\|^2_{L^2} \\ &+ \frac{1}{\lambda_c^2}\|\partial_x[w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c]\|^2_{L^2}\Big).\end{aligned}$$

We deduce that

$$\|w(t)-Q_{c_+}(\cdot-\rho(t))\|_{H^1}^2 \geqslant \begin{cases} \lambda_c^{\frac{p-5}{p-1}}\|w_c(t/\lambda_c^3,\cdot+\rho(t)/\lambda_c)-Q_c\|_{H^1}^2 & \text{if } \lambda_c \leqslant 1, \\ \lambda_c^{-\frac{p+3}{p-1}}\|w_c(t/\lambda_c^3,\cdot+\rho(t)/\lambda_c)-Q_c\|_{H^1}^2 & \text{if } \lambda_c \geqslant 1, \end{cases}$$

and so $\lim_{t\to+\infty}\|w_c(t/\lambda_c^3,\cdot+\rho(t)/\lambda_c)-Q_c\|_{H^1}=0$ in both cases by Theorem 3.13. We finally obtain (i) if we take $\rho_c(t)=\frac{\rho(\lambda_c^3 t)}{\lambda_c}$. For (ii), if we suppose that there exist $c'>0$ and $x_0\in\mathbb{R}$ such that $w_c(0,\cdot+\rho_c(0))=Q_{c'}(\cdot+x_0)$, then we get

$$w_0 = Q_{\frac{c'c_+}{c}}\left(\cdot+\left(\sqrt{\frac{c}{c_+}}x_0-\rho_0\right)\right),$$

which is a contradiction with Remark 3.5. □

4 Construction and uniqueness of a family of special solutions via the contraction principle

In this section, we prove Theorem 1.1. The proof is an extension to (gKdV) of the method by fixed point developed in [7, 8] for the nonlinear Schrödinger equation. To adapt the method to (gKdV), we use first information on the spectrum of the linearized operator around $Q(\cdot-t)$ due to [20] (see Proposition 4.2 in the present paper). Secondly, we rely on the Cauchy theory for (gKdV) developed in [12, 13]. Indeed, one of the main difficulties is the lack of a derivative due to the equation, but compensated by a smoothing effect already used in [12, 13].

4.1 Preliminary estimates for the Cauchy problem

Theorem 3.5 of [12] and Proposition 2.3 of [13] are summed up and adapted to our situation in Proposition 4.1 below. We note $W(t)$ the semigroup associated to the linear equation $\partial_t u+\partial_x^3 u=0$.

Notation. Let $I\subset\mathbb{R}$ be an interval, $1\leqslant p,q\leqslant\infty$ and $g: I\times\mathbb{R}\to\mathbb{R}$. Then define

$$\|g\|_{L_x^pL_I^q}=\left(\int_{-\infty}^{+\infty}\left(\int_I|g(t,x)|^q\,dt\right)^{p/q}dx\right)^{1/p},\ \|g\|_{L_I^qL_x^p}=\left(\int_I\left(\int_{-\infty}^{+\infty}|g(t,x)|^p\,dx\right)^{q/p}dt\right)^{1/q}$$

and $L_x^pL_I^q=\{g\mid \|g\|_{L_x^pL_I^q}<+\infty\}$ and $L_I^qL_x^p=\{g\mid \|g\|_{L_I^qL_x^p}<+\infty\}$. Finally, denote $L_x^pL_t^q=L_x^pL_{\mathbb{R}}^q$ and $L_t^qL_x^p=L_{\mathbb{R}}^qL_x^p$.

Proposition 4.1. *There exists $C>0$ such that, for all $g\in L_x^1L_t^2$ and all $T\in\mathbb{R}$,*

$$\left\|\frac{\partial}{\partial x}\int_t^{+\infty}W(t-t')g(t',x)\,dt'\right\|_{L^\infty_{[T,+\infty)}L_x^2}\leqslant C\|g\|_{L_x^1L^2_{[T,+\infty)}}, \tag{4.1}$$

$$\left\|\frac{\partial}{\partial x}\int_t^{+\infty}W(t-t')g(t',x)\,dt'\right\|_{L_x^5L^{10}_{[T,+\infty)}}\leqslant C\|g\|_{L_x^1L^2_{[T,+\infty)}}. \tag{4.2}$$

Proof. (i) Inequality (4.1) comes from the dual inequality of (3.6) in [12], *i.e.*

$$\left\| \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} W(-t')g(t',x)\,dt' \right\|_{L^2_x} \leqslant C\|g\|_{L^1_x L^2_t}.$$

Let $t \geqslant T$. We get, for $\tilde{g}(t',x) = \mathbf{1}_{[t,+\infty)}(t')g(t',x)$,

$$\left\| \frac{\partial}{\partial x} \int_{t}^{+\infty} W(-t')g(t',x)\,dt' \right\|_{L^2_x} = \left\| \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} W(-t')\tilde{g}(t',x)\,dt' \right\|_{L^2_x} \leqslant C\|g\|_{L^1_x L^2_{[T,+\infty)}}$$

and so the desired inequality since W is unitary on L^2.

(ii) Inequality (4.2) comes from the inequalities (2.6) and (2.8) of [13] with the admissible triples $(p_1,q_1,\alpha_1) = (5,10,0)$ and $(p_2,q_2,\alpha_2) = (\infty,2,1)$. In fact, if we combine (2.6) cut in time with $[0,+\infty)$ and (2.8), we get

$$\left\| \frac{\partial}{\partial x} \int_{t}^{+\infty} W(t-t')g(t',x)\,dt' \right\|_{L^5_x L^{10}_t} \leqslant C\|g\|_{L^1_x L^2_t}.$$

If we apply it to $\tilde{g}(t',x) = \mathbf{1}_{[T,+\infty)}(t')g(t',x)$, we reach the desired inequality since

$$\left\| \frac{\partial}{\partial x} \int_{t}^{+\infty} W(t-t')g(t',x)\,dt' \right\|_{L^5_x L^{10}_{[T,+\infty)}} \leqslant \left\| \frac{\partial}{\partial x} \int_{t}^{+\infty} W(t-t')\tilde{g}(t',x)\,dt' \right\|_{L^5_x L^{10}_t}. \qquad \square$$

4.2 Preliminary results on the linearized equation

4.2.1 Linearized equation

The linearized equation appears if one considers a solution of (gKdV) close to the soliton $Q(x-t)$. More precisely, if $u(t,x) = Q(x-t) + h(t,x-t)$ satisfies (gKdV), then h satisfies

$$\partial_t h + \mathcal{L}h = R(h) \tag{4.3}$$

where $\mathcal{L}a = -(La)_x$, $La = -\partial_x^2 a + a - pQ^{p-1}a$ is defined in Section 2.1, and

$$R(h) = -\partial_x \left(\sum_{k=2}^{p} \binom{p}{k} Q^{p-k} h^k \right).$$

The spectrum of $\mathcal{L}$ has been calculated by Pego and Weinstein [20], and their results are summed up here for reader's convenience.

Proposition 4.2 ([20]). *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}) = i\mathbb{R} \quad \textit{and} \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \ \textit{with}\ e_0 > 0.$$

Moreover, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions $\mathcal{Y}_+$ and $\mathcal{Y}_- = \check{\mathcal{Y}}_+$ which have an exponential decay at infinity, and the null space of $\mathcal{L}$ is spanned by Q'.

4.2.2 Exponential decay

Exponential decay of $\mathcal{Y}_+$ has been proved in [20], but a generalization of this fact to a larger family of functions will be necessary in the proof of Proposition 4.6. For $\lambda > 0$, consider the operator A_λ defined on L^2 by $A_\lambda u = u''' - u' - \lambda u$, and the characteristic equation of $A_\lambda u = 0$,

$$f_\lambda(x) := x^3 - x - \lambda = 0.$$

Note σ_1^λ, σ_2^λ, σ_3^λ the roots of f_λ, eventually complex, and sorted by their real part. A simple study of f_λ shows that σ_3^λ is always real, $\sigma_3^\lambda > \frac{1}{\sqrt{3}}$, and $(\sigma_3^\lambda)_{\lambda>0}$ is increasing. Moreover, we have the three cases:

(a) If $\lambda > \frac{2}{3\sqrt{3}}$, then σ_1^λ and σ_2^λ are two conjugate roots such that $\operatorname{Re}\sigma_1^\lambda = \operatorname{Re}\sigma_2^\lambda = -\frac{\sigma_3^\lambda}{2}$.

(b) If $\lambda = \frac{2}{3\sqrt{3}}$, then $\sigma_1^\lambda = \sigma_2^\lambda = -\frac{1}{\sqrt{3}}$ and $\sigma_3^\lambda = \frac{2}{\sqrt{3}}$.

(c) If $\lambda < \frac{2}{3\sqrt{3}}$, then σ_1^λ, σ_2^λ are real and $\sigma_1^\lambda \in \left(-\sqrt{3}, -\frac{1}{\sqrt{3}}\right)$, $\sigma_2^\lambda \in \left(-\frac{1}{\sqrt{3}}, 0\right)$. Moreover, $(\sigma_2^\lambda)_\lambda$ is decreasing, and in particular $\sigma_2^\lambda \nearrow 0$ when $\lambda \searrow 0$.

This analysis allows us to define

$$\mu = \frac{1}{4}\min_{\lambda \geqslant e_0}(\sigma_3^\lambda, -\operatorname{Re}\sigma_2^\lambda, e_0, 1) > 0$$

and

$$\mathcal{H} = \{f \in H^\infty(\mathbb{R}) \mid \forall j \geqslant 0, \exists C_j \geqslant 0, \forall x \in \mathbb{R},\ |f^{(j)}(x)| \leqslant C_j e^{-\mu|x|}\}.$$

Lemma 4.3. *If $u \in L^2$ and $f \in \mathcal{H}$ satisfy $u''' - u' - \lambda u = f$ with $\lambda \geqslant e_0$, then $u \in \mathcal{H}$.*

Proof. First notice that $u \in H^\infty(\mathbb{R})$ by a simple bootstrap argument. Moreover, the method of variation of constants gives us

$$u(x) = Ae^{\sigma_3^\lambda x}\int_x^{+\infty} e^{-\sigma_3^\lambda s} f(s)\,ds + Be^{\sigma_2^\lambda x}\int_{-\infty}^x e^{-\sigma_2^\lambda s} f(s)\,ds + Ce^{\sigma_1^\lambda x}\int_{-\infty}^x e^{-\sigma_1^\lambda s} f(s)\,ds$$

with $A, B, C \in \mathbb{C}$, if we suppose that $\lambda \neq \frac{2}{3\sqrt{3}}$. We can also notice that u' has the same form as u, except for three terms in $f(x)$ which appear, and which have the expected decay by hypothesis, and so on for $u^{(j)}$ for $j \geqslant 2$. Hence, we only have to check exponential decay for u, and we write

$$|u(x)| \leqslant A'e^{\sigma_3^\lambda x}\int_x^{+\infty} e^{-\sigma_3^\lambda s}|f(s)|\,ds + B'e^{\operatorname{Re}\sigma_2^\lambda x}\int_{-\infty}^x e^{-\operatorname{Re}\sigma_2^\lambda s}|f(s)|\,ds$$
$$+ C'e^{\operatorname{Re}\sigma_1^\lambda x}\int_{-\infty}^x e^{-\operatorname{Re}\sigma_1^\lambda s}|f(s)|\,ds.$$

By changing x in $-x$ and by the definition of μ, it is enough to show that, if

$$v(x) = e^{-ax}\int_{-\infty}^x e^{as}e^{-\mu|s|}\,ds$$

with $a \geqslant 2\mu$, then $v(x) \leqslant e^{-\mu|x|}$. Notice that one half could also have replaced one quarter in the definition of μ, but this gain of 2 allows us to treat the case $\lambda = \frac{2}{3\sqrt{3}}$ (not written here for brevity), which makes appear a polynomial in front of the exponential in the last two terms of the expression of u. Finally, we conclude in both cases, since $a - \mu \geqslant \mu > 0$:

- If $x < 0$, then $v(x) \leqslant e^{-ax} \int_{-\infty}^{x} e^{as} e^{\mu s}\, ds = Ce^{-ax} \cdot e^{(a+\mu)x} = Ce^{\mu x} = Ce^{-\mu|x|}$.
- If $x \geqslant 0$, then $v(x) \leqslant e^{-ax} \int_{-\infty}^{x} e^{as} e^{-\mu s}\, ds = Ce^{-ax} \cdot e^{(a-\mu)x} = Ce^{-\mu x} = Ce^{-\mu|x|}$.

The case $\lambda = \frac{2}{3\sqrt{3}}$ is treated similarly. □

Corollary 4.4. *$\mathcal{Y}_+, \mathcal{Y}_- \in \mathcal{H}$.*

Proof. Since $\mathcal{Y}_- = \check{\mathcal{Y}}_+$, it is enough to show that $\mathcal{Y}_+ \in \mathcal{H}$. But by definition of $\mathcal{Y}_+$ in [20], we have $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$ with $\mathcal{Y}_+ \in L^2$, *i.e.*

$$\mathcal{Y}_+''' - \mathcal{Y}_+' - e_0\mathcal{Y}_+ = -p\partial_x(Q^{p-1}\mathcal{Y}_+) = -p(p-1)Q'Q^{p-2}\mathcal{Y}_+ - pQ^{p-1}\mathcal{Y}_+'.$$

By a bootstrap argument, we have $\mathcal{Y}_+ \in H^\infty(\mathbb{R})$, and in particular $\mathcal{Y}_+^{(j)} \in L^\infty(\mathbb{R})$ for all $j \geqslant 0$. If we denote $f(x) = -p(p-1)Q'Q^{p-2}\mathcal{Y}_+ - pQ^{p-1}\mathcal{Y}_+'$, then by exponential decay of $Q^{(j)}$ for all $j \geqslant 0$ and by definition of μ, we have $|f^{(j)}(x)| \leqslant Ce^{-(p-1)|x|} \leqslant Ce^{-\mu|x|}$ and so $f \in \mathcal{H}$. It is enough to apply Lemma 4.3 with $\lambda = e_0$ to conclude. □

4.3 Existence of special solutions

We now prove the following result, which is the first part of Theorem 1.1.

Proposition 4.5. *Let $A \in \mathbb{R}$. If $t_0 = t_0(A)$ is large enough, then there exists a solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of* (gKdV) *such that*

$$\forall s \in \mathbb{R}, \exists C > 0, \forall t \geqslant t_0, \quad \|U^A(t, \cdot + t) - Q - Ae^{-e_0 t}\mathcal{Y}_+\|_{H^s} \leqslant Ce^{-2e_0 t}. \tag{4.4}$$

4.3.1 A family of approximate solutions

The following proposition is similar to [8, Proposition 3.4], except for the functional space, which is not the Schwartz space but the space $\mathcal{H}$ described above.

Proposition 4.6. *Let $A \in \mathbb{R}$. There exists a sequence $(\mathcal{Z}_j^A)_{j\geqslant 1}$ of functions of $\mathcal{H}$ such that $\mathcal{Z}_1^A = A\mathcal{Y}_+$, and if $k \geqslant 1$ and $\mathcal{V}_k^A = \sum_{j=1}^{k} e^{-je_0 t}\mathcal{Z}_j^A$, then*

$$\partial_t \mathcal{V}_k^A + \mathcal{L}\mathcal{V}_k^A = R(\mathcal{V}_k^A) + \varepsilon_k^A(t), \quad \text{where} \quad \varepsilon_k^A(t) = \sum_{j=k+1}^{pk} e^{-je_0 t} g_{j,k}^A, \quad g_{j,k}^A \in \mathcal{H}, \tag{4.5}$$

and R is defined in (4.3).

Proof. The proof is very similar to the one in [8], and we write it there for reader's convenience. We prove this proposition by induction, and for brevity, we omit the superscript A.

Define $\mathcal{Z}_1 := A\mathcal{Y}_+$ and $\mathcal{V}_1 := e^{-e_0 t}\mathcal{Z}_1$. Then, by the explicit definition of R in (4.3),

$$\partial_t \mathcal{V}_1 + \mathcal{L}\mathcal{V}_1 - R(\mathcal{V}_1) = -R(\mathcal{V}_1) = -R(Ae^{-e_0 t}\mathcal{Y}_+) = \sum_{j=2}^{p} e^{-je_0 t} A^j \binom{p}{j} \partial_x[Q^{p-j}\mathcal{Y}_+^j]$$

which yields (4.5) for $k = 1$, since $\mathcal{Y}_+, Q \in \mathcal{H}$ by Corollary 4.4 and Claim 2.1.

Let $k \geqslant 1$ and assume that $\mathcal{Z}_1, \dots, \mathcal{Z}_k$ are known with the corresponding $\mathcal{V}_k$ satisfying (4.5). Now let $\mathcal{U}_{k+1} := g_{k+1,k} \in \mathcal{H}$, so that

$$\partial_t \mathcal{V}_k + \mathcal{L}\mathcal{V}_k = R(\mathcal{V}_k) + e^{-(k+1)e_0 t}\mathcal{U}_{k+1} + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k},$$

and define $\mathcal{Z}_{k+1} := -(\mathcal{L} - (k+1)e_0)^{-1}\mathcal{U}_{k+1}$. Remark that $\mathcal{Z}_{k+1}$ is well defined since $(k+1)e_0$ is not in the spectrum of $\mathcal{L}$ by Proposition 4.2, and moreover $\mathcal{Z}_{k+1} \in \mathcal{H}$. Indeed, we have

$$\mathcal{Z}_{k+1}''' - \mathcal{Z}_{k+1}' - (k+1)e_0\mathcal{Z}_{k+1} = -\mathcal{U}_{k+1} - p(p-1)Q'Q^{p-2}\mathcal{Z}_{k+1} - pQ^{p-1}\mathcal{Z}_{k+1}' \in \mathcal{H}$$

by exponential decay of $Q^{(j)}$ for all $j \geqslant 0$ and since $\mathcal{Z}_{k+1}^{(j)} \in H^\infty(\mathbb{R})$ by a bootstrap argument. Hence, $\mathcal{Z}_{k+1} \in \mathcal{H}$ by Lemma 4.3 applied with $\lambda = (k+1)e_0 \geqslant e_0$.

Then, we have

$$\partial_t \left(\mathcal{V}_k + e^{-(k+1)e_0 t}\mathcal{Z}_{k+1}\right) + \mathcal{L}\left(\mathcal{V}_k + e^{-(k+1)e_0 t}\mathcal{Z}_{k+1}\right) = R(\mathcal{V}_k) + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k}.$$

Denote $\mathcal{V}_{k+1} := \mathcal{V}_k + e^{-(k+1)e_0 t}\mathcal{Z}_{k+1}$. Thus, we have

$$\partial_t \mathcal{V}_{k+1} + \mathcal{L}\mathcal{V}_{k+1} - R(\mathcal{V}_{k+1}) = R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k}.$$

We conclude the proof by evaluating

$$\begin{aligned} R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) &= R(\mathcal{V}_k) - R(\mathcal{V}_k + e^{-(k+1)e_0 t}\mathcal{Z}_{k+1}) \\ &= \partial_x \left[\sum_{j=2}^{p} \binom{p}{j} Q^{p-j}\left(\left(\mathcal{V}_k + e^{-(k+1)e_0 t}\mathcal{Z}_{k+1}\right)^j - \mathcal{V}_k^j\right)\right] = \sum_{j=k+2}^{p(k+1)} e^{-je_0 t}\tilde{g}_{j,k}, \end{aligned}$$

which yields (4.5) for $k+1$, and so completes the proof. □

4.3.2 Construction of special solutions

We now prove Proposition 4.5, following the same three steps as in [8]. The main difference comes from step 2, because of the derivative in the error term which forces us to use the sharp smoothing effect developed in [12]. Let $A \in \mathbb{R}$ and $s \geqslant 1$ be an integer. Write

$$U^A(t, x+t) = Q(x) + h^A(t, x).$$

First, by a fixed point argument, we construct a solution $h^A \in C^0([t_k, +\infty), H^s)$ of (4.3) for k and t_k large and such that

$$\forall T \geqslant t_k,\ \|(h^A - \mathcal{V}_k)(T)\|_{H^s} \leqslant e^{-(k+\frac{1}{2})e_0 T}. \tag{4.6}$$

Next, the same arguments as in [8] show that h^A does not depend on s and k. For brevity, we omit the superscript A.

Step 1. Reduction to a fixed point problem. If we set $\tilde{h}(t,x) = h(t,x-t)$, equation (4.3) can be written as

$$\partial_t \tilde{h} + \partial_x^3 \tilde{h} = -S(\tilde{h}), \quad S(\tilde{h}) = \frac{\partial}{\partial x}\left[\sum_{k=1}^{p}\binom{p}{k} Q^{p-k}(x-t)\tilde{h}^k\right]. \tag{4.7}$$

Moreover, we have, by (4.5), $\varepsilon_k(t) = \partial_t \mathcal{V}_k + \partial_x^3 \mathcal{V}_k - \partial_x \mathcal{V}_k + \partial_x \left[\sum_{j=1}^{p} \binom{p}{j} Q^{p-j} \mathcal{V}_k^j\right]$. Now let $v(t,x) = (h - \mathcal{V}_k)(t, x-t)$ and subtract the previous equation from (4.7), so that

$$\partial_t v + \partial_x^3 v = -S[v + \mathcal{V}_k(t,x-t)] + S[\mathcal{V}_k(t,x-t)] - \varepsilon_k(t,x-t).$$

For notation simplicity, we drop the space argument $(x-t)$ for the moment. Then, by Duhamel's formula, the equation can be written as

$$v(t) = \mathcal{M}(v)(t) := \int_t^{+\infty} W(t-t')\left[S(\mathcal{V}_k(t') + v(t')) - S(\mathcal{V}_k(t')) + \varepsilon_k(t')\right] dt'. \tag{4.8}$$

Note that (4.6) is equivalent to $\|v(T)\|_{H^s} \leqslant e^{-(k+\frac{1}{2})e_0 T}$ for $T \geqslant t_k$. In other words, defining

$$\begin{cases} N_1(v) = \sup_{T\geqslant t_k} e^{(k+\frac{1}{2})e_0 T}\|v(T)\|_{H^s}, \\ N_2(v) = \sum_{s'=0}^{s} \sup_{T\geqslant t_k} e^{(k+\frac{1}{2})e_0 T}\|\partial^{s'} v\|_{L_x^5 L^{10}_{[T,+\infty)}}, \\ \Lambda(v) = \Lambda_{t_k,k,s}(v) = \max(N_1(v), N_2(v)), \end{cases}$$

it is enough to show that $\mathcal{M}$ is a contraction on B defined by

$$B = B(t_k,k,s) = \left\{v \in C^0([t_k,+\infty), H^s) \mid \Lambda(v) \leqslant 1\right\}.$$

Remark 4.7. The choice of the two norms N_1 and N_2 is related to the fact that global well-posedness of supercritical (gKdV) with initial data small in H^1 can be proved with the two norms $\widetilde{N_1}(v) = \sup_{t\in\mathbb{R}}\|v(t)\|_{H^1}$ and $\widetilde{N_2}(v) = \|v\|_{L_x^5 L_t^{10}} + \|\partial_x v\|_{L_x^5 L_t^{10}}$, following [13]. We could also have used other norms from [12].

Step 2. Contraction argument. We show that $\mathcal{M}$ is a contraction on B for $s \geqslant 1$ and k, t_k sufficiently large. Throughout this proof, we denote by C a constant depending only on s, and C_k a constant depending on s and k. To estimate $N_1(\mathcal{M}(v))$ and $N_2(\mathcal{M}(v))$, we have to explicit

$$\begin{aligned} S(\mathcal{V}_k + v) - S(\mathcal{V}_k) &= \frac{\partial}{\partial x}\left[\sum_{i=1}^{p}\binom{p}{i} Q^{p-i}\left((\mathcal{V}_k + v)^i - \mathcal{V}_k^i\right)\right] \\ &= \frac{\partial}{\partial x}\left(pQ^{p-1}v\right) + \frac{\partial}{\partial x}\left[\sum_{i=2}^{p}\binom{p}{i} Q^{p-i} v \cdot \sum_{l=1}^{i}\binom{i}{l}\mathcal{V}_k^{i-l} v^{l-1}\right] \\ &= p\frac{\partial \mathbf{I}}{\partial x} + \sum_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}\frac{\partial \mathbf{II}_{\alpha,\beta,\gamma}}{\partial x}, \end{aligned}$$

where $\mathbf{I} = Q^{p-1}v$ and $\mathbf{II}_{\alpha,\beta,\gamma} = Q^\alpha \mathcal{V}_k^\beta v^\gamma$, with: $\gamma \geqslant 1$, $\beta+\gamma \geqslant 2$, $\alpha+\beta+\gamma = p \geqslant 6$. We can now write

$$\begin{aligned} \partial^s \mathcal{M}(v) = p\int_t^{+\infty} W(t-t')\frac{\partial}{\partial x}\left[\partial^s(\mathbf{I})\right] dt' + \sum_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}\int_t^{+\infty} W(t-t')\frac{\partial}{\partial x}\left[\partial^s(\mathbf{II}_{\alpha,\beta,\gamma})\right] dt' \\ + \int_t^{+\infty} W(t-t')\partial^s \varepsilon_k(t')\, dt'. \end{aligned}$$

By (4.1) and (4.2), we obtain

$$\max\left(\|\partial^s\mathcal{M}(v)(T)\|_{L^2_x}, \|\partial^s\mathcal{M}(v)\|_{L^5_xL^{10}_{[T,+\infty)}}\right) \leqslant C\|\partial^{s-1}\varepsilon_k\|_{L^1_xL^2_{[T,+\infty)}} + C\|\partial^s(\mathbf{I})\|_{L^1_xL^2_{[T,+\infty)}} + \sum_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}\|\partial^s(\mathbf{II}_{\alpha,\beta,\gamma})\|_{L^1_xL^2_{[T,+\infty)}}. \quad (4.9)$$

We treat the terms ε_k, $\mathbf{I}$, $\mathbf{II}_{\alpha,\beta,\gamma}$ for $\alpha = p-2, \beta = \gamma = 1$, and for $\alpha = \beta = 0, \gamma = p$. All other terms can be treated similarly: for example, $\mathbf{II}_{0,p-1,1}$ can be treated like $\mathbf{II}_{p-2,1,1}$, etc.

For $\mathbf{I}$, since Q and its derivatives have the same decay, it is enough to estimate the term $\tilde{\mathbf{I}} = \|Q^{p-1}\partial^s v\|_{L^1_xL^2_{[T,+\infty)}} \leqslant C\|e^{-|x-t|}\partial^s v\|_{L^1_xL^2_{[T,+\infty)}}$:

$$\begin{aligned}\tilde{\mathbf{I}} &\leqslant C\|e^{x-t}\partial^s v\|_{L^1_{(-\infty,T]}L^2_{[T,+\infty)}} + C\|e^{t-x}\partial^s v\|_{L^1_{[T,+\infty)}L^2_{[T,x]}} + C\|e^{x-t}\partial^s v\|_{L^1_{[T,+\infty)}L^2_{[x,+\infty)}}\\ &\leqslant C\sqrt{\int_{-\infty}^T e^{2x}\,dx}\sqrt{\int_x\int_T^{+\infty} e^{-2t}(\partial^s v)^2\,dt\,dx} + C\sqrt{\int_T^{+\infty} e^{-2x}\,dx}\sqrt{\int_x\int_T^{+\infty} e^{2t}(\partial^s v)^2\,dt\,dx}\\ &\quad + C\sqrt{\int_T^{+\infty} e^{-2x}\,dx}\sqrt{\int_T^{+\infty}\int_x^{+\infty} e^{4x-2t}(\partial^s v)^2\,dt\,dx}\end{aligned}$$

by the Cauchy-Schwarz inequality. Now, by Fubini's theorem, and since $4x - 2t \leqslant 2t$ in the last integral, we get

$$\begin{aligned}\tilde{\mathbf{I}} &\leqslant Ce^T N_1(v)\sqrt{\int_T^{+\infty} e^{-(2k+1)e_0t-2t}\,dt} + 2Ce^{-T}N_1(v)\sqrt{\int_T^{+\infty} e^{-(2k+1)e_0t+2t}\,dt}\\ &\leqslant Ce^T N_1(v)\frac{e^{-(k+\frac{1}{2})e_0T-T}}{\sqrt{(2k+1)e_0+2}} + 2Ce^{-T}N_1(v)\frac{e^{-(k+\frac{1}{2})e_0T+T}}{\sqrt{(2k+1)e_0-2}} \leqslant CN_1(v)\frac{e^{-(k+\frac{1}{2})e_0T}}{\sqrt{k}}.\end{aligned}$$

Note that, since k will be chosen large at the end of the argument, we can suppose $(2k+1)e_0 > 2$.

For $\mathbf{II}_{p-2,1,1}$, we treat similarly the term $\widetilde{\mathbf{II}}_{p-2,1,1} = \|Q^{p-2}\mathcal{V}_k\partial^s v\|_{L^1_xL^2_{[T,+\infty)}}$, since $\mathcal{V}_k$ and its derivatives have the same decay. In fact, we have by Hölder's inequality

$$\widetilde{\mathbf{II}}_{p-2,1,1} \leqslant C\|\partial^s v\|_{L^5_xL^{10}_{[T,+\infty)}}\|\mathcal{V}_k\|_{L^{5/4}_xL^{5/2}_{[T,+\infty)}} \leqslant CN_2(v)e^{-(k+\frac{1}{2})e_0T}\|\mathcal{V}_k\|_{L^{5/4}_xL^{5/2}_{[T,+\infty)}}$$

By the definition of $\mathcal{V}_k$ in Proposition 4.6, we have, noting $e_0' = \frac{5}{2}e_0$ and $\mu' = \frac{5}{2}\mu$,

$$\begin{aligned}\|\mathcal{V}_k\|^{5/4}_{L^{5/4}_xL^{5/2}_{[T,+\infty)}} &\leqslant C_k\|e^{-e_0t}e^{-\mu|x-t|}\|^{5/4}_{L^{5/4}_xL^{5/2}_{[T,+\infty)}}\\ &\leqslant C_k\int_{-\infty}^T\sqrt{\int_T^{+\infty} e^{-e_0't}e^{-\mu't}e^{\mu'x}\,dt}\,dx + C_k\int_T^{+\infty}\sqrt{\int_T^x e^{-e_0't}e^{\mu't}e^{-\mu'x}\,dt}\,dx\\ &\quad + C_k\int_T^{+\infty}\sqrt{\int_x^{+\infty} e^{-e_0't}e^{-\mu't}e^{\mu'x}\,dt}\,dx\\ &\leqslant C_ke^{\frac{\mu'}{2}T}\sqrt{\int_T^{+\infty} e^{-(e_0'+\mu')t}\,dt} + C_ke^{-\frac{\mu'}{2}T}\sqrt{\int_T^{+\infty} e^{(\mu'-e_0')t}\,dt}\\ &\quad + C_k\int_T^{+\infty} e^{\frac{\mu'}{2}x}\sqrt{\int_x^{+\infty} e^{-(e_0'+\mu')t}\,dt}\,dx\\ &\leqslant 3C_ke^{-\frac{e_0'}{2}T} \quad \text{since } \mu < e_0 \text{ by definition of } \mu.\end{aligned}$$

We finally deduce that $\|\mathcal{V}_k\|_{L_x^{5/4}L^{5/2}_{[T,+\infty)}} \leqslant C_k e^{-e_0 T}$, and so $\widetilde{\mathbf{II}}_{p-2,1,1} \leqslant C_k N_2(v) e^{-(k+\frac{3}{2})e_0 T}$.

For $\mathbf{II}_{0,0,p} = v^p$, first remark that

$$\partial^s(v^p) = p\partial^{s-1}(\partial v \cdot v^{p-1}) = p\partial^s v \cdot v^{p-1} + p\sum_{k=0}^{s-2}\binom{s-1}{k}\partial^{k+1}v \cdot \partial^{s-1-k}(v^{p-1}),$$

where each term of the sum is a product of p terms like $\partial^{s_j}v$ with $s_j \leqslant s-1$. Since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we can estimate the first term thanks to Hölder's inequality, by

$$\begin{aligned}\|\partial^s v \cdot v^{p-1}\|_{L_x^1 L^2_{[T,+\infty)}} &\leqslant \|v\|^{p-5}_{L^\infty_{[T,+\infty)}L^\infty_x} \cdot \|\partial^s v\|_{L_x^5 L^{10}_{[T,+\infty)}} \cdot \|v\|^4_{L_x^5 L^{10}_{[T,+\infty)}} \\ &\leqslant C e^{-p(k+\frac{1}{2})e_0 T} N_1(v)^{p-5} N_2(v)^5.\end{aligned}$$

The other terms in the sum can be treated in the same way, and more simply since we can choose any $(p-5)$ terms to take out in $L^\infty_{[T,+\infty)}L^\infty_x$ norm, and any 5 others left in $L_x^5 L^{10}_{[T,+\infty)}$ norm.

For ε_k, we deduce, by a similar calculation like above and by the expression of ε_k in (4.5), that

$$\|\partial^{s-1}\varepsilon_k\|_{L_x^1 L^2_{[T,+\infty)}} \leqslant C_k \int_{\mathbb{R}} \sqrt{\int_T^{+\infty} e^{-2(k+1)e_0 t} e^{-2\mu|x-t|}\, dt}\, dx \leqslant C'_k e^{-(k+1)e_0 T}.$$

Summarizing from (4.9), we have shown

$$\begin{aligned}\max &\left(e^{(k+\frac{1}{2})e_0 T}\|\mathcal{M}(v)(T)\|_{H^s}, \sum_{s'=0}^{s} e^{(k+\frac{1}{2})e_0 T}\|\partial^{s'}v\|_{L_x^5 L^{10}_{[T,+\infty)}}\right) \\ &\leqslant C_k e^{-\frac{e_0}{2}T} + \frac{CN_1(v)}{\sqrt{k}} + C_k N_2(v) e^{-e_0 T} + Ce^{-(p-1)(k+\frac{1}{2})e_0 T} N_1(v)^{p-5} N_2(v)^5.\end{aligned}$$

Since $v \in B(t_k, k, s)$, *i.e.* $\Lambda(v) \leqslant 1$, we have

$$\Lambda(\mathcal{M}(v)) \leqslant C_k e^{-\frac{e_0}{2}t_k} + \left(\frac{C}{\sqrt{k}} + C_k e^{-e_0 t_k}\right)\Lambda(v) \leqslant \left(\frac{C}{\sqrt{k}} + C_k e^{-\frac{e_0}{2}t_k}\right).$$

First, choose k so that $\frac{C}{\sqrt{k}} \leqslant \frac{1}{2}$, then take t_k such that $C_k e^{-\frac{e_0}{2}t_k} \leqslant \frac{1}{2}$. Then $\mathcal{M}$ maps $B = B(t_k, k, s)$ into itself.

It remains to show that $\mathcal{M}$ is a contraction on B. But for $v, w \in B$, we have

$$\mathcal{M}(v) - \mathcal{M}(w) = \int_t^{+\infty} W(t-t')\Big[S(\mathcal{V}_k(t') + v(t')) - S(\mathcal{V}_k(t') + w(t'))\Big]dt'$$

and

$$\begin{aligned}S(\mathcal{V}_k + v) - S(\mathcal{V}_k + w) &= \frac{\partial}{\partial x}\left[\sum_{j=1}^{p}\binom{p}{j}Q^{p-j}\left[(\mathcal{V}_k + v)^j - (\mathcal{V}_k + w)^j\right]\right] \\ &= \frac{\partial}{\partial x}\sum_{j=1}^{p}\binom{p}{j}Q^{p-j}(v-w)\sum_{i=1}^{j-1}(\mathcal{V}_k + v)^i(\mathcal{V}_k + w)^{j-i} \\ &= p\frac{\partial}{\partial x}\Big[Q^{p-1}(v-w)\Big] + \frac{\partial}{\partial x}\Big[(v-w)\cdot\sum_{\alpha,\beta,\gamma,\delta} C_{\alpha,\beta,\gamma,\delta}Q^\alpha \mathcal{V}_k^\beta v^\gamma w^\delta\Big].\end{aligned}$$

Under this form, a similar calculation like above allows us to conclude: the first term is treated like **I**, and each $Q^\alpha \mathcal{V}_k^\beta v^\gamma w^\delta$ can be treated like $\mathbf{II}_{\alpha,\beta,\gamma}$ if we systematically take out the term $\Lambda(v-w)$ by Hölder's inequality. Hence we get, as there is no term in ε_k,

$$\Lambda(\mathcal{M}(v)-\mathcal{M}(w)) \leqslant \left(\frac{C}{\sqrt{k}} + C_k e^{-e_0 t_k}\right)\Lambda(v-w).$$

Choosing if necessary a larger k, then a larger t_k, we may assume that $\frac{C}{\sqrt{k}} < \frac{1}{2}$ and $C_k e^{-e_0 t_k} \leqslant \frac{1}{2}$, showing that $\mathcal{M}$ is a contraction on B. Hence, step 2 is complete.

Step 3. End of the proof. By the previous step with $s=1$, there exist k_0 and t_0 such that there exists a unique solution U^A of (gKdV) satisfying $U^A \in C^0([t_0,+\infty),H^1)$ and

$$\Lambda_{t_0,k_0,1}\Big(U^A(t,x) - Q(x-t) - \mathcal{V}_{k_0}^A(t,x-t)\Big) \leqslant 1. \tag{4.10}$$

Note that the fixed point argument still holds taking a larger t_0, and so the uniqueness remains valid, for any $t_0' \geqslant t_0$, in the class of solutions of (gKdV) in $C^0([t_0',+\infty),H^1)$ satisfying (4.10).

Finally, we can show Proposition 4.5. Since U^A is a solution of (gKdV), it is sufficient to show that $U^A \in C^0([t_0,+\infty),H^s)$ for any s, since the smoothness in time will follow from the equation. Let $s \geqslant 1$. By step 2, if k_s is large enough, there exist t_s and $\widetilde{U}^A \in C^0([t_s,+\infty),H^s)$ such that

$$\Lambda_{t_s,k_s,s}\Big(\widetilde{U}^A(t,x) - Q(x-t) - \mathcal{V}_{k_s}^A(t,x-t)\Big) \leqslant 1.$$

Of course, we may choose $k_s \geqslant k_0+1$. But by construction of $\mathcal{V}_k^A$ in Proposition 4.6, we have $\mathcal{V}_{k_s}^A(t,x-t) - \mathcal{V}_{k_0}^A(t,x-t) = \sum_{j=k_0+1}^{k_s} e^{-je_0t}\mathcal{Z}_j^A(x-t)$ where $\mathcal{Z}_j^A \in \mathcal{H}$, and so, by similar calculation like in step 2,

$$\Lambda_{t_s,k_0,s}\Big(\mathcal{V}_{k_s}^A(t,x-t) - \mathcal{V}_{k_0}^A(t,x-t)\Big) \leqslant Ce^{-\frac{e_0}{2}t_s} \leqslant \frac{1}{2}$$

for t_s large enough. Moreover, we have, by definition of Λ (and since $k_0 \leqslant k_s - 1$),

$$\Lambda_{t_s,k_0,s}(u) \leqslant e^{-e_0t_s}\Lambda_{t_s,k_s,s}(u).$$

Thus, if we choose t_s large enough such that $e^{-e_0t_s} \leqslant \frac{1}{2}$, we get by triangle inequality

$$\Lambda_{t_s,k_0,1}\Big(\widetilde{U}^A(t,x)-Q(x-t)-\mathcal{V}_{k_0}^A(t,x-t)\Big) \leqslant \Lambda_{t_s,k_0,s}\Big(\widetilde{U}^A(t,x)-Q(x-t)-\mathcal{V}_{k_0}^A(t,x-t)\Big)$$
$$\leqslant \Lambda_{t_s,k_0,s}\Big(\widetilde{U}^A(t,x)-Q(x-t)-\mathcal{V}_{k_s}^A(t,x-t)\Big)+\Lambda_{t_s,k_0,s}\Big(\mathcal{V}_{k_s}^A(t,x-t)-\mathcal{V}_{k_0}^A(t,x-t)\Big) \leqslant 1.$$

In particular, $\widetilde{U}^A$ satisfies (4.10) for large t_s. By the uniqueness in the fixed point argument, we have $U^A = \widetilde{U}^A$, which shows that $U^A \in C^0([t_s,+\infty),H^s)$. By the persistence of regularity of the (gKdV) equation, $U^A \in C^0([t_0,+\infty),H^s)$, with $s \geqslant 1$. In particular, by compactness on $[t_0,t_s]$, there exists $C=C(s)$ such that

$$\forall t \geqslant t_0, \quad \|U^A(t,x) - Q(x-t) - \mathcal{V}_{k_0}^A(t,x-t)\|_{H^s} \leqslant Ce^{-(k_0+\frac{1}{2})e_0t}$$

and so (4.4) follows, which achieves the proof of Proposition 4.5.

4.4 Uniqueness

Now, the special solution U^A being constructed, we prove its uniqueness, in the sense of the following proposition, which implies the second part of Theorem 1.1.

Proposition 4.8. *Let u be a solution of* (gKdV) *such that*

$$\inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \xrightarrow[t\to+\infty]{} 0. \tag{4.11}$$

Then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for all $t \geqslant t_0$, where U^A is the solution of (gKdV) *defined in Proposition 4.5.*

The proof of Proposition 4.8 proceeds in four steps. First, we improve condition (4.11) into an exponential convergence and we control the translation parameter, then we improve the exponential convergence up to any order, and finally we adapt step 3 of [8] to (gKdV) to conclude the proof. A crucial argument for the first and third steps is the coercivity of $(L\cdot, \cdot)$ under orthogonality to eigenfunctions of the adjoint of $\mathcal{L}$, proved in [6].

4.4.1 Adjoint of $\mathcal{L}$

We recall that L is defined by $La = -\partial_x^2 a + a - pQ^{p-1}a$ and $\mathcal{L}$ by $\mathcal{L} = -\partial_x L$. In particular, the adjoint of $\mathcal{L}$ is $L\partial_x$. Moreover, $\mathcal{L}$ has two eigenfunctions $\mathcal{Y}_\pm$, with $\mathcal{L}\mathcal{Y}_\pm = \pm e_0 \mathcal{Y}_\pm$ where $e_0 > 0$.

Lemma 4.9. *Let $Z_\pm = L\mathcal{Y}_\pm$. Then the following properties hold.*

(i) $Z_\pm$ are two eigenfunctions of $L\partial_x$: $L(\partial_x Z_\pm) = \mp e_0 Z_\pm$.

(ii) $(\mathcal{Y}_+, Z_+) = (\mathcal{Y}_-, Z_-) = 0$ and $(Z_+, Q') = (Z_-, Q') = 0$.

(iii) There exists $\sigma_1 > 0$ such that, for all $v \in H^1$ such that $(v, Z_+) = (v, Z_-) = (v, Q') = 0$, $(Lv, v) \geqslant \sigma_1 \|v\|_{H^1}^2$.

(iv) One has $(\mathcal{Y}_+, Z_-) \neq 0$ and $(Q', \mathcal{Y}'_+) \neq 0$. Hence, one can normalize $\mathcal{Y}_\pm$ and $Z_\pm$ to have

$$(\mathcal{Y}_+, Z_-) = (\mathcal{Y}_-, Z_+) = 1, \quad (Q', \mathcal{Y}'_+) > 0 \quad \textit{and still} \quad L\mathcal{Y}_\pm = Z_\pm.$$

(v) There exist $\sigma_2 > 0$ and $C > 0$ such that, for all $v \in H^1$,

$$(Lv, v) \geqslant \sigma_2 \|v\|_{H^1}^2 - C(v, Z_+)^2 - C(v, Z_-)^2 - C(v, Q')^2. \tag{4.12}$$

Proof. (i) It suffices to apply L to the equality $-\partial_x(L\mathcal{Y}_\pm) = \pm e_0 \mathcal{Y}_\pm$.

(ii) We have $(\mathcal{Y}_\pm, Z_\pm) = \mp\frac{1}{e_0}(\partial_x(L\mathcal{Y}_\pm), L\mathcal{Y}_\pm) = 0$ and $(Z_\pm, Q') = (L\mathcal{Y}_\pm, Q') = (\mathcal{Y}_\pm, LQ') = 0$ since L is self-adjoint and $LQ' = 0$.

(iii) This fact is assertion (7) proved in [6].

(iv) If we had $(\mathcal{Y}_+, Z_-) = (Z_+, \mathcal{Y}_-) = 0$, then by (ii) we would have in fact $(\mathcal{Y}_+ + \mathcal{Y}_-) \perp Z_+, Z_-, Q'$ since Q' is odd and $\mathcal{Y}_+ + \mathcal{Y}_-$ is even, and so, by (iii), $(L(\mathcal{Y}_+ + \mathcal{Y}_-), \mathcal{Y}_+ + \mathcal{Y}_-) \geqslant \sigma_1 \|\mathcal{Y}_+ + \mathcal{Y}_-\|_{H^1}^2$. But $(L(\mathcal{Y}_+ + \mathcal{Y}_-), \mathcal{Y}_+ + \mathcal{Y}_-) = (L\mathcal{Y}_+, \mathcal{Y}_+) + (L\mathcal{Y}_-, \mathcal{Y}_-) + 2(L\mathcal{Y}_+, \mathcal{Y}_-) = (Z_+, \mathcal{Y}_+) + (Z_-, \mathcal{Y}_-) + 2(Z_+, \mathcal{Y}_-) = 0$, and so we would get $\|\mathcal{Y}_+ + \mathcal{Y}_-\|_{H^1} = 0$, *i.e.* $\mathcal{Y}_+ = -\mathcal{Y}_-$, which is a contradiction with the independence of the family $(\mathcal{Y}_+, \mathcal{Y}_-)$.

Similarly, if we had $(Q', \mathcal{Y}_+') = 0$, we would have $(Q'', \mathcal{Y}_+) = 0$. Moreover, we have $(Q, \mathcal{Y}_+) = -\frac{1}{e_0}(Q, (L\mathcal{Y}_+)') = \frac{1}{e_0}(LQ', \mathcal{Y}_+) = 0$, and so we would have

$$(Q, Z_+) = (Q, L\mathcal{Y}_+) = (LQ, \mathcal{Y}_+) = (-Q'' + Q - pQ^p, \mathcal{Y}_+) = -p(Q - Q'', \mathcal{Y}_+) = 0.$$

But we would also have $(Q, Z_-) = 0$ as Q is even and $Z_- = \check{Z}_+$. Since $(Q, Q') = 0$, we would finally have $(LQ, Q) \geqslant \sigma_1 \|Q\|_{H^1}^2$ by (iii). But a straightforward calculation gives $(LQ, Q) = (1-p)\int Q^{p+1} < 0$, and so a contradiction.

Finally, if we note $\eta = (\mathcal{Y}_+, Z_-) \neq 0$, then the normalization $\widetilde{\mathcal{Y}_-} = \frac{1}{\eta}\mathcal{Y}_-$, $\widetilde{Z_-} = \frac{1}{\eta}Z_- = L\widetilde{\mathcal{Y}_-}$ satisfies the required properties if $(Q', \mathcal{Y}_+') > 0$. Otherwise, it suffices to change $\mathcal{Y}_\pm$ and $Z_\pm$ in $-\mathcal{Y}_\pm$ and $-Z_\pm$ respectively.

(v) Let $v \in H^1$, and decompose it as

$$v = \alpha\mathcal{Y}_+ + \beta\mathcal{Y}_- + \gamma Q' + v_\perp$$

with $\alpha = (v, Z_-)$, $\beta = (v, Z_+)$, $\gamma = \|Q'\|_{L^2}^{-2}[(v, Q') - \alpha(\mathcal{Y}_+, Q') - \beta(\mathcal{Y}_-, Q')]$ and $v_\perp$ orthogonal to Z_+, Z_-, Q' by the previous normalization. We have, by straightforward calculation, $(Lv, v) = (Lv_\perp, v_\perp) + 2\alpha\beta$, and $(Lv_\perp, v_\perp) \geqslant \sigma_1 \|v_\perp\|_{H^1}^2$ by (iii), so we have $(Lv, v) \geqslant \sigma_1 \|v_\perp\|_{H^1}^2 - \alpha^2 - \beta^2$. Finally, we have by the previous decomposition of v that

$$\|v\|_{H^1}^2 \leqslant C(\alpha^2 + \beta^2 + \gamma^2 + \|v_\perp\|_{H^1}^2) \leqslant C'(\alpha^2 + \beta^2 + (v, Q')^2 + \|v_\perp\|_{H^1}^2)$$

and so $(Lv, v) \geqslant \sigma_1 \left[\frac{\|v\|_{H^1}^2}{C'} - \alpha^2 - \beta^2 - (v, Q')^2\right] - \alpha^2 - \beta^2$, as desired. □

4.4.2 Step 1: Improvement of the decay at infinity

We begin the proof of Proposition 4.8 here: let u be a solution of (gKdV) satisfying (4.11).

- By Lemma 2.10, we can write $\varepsilon(t, x) = u(t, x + x(t)) - Q(x)$ for $t \geqslant t_0$ with t_0 large enough, where ε satisfies $\|\varepsilon(t)\|_{H^1} \longrightarrow 0$ and $\varepsilon(t) \perp Q'$ for all $t \geqslant t_0$. We recall that we have, by Proposition 2.15,

$$\varepsilon_t - (L\varepsilon)_x = (x' - 1)(Q + \varepsilon)_x + R(\varepsilon) \tag{4.13}$$

where $\|R(\varepsilon)\|_{L^1} \leqslant C\|\varepsilon\|_{H^1}^2$ and $|x' - 1| \leqslant C\|\varepsilon\|_{H^1}$.

- Now consider

$$\alpha_+(t) = \int Z_+\varepsilon(t), \qquad \alpha_-(t) = \int Z_-\varepsilon(t)$$

where $Z_\pm$ are defined in Lemma 4.9. Since $\|\varepsilon(t)\|_{H^1} \longrightarrow 0$, we have of course $\alpha_\pm(t) \longrightarrow 0$. The two remaining points will be to show that $\alpha_\pm(t)$ control $\|\varepsilon(t)\|_{H^1}$, and have exponential decay at infinity.

- First, we recall that, by linearization of Weinstein's functional (Lemma 2.4),

$$F(Q+\varepsilon) = F(Q) + \frac{1}{2}(L\varepsilon,\varepsilon) + K(\varepsilon)$$

where $|K(\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^3$. But $F(Q+\varepsilon) - F(Q)$ is a constant which tends to 0 at infinity in time, and so is null, hence we get $|(L\varepsilon,\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^3$. We now use (4.12), which gives, since $(\varepsilon, Q') = 0$,

$$(L\varepsilon,\varepsilon) \geqslant \sigma_2\|\varepsilon(t)\|_{H^1}^2 - C\alpha_+^2(t) - C\alpha_-^2(t)$$

and so $\sigma_2\|\varepsilon(t)\|_{H^1}^2 - C\alpha_+^2(t) - C\alpha_-^2(t) - C'\|\varepsilon(t)\|_{H^1}^3 \leqslant 0$. For t_0 chosen possibly larger, we conclude that

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant C(\alpha_+^2(t) + \alpha_-^2(t)).$$

- We have now to obtain exponential decay of $\alpha_\pm$ to conclude the first step. If we multiply (4.13) by Z_+ and integrate, we obtain

$$\alpha_+'(t) - e_0\alpha_+(t) = (x'-1)\int (Q+\varepsilon)_x Z_+ + \int R(\varepsilon)Z_+ = (x'-1)\int \varepsilon_x Z_+ + \int R(\varepsilon)Z_+$$

by integrating by parts and using (i) and (ii) of Lemma 4.9. By the controls of $|x'-1|$ and $R(\varepsilon)$, we get $|\alpha_+' - e_0\alpha_+| \leqslant C\|\varepsilon\|_{H^1}^2 \leqslant C(\alpha_+^2 + \alpha_-^2)$. Doing similarly with Z_-, we have finally the differential system

$$\begin{cases} |\alpha_+' - e_0\alpha_+| \leqslant C(\alpha_+^2 + \alpha_-^2), & (4.14)\\ |\alpha_-' + e_0\alpha_-| \leqslant C(\alpha_+^2 + \alpha_-^2). & (4.15)\end{cases}$$

- Now define $h(t) = \alpha_+(t) - M\alpha_-^2(t)$ where M is a large constant to define later. Multiplying (4.15) by $|\alpha_-|$ (which can of course be taken less than 1), we get

$$\begin{aligned} h'(t) &= \alpha_+'(t) - 2M\alpha_-(t)\alpha_-'(t)\\ &\geqslant e_0\alpha_+ - C(\alpha_+^2+\alpha_-^2) + 2Me_0\alpha_-^2 - 2CM|\alpha_-|(\alpha_+^2+\alpha_-^2)\\ &\geqslant e_0 h + 3Me_0\alpha_-^2 - 2Ch^2 - 2CM^2\alpha_-^4 - C^*\alpha_-^2\\ &\quad - 4CMh^2 - 4CM^3|\alpha_-|^5 - 2CM|\alpha_-|^3\end{aligned}$$

since $\alpha_+^2 = \left(h + M\alpha_-^2\right)^2 \leqslant 2(h^2 + M^2\alpha_-^4)$. We now fix $M = \frac{C^*}{e_0}$, so that

$$h' \geqslant e_0 h - 2Ch^2 - 4CMh^2 + \alpha_-^2\left(2Me_0 - 2CM^2\alpha_-^2 - 4CM^3|\alpha_-|^3 - 2CM|\alpha_-|\right).$$

Then, for t large enough, the expression in parenthesis is positive, and so

$$h' \geqslant e_0 h - c_M h^2.$$

Now take t_0 large enough such that, for $t \geqslant t_0$, we have $c_M h^2 \leqslant \frac{e_0}{2}|h|$, and suppose for the sake of contradiction that there exists $t_1 \geqslant t_0$ such that $h(t_1) > 0$. Define $T = \sup\{t \geqslant t_1 \mid h(t) > 0\}$ and suppose that $T < +\infty$. As we have $h'(t) \geqslant e_0\left(h(t) - \frac{|h(t)|}{2}\right)$ for all $t \geqslant t_0$ and of course $h(T) = 0$, we would have in particular $h'(T) \geqslant 0$, so h increasing near T, and so $h(t) \leqslant 0$ for $t \in [T-\varepsilon, T]$, which would be in contradiction with the definition of T. Hence we have $T = +\infty$, and so $h(t) > 0$ for all $t \geqslant t_1$. Consequently, we would have $h'(t) \geqslant \frac{e_0}{2}h(t)$ for all $t \geqslant t_1$, and so $h(t) \geqslant Ce^{\frac{e_0}{2}t}$, which would be a contradiction with $\lim_{t\to+\infty} h(t) = 0$. Therefore, we have $h(t) \leqslant 0$ for all $t \geqslant t_0$. Since $-\alpha_+$ satisfies the same differential system, we obtain by the same technique, for all $t \geqslant t_0$, $|\alpha_+(t)| \leqslant M\alpha_-^2(t)$.

- Reporting this estimate in (4.15), we obtain

$$|\alpha_-'(t) + e_0\alpha_-(t)| \leqslant C\alpha_-^2(t) \leqslant \frac{e_0}{10}|\alpha_-(t)|$$

for t large enough. In other words, we have $|(e^{e_0t}\alpha_-(t))'| \leqslant \frac{e_0}{10}|e^{e_0t}\alpha_-(t)|$, and so, by integration, $|\alpha_-(t)| \leqslant Ce^{-\frac{9}{10}e_0t}$. Using a bootstrap argument, we obtain $|\alpha_-'(t) + e_0\alpha_-(t)| \leqslant Ce^{-\frac{9}{10}e_0t}|\alpha_-(t)|$, and so, we get $|e^{e_0t}\alpha_-(t)| \leqslant C$ for all $t \geqslant t_0$, still by integration, *i.e.* $|\alpha_-(t)| \leqslant Ce^{-e_0t}$. By the previous point, we also obtain

$$|\alpha_+(t)| \leqslant Ce^{-2e_0t} \tag{4.16}$$

and finally, $\|\varepsilon(t)\|_{H^1}^2 \leqslant C(\alpha_+^2(t) + \alpha_-^2(t)) \leqslant Ce^{-2e_0t}$.

For clarity, we summarize the results obtained so far.

Lemma 4.10. *If u is a solution of* (gKdV) *which satisfies*

$$\inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \xrightarrow[t\to+\infty]{} 0,$$

then there exist a C^1 map $x : t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $C > 0$ such that

$$\forall t \geqslant t_0, \quad \|u(t, \cdot + x(t)) - Q\|_{H^1} \leqslant Ce^{-e_0t}.$$

4.4.3 Step 2: Removing modulation

- From the previous point, we have in fact $|(e^{e_0t}\alpha_-(t))'| \leqslant Ce^{-e_0t} \in L^1([t_0, +\infty))$, and so there exists

$$\lim_{t\to+\infty} e^{e_0t}\alpha_-(t) =: A \in \mathbb{R}$$

with $|e^{e_0t}\alpha_-(t) - A| \leqslant Ce^{-e_0t}$ for $t \geqslant t_0$ by integration. Similarly, since $|x'(t) - 1| \leqslant C\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_0t}$, then $\exists \lim_{t\to+\infty} x(t) - t =: x_0 \in \mathbb{R}$ with $|x(t) - t - x_0| \leqslant Ce^{-e_0t}$.

- Now consider the special solution U^A constructed in Proposition 4.5, defined for a t_0 chosen possibly larger, and still write $U^A(t, x + t) = Q(x) + h^A(t, x)$. Let

$$v(t,x) = u(t, x + t + x_0) - Q(x) - h^A(t,x) = u(t, x + t + x_0) - U^A(t, x + t).$$

So we want to prove that $v = 0$ to complete the proof of Proposition 4.8. We first give estimates on v using the previous estimates on ε.

- Since $v(t,x) = \varepsilon(t, x - (x(t) - t - x_0)) - h^A(t,x) + Q(x - (x(t) - t - x_0)) - Q(x)$, we simply obtain exponential decay for v for t_0 large enough, by Lemma 2.11 and exponential decay of h^A, if we write

$$\begin{aligned}\|v(t)\|_{H^1} &\leqslant \|\varepsilon(t)\|_{H^1} + \|h^A(t)\|_{H^1} + \|Q - Q(\cdot - (x(t) - t - x_0))\|_{H^1} \\ &\leqslant Ce^{-e_0t} + C|x(t) - t - x_0| \leqslant Ce^{-e_0t}.\end{aligned}$$

- Moreover, we can write

$$u(t,x) = Q(x-x(t))+\varepsilon(t,x-x(t)) = Q(x-t-x_0)+h^A(t,x-t-x_0)+v(t,x-t-x_0).$$

If we denote $\omega(t,x) = Q(x-(x(t)-t-x_0)) - Q(x) - (x(t)-t-x_0)Q'(x)$, we have $\|\omega(t)\|_{L^\infty} \leqslant C(x(t)-t-x_0)^2 \leqslant Ce^{-2e_0t}$ by Taylor-Lagrange inequality, and

$$v(t,x) = (x(t)-t-x_0)Q'(x) - h^A(t,x) + \varepsilon(t,x-(x(t)-t-x_0)) + \omega(t,x).$$

Moreover, we have, for all $x \in \mathbb{R}$ and $t \geqslant t_0$,

$$\begin{aligned}|\varepsilon(t,x-(x(t)-t-x_0)) - \varepsilon(t,x)| &= \left|\int_x^{x-(x(t)-t-x_0)} \partial_x\varepsilon(t,s)\,ds\right| \\ &\leqslant \sqrt{|x(t)-t-x_0|}\cdot\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\frac{3}{2}e_0t}\end{aligned}$$

by the Cauchy-Schwarz inequality. We have finally

$$v(t,x) = (x(t)-t-x_0)Q'(x) - h^A(t,x) + \varepsilon(t,x) + \omega(t,x) \tag{4.17}$$

where ω satisfies $\|\omega(t)\|_{L^\infty} \leqslant Ce^{-\frac{3}{2}e_0t}$.

- Following the proof of (v) in Lemma 4.9, we now decompose

$$v(t,x) = \alpha_+^A(t)\mathcal{Y}_-(x) + \alpha_-^A(t)\mathcal{Y}_+(x) + \beta(t)Q'(x) + v_\perp(t,x) \tag{4.18}$$

with

$$\alpha_+^A(t) = \int Z_+v(t),\ \alpha_-^A(t) = \int Z_-v(t)$$

and

$$\beta(t) = \|Q'\|_{L^2}^{-2}\int\Big(v(t) - \alpha_+^A(t)\mathcal{Y}_- - \alpha_-^A(t)\mathcal{Y}_+\Big)Q'.$$

Hence, we have $(v_\perp, Q') = (v_\perp, Z_+) = (v_\perp, Z_-) = 0$, and so, by (iii) of Lemma 4.9,

$$(Lv_\perp, v_\perp) \geqslant \sigma_1\|v_\perp\|_{H^1}^2. \tag{4.19}$$

- Multiplying (4.17) by $Z_\pm$, we obtain estimates on $\alpha_\pm^A$. Indeed, since $(Z_\pm, Q') = 0$, we have

$$\alpha_\pm^A = -(h^A, Z_\pm) + \alpha_\pm + (\omega, Z_\pm).$$

But $|(h^A, Z_+)| \leqslant Ce^{-2e_0t}$ since $(\mathcal{Y}_+, Z_+) = 0$, and $|\alpha_+| \leqslant Ce^{-2e_0t}$ by (4.16), and so $|\alpha_+^A| \leqslant Ce^{-\frac{3}{2}e_0t}$. Similarly, $(\mathcal{Y}_+, Z_-) = 1$ implies that $|(h^A, Z_-) - Ae^{-e_0t}| \leqslant Ce^{-2e_0t}$, and since $|\alpha_- - Ae^{-e_0t}| \leqslant Ce^{-2e_0t}$, we also get $|\alpha_-^A| \leqslant Ce^{-\frac{3}{2}e_0t}$. To sum up this step, we have (4.18) with the following estimates, for $t \geqslant t_0$,

$$|\alpha_+^A(t)| \leqslant Ce^{-\frac{3}{2}e_0t},\quad |\alpha_-^A(t)| \leqslant Ce^{-\frac{3}{2}e_0t},\quad \|v(t)\|_{H^1} \leqslant Ce^{-e_0t}. \tag{4.20}$$

In (4.20), it is essential to have obtained estimates better than Ce^{-e_0t} for $\alpha_\pm^A$ (see next step).

4.4.4 Step 3: Exponential decay at any order

- We want to prove in this section that v decays exponentially at any order to 0. In other words, we prove that

$$\forall \gamma > 0, \exists C_\gamma > 0, \forall t \geqslant t_0, \quad \|v(t)\|_{H^1} \leqslant C_\gamma e^{-\gamma t}.$$

It has been proved for $\gamma = e_0$, so that it is enough to prove it by induction on $\gamma \geqslant e_0$. Suppose that $\|v(t)\|_{H^1} \leqslant Ce^{-\gamma t}$, and let us prove that it implies $\|v(t)\|_{H^1} \leqslant C' e^{-(\gamma+\frac{1}{2}e_0)t}$.

- Since u and U^A are solutions of (gKdV), v satisfies

$$\partial_t v - \partial_x v + \partial_x^3 v + \partial_x \left[\left(Q + h^A + v\right)^p - \left(Q + h^A\right)^p \right] = 0. \tag{4.21}$$

But

$$\begin{aligned}\left(Q + h^A + v\right)^p - \left(Q + h^A\right)^p &= p\left(Q + h^A\right)^{p-1} v + \sum_{k=2}^{p} \binom{p}{k} \left(Q + h^A\right)^{p-k} v^k \\ &= pQ^{p-1} v + \omega_1(t,x) v + \omega_2(t,x) v^2\end{aligned}$$

where

$$\omega_1(t,x) = \sum_{k=1}^{p-1} p \binom{p-1}{k} Q^{p-1-k} \left(h^A\right)^k$$

and

$$\omega_2(t,x) = \sum_{k=2}^{p} \binom{p}{k} \left(Q + h^A\right)^{p-k} v^{k-2}.$$

Since $\|h^A(t)\|_{L^\infty} \leqslant C\|h^A(t)\|_{H^1} \leqslant Ce^{-e_0 t}$ and $\|v(t)\|_{L^\infty} \leqslant C\|v(t)\|_{H^1} \leqslant C$, we have the estimates

$$\|\omega_1(t)\|_{L^\infty} \leqslant Ce^{-e_0 t}, \quad \|\omega_2(t)\|_{L^\infty} \leqslant C, \tag{4.22}$$

and (4.21) can be rewritten

$$\partial_t v + \mathcal{L} v + \partial_x [\omega_1(t,x) v] + \partial_x [\omega_2(t,x) v^2] = 0. \tag{4.23}$$

- If we multiply (4.23) by Z_+ and integrate, we get $\alpha_+^{A\prime} - e_0 \alpha_+^A = \int \omega_1 v Z_+' + \int \omega_2 v^2 Z_+'$, and so

$$\begin{aligned}|\alpha_+^{A\prime} - e_0 \alpha_+^A| &\leqslant \|\omega_1(t)\|_{L^\infty} \|v(t)\|_{L^\infty} \|Z_+'\|_{L^1} + \|\omega_2(t)\|_{L^\infty} \|v(t)\|_{L^\infty}^2 \|Z_+'\|_{L^1} \\ &\leqslant Ce^{-(\gamma+e_0)t} + Ce^{-2\gamma t} \leqslant Ce^{-(\gamma+e_0)t}.\end{aligned}$$

Consequently, we have $|(e^{-e_0 t}\alpha_+^A)'| \leqslant Ce^{-(\gamma+2e_0)t}$, and since $e^{-e_0 t}\alpha_+^A(t) \xrightarrow[t\to+\infty]{} 0$ by (4.20), we get by integration $|\alpha_+^A(t)| \leqslant Ce^{-(\gamma+e_0)t}$.

Multiplying (4.23) by Z_-, we obtain similarly $|\alpha_-^{A\prime} + e_0 \alpha_-^A| \leqslant Ce^{-(\gamma+e_0)t}$, and so $|\alpha_-^A(t)| \leqslant Ce^{-(\gamma+e_0)t}$, since $|e^{e_0 t}\alpha_-^A(t)| \leqslant Ce^{-\frac{1}{2}e_0 t} \xrightarrow[t\to+\infty]{} 0$ still by (4.20).

- We want now to estimate $|(Lv,v)|$. To do this, we rewrite (4.21) as
$$\partial_t v + \partial_x\left[\partial_x^2 v - v + \left(Q+h^A+v\right)^p - \left(Q+h^A\right)^p\right] = 0,$$
multiply this equality by the expression in the brackets and integrate, to obtain $\int \partial_t v \cdot \left[\partial_x^2 v - v + \left(Q+h^A+v\right)^p - \left(Q+h^A\right)^p\right] = 0$. In other words, if we define
$$F(t) = \frac{1}{2}\int v_x^2 + \frac{1}{2}\int v^2 - \int \frac{1}{p+1}\left(Q+h^A+v\right)^{p+1} + \int v\left(h^A+Q\right)^p + \int \frac{1}{p+1}\left(h^A+Q\right)^{p+1},$$
we have $F'(t) = -\int \partial_t h^A \cdot \left[\left(Q+h^A+v\right)^p - \left(Q+h^A\right)^p - pv\left(Q+h^A\right)^{p-1}\right]$.
But h^A satisfies (4.3) by definition, so $\partial_t h^A = -\partial_x^3 h^A + \partial_x h^A - p\partial_x(Q^{p-1}h^A) + R(h^A)$. Moreover, by Proposition 4.5, there exists $C>0$ such that, for all $t \geqslant t_0$, we have $\|h^A(t)\|_{H^4} \leqslant Ce^{-e_0 t}$. We deduce that
$$\|\partial_t h^A\|_{L^\infty} \leqslant C\|\partial_t h^A\|_{H^1} \leqslant C\|h^A(t)\|_{H^4} \leqslant Ce^{-e_0 t}.$$
Therefore, $|F'(t)| \leqslant C\|\partial_t h^A\|_{L^\infty}\|v(t)\|_{L^2}^2 \leqslant Ce^{-(2\gamma+e_0)t}$, and so $|F(t)| \leqslant Ce^{-(2\gamma+e_0)t}$ by integration, since $\lim_{t\to+\infty} F(t) = 0$. Moreover, by developing $\left(Q+h^A+v\right)^{p+1}$ in the expression of F, we get
$$\begin{aligned} F(t) &= \frac{1}{2}\int (v_x^2+v^2) - \frac{p}{2}\int \left(Q+h^A\right)^{p-1} v^2 - \frac{1}{p+1}\sum_{k=3}^{p+1}\binom{p+1}{k}\int \left(Q+h^A\right)^{p+1-k} v^k \\ &= \frac{1}{2}(Lv,v) - \frac{1}{2}\int \omega_1(t,x)v^2 - \int \widetilde{\omega_2}(t,x)v^3 \end{aligned}$$
where ω_1 defined above and $\widetilde{\omega_2}(t,x) = \frac{1}{p+1}\sum_{k=3}^{p+1}\binom{p+1}{k}\left(Q+h^A\right)^{p+1-k} v^{k-3}$ satisfy the estimates $\|\omega_1(t)\|_{L^\infty} \leqslant Ce^{-e_0 t}$ and $\|\widetilde{\omega_2}(t)\|_{L^\infty} \leqslant C$. Hence, we have
$$\begin{aligned} \left|F(t) - \frac{1}{2}(Lv,v)\right| &\leqslant \frac{1}{2}\|\omega_1(t)\|_{L^\infty}\|v(t)\|_{L^2}^2 + \|\widetilde{\omega_2}(t)\|_{L^\infty}\|v(t)\|_{H^1}^3 \\ &\leqslant Ce^{-(2\gamma+e_0)t} + Ce^{-3\gamma t} \leqslant Ce^{-(2\gamma+e_0)t}. \end{aligned}$$
Thus, we finally obtain $|(Lv,v)| \leqslant Ce^{-(2\gamma+e_0)t}$.

- The previous points allow us to estimate $\|v_\perp\|_{H^1}$. Indeed, we have, by straightforward calculation from (4.18), the identity
$$(Lv,v) = (Lv_\perp, v_\perp) + 2\alpha_+^A\alpha_-^A,$$
and so $|(Lv_\perp,v_\perp)| \leqslant |(Lv,v)| + 2|\alpha_+^A|\cdot|\alpha_-^A| \leqslant Ce^{-(2\gamma+e_0)t} + Ce^{-(2\gamma+2e_0)t} \leqslant Ce^{-(2\gamma+e_0)t}$. But from (4.19), we deduce that $\sigma_1\|v_\perp\|_{H^1}^2 \leqslant Ce^{-(2\gamma+e_0)t}$, and so $\|v_\perp\|_{H^1} \leqslant Ce^{-(\gamma+\frac{1}{2}e_0)t}$.

- To conclude this step, it is now enough to estimate $|\beta(t)|$, since the conclusion will immediately follow from decomposition (4.18). To do this, we first multiply (4.23) by Q' and integrate, so that
$$\begin{aligned} |(\partial_t v, Q') + (\mathcal{L}v, Q')| &\leqslant \|\omega_1(t)\|_{L^\infty}\|v(t)\|_{L^\infty}\|Q''\|_{L^1} + \|\omega_2(t)\|_{L^\infty}\|v(t)\|_{L^\infty}^2\|Q''\|_{L^1} \\ &\leqslant Ce^{-(\gamma+e_0)t} + Ce^{-2\gamma t} \leqslant Ce^{-(\gamma+e_0)t}. \end{aligned}$$

Moreover, by applying $\mathcal{L}$ to (4.18), we get $\mathcal{L}v = -e_0\alpha_+^A\mathcal{Y}_- + e_0\alpha_-^A\mathcal{Y}_+ + \mathcal{L}v_\perp$, and so

$$\begin{aligned}\|Q'\|_{L^2}^2\beta'(t) &= (\partial_t v - \alpha_+^{A'}\mathcal{Y}_- - \alpha_-^{A'}\mathcal{Y}_+, Q')\\ &= (\partial_t v + \mathcal{L}v, Q') - (-e_0\alpha_+^A\mathcal{Y}_- + e_0\alpha_-^A\mathcal{Y}_+ + \alpha_+^{A'}\mathcal{Y}_- + \alpha_-^{A'}\mathcal{Y}_+, Q') - (\mathcal{L}v_\perp, Q')\\ &= (\partial_t v + \mathcal{L}v, Q') - (\alpha_+^{A'} - e_0\alpha_+^A)(\mathcal{Y}_-, Q') - (\alpha_-^{A'} + e_0\alpha_-^A)(\mathcal{Y}_+, Q') + (v_\perp, LQ'').\end{aligned}$$

Finally, we obtain, thanks to all previous estimates,

$$\begin{aligned}|\beta'(t)| &\leqslant C|(\partial_t v + \mathcal{L}v, Q')| + C|\alpha_+^{A'} - e_0\alpha_+^A| + C|\alpha_-^{A'} + e_0\alpha_-^A| + C\|v_\perp\|_{L^2}\\ &\leqslant Ce^{-(\gamma+e_0)t} + Ce^{-(\gamma+e_0)t} + Ce^{-(\gamma+e_0)t} + Ce^{-(\gamma+\frac{1}{2}e_0)t} \leqslant Ce^{-(\gamma+\frac{1}{2}e_0)t},\end{aligned}$$

and so $|\beta(t)| \leqslant Ce^{-(\gamma+\frac{1}{2}e_0)t}$ by integration.

4.4.5 Step 4: Conclusion of uniqueness argument by contraction

- The final argument, which corresponds to step 3 in [8], is an argument of contraction in short time. In other words, we want to reproduce the contraction argument developed in Section 4.3.2 on a short interval of time, with suitable norms.

  Define $w(t,x) = v(t,x-t)$, so that (4.21) can be rewritten

  $$\partial_t w + \partial_x^3 w = -\partial_x\left[\left(Q(x-t) + h^A(t,x-t) + w\right)^p - \left(Q(x-t) + h^A(t,x-t)\right)^p\right].$$

  If we denote $\Omega_w(t,x) = \sum_{k=1}^p \binom{p}{k}\left(Q(x-t) + h^A(t,x-t)\right)^{p-k} w^k(t,x)$, then the equation of w can be rewritten

  $$\partial_t w + \partial_x^3 w = -\partial_x(\Omega_w).$$

  Moreover, we have, by previous steps,

  $$\forall\gamma > 0, \exists C_\gamma > 0, \forall t \geqslant t_0, \quad \|w(t)\|_{H^1} \leqslant C_\gamma e^{-\gamma t}.$$

- Now let $t_1 \geqslant t_0$, $\tau > 0$ to fix later, and $I = (t_1, t_1+\tau)$. Moreover, consider the nonlinear equation in $\tilde{w}$

  $$\begin{cases}\partial_t\tilde{w} + \partial_x^3\tilde{w} = -\partial_x(\Omega_{\tilde{w}}),\\ \tilde{w}(t_1+\tau) = w(t_1+\tau).\end{cases} \tag{4.24}$$

  Note that w is of course a solution of (4.24), associated to a solution u of (gKdV) in the sense of [12].

- Then, for $t \in I$, we have the Duhamel's formula

  $$\tilde{w}(t) = \mathcal{M}^I(\tilde{w})(t) := W(t-t_1-\tau)w(t_1+\tau) + \int_t^{t_1+\tau} W(t-t')\partial_x[\Omega_{\tilde{w}}(t')]\,dt'.$$

  Similarly as in Section 4.3.2, we consider

  $$\begin{cases}N_1^I(\tilde{w}) = \sup_{t\in I}\|\tilde{w}(t)\|_{H^1}, \ N_2^I(\tilde{w}) = \|\tilde{w}\|_{L_x^5L_I^{10}} + \|\partial_x\tilde{w}\|_{L_x^5L_I^{10}},\\ \Lambda^I(\tilde{w}) = \max(N_1^I(\tilde{w}), N_2^I(\tilde{w})),\end{cases}$$

and we prove that for t_1 large enough, τ small enough independently of t_1, and $K > 1$ to determine, $\widetilde{w} \mapsto \mathcal{M}^I(\widetilde{w})$ is a contraction on

$$B = \{\widetilde{w} \in C^0(I, H^1) \mid \Lambda^I(\widetilde{w}) \leqslant 3K\|w(t_1+\tau)\|_{H^1}\}.$$

In other words, we want to estimate $\Lambda^I(\mathcal{M}^I(\widetilde{w}))$ in terms of $\Lambda^I(\widetilde{w})$, and as in Section 4.3.2, we estimate only the term

$$\partial_x \mathcal{M}^I(\widetilde{w})(t) = W(t-t_1-\tau)\partial_x w(t_1+\tau) + \frac{\partial}{\partial x}\int_t^{t_1+\tau} W(t-t')\partial_x[\Omega_{\widetilde{w}}(t')]\,dt'$$

in $L_I^\infty L_x^2$ and $L_x^5 L_I^{10}$ norms. The term $\mathcal{M}^I(\widetilde{w})(t)$ is treated similarly.

- Firstly, for the linear term, we have

$$\begin{cases} \|W(t-t_1-\tau)\partial_x w(t_1+\tau)\|_{L^2} = \|\partial_x w(t_1+\tau)\|_{L^2} \leqslant \|w(t_1+\tau)\|_{H^1}, \\ \|W(t-t_1-\tau)\partial_x w(t_1+\tau)\|_{L_x^5 L_I^{10}} \leqslant C\|\partial_x w(t_1+\tau)\|_{L^2} \leqslant C\|w(t_1+\tau)\|_{H^1}, \end{cases}$$

since W is unitary on L^2 and $\|W(t)u_0\|_{L_x^5 L_t^{10}} \leqslant C\|u_0\|_{L^2}$, which is the linear estimate (2.3) of [13].

- For the nonlinear term, we have to use estimates similar to (4.1) and (4.2). We obtain easily by a similar proof that, for all $g \in L_x^1 L_I^2$,

$$\left\|\frac{\partial}{\partial x}\int_t^{t_1+\tau} W(t-t')g(t',x)\,dt'\right\|_{L_I^\infty L_x^2} + \left\|\frac{\partial}{\partial x}\int_t^{t_1+\tau} W(t-t')g(t',x)\,dt'\right\|_{L_x^5 L_I^{10}} \leqslant C\|g\|_{L_x^1 L_I^2}.$$

Hence, we get

$$\begin{cases} \left\|\frac{\partial}{\partial x}\int_t^{t_1+\tau} W(t-t')\partial_x[\Omega_{\widetilde{w}}(t')]\,dt'\right\|_{L_I^\infty L_x^2} \leqslant C\|\partial_x(\Omega_{\widetilde{w}})\|_{L_x^1 L_I^2}, \\ \left\|\frac{\partial}{\partial x}\int_t^{t_1+\tau} W(t-t')\partial_x[\Omega_{\widetilde{w}}(t')]\,dt'\right\|_{L_x^5 L_I^{10}} \leqslant C\|\partial_x(\Omega_{\widetilde{w}})\|_{L_x^1 L_I^2}. \end{cases}$$

We deduce that we only have to estimate $\|\partial_x(\Omega_{\widetilde{w}})\|_{L_x^1 L_I^2}$. There are many terms to estimate, so as in Section 4.3.2, we only treat three typical terms: $\mathbf{A} = \|\partial_x\widetilde{w}\cdot\widetilde{w}^4\cdot\widetilde{w}^{p-5}\|_{L_x^1 L_I^2}$, $\mathbf{B} = \|\partial_x\widetilde{w}\cdot(h^A)^{p-1}(t,x-t)\|_{L_x^1 L_I^2}$, and the term $\mathbf{D} = \|\partial_x\widetilde{w}\cdot Q^{p-1}(x-t)\|_{L_x^1 L_I^2}$.

For $\mathbf{A}$, we have, by Hölder's inequality,

$$\mathbf{A} \leqslant \|\widetilde{w}\|_{L_I^\infty L_x^\infty}^{p-5}\|\partial_x\widetilde{w}\|_{L_x^5 L_I^{10}}\|\widetilde{w}\|_{L_x^5 L_I^{10}}^4 \leqslant Ce^{-e_0 t_1}N_2^I(\widetilde{w})^5 \leqslant C'e^{-e_0 t_1}N_2^I(\widetilde{w}).$$

Indeed, we have

$$\Lambda^I(\widetilde{w}) \leqslant 3K\|w(t_1+\tau)\|_{H^1} \leqslant Ce^{-e_0 t_1} \leqslant 1$$

for t_1 large enough, by exponential decay of w in H^1. In particular, we have $N_2^I(\widetilde{w}) \leqslant 1$ and $\|\widetilde{w}\|_{L_I^\infty L_x^\infty}^{p-5} \leqslant CN_1^I(\widetilde{w})^{p-5} \leqslant Ce^{-e_0 t_1}$ since $p-5 \geqslant 1$.

For $\mathbf{B}$, we write similarly

$$\mathbf{B} \leqslant \|h^A\|_{L_I^\infty L_x^\infty}^{p-5}\|\partial_x\widetilde{w}\|_{L_x^5 L_I^{10}}\|h^A(t,x-t)\|_{L_x^5 L_I^{10}}^4.$$

Moreover, we have, by construction of h^A (see Section 4.3.2), $\|h^A\|^{p-5}_{L^\infty_I L^\infty_x} \leqslant Ce^{-e_0 t_1}$ since $\|h^A(t)\|_{H^1} \leqslant Ce^{-e_0 t} \leqslant Ce^{-e_0 t_1}$ for $t \geqslant t_1$ and $p-5 \geqslant 1$, and

$$\begin{aligned}
\|h^A(t,x-t)\|_{L^5_x L^{10}_I} &\leqslant \|h^A(t,x-t)\|_{L^5_x L^{10}_{[t_1,+\infty)}} \\
&\leqslant \|(h^A - \mathcal{V}^A_{k_0})(t,x-t)\|_{L^5_x L^{10}_{[t_1,+\infty)}} + \|V^A_{k_0}(t,x-t)\|_{L^5_x L^{10}_{[t_1,+\infty)}} \\
&\leqslant Ce^{-(k_0+\frac{1}{2})e_0 t_1} + Ce^{-e_0 t_1} \leqslant Ce^{-e_0 t_1}.
\end{aligned}$$

Note that the estimate $\|V^A_{k_0}(t,x-t)\|_{L^5_x L^{10}_{[t_1,+\infty)}} \leqslant Ce^{-e_0 t_1}$ follows from the paragraph on $\mathbf{II}_{p-2,1,1}$ in Section 4.3.2.

For $\mathbf{D}$, we use exponential decay of Q to write

$$\begin{aligned}
\mathbf{D} &\leqslant C\int_{\mathbb{R}}\sqrt{\int_I e^{-2|x-t|}(\partial_x\tilde{w})^2\,dtdx} \leqslant C\int_{-\infty}^{t_1} e^x\sqrt{\int_I e^{-2t}(\partial_x\tilde{w})^2\,dtdx} \\
&+ C\int_{t_1+\tau}^{+\infty} e^{-x}\sqrt{\int_I e^{2t}(\partial_x\tilde{w})^2\,dtdx} + C\int_I\sqrt{\int_I(\partial_x\tilde{w})^2\,dtdx} = \mathbf{D_1}+\mathbf{D_2}+\mathbf{D_3}.
\end{aligned}$$

But by the Cauchy-Schwarz inequality, we get

$$\begin{cases}
\mathbf{D_1} \leqslant Ce^{t_1}\sqrt{\int_I e^{-2t}\int_{\mathbb{R}}(\partial_x\tilde{w})^2\,dx\,dt} \leqslant Ce^{t_1}N_1^I(\tilde{w})\sqrt{\int_I e^{-2t}\,dt} \leqslant C\sqrt{\tau}N_1^I(\tilde{w}), \\
\mathbf{D_2} \leqslant Ce^{-(t_1+\tau)}N_1^I(\tilde{w})\sqrt{\int_I e^{2t}\,dt} \leqslant C\sqrt{\tau}N_1^I(\tilde{w}), \\
\mathbf{D_3} \leqslant C\sqrt{\tau}\sqrt{\int_I\int_I(\partial_x\tilde{w})^2\,dx\,dt} \leqslant C\tau N_1^I(\tilde{w}).
\end{cases}$$

Hence, we obtain $\mathbf{D} \leqslant C\sqrt{\tau}N_1^I(\tilde{w})$.

- In conclusion, we have shown that there exist $K, C_1, C_2 > 0$ such that

$$\Lambda^I(\mathcal{M}^I(\tilde{w})) \leqslant K\Big[\|w(t_1+\tau)\|_{H^1} + C_1e^{-e_0 t_1}\Lambda^I(\tilde{w}) + C_2\sqrt{\tau}\Lambda^I(\tilde{w})\Big].$$

Now fix $\tau = \frac{1}{9C_2^2K^2}$ and t_1 such that $C_1e^{-e_0 t_1} \leqslant \frac{1}{3K}$, so we get

$$\Lambda^I(\mathcal{M}^I(\tilde{w})) \leqslant K\|w(t_1+\tau)\|_{H^1} + \frac{2}{3}\Lambda^I(\tilde{w}).$$

We conclude that $\mathcal{M}^I$ maps B into itself for this choice of t_1, τ, K. We prove similarly that $\mathcal{M}^I$ is a contraction on B, and so there exists a unique solution $\tilde{w} \in B$ of (4.24).

- Now we identify w and $\tilde{w}$. It is well-known for (gKdV) that for regular solutions (H^2), uniqueness holds by energy method. Since w and $\tilde{w}$ are both obtained by fixed point, we get $w = \tilde{w}$ by continuous dependence, persistence of regularity and density. In particular, $w \in B$, and so

$$\|w(t_1)\|_{H^1} \leqslant N_1^I(w) \leqslant \Lambda^I(w) \leqslant 3K\|w(t_1+\tau)\|_{H^1}.$$

To conclude the proof, we fix $t \geqslant t_1$, and we remark that a simple iteration argument and the exponential decay at any order of w show that, for all $n \in \mathbb{N}$, we have

$$\|w(t)\|_{H^1} \leqslant (3K)^n \|w(t+n\tau)\|_{H^1} \leqslant C_\gamma (3K)^n e^{-\gamma t} e^{-\gamma n\tau} = C_\gamma e^{-\gamma t}\left(3Ke^{-\gamma\tau}\right)^n.$$

We finally choose γ large enough so that $3Ke^{-\gamma\tau} \leqslant \frac{1}{2}$. Thus,

$$\|w(t)\|_{H^1} \leqslant \frac{C}{2^n} \xrightarrow[n\to+\infty]{} 0,$$

i.e. $\|w(t)\|_{H^1} = 0$. This finishes the proof of Proposition 4.8.

4.5 Corollaries and remarks

Corollary 4.11. *Let $c > 0$.*

1. *There exists a one-parameter family $(U_c^A)_{A\in\mathbb{R}}$ of solutions of* (gKdV) *such that*

$$\forall A \in \mathbb{R}, \exists t_0 \in \mathbb{R}, \forall s \in \mathbb{R}, \exists C > 0, \forall t \geqslant t_0, \quad \|U_c^A(t, \cdot + ct) - Q_c\|_{H^s} \leqslant Ce^{-e_0 c^{3/2} t}.$$

2. *If u_c is a solution of* (gKdV) *such that $\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u_c(t) - Q_c(\cdot - y)\|_{H^1} = 0$, then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u_c(t) = U_c^A(t, \cdot - x_0)$ for all $t \geqslant t_0$.*

Proof. The proof, based on the scaling invariance, is very similar to the proof of Corollary 3.14. We recall that if $u(t,x)$ is a solution of (gKdV), then $\lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$ with $\lambda > 0$ is also a solution.

1. We define U_c^A by $U_c^A(t,x) = c^{\frac{1}{p-1}} U^A(c^{3/2}t, \sqrt{c}x)$, where U^A is defined in Theorem 1.1. Since $U^A(c^{3/2}t, \sqrt{c}x + c^{3/2}t) = Q(\sqrt{c}x) + Ae^{-e_0 c^{3/2} t}\mathcal{Y}_+(\sqrt{c}x) + O(e^{-2e_0 c^{3/2} t})$ and $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$, U_c^A satisfies

$$U_c^A(t, x+ct) = Q_c(x) + Ac^{\frac{1}{p-1}} e^{-e_0 c^{3/2} t}\mathcal{Y}_+(\sqrt{c}x) + O(e^{-2e_0 c^{3/2} t}).$$

2. Let u be the solution of (gKdV) defined by $u(t,x) = c^{-\frac{1}{p-1}} u_c\left(\frac{t}{c^{3/2}}, \frac{x}{\sqrt{c}}\right)$. Then, we have

$$u(t,x) - Q(x-y) = c^{-\frac{1}{p-1}} u_c\left(\frac{t}{c^{3/2}}, \frac{x}{\sqrt{c}}\right) - c^{-\frac{1}{p-1}} Q_c\left(\frac{x-y}{\sqrt{c}}\right)$$

for all $y \in \mathbb{R}$, and so, as in the proof of Corollary 3.14,

$$\inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \leqslant K(c) \inf_{y\in\mathbb{R}} \left\| u_c\left(\frac{t}{c^{3/2}}\right) - Q_c\left(\cdot - \frac{y}{\sqrt{c}}\right)\right\|_{H^1} \xrightarrow[t\to+\infty]{} 0.$$

Therefore, by Theorem 1.1, there exist $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t,x) = U^A(t, x - x_0)$, and so finally $u_c(t,x) = U_c^A\left(t, x - \frac{x_0}{\sqrt{c}}\right)$. □

Proposition 4.12. *Up to translations in time and in space, there are only three special solutions: U^1, U^{-1} and Q. More precisely, one has (for t large enough):*

(a) If $A > 0$, then $U^A(t) = U^1(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.

(b) If $A = 0$, then $U^0(t) = Q(\cdot - t)$.

(c) If $A < 0$, then $U^A(t) = U^{-1}(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.

Proof. (a) Let $A > 0$ and denote $t_A = -\frac{\ln A}{e_0}$. Then, by Proposition 4.5,

$$\begin{aligned} U^1(t + t_A, x + t + t_A) &= Q(x) + e^{-e_0(t+t_A)}\mathcal{Y}_+(x) + O(e^{-2e_0t}) \\ &= Q(x) + Ae^{-e_0t}\mathcal{Y}_+(x) + O(e^{-2e_0t}). \end{aligned}$$

In particular, we have $\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|U^1(t + t_A) - Q(\cdot - y)\|_{H^1} = 0$, and so by Proposition 4.8, there exist $\widetilde{A} \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $U^1(t+t_A) = U^{\widetilde{A}}(t, \cdot - x_0)$. But still by Proposition 4.5, we have $U^1(t+t_A, x+t+t_A) = U^{\widetilde{A}}(t, x+t+t_A - x_0) = Q(x + t_A - x_0) + \widetilde{A}e^{-e_0t}\mathcal{Y}_+(x + t_A - x_0) + O(e^{-2e_0t})$, and so

$$Q(x+t_A-x_0)+\widetilde{A}e^{-e_0t}\mathcal{Y}_+(x+t_A-x_0)+O(e^{-2e_0t}) = Q(x)+Ae^{-e_0t}\mathcal{Y}_+(x)+O(e^{-2e_0t}).$$

The first order imposes $x_0 = t_A$, since $\|Q - Q(\cdot + t_A - x_0)\|_{H^1} \leqslant Ce^{-e_0t}$ and so Lemma 2.11 applies for t large. Similarly, the second order imposes $\widetilde{A} = A$, as expected.

(b) Since $\inf_{y\in\mathbb{R}} \|Q(\cdot - t) - Q(\cdot - y)\|_{H^1} = 0$, Proposition 4.8 applies, so there exist $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $Q(x - t) = U^A(t, x - x_0)$. Hence, we have, by Proposition 4.5,

$$U^A(t, x + t) = Q(x - x_0) = Q(x) + Ae^{e_0t}\mathcal{Y}_+(x) + O(e^{-2e_0t}).$$

As in the previous case, it follows first that $x_0 = 0$, then $A = 0$, and so the result.

(c) For $A < 0$, the proof is exactly the same as $A > 0$, with $-A$ instead of A. □

We conclude this paper by two remarks, based on the following claim. The first one is the fact that $U^{-1}(t)$ is defined for all $t \in \mathbb{R}$, and the second one is the identification of the special solution $w(t)$ constructed in Section 3 among the family (U^A) constructed in Section 4.

Claim 4.13. *For all $c > 0$, $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q_c'\|_{L^2}^2$ has the sign of A as long as $U_c^A(t)$ exists.*

Proof.
- From Corollary 4.11, we have

$$\partial_x U_c^A(t, x + ct) = Q_c'(x) + Ac^{\frac{p+1}{2(p-1)}}e^{-e_0c^{3/2}t}\mathcal{Y}_+'(\sqrt{c}x) + O(e^{-2e_0c^{3/2}t}),$$

and so

$$\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q_c'\|_{L^2}^2 = 2Ac^{\frac{p+1}{2(p-1)}}e^{-e_0c^{3/2}t}\int Q_c'(x)\mathcal{Y}_+'(\sqrt{c}x)\,dx + O(e^{-2e_0c^{3/2}t}).$$

But $\int Q_c'(x)\mathcal{Y}_+'(\sqrt{c}x)\,dx = c^{\frac{1}{p-1}}\int Q'(y)\mathcal{Y}_+'(y)\,dy > 0$ by the substitution $y = \sqrt{c}x$ and the normalization chosen in Lemma 4.9, and so $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q_c'\|_{L^2}^2$ has the sign of A for t large enough.

- It remains to show that this fact holds as long as $U_c^A(t)$ exists. For example, suppose that $A > 0$ and so $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q_c'\|_{L^2}^2 > 0$ for $t \geqslant t_1$, and suppose for the sake of contradiction that there exists $T < t_1$ such that $U^A(T)$ is defined and $\|\partial_x U_c^A(T)\|_{L^2}^2 = \|Q_c'\|_{L^2}^2$. Since $\|U_c^A(t, \cdot + ct) - Q_c\|_{H^1} \longrightarrow 0$, we also have, by (1.1) and (1.2), $\|U_c^A(T)\|_{L^2} = \|Q_c\|_{L^2}$ and $E(U_c^A(T)) = E(Q_c)$. In other words, we would get by scaling

$$\|U^A(T)\|_{L^2} = \|Q\|_{L^2}, \ \|\partial_x U^A(T)\|_{L^2} = \|Q'\|_{L^2} \ \text{ and } \ E(U^A(T)) = E(Q).$$

But the two last identities give in particular $\int U^A(T)^{p+1} = \int Q^{p+1}$, and so, by (1.4),

$$\begin{aligned}\|U^A(T)\|_{L^{p+1}}^{p+1} \geqslant \|Q\|_{L^{p+1}}^{p+1} &= C_{\mathrm{GN}}(p)\|Q'\|_{L^2}^{\frac{p-1}{2}}\|Q\|_{L^2}^{\frac{p+3}{2}} \\ &\geqslant C_{\mathrm{GN}}(p)\|\partial_x U^A(T)\|_{L^2}^{\frac{p-1}{2}}\|U^A(T)\|_{L^2}^{\frac{p+3}{2}}.\end{aligned}$$

Still by (1.4), we get $(\lambda_0, a_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}$ such that $U^A(T,x) = a_0 Q(\lambda_0 x + b_0)$. But $\|U^A(T)\|_{L^2} = \|Q\|_{L^2}$ and $\|\partial_x U^A(T)\|_{L^2} = \|Q'\|_{L^2}$ impose $\lambda_0 = 1$ and $a_0 \in \{-1, 1\}$. Thus, by uniqueness in (gKdV), $U^A(t,x) = \pm Q(x - t + T + b_0)$ for all $t \geqslant T$. In particular, $\|\partial_x U_c^A(t)\|_{L^2}^2 = \|Q_c'\|_{L^2}^2$ for $t \geqslant t_1$, which is a contradiction. The cases $A = 0$ and $A < 0$ are treated similarly. □

Remark 4.14. Let us now notice that U^{-1} is globally defined, *i.e.* $U^{-1}(t)$ exists for all $t \in \mathbb{R}$. By the blow up criterion and the mass conservation, it is enough to remark that $\|\partial_x U^{-1}(t)\|_{L^2}$ is bounded uniformly on its interval of existence, which is an immediate consequence of Claim 4.13, since $\|\partial_x U^{-1}(t)\|_{L^2} < \|Q'\|_{L^2}$ for all t.

Remark 4.15. As noticed in Remark 2.13, we can chose $\lambda_n = 1 - \frac{1}{n}$ in the definition of $u_{0,n}$ in Section 3. We still call $w(t)$ the special solution obtained by this method for this new initial data. In this remark, we prove that $w = U_{c_+}^{-1}$ up to translations in time and in space. We do not know if U^1 can be obtained similarly by a compactness method. We recall that $u_{0,n}(x) = \lambda_n Q(\lambda_n^2 x)$, $u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup \check{w}_0 \neq Q_{c_+}$ and $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \longrightarrow 0$.

- First note that $\int u_{0,n}'^2 = \lambda_n^4 \int Q'^2 < \int Q'^2$ for $n \geqslant 2$, and let us prove that $\|\partial_x(u_n(T_n))\|_{L^2} < \|Q'\|_{L^2}$ for n large enough. Otherwise, there would exist n large and $T \in [0, T_n]$ such that $\|\partial_x(u_n(T))\|_{L^2} = \|Q'\|_{L^2}$ and $E(u_{0,n}) < E(Q)$. But we have, by (1.2),

$$\begin{aligned}E(u_{0,n}) = E(u_n(T)) &= \frac{1}{2}\int (\partial_x(u_n(T)))^2 - \frac{1}{p+1}\int u_n^{p+1}(T) \\ &= \frac{1}{2}\int Q'^2 - \frac{1}{p+1}\int u_n^{p+1}(T) < E(Q) = \frac{1}{2}\int Q'^2 - \frac{1}{p+1}\int Q^{p+1}.\end{aligned}$$

Hence, as $\|u_n(T)\|_{L^2} = \|u_{0,n}\|_{L^2} = \|Q\|_{L^2}$ by (1.1),

$$\begin{aligned}\|u_n(T)\|_{L^{p+1}}^{p+1} \geqslant \int u_n^{p+1}(T) > \int Q^{p+1} &= C_{\mathrm{GN}}(p)\Big(\int Q'^2\Big)^{\frac{p-1}{4}}\Big(\int Q^2\Big)^{\frac{p+3}{4}} \\ &= C_{\mathrm{GN}}(p)\Big(\int (\partial_x(u_n(T)))^2\Big)^{\frac{p-1}{4}}\Big(\int u_n^2(T)\Big)^{\frac{p+3}{4}},\end{aligned}$$

which would be a contradiction with the Gagliardo-Nirenberg inequality (1.3).

- Since $u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup \check{w}_0$ in H^1, we obtain $\|\partial_x w_0\|_{L^2} \leqslant \|Q'\|_{L^2}$ and $\|w_0\|_{L^2} \leqslant \|Q\|_{L^2}$ by weak convergence. But $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \longrightarrow 0$ implies, by (1.1) and (2.1), that

$$\|w_0\|_{L^2}^2 = \|w(t)\|_{L^2}^2 = \|Q_{c_+}\|_{L^2}^2 = c_+^{\frac{5-p}{2(p-1)}} \|Q\|_{L^2}^2 \leqslant \|Q\|_{L^2}^2,$$

thus $c_+ \geqslant 1$, and so $\|\partial_x w_0\|_{L^2}^2 \leqslant \|Q'\|_{L^2}^2 = c_+^{-\frac{p+3}{2(p-1)}} \|Q'_{c_+}\|_{L^2}^2 \leqslant \|Q'_{c_+}\|_{L^2}^2$ by (2.1).

- Finally, since $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \longrightarrow 0$, Corollary 4.11 applies, and so there exists $A \in \mathbb{R}$ such that $w = U_{c_+}^A$ up to a translation in space. But the conclusion of the previous point and Claim 4.13 impose $A < 0$ (note that $A \neq 0$ since $w_0 \neq Q_{c_+}$), *i.e.* $w = U_{c_+}^{-1}$ up to translations in time and in space by Proposition 4.12.

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Part III

Multi-soliton solutions for the supercritical gKdV equations

Summary

Abstract

For the L^2 subcritical and critical (gKdV) equations, Martel [11] proved the existence and uniqueness of multi-solitons. Recall that for any N given solitons, we call multi-soliton a solution of (gKdV) which behaves as the sum of these N solitons asymptotically as $t \to +\infty$. More recently, for the L^2 supercritical case, Côte, Martel and Merle [4] proved the existence of at least one multi-soliton. In the present paper, as suggested by a previous work concerning the one soliton case [3], we first construct an N-parameter family of multi-solitons for the supercritical (gKdV) equation, for N arbitrarily given solitons, and then prove that any multi-soliton belongs to this family. In other words, we obtain a complete classification of multi-solitons for (gKdV).

1 Introduction

1.1 The generalized Korteweg-de Vries equation

We consider the generalized Korteweg-de Vries equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases} \tag{gKdV}$$

where $(t,x) \in \mathbb{R}^2$ and $p \geqslant 2$ is an integer. The following quantities are formally conserved for solutions of (gKdV):

$$\int u^2(t) = \int u^2(0) \quad \text{(mass)},$$
$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \quad \text{(energy)}.$$

Kenig, Ponce and Vega [10] have shown that the local Cauchy problem for (gKdV) is well-posed in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ of (gKdV) satisfying $u(0) = u_0$ which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. Moreover, if $T^* \geqslant T$ is the maximal time of existence of u, then either $T^* = +\infty$, which means that $u(t)$ is a global solution, or $T^* < +\infty$ and then $\|u(t)\|_{H^1} \to +\infty$ as $t \uparrow T^*$ ($u(t)$ is a finite time blow up solution). Throughout this paper, when referring to an H^1 solution of (gKdV), we mean a solution in the above sense. Finally, if $u_0 \in H^s(\mathbb{R})$ for some $s \geqslant 1$, then $u(t) \in H^s(\mathbb{R})$ for all $t \in [0, T^*)$.

In the case where $2 \leqslant p < 5$, it is standard that all solutions in H^1 are global and uniformly bounded by the energy and mass conservations and the Gagliardo-Nirenberg inequality. In the case $p = 5$, the existence of finite time blow up solutions was proved by Merle [17] and Martel and Merle [12]. Therefore, $p = 5$ is the critical exponent for the long time behavior of solutions of (gKdV). For $p > 5$, the existence of blow up solutions is an open problem.

We recall that a fundamental property of (gKdV) equations is the existence of a family of explicit traveling wave solutions. Let Q be the only solution (up to translations) of

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q'' + Q^p = Q, \quad \text{i.e. } Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}$$

For all $c_0 > 0$ and $x_0 \in \mathbb{R}$,

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0)$$

is a solution of (gKdV), where $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$. We call these solutions by solitons though they are known to be solitons only for $p = 2, 3$ (in the sense that they are stable by interaction).

It is well-known that the stability properties of a soliton solution depend on the sign of $\frac{d}{dc}\int Q_c^2{}_{|c=c_0}$. Since $\int Q_c^2 = c^{\frac{5-p}{2(p-1)}} \int Q^2$, we distinguish the following three cases.

- For $p < 5$ (L^2 subcritical case), solitons are stable and asymptotically stable in H^1 in some suitable sense: see Cazenave and Lions [2], Weinstein [23], Grillakis, Shatah and Strauss [7] for orbital stability; and Pego and Weinstein [20], Martel and Merle [13] for asymptotic stability.
- For $p = 5$ (L^2 critical case), solitons are unstable, and blow up occurs for a large class of solutions initially arbitrarily close to a soliton, see [12, 17]. Moreover, for both critical and subcritical cases, previous works imply the following asymptotic classification result: if u is a solution of (gKdV) such that $\lim_{t\to+\infty} \|u(t) - Q(\cdot - t)\|_{H^1} = 0$, then $u(t) = Q(\cdot - t)$ for t large enough.
- For $p > 5$ (L^2 supercritical case), solitons are unstable (see Grillakis, Shatah and Strauss [7] and Bona, Souganidis and Strauss [1]). In particular, the previous asymptotic classification result does not hold in this case. More precisely, we have the following theorem.

Theorem 1.1 ([3]). *Let $p > 5$.*

(i) There exists a one-parameter family $(U^A)_{A\in\mathbb{R}}$ of solutions of (gKdV) *such that, for all $A \in \mathbb{R}$,*

$$\lim_{t\to+\infty} \|U^A(t, \cdot + t) - Q\|_{H^1} = 0,$$

and if $A' \in \mathbb{R}$ satisfies $A' \neq A$, then $U^{A'} \neq U^A$.

(ii) Conversely, if u is a solution of (gKdV) *such that*

$$\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0,$$

then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for $t \geqslant t_0$.

We recall that this result was an adaptation to (gKdV) of previous works, concerning the nonlinear Schrödinger equation, of Duyckaerts and Merle [5] and Duyckaerts and Roudenko [6]. The purpose of this paper is to extend Theorem 1.1 to multi-solitons.

1.2 Multi-solitons

Now, we focus on multi-soliton solutions. Given $2N$ parameters defining $N \geqslant 2$ solitons with different speeds,

$$0 < c_1 < \cdots < c_N, \quad x_1, \ldots, x_N \in \mathbb{R}, \tag{1.1}$$

we set

$$R_j(t) = R_{c_j,x_j}(t) \quad \text{and} \quad R(t) = \sum_{j=1}^{N} R_j(t),$$

and we call multi-soliton a solution $u(t)$ of (gKdV) such that

$$\|u(t) - R(t)\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty. \tag{1.2}$$

Let us recall known results on multi-solitons.

- For $p = 2$ and 3 (KdV and mKdV), multi-solitons (in a stronger sense) are well-known to exist for any set of parameters (1.1), as a consequence of the inverse scattering method (see for example Miura [18]).
- In the L^2 subcritical and critical cases, *i.e.* for (gKdV) with $p \leqslant 5$, Martel [11] constructed multi-solitons for any set of parameters (1.1). The proof in [11] follows the strategy of Merle [16] (compactness argument) and relies on monotonicity properties developed in [13] (see also [15]). Recall that Martel, Merle and Tsai [15] proved stability and asymptotic stability of a sum of N solitons for large time for the subcritical case. A refined version of the stability result of [15] shows that, for a given set of parameters, there exists a *unique* multi-soliton solution satisfying (1.2), see Theorem 1 in [11].
- In the L^2 supercritical case, *i.e.* in a situation where solitons are known to be unstable, Côte, Martel and Merle [4] have recently proved the existence of at least *one* multi-soliton solution for (gKdV):

Theorem 1.2 ([4]). *Let $p > 5$ and $N \geqslant 2$. Let $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $\varphi \in C([T_0, +\infty), H^1)$ of (gKdV) such that*

$$\forall t \in [T_0, +\infty), \quad \|\varphi(t) - R(t)\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

Recall that, with respect to [11, 15], the proof of Theorem 1.2 relies on an additional topological argument to control the unstable nature of the solitons. Moreover, note that no uniqueness result is proved in [4], contrary to the subcritical and critical cases in [11]. In fact, the objective of this paper is to prove uniqueness up to N parameters, as suggested by Theorem 1.1.

1.3 Main result and outline of the paper

The whole paper is devoted to prove the following theorem of existence and uniqueness of a family of multi-solitons for the supercritical (gKdV) equation.

Theorem 1.3. *Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \dots, x_N \in \mathbb{R}$. Denote $R = \sum_{j=1}^N R_j$ with $R_j = R_{c_j,x_j}$.*

1. *There exists an N-parameter family $(\varphi_{A_1,\dots,A_N})_{(A_1,\dots,A_N)\in\mathbb{R}^N}$ of solutions of* (gKdV) *such that, for all $(A_1, \dots, A_N) \in \mathbb{R}^N$,*
$$\lim_{t\to+\infty} \|\varphi_{A_1,\dots,A_N}(t) - R(t)\|_{H^1} = 0,$$
and if $(A'_1, \dots, A'_N) \neq (A_1, \dots, A_N)$, then $\varphi_{A'_1,\dots,A'_N} \neq \varphi_{A_1,\dots,A_N}$.

2. *Conversely, if u is a solution of* (gKdV) *such that $\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0$, then there exists $(A_1, \dots, A_N) \in \mathbb{R}^N$ such that $u = \varphi_{A_1,\dots,A_N}$.*

Remark 1.4. The convergence of $\varphi_{A_1,\dots,A_N}$ to R in Theorem 1.3 is actually exponential in time, as in Theorem 1.2. See the proof of Theorem 1.3 at the beginning of Section 3 for more details.

Remark 1.5. For the nonlinear Schrödinger equation, the question of the classification of multi-solitons as in Theorem 1.3 is open. In fact, even for subcritical and critical cases, no general uniqueness result has been proved yet (see general existence results in [16, 21, 22, 14, 4]).

The paper is organized as follows. In the next section, we briefly recall some well-known results on solitons, multi-solitons, and on the linearized equation. One of the most important facts about the linearized equation, also strongly used in [4, 3], is the determination by Pego and Weinstein [19] of the spectrum of the linearized operator $\mathcal{L}$ around the soliton $Q(x-t)$: $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and Y^-. Indeed, $Y^\pm$ allow to control the negative directions of the linearized energy around a soliton (see Lemma 2.5). Moreover, by a simple scaling argument, we determine eigenvalues of the linearized operator around Q_{c_j}: $\pm e_j = \pm c_j^{3/2} e_0$ are eigenvalues with eigenfunctions $Y_j^\pm$ (see Notation 2.6 for precise definitions).

In Section 3, we construct the family $(\varphi_{A_1,\dots,A_N})$ described in Theorem 1.3. To do this, we first claim Proposition 3.1, which is the new key point of the proof of the multi-existence result, and can be summarized as follows. *Let φ be a multi-soliton given by Theorem 1.2, $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$. Then there exists a solution $u(t)$ of* (gKdV) *such that*
$$\|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$
for t large and for some small $\gamma > 0$. This means that, similarly as in [3] for one soliton, we can perturb the multi-soliton φ locally around *one* given soliton at the order $e^{-e_j t}$. Since $e_1 < \cdots < e_N$, $\varphi_{A_1,\dots,A_N}$ has to be constructed by iteration, from $j = 1$ to $j = N$. Indeed, it is not significant to perturb φ at order e_j before order e_{j-1}, since $e_j > e_{j-1} + \gamma$. Similarly, it seems that there exists no simple way to compare $\varphi_{A_1,\dots,A_N}$ to φ. Finally, to prove Proposition 3.1, we rely on refinements of arguments developed in [4], in particular the topological argument to control the unstable directions.

In Section 4, we classify all multi-solitons in terms of the family which was constructed in Section 3. Once again, it appears that the identification of the solution has to be done step by step (after an improvement of the convergence rate, as in [3]), from order e_1 to order e_N. In this section, we strongly use special monotonicity properties of (gKdV), in particular, to prove that any multi-soliton converges exponentially (Section 4.1). Such arguments are not known for the nonlinear Schrödinger equations.

Finally, recall that in the one soliton case for (gKdV) [3], a construction of a family of approximate solutions of the linearized equation and fixed point arguments were used (among other things), as in the one soliton case for the nonlinear Schrödinger equation [6]. For multi-solitons, since the construction of approximate solutions is not natural (because of the interactions between solitons), we propose in this paper an alternate approach based only on compactness and energy methods.

2 Preliminary results

2.1 Notation and first properties of the solitons

Notation 2.1. They are available in the whole paper.

(a) $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R})$ scalar product.

(b) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1+\xi^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R})\}$, and in particular, $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.

(c) We denote $\partial_x v = v_x$ the partial derivative of v with respect to x.

(d) All numbers C, K appearing in inequalities are real constants (with respect to the context) strictly positive, which may change in each step of an inequality.

Claim 2.2. *For all $c > 0$, one has:*

(i) $Q_c > 0$, *Q_c is even, Q_c is C^∞, and* $Q_c'(x) < 0$ *for all* $x > 0$.

(ii) For all $j \geqslant 0$, there exists $C_j > 0$ such that $Q_c^{(j)}(x) \sim C_j e^{-\sqrt{c}|x|}$ as $|x| \to +\infty$. For all $j \geqslant 0$, there exists $C_j' > 0$ such that $|Q_c^{(j)}(x)| \leqslant C_j' e^{-\sqrt{c}|x|}$ for all $x \in \mathbb{R}$.

(iii) $Q_c'' + Q_c^p = cQ_c$.

Proof. It is an immediate consequence of the formula of Q and the scaling relation $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$. □

2.2 Linearized equation

Let $c > 0$.

2.2.1 Linearized operator around Q_c

The linearized equation appears if one considers a solution of (gKdV) close to the soliton $Q_c(x-ct)$. More precisely, if $u_c(t,x) = Q_c(x-ct) + h_c(t, x-ct)$ satisfies (gKdV), then h_c satisfies

$$\partial_t h_c + \mathcal{L}_c h_c = O(h_c^2),$$

where

$$\mathcal{L}_c a = -\partial_x(L_c a) \quad \text{and} \quad L_c a = -\partial_x^2 a + ca - pQ_c^{p-1} a.$$

The spectrum of $\mathcal{L}_c$ has been calculated by Pego and Weinstein [19] for $c = 1$. Their results are summed up in the following proposition for the reader's convenience.

Proposition 2.3 ([19]). *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}) = i\mathbb{R} \quad \textit{and} \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \ \textit{with}\ e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and Y^- which have an exponential decay at infinity, and the null space of $\mathcal{L}$ is spanned by Q'.

This result is extended to $\mathcal{L}_c$ in Corollary 2.4 by a simple scaling argument. Indeed, we recall that if u is a solution of (gKdV), then for all $\lambda > 0$, $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$ is also a solution.

Corollary 2.4. *Let $\sigma(\mathcal{L}_c)$ be the spectrum of the operator $\mathcal{L}_c$ defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L}_c)$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}_c) = i\mathbb{R} \quad \textit{and} \quad \sigma(\mathcal{L}_c) \cap \mathbb{R} = \{-e_c, 0, e_c\} \quad \textit{where} \quad e_c = c^{3/2} e_0 > 0.$$

Furthermore, e_c and $-e_c$ are simple eigenvalues of $\mathcal{L}_c$ with eigenfunctions Y_c^+ and Y_c^-, where

$$Y_c^\pm(x) = c^{-1/2} Y^\pm(\sqrt{c}x),$$

and the null space of $\mathcal{L}_c$ is spanned by Q'_c.

2.2.2 Adjoint of $\mathcal{L}_c$

We recall that Lemma 4.9 in [3], under a suitable normalization of $Y^\pm$, shows important properties of the adjoint of $\mathcal{L}$. With the same normalization and by Corollary 2.4, we obtain the following lemma by a simple scaling argument. Recall that assertion (v) is proved in [4] for $c = 1$.

Lemma 2.5. *Let $Z_c^\pm = L_c Y_c^\pm$. Then the following properties hold.*

(i) $Z_c^\pm$ are two eigenfunctions of $L_c\partial_x$: $L_c(\partial_x Z_c^\pm) = \mp e_c Z_c^\pm$.

(ii) There exists $\eta_0 > 0$ such that, for all $x \in \mathbb{R}$,

$$|Y_c^\pm(x)| + |\partial_x Y_c^\pm(x)| + |Z_c^\pm(x)| + |\partial_x Z_c^\pm(x)| \leqslant Ce^{-\eta_0\sqrt{c}|x|}.$$

(iii) $(Y_c^+, Z_c^+) = (Y_c^-, Z_c^-) = 0$ and $(Z_c^+, Q'_c) = (Z_c^-, Q'_c) = 0$.

(iv) $(Y_c^+, Z_c^-) = (Y_c^-, Z_c^+) = 1$ *and* $(Q_c', \partial_x Y_c^+) > 0$.

(v) There exists $\widetilde{\sigma_c} > 0$ *such that, for all* $v_c \in H^1$ *such that* $(v_c, Z_c^+) = (v_c, Z_c^-) = (v_c, Q_c') = 0$, $(L_c v_c, v_c) \geqslant \widetilde{\sigma_c} \|v_c\|_{H^1}^2$.

(vi) There exist $\sigma_c > 0$ *and* $C > 0$ *such that, for all* $v_c \in H^1$,

$$(L_c v_c, v_c) \geqslant \sigma_c \|v_c\|_{H^1}^2 - C(v_c, Z_c^+)^2 - C(v_c, Z_c^-)^2 - C(v_c, Q_c')^2.$$

2.3 Multi-solitons results

A set of parameters (1.1) being given, we adopt the following notation.

Notation 2.6. For all $j \in [\![1, N]\!]$, define:

(i) $R_j(t,x) = Q_{c_j}(x - c_j t - x_j)$, where $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$.

(ii) $Y_j^{\pm}(t,x) = Y_{c_j}^{\pm}(x - c_j t - x_j)$, where $Y_c^{\pm}(x) = c^{-1/2} Y^{\pm}(\sqrt{c}x)$.

(iii) $Z_j^{\pm}(t,x) = Z_{c_j}^{\pm}(x - c_j t - x_j)$, where $Z_c^{\pm} = L_c Y_c^{\pm}$.

(iv) $e_j = e_{c_j}$, where $e_c = c^{3/2} e_0$.

Now, to estimate interactions between solitons, we denote the small parameters

$$\sigma_0 = \min\{\eta_0\sqrt{c_1}, e_0^{2/3} c_1, c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\} \quad \text{and} \quad \gamma = \frac{\sigma_0^{3/2}}{10^6}. \tag{2.1}$$

From [11], it appears that γ is a suitable parameter to quantify interactions between solitons in large time. For instance, we have, for $j \neq k$ and all $t \geqslant 0$,

$$\int R_j(t) R_k(t) + |(R_j)_x(t)||(R_k)_x(t)| \leqslant C e^{-10\gamma t}. \tag{2.2}$$

From the definition of σ_0 and Lemma 2.5, such an inequality is also true for $Y_j^{\pm}$ and $Z_j^{\pm}$.

Moreover, since σ_0 has the same definition as in [4], then from their Remark 1, Theorem 1.2 can be rewritten as follows. *There exist* $T_0 \in \mathbb{R}$ *and* $\varphi \in C([T_0, +\infty), H^1)$ *such that, for all* $s \geqslant 1$, *there exists* $A_s > 0$ *such that, for all* $t \geqslant T_0$,

$$\|\varphi(t) - R(t)\|_{H^s} \leqslant A_s e^{-4\gamma t}. \tag{2.3}$$

3 Construction of a family of multi-solitons

In this section, we prove the first point of Theorem 1.3 as a consequence of the following crucial Proposition 3.1. Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. We still denote $R = \sum_{k=1}^N R_k$.

Proposition 3.1. *Let φ be a multi-soliton solution satisfying* (2.3). *Let $j \in [\![1,N]\!]$ and $A_j \in \mathbb{R}$. Then there exist $t_0 > 0$ and a solution $u \in C([t_0,+\infty),H^1)$ of* (gKdV) *such that*

$$\forall t \geqslant t_0, \quad \|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}. \tag{3.1}$$

Before proving this proposition, let us show how this proposition implies the first point of Theorem 1.3.

Proof of 1. of Theorem 1.3. Let $(A_1,\dots,A_N) \in \mathbb{R}^N$.

(i) Consider φ_{A_1} the solution of (gKdV) given by Proposition 3.1 applied with φ given by Theorem 1.2. Thus, there exists $t_0 > 0$ such that

$$\forall t \geqslant t_0, \quad \|\varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^+(t)\|_{H^1} \leqslant e^{-(e_1+\gamma)t}.$$

Now remark that φ_{A_1} is also a multi-soliton, which satisfies (2.3) by the definition of γ and the same techniques used in [11, Section 3.4] to improve the estimate in higher order Sobolev norms. Hence, we can apply Proposition 3.1 with φ_{A_1} instead of φ, so that we obtain φ_{A_1,A_2} such that

$$\forall t \geqslant t_0', \quad \|\varphi_{A_1,A_2}(t) - \varphi_{A_1}(t) - A_2 e^{-e_2 t} Y_2^+(t)\|_{H^1} \leqslant e^{-(e_2+\gamma)t}.$$

Similarly, for all $j \in [\![2,N]\!]$, we construct by induction a solution $\varphi_{A_1,\dots,A_j}$ such that

$$\forall t \geqslant t_0, \quad \|\varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}. \tag{3.2}$$

Observe finally that $\varphi_{A_1,\dots,A_N}$ constructed by this way satisfies (2.3).

(ii) Let $(A_1',\dots,A_N') \in \mathbb{R}^N$ be such that $(A_1',\dots,A_N') \neq (A_1,\dots,A_N)$, and suppose in the sake of contradiction that $\varphi_{A_1',\dots,A_N'} = \varphi_{A_1,\dots,A_N}$. Denote $i_0 = \min\{i \in [\![1,N]\!] \mid A_i' \neq A_i\}$. Hence, we have $A_i' = A_i$ for $i \in [\![1,i_0-1]\!]$, $A_{i_0}' \neq A_{i_0}$ and, from the construction of $\varphi_{A_1,\dots,A_N}$,

$$\begin{aligned}\varphi_{A_1,\dots,A_N}(t) &= \varphi_{A_1,\dots,A_{N-1}}(t) + A_N e^{-e_N t} Y_N^+(t) + z_N(t)\\ &= \varphi_{A_1,\dots,A_{N-2}}(t) + A_{N-1} e^{-e_{N-1} t} Y_{N-1}^+(t) + A_N e^{-e_N t} Y_N^+(t) + z_{N-1}(t) + z_N(t)\\ &= \cdots = \varphi_{A_1,\dots,A_{i_0-1}}(t) + A_{i_0} e^{-e_{i_0} t} Y_{i_0}^+(t) + \sum_{k>i_0} A_k e^{-e_k t} Y_k^+(t) + \sum_{k\geqslant i_0} z_k(t),\end{aligned}$$

where z_k satisfies $\|z_k(t)\|_{H^1} \leqslant e^{-(e_k+\gamma)t}$ for $t \geqslant t_0$ and each $k \geqslant i_0$. Similarly, we get

$$\varphi_{A_1',\dots,A_N'}(t) = \varphi_{A_1',\dots,A_{i_0-1}'}(t) + A_{i_0}' e^{-e_{i_0} t} Y_{i_0}^+(t) + \sum_{k>i_0} A_k' e^{-e_k t} Y_k^+(t) + \sum_{k\geqslant i_0} \widetilde{z_k}(t),$$

and so using $\varphi_{A_1',\dots,A_N'} = \varphi_{A_1,\dots,A_N}$ and $\varphi_{A_1',\dots,A_{i_0-1}'} = \varphi_{A_1,\dots,A_{i_0-1}}$, we obtain

$$e^{-e_{i_0} t} |A_{i_0} - A_{i_0}'| \leqslant C e^{-(e_{i_0}+\gamma)t}$$

for $t \geqslant t_0$, thus, $|A_{i_0} - A_{i_0}'| \leqslant C e^{-\gamma t}$, and so $A_{i_0}' = A_{i_0}$ by letting $t \to +\infty$, which is a contradiction and concludes the proof. □

Now, the only purpose of the rest of this section is to prove Proposition 3.1. Let $j \in [\![1,N]\!]$ and $A_j \in \mathbb{R}$. We want to construct a solution u of (gKdV) such that

$$z(t,x) = u(t,x) - \varphi(t,x) - A_j e^{-e_j t} Y_j^+(t,x)$$

satisfies $\|z(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}$ for $t \geqslant t_0$ with t_0 large enough.

3.1 Equation of z

Since u is a solution of (gKdV) and also φ is (and this fact is crucial for the whole proof), we get

$$\partial_t z + \partial_x^3 z + \partial_x[(\varphi + A_j e^{-e_j t} Y_j^+ + z)^p - \varphi^p] + A_j e^{-e_j t}[\partial_x^3 Y_j^+ - c_j \partial_x Y_j^+ - e_j Y_j^+] = 0.$$

But from Corollary 2.4, we have

$$\mathcal{L}_{c_j} Y_{c_j}^+ = e_j Y_{c_j}^+ = \partial_x^3 Y_{c_j}^+ - c_j \partial_x Y_{c_j}^+ + p\partial_x(Q_{c_j}^{p-1} Y_{c_j}^+)$$

and so, following Notation 2.6, we can rewrite the equation for z as

$$\partial_t z + \partial_x^3 z + \partial_x[(\varphi + A_j e^{-e_j t} Y_j^+ + z)^p - \varphi^p - pA_j e^{-e_j t} R_j^{p-1} Y_j^+] = 0. \tag{3.3}$$

This can also be written

$$\begin{aligned} \partial_t z + \partial_x\left[\partial_x^2 z + p\varphi^{p-1} z\right] &+ p\partial_x\left[((\varphi + A_j e^{-e_j t} Y_j^+)^{p-1} - \varphi^{p-1}) \cdot z\right] \\ &+ \partial_x\left[(\varphi + A_j e^{-e_j t} Y_j^+ + z)^p - (\varphi + A_j e^{-e_j t} Y_j^+)^p - p(\varphi + A_j e^{-e_j t} Y_j^+)^{p-1} z\right] \\ &= -\partial_x\left[(\varphi + A_j e^{-e_j t} Y_j^+)^p - \varphi^p - pA_j e^{-e_j t} Y_j^+ R_j^{p-1}\right]. \end{aligned}$$

Finally, if we denote

$$\begin{cases} \omega_1 = p[(\varphi + A_j e^{-e_j t} Y_j^+)^{p-1} - \varphi^{p-1}], \\ \omega(z) = (\varphi + A_j e^{-e_j t} Y_j^+ + z)^p - (\varphi + A_j e^{-e_j t} Y_j^+)^p - p(\varphi + A_j e^{-e_j t} Y_j^+)^{p-1} z, \\ \Omega = (\varphi + A_j e^{-e_j t} Y_j^+)^p - \varphi^p - pA_j e^{-e_j t} Y_j^+ R_j^{p-1}, \end{cases}$$

we obtain the shorter form of the equation of z:

$$\partial_t z + \partial_x\left[\partial_x^2 z + p\varphi^{p-1} z\right] + \partial_x[\omega_1 \cdot z] + \partial_x[\omega(z)] = -\partial_x \Omega. \tag{3.4}$$

Note that the term $\omega(z)$ is the nonlinear term in z, and that ω_1 satisfies, for all $s \geqslant 0$, $\|\omega_1(t)\|_{H^s} \leqslant C_s e^{-e_j t}$ for all $t \geqslant T_0$. Moreover, the source term Ω satisfies

$$\forall s \geqslant 1, \exists C_s > 0, \forall t \geqslant T_0, \quad \|\Omega(t)\|_{H^s} \leqslant C_s e^{-(e_j + 4\gamma)t}. \tag{3.5}$$

Indeed, if we write Ω under the form

$$\begin{aligned} \Omega = &\left[(\varphi + A_j e^{-e_j t} Y_j^+)^p - \varphi^p - p\varphi^{p-1} A_j e^{-e_j t} Y_j^+\right] \\ &+ pA_j e^{-e_j t} Y_j^+(\varphi^{p-1} - R^{p-1}) + pA_j e^{-e_j t} Y_j^+(R^{p-1} - R_j^{p-1}), \end{aligned}$$

we deduce from (2.3), (2.2) and the definition of γ (2.1) that

$$\|\Omega(t)\|_{H^s} \leqslant Ce^{-2e_j t} + Ce^{-e_j t}\|\varphi(t) - R(t)\|_{H^s} + Ce^{-e_j t} \cdot e^{-4\gamma t} \leqslant Ce^{-(e_j + 4\gamma)t}.$$

3.2 Compactness argument assuming uniform estimates

To prove Proposition 3.1, we follow the strategy of [11, 4]. Let $S_n \to +\infty$ be an increasing sequence of time, $\mathfrak{b}_n = (b_{n,k})_{j<k\leqslant N} \in \mathbb{R}^{N-j}$ be a sequence of parameters to be determined, and let u_n be the solution of

$$\begin{cases} \partial_t u_n + \partial_x[\partial_x^2 u_n + u_n^p] = 0, \\ u_n(S_n) = \varphi(S_n) + A_j e^{-e_j S_n} Y_j^+(S_n) + \sum_{k>j} b_{n,k} Y_k^+(S_n). \end{cases} \tag{3.6}$$

Notation 3.2. (i) $\mathbb{R}^N$ is equipped with the ℓ^2 norm, simply denoted $\|\cdot\|$.

(ii) $B_{\mathcal{B}}(P,r)$ is the closed ball of the Banach space $\mathcal{B}$, centered at P and of radius $r \geqslant 0$. If $P = 0$, we simply write $B_{\mathcal{B}}(r)$.

(iii) $\mathbb{S}_{\mathbb{R}^N}(r)$ denotes the sphere of radius r in $\mathbb{R}^N$.

Proposition 3.3. *There exist $n_0 \geqslant 0$ and $t_0 > 0$ (independent of n) such that the following holds. For each $n \geqslant n_0$, there exists $\mathfrak{b}_n \in \mathbb{R}^{N-j}$ with $\|\mathfrak{b}_n\| \leqslant 2e^{-(e_j+2\gamma)S_n}$, and such that the solution u_n of* (3.6) *is defined on the interval $[t_0, S_n]$, and satisfies*

$$\forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Assuming this proposition and the following lemma of weak continuity of the flow, we can deduce the proof of Proposition 3.1. The proof of Proposition 3.3 is postponed to the next section, whereas the proof of Lemma 3.4 is postponed to Appendix A.

Lemma 3.4. *Suppose that $z_{0,n} \rightharpoonup z_0$ in H^1, and that there exists $T > 0$ such that the solution $z_n(t)$ corresponding to initial data $z_{0,n}$ exists for $t \in [0,T]$ and $\sup_{t\in[0,T]} \|z_n(t)\|_{H^1} \leqslant K$. Then, for all $t \in [0,T]$, the solution $z(t)$ corresponding to initial data z_0 exists, and $z_n(T) \rightharpoonup z(T)$ in H^1.*

Remark 3.5. Note that the proof of Lemma 3.4 strongly relies on the Cauchy theory in H^s with $s < 1$, developed in [10]. Thus, this argument is quite similar to the compactness argument developed in [4] or [11].

Proof of Proposition 3.1 assuming Proposition 3.3. We may assume $n_0 = 0$ in Proposition 3.3 without loss of generality. It follows from this proposition that there exists a sequence $u_n(t)$ of solutions of (gKdV), defined on $[t_0, S_n]$, such that the following uniform estimates hold:

$$\forall n \geqslant 0, \forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

In particular, there exists $C_0 > 0$ such that $\|u_n(t_0)\|_{H^1} \leqslant C_0$ for all $n \geqslant 0$. Thus, there exists $u_0 \in H^1(\mathbb{R})$ such that $u_n(t_0) \rightharpoonup u_0$ in H^1 weak (after passing to a subsequence). Now, consider u solution of

$$\begin{cases} \partial_t u + \partial_x[\partial_x^2 u + u^p] = 0, \\ u(t_0) = u_0. \end{cases}$$

Let $T \geqslant t_0$. For n such that $S_n > T$, $u_n(t)$ is well defined for all $t \in [t_0, T]$, and moreover, $\|u_n(t)\|_{H^1} \leqslant C$. By Lemma 3.4, we have $u_n(T) \rightharpoonup u(T)$ in H^1. As

$$\|u_n(T) - \varphi(T) - A_j e^{-e_j T} Y_j^+(T)\|_{H^1} \leqslant e^{-(e_j+\gamma)T},$$

we finally obtain, by weak convergence, $\|u(T) - \varphi(T) - A_j e^{-e_j T} Y_j^+(T)\|_{H^1} \leqslant e^{-(e_j+\gamma)T}$. Thus, u is a solution of (gKdV) which satisfies (3.1). □

3.3 Proof of Proposition 3.3

The proof proceeds in several steps. For the sake of simplicity, we will drop the index n for the rest of this section (except for S_n). As Proposition 3.3 is proved for given n, this should not be a source of confusion. Hence, we will write u for u_n, z for z_n, $\mathfrak{b}$ for $\mathfrak{b}_n$, etc. We possibly drop the first terms of the sequence S_n, so that, for all n, S_n is large enough for our purposes.

From (3.4), the equation satisfied by z is

$$\begin{cases} \partial_t z + \partial_x[\partial_x^2 z + p\varphi^{p-1} z] + \partial_x[\omega_1 \cdot z] + \partial_x[\omega(z)] = -\partial_x \Omega, \\ z(S_n) = \sum_{k>j} b_k Y_k^+(S_n). \end{cases} \tag{3.7}$$

Moreover, for all $k \in [\![1, N]\!]$, we denote

$$\alpha_k^{\pm}(t) = \int z(t) \cdot Z_k^{\pm}(t).$$

In particular, we have

$$\alpha_k^{\pm}(S_n) = \sum_{l>j} b_l \int Y_l^+(S_n) \cdot Z_k^{\pm}(S_n).$$

Finally, we denote $\boldsymbol{\alpha}^-(t) = (\alpha_k^-(t))_{j<k\leqslant N}$.

3.3.1 Modulated final data

Lemma 3.6. *For $n \geqslant n_0$ large enough, the following holds. For all $\mathfrak{a}^- \in \mathbb{R}^{N-j}$, there exists a unique $\mathfrak{b} \in \mathbb{R}^{N-j}$ such that $\|\mathfrak{b}\| \leqslant 2\|\mathfrak{a}^-\|$ and $\boldsymbol{\alpha}^-(S_n) = \mathfrak{a}^-$.*

Proof. Consider the linear application

$$\begin{array}{rccc} \Phi : & \mathbb{R}^{N-j} & \to & \mathbb{R}^{N-j} \\ & \mathfrak{b} = (b_l)_{j<l\leqslant N} & \mapsto & (\alpha_k^-(S_n))_{j<k\leqslant N}. \end{array}$$

From the normalization of Lemma 2.5, its matrix in the canonical basis is

$$\operatorname{Mat}\Phi = \begin{pmatrix} 1 & \int Y_{j+2}^+ Z_{j+1}^-(S_n) & \cdots & \int Y_{j+N}^+ Z_{j+1}^-(S_n) \\ \int Y_{j+1}^+ Z_{j+2}^-(S_n) & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \int Y_{j+1}^+ Z_{j+N}^-(S_n) & \cdots & \cdots & 1 \end{pmatrix}.$$

But from (2.2), we have, for $k \neq l$,

$$\left|\int Y_l^{\pm} Z_k^{\pm}(S_n)\right| \leqslant C_0 e^{-\gamma S_n}$$

with C_0 independent of n, and so by taking n_0 large enough, we have $\Phi = \mathrm{Id} + A_n$ where $\|A_n\| \leqslant \frac{1}{2}$. Thus, Φ is invertible and $\|\Phi^{-1}\| \leqslant 2$. Finally, for a given $\mathfrak{a}^- \in \mathbb{R}^{N-j}$, it is enough to define $\mathfrak{b}$ by $\mathfrak{b} = \Phi^{-1}(\mathfrak{a}^-)$ to conclude the proof of Lemma 3.6. □

Claim 3.7. *The following estimates at S_n hold:*

- $|\alpha_k^+(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in [\![1, N]\!]$,
- $|\alpha_k^-(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in [\![1, j]\!]$,
- $\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\|$.

3.3.2 Equations on $\alpha_k^{\pm}$

Let $t_0 > 0$ independent of n to be determined later in the proof, $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ to be chosen, $\mathfrak{b}$ be given by Lemma 3.6 and u be the corresponding solution of (3.6). We now define the maximal time interval $[T(\mathfrak{a}^-), S_n]$ on which suitable exponential estimates hold.

Definition 3.8. Let $T(\mathfrak{a}^-)$ be the infimum of $T \geqslant t_0$ such that, for all $t \in [T, S_n]$, both following properties hold:

$$e^{(e_j+\gamma)t} z(t) \in B_{H^1}(1) \quad \text{and} \quad e^{(e_j+2\gamma)t}\boldsymbol{\alpha}^-(t) \in B_{\mathbb{R}^{N-j}}(1). \tag{3.8}$$

Observe that Proposition 3.3 is proved if we can find $\mathfrak{a}^-$ such that $T(\mathfrak{a}^-) = t_0$, for all n. The rest of the proof is devoted to prove the existence of such a value of $\mathfrak{a}^-$.

First, we prove the following estimate on $\alpha_k^{\pm}$.

Claim 3.9. *For all $k \in [\![1, N]\!]$ and all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\left|\frac{d}{dt}\alpha_k^{\pm}(t) \mp e_k \alpha_k^{\pm}(t)\right| \leqslant C_0 e^{-4\gamma t}\|z(t)\|_{H^1} + C_1\|z(t)\|_{H^1}^2 + C_2 e^{-(e_j+4\gamma)t}. \tag{3.9}$$

Proof. Using the equation of z (3.7), we first compute

$$\begin{aligned}
\frac{d}{dt}\alpha_k^{\pm}(t) &= \int z_t Z_k^{\pm} + \int z Z_{kt}^{\pm} \\
&= \int (z_{xx} + p\varphi^{p-1} z) Z_{kx}^{\pm} + \int \omega_1 z Z_{kx}^{\pm} + \int \omega(z) Z_{kx}^{\pm} + \int \Omega Z_{kx}^{\pm} - c_k \int z Z_{kx}^{\pm} \\
&= \int (z_{xx} - c_k z + pR_k^{p-1} z) Z_{kx}^{\pm} + p\int (\varphi^{p-1} - R_k^{p-1}) z Z_{kx}^{\pm} + \int (\omega_1 z + \omega(z) + \Omega) Z_{kx}^{\pm}.
\end{aligned}$$

But from (i) of Lemma 2.5, we have

$$\begin{aligned}
\int (z_{xx} - c_k z + pR_k^{p-1} z) Z_{kx}^{\pm} &= (-L_{c_k} z(t, \cdot + c_k t + x_k), \partial_x Z_{c_k}^{\pm}) \\
&= (z(t, \cdot + c_k t + x_k), -L_{c_k}(\partial_x Z_{c_k}^{\pm})) = \pm e_k (z(t, \cdot + c_k t + x_k), Z_{c_k}^{\pm}) = \pm e_k \alpha_k^{\pm}.
\end{aligned}$$

Finally, from (2.3) and (3.5), we have the following estimates:

- $|\int(\varphi^{p-1} - R_k^{p-1})zZ_{kx}^{\pm}| \leqslant C\|\varphi - R\|_{L^\infty}\|z\|_{L^\infty} + Ce^{-4\gamma t}\|z\|_{L^2} \leqslant Ce^{-4\gamma t}\|z\|_{H^1}$,
- $|\int \omega_1 z Z_{kx}^{\pm}| \leqslant \|\omega_1\|_{L^\infty}\|z\|_{L^\infty}\|Z_{kx}^{\pm}\|_{L^1} \leqslant Ce^{-e_j t}\|z\|_{H^1} \leqslant Ce^{-4\gamma t}\|z\|_{H^1}$,
- $|\int \omega(z)Z_{kx}^{\pm}| \leqslant C\|z\|_{L^2}^2 \leqslant C\|z\|_{H^1}^2$,
- $|\int \Omega Z_{kx}^{\pm}| \leqslant C\|\Omega\|_{L^\infty} \leqslant Ce^{-(e_j+4\gamma)t}$,

which conclude the proof of the claim. □

3.3.3 Control of the stable directions

We estimate here $\alpha_k^+(t)$ for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. From (3.9) and (3.8), we have

$$\left|\frac{d}{dt}\alpha_k^+(t) - e_k\alpha_k^+(t)\right| \leqslant C_0 e^{-(e_j+5\gamma)t} + C_1 e^{-2(e_j+\gamma)t} + C_2 e^{-(e_j+4\gamma)t} \leqslant K_2 e^{-(e_j+4\gamma)t}.$$

Thus, $|(e^{-e_k s}\alpha_k^+(s))'| \leqslant K_2 e^{-(e_j+e_k+4\gamma)s}$, and so by integration on $[t, S_n]$, we get $|e^{-e_k S_n}\alpha_k^+(S_n) - e^{-e_k t}\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+e_k+4\gamma)t}$, and so

$$|\alpha_k^+(t)| \leqslant e^{e_k(t-S_n)}|\alpha_k^+(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But from Claim 3.7 and Lemma 3.6, we have

$$\begin{aligned} e^{e_k(t-S_n)}|\alpha_k^+(S_n)| &\leqslant |\alpha_k^+(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\| \\ &\leqslant Ce^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in [\![1, N]\!], \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.10}$$

3.3.4 Control of the unstable directions for $k \leqslant j$

We estimate here $\alpha_k^-(t)$ for all $k \in [\![1, j]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. Note first that, as in the previous paragraph, we get, for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$,

$$\left|\frac{d}{dt}\alpha_k^-(t) + e_k\alpha_k^-(t)\right| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.11}$$

Now suppose $k \leqslant j$, which implies $e_k \leqslant e_j$. Since $|(e^{e_k s}\alpha_k^-(s))'| \leqslant K_2 e^{(e_k-e_j-4\gamma)s}$, we obtain, by integration on $[t, S_n]$,

$$|\alpha_k^-(t)| \leqslant e^{e_k(S_n-t)}|\alpha_k^-(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But again from Claim 3.7 and Lemma 3.6, we have

$$\begin{aligned} e^{e_k(S_n-t)}|\alpha_k^-(S_n)| &\leqslant K_2 e^{e_k(S_n-t)}e^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} = K_2 e^{e_k(S_n-t)}e^{-(e_j+4\gamma)S_n} \\ &\leqslant K_2 e^{(S_n-t)(e_k-e_j)}e^{-e_j t}e^{-4\gamma S_n} \leqslant K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in [\![1, j]\!], \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^-(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.12}$$

3.3.5 Monotonicity property of the energy

We follow here the same strategy as in [11, Section 4] to estimate the energy backwards. Since calculations are long and technical, we refer to [11] for more details.

We define the function

$$\psi(x) = \frac{2}{\pi}\arctan(\exp(-\sqrt{\sigma_0}x/2)),$$

so that $\lim_{x\to+\infty}\psi(x) = 0$, $\lim_{x\to-\infty}\psi(x) = 1$, and for all $x \in \mathbb{R}$, $\psi(-x) = 1 - \psi(x)$. Note that, by a direct calculation, we have $|\psi'''(x)| \leqslant \frac{\sigma_0}{4}|\psi'(x)|$. Moreover, we set

$$h(t,x) = \frac{1}{c_N} + \sum_{k=1}^{N-1}\left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right)\psi\left(x - \frac{c_k + c_{k+1}}{2}t - \frac{x_k + x_{k+1}}{2}\right).$$

Observe that the function h takes values close to $\frac{1}{c_k}$ for x close to $c_k t + x_k$, and has large variations only in regions far away from the solitons (for instance we have, for all $k \in [\![1,N]\!]$ and $t \geqslant 0$, $\|R_k(t)h_x(t)\|_{L^\infty} \leqslant Ce^{-4\gamma t}$). We also define a quantity related to the energy for z:

$$H(t) = \int\left\{\Big(z_x^2(t,x) - F(t,z(t,x))\Big)h(t,x) + z^2(t,x)\right\}dx,$$

where

$$F(t,z) = 2\left[\frac{(\varphi + v_j + z)^{p+1}}{p+1} - \frac{(\varphi + v_j)^{p+1}}{p+1} - (\varphi + v_j)^p z\right],$$

and $v_j(t,x) = A_j e^{-e_j t}Y_j^+(t,x)$.

Lemma 3.10. *For all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\frac{dH}{dt}(t) \geqslant -C_0\|z(t)\|_{H^1}^3 - C_1 e^{-2\gamma t}\|z(t)\|_{H^1}^2 - C_2 e^{-(e_j+3\gamma)t}\|z(t)\|_{H^1}.$$

Proof. Since $\frac{\partial F}{\partial z} = 2[(\varphi + v_j + z)^p - (\varphi + v_j)^p]$, we can first compute

$$\begin{aligned}\frac{dH}{dt} = \int(z_x^2 - F(z))h_t - 2\int z_t\big[(\varphi + v_j + z)^p - (\varphi + v_j)^p\big]h + 2\int z_{xt}z_x h + 2\int z_t z \\ - 2\int(\varphi + v_j)_t\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1}z\Big]h.\end{aligned}$$

Moreover, $2\int z_{xt}z_x h = -2\int z_t(z_{xx}h + z_x h_x)$, thus

$$\begin{aligned}\frac{dH}{dt} = \int(z_x^2 - F(z))h_t - 2\int z_t\big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p\big]h + 2\int z_t(z - z_x h_x) \\ - 2\int(\varphi + v_j)_t\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1}z\Big]h.\end{aligned}$$

Now we replace z_t by the equation that it satisfies, which can be written, from (3.3),

$$z_t + \Big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p\Big]_x = -\Omega_x.$$

Using multiple integrations by parts, we finally obtain

$$\frac{dH}{dt} = \int (z_x^2 - F(z))h_t + \int z_x^2 h_{xxx} \tag{3.13}$$
$$+ 2\int z_x h_x \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p\Big]_x \tag{3.14}$$
$$- 2\int z\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p\Big]_x$$
$$- 2\int \varphi_t h\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1}z\Big] \tag{3.15}$$
$$- 2\int z\Omega_x + 2\int zh\Omega_{xxx} + 2\int zh_x\Omega_{xx}$$
$$+ 2\int h\Omega_x\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p\Big] \tag{3.16}$$
$$- 2\int hv_{jt}\Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1}z\Big] \tag{3.17}$$
$$- \int \Big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p\Big]^2 h_x - 2\int z_{xx}^2 h_x. \tag{3.18}$$

To conclude, we estimate each term of this equality.

- First note that (3.18) $\geqslant 0$ since $h_x < 0$.
- (3.13): By the expression of h and $|\psi'''| \leqslant \frac{\sigma_0}{4}|\psi'|$, we see after direct calculation that $h_t \geqslant \sigma_0|h_x| \geqslant 4|h_{xxx}|$, thus
$$(3.13) \geqslant \frac{3}{4}\int z_x^2 h_t - \int F(z)h_t \geqslant -\int |F(z)|h_t.$$
Moreover, since $\|Rh_t\|_{L^\infty} \leqslant Ce^{-4\gamma t}$, and
$$|F(z)| \leqslant C|z|^{p+1} + Cz^2|\varphi + v_j|^{p-1} \leqslant C\|z\|_{L^\infty}^{p-1}z^2 + Cz^2(|\varphi|^{p-1} + |v_j|^{p-1})$$
$$\leqslant C\|z\|_{L^\infty}z^2 + Cz^2|\varphi - R|^{p-1} + Cz^2|R|^{p-1} + Cz^2\|v_j\|_{L^\infty},$$
we have $\int |F(z)|h_t \leqslant C_0\|z\|_{H^1}^3 + C_1e^{-2\gamma t}\|z\|_{H^1}^2$.
- For (3.17), first note that $\|v_{jt}\|_{L^\infty} \leqslant Ce^{-e_j t}$, and so
$$|(3.17)| \leqslant C\|v_{jt}\|_{L^\infty}\|z\|_{L^2}^2 \leqslant C_1e^{-2\gamma t}\|z\|_{H^1}^2.$$
- $|(3.16)| \leqslant C\|\Omega\|_{H^3}\|z\|_{L^2} \leqslant C_2e^{-(e_j+4\gamma)t}\|z\|_{H^1}$ by (3.5).
- To estimate (3.14), we develop it as
$$\frac{1}{2}(3.14) = \int z_x h_x \sum_{k=1}^{p}\binom{p}{k}\Big[(\varphi + v_j)^{p-k}z^k\Big]_x = \sum_{k=1}^{p-1}\binom{p}{k}k\int z_x^2 z^{k-1}(\varphi + v_j)^{p-k}h_x$$
$$+ \sum_{k=1}^{p-1}\binom{p}{k}(p-k)\int (\varphi + v_j)_x(\varphi + v_j)^{p-k-1}h_x z_x z^k + p\int z_x^2 z^{p-1}h_x.$$
Since $|\varphi_x h_x| + |\varphi h_x| \leqslant Ce^{-2\gamma t}$ and $|v_{jx}| + |v_j| \leqslant Ce^{-e_j t}$, we find
$$|(3.14)| \leqslant C_1e^{-2\gamma t}\|z\|_{H^1}^2 + C_0\|z\|_{H^1}^3.$$

- We finally estimate (3.15) to conclude. The key point to control it is that, locally around $x = c_k t + x_k$, φ behaves as a solitary wave of speed c_k. More precisely, we strongly use the estimate $\|\varphi_t h + \varphi_x\|_{L^\infty} \leqslant Ce^{-2\gamma t}$, proved in [11]. Note that the proof uses the H^4 norm of the difference $\varphi - R$, *i.e.* (2.3). Now, we compute

$$-\frac{1}{2}(3.15) = \int z\Big[(\varphi+v_j+z)^p - (\varphi+v_j)^p - p(\varphi+v_j)^{p-1}z\Big]_x$$
$$+\int \varphi_t h\Big[(\varphi+v_j+z)^p - (\varphi+v_j)^p - p(\varphi+v_j)^{p-1}z - \frac{p(p-1)}{2}(\varphi+v_j)^{p-2}z^2\Big]$$
$$-p\int(\varphi+v_j)^{p-1}z_x z + \frac{p(p-1)}{2}\int \varphi_t h(\varphi+v_j)^{p-2}z^2 = \mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}.$$

First notice that $|\mathbf{I}| + |\mathbf{II}| \leqslant C_0\|z\|_{H^1}^3$. Moreover, an integration by parts gives

$$\mathbf{III}+\mathbf{IV} = \frac{p}{2}\int z^2(p-1)(\varphi_x+v_{jx})(\varphi+v_j)^{p-2} + \frac{p(p-1)}{2}\int\varphi_t h(\varphi+v_j)^{p-2}z^2$$
$$= \frac{p(p-1)}{2}\int z^2(\varphi+v_j)^{p-2}(\varphi_x+\varphi_t h) + \frac{p(p-1)}{2}\int z^2 v_{jx}(\varphi+v_j)^{p-2},$$

thus

$$|\mathbf{III}+\mathbf{IV}| \leqslant C\|\varphi_x+\varphi_t h\|_{L^\infty}\|z\|_{L^2}^2 + C\|v_{jx}\|_{L^\infty}\|z\|_{L^2}^2 \leqslant Ce^{-2\gamma t}\|z\|_{H^1}^2 + Ce^{-e_j t}\|z\|_{H^1}^2,$$

and so finally $|(3.15)| \leqslant C_0\|z\|_{H^1}^3 + C_1 e^{-2\gamma t}\|z\|_{H^1}^2$. □

We can now prove that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\int \Big(z_x^2(t) - pR^{p-1}(t)z^2(t)\Big)h(t) + z^2(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}. \tag{3.19}$$

Indeed, from Lemma 3.10 and estimates (3.8), we deduce that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\frac{dH}{dt}(t) \geqslant -C_0 e^{-3(e_j+\gamma)t} - C_1 e^{-2\gamma t}e^{-2(e_j+\gamma)t} - C_2 e^{-(e_j+3\gamma)t}e^{-(e_j+\gamma)t} \geqslant -K_1 e^{-2(e_j+2\gamma)t}.$$

Thus, by integration on $[t, S_n]$, we obtain $H(S_n) - H(t) \geqslant -K_1 e^{-2(e_j+2\gamma)t}$, and so

$$H(t) \leqslant H(S_n) + K_1 e^{-2(e_j+2\gamma)t}.$$

But from Claim 3.7 and Lemma 3.6, we have

$$H(S_n) \leqslant |H(S_n)| \leqslant C\|z(S_n)\|_{H^1}^2 \leqslant C\|\mathfrak{b}\|^2 \leqslant C\|\mathfrak{a}^-\|^2$$
$$\leqslant Ce^{-2(e_j+2\gamma)S_n} \leqslant Ce^{-2(e_j+2\gamma)t},$$

and so

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad H(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}.$$

Finally, since

$$|F(z) - pR^{p-1}z^2| \leqslant |F(z) - p(\varphi+v_j)^{p-1}z^2| + p|((\varphi+v_j)^{p-1} - \varphi^{p-1})z^2|$$
$$+ p|(\varphi^{p-1} - R^{p-1})z^2| \leqslant C_0|z|^3 + C_1 e^{-2\gamma t}|z|^2,$$

we easily obtain (3.19) from the definition of H.

3.3.6 Control of the R_{kx} directions

Define $\tilde{z}(t) = z(t) + \sum_{k=1}^{N} a_k(t) R_{kx}(t)$, where $a_k(t) = -\frac{\int z(t)R_{kx}(t)}{\int (Q'_{c_k})^2}$, so that, by (2.2),

$$\left| \int \tilde{z} R_{kx} \right| \leqslant C e^{-\gamma t} \|z\|_{H^1}, \tag{3.20}$$

and there exist $C_1, C_2 > 0$ such that

$$C_1 \|z\|_{H^1} \leqslant \|\tilde{z}\|_{H^1} + \sum_{k=1}^{N} |a_k| \leqslant C_2 \|z\|_{H^1}. \tag{3.21}$$

As in [11, Section 4], we find

$$\int \left[(\tilde{z}_x^2 - pR^{p-1}\tilde{z}^2)h + \tilde{z}^2 \right] \leqslant \int \left[(z_x^2 - pR^{p-1}z^2)h + z^2 \right] + Ce^{-2\gamma t} \|z\|_{H^1}^2.$$

Using (3.19), we deduce that

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \int \left(\tilde{z}_x^2(t) - pR^{p-1}(t)\tilde{z}^2(t) \right) h(t) + \tilde{z}^2(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}. \tag{3.22}$$

Now, from the property of coercivity (vi) in Lemma 2.5, and since h takes values close to $\frac{1}{c_k}$ for x close to $c_k t + x_k$, we obtain, by simple localization arguments (see [15, Lemma 4] for details), that there exists $\lambda_2 > 0$ such that

$$\int (\tilde{z}_x^2 - pR^{p-1}\tilde{z}^2)h + \tilde{z}^2 \geqslant \lambda_2 \|\tilde{z}\|_{H^1}^2 - \frac{1}{\lambda_2} \sum_{k=1}^{N} \left[\left(\int \tilde{z} R_{kx} \right)^2 + \left(\int \tilde{z} Z_k^+ \right)^2 + \left(\int \tilde{z} Z_k^- \right)^2 \right].$$

Moreover, gathering all previous estimates, we have for all $t \in [T(\mathfrak{a}^-), S_n]$:

(a) For all $k \in [\![1, N]\!]$, $(\int \tilde{z} R_{kx})^2 \leqslant Ce^{-2\gamma t} \|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (3.20).

(b) For all $k \in [\![1, N]\!]$, $\left(\int \tilde{z} Z_k^+\right)^2 \leqslant 2(\alpha_k^+)^2 + Ce^{-2\gamma t} \|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (iii) of Lemma 2.5, (3.10) and (2.2).

(c) For all $k \in [\![1, j]\!]$, $\left(\int \tilde{z} Z_k^-\right)^2 \leqslant 2(\alpha_k^-)^2 + Ce^{-2\gamma t} \|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (iii) of Lemma 2.5, (3.12) and (2.2).

(d) For all $k \in [\![j+1, N]\!]$, $\left(\int \tilde{z} Z_k^-\right)^2 \leqslant 2(\alpha_k^-)^2 + Ce^{-2\gamma t} \|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (3.8).

Finally, we have proved that there exists $K > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\|\tilde{z}(t)\|_{H^1} \leqslant K e^{-(e_j+2\gamma)t}.$$

We want now to prove the same estimate for z.

Lemma 3.11. *There exists $K_0 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\|z(t)\|_{H^1} \leqslant K_0 e^{-(e_j+2\gamma)t}.$$

Proof. By (3.21), it is enough to prove this estimate for $|a_k(t)|$ with $k \in [\![1, N]\!]$ fixed. To do this, write first the equation of $\tilde{z}$ from the equation of z (3.4):

$$\begin{aligned}
&\tilde{z}_t + (\tilde{z}_{xx} + p\varphi^{p-1}\tilde{z})_x \\
&= z_t + \sum_{l=1}^N a_l R_{lxt} + \sum_{l=1}^N a_l' R_{lx} + z_{xxx} + \sum_{l=1}^N a_l R_{lxxx} + p\sum_{l=1}^N a_l (R_{lx}\varphi^{p-1})_x + p(\varphi^{p-1}z)_x \\
&= -(\omega_1 \cdot z)_x - (\omega(z))_x - \Omega_x + \sum_{l=1}^N a_l' R_{lx} + \sum_{l=1}^N a_l \Big[-c_l R_{lx} + R_{lxxx} + p\varphi^{p-1} R_{lx} \Big]_x .
\end{aligned}$$

Then multiply this equation by R_{kx} and integrate, so that we obtain

$$\begin{aligned}
&\int \tilde{z}_t R_{kx} - \int (\tilde{z}_{xx} + p\varphi^{p-1}\tilde{z}) R_{kxx} = a_k' \int R_{kx}^2 + \sum_{l \neq k} a_l' \int R_{lx} R_{kx} \\
&+ \sum_{l=1}^N a_l \int \Big[R_{lxxx} - c_l R_{lx} + p\varphi^{p-1} R_{lx} \Big]_x R_{kx} + \int \omega_1 z R_{kxx} + \int \omega(z) R_{kxx} + \int \Omega R_{kxx}.
\end{aligned}$$

But from (2.3) and (iii) of Claim 2.2, we have

$$\begin{aligned}
\|(R_{lxxx} - c_l R_{lx} + p\varphi^{p-1} R_{lx})_x\|_{L^\infty} &\leqslant p\|R_{lx}(\varphi^{p-1} - R_l^{p-1})\|_{H^2} \\
&\leqslant C\|\varphi - R\|_{H^2} + p\|R_{lx}(R^{p-1} - R_l^{p-1})\|_{H^2} \leqslant Ce^{-2\gamma t},
\end{aligned}$$

and consequently

$$\begin{aligned}
|a_k'| \leqslant C\left|\int \tilde{z}_t R_{kx}\right| + C\|\tilde{z}\|_{L^2} + Ce^{-\gamma t} \sum_{l \neq k} |a_l'| + Ce^{-2\gamma t} \sum_{l=1}^N |a_l| \\
+ Ce^{-e_j t}\|z\|_{L^2} + C\|z\|_{L^2}^2 + C\|\Omega\|_{L^2}.
\end{aligned}$$

Moreover, from $\int \tilde{z} R_{kx} = \sum_{l \neq k} a_l \int R_{lx} R_{kx}$, we deduce that

$$\begin{aligned}
\frac{d}{dt}\int \tilde{z} R_{kx} &= \sum_{l \neq k} a_l' \int R_{kx} R_{lx} + \sum_{l \neq k} a_l \int (-c_l R_{lxx} R_{kx} - c_k R_{lx} R_{kxx}) \\
&= \int \tilde{z}_t R_{kx} + \int \tilde{z}(-c_k R_{kxx}),
\end{aligned}$$

and so

$$\left|\int \tilde{z}_t R_{kx}\right| \leqslant C\|\tilde{z}\|_{H^1} + Ce^{-\gamma t} \sum_{l \neq k} |a_l'| + Ce^{-2\gamma t} \sum_{l=1}^N |a_l|.$$

Gathering previous estimates, we have, from (3.21) and (3.5),

$$\begin{aligned}
|a_k'| &\leqslant C\|\tilde{z}\|_{H^1} + C_4 e^{-\gamma t} \sum_{l \neq k} |a_l'| + Ce^{-2\gamma t}\|z\|_{H^1} + C\|z\|_{H^1}^2 + C\|\Omega\|_{L^2} \\
&\leqslant Ke^{-(e_j+2\gamma)t} + C_4 e^{-\gamma t} \sum_{l \neq k} |a_l'| + Ce^{-2\gamma t} e^{-(e_j+\gamma)t} + Ce^{-2(e_j+\gamma)t} + Ce^{-(e_j+4\gamma)t}.
\end{aligned}$$

Finally, if we choose t_0 large enough so that $C_4 e^{-\gamma t_0} \leqslant \frac{1}{N}$, we obtain for all $s \in [T(\mathfrak{a}^-), S_n]$,

$$|a_k'(s)| \leqslant Ke^{-(e_j+2\gamma)s}.$$

By integration on $[t, S_n]$ with $t \in [T(\mathfrak{a}^-), S_n]$, we get $|a_k(t)| \leqslant |a_k(S_n)| + Ke^{-(e_j+2\gamma)t}$. But from Claim 3.7 and Lemma 3.6, we have

$$|a_k(S_n)| \leqslant C\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\| \leqslant C\|\mathfrak{a}^-\| \leqslant Ce^{-(e_j+2\gamma)S_n} \leqslant Ce^{-(e_j+2\gamma)t},$$

and so finally,

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad |a_k(t)| \leqslant Ke^{-(e_j+2\gamma)t}.$$

□

3.3.7 Control of the unstable directions for $k > j$ by a topological argument

Lemma 3.11 being proved, we choose t_0 large enough so that $K_0 e^{-\gamma t_0} \leqslant \frac{1}{2}$. Therefore, we have

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \|z(t)\|_{H^1} \leqslant \frac{1}{2} e^{-(e_j+\gamma)t}.$$

We can now prove the following final lemma, which concludes the proof of Proposition 3.3.

Lemma 3.12. *For t_0 large enough, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$.*

Proof. For the sake of contradiction, suppose that, for all $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$, $T(\mathfrak{a}^-) > t_0$. As $e^{(e_j+\gamma)T(\mathfrak{a}^-)} z(T(\mathfrak{a}^-)) \in B_{H^1}(1/2)$, then by definition of $T(\mathfrak{a}^-)$ and continuity of the flow, we have

$$e^{(e_j+2\gamma)T(\mathfrak{a}^-)} \alpha^-(T(\mathfrak{a}^-)) \in \mathbb{S}_{\mathbb{R}^{N-j}}(1). \tag{3.23}$$

Now, let $T \in [t_0, T(\mathfrak{a}^-)]$ be close enough to $T(\mathfrak{a}^-)$ such that z is defined on $[T, S_n]$, and by continuity,

$$\forall t \in [T, S_n], \quad \|z(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

We can now consider, for $t \in [T, S_n]$,

$$\mathcal{N}(t) = \mathcal{N}(\alpha^-(t)) = \|e^{(e_j+2\gamma)t} \alpha^-(t)\|^2.$$

To calculate $\mathcal{N}'$, we start from estimate (3.11):

$$\forall k \in [\![j+1, N]\!], \forall t \in [T, S_n], \quad \left|\frac{d}{dt}\alpha_k^-(t) + e_k \alpha_k^-(t)\right| \leqslant K_2' e^{-(e_j+4\gamma)t}.$$

Multiplying by $|\alpha_k^-(t)|$, we obtain

$$\left|\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + e_k \alpha_k^-(t)^2\right| \leqslant K_2' e^{-(e_j+4\gamma)t} |\alpha_k^-(t)|,$$

and thus

$$2\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + 2e_{j+1}\alpha_k^-(t)^2 \leqslant 2\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + 2e_k\alpha_k^-(t)^2 \leqslant K_2 e^{-(e_j+4\gamma)t}|\alpha_k^-(t)|.$$

By summing on $k \in [\![j+1, N]\!]$, we get

$$(\|\alpha^-(t)\|^2)' + 2e_{j+1}\|\alpha^-(t)\|^2 \leqslant K_2 e^{-(e_j+4\gamma)t}\|\alpha^-(t)\|.$$

Therefore, we can estimate

$$\mathcal{N}'(t) = (e^{2(e_j+2\gamma)t}\|\boldsymbol{\alpha}^-(t)\|^2)' = e^{2(e_j+2\gamma)t}\left[2(e_j+2\gamma)\|\boldsymbol{\alpha}^-(t)\|^2 + (\|\boldsymbol{\alpha}^-(t)\|^2)'\right]$$
$$\leqslant e^{2(e_j+2\gamma)t}\left[2(e_j+2\gamma)\|\boldsymbol{\alpha}^-(t)\|^2 - 2e_{j+1}\|\boldsymbol{\alpha}^-(t)\|^2 + K_2 e^{-(e_j+4\gamma)t}\|\boldsymbol{\alpha}^-(t)\|\right].$$

Hence, we have, for all $t \in [T, S_n]$,

$$\mathcal{N}'(t) \leqslant -\theta \cdot \mathcal{N}(t) + K_2 e^{e_j t}\|\boldsymbol{\alpha}^-(t)\|,$$

where $\theta = 2(e_{j+1} - e_j - 2\gamma) > 0$ by definition of γ (2.1). In particular, for all $\tau \in [T, S_n]$ satisfying $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\theta + K_2 e^{e_j\tau}\|\boldsymbol{\alpha}^-(\tau)\| = -\theta + K_2 e^{e_j\tau}e^{-(e_j+2\gamma)\tau} = -\theta + K_2 e^{-2\gamma\tau} \leqslant -\theta + K_2 e^{-2\gamma t_0}.$$

Now, we fix t_0 large enough so that $K_2 e^{-2\gamma t_0} \leqslant \frac{\theta}{2}$, and so, for all $\tau \in [T, S_n]$ such that $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\frac{\theta}{2}. \tag{3.24}$$

In particular, by (3.23), we have $\mathcal{N}'(T(\mathfrak{a}^-)) \leqslant -\frac{\theta}{2}$.

First consequence: $\mathfrak{a}^- \mapsto T(\mathfrak{a}^-)$ is continuous. Indeed, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\mathcal{N}(T(\mathfrak{a}^-) - \varepsilon) > 1 + \delta$ and $\mathcal{N}(T(\mathfrak{a}^-) + \varepsilon) < 1 - \delta$. Moreover, by definition of $T(\mathfrak{a}^-)$ and (3.24), there can not exist $\tau \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$ such that $\mathcal{N}(\tau) = 1$, and so by choosing δ small enough, we have, for all $t \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$, $\mathcal{N}(t) < 1 - \delta$. But from continuity of the flow, there exists $\eta > 0$ such that, for all $\tilde{\mathfrak{a}}^-$ satisfying $\|\tilde{\mathfrak{a}}^- - \mathfrak{a}^-\| \leqslant \eta$, we have

$$\forall t \in [T(\mathfrak{a}^-) - \varepsilon, S_n], \quad |\mathcal{N}(\tilde{\boldsymbol{\alpha}}^-(t)) - \mathcal{N}(\boldsymbol{\alpha}^-(t))| \leqslant \delta/2.$$

We finally deduce that $T(\mathfrak{a}^-) - \varepsilon \leqslant T(\tilde{\mathfrak{a}}^-) \leqslant T(\mathfrak{a}^-) + \varepsilon$, as expected.

Second consequence: We can define the map

$$\begin{array}{rccl} \mathcal{M} : & B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n}) & \to & \mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n}) \\ & \mathfrak{a}^- & \mapsto & e^{-(e_j+2\gamma)(S_n - T(\mathfrak{a}^-))}\boldsymbol{\alpha}^-(T(\mathfrak{a}^-)). \end{array}$$

Note that $\mathcal{M}$ is continuous by the previous point. Moreover, let $\mathfrak{a}^- \in \mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$. As $\mathcal{N}'(S_n) \leqslant -\frac{\theta}{2}$ by (3.24), we deduce by definition of $T(\mathfrak{a}^-)$ that $T(\mathfrak{a}^-) = S_n$, and so $\mathcal{M}(\mathfrak{a}^-) = \mathfrak{a}^-$. In other words, $\mathcal{M}$ restricted to $\mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ is the identity. But the existence of such a map $\mathcal{M}$ contradicts Brouwer's fixed point theorem.

In conclusion, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$. □

4 Classification of multi-solitons

This section is devoted to prove the second assertion of Theorem 1.3. Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R = \sum_{j=1}^{N} R_{c_j,x_j}$ and φ the multi-soliton given by Theorem 1.2. Let u be a solution of (gKdV), defined on $[t_1, +\infty)$ with $t_1 > 0$ large, satisfying

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0. \tag{4.1}$$

The main idea of the proof is to improve (4.1), by giving the asymptotic behavior of $u(t)$ at infinity at any exponential rate. This has to be done in several steps.

4.1 Convergence at exponential rate γ

We first improve condition (4.1) into an exponential convergence, with a small rate $\gamma > 0$, where γ is defined by (2.1).

Lemma 4.1. *Let $\varepsilon = u - \varphi$. Then there exist $C, t_0 > 0$ such that, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma t}$.*

Proof. Step 1: Modulation. Denote $v = u - R$, so that $\|v(t)\|_{H^1} \to 0$ as $t \to +\infty$ by (4.1). Therefore, by a standard lemma of modulation (see for example [11, Lemma 2]), for t_0 large enough, there exist N functions $y_j : [t_0, +\infty) \to \mathbb{R}$ of class C^1 such that $w = u - \widetilde{R}$, where $\widetilde{R} = \sum \widetilde{R}_j$ and $\widetilde{R}_j(t) = R_j(t, \cdot - y_j(t))$, satisfies

$$\begin{cases} \forall j \in [\![1, N]\!], \quad \int w(t)(\widetilde{R}_j)_x(t) = 0, \\ \|w(t)\|_{H^1} + \sum_{j=1}^N |y_j(t)| \leqslant C\|v(t)\|_{H^1}, \\ \forall j \in [\![1, N]\!], \quad |y_j'(t)| \leqslant C\|w(t)\|_{H^1} + Ce^{-\gamma t}. \end{cases}$$

Note that the first two facts are a simple consequence of the implicit function theorem, while the last estimate comes from the equation satisfied by w,

$$\partial_t w + \partial_x^3 w = \sum_{k=1}^N y_k' \partial_x(\widetilde{R}_k) - \partial_x \left((w + \widetilde{R})^p - \sum_{k=1}^N \widetilde{R}_k^p \right),$$

multiplied by $(\widetilde{R}_j)_x$ and integrated. Similarly, if we denote $\widetilde{Z}_j^{\pm}(t) = Z_j^{\pm}(t, \cdot - y_j(t))$ and $\widetilde{\alpha}_j^{\pm}(t) = \int w(t)\widetilde{Z}_j^{\pm}(t)$, the equation of w multiplied by $\widetilde{Z}_j^{\pm}$ leads to

$$\forall t \geqslant t_0, \quad \left| \frac{d}{dt}\widetilde{\alpha}_j^{\pm}(t) \mp e_j \widetilde{\alpha}_j^{\pm}(t) \right| \leqslant C\|w(t)\|_{H^1}^2 + Ce^{-2\gamma t}. \tag{4.2}$$

Step 2: Monotonicity. We use again the function ψ introduced in Section 3.3.5. Following [11], we introduce moreover $\psi_N \equiv 1$ and, for $j \in [\![1, N-1]\!]$,

$$m_j(t) = \frac{c_j + c_{j+1}}{2} t + \frac{x_j + x_{j+1}}{2}, \quad \psi_j(t) = \psi(x - m_j(t)),$$

and

$$\phi_1 \equiv \psi_1, \quad \phi_N \equiv 1 - \psi_{N-1}, \quad \phi_j \equiv \psi_j - \psi_{j-1} \quad \text{for } j \in [\![2, N-1]\!].$$

We also define some local quantities related to L^2 mass and energy:

$$M_j(t) = \int u^2(t)\phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2}u_x^2(t) - \frac{1}{p+1}u^{p+1}(t) \right) \phi_j(t),$$

and

$$\widetilde{E}_j(t) = E_j(t) + \frac{\sigma_0}{100} M_j(t).$$

Then, by (4.1) and monotonicity results on the quantities $t \mapsto \sum_{k=1}^j M_k(t)$ and $t \mapsto \sum_{k=1}^j E_k(t)$, we have, for all $t \geqslant t_0$ and all $j \in [\![1, N]\!]$, following Lemmas 1 and 3 of [11],

$$\begin{cases} \sum_{k=1}^j \left(\int Q_{c_k}^2 - M_k(t) \right) \geqslant -K_2 e^{-2\gamma t}, & (4.3) \\ \sum_{k=1}^j \left(E(Q_{c_k}) + \frac{\sigma_0}{100} \int Q_{c_k}^2 - \widetilde{E}_k(t) \right) \geqslant -K_2 e^{-2\gamma t}, & (4.4) \end{cases}$$

and

$$\left|\left(E_j(t)+\frac{c_j}{2}M_j(t)\right)-\left(E(Q_{c_j})+\frac{c_j}{2}\int Q_{c_j}^2\right)-\frac{1}{2}H_j(t)\right|\leqslant K_4e^{-2\gamma t}+K_4\|w(t)\|_{H^1}\int w^2\phi_j, \tag{4.5}$$

where $H_j(t)=\int\left(w_x^2(t)+c_jw^2(t)-p\widetilde{R}_j^{p-1}(t)w^2(t)\right)\phi_j(t)$. But if we write

$$\sum_{j=1}^N\frac{1}{c_j^2}\left(E_j+\frac{c_j}{2}M_j\right)=\sum_{j=1}^{N-1}\left[\left(\frac{1}{c_j^2}-\frac{1}{c_{j+1}^2}\right)\sum_{k=1}^j\widetilde{E}_k\right]+\frac{1}{2c_N}\left(1-\frac{\sigma_0}{50c_N}\right)\sum_{k=1}^N M_k$$
$$+\frac{1}{c_N^2}\sum_{k=1}^N\widetilde{E}_k+\sum_{j=1}^{N-1}\left[\frac{1}{2}\left(\frac{1}{c_j}-\frac{1}{c_{j+1}}\right)\left(1-\frac{\sigma_0}{50}\left(\frac{1}{c_j}+\frac{1}{c_{j+1}}\right)\right)\sum_{k=1}^j M_k\right],$$

and similarly

$$\sum_{j=1}^N\frac{1}{c_j^2}\left(E(Q_{c_j})+\frac{c_j}{2}\int Q_{c_j}^2\right)=\sum_{j=1}^{N-1}\left[\left(\frac{1}{c_j^2}-\frac{1}{c_{j+1}^2}\right)\sum_{k=1}^j\left(E(Q_{c_k})+\frac{\sigma_0}{100}\int Q_{c_k}^2\right)\right]$$
$$+\sum_{j=1}^{N-1}\left[\frac{1}{2}\left(\frac{1}{c_j}-\frac{1}{c_{j+1}}\right)\left(1-\frac{\sigma_0}{50}\left(\frac{1}{c_j}+\frac{1}{c_{j+1}}\right)\right)\sum_{k=1}^j\int Q_{c_k}^2\right]$$
$$+\frac{1}{c_N^2}\sum_{k=1}^N\left(E(Q_{c_k})+\frac{\sigma_0}{100}\int Q_{c_k}^2\right)+\frac{1}{2c_N}\left(1-\frac{\sigma_0}{50c_N}\right)\sum_{k=1}^N\int Q_{c_k}^2,$$

and if we remark that all coefficients in these decompositions are positive, we obtain, by (4.3) and (4.4),

$$\sum_{j=1}^N\frac{1}{c_j^2}\left(E_j(t)+\frac{c_j}{2}M_j(t)\right)-\sum_{j=1}^N\frac{1}{c_j^2}\left(E(Q_{c_j})+\frac{c_j}{2}\int Q_{c_j}^2\right)\leqslant Ce^{-2\gamma t}.$$

Therefore, we have, by (4.5),

$$\frac{1}{2}\sum_{j=1}^N\frac{1}{c_j^2}H_j(t)\leqslant\sum_{j=1}^N\frac{1}{c_j^2}\left(E_j(t)+\frac{c_j}{2}M_j(t)\right)-\sum_{j=1}^N\frac{1}{c_j^2}\left(E(Q_{c_j})+\frac{c_j}{2}\int Q_{c_j}^2\right)$$
$$+K_4\sum_{j=1}^N\frac{1}{c_j^2}e^{-2\gamma t}+K_4\|w(t)\|_{H^1}\sum_{j=1}^N\frac{1}{c_j^2}\int w^2\phi_j$$
$$\leqslant C_1e^{-2\gamma t}+\frac{K_4}{\sigma_0^2}\|w(t)\|_{H^1}\int w^2\sum_{j=1}^N\phi_j$$

since $\phi_j\geqslant 0$. Finally, as $\sum_{j=1}^N\phi_j\equiv 1$, we obtain

$$\sum_{j=1}^N\frac{1}{c_j^2}H_j(t)\leqslant C_1e^{-2\gamma t}+C_2\|w(t)\|_{H^1}^3. \tag{4.6}$$

Step 3: Coercivity. Now, from the property of coercivity (vi) in Lemma 2.5 and by standard localization arguments (as in Section 3), we have

$$\sum_{j=1}^N\frac{1}{c_j^2}H_j(t)\geqslant\lambda_c\|w(t)\|_{H^1}^2-\frac{1}{\lambda_c}\sum_{j=1}^N\left(\int w(t)(\widetilde{R}_j)_x(t)\right)^2-\frac{1}{\lambda_c}\sum_{j,\pm}\left(\int w(t)\widetilde{Z}_j^{\pm}(t)\right)^2.$$

As $\int w(t)(\tilde{R}_j)_x(t) = 0$ and $\tilde{\alpha}_j^{\pm}(t) = \int w(t)\tilde{Z}_j^{\pm}(t)$, we obtain, by (4.6),

$$\lambda_c \|w(t)\|_{H^1}^2 \leqslant C_1 e^{-2\gamma t} + C_2 \|w(t)\|_{H^1}^3 + C_3 \|\tilde{\alpha}(t)\|^2,$$

where $\tilde{\alpha}(t) = (\tilde{\alpha}_j^{\pm}(t))_{j,\pm}$. For t_0 large enough so that $C_2\|w(t)\|_{H^1} \leqslant \frac{\lambda_c}{2}$, we obtain

$$\forall t \geqslant t_0, \quad \|w(t)\|_{H^1}^2 \leqslant C_1 \|\tilde{\alpha}(t)\|^2 + C_2 e^{-2\gamma t}. \tag{4.7}$$

Step 4: Exponential decay of $\tilde{\alpha}$. From (4.2) and (4.7), we have, for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$,

$$\left| \frac{d}{dt} \tilde{\alpha}_j^{\pm}(t) \mp e_j \tilde{\alpha}_j^{\pm}(t) \right| \leqslant C_1 \|\tilde{\alpha}(t)\|^2 + C_2 e^{-2\gamma t}.$$

We follow here the strategy of [3, Section 4.4.2]. Define $A(t) = \sum_{j=1}^N \tilde{\alpha}_j^+(t)^2$ and $B(t) = \sum_{j=1}^N \tilde{\alpha}_j^-(t)^2$, and let us prove that $A(t) \leqslant B(t) + Le^{-2\gamma t}$ for L large enough. First, we have, by multiplying the previous estimate by $|\tilde{\alpha}_j^+(t)|$ (that we can of course suppose less than 1),

$$\tilde{\alpha}_j^+(t) \frac{d}{dt} \tilde{\alpha}_j^+(t) \geqslant e_j \tilde{\alpha}_j^+(t)^2 - C_1 |\tilde{\alpha}_j^+(t)| \cdot \|\tilde{\alpha}(t)\|^2 - C_2 e^{-2\gamma t},$$

and so, by summing,

$$A'(t) \geqslant 2e_1 A(t) - C_1 \|\tilde{\alpha}(t)\|^3 - C_2 e^{-2\gamma t}.$$

Similarly, we obtain

$$B'(t) \leqslant -2e_1 B(t) + C_1 \|\tilde{\alpha}(t)\|^3 + C_2 e^{-2\gamma t}. \tag{4.8}$$

Now, let $h(t) = A(t) - B(t) - Le^{-2\gamma t}$ with L to be determined later. We have of course $h(t) \to 0$ as $t \to +\infty$, and by the previous estimates, we can calculate

$$\begin{aligned} h'(t) &= A'(t) - B'(t) + 2L\gamma e^{-2\gamma t} \\ &\geqslant 2e_1 A(t) + 2e_1 B(t) - C_1 \|\tilde{\alpha}(t)\|^3 - C_2 e^{-2\gamma t} \\ &\geqslant 2e_1 h(t) + 4e_1 B(t) - C_1 \|\tilde{\alpha}(t)\|^3 - C_2 e^{-2\gamma t} + 2Le_1 e^{-2\gamma t}. \end{aligned}$$

Since $\|\tilde{\alpha}(t)\|^2 = A(t) + B(t) = h(t) + 2B(t) + Le^{-2\gamma t}$, we get

$$h'(t) \geqslant h(t)(2e_1 - C_1\|\tilde{\alpha}(t)\|) + B(t)(4e_1 - 2C_1\|\tilde{\alpha}(t)\|) + e^{-2\gamma t}(2Le_1 - C_2 - C_1 L \|\tilde{\alpha}(t)\|).$$

Now, choose t_0 large enough so that $C_1\|\tilde{\alpha}(t)\| \leqslant \frac{e_1}{2}$ for $t \geqslant t_0$, and fix $L = \frac{C_2}{e_1}$. Therefore, we have, for all $t \geqslant t_0$ such that $h(t) \geqslant 0$, $h'(t) \geqslant e_1 h(t)$. Hence, if there exists $T \geqslant t_0$ such that $h(T) \geqslant 0$, then $h(t) \geqslant 0$ for all $t \geqslant T$, and thus $h(t) \geqslant Ce^{e_1 t}$, which would be in contradiction with $\lim_{t\to+\infty} h(t) = 0$. So we have proved that $h(t) \leqslant 0$ for all $t \geqslant t_0$, as expected.

Now, from (4.8) and the choice of t_0 to have $C_1\|\tilde{\alpha}(t)\| \leqslant \frac{e_1}{2}$ for all $t \geqslant t_0$, it comes

$$B'(t) + 2e_1 B(t) \leqslant e_1 B(t) + \left(\frac{Le_1}{2} + C_2 \right) e^{-2\gamma t},$$

and so $B'(t) + e_1 B(t) \leqslant Ke^{-2\gamma t}$.

Therefore, $(e^{e_1 s}B(s))' \leqslant Ke^{(e_1-2\gamma)s}$ for $s \geqslant t_0$, and so by integration on $[t_0, t]$,

$$e^{e_1 t}B(t) - e^{e_1 t_0}B(t_0) \leqslant Ke^{(e_1-2\gamma)t},$$

since $e_1 - 2\gamma > 0$. We deduce that

$$B(t) \leqslant Ke^{-2\gamma t} + K'e^{-e_1 t} \leqslant Ke^{-2\gamma t}.$$

Finally, we also have by the previous point $A(t) \leqslant K'e^{-2\gamma t}$, and so

$$\forall t \geqslant t_0, \quad \|\tilde{\alpha}(t)\|^2 \leqslant Ce^{-2\gamma t}.$$

Step 5: Conclusion. By (4.7), we deduce that $\|w(t)\|_{H^1} \leqslant Ce^{-\gamma t}$, and from the estimate on $|y_j'|$, we have, for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$, $|y_j(t)| \leqslant Ce^{-\gamma t}$, by integration and the fact that $y_j(t) \to 0$ as $t \to +\infty$. To conclude, write

$$\varepsilon = u - \varphi = w + \tilde{R} - \varphi = w - (\varphi - R) + (\tilde{R} - R),$$

so that

$$\|\varepsilon(t)\|_{H^1} \leqslant \|w(t)\|_{H^1} + \|(\varphi - R)(t)\|_{H^1} + \|(\tilde{R} - R)(t)\|_{H^1} \leqslant Ce^{-\gamma t} + \|(\tilde{R} - R)(t)\|_{H^1}.$$

But we have

$$\|(\tilde{R} - R)(t)\|_{H^1} \leqslant \sum_{j=1}^{N} \|R_j(t, \cdot - y_j(t)) - R_j(t)\|_{H^1} \leqslant C\sum_{j=1}^{N} |y_j(t)| \leqslant Ce^{-\gamma t},$$

and so finally, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma t}$. □

4.2 Convergence at exponential rate e_1

Now, we further improve the convergence of the previous lemma, with an exponential rate $e_1 \gg \gamma$. The proof will mainly use arguments developed in [11, Section 4], where the starting point is also a convergence of $\|\varepsilon(t)\|_{H^1}$ at a small exponential rate. Note that it is impossible to obtain the improvement of the convergence on $w = u - \tilde{R}$ considered in the previous section, because of the source terms $Ce^{-2\gamma t}$ as in (4.2) for example. That is why the analysis will be done, in this section, on ε (difference between *two solutions*), to obtain the following lemma.

Lemma 4.2. *There exist $C, t_0 > 0$ such that, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_1 t}$.*

Proof. Step 1: Estimates. We follow the same strategy as in Section 3.3. First, from the equation of ε,

$$\varepsilon_t + (\varepsilon_{xx} + (\varphi + \varepsilon)^p - \varphi^p)_x = 0,$$

we can estimate $\alpha_j^{\pm}(t) = \int \varepsilon(t)Z_j^{\pm}(t)$ for $j \in [\![1, N]\!]$ and $t \geqslant t_0$. Indeed, we have

$$\begin{aligned}
\frac{d}{dt}\alpha_j^{\pm}(t) &= \int \varepsilon_t Z_j^{\pm} + \int \varepsilon Z_{jt}^{\pm} = \int \left(\varepsilon_{xx} + (\varphi + \varepsilon)^p - \varphi^p\right) Z_{jx}^{\pm} - c_j \int \varepsilon Z_{jx}^{\pm} \\
&= \int \left[\varepsilon_{xx} - c_j\varepsilon + \sum_{k=1}^{p} \binom{p}{k} \varphi^{p-k}\varepsilon^k\right] Z_{jx}^{\pm} \\
&= \int \left[\varepsilon_{xx} - c_j\varepsilon + pR_j^{p-1}\varepsilon\right] Z_{jx}^{\pm} + p\int (\varphi^{p-1} - R_j^{p-1})\varepsilon Z_{jx}^{\pm} + \sum_{k=2}^{p} \binom{p}{k} \int \varphi^{p-k}\varepsilon^k Z_{jx}^{\pm} \\
&= \mathbf{I} + \mathbf{II} + \mathbf{III}.
\end{aligned}$$

But we have $\mathbf{I} = \pm e_j \alpha_j^{\pm}(t)$ (see proof of (3.9)), $|\mathbf{II}| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}$ and $|\mathbf{III}| \leqslant C\|\varepsilon(t)\|_{H^1}^2$, and so, for all $t \geqslant t_0$ and all $j \in [\![1,N]\!]$,

$$\left|\frac{d}{dt}\alpha_j^{\pm}(t) \mp e_j\alpha_j(t)\right| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}. \tag{4.9}$$

To control the R_{jx} directions, we proceed exactly as in Section 3.3.6. Define

$$\tilde{\varepsilon}(t) = \varepsilon(t) + \sum_{j=1}^{N} a_j(t)R_{jx}(t),$$

where $a_j(t) = -\frac{\int \varepsilon(t)R_{jx}(t)}{\int (Q'_{c_j})^2}$, so that $|\int \tilde{\varepsilon}(t)R_{jx}(t)| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}$ and

$$C_1\|\varepsilon\|_{H^1} \leqslant \|\tilde{\varepsilon}\|_{H^1} + \sum_{j=1}^{N}|a_j| \leqslant C_2\|\varepsilon\|_{H^1}. \tag{4.10}$$

As $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma t}$, we have exactly as in [11], for all $t \geqslant t_0$, by monotonicity arguments,

$$\int \left[\varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t)\right]h(t) + \varepsilon^2(t) \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2,$$

where h is defined in Section 3.3.5. We also have, from [11],

$$\int (\tilde{\varepsilon}_x^2 - pR^{p-1}\tilde{\varepsilon}^2)h + \tilde{\varepsilon}^2 \leqslant \int \left[(\varepsilon_x^2 - pR^{p-1}\varepsilon^2)h + \varepsilon^2\right] + Ce^{-2\gamma t}\sum_{j=1}^{N} a_j^2 + Ce^{-2\gamma t}\|\varepsilon\|_{H^1}^2,$$

and thus

$$\int \left[\tilde{\varepsilon}_x^2(t) - pR^{p-1}(t)\tilde{\varepsilon}^2(t)\right]h(t) + \tilde{\varepsilon}^2(t) \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2.$$

But as in Section 3.3.5, a localization argument of the property of coercivity (vi) in Lemma 2.5 leads to

$$\int (\tilde{\varepsilon}_x^2 - pR^{p-1}\tilde{\varepsilon}^2)h + \tilde{\varepsilon}^2 \geqslant \lambda_2\|\tilde{\varepsilon}\|_{H^1}^2 - \frac{1}{\lambda_2}\sum_{j=1}^{N}\left[\left(\int \tilde{\varepsilon}R_{jx}\right)^2 + \left(\int \tilde{\varepsilon}Z_j^+\right)^2 + \left(\int \tilde{\varepsilon}Z_j^-\right)^2\right].$$

Since $(\int \tilde{\varepsilon}R_{jx})^2 \leqslant Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2$ and $\left(\int \tilde{\varepsilon}Z_j^{\pm}\right)^2 \leqslant 2(\alpha_j^{\pm})^2 + Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2$, we have

$$\lambda_2\|\tilde{\varepsilon}\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2 + Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2 + C\sum_{j=1}^{N}(\alpha_j^+)^2 + C\sum_{j=1}^{N}(\alpha_j^-)^2.$$

By denoting $\boldsymbol{\alpha}(t) = (\alpha_j^{\pm}(t))_{j,\pm}$, we thus have

$$\|\tilde{\varepsilon}(t)\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2 + C\|\boldsymbol{\alpha}(t)\|^2. \tag{4.11}$$

Finally, to estimate $|a_j(t)|$ for all $j \in [\![1,N]\!]$, we follow the strategy and some calculation from the proof of Lemma 3.11. First, write the equation satisfied by $\tilde{\varepsilon}$:

$$\begin{aligned}
&\tilde{\varepsilon}_t + (\tilde{\varepsilon}_{xx} + p\varphi^{p-1}\tilde{\varepsilon})_x \\
&= \varepsilon_t + \varepsilon_{xxx} + p(\varphi^{p-1}\varepsilon)_x + \sum_{k=1}^N a_k R_{kxt} + \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k R_{kxxx} + p\sum_{k=1}^N a_k (R_{kx}\varphi^{p-1})_x \\
&= -[(\varphi+\varepsilon)^p - \varphi^p]_x + p(\varphi^{p-1}\varepsilon)_x + \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k \Big[-c_k R_{kx} + R_{kxxx} + p\varphi^{p-1}R_{kx}\Big]_x \\
&= \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k \Big[R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx}\Big]_x - \Big[(\varphi+\varepsilon)^p - \varphi^p - p\varphi^{p-1}\varepsilon\Big]_x.
\end{aligned}$$

Then multiply by R_{jx} and integrate, so that

$$\begin{aligned}
\int \tilde{\varepsilon}_t R_{jx} - \int (\tilde{\varepsilon}_{xx} + p\varphi^{p-1}\tilde{\varepsilon})R_{jxx} &= a_j' \int R_{jx}^2 + \sum_{k\neq j} a_k' \int R_{kx}R_{jx} \\
&+ \sum_{k=1}^N a_k \int \Big[R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx}\Big]_x R_{jx} + \int \Big[(\varphi^p+\varepsilon)^p - \varphi^p - p\varphi^{p-1}\varepsilon\Big] R_{jxx}.
\end{aligned}$$

As $\|(R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx})_x\|_{L^\infty} \leqslant Ce^{-\gamma t}$, we obtain

$$|a_j'(t)| \leqslant C\left|\int \tilde{\varepsilon}_t(t)R_{jx}(t)\right| + Ce^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\varepsilon(t)\|_{H^1}^2 + C\|\tilde{\varepsilon}(t)\|_{H^1}.$$

Moreover, we still have

$$\left|\int \tilde{\varepsilon}_t(t)R_{jx}(t)\right| \leqslant C\|\tilde{\varepsilon}(t)\|_{H^1} + Ce^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1},$$

and so

$$|a_j'(t)| \leqslant C_1 e^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\varepsilon(t)\|_{H^1}^2 + C\|\tilde{\varepsilon}(t)\|_{H^1}.$$

Finally, choose t_0 large enough such that $C_1 e^{-\gamma t_0} \leqslant \frac{1}{N}$, so that we obtain, for all $j \in [\![1,N]\!]$ and all $t \geqslant t_0$,

$$|a_j'(t)| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\tilde{\varepsilon}(t)\|_{H^1}. \tag{4.12}$$

Step 2: Induction. With estimates (4.9) to (4.12), we can now improve exponential convergence of ε by a bootstrap argument. We recall that we have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma_0 = \gamma$ already. Now, we prove that if $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma \leqslant \gamma_0 < e_1 - \gamma$, then $\|\varepsilon(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$. So, suppose that $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma \leqslant \gamma_0 < e_1 - \gamma$.

(a) From (4.9), we get, for all $j \in [\![1,N]\!]$, $|(e^{-e_j t}\alpha_j^+(t))'| \leqslant Ce^{-(e_j+\gamma_0+\gamma)t}$, and so by integration on $[t,+\infty)$, $|\alpha_j^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, since $\alpha_j^+(t) \to 0$ as $t \to +\infty$.

(b) Still from (4.9), we get, for all $j \in [\![1,N]\!]$, $|(e^{e_j t}\alpha_j^-(t))'| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$. As $e_j - \gamma - \gamma_0 \geqslant e_1 - \gamma - \gamma_0 > 0$, we obtain, by integration on $[t_0,t]$, $|e^{e_j t}\alpha_j^-(t) - e^{e_j t_0}\alpha_j^-(t_0)| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$, and so

$$|\alpha_j^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} + Ce^{-e_j t} \leqslant Ce^{-(\gamma_0+\gamma)t}.$$

(c) Therefore, we have $\|\alpha(t)\|^2 \leqslant Ce^{-2(\gamma_0+\gamma)t}$, and so, we obtain $\|\tilde{\varepsilon}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$ by (4.11).

(d) From (4.12), we deduce that, for all $j \in [\![1,N]\!]$, $|a_j'(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, and so, by integration on $[t,+\infty)$, $|a_j(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, since $a_j(t) \to 0$ as $t \to +\infty$.

(e) Finally, from (4.10), we have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$, as expected.

Step 3: Conclusion. We apply the previous induction until we have $e_1 - \gamma < \gamma_0 < e_1$. Note that if $\gamma_0 = e_1 - \gamma$, then the estimate is still true for $\gamma_0 = e_1 - \frac{3}{2}\gamma < e_1 - \gamma$, and so for $\gamma_0 = e_1 - \frac{1}{2}\gamma > e_1 - \gamma$ by the previous step. Now we follow the scheme of step 2. We still have, for all $j \in [\![1,N]\!]$, $|\alpha_j^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_1 t}$, and $|(e^{e_j t}\alpha_j^-(t))'| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$. In particular, for $j=1$, we have

$$|(e^{e_1 t}\alpha_1^-(t))'| \leqslant Ce^{(e_1-\gamma-\gamma_0)t} \in L^1([t_0,+\infty)),$$

since $e_1 - \gamma - \gamma_0 < 0$. Hence, there exists $A_1 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} e^{e_1 t}\alpha_1^-(t) = A_1, \tag{4.13}$$

and $|e^{e_1 t}\alpha_1^-(t) - A_1| \leqslant Ce^{(e_1-\gamma-\gamma_0)t}$, and so $|\alpha_1^-(t)| \leqslant Ce^{-e_1 t}$. For $j \geqslant 2$, since $e_j - \gamma - \gamma_0 > e_2 - \gamma - e_1 > 0$ by definition of γ, we still obtain, by integration on $[t_0,t]$, $|\alpha_j^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_1 t}$. As in step 2, it follows $\|\alpha(t)\|^2 \leqslant Ce^{-2e_1 t}$, then $\|\tilde{\varepsilon}(t)\|_{H^1} \leqslant Ce^{-e_1 t}$ by (4.11), $|a_j(t)| \leqslant Ce^{-e_1 t}$ for all $j \in [\![1,N]\!]$ by (4.12), and finally $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_1 t}$ by (4.10), as expected. □

Remark 4.3. Note that we can not obtain a better rate than e_1 for the estimate on $\|\varepsilon(t)\|_{H^1}$ if $A_1 \neq 0$. Therefore, to continue improving the rate of convergence, we have to consider a refined asymptotic expansion of u, which is the object of the next section.

4.3 Identification of the solution

We now prove the following proposition by induction, following the strategy of the previous section. We identify u among the family $(\varphi_{A_1,\dots,A_N})$ constructed in Section 3. We recall that this family was constructed thanks to the subfamilies $(\varphi_{A_1,\dots,A_j})$, which satisfy (3.2) for all $j \in [\![1,N]\!]$:

$$\forall t \geqslant t_0, \quad \|\varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Proposition 4.4. *For all $j \in [\![1,N]\!]$, there exist $t_0, C > 0$ and $(A_1,\dots,A_j) \in \mathbb{R}^j$ such that, defining $\varepsilon_j(t) = u(t) - \varphi_{A_1,\dots,A_j}(t)$, one has*

$$\forall t \geqslant t_0, \quad \|\varepsilon_j(t)\|_{H^1} \leqslant Ce^{-e_j t}.$$

Moreover, defining $\alpha_{j,k}^\pm(t) = \int \varepsilon_j(t) Z_k^\pm(t)$ for all $k \in [\![1,N]\!]$, one has

$$\forall k \in [\![1,j]\!], \quad \lim_{t\to+\infty} e^{e_k t}\alpha_{j,k}^-(t) = 0.$$

Remark 4.5. As $\varepsilon_1 = u - \varphi_{A_1} = \varepsilon + (\varphi - \varphi_{A_1})$, we have

$$\|\varepsilon_1(t)\|_{H^1} \leqslant \|\varepsilon(t)\|_{H^1} + \|\varphi(t) - \varphi_{A_1}(t)\|_{H^1} \leqslant Ce^{-e_1 t}$$

by Lemma 4.2 and (3.2). Moreover, defining z_1 by $z_1(t) = \varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^+(t)$, we have

$$\begin{aligned}\alpha_{1,1}^-(t) &= \int \varepsilon_1(t) Z_1^-(t) = \int \varepsilon(t) Z_1^-(t) - A_1 e^{-e_1 t} \int Y_1^+(t) Z_1^-(t) - \int z_1(t) Z_1^-(t) \\ &= \alpha_1^-(t) - A_1 e^{-e_1 t} - \int z_1(t) Z_1^-(t)\end{aligned}$$

by definition of α_1^- in the previous section and by normalization (iv) of Lemma 2.5. As $\|z_1(t)\|_{H^1} \leqslant e^{-(e_1+\gamma)t}$, we finally deduce, by (4.13),

$$|e^{e_1 t}\alpha_{1,1}^-(t)| \leqslant |e^{e_1 t}\alpha_1^-(t) - A_1| + Ce^{-\gamma t} \xrightarrow[t\to+\infty]{} 0.$$

Therefore, Proposition 4.4 is proved for $j = 1$.

Proof of Proposition 4.4. By Remark 4.5, it is enough to prove the inductive step: we suppose the assertion true for $j-1$ with $j \geqslant 2$, and we prove it for j. So, suppose that there exist $t_0, C > 0$ and $(A_1, \ldots, A_{j-1}) \in \mathbb{R}^{j-1}$ such that $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-e_{j-1}t}$ for all $t \geqslant t_0$, and moreover, for all $k \in [\![1, j-1]\!]$, $e^{e_k t}\alpha_{j-1,k}^-(t) \to 0$ as $t \to +\infty$.

Step 1: Another induction. Following the proof of Lemma 4.2, we prove that if $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $e_{j-1} \leqslant \gamma_0 < e_j - \gamma$, then $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$. But, as φ_{A_1} is a soliton like φ, estimates (4.9) to (4.12) of the previous section hold. In other words, we have, with obvious notation, for all $t \geqslant t_0$,

$$\begin{cases}\forall k \in [\![1,N]\!], \quad \left|\frac{d}{dt}\alpha_{j-1,k}^\pm(t) \mp e_k \alpha_{j-1,k}^\pm(t)\right| \leqslant Ce^{-\gamma t}\|\varepsilon_{j-1}(t)\|_{H^1}, \\ \|\tilde{\varepsilon}_{j-1}(t)\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon_{j-1}(t')\|_{H^1}^2 + C\|\boldsymbol{\alpha}_{j-1}(t)\|^2, \\ \forall k \in [\![1,N]\!], \quad |a_{j-1,k}'(t)| \leqslant Ce^{-\gamma t}\|\varepsilon_{j-1}(t)\|_{H^1} + C\|\tilde{\varepsilon}_{j-1}(t)\|_{H^1}, \\ \|\varepsilon_{j-1}(t)\|_{H^1} \leqslant C\|\tilde{\varepsilon}_{j-1}(t)\|_{H^1} + C\sum_{k=1}^N |a_{j-1,k}(t)|.\end{cases}$$

From these estimates, we deduce the following steps as in the previous section.

(a) For all $k \in [\![1,N]\!]$, $|\alpha_{j-1,k}^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$.

(b) For all $k \in [\![1, j-1]\!]$, we have $|(e^{e_k t}\alpha_{j-1,k}^-(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$. As $e_k - \gamma_0 - \gamma \leqslant e_{j-1} - \gamma_0 - \gamma \leqslant -\gamma < 0$ and $e^{e_k t}\alpha_{j-1,k}^-(t) \to 0$ as $t \to +\infty$ by hypothesis, we deduce by integration on $[t, +\infty)$ that $|e^{e_k t}\alpha_{j-1,k}^-(t)| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$, and so $|\alpha_{j-1,k}^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$.

(c) For all $k \in [\![j, N]\!]$, we still have $|(e^{e_k t}\alpha_{j-1,k}^-(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$. As $e_k - \gamma_0 - \gamma \geqslant e_j - \gamma_0 - \gamma > 0$, we deduce that $|e^{e_k t}\alpha_{j-1,k}^-(t) - e^{e_k t_0}\alpha_{j-1,k}^-(t_0)| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$ by integration on $[t_0, t]$, and so

$$|\alpha_{j-1,k}^-(t)| \leqslant Ce^{-e_k t} + Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-(\gamma_0+\gamma)t}.$$

(d) Hence, we have $\|\boldsymbol{\alpha}_{j-1}(t)\|^2 \leqslant Ce^{-2(\gamma_0+\gamma)t}$. It follows $\|\tilde{\varepsilon}_{j-1}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$, $|a_{j-1,k}(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$ by integration, and finally $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$, as expected.

Step 2: Identification of A_j. We apply the previous induction until we have γ_0 such that $e_j - \gamma < \gamma_0 < e_j$. Moreover, with the same scheme, we obtain the following estimates.

(a) For all $k \in [\![1,N]\!]$, $|\alpha^+_{j-1,k}(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_jt}$, and we still have

$$|(e^{e_kt}\alpha^-_{j-1,k}(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}.$$

(b) For all $k \in [\![1,j-1]\!]$, we still have $|\alpha^-_{j-1,k}(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_jt}$.

(c) For $k = j$, we have $|(e^{e_jt}\alpha^-_{j-1,j}(t))'| \leqslant Ce^{(e_j-\gamma_0-\gamma)t} \in L^1([t_0,+\infty))$ as $e_j-\gamma_0-\gamma<0$. Thus, there exists $A_j \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} e^{e_jt}\alpha^-_{j-1,j}(t) = A_j,$$

and moreover $|e^{e_jt}\alpha^-_{j-1,j}(t) - A_j| \leqslant Ce^{(e_j-\gamma_0-\gamma)t}$. Hence, we have $|\alpha^-_{j-1,j}(t)| \leqslant Ce^{-e_jt}$.

(d) For all $k \in [\![j+1,N]\!]$, we have $e_k - \gamma_0 - \gamma > e_{j+1} - e_j - \gamma > 0$, thus by integration on $[t_0,t]$, we get $|\alpha^-_{j-1,k}(t)| \leqslant Ce^{-e_kt} + Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_jt}$.

(e) We now have $\|\boldsymbol{\alpha}_{j-1}(t)\|^2 \leqslant Ce^{-2e_jt}$, and so as in the first step, we conclude that $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-e_jt}$.

Step 3: Conclusion. To conclude the induction, we write

$$\begin{aligned}\varepsilon_j(t) = u(t) - \varphi_{A_1,\dots,A_j}(t) &= \varepsilon_{j-1}(t) + [\varphi_{A_1,\dots,A_{j-1}}(t) - \varphi_{A_1,\dots,A_j}(t)]\\ &= \varepsilon_{j-1}(t) - A_je^{-e_jt}Y_j^+(t) - z_j(t),\end{aligned}$$

where $z_j(t) = \varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_je^{-e_jt}Y_j^+(t)$ satisfies $\|z_j(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}$ by (3.2). Thus, we first have

$$\|\varepsilon_j(t)\|_{H^1} \leqslant \|\varepsilon_{j-1}(t)\|_{H^1} + Ce^{-e_jt} + \|z_j(t)\|_{H^1} \leqslant Ce^{-e_jt}.$$

Moreover, we find

$$\alpha^-_{j,k}(t) = \int \varepsilon_j(t)Z_k^-(t) = \alpha^-_{j-1,k}(t) - A_je^{-e_jt}\int Y_j^+(t)Z_k^-(t) - \int z_j(t)Z_k^-(t).$$

Therefore, for all $k \in [\![1,j-1]\!]$, we have $|\alpha^-_{j,k}(t)| \leqslant |\alpha^-_{j-1,k}(t)| + Ce^{-e_jt} + Ce^{-(e_j+\gamma)t}$, and so

$$e^{e_kt}|\alpha^-_{j,k}(t)| \leqslant e^{e_kt}|\alpha^-_{j-1,k}(t)| + Ce^{-(e_j-e_k)t} \xrightarrow[t\to+\infty]{} 0.$$

Finally, for $k = j$, we have by the normalization (iv) of Lemma 2.5, $\alpha^-_{j,j}(t) = \alpha^-_{j-1,j}(t) - A_je^{-e_jt} - \int z_j(t)Z_j^-(t)$, and so

$$e^{e_jt}|\alpha^-_{j,j}(t)| \leqslant |e^{e_jt}\alpha^-_{j-1,j}(t) - A_j| + Ce^{-\gamma t} \xrightarrow[t\to+\infty]{} 0,$$

which achieves the proof of Proposition 4.4. □

Corollary 4.6. *There exist $(A_1,\dots,A_N) \in \mathbb{R}^N$ and $C, t_0 > 0$ such that, defining $z(t) = u(t) - \varphi_{A_1,\dots,A_N}(t)$, we have $\|z(t)\|_{H^1} \leqslant Ce^{-2e_Nt}$ for all $t \geqslant t_0$.*

Proof. Applying Proposition 4.4 with $j = N$, we obtain $(A_1, \dots, A_N) \in \mathbb{R}^N$ and $C, t_0 > 0$ such that $\|z(t)\|_{H^1} \leqslant Ce^{-e_N t}$ for all $t \geqslant t_0$. Moreover, if we set

$$\alpha_k^\pm(t) = \int z(t) Z_k^\pm(t)$$

for all $k \in [\![1, N]\!]$, we have $e^{e_k t}\alpha_k^-(t) \to 0$ as $t \to +\infty$. But, as in the previous proof, it easily follows that if $\|z(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma_0 \geqslant e_N$, then $\|z(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$, and we apply this induction until we have $\gamma_0 = 2e_N$. □

4.4 Uniqueness

Finally, we prove the following proposition, which achieves the proof of Theorem 1.3. Note that its proof is based on the schemes developed above, and on arguments developed in [11, Section 4].

Proposition 4.7. *There exists $t_0 > 0$ such that, for all $t \geqslant t_0$, $z(t) = 0$.*

Proof. We start from the conclusion of Corollary 4.6, we set

$$\theta(t) = \sup_{t' \geqslant t} e^{e_N t'} \|z(t')\|_{H^1},$$

well defined and decreasing, and we prove that $\theta = 0$. Indeed, with obvious notation, we still have the following estimates, for all $t \geqslant t_0$,

$$\begin{cases} \forall k \in [\![1, N]\!], & \left|\frac{d}{dt}\alpha_k^\pm(t) \mp e_k \alpha_k^\pm(t)\right| \leqslant Ce^{-\gamma t}\|z(t)\|_{H^1}, \\ \forall k \in [\![1, N]\!], & |a_k'(t)| \leqslant Ce^{-\gamma t}\|z(t)\|_{H^1} + C\|\tilde{z}(t)\|_{H^1}, \\ \|z(t)\|_{H^1} \leqslant C\|\tilde{z}(t)\|_{H^1} + C\sum_{k=1}^N |a_k(t)|. \end{cases}$$

Moreover, if we define H_0 as in [11] by

$$H_0(t) = \int \left\{ \left(z_x^2(t,x) - F_0(t, z(t,x)) \right) h(t,x) + z^2(t,x) \right\} dx,$$

where

$$F_0(t,z) = 2\left[\frac{(\varphi_{A_1,\dots,A_N}(t) + z)^{p+1}}{p+1} - \frac{\varphi_{A_1,\dots,A_N}^{p+1}(t)}{p+1} - \varphi_{A_1,\dots,A_N}^p(t) z \right].$$

and h is defined in Section 3.3.5, we also have $\frac{dH_0}{dt}(t) \geqslant -Ce^{-2\gamma t}\|z(t)\|_{H^1}^2$. Now, we want to prove that $\theta(t) = 0$, for $t \geqslant t_0$ with t_0 large enough. Let $t \geqslant t_0$.

First, we have, for all $k \in [\![1, N]\!]$, $\left|\frac{d}{dt}\alpha_k^\pm(t) \mp e_k\alpha_k^\pm(t)\right| \leqslant Ce^{-\gamma t}e^{-e_N t}\theta(t)$, and thus, for all $s \geqslant t$,

$$\left| \frac{d}{dt}\alpha_k^\pm(s) \mp e_k \alpha_k^\pm(s) \right| \leqslant Ce^{-(e_N+\gamma)s}\theta(t).$$

Hence, we have $|(e^{-e_k s}\alpha_k^+(s))'| \leqslant Ce^{-(e_N+e_k+\gamma)s}\theta(t)$, and so, by integration on $[t, +\infty)$,

$$|\alpha_k^+(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t).$$

Similarly, we have $|(e^{e_k s}\alpha_k^-(s))'| \leqslant Ce^{-(e_N-e_k+\gamma)s}\theta(t)$, and since $e_N - e_k + \gamma \geqslant \gamma > 0$ and $e^{e_k t}\alpha_k^-(t) \to 0$ as $t \to +\infty$, we also get, by integration on $[t, +\infty)$,

$$|\alpha_k^-(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t).$$

We thus have $\|\alpha(t)\|^2 \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t)$. But we also have, for $s \geqslant t$,

$$\frac{dH_0}{dt}(s) \geqslant -Ce^{-2\gamma s}\|z(s)\|_{H^1}^2 = -Ce^{-2(e_N+\gamma)s}(e^{e_N s}\|z(s)\|_{H^1})^2 \geqslant -Ce^{-2(e_N+\gamma)s}\theta^2(t),$$

and so, by integration on $[t, +\infty)$, $H_0(t) \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t)$. As in the proof of Lemma 4.2, we deduce that

$$\|\tilde{z}(t)\|_{H^1}^2 \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t) + C\|\alpha(t)\|^2 \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t),$$

and so $\|\tilde{z}(t)\|_{H^1} \leqslant Ce^{-(e_N+\gamma)t}\theta(t)$. But, for all $k \in [\![1, N]\!]$ and all $s \geqslant t$, we have

$$|a_k'(s)| \leqslant Ce^{-\gamma s}\|z(s)\|_{H^1} + C\|\tilde{z}(s)\|_{H^1} \leqslant Ce^{-(e_N+\gamma)s}\theta(s) \leqslant Ce^{-(e_N+\gamma)s}\theta(t),$$

and so, by integration on $[t, +\infty)$, $|a_k(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t)$.

Finally, we proved that there exists $C^* > 0$ such that $\|z(t)\|_{H^1} \leqslant C^* e^{-(e_N+\gamma)t}\theta(t)$, for all $t \geqslant t_0$. Now fix $t \geqslant t_0$. We have, for all $t' \geqslant t$,

$$e^{e_N t'}\|z(t')\|_{H^1} \leqslant C^* e^{-\gamma t'}\theta(t') \leqslant C^* e^{-\gamma t_0}\theta(t),$$

and thus $\theta(t) \leqslant C^* e^{-\gamma t_0}\theta(t)$. Choosing t_0 large enough so that $C^* e^{-\gamma t_0} \leqslant \frac{1}{2}$, we obtain $\theta(t) \leqslant \frac{1}{2}\theta(t)$, so $\theta(t) \leqslant 0$, and so finally $\theta(t) = 0$, as expected. □

A Appendix

Proof of Lemma 3.4. The scheme of the proof is quite similar to the proof of [9, Theorem 5], and uses moreover some arguments developed in [11, section 3.4]. Let $T^* = T^*(\|z_0\|_{H^{\frac{3}{4}}}) > 0$ be the maximum time of existence of the solution $z(t)$ associated to z_0. We distinguish two cases, whether $T < T^*$ or not, and we show that this last case is in fact impossible.

First case. Suppose that $T < T^*$, and let us show that $z_n(T) \rightharpoonup z(T)$ in H^1. Since C_0^∞ is dense in H^{-1} and $\|z_n(T) - z(T)\|_{H^1} \leqslant \|z_n(T)\|_{H^1} + \|z(T)\|_{H^1} \leqslant K'$, it is enough to show that $z_n(T) \to z(T)$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$. So let $g \in C_0^\infty(\mathbb{R})$ and $\varepsilon > 0$, and let us show the lemma in three steps, using an H^3 regularization.

Step 1. For $N \gg 1$ to fix later, we define $z_{0,n}^N$ and z_0^N by

$$\begin{cases} \widehat{z_{0,n}^N}(\xi) = \mathbb{1}_{[-N,N]}(\xi)\widehat{z_{0,n}}(\xi), \\ \widehat{z_0^N}(\xi) = \mathbb{1}_{[-N,N]}(\xi)\widehat{z_0}(\xi). \end{cases}$$

In particular, $z_{0,n}^N$ and z_0^N belong to H^3, and $z_{0,n}^N \to z_0^N$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$, since Fourier transform is continuous in $\mathcal{D}'(\mathbb{R})$. Moreover, since $(z_{0,n})$ is uniformly bounded in H^1 by Banach-Steinhaus' theorem, we have $\|z_{0,n}^N\|_{H^3} \leqslant C(N)\|z_{0,n}\|_{H^1} \leqslant C(N)$, and

$$\begin{aligned}\|z_{0,n}^N - z_{0,n}\|_{H^{\frac{3}{4}}}^2 &= \int_{|\xi|\geqslant N} \left(1+\xi^2\right)^{3/4}|\widehat{z_{0,n}}(\xi)|^2\,d\xi \leqslant 2^{3/4}\int_{|\xi|\geqslant N}|\xi|^{3/2}\cdot|\widehat{z_{0,n}}(\xi)|^2\,d\xi \\ &\leqslant \frac{2^{3/4}}{\sqrt{N}}\int_{|\xi|\geqslant N}\xi^2|\widehat{z_{0,n}}(\xi)|^2\,d\xi \leqslant \frac{2^{3/4}}{\sqrt{N}}\|z_{0,n}\|_{H^1}^2 \leqslant \frac{C}{\sqrt{N}}.\end{aligned}$$

Thus, $z_{0,n}^N \to z_{0,n}$ as $N \to +\infty$ in $H^{\frac{3}{4}}$ uniformly in n. If we call $z_n^N(t)$ the solution corresponding to initial data $z_{0,n}^N$, and since $\|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant \|z_n(t)\|_{H^1} \leqslant K$, we deduce that

$$\sup_{t\in[0,T]} \|z_n^N(t) - z_n(t)\|_{H^{\frac{3}{4}}} \leqslant C\|z_{0,n}^N - z_{0,n}\|_{H^{\frac{3}{4}}}$$

for N large enough, by applying [10, Corollary 2.18] with $s = \frac{3}{4} > \frac{p-5}{2(p-1)}$ and $T = T_K = T(\|z_n(t)\|_{H^{\frac{3}{4}}})$. As a consequence, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|z_n^N(t)\|_{H^{\frac{3}{4}}} &\leqslant \sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} + C\|z_{0,n}^N\|_{H^{\frac{3}{4}}} + C\|z_{0,n}\|_{H^{\frac{3}{4}}}\\ &\leqslant \sup_{t\in[0,T]} \|z_n(t)\|_{H^1} + 2C\|z_{0,n}\|_{H^1} \leqslant C.\end{aligned}$$

Similarly, as $\sup_{t\in[0,T]} \|z(t)\|_{H^1} \leqslant K'$ by hypothesis, we also obtain, for N large enough,

$$\sup_{t\in[0,T]} \|z^N(t) - z(t)\|_{H^{\frac{3}{4}}} \leqslant C'\|z_0^N - z_0\|_{H^{\frac{3}{4}}},$$

where $z^N(t)$ is the solution corresponding to initial data z_0^N. Notice that C and C' are independent of n, and that by propagation of the regularity, we have $z_n^N(t), z^N(t) \in H^3$ for all $t \in [0,T]$. Finally, we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\int (z_n(T) - z(T))g - \int (z_n^N(T) - z^N(T))g\right| \leqslant \left|\int (z_n(T) - z_n^N(T))g\right|\\ &+\left|\int (z(T) - z^N(T))g\right| \leqslant (\|z_n(T) - z_n^N(T)\|_{L^2} + \|z(T) - z^N(T)\|_{L^2})\|g\|_{L^2} \leqslant \frac{C}{\sqrt[4]{N}} \leqslant \frac{\varepsilon}{2}\end{aligned}$$

for N large enough, and we now fix it to this value.

Step 2. Now N is fixed, we forget it and the situation amounts to: $z_n(t), z(t) \in H^3$ for all $t \in [0,T]$, $\sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant C$, $\|z_{0,n}\|_{H^3} \leqslant C'$ (with C and C' independent of n) and $z_{0,n} \to z_0$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$. The aim of this step is to show consecutively that $z_n(t)$ is uniformly bounded in H^1, H^2 and H^3, and finally, z_n is uniformly bounded in $H^1([0,T] \times \mathbb{R})$.

Since $\sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant C$ and $H^{\frac{3}{4}}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ continuously, we have

$$\sup_{t\in[0,T]} \|z_n(t)\|_{L^\infty} \leqslant C \quad \text{and} \quad \sup_{t\in[0,T]} \|z_n(t)\|_{L^2} \leqslant C.$$

But energy conservation gives, for all $t \in [0,T]$,

$$\frac{1}{2}\int (\partial_x z_n(t))^2 - \frac{1}{p+1}\int z_n(t)^{p+1} = \frac{1}{2}\int (\partial_x z_{0,n})^2 - \frac{1}{p+1}\int z_{0,n}^{p+1}.$$

We deduce that

$$\left|\int (\partial_x z_n(t))^2\right| \leqslant C\|z_n(t)\|_{L^\infty}^{p-1}\|z_n(t)\|_{L^2}^2 + C\|z_{0,n}\|_{H^1}^2 + C\|z_{0,n}\|_{H^1}^{p+1} \leqslant C,$$

and so $\sup_{t\in[0,T]} \|z_n(t)\|_{H^1} \leqslant C$.

To estimate $\|z_n(t)\|_{H^2}$, we use the "modified energy" as in [11, Section 3.4] (see also [8]). If we denote z_n by z for a short moment, and if we also define $G_2(t) = \int \left(z_{xx}^2(t) - \frac{5p}{3}z_x^2(t)z^{p-1}(t)\right)$ for $t \in [0,T]$, we have the identity

$$G_2'(t) = \frac{1}{12}p(p-1)(p-2)(p-3)\int z_x^5(t)z^{p-4}(t) + \frac{5}{3}p^2(p-1)\int z_x^3(t)z^{2p-3}(t).$$

But Gagliardo-Nirenberg inequalities give, for all $k \geqslant 2$,

$$\int |u_x|^k \leqslant C\Big(\int u_x^2\Big)^{\frac{k+2}{4}} \Big(\int u_{xx}^2\Big)^{\frac{k-2}{4}},$$

and since $\sup_{t\in[0,T]} \|z(t)\|_{L^\infty} \leqslant C$, we have

$$\begin{aligned} G_2'(t) &\leqslant C\|z(t)\|_{L^\infty}^{p-4} \int |z_x(t)|^5 + C'\|z(t)\|_{L^\infty}^{2p-3} \int |z_x(t)|^3 \\ &\leqslant C\Big(\int z_x^2(t)\Big)^{7/4} \Big(\int z_{xx}^2(t)\Big)^{3/4} + C'\Big(\int z_x^2(t)\Big)^{5/4} \Big(\int z_{xx}^2(t)\Big)^{1/4} \\ &\leqslant C\Big(\int z_{xx}^2(t)\Big)^{3/4} + C'\Big(\int z_{xx}^2(t)\Big)^{1/4}. \end{aligned}$$

Since $a \leqslant a^{4/3} + 1$ and $a \leqslant a^4 + 1$ for $a \geqslant 0$, we deduce that for some $C, D > 0$ (still independent of n), we have, for all $s \in [0,T]$,

$$G_2'(s) \leqslant C\left(\int z_{xx}^2(s)\right) + D.$$

Now, for $t \in [0,T]$, we integrate between 0 and t, and we obtain

$$G_2(t) - G_2(0) \leqslant C\int_0^t \|z_{xx}(s)\|_{L^2}^2\, ds + Dt.$$

Moreover, by definition of G_2,

$$\begin{aligned} \|z_{xx}(t)\|_{L^2}^2 &\leqslant \frac{5p}{3}\left|\int z_x^2(t) z^{p-1}(t)\right| + \frac{5p}{3}\left|\int z_x^2(0) z^{p-1}(0)\right| \\ &\quad + \|z_{xx}(0)\|_{L^2}^2 + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\, ds + DT \\ &\leqslant C\|z(t)\|_{H^1}^{p+1} + C\|z(0)\|_{H^1}^{p+1} + \|z(0)\|_{H^2} + DT + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\, ds \\ &\leqslant B + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\, ds. \end{aligned}$$

Finally, we obtain by Grönwall's lemma that, for all $t \in [0,T]$,

$$\|z_{xx}(t)\|_{L^2}^2 \leqslant Be^{Ct} \leqslant Be^{CT}.$$

We can conclude that $\sup_{t\in[0,T]} \|z_n(t)\|_{H^2} \leqslant C$ with $C > 0$ independent of n.

For a uniform bound in H^3, we use the same arguments as for H^2. In fact, it is easier, since we have, by straightforward calculation (we forget again n for a while),

$$\begin{aligned} \frac{d}{dt}\int z_{xxx}^2(t) &= -7p(p-1)\int z_{xxx}^2(t) z_x(t) z^{p-2}(t) \\ &\quad + 14p(p-1)(p-2)\int z_{xx}^3(t) z_x(t) z^{p-3}(t) \\ &\quad + 14p(p-1)(p-2)(p-3)\int z_{xx}^2(t) z_x^3(t) z^{p-4}(t) \\ &\quad + 2p(p-1)(p-2)(p-3)(p-4)\int z_{xx}(t) z_x^5(t) z^{p-5}(t). \end{aligned}$$

But we have now $\sup_{t\in[0,T]} \|z_x(t)\|_{L^\infty} \leqslant C \sup_{t\in[0,T]} \|z_x(t)\|_{H^1} \leqslant C \sup_{t\in[0,T]} \|z(t)\|_{H^2} \leqslant C$, and still $\sup_{t\in[0,T]} \|z(t)\|_{L^\infty} \leqslant C$, so

$$\frac{d}{dt}\int z_{xxx}^2(t) \leqslant A\int z_{xxx}^2(t) + B\int |z_{xx}(t)|^3 + C\int z_{xx}^2(t) + D\int |z_{xx}(t)||z_x(t)|.$$

Using a Gagliardo-Nirenberg inequality for the second term and the Cauchy-Schwarz one for the last term, we obtain

$$\begin{aligned}\frac{d}{dt}\int z_{xxx}^2(t) &\leqslant A\int z_{xxx}^2(t) + B'\Big(\int z_{xx}^2(t)\Big)^{5/4}\Big(\int z_{xxx}^2(t)\Big)^{1/4} \\ &\quad + C\|z(t)\|_{H^2}^2 + D\|z_{xx}(t)\|_{L^2}\|z_x(t)\|_{L^2} \\ &\leqslant A\int z_{xxx}^2(t) + B''\int z_{xxx}^2(t) + B'' + C' + D\|z(t)\|_{H^2}^2 \\ &\leqslant A'\int z_{xxx}^2(t) + D'.\end{aligned}$$

Now, if we integrate this inequality between 0 and $t \in [0,T]$, we get

$$\begin{aligned}\|z_{xxx}(t)\|_{L^2}^2 &\leqslant \|z_{xxx}(0)\|_{L^2}^2 + A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D't \\ &\leqslant \|z(0)\|_{H^3}^2 + A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D'T \\ &\leqslant A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D'',\end{aligned}$$

and we conclude again by Grönwall's lemma that $\|z_{xxx}(t)\|_{L^2}^2 \leqslant D''e^{A't} \leqslant D''e^{A'T}$. Finally, we have $\sup_{t\in[0,T]} \|z_n(t)\|_{H^3} \leqslant C$, as excepted.

As $z_{nt}(t) = -z_{nxxx}(t) - pz_{nx}(t)z_n^{p-1}(t)$, then we have, for all $t \in [0,T]$,

$$\|z_{nt}(t)\|_{L^2} \leqslant \|z_{nxxx}(t)\|_{L^2} + p\|z_n(t)\|_{L^\infty}^{p-1}\|z_{nx}\|_{L^2} \leqslant \|z_n(t)\|_{H^3} + C\|z_n(t)\|_{H^1}^p \leqslant C.$$

We deduce that (z_n) is uniformly bounded in $H^1([0,T]\times\mathbb{R})$, thus there exists $\tilde{z}$ such that $z_n \rightharpoonup \tilde{z}$ weakly in $H^1([0,T]\times\mathbb{R})$ (after passing to a subsequence), and in particular strongly on compacts in $L^2([0,T]\times\mathbb{R})$. Moreover, since $\sup_t \|z_n(t)\|_{H^3} \leqslant C$, we have $\sup_t \|\tilde{z}(t)\|_{H^3} \leqslant C$.

Step 3. This step is very similar to the first one of the proof of [9, Theorem 5]. We recall that we want to prove $\int (z_n(T) - z(T))g \to 0$ as $n \to +\infty$. Let $w_n = z_n - z$. The equation satisfied by w_n is $w_{nt} + w_{nxxx} + (z_n^p - z^p)_x = 0$, and moreover

$$\begin{aligned}(z_n^p - z^p)_x &= pz_{nx}z_n^{p-1} - pz_xz^{p-1} = p[(z_{nx} - z_x)z_n^{p-1} + z_x(z_n^{p-1} - z^{p-1})] \\ &= p\left[w_{nx}z_n^{p-1} + z_x(z_n - z)\sum_{k=0}^{p-2} z_n^k z^{p-2-k}\right].\end{aligned}$$

If we define $S(u,v) = \sum_{k=0}^{p-2} v^k u^{p-2-k}$, the equation satisfied by w_n can be written

$$\begin{cases} w_{nt} + w_{nxxx} + pz_n^{p-1}w_{nx} + pz_xS(z,z_n)w_n = 0, \\ w_n(0) = \psi_n = z_{0,n} - z_0. \end{cases}$$

Now, consider $v(t)$ the solution of

$$\begin{cases} v_t + v_{xxx} + p(\tilde{z}^{p-1}v)_x + pz_x S(z,\tilde{z})v = 0, \\ v(T) = g. \end{cases}$$

First notice that $\sup_t \|v\|_{L^2} \leqslant C$ by an energy method. Indeed, we have by direct calculation

$$\frac{d}{dt}\int v^2 = -p\int v^2 \left[(p-1)\tilde{z}_x\tilde{z}^{p-2} + 2z_x S(z,\tilde{z})\right].$$

But we have $\sup_t \|\tilde{z}_x(t)\|_{L^\infty} \leqslant \sup_t \|\tilde{z}(t)\|_{H^2} \leqslant C$, and similarly $\sup_t \|\tilde{z}(t)\|_{L^\infty} \leqslant C$, $\sup_t \|z_x(t)\|_{L^\infty} \leqslant C$ and $\sup_t \|S(z(t),\tilde{z}(t))\|_{L^\infty} \leqslant C$, and so

$$-\frac{d}{ds}\int v^2(s) \leqslant C\int v^2(s).$$

By integration between $t \in [0,T]$ and T, we obtain

$$\|v(t)\|_{L^2}^2 - \|v(T)\|_{L^2}^2 \leqslant C\int_t^T \|v(s)\|_{L^2}^2\, ds,$$

and so $\|v(t)\|_{L^2}^2 \leqslant \|g\|_{L^2}^2 + C\int_t^T \|v(s)\|_{L^2}^2\, ds$. We conclude, by Grönwall's lemma, that

$$\|v(t)\|_{L^2}^2 \leqslant \|g\|_{L^2}^2 e^{C(T-t)} \leqslant \|g\|_{L^2}^2 e^{CT} = K.$$

Now, we write

$$\int w_n(T,x)g(x)\,dx - \int \psi_n(x)v(0,x)\,dx = \int_0^T\int w_{nt}v + \int_0^T\int w_n v_t = \mathbf{I} + \mathbf{II}$$

with

$$\begin{cases} \mathbf{I} = \int_0^T\int w_n\left[v_{xxx} + p(vz_n^{p-1})_x - pz_x S(z,z_n)v\right], \\ \mathbf{II} = \int_0^T\int w_n\left[-v_{xxx} - p(v\tilde{z}^{p-1})_x + pz_x S(z,\tilde{z})v\right], \end{cases}$$

and so

$$\begin{aligned} \mathbf{I}+\mathbf{II} &= p\int_0^T\int w_n[v(z_n^{p-1} - \tilde{z}^{p-1})]_x + p\int_0^T\int w_n z_x v[S(z,\tilde{z}) - S(z,z_n)] \\ &= -p\int_0^T\int w_{nx}v(z_n^{p-1} - \tilde{z}^{p-1}) - p\int_0^T\int w_n z_x v\sum_{k=1}^{p-2} z^{p-2-k}(z_n^k - \tilde{z}^k) \\ &= -p\int_0^T\int w_{nx}v(z_n - \tilde{z})S(\tilde{z},z_n) - p\int_0^T\int w_n z_x v(z_n - \tilde{z})S'(z,\tilde{z},z_n) \\ &= -p\int_0^T\int [w_{nx}S(\tilde{z},z_n) + w_n z_x S'(z,\tilde{z},z_n)]v(z_n - \tilde{z}), \end{aligned}$$

where $S(\tilde{z},z_n) = \sum_{k=0}^{p-2} \tilde{z}^{p-2-k}z_n^k$ and $S'(z,\tilde{z},z_n) = \sum_{k=1}^{p-2}\sum_{l=0}^{k-1} z^{p-2-k}\tilde{z}^{k-1-l}z_n^l$ both satisfy

$$\sup_{t\in[0,T]} \|S(\tilde{z},z_n)\|_{L^\infty} \leqslant C \quad \text{and} \quad \sup_{t\in[0,T]} \|S'(z,\tilde{z},z_n)\|_{L^\infty} \leqslant C.$$

As $\psi_n \rightharpoonup 0$ in L^2 and $v(0) \in L^2$, then, for n large enough, $|\int \psi_n(x)v(0,x)\,dx| \leqslant \frac{\varepsilon}{4}$. Therefore, it is enough to conclude to show that, for n large enough, $|\mathbf{I}+\mathbf{II}| \leqslant \frac{\varepsilon}{4}$. But

$$\sup_t \|w_{nx}S(\tilde{z},z_n) + w_n z_x S'(z,\tilde{z},z_n)\|_{L^\infty} \leqslant C,$$

and $\sup_t \|z_n - \tilde{z}\|_{L^2} \leqslant C\|z_n - \tilde{z}\|_{H^1(]0,T[\times\mathbb{R})} \leqslant C$, $\sup_t \|v\|_{L^2} \leqslant C$. Hence, there exists $R > 0$ such that

$$\left| -p \int_0^T \int_{|x|>R} [w_{nx}S(\tilde{z}, z_n) + w_n z_x S'(z, \tilde{z}, z_n)] v(z_n - \tilde{z}) \right| \leqslant \frac{\varepsilon}{8}.$$

And finally, by Cauchy-Schwarz inequality, we have

$$\left| -p \int_0^T \int_{|x|\leqslant R} [w_{nx}S(\tilde{z}, z_n) + w_n z_x S'(z, \tilde{z}, z_n)] v(z_n - \tilde{z}) \right| \leqslant C \int_0^T \int_{|x|\leqslant R} |z_n - \tilde{z}||v|$$
$$\leqslant C \left(\int_0^T \int_{|x|\leqslant R} |z_n - \tilde{z}|^2 \right)^{1/2} \left(\int_0^T \int_{|x|\leqslant R} v^2 \right)^{1/2} \leqslant C \left(\int_0^T \int_{|x|\leqslant R} |z_n - \tilde{z}|^2 \right)^{1/2} \leqslant \frac{\varepsilon}{8}$$

for n large enough, which concludes the first case.

Second case. Suppose that $T^* \leqslant T$ and let us show that it implies a contradiction. Indeed, there would exist $T' < T^*$ such that $\|z(T')\|_{H^{\frac{3}{4}}} \geqslant 2K$ (where K is the same constant as in the hypothesis of the lemma). But we can apply the first case with T replaced by T', so that $z_n(T') \rightharpoonup z(T')$ in H^1, and since $\|z_n(T')\|_{H^1} \leqslant K$, we would obtain by weak convergence $\|z(T')\|_{H^{\frac{3}{4}}} \leqslant \|z(T')\|_{H^1} \leqslant K$, and so the desired contradiction and the end of the proof of the lemma. □

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Part IV

Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension

Summary

Abstract

For the L^2 supercritical generalized Korteweg-de Vries equation, we proved in [2] the existence and uniqueness of an N-parameter family of N-solitons. Recall that, for any N given solitons, we call N-soliton a solution of the equation which behaves as the sum of these N solitons asymptotically as $t \to +\infty$. In the present paper, we also construct an N-parameter family of N-solitons for the supercritical nonlinear Schrödinger equation, in dimension 1 for the sake of simplicity. Nevertheless, we do not obtain any classification result; but recall that, even in subcritical and critical cases, no general uniqueness result has been proved yet.

1 Introduction

1.1 The nonlinear Schrödinger equation

We consider the L^2 supercritical focusing nonlinear Schrödinger equation in one dimension:

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases} \tag{NLS}$$

where $(t,x) \in \mathbb{R}^2$, $p > 5$ is real, and u is a complex-valued function. Recall first that Ginibre and Velo [6] proved that (NLS) is locally well-posed in $H^1(\mathbb{R})$ for $p > 1$: for any $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a unique maximal solution $u \in C([0,T), H^1(\mathbb{R}))$ of (NLS). Moreover, either $T = +\infty$ or $T < +\infty$ and then $\lim_{t\to T} \|\partial_x u(t)\|_{L^2} = +\infty$. It is also well-known that H^1 solutions of (NLS) satisfy the following three conservation laws: for all $t \in [0,T)$,

$$M(u(t)) = \int |u(t)|^2 = M(u_0) \quad \text{(mass)},$$
$$E(u(t)) = \frac{1}{2}\int |\partial_x u(t)|^2 - \frac{1}{p+1}\int |u(t)|^{p+1} = E(u_0) \quad \text{(energy)},$$
$$P(u(t)) = \operatorname{Im} \int \partial_x u(t)\bar{u}(t) = P(u_0) \quad \text{(momentum)}.$$

Recall also that (NLS) admits the following symmetries.

- Space-time translation invariance: if $u(t,x)$ satisfies (NLS), then for any $t_0, x_0 \in \mathbb{R}$, $w(t,x) = u(t-t_0, x-x_0)$ also satisfies (NLS).
- Scaling invariance: if $u(t,x)$ satisfies (NLS), then for any $\lambda > 0$, $w(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ also satisfies (NLS).
- Phase invariance: if $u(t,x)$ satisfies (NLS), then for any $\gamma_0 \in \mathbb{R}$, $w(t,x) = u(t,x)e^{i\gamma_0}$ also satisfies (NLS).
- Galilean invariance: if $u(t,x)$ satisfies (NLS), then for any $v_0 \in \mathbb{R}$, $w(t,x) = u(t, x - v_0 t)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t)}$ also satisfies (NLS).

We now consider solitary waves of (NLS), in other words solutions of the form $u(t,x) = e^{ic_0t}Q_{c_0}(x)$, where $c_0 > 0$ and Q_{c_0} is solution of

$$Q_{c_0} > 0, \quad Q_{c_0} \in H^1(\mathbb{R}), \quad Q''_{c_0} + Q^p_{c_0} = c_0 Q_{c_0}. \tag{1.1}$$

Recall that such positive solution of (1.1) exists and is unique up to translations, and is moreover the solution of a variational problem: we call Q_{c_0} the solution of (1.1) which is even, and we denote $Q := Q_1$. By the symmetries of (NLS), for any $\gamma_0, v_0, x_0 \in \mathbb{R}$,

$$R_{c_0,\gamma_0,v_0,x_0}(t,x) = Q_{c_0}(x - v_0 t - x_0)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t + c_0 t + \gamma_0)}$$

is a solitary wave of (NLS), moving on the line $x = v_0 t + x_0$, that we also call *soliton.*

Finally recall that, in the supercritical case $p > 5$, solitons are *unstable* (see [8]). A striking illustration of this fact is the following result of Duyckaerts and Roudenko [5] (adapted from a previous work of Duyckaerts and Merle [4]), obtained for the 3d focusing cubic nonlinear Schrödinger equation (cNLS-3d), which is also L^2 supercritical and H^1 subcritical as in our case.

Proposition 1.1 ([5]). *Let $A \in \mathbb{R}$. If $t_0 = t_0(A) > 0$ is large enough, then there exists a radial solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of (cNLS-3d) such that*

$$\forall b \in \mathbb{R}, \exists C > 0, \forall t \geqslant t_0, \quad \|U^A(t) - e^{it}Q - Ae^{(i-e_0)t}Y^+\|_{H^b} \leqslant Ce^{-2e_0 t},$$

where $e_0 > 0$ and $Y^+ \neq 0$ is in the Schwartz space $\mathcal{S}$.

In particular, $U^A(t) \neq e^{it}Q$ if $A \neq 0$, whereas $\lim_{t\to+\infty} \|U^A(t) - e^{it}Q\|_{H^1} = 0$. Note that, in the subcritical and critical cases $p \leqslant 5$, no such special solutions $U^A(t)$ can exist, due to a variational characterization of Q. Indeed, if $\lim_{t\to+\infty} \|u(t) - e^{it}Q\|_{H^1} = 0$, then $u(t) = e^{it}Q$ in this case. The purpose of this paper is to extend Proposition 1.1 to multi-solitons.

1.2 Multi-solitons

Now, we focus on multi-soliton solutions. Given $4N$ parameters defining $N \geqslant 2$ solitons with different speeds,

$$v_1 < \cdots < v_N, \quad c_1, \ldots, c_N \in \mathbb{R}_+^*, \quad \gamma_1, \ldots, \gamma_N \in \mathbb{R}, \quad x_1, \ldots, x_N \in \mathbb{R}, \tag{1.2}$$

we set

$$R_j(t) = R_{c_j,\gamma_j,v_j,x_j}(t) \quad \text{and} \quad R(t) = \sum_{j=1}^N R_j(t),$$

and we call N-soliton a solution $u(t)$ of (NLS) such that

$$\|u(t) - R(t)\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

Let us recall known results on multi-solitons.

- In the L^2 subcritical and critical cases, *i.e.* for (NLS) with $p \leqslant 5$, there exists a large literature on the problem of existence of multi-solitons and on their properties. Merle [12] first established an existence result in the critical case, as a consequence of a blow up result and the conformal invariance. This result was extended by Martel and Merle [10] to the subcritical case, using arguments developed by Martel, Merle and Tsai [11] for the stability in H^1 of solitons. Nevertheless, we recall that no general uniqueness result has been proved, contrarily to the generalized Korteweg-de Vries (gKdV) equation (see [9]).

  For other stability and asymptotic stability results on multi-solitons of some nonlinear Schrödinger equations, see [13, 14, 15].

- In the L^2 supercritical case, *i.e.* in a situation where solitons are known to be unstable, Côte, Martel and Merle [3] have recently proved the existence of at least one multi-soliton solution for (NLS):

Theorem 1.2 ([3]). *Let $p > 5$ and $N \geqslant 2$. Let $v_1 < \cdots < v_N$, $(c_1, \ldots, c_N) \in (\mathbb{R}_+^*)^N$, $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ and $(x_1, \ldots, x_N) \in \mathbb{R}^N$. There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $\varphi \in C([T_0, +\infty), H^1)$ of* (NLS) *such that*

$$\forall t \in [T_0, +\infty), \quad \|\varphi(t) - R(t)\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

Recall that, with respect to [10, 11], the proof of Theorem 1.2 relies on an additional topological argument to control the unstable nature of the solitons. Finally, recall that Theorem 1.2 was also obtained for the L^2 supercritical gKdV equation, and has been a crucial starting point in [2] to obtain the multi-existence and the classification of multi-solitons. It is a similar multi-existence result that we propose to prove in this paper.

1.3 Main result and outline of the paper

The whole paper is devoted to prove the following theorem of existence of a family of multi-solitons for the supercritical (NLS) equation.

Theorem 1.3. *Let $p > 5$, $N \geqslant 2$, $v_1 < \cdots < v_N$, $(c_1, \ldots, c_N) \in (\mathbb{R}_+^*)^N$, $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ and $(x_1, \ldots, x_N) \in \mathbb{R}^N$. Denote $R = \sum_{j=1}^N R_{c_j,\gamma_j,v_j,x_j}$.*

Then there exist $\gamma > 0$ and an N-parameter family $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ of solutions of (NLS) *such that, for all $(A_1, \ldots, A_N) \in \mathbb{R}^N$, there exist $C > 0$ and $t_0 > 0$ such that*

$$\forall t \geqslant t_0, \ \|\varphi_{A_1,\ldots,A_N}(t) - R(t)\|_{H^1} \leqslant Ce^{-\gamma t},$$

and if $(A_1', \ldots, A_N') \neq (A_1, \ldots, A_N)$, then $\varphi_{A_1',\ldots,A_N'} \neq \varphi_{A_1,\ldots,A_N}$.

Remark 1.4. As underlined above, the question of the classification of multi-solitons is open for the (NLS) equation, even in the subcritical case, while it was obtained in [2] for the supercritical gKdV equation, and in [9] for the subcritical and critical cases. Although we expect that the family constructed in Theorem 1.3 characterizes all multi-solitons, the lack of monotonicity properties such as for the gKdV equation does not allow to prove it for now.

The paper is organized as follows. In the next section, we briefly recall some well-known results on multi-solitons and on the linearized equation. One of the most important facts about the linearized equation, also strongly used in [5, 3], is the determination of the spectrum of the linearized operator $\mathcal{L}$ around the soliton $e^{it}Q$ (proved in [16] and [7]): $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and Y^-. Indeed, $Y^\pm$ allow to control the negative directions of the linearized energy around a soliton (see Proposition 2.4). Moreover, by a simple scaling argument, we determine the eigenvalues of the linearized operator around $e^{ic_jt}Q_{c_j}$, and in particular $\pm e_j = \pm c_j^{3/2} e_0$ are simple eigenvalues with eigenfunctions $Y_j^\pm$ (see Notation 2.7 for precise definitions).

In Section 3, we construct the family $(\varphi_{A_1,\dots,A_N})$ described in Theorem 1.3. To do this, we first claim Proposition 3.1, which is the key point of the proof of the multi-existence result as in [2], and can be summarized as follows. *Let φ be a multi-soliton given by Theorem 1.2, $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$. Then there exists a solution $u(t)$ of* (NLS) *such that*

$$\|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$

for t large and for some small $\gamma > 0$. This means that, similarly as in [5] for one soliton, we can perturb the multi-soliton φ locally around *one* given soliton at the order $e^{-e_j t}$. Since it is not significant to perturb φ at order e_j before order e_k if $e_j > e_k$, the construction of $\varphi_{A_1,\dots,A_N}$ has to be done following values (possibly equal) of e_j.

Finally, to prove Proposition 3.1, we follow the strategy of the proof of the similar proposition in [2], except for the monotonicity property of the energy which does not hold for the (NLS) equation. If this property of monotonicity was necessary to obtain the classification, we prove that a slightly different functional estimated regardless its sign is sufficient to reach our purpose. We also rely on refinements of arguments developed in [3], in particular the topological argument to control the unstable directions.

2 Preliminary results

Notation 2.1. They are available in the whole paper.

(a) We denote $\partial_x v = v_x$ the partial derivative of v with respect to x.

(b) For $h \in \mathbb{C}$, we denote $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$.

(c) For $f, g \in L^2$, $(f, g) = \operatorname{Re} \int f\bar{g}$ denotes the real scalar product.

(d) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1+\xi^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R})\}$, and particularly $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.

(e) If a and b are two functions of t and if b is positive, we write $a = O(b)$ when there exists a constant $C > 0$ independent of t such that $|a(t)| \leqslant Cb(t)$ for all t.

2.1 Linearized operator around a stationary soliton

The linearized equation appears if one considers a solution of (NLS) close to the stationary soliton $e^{it}Q$. More precisely, if $u(t,x) = e^{it}(Q(x) + h(t,x))$ satisfies (NLS), then h satisfies $\partial_t h + \mathcal{L}h = O(h^2)$, where the operator $\mathcal{L}$ is defined for $v = v_1 + iv_2$ by

$$\mathcal{L}v = -L_- v_2 + iL_+ v_1,$$

and the self-adjoint operators L_+ and L_- are defined by

$$L_+ v_1 = -\partial_x^2 v_1 + v_1 - pQ^{p-1}v_1, \quad L_- v_2 = -\partial_x^2 v_2 + v_2 - Q^{p-1}v_2.$$

The spectral properties of $\mathcal{L}$ are well-known (see [7, 16] for instance), and summed up in the following proposition.

Proposition 2.2 ([7, 16]). *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and let $\sigma_{\mathrm{ess}}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \{i\xi \ ; \ \xi \in \mathbb{R}, |\xi| \geqslant 1\}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\} \quad \text{with } e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and $Y^- = \overline{Y^+}$ which have an exponential decay at infinity. Finally, the null space of $\mathcal{L}$ is spanned by $\partial_x Q$ and iQ, and as a consequence, the null space of L_+ is spanned by $\partial_x Q$ and the null space of L_- is spanned by Q.

Remark 2.3. By standard ODE techniques, we can quantify the exponential decay of $Y^\pm$ and $\partial_x Y^\pm$ at infinity. In fact, there exist $\eta_0 > 0$ and $C > 0$ such that, for all $x \in \mathbb{R}$,

$$|Y^\pm(x)| + |\partial_x Y^\pm(x)| \leqslant Ce^{-\eta_0|x|}.$$

Moreover, $\mathcal{L}$, L_+ and L_- satisfy some properties of positivity or coercivity. The following proposition sums up the two properties useful for our purpose. Note that the first one is proved in [16], while the second one is proved in [4, 5].

Proposition 2.4 ([16, 5]). *(i) For all $f \in H^1 \setminus \{\lambda Q \ ; \ \lambda \in \mathbb{R}\}$ real-valued, one has $\int (L_- f)f > 0$.*

(ii) There exists $\kappa_0 > 0$ such that, for all $v = v_1 + iv_2 \in H^1$,

$$(L_+ v_1, v_1) + (L_- v_2, v_2) \geqslant \frac{1}{\kappa_0}\|v\|_{H^1}^2 - \kappa_0 \left[\left(\int \partial_x Q v_1\right)^2 + \left(\int Q v_2\right)^2 + \left(\operatorname{Im}\int Y^+ \bar{v}\right)^2 + \left(\operatorname{Im}\int Y^- \bar{v}\right)^2\right].$$

Finally, we extend Proposition 2.2 to the operator $\mathcal{L}_c$ linearized around the stationary soliton $e^{ict}Q_c(x)$, by a simple scaling argument. In fact, we recall that if u is a solution of (NLS), then $w(t,x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$ is also a solution, and moreover, we have $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$ for all $c > 0$.

Corollary 2.5. *Let $c > 0$. For $v = v_1 + iv_2$, $\mathcal{L}_c$ is defined by $\mathcal{L}_c v = -L_{c-}v_2 + iL_{c+}v_1$, where*

$$L_{c+}v_1 = -\partial_x^2 v_1 + cv_1 - pQ_c^{p-1}v_1 \quad \text{and} \quad L_{c-}v_2 = -\partial_x^2 v_2 + cv_2 - Q_c^{p-1}v_2.$$

Moreover, the spectrum $\sigma(\mathcal{L}_c)$ of $\mathcal{L}_c$ satisfies

$$\sigma(\mathcal{L}_c) \cap \mathbb{R} = \{-e_c, 0, +e_c\}, \text{ where } e_c = c^{3/2}e_0 > 0.$$

Finally, e_c and $-e_c$ are simple eigenvalues of $\mathcal{L}_c$ with eigenfunctions Y_c^+ and Y_c^-, where

$$Y_c^+(x) = c^{1/4}Y^+(\sqrt{c}x) \quad \text{and} \quad Y_c^- = \overline{Y_c^+},$$

and the null space of $\mathcal{L}_c$ is spanned by $\partial_x Q_c$ and iQ_c.

Claim 2.6. *One can normalize $Y^\pm$ so that*

$$-\operatorname{Im}\int (Y^+)^2 = 1 \quad \text{and still} \quad Y^- = \overline{Y^+}.$$

Proof. Denote $Y_1 = \operatorname{Re} Y^+$, $Y_2 = \operatorname{Im} Y^+$. Thus, we have $Y^+ = Y_1 + iY_2$, $Y^- = Y_1 - iY_2$, and

$$L_+Y_1 = e_0Y_2, \ L_-Y_2 = -e_0Y_1.$$

Now, suppose that there exists $\lambda \in \mathbb{R}$ such that $Y_2 = \lambda Q$. Then, we would have $L_-Y_2 = -e_0Y_1 = \lambda L_-Q = 0$, and so $Y_1 = 0$. But it would imply $L_+Y_1 = 0 = e_0Y_2$, and so $Y_2 = 0$, which would be a contradiction. Therefore, by (i) of Proposition 2.4, we have $\int(L_-Y_2)Y_2 = -e_0\int Y_1Y_2 > 0$. Hence, since $\operatorname{Im}\int (Y^+)^2 = 2\int Y_1Y_2$, we normalize $Y^\pm$ by taking

$$\widetilde{Y^+} = \frac{Y^+}{\sqrt{-2\int Y_1Y_2}}, \ \widetilde{Y^-} = \overline{\widetilde{Y^+}}.$$ □

2.2 Multi-solitons results

A set of parameters (1.2) being given, we adopt the following notation.

Notation 2.7. For all $j \in [\![1, N]\!]$, define:

(i) $\lambda_j(t,x) = x - v_jt - x_j$ and $\theta_j(t,x) = \frac{1}{2}v_jx - \frac{1}{4}v_j^2t + c_jt + \gamma_j$.

(ii) $R_j(t,x) = Q_{c_j}(\lambda_j(t,x))e^{i\theta_j(t,x)}$, where $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$.

(iii) $Y_j^\pm(t,x) = Y_{c_j}^\pm(\lambda_j(t,x))e^{i\theta_j(t,x)}$, where $Y_c^\pm(x) = c^{1/4}Y^\pm(\sqrt{c}x)$.

(iv) $e_j = e_{c_j}$, where $e_c = c^{3/2}e_0$.

Now, to estimate interactions between solitons, we denote $c_{\min} = \min\{c_k \ ; \ k \in [\![1, N]\!]\}$, and the small parameters

$$\sigma_0 = \min\{\eta_0\sqrt{c_{\min}}, e_0^{2/3}c_{\min}, c_{\min}, v_2 - v_1, \dots, v_N - v_{N-1}\} \quad \text{and} \quad \gamma = \frac{\sigma_0^{3/2}}{10^6}. \qquad (2.1)$$

From [10], it appears that γ is a suitable parameter to quantify interactions between solitons in large time. For instance, we have, for $j \neq k$ and all $t \geqslant 0$,

$$\int |R_j(t)||R_k(t)| + |(R_j)_x(t)||(R_k)_x(t)| \leqslant Ce^{-10\gamma t}. \tag{2.2}$$

From the definition of σ_0 and Remark 2.3, such an inequality is also true for $Y_j^{\pm}$.

Moreover, since σ_0 has the same definition as in [3], Theorem 1.2 can be rewritten as follows. *There exist $T_0 \in \mathbb{R}$, $C > 0$ and $\varphi \in C([T_0, +\infty), H^1)$ such that, for all $t \geqslant T_0$,*

$$\|\varphi(t) - R(t)\|_{H^1} \leqslant Ce^{-4\gamma t}. \tag{2.3}$$

3 Construction of a family of multi-solitons

In this section, we prove Theorem 1.3 as a consequence of the following crucial Proposition 3.1. Let $p > 5$, $N \geqslant 2$, a set of parameters (1.2), and denote $R = \sum_{k=1}^{N} R_k$.

Proposition 3.1. *Let φ be a multi-soliton solution satisfying* (2.3). *Let $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$. Then there exist $t_0 > 0$ and a solution $u \in C([t_0, +\infty), H^1)$ of* (NLS) *such that*

$$\forall t \geqslant t_0, \quad \|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}. \tag{3.1}$$

Before proving this proposition, let us show how it implies Theorem 1.3.

Proof of Theorem 1.3. Let $(A_1, \ldots, A_N) \in \mathbb{R}^N$. Denote σ the permutation of $[\![1, N]\!]$ which satisfies

$$c_{\sigma(1)} \leqslant \cdots \leqslant c_{\sigma(N)}, \text{ and } \sigma(i) < \sigma(j) \text{ if } c_{\sigma(i)} = c_{\sigma(j)} \text{ and } i < j.$$

(i) Consider $\varphi_{A_{\sigma(1)}}$ the solution of (NLS) given by Proposition 3.1 applied with φ given by Theorem 1.2. Thus, there exists $t_0 > 0$ such that

$$\forall t \geqslant t_0, \quad \|\varphi_{A_{\sigma(1)}}(t) - \varphi(t) - A_{\sigma(1)} e^{-e_{\sigma(1)} t} Y_{\sigma(1)}^+(t)\|_{H^1} \leqslant e^{-(e_{\sigma(1)}+\gamma)t}.$$

Now, remark that $\varphi_{A_{\sigma(1)}}$ is also a multi-soliton which satisfies (2.3). Hence, we can apply Proposition 3.1 with $\varphi_{A_{\sigma(1)}}$ instead of φ, so that we obtain $\varphi_{A_{\sigma(1)}, A_{\sigma(2)}}$ such that

$$\forall t \geqslant t_0', \quad \|\varphi_{A_{\sigma(1)}, A_{\sigma(2)}}(t) - \varphi_{A_{\sigma(1)}}(t) - A_{\sigma(2)} e^{-e_{\sigma(2)} t} Y_{\sigma(2)}^+(t)\|_{H^1} \leqslant e^{-(e_{\sigma(2)}+\gamma)t}.$$

Similarly, for all $j \in [\![2, N]\!]$, we construct by induction a solution $\varphi_{A_{\sigma(1)}, \ldots, A_{\sigma(j)}}$ such that, for all $t \geqslant t_0$,

$$\|\varphi_{A_{\sigma(1)}, \ldots, A_{\sigma(j)}}(t) - \varphi_{A_{\sigma(1)}, \ldots, A_{\sigma(j-1)}}(t) - A_{\sigma(j)} e^{-e_{\sigma(j)} t} Y_{\sigma(j)}^+(t)\|_{H^1} \leqslant e^{-(e_{\sigma(j)}+\gamma)t}.$$

Note finally that $\varphi_{A_1, \ldots, A_N} := \varphi_{A_{\sigma(1)}, \ldots, A_{\sigma(N)}}$ constructed by this way satisfies (2.3).

(ii) Let $(A'_1, \dots, A'_N) \in \mathbb{R}^N$ be such that $\varphi_{A'_1,\dots,A'_N} = \varphi_{A_1,\dots,A_N}$, and let us show that it implies $(A'_1, \dots, A'_N) = (A_1, \dots, A_N)$. In fact, we prove by induction on j that $A_{\sigma(j)} = A'_{\sigma(j)}$ for all $j \in [\![1, N]\!]$. For $j = 1$, first note that, from the construction of $\varphi_{A_1,\dots,A_N}$, the hypothesis means $\varphi_{A'_{\sigma(1)},\dots,A'_{\sigma(N)}} = \varphi_{A_{\sigma(1)},\dots,A_{\sigma(N)}}$, and moreover

$$\begin{aligned}\varphi_{A_{\sigma(1)},\dots,A_{\sigma(N)}}(t) &= \varphi_{A_{\sigma(1)},\dots,A_{\sigma(N-1)}}(t) + A_{\sigma(N)} e^{-e_{\sigma(N)}t} Y^+_{\sigma(N)}(t) + z_{\sigma(N)}(t) \\ &= \cdots = \varphi(t) + \sum_{k=1}^{N} A_{\sigma(k)} e^{-e_{\sigma(k)}t} Y^+_{\sigma(k)}(t) + \sum_{k=1}^{N} z_{\sigma(k)}(t),\end{aligned}$$

where $z_{\sigma(k)}$ satisfies $\|z_{\sigma(k)}(t)\|_{H^1} \leqslant e^{-(e_{\sigma(k)}+\gamma)t}$ for $t \geqslant t_0$ and each $k \in [\![1, N]\!]$. Similarly, we get

$$\varphi_{A'_{\sigma(1)},\dots,A'_{\sigma(N)}}(t) = \varphi(t) + \sum_{k=1}^{N} A'_{\sigma(k)} e^{-e_{\sigma(k)}t} Y^+_{\sigma(k)}(t) + \sum_{k=1}^{N} \widetilde{z_{\sigma(k)}}(t),$$

and so, by difference, we have

$$\begin{aligned}(A_{\sigma(1)} - A'_{\sigma(1)}) e^{-e_{\sigma(1)}t} Y^+_{\sigma(1)}(t) = &\sum_{k=2}^{N} (A'_{\sigma(k)} - A_{\sigma(k)}) e^{-e_{\sigma(k)}t} Y^+_{\sigma(k)}(t) \\ &+ \sum_{k=1}^{N} \widetilde{z_{\sigma(k)}}(t) - z_{\sigma(k)}(t).\end{aligned}$$

Now, if we multiply this equality by $Y^+_{\sigma(1)}(t)$, integrate, and take the imaginary part of it, we obtain, by Claim 2.6 and (2.2),

$$|A_{\sigma(1)} - A'_{\sigma(1)}| e^{-e_{\sigma(1)}t} \leqslant C e^{-(e_{\sigma(1)}+\gamma)t},$$

and so $A_{\sigma(1)} = A'_{\sigma(1)}$ by taking $t \to +\infty$. For the inductive step from $j - 1$ to j, we write similarly

$$\begin{aligned}\varphi_{A_{\sigma(1)},\dots,A_{\sigma(N)}}(t) &= \varphi_{A_{\sigma(1)},\dots,A_{\sigma(j-1)}}(t) + \sum_{k=j}^{N} A_{\sigma(k)} e^{-e_{\sigma(k)}t} Y^+_{\sigma(k)}(t) + \sum_{k=j}^{N} z_{\sigma(k)}(t) \\ &= \varphi_{A_{\sigma(1)},\dots,A_{\sigma(j-1)}}(t) + \sum_{k=j}^{N} A'_{\sigma(k)} e^{-e_{\sigma(k)}t} Y^+_{\sigma(k)}(t) + \sum_{k=j}^{N} \widetilde{z_{\sigma(k)}}(t),\end{aligned}$$

and we finally obtain $A_{\sigma(j)} = A'_{\sigma(j)}$ as expected, by taking the difference of these two expressions, multiplying by $Y^+_{\sigma(j)}(t)$, integrating and taking the imaginary part of it. □

Now, the only purpose of the rest of the paper is to prove Proposition 3.1. Let $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$, and denote $r_j(t, x) = A_j e^{-e_j t} Y^+_j(t, x) = A_j e^{-e_j t} Y^+_{c_j}(\lambda_j(t, x)) e^{i\theta_j(t,x)}$. We want to construct a solution u of (NLS) such that

$$z(t, x) = u(t, x) - \varphi(t, x) - r_j(t, x)$$

satisfies $\|z(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}$ for $t \geqslant t_0$ with t_0 large enough.

3.1 Equation of z

Since u is a solution of (NLS) and also φ is (and this fact is crucial for the whole proof), we get

$$i\partial_t z+\partial_x^2 z+|\varphi+r_j+z|^{p-1}(\varphi+r_j+z)-|\varphi|^{p-1}\varphi+A_j e^{-e_j t}e^{i\theta_j}[\partial_x^2 Y_{c_j}^+-c_j Y_{c_j}^+-ie_j Y_{c_j}^+](\lambda_j)=0.$$

But from Corollary 2.5, we have

$$\mathcal{L}_{c_j} Y_{c_j}^+ = e_j Y_{c_j}^+ = e_j Y_{c_j,1}^+ + ie_j Y_{c_j,2}^+ = -L_- Y_{c_j,2}^+ + iL_+ Y_{c_j,1}^+$$

where $Y_{c_j,1}^+ = \operatorname{Re} Y_{c_j}^+$ and $Y_{c_j,2}^+ = \operatorname{Im} Y_{c_j}^+$, and so

$$\partial_x^2 Y_{c_j}^+ - c_j Y_{c_j}^+ + iQ_{c_j}^{p-1} Y_{c_j,2}^+ + pQ_{c_j}^{p-1} Y_{c_j,1}^+ = ie_j Y_{c_j}^+. \tag{3.2}$$

Therefore, we get the following equation for z:

$$i\partial_t z+\partial_x^2 z+|\varphi+r_j+z|^{p-1}(\varphi+r_j+z)-|\varphi|^{p-1}\varphi = A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j}[pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j). \tag{3.3}$$

By developing the nonlinearity, we find

$$\begin{aligned}|\varphi+r_j+z|^{p-1}(\varphi+r_j+z)-|\varphi|^{p-1}\varphi &= |\varphi+r_j|^{p-1}(\varphi+r_j)-|\varphi|^{p-1}\varphi+\omega(z)\\ &\quad+(p-1)|\varphi+r_j|^{p-3}(\varphi+r_j)\operatorname{Re}((\overline{\varphi}+\overline{r_j})z)+|\varphi+r_j|^{p-1}z,\end{aligned}$$

where $\omega(z)$ satisfies $|\omega(z)|\leqslant C|z|^2$ for $|z|\leqslant 1$. Hence, we can rewrite (3.3) as

$$i\partial_t z+\partial_x^2 z+(p-1)|\varphi+r_j|^{p-3}(\varphi+r_j)\operatorname{Re}((\overline{\varphi}+\overline{r_j})z)+|\varphi+r_j|^{p-1}z+\omega(z)=-\Omega,$$

where

$$\Omega = |\varphi+r_j|^{p-1}(\varphi+r_j)-|\varphi|^{p-1}\varphi - A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j)e^{i\theta_j}[pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j). \tag{3.4}$$

Finally, the equation of z can be written in the shorter form

$$i\partial_t z+\partial_x^2 z+(p-1)|\varphi|^{p-3}\varphi\operatorname{Re}(\overline{\varphi}z)+|\varphi|^{p-1}z+\omega_1\cdot z+\omega(z)=-\Omega, \tag{3.5}$$

where ω_1 satisfies $\|\omega_1(t)\|_{L^2}\leqslant Ce^{-e_j t}$ for all $t\geqslant T_0$. We finally estimate the source term Ω in the following lemma, that we prove in Appendix A.

Lemma 3.2. *There exists $C>0$ such that, for all $t\geqslant T_0$, $\|\Omega(t)\|_{H^1}\leqslant Ce^{-(e_j+4\gamma)t}$.*

3.2 Compactness argument assuming uniform estimates

To prove Proposition 3.1, we follow the strategy of [10, 3]. We first need some notation for our purpose.

Notation 3.3. (i) Denote $J=\{k\in[\![1,N]\!] \mid c_k\leqslant c_j\}$, $K=\{k\in[\![1,N]\!] \mid c_k>c_j\}$ and $k_0=\sharp K$.

(ii) $\mathbb{R}^{k_0}$ is equipped with the ℓ^2 norm, simply denoted $\|\cdot\|$.

(iii) $\mathbb{S}_{\mathbb{R}^{k_0}}(r)$ denotes the sphere of radius $r\geqslant 0$ in $\mathbb{R}^{k_0}$.

(iv) $B_{\mathcal{B}}(r)$ is the closed ball of the Banach space $\mathcal{B}$, centered at 0 and of radius r.

Let $S_n \to +\infty$ be an increasing sequence of time, $\mathfrak{b}_n = (b_{n,k})_{k\in K} \in \mathbb{R}^{k_0}$ be a sequence of parameters to be determined, and let u_n be the solution of

$$\begin{cases} i\partial_t u_n + \partial_x^2 u_n + |u_n|^{p-1}u_n = 0, \\ u_n(S_n) = \varphi(S_n) + A_j e^{-e_j S_n} Y_j^+(S_n) + \sum_{k\in K} b_{n,k} Y_k^+(S_n). \end{cases} \tag{3.6}$$

Proposition 3.4. *There exist $n_0 \geqslant 0$ and $t_0 > 0$ (independent of n) such that the following holds. For each $n \geqslant n_0$, there exists $\mathfrak{b}_n \in \mathbb{R}^{k_0}$ with $\|\mathfrak{b}_n\| \leqslant 2e^{-(e_j+2\gamma)S_n}$, and such that the solution u_n of (3.6) is defined on the interval $[t_0, S_n]$, and satisfies*

$$\forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Assuming this key proposition of uniform estimates, we can sketch the proof of Proposition 3.1, relying on compactness arguments developed in [10, 3]. The proof of Proposition 3.4 is postponed to the next section.

Sketch of the proof of Proposition 3.1 assuming Proposition 3.4. From Proposition 3.4, there exists a sequence $u_n(t)$ of solutions to (NLS), defined on $[t_0, S_n]$, such that the following uniform estimates hold:

$$\forall n \geqslant n_0, \forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

In particular, there exists $C_0 > 0$ such that $\|u_n(t_0)\|_{H^1} \leqslant C_0$ for all $n \geqslant n_0$. Thus, there exists $u_0 \in H^1(\mathbb{R})$ such that $u_n(t_0) \rightharpoonup u_0$ in H^1 weak (after passing to a subsequence). Moreover, using the compactness result [10, Lemma 2], we can suppose that $u_n(t_0) \to u_0$ in L^2 strong, and so in H^{s_p} strong by interpolation, where $0 \leqslant s_p < 1$ is an exponent for which local well-posedness and continuous dependence hold, according to a result of Cazenave and Weissler [1]. Now, consider u solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, \\ u(t_0) = u_0. \end{cases}$$

Fix $t \geqslant t_0$. For n large enough, we have $S_n > t$, so $u_n(t)$ is defined and by continuous dependence of the solutions of (NLS) upon the initial data, we have $u_n(t) \to u(t)$ in H^{s_p} strong. By the uniform H^1 bound, we also obtain $u_n(t) \rightharpoonup u(t)$ in H^1 weak. As

$$\|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$

we finally obtain, by weak convergence, $\|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}$. Thus, u is a solution of (NLS) which satisfies (3.1). □

3.3 Proof of Proposition 3.4

The proof proceeds in several steps. For the sake of simplicity, we will drop the index n for the rest of this section (except for S_n). As Proposition 3.4 is proved for given n, this should not be a source of confusion. Hence, we will write u for u_n, z for z_n, $\mathfrak{b}$ for $\mathfrak{b}_n$, etc. We possibly drop the first terms of the sequence S_n, so that, for all n, S_n is large enough for our purposes.

From (3.5), the equation satisfied by z is

$$\begin{cases} i\partial_t z + \partial_x^2 z + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z + \omega_1 \cdot z + \omega(z) = -\Omega, \\ z(S_n) = \sum_{k\in K} b_k Y_k^+(S_n). \end{cases} \tag{3.7}$$

Moreover, for all $k \in [\![1, N]\!]$, we denote

$$\alpha_k^\pm(t) = \operatorname{Im}\int \bar{z}(t)\cdot Y_k^\pm(t).$$

In particular, we have

$$\alpha_k^\pm(S_n) = -\sum_{l\in K} b_l \operatorname{Im}\int Y_{c_k}^\mp(\lambda_k(S_n)) Y_{c_l}^+(\lambda_l(S_n)) e^{-i\theta_k(S_n)} e^{i\theta_l(S_n)}.$$

Finally, we denote $\boldsymbol{\alpha}^-(t) = (\alpha_k^-(t))_{k\in K}$.

3.3.1 Modulated final data

Lemma 3.5. *For $n \geqslant n_0$ large enough, the following holds. For all $\mathfrak{a}^- \in \mathbb{R}^{k_0}$, there exists a unique $\mathfrak{b} \in \mathbb{R}^{k_0}$ such that $\|\mathfrak{b}\| \leqslant 2\|\mathfrak{a}^-\|$ and $\boldsymbol{\alpha}^-(S_n) = \mathfrak{a}^-$.*

Proof. Consider the linear application

$$\begin{array}{rccc} \Phi: & \mathbb{R}^{k_0} & \to & \mathbb{R}^{k_0} \\ & \mathfrak{b} = (b_l)_{l\in K} & \mapsto & (\alpha_k^-(S_n))_{k\in K}. \end{array}$$

If we denote $(\sigma_1, \ldots, \sigma_{k_0})$ the canonical basis of $\mathbb{R}^{k_0}$, then, by the normalization of Claim 2.6 and the definition of Y_c^+ in Corollary 2.5, we have, for all $k \in [\![1, k_0]\!]$,

$$(\Phi(\sigma_k))_k = -\operatorname{Im}\int \left(Y_{c_k}^+\right)^2 = -\operatorname{Im}\int \left(Y^+\right)^2 = 1.$$

Moreover, from (2.2), there exists $C_0 > 0$ independent of n such that, for $l \neq k$,

$$|(\Phi(\sigma_k))_l| \leqslant \int \left|Y_{c_l}^+(\lambda_l(S_n))\right|\left|Y_{c_k}^+(\lambda_k(S_n))\right| \leqslant C_0 e^{-\gamma S_n}.$$

Thus, by taking n_0 large enough, we have $\Phi = \mathrm{Id} + A_n$ where $\|A_n\| \leqslant \frac{1}{2}$, so Φ is invertible and $\|\Phi^{-1}\| \leqslant 2$. Finally, for a given $\mathfrak{a}^- \in \mathbb{R}^{k_0}$, it is enough to define $\mathfrak{b}$ by $\mathfrak{b} = \Phi^{-1}(\mathfrak{a}^-)$ to conclude the proof of Lemma 3.5. □

Claim 3.6. *The following estimates at S_n hold:*

- $|\alpha_k^+(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in [\![1, N]\!]$, *since* $\operatorname{Im}\int Y_{c_k}^- Y_{c_k}^+ = \operatorname{Im}\int |Y_{c_k}^+|^2 = 0$.
- $|\alpha_k^-(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in J$.
- $\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\|$.

3.3.2 Equations on $\alpha_k^{\pm}$

Let $t_0 > 0$ independent of n to be determined later in the proof, $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ to be chosen, $\mathfrak{b}$ be given by Lemma 3.5 and u be the corresponding solution of (3.6). We now define the maximal time interval $[T(\mathfrak{a}^-), S_n]$ on which suitable exponential estimates hold.

Definition 3.7. Let $T(\mathfrak{a}^-)$ be the infimum of $T \geqslant t_0$ such that, for all $t \in [T, S_n]$, both following properties hold:

$$e^{(e_j+\gamma)t} z(t) \in B_{H^1}(1) \quad \text{and} \quad e^{(e_j+2\gamma)t} \alpha^-(t) \in B_{\mathbb{R}^{k_0}}(1). \tag{3.8}$$

Observe that Proposition 3.4 is proved if we can find $\mathfrak{a}^-$ such that $T(\mathfrak{a}^-) = t_0$, for all n. The rest of the proof is devoted to prove the existence of such a value of $\mathfrak{a}^-$.

First, we prove the following estimate on $\alpha_k^{\pm}$.

Claim 3.8. *For all $k \in [\![1, N]\!]$ and all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\left| \frac{d}{dt} \alpha_k^{\pm}(t) \mp e_k \alpha_k^{\pm}(t) \right| \leqslant C_0 e^{-4\gamma t} \|z(t)\|_{H^1} + C_1 \|z(t)\|_{H^1}^2 + C_2 e^{-(e_j+4\gamma)t}. \tag{3.9}$$

Proof. Following Notation 2.7, we compute

$$\begin{aligned}
\frac{d}{dt}\alpha_k^{\pm}(t) &= -\frac{d}{dt} \operatorname{Im} \int \overline{Y_k^{\pm}}(t) z(t) = -\frac{d}{dt} \operatorname{Im} \int Y_{c_k}^{\mp}(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z(t) \\
&= -\operatorname{Im} \int \left[-v_k \partial_x Y_{c_k}^{\mp} - i(c_k - \frac{1}{4}v_k^2) Y_{c_k}^{\mp} \right] (x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z(t) \\
&\quad - \operatorname{Im} \int Y_{c_k}^{\mp}(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z_t.
\end{aligned}$$

Moreover, using the equation of z (3.7) and an integration by parts, we find for the second term

$$\begin{aligned}
&- \operatorname{Im} \int Y_{c_k}^{\mp}(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z_t \\
&= - \operatorname{Im} \int Y_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \times i \left[\partial_x^2 z + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z + \omega_1 \cdot z + \omega(z) + \Omega \right] \\
&= - \operatorname{Im} \int i z e^{-i\theta_k} \left[\partial_x^2 Y_{c_k}^{\mp} - i v_k \partial_x Y_{c_k}^{\mp} - \frac{v_k^2}{4} Y_{c_k}^{\mp} \right] (\lambda_k) \\
&\quad - \operatorname{Im} \int i Y_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[(p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z \right] \\
&\quad - \operatorname{Im} \int i Y_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[\omega_1 \cdot z + \omega(z) + \Omega \right].
\end{aligned}$$

Using the estimate $\|\omega_1(t)\|_{L^2} \leqslant C e^{-e_j t}$ and Lemma 3.2, we find for the last term

$$\left| - \operatorname{Im} \int i Y_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[\omega_1 \cdot z + \omega(z) + \Omega \right] \right| \leqslant C e^{-e_j t} \|z\|_{H^1} + C \|z\|_{H^1}^2 + C e^{-(e_j+4\gamma)t}.$$

From the definition of γ (2.1), we deduce that

$$\begin{aligned}
\frac{d}{dt}\alpha_k^{\pm}(t) &= - \operatorname{Im} \int i z e^{-i\theta_k} \left[\partial_x^2 Y_{c_k}^{\mp} - c_k Y_{c_k}^{\mp} \right] (\lambda_k) \\
&\quad - \operatorname{Im} \int i Y_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[(p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z \right] \\
&\quad + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}).
\end{aligned}$$

Now, from (3.2), we find

$$- \operatorname{Im} \int ize^{-i\theta_k} \left[\partial_x^2 Y_{c_k}^{\mp} - c_k Y_{c_k}^{\mp}\right](\lambda_k) \\ = - \operatorname{Im} \int ize^{-i\theta_k} \left[\mp ie_k Y_{c_k}^{\mp} - iQ_{c_k}^{p-1} Y_{c_k,2}^{\mp} - pQ_{c_k}^{p-1} Y_{c_k,1}^{\mp}\right](\lambda_k),$$

and, as in the proof of Lemma 3.2, we also find

$$- \operatorname{Im} \int iY_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[(p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z\right] \\ = - \operatorname{Im} \int iY_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[(p-1)|R_k|^{p-3} R_k \operatorname{Re}(\overline{R_k} z) + |R_k|^{p-1} z\right] + O(e^{-4\gamma t}\|z\|_{H^1}).$$

Hence, we have

$$\frac{d}{dt}\alpha_k^{\pm}(t) = \pm\left(- \operatorname{Im} \int ze^{-i\theta_k} Y_{c_k}^{\mp}(\lambda_k)\right) + \operatorname{Im} \int ize^{-i\theta_k} \left[iQ_{c_k}^{p-1} Y_{c_k,2}^{\mp} + pQ_{c_k}^{p-1} Y_{c_k,1}^{\mp}\right](\lambda_k) \\ - \operatorname{Im} \int iY_{c_k}^{\mp}(\lambda_k) e^{-i\theta_k} \left[(p-1)Q_{c_k}^{p-2}(\lambda_k) e^{i\theta_k} \operatorname{Re}[Q_{c_k}(\lambda_k) e^{-i\theta_k} z] + Q_{c_k}^{p-1}(\lambda_k) z\right] \\ + O(e^{-4\gamma t}\|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}).$$

Finally, if we denote $z_1 = \operatorname{Re}(ze^{-i\theta_k})$ and $z_2 = \operatorname{Im}(ze^{-i\theta_k})$, we find

$$\frac{d}{dt}\alpha_k^{\pm}(t) = \pm e_k \alpha_k^{\pm}(t) + O(e^{-4\gamma t}\|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) \\ + \operatorname{Re} \int (z_1 + iz_2) \left[iQ_{c_k}^{p-1}(\lambda_k) Y_{c_k,2}^{\mp}(\lambda_k) + pQ_{c_k}^{p-1}(\lambda_k) Y_{c_k,1}^{\mp}(\lambda_k)\right] \\ - \operatorname{Re} \int Y_{c_k}^{\mp}(\lambda_k) Q_{c_k}^{p-1}(\lambda_k)(z_1 + iz_2) - \operatorname{Re} \int (p-1) Y_{c_k}^{\mp}(\lambda_k) Q_{c_k}^{p-1}(\lambda_k) z_1 \\ = \pm e_k \alpha_k^{\pm}(t) + O(e^{-4\gamma t}\|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) \\ + p\int z_1 Q_{c_k}^{p-1}(\lambda_k) Y_{c_k,1}^{\mp}(\lambda_k) - \int z_2 Q_{c_k}^{p-1}(\lambda_k) Y_{c_k,2}^{\mp}(\lambda_k) - \int Y_{c_k,1}^{\mp}(\lambda_k) Q_{c_k}^{p-1}(\lambda_k) z_1 \\ + \int Y_{c_k,2}^{\mp}(\lambda_k) Q_{c_k}^{p-1}(\lambda_k) z_2 - (p-1)\int Y_{c_k,1}^{\mp}(\lambda_k) Q_{c_k}^{p-1}(\lambda_k) z_1 \\ = \pm e_k \alpha_k^{\pm}(t) + O(e^{-4\gamma t}\|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}),$$

since all other terms cancel. □

3.3.3 Control of the stable directions

We estimate here $\alpha_k^+(t)$ for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. From (3.9) and (3.8), we have

$$\left|\frac{d}{dt}\alpha_k^+(t) - e_k \alpha_k^+(t)\right| \leqslant C_0 e^{-(e_j+5\gamma)t} + C_1 e^{-2(e_j+\gamma)t} + C_2 e^{-(e_j+4\gamma)t} \leqslant K_2 e^{-(e_j+4\gamma)t}$$

Thus, $|(e^{-e_k s}\alpha_k^+(s))'| \leqslant K_2 e^{-(e_j+e_k+4\gamma)s}$, and so, by integration on $[t, S_n]$, we get $|e^{-e_k S_n}\alpha_k^+(S_n) - e^{-e_k t}\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+e_k+4\gamma)t}$, which gives

$$|\alpha_k^+(t)| \leqslant e^{e_k(t-S_n)}|\alpha_k^+(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But from Claim 3.6 and Lemma 3.5, we have

$$e^{e_k(t-S_n)}|\alpha_k^+(S_n)| \leqslant |\alpha_k^+(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\| \leqslant Ce^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)t},$$

and so finally

$$\forall k \in [\![1,N]\!], \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.10}$$

3.3.4 Control of the unstable directions for $k \in J$

We estimate here $\alpha_k^-(t)$ for all $k \in J$ and $t \in [T(\mathfrak{a}^-), S_n]$. Note first that, as in the previous paragraph, we get, for all $k \in [\![1,N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$,

$$\left| \frac{d}{dt}\alpha_k^-(t) + e_k \alpha_k^-(t) \right| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.11}$$

Now suppose $k \in J$, which implies $e_k \leqslant e_j$. Since $|(e^{e_k s}\alpha_k^-(s))'| \leqslant K_2 e^{(e_k-e_j-4\gamma)s}$, we obtain, by integration on $[t, S_n]$,

$$|\alpha_k^-(t)| \leqslant e^{e_k(S_n-t)}|\alpha_k^-(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But again from Claim 3.6 and Lemma 3.5, we have

$$e^{e_k(S_n-t)}|\alpha_k^-(S_n)| \leqslant K_2 e^{e_k(S_n-t)}e^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} = K_2 e^{e_k(S_n-t)}e^{-(e_j+4\gamma)S_n} \leqslant K_2 e^{(S_n-t)(e_k-e_j)}e^{-e_j t}e^{-4\gamma S_n} \leqslant K_2 e^{-(e_j+4\gamma)t},$$

and so finally

$$\forall k \in J, \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^-(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.12}$$

3.3.5 Localized Weinstein's functional

We follow here the same strategy as in [11, 10, 3] to estimate the energy backwards. For this, we define the function ψ by

$$\psi(x) = 0 \text{ for } x \leqslant -1, \quad \psi(x) = 1 \text{ for } x \geqslant 1, \quad \psi(x) = \frac{1}{c_0}\int_{-1}^{x} e^{-\frac{1}{1-y^2}}dy \quad \text{for } x \in (-1,1),$$

where $c_0 = \int_{-1}^{1} e^{-\frac{1}{1-y^2}}\,dy$. Hence, $\psi \in C^\infty(\mathbb{R})$ is non-decreasing and $0 \leqslant \psi \leqslant 1$. Moreover, we define, for all $k \in [\![2,N]\!]$, $m_k(t) = \frac{1}{2}\left[(v_k + v_{k-1})t + x_k + x_{k-1}\right]$, and

$$\psi_k(t,x) = \psi\left[\frac{1}{\sqrt{t}}(x - m_k(t))\right], \quad \psi_1 \equiv 1.$$

Moreover, we set

$$h_1(t,x) = \left(c_1 + \frac{v_1^2}{4}\right) + \sum_{k=2}^{N}\left[\left(c_k + \frac{v_k^2}{4}\right) - \left(c_{k-1} + \frac{v_{k-1}^2}{4}\right)\right]\psi_k(t,x),$$

$$h_2(t,x) = v_1 + \sum_{k=2}^{N}(v_k - v_{k-1})\psi_k(t,x).$$

Observe that the functions h_1 and h_2 take values close to $c_k + \frac{v_k^2}{4}$ and v_k respectively, for x close to $v_k t + x_k$, and have large variations only in regions far away from the solitons. To quantify these facts (see Lemma 3.9), we introduce the functions ϕ_k, defined for $k \in [\![1, N-1]\!]$ by

$$\phi_k = \psi_k - \psi_{k+1}, \quad \phi_N = \psi_N.$$

Hence, we have $\phi_k \geqslant 0$ and $\sum_{k=1}^N \phi_k \equiv 1$, and by an Abel's transform, we also have

$$h_1 \equiv \sum_{k=1}^{N} \left(c_k + \frac{v_k^2}{4} \right) \phi_k \quad \text{and} \quad h_2 \equiv \sum_{k=1}^{N} v_k \phi_k.$$

Lemma 3.9. *(i) For all $k \in [\![1, N]\!]$, $(|R_k| + |R_{kx}|)|\phi_k - 1| \leqslant Ce^{-4\gamma t} e^{-\sqrt{\sigma_0}|x - v_k t|}$.*

(ii) For all $k, l \in [\![1, N]\!]$ such that $l \neq k$, $(|R_k| + |R_{kx}|)\phi_l \leqslant Ce^{-4\gamma t} e^{-\sqrt{\sigma_0}|x - v_k t|}$.

(iii) For all $k \in [\![1, N]\!]$, $\|\phi_{kx}\|_{L^\infty} + \|\phi_{kxx}\|_{L^\infty} + \|\phi_{kt}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$.

(iv) One has $\|h_{1x}\|_{L^\infty} + \|h_{2x}\|_{L^\infty} + \|h_{1xx}\|_{L^\infty} + \|h_{2xx}\|_{L^\infty} + \|h_{1t}\|_{L^\infty} + \|h_{2t}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$, and, for all $k \in [\![1, N]\!]$,

$$\left| h_1 - \left(c_k + \frac{v_k^2}{4} \right) \right| (|R_k| + |R_{kx}|) \leqslant Ce^{-4\gamma t} e^{-\sqrt{\sigma_0}|x - v_k t|},$$

$$|h_2 - v_k|(|R_k| + |R_{kx}|) \leqslant Ce^{-4\gamma t} e^{-\sqrt{\sigma_0}|x - v_k t|}.$$

Proof. See Appendix A. □

Now, we define a quantity related to the energy for z, by

$$H(t) = \int |\partial_x z|^2 - \frac{2}{p+1} \int |\varphi + r_j + z|^{p+1} - |\varphi + r_j|^{p+1} - (p+1)|\varphi + r_j|^{p-1} \operatorname{Re}[(\overline{\varphi} + \overline{r_j}) z] + \int h_1 |z|^2 - \operatorname{Im} \int h_2 \bar{z} \partial_x z. \quad (3.13)$$

The following estimate of the variation of H is the main new point of this paper, and as its proof is long and technical, it is postponed to Appendix B.

Proposition 3.10. *For all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\left| \frac{dH}{dt}(t) \right| \leqslant \frac{C_0}{\sqrt{t}} \|z(t)\|_{H^1}^2 + C_1 e^{-(e_j + 4\gamma)t} \|z(t)\|_{H^1} + C_2 \|z(t)\|_{H^1}^3.$$

We can now prove that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\mathcal{H}[z](t) := \int |\partial_x z|^2 - |R|^{p-1}|z|^2 - (p-1)\big(\operatorname{Re}(\overline{R} z)\big)^2 |R|^{p-3} + h_1|z|^2 - \operatorname{Im} h_2 \bar{z} \partial_x z$$

satisfies

$$\mathcal{H}[z](t) \leqslant \frac{K_1}{\sqrt{t}} e^{-2(e_j + \gamma)t}. \quad (3.14)$$

Indeed, from Proposition 3.10 and estimates (3.8), we deduce that, for all $s \in [t, S_n]$,

$$\left| \frac{dH}{ds}(s) \right| \leqslant \frac{C_0}{\sqrt{s}} e^{-2(e_j + \gamma)s} + C_1 e^{-3\gamma s} e^{-2(e_j + \gamma)s} + C_2 e^{-3(e_j + \gamma)s} \leqslant \frac{K_1}{\sqrt{t}} e^{-2(e_j + \gamma)s}.$$

Thus, by integration on $[t, S_n]$, we obtain $|H(t) - H(S_n)| \leqslant \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}$, and so

$$H(t) \leqslant |H(S_n)| + \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}.$$

But from Claim 3.6 and Lemma 3.5, we have

$$\begin{aligned} |H(S_n)| &\leqslant C\|z(S_n)\|_{H^1}^2 \leqslant C\|\mathfrak{b}\|^2 \leqslant C\|\mathfrak{a}^-\|^2 \\ &\leqslant Ce^{-2(e_j+2\gamma)S_n} \leqslant Ce^{-2(e_j+2\gamma)t}, \end{aligned}$$

and so

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad H(t) \leqslant \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}.$$

Finally, expanding $|\varphi + r_j + z|^{p+1} = \left[|\varphi + r_j|^2 + 2\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z] + |z|^2\right]^{\frac{p+1}{2}}$, we find

$$\begin{aligned} \Big| |\varphi + r_j + z|^{p+1} - |\varphi + r_j|^{p+1} - (p+1)\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]|\varphi + r_j|^{p-1} \\ - \left(\frac{p+1}{2}\right)|z|^2|\varphi + r_j|^{p-1} - \frac{(p+1)(p-1)}{2}(\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2|\varphi + r_j|^{p-3}\Big| \leqslant C|z|^3, \end{aligned}$$

and so, from the definition of H (3.13),

$$\begin{aligned} \int |\partial_x z|^2 - \int |\varphi + r_j|^{p-1}|z|^2 - (p-1)\int (\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2|\varphi + r_j|^{p-3} \\ + \int h_1|z|^2 - \operatorname{Im}\int h_2\bar{z}\partial_x z \leqslant \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}. \end{aligned}$$

Using (2.3), we easily obtain (3.14) by similar techniques used in the proof of Lemma 3.2 in Appendix A to replace $(\varphi + r_j)$ by R plus an exponentially small error term.

3.3.6 Control of the directions of null energy

Define $\tilde{z}(t) = z(t) + \sum_{k=1}^{N} \beta_k(t) iR_k(t) + \sum_{k=1}^{N} \gamma_k(t)\partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}$, where

$$\beta_k(t) = -\frac{\operatorname{Re}\int iR_k\bar{z}}{\|Q_{c_k}\|_{L^2}^2} = \frac{\operatorname{Im}\int R_k\bar{z}}{\|Q_{c_k}\|_{L^2}^2} \quad \text{and} \quad \gamma_k(t) = -\frac{\operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\bar{z}}{\|\partial_x Q_{c_k}\|_{L^2}^2}.$$

First, note that there exist $C_1, C_2 > 0$ such that

$$C_1\|z\|_{H^1} \leqslant \|\tilde{z}\|_{H^1} + \sum_{k=1}^{N}(|\beta_k| + |\gamma_k|) \leqslant C_2\|z\|_{H^1}. \tag{3.15}$$

Moreover, by this choice of parameters, we have, for all $k \in [\![1, N]\!]$,

$$\left|\operatorname{Re}\int -i\overline{R_k}\tilde{z}\right| \leqslant Ce^{-\gamma t}\|z\|_{H^1}, \quad \left|\operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\overline{\tilde{z}}\right| \leqslant Ce^{-\gamma t}\|z\|_{H^1}. \tag{3.16}$$

Indeed, by (2.2), we have

$$\begin{aligned}\operatorname{Re}\int -i\overline{R_k}\tilde{z} &= \operatorname{Im}\int \overline{R_k}\left[z(t)+\sum_{l=1}^{N}\beta_l(t)iR_l(t)+\sum_{l=1}^{N}\gamma_l(t)\partial_x Q_{c_l}(\lambda_l)e^{i\theta_l}\right]\\ &= \operatorname{Im}\int \overline{R_k}z+\beta_k(t)\operatorname{Re}\int |R_k|^2+\gamma_k(t)\operatorname{Im}\int Q_{c_k}\partial_x Q_{c_k}+O(e^{-\gamma t}\|z\|_{H^1})\\ &= \operatorname{Im}\int \overline{R_k}z+\operatorname{Im}\int R_k\bar{z}+O(e^{-\gamma t}\|z\|_{H^1})=O(e^{-\gamma t}\|z\|_{H^1}),\end{aligned}$$

and similarly,

$$\begin{aligned}&\operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\overline{\tilde{z}}\\ &= \operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\bar{z}+\beta_k(t)\operatorname{Im}\int Q_{c_k}\partial_x Q_{c_k}+\gamma_k(t)\operatorname{Re}\int |\partial_x Q_{c_k}|^2+O(e^{-\gamma t}\|z\|_{H^1})\\ &= \operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\bar{z}-\operatorname{Re}\int \partial_x Q_{c_k}(\lambda_k)e^{i\theta_k}\bar{z}+O(e^{-\gamma t}\|z\|_{H^1})=O(e^{-\gamma t}\|z\|_{H^1}).\end{aligned}$$

Now, we compare the functionals $\mathcal{H}[\tilde{z}]$ and $\mathcal{H}[z]$ in the following lemma, that we prove in Appendix A.

Lemma 3.11. *For all $t\in[T(\mathfrak{a}^-),S_n]$, one has*

$$\mathcal{H}[\tilde{z}](t)\leqslant \mathcal{H}[z](t)+\frac{C}{\sqrt{t}}\|z\|_{H^1}^2.$$

By (3.14) and (3.8), we deduce that

$$\forall t\in[T(\mathfrak{a}^-),S_n],\quad \mathcal{H}[\tilde{z}](t)\leqslant \frac{K_1}{\sqrt{t}}e^{-2(e_j+\gamma)t}. \tag{3.17}$$

Now, from the property of coercivity (ii) in Proposition 2.4, and by the definitions of h_1 and h_2, we obtain, by simple localization arguments (see [11, Appendix B] for details), that there exists $\kappa_1>0$ such that

$$\begin{aligned}\mathcal{H}[\tilde{z}](t)\geqslant \frac{1}{\kappa_1}\|\tilde{z}\|_{H^1}^2-\kappa_1\sum_{k=1}^{N}\Bigg[&\left(-\operatorname{Im}\int \tilde{z}\overline{Y_k^+}\right)^2+\left(-\operatorname{Im}\int \tilde{z}\overline{Y_k^-}\right)^2\\ &+\left(\operatorname{Re}\int \tilde{z}(-i\overline{R_k})\right)^2+\left(\operatorname{Re}\int \tilde{z}\partial_x Q_{c_k}(\lambda_k)e^{-i\theta_k}\right)^2\Bigg].\end{aligned} \tag{3.18}$$

To justify heuristically this inequality, we compute, for $k\in[\![1,N]\!]$, the localized version $\mathcal{H}_k[z]$ of $\mathcal{H}[z]$ (it would be the same for $\tilde{z}$), defined by

$$\mathcal{H}_k[z]=\int |\partial_x z|^2-|R_k|^{p-1}|z|^2-(p-1)\left(\operatorname{Re}(\overline{R_k}z)\right)^2|R_k|^{p-3}+\left(c_k+\frac{v_k^2}{4}\right)|z|^2-v_k\operatorname{Im}\bar{z}\partial_x z.$$

In fact, if we denote $[e^{-i\theta_k}z](\cdot+v_kt+x_k)=z_1+iz_2$, *i.e.* $z=e^{i\theta_k}(z_1+iz_2)(\lambda_k)$, then we have $\partial_x z=\frac{iv_k}{2}e^{i\theta_k}(z_1+iz_2)(\lambda_k)+e^{i\theta_k}(\partial_x z_1+i\partial_x z_2)(\lambda_k)$, and so, by (ii) of Proposition 2.4, we find

$$
\begin{aligned}
\mathcal{H}_k[z] &= \int \Big(-\frac{v_k}{2}z_2 + \partial_x z_1\Big)^2(\lambda_k) + \int \Big(\frac{v_k}{2}z_1 + \partial_x z_2\Big)^2(\lambda_k) \\
&\quad - \int Q_{c_k}^{p-1}(\lambda_k)(z_1^2 + z_2^2)(\lambda_k) - (p-1)\int Q_{c_k}^{p-1}(\lambda_k) z_1^2(\lambda_k) \\
&\quad + \int \Big(c_k + \frac{v_k^2}{4}\Big)(z_1^2 + z_2^2)(\lambda_k) - v_k \int \Big(\frac{v_k}{2}z_1^2 + z_1\partial_x z_2 + \frac{v_k}{2}z_2^2 - z_2\partial_x z_1\Big)(\lambda_k) \\
&= \int (\partial_x z_1)^2 + c_k z_1^2 - pQ_{c_k}^{p-1} z_1^2 + \int (\partial_x z_2)^2 + c_k z_2^2 - Q_{c_k}^{p-1} z_2^2 \\
&= (L_{c_k +} z_1, z_1) + (L_{c_k -} z_2, z_2) \\
&\geqslant \frac{1}{\kappa_0}\|z\|_{H^1}^2 - \kappa_0 \Big[\Big(\int \partial_x Q_{c_k} z_1\Big)^2 + \Big(\int Q_{c_k} z_2\Big)^2 + \Big(\mathrm{Im}\int Y_k^+ \bar{z}\Big)^2 + \Big(\mathrm{Im}\int Y_k^- \bar{z}\Big)^2\Big].
\end{aligned}
$$

Now, we return to (3.18), and we estimate each term of the sum, for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. First, by (3.16), we have

$$
\Big(\mathrm{Re}\int \tilde{z}(-i\overline{R_k})\Big)^2 + \Big(\mathrm{Re}\int \tilde{z}\partial_x Q_{c_k}(\lambda_k)e^{-i\theta_k}\Big)^2 \leqslant Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2\gamma t}e^{-2(e_j+\gamma)t}.
$$

Second, denoting $Y_1 = \mathrm{Re}\, Y^+$ and $Y_2 = \mathrm{Im}\, Y^+$ again, we have

$$
\begin{aligned}
-\mathrm{Im}\int \overline{Y_k^+}(t)\tilde{z}(t) &= \alpha_k^+(t) - \beta_k(t)\,\mathrm{Re}\int Q_{c_k}(\lambda_k)(Y_{c_k,1}^+ - iY_{c_k,2}^+)(\lambda_k) \\
&\quad - \gamma_k(t)\,\mathrm{Im}\int \partial_x Q_{c_k}(\lambda_k)(Y_{c_k,1}^+ - iY_{c_k,2}^+)(\lambda_k) + O(e^{-\gamma t}\|z\|_{H^1}) \\
&= \alpha_k^+(t) - C\beta_k(t)\int QY_1 + C\gamma_k(t)\int \partial_x QY_2 + O(e^{-\gamma t}\|z\|_{H^1}).
\end{aligned}
$$

But by definition of Y^+, we recall that $L_+ Y_1 = e_0 Y_2$ and $L_- Y_2 = -e_0 Y_1$, and so

$$
\begin{aligned}
-\mathrm{Im}\int \overline{Y_k^+}(t)\tilde{z}(t) &= \alpha_k^+(t) + \frac{C\beta_k(t)}{e_0}\int Q(L_- Y_2) + \frac{C\gamma_k(t)}{e_0}\int \partial_x Q(L_+ Y_1) + O(e^{-\gamma t}\|z\|_{H^1}) \\
&= \alpha_k^+(t) + C'\beta_k(t)\int (L_- Q)Y_2 + C'\gamma_k(t)\int L_+(\partial_x Q)Y_1 + O(e^{-\gamma t}\|z\|_{H^1}) \\
&= \alpha_k^+(t) + O(e^{-\gamma t}\|z\|_{H^1}),
\end{aligned}
$$

since $L_\pm$ are self-adjoint, and moreover $L_- Q = 0$ and $L_+(\partial_x Q) = 0$ by Proposition 2.2. Hence, by (3.10), we find, for all $k \in [\![1, N]\!]$,

$$
\begin{aligned}
\Big(-\mathrm{Im}\int \tilde{z}\overline{Y_k^+}\Big)^2 &\leqslant 2(\alpha_k^+)^2 + Ce^{-2\gamma t}\|z\|_{H^1}^2 \\
&\leqslant Ce^{-2(e_j+4\gamma)t} + Ce^{-2\gamma t}e^{-2(e_j+\gamma)t} \leqslant Ce^{-2\gamma t}e^{-2(e_j+\gamma)t}.
\end{aligned}
$$

Completely similarly, we find, for all $k \in [\![1, N]\!]$,

$$
\Big(-\mathrm{Im}\int \tilde{z}\overline{Y_k^-}\Big)^2 \leqslant 2(\alpha_k^-)^2 + Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2\gamma t}e^{-2(e_j+\gamma)t},
$$

using (3.12) for $k \in J$, and (3.8) for $k \in K$.

Finally, gathering all estimates from (3.17), we have proved that there exists $\widetilde{K_0} > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$
\|\tilde{z}(t)\|_{H^1} \leqslant \frac{\widetilde{K_0}}{t^{1/4}}e^{-(e_j+\gamma)t}.
$$

We want now to prove the same estimate for z, and so we have to control the parameters $\beta_k(t)$ and $\gamma_k(t)$ introduced above.

3.3.7 Improvement of the decay of z

Lemma 3.12. *There exists $K_0 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\|z(t)\|_{H^1} \leqslant \frac{K_0}{t^{1/4}} e^{-(e_j+\gamma)t}.$$

Proof. By (3.15), it is enough to prove this estimate for $|\beta_k(t)| + |\gamma_k(t)|$ with $k \in [\![1, N]\!]$ fixed. To do this, write first the equation of $\tilde{z}$, from the equation of z (3.5),

$$\begin{aligned}
&i\partial_t \tilde{z} + \partial_x^2 \tilde{z} + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi}\tilde{z}) + |\varphi|^{p-1}\tilde{z} \\
&= i\partial_t z - \sum \beta_l' R_l - \sum \beta_l \left[-v_l \partial_x Q_{c_l} + i\left(c_l - \frac{v_l^2}{4}\right) Q_{c_l}\right](\lambda_l) e^{i\theta_l} + i\sum \gamma_l' \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} \\
&\quad + i\sum \gamma_l \left[-v_l \partial_x^2 Q_{c_l} + i\left(c_l - \frac{v_l^2}{4}\right) \partial_x Q_{c_l}\right](\lambda_l) e^{i\theta_l} + \partial_x^2 z \\
&\quad + i\sum \beta_l \left[\partial_x^2 Q_{c_l} + i v_l \partial_x Q_{c_l} - \frac{v_l^2}{4} Q_{c_l}\right](\lambda_l) e^{i\theta_l} \\
&\quad + \sum \gamma_l \left[\partial_x^3 Q_{c_l} + i v_l \partial_x^2 Q_{c_l} - \frac{v_l^2}{4} \partial_x Q_{c_l}\right](\lambda_l) e^{i\theta_l} \\
&\quad + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi} z) + (p-1)|\varphi|^{p-3}\varphi \sum \beta_l \operatorname{Re}(i\overline{\varphi} R_l) + |\varphi|^{p-1} z \\
&\quad + (p-1)|\varphi|^{p-3}\varphi \sum \gamma_l \operatorname{Re}(\overline{\varphi}\partial_x Q_{c_l}(\lambda_l) e^{i\theta_l}) + \sum \beta_l i |\varphi|^{p-1} R_l + \sum \gamma_l |\varphi|^{p-1} \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l},
\end{aligned}$$

and so, since $\partial_x^2 Q_{c_l} + Q_{c_l}^p = c_l Q_{c_l}$, we find

$$\begin{aligned}
&i\partial_t \tilde{z} + \partial_x^2 \tilde{z} + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi}\tilde{z}) + |\varphi|^{p-1}\tilde{z} \\
&= -\omega_1 \cdot z - \omega(z) - \Omega - \sum \beta_l' R_l + i\sum \gamma_l' \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} \\
&\quad - i\sum \beta_l Q_{c_l}^p(\lambda_l) e^{i\theta_l} - p\sum \gamma_l \partial_x Q_{c_l}(\lambda_l) Q_{c_l}^{p-1}(\lambda_l) e^{i\theta_l} \\
&\quad - (p-1)\sum \beta_l |\varphi|^{p-3}\varphi \operatorname{Im}(\overline{\varphi} R_l) + (p-1)\sum \gamma_l |\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi}\partial_x Q_{c_l}(\lambda_l) e^{i\theta_l}) \\
&\quad + i\sum \beta_l |\varphi|^{p-1} Q_{c_l}(\lambda_l) e^{i\theta_l} + \sum \gamma_l |\varphi|^{p-1} \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} \\
&= -\omega_1 \cdot z - \omega(z) - \Omega - \sum \beta_l' R_l + i\sum \gamma_l' \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} - (p-1)\sum \beta_l |\varphi|^{p-3}\varphi \operatorname{Im}(\overline{\varphi} R_l) \\
&\quad + i\sum \beta_l e^{i\theta_l} Q_{c_l}(\lambda_l)[|\varphi|^{p-1} - Q_{c_l}^{p-1}(\lambda_l)] \\
&\quad + \sum \gamma_l \Big[|\varphi|^{p-1} \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi}\partial_x Q_{c_l}(\lambda_l) e^{i\theta_l}) \\
&\qquad - p\partial_x Q_{c_l}(\lambda_l) Q_{c_l}^{p-1}(\lambda_l) e^{i\theta_l} \Big].
\end{aligned}$$

Then, multiply this equation by $\overline{R_k}$, integrate, and take the real part of it, so that we obtain, by (2.2), (2.3) and Lemma 3.2,

$$\begin{aligned}
-\operatorname{Im}\int \partial_t \tilde{z}\overline{R_k} + O(\|\tilde{z}\|_{L^2}) &= O(e^{-e_j t}\|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) - C\beta_k' \\
&\quad + \sum_{l\neq k} (\beta_l' + \gamma_l') O(e^{-\gamma t}) + \sum \beta_l O(e^{-\gamma t}) + \sum \gamma_l O(e^{-\gamma t}).
\end{aligned}$$

In other words, we have, by (3.15) and (3.8),

$$|\beta_k'| \leqslant C\left|\operatorname{Im}\int \partial_t \tilde{z}\overline{R_k}\right| + Ce^{-\gamma t}\sum_{l\neq k}(|\beta_l'| + |\gamma_l'|) + \frac{C}{t^{1/4}} e^{-(e_j+\gamma)t}.$$

Moreover, from

$$\operatorname{Im}\int \tilde{z}\overline{R_k} = \sum_{l\neq k}\beta_l \operatorname{Im}\int iR_l\overline{R_k} + \sum_{l\neq k}\gamma_l \operatorname{Im}\int \partial_x Q_{c_l}(\lambda_l)e^{i\theta_l}\overline{R_k},$$

we deduce that

$$\begin{aligned}\frac{d}{dt}\operatorname{Im}\int \tilde{z}\overline{R_k} &= \sum_{l\neq k}(\beta_l' + \gamma_l')O(e^{-\gamma t}) + \sum_{l\neq k}(\beta_l+\gamma_l)O(e^{-\gamma t})\\ &= \operatorname{Im}\int \partial_t\tilde{z}\overline{R_k} + \operatorname{Im}\int \tilde{z}\partial_t\overline{R_k},\end{aligned}$$

and so, as $\partial_t R_k = -v_k\partial_x R_k + i\left(c_k + \frac{v_k^2}{4}\right)R_k$,

$$\left|\operatorname{Im}\int \partial_t\tilde{z}\overline{R_k}\right| \leqslant C\|\tilde{z}\|_{H^1} + Ce^{-\gamma t}\sum_{l\neq k}(|\beta_l'|+|\gamma_l'|) + Ce^{-\gamma t}\sum_{l\neq k}(|\beta_l|+|\gamma_l|).$$

Gathering previous estimates, we find

$$|\beta_k'| \leqslant Ce^{-\gamma t}\sum_{l\neq k}(|\beta_l'|+|\gamma_l'|) + \frac{C}{t^{1/4}}e^{-(e_j+\gamma)t}.$$

Completely similarly, if we multiply the equation on $\tilde{z}$ by $\partial_x Q_{c_k}(\lambda_k)e^{-i\theta_k}$, integrate and take the imaginary part of it, we find

$$|\gamma_k'| \leqslant Ce^{-\gamma t}\sum_{l\neq k}(|\beta_l'|+|\gamma_l'|) + \frac{C}{t^{1/4}}e^{-(e_j+\gamma)t}.$$

Hence, we have proved that there exist $C_3, C_4 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$|\beta_k'| + |\gamma_k'| \leqslant C_3 e^{-\gamma t}\sum_{l\neq k}(|\beta_l'|+|\gamma_l'|) + \frac{C_4}{t^{1/4}}e^{-(e_j+\gamma)t}.$$

Finally, if we choose t_0 large enough so that $C_3 e^{-\gamma t_0} \leqslant \frac{1}{N}$, we obtain, for all $s \in [t, S_n]$, with $t \in [T(\mathfrak{a}^-), S_n]$,

$$|\beta_k'(s)| + |\gamma_k'(s)| \leqslant \frac{C}{t^{1/4}}e^{-(e_j+\gamma)s}.$$

By integration on $[t, S_n]$, we get $|\beta_k(t)| + |\gamma_k(t)| \leqslant |\beta_k(S_n)| + |\gamma_k(S_n)| + \frac{C}{t^{1/4}}e^{-(e_j+\gamma)t}$. But from Claim 3.6, Lemma 3.5 and (3.15), we have

$$|\beta_k(S_n)| + |\gamma_k(S_n)| \leqslant C\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\| \leqslant C\|\mathfrak{a}^-\| \leqslant Ce^{-(e_j+2\gamma)S_n} \leqslant Ce^{-(e_j+2\gamma)t},$$

and so finally,

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad |\beta_k(t)| + |\gamma_k(t)| \leqslant \frac{C}{t^{1/4}}e^{-(e_j+\gamma)t}. \qquad \square$$

3.3.8 Control of the unstable directions for $k \in K$ by a topological argument

Lemma 3.12 being proved, we choose t_0 large enough so that $\frac{K_0}{t_0^{1/4}} \leqslant \frac{1}{2}$. Therefore, we have

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \|z(t)\|_{H^1} \leqslant \frac{1}{2} e^{-(e_j+\gamma)t}.$$

We can now prove the following final lemma, which concludes the proof of Proposition 3.4. Note that its proof is very similar to the one in [2], by the common choice of notation, but it is reproduced here for the reader's convenience.

Lemma 3.13. *For t_0 large enough, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$.*

Proof. For the sake of contradiction, suppose that, for all $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$, $T(\mathfrak{a}^-) > t_0$. As $e^{(e_j+\gamma)T(\mathfrak{a}^-)} z(T(\mathfrak{a}^-)) \in B_{H^1}(1/2)$, then, by definition of $T(\mathfrak{a}^-)$ and continuity of the flow, we have

$$e^{(e_j+2\gamma)T(\mathfrak{a}^-)} \boldsymbol{\alpha}^-(T(\mathfrak{a}^-)) \in \mathbb{S}_{\mathbb{R}^{k_0}}(1). \tag{3.19}$$

Now, let $T \in [t_0, T(\mathfrak{a}^-)]$ be close enough to $T(\mathfrak{a}^-)$ such that z is defined on $[T, S_n]$, and by continuity,

$$\forall t \in [T, S_n], \quad \|z(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

We can now consider, for $t \in [T, S_n]$,

$$\mathcal{N}(t) = \mathcal{N}(\boldsymbol{\alpha}^-(t)) = \|e^{(e_j+2\gamma)t} \boldsymbol{\alpha}^-(t)\|^2.$$

To calculate $\mathcal{N}'$, we start from estimate (3.11):

$$\forall k \in K, \forall t \in [T, S_n], \quad \left| \frac{d}{dt} \alpha_k^-(t) + e_k \alpha_k^-(t) \right| \leqslant K_2' e^{-(e_j+4\gamma)t}.$$

Multiplying by $|\alpha_k^-(t)|$, we obtain

$$\left| \alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + e_k \alpha_k^-(t)^2 \right| \leqslant K_2' e^{-(e_j+4\gamma)t} |\alpha_k^-(t)|,$$

and thus

$$2\alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + 2e_{\min} \alpha_k^-(t)^2 \leqslant 2\alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + 2e_k \alpha_k^-(t)^2 \leqslant K_2' e^{-(e_j+4\gamma)t} |\alpha_k^-(t)|,$$

where $e_{\min} = \min\{e_k \ ; \ k \in K\}$. By summing on $k \in K$, we get

$$(\|\boldsymbol{\alpha}^-(t)\|^2)' + 2e_{\min} \|\boldsymbol{\alpha}^-(t)\|^2 \leqslant K_3 e^{-(e_j+4\gamma)t} \|\boldsymbol{\alpha}^-(t)\|.$$

Therefore, we can estimate

$$\begin{aligned} \mathcal{N}'(t) &= (e^{2(e_j+2\gamma)t} \|\boldsymbol{\alpha}^-(t)\|^2)' = e^{2(e_j+2\gamma)t} \left[2(e_j + 2\gamma) \|\boldsymbol{\alpha}^-(t)\|^2 + (\|\boldsymbol{\alpha}^-(t)\|^2)' \right] \\ &\leqslant e^{2(e_j+2\gamma)t} \left[2(e_j + 2\gamma) \|\boldsymbol{\alpha}^-(t)\|^2 - 2e_{\min} \|\boldsymbol{\alpha}^-(t)\|^2 + K_3 e^{-(e_j+4\gamma)t} \|\boldsymbol{\alpha}^-(t)\| \right]. \end{aligned}$$

Hence, we have, for all $t \in [T, S_n]$,

$$\mathcal{N}'(t) \leqslant -\theta \cdot \mathcal{N}(t) + K_3 e^{e_j t} \|\alpha^-(t)\|,$$

where $\theta = 2(e_{\min} - e_j - 2\gamma) > 0$ by the definitions of γ (2.1) and of the set K. In particular, for all $\tau \in [T, S_n]$ satisfying $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\theta + K_3 e^{e_j \tau} \|\alpha^-(\tau)\| = -\theta + K_3 e^{e_j \tau} e^{-(e_j+2\gamma)\tau} = -\theta + K_3 e^{-2\gamma\tau} \leqslant -\theta + K_3 e^{-2\gamma t_0}.$$

Now, we finally fix t_0 large enough so that $K_3 e^{-2\gamma t_0} \leqslant \frac{\theta}{2}$, and so, for all $\tau \in [T, S_n]$ such that $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\frac{\theta}{2}. \tag{3.20}$$

In particular, by (3.19), we have $\mathcal{N}'(T(\mathfrak{a}^-)) \leqslant -\frac{\theta}{2}$.

First consequence: $\mathfrak{a}^- \mapsto T(\mathfrak{a}^-)$ is continuous. Indeed, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\mathcal{N}(T(\mathfrak{a}^-) - \varepsilon) > 1 + \delta$ and $\mathcal{N}(T(\mathfrak{a}^-) + \varepsilon) < 1 - \delta$. Moreover, by definition of $T(\mathfrak{a}^-)$ and (3.20), there can not exist $\tau \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$ such that $\mathcal{N}(\tau) = 1$, and so by choosing δ small enough, we have, for all $t \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$, $\mathcal{N}(t) < 1 - \delta$. But from continuity of the flow, there exists $\eta > 0$ such that, for all $\tilde{\mathfrak{a}}^-$ satisfying $\|\tilde{\mathfrak{a}}^- - \mathfrak{a}^-\| \leqslant \eta$, we have

$$\forall t \in [T(\mathfrak{a}^-) - \varepsilon, S_n], \quad |\mathcal{N}(\tilde{\alpha}^-(t)) - \mathcal{N}(\alpha^-(t))| \leqslant \delta/2.$$

We finally deduce that $T(\mathfrak{a}^-) - \varepsilon \leqslant T(\tilde{\mathfrak{a}}^-) \leqslant T(\mathfrak{a}^-) + \varepsilon$, as expected.

Second consequence: We can define the map

$$\begin{array}{rccl} \mathcal{M} : & B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n}) & \to & \mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n}) \\ & \mathfrak{a}^- & \mapsto & e^{-(e_j+2\gamma)(S_n - T(\mathfrak{a}^-))}\alpha^-(T(\mathfrak{a}^-)). \end{array}$$

Note that $\mathcal{M}$ is continuous by the previous point. Now, let $\mathfrak{a}^- \in \mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$. As $\mathcal{N}'(S_n) \leqslant -\frac{\theta}{2}$ by (3.20), we deduce by definition of $T(\mathfrak{a}^-)$ that $T(\mathfrak{a}^-) = S_n$, and so $\mathcal{M}(\mathfrak{a}^-) = \mathfrak{a}^-$. In other words, $\mathcal{M}$ restricted to $\mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ is the identity. But the existence of such a map $\mathcal{M}$ contradicts Brouwer's fixed point theorem.

In conclusion, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$. □

A Appendix

Proof of Lemma 3.2. First, we calculate

$$\begin{aligned} |R_j|^{p-1} r_j &+ (p-1)|R_j|^{p-3} R_j \operatorname{Re}(\overline{R_j} r_j) \\ &= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j)[Y_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j) e^{i\theta_j} \\ &\quad + (p-1) Q_{c_j}^{p-2}(\lambda_j) e^{i\theta_j} \operatorname{Re}[A_j e^{-e_j t} Q_{c_j}(Y_{c_j,1}^+ + iY_{c_j,2}^+)](\lambda_j) \\ &= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j}[Y_{c_j,1}^+ + iY_{c_j,2}^+ + (p-1)Y_{c_j,1}^+](\lambda_j) \\ &= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j}[pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j). \end{aligned}$$

Hence, from the expression of Ω (3.4), it can be written

$$\Omega = |\varphi + r_j|^{p-1}(\varphi + r_j) - |\varphi|^{p-1}\varphi - |R_j|^{p-1}r_j - (p-1)|R_j|^{p-3}R_j \operatorname{Re}(\overline{R_j}r_j).$$

We can now estimate $\|\Omega\|_{H^1}$, and we estimate $\|\partial_x\Omega\|_{L^2}$ for example, the term $\|\Omega\|_{L^2}$ being similar and easier. To do this, we write

$$\begin{aligned}
\Omega_x &= (p-1)\operatorname{Re}[(\varphi_x + r_{jx})(\overline{\varphi} + \overline{r_j})]|\varphi + r_j|^{p-3}(\varphi + r_j) + |\varphi + r_j|^{p-1}(\varphi_x + r_{jx}) \\
&\quad - (p-1)\operatorname{Re}(\varphi_x\overline{\varphi})|\varphi|^{p-3}\varphi - |\varphi|^{p-1}\varphi_x - (p-1)\operatorname{Re}(R_{jx}\overline{R_j})|R_j|^{p-3}r_j - |R_j|^{p-1}r_{jx} \\
&\quad - (p-1)(p-3)\operatorname{Re}(R_{jx}\overline{R_j})|R_j|^{p-5}R_j\operatorname{Re}(\overline{R_j}r_j) - (p-1)|R_j|^{p-3}R_{jx}\operatorname{Re}(\overline{R_j}r_j) \\
&\quad - (p-1)|R_j|^{p-3}R_j\operatorname{Re}(\overline{R_{jx}}r_j) - (p-1)|R_j|^{p-3}R_j\operatorname{Re}(\overline{R_j}r_{jx}) \\
&= (p-1)\operatorname{Re}(\varphi_x\overline{\varphi})\left[|\varphi + r_j|^{p-3}(\varphi + r_j) - |\varphi|^{p-3}\varphi - (p-3)\varphi\operatorname{Re}(\overline{\varphi}r_j)|\varphi|^{p-5} - |\varphi|^{p-3}r_j\right] \\
&\quad + (p-1)(p-3)\left[\operatorname{Re}(\varphi_x\overline{\varphi})\operatorname{Re}(\overline{\varphi}r_j)|\varphi|^{p-5}\varphi - \operatorname{Re}(R_{jx}\overline{R_j})\operatorname{Re}(\overline{R_j}r_j)|R_j|^{p-5}R_j\right] \\
&\quad + (p-1)r_j\left[\operatorname{Re}(\varphi_x\overline{\varphi})|\varphi|^{p-3} - \operatorname{Re}(R_{jx}\overline{R_j})|R_j|^{p-3}\right] \\
&\quad + (p-1)\left[\operatorname{Re}(\varphi_x\overline{r_j})|\varphi + r_j|^{p-3}(\varphi + r_j) - \operatorname{Re}(\overline{R_{jx}}r_j)|R_j|^{p-3}R_j\right] \\
&\quad + (p-1)\left[\operatorname{Re}(r_{jx}\overline{\varphi})|\varphi + r_j|^{p-3}(\varphi + r_j) - \operatorname{Re}(r_{jx}\overline{R_j})|R_j|^{p-3}R_j\right] \\
&\quad + (p-1)\operatorname{Re}(r_{jx}\overline{r_j})|\varphi + r_j|^{p-3}(\varphi + r_j) + r_{jx}\left[|\varphi + r_j|^{p-1} - |R_j|^{p-1}\right] \\
&\quad + \varphi_x\left[|\varphi + r_j|^{p-1} - |\varphi|^{p-1} - (p-1)\operatorname{Re}(\overline{\varphi}r_j)|\varphi|^{p-3}\right] \\
&\quad + (p-1)\left[\operatorname{Re}(\overline{\varphi}r_j)\varphi_x|\varphi|^{p-3} - \operatorname{Re}(\overline{R_j}r_j)R_{jx}|R_j|^{p-3}\right].
\end{aligned}$$

To estimate all these terms in L^2 norm, we use the facts that φ is equal to R plus a small error term according to (2.3), that R multiplied by a term moving on the line $x = v_jt + x_j$ (like r_j) is equal to R_j plus a small error term according to (2.2), and finally that r_j is at order e^{-e_jt}. To illustrate this, we estimate the first two terms **I** and **II**, for example, as all other terms can be treated similarly. For **I**, we simply remark that

$$\|\mathbf{I}\|_{L^2} \leqslant C\|r_j\|_{L^2}^2 \leqslant Ce^{-2e_jt} \leqslant Ce^{-(e_j+4\gamma)t}$$

by the definition of γ (2.1). For **II**, we decompose it as

$$\begin{aligned}
\frac{1}{(p-1)(p-3)}\mathbf{II} &= \operatorname{Re}[(\varphi_x - R_x)\overline{\varphi}]\operatorname{Re}(\overline{\varphi}r_j)|\varphi|^{p-5}\varphi + \operatorname{Re}(R_x(\overline{\varphi} - \overline{R})\operatorname{Re}(\overline{\varphi}r_j)|\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}(R_x\overline{R})\operatorname{Re}[(\overline{\varphi} - \overline{R})r_j]|\varphi|^{p-5}\varphi + \operatorname{Re}[(R_x - R_{jx})\overline{R}]\operatorname{Re}(\overline{R}r_j)|\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}[R_{jx}(\overline{R} - \overline{R_j})]\operatorname{Re}(\overline{R}r_j)|\varphi|^{p-5}\varphi + \operatorname{Re}[R_{jx}\overline{R_j}]\operatorname{Re}[(\overline{R} - \overline{R_j})r_j]|\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}(R_{jx}\overline{R_j})\operatorname{Re}(\overline{R_j}r_j)\left[|\varphi|^{p-5}\varphi - |R_j|^{p-5}R_j\right].
\end{aligned}$$

Since $\|\varphi - R\|_{H^1} \leqslant Ce^{-4\gamma t}$ by (2.3), the first three terms are bounded in L^2 norm by $Ce^{-(e_j+4\gamma)t}$. Moreover, by (2.2), the next three terms are also bounded in L^2 norm by $Ce^{-(e_j+4\gamma)t}$. Finally, for the last term, we write

$$|\varphi|^{p-5}\varphi - |R_j|^{p-5}R_j = (|\varphi|^{p-5}\varphi - |R|^{p-5}R) + (|R|^{p-5}R - |R_j|^{p-5}R_j),$$

so that, since $p > 5$, we can conclude similarly that $\|\mathbf{II}\|_{L^2} \leqslant Ce^{-(e_j+4\gamma)t}$. □

Proof of Lemma 3.9. (i) For $k \in [\![1, N]\!]$, we have

$$(|R_k| + |R_{kx}|)|\phi_k - 1| \leqslant Ce^{-\sqrt{c_k}|x-v_kt|}[1 + \psi_{k+1} - \psi_k] \\ \leqslant Ce^{-\sqrt{\sigma_0}|x-v_kt|} \cdot e^{-\sqrt{\sigma_0}|x-v_kt|}[1 + \psi_{k+1} - \psi_k].$$

But, if $x < m_k(t) + \sqrt{t}$, then

$$e^{-\sqrt{\sigma_0}|x-v_kt|} \leqslant Ce^{\sqrt{\sigma_0}x}e^{-\sqrt{\sigma_0}v_kt} \leqslant Ce^{\frac{1}{2}\sqrt{\sigma_0}(v_k+v_{k-1}-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leqslant Ce^{-\frac{1}{4}\sigma_0^{3/2}t},$$

and similarly, if $x > m_{k+1}(t) - \sqrt{t}$, then

$$e^{-\sqrt{\sigma_0}|x-v_kt|} \leqslant Ce^{-\sqrt{\sigma_0}x}e^{\sqrt{\sigma_0}v_kt} \leqslant Ce^{-\frac{1}{2}\sqrt{\sigma_0}(v_{k+1}-v_k-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leqslant Ce^{-\frac{1}{4}\sigma_0^{3/2}t}.$$

As $\phi_k \equiv 1$ for $m_k(t) + \sqrt{t} \leqslant x \leqslant m_{k+1}(t) - \sqrt{t}$, the conclusion follows from (2.1).

(ii) For $l, k \in [\![1, N]\!]$ such that $l \neq k$, we have

$$(|R_k| + |R_{kx}|)\phi_l \leqslant Ce^{-\sqrt{c_k}|x-v_kt|}[\psi_l - \psi_{l+1}]\mathbb{1}_{\{x>m_l(t)-\sqrt{t}\}}\mathbb{1}_{\{x<m_{l+1}(t)+\sqrt{t}\}} \\ \leqslant Ce^{-\sqrt{\sigma_0}|x-v_kt|} \cdot e^{-\sqrt{\sigma_0}|x-v_kt|}\mathbb{1}_{\{x>m_l(t)-\sqrt{t}\}}\mathbb{1}_{\{x<m_{l+1}(t)+\sqrt{t}\}}.$$

But, if $k > l$, then

$$e^{-\sqrt{\sigma_0}|x-v_kt|}\mathbb{1}_{\{x>m_l(t)-\sqrt{t}\}}\mathbb{1}_{\{x<m_{l+1}(t)+\sqrt{t}\}} \leqslant e^{\sqrt{\sigma_0}x}e^{-\sqrt{\sigma_0}v_kt}\mathbb{1}_{\{x<m_{l+1}(t)+\sqrt{t}\}} \\ \leqslant Ce^{\frac{1}{2}\sqrt{\sigma_0}(v_{l+1}+v_l-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leqslant Ce^{-\frac{1}{4}\sigma_0^{3/2}t},$$

and similarly, if $k < l$, then

$$e^{-\sqrt{\sigma_0}|x-v_kt|}\mathbb{1}_{\{x>m_l(t)-\sqrt{t}\}}\mathbb{1}_{\{x<m_{l+1}(t)+\sqrt{t}\}} \leqslant Ce^{-\sqrt{\sigma_0}x}e^{\sqrt{\sigma_0}v_kt}\mathbb{1}_{\{x>m_l(t)-\sqrt{t}\}} \\ \leqslant Ce^{-\frac{1}{2}\sqrt{\sigma_0}(v_l+v_{l-1}-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leqslant Ce^{-\frac{1}{4}\sigma_0^{3/2}t},$$

and the conclusion follows again from the definition of γ (2.1).

(iii) For $k \in [\![1, N]\!]$, it suffices to prove $\|\psi_{kx}\|_{L^\infty} + \|\psi_{kxx}\|_{L^\infty} + \|\psi_{kt}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$. The first two inequalities are obvious since $\psi_{kx}(t,x) = \frac{1}{\sqrt{t}}\psi'\left[\frac{1}{\sqrt{t}}(x - m_k(t))\right]$ and so $\|\psi_{kx}\|_{L^\infty} \leqslant \frac{1}{\sqrt{t}}\|\psi'\|_{L^\infty}$, and similarly $\|\psi_{kxx}\|_{L^\infty} \leqslant \frac{1}{t}\|\psi''\|_{L^\infty}$. For the last one, we write

$$\psi_k(t,x) = \psi\left[\frac{x - \frac{1}{2}(x_k + x_{k-1})}{\sqrt{t}} - \frac{1}{2}(v_k + v_{k-1})\sqrt{t}\right],$$

so that

$$\psi_{kt}(t,x) = \left[-\frac{1}{2}\left(\frac{x - \frac{x_k+x_{k-1}}{2}}{t^{3/2}}\right) - \frac{1}{4}\left(\frac{v_k + v_{k-1}}{\sqrt{t}}\right)\right] \cdot \psi'\left[\frac{1}{\sqrt{t}}(x - m_k(t))\right].$$

But $\mathrm{supp}(\psi') = [-1, 1]$, and for x such that $|x - m_k(t)| \leqslant \sqrt{t}$, we have $\left|x - \frac{x_k+x_{k-1}}{2}\right| \leqslant Ct$, so finally $\|\psi_{kt}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}\|\psi'\|_{L^\infty}$.

(iv) Since $h_1 \equiv \sum_{k=1}^N \left(c_k + \frac{v_k^2}{4}\right)\phi_k$ and $h_2 \equiv \sum_{k=1}^N v_k\phi_k$ have a similar form, it is clear that it suffices to prove the inequalities for h_2, for example. Moreover, the first inequalities are obvious by (iii). Finally, for the last inequality, we write

$$\begin{aligned}|h_2 - v_k|(|R_k| + |R_{kx}|) &= \left|\sum_{l=1}^N v_l\phi_l - v_k\right|(|R_k| + |R_{kx}|)\\ &\leqslant v_k|\phi_k - 1|(|R_k| + |R_{kx}|) + \sum_{l\neq k} v_l\phi_l(|R_k| + |R_{kx}|) \leqslant Ce^{-4\gamma t}e^{-\sqrt{\sigma_0}|x-v_kt|}\end{aligned}$$

by (i) and (ii), which concludes the proof. □

Proof of Lemma 3.11. To compare $\mathcal{H}[\tilde{z}]$ and $\mathcal{H}[z]$, we replace $\tilde{z}$ in $\mathcal{H}[\tilde{z}]$ by its definition,

$$\tilde{z} = z + \sum_{k=1}^N \beta_k iQ_{c_k}(\lambda_k)e^{i\theta_k} + \sum_{k=1}^N \gamma_k\partial_x Q_{c_k}(\lambda_k)e^{i\theta_k},$$

dropping the argument λ_k for this proof, which would not be a source of confusion since there is no time derivative. Hence, we compute

$$\begin{aligned}\mathcal{H}[\tilde{z}] &= \int \partial_x\tilde{z}\cdot\overline{\partial_x\tilde{z}} - \operatorname{Im} h_2\partial_x\tilde{z}\cdot\bar{\tilde{z}} + (h_1 - |R|^{p-1})\tilde{z}\cdot\bar{\tilde{z}} - (p-1)\big(\operatorname{Re}(\overline{R}\tilde{z})\big)^2|R|^{p-3}\\ &= \int \left[\partial_x z + \sum\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} + i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{i\theta_k}\right]\\ &\qquad\times\left[\partial_x\bar{z} + \sum\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} - i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{-i\theta_k}\right]\\ &\quad - \int h_2\operatorname{Im}\left[\partial_x z + \sum\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} + i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{i\theta_k}\right]\\ &\qquad\times\left[\bar{z} + \sum(\gamma_k\partial_x Q_{c_k} - i\beta_kQ_{c_k})e^{-i\theta_k}\right]\\ &\quad + \int (h_1 - |R|)^{p-1}\left[z + \sum(\gamma_k\partial_x Q_{c_k} + i\beta_kQ_{c_k})e^{i\theta_k}\right]\\ &\qquad\times\left[\bar{z} + \sum(\gamma_k\partial_x Q_{c_k} - i\beta_kQ_{c_k})e^{-i\theta_k}\right]\\ &\quad - \int (p-1)|R|^{p-3}\left[\operatorname{Re}(\overline{R}z) - \sum\beta_k\operatorname{Im}(R_k\overline{R}) + \sum\gamma_k\operatorname{Re}(\partial_x Q_{c_k}e^{i\theta_k}\overline{R})\right]^2.\end{aligned}$$

Developing in terms of z, we find

$$\begin{aligned}\mathcal{H}[\tilde{z}] &= \int |\partial_x z|^2 + 2\operatorname{Re}\int \partial_x z\cdot\sum\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} - i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{-i\theta_k}\\ &\quad + \sum_{k,l}\int\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} + i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{i\theta_k}\\ &\qquad\times\left(\gamma_l\partial_x^2 Q_{c_l} - \frac{\beta_l}{2}v_lQ_{c_l} - i\partial_x Q_{c_l}(\beta_l + \frac{1}{2}v_l\gamma_l)\right)e^{-i\theta_l}\\ &\quad - \operatorname{Im}\int h_2\partial_x z\cdot\bar{z} - \operatorname{Im}\int h_2\partial_x z\cdot\sum(\gamma_k\partial_x Q_{c_k} - i\beta_kQ_{c_k})e^{-i\theta_k}\\ &\quad + \operatorname{Im}\int h_2 z\cdot\sum\left(\gamma_k\partial_x^2 Q_{c_k} - \frac{\beta_k}{2}v_kQ_{c_k} - i\partial_x Q_{c_k}(\beta_k + \frac{1}{2}v_k\gamma_k)\right)e^{-i\theta_k}\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k,l}\operatorname{Im}\int h_2\left(\gamma_k\partial_x^2Q_{c_k}-\frac{\beta_k}{2}v_kQ_{c_k}+i\partial_xQ_{c_k}(\beta_k+\frac{1}{2}v_k\gamma_k)\right)e^{i\theta_k}\\
&\qquad\times(\gamma_l\partial_xQ_{c_l}-i\beta_lQ_{c_l})e^{-i\theta_l}\\
&+\int(h_1-|R|^{p-1})|z|^2+2\operatorname{Re}\int(h_1-|R|^{p-1})z\cdot\sum(\gamma_k\partial_xQ_{c_k}-i\beta_kQ_{c_k})e^{-i\theta_k}\\
&+\sum_{k,l}\int(h_1-|R|^{p-1})(\gamma_k\partial_xQ_{c_k}+i\beta_kQ_{c_k})e^{i\theta_k}(\gamma_l\partial_xQ_{c_l}-i\beta_lQ_{c_l})e^{-i\theta_l}\\
&-(p-1)\int|R|^{p-3}\big(\operatorname{Re}(\overline{R}z)\big)^2-(p-1)\int|R|^{p-3}\sum_{k,l}\beta_k\beta_l\operatorname{Im}(R_k\overline{R})\operatorname{Im}(R_l\overline{R})\\
&-(p-1)\int|R|^{p-3}\sum_{k,l}\gamma_k\gamma_l\operatorname{Re}(\partial_xQ_{c_k}e^{i\theta_k}\overline{R})\operatorname{Re}(\partial_xQ_{c_l}e^{i\theta_l}\overline{R})\\
&+2(p-1)\int|R|^{p-3}\operatorname{Re}(\overline{R}z)\sum\beta_k\operatorname{Im}(R_k\overline{R})\\
&-2(p-1)\int|R|^{p-3}\operatorname{Re}(\overline{R}z)\sum\gamma_k\operatorname{Re}(\partial_xQ_{c_k}e^{i\theta_k}\overline{R})\\
&+2(p-1)\int|R|^{p-3}\sum_{k,l}\beta_k\gamma_l\operatorname{Im}(R_k\overline{R})\operatorname{Re}(\partial_xQ_{c_l}e^{i\theta_l}\overline{R}).
\end{aligned}
$$

Now, first remark that $\operatorname{Im}(R_k\overline{R})=\sum_{q\neq k}\operatorname{Im}(R_k\overline{R_q})$, and so, by (2.2), all integrals containing this term are in $O(e^{-\gamma t}\|z\|_{H^1}^2)$. Moreover, still by (2.2), all double sums on k,l have their terms in $O(e^{-\gamma t}\|z\|_{H^1}^2)$ whenever $k\neq l$. Note finally that all terms composing $\mathcal{H}[z]$ appear. Hence, with an integration by parts to make ∂_xz disappear, we have

$$
\begin{aligned}
H[\tilde{z}]&=\int|\partial_xz|^2-\operatorname{Im}h_2\partial_xz\cdot\bar{z}+(h_1-|R|^{p-1})|z|^2-(p-1)|R|^{p-3}\big(\operatorname{Re}(\overline{R}z)\big)^2\\
&-2\sum\operatorname{Re}\int ze^{-i\theta_k}\Big[\Big(\gamma_k\partial_x^3Q_{c_k}-\beta_kv_k\partial_xQ_{c_k}-\frac{1}{4}\gamma_kv_k^2\partial_xQ_{c_k}\Big)\\
&\qquad\qquad+i\Big(-v_k\gamma_k\partial_x^2Q_{c_k}-\beta_k\partial_x^2Q_{c_k}+\frac{1}{4}v_k^2\beta_kQ_{c_k}\Big)\Big]\\
&+\sum\int\Big(\gamma_k\partial_x^2Q_{c_k}-\frac{\beta_k}{2}v_kQ_{c_k}\Big)^2+\Big(\beta_k+\frac{1}{2}v_k\gamma_k\Big)^2(\partial_xQ_{c_k})^2+O(e^{-\gamma t}\|z\|_{H^1}^2)\\
&+\sum\operatorname{Im}\int z\partial_xh_2(\gamma_k\partial_xQ_{c_k}-i\beta_kQ_{c_k})e^{-i\theta_k}\\
&+2\sum\operatorname{Im}\int h_2ze^{-i\theta_k}\Big[\Big(\gamma_k\partial_x^2Q_{c_k}-\frac{\beta_k}{2}v_kQ_{c_k}\Big)-i\partial_xQ_{c_k}\Big(\beta_k+\frac{1}{2}v_k\gamma_k\Big)\Big]\\
&-\sum\int h_2\gamma_k(\beta_k+\frac{1}{2}v_k\gamma_k)(\partial_xQ_{c_k})^2+\sum\int h_2\beta_kQ_{c_k}(\gamma_k\partial_x^2Q_{c_k}-\frac{\beta_k}{2}v_kQ_{c_k})\\
&+2\sum\operatorname{Re}\int(h_1-|R|^{p-1})ze^{-i\theta_k}(\gamma_k\partial_xQ_{c_k}-i\beta_kQ_{c_k})\\
&+\sum\int(h_1-|R|^{p-1})(\gamma_k^2(\partial_xQ_{c_k})^2+\beta_k^2Q_{c_k}^2)\\
&-(p-1)\sum\int|R|^{p-3}\gamma_k^2Q_{c_k}^2(\partial_xQ_{c_k})^2-2(p-1)\sum\operatorname{Re}\int|R|^{p-3}ze^{-i\theta_k}\gamma_kQ_{c_k}^2\partial_xQ_{c_k}.
\end{aligned}
$$

We now use notation $z_{1,k}=\operatorname{Re}(z^{-i\theta_k})$ and $z_{2,k}=\operatorname{Im}(z^{-i\theta_k})$ again. Moreover, recall that we have $\|\partial_xh_2\|_{L^\infty}\leqslant\frac{C}{\sqrt{t}}$ by (iv) of Lemma 3.9, and $\partial_x^2Q_{c_k}+Q_{c_k}^p=c_kQ_{c_k}$ by (1.1).

Thus, we find

$$
\begin{aligned}
\mathcal{H}[\tilde{z}] &= \mathcal{H}[z] + O(t^{-1/2}\|z\|_{H^1}^2) \\
&\quad + \sum \int z_{1,k}[-2c_k\gamma_k\partial_x Q_{c_k} + 2p\gamma_k\partial_x Q_{c_k}Q_{c_k}^{p-1} + 2\beta_k v_k \partial_x Q_{c_k} \\
&\qquad + \frac{1}{2}\gamma_k v_k^2 \partial_x Q_{c_k} - 2h_2\beta_k\partial_x Q_{c_k} - h_2\gamma_k v_k \partial_x Q_{c_k} \\
&\qquad + 2h_1\gamma_k\partial_x Q_{c_k} - 2\gamma_k\partial_x Q_{c_k}Q_{c_k}^{p-1} - 2(p-1)\gamma_k\partial_x Q_{c_k}Q_{c_k}^{p-1}] \qquad (A.1) \\
&\quad + \sum \int z_{2,k}[-2\gamma_k v_k c_k Q_{c_k} + 2\gamma_k v_k Q_{c_k}^p - 2\beta_k c_k Q_{c_k} + 2\beta_k Q_{c_k}^p + \frac{1}{2}\beta_k v_k^2 Q_{c_k} \\
&\qquad + 2h_2\gamma_k c_k Q_{c_k} - 2h_2\gamma_k Q_{c_k}^p - h_2\beta_k v_k Q_{c_k} + 2h_1\beta_k Q_{c_k} - 2\beta_k Q_{c_k}^p] \qquad (A.2) \\
&\quad + \sum \int \left(\gamma_k c_k Q_{c_k} - \gamma_k Q_{c_k}^p - \frac{\beta_k}{2}v_k Q_{c_k}\right)^2 + \left(\beta_k + \frac{1}{2}v_k\gamma_k\right)^2 (\partial_x Q_{c_k})^2 \\
&\quad - \sum \int h_2\gamma_k(\beta_k + \frac{1}{2}v_k\gamma_k)(\partial_x Q_{c_k})^2 + \sum \int h_2\beta_k Q_{c_k}(\gamma_k c_k Q_{c_k} - \gamma_k Q_{c_k}^p - \frac{\beta_k}{2}v_k Q_{c_k}) \\
&\quad + \sum \int h_1[\gamma_k^2(\partial_x Q_{c_k})^2 + \beta_k^2 Q_{c_k}^2] - \sum \int Q_{c_k}^{p-1}[\gamma_k^2(\partial_x Q_{c_k})^2 + \beta_k^2 Q_{c_k}^2] \\
&\quad - \sum \int (p-1)\gamma_k^2 Q_{c_k}^{p-1}(\partial_x Q_{c_k})^2. \qquad (A.3)
\end{aligned}
$$

To conclude, we estimate the term (A.1) involving $z_{1,k}$, the term (A.2) involving $z_{2,k}$, and finally the source term (A.3). For (A.1), we write

$$
(A.1) = \sum \int z_{1,k}\gamma_k\partial_x Q_{c_k}(-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1) + 2\sum \int z_{1,k}\beta_k\partial_x Q_{c_k}(v_k - h_2),
$$

and $-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1 = 2(h_1 - c_k - \frac{v_k^2}{4}) + v_k(v_k - h_2)$, so that, by (iv) of Lemma 3.9, we have (A.1) $= O(e^{-\gamma t}\|z\|_{H^1}^2)$. Similarly, we write

$$
\begin{aligned}
(A.2) = 2\sum \int z_{2,k}\gamma_k c_k Q_{c_k}(h_2 - v_k) + 2\sum \int z_{2,k}\gamma_k Q_{c_k}^p(v_k - h_2) \\
+ \sum \int z_{2,k}\beta_k Q_{c_k}(-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1),
\end{aligned}
$$

and we also conclude that (A.2) $= O(e^{-\gamma t}\|z\|_{H^1}^2)$. For the last term, we expand it as

$$
\begin{aligned}
(A.3) = \sum \int & \beta_k\gamma_k c_k Q_{c_k}^2(h_2 - v_k) + \beta_k\gamma_k Q_{c_k}^{p+1}(v_k - h_2) + \beta_k\gamma_k(\partial_x Q_{c_k})^2(v_k - h_2) \\
+ \sum \int & \gamma_k^2 c_k^2 Q_{c_k}^2 + \gamma_k^2 Q_{c_k}^{2p} + \frac{\beta_k^2}{4}v_k^2 Q_{c_k}^2 - 2\gamma_k^2 c_k Q_{c_k}^{p+1} + \beta_k^2(\partial_x Q_{c_k})^2 \\
& + \frac{1}{4}\gamma_k^2 v_k^2(\partial_x Q_{c_k})^2 - \frac{1}{2}h_2\gamma_k^2 v_k(\partial_x Q_{c_k})^2 - \frac{1}{2}h_2\beta_k^2 v_k Q_{c_k}^2 \\
& + h_1\gamma_k^2(\partial_x Q_{c_k})^2 + h_1\beta_k^2 Q_{c_k}^2 - \beta_k^2 Q_{c_k}^{p+1} - p\gamma_k^2 Q_{c_k}^{p-1}(\partial_x Q_{c_k})^2.
\end{aligned}
$$

Note that the first sum is in $O(e^{-\gamma t}\|z\|_{H^1}^2)$ as above. Hence, with several integrations by parts and using $\partial_x^2 Q_{c_k} = c_k Q_{c_k} - Q_{c_k}^p$, we find

$$
\begin{aligned}
(A.3) = \sum \int & \gamma_k^2 c_k^2 Q_{c_k}^2 + \gamma_k^2 Q_{c_k}^{2p} + \frac{\beta_k^2}{4}v_k^2 Q_{c_k}^2 - 2\gamma_k^2 c_k Q_{c_k}^{p+1} - \beta_k^2 Q_{c_k}(c_k Q_{c_k} - Q_{c_k}^p) \\
& - \frac{1}{4}\gamma_k^2 v_k^2 Q_{c_k}(c_k Q_{c_k} - Q_{c_k}^p) + \frac{1}{2}h_2\gamma_k^2 v_k Q_{c_k}(c_k Q_{c_k} - Q_{c_k}^p) - \frac{1}{2}h_2\beta_k^2 v_k Q_{c_k}^2 \\
& - h_1\gamma_k^2 Q_{c_k}(c_k Q_{c_k} - Q_{c_k}^p) + h_1\beta_k^2 Q_{c_k}^2 - \beta_k^2 Q_{c_k}^{p+1} + \gamma_k^2 Q_{c_k}^p(c_k Q_{c_k} - Q_{c_k}^p) \\
+ O(e^{-\gamma t}\|z\|_{H^1}^2)
\end{aligned}
$$

$$= -\frac{1}{2}\sum\int \gamma_k^2 c_k Q_{c_k}^2(-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1)$$
$$+ \frac{1}{2}\sum\int \beta_k^2 Q_{c_k}^2(-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1)$$
$$+ \frac{1}{2}\sum\int \gamma_k^2 Q_{c_k}^{p+1}(-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1) + O(e^{-\gamma t}\|z\|_{H^1}^2),$$

and so we can conclude as above that (A.3) $= O(e^{-\gamma t}\|z\|_{H^1}^2)$. Finally, we proved that $\mathcal{H}[\tilde{z}] = \mathcal{H}[z] + O(t^{-1/2}\|z\|_{H^1}^2)$, as expected. □

B Appendix

We prove here Proposition 3.10. To do this, we first need a lemma quantifying the fact that φ almost satisfies a transport equation similar to those satisfied by the solitons. Note finally that, since φ_t takes values in H^{-1}, all integrals in this appendix may be seen as the dual bracket $\langle\cdot,\cdot\rangle_{H^1,H^{-1}}$.

Lemma B.1. *There exists $C > 0$ such that, for all $t \geqslant T_0$,*

$$\|\varphi_t + h_2\varphi_x - ih_1\varphi\|_{H^{-1}} \leqslant Ce^{-4\gamma t}.$$

Remark B.2. To find the transport equation almost satisfied by φ, it suffiees to compute an exact relation for R_k with $k \in [\![1, N]\!]$. In fact, as

$$R_k(t,x) = Q_{c_k}(x - v_k t - x_k)e^{i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)},$$

we have $R_{kt} = [-v_k\partial_x Q_{c_k} + i(c_k - \frac{1}{4}v_k^2)Q_{c_k}](\lambda_k)e^{i\theta_k}$ and $R_{kx} = [\partial_x Q_{c_k} + \frac{i}{2}v_k Q_{c_k}](\lambda_k)e^{i\theta_k}$, and so

$$R_{kt} + v_k R_{kx} - i\left(c_k + \frac{v_k^2}{4}\right)R_k = 0.$$

Proof of Lemma B.1. Let $f \in H^1$ and compute

$$\int(\varphi_t + h_2\varphi_x - ih_1\varphi)f = \int(i\varphi_{xx} + i|\varphi|^{p-1}\varphi + h_2\varphi_x - ih_1\varphi)f$$
$$= i\int(\varphi_{xx} - R_{xx})f + i\int(|\varphi|^{p-1}\varphi - |R|^{p-1}R)f + \int h_2(\varphi_x - R_x)f - i\int h_1(\varphi - R)f$$
$$+ i\int(R_{xx} + |R|^{p-1}R - ih_2R_x - h_1R)f$$
$$= -i\int(\varphi_x - R_x)f_x + i\int(|\varphi|^{p-1}\varphi - |R|^{p-1}R)f + \int h_2(\varphi_x - R_x)f - i\int h_1(\varphi - R)f$$
$$+ i\sum_{k=1}^N\int(R_{kxx} + |R_k|^{p-1}R_k - ih_2R_{kx} - h_1R_k)f + i\sum_{k=1}^N\int R_k(|R|^{p-1} - |R_k|^{p-1})f$$
$$= \mathbf{I} + \mathbf{II} + \mathbf{III}.$$

First note that, by (2.3), $|\mathbf{I}| \leqslant C\|\varphi - R\|_{H^1}\|f\|_{H^1} \leqslant Ce^{-4\gamma t}\|f\|_{H^1}$. Moreover, by (2.2), we also have $|\mathbf{III}| \leqslant Ce^{-4\gamma t}\|f\|_{L^2}$. For the last term, we first compute

$$\begin{cases} R_k = Q_{c_k}(\lambda_k)e^{i\theta_k}, \ R_{kx} = (\partial_x Q_{c_k} + \frac{i}{2}v_k Q_{c_k})(\lambda_k)e^{i\theta_k}, \\ R_{kxx} = (\partial_x^2 Q_{c_k} + iv_k\partial_x Q_{c_k} - \frac{v_k^2}{4}Q_{c_k})(\lambda_k)e^{i\theta_k}. \end{cases}$$

Then, using $\partial_x^2 Q_{c_k} = c_k Q_{c_k} - Q_{c_k}^p$, we obtain

$$\mathbf{II} = i\sum_{k=1}^{N}\int\left[\left(c_k - \frac{v_k^2}{4} - h_1\right)R_k + iv_kR_{kx} + \frac{v_k^2}{2}R_k - ih_2R_{kx}\right]f$$
$$= i\sum_{k=1}^{N}\int\left(c_k + \frac{v_k^2}{4} - h_1\right)R_kf + \sum_{k=1}^{N}\int(h_2 - v_k)R_{kx}f.$$

Therefore, by (iv) of Lemma 3.9, we also have $|\mathbf{II}| \leqslant Ce^{-4\gamma t}\|f\|_{L^2}$, which concludes the proof of Lemma B.1. □

Proof of Proposition 3.10. First recall that, from Section 3.1, the equation of z can be written

$$iz_t + z_{xx} + |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) = -\Omega,$$

where $r_j(t,x) = A_je^{-e_jt}Y_j^+(t,x)$ and Ω satisfies $\|\Omega\|_{H^1} \leqslant Ce^{-(e_j+4\gamma)t}$ by Lemma 3.2.

From the definition of H (3.13), we now compute, using integrations by parts,

$$\begin{aligned}H'(t) &= 2\operatorname{Re}\int z_{tx}\bar{z}_x - 2\operatorname{Re}\int(\varphi + r_j + z)_t|\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z})\\
&\quad + 2\operatorname{Re}\int(\varphi + r_j)_t|\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j})\\
&\quad + 2(p-1)\operatorname{Re}\int(\varphi + r_j)_t|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j})\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\\
&\quad + 2\int|\varphi + r_j|^{p-1}\operatorname{Re}[(\overline{\varphi} + \overline{r_j})_tz] + 2\int|\varphi + r_j|^{p-1}\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z_t]\\
&\quad + \int h_{1t}|z|^2 + 2\operatorname{Re}\int h_1z_t\bar{z} - \operatorname{Im}\int h_{2t}z_x\bar{z} - \operatorname{Im}\int h_2z_{tx}\bar{z} - \operatorname{Im}\int h_2z_x\bar{z}_t\\
&= -2\operatorname{Re}\int z_t\left[\bar{z}_{xx} + |\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) - |\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j})\right]\\
&\quad - 2\operatorname{Re}\int(\varphi + r_j)_t\Big[|\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z})\\
&\qquad - |\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) - (p-1)|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j})\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\Big]\\
&\quad + 2\operatorname{Re}\int h_1z_t\bar{z} + 2\operatorname{Im}\int h_2\bar{z}_xz_t + \operatorname{Im}\int h_{2x}z_t\bar{z} + \int h_{1t}|z|^2 - \operatorname{Im}\int h_{2t}z_x\bar{z}.\end{aligned}$$

But from (iv) of Lemma 3.9, we have $\|h_{1t}\|_{L^\infty} + \|h_{2t}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$, and so

$$\left|\int h_{1t}|z|^2 - \operatorname{Im}\int h_{2t}z_x\bar{z}\right| \leqslant \frac{C}{\sqrt{t}}\|z\|_{H^1}^2.$$

Moreover, by expanding $|\varphi + r_j + z|^{p-1} = [|\varphi + r_j + z|^2]^{\frac{p-1}{2}}$, and as $\|r_{jt}\|_{L^\infty} \leqslant Ce^{-e_jt}$, we have

$$\Big|-2\operatorname{Re}\int r_{jt}\Big[|\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) - |\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z})$$
$$-(p-1)|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j})\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\Big]\Big| \leqslant Ce^{-4\gamma t}\|z\|_{H^1}^2.$$

Hence, replacing z_t by its equation, we find

$$\begin{aligned}
H'(t) = & -2\operatorname{Im}\int \overline{\Omega}\left[z_{xx} + |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right] \\
& - 2\operatorname{Re}\int \varphi_t \left[|\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) \right. \\
& \qquad \left. - |\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) - (p-1)|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j}) \operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\right] \\
& - 2\operatorname{Im}\int h_1 \bar{z} z_{xx} - 2\operatorname{Im}\int h_1 \bar{z}[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)] \\
& + 2\operatorname{Re}\int h_2 \bar{z}_x z_{xx} + \operatorname{Re}\int h_{2x} \bar{z} z_{xx} \\
& - 2\operatorname{Re}\int h_2 \bar{z}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right]_x \\
& - \operatorname{Re}\int h_{2x} \bar{z}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right] \\
& - 2\operatorname{Im}\int h_1 \Omega \bar{z} + \operatorname{Re}\int (2h_2 \bar{z}_x + h_{2x}\bar{z})\Omega + O(t^{-1/2}\|z\|_{H^1}^2).
\end{aligned}$$

We can already estimate several terms in this expression. For the first term, for example, we have, by an integration by parts,

$$\left|-2\operatorname{Im}\int \overline{\Omega} z_{xx}\right| = \left|2\operatorname{Im}\int \overline{\Omega}_x z_x\right| \leqslant C\|\Omega\|_{H^1}\|z\|_{H^1} \leqslant Ce^{-(e_j+4\gamma)t}\|z\|_{H^1}.$$

Similarly, we have

$$\left|-2\operatorname{Im}\int \overline{\Omega}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right]\right| \leqslant Ce^{-(e_j+4\gamma)t}\|z\|_{H^1},$$

$$\left|-2\operatorname{Im}\int h_1\Omega\bar{z} + \operatorname{Re}\int (2h_2\bar{z}_x + h_{2x}\bar{z})\Omega\right| \leqslant Ce^{-(e_j+4\gamma)t}\|z\|_{H^1}.$$

Then, another integration by parts gives

$$-2\operatorname{Im}\int h_1 \bar{z} z_{xx} = 2\operatorname{Im}\int h_1|z_x|^2 + 2\operatorname{Im}\int h_{1x}\bar{z}z_x = 2\operatorname{Im}\int h_{1x}\bar{z}z_x,$$

and so, as $\|h_{1x}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$ by Lemma 3.9, $|-2\operatorname{Im}\int h_1\bar{z}z_{xx}| \leqslant \frac{C}{\sqrt{t}}\|z\|_{H^1}^2$. As we also have $\|h_{2x}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$, we can estimate

$$\left|-\operatorname{Re}\int h_{2x}\bar{z}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right]\right| \leqslant \frac{C}{\sqrt{t}}\|z\|_{H^1}^2.$$

Finally, we can also estimate

$$\begin{aligned}
2\operatorname{Re}\int h_2\bar{z}_x z_{xx} + \operatorname{Re}\int h_{2x}\bar{z}z_{xx} &= -\int h_{2x}|z_x|^2 - \operatorname{Re}\int z_x(h_{2xx}\bar{z} + h_{2x}\bar{z}_x) \\
&= -2\int h_{2x}|z_x|^2 - \operatorname{Re}\int h_{2xx}z_x\bar{z}.
\end{aligned}$$

Indeed, since $\|h_{2x}\|_{L^\infty} + \|h_{2xx}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$ by Lemma 3.9, we have

$$\left|2\operatorname{Re}\int h_2\bar{z}_x z_{xx} + \operatorname{Re}\int h_{2x}\bar{z}z_{xx}\right| \leqslant \frac{C}{\sqrt{t}}\|z\|_{H^1}^2.$$

Gathering all previous estimates, we have proved that

$$-\frac{1}{2}H'(t) = \mathbf{I} + \mathbf{II} + \mathbf{III} + O(e^{-(e_j+4\gamma)t}\|z\|_{H^1}) + O(t^{-1/2}\|z\|_{H^1}^2),$$

where

$$\begin{cases} \mathbf{I} = \operatorname{Re}\int h_2\bar{z}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right]_x, \\ \mathbf{II} = \operatorname{Im}\int h_1\bar{z}\left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j)\right], \\ \mathbf{III} = \operatorname{Re}\int \varphi_t\left[|\varphi + r_j + z|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z}) - |\varphi + r_j|^{p-1}(\overline{\varphi} + \overline{r_j} + \bar{z})\right. \\ \qquad\qquad \left. -(p-1)|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j})\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\right]. \end{cases}$$

The purpose is now to make appear quadratic terms in z in these expressions. For **II** and **III**, we simply write

$$\mathbf{II} = -\operatorname{Re}\int ih_1\bar{z}\left[|\varphi + r_j|^{p-1}z + (p-1)(\varphi + r_j)|\varphi + r_j|^{p-3}\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\right] + O(\|z\|_{H^1}^3)$$

and

$$\begin{aligned} \mathbf{III} = \operatorname{Re}\int \varphi_t&\left[\left(\frac{p-1}{2}\right)|z|^2|\varphi + r_j|^{p-3}(\overline{\varphi} + \overline{r_j}) + (p-1)\bar{z}|\varphi + r_j|^{p-3}\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\right. \\ &\left. + \frac{(p-1)(p-3)}{2}(\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2|\varphi + r_j|^{p-5}(\overline{\varphi} + \overline{r_j})\right] + O(\|z\|_{H^1}^3). \end{aligned}$$

For **I**, we have to compute

$$\begin{aligned} \mathbf{I} = \operatorname{Re}\int \bar{z}h_2 &\left\{(p-1)|\varphi + r_j + z|^{p-3}\operatorname{Re}[(\varphi + r_j + z)_x(\overline{\varphi} + \overline{r_j} + \bar{z})](\varphi + r_j + z)\right. \\ &+ |\varphi + r_j + z|^{p-1}(\varphi + r_j + z)_x - (p-1)|\varphi + r_j|^{p-3}\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j})](\varphi + r_j) \\ &\left. - |\varphi + r_j|^{p-1}(\varphi + r_j)_x\right\} \\ = \operatorname{Re}\int \bar{z}h_2 &\left\{(p-1)\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j})](\varphi + r_j)\left[|\varphi + r_j + z|^{p-3} - |\varphi + r_j|^{p-3}\right]\right. \\ &+ (p-1)z|\varphi + r_j + z|^{p-3}\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j}) + (\varphi + r_j)_x\bar{z} + z_x(\overline{\varphi} + \overline{r_j}) + z_x\bar{z}] \\ &+ (p-1)(\varphi + r_j)|\varphi + r_j + z|^{p-3}\operatorname{Re}[(\varphi + r_j)_x\bar{z} + z_x(\overline{\varphi} + \overline{r_j}) + z_x\bar{z}] \\ &\left. + (\varphi + r_j)_x\left[|\varphi + r_j + z|^{p-1} - |\varphi + r_j|^{p-1}\right] + |\varphi + r_j + z|^{p-1}z_x\right\} \\ = \operatorname{Re}\int \bar{z}h_2 &\left\{(p-1)(p-3)|\varphi + r_j|^{p-5}(\varphi + r_j)\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j})]\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z]\right. \\ &+ (p-1)z|\varphi + r_j|^{p-3}\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j})] \\ &+ (p-1)(\varphi + r_j)|\varphi + r_j|^{p-3}\operatorname{Re}[(\varphi + r_j)_x\bar{z} + z_x(\overline{\varphi} + \overline{r_j})] \\ &\left. + (p-1)(\varphi + r_j)_x|\varphi + r_j|^{p-3}\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z] + |\varphi + r_j|^{p-1}z_x\right\} + O(\|z\|_{H^1}^3). \end{aligned}$$

In the last expression, we integrate by parts the following two terms. First, we have

$$\begin{aligned} \operatorname{Re}\int \bar{z}h_2 &\cdot (p-1)(\varphi + r_j)|\varphi + r_j|^{p-3}\operatorname{Re}[(\varphi + r_j)_x\bar{z} + z_x(\overline{\varphi} + \overline{r_j})] \\ &= (p-1)\int \operatorname{Re}[z(\overline{\varphi} + \overline{r_j})]\operatorname{Re}[z(\overline{\varphi} + \overline{r_j})]_x h_2|\varphi + r_j|^{p-3} \\ &= -\left(\frac{p-1}{2}\right)\int (\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2 h_{2x}|\varphi + r_j|^{p-3} \\ &\quad - \frac{(p-1)(p-3)}{2}\int (\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2|\varphi + r_j|^{p-5}h_2\operatorname{Re}[(\varphi + r_j)_x(\overline{\varphi} + \overline{r_j})]. \end{aligned}$$

Second, we have similarly

$$\begin{aligned}
&\operatorname{Re}\int \bar{z}h_2 z_x|\varphi+r_j|^{p-1} \\
&= -\frac{1}{2}\int |z|^2\left[h_{2x}|\varphi+r_j|^{p-1}+h_2(p-1)|\varphi+r_j|^{p-3}\operatorname{Re}[(\varphi+r_j)_x(\overline{\varphi}+\overline{r_j})]\right] \\
&= -\frac{1}{2}\int |z|^2 h_{2x}|\varphi+r_j|^{p-1} - \left(\frac{p-1}{2}\right)\operatorname{Re}\int h_2(\varphi+r_j)_x(\overline{\varphi}+\overline{r_j})|\varphi+r_j|^{p-3}|z|^2.
\end{aligned}$$

Therefore, as $\|h_{2x}\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}}$, we have obtained

$$\begin{aligned}
-\frac{1}{2}H'(t) &= O(e^{-(e_j+4\gamma)t}\|z\|_{H^1}) + O(t^{-1/2}\|z\|_{H^1}^2) + O(\|z\|_{H^1}^3) \\
&+ \frac{(p-1)(p-3)}{2}\int (\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z])^2|\varphi+r_j|^{p-5}h_2\operatorname{Re}[(\varphi+r_j)_x(\overline{\varphi}+\overline{r_j})] \\
&+ \left(\frac{p-1}{2}\right)\int h_2|z|^2|\varphi+r_j|^{p-3}\operatorname{Re}[(\varphi+r_j)_x(\overline{\varphi}+\overline{r_j})] \\
&+ (p-1)\operatorname{Re}\int \bar{z}h_2(\varphi+r_j)_x|\varphi+r_j|^{p-3}\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z] \\
&+ \frac{(p-1)(p-3)}{2}\operatorname{Re}\int \varphi_t(\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z])^2|\varphi+r_j|^{p-5}(\overline{\varphi}+\overline{r_j}) \\
&+ \left(\frac{p-1}{2}\right)\operatorname{Re}\int \varphi_t|z|^2|\varphi+r_j|^{p-3}(\overline{\varphi}+\overline{r_j}) \\
&+ (p-1)\operatorname{Re}\int \varphi_t\bar{z}|\varphi+r_j|^{p-3}\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z] \\
&- (p-1)\operatorname{Re}\int ih_1\bar{z}(\varphi+r_j)|\varphi+r_j|^{p-3}\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z].
\end{aligned}$$

Finally, collecting similar terms in a single integral, we get, as $\|r_j\|_{H^1} \leqslant Ce^{-e_j t}$,

$$\begin{aligned}
-\frac{1}{2}H'(t) &= O(e^{-(e_j+4\gamma)t}\|z\|_{H^1}) + O(t^{-1/2}\|z\|_{H^1}^2) + O(\|z\|_{H^1}^3) \\
&+ \frac{(p-1)(p-3)}{2}\operatorname{Re}\int \overline{\varphi}|\varphi+r_j|^{p-5}(\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z])^2\left[\varphi_t+h_2\varphi_x-ih_1\varphi\right] \\
&+ \left(\frac{p-1}{2}\right)\operatorname{Re}\int |z|^2\overline{\varphi}|\varphi+r_j|^{p-3}\left[\varphi_t+h_2\varphi_x-ih_1\varphi\right] \\
&+ (p-1)\operatorname{Re}\int \bar{z}|\varphi+r_j|^{p-3}\operatorname{Re}[(\overline{\varphi}+\overline{r_j})z]\left[\varphi_t+h_2\varphi_x-ih_1\varphi\right] \\
&= O(e^{-(e_j+4\gamma)t}\|z\|_{H^1}) + O(t^{-1/2}\|z\|_{H^1}^2) + O(\|z\|_{H^1}^3),
\end{aligned}$$

since $\|\varphi_t+h_2\varphi_x-ih_1\varphi\|_{H^{-1}} \leqslant Ce^{-4\gamma t}$ by Lemma B.1 and the three terms in front of $\varphi_t+h_2\varphi_x-ih_1\varphi$ are bounded in H^1 by $\|z\|_{H^1}^2$, which concludes the proof of Proposition 3.10. □

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