MANUELA VALERIA BANICA

Equation de Schrödinger en milieu inhomogène

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Manuela Valeria Banica

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M. Thierry CAZENAVE, rapporteur
M. Patrick GÉRARD, directeur de thèse
M. Guy MÉTIVIER, rapporteur
M. Fabrice PLANCHON, examinateur
M. Jean-Claude SAUT, examinateur
M. Maciej ZWORSKI, examinateur

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Chapitre 1

Introduction

1.1 Introduction à l'équation de Schrödinger sur $\mathbb{R}^n$

1.1.1 Caractéristiques de l'équation

L'étude de l'équation de Schrödinger non-linéaire

$$\begin{cases} i\partial_t u + \Delta u = V'(|u|^2)u, \\ u(0) = u_0, \end{cases}$$

où u est une fonction en temps et en espace, à valeurs complexes, et V est une fonction réelle avec une croissance contrôlée à l'infini, a été motivée par de nombreux problèmes venant de la Physique. On la dérive essentiellement en théorie quantique des champs et en optique non-linéaire (voir la monographie récente sur le sujet [53], écrite par Sulem&Sulem).

Cette équation a un caractère conservatif, au sens où deux quantités sont constantes au long du temps : la masse

$$\int |u|^2 dx,$$

et l'énergie

$$\int |\nabla u|^2 dx + \int V(|u|^2) dx.$$

De plus, elle a des propriétés de dispersion (voir §1.1.2), d'effet régularisant, mais non-analytique, et une vitesse infinie de propagation (voir le livre de Cazenave [12]). Le problème de Cauchy associé est délicat à traiter et beaucoup de questions restent ouvertes.

1.1.2 Propriétés dispersives de l'équation linéaire

La solution de l'équation de Schrödinger homogène sur $\mathbb{R}^n$

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u(0) = u_0 \in \mathbb{L}^2(\mathbb{R}^n), \end{cases}$$

a la transformée de Fourier

$$\widehat{u}(\xi) = e^{-it|\xi|^2}\widehat{u_0}(\xi).$$

Avec la formule de Plancherel, on remarque encore une fois la conservation de la masse. La solution s'écrit, en utilisant la transformée de Fourier,

$$u(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-it|\xi|^2+ix\xi}\widehat{u_0}(\xi)d\xi,$$

et, toujours par la formule de Plancherel,

$$u(t,x) = \frac{1}{(4\pi it)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy,$$

d'où l'on déduit l'estimation

$$\|u(t)\|_\infty \leq \frac{1}{|4\pi t|^{\frac{n}{2}}}\|u_0\|_1,$$

appelée estimation de dispersion. Comme la norme $\mathbb{L}^2$ se conserve, par interpolation, on obtient que pour tout p entre 2 et l'infini

$$\|u(t)\|_p \leq \frac{1}{|4\pi t|t^{\frac{n}{2}-\frac{n}{p}}}\|u_0\|_{\overline{p}},$$

où on désigne par $\overline{p}$ l'exposant conjugué de p par la relation $\frac{1}{p}+\frac{1}{\overline{p}}=1$.

A partir de cette propriété, en utilisant l'inégalité de Hardy-Littlewood-Sobolev et des arguments de dualité ([55]), on obtient les estimations de Strichartz généralisées

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|u(t,x)|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} = \|u\|_{\mathbb{L}^p(\mathbb{R},\mathbb{L}^q(\mathbb{R}^n))} \leq C\|u_0\|_2$$

où $(p,q)\in[2,\infty]\times[2,\infty]$ définit un couple admissible pour l'équation de Schrödinger, c'est à dire

$$\frac{2}{p}+\frac{n}{q}=\frac{n}{2},$$

et $(p,q)\neq(2,\infty)$ si $n=2$. La première condition est nécessaire pour avoir l'estimation de Strichartz, car si u est solution de l'équation, alors $u_\lambda(t,x)=u(\lambda^2 t,\lambda x)$ est une famille de solutions. Pour l'équation en dimension 2, Montgomery-Smith a montré que l'estimation $\mathbb{L}^2\mathbb{L}^\infty$ n'est pas vérifiée ([42]).

Ces estimations nous donnent de l'information sur la régularité des solutions. Elles ont été introduites en 1977 par Strichartz pour des équations linéaires à coefficients constants ([52]), dans le cas

$$p=q=2+\frac{4}{n},$$

appelé exposant de Strichartz. On obtient par exemple que la solution de l'équation de Schrödinger 1-dimensionnelle appartient à $\mathbb{L}^6$ en temps et espace, ce qui améliore l'information usuelle $\mathbb{L}^\infty(\mathbb{L}^2)$. Ces inégalités ont été généralisées ensuite à des couples admissibles quelconques avec $p>2$ par Ginibre et Velo ([20]) et par Keel et Tao dans [28] pour $p=2$.

Pour l'équation inhomogène Yajima ([60]) et Cazenave et Weissler ([14]) ont obtenu le résultat suivant : si (p,q) et $(\tilde{p},\tilde{q})$ sont deux couples admissibles pour l'équation de Schrödinger et $f\in\mathbb{L}^{\overline{p}}([0,T],\mathbb{L}^{\overline{q}}(\mathbb{R}^n))$, l'unique solution de

$$\begin{cases} i\partial_t v+\Delta v=f, \\ v(0)=0, \end{cases}$$

appartient à $\mathcal{C}([0,T],\mathbb{L}^2)\cap\mathbb{L}^{\tilde{p}}([0,T],\mathbb{L}^{\tilde{q}}(\mathbb{R}^n))$ et vérifie

$$\|v\|_{\mathbb{L}^{\tilde{p}}([0,T],\mathbb{L}^{\tilde{q}}(\mathbb{R}^n))} \leq C\|f\|_{\mathbb{L}^{\overline{p}}([0,T],\mathbb{L}^{\overline{q}}(\mathbb{R}^n))}$$

avec C constante ne dépendant pas du temps T et des couples admissibles choisis.

De manière similaire, pour l'équation des ondes

$$\begin{cases} \partial_t^2 u - \Delta u = f \in \mathbb{L}^1([0,T], \mathbb{L}^2(\mathbb{R}^n)), \\ u(0) = u_0 \in \mathbb{H}^1(\mathbb{R}^n), \\ \partial_t u(0) = u_1 \in \mathbb{L}^2(\mathbb{R}^n), \end{cases}$$

en dimension plus grande ou égale à 3, on a les estimations

$$\|u\|_{\mathbb{L}^p([0,T],\mathbb{L}^q(\mathbb{R}^n))} \leq C(\|u_0\|_{\mathbb{H}^1} + \|u_1\|_{\mathbb{L}^2} + \|f\|_{\mathbb{L}^1([0,T],\mathbb{L}^2(\mathbb{R}^n))}),$$

où (p,q) forment un couple admissible pour l'équation des ondes :

$$\begin{cases} \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - 1, \\ p \geq 2, \end{cases}$$

et $(p,q) \neq (2,\infty)$ en dimension 3 ([21].

Ces inégalités, par les informations qu'elles donnent sur l'effet régularisant des équations dispersives, permettent de contrôler certaines non-linéarités afin d'obtenir des résultats d'existence pour les équations non-linéaires (voir §1.1.3).

L'étude des équations posées sur des domaines a motivé la recherche de résultats de dispersion en fonction du domaine ([48],[5]).

Récemment, on s'est intéressé aux estimations de type Strichartz pour des opérateurs à coefficients peu réguliers (voir §1.2), et sur des variétés (voir §1.3).

1.1.3 Résultats d'existence locale pour l'équation non-linéaire

Un grand nombre d'équations aux dérivées partielles non-linéaires, notamment des équations d'évolution de la Physique (Schrödinger, ondes, transport, KdV), se traitent en s'appuyant sur leur caractère dispersif. Pour une équation de type Schrödinger non-linéaire

$$\begin{cases} i\partial_t u + Au = F(u), \\ u(0) = u_0, \end{cases}$$

si l'on note $U(t) = e^{itA}$, l'existence de solutions, que l'on peut écrire sous la forme intégrale

$$u(t,x) = U(t)u_0 - i\int_0^t U(t-s)F(u(s))ds,$$

peut être démontrée par des méthodes de point fixe, à l'aide des inégalités de Strichartz.

Ainsi, on obtient que le problème de Cauchy pour l'équation de Schrödinger sur $\mathbb{R}^n$ avec non-linéarité polynomiale

$$(S_p)\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1}u = 0, \\ u(0) = u_0, \end{cases}$$

est localement bien posé dans $\mathbb{H}^1$ pour $p < 1 + \frac{4}{n-2}$. Le théorème a été prouvé, dans différents cas, par Ginibre et Velo ([19]), Cazenave ([13]), et dans un cadre plus général, par Kato ([26]). En fonction du caractère focalisant ou défocalisant de l'équation, c'est à dire du signe de la non-linéarité, et en fonction de la puissance considérée, les résultats d'existence locale peuvent être ou non globaux en temps (voir §1.1.4). Les cas sur-critiques restent ouverts.

On rappelle les propriétés conservatives énoncées dans §1.1.1 pour ce cas particulier de non-linéarité polynomiale. La masse

$$\int_{\mathbb{R}^n} |u|^2 dx,$$

et l'énergie

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx \mp \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx,$$

sont conservées au cours du temps.

1.1.4 Résultats d'existence globale pour l'équation non-linéaire

On considère les équations de Schrödinger non-linéaires sur $\mathbb{R}^n$

$$(S_p^{\pm})\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1}u = 0, \\ u(0) = u_0. \end{cases}$$

Le problème de Cauchy est localement bien posé dans $\mathbb{H}^1$ pour $p < 1 + \frac{4}{n-2}$ (voir §1.2.1.).

Dans le cas défocalisant, c'est-à-dire quand le coefficient de la non-linéarité est négatif, la conservation en temps de l'énergie

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx$$

entraîne le contrôle de la norme $\mathbb{H}^1$, donc on a l'existence globale des solutions de (S_p^-), pour tout $p < 1 + \frac{4}{n-2}$.

L'analyse de l'équation focalisante (S_p^+), c'est-à-dire quand le coefficient de la non-linéarité est positif, est plus complexe.

L'inégalité de Gagliardo-Nirenberg

$$\|v\|_{p+1}^{p+1} \leq C_{p+1}\|v\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla v\|_2^{(p-1)\frac{n}{2}}$$

implique que l'énergie de la solution u de l'équation (S_p^+),

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx,$$

est minorée par

$$\|\nabla u\|_2^2\left(\frac{1}{2} - \frac{C_{p+1}}{p+1}\|u\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla u\|_2^{(p-1)\frac{n}{2}-2}\right).$$

En conséquence, si $p < 1+\frac{4}{n}$, compte tenu aussi de la conservation de la masse, le gradient est contrôlé par l'énergie, donc la solution n'explose pas et on a bien l'existence globale.

La puissance $p = 1 + \frac{4}{n}$ est une puissance critique, c'est-à dire que la non-linéarité est assez puissante pour générer des solutions explosant en temps fini. Mais, même dans ce cas, on a un résultat d'existence globale pour les données initiales de masse petite.

Ainsi, dans le cas $p = 1 + \frac{4}{n}$, si la masse est assez petite pour que

$$\frac{C_{2+\frac{4}{n}}}{2+\frac{4}{n}}\|u\|_2^{\frac{4}{n}} < \frac{1}{2},$$

l'énergie qui se conserve contrôle le gradient et on a de nouveau l'existence globale des solutions de (S_p^+).

Pour cette valeur de p, Weinstein a donné une version précisée de l'inégalité de Gagliardo-Nirenberg ([57]). Par des méthode variationnelles utilisant le lemme de concentration-compacité de Lions ([35], [36]), on obtient l'existence d'un minimiseur Q de la constante optimale de l'inégalité de Gagliardo-Nirenberg

$$\frac{1}{C_{2+\frac{4}{n}}} = \inf_{v\in\mathbb{H}^1(\mathbb{R}^n)} \frac{\|v\|_2^{\frac{4}{n}}\|\nabla v\|_2^2}{\|v\|_{2+\frac{4}{n}}^{2+\frac{4}{n}}}.$$

Ce minimiseur vérifie l'équation

$$\Delta Q + Q^{1+\frac{4}{n}} = \frac{2}{n}Q.$$

Une telle fonction positive, appelée état fondamental de l'équation de Schrödinger non-linéaire, est radiale, exponentiellement décroissante à l'infini et régulière. Récemment, Kwong a montré qu'elle est unique à translation près ([33]). De plus, elle vérifie les identités de Pohozaev

$$\begin{cases} \|\nabla Q\|_2^2 - \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + \frac{2}{n}\|Q\|_2^2 = 0, \\ (n-2)\|\nabla Q\|_2^2 - \frac{n^2}{n+2}\|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + 2\|Q\|_2^2 = 0, \end{cases}$$

qui entraînent les relations suivantes entre les normes de Q,

$$\begin{cases} \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} = \frac{n+2}{n}\|Q\|_2^2, \\ \|\nabla Q\|_2^2 = \|Q\|_2^2. \end{cases}$$

La valeur optimale de la constante de Gagliardo-Nirenberg est alors

$$C_{2+\frac{4}{n}} = \frac{n+2}{n}\frac{1}{\|Q\|_2^{\frac{4}{n}}}.$$

En conclusion, quand $p = 1 + \frac{4}{n}$, on a l'existence globale des solutions de (S_p^+) avec des données initiales de masse plus petite que celle de l'état fondamental

$$\|u\|_2 < \|Q\|_2.$$

La masse $\|Q\|_2$ est une masse critique, c'est-à dire qu'il existe des solutions de masse égale à celle de l'état fondamental, et qui explosent en temps fini (voir §1.1.5).

Finalement, remarquons que à partir de l'état fondamental, on peut construire des solutions stationnaires globales sur $\mathbb{R}^n$, de la forme

$$e^{it}Q(x).$$

1.1.5 Solutions explosant en temps fini

Dans le suite on va considérer la puissance critique $p = 1 + \frac{4}{n}$. Dans ce cas, la transformation pseudo-conforme d'une solution u de l'équation (S_p),

$$\frac{1}{t^{\frac{n}{2}}} e^{i\frac{|x|^2}{4t}} u\left(-\frac{1}{t}, \frac{x}{t}\right),$$

est aussi une solution de (S_p). Donc, à partir des solutions stationnaires mentionnées précédemment, on peut construire pour tout T positif des solutions explicites explosant au temps T,

$$u(t,x) = \frac{e^{\frac{i}{T-t}}}{(T-t)^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4(T-t)}} Q\left(\frac{x}{T-t}\right).$$

De plus, Merle a prouvé dans [38] que toutes les solutions explosives sur $\mathbb{R}^n$, de masse critique $\|Q\|_2$ sont de ce type, modulo les invariants de l'équation. Sa preuve est basée sur un résultat de concentration de Weinstein ([58]) et sur l'étude du moment de premier ordre

$$f(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 x dx,$$

et du viriel

$$g(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 |x|^2 dx$$

associés à une solution u de l'équation (S_p). On utilise les propriétés conservatives de ces deux quantités sur $\mathbb{R}^n$ dans le cas de la puissance critique $1 + \frac{4}{n}$. Le moment de premier ordre a la dérivée constante en temps

$$\partial_t^2 f = 0,$$

et g vérifie l'identité du viriel ([12])

$$\partial_t^2 g = 16E(u) - 4\frac{n(p-1)-4}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx = 16E(u).$$

Dans certains cas de masse sur-critique, des études récentes ont été faites par Merle et Raphaël sur la vitesse et le profile d'explosion ([39], [40]).

Pour l'équation (S_p) avec une non-linéarité sur-critique, Zakharov [61] et Glassey [22] ont prouvé le caractère explosif en temps fini des solutions à énergie négative. Le même résultat pour les solutions d'énergie positive ou nulle est vrai sous certaines conditions sur les dérivées en temps du viriel ([53]). La preuve est basée sur une majoration du viriel en fonction de ses dérivées, qui implique l'annulation du viriel à un temps fini T. Comme la masse reste constante, la solution doit exploser au temps T.

1.2 L'équation en milieu inhomogène

Les résultats concernant les inégalités de Strichartz pour des équations d'évolution en milieu inhomogène sont très récents. En 1990 Kapitanski a montré que l'équation des ondes à coefficients $\mathcal{C}^\infty$ vérifie ces inégalités ([23]). On s'est beaucoup intéressé aux estimations de type Strichartz que l'on peut avoir pour des opérateurs d'ondes à coefficients peu réguliers ([3],[31],[54]). Smith ([46]) a montré qu'on peut même affaiblir leur régularité jusqu'à $\mathcal{C}^{1,1}$. Si les coefficients sont dans l'espace de Hölder $\mathcal{C}^{1,\alpha}$, avec $0 < \alpha < 1$, Smith et Sogge ont donné un contre-exemple en dimension plus grande que 3 et des données initiales dans $\mathbb{H}^{\frac{1}{2}} \times \mathbb{H}^{-\frac{1}{2}}$ ([47]). Dans [4], on montre qu'on peut construire, en dimension plus grande que 2, de la même manière, un contre-exemple pour les ondes à données initiales dans $\mathbb{H}^s \times \mathbb{H}^{s-1}$ avec $\frac{1}{2} \leq s < 1$, et deux autres contre-exemples pour l'équation de Schrödinger sous forme non-conservative et respectivement conservative, les deux à coefficients $\mathcal{C}^{1,\alpha}$.

L'équation de Schrödinger pose plus de difficultés que l'équation des ondes, à cause de l'absence de la propriété de vitesse finie de propagation. Une façon de "remplacer" cette propriété est d'imposer une condition de non-capture des trajectoires. On dispose de plusieurs résultats d'existence et d'effet régularisant pour des coefficients réguliers, constants à l'infini et satisfaisant une condition de non-capture ([16],[17],[24],[25],[29]). Staffilani et Tataru ([51]) ont prouvé les inégalités de Strichartz pour des métriques perturbation à support compact de la métrique euclidienne, satisfaisant une condition de non-capture, mais avec des coefficients de régularité plus faible, seulement $\mathcal{C}^2$. En fait, pour avoir le problème de Cauchy bien posé pour l'équation de Schrödinger non-linéaire, la condition de non-capture n'est pas nécessaire. Dans un article récent, Burq, Gérard et Tzvetkov ([7]) ont montré les estimations de Strichartz avec perte fractionnaire de derivées pour toute métrique régulière sur $\mathbb{R}^d$ avec estimations uniformes à l'infini, sans condition géométrique. Ces nouvelles estimations entraînent des résultats positifs pour le problème de Cauchy local et global (voir §1.4.1.).

Dans le première partie de la thèse (§2) on étudie l'existence des inégalités de dispersion et de Strichartz pour l'équation de Schrödinger sur $\mathbb{R}$

$$(S_a) \begin{cases} i\,\partial_t u + \partial_x a(x) \partial_x u = 0, \\ u(0) = u_0 \in \mathbb{L}^2(\mathbb{R}), \end{cases}$$

pour certains coefficients $a(x)$ peu réguliers, sans condition géométrique de non-capture.

On montre la dispersion locale dans le cas des coefficients positifs laminaires, c'est-à-dire fonctions en escalier avec un nombre fini de singularités. On remarque dans cette situation l'existence de trajectoires captées.

Théorème 1. *(cf Th.* 2.1.1*) On considère la partition de la droite réelle*

$$-\infty = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \infty,$$

et une fonction en escalier

$$a(x) = b_i^{-2} \text{ pour } x \in (x_{i-1}, x_i),$$

où b_i sont des nombres strictement positifs.

La solution de l'équation de Schrödinger (S_a) satisfait à l'inégalité de dispersion

$$\|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C_n}{\sqrt{t}}\|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

et les inégalités de Strichartz

$$\|u\|_{\mathbb{L}^p(\mathbb{R},\mathbb{L}^q(\mathbb{R}))} \leq C_n\|u_0\|_{\mathbb{L}^2(\mathbb{R})},$$

pour tout couple (p,q) vérifiant

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{2}.$$

La preuve se base sur la représentation de la solution en utilisant la résolvante de l'opérateur $-\partial_x a(x)\partial_x$. La résolvante est calculée explicitement et exprimée en termes de séries d'exponentielles. Pour obtenir la dispersion globale, on étudie ces séries dans le cadre de la théorie des fonctions presque-périodiques de Wiener.

On obtiendrait un résultat similaire pour l'opérateur

$$i\,\partial_t + \frac{1}{\rho(x)}\partial_x a(x)\partial_x,$$

où $\rho(x)$ est une fonction en escalier du même type que $a(x)$.

De plus, si $v(t,x)$ est la solution du système d'ondes associé sur $\mathbb{R}$

$$\begin{cases} \partial_t^2 v - \partial_x a(x)\partial_x v = 0, \\ v(0) = u_0 \in \mathbb{L}^2(\mathbb{R}), \\ \partial_t v(0) = 0, \end{cases}$$

la même méthode nous donne l'estimation

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |v(t,x)|dt \leq C_n\|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

La dispersion n'est pas vérifiée si le coefficient est un fonction en escalier périodique. En utilisant le modèle de Krönig-Penney, classique en mécanique quantique ([45]), on prouve l'absence de la dispersion locale dans le cas des coefficients périodiques fonctions en escalier, prenant juste deux valeurs.

Théorème 2. *(cf Th. 2.1.2) Soit $x_0 \in (0,1)$ et b_0 et b_1 des nombres positifs satisfaisant la condition $b_0x_0 = b_1(1-x_0)$. On considère la fonction 1-périodique*

$$a(x) = \begin{cases} b_0^{-2} \text{ pour } x \in [0,x_o), \\ b_1^{-2} \text{ pour } x \in [x_o,1). \end{cases}$$

L'inégalité de dispersion locale n'est pas vérifiée pour l'équation de Schrödinger (S_a).

La démonstration est basée sur la représentation de la solution par la décomposition de Floquet.

Le fait que la fonction a ne soit pas très oscillante à l'infini semble donc essentiel pour avoir la dispersion. En utilisant la méthode de Avellaneda, Bardos et Rauch de [2], on obtient des contre-exemples pour la dispersion et pour les inégalités de Strichartz globales dans le cas de certains coefficients continus oscillants à l'infini.

De plus, comme Castro et Zuazua l'ont montré récemment dans [11], même si les coefficients sont constants à l'infini, mais peu réguliers ($C^{0,\alpha}$) et localement très oscillants, les inégalités de Strichartz ne sont pas vérifiées.

Tous ces résultats suggèrent la conjecture que l'équation de Schrödinger en dimension 1, à coefficients BV compris entre deux nombres strictement positifs, vérifie les inégalités de dispersion.

1.3 L'équation posée sur des variétés compactes

1.3.1 Résultats d'existence locale pour l'équation non-linéaire

On considère l'équation de Schrödinger non-linéaire

$$\begin{cases} i\partial_t u + \Delta u = F(u), \\ u(0) = u_0 \in \mathbb{L}^2(M), \end{cases}$$

posée sur une variété riemannienne compacte (M, g). Ici Δ est l'opérateur de Laplace-Beltrami associé à la métrique g. Récemment, on a constaté que la géométrie de la variété influence la dynamique de l'équation.

L'équation ne garde pas exactement les mêmes propriétés dispersives qu'elle a sur $\mathbb{R}^n$, mais de nouvelles estimations de type Strichartz sont valables. Dans [7], Burq, Gérard et Tzvetkov montrent que la solution de l'équation homogène vérifie

$$\|u\|_{\mathbb{L}^p(I,\mathbb{L}^q(M))} \leq C\|u_0\|_{\mathbb{H}^{\frac{1}{p}}(M)},$$

où (p, q) est un couple admissible dans le sens défini en §1.1.2, et I est un intervalle de temps fini. De plus, ces estimations sont optimales dans le cas de la sphère pour $p = 2$. La perte fractionnaire de dérivées n'empêche pas l'obtention de résultats d'existence locale pour l'équation avec une non-linéarité polynomiale, à données initiales peu régulières. De plus, sur des surfaces dans le cas des non-linéarités polynomiales défocalisantes, et sur des variétés de dimension 3 dans le cas des non-linéarités cubiques défocalisantes, on montre l'existence globale dans l'espace d'énergie $\mathbb{H}^1$.

1.3.2 Phénomènes d'instabilité

Des phénomènes d'instabilité apparaissent pour l'équation posée sur une variété riemannienne compacte, même dans le cas défocalisant.

D'une part, Burq, Gérard et Tzvetkov ont montré dans [8] que le flot de l'équation de Schrödinger cubique défocalisante sur $\mathbb{S}^2$

$$\begin{cases} i\partial_t u + \Delta_{S^2} u = |u|^2 u, \\ u(0) = u_0 \in \mathbb{H}^s(\mathbb{S}^2), \end{cases}$$

n'est pas uniformément continu sur les bornés de $\mathbb{H}^s$ pour $s \in [0, \frac{1}{4}[$, c'est à dire pour des indices de régularité de Sobolev plus grands que zéro, qui est l'indice de scaling. Des résultats similaires sont valables sur des domaines de $\mathbb{R}^2$ ([9]). Pour $\mathbb{S}^2$, les mêmes auteurs ont prouvé récemment que l'indice $\frac{1}{4}$ est le niveau de régularité critique, c'est à dire que pour $s > \frac{1}{4}$ le problème de Cauchy est bien posé ([10]).

D'autre part, Cazenave et Weissler [14], et Bourgain [5], ont établi le caractère bien posé du problème de Cauchy dans $\mathbb{H}^s$ sur $\mathbb{R}^2$, et sur $\mathbb{T}^2$ respectivement, pour tout s strictement positif. De plus, pour $\mathbb{T}^2$, le flot n'est pas uniformément continu pour s négatif ([8]), donc 0 est l'indice de régularité critique.

On conclut donc que la géométrie de la variété joue un rôle important dans la dynamique de l'équation. On peut penser que la courbure positive est à l'origine des phénomènes d'instabilité.

Remarquons que ces résultats ne sont pas en contradiction avec ceux positifs concernant l'équation des ondes, étant donné que l'équation Schrödinger ne possède pas la propriété de vitesse finie de propagation.

Les phénomènes d'instabilité apparaissent pour une large classe d'équations dispersives posées sur $\mathbb{R}^n$. Dans l'article récent [30], Kenig, Ponce et Vega ont étudié les propriétés des équations non-linéaires focalisantes de Schrödinger et Korteweg-de Vries, à données initiales peu régulières. Ensuite, dans [15], Christ, Colliander et Tao ont étendu cette étude aux analogues défocalisants de ces équations. Des phénomènes d'instabilité apparaissent aussi pour l'équation des ondes défocalisante sur $\mathbb{R}^3$, avec une non-linéarité sur-critique, comme Lebeau l'a prouvé dans [34] (voir aussi [41]). Koch et Tzvetkov ont récemment montré dans [32] que le flot de l'équation de Benjamin-Ono n'est pas uniformément continu dans $\mathbb{H}^s$ pour les s positifs. Tous ces résultats sont obtenus en construisant des familles de solutions exactes avec des estimations qui contredisent le caractère bien posé du problème de Cauchy.

Pour obtenir dans [8] le résultat d'instabilité sur $\mathbb{S}^2$, l'évolution de certaines harmoniques sphériques, concentrées sur des géodésiques, a été étudiée comme suit.

Soit ψ_n l'harmonique sphérique de poids principal, normalisée dans $\mathbb{H}^s$, obtenue par restriction à la sphère du polynôme harmonique sur $\mathbb{R}^3$

$$\psi_n(x_1, x_2, x_3) = n^{\frac{1}{4}-s}(x_1 + ix_2)^n.$$

On constate que, lorsque n tend vers l'infini, ψ_n se concentre sur le grand cercle $x_1^2 + x_2^2 = 1$. Les normes $\mathbb{L}^p$ de ψ s'estiment par la méthode de Laplace

$$\begin{cases} \|\psi_n\|_\infty \approx n^{\frac{1}{4}-s}, \\ \|\psi_n\|_2 \approx n^{-s}, \\ \|\psi_n\|_4^4 \approx n^{\frac{1}{2}-4s}, \\ \|\psi_n\|_6^3 \approx n^{\frac{1}{2}-3s}. \end{cases}$$

Les équivalents sont considérés quand n tend vers l'infini :

$$f_n \approx g_n \iff \exists\, c, C \in \mathbb{R}^+, c\, g_n \leq f_n \leq C g_n.$$

$$f_n \lesssim g_n \iff \exists\, C \in \mathbb{R}^+, f_n \leq C g_n.$$

On considère maintenant l'équation de Schrödinger

$$(S_{\mathbb{S}}) \begin{cases} i\partial_t u + \Delta_{\mathbb{S}^2} u = |u|^2 u, \\ u_n(0) = \kappa_n \psi_n(x), \end{cases}$$

où κ_n est un nombre entre $\frac{1}{2}$ et 1. Pour tout réel α, la rotation R_α définie sur $\mathbb{R}^3$ par

$$R_\alpha(x_1, x_2, x_3) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha, x_3)$$

vérifie la relation

$$\psi(R_\alpha(x)) = e^{in\alpha}\psi(x).$$

Alors, grâce à l'unicité de la solution pour le même problème de Cauchy $(S_{\mathbb{S}})$ avec donnée initiale $e^{in\alpha}\psi(x)$, on obtient l'identité

$$u(t, R_\alpha(x)) = e^{in\alpha} u(t, x).$$

En utilisant ce fait et un lemme algébrique sur les harmoniques sphériques, dans [8] on montre que la solution u peut être décomposée sur ψ_n et sur $\{h_{n+j}\}_{j\geq 1}$, les harmoniques sphériques d'ordre $n+j$ vérifiant

$$h_{n+j}(R_\alpha(x)) = e^{in\alpha} h_{n+j}(x).$$

On va considérer ces harmoniques sphériques normalisées dans $\mathbb{L}^2$.

Soit $\omega_n \psi_n$ la projection orthonormale de $|\psi_n|^2\psi_n$ sur l'espace engendré par ψ_n, et r_n le reste de la projection

$$|\psi_n|^2\psi_n = \omega_n \psi_n + r_n.$$

Par les mêmes arguments que précédemment, r_n s'exprime seulement en fonction des h_{n+j}. On écrit la solution de (S) sous la forme

$$u_n(t, x) = \kappa_n e^{-it(n(n+1)+\kappa_n^2\omega_n)}((1 + \widetilde{z_n}(t))\psi_n(x) + q_n(t, x)),$$

avec q_n combinaison des h_{n+j}.

Pour $s \in]\frac{3}{20}, \frac{1}{4}[$, il est établi dans [8] que la norme $\mathbb{H}^s$ de $q_n(t)$ est négligeable par rapport à celle de ψ_n, et que $|\widetilde{z_n}(t)|$ tend vers 0 quand n tend vers l'infini. Ces résultats impliquent que la solution se comporte comme la donnée initiale ψ_n avec un coefficient oscillant de type exponentiel. Sachant que ω_n tend vers l'infini, un bon choix de κ_n entraîne un important déphasage entre les solutions u_n et donc le problème de Cauchy pour l'équation $(S_{\mathbb{S}})$ est mal posé dans $\mathbb{H}^s(\mathbb{S}^2)$, au sens où le flot n'est pas uniformément continu sur les bornés de $\mathbb{H}^s$.

Le but de la deuxième partie de la thèse (§3) est de faire une analyse plus poussée de ces solutions. On donne des équivalents pour $|\widetilde{z_n}|$ et pour $\|q(t)\|_{H^s}$. En particulier ces résultats montrent que même dans le reste $\widetilde{z_n}\psi_n + q_n$, la dynamique orthogonale à ψ_n est faible. On obtient aussi un ansatz plus précis pour ces solutions par rapport aux harmoniques sphériques h_{n+j}.

On va ignorer dans la suite les indices n des fonctions introduites précédemment. On définit

$$\alpha_j = 2nj + j^2 + j - \kappa^2\omega \ , \ k_{j,l} = \kappa^2 < \overline{h_{n+l}}\,\psi^2, h_{n+j} >,$$

$$\mu_j = \sqrt{(3k_{j,j} + \alpha_j)(k_{j,j} + \alpha_j)}.$$

On considère l'opérateur

$$A = -\Delta - n(n+1)$$

et l'opérateur M défini sur l'espace engendré par les h_{n+j} comme suit

$$M(h_{n+j}) = \mu_j h_{n+j}.$$

Théorème 3. *(cf Th. 3.1.1) Soit $T > 0$. Pour tout $s \in [0, \frac{1}{4}[$ et $t \in [0, T]$, la solution de $(S_{\mathbb{S}})$ est*

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)}(z(t)\psi(x) + q(t,x)),$$

avec les estimation précises

$$\begin{cases} \sup_{0<t<T} \|q(t)\|_{H^{\frac{1}{2}}} \approx n^{-3s}, \\ |z(t) - 1| \approx t\, n^{-4s}. \end{cases}$$

(dans le deuxième équivalent, t est present pour inclure le cas $t = 0$ quand $z(0) = 1$)
De plus,

i) Pour $s \in [0, \frac{1}{4}[$ le coefficient de ψ est

$$z(t) = e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<A^{-1}r,r>} + O(n^{-\frac{1}{2}-6s}).$$

ii) Pour $s \in]\frac{1}{12}, \frac{1}{4}[$ on a l'ansatz plus précis

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)}\left(e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<A^{-1}r,r>}\psi(x) - i\int_0^t e^{isM} r ds\right) + \widetilde{u}(t,x),$$

avec

$$\|\widetilde{u}(t)\|_{H^s} \ll n^{-\frac{1}{2}-2s}.$$

La preuve est d'abord basée sur une exploitation des lois de conservation plus précise que dans [8]. On obtient ainsi des majorations pour la norme $\mathbb{L}^2$ de q, et pour $|z-1|$. De plus, en utilisant les estimées de Sogge sur les harmoniques sphériques ([49], [50])

$$\|h_m\|_p \leq C m^{\frac{1}{4}-\frac{1}{2p}} \text{ pour } 2 \leq p \leq 6,$$

on déduit des majorations pour les normes $\mathbb{L}^p$ de q meilleures que celles qu'on a par interpolation entre $\mathbb{L}^2$ et $\mathbb{H}^1$. Ces majorations nous permettent aussi d'avoir des informations sur les coefficients des harmoniques sphériques h_{n+l} dans la solution u. En utilisant dans l'étude des équations de z et de q toutes ces estimations et le calcul des équivalents de certains produit scalaires d'harmoniques sphériques, on obtient la description de z. Il s'ensuit alors que les majorations trouvées auparavant sont optimales. Enfin, l'ansatz précisé dans ii) s'obtient en projetant l'équation $(S_{\mathbb{S}})$ sur chaque mode, et en analysant le système obtenu en tirant parti de la distance importante entre deux valeurs propres consécutives du laplacien.

Remarques:

i) On va montrer que

$$\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r > \approx n^{-4s},$$

donc l'oscillation de la solution est plus forte quand s decroît vers zéro, c'est-à dire quand l'amplitude de la donnée initiale croît plus rapidement.

ii) La partie linéaire qui vient de la non-linéarité cubique a une contribution essentielle dans l'ansatz de la solution. Du fait de cette non-linéarité, l'opérateur M est défini en termes de μ_j au lieu de α_j, et la dynamique orthogonale à ψ

$$-i\int_0^t e^{isM} r ds$$

vérifie une équation dépendant de M

$$\begin{cases} i\partial_t v + Mv + ir = 0, \\ v(0) = 0. \end{cases}$$

iii) Dans le cas $s = \frac{1}{4}$ il n'est pas connu si le flot est ou non uniformément continu.

1.4 L'équation posée sur un domaine de $\mathbb{R}^n$

On considère l'équation de Schrödinger non-linéaire posée sur un domaine régulier Ω de $\mathbb{R}^n$, avec condition de Dirichlet

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \\ u_{|\mathbb{R}\times\partial\Omega} = 0, \\ u(0) = u_0. \end{cases}$$

Remarquons d'abord que les conservations de la masse et de l'énergie des solutions restent vraies. Le problème de Cauchy est localement bien posé dans $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ en dimension 2 et 3. En dimension 2, pour des non-linéarités au plus cubiques, Vladimirov [56] et Ogawa et Ozawa [43] ont montré le caractère bien posé du problème de Cauchy dans $\mathbb{H}_0^1(\Omega)$, mais sans condition d'uniforme continuité du flot sur les bornés de $\mathbb{H}_0^1(\Omega)$. Pour des non-linéarités plus que cubiques en dimension 2, ou pour n'importe quelle puissance de p, en dimension supérieure à 2, le problème de Cauchy posé dans $\mathbb{H}_0^1(\Omega)$ reste ouvert.

Pour l'équation avec puissance $p < 1 + \frac{4}{n}$, on montre comme sur $\mathbb{R}^n$ l'existence globale des solutions $\mathbb{H}_0^1(\Omega)$ (voir §1.1.4). Pour l'équation avec puissance $p \geq 1 + \frac{4}{n}$, posée sur un domaine étoilé de $\mathbb{R}^n$, Kavian a prouvé l'explosion en temps fini des solutions $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ à énergie négative ou à énergie positive mais avec certaines conditions sur les dérivées du viriel ([27]). Sa preuve reprend celle sur $\mathbb{R}^n$ (voir §1.1.5), en estimant à l'aide de la condition géométrique sur Ω les termes de bord qui apparaissent dans la dérivée seconde du viriel.

À partir de maintenant on considère l'équation cubique focalisante sur Ω

$$(S_\Omega) \quad \begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, \\ u_{|\mathbb{R}\times\partial\Omega} = 0, \\ u(0) = u_0. \end{cases}$$

Le problème de Cauchy est donc localement bien posé dans $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ et aussi dans $\mathbb{H}_0^1(\Omega)$, mais sans condition d'uniforme continuité du flot. Les inégalités de Strichartz usuelles ne sont pas vérifiées, et la perte de dérivées est encore plus forte que sur une variété compacte ([9]).

Comme dans le cas du plan entier, pour des données initiales de masse

$$\|u_0\|_2 < \|Q\|_2,$$

le problème de Cauchy est globalement bien posé dans $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$. La preuve, due à Brézis et Gallouët utilise des estimations de type logarithmique ([6]). Ce résultat a été étendu, modulo l'uniforme continuité du flot, au cas de l'espace naturel $\mathbb{H}_0^1(\Omega)$ ([56],[43],[12]).

La masse critique pour l'explosion est $\|Q\|_2$, comme pour l'équation posée sur $\mathbb{R}^2$ tout entier. Plus précisément, on a le résultat suivant.

Théorème. (Burq-Gérard-Tzvetkov [9]) *Soit Ω un domaine régulier borné de $\mathbb{R}^2$. Soit $x_0 \in \Omega$ et $\psi \in C_0^\infty$ une fonction égale à 1 autour de x_0. Alors il existe κ et α_0 nombres positifs tels que pour tout $\alpha > \alpha_0$, il existe un temps T_α et une fonction r_α définie sur $[0, T_\alpha[\times\Omega$ vérifiant*

$$\|r_\alpha(t)\|_{\mathbb{H}^2(\Omega)} \leq ce^{-\frac{\kappa}{T_\alpha - t}},$$

tel que

$$u(t,x) = \psi(x)\frac{e^{\frac{i}{\alpha^2(T_\alpha - t)}}}{\alpha(T_\alpha - t)} e^{-i\frac{|x-x_0|^2}{4\alpha(T_\alpha - t)}} Q\left(\frac{x - x_0}{\alpha(T_\alpha - t)}\right) + r_\alpha(t,x),$$

soit solution de (S_Ω) de masse critique, explosant en x_0 au temps T_α avec une vitesse $\frac{1}{T_\alpha - t}$.

La preuve, suivant une idée de Ogawa et Tsutsumi ([44]), est basée sur une méthode de point fixe qui permet de compléter la localisation en x_0 de la solution explicite de l'équation sur $\mathbb{R}^2$ explosant en x_0, à une solution explosive de l'équation sur Ω. Le théorème implique donc l'existence des solutions explosives en tout point de Ω. De plus, la preuve est valable sur le tore $\mathbb{T}^2$ et sur une plus grande classe d'ensembles du plan, qui vérifient la propriété de 2-prolongement, de $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ à $\mathbb{H}^2(\mathbb{R}^2)$, et pour lesquels le domaine du Laplacien

$$D(-\Delta_\Omega) = \{u \in \mathbb{H}_0^1(\Omega), \Delta u \in \mathbb{L}^2(\Omega)\},$$

est $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$. De tels ensembles sont par exemple les domaines avec frontière compacte lisse et les polygones convexes bornés ou non-bornés.

Comme dans le cas de $\mathbb{R}^n$, le lemme suivant, dû à Weinstein, nous donne le comportement des solutions explosives de masse critique sur un domaine.

Lemme. (Weinstein [58]) *Soit $u_k \in \mathbb{H}^1(\mathbb{R}^n)$ une suite de fonctions de masse critique vérifiant*

$$\begin{cases} \beta_k = \|\nabla u_k\|_2 \underset{k\to\infty}{\longrightarrow} \infty, \\ E(u_k) \underset{k\to\infty}{\longrightarrow} c < \infty. \end{cases}$$

Alors il existe des points $x_k \in \mathbb{R}^d$ et $\theta_k \in \mathbb{R}$ tels que dans $\mathbb{H}^1(\mathbb{R}^n)$

$$\frac{e^{i\theta_k}}{\beta_k^{\frac{n}{2}}} u_k\left(\frac{x}{\beta_k} + x_n\right) \underset{k\to\infty}{\longrightarrow} \frac{1}{\omega^{\frac{n}{2}}} Q(\omega x),$$

où $\omega = \|\nabla Q\|_2$.

Soit u une solution de (S_Ω) qui explose au temps fini T, c'est-à-dire

$$\lambda(t) = \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \underset{t\to T}{\longrightarrow} \infty.$$

On va considérer u étant prolongée par zero en dehors de Ω. En combinant le lemme précédent, appliqué à des familles $u_k = u(t_k)$, pour des suites t_k qui convergent vers T, avec le résultat de Kwong sur l'unicité de l'état fondamental ([33]), on obtient l'existence de nombres réels $\theta(t)$ et des points $x(t) \in \mathbb{R}^2$ tels que dans $\mathbb{H}^1(\mathbb{R}^2)$

$$\frac{e^{i\theta(t)}}{\lambda(t)} u\left(t, \frac{x}{\lambda(t)} + x(t)\right) \underset{t\to T}{\longrightarrow} Q(x), \tag{1.1}$$

où u est prolongée par zéro en dehors de Ω. Alors, dans l'espace des distributions,

$$|u(t, \cdot + x(t))|^2 \underset{t\to T}{\longrightarrow} \|Q\|_2^2\, \delta_0.$$

Dans la troisième partie de la thèse (§4) on fait une analyse des solutions explosives sur Ω, de masse critique. On obtient les résultats suivants.

Théorème 4. *(cf Th. 4.1.1) Soit u une solution $C([0,T[, \mathbb{H}_0^1)$ de l'équation de Schrödinger (S_Ω), de masse critique et explosant au temps fini T.*

i) Pour des domaines bornés, la vitesse d'explosion est minorée par

$$\frac{1}{T-t} \lesssim \|\nabla u(t)\|_2.$$

ii) S'il existe des solutions u explosant au temps T fini sur le bord de Ω*, c'est-à-dire si le paramètre de concentration* $x(t)$ *converge quand* $t \to T$ *vers un point du bord, alors la vitesse d'explosion vérifie*

$$\lim_{t\to T}(T-t)\|\nabla u(t)\|_2 = \infty.$$

La difficulté principale pour l'équation de Schrödinger posée sur un domaine est que la conservation de la dérivée du moment de premier ordre et l'identité du viriel ne sont plus vérifiées.

Pour surmonter cette difficulté, on va utiliser systématiquement dans la preuve du théorème une inégalité de type Cauchy-Schwarz. Plus précisément, on montre que si v est une fonction de $\mathbb{H}^1(\mathbb{R}^2)$, de masse critique ou sous-critique, alors

$$\left|\int \Im(v\nabla\overline{v})\nabla\theta dx\right| \leq \left(2E(v)\int |v|^2|\nabla\theta|^2 dx\right)^{\frac{1}{2}}$$

pour toute fonction θ réelle. Cette inégalité permet d'estimer le viriel, que l'on prend localisé si Ω est non-borné (voir la remarque ci-dessous). La minoration de la vitesse d'explosion obtenue est la même que celle trouvée par Antonini sur le tore ([1]).

En suivant l'approche de Weinstein dans [59] et le résultat récent de Maris dans [37], on analyse la convergence vers l'état fondamental des modulations de la solution (1.1), et on obtient, pour des domaines bornés, les informations supplémentaires suivantes.

Proposition 1. *(cf Prop. 4.1.1) i) La vitesse d'explosion vérifie*

$$\int |u(t)|^2|x-x(t)|^2 dx \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

ii) Le paramètre de concentration $x(t)$ *peut être choisi comme le moment du premier ordre*

$$x(t) = \frac{\int |u(t)|^2 x dx}{\|Q\|_2^2}.$$

Corrolaire. *Si l'équation* Ω *et les donées de Cauchy sont considérées être invariantes par rotations, alors* $x(t)$ *peut être choisi* 0*, et de plus,*

$$g(t) \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

Remarque. *Pour des domaines non-bornés, si la solution se concentre en un seul point, c'est-à-dire si* $x(t)$ *converge quand* $t \to T$*, alors la première affirmation du Théorème 4 est vraie, ainsi que les affirmations de la Proposition 1, pour le viriel et le moment du premier ordre localisés au point d'explosion.*

On ne connaît aucun exemple de solution de l'équation de Schrödinger non-linéaire qui explose avec une vitesse plus grande que $\frac{1}{T-t}$, ni dans le cas de masse sur-critique, ni dans le cas d'une non-linéarité sur-critique.

On s'attend donc à ce que la vitesse croisse exactement comme $\frac{1}{T-t}$ et que les profils d'explosions soient les mêmes que ceux sur $\mathbb{R}^2$, modulo des fonctions exponentiellement décroissantes dans $\mathbb{H}^1$.

Le fait que la vitesse d'explosion au bord croisse strictement plus vite que $\frac{1}{T-t}$ étant peu probable, on s'attend aussi à ne pas avoir des solutions de masse critique explosant sur le bord d'un domaine. Le résultat suivant confirme cette conjecture pour certaines géométries simples.

Théorème 5. *(cf Th. 4.1.2) Si Ω est un demi-plan ou un secteur de plan, alors il n'existe pas de solution de masse critique explosant en temps fini sur la frontière du demi-plan ou dans le coin du secteur respectivement.*

1.5 Organisation des chapitres

Le chapitre §2 contient les premiers résultats de la thèse, qui font l'objet d'un article à paraître dans SIAM Journal of Mathematical Analysis. Ces résultats ont été présentés dans la sous-section §1.2. Le contenu du chapitre suivant §3, qui va être publié dans le Journal de Mathématiques Pures et Appliquées, a été introduit dans la sous-section §1.3.2. Enfin, le chapitre §4 contient les derniers résultats de la thèse, présentés dans la sous-section §1.4 de ce chapitre introductif.

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Chapitre 2

Dispersion and Strichartz inequalities for Schrödinger equations with singular coefficients

Abstract[1]. In this paper we prove the global dispersion and the Strichartz inequalities for a class of one-dimensional Schrödinger equations with step function coefficients having a finite number of discontinuities. The local and global dispersion and Strichartz inequalities are discussed for certain Schrödinger equations with low regularity coefficients oscillating at infinity.
2000 Mathematics Subject Classification. 35J10, 35R05, 35B45, 35CXX
Keywords. Schrödinger equation, nonsmooth coefficients, dispersion and Strichartz inequalities, Bloch waves

2.1 Introduction

Strichartz estimates ([8],[14]) are an important tool for the understanding of nonlinear evolution equations. In the study of the dispersive properties of the Schrödinger equation with variable coefficients, the absence of the property of finite speed of propagation raises more difficulties than in the case of the wave equation. A way to "replace" this property is to impose a non-trapping condition on the trajectories. There are many results of well-posedness and smoothing effect for Schrödinger operators with smooth coefficients which are asymptotically flat and satisfy a non-trapping condition ([5],[6],[9],[10]). Staffilani and Tataru ([13]) proved the Strichartz estimates under the same conditions, but for lower regularity coefficients, only of C^2-class. However, in order to have wellposedness for NLS, the nontrapping condition can be dropped. In their recent paper ([2]), Burq, Gérard and Tzvetkov have obtained Strichartz estimates with fractional loss of derivative for metrics on $\mathbb{R}^d$ with uniformity assumptions at infinity, without geometric conditions. These new dispersive estimates imply local and global existence results for the Cauchy problem.

In this paper we study the dispersion property and the Strichartz inequalities for the one-dimensional Schrödinger equation

$$(S)\left\{\begin{array}{c}(i\,\partial_t+\partial_x a(x)\partial_x)\,u(t,x)=0 \text{ for } (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x)=u_0(x)\in\mathbb{L}^2(\mathbb{R})\end{array}\right.$$

for certain rough coefficients $a(x)$ without any geometric nontrapping condition.

In §2.2 we prove the global dispersion in the case of positive lamina coefficients, i.e. step functions with a finite number of singularities. Let us note in this situation the existence of trapped trajectories.

Theorem 2.1.1. *Consider a partition of the real axis*

$$-\infty=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=\infty$$

and a step function

$$a(x)=b_i^{-2} \text{ for } x\in(x_{i-1},x_i),$$

[1]to appear in SIAM Journal of Mathematical Analysis

where b_i are positive numbers.
The solution of the Schrödinger equation (S) satisfies the dispersion inequality

$$\|u(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C_n}{\sqrt{t}}\|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

and the Strichartz inequalities

$$\|u\|_{\mathbb{L}^p(\mathbb{R},\mathbb{L}^q(\mathbb{R}))} \leq C_n\|u_0\|_{\mathbb{L}^2(\mathbb{R})}$$

for every pair (p,q) verifying

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{2}.$$

The proof consists of writing the solution using the resolvent of the operator $-\partial_x a(x)\partial_x$. The resolvent is calculated and expressed in terms of series of exponentials. In order to get the global dispersion, we discuss these series within the framework of the theory of Wiener's almost periodic functions.

We can also prove a similar result for the operator

$$i\,\partial_t + \frac{1}{\rho(x)}\partial_x a(x)\partial_x\,,$$

where $\rho(x)$ is a step function of the same type as $a(x)$.

Moreover, if $v(t,x)$ is the solution of the associated wave system

$$(O)\begin{cases} (\partial_t^2 - \partial_x a(x)\partial_x)\, v(t,x) = 0 \text{ for } x \in \mathbb{R}, \\ v(0,x) = u_0(x) \in \mathbb{L}^2(\mathbb{R}), \\ \partial_t v(0,x) = 0, \end{cases}$$

the same method gives us the following estimate :

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |v(t,x)|dt \leq C_n\|u_0\|_{\mathbb{L}^1(\mathbb{R})}.$$

The dispersion is not satisfied if the step function coefficients are periodic. In §2.3, by using the Krönig-Penney model, we show that the local dispersion fails in the case of 2-valued periodic step-function coefficients.

Theorem 2.1.2. *Let $x_0 \in (0,1)$ and let b_0, b_1 be positive numbers satisfying $b_0 x_0 = b_1(1-x_0)$. Consider the 1-periodic function*

$$a(x) = \begin{cases} b_0^{-2} \text{ for } x \in [0,x_0), \\ b_1^{-2} \text{ for } x \in [x_0,1). \end{cases}$$

The local dispersion estimate fails for the Schrödinger equation (S).

The proof is based on the representation of the solution by its Floquet decomposition.

The fact that the coefficient a is not very oscillating at infinity seems to be essential for having the dispersion. Applying the method used by Avellaneda, Bardos and Rauch in [1], we can construct counterexamples for the global dispersion and Strichartz's inequalities in the case of certain continuous coefficients oscillating at infinity.

Also, as Castro and Zuazua have recently shown in [4], even if the coefficients are flat at infinity, but rough ($C^{0,\alpha}$) and locally very oscillating, the local Strichartz inequalities fail.

All these results suggest the conjecture that the one-dimensional Schrödinger equations with strictly positive BV coefficients satisfy the dispersion property.

I thank my advisor Patrick Gérard for having guided this work.

2.2 Laminar media

2.2.1 Representation of the resolvent of $-\partial_x a(x)\partial_x$

The operator $-\partial_x a(x)\partial_x$, defined from

$$\{h \in \mathbb{H}^1(\mathbb{R}), a\,\partial_x h \in \mathbb{H}^1(\mathbb{R})\}$$

to $\mathbb{L}^2(\mathbb{R})$, is self-adjoint. For $\omega > 0$ let R_ω be its resolvent

$$R_\omega g = (-\partial_x a(x)\partial_x + \omega^2 I)^{-1} g.$$

In order to obtain the expression of the resolvent on the intervals where a is constant, the second-order equations

$$\frac{1}{b_i^2}(R_\omega g)'' = \omega^2 R_\omega g - g$$

must be solved. Then, for $x \in (x_{i-1}, x_i)$, we have

$$R_\omega g(x) = c_{2i-1}e^{\omega b_i x} + c_{2i}e^{-\omega b_i x} + \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_i e^{-\omega b_i |x-y|}dy.$$

Since $R_\omega g$ belongs to $\mathbb{L}^2(\mathbb{R})$ the coefficients c_2 and c_{2n-1} are zero. The conditions of continuity of $R_\omega g$ and of $a\,\partial_x R_\omega g$ at the points x_i give a system of $2n-2$ equations on the c_i's. The matrix D_n of this system is

$$\begin{pmatrix}
e^{\omega b_1 x_1} & -e^{\omega b_2 x_1} & -e^{-\omega b_2 x_1} & 0 & 0 & 0 & 0 & 0 \\
b_2 e^{\omega b_1 x_1} & -b_1 e^{\omega b_2 x_1} & b_1 e^{-\omega b_2 x_1} & 0 & 0 & 0 & 0 & 0 \\
0 & e^{\omega b_2 x_2} & e^{-\omega b_2 x_2} & -e^{\omega b_3 x_2} & -e^{-\omega b_3 x_2} & 0 & 0 & 0 \\
0 & b_3 e^{\omega b_2 x_2} & -b_3 e^{-\omega b_2 x_2} & -b_2 e^{\omega b_3 x_2} & b_2 e^{-\omega b_3 x_2} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & e^{\omega b_{n-1} x_{n-1}} & e^{-\omega b_{n-1} x_{n-1}} & -e^{-\omega b_n x_{n-1}} \\
0 & 0 & 0 & 0 & 0 & b_n e^{\omega b_{n-1} x_{n-1}} & -b_n e^{-\omega b_{n-1} x_{n-1}} & b_{n-1} e^{-\omega b_n x_{n-1}}
\end{pmatrix}.$$

The right-hand side of the system is

$$T_n = \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \end{pmatrix},$$

with

$$t_i = \begin{pmatrix} \int_{-\infty}^{\infty} \frac{g(y)}{2\omega}(-b_i e^{-\omega b_i |x_i - y|} + b_{i+1} e^{-\omega b_{i+1}|x_i - y|})dy \\ \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_{i+1} b_i (-e^{-\omega b_i |x_i - y|} + e^{-\omega b_{i+1}|x_i - y|})\text{sign}(x_i - y)dy \end{pmatrix}.$$

Therefore we obtain the following lemma.

Lemma 2.2.1. *The resolvent $R_\omega g(x)$ is on each interval (x_i, x_{i+1}) a finite sum of terms :*

$$R_\omega g(x) = \sum_{finite} C e^{\omega \beta(x)} \int_{I(x_i)} \frac{g(y)}{2\omega} \frac{e^{\pm \omega b_i y}}{\det D_n(\omega)} dy + \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_i e^{-\omega b_i |x-y|} dy,$$

where $\beta(x)$ are real functions depending of $\{x_i, b_i\}$, C is a constant depending of $\{b_i\}$ and bounded by $(\max b_i^{-2})^n$, and $I(x_i)$ is either $(-\infty, x_i)$ or (x_i, ∞).

Let $\widetilde{D}_n$ be the same matrix as D_n, with the last two terms of the last column replaced by

$$\begin{pmatrix} -e^{\omega b_n x_{n-1}} \\ -b_{n-1} e^{\omega b_n x_{n-1}} \end{pmatrix}.$$

The development of the determinants of D_n and $\widetilde{D}_n$ with respect to the last column gives the following induction relations:

$$\begin{cases} \det D_n = e^{-\omega b_n x_{n-1}}[(b_{n-1} - b_n)e^{-\omega b_{n-1} x_{n-1}} \det \widetilde{D_{n-1}} - \\ \qquad -(b_{n-1} + b_n)e^{\omega b_{n-1} x_{n-1}} \det D_{n-1}], \\ \det \widetilde{D_n} = e^{\omega b_n x_{n-1}}[(b_{n-1} - b_n)e^{\omega b_{n-1} x_{n-1}} \det D_{n-1} - \\ \qquad -(b_{n-1} + b_n)e^{-\omega b_{n-1} x_{n-1}} \det \widetilde{D_{n-1}}]. \end{cases}$$

Let us define for $n \geq m \geq 2$

$$Q_m(\omega) = e^{-2\omega b_m x_m} \frac{\det \widetilde{D_m}}{\det D_m}.$$

By denoting

$$d_{m-1} = \frac{b_{m-1} - b_m}{b_{m-1} + b_m},$$

we have for $n \geq 3$

$$\det D_n(\omega) = (b_1 + b_2)e^{-\omega(b_2 - b_1)x_1} \prod_{i=2..n-1} (b_i + b_{i+1}) e^{\omega(b_i - b_{i+1})x_i}(1 - d_i Q_i(\omega)) \tag{2.1}$$

and for $n = 2$

$$\det D_2(\omega) = (b_1 + b_2)e^{-\omega(b_2-b_1)x_1}. \tag{2.2}$$

Also, we obtain an induction formula on the Q_m's:

$$Q_m(\omega) = e^{-2\omega b_m(x_m-x_{m-1})}\frac{-d_{m-1} + Q_{m-1}(\omega)}{1 - d_{m-1}Q_{m-1}(\omega)}. \tag{2.3}$$

Note that a Moebius transform on the unit disc occurs in this expression.
Let $\epsilon_n > 0$ be such that for every complex ω with

$$\Re\omega > -\epsilon_n,$$

the estimate

$$|Q_2(\omega)| = |d_1 e^{-2\omega b_2(x_2-x_1)}| < 1$$

holds and gives by induction

$$|Q_m(\omega)| < 1.$$

Hence $(\det D_n(\omega))^{-1}$ is uniformly bounded and well defined in this region, which contains the imaginary axis. Therefore, in view of Lemma 2.2.1, $\omega R_\omega u_0(x)$ can be analytically continued and we can use the following spectral theory lemma.

Lemma 2.2.2. *The solution of the Schrödinger equation* (S) *verifies*

$$u(t,x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}u_0(x)\frac{d\tau}{\pi}.$$

2.2.2 The algebra of Wiener's almost-periodic functions

Let us recall the structure of the Banach algebra of Wiener's almost-periodic functions:

$$B = \left\{h : \mathbb{R} \mapsto \mathbb{C}\,,\ h(t) = \sum_{\lambda\in\mathbb{R}} c(\lambda)e^{i\lambda t} \text{ with } \|h\|_B = \sum_{\lambda\in\mathbb{R}} |c(\lambda)| < \infty\right\}.$$

We will use the following classical theorem, which generalize Wiener's theorem to almost-periodic functions (see [7]).

Theorem 2.2.1. *(Cameron [3], Pitt [12]) If h is an element of B and F is an analytic function, regular on the closure of the set of values of h, then $F \circ h$ is also in B.*

Clearly, $\det D_n(i\tau)$ is an element of the algebra B. In the previous subsection we have proved that $(\det D_n(\omega))^{-1}$ is uniformly bounded and well defined in a region which contains the imaginary axis. Therefore, by using Theorem 2.2.1, it follows that

$$\|(\det D_n(i\tau))^{-1}\|_B < K_n, \tag{2.4}$$

where K_n is a constant depending on the b_i's.

2.2.3 The dispersion inequality

By using Lemma 2.2.1 and Lemma 2.2.2, the solution of (S) is for $x \in (x_i, x_{i+1})$:

$$u(t,x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau \left(\sum_{finite} Ce^{i\tau\beta(x)} \int_{I(x_i)} \frac{u_0(y)}{2i\tau} \frac{e^{\pm i\tau b_i y}}{\det D_n(i\tau)} dy + \int_{-\infty}^{\infty} \frac{u_0(y)}{2i\tau} b_i e^{-i\tau b_i |x-y|} dy \right) \frac{d\tau}{\pi}$$

$$= \int_{I(x_i)} u_0(y) \int_{-\infty}^{\infty} e^{it\tau^2} \left(C \sum_{finite} e^{i\tau(\beta(x) \pm b_i y)} \frac{1}{\det D_n(i\tau)} + b_i e^{-i\tau b_i |x-y|} \right) d\tau \frac{dy}{2i\pi},$$

where $\beta(x)$ are functions depending of $\{x_i, b_i\}$, C is a constant depending of $\{b_i\}$ and bounded by $(\max b_i^{-2})^n$, and $I(x_i)$ is either $(-\infty, x_i)$, (x_i, ∞) or $(-\infty, \infty)$. Since for a real α

$$\left| \int_{-\infty}^{\infty} e^{it\tau^2} e^{i\tau\alpha} d\tau \right| = \left| \frac{e^{-i\frac{\alpha^2}{4t}}}{\sqrt{t}} \int_{-\infty}^{\infty} e^{i\tau^2} d\tau \right| = \frac{C}{\sqrt{t}},$$

it follows that

$$\sup_{x \in (x_i, x_{i+1})} |u(t,x)| \leq \|u_0\|_{\mathbb{L}^1(\mathbb{R})} \frac{C}{\sqrt{t}} \|(\det D_n(i\xi))^{-1}\|_B.$$

Then (2.4) implies that

$$\sup_x |u(t,x)| \leq \frac{C_n}{\sqrt{t}} \|u_0\|_{\mathbb{L}^1(\mathbb{R})},$$

so the dispersion inequality for the Schrödinger equation (S) is satisfied.

Remark 2.2.1. *The finite sum in* (2.2.1) *contains* $n2^n$ *terms of the type* J_i. *Therefore, by estimating the solution as above, term by term, we cannot obtain the dispersion for equation* (S) *if* $a(x)$ *has an infinite number of steps. Therefore the method is too rough to prove the dispersion for an arbitrary strictly positive* BV *coefficient* $a(x)$.

Strichartz inequalities follow from the dispersion inequality by the classical duality argument TT^* ([15]) and so the proof of Theorem 2.1.1 is complete.

Since we can express the solution of the wave equation (O) as

$$v(t,x) = \int_{-\infty}^{\infty} e^{it\tau} R_{i\tau} u_0(x) i\tau \frac{d\tau}{2\pi},$$

the property

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |v(t,x)| dt \leq C \|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

follows similarly to the dispersion inequality for the solution of (S).

2.3 Periodic laminar media

2.3.1 General theory of periodic-coefficient equations

Let θ be a number in $[0, 2\pi]$ and consider the operator on $\mathbb{L}^2(\mathbb{S}^1)$

$$A_\theta = -(i\theta + \partial_x)a(x)(i\theta + \partial_x).$$

This operator is self-adjoint with a compact resolvent, hence the eigenvalues form a sequence of strictly positive numbers $\{\omega^2_{\theta,n}\}_{n\in\mathbb{N}}$. Moreover, the set of the corresponding eigenfunctions $p_n(\theta, x)$ is an orthonormal basis of $\mathbb{L}^2(\mathbb{S}^1)$.

Let us give a way to construct the elements of this basis. Finding the eigenfunction $p_n(\theta, x)$ is equivalent to finding the function

$$\Psi_n(\theta, x) = e^{i\theta x} p_n(\theta, x)$$

that satisfies

$$(H_{\theta,n}) \qquad -\partial_x a(x)\partial_x \Psi_n(\theta, x) = \omega^2_{\theta,n} \Psi_n(\theta, x).$$

Note that this new function has the quasi-periodic property

$$\Psi_n(\theta, x+1) = e^{i\theta} \Psi_n(\theta, x).$$

Equation $(H_{\theta,n})$ is of the type

$$(H) \qquad -\partial_x a(x)\partial_x \Psi(x) = \lambda^2 \Psi(x)$$

on

$$\{\Psi \in \mathbb{H}^1_{loc}(\mathbb{R}),\, a\,\partial_x \Psi \in \mathbb{H}^1_{loc}(\mathbb{R})\}.$$

This equation can be treated similarly to Hill's equation ([11]). Let T be an operator acting on the solutions's space as follows :

$$T(\Psi)(x) = \Psi(x+1).$$

On the one hand, the eigenvalues of T verify

$$x^2 - x\mathrm{Tr}(T) + \det T = 0.$$

On the other hand, the generalized Wronskian

$$W = \Psi_1 a\partial_x \Psi_2 - \Psi_2 a \partial_x \Psi_1$$

associated with (Ψ_1, Ψ_2), a normalized basis of solutions of (H), i.e.

$$\Psi_1(0) = (a\,\partial_x \Psi_2)(0) = 1 \quad , \quad (a\,\partial_x \Psi_1)(0) = \Psi_2(0) = 0,$$

is constant. Therefore

$$\det T = W(1) = W(0) = 1,$$

and the eigenvalues are $e^{i\xi}$ and $e^{-i\xi}$ for some complex ξ. If $|\mathrm{Tr}(T)|$ is larger than 2, then ξ is purely imaginary and there exists a basis of solutions of exponential growth. In this case λ^2 belongs to an instability interval of the equation. Otherwise, if $|\mathrm{Tr}(T)|$ is less than or equal to 2, ξ is real and λ^2 belongs to a stability interval. Moreover, if $\xi \in \pi\mathbb{Z}$, periodic solutions exist. If $\xi \in \mathbb{R} \setminus \pi\mathbb{Z}$, the existence of a basis of quasi-periodic solutions is assured.

So, the eigenvalues of A_θ are exactly the values λ^2 for which the operator T associated with (H) admits $e^{i\theta}$ and $e^{-i\theta}$ as eigenvalues. If $\theta \in (0, \pi) \cup (\pi, 2\pi)$, then these eigenvalues are simple. Therefore, in order to construct the $\mathbb{L}^2(\mathbb{S}^1)$ basis made of the eigenfunctions of A_θ, one has to find all λ for which the operator T associated with (H) verifies

$$\mathrm{Tr} T = 2\cos\theta.$$

For such a λ, we consider (Ψ_1, Ψ_2) a normalized basis of solutions of (H). If $\Psi_2(1) \neq 0$, then

$$\Psi(x) = \Psi_1(x) - \frac{\Psi_1(1) - e^{i\theta}}{\Psi_2(1)} \Psi_2(x) \tag{2.5}$$

is a solution of (H) and an eigenfunction of T for the eigenvalue $e^{i\theta}$. Finally,

$$p(x) = \Psi(x) e^{-i\theta x}$$

is an eigenfunction of the operator A_θ, associated with the eigenvalue λ^2.

2.3.2 Representation of solutions

In order to find the representation of the solution of (S), we decompose the initial data as follows:

$$\begin{aligned} u_0(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi}\, \widehat{u_0}(\xi) d\xi = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} e^{ix\xi}\, \widehat{u_0}(\xi) d\xi \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_0^{2\pi} e^{i(2k\pi+\theta)x}\, \widehat{u_0}(2k\pi + \theta) d\theta. \end{aligned}$$

Thus u_0 can be written

$$u_0(x) = \frac{1}{2\pi} \int_0^{2\pi} v(\theta, x)\, d\theta,$$

with

$$v(\theta, x) = \sum_{k\in\mathbb{Z}} e^{i(2k\pi+\theta)x}\, \widehat{u_0}(2k\pi + \theta). \tag{2.6}$$

Moreover,

$$||u_0||^2_{\mathbb{L}^2(\mathbb{R})} = \frac{1}{2\pi}||\widehat{u_0}||^2_{\mathbb{L}^2(\mathbb{R})} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi} |\widehat{u_0}(x)|^2 dx = \sum_{k\in\mathbb{Z}}\int_0^{2\pi} |\widehat{u_0}(2k\pi+\theta)|^2 d\theta$$

$$= \int_0^{2\pi}\int_0^1 |e^{-i\theta x} v(\theta,x)|^2 dx d\theta = \int_0^{2\pi}\int_0^1 |v(\theta,x)|^2 dx d\theta. \tag{2.7}$$

Since v satisfies the quasi-periodicity property

$$v(\theta, x+1) = e^{i\theta} v(\theta, x),$$

then $v(\theta,x)e^{-i\theta x}$ is 1-periodic. Therefore we can decompose it with respect to the $\mathbb{L}^2(\mathbb{S}^1)$ basis of eigenfunctions of the operator A_θ introduced in §2.3.1. If $\theta \in (0,\pi)\cup(\pi,2\pi)$, the eigenvalues of A_θ are simple and we can write

$$v(\theta,x)e^{-i\theta x} = \sum_{n\in\mathbb{N}} c_n(\theta) p_n(\theta,x);$$

that is,

$$v(\theta,x) = \sum_{n\in\mathbb{N}} c_n(\theta)\Psi_n(\theta,x). \tag{2.8}$$

Finally,

$$u(t,x) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{N}} e^{it\omega^2_{\theta,n}} c_n(\theta)\Psi_n(\theta,x) d\theta \tag{2.9}$$

is the solution of the Schrödinger equation (S). Moreover, using the above link (2.7) between the $\mathbb{L}^2$ norms of the initial datum u_0 and of v,

$$||u_0||^2_{\mathbb{L}^2(\mathbb{R})} = \sum_{n\in\mathbb{N}} ||c_n||^2_{\mathbb{L}^2(0,2\pi)}.$$

Let us now express the solution u in terms of the initial datum u_0. By using the definitions (2.6) and (2.8),

$$c_n(\theta) = < v(\theta,\cdot), \Psi_n(\theta,\cdot) > = \sum_{k\in\mathbb{Z}} \widehat{u_0}(2k\pi+\theta) < e^{i(2k\pi+\theta)\cdot}, \Psi_n(\theta,\cdot) > .$$

Since $e^{-i\theta x}\Psi_n(\theta,x)$ is 1-periodic, its Fourier decomposition contains only even exponentials

$$e^{-i\theta x}\Psi_n(\theta,x) = \sum_{k\in\mathbb{Z}} d_{n,k}(\theta) e^{i2\pi kx}.$$

Therefore

$$c_n(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u_0}(2k\pi+\theta)\overline{d}_{n,k}(\theta) = \int_{-\infty}^{\infty} u_0(y)e^{-iy\theta}\sum_{k\in\mathbb{Z}} e^{-i2k\pi y}\overline{d}_{n,k}(\theta)dy$$

$$= \int_{-\infty}^{\infty} u_0(y)\overline{\Psi}_n(\theta,y)dy.$$

In conclusion, for any initial datum u_0, the solution of the Schrödinger equation (S) is

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(y)\int_0^{2\pi}\sum_{n\in\mathbb{N}} e^{it\omega_{\theta,n}^2}\Psi_n(\theta,x)\overline{\Psi}_n(\theta,y)d\theta dy.$$

2.3.3 Explicit solutions for the Krönig-Penney model

Let

$$a(x) = \begin{cases} b_0^{-2} \text{ for } x\in[0,x_0), \\ b_1^{-2} \text{ for } x\in[x_0,1) \end{cases}$$

as defined in the statement of Theorem 2.1.2. Fix $\theta\in(0,\pi)\cup(\pi,2\pi)$. Following the approach presented in §2.3.1., in this subsection we will explicitly find the functions $\Psi_n(\theta,x)$, associated with the eigenfunctions $p_n(\theta,x)$ of the operator A_θ.

The basis of normalized solutions associated with the equation (H) is

$$\begin{cases} \Psi_1(x) = \begin{cases} \frac{1}{2}e^{i\lambda b_0 x} + \frac{1}{2}e^{-i\lambda b_0 x} \text{ for } x\in(0,x_0), \\ a_j^1 e^{i\lambda b_1 x} + b_j^1 e^{-i\lambda b_1 x} \text{ for } x\in(x_0,1), \end{cases} \\ \Psi_2(x) = \begin{cases} -\frac{ib_0}{2\lambda}e^{i\lambda b_0 x} + \frac{ib_0}{2\lambda}e^{-i\lambda b_0 x} \text{ for } x\in(0,x_0), \\ a_j^2 e^{i\lambda b_1 x} + b_j^2 e^{-i\lambda b_1 x} \text{ for } x\in(x_0,1) \end{cases} \end{cases}$$

with

$$\begin{cases} a_j^1 = \frac{1}{4b_0}[(b_0+b_1)e^{i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{-i\lambda x_0(b_0+b_1)}], \\ b_j^1 = \frac{1}{4b_1}[(b_0+b_1)e^{-i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{i\lambda x_0(b_0+b_1)}], \\ a_j^2 = \frac{i}{4\lambda}[-(b_0+b_1)e^{i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{-i\lambda x_0(b_0+b_1)}], \\ b_j^2 = \frac{i}{4\lambda}[(b_0+b_1)e^{-i\lambda x_0(b_0-b_1)} - (b_0-b_1)e^{i\lambda x_0(b_0+b_1)}]. \end{cases}$$

The trace of the shift operator T is

$$\mathrm{Tr}T = \Psi_1(1) + \frac{1}{b_1^2}\partial_x\Psi_2(1).$$

One can calculate

$$\mathrm{Tr}(T) = (r+1)\cos[\lambda(x_0b_0+(1-x_0)b_1)] - (r-1)\cos[\lambda(x_0b_0-(1-x_0)b_1)],$$

where

$$r = \frac{b_0^2+b_1^2}{2b_0b_1}.$$

By setting the conditions

$$\mathrm{Tr}(T) = 2\cos\theta\,, \quad x_0 b_0 = (1-x_0)b_1,$$

it follows that

$$2\cos\theta = (r+1)\cos(\lambda 2x_0b_0) - (r-1).$$

Hence we have

$$\lambda \in \left\{ \frac{2\pi j + f(\theta)}{2x_0b_0}, j \in \mathbb{Z} \right\},$$

where $f(\theta)$ is the analytic function

$$f(\theta) = \arccos\frac{r-1+2\cos\theta}{r+1}.$$

As the solutions Ψ_1 and Ψ_2 are the same for λ and for $-\lambda$, we have to check if there exist different integers j and k such that

$$2\pi j + f(\theta) = \pm(2\pi k + f(\theta)).$$

If this is true, it follows that

$$j + k = \frac{f(\theta)}{\pi}.$$

Since $r > 1$ gives $f(\theta) < \pi$ and $\theta \neq 0$ gives $f(\theta) \neq 0$, then j and k must satisfy

$$0 < |j+k| < 1.$$

In conclusion, the values

$$\left| \frac{2\pi j + f(\theta)}{2x_0b_0} \right|$$

are different, so we can consider the eigenvalues of the operator A_θ indexed by $j \in \mathbb{Z}$ as follows:

$$\omega_{\theta,j} = \frac{2\pi j + f(\theta)}{2x_0b_0}. \tag{2.10}$$

Note that since θ has been fixed in $(0,\pi)\cup(\pi,2\pi)$,

$$\omega_{\theta,j} \neq 0 \text{ for all } j \in \mathbb{Z}.$$

By using (2.5), we obtain a quasi-periodic solution for equation $(H_{\theta,j})$

$$\widetilde{\Psi}_j(\theta,x) = \left(\frac{1}{2} + h_j(\theta)\right) e^{i\omega_{\theta,j}b_0x} + \left(\frac{1}{2} - h_j(\theta)\right) e^{-i\omega_{\theta,j}b_0x} \text{ for } x \in (0,x_0) \tag{2.11}$$

with

$$h_j(\theta) = i\frac{(b_0+b_1)\cos(2\omega_{\theta,j}b_0x_0) + (b_0-b_1) - e^{i\theta}}{(b_0+b_1)\sin(2\omega_{\theta,j}b_0x_0)}.$$

The definition (2.10) of $\omega_{\theta,j}$ gives

$$h_j(\theta) = h(\theta) = i\frac{(b_0+b_1)\cos f(\theta) + (b_0-b_1) - e^{i\theta}}{(b_0+b_1)\sin f(\theta)}.$$

Then we can calculate for $x \in (0, x_0)$

$$\widetilde{\Psi}_j(\theta,x) = \cos(\omega_{\theta,j}\, b_0 x) + 2h(\theta)\sin(\omega_{\theta,j}\, b_0 x),$$

and for $x \in (x_0, 1)$

$$\widetilde{\Psi}_j(\theta,x) = \left(a_j^1 - a_j^2\, h(\theta)\frac{2\omega_{\theta,j}}{ib_0}\right) e^{i\omega_{\theta,j} b_1 x} + \left(b_j^1 - b_j^2\, h(\theta)\frac{2\omega_{\theta,j}}{ib_0}\right) e^{-i\omega_{\theta,j} b_1 x}$$

$$= \frac{b_0+b_1}{4b_0}(1+2h(\theta))e^{i\omega_{\theta,j}(x_0(b_0-b_1)+b_1 x)} + \frac{b_0-b_1}{4b_0}(1-2h(\theta))e^{-i\omega_{\theta,j}(x_0(b_0+b_1)-b_1 x)}$$

$$+\frac{b_0+b_1}{4b_0}(1-2h(\theta))e^{-i\omega_{\theta,j}(x_0(b_0-b_1)+b_1 x)} + \frac{b_0-b_1}{4b_0}(1+2h(\theta))e^{i\omega_{\theta,j}(x_0(b_0+b_1)-b_1 x)}.$$

It follows that

$$\int_0^1 |\widetilde{\Psi}_j(\theta,x)|^2 dx = \alpha_j(\theta) = \beta(\theta) + \frac{\gamma(\theta)}{2\pi j + f(\theta)},$$

with $\beta(\theta)$ strictly positive. Let $\Psi_j(\theta,x)$ be the $\mathbb{L}^2$ normalization of $\widetilde{\Psi}_j(\theta,x)$:

$$\Psi_j(\theta,x) = \frac{\widetilde{\Psi}_j(\theta,x)}{\sqrt{\alpha_j(\theta)}}.$$

We are now in the context described in §2.3.2.

2.3.4 The failure of local dispersion

Let $\mathcal{X}$ be a 2π-periodic function whose restriction to $(0,2\pi)$ is $\mathcal{C}_0^\infty$. One can write

$$\mathcal{X}(\xi) = \sum_{k\in\mathbb{Z}} s_k e^{ik\xi}.$$

Let v_0 be the Fourier localization outside $2\pi\mathbb{Z}$ points of the initial data u_0

$$\widehat{v_0}(\xi) = \widehat{u_0}(\xi)\mathcal{X}(\xi).$$

By applying Plancherel's theorem one has

$$v_0(x) = \int_{-\infty}^{\infty} e^{ix\xi}\widehat{u_0}(\xi)\mathcal{X}(\xi)\frac{d\xi}{2\pi} = \sum_{k\in\mathbb{Z}} u_0(x+k)s_k.$$

Since $\mathcal{X}|_{(0,2\pi)}$ is in $\mathcal{C}_0^\infty$,

$$\sum_{k\in\mathbb{Z}} |s_k| = S < \infty,$$

so the localization preserves the regularity $\mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ with

$$\begin{cases} \|v_0\|_{\mathbb{L}^1(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^1(\mathbb{R})}, \\ \|v_0\|_{\mathbb{L}^2(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^2(\mathbb{R})}. \end{cases}$$

For such an initial datum v_0, the coefficients $c_j(\theta)$ defined in §2.3.2 are:

$$c_j(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u_0}(2k\pi+\theta)\mathcal{X}(2k\pi+\theta)\overline{d}_{j,k}(\theta)$$

$$= \mathcal{X}(\theta)\int_{-\infty}^{\infty} u_0(y)e^{-iy\theta}\sum_{k\in\mathbb{Z}} e^{-i2k\pi y}\overline{d}_{j,k}(\theta)dy = \mathcal{X}(\theta)\int_{-\infty}^{\infty} u_0(y)\overline{\Psi}_{\theta,j}(y)dy.$$

Then, by the representation formula (2.9), the solution $v(t,x)$ of the equation (S) with initial datum v_0 can be written as

$$v(t,x) = \int_{-\infty}^{\infty} u_0(y)K_t(x,y)dy,$$

where

$$K_t(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{j\in\mathbb{Z}} e^{it\omega_{\theta,j}^2}\Psi_{\theta,j}(x)\overline{\Psi}_{\theta,j}(y)\mathcal{X}(\theta)d\theta.$$

Since

$$\|v_0\|_{\mathbb{L}^1(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^1(\mathbb{R})},$$

in order to have the dispersion inequality

$$\|v(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}\|v_0\|_{\mathbb{L}^1(\mathbb{R})},$$

one must have

$$\left\|\int_{-\infty}^{\infty} u_0(y)K_t(\cdot,y)dy\right\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}\|u_0\|_{\mathbb{L}^1(\mathbb{R})}.$$

Since the dual of $\mathbb{L}^1$ is $\mathbb{L}^\infty$, the function K_t must verify

$$\|K_t\|_{\mathbb{L}^\infty(x,y)} \leq \frac{C}{\sqrt{t}}.$$

Lemma 2.3.1. *There exist times t, arbitrarily small, for which K_t is not a $\mathbb{L}^\infty(x,y)$ function.*

Proof. Let us change t in $\frac{t}{4b_0^2x_0^2}$ and x in $\frac{x}{2x_0}$. By using definition (2.10) of $\omega_{\theta,j}$ and formula (2.11) for $\widetilde{\Psi}_j(\theta,x)$, we have that $K_t(x,y)$ is, for $x < x_0$, equal to

$$\frac{1}{4\pi}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} e^{it(2\pi j+f(\theta))^2}\left(e^{ix(2\pi j+f(\theta))}(1+2h(\theta))+e^{-ix(2\pi j+f(\theta))}(1-2h(\theta))\right)$$

$$\times\left(e^{-iy(2\pi j+f(\theta))}(1+2\overline{h}(\theta))+e^{iy(2\pi j+f(\theta))}(1-2\overline{h}(\theta))\right)\frac{\mathcal{X}(\theta)}{\alpha_j(\theta)}d\theta.$$

It follows that the kernel is the sum of four terms of the following type:

$$J_t(x,y)=\frac{1}{4\pi}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} e^{it(2\pi j+f(\theta))^2}e^{i(x-y)(2\pi j+f(\theta))}(1+2h(\theta))(1+2\overline{h}(\theta))\frac{\mathcal{X}(\theta)}{\alpha_j(\theta)}d\theta.$$

In view of the forthcoming applications of the stationary phase formula, we can consider that $J_t(x,y)$ is, modulo a $\mathbb{L}^\infty$ function, the same sum as above, with α_0 replaced by α_1. Since $|f(\theta)|<\pi$, one can choose a function $\alpha_\xi(\theta)$ which is strictly positive, bounded, and $\mathcal{C}^\infty$ with respect to the variable ξ, such that

$$\alpha_\xi(\theta)=\beta(\theta)+\frac{\gamma(\theta)}{\xi+f(\theta)}\ \text{for}\ |\xi|>\pi.$$

This allows us to apply the Poisson formula, so $J_t(x,y)$ can be written as

$$\frac{1}{2}\sum_{l\in\mathbb{Z}}\int_{-\infty}^{\infty}\int_0^{2\pi} e^{i\xi l}e^{it(\xi+f(\theta))^2}e^{i(x-y)(\xi+f(\theta))}(1+2h(\theta))(1+2\overline{h}(\theta))\frac{\mathcal{X}(\theta)}{\alpha_\xi(\theta)}d\theta d\xi.$$

By changing $\xi+f(\theta)$ into ζ,

$$J_t(x,y)=\frac{1}{2}\sum_{l\in\mathbb{Z}}\int_0^{2\pi} e^{-if(\theta)l}\,I_l(t,x-y,\theta)\,d\theta,$$

where

$$I_l(t,x-y,\theta)=\mathcal{X}(\theta)(1+2h(\theta))(1+2\overline{h}(\theta))\int_{-\infty}^{\infty} e^{it\zeta^2}e^{i(x-y+l)\zeta}\frac{d\zeta}{\alpha_{\zeta-f(\theta)}(\theta)}$$

verifies

$$|\partial\theta^k I_l(t,x-y,\theta)|\le C\ \text{for all}\ k\in\mathbb{N}.$$

The only critical point of $f|_{(0,2\pi)}$ is π, which is nondegenerate, so we can apply the stationary phase formula for large l. In view of the definition of $\alpha_\zeta(\pi)$, $J_t(x,y)$ is modulo a $\mathbb{L}^\infty$ function

$$J_t(x,y)=\sum_{l\in\mathbb{Z}^*}\left(\frac{e^{-if(\pi)l}}{\sqrt{|l|}}\,I_l(t,x-y)\,\frac{1}{2}\mathcal{X}(\pi)(1+2h(\pi))(1+2\overline{h}(\pi))+O(|l|^{-\frac{3}{2}})\right)$$

with

$$I_l(t, x-y) = \int_{-\infty}^{\infty} e^{it\zeta^2} e^{i(x-y+l)\zeta} \frac{d\zeta}{\beta(\pi) + \frac{\gamma(\pi)}{\zeta}}.$$

For writting the remainder term $O(|l|^{-\frac{3}{2}})$ we have used the following known result, applied here for $\alpha = f(\pi) \in (0, \pi)$.

Lemma 2.3.2. *The sum of exponentials*

$$F(\alpha) = \sum_{l \in \mathbb{Z}^*} \frac{e^{-i\alpha l}}{\sqrt{|l|}}$$

blows up as

$$\frac{1}{\sqrt{|\alpha|}}$$

if α tends to zero, and otherwise the sum is finite.

By changing ζ in $\frac{x-y+l}{\sqrt{t}}$ and by considering that (x, y) lies in a compact set, we have

$$I_l(t, x-y) = \frac{x-y+l}{\sqrt{t}} \int_{-\infty}^{\infty} e^{i(x-y+l)^2(\zeta^2 + \frac{\zeta}{\sqrt{t}})} \frac{d\zeta}{\beta(\pi) + \frac{\gamma(\pi)\sqrt{t}}{(x-y+l)\zeta}}.$$

The stationary phase formula applied again for $\zeta = -\frac{1}{2\sqrt{t}}$ gives

$$I_l(t, x-y) = \frac{1}{\sqrt{t}} e^{-i\frac{(x-y+l)^2}{4t}} \frac{1}{\beta(\pi) - \frac{2\gamma(\pi)t}{x-y+l}} + \frac{O((x-y+l)^{-2})}{\sqrt{t}}.$$

Thus, modulo a $\mathbb{L}^\infty$ function, we obtain that

$$J_t(x, y) = \frac{C}{\sqrt{t}} \sum_{l \in \mathbb{Z}^*} \frac{e^{-if(\pi)l}}{\sqrt{|l|}} e^{-i\frac{(x-y+l)^2}{4t}},$$

with $C \neq 0$. Let t verify

$$\frac{1}{4t} \in 2\pi\mathbb{Z}.$$

Note that t can be chosen arbitrary small. Also,

$$J_t(x, y) = \frac{Ce^{-i\frac{(x-y)^2}{4t}}}{\sqrt{t}} \sum_{l \in \mathbb{Z}^*} \frac{e^{-i(\frac{x-y}{2t} + f(\pi))l}}{\sqrt{|l|}}.$$

It follows then that $K_t(x, y)$ is, modulo a $\mathbb{L}^\infty$ function,

$$\frac{e^{-i\frac{(x-y)^2}{4t}}}{\sqrt{t}} \left(C_1 F\left(\frac{x-y}{2t} + f(\pi)\right) + C_2 F\left(-\frac{x-y}{2t} + f(\pi)\right)\right)$$

$$+\frac{e^{-i\frac{(x+y)^2}{4t}}}{\sqrt{t}}\left(C_3\,F\left(\frac{x+y}{2t}+f(\pi)\right)+C_4\,F\left(-\frac{x+y}{2t}+f(\pi)\right)\right).$$

Since $f(\pi) \neq 0$, in view of the behavior of F presented in Lemma 2.3.2, the kernel $K_t(x,y)$ is not in $\mathbb{L}^\infty(x,y)$. □

In conclusion the local dispersion for the Schrödinger equation (S) fails and Theorem 2.1.2 is proved.

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Chapitre 3

On the nonlinear Schrödinger dynamics on $\mathbb{S}^2$

Abstract[1]. We analyze the evolution of the highest weight spherical harmonics by the nonlinear Schrödinger equation on $\mathbb{S}^2$. Sharp estimates are proved for the dynamics parallel and orthogonally to the initial data. Also, we give an ansatz of the solution with respect to the spherical harmonics.
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Keywords. Nonlinear Schrödinger, eigenfunctions, ansatz

3.1 Introduction

The nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = F(u)$$

is motivated by many questions raised in Physics (see the recent survey of the subject [17]). On $\mathbb{R}^d$ the Cauchy problem has been largely studied in the past twenty years. In the one-dimensional case the Sobolev embedding suffices to have well-posedness in the energy space. Unfortunately in higher dimensions this argument is no longer effective. The Strichartz estimates ([18]) were then successfully exploited in order to get existence and regularity results ([8],[10], [19]).

For the same problem on a compact Riemannian manifold, with Δ being the associated Laplace-Beltrami operator, it appears that the geometry influences the dynamics of the equation.

The Strichartz estimates with fractional loss of derivative have been proved by Burq, Gérard and Tzvetkov in [2]. If we consider the low regularity equation with defocusing polynomial nonlinearity, these estimates imply local existence results. Moreover, on surfaces in the case of defocusing polynomial nonlinearities and on three-manifolds in the case of defocusing cubic nonlinearities the global existence in the energy space $\mathbb{H}^1$ follows.

However, instability phenomena appear, even in the defocusing case.

On the one hand the same authors proved in [3] that the flow map of the cubic defocusing Schrödinger equation on the sphere $\mathbb{S}^2$

$$\begin{cases} i\partial_t u + \Delta_{S^2} u = |u|^2 u, \\ u(0,x) \in \mathbb{H}^s(S^2) \end{cases}$$

is not uniformly continuous for $s \in [0, \frac{1}{4}[$, that is for Sobolev regularity indices greater than zero, which is the scaling index. Similar results hold also on a plane domain [4]. For $\mathbb{S}^2$, the same authors have proved recently that the index $\frac{1}{4}$ is the critical regularity index, that is for $s > \frac{1}{4}$, the Cauchy problem is well posed ([5]).

On the other hand Cazenave-Weissler [6] and Bourgain [1] proved that the Cauchy problem is $\mathbb{H}^\epsilon$ well-posed on $\mathbb{R}^2$ and on $\mathbb{T}^2$ respectively for all positive ϵ. Moreover, on $\mathbb{T}^2$, the flow is not uniformly continuous for s negative ([3]), therefore zero is the critical regularity index.

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Hence these results point out the importance played by the geometry of the manifold in the dynamics of the equation. This is not in contradiction with the positive results on the wave operator, since the Schrödinger equation does not enjoy the property of finite speed of propagation.

Instability phenomena appear for a large class of dispersive equations. In the recent paper [11], Kenig, Ponce and Vega have studied the low regularity properties of the focusing nonlinear Schrödinger and Korteweg-de Vries equations. Then, in [7], Christ, Colliander and Tao have extended this study to the defocusing analogues of these equations. Instability phenomena appear also for the defocusing wave equation in $\mathbb{R}^3$, with supercritical nonlinearity, as Lebeau has shown in [13] (see also [14]). Koch and Tzvetkov have shown in [12] that the flow of the Benjamin-Ono equation fails to be uniformly continuous on $\mathbb{H}^s$ for all positive s. All these results are obtained by constructing families of exact solutions of the equations, that contradict the well-posedness.

In order to obtain in [3] the instability result on $\mathbb{S}^2$, the evolution of certain spherical harmonics, concentrated on geodesics, is studied as follows. Let ψ_n be the $\mathbb{H}^s$-normalized spherical harmonic obtained by restricting to the sphere the harmonic polynomial

$$\psi_n(x_1, x_2, x_3) = n^{\frac{1}{4}-s}(x_1 + ix_2)^n.$$

Let us notice that when n tends to infinity, ψ_n concentrates on the circle $x_1^2 + x_2^2 = 1$. By direct calculus one can estimates the $\mathbb{L}^p$ norms of ψ

$$\begin{cases} \|\psi_n\|_\infty \approx n^{\frac{1}{4}-s}, \\ \|\psi_n\|_2 \approx n^{-s}, \\ \|\psi_n\|_4^4 \approx n^{\frac{1}{2}-4s}, \\ \|\psi_n\|_6^3 \approx n^{\frac{1}{2}-3s}. \end{cases} \tag{3.1}$$

The equivalents are considered as n goes to infinity, and so shall be in all the paper :

$$f_n \approx g_n \iff \exists\, c, C \in \mathbb{R}^+, c g_n \leq f_n \leq C g_n.$$

$$f_n \lesssim g_n \iff \exists\, C \in \mathbb{R}^+, f_n \leq C g_n.$$

Consider now the Schrödinger equation

$$(S) \begin{cases} i\partial_t u + \Delta_{\mathbb{S}^2} u = |u|^2 u, \\ u_n(0, x) = \kappa_n \psi_n(x), \end{cases}$$

where κ_n is a number between $\frac{1}{2}$ and 1. For every real α the rotation R_α defined on $\mathbb{R}^3$ by

$$R_\alpha(x_1, x_2, x_3) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha, x_3)$$

verifies the relation

$$\psi(R_\alpha(x)) = e^{in\alpha}\psi(x).$$

Then the uniqueness of the solution for the same Cauchy problem (S) with initial data $e^{in\alpha}\psi(x)$ gives us the identity

$$u(t, R_\alpha(x)) = e^{in\alpha}u(t,x).$$

Using this fact and an algebraic lemma on the spherical harmonics, in [3] it is shown that the solution u can be decomposed only on ψ_n and on $\{h_{n+j}\}_{j\geq 1}$, the spherical harmonics of order $n+j$ satisfying

$$h_{n+j}(R_\alpha(x)) = e^{in\alpha}h_{n+j}(x).$$

We shall consider these spherical harmonics to be normalized in $\mathbb{L}^2$.
Let $\omega_n\psi_n$ be the orthonormal projection of $|\psi_n|^2\psi_n$ on the space spanned by ψ_n, and r_n the remainder term of this projection

$$|\psi_n|^2\psi_n = \omega_n\psi_n + r_n.$$

By the same arguments above, r_n express only in terms of h_{n+j}'s. One can write the solution of (S)

$$u_n(t,x) = \kappa_n e^{-it(n(n+1)+\kappa_n^2\omega_n)}((1+\widetilde{z_n}(t))\psi_n(x) + q_n(t,x)),$$

with q_n only in terms of h_{n+j}'s.

For $s \in]\frac{3}{20}, \frac{1}{4}[$, it is shown in ([3]) that the $\mathbb{H}^s$ norm of $q_n(t)$ is negligible with respect to the one of ψ_n, and $|\tilde{z}(t)|$ tends to 0 when n tends to infinity. These results imply that the solution behaves like the initial data ψ_n with an oscillating exponential type coefficient. Knowing that ω_n tends to infinity, a good choice of a bounded sequence κ_n gives an important dephasing between the solutions u_n, so the Cauchy problem for the equation (S_Ω) is ill-posed on $\mathbb{H}^s(\mathbb{S}^2)$, in the sense that the flow is not uniformly continuous on bounded sets of $\mathbb{H}^s$.

The purpose of this paper is to provide a further analysis of these solutions. We prove sharp estimates for $|\widetilde{z_n}|$ and for $\|q(t)\|_{H^s}$. In particular these results point out that even in the remainder part $\widetilde{z_n}\psi_n + q_n$, the dynamics orthogonally to ψ_n is weak. We also obtain an ansatz of the solution with respect to the spherical harmonics h_{n+j}.

For simplicity, the indices n of the functions defined above will be ignored from now on. We define

$$\alpha_j = 2nj + j^2 + j - \kappa^2\omega \ , \ k_{j,l} = \kappa^2 < \overline{h_{n+l}}\,\psi^2, h_{n+j} >,$$

$$\mu_j = \sqrt{(3k_{j,j}+\alpha_j)(k_{j,j}+\alpha_j)}.$$

Consider the operator

$$A = -\Delta - n(n+1)$$

and the operator M defined on the space spanned by the h_{n+j}'s by

$$M(h_{n+j}) = \mu_j h_{n+j}.$$

Theorem 3.1.1. *Let $T > 0$. For every $s \in [0, \frac{1}{4}[$, for $t \in [0, T]$, the solution of (S) is*

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)}(z(t)\psi(x) + q(t,x)),$$

with the sharp estimates

$$\begin{cases} \sup_{0<t<T} \|q(t)\|_{H^{\frac{1}{2}}} \approx n^{-3s}, \\ |z(t) - 1| \approx tn^{-4s}. \end{cases}$$

(in the second equivalent, t is present in order to include the case $t = 0$, when $z(0) = 1$) Moreover,

i) For $s \in [0, \frac{1}{4}[$ the coefficient of ψ is

$$z(t) = e^{-it\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r >} + O(n^{-\frac{1}{2}-6s}).$$

ii) For $s \in]\frac{1}{12}, \frac{1}{4}[$ we have the ansatz

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)} \left(e^{-it\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r >} \psi(x) - i \int_0^t e^{isM} r ds \right) + \widetilde{u}(t,x),$$

with

$$\|\widetilde{u}(t)\|_{H^s} \ll n^{-\frac{1}{2}-2s}.$$

The proof is based on a further exploitation of the conservation laws than in [3]. Also, by using Sogge's estimates on the spherical harmonics ([15], [16])

$$\|h_m\|_p \leq C m^{\frac{1}{4}-\frac{1}{2p}} \text{ for } 2 \leq p \leq 6, \tag{3.2}$$

we give upper bounds for the $\mathbb{L}^p$ norms of q better than the ones obtained by interpolation between $\mathbb{L}^2$ and $\mathbb{H}^1$. By using all these estimates in the study of the equations of z and q, we have the description of z. It follows then that the upper bounds founded before are sharp. Finally, we obtain the ansatz by analyzing the system obtained by projecting the equation (S) on each mode, and by using the important distance between two consecutive eigenvalues of the laplacian.

Remark 3.1.1. *It will be shown that*

$$\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r > \approx n^{-4s},$$

so the oscillation of the solution is stronger as s decreases to zero, that is if the amplitude of the initial data grows faster.

Remark 3.1.2. *The linear part that comes from the cubic nonlinearity of the equation has an essential contribution in the ansatz of the solution. From it the operator M is defined in terms of μ_j instead of α_j, and the effective dynamics orthogonally to ψ*

$$-i \int_0^t e^{isM} r ds$$

verifies an equation depending on M

$$\begin{cases} (i\partial_t + M)v(t,x) + ir(x) = 0, \\ v(0,x) = 0. \end{cases}$$

Remark 3.1.3. *In the case $s = \frac{1}{4}$, it is not known if the flow is uniformly continuous or not.*

The paper is organized as follows. In §3.2.1, by using the energy laws, we find upper bounds for the norms of q, for $|z-1|$ and for the coefficients of the spherical harmonics h_{n+l} in the solution u. In §3.2.2 sharp estimates are given for some particular scalar product of spherical harmonics. In §3.2.3, by using these estimates in the study of the equation verified by $z(t)$, we get the description of $z(t)$. In §3.2.4 this description implies that the upper bounds obtained previously on $\|q\|_{H^s}$ and on $|z-1|$ are sharp. By projecting the equation (S) on the space spanned by h_{n+j}, we describe in §3.3 the ansatz of the solution with respect to the spherical harmonics.

I thank my advisor Patrick Gérard for having guided this work.

3.2 Estimates on the solution

3.2.1 Upper bounds on norms of q and on $|z-1|$

The conservation laws of the equation (S) are

$$\begin{cases} |z(t)|^2\|\psi\|_2^2 + \|q(t)\|_2^2 = \|\psi\|_2^2, \\ |z(t)|^2\|\nabla\psi\|_2^2 + \|\nabla q(t)\|_2^2 + \frac{1}{2\kappa^2}\|u(t)\|_4^4 = \|\nabla\psi\|_2^2 + \frac{\kappa^2}{2}\|\psi\|_4^4. \end{cases}$$

By subtracting from the second conservation law the first one multiplied by $n(n+1)$ we obtain

$$\|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 = \frac{\kappa^2}{2}\|\psi\|_4^4 - \frac{1}{2\kappa^2}\|u(t)\|_4^4. \tag{3.3}$$

As mentioned in the introduction, one can decompose

$$q(t,x) = \sum_{j\geq 1} z_j(t)h_{n+j}(x).$$

Obviously,

$$\sum_{j\geq 1}(n+j)|z_j(t)|^2 \leq \sum_{j\geq 1}((j+n)(j+n+1) - n(n+1))|z_j(t)|^2,$$

so

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2,$$

and the identity (3.3) gives

$$\|q(t)\|^2_{H^{\frac{1}{2}}} \lesssim \frac{\kappa^2}{2}\|\psi\|_4^4 - \frac{\kappa^2}{2}\|z(t)\psi + q(t)\|_4^4.$$

The numbers κ will be chosen to be bounded with respect to n, so

$$\|q(t)\|^2_{H^{\frac{1}{2}}} \lesssim \|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4, \tag{3.4}$$

and by using the estimates (3.1) on the norms of ψ,

$$\|q(t)\|_{H^{\frac{1}{2}}} \lesssim \|\psi\|_4^2 \lesssim n^{\frac{1}{4}-2s}.$$

Then one has a first upper bound on the $\mathbb{L}^2$ and on the $\mathbb{H}^s$ norm of q

$$\begin{cases} \|q(t)\|_2 \lesssim n^{-\frac{1}{4}-2s}, \\ \|q(t)\|_{H^s} \lesssim n^{-\frac{1}{4}-s}. \end{cases} \tag{3.5}$$

By exploiting further the inequality (3.4), better estimations on $q(t)$ are found, namely the ones claimed in Theorem 3.1.1.

Lemma 3.2.1. *For $s \in [0, \frac{1}{4}[$ the norms of q are upper-bounded by*

$$\begin{cases} \|q(t)\|_{H^{\frac{1}{2}}} \lesssim n^{-3s}, \\ \|q(t)\|_2 \lesssim n^{-\frac{1}{2}-3s}, \\ \|q(t)\|_{H^s} \lesssim n^{-\frac{1}{2}-2s}. \end{cases} \tag{3.6}$$

Proof. By developing the right-hand side of (3.4) and by neglecting the negative terms,

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim \int |\psi|^4 \left|1 - |z(t)|^4\right| + |z(t)|^3|\psi^3 q(t)| + |z(t)||\psi q^3(t)|.$$

The conservation of the mass gives

$$1 - |z(t)|^2 = \frac{\|q(t)\|_2^2}{\|\psi\|_2^2}, \tag{3.7}$$

and the preliminary estimates (3.5) obtained above on $q(t)$ ensures us that

$$|z(t)|^2 = 1 + O(n^{-\frac{1}{2}-2s}) \approx 1. \tag{3.8}$$

Thus one can write

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim \int |\psi|^4 \frac{\|q(t)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(t)| + |\psi q^3(t)|.$$

The terms in the right side can be estimated as follows

$$\int |\psi q^3| \lesssim \|\psi\|_\infty \|q\|_3^3 \lesssim n^{\frac{1}{4}-s}\|q\|^3_{H^{\frac{1}{3}}} \lesssim n^{-\frac{1}{4}-s}\|q\|^3_{H^{\frac{1}{2}}},$$

$$\int |\psi^3 q| \lesssim \|\psi\|_6^3 \|q\|_2 \lesssim n^{-3s} \|q\|_{H^{\frac{1}{2}}},$$

and

$$\int |\psi|^4 \frac{\|q\|_2^2}{\|\psi\|_2^2} \lesssim \|\psi\|_4^4 n^{2s} \|q\|_2^2 \lesssim n^{-\frac{1}{2}-2s} \|q\|_{H^{\frac{1}{2}}}^2.$$

So for n large enough

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim n^{-3s}\|q(t)\|_{H^{\frac{1}{2}}} + n^{-\frac{1}{2}-2s}\|q(t)\|_{H^{\frac{1}{2}}}^2 + n^{-\frac{1}{4}-s}\|q(t)\|_{H^{\frac{1}{2}}}^3.$$

By using (3.4),

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim n^{-3s}\|q(t)\|_{H^{\frac{1}{2}}} + n^{-\frac{1}{2}-2s}\|q(t)\|_{H^{\frac{1}{2}}}^2 + n^{-\frac{1}{4}-s}\|q(t)\|_{H^{\frac{1}{2}}}^3.$$

Since

$$\|q(0)\|_{H^{\frac{1}{2}}} = 0,$$

the term that gives the behavior of the right-hand side is $n^{-3s}\|q(t)\|_{H^{\frac{1}{2}}}$, and the claimed better estimations (3.6) are obtained. In particular, we also obtain

$$\int |\psi|^4 \frac{\|q(t)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(t)| + |\psi q^3(t)| \lesssim n^{-3s}\|q(t)\|_{H^{\frac{1}{2}}}, \tag{3.9}$$

and by (3.3)

$$\|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 \approx \|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim n^{-6s} \tag{3.10}$$

□

These new estimates on q will imply the ones on z mentioned in Theorem 3.1.1.

Lemma 3.2.2. *For $s \in [0, \frac{1}{4}[$ the coefficient z verifies*

$$|z(t) - 1| \lesssim t n^{-4s}.$$

Proof. The function

$$c(t) := e^{-it(n(n+1)+\kappa^2\omega)}$$

verifies

$$i\partial_t c - n(n+1)c = \kappa^2\omega|c|^2 c.$$

Then, the projection of the equation (S) on the space spanned by ψ is

$$\begin{cases} (i\partial_t z + \kappa^2\omega z)\|\psi\|_2^2 = \kappa^2 \int |z\psi + q|^2 (z\psi + q)\overline{\psi}, \\ z(0) = 1. \end{cases}$$

Since

$$\omega = \frac{\|\psi\|_4^4}{\|\psi\|_2^2},$$

the equation of $z-1$ writes

$$\begin{cases} i\partial_t(z-1) = \frac{\kappa^2}{\|\psi\|_2^2}\left(\int |z\psi+q|^2(z\psi+q)\overline{\psi} - \int z|\psi|^4\right), \\ (z-1)(0)=0. \end{cases} \tag{3.11}$$

By integrating in time and by developing the right-hand side of the equation,

$$|z(t)-1| \lesssim \frac{1}{\|\psi\|_2^2}\int_0^t \int |\psi|^4|z(t)|||z(t)|^2-1|+|z(t)|^2|\psi^3 q(\tau)|+|z(t)||\psi^2 q^2(\tau)|+|\psi q^3(\tau)|dxd\tau.$$

By using again the informations (3.7), (3.8) on z, we have

$$|z(t)-1| \lesssim \frac{1}{\|\psi\|_2^2}\int_0^t \int |\psi|^4\frac{\|q(\tau)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(\tau)| + |\psi^2 q^2(\tau)| + |\psi q^3(\tau)|dxd\tau.$$

The square term on the right-hand side can be upper bounded by estimates (3.1) and (3.6)

$$\int \psi^2 q^2(\tau) \lesssim \|\psi\|_\infty^2\|q(\tau)\|_2 n^{-\frac{1}{2}}\|q(\tau)\|_{\mathbb{H}^{\frac{1}{2}}} \lesssim n^{\frac{1}{2}-2s}n^{-\frac{1}{2}-3s}n^{-\frac{1}{2}}\|q(\tau)\|_{\mathbb{H}^{\frac{1}{2}}} \ll n^{-3s}\|q(\tau)\|_{H^{\frac{1}{2}}},$$

and the others have been upper bounded in (3.9), therefore

$$|z(t)-1| \lesssim n^{2s}tn^{-3s}\sup_{0<\tau<T}\|q(\tau)\|_{H^{\frac{1}{2}}} \lesssim tn^{-4s}, \tag{3.12}$$

and Lemma 3.2.2 is proved. □

Finally, let us prove some estimates on the norms of q and on the coefficients z_j of the spherical harmonics in q that will be used in the next sections. We will use Sogge's estimates (3.2) on the spherical harmonics in order to obtain better estimations on the $\mathbb{L}^p$ norms of q than the ones given by interpolation between $\mathbb{L}^2$ and $\mathbb{H}^1$.

Lemma 3.2.3. *One has*

$$\left(\sum_{j\geq 1} j|z_j(t)|^2\right)^{\frac{1}{2}} \lesssim n^{-\frac{1}{2}-3s}, \tag{3.13}$$

and

$$\begin{cases} \|q(t)\|_4^2 \leq n^{-\frac{3}{4}-6s}\log n, \\ \|q(t)\|_6^3 \leq n^{-1-9s}(\log n)^{\frac{3}{2}}. \end{cases} \tag{3.14}$$

Proof. By repeating the argument on the conservation laws, one has

$$n\sum_{j\geq 1} j|z_j(t)|^2 \leq \sum_{j\geq 1}((j+n)(j+n+1)-n(n+1))|z_j(t)|^2$$

$$\leq \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2,$$

so (3.13) is obtained by using (3.10).
Let us decompose

$$q(t,x) = \sum_{j=1..n^\alpha} z_j(t)h_{n+j}(x) + q_\alpha(t,x),$$

where α is a positive number to be fixed later. Here q_α is the part of q containing only the spherical harmonics of order greater than $n + n^\alpha$.
The same argument before, for α smaller or equal to 1, gives

$$nn^\alpha \|q_\alpha(t)\|_2^2 \leq \|\nabla q_\alpha(t)\|_2^2 - n(n+1)\|q_\alpha(t)\|_2^2$$

$$\leq \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 \lesssim n^{-6s}.$$

Then

$$\begin{cases} \|q_\alpha(t)\|_2 \lesssim n^{-\frac{1}{2}-3s-\frac{\alpha}{2}}, \\ \|\nabla q_\alpha(t)\|_2 \lesssim n^{\frac{1}{2}-3s-\frac{\alpha}{2}}, \end{cases} \tag{3.15}$$

and by interpolation

$$\begin{cases} \|q_\alpha(t)\|_4 \lesssim n^{-3s-\frac{\alpha}{2}}, \\ \|q_\alpha(t)\|_6 \lesssim n^{\frac{1}{6}-3s-\frac{\alpha}{2}}. \end{cases}$$

Now we are able to estimate the $\mathbb{L}^p$ norms of q

$$\|q(t)\|_4 \leq \sum_{l=1}^{n^\alpha} |z_l(t)|\|h_{n+l}\|_4 + \|q_\alpha(t)\|_4 \leq \left(\sum_{j\geq 1}^{n^\alpha} j|z_j|^2\right)^{\frac{1}{2}} \left(\sum_{j\geq 1}^{n^\alpha} \frac{\|h_{n+j}\|_4^2}{j}\right)^{\frac{1}{2}} + n^{-3s-\frac{\alpha}{2}}.$$

Sogge's estimates (3.2) on the norms of the h_{n+j}'s and the relation (3.13) imply

$$\|q(t)\|_4^2 \lesssim n^{-1-6s} n^{\frac{1}{4}} \log n + n^{-6s-\alpha} \leq n^{-\frac{3}{4}-6s} \log n$$

if α is chosen larger enough. Similarly,

$$\|q(t)\|_6 \leq \sum_{l=1}^{n^\alpha} |z_l(t)|\|h_{n+l}\|_6 + \|q_\alpha(t)\|_6 \leq \left(\sum_{j\geq 1}^{n^\alpha} j|z_j(t)|^2\right)^{\frac{1}{2}} \left(\sum_{j\geq 1}^{n^\alpha} \frac{\|h_{n+j}\|_6^2}{j}\right)^{\frac{1}{2}} + n^{\frac{1}{6}-3s-\frac{\alpha}{2}},$$

and

$$\|q(t)\|_6^3 \lesssim n^{-\frac{3}{2}-9s} n^{\frac{1}{2}} (\log n)^{\frac{3}{2}} + n^{\frac{1}{2}-9s-\frac{3\alpha}{2}} \leq n^{-1-9s} (\log n)^{\frac{3}{2}}$$

if $\alpha = 1$.

□

3.2.2 Sharp estimates for $< |\psi|^2\psi, h_{n+2} >$

Lemma 3.2.4. *We have the sharp estimate*

$$< |\psi|^2\psi, h_{n+2} > \approx n^{\frac{1}{2}-3s}.$$

Proof. In polar coordinates

$$x_1 = \sin\theta\cos\phi \ , \ x_2 = \sin\theta\sin\phi \ , \ x_3 = \cos\theta,$$

the spherical harmonics h_{n+l} can be written in terms of the associated Legendre functions ([9])

$$h_{n+l}(\phi,\theta) = c_j e^{in\phi} P^n_{n+l}(\cos(\theta)),$$

where c_l is the coefficient of the $\mathbb{L}^2$ normalization

$$c_l = \sqrt{\frac{l!(n+l)}{2\pi(2n+l)!}}.$$

Since ψ is the restriction of $(x_1 + ix_2)^n$ to the sphere, we can calculate

$$< |\psi|^2\psi, h_{n+l} > = c_l \int_0^{2\pi}\int_0^{\pi} n^{\frac{3}{4}-3s}\sin^{3n}(\theta)e^{in\phi}\sin(\theta)\, e^{-in\phi}P^n_{n+l}(\cos(\theta))d\theta d\phi$$

$$= 2\pi c_l n^{\frac{3}{4}-3s}\int_0^{\pi} P^n_{n+l}(\cos(\theta))\sin^{3n+1}(\theta)d\theta.$$

A way to write the associated Legendre functions P^n_{n+l} is ([9])

$$P^n_{n+l}(\cos(\theta)) = (-1)^n\frac{(2n+l)!}{2^n n! l!}\sin^n(\theta)\left(\cos^l(\theta) - \frac{l(l-1)}{2(2n+2)}\cos^{l-2}(\theta)\sin^2(\theta)\right.$$
$$\left. + \frac{l(l-1)(l-3)(l-4)}{2\cdot 4(2n+2)(2n+4)}\cos^{l-4}(\theta)\sin^4(\theta) - \dots\right);$$

the sum ends when the power of the cosinus, decreasing each time by 2, becomes 1 or 0. In particular, this formula implies that for odd l

$$< |\psi|^2\psi, h_{n+l} > = \sum_{k=0}^{[\frac{l}{2}]}\widetilde{c_{l,k}}\int_0^{\pi}\cos^{l-2k}(\theta)\sin^{3n+1}(\theta)d\theta = 0.$$

Let us also remark that $< |\psi|^2\psi, h_{n+l} >$ is a real number.
For $l = 2$, Stirling's formula gives us the sharp estimate

$$< |\psi|^2\psi, h_{n+2} > \approx n^{\frac{5}{4}} n^{\frac{3}{4}-3s}\int_0^{\pi}\sin^{4n+1}(\theta)\left(1 - \frac{2n+3}{2n+2}\sin^2(\theta)\right)d\theta$$

$$\approx n^{2-3s}\frac{1}{\sqrt{4n+1}}\frac{2n}{(2n+2)(4n+3)} \approx n^{\frac{1}{2}-3s}.$$

$\square$

Since

$$\|r\|_2 \leq \|\psi\|_6^3 \leq n^{\frac{1}{2}-3s},$$

then we also get the sharp estimate of r in $\mathbb{L}^2$

$$\|r\|_2 = \left(\sum_{l\geq 1} | < h_{n+l}, |\psi^2|\psi > |^2\right)^{\frac{1}{2}} \approx n^{\frac{1}{2}-3s}.$$

Notice that we also get the equivalent

$$< A^{-1}r, r > \approx n^{-6s}. \tag{3.16}$$

3.2.3 The description of $z(t)$

The equation (3.11) verified by z can be developed

$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(z(|z|^2-1)\|\psi\|_4^4 + 2|z|^2|\psi|^2\overline{\psi}q + z^2|\psi|^2\psi\overline{q} + 2z|\psi|^2|q|^2 + \overline{z}(\overline{\psi})^2q^2 + \overline{\psi}|q|^2q).$$

On the one hand the identity (3.7) on z and the estimates (3.6) on q give

$$|z|^2 = 1 - \frac{\|q\|_2^2}{\|\psi\|_2^2} \lesssim 1 + n^{-1-4s}. \tag{3.17}$$

On the other hand q and ψ are orthogonal, so

$$\int |\psi|^2\overline{\psi}q = \omega\int\overline{\psi}q + \int\overline{r}q =< q, r > .$$

Then one has

$$i\partial_t z - \frac{\kappa^2}{\|\psi\|_2^2}(2 < q, r > + z^2 < \overline{q}, \overline{r} >) \lesssim \frac{\kappa^2}{\|\psi\|_2^2}\int |\psi|^4 n^{-1-4s} + |\psi^2 q^2| + |\psi q^3| + n^{-1-4s}|qr|.$$

By using the estimates (3.1) on ψ and (3.6), (3.14) on q, the terms on the right side are upper bounded as follows

$$\int |\psi|^4 n^{-1-4s} \lesssim n^{-\frac{1}{2}-8s},$$

$$\int |\psi^2 q^2| \leq \|\psi\|_\infty^2\|q\|_2^2 \lesssim n^{-\frac{1}{2}-8s},$$

$$\int |\psi q^3| \lesssim \|\psi\|_2\|q\|_6^3 \lesssim n^{-1-10s}(\log n)^{\frac{3}{2}},$$

and

$$\int |qr| n^{-1-4s} \lesssim \|\psi\|_6^3\|q\|_2 n^{-1-4s} \lesssim n^{-1-10s}.$$

So, the equation of z becomes

$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(2 < q, r > +z^2 < \overline{q}, \overline{r} >) + O(n^{-\frac{1}{2}-6s}).$$

The equation of q is given by the projection of the equation (S) on the space spanned by the h_{n+j}'s

$$i\partial_t q - Aq = \kappa^2 \Pi \left(|z\psi + q|^2 (z\psi + q) + \omega q \right),$$

where Π is the associated projector. One can write

$$< q, r > = < Aq, A^{-1} r > = - < \kappa^2 \Pi \left(|z\psi + q|^2 (z\psi + q) + \omega q \right), A^{-1} r > + < i\partial_t q, A^{-1} r > .$$

The operator A^{-1} induces a decay of n^{-1} so the first term can be estimated as follows

$$\int |\psi^2 q A^{-1} r| \lesssim n^{-1} \|\psi\|_\infty^2 \|q\|_2 \|r\|_2 \lesssim n^{-\frac{1}{2}-8s},$$

$$\int |q^3 A^{-1} r| \lesssim n^{-1} \|q\|_6^3 \|r\|_2 \lesssim n^{-\frac{3}{2}-12s} (\log n)^{\frac{3}{2}},$$

$$\omega \int |q A^{-1} r| \lesssim \frac{\|\psi\|_4^4}{\|\psi\|_2^2} n^{-1} \|q\|_2 \|r\|_2 \lesssim n^{-\frac{1}{2}-8s},$$

and

$$< |\psi|^2 \psi, A^{-1} r > = < r, A^{-1} r > \approx n^{-6s}.$$

For the last term the equivalent is given by (3.16). His coefficient is $|z|^2 z$, so using the behavior (3.17) of z,

$$|z|^2 z < |\psi|^2 \psi, A^{-1} r > = z < r, A^{-1} r > +O(n^{-1-10s}). \tag{3.18}$$

Therefore

$$< q, r > = -z\kappa^2 < r, A^{-1} r > +i\partial_t < q, A^{-1} r > +O(n^{-\frac{1}{2}-8s}).$$

Noticing that $< r, A^{-1} r >$ is a real number, and using (3.18), the equation of z can be written now

$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(-iz3\kappa^2 < r, A^{-1} r > +2i\partial_t < q, A^{-1} r > -iz^2 \partial_t < \overline{q}, A^{-1} \overline{r} >) + O(n^{-\frac{1}{2}-6s}).$$

The value of z at zero is 1 so the Duhamel formula implies

$$\left| z(t) - e^{-it \frac{3\kappa^4}{\|\psi\|_2^2} < r, A^{-1} r >} \right| \lesssim n^{2s} \left| \int_0^t e^{-i(t-s) \frac{3\kappa^4}{\|\psi\|_2^2} < r, A^{-1} r >} \partial_s < q, A^{-1} r > ds \right|$$

$$+ n^{2s} \left| \int_0^t e^{-i(t-s) \frac{3\kappa^4}{\|\psi\|_2^2} < r, A^{-1} r >} z^2(s) \partial_s < \overline{q}, A^{-1} \overline{r} > ds \right| + O(n^{-\frac{1}{2}-6s}).$$

By integration by parts in the first term

$$\int_0^t e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>}\partial_s < q, A^{-1}r > ds \lesssim (1+n^{2s}n^{-1}\|r\|_2^2)n^{-1}\|q\|_2\|r\|_2$$

$$\lesssim (1+n^{-4s})n^{-1-6s} \lesssim n^{-1-6s}.$$

Let us notice that from (3.11), the derivative of z has the same upper bound as $z-1$, that is n^{-4s}. This fact, together with the behavior (3.17) of z, gives by integrations by parts the same upper bound for the second term as for the first one.
Therefore the description of the coefficient of ψ in the solution u is

$$z(t) = e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>} + O(n^{-\frac{1}{2}-6s}), \tag{3.19}$$

and the assertion i) of the Theorem 3.1.1 is proved.

3.2.4 The exact growth of $\|q\|_{H^s}$ and of $|z-1|$

For $s>0$ the result (3.19) of the former section implies that

$$z(t)-1 = -i\frac{3\kappa^4 t}{\|\psi\|_2^2} < A^{-1}r, r > +O(n^{-8s}),$$

and the equivalent (3.16) gives the exact growth

$$|z(t)-1| \approx tn^{-4s}.$$

The link (3.12) between the estimates of $z-1$ and q

$$|z(t)-1| \lesssim tn^{-s}\sup_{0<\tau<T}\|q(\tau)\|_{H^{\frac{1}{2}}} \lesssim tn^{-4s},$$

implies that

$$\sup_{0<\tau<T}\|q(\tau)\|_{\mathbb{H}^{\frac{1}{2}}} \approx n^{-3s}.$$

If $s=0$ then by (3.16)

$$\lim_{n\to\infty}\frac{3\kappa^4 t}{\|\psi\|_2^2} < A^{-1}r, r > \neq 0,$$

and by using the description (3.19) of z,

$$|z(t)-1| \approx t.$$

By the same arguments above

$$\sup_{0<t<T}\|q(t)\|_{\mathbb{H}^{\frac{1}{2}}} \approx 1,$$

and the equivalents claimed in the begining of Theorem 3.1.1 are proved.

Remark 3.2.1. *As a consequence, fir $t > 0$,*

$$\|(z(t)-1)\psi\|_{\mathbb{H}^s} \approx tn^{-4s} \gg tn^{-\frac{1}{2}-2s} \approx \sup_{0<t<T} \|q(t)\|_{\mathbb{H}^{\frac{1}{2}}} n^{-\frac{1}{2}+s} \gtrsim \|q(t)\|_{\mathbb{H}^s}.$$

This shows that the main part in the remainder term in the evolution of ψ by the equation (S) remains parallel to ψ.

3.3 The ansatz of the solution

3.3.1 The equations of the z_j's

We recall the notations done in the introduction

$$\alpha_j = 2nj + j^2 + j - \kappa^2\omega \ , \ k_{j,l} = \kappa^2 < \overline{h_{n+l}}\psi^2, h_{n+j} >,$$

$$\mu_j = \sqrt{(3k_{j,j}+\alpha_j)(k_{j,j}+\alpha_j)}.$$

The equation of z_j is obtained by taking the scalar product of the equation (S) with the spherical harmonic h_{n+j}

$$\begin{cases} i\partial_t z_j - \alpha_j z_j = \kappa^2 < |z\psi+q|^2(z\psi+q), h_{n+j} >, \\ z_j(0) = 0. \end{cases}$$

Let α be a number smaller than 1. The equation can be written

$$i\partial_t z_j = \alpha_j z_j + \sum_{l=1}^{n^\alpha}(2z_l + \overline{z_l})k_{j,l} + r_j,$$

where

$$r_j = \kappa^2 < |z\psi+q|^2(z\psi+q), h_{n+j} > - \sum_{l=1}^{n^\alpha}(2z_l+\overline{z_l})k_{j,l}.$$

Notice that r_j does not contain linear terms in z_l's. Consider now the system of equations of the real and imaginary parts of z_j

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}' = \begin{pmatrix} 0 & \alpha_j + k_{j,j} \\ -\alpha_j - 3k_{j,j} & 0 \end{pmatrix} \begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}$$

$$+\sum_{l\neq j}^{n^\alpha} k_{j,l} \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} \Re z_l \\ \Im z_l \end{pmatrix} + \begin{pmatrix} \Im r_j \\ -\Re r_j \end{pmatrix}.$$

The eigenvalues of the first matrix on the righthandside are $\pm i\mu_j$. Notice that

$$\begin{pmatrix} 0 & \alpha_j + k_{j,j} \\ -\alpha_j - 3k_{j,j} & 0 \end{pmatrix} = B_j^{-1} A_j B_j,$$

where A_j is the diagonal matrix

$$A_j = \begin{pmatrix} -i\mu_j & 0 \\ 0 & i\mu_j \end{pmatrix},$$

and

$$B_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\frac{\mu_j}{\alpha_j+3k_{j,j}} \\ -i\frac{\mu_j}{\alpha_j+k_{j,j}} & 1 \end{pmatrix}.$$

Set

$$A_{j,l} = \frac{k_{j,l}}{2} B_j \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} B_l^{-1}$$

and

$$\begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix} = B_j \begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}.$$

The system in the new variables is

$$\begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix}' = A_j \begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix} + \sum_{l \neq j}^{n^\alpha} A_{j,l} \begin{pmatrix} d_l \\ \tilde{d}_l \end{pmatrix} - B_j \begin{pmatrix} \Im r_j \\ -\Re r_j \end{pmatrix}.$$

By performing a second change of variable

$$\begin{pmatrix} f_j \\ \tilde{f}_j \end{pmatrix} = e^{-tA_j} \begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix},$$

the system becomes

$$\begin{pmatrix} f_j \\ \tilde{f}_j \end{pmatrix}' = \sum_{l \neq j}^{n^\alpha} e^{-tA_j} A_{j,l} e^{tA_l} \begin{pmatrix} f_l \\ \tilde{f}_l \end{pmatrix} - R'_{j,j}(t),$$

with

$$R_{j,l}(t) = \int_0^t e^{-\tau A_j} B_l \begin{pmatrix} \Im r_l \\ -\Re r_l \end{pmatrix} d\tau.$$

The integration in time gives

$$\begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} = \sum_{l \neq j}^{n^\alpha} \int_0^t e^{-\tau A_j} A_{j,l} e^{\tau A_l} \begin{pmatrix} f_l(\tau) \\ \tilde{f}_l(\tau) \end{pmatrix} d\tau - R_{j,j}(t).$$

Lemma 3.3.1. *For $j \neq l$ there are matrices $M_{j,l}(t)$ and $B_{j,l}$ verifying the relation*

$$M_{j,l}(t) = \int_0^t e^{-\tau A_j} A_{j,l} e^{\tau A_l} d\tau = e^{-tA_j} B_{j,l} e^{tA_l} - B_{j,l}$$

and the estimates

$$|M_{j,l}(t)| \approx |B_{j,l}| \lesssim \frac{n^{\frac{1}{2}-2s}}{n|j-l|}.$$

Proof. Finding $B_{j,l}$ is equivalent to solving the matrix equation

$$A_{j,l} = (B_{j,l} A_l - A_j B_{j,l}).$$

Let us denote

$$B_{j,l} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

In view of the expression of A_l, the equation becomes

$$A_{j,l} = (\mu_j - \mu_l) \begin{pmatrix} x & -y \\ z & -t \end{pmatrix}.$$

Thus the existence of $B_{j,l}$ is proved and estimates on it can be found as follows. Since $k_{j,l} \lesssim n^{\frac{1}{2}-2s}$, we have $\mu_j \approx \alpha_j$, so

$$|B_j| \approx 1 \quad , \quad |A_{j,l}| \lesssim n^{\frac{1}{2}-2s},$$

and consequently

$$|M_{j,l}(t)| \approx |B_{j,l}| \lesssim \frac{|A_{j,l}|}{|\mu_j - \mu_l|} \lesssim \frac{n^{\frac{1}{2}-2s}}{n|j-l|}.$$

□

Since $f_j(0) = \widetilde{f}_j(0) = 0$, after integrating by parts

$$\begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} = \sum_{l \neq j}^{n^\alpha} M_{j,l}(t) \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} - \sum_{l \neq j}^{n^\alpha} \int_0^t M_{j,l}(\tau) \begin{pmatrix} f_l(\tau) \\ \widetilde{f}_l(\tau) \end{pmatrix}' d\tau - R_{j,j}(t).$$

Using the expression of f_l' and $\widetilde{f}_l'$ we obtain

$$(Rel_j) \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} = \sum_{l \neq j}^{n^\alpha} M_{j,l}(t) \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} - \sum_{l \neq j}^{n^\alpha} \int_0^t M_{j,l}(\tau) \sum_{k \neq l}^{n^\alpha} e^{-\tau A_l} A_{l,k} e^{\tau A_k} \begin{pmatrix} f_k(\tau) \\ \widetilde{f}_k(\tau) \end{pmatrix} d\tau$$

$$- \sum_{l \neq j}^{n^\alpha} \left(\int_0^t e^{-\tau A_j} B_{j,l} B_l \begin{pmatrix} \Im r_l \\ -\Re r_l \end{pmatrix} d\tau + B_{j,l} R_{l,l}(t) \right) - R_{j,j}(t).$$

Since

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = e^{tA_j} B_j^{-1} \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix},$$

(Rel_j) is the searched relation between the z_j's.

3.3.2 Estimates on the source terms $R_{j,l}$

Lemma 3.3.2. *Let $s \in]\frac{1}{12}, \frac{1}{4}[$, and let $\alpha \in]1-4s, 8s[$. Then we have the estimates*

$$\begin{cases} |R_{2,2}| \approx n^{-\frac{1}{2}-3s}, \\ |R_{j,l}| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \text{ if } j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}}, \\ |R_{j,l}| \lesssim n^{-5s-\frac{\alpha}{2}} \text{ for the other } j \ll n^{\alpha}. \end{cases} \tag{3.20}$$

Proof. Let us recall the expression of r_l

$$r_l = \kappa^2 < |z|^2 z|\psi|^2\psi + 2(|z|^2 q - q + q_\alpha)|\psi|^2 + (z^2\overline{q} - \overline{q} + \overline{q_\alpha})\psi^2 + 2z\psi|q|^2 + \overline{z}\overline{\psi}q^2 + |q|^2 q, h_{n+l} > .$$

Since $|B_l| \approx 1$ one can estimate

$$\left| R_{j,l} - < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| \lesssim \left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j}(|z(\tau)|^2 z(\tau) - 1) d\tau \right|$$

$$+ \int (|\psi|^2(|q||z^2 - 1| + |q_\alpha|) + |\psi q^2| + |q|^2 q)|h_{n+l}|.$$

By using Cauchy-Schwarz's inequality and (3.1)

$$| < |\psi|^2\psi, h_{n+l} > | \lesssim n^{\frac{1}{2}-3s},$$

with equivalence for $l = 2$, and cancellation for odd l, by Lemma 3.2.4. Therefore

$$\left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| \lesssim \frac{n^{\frac{1}{2}-3s}}{nj}, \tag{3.21}$$

also with equivalence for $l = 2$. Note that from (3.11) one has the same estimate for $(z-1)'$ as for $z - 1$. Integration by parts gives

$$\left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j}(|z(\tau)|^2 z(\tau) - 1) d\tau \right| \ll \frac{n^{\frac{1}{2}-3s}}{nj}.$$

By using the upper bounds (3.14) and (3.15) the other terms can be estimated as follows

$$\int |\psi^2 q(z^2 - 1) h_{n+l}| \leq \sup_{0<t<T} |z - 1| \|\psi\|_\infty^2 \|q\|_2 \lesssim n^{-9s},$$

$$\int |\psi^2 q_\alpha h_{n+l}| \leq n^{\frac{1}{2}-2s} \|q_\alpha\|_2 \lesssim n^{-5s-\frac{\alpha}{2}},$$

$$\int |\psi q^2 h_{n+l}| \leq \|\psi\|_\infty \|q\|_4^2 \lesssim n^{-\frac{1}{2}-7s} \log n \ll n^{-5s-\frac{\alpha}{2}},$$

and

$$\int |q^3 h_{n+l}| \leq \|q\|_6^3 \lesssim n^{-1-9s}(\log n)^{\frac{3}{2}} \ll n^{-5s-\frac{\alpha}{2}}.$$

Therefore

$$|R_{j,l}| \lesssim \left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| + O(n^{-5s-\frac{\alpha}{2}}) + O(n^{-9s}).$$

In view of (3.21), in order to have the first term as dominant for small j, we choose

$$\alpha \in]1-4s, 8s[.$$

Indeed

$$n^{-5s-\frac{\alpha}{2}} \ll \frac{n^{\frac{1}{2}-3s}}{nj}$$

for all $j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}}$, and

$$n^{-9s} \ll n^{-5s-\frac{\alpha}{2}}$$

for all j. This condition on α implies the restriction $s \in]\frac{1}{12}, \frac{1}{4}[$. In conclusion, for α chosen in $]1-4s, 8s[$, we have the estimates (3.20). □

Remark 3.3.1. *The restriction $s \in]\frac{1}{12}, \frac{1}{4}[$ is due to the presence in the source terms $R_{j,l}$ of the linear terms in z_j's, that have only the decay $O(n^{-9s})$. These terms come from*

$$< z^2 \overline{q} \psi^2, h_{n+l} >,$$

and have variable coefficients. If we consider them in the linear part of the system of the z_j's, we are unable to obtain the decay estimates claimed in Theorem 3.1.1.

3.3.3 Estimates on the z_j's

Since $|e^{tA_j} B_j| \approx 1$

$$\left| \begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} \right| \approx \left| \begin{pmatrix} \Re z_j(t) \\ \Im z_j(t) \end{pmatrix} \right|.$$

The relation (Rel_j) gives

$$\left| \begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} + R_{j,j}(t) \right| \lesssim \sum_{l\neq j}^{n^\alpha} |M_{j,l}||z_j| - \sum_{l\neq j}^{n^\alpha} |M_{j,l}| \sum_{k\neq l}^{n^\alpha} |A_{l,k}||z_l| - \sum_{l\neq j}^{n^\alpha} |B_{j,l}|(|R_{j,l}| + |R_{l,l}|).$$

Then the estimate (3.13) on the z_l's, Lemma 3.3.1 and (3.20) imply that the term on the right is upper bounded by

$$n^{-\frac{1}{2}-2s} \log n (n^{-\frac{1}{2}-3s} + n^{\frac{1}{2}-2s} n^{-\frac{1}{2}-3s} \log n^{\frac{1}{2}} + (n^{-\frac{1}{2}-3s} + n^{-5s-\frac{\alpha}{2}})).$$

Therefore, since α is smaller than $1+4s$,

$$\left|\begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} + R_{j,j}(t)\right| \lesssim n^{-1-5s}\log n + n^{-\frac{1}{2}-7s}\log n \ll n^{-5s-\frac{\alpha}{2}}. \tag{3.22}$$

By using again (3.20) we get the behavior of $|f_j|$ and implicitly the one of $|z_j|$

$$\begin{cases} |z_2| \approx n^{-\frac{1}{2}-3s} \\ |z_j| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \text{ if } j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}} \\ |z_j| \lesssim n^{-5s-\frac{\alpha}{2}} \text{ for the other } j \ll n^{\alpha} \end{cases}$$

If α is taken to be $8s-2\epsilon$ with ϵ small and positive, these estimates become

$$\begin{cases} |z_2| \approx n^{-\frac{1}{2}-3s} \\ |z_j| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \text{ if } j \ll n^{-\frac{1}{2}+6s-\epsilon} \\ |z_j| \lesssim n^{-9s+\epsilon} \text{ for the other } j \ll n^{8s-2\epsilon} \end{cases}$$

3.3.4 The ansatz

As was proved in §3.2.2,

$$< |\psi|^2\psi, h_{n+j} >=< r, h_{n+j} >$$

is a real number independent of time. As a consequence, using (3.22) and the analysis of the source terms $R_{j,l}$ done in §3.3.2 for $\alpha = 8s-2\epsilon$, one can write for all $j \ll n^{8s-2\epsilon}$

$$e^{-tA_j}B_j\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = O(n^{-9s+\epsilon}) - \int_0^t e^{-\tau A_j}d\tau B_j\begin{pmatrix} 0 \\ < r, h_{n+j} > \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = O(n^{-9s+\epsilon}) - B_j^{-1}\int_0^t e^{\tau A_j}d\tau B_j\begin{pmatrix} 0 \\ < r, h_{n+j} > \end{pmatrix}.$$

Using the explicit form of A_j and B_j

$$z_j(t) = O(n^{-9s+\epsilon}) - i\frac{< r, h_{n+j} >}{2}\int_0^t e^{i\tau\mu_j}\left(\frac{\mu_j}{3k_{j,j}+\alpha_j}+1\right) + e^{-i\tau\mu_j}\left(\frac{\mu_j}{3k_{j,j}+\alpha_j}-1\right)d\tau =$$

$$= O(n^{-9s+\epsilon}) - i < r, h_{n+j} > \int_0^t \frac{\mu_j}{3k_{j,j}+\alpha_j}\cos\tau\mu_j + i\sin\tau\mu_j d\tau.$$

Let

$$\beta \in \left]0, 6s-\frac{1}{2}\right[$$

and q_β the part of q containing only the spherical harmonics of order greater than $n+n^\beta$. Hence $q(t,x)$ can be decomposed as follows

$$q(t,x) = -i\int_0^t e^{i\tau M}r\,d\tau + i\sum_{j=n^\beta..\infty} <r,h_{n+j}> \int_0^t e^{i\tau\mu_j}d\tau\, h_{n+j}(x) + \widetilde{q}(t,x) + q_\beta(t,x),$$

where

$$\widetilde{q}(t,x) = \sum_{j=2..n^\beta}\left(O(n^{-9s+\epsilon}) - i <r,h_{n+j}> \left(\frac{\mu_j}{3k_{j,j}+\alpha_j} - 1\right)\int_0^t \cos\tau\mu_j d\tau\right) h_{n+j}(x).$$

As was proved in §3.2.2, the upper bound $n^{\frac{1}{2}-3s}$ of $<r,h_{n+j}>$ is an equivalent for $j=2$ so the $\mathbb{H}^s$ norm of the principal part of q is

$$\left\|\int_0^t e^{i\tau M}r\,d\tau\right\|_{\mathbb{H}^s} \approx \left(\sum_{j=2..\infty}\frac{|<r,h_{n+j}>|^2}{(nj+j^2)^2}(n+j)^{2s}\right)^{\frac{1}{2}} \approx n^{-\frac{1}{2}-2s}.$$

Similarly

$$\left\|\sum_{j=n^\beta..\infty}|<r,h_{n+j}>|\int_0^t e^{i\tau\mu_j}d\tau\, h_{n+j}\right\|_{\mathbb{H}^s} \lesssim \sum_{j=n^\beta..\infty}\frac{<r,h_{n+j}>}{nj+j^2}(n+j)^s \ll n^{-\frac{1}{2}-2s}.$$

Using the upper bounds (3.15) on the norms of q_β

$$\|q_\beta\|_{\mathbb{H}^s} \le \|q_\beta\|_2^{1-s}\|q_\beta\|_{\mathbb{H}^1}^s \lesssim n^{-\frac{1}{2}-2s-\frac{\beta}{2}} \ll n^{-\frac{1}{2}-2s}.$$

Finally, one can estimate the $\mathbb{H}^s$ norm of $\widetilde{q}$ as follows

$$\|\widetilde{q}\|_{\mathbb{H}^s} \lesssim n^\beta n^{-9s+\epsilon}n^s + \|r\|_2\sum_{j=2..n^\beta}\left(\frac{1}{3k_{j,j}+\alpha_j} - \frac{1}{\mu_j}\right)(n+j)^s \lesssim$$

$$\lesssim n^{\beta-8s+\epsilon} + n^{\frac{1}{2}-3s}\sum_{j=2..n^\beta}\frac{|k_{j,j}|}{\alpha_j^2}(n+j)^s.$$

The estimates

$$|k_{j,j}| \le \|\psi\|_\infty^2 = n^{\frac{1}{2}-2s} \ll nj+j^2 \approx \alpha_j$$

imply that

$$\|\widetilde{q}\|_{\mathbb{H}^s} \lesssim n^{\beta-8s+\epsilon} + n^{1-5s}\sum_{j=2..n^\beta}\frac{(n+j)^s}{(nj+j^2)^2} \lesssim$$

$$\lesssim n^{-\frac{1}{2}-2s}(n^{\beta+\frac{1}{2}-6s+\epsilon} + n^{-\frac{1}{2}-2s}) \ll n^{-\frac{1}{2}-2s}.$$

Therefore

$$\|q + i\int_0^t e^{i\tau M}r\,d\tau\|_{\mathbb{H}^s} \ll n^{-\frac{1}{2}-2s},$$

and the proof of Theorem 3.1.1 is complete.

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Chapitre 4

Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain

Abstract[1] In this paper we concentrate on the analysis of the critical mass blowing-up solutions for the cubic focusing Schrödinger equation with Dirichlet boundary conditions, posed on a plane domain. We bound from below the blow-up rate for bounded and unbounded domains. If the blow-up occurs on the boundary, the blow-up rate is proved to grow faster than $(T-t)^{-1}$, the expected one. Moreover, we show that blow-up cannot occur on the boundary, under certain geometric conditions on the domain.
2000 Mathematics Subject Classification. 35Q55, 35B33 35B40 35Q40
Keywords. Nonlinear Schrödinger, blow-up

4.1 Introduction

Consider the nonlinear Schrödinger equation on $\mathbb{R}^n$

$$(S)\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \\ u(0) = u_0. \end{cases}$$

The associated Cauchy problem is locally well posed in $\mathbb{H}^1$ for $p < 1 + \frac{4}{n-2}$ ([6], [8]).

The Gagliardo-Nirenberg inequality

$$\|v\|_{p+1}^{p+1} \leq C_{p+1}\|v\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla v\|_2^{(p-1)\frac{n}{2}}$$

implies that the energy of the solution u of the equation (S),

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx,$$

is bounded from below by

$$\|\nabla u\|_2^2\left(\frac{1}{2} - \frac{C_{p+1}}{p+1}\|u\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla u\|_2^{(p-1)\frac{n}{2}-2}\right).$$

As a consequence, if $p < 1 + \frac{4}{n}$, since the mass is conserved, the gradient of u is controlled by the energy. Therefore the solution does not blow up and global existence occurs.

The power $p = 1 + \frac{4}{n}$ is a critical power, in the sense that the nonlinearity is strong enough to generate solutions blowing up in a finite time. However, even in this case, we have a global result for small initial conditions.

Indeed, in the case $p = 1 + \frac{4}{n}$, if the mass of the initial condition is small enough so that

$$\frac{C_{2+\frac{4}{n}}}{2+\frac{4}{n}}\|u\|_2^{\frac{4}{n}} < \frac{1}{2},$$

then the energy controls the gradient and again, the global existence is proved for the equation (S).

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[1] in preparation

For this particular value of p, Weinstein has given a sharpening of the Gagliardo-Nirenberg inequality ([25]). By variational methods using Lions concentration-compacity lemma ([11], [12]), he obtained the existence of a minimizer Q for the optimal constant of Gagliardo-Nirenberg's inequality

$$\frac{1}{C_{2+\frac{4}{n}}} = \inf_{v \in \mathbb{H}^1(\mathbb{R}^n)} \frac{\|v\|_2^{\frac{4}{n}} \|\nabla v\|_2^2}{\|v\|_{2+\frac{4}{n}}^{2+\frac{4}{n}}}.$$

This minimizer verifies the Euler-Lagrange equation

$$\Delta Q + Q^{1+\frac{4}{n}} = \frac{2}{n} Q.$$

Such a positive function, called ground state of the nonlinear Schrödinger equation, is radial, exponentially decreasing at infinity and regular. Recently, Kwong has shown that it is unique up to a translation ([10]). Moreover, it verifies Pohozaev's identities

$$\begin{cases} \|\nabla Q\|_2^2 - \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + \frac{2}{n}\|Q\|_2^2 = 0, \\ (n-2)\|\nabla Q\|_2^2 - \frac{n^2}{n+2}\|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + 2\|Q\|_2^2 = 0, \end{cases}$$

which lead to the following relations between the norms of Q

$$\begin{cases} \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} = \frac{n+2}{n}\|Q\|_2^2, \\ \|\nabla Q\|_2^2 = \|Q\|_2^2. \end{cases} \tag{4.1}$$

Then the optimal value for the constant of the Gagliardo-Nirenberg inequality is

$$C_{2+\frac{4}{n}} = \frac{n+2}{n} \frac{1}{\|Q\|_2^{\frac{4}{n}}}.$$

In conclusion, if $p = 1 + \frac{4}{n}$, the solutions of the equation (S) with initial condition of mass smaller than the one of the ground state

$$\|u\|_2 < \|Q\|_2,$$

are global in time.

The mass $\|Q\|_2$ is critical, in the sense that we can construct as follows solutions of mass equal to $\|Q\|_2$, which blows up in finite time. Since $p = 1 + \frac{4}{n}$, the pseudo-conformal transform of a solution u of (S)

$$\frac{1}{t^{\frac{n}{2}}} e^{i\frac{|x|^2}{4t}} u\left(-\frac{1}{t}, \frac{x}{t}\right),$$

is also a solution of (S) ([4]). So, from a stationary solution on $\mathbb{R}^n$

$$e^{it} Q(x),$$

for all positive T,

$$u(t,x) = \frac{e^{\frac{i}{T-t}}}{(T-t)^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4(T-t)}} Q\left(\frac{x}{T-t}\right),$$

is a solution blowing-up at the time T. Moreover, Merle proved in [15] that all blowing up solutions on $\mathbb{R}^n$ with critical mass $\|Q\|_2$ are of this type, up to the invariants of the equation. The proof is based on a result of concentration of Weinstein ([26], see Lemma 4.1.1) and on the study of the first order momentum

$$f(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 x dx,$$

and of the virial

$$g(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 |x|^2 dx$$

associated to a solution u of the equation (S). The conservative properties of these two quantities on $\mathbb{R}^n$, in the case of the critical power $1+\frac{4}{n}$, play an important role in Merle's proof. The derivative of the first order momentum is constant in time

$$\partial_t^2 f = 0,$$

and g satisfies the virial identity ([4])

$$\partial_t^2 g = 16E(u) - 4\frac{n(p-1)-4}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx = 16E(u).$$

In certain cases of initial conditions with mass larger than $\|Q\|_2$ recent achievements were done by Merle and Raphaël, concerning the blow-up rate and the blow-up profile ([17], [18]).

For the equation (S_p) with $p \geq 1+\frac{4}{n}$, Zakharov [28] and Glassey [7] had obtained that the solutions of negative energy are blowing up in finite time. The same result for solutions of positive or null energy is valid under certains conditions on the derivatives of the virial ([23]). The proof is based on an upper bound of the virial in terms of its first and second derivative, which implies the cancellation of the virial at a finite time T. Since the mass is conserved, it follows that the solution must blow up at the time T.

In this paper we are concerned with the nonlinear Schrödinger equation posed on a regular domain Ω of $\mathbb{R}^n$, with Dirichlet boundary conditions

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \\ u_{|\mathbb{R}\times\partial\Omega} = 0, \\ u(0) = u_0. \end{cases}$$

The Cauchy problem is locally well posed on $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ in dimension 2 and 3. In dimension 2, for nonlinearities less than cubic, Vladimirov [24] and Ogawa and Ozawa [20] have shown the well-posedness of the Cauchy problem on $\mathbb{H}_0^1(\Omega)$, but without the uniform continuity of the flow on bounded sets of $\mathbb{H}^1_0(\Omega)$. For nonlinearities stronger than cubic in dimension 2, or for any power nonlinearity p, in dimension higher than 2, the Cauchy problem on $\mathbb{H}^1_0(\Omega)$ is open.

For the equation with power $p < 1 + \frac{4}{n}$, one can show as for the case $\mathbb{R}^n$ that the $\mathbb{H}^1_0(\Omega)$ solutions are global in time. For the equation with power $p \geq 1 + \frac{4}{n}$, posed on a star-shaped domain of $\mathbb{R}^n$, Kavian has proved the blow-up in finite time of the $\mathbb{H}^2 \cap \mathbb{H}^1_0(\Omega)$ solutions of negative energy or of positive energy but under some conditions on the first and second derivatives of the virial ([9]). His proof follows the one on $\mathbb{R}^n$ ([7]), by estimating via the geometric condition on Ω the boundary terms which appear in the second derivative of the virial.

From now on we will analyze the cubic equation on Ω

$$(S_\Omega) \quad \begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, \\ u_{|\mathbb{R}\times\partial\Omega} = 0, \\ u(0) = u_0. \end{cases}$$

Let us first notice that the conservations of the mass and of the energy of the solutions are still valid. The Cauchy problem is locally well posed on $\mathbb{H}^2 \cap \mathbb{H}^1_0(\Omega)$ and also on $\mathbb{H}^1_0(\Omega)$, but without uniform continuity of the flow. The usual Strichartz inequalities are no longer valid and the loss of derivatives is stronger than in the case of a compact manifold ([3]).

As in the case of the plane, for initial conditions with mass smaller than the one of the ground state, the Cauchy problem is globally well-posed on $\mathbb{H}^2 \cap \mathbb{H}^1_0(\Omega)$. The proof, given by Brézis and Gallouët, is based on logarithmic type estimates ([2]). This result has been extended to the natural space $\mathbb{H}^1_0(\Omega)$, up to the uniform continuity of the flow ([24],[20],[4]).

The critical mass for blow-up is $\|Q\|_2$, as in the case of the equation posed on $\mathbb{R}^2$. More precisely, the following result holds.

Theorem 4.1.1. *(Burq-Gérard-Tzvetkov [3]) Let Ω be a regular bounded domain of $\mathbb{R}^2$. Let $x_0 \in \Omega$ and $\psi \in C_0^\infty$ be a function equal to 1 near x_0. Then there exist positive numbers κ and α_0 such that for all $\alpha > \alpha_0$, there exists a time T_α and a function r_α defined on $[0, T_\alpha[\times\Omega$, satisfying*

$$\|r_\alpha(t)\|_{\mathbb{H}^2(\Omega)} \leq ce^{-\frac{\kappa}{T_\alpha - t}},$$

such that

$$u(t,x) = \psi(x)\frac{e^{\frac{i}{\alpha^2(T_\alpha - t)}}}{\alpha(T_\alpha - t)}e^{-i\frac{|x-x_0|^2}{4\alpha(T_\alpha - t)}}Q\left(\frac{x - x_0}{\alpha(T_\alpha - t)}\right) + r_\alpha(t,x),$$

is a critical mass solution of (S_Ω), blowing up at x_0 at the time T_α with the blow-up rate $\frac{1}{T_\alpha - t}$.

The proof, following an idea of Ogawa and Tsutsumi ([21]), is based on a fixed point method which allows to complete the cut-off of the explicit blowing up solution on $\mathbb{R}^2$ at x_0 to a blowing up solution on Ω at x_0. Theorem 4.1.1 implies in particular that at every point of Ω there are explosive solutions. Moreover, the proof is still valid for the torus $\mathbb{T}^2$

and for a larger class of subsets of the plane, which satisfy the property of 2-continuation, from $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ to $\mathbb{H}^2(\mathbb{R}^2)$, and for which the Laplacian domain

$$D(-\Delta_\Omega) = \{u \in \mathbb{H}_0^1(\Omega), \Delta u \in \mathbb{L}^2(\Omega)\},$$

is $\mathbb{H}^2 \cap \mathbb{H}_0^1$. Such subsets are for example the domains with compact regular boundary and convex polygons bounded or unbounded.

As in the $\mathbb{R}^n$ case, the following lemma, due to Weinstein, will give us the general behavior of a blowing-up solution of critical mass on a domain.

Lemma 4.1.1. *(Weinstein [26]) Let $u_k \in \mathbb{H}^1(\mathbb{R}^n)$ be a sequence of functions of critical mass satisfying*

$$\begin{cases} \beta_k = \|\nabla u_k\|_2 \underset{k\to\infty}{\longrightarrow} \infty, \\ E(u_k) \underset{k\to\infty}{\longrightarrow} c < \infty. \end{cases}$$

Then there exist points $x_k \in \mathbb{R}^d$ and $\theta_k \in \mathbb{R}$ such that in $\mathbb{H}^1(\mathbb{R}^n)$

$$\frac{e^{i\theta_k}}{\beta_k^{\frac{n}{2}}} u_k\left(\frac{x}{\beta_k} + x_n\right) \underset{k\to\infty}{\longrightarrow} \frac{1}{\omega^{\frac{n}{2}}} Q(\omega x),$$

where $\omega = \|\nabla Q\|_2$.

Let u be a solution of (S_Ω) that blows up at the finite time T, that is

$$\lambda(t) = \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \underset{t\to T}{\longrightarrow} \infty.$$

Consider u to be extended by zero outside Ω. By combining Lemma 4.1.1 for families $u_k = u(t_k)$ with t_k sequences convergent to T with the result of Kwong on the uniqueness of the ground state ([10]), there exist $\theta(t)$ real numbers and $x(t) \in \mathbb{R}^2$ such that in $\mathbb{H}^1(\mathbb{R}^2)$

$$\frac{e^{i\theta(t)}}{\lambda(t)} u\left(t, \frac{x}{\lambda(t)} + x(t)\right) \underset{t\to T}{\longrightarrow} Q(x). \tag{4.2}$$

Then, in the space of distributions,

$$|u(t, \cdot + x(t))|^2 \underset{t\to T}{\longrightarrow} \|Q\|_2^2 \, \delta_0. \tag{4.3}$$

In this paper we concentrate on the further analysis of the blowing-up solutions with critical mass on a plane domain. The results are the following.

Theorem 4.1.2. *Let u be a $C([0,T[, \mathbb{H}_0^1)$ solution of the Schrödinger equation (S_Ω), which has critical mass and blows up at the finite time T.*

i) For bounded domains, the blowing-up rate is lower bounded by

$$\frac{1}{T-t} \lesssim \|\nabla u(t)\|_2.$$

ii) If there exist solutions u of critical mass blowing up at a finite time T on the boundary of Ω, that is if the concentration parameter $x(t)$ converges as $t \to T$ to a point on the boundary, then the blowing-up rate satisfies

$$\lim_{t \to T}(T - t)\|\nabla u(t)\|_2 = \infty.$$

The main difficulty for the Schrödinger equation posed on a domain is that the conservation of the derivative of the first momentum and the virial identity fail.

In order to avoid this difficulty, we shall use systematically in the proof of Theorem 4.1.2 a Cauchy-Schwarz type inequality derived from Weinstein's inequality. Precisely, we show that if v is a $\mathbb{H}^1(\mathbb{R}^2)$ function of critical or subcritical mass, then

$$\left|\int \Im(v \nabla \bar{v}) \nabla \theta dx\right| \leq \left(2E(v) \int |v|^2 |\nabla \theta|^2 dx\right)^{\frac{1}{2}}$$

for all real function θ. This inequality allows us to estimate the virial, that we shall assume to be localized if Ω is unbounded (see Remark 4.1.1). The lower bound for the blowing-up rate is the same as the one found by Antonini on the torus ([1]).

By following the approach of Weinstein in [27], and the recent results of Maris in [14], we analyze the convergence to the ground state of the modulations of the solutions (4.2), and we obtain, for bounded domains, the following additional informations.

Proposition 4.1.1. *i) The blow-up rate verifies*

$$\int |u(t,x)|^2 |x - x(t)|^2 dx \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

ii) The concentration parameter $x(t)$ can be chosen to be as the first order momentum

$$x(t) = \frac{\int |u(t,x)|^2 x dx}{\|Q\|_2^2}.$$

Corollary 4.1.1. *If the equation (S_Ω) is considered to be invariant under rotations, then $x(t)$ can be chosen 0, and we have*

$$g(t) \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

Remark 4.1.1. *For unbounded domains, if the solution concentrates at one point, that is if $x(t)$ converges as $t \to T$, then the first assertion of Theorem 4.1.2 is true, and so are the assertions of Proposition 4.1.1, for the virial and the first order momentum localized at the blow-up point.*

There is no known example of a solution of nonlinear Schrödinger equation with a blow-up rate larger than $\frac{1}{T-t}$, neither in the case of supercritical mass, nor in the case of supercritical nonlinearities.

Therefore we expect that the blowing-up rate grows exactly like $\frac{1}{T-t}$ and that the profiles are the ones on $\mathbb{R}^2$ modulo an exponentially decreasing in $\mathbb{H}^1$ function.

Since it is not likely that the blowing-up rate at the boundary grows strictly faster than $\frac{1}{T-t}$, we also expect that there are no solutions blowing-up on the boundary of a domain. This is confirmed for certain simple cases by the following result.

Theorem 4.1.3. *If Ω is a half-plane or a plane sector, then there are no solutions blowing-up in a finite time on the boundary of the half-plane or in the corner of the sector respectively.*

Indeed, under these geometric hypotheses on Ω, the boundary terms which appear in the second derivative of the virial associated to a blowing-up solution of (S_Ω) cancel, so we have, as on $\mathbb{R}^n$, the virial identity

$$\partial_t^2 g = 16E(u).$$

The proof then follows the one by Merle in [15] for the equation posed on $\mathbb{R}^n$, and we obtain that all explosive solutions on Ω must be of the type

$$\frac{e^{\frac{i}{T-t}}}{(T-t)^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4(T-t)}} Q\left(\frac{x}{T-t}\right),$$

up to the invariants of the equation. Therefore we arrive at a contradiction by looking at the support of the solution.

The paper is organized as follows. The first section §4.2 contains some results on general domains. We prove a Cauchy-Schwarz type inequality for critical and subcritical mass functions, which we will use to show Theorem 4.1.2. The nature of the convergence to the ground state of the modulations of the solutions is analyzed, by spectral theory techniques, and will be used to prove Theorem 4.1.3 and Proposition 4.1.1. Moreover, we calculate the derivatives in time for a virial type function. In §4.3, by studying the virial, the lower-bound of the blowing-up rate is proved for bounded domains Ω. In this section, we also give the proof of Proposition 4.1.1. In §4.4, by introducing a localized virial, we find the same lower-bound for the blowing-up rate for unbounded domains. The last section §4.5 contains the results regarding the explosion on the boundary of Ω.

I thank my advisor Patrick Gérard for having introduced me to this beautiful subject and for having guided this work.

4.2 Results on general domains

4.2.1 A Cauchy-Schwarz inequality for subcritical mass functions

Lemma 4.2.1. *Let θ be a real valued function. All $v \in \mathbb{H}^1(\mathbb{R}^2)$ with critical or subcritical mass satisfy*

$$(*) \left|\int \Im(v\nabla\overline{v})\nabla\theta dx\right| \leq \left(2E(v)\int |v|^2|\nabla\theta|^2 dx\right)^{\frac{1}{2}}.$$

Proof. The precised version of the Gagliardo-Niremberg inequality, presented in the introduction, is, for function w in $\mathbb{H}^1(\mathbb{R}^2)$,

$$\|w\|_4^4 \leq \frac{2}{\|Q\|_2^2}\|w\|_2^2\|\nabla w\|_2^2.$$

As a consequence, if

$$\|w\|_2 \leq \|Q\|_2,$$

then the energy of w is positive.

Therefore on one hand,

$$0 \leq E(e^{i\alpha\theta}v)$$

for every real number α and for all real function θ, since $e^{i\alpha\theta}v$ is still a function of critical or subcritical mass. On the other hand

$$E(e^{i\alpha\theta}v) = \frac{1}{2}\int |i\alpha\nabla\theta\, v + \nabla v|^2 dx - \frac{1}{4}\int |v|^4 dx$$

$$= \frac{\alpha^2}{2}\int |v|^2|\nabla\theta|^2 dx - \alpha\int \Im(v\nabla\overline{v})\nabla\theta dx + E(v)$$

Thus the discriminant of the equation in α must be non positive and we obtain the claimed Cauchy-Schwarz type inequality $(*)$.

□

4.2.2 The concentration of the solution

In this subsection we shall give a finer description of a critical mass blowing-up solution u of (S_Ω), by following the approach of Weinstein in [27].

In order to deal with real functions, we shall analyze the modulus of u. However, the same arguments below can be used to get the corresponding results on u (see Remark 4.2.2).

One can write the convergence (4.2)

$$u(t,x) = e^{-i\theta(t)}\lambda(t)(Q + R(t))(\lambda(t)(x - x(t))),$$

with R a complex function such that

$$\|R(t)\|_{\mathbb{H}^1(\mathbb{R}^2)} \underset{t\to T}{\longrightarrow} 0.$$

Since the modulus is a continuous function on $\mathbb{H}^1(\mathbb{R}^2)$ ([13]), this implies

$$|u(t,x)| = \lambda(t)|Q + R(t)|(\lambda(t)(x - x(t))) = \lambda(t)(Q + \tilde{\tilde{R}}(t))(\lambda(t)(x - x(t))),$$

with $\tilde{\tilde{R}}$ a real function strongly converging to 0 in $\mathbb{H}^1(\mathbb{R}^2)$.

Let us set

$$\tilde{\lambda}(t) = \frac{\|\nabla|u(t)|\|_2}{\|\nabla Q\|_2}.$$

By noticing that $|u(t)|$ is also of critical mass, its energy is positive, and

$$0 \leq \|\nabla u(t)\|_2^2 - \|\nabla|u(t)|\|_2^2 = 2E(u) - 2E(|u(t)|) \leq 2E(u), \tag{4.4}$$

which implies

$$(\lambda(t) + \tilde{\lambda}(t))(\lambda(t) - \tilde{\lambda}(t)) = O(1).$$

Since $0 \leq \tilde{\lambda}(t) \leq \lambda(t)$,

$$\lambda(t) - \tilde{\lambda}(t) = O\left(\frac{1}{\lambda(t)}\right),$$

and we have

$$|u(t,x)| = \tilde{\lambda}(t)(Q + \tilde{R}(t))(\tilde{\lambda}(t)(x - x(t))), \tag{4.5}$$

with $\tilde{R}(t)$ a real function such that

$$\|\tilde{R}(t)\|_{\mathbb{H}^1(\mathbb{R}^2)} \underset{t\to T}{\longrightarrow} 0.$$

In this subsection, we shall show

Proposition 4.2.1. *The remainder term $\tilde{R}$ has the decay*

$$\|\tilde{R}(t)\|_{\mathbb{H}^1} \leq \frac{\tilde{C}}{\tilde{\lambda}(t)} \leq \frac{C}{\lambda(t)}. \tag{4.6}$$

Remark 4.2.1. *Merle has pointed out recently that the same result was already proved in [16], by a slightly different method. However, for the sake of completness, we give here the following proof.*

Proof. The fact that u is of critical mass gives us

$$\int \tilde{R}^2 = -2\int Q\tilde{R}, \tag{4.7}$$

and the choice of $\tilde{\lambda}$ implies

$$\int |\nabla\tilde{R}|^2 = -2\int \nabla Q\nabla\tilde{R}.$$

Let us calculate the energy of $|u|$,

$$\frac{2E(|u|)}{\tilde{\lambda}^2} = \int |\nabla Q + \nabla\tilde{R}|^2 - \frac{1}{2}\int (Q + \tilde{R})^4.$$

The energy of Q is zero, so

$$\frac{2E(|u|)}{\tilde{\lambda}^2} = \int |\nabla \tilde{R}|^2 + 2\nabla Q \nabla \tilde{R} - \frac{\tilde{R}^4}{2} - 2Q\tilde{R}^3 - 3Q^2\tilde{R}^2 - 2Q^3\tilde{R}.$$

The ground state Q verifies the equation

$$\Delta Q + Q^3 = Q,$$

and therefore, by using the relation (4.7) on $\tilde{R}$,

$$\int 2\nabla Q \nabla \tilde{R} - 2Q^3\tilde{R} = -2\int Q\tilde{R} = \int \tilde{R}^2.$$

So finally

$$< L\tilde{R}, \tilde{R} >= \frac{2E(|u|)}{\tilde{\lambda}^2} + \frac{1}{2}\int \tilde{R}^4 + \int 2Q\tilde{R}^3,$$

where L is the operator

$$L = -\Delta + (1 - 3Q^2).$$

Since $\tilde{R}$ tends to 0 in $\mathbb{H}^1$, by using the Sobolev embeddings, the cubic and quadratic terms in $\tilde{R}$ are negligible with respect to the $\mathbb{H}^1$ norm of $\tilde{R}$. Also, the energy of $|u|$ is bounded by the constant energy of u, so for having (4.6) it is sufficient to prove the existence of a positive constant δ such that for t close enough to T

$$\delta \|\tilde{R}(t)\|^2_{\mathbb{H}^1} \leq < L\tilde{R}(t), \tilde{R}(t) > .$$

Remark 4.2.2. *The initial complex function R can be analyzed by the same manner, and one has*

$$< L_-\Im R, \Im R > + < L\Re R, \Re R > \leq \frac{2E(u)}{\lambda^2} + \frac{1}{2}\int |R|^4 + \int 2Q|R|^3,$$

where L_- is the operator

$$L_- = -\Delta + (1 - Q).$$

This operator is non-negative and its kernel is spanned by Q. So once the decay (4.6) is obtained, by decomposing $\Im R$ with respect to Q, we also have

$$\|R(t)\|_{\mathbb{H}^1} \leq \frac{C}{\lambda(t)}.$$

Following the ideas of Weinstein in [27], we shall look for the nature of the negative eigenvalues of L.

Lemma 4.2.2. *The second eigenvalue of L is 0.*

Proof. Let us consider the functional

$$J(f) = \frac{\|u\|_2^2 \|\nabla u\|_2^2}{\|u\|_4^4},$$

which is minimized by Q (see the introduction). Then, for a test function f,

$$\partial_\epsilon^2 J(Q + \epsilon f)|_{\epsilon=0} \geq 0,$$

and so, by using (4.1) in the calculus of this second derivative, one has

$$2\|Q\|_2^2 < Lf, f > \geq -8 < Q, f >< \nabla Q, \nabla f > .$$

If we take f to be orthogonal to Q, then

$$< Lf, f > \geq 0,$$

and by the Min-Max Principle ([22]), the second eigenvalue of L is non-negative. By noticing that the two partial derivatives of Q verify

$$L\partial_i Q = 0,$$

we obtain that 0 is an eigenvalue of L of order grater than one, so the first eigenvalue is negative. Therefore the second eigenvalue of L is 0. □

We shall use the following theorem.

Theorem 4.2.1. *(M. Maris [14]). Let $g \in C^1([0,\infty))$, with $g(0) = 0$, $g'(0) > 0$ and $|g'(s) - g'(0)| \leq C|s|^\alpha$, for small s and some $C, \alpha > 0$. Let $a_0 = \sup\{a > 0 | g(s) > 0, \forall s \in (0,a)\}$, and let u_0 be a ground state of the operator*

$$-\Delta u + g(u).$$

We define

$$I(u, \lambda) = \lambda u g'(u) - (\lambda + 2) g(u),$$

and we will make the following assumptions : $a_0 < u_0(0)$ and there exists a continuous function $\lambda : (a_0, u_0(0)] \to (0, \infty)$ such that for any $U \in (a_0, u_0(0)]$ we have

$$\begin{cases} I(u, \lambda(U)) \leq 0, \forall u \in [0, U], \\ I(u, \lambda(U)) \geq 0, \forall u \in [U, u_0(0)]. \end{cases}$$

Then

$$Ker(-\Delta + g'(u_0)) = \{\partial_1 u_0, \partial_2 u_0\}.$$

Next we show that the operator L verifies the hypothesis of the theorem.

Lemma 4.2.3. *The kernel of L has dimension 2.*

Proof. In we take function g to be

$$g(s) = s - s^3,$$

then $a_0 = 1$, the ground state u_0 is Q,

$$-\Delta + g'(u_0) = L,$$

and

$$I(u, \lambda) = 2u((1-\lambda)u^2 - 1).$$

Let us consider the integral of g,

$$G(s) = \frac{s^2}{2} - \frac{s^4}{4}.$$

By using the relation (4.1) between the $\mathbb{L}^2$ and the $\mathbb{L}^4$ norms of Q

$$\int G(Q(x))dx = 0.$$

The positivity of $G(s)$ on $[0, \sqrt{2}[$ implies the existence of points x such that $Q(x) > \sqrt{2}$, and in particular $Q(x) > 1$. Let us recall that Q is a radial positive decreasing function. It follows that $Q(0) > 1$, and the first assumption of the theorem 4.2.1 is satisfied. The second assumption is satisfied for the function

$$\lambda(U) = 1 - \frac{1}{U^2},$$

and we can conclude that

$$KerL = \{\partial_1 Q, \partial_2 Q\}.$$

□

We return now to the study of $\tilde{R}$. We impose a choice of $x(t)$ which will yield an orthogonality property of $\tilde{R}$. Since

$$\frac{1}{\tilde{\lambda}(t)}|u|\left(t, \frac{x}{\tilde{\lambda}(t)} + x(t)\right) \underset{t\to T}{\longrightarrow} Q(x),$$

we can choose $x(t)$ such that the functional

$$I(z) = \left\|\frac{1}{\tilde{\lambda}(t)}|u|\left(t, \frac{\cdot + z}{\tilde{\lambda}(t)} + x(t)\right) - Q(\cdot)\right\|^2_{\mathbb{H}^1}$$

reaches its minimum for $z = 0$. By using (4.5), this implies that the derivative in z of

$$\partial_z \|(Q + \tilde{R}(t))(\cdot) - Q(\cdot - z)\|^2_{\mathbb{H}^1},$$

must be zero at $z = 0$. It follows that

$$\int \tilde{R}\,\partial_i Q + \int \nabla\tilde{R}\,\partial_i \nabla Q = 0.$$

One can then integrate by parts and obtain

$$\int \tilde{R}\,\partial_i Q - \int \tilde{R}\,\partial_i \Delta Q = 0.$$

By recalling that the ground state Q verifies

$$\Delta Q + Q^3 = Q,$$

it follows that $\tilde{R}$ has the orthogonality property

$$< \partial_i Q^3, \tilde{R} >= 0. \tag{4.8}$$

Let us recall that for having the decay property (4.6) of $\tilde{R}$, it is sufficient to prove that the operator L controls its $\mathbb{H}^1$ norm.

Lemma 4.2.4. *There exist a positive constant δ such that for t close enough to T,*

$$\delta\|\tilde{R}(t)\|^2_{\mathbb{H}^1} \leq < L\tilde{R}(t), \tilde{R}(t) > .$$

Proof. We denote by $R_{\shortparallel}$ the projection of $\tilde{R}$ on the space spanned by Q, and by $R_\perp$ the remainder term, orthogonal to Q. Since the operator L is self-adjoint,

$$< L\tilde{R}, \tilde{R} >=< LR_{\shortparallel}, R_{\shortparallel} > +2 < LR_{\shortparallel}, R_\perp > + < LR_\perp, R_\perp > .$$

The first term reads

$$< LR_{\shortparallel}, R_{\shortparallel} >=< LQ, Q > \frac{< Q, \tilde{R} >^2}{\|Q\|_2^4},$$

and by using (4.7)

$$< LR_{\shortparallel}, R_{\shortparallel} >= C\|\tilde{R}\|_2^4.$$

The second term is

$$< LR_{\shortparallel}, R_\perp >= \frac{< Q, \tilde{R} >}{\|Q\|_2^2} < LQ, R_\perp >,$$

and since $LQ = -2Q^3$, by using the Cauchy-Schwarz inequality,

$$< LR_\perp, R_{\shortparallel} >= -2\frac{< Q, \tilde{R} >}{\|Q\|_2^2} < Q^3 . R_\perp >\leq C\|\tilde{R}\|_2^3.$$

Now we have to estimate the third term. Let us notice that the orthogonality relation (4.8) yields

$$< \partial_i Q^3, R_\perp >= 0.$$

We will show that

$$\inf_{f\in^{\perp}\{Q,\partial_i Q^3\}} \frac{< Lf, f >}{\|f\|_2^2} = I > 0.$$

From the proof of Lemma 4.2.2 we have $I \geq 0$. Consider now a sequence of functions f_j, normalized in $\mathbb{L}^2$, which minimize I

$$< Lf_j, f_j > \underset{j\to\infty}{\longrightarrow} I.$$

The gradients of f_j are also bounded in $\mathbb{L}^2$, so we can extract a subsequence converging weakly in $\mathbb{H}^1$ to a function f

$$f_{j_n} \rightharpoonup f.$$

In particular,

$$< f_{j_n}^2, Q^2 > \underset{n\to\infty}{\longrightarrow} < f^2, Q^2 >,$$

and it follows that f is a minimizer for I,

$$Lf = If.$$

If $I = 0$, then f must be in the kernel of L. Lemma 4.2.3 ensures us that the kernel contains only the derivatives of Q, and since f is orthogonal to the derivatives of Q^3, it follows that $f = 0$. This is in contradiction with the positive $\mathbb{L}^2$ norm of f, so $I > 0$.

Therefore, since $R_\perp$ is orthogonal to Q and to the two derivatives of Q^3,

$$< LR_\perp, R_\perp > \geq I\|R_\perp\|_2^2 = I(\|\tilde{R}\|_2^2 - \|R_{\shortparallel}\|_2^2).$$

Arguing as for the first term,

$$\|R_{\shortparallel}\|_2^2 \leq C\|\tilde{R}\|_2^4,$$

and we finally have

$$< L\tilde{R}, \tilde{R} > \geq I\|\tilde{R}\|_2^2 - C\|\tilde{R}\|_2^4 - C\|\tilde{R}\|_2^3.$$

Since $\tilde{R}$ tends to 0 in $\mathbb{L}^2$ norm, there exist a positive constant C such that for t close enough to T,

$$< L\tilde{R}, \tilde{R} > \geq C\|\tilde{R}\|_2^2.$$

For a positive number ϵ,

$$< L\tilde{R}, \tilde{R} >= \epsilon\left(\int |\nabla \tilde{R}|^2 + \int (1 - 3Q^2)\tilde{R}^2\right) + (1-\epsilon) < L\tilde{R}, \tilde{R} >,$$

so, using the control of the $\mathbb{L}^2$ norm by L and the boundeness of Q,

$$< L\tilde{R}, \tilde{R} > \geq \epsilon\left(\int |\nabla \tilde{R}|^2 - C_Q \int \tilde{R}^2\right) + (1-\epsilon)C\|\tilde{R}\|_2^2.$$

By choosing ϵ small enough to have

$$(1-\epsilon)C - \epsilon C_Q > 0,$$

we get the existence of a positive constant δ such that

$$< L\tilde{R}, \tilde{R} > \geq \delta \|\tilde{R}\|^2_{\mathbb{H}^1}.$$

□

Therefore the proof of Proposition 4.2.1 is complete. □

Finally, let us give the following property of decay of the solution.

Lemma 4.2.5. *Let u be a critical mass solution of (S_Ω), blowing up at the finite time T, at one point $x_0 \in \Omega$, which means that the concentration parameter $x(t)$ converges to x_0. Then, the gradient of $u(t)$ restricted outside any neighborhood V of x_0 satisfies*

$$\sup_{t\in[0,T[} \int_{^cV} |\nabla u(t)|^2 dx < \infty.$$

Proof. The inequality (4.4) implies

$$\sup_{t\in[0,T[} \int_{^cV} |\nabla u(t)|^2 dx \leq 2E(u) + \sup_{t\in[0,T[} \int_{^cV} |\nabla |u(t)||^2 dx.$$

By using (4.5),

$$\int_{^cV} |\nabla |u(t)||^2 dx = \tilde{\lambda}^2(t) \int_{^c\tilde{\lambda}(t)(V-x(t))} |\nabla Q + \nabla \tilde{R}(t)|^2.$$

Since $x(t)$ converges to x_0 and Q is exponentially decreasing,

$$\tilde{\lambda}^2(t) \int_{^c\tilde{\lambda}(t)(V-x(t))} |\nabla Q|^2 = o(1).$$

Then it follows that

$$\int_{^cV} |\nabla |u(t)||^2 dx \lesssim \tilde{\lambda}^2(t) \int_{^c\tilde{\lambda}(t)(V-x(t))} |\nabla \tilde{R}|^2,$$

and the decay of R (4.6) implies

$$\int_{^cV} |\nabla |u(t)||^2 dx = O(1),$$

so the lemma is proved. □

Remark 4.2.3. *Another proof of this lemma can be done by using the appoach of Merle in [15]. However, we shall need the full strength of Proposition 4.2.1 later in §4.3.3 and §4.3.4.*

4.2.3 Derivatives of virial type functions

Let u be a solution of (S_Ω) and let h be a $\mathcal{C}^\infty(\mathbb{R}^2)$ function with bounded first and second derivatives. Then, by using the fact that u satisfies (S_Ω), we obtain

$$\partial_t \int_\Omega |u(t)|^2 h dx = 2\int_\Omega \Re\left(u(t)\overline{u}_t(t)\right) h dx = 2\int_\Omega \Im\left(u(t)\Delta\overline{u}_t(t)\right) h dx.$$

Since u cancels on the boundary of Ω, by integration by parts

$$\partial_t \int_\Omega |u(t)|^2 h dx = -2\int_\Omega \Im\left(u(t)\nabla\overline{u}_t(t)\right)\nabla h dx. \tag{4.9}$$

By using again the equation (S_Ω)

$$\partial_t{}^2 \int_\Omega |u|^2 h = -2\int_\Omega \Re\left((\Delta u + |u|^2 u)\nabla\overline{u}\right)\nabla h + 2\int_\Omega \Re\left(u\nabla(\Delta\overline{u} + |u|^2\overline{u})\right)\nabla h$$

$$= \int_\Omega -|u|^2\Delta^2 h - |u|^4\Delta h + 2|\nabla u|^2\Delta h - 4\Re\left(\Delta u\nabla\overline{u}\right)\nabla h.$$

It follows that

$$\partial_t{}^2 \int_\Omega |u|^2 h = \int_\Omega -|u|^2\Delta^2 h - |u|^4\Delta h + 4\Re\sum_{i,j}\partial_i u\partial_j\overline{u}\partial_{ij}h - 2\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2\frac{\partial h}{\partial\nu}d\nu.$$

Therefore, by making the energy of the solution appear, we have the following identity.

Lemma 4.2.6. *For a solution u of (S_Ω) and a $\mathcal{C}^\infty(\mathbb{R}^2)$ function h with bounded derivatives $\partial_{ij}h$ and $\Delta^2 h$, we have*

$$\partial_t{}^2 \int_\Omega |u|^2 h = 16E(u) - \int_\Omega (2|\nabla u|^2 - |u|^4)(4 - \Delta h) - \int_\Omega |u|^2\Delta^2 h$$

$$+4\Re\sum_{i,j}\partial_i u\partial_j\overline{u}\partial_{ij}h - 2|\nabla u|^2\Delta h - 2\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2\frac{\partial h}{\partial\nu}d\sigma.$$

Corollary 4.2.1. *For a solution u of (S_Ω) and a $\mathcal{C}^\infty(\mathbb{R}^2)$ function h equal to $|x|^2$ on $B(0,R)$, with bounded derivatives $\partial_{ij}h$ and $\Delta^2 h$, we have the estimate*

$$\left|\partial_t^2 \int_\Omega |u(t)|^2 h\right| \leq 16E(u) + C\int_{|x|\geq R}(|u(t)|^2 + |\nabla u(t)|^2)dx$$

$$+\int_{\partial\Omega}\left|\frac{\partial u(t)}{\partial\nu}\right|^2\left|\frac{\partial h}{\partial\nu}\right|d\sigma.$$

4.3 The blow-up rate on bounded plane domains

4.3.1 The convergence of the concentration points $x(t)$

Lemma 4.3.1. *Let Ω be a bounded domain and let u be a critical mass solution of (S_Ω), blowing up at the finite time T. Then the concentration parameter $x(t)$ has a limit at the time T.*

Proof. From (4.3) it follows that for a test function ψ,

$$\int_{\Omega - x(t)} |u(t, x + x(t))|^2 \psi(x) dx \underset{t \to T}{\longrightarrow} \|Q\|_2^2 \psi(0).$$

If ψ is chosen such that $\psi(0) \neq 0$ then, since the set Ω is bounded, it follows that

$$\limsup_{t \to T} |x(t)| < \infty. \tag{4.10}$$

The first order momentum

$$f(t) = \int_\Omega |u(t, x)|^2 x dx,$$

stays finite in time since Ω is bounded and u conserves its mass. By using the formula (4.9) for vector-valued functions h, one can calculate the derivative

$$f'(t) = -2 \int_\Omega \Im(u(t) \nabla \overline{u}(t))dx.$$

The inequality $(*)$ in the special case $\theta_i(x) = x_i$ implies that this derivative is bounded in time

$$|f'(t)|^2 \leq 4 \sum_{i \in \{1,2\}} \left| \int_\Omega \Im(u(t) \nabla \overline{u}(t)) \nabla \theta_i dx \right|^2 \leq 16 E(u) \|u\|_2^2.$$

Therefore f admits a limit at the time T. Let us define x_0 by

$$f(T) = x_0 \|Q\|_2^2.$$

Using the convergence (4.3) and (4.10) which implies that $\Omega - x(t)$ is a uniformly bounded set, one has

$$f(t) - x(t) \|Q\|_2^2 = \int_{\Omega - x(t)} |u(t, x + x(t))|^2 x dx \underset{t \to T}{\longrightarrow} 0.$$

Therefore the point x_0 is the limit of $x(t)$, and the square of the solution behaves like a Dirac function

$$|u(t, \cdot)|^2 \underset{t \to T}{\longrightarrow} \|Q\|_2^2 \, \delta_{x_0}. \tag{4.11}$$

$\square$

In the following, we shall suppose, up to a translation, that the solution blows up at the point $0 \in \overline{\Omega}$.

4.3.2 Lower bound for the blow-up rate

The derivative in time of the the virial of u,

$$g(t) = \int_{\Omega} |u(t,x)|^2 |x|^2 dx,$$

can be calculated with the formula (4.9) with $h(x) = |x|^2$, and

$$g'(t) = -4 \int_{\Omega} \Im(u(t)\nabla\overline{u}(t))x dx.$$

Therefore the inequality (*) in the case $\theta(x) = |x|^2$ implies that

$$|g'(t)| \leq 4\sqrt{2E(u)g(t)}.$$

The concentration result (4.11) of the former subsection gives

$$g(T) = 0,$$

and one can now write

$$\sqrt{g(t)} = -\int_t^T \frac{g'(\tau)}{2\sqrt{g(\tau)}} d\tau \leq \int_t^T 2\sqrt{2E(u)} = 2\sqrt{2E(u)}(T-t),$$

and obtain

$$g(t) \leq 8E(u)(T-t)^2.$$

Then the uncertainty principle

$$\left(\int_{\mathbb{R}^2} |u|^2\right)^2 \leq \left(\int_{\mathbb{R}^2} |u|^2|x|^2\right)\left(\int_{\mathbb{R}^2} |\nabla u|^2\right)$$

gives us a lower bound of the blow-up rate

$$\frac{\|Q\|_2^2}{2\sqrt{2E(u)}(T-t)} \leq \|\nabla u(t)\|_2,$$

so the first assertion of Theorem 4.1.2 is proved.

4.3.3 Equivalence between the virial and the blow-up rate

By using (4.5),

$$\int_{\Omega} |u(t)|^2 |x - x(t)|^2 dx = \frac{1}{\tilde{\lambda}^2(t)} \int_{\tilde{\lambda}(t)(\Omega - x(t))} (Q + \tilde{R}(t))^2 |x|^2 dx.$$

Since $x(t)$ tends to 0 and Q is exponentially decreasing,

$$\frac{1}{\tilde{\lambda}^2}\int_{\tilde{\lambda}(t)(\Omega-x(t))} Q^2|x|^2dx = O\left(\frac{1}{\tilde{\lambda}^2(t)}\right),$$

so

$$\int_\Omega |u(t)|^2|x-x(t)|^2dx \lesssim \frac{1}{\tilde{\lambda}^2(t)}\int_{\tilde{\lambda}(t)(\Omega-x(t))} \tilde{R}^2(t)|x|^2dx + \frac{1}{\tilde{\lambda}^2(t)}.$$

The domain Ω is considered bounded, so one can write

$$\int_\Omega |u(t)|^2|x-x(t)|^2dx \lesssim |\Omega|^2\int \tilde{R}^2(t)dx + \frac{1}{\tilde{\lambda}^2(t)},$$

and by using the decay (4.6) of $\tilde{R}$, we obtain

$$\int_\Omega |u(t)|^2|x-x(t)|^2dx \lesssim \frac{1}{\lambda^2(t)}.$$

As we did in the previous subsection, by the uncertainy principle for $u(t,x+x(t))$,

$$\|u\|_2^4 \lesssim \lambda^2(t)\int |u(t)|^2|x-x(t)|^2dx,$$

and so the first assertion of Proposition 4.1.1 follows,

$$\int_\Omega |u(t)|^2|x-x(t)|^2dx \approx \frac{1}{\lambda^2(t)}.$$

4.3.4 A differentiable choice for $x(t)$

Let us set

$$y(t) = \frac{\int |u(t)|^2xdx}{\|Q\|^2}.$$

By using the conservation of the mass, which is critical,

$$x(t)-y(t) = \frac{1}{\|Q\|_2^2}\int |u(t)|^2(x-x(t))dx.$$

Then by (4.5) one has

$$x(t)-y(t) = \frac{1}{\tilde{\lambda}(t)\|Q\|_2^2}\int_{\tilde{\lambda}(t)(\Omega-x(t))} (Q+R(t))^2xdx.$$

Therefore, by the same arguments as in the previous subsection, and the by using the fact that since Q is radially symmetric,

$$\int Q^2(x)xdx = 0,$$

then

$$|x(t) - y(t)| \leq \frac{C}{\tilde{\lambda}^2(t)}.$$

If we define S by

$$|u(t,x)| = \tilde{\lambda}(Q + \tilde{R}(t))(\tilde{\lambda}(x - x(t))) = \tilde{\lambda}(Q + S(t))(\tilde{\lambda}(x - y(t))),$$

one has

$$\|S(t)\|_{\mathbb{H}^1} \leq 2\|\tilde{R}(t)\|_{\mathbb{H}^1} + \|Q(\cdot + \lambda(x(t) - y(t)) - Q(\cdot)\|_{\mathbb{H}^1}.$$

The decay of the difference between $x(t)$ and $y(t)$, together with (4.6), implies

$$\|S(t)\|_{\mathbb{H}^1} \leq \frac{C}{\tilde{\lambda}(t)}.$$

So, by changing $x(t)$ into

$$\frac{\int |u(t)|^2 x dx}{\|Q\|^2},$$

we have the convergence corresponding to (4.5)

$$|u(t,x)| = \tilde{\lambda}(t)(Q + S(t))(\tilde{\lambda}(t)(x - y(t))),$$

with S decreasing in $\mathbb{H}^1$ as does R, and so the second assertion of Proposition 4.1.1 follows.

The interest of this choice of the concentration parameter is that $y(t)$ is a differentiable function, and, moreover, in the radial case we obtain Corollary 4.1.1.

4.4 The blow-up rate on unbounded plane domains

Consider now the equation (S_Ω) on an unbounded domain of the plane or on a surface. Let u be a critical mass solution that blows up in an interior point x_0 of Ω, that is

$$x(t) \underset{t \to T}{\longrightarrow} x_0.$$

Modulo a translation, we can suppose that x_0 is zero and so,

$$|u(t,x)|^2 \underset{t \to T}{\longrightarrow} \|Q\|_2^2 \, \delta_0.$$

Let ϕ be a C_0^∞ function, equal to 1 on $B(0,R)$. Let us introduce the localized virial of the solution

$$g_\phi(t) = \int |u(t,x)|^2 \phi^2(x) |x|^2 dx.$$

Then, using (4.9) with $h(x) = \phi^2(x)|x|^2$, one has

$$g'_\phi(t) = -2 \int \Im(u(t)\nabla\bar{u}(t))\nabla(\phi^2|x|^2) dx.$$

The inequality $(*)$ with $\theta(x) = \phi^2(x)|x|^2$ gives us

$$|g'_\phi(t)|^2 \leq 8E(u) \int |u|^2 |\nabla(\phi^2|x|^2)|^2 dx$$

Since $\nabla(\phi^2|x|^2)$ is a $\mathcal{C}_o^\infty(\mathbb{R}^2)$ function cancelling at 0, and since the square of $|u|$ behaves like a Dirac distribution, it follows that

$$g'_\phi(T) = 0.$$

Then, as in the former section, and using the existence of a positive constant C such that

$$|\nabla(\phi^2|x|^2)|^2 \leq C\psi^2|x|^2,$$

one has

$$g_\phi(t) \lesssim (T-t)^2.$$

The uncertainty principle reads

$$\left(\int |u|^2\phi^2 dx\right)^2 \leq \left(\int |u|^2\phi^2|x|^2 dx\right)\left(\int |\nabla(u\phi)|^2 dx\right).$$

By integrating by parts the last term and by using the fact that ϕ is equal to 1 on $B(0,R)$, it follows that

$$\left(\int_{B(0,R)} |u(t)|^2\right)^2 \leq g_\phi(t)\left(\int |\nabla u|^2\phi^2 dx - \int |u|^2\phi\Delta\phi dx\right).$$

Since ϕ is a $\mathcal{C}_0^\infty$ function,

$$\left(\int_{B(0,R)} |u(t)|^2\right)^2 \leq g_\phi(t)\left(C\int |\nabla u|^2 dx - \int |u|^2\phi\Delta\phi dx\right).$$

On the one hand the $\mathbb{L}^2$ norm of u is conserved. On the other hand, the behavior of $|u|^2$ as a Dirac distribution implies that the norm of its restriction outside a neighborhood of zero tends to 0 in time. So we have

$$\begin{cases} \int_{B(0,R)} |u(t)|^2 = O(1), \\ \int |u(t)|^2\phi\Delta\phi dx = o(1), \end{cases}$$

and since g_ϕ is bounded in time,

$$1 \lesssim \sqrt{g_\phi(t)}\|\nabla u(t)\|_2$$

Then the decay of g_ϕ gives us the lower bound of the blow-up speed

$$\frac{1}{T-t} \lesssim \|\nabla u(t)\|_2,$$

and the first assertion of Theorem 4.1.2 is completely proved.

4.5 Blow-up on the boundary

4.5.1 Necessary condition for blow-up on the boundary

Let us first introduce a notion of limit of sets, as in [5].

Definition 4.5.1. *A sequence of open sets M_m is said to tend to an open set M of $\mathbb{R}^2$ if the following conditions are verified.*
i) For all compact $K \subset M$, there exist $n_K \in \mathbb{N}$, such that for all $n \geq n_K$, $K \subset M_n$.
ii) For all compact $K \subset^c \overline{M}$, there exist $n_K \in \mathbb{N}$, such that for all $n \geq n_K$, $K \subset^c \overline{M_n}$.

Let us suppose that there exists an explosive solution u of the equation (S_Ω) at $0 \in \partial\Omega$. The convergence (4.2) implies that

$$\lambda(t)(\Omega - x(t)) \underset{t \to T}{\longrightarrow} \mathbb{R}^2.$$

As in [5], the limit set depends on the position of $x(t)$ with respect to the boundary of Ω. If there is a positive number C such that for all t

$$\lambda(t) d(x(t), \partial\Omega) \leq C,$$

then $\lambda(t)(\Omega - x(t))$ tends to a half-plane and blow-up cannot occur. Also, if

$$\lambda(t) d(x(t), \partial\Omega) \underset{t \to T}{\longrightarrow} \infty,$$

and $x(t)$ is not in Ω, then $\lambda(t)(\Omega - x(t))$ is a set that moves to infinity and does not cover at the limit time the whole plane. Therefore the only possibility to have explosion on the boundary is that $x(t) \in \Omega$ and

$$\lambda(t) d(x(t), \partial\Omega) \underset{t \to T}{\longrightarrow} \infty.$$

In particular, since 0 is on the boundary,

$$|\lambda(t) x(t)| \underset{t \to T}{\longrightarrow} \infty. \tag{4.12}$$

We have

$$|x(t)|^2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 \leq 2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x - x(t)|^2 + 2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x|^2.$$

On the one hand, by using the Weinstein relation (4.2), one has

$$|x(t)|^2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 \approx |x(t)|^2.$$

On the other hand, using again (4.2),

$$\int_{B(x(t),\frac{C}{\lambda(t)})} |u|^2|x - x(t)|^2 \lesssim \frac{1}{\lambda(t)^2}.$$

In view of (4.12), these two facts imply

$$|x(t)|^2 \lesssim \int_{B(x(t),\frac{C}{\lambda(t)})} |u|^2|x|^2 \lesssim g_\psi,$$

where g_ψ is the localized virial function defined in §4.4.2. In the same subsection it was proved that

$$g_\psi \lesssim (T-t)^2,$$

so it follows that

$$|x(t)| \lesssim T - t.$$

By using again (4.12),

$$\frac{1}{T-t} \ll \lambda(t),$$

and the second assertion of Theorem 4.1.1 is proved.

4.5.2 Results of non-explosion

From now on we assume that Ω be a half plane whose boundary contains 0 or a plane sector with corner 0. Suppose there exists an explosive solution u of critical mass such that u behaves like a Dirac mass at 0.

For a radial function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, the result of Lemma 4.2.6 becomes

$$\partial_t{}^2 \int_\Omega |u|^2 f = 16E(u) - \int_\Omega (2|\nabla u|^2 - |u|^4)(4 - \Delta f) - \int_\Omega |u|^2 \Delta^2 f$$

$$+4\Re \sum_{i,j} \partial_i u \partial_j \bar{u} \partial_{ij} f - 2|\nabla u|^2 \Delta f, \tag{4.13}$$

since from the choice of Ω

$$x.\nu = 0 \text{ on } \partial\Omega.$$

It follows that for a radial function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, equal to $|x|^2$ on $B(0,R)$, with bounded derivatives $\partial_{i,j} f$ and $\Delta^2 f$, the estimate of Corollary 4.2.1 becomes

$$\left|\partial_t^2 \int_\Omega |u(t)|^2 f\right| \le 16E(u) + C\int_{|x|\ge R} |u(t)|^2 + |\nabla u(t)|^2. \tag{4.14}$$

Arguing as in [15], we obtain the following lemmas.

Lemma 4.5.1. *The initial condition is of finite variance*

$$\int_\Omega |u_0|^2|x|^2 dx < \infty.$$

Proof. Let us consider ψ a $C_0^\infty(\mathbb{R})$ positive radial function which is equal to $|x|^2$ on $B(0,1)$. Notice that

$$|\nabla\psi|^2 \leq C\psi.$$

For all entire n, we introduce the localized virial functions

$$g_n(t) = \int_\Omega |u(t)|^2 \psi_n dx,$$

where

$$\psi_n(x) = n^2 \psi\left(\frac{x}{n}\right).$$

The Taylor formula in zero for the function $g_n(t)$ gives us

$$|g_n(t) - g_n(0)| \leq t|g_n'(0)| + C \sup_t |g_n''(t)|.$$

Since ψ_n are equal to $|x|^2$ on $B(0,1)$, and the derivatives $\partial_{ij}\psi_n$ and $\Delta^2\psi_n$ are uniformly bounded, we can estimate by (4.14)

$$|g_n''(t) - 16E(u)| \leq C \int_{|x|>1} (|u(t)|^2 + |\nabla u(t)|^2) dx.$$

Then, in view of Lemma 4.2.5, the quantity $g_n''(t)$ is bounded uniformly on n. So we have

$$|g_n(t) - g_n(0)| \leq T|g_n'(0)| + C.$$

By using the inequality $(*)$,

$$|g_n'(t)| = \left|\int_\Omega \Im(u(t)\nabla\overline{u}(t))\nabla\psi_n\right| \leq \left(2E(u)\int_\Omega |u(t)|^2|\nabla\psi_n|^2\right)^{\frac{1}{2}}.$$

The choice of ψ_n gives us

$$|\nabla\psi_n|^2 \leq \psi_n,$$

and it follows that

$$|g_n'(t)| \leq C\left(\int_\Omega |u(t)|^2\psi_n\right)^{\frac{1}{2}} = C\sqrt{g_n(t)}.$$

Therefore

$$g_n(0) - 2C\sqrt{g_n(0)} - C \leq g_n(t)$$

for any time t, and

$$g_n(0) - 2C\sqrt{g_n(0)} - C \leq \lim_{t\to T} g_n(t).$$

The concentration of the solution as a Dirac distribution implies that for fixed n

$$\lim_{t\to T} g_n(t) = 0,$$

and therefore

$$\lim_{r\to\infty}(g_n(0) - 2C\sqrt{g_n(0)} - C) \leq 0.$$

As a consequence, $g_n(0)$ is bounded as n tends to infinity. Since the supports of ψ_n cover Ω when n tends to infinity, it follows that the initial condition is of finite variance

$$\int_\Omega |u_0|^2|x|^2dx < \infty.$$

□

Remark 4.5.1. *When Ω is a bounded domain, and u is a critical mass function blowing up at a point of Ω or of its boundary, it is easy to see that the initial condition is of finite variance*

$$\int_\Omega |u|^2|x|^2dx \leq |\Omega|^2\|u\|_2^2 < \infty.$$

Lemma 4.5.2. *The limit in time of the virial function is*

$$g(T) = 0.$$

Proof. Let us consider a C^∞ positive function ϕ which is null on $B(0,1)$ and verifies

$$\frac{|x|}{2} \leq \phi(x) \leq |x|,$$

on ${}^cB(0,2)$. Suppose also that the derivatives $\partial_{ij}\psi$ and $\Delta^2\psi$ are bounded. We denote

$$\phi_n(x) = n\,\phi\left(\frac{x}{n}\right),$$

so ϕ_n are supported on ${}^cB(0,n)$ and verify

$$\frac{|x|}{2} \leq \phi_n(x) \leq |x|$$

on ${}^cB(0,2n)$.

Taylor's formula together with (4.9) and the estimate (4.14) gives us

$$\int |u(t)|^2\phi_n \leq \int_{|x|>n} |u_0|^2|x|^2 + T\left|\int_{|x|>n} \Im(u_0\nabla\overline{u_0})\nabla\phi_n^2\right|$$

$$+C(T-t)^2 + C\int_0^T (T-\tau)\int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2)d\tau. \tag{4.15}$$

The Lemma 4.5.1 ensures us that the initial data is of finite variance, therefore

$$\int_{|x|>n} |u_0|^2|x|^2 \underset{n\to\infty}{\longrightarrow} 0,$$

Also, by Lemma 4.2.5,

$$\left|\int_{|x|>n} \Im(u_0 \nabla \overline{u_0}) \nabla \phi_n^2\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Then, using again Lemma 4.2.5 and the conservation of the mass, for all τ and for all n there exist a positive constant C such that

$$\int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) \leq C.$$

One also has, for every τ,

$$\int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) \underset{n\to\infty}{\longrightarrow} 0.$$

Then by the dominated convergence theorem

$$\int_0^T (T-\tau) \int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) d\tau \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore it follows from (4.15) that for all t,

$$\int |u(t)|^2 \phi_n^2 \leq \epsilon(n) + C(T-t)^2,$$

with

$$\epsilon(n) \underset{n\to\infty}{\longrightarrow} 0.$$

On the one hand, in view of the choice of ϕ_n, this gives us

$$\int_{|x|>2n} |u(t)|^2 |x|^2 \leq 2\epsilon(n) + C(T-t)^2.$$

On the other hand, for fixed n, the concentration of the solution as a Dirac distribution implies

$$\lim_{t\to T} \int_{|x|<2n} |u(t)|^2 |x|^2 = 0.$$

Therefore, for all n

$$\lim_{t\to T} \int |u(t)|^2 |x|^2 \leq \epsilon(n).$$

By letting n to tend to infinity one has

$$\lim_{t\to T} \int |u(t)|^2 |x|^2 = 0,$$

that is

$$g(T) = 0,$$

and the Lemma 4.5.2 is proved. □

This lemma and the same arguments as in §4.2.3 give us also

$$g'(T) = 0.$$

By using the formula (4.13) with $f(x) = |x|^2$, the second derivative of the virial is exactly

$$g''(t) = 16E(u).$$

Then it follows that

$$g(t) = 8E(u)(T-t)^2,$$

and by the same calculation as in §4.2.1

$$E(e^{i\frac{|x|^2}{4(T-t)}}u(t,x)) = E(u) + \frac{1}{4(T-t)}g'(t) + \frac{1}{16(T-t)^2}g(t) = 0.$$

For fixed t, by the variational characterization of the ground state Q, there exists real numbers θ and ω such that

$$u(t,x) = e^{-i\frac{|x|^2}{4(T-t)}}e^{i\theta}\omega Q(\omega(x-x_0))$$

for some $x_0 \in \mathbb{R}^2$ ([4]). This means that the support of u is the entire $\mathbb{R}^2$ that is a contradiction, and the proof of Theorem 4.1.3 is complete.

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