MIHAI MARIŞ

Sur quelques problèmes elliptiques non-linéaires

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Mihai MARIŞ

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M.	Fabrice BETHUEL,	rapporteur
Mme	Anne de BOUARD,	examinatrice
M.	Vincent HAKIM,	invité
M.	Bernard HELFFER,	examinateur
M.	Louis JEANJEAN,	rapporteur
M.	Franck MERLE,	examinateur
M.	Jean-Claude SAUT,	directeur

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Abstract

In this thesis we study particular solutions for some nonlinear dispersive partial differential equations which appear in Physics, such as the nonlinear Schrödinger equation, the Benney-Luke equation or the Benjamin-Ono equation. We are particularly interested in the stationary waves and in the travelling waves of these equations. This gives nonlinear elliptic problems in the whole space. Solitary and travelling waves for the considered equations have been observed in experiments and in numerical simulations. In some cases, these solutions seem to play an important role in the general dynamics of the corresponding evolution equations.

In the first chapter we prove the analyticity and we find the optimal algebraic decay rate at infinity of solitary waves to the Benney-Luke equation and to the generalized Benjamin-Ono equation.

The second chapter is devoted to the proof of existence of stationary solutions for a nonlinear Schrödinger equation with potential in one dimension which describes the flow of a fluid past an obstacle.

In the third chapter we prove the existence of unstationary bubbles for the nonlinear Schrödinger equation in space dimension greather or equal than 4 by using a new and general result in critical point theory.

Key words: Nonlinear elliptic equation, existence of solutions, regularity, decay at infinity, nonlinear Schrödinger equation, stationary waves, travelling waves, Mountain-Pass Theorem.

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Contents

Introduction

Les travaux présentés dans cette thèse portent sur l'étude des solutions particulières de certaines équations aux dérivées partielles dispersives issues de la physique, comme par exemple l'équation de Schrödinger, l'équation de Benney-Luke ou l'équation de Benjamin-Ono. Les solutions étudiées sont de type *ondes stationnaires* (intuitivement, il s'agit d'un profil qui tourne périodiquement en temps) ou *ondes progressives* (i. e. un profil qui se déplace à vitesse constante dans une certaine direction de l'espace). Ceci nous conduit à des problèmes elliptiques non-linéaires dans l'espace $\mathbf{R}^N$ tout entier. Des solutions de type onde progressive ou bien onde stationnaire pour les équations considérées ont été observées dans les expérimentations ou dans les calculs numériques. Dans certains cas, elles semblent jouer un rôle important dans la dynamique générale des équations d'évolution correspondantes.

Le premier chapitre est consacré à l'étude des propriétés qualitatives des solutions de quelques équations elliptiques non-linéaires dans $\mathbf{R}^N$. Plus précisément, on s'intéresse à la régularité et à la décroissance à l'infini. Pour démontrer la régularité dans les espaces de Sobolev $W^{k,p}$ on utilise la théorie classique des multiplicateurs de Fourier. Il s'avère que les solutions étudiées sont même plus régulières que C^∞ : elles sont analytiques. Pour prouver ce fait, on fait utilise la technique suivante: tout d'abord on estime par récurrence $|\xi|^k\widehat{u}$, $k \in \mathbf{N}$ (u étant la solution et $\widehat{u}$ sa transformée de Fourier) dans une certaine norme (le plus souvent, la norme L^2). Les majorations obtenues permettent d'estimer la norme de $e^{\sigma|\xi|}\widehat{u}$ pour un certain $\sigma > 0$. En utilisant la théorie de Paley-Wiener on en déduit l'existence d'un prolongement analytique de u dans une N-bande $\{(z_1, \dots, z_N) \in \mathbf{C}^N \mid |Im(z_i)| < \sigma\}$. Cette méthode a été utilisée par J. L. Bona et Yi A. Li pour des problèmes en dimension 1 de l'espace dans [26]. Ils ont obtenu un résultat général qui implique l'analyticité des ondes stationnaires et des ondes progressives d'un grand nombre d'équations à une variable d'espace. L'intérêt de notre travail a été d'adapter ces techniques à quelques problèmes en dimension supérieure.

Pour obtenir des résultats de décroissance à l'infini, on dispose d'une méthode assez générale qui consiste à transformer l'équation aux dérivées partielles en une équation de convolution

$$u = k \star G(u).$$

Ensuite, la preuve comporte généralement trois étapes:

1. Trouver le taux de décroissance du noyau de convolution k. Parfois on connaît même explicitement ce noyau, mais il existe aussi des cas où on a besoin d'estimations assez délicates.

2. Obtenir une première estimation (non-optimale) de u. Cela se fait dans certains cas à l'aide même de l'équation de convolution, en exploitant la décroissance de k et le comportement surlinéaire de G. Dans d'autres cas, il faut multiplier l'équation satisfaite par u par des multiplicateurs de Pohozaev bien choisis. Le plus souvent on trouve à cette étape une estimation intégrale.

3. Améliorer l'estimation initiale par un argument de "boot-strap" en utilisant l'équation de convolution et passer à une estimation ponctuelle. En général, on peut s'attendre à ce que les solutions décroîssent au moins aussi vite que le noyau de convolution.

Cette méthode a aussi été utilisée par J. Bona et Yi Li pour des problèmes unidimensionnels. Ils ont prouvé un théorème général de décroissance qui s'applique aux solutions d'une large classe d'équations. Une variante de la technique que nous venons de décrire a été employée par A. de Bouard et J.-C.Saut pour démontrer la décroissance à l'infini des ondes solitaires de l'équation de Kadomtsev-Petviashvili (KP-I) en dimension 2 et 3.

Dans la section 1.2 nous étudions les ondes solitaires de l'équation de Benney-Luke. Cette équation décrit la propagation des vagues longues avec une petite amplitude dans l'eau en présence d'une tension superficielle. Modulo un changement d'échelle, une onde solitaire est une fonction de deux variables x, y qui satisfait l'équation

$$\begin{array}{c}(c^2-1)u_{xx} - u_{yy} + (a - bc^2)u_{xxxx} + (2a - bc^2)u_{xxyy} + au_{yyyy} - \\ c(3u_x u_{xx} + u_x u_{yy} + 2u_y u_{xy}) = 0 \ .\end{array} \tag{1}$$

où les paramètres a, b, c sont positifs et $c^2 < \min(1, \frac{a}{b})$. L'existence des ondes solitaires dans un espace de Sobolev a été démontrée par R. L. Pego et J. R. Quintero en 1998 en utilisant la méthode de concentration-compacité (v. [39]). Nous étudions les propriétés qualitatives des ondes qu'ils ont trouvées. En ce qui concerne leur régularité, on obtient les résultats suivants:

Théorème 1. *Soit u une solution de (1). Alors:*

a) $u \in W^{k,p}(\mathbf{R}^2)$ pour tout $k \in \mathbf{N}$ et pour tout $p \in]2, \infty]$;

b) $u_x, u_y \in W^{k,p}(\mathbf{R}^2)$ pour tout $k \in \mathbf{N}$ et pour tout $p \in]1, \infty]$.

Théorème 2 *Soit u une solution de (1). Il existe $\sigma > 0$ et une fonction $U(z_1, z_2)$ définie et holomorphe dans le domaine*

$$\Omega_\sigma = \{(z_1, z_2) \in \mathbf{C}^2 \ | \ |Im(z_1)| < \sigma, \ |Im(z_2)| < \sigma\}$$

telle que $U(x, y) = u(x, y)$, $\forall (x, y) \in \mathbf{R}^2$.

Le taux algébrique de décroissance des ondes solitaires est donné par

Théorème 3. *Soit u une solution de (1). Alors:*

a) $r^2 D^\alpha u \in L^\infty(\mathbf{R}^2)$, $\forall \alpha \in \mathbf{N}^2$, $|\alpha| \geq 1$ où $r = \sqrt{x^2 + y^2}$;

b) $ru \in L^\infty(\mathbf{R}^2)$.

Il est intéressant de remarquer que $u \notin L^2(\mathbf{R}^2)$ et $u_x, u_y \notin L^1(\mathbf{R}^2)$, donc les taux de décroissance donnés par le Théorème 3 pour u et u_x, u_y sont optimaux.

La preuve du Théorème 3 se fait selon le schéma déjà préseté et comporte plusieurs étapes :

1. À l'aide des multiplicateurs de Pohozaev bien choisis on obtient les estimations intégrales

$$\int_{\mathbf{R}^2} r^2 |\nabla^2 u|^2 dxdy < \infty \quad \text{et} \quad \int_{\mathbf{R}^2} r^2 |\nabla^3 u|^2 dxdy < \infty.$$

2. Soit $Q(\xi_1, \xi_2) = (1-c^2)\xi_1^2 + \xi_2^2 + (a - bc^2)\xi_1^4 + (2a - bc^2)\xi_1^2\xi_2^2 + a\xi_2^4$ le symbole de la partie linéaire de l'équation (1) et soit $g = 3u_x u_{xx} + u_x u_{yy} + 2u_y u_{xy}$ On écrit (1) sous la forme

$$u_x = ich_1 \star g, \text{ respectivement } u_y = ich_2 \star g, \qquad \text{où } h_i = \mathcal{F}^{-1}\left(\frac{\xi_i}{Q(\xi_1, \xi_2)}\right).$$

3. On montre que $rh_i \in L^\infty(\mathbf{R}^2)$.

4. À l'aide de l'inégalité $|ru_x| \leq C(|rh_1| \star |g| + |h_1| \star |rg|)$ on prouve que $ru_x \in L^\infty(\mathbf{R}^2)$ et $ru_y \in L^\infty(\mathbf{R}^2)$.

5. Soit $k_{ij} = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{Q(\xi_1, \xi_2)}\right)$ et $\phi_1 = \frac{3}{2}u_x^2 + \frac{1}{2}u_y^2$, $\phi_2 = u_x u_y$. L'équation (1) implique

$$u_x = -ck_{11} \star \varphi_1 - ck_{12} \star \varphi_2, \quad u_y = -ck_{21} \star \varphi_1 - ck_{22} \star \varphi_2.$$

6. On prouve que $\widehat{k_{ij}} \in H^s(\mathbf{R}^2)$, $0 \leq s < 1$, $k_{ij} \in L^q(\mathbf{R}^2)$, $1 < q \leq 2$ et $r^2 k_{ij} \in L^\infty(\mathbf{R}^2)$.

7. On obtient d'abord que $r^{1+\delta}u_x, r^{1+\delta}u_y \in L^\infty(\mathbf{R}^2)$, $\forall \delta \in [0, 1)$, ensuite que $r^2 u_x, r^2 u_y \in L^\infty(\mathbf{R}^2)$.

8. On démontre b) en utilisant l'identité $u = ich_1 \star \varphi_1 + ich_2 \star \varphi_2$.

Dans la section 1.3 on étudie les ondes solitaires d'une généralisation en dimension 2 de l'équation de Benjamin-Ono

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0. \tag{2}$$

Cette équation apparaît dans plusieurs problèmes physiques. Les ondes solitaires sont des solutions de type $A(x, y, t) = u(x - ct, y)$. Après un changement d'échelle, ces sont des solutions de l'équation

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \quad \text{dans } \mathbf{R}^2. \tag{3}$$

Il est facile de montrer l'existence des ondes solitaires dans $H^{\frac{1}{2}}(\mathbf{R}^2)$. On peut utiliser soit le principe de concentration-compacité de P.-L. Lions, soit une méthode alternative à ce principe dûe à O. Lopes. Les solutions obtenues ont des propriétés variationnelles remarquables: elles sont des minimiseurs de l'"énergie" associée à (2) sous la contrainte "charge"= constante. Nous avons obtenu le résultat suivant sur la régularité des ondes solitaires:

Théorème 4. *Soit* $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ *une solution de (3). Alors:*

a) $u \in W^{k,p}(\mathbf{R}^2)$, $\forall k \in \mathbf{N}$, $\forall p \in [1,\infty]$. *En particulier,* u *est une fonction régulière qui tend vers zéro à l'infini.*

b) Il existe $\sigma > 0$ *et une fonction* $U(z_1, z_2)$ *définie et holomorphe dans le domaine*

$$\Omega_\sigma = \{(z_1, z_2) \in \mathbf{C}^2 \mid |Im(z_1)| < \sigma, \ |Im(z_2)| < \sigma\}$$

telle que $U(x,y) = u(x,y)$, $\forall (x,y) \in \mathbf{R}^2$.

On a montré aussi un résultat de décroissance à l'infini des solutions d'une généralisation de l'équation (3). Plus précisément, considérons l'équation

$$(1 + (-\Delta)^{\frac{1}{2}})u = g(u) \qquad \text{dans } \mathbf{R}^N, \tag{4}$$

où $g : \mathbf{C} \longrightarrow \mathbf{C}$ est continue et il existe $\gamma > 1, C > 0$ tels que $|g(z)| \leq C|z|^\gamma$. On a :

Théorème 5. *Soit* u *une solution de (4). Supposons que:*

- *soit* $u \in L^p(\mathbf{R}^N)$ *pour un* $p \in](\gamma - 1)N, \infty[$, $p \geq \gamma$,
- *soit* $u \in L^\infty(\mathbf{R}^N)$ *et* $\lim_{|x| \to \infty} u(x) = 0$.

Alors $|x|^{N+1}u(x) \in L^\infty(\mathbf{R}^N)$.

Pour démontrer le Théorème 5 on écrit (4) sous la forme

$$u = k \star g(u)$$

avec $k = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|}\right)$. Ensuite on estime le noyau de convolution k. On trouve qu'il se comporte comme $\frac{1}{|x|^{N-1}}$ pour $|x|$ petit, respectivement comme $\frac{1}{|x|^{N+1}}$ pour $|x|$ grand ($N \geq 2$). En utilisant l'équation de convolution on montre l'"estimation initiale"

$$|x|^\beta u(x) \in L^q(\mathbf{R}^N), \quad \forall q \in]N, \infty[, \ \ \forall \beta \in [0, 1[.$$

Ce premier résultat de décroissance est amélioré après par un argument de "bootstrap" et on aboutit à la conclusion du théorème.

Le Théorème 5 implique que les solutions de (3) dans $\mathbf{R}^N$ décroîssent comme $\frac{1}{|x|^{N+1}}$; ce taux algérique est optimal car on ne peut pas avoir $|x|u \in L^1(\mathbf{R}^N)$ à cause du fait que $\widehat{x_i u} = i\partial_{\xi_i}\widehat{u}$ sont des fonctions discontinues à l'origine.

Dans le deuxième chapitre on étudie une équation de Schrödinger en dimension 1, avec potentiel. Celle-ci décrit, par exemple, l'écoulement derrière un obstacle immobile d'un fluïde de SNL injecté avec une vitesse constante v à l'infini. Elle peut aussi être interprétée comme l'équation du mouvement avec une vitesse constante v d'une impurité dans un fluïde au repos à l'infini. L'équation est la suivante:

$$iA_t - ivA_x = -A_{xx} - A + |A|^2 A + U(x)A. \tag{5}$$

Le potentiel U est une mesure positive et modélise l'obstacle (respectivement l'impurité). On cherche des solutions stationnaires (i.e. qui ne dépendent pas de t) dont le module tend vers 1 à $\pm\infty$. On a des raisons heuristiques de croire que de telles solutions n'existent pas si $v > \sqrt{2}$ (à l'échelle de l'équation, $\sqrt{2}$ représente la vitesse du son dans le milieu considéré).

L'équation (5) a été étudiée par V. Hakim dans [23]. Dans le cas où le potentiel est une masse de Dirac, il a montré l'existence d'une vitesse critique v_c telle que pour $v < v_c$ il existe deux solutions stationnaires de (5), l'une étant stable et l'autre instable. En utilisant des développements asymptotyques formels et des calculs numériques, il a montré qu'un phénomène similaire a lieu pour des potentiels petits et pour des potentiels qui varient lentement (i.e. de la forme $U(\varepsilon x)$, ε petit). Dans tous ces cas les deux solutions stationnaires s'identifient lorsque $v = v_c$ et disparaîssent pour $v > v_c$. La vitesse critique dépend du potentiel et est plus petite que la vitesse du son. V. Hakim a aussi étudié numériquement les solutions de (5) qui dépendent du temps lorsque $v > v_c$.

Notre but a été de démontrer rigoureusement une partie de ces résultats.

Dans le cas $U = g\delta$ (où δ est la masse de Dirac) l'équation (5) peut être intégrée et on obtient explicitement les solutions stationnaires. Pour une vitesse v donnée, on trouve deux solutions distinctes si $g < \phi(v)$; ces solutions s'identifient lorsque $g = \phi(v)$ et disparaîssent quand $g > \phi(v)$. Ici $\phi(v)$ est une fonction positive qu'on sait déterminer.

Dans le cas général, on a démontré l'existence de deux fonctions positives $\phi_1(v)$ et $\phi_2(v)$ avec les propriétés suivantes:

1° Si $||U|| \leq \phi_1(v)$, (5) possède au moins une solution stationnaire qui est un minimiseur de l'énergie associée sur un certain ouvert de $H^1(\mathbf{R})$. $||U||$ représente la variation totale de U.

2° Si $||U|| \leq \phi_2(v)$ et si U est à support compact, (5) admet une deuxième solution stationnaire.

La preuve de 1° est classique. La démonstration de 2° est plus délicate et repose sur une variante du Lemme du Col due à Ghoussoub et Preiss. La difficulté essentielle est d'obtenir des informations assez précises sur les suites de Palais-Smale pour déduire leur convergence et pour montrer que cette deuxième solution est différente de celle obtenue par minimisation au 1°.

Dans la section 6 on montre l'existence des ondes progressives non-triviales du système

$$(6) \qquad \left\{ \begin{array}{rcl} 2i\varepsilon^2\psi_t & = & -\varepsilon^2\psi_{xx} + (|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \ , \\ 2i\varepsilon^2\delta\varphi_t & = & -\varepsilon^2\varphi_{xx} + (q^2|\psi|^2 - \varepsilon^2 k_M^2)\varphi \end{array} \right. \qquad x \in \mathbf{R},\ t \in \mathbf{R},$$

où ψ et φ sont des fonctions complexes et vérifient les conditions au bord $\psi \longrightarrow 1$, $\varphi \longrightarrow 0$ lorsque $x \longrightarrow \pm\infty$. Ce système a été introduit par Gross et Clark dans les années soixante et intervient dans l'étude du mouvement d'une impurité dans un condensat de Bose. Le système a été étudié ensuite par J. Grant and P. H. Roberts dans [22]. En utilisant des développements asymptotiques formels et des calculs numériques, ils ont trouvé le rayon effectif et la masse induite de l'impurité.

On remarque qu'il existe des solutions de (6) de la forme $\psi(x,t) = A(x - vt)$, $\varphi(x,t) \equiv 0$, où A est (après un changement d'échelle) une solution stationnaire de (5) avec $U = 0$. Ces sont les ondes progressives triviales de (6). Nous obtenons par une technique classique de bifurcation une courbe régulière d'ondes progressives non-triviales.

Notons qu'il serait intéressant d'étudier le système (6) en dimension 2 ou 3 d'espace (bien sûr, dans ces cas il faudrait remplacer les dérivées secondes par le laplacien); c'est dans ces dimensions qu'il décrit des phénomènes physiques réels. L'existence des ondes progressives en dimension plus grande que 1 reste un problème ouvert.

Dans le troisième chapitre nous nous intéressons à l'existence des ondes progressives (appellées également "bulles instationnaires") pour l'équation de Schrödinger non-linéaire

$$i\frac{\partial \varphi}{\partial t} + \Delta\varphi + F(|\varphi|^2)\varphi = 0 \quad \text{dans } \mathbf{R}^N \tag{7}$$

où φ est une fonction complexe qui satisfait la "condition aux limites" $|\varphi| \longrightarrow r_0$ quand $|x| \longrightarrow \infty$ et $r_0 > 0$ est tel que $F(r_0^2) = 0$. Le cas modèle est celui de l'équation de Schrödinger "$\psi^3 - \psi^5$"

$$i\frac{\partial \psi}{\partial t} + \Delta\psi - \alpha_1\psi + \alpha_3|\psi|^2\psi - \alpha_5|\psi|^4\psi = 0 \quad \text{dans } \mathbf{R}^3, \tag{8}$$

avec α_1, α_2, $\alpha_3 > 0$ et $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$.

Les équations (7) et (8) apparaîssent dans une large variété de problèmes physiques, voir [3].

Il existe des solutions de (7) de la forme $\varphi(t,x) = e^{i\omega t}\psi(x)$, qu'on appelle des *bulles stationnaires.* Ce type de solutions a été étudié dans [13], où on a montré par une analyse spectrale délicate qu'elles sont instables.

Nous nous intéressons à un autre type de solutions particulières, les *bulles instationnaires*, qui sont les solutions de la forme $\varphi(t,x) = \Phi(x_1 - ct, x_2, \ldots, x_N)$.

Il a été démontré qu'en dimensioin 1 il existe des solutions localisées de type bulles instationnaires qui se déplacent avec une vitesse c et ont la forme $\varphi(t,x) = \Phi(x - ct)$. La condition aux limites devient dans ce cas $\lim_{x \to \pm\infty} \Phi(x) = r_0 e^{\mp i\mu}$, où μ dépend de c et $\mu = 0$ quand $c = 0$.

L'existence des bulles instationnaires en dimension 2 et 3 a été annoncée par Zhiwu Lin dans [28] en 1999. Il fait appel à la formulation hydrodynamique de l'équation de Schrödinger non linéaire et il a cherché des ondes progressives de la forme $\sqrt{\rho}e^{i\theta}$ avec $\rho \in H^2(\mathbf{R}^N)$ et $\theta \in \dot{H}^2(\mathbf{R}^N)$, $N = 2, 3$. Ensuite il a appliqué la méthode de Lyapounov-Schmidt aux équations en ρ et θ (comme, par exemple, dans [20]).

Si la nonlinéarité F est d'un type différent ("$\psi - \psi^3$"), (7) devient l'équation de Gross-Pitaevskii

$$i\psi_t + \Delta\psi + \psi - |\psi|^2\psi = 0.$$

Il a été démontré par F. Bethuel et J.-C. Saut que cette équation possède des ondes progressives de vitesse c si c est suffisament petite (voir [11]). Les ondes solitaires qu'ils ont trouvées ont deux vortex de degrés ± 1, situés à une distance de l'ordre de $\frac{1}{c}$ quand $c \longrightarrow 0$. La preuve est très délicate et utilise des méthodes variationnelles (le lemme du col de Ghoussoub et Preiss) et des techniques développées pour l'étude des vortex dans l'équation de Ginzburg-Landau.

Les conditions que nous imposons sur la nonlinéarité F sont très proches des hypothèses faites dans [13] ou dans [28]. Sous ces conditions, le problème stationnaire

$$\Delta\varphi + F(|\varphi|^2)\varphi = 0 \tag{9}$$

admet en dimension $N \geq 3$ une solution de la forme $\varphi = r_0 - u$ avec $u \in H^1(\mathbf{R}^N)$ qui minimise parmi toutes les solutions de (9) l'énergie associée au problème, $E(\varphi) = E(r_0 - u) = \int_{\mathbf{R}^N} |\nabla u|^2 dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2) dx$, où $V(s) = \int_s^{r_0^2} F(\tau) d\tau$. On va appeler cette solution un *état fondamental* et on la note u_0. L'état fondamental a des propriétés tout à fait remarquables: c'est une fonction régulière, à symétrie radiale et décroît exponentiellement à l'infini. Le résultat principal que nous obtenons est le suivant:

Théorème. *Soit $N \geq 4$. Il existe $c_0 > 0$ tel que pour tout $|c| \leq c_0$, l'équation (7) possède une solution de type bulle instationnaire $\varphi(x,t) = \Phi_c(x_1 - ct, x_2, \ldots, x_N)$ où Φ_c est de la forme $\Phi_c = r_0 - u_{1,c} - iu_{2,c}$ avec $u_{1,c} \in H^1(\mathbf{R}^N)$, $u_{2,c} \in D^{1,2}(\mathbf{R}^N)$. De plus, $u_{1,c} \longrightarrow u_0$ dans $H^1(\mathbf{R}^N)$ et $u_{2,c} \longrightarrow 0$ dans $D^{1,2}(\mathbf{R}^N)$ quand $c \longrightarrow 0$ et les fonctions $u_{1,c}$, ${}_{2,c}$ ont une symétrie radiale dans les variables $x_2, \ldots, x_N$.*

Afin de démontrer ce théorème, nous regardons les ondes progresives qui se déplacent avec une vitesse c petite comme des points critiques d'une certaine fonctionnelle E_c qui est une perturbation de la fonctionnelle d'énergie E. Cependant, les résultats classiques de la théorie des points critiques ne s'appliquent pas à la fonctionnelle E_c car il semble extrêmement difficile de mettre en évidence un changement de topologie au niveau global dans les ensembles de niveau de E_c. Pour surmonter cette difficulté, nous démontrons un résultat général dans la théorie des points critiques qui est une variante locale du théorème du point selle de Rabinowitz. Cependant, ce résultat abstrait nous donne l'existence des points critiques seulement pour E_c restreinte au complément orthogonal du noyau d'un opérateur linéaire A qui est le linéarisé de $-\Delta u + F(|r_0 - u|^2)(r_0 - u)$ autour de u_0. Il est évident que les dérivées de u_0, $\frac{\partial u_0}{\partial x_i}$, appartiennent à ce noyau. Une autre difficulté importante a été de montrer que le noyau est engendré exactement par ces dérivées. On conclût ensuite grâce à l'invariance de E_c par translations dans $\mathbf{R}^N$ que les "points critiques" trouvés sont des vrais points critiques (c'est à dire la différentielle de E_c en ces points s'annulle dans toutes les directions de l'espace fonctionnel considéré).

Chapter 1

Qualitative properties of solutions of some nonlinear elliptic equations

1.1 Introduction

This chapter is essentially devoted to the study of two kinds of properties of solutions of elliptic equations in $\mathbf{R}^N$: the regularity and the asymptotic decay at infinity. We present several results and methods that can be applied to the solitary and the standing waves of some types of equations.

In order to obtain $W^{k,p}$ regularity, we use the classical theory of Fourier multipliers. In some cases it turns out that the solutions (and especially the solitons) are even more regular than C^∞: they are analytic. This phenomenon is strongly related to the properties of the symbol of the differential operator (the ellipticity is needed in proofs) as well as to the nonlinearity. Essentially the method that we use below to prove the analyticity of a solution u of an equation in the entire space is the following: by an inductive argument we first estimate $|\xi|^k\widehat{u}$, $k \geq 0$ in a suitable norm (usually, in the L^2-norm). The bounds obtained allow us to estimate the norm of $e^{\sigma|\xi|}\widehat{u}$ for a positive σ. Finally, the Paley-Wiener theory gives an analytic extension of u in a symmetric N-strip $\{(z_1, \cdots, z_n) \in \mathbf{C}^N |\ \ |Im(z_i)| < \sigma\}$. A detailed description of this method and a general result which implies the analyticity of solitons of a large class of equations in dimension 1 can be found in [26].

The decay at infinity of solitary waves was systematically studied in one dimension by Bona and Li (see [12]). They wrote a differential equation as a convolution equation and then found the decay rate at infinity of solutions. They obtained a general result which shows that the decay of solutions is related to the decay of the convolution kernel k (or, equivalently, to the regularity of its Fourier transform) and to the nonlinearity present in the equation. It is proved that if the Fourier transform $\hat{k} \in H^s(\mathbf{R})$ for some $s > \frac{1}{2}$, the solution decays at least as fast as the kernel k itself. The main results of Bona and Li are the following:

Theorem 1.1 ([12]) *Suppose that* $f \in L^\infty(\mathbf{R})$ *with* $\lim_{|x|\longrightarrow\infty} f(x) = 0$ *is a solution of* *the convolution equation*

$$f = k \star G(f), \tag{1.1}$$

where G and k satisfy the following conditions:

i) G is measurable, bounded on any compact interval and there exist $r > 1$, $C > 0$ such that

$$|G(u)| \le C|u|^r , \quad \forall u \in [-1,1] ; \tag{1.2}$$

ii) k is measurable and $\hat{k} \in H^s(\mathbf{R})$ for some $s > \frac{1}{2}$.
Then $f \in L^1 \cap L^\infty(\mathbf{R})$ and $|x|^l f(x) \in L^2 \cap L^\infty(\mathbf{R})$ for all $l \in [0, s]$. In particular, $\hat{f} \in H^s(\mathbf{R})$.

Corollary 1.2 ([12]) *In addition to the assumptions of Theorem 1.1, suppose that there exists $\sigma_0 > 0$ such that*

$$e^{\sigma|x|}k(x) \in L^2(\mathbf{R}) , \quad \forall\sigma \in [0, \sigma_0).$$

Then $e^{\sigma|x|}f(x) \in L^2 \cap L^\infty(\mathbf{R})$, $\forall \sigma \in [0, \sigma_0)$.

Theorem 1.3 ([12]) *Suppose that $f = k \star G(f)$, where f, k and G satisfy the assumptions in Theorem 1.1. Suppose also that there is a constant $m > 1$ such that $\lim_{x \longrightarrow \pm\infty} |x|^m k(x) = C_\pm$. Then it follows that*

$$\lim_{x \longrightarrow \pm\infty} |x|^m f(x) = C_\pm \int_{-\infty}^{\infty} G(f(t))dt.$$

Theorem 1.4 ([12]) *Under the assumptions of Corollary 1.2, suppose that for some $\sigma_0 > 0$,*

$$\lim_{x \longrightarrow \pm\infty} e^{\sigma_0|x|}k(x) = C_\pm.$$

Then $e^{\sigma_0|x|}f(x) \in L^\infty(\mathbf{R})$ and

$$\lim_{x \longrightarrow \pm\infty} e^{\sigma_0|x|}f(x) = C_\pm \int_{-\infty}^{\infty} e^{\pm\sigma_0|t|}G(f(t))dt.$$

Bona and Li applied these results to a large class of equations, including those of solitary-waves (i.e. solutions of the form $u(x,t) = \varphi(x - ct)$) of the generalized Korteweg - de Vries (KdV) equation

$$u_t + u_x + F(u)_x - (Mu)_x = 0 \tag{1.3}$$

and of the regularized long-waves (RLW) equation

$$u_t + u_x + F(u)_x + (Mu)_t = 0 \tag{1.4}$$

as well as to the stationary waves (i.e. solutions of the form $u(x,t) = e^{i\omega t}\varphi(x)$) of the nonlinear Schrödinger equation

$$iu_t - Mu + F(|u|)u = 0, \tag{1.5}$$

where M is a Fourier operator, that is $\widehat{Mv}(\xi) = \alpha(\xi)\widehat{v}(\xi)$. They found the decay rate at infinity of these types of solutions under general assumptions on the symbol α of M and on the nonlinearity F.

There also exist several results about the decay of solutions of some particular equations in higher dimensions (see [15], citeM1, citeM2). Although a general theory containing all this results is not available, a careful analysis of the proofs shows that in fact they all use the same method which we describe below.

For simplicity, consider a nonlinear equation of the form

$$Au = F(u), \tag{1.6}$$

where A is a Fourier operator, i.e. $\widehat{Au}(\xi) = \alpha(\xi)\widehat{u}(\xi)$. In order to get decay, some conditions have to be imposed. We assume that $k = \mathcal{F}^{-1}\left(\frac{1}{\alpha}\right)$ is a function which decays "sufficiently fast" at infinity. We also suppose that F has a superlinear growth, that is there exists $r > 1$ such that

$$|F(u)| \leq C|u|^r . \tag{1.7}$$

At least formally, equation (1.6) can be written as $\alpha(\xi)\widehat{u}(\xi) = \widehat{F(u)}(\xi)$ or equivalently $\widehat{u}(\xi) = \frac{1}{\alpha(\xi)}\widehat{F(u)}(\xi)$, that is

$$u = k \star F(u). \tag{1.8}$$

Thus we have transformed equation (1.6) into a nonlinear convolution equation. Next, the proof is usually divided into three steps:

1. Find the decay rate of the convolution kernel k. Sometimes this may be very easy and there are even cases where k is known explicitly, but there are also cases where a quite long and complicated calculation is needed.

2. Obtain a first decay estimate on u (usually, an integral bound). In some cases, this can be done by multiplying the equation of u by suitable Pohozaev-type multipliers and integrating, while in other cases we can get this directly from the convolution equation (1.8) using the decay of k and the superlinear growth of F near 0.

3. The estimate obtained in step 2 is improved by a boot-strap argument. In general, one can expect that the solution u decays at least as fast as k.

Remark 1.5 The fact that F has a superlinear growth is very important. The solutions of a linear equation do not necessarily decay even if the convolution kernel k decays very fast. For example, in the "limit case" when k is the Dirac mass δ (which decays as fast as one could imagine, that is vanishes outside $\{0\}$!), taking $F(x) = x$ equation (1.8) becomes $\delta \star u = u$ and this is satisfied by any "reasonable" function u (for example, by any function that lies in some Sobolev space).

On the other hand, one may ask if the solutions of (1.6) or (1.8) decay faster if the exponent r in (1.7) is greater. The answer is negative even in dimension one. For example, consider the solitary waves $u(x,t) = \varphi_c(x - ct)$ of the KdV equation

$$u_t + u_x + u^p u_x - u_{xxx} = 0.$$

It can be seen that φ_c behaves at infinity like $e^{-\sqrt{c-1}|x|}$, and this for all values of $p > 0$.

Remark 1.6 Some obstructions may occur sometimes and prevent the solution of an equation to decay too fast. One example is given by Theorem 1.3 or Theorem 1.4. If $C_+ \neq 0$ or $C_- \neq 0$ and we know *à priori* that $\int_{-\infty}^{\infty} G(f(t))dt \neq 0$ (respectively, that $\int_{-\infty}^{\infty} e^{\pm\sigma_0 t} G(f(t))dt \neq 0$), then f decays exactly as $|x|^{-m}$ (respectively, as $e^{-\sigma_0|x|}$) and not faster.

Another simple (and related) example is the following: suppose that $u \in L^r \cap L^\infty(\mathbf{R}^N)$ satisfies (1.6) (or, equivalently, (1.8)) and $\int_{\mathbf{R}^N} F(u)dx \neq 0$. If $\widehat{k}(\xi) = \frac{1}{\alpha(\xi)}$ is not continuous at the origin, then u cannot belong to $L^1(\mathbf{R}^N)$. In particular, we cannot have an estimate of the type $|u(x)| \leq \frac{C}{|x|^{N+\eta}}$ for a positive η. Indeed, suppose that $u \in L^1(\mathbf{R}^N)$. Clearly $F(u) \in L^1(\mathbf{R}^N)$. Hence $\widehat{u}$ and $\widehat{F(u)}$ are continuous. The equation gives $\widehat{u} = \widehat{k}\widehat{F(u)}$. But this is absurd because $\widehat{k}$ is not continuous at zero and $\widehat{F(u)}(0) = \int_{\mathbf{R}^N} F(u)dx \neq 0$.

A variant of the method described above was applied by A. de Bouard and J.-C. Saut to prove the decay of the solitary-waves $v(x_1, \ldots, x_N, t) = u(x_1 - ct, x_2, \ldots, x_N)$ to the Kadomtsev-Petviashvili (KP-I) equation

$$v_t + |v|^p v_{x_1} + v_{x_1x_1x_1} - D_{x_1}^{-1}\Delta^\perp v = 0 \quad \text{in } \mathbf{R}^N, \quad N = 2,3, \tag{1.9}$$

where $D_{x_1}^{-1}h(x_1, \ldots, x_N) = \int_{-\infty}^{x_1} h(s, x_2, \ldots, x_N)ds$ and $\Delta^\perp = \partial_{x_2x_2} + \ldots \partial_{x_Nx_N}$. The equation satisfied by the solitary waves is, modulo a scale change,

$$-\Delta u + \partial_{x_1}^4 u + \frac{1}{p+1}(u^{p+1})_{x_1x_1} = 0. \tag{1.10}$$

It was proved in [14] that solitary waves exist in the space $Y =$ closure of $\partial_{x_1} C_0^\infty(\mathbf{R}^N)$ for the norm $||\partial_{x_1}\varphi||_Y^2 = ||\nabla\varphi||_{L^2}^2 + ||\partial_{x_1}^2\varphi||_{L^2}^2$ if and only if $1 \leq p < 4$ if $N = 2$, respectively $1 \leq p < \frac{4}{3}$ if $N = 3$. Moreover, these solutions are radially symmetric in $y = (x_2, \ldots, x_N)$ and belong to $H^s(\mathbf{R}^N)$ for all s (see [15]). The following (optimal) result about the decay of the solitary waves was proved in the 2-dimensional case:

Theorem 1.7 ([15]) *Any non-trivial solitary-wave of (1.9) satisfies*

$$r^2 u \in L^\infty(\mathbf{R}^2), \quad \textit{where } r^2 = x_1^2 + x_2^2.$$

The proof of Theorem 1.7 consists in the following sequence of steps:

1. $\int_{\mathbf{R}^2} r^2(|\nabla u|^2 + |u_{x_1x_1}|^2)dx < \infty$ (use Pohozaev multipliers).
2. Write $u = ih \star (u^p u_x)$, where $h = \mathcal{F}^{-1}(\frac{\xi_1}{|\xi|^2 + \xi_1^4})$.
3. Show that $h(x_1, x_2)| \leq \frac{C}{r}$.
4. $ru \in L^\infty(\mathbf{R}^2)$ (use 2, 3 and Young's inequality).
5. Write $u = -\frac{1}{p+1} k \star u^{p+1}$, where $k = \mathcal{F}^{-1}\left(\frac{\xi_1^2}{|\xi|^2 + \xi_1^4}\right)$.
6. Show that $\widehat{k} \in H^s(\mathbf{R}^2)$, $\forall s \in [0,1)$ and $r^2 k \in L^\infty(\mathbf{R}^2)$.

7. Prove that $r^\delta u \in L^2(\mathbf{R}^2)$ for all $\delta \in [0,1)$ (use 5, 6 and Young's inequality).

8. Prove that $r^{1+\delta}\nabla u \in L^2(\mathbf{R}^2)$ and $r^{1+\delta}u_{x_1x_1} \in L^2(\mathbf{R}^2)$ for all $\delta \in [0,1)$ (use the equation and Pohozaev-type multipliers).

9. Conclusion: $|r^2 u| \le C[(r^2 k) \star |u|^{p+1} + k \star (r^2 u^{p+1})] \in L^\infty(\mathbf{R}^2)$.

In dimension $N = 3$, a similar proof gives

Theorem 1.8 [15] *Any nontrivial solitary wave of (1.9) satisfies*

$$r^\delta u \in L^2(\mathbf{R}^3), \quad \forall \delta \in [0, \frac{3}{2}), \qquad \textit{where } r^2 = x_1^2 + x_2^2 + x_3^2.$$

In the next sections we show how the method presented above can be applied to the solitary waves of the Benney-Luke equation and to the solitary waves of the generalized Benjamin-Ono (BO) equation.

1.2 Analyticity and decay properties of the solitary waves to the Benney-Luke equation

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1.2.1 Introduction

In a recent paper [39] PEGO and QUINTERO studied the propagation of long water waves with small amplitude. They showed that in the presence of a surface tension, the propagation of such waves is governed by the following equation originally derived by BENNEY and LUKE (see [5]):

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon(\Phi_t\Delta\Phi + (\nabla\Phi)_t^2) = 0 \ . \tag{1.1}$$

Here a and b are positive and satisfy $a - b = \sigma - \frac{1}{3}$ where σ is the Bond number, while the parameters ε and μ are supposed to be small.

Pego and Quintero looked for traveling-wave solutions of (1.1), that is solutions of the form

$$\Phi(x, y, t) = \frac{\sqrt{\mu}}{\varepsilon} u\left(\frac{x - ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right) .$$

The scaling was introduced here to eliminate ε and μ. A traveling-wave profile u should satisfy the equation

$$\begin{aligned}(c^2 - 1)u_{xx} - u_{yy} + (a - bc^2)u_{xxxx} + (2a - bc^2)u_{xxyy} + au_{yyyy} - \\ c(3u_x u_{xx} + u_x u_{yy} + 2u_y u_{xy}) = 0 \ .\end{aligned} \tag{1.2}$$

The energy associated to u is

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^2} (1 + c^2)u_x^2 + u_y^2 + (a + bc^2)u_{xx}^2 + (2a + bc^2)u_{xy}^2 + au_{yy}^2 \ dxdy \ . \tag{1.3}$$

It was proved in [39] by the means of the concentration-compactness method that if the wave speed c satisfies $c^2 < \min(1, \frac{a}{b})$, then there exist non-trivial finite energy solutions of (1.2) in a space $\mathcal{V}$, where $\mathcal{V}$ is the completion of $C_0^\infty(\mathbf{R}^2)$ for the norm

$$\| \varphi \|_{\mathcal{V}}^2 = \int_{\mathbf{R}^2} \varphi_x^2 + \varphi_y^2 + \varphi_{xx}^2 + 2\varphi_{xy}^2 + \varphi_{yy}^2 \ dxdy \ .$$

The Benney-Luke equation reduces formally to the Kadomtsev-Petviashvili (KP) equation after a suitable renormalization. Indeed, putting $\tau = \frac{\varepsilon t}{2}$, $X = x - t$, $Y = \varepsilon^{\frac{1}{2}} y$ and $\Phi(x, y, t) = f(X, Y, \tau)$, neglecting $O(\varepsilon)$ terms we find that $\eta = f_X$ satisfies the KP equation

$$(\eta_\tau - (\sigma - \frac{1}{3})\eta_{XXX} + 3\eta\eta_X)_X + \eta_{YY} = 0 \ . \tag{1.4}$$

DE BOUARD and SAUT proved (see [14]) that finite energy solitary waves exist for the KP equation when $\sigma > \frac{1}{3}$ (the KP-I case).

Moreover, let $\sigma > \frac{1}{3}$ (that is, $a > b$), $\varepsilon = 1 - c^2$ and let u_ε be the corresponding solution of (1.2) obtained in [39]. Then if $\varepsilon \longrightarrow 0$, there exists a sequence (ε_j) such that (u_{ε_j}) converges (after a suitable renormalization) to a distribution $v_0 \in \mathcal{D}'(\mathbf{R}^2)$ and $\partial_x v_0$ is a nontrivial solitary wave of the KP equation (see [39]).

It is known (see DE BOUARD and SAUT [15]) that the solitary waves of the KP equation are smooth and decay at infinity with an optimal algebraic rate ($\frac{1}{r^2}$ in dimension 2).

It is then natural to ask whether the Benney-Luke solitary waves have the same properties. The aim of this paper is to give an answer to this question.

We suppose throughout that the parameters a, b, c appearing in (1.2) satisfy: $a > 0$ and if $b > 0$, then $c^2 < \min(1, \frac{a}{b})$.

Our method follows very closely the ideas developed in [15].

This paper is organized as follows: in the next section we prove that the Benney-Luke solitary waves are analytic functions. Section 3 contains our main result about the decay at infinity of such waves. We give an algebraic decay rate which is optimal for the solutions of (1.2) and their first order derivatives. In Section 4 we state some integral identities satisfied by these solitary waves. Some technical facts about the Fourier transform that we use in proofs are treated in an Appendix.

1.2.2 Analyticity

The aim of this section is to prove that any solution $u \in \mathcal{V}$ of (1.2) is an analytic function and tends to zero at infinity as well as all its derivatives. We begin with the following result:

Theorem 2.1 *Let $u \in \mathcal{V}$ be a solution of (1.2). Then*

a) $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in]2, \infty]$;

b) $u_x,\ u_y \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in]1, \infty]$.

Proof. We make extensively use of the following theorem on Fourier multipliers due to LIZORKIN:

Theorem 2.2 ([30]) *Let $\Phi : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a C^n function for $|\xi_j| > 0$, $j = 1, \dots, n$. Assume that*

$$\xi_1^{k_1} \dots \xi_n^{k_n} \frac{\partial^k \Phi}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \in L^\infty(\mathbf{R}^n) \ ,$$

with $k_i = 0$ or 1, $k = k_1 + \dots k_n = 0, 1, \dots, n$. Then $\Phi \in M_q(\mathbf{R}^n)$ for $1 < q < \infty$, i.e. Φ is a Fourier multiplier on $L^q(\mathbf{R}^n)$.

We have $u_x,\ u_y \in H^1(\mathbf{R}^2) \subset L^p(\mathbf{R}^2)$ for all $p \in [2, \infty[$ by the Sobolev imbedding theorem. The nonlinearity can be written as $\partial_x(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + \partial_y(u_x u_y)$. Let $Q(\xi_1, \xi_2) = (1 - c^2)\xi_1^2 + \xi_2^2 + (a - bc^2)\xi_1^4 + (2a - bc^2)\xi_1^2\xi_2^2 + a\xi_2^4$. Equation (1.2) gives

$$Q(\xi_1, \xi_2)\widehat{u_x} = -\xi_1^2 c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \xi_1\xi_2 c\mathcal{F}(u_x u_y)$$

and

$$Q(\xi_1, \xi_2)\widehat{u_y} = -\xi_1\xi_2 c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \xi_2^2 c\mathcal{F}(u_x u_y) \ .$$

The Theorem 2.2 implies that $u_x,\ u_y \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$. We have:

$$Q(\xi_1, \xi_2)\widehat{D^\alpha u} = i\xi_1(i\xi)^\alpha c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + i\xi_2(i\xi)^\alpha c\mathcal{F}(u_x u_y) \ .$$

By Theorem 2.2, $D^\alpha u \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$ if $|\alpha| = 2, 3$. In particular, $u_x,\ u_y \in W^{2,p}(\mathbf{R}^2) \subset C^1 \cap L^\infty(\mathbf{R}^2)$ and for $|\alpha| = 2$, $D^\alpha u \in W^{1,p}(\mathbf{R}^2) \subset C^0 \cap L^\infty(\mathbf{R}^2)$ by the Sobolev imbedding theorem applied for a $p > 2$.

The rest of the proof follows easily by induction. Suppose that all the derivatives of u of order $1, 2, \ldots, n-1$ are in $C^0 \cap L^\infty \cap L^p(\mathbf{R}^2)$ and the n^{th} order derivatives are in $L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Let $\alpha \in \mathbf{N}^2$ with $|\alpha| = n+1$ and $\beta \leq \alpha$ with $|\alpha - \beta| = 2$. Then

$$Q(\xi_1, \xi_2)\widehat{D^\alpha u} = i\xi_1 (i\xi)^{\alpha-\beta} c\mathcal{F}(D^\beta(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)) + i\xi_2 (i\xi)^{\alpha-\beta} c\mathcal{F}(D^\beta(u_x u_y)) .$$

Again by Theorem 2.2 we obtain $D^\alpha u \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. The Sobolev imbedding theorem gives us $D^{\alpha'} u \in C^0 \cap L^\infty(\mathbf{R}^2)$ if $|\alpha'| = n$. This finishes the induction and the proof of part b).

Since $u_x \ u_y \in L^p(\mathbf{R}^2)$ for $p \in]1, \infty]$, Theorem 14.4, p. 295 of [9] yields $u \in L^q(\mathbf{R}^2)$ for all $q \in]2, \infty[$ and

$$\| \ u \ \|_{L^q} \leq C_p \ \| \ \nabla u \ \|_{L^p} \quad , \text{ where } \frac{1}{p} = \frac{1}{2} + \frac{1}{q} \ .$$

Hence $u \in W^{k,q}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $q \in]2, \infty[$. Consequently u is a C^∞ function, it is bounded and tends to zero at infinity.

The Theorem 2.1 is proved. □

Remark 2.3 If $u \in \mathcal{V}$ is a nontrivial solution of (1.2), then u_x and u_y are not in $L^1(\mathbf{R}^2)$.

Proof. We argue by contradiction. Suppose $u_x \in L^1(\mathbf{R}^2)$. Then $\widehat{u_x}$ is a continuous function. But $\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)$ and $\mathcal{F}(u_x u_y)$ are also continuous functions and

$$\widehat{u_x}(\xi_1, \xi_2) = -\frac{\xi_1^2}{Q(\xi_1, \xi_2)} c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \frac{\xi_1 \xi_2}{Q(\xi_1, \xi_2)} c\mathcal{F}(u_x u_y) . \tag{2.2}$$

For a fixed $\lambda \in \mathbf{R}$ we put $\xi_2 = \lambda \xi_1$ and let $\xi_1 \longrightarrow 0$ in (2.2). We obtain

$$\widehat{u_x}(0,0) = -\frac{c}{1 - c^2 + \lambda^2} \int_{\mathbf{R}^2} (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) \ dxdy - \frac{c\lambda}{1 - c^2 + \lambda^2} \int_{\mathbf{R}^2} u_x u_y \ dxdy \ .$$

Since this is true for all $\lambda \in \mathbf{R}$ we deduce that

$$\int_{\mathbf{R}^2} (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) \ dxdy = \int_{\mathbf{R}^2} u_x u_y \ dxdy = 0$$

which implies that u is constant, contrary to the assumption. The same argument applies to u_y. □

Remark 2.4 If $u \in \mathcal{V}$ is a nontrivial solution of (1.2) and $r^{\frac{1}{2}} u_x$, $r^{\frac{1}{2}} u_y \in L^2(\mathbf{R}^2)$ where $r = \sqrt{x^2 + y^2}$ (we shall see in the next section that this is always the case), then u cannot belong to $L^2(\mathbf{R}^2)$.

Proof. Assume $u \in L^2(\mathbf{R}^2)$. Then $\widehat{u} \in L^2(\mathbf{R}^2)$ and

$$\widehat{u}(\xi_1, \xi_2) = \frac{i\xi_1}{Q(\xi_1, \xi_2)} c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + \frac{i\xi_2}{Q(\xi_1, \xi_2)} c\mathcal{F}(u_x u_y) . \tag{2.3}$$

The fact that $r^{\frac{1}{2}}u_x$, $r^{\frac{1}{2}}u_y \in L^2(\mathbf{R}^2)$ implies that $g_1 = c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)$ and $g_2 = c\mathcal{F}(u_x u_y)$ are C^1 functions. The equation (2.3) can be written as

$$\widehat{u}(\xi_1, \xi_2) = \left[\frac{i\xi_1}{Q(\xi_1,\xi_2)}(g_1(\xi_1,\xi_2) - g_1(0,0)) + \frac{i\xi_2}{Q(\xi_1,\xi_2)}(g_2(\xi_1,\xi_2) - g_2(0,0))\right] + \left[\frac{i\xi_1}{Q(\xi_1,\xi_2)}g_1(0,0) + \frac{i\xi_2}{Q(\xi_1,\xi_2)}g_2(0,0)\right]. \tag{2.4}$$

Since g_1 and g_2 are locally Lipschitz functions, the first term in the right hand side of (2.4) is bounded for $\xi \in B_{\mathbf{R}^2}(0,1)$. This forces

$$\frac{i\xi_1}{Q(\xi_1,\xi_2)}g_1(0,0) + \frac{i\xi_2}{Q(\xi_1,\xi_2)}g_2(0,0) \in L^2(B_{\mathbf{R}^2}(0,1)) .$$

But $g_1(0,0) = c\int_{\mathbf{R}^2}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)dxdy > 0$, so it suffices to show that

$$\frac{a\xi_1 + b\xi_2}{Q(\xi_1,\xi_2)} \notin L^2(B_{\mathbf{R}^2}(0,1))$$

if a, $b \in \mathbf{R}$, $a \neq 0$ to obtain a contradiction.

For ξ varying in a bounded set K there exists $m_K > 0$ such that $Q(\xi) \leq m_K|\xi|^2$. We make the change of variables $\xi_1' = a\xi_1 + b\xi_2$, $\xi_2' = \xi_2$, $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. We have:

$$\begin{aligned} \int_{B_{\mathbf{R}^2}(0,1)} \frac{(a\xi_1 + b\xi_2)^2}{Q(\xi_1,\xi_2)^2}d\xi &= \int_{AB_{\mathbf{R}^2}(0,1)} \frac{(\xi_1')^2}{Q(A^{-1}(\xi_1',\xi_2'))^2}|det(A)|^{-1}d\xi' \\ &\geq C\int_{AB_{\mathbf{R}^2}(0,1)} \frac{(\xi_1')^2}{|\xi'|^4}d\xi' = \infty . \ \square \end{aligned}$$

We prove now that any solution $u \in \mathcal{V}$ of (1.2) is an analytic function. The proof relies on the Paley-Wiener theory. We borrowed the ideas developed by Li and Bona in [26].

Let $u \in \mathcal{V}$ be a solution of (2.1). By Theorem 2.1 we have $|\xi|(1+|\xi|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbf{R}^2)$ for all m. We take $m > 1$ and apply thy Cauchy-Schwarz inequality to get

$$\int_{\mathbf{R}^2}|\xi||\widehat{u}(\xi)|d\xi \leq \left(\int_{\mathbf{R}^2}|\xi|^2(1+|\xi|^2)^m|\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^2}(1+|\xi|^2)^{-m}d\xi\right)^{\frac{1}{2}} < \infty.$$

Hence $|\xi|\widehat{u} \in L^1(\mathbf{R}^2)$. Equation (2.3) gives us

$$\begin{aligned} |\xi|\widehat{u}(\xi_1,\xi_2) &= \frac{ic\xi_1|\xi|}{Q(\xi_1,\xi_2)}\left(\frac{3}{2}(i\xi_1\widehat{u}) \star (i\xi_1\widehat{u}) + \frac{1}{2}(i\xi_2\widehat{u}) \star (i\xi_2\widehat{u})\right) \\ &+ \frac{ic\xi_2|\xi|}{Q(\xi_1,\xi_2)}(i\xi_1\widehat{u}) \star (i\xi_2\widehat{u}) \end{aligned}$$

from which we infer that

$$|\xi||\widehat{u}| \leq \frac{3c|\xi|^2}{Q(\xi_1,\xi_2)}(|\xi||\widehat{u}|)\star(|\xi||\widehat{u}|). \tag{2.5}$$

Let $M = \max\limits_{i=2,3,4}\left(\sup\limits_{(\xi_1,\xi_2)\neq(0,0)}\left(\frac{3c|\xi|^i}{Q(\xi_1,\xi_2)}\right)\right)$. Obviously $M<\infty$. We note $\psi(\xi_1,\xi_2) = M|\xi|\cdot|\widehat{u}(\xi_1,\xi_2)|$. Then $\psi\geq 0$, $\psi\in L^1(\mathbf{R}^2)$ and the inequality (2.5) gives

$$\psi\leq\psi\star\psi,\quad |\xi|\psi\leq\psi\star\psi\quad\text{and}\quad|\xi|^2\psi\leq\psi\star\psi. \tag{2.6}$$

For an integrable function f we define $\mathcal{C}_1 f = f$ and for $n>1$, $\mathcal{C}_n f(x) = (f\star(\mathcal{C}_{n-1}f))(x)$. We have

Lemma 2.5 *The function ψ introduced above satisfies*

$$|\xi|^k\psi \leq (\frac{k}{2}+1)^{k-1}\mathcal{C}_{2([\frac{k}{2}]+1)}\psi \tag{2.7}$$

where $[x]$ denotes the greatest integer less or equal than x.

Proof. We proceed by induction on k. From (2.6) it follows that (2.7) holds for $k=1,2,3$. Notice that the first of the inequalities (2.6) implies that $\mathcal{C}_p\psi\leq\mathcal{C}_r\psi$ if $p\leq r$. We suppose that (2.7) is valid up to order k and prove that it is valid for $k+2$. We have:

$$\begin{aligned}
|\xi|^{k+2}\psi(\xi) &\leq |\xi|^k(\psi\star\psi)(\xi) = \int_{\mathbf{R}^2}|\xi|^k\psi(\xi-\zeta)\cdot\psi(\zeta)d\zeta\\
&\leq \int_{\mathbf{R}^2}(|\xi-\zeta|+|\zeta|)^k\psi(\xi-\zeta)\cdot\psi(\zeta)d\zeta\\
&= \int_{\mathbf{R}^2}\sum_{i=0}^{k}C_k^i(|\xi-\zeta|^i\psi(\xi-\zeta))\cdot(|\zeta|^{k-i}\psi(\zeta))d\zeta\\
&= \sum_{i=0}^{k}C_k^i(|\cdot|^i\psi)\star(|\cdot|^{k-i}\psi)(\xi)
\end{aligned}$$

where $C_k^i = \dfrac{k!}{i!(k-i)!}$ is the binomial coefficient. Using the induction hypothesis, the last sum is majorized by

$$\begin{aligned}
&\sum_{i=0}^{k}C_k^i\left((\frac{i}{2}+1)^{i-1}\mathcal{C}_{2([\frac{i}{2}]+1)}\psi\right)\star\left((\frac{k-i}{2}+1)^{k-i-1}\mathcal{C}_{2([\frac{k-i}{2}]+1)}\psi\right)\\
&\quad=\sum_{i=0}^{k}C_k^i(\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1}\mathcal{C}_{2([\frac{i}{2}]+[\frac{k-i}{2}]+2)}\psi\\
&\quad\leq\left(\sum_{i=0}^{k}C_k^i(\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1}\right)\mathcal{C}_{2([\frac{k+2}{2}]+1)}\psi.
\end{aligned}$$

We use a specialization of the Abel identity (see [44], p. 26)

$$\sum_{i=0}^{k}C_k^i(x_1+i)^{i-1}(x_2+k-i)^{k-i-1} = \frac{1}{x_1x_2}(x_1+x_2)(x_1+x_2+k)^{k-1}$$

for $x_1 = x_2 = 2$ to obtain

$$n\sum_{i=0}^{k} C_k^i(\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1} = \frac{1}{2^{k-2}}\sum_{i=0}^{k} C_k^i(2+i)^{i-1}(2+k-i)^{k-i-1}$$
$$= \frac{(4+k)^{k-1}}{2^{k-2}} = 2(2+\frac{k}{2})^{k-1} \leq \left(\frac{k+2}{2}+1\right)^{k+1}.$$

Hence (2.7) holds for $k+2$ and the Lemma is proved. □

Theorem 2.6 *Let $u \in \mathcal{V}$ be a solution of (1.2). Then there exists $\sigma > 0$ and an holomorphic function U of two complex variables z_1, z_2 defined in the domain*

$$\Omega_\sigma = \{(z_1, z_2) \in \mathbf{C}^2 \quad | \quad |Im(z_1)| < \sigma, \quad |Im(z_2)| < \sigma\}$$

such that $U(x,y) = u(x,y)$ for all $(x,y) \in \mathbf{R}^2$.

Proof. It is easily seen from (2.3) that $|\widehat{u}(\xi)| \leq \dfrac{C}{|\xi|}$ for $0 < |\xi| \leq 1$ and $|\widehat{u}(\xi)| \leq \dfrac{C}{|\xi|^3}$ for $|\xi| \geq 1$, so $\widehat{u} \in L^1(\mathbf{R}^2)$.

Keeping the notation introduced above and using Lemma 2.5 we infer that for $k \geq 1$,

$$|\xi|^k|\widehat{u}(\xi)| = \frac{1}{M}|\xi|^{k-1}\cdot M|\xi|\cdot|\widehat{u}(\xi)| = \frac{1}{M}|\xi|^{k-1}\psi(\xi)$$
$$\leq \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2} \mathcal{C}_{2([\frac{k-1}{2}]+1)}\psi(\xi)$$
$$\leq \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2} \mathcal{C}_{k+1}\psi(\xi)$$
$$\leq \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2} ||\psi||_{L^2}\cdot||\mathcal{C}_k\psi||_{L^2}$$
$$\leq \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2} ||\psi||_{L^2}^2\cdot||\psi||_{L^1}^{k-1}.$$

Put $a_k = \dfrac{(\frac{k-1}{2}+1)^{k-2}||\psi||_{L^2}^2\cdot||\psi||_{L^1}^{k-1}}{Mk!}$. Then

$$\frac{a_{k+1}}{a_k} = \frac{1}{2}||\psi||_{L^1}\cdot\left(\frac{k+2}{k+1}\right)^{k-1} \longrightarrow \frac{e}{2}||\psi||_{L^1} \quad \text{as} \quad k \longrightarrow \infty.$$

Let $\sigma = \dfrac{2}{e||\psi||_{L^1}}$. The series $\sum_{k=1}^{\infty} a_k s^k$ converges absolutely for $|s| < \sigma$; we denote by $C(s)$ its sum. Fix $\sigma_1 \in\,]\,0, \sigma\,[$ and choose $\sigma_2 \in\,]\,\sigma_1, \sigma\,[$. One has

$$e^{\sigma_2|\xi|}|\widehat{u}(\xi)| \leq \sum_{k=0}^{\infty}\frac{\sigma_2^k|\xi|^k}{k!}|\widehat{u}(\xi)| \leq |\widehat{u}(\xi)| + \sum_{k=1}^{\infty}\sigma_2^k a_k = |\widehat{u}(\xi)| + C(\sigma_2).$$

Hence

$$e^{\sigma_1|\xi|}|\widehat{u}(\xi)| \leq e^{-(\sigma_2-\sigma_1)|\xi|}|\widehat{u}(\xi)| + e^{-(\sigma_2-\sigma_1)|\xi|}C(\sigma_2).$$

It follows that $e^{\sigma_1|\cdot|}\widehat{u} \in L^1(\mathbf{R}^2)$ for all $\sigma_1 < \sigma$. We define the function

$$U(z_1, z_2) = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{i(z_1\xi_1+z_2\xi_2)}\widehat{u}(\xi_1,\xi_2)d\xi_1 d\xi_2.$$

By the Paley-Wiener Theorem, U is well defined and analytic in Ω_σ and the Plancherel's Theorem implies that $U(x,y) = u(x,y)$ for all $(x,y) \in \mathbf{R}^2$. This proves the Theorem 2.6. □

1.2.3 Decay properties

We prove in this section that all the solutions in $\mathcal{V}$ of (1.2) decay at infinity as $\frac{1}{r}$ and their derivatives decay as $\frac{1}{r^2}$.

From (2.3) we deduce that

$$u = ic\mathcal{F}^{-1}\left(\frac{\xi_1}{Q(\xi_1,\xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + ic\mathcal{F}^{-1}\left(\frac{\xi_2}{Q(\xi_1,\xi_2)}\right) \star (u_x u_y) \ . \tag{3.1}$$

(2.2) gives us

$$u_x = -c\mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - c\mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) \star (u_x u_y) \tag{3.2}$$

and similarly

$$u_y = -c\mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - c\mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right) \star (u_x u_y) \ . \tag{3.3}$$

As we have mentioned in Introduction, our method was inspired by the work of DE BOUARD and SAUT [15]. This idea had already been used by BONA and LI (see [12]). It is based on the study of the convolution equations (3.1), (3.2), (3.3).

We begin with an integral estimate.

Theorem 3.1 *Let $u \in \mathcal{V}$ be a solution of (1.2). Then*

$$\int_{\mathbf{R}^2} (x^2+y^2)|\nabla^2 u|^2 dxdy < \infty \tag{3.4}$$

and

$$\int_{\mathbf{R}^2} (x^2+y^2)|\nabla^3 u|^2 dxdy < \infty \ . \tag{3.5}$$

Proof. Fix a function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = |x|$ for $|x| > 1$, $\varphi(0) = 0$, φ decrease on $]-\infty, 0]$ and increase on $[0,\infty[$. We put

$$\chi_n(x) = e^{-\varphi(\frac{x}{n})} \ .$$

We multiply (1.2) by $x^2\chi_n(x)u_{xx}$ and integrate over $\mathbf{R}^2$. Using several integrations by parts we have

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xxxx}\,dxdy = \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xx}u_{xxx}\,dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xxx}^2dxdy \\
&= \frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_{xx}^2\,dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xxx}^2dxdy\ ;
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{yy}\,dxdy = \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_xu_{yy}\,dxdy + \chi_n(x)x^2u_xu_{xyy}dxdy \\
&= \int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_y\,dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2dxdy \\
&= -\frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_y^2\,dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2dxdy\ ;
\end{aligned}
$$

$$
\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xxyy}\,dxdy = -\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xxy}^2dxdy\ ;
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{yyyy}\,dxdy = \\
&= \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xxyy}u_{yy}dxdy \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xyy}u_{yy} + \chi_n(x)x^2u_{xyy}^2\,dxdy \\
&= \frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_{yy}^2\,dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xyy}^2dxdy\ ;
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_xu_{yy}\,dxdy = \\
&= -\frac{1}{2}\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_x^2u_{yy} + \chi_n(x)x^2u_x^2u_{yyx}\,dxdy \\
&= \int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\,dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\,dxdy\ .
\end{aligned}
$$

Finally we get

(3.6)

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2[(1-c^2)u_{xx}^2 + u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2]\,dxdy \\
&\quad +3c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x\,dxdy + c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\,dxdy \\
&\quad +c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\,dxdy + 2c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y\,dxdy \\
&= \int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)\left[\frac{a-bc^2}{2}u_{xx}^2 + \frac{a}{2}u_{yy}^2 + \frac{1}{2}u_y^2\right]\,dxdy\ .
\end{aligned}
$$

Since $\chi_n'(x) = -\frac{1}{n}\varphi'\left(\frac{x}{n}\right)e^{-\varphi(\frac{x}{n})}$, there exists a constant $k > 0$ such that $|x\chi_n'(x)| \le k\chi_n(x)^{\frac{1}{2}}$ for all $x \in \mathbf{R}$ and $n \ge 1$. We have

$$\begin{aligned}
&|\partial_x(\chi_n(x)x^2)u_{xy}u_xu_y| \\
&\le |\chi_n'(x)x^2u_{xy}u_xu_y| + 2|\chi_n(x)xu_{xy}u_xu_y| \\
&\le k\chi_n(x)^{\frac{1}{2}}|xu_{xy}u_xu_y| + 2\chi_n(x)|xu_{xy}u_xu_y| \\
&\le \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x| \,.
\end{aligned}$$

and

$$2|(\chi_n(x)x^2)u_{xx}u_{xy}u_y| \le \chi_n(x)x^2(u_{xx}^2 + u_{xy}^2)|u_y| \,.$$

Let $\varepsilon \in \,]\,0,1\,[$. Since u_x and u_y tend to 0 as $r \longrightarrow \infty$, there exists $R_\varepsilon > 0$ such that $|u_x(x,y)| < \varepsilon$ and $|u_y(x,y)| < \varepsilon$ if $|(x,y)| > R_\varepsilon$. Then

$$\begin{aligned}
\left| c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y \, dxdy \right| &\le c\int_{\mathbf{R}^2} \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x| \, dxdy \\
&\le c\int_{B(0,R_\varepsilon)} \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x| \, dxdy \\
&\quad + c\varepsilon\frac{k+2}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2 + u_y^2 \, dxdy \\
&\le C(\varepsilon) + c\cdot\frac{k+2}{2}\cdot\varepsilon\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2 \, dxdy
\end{aligned}$$

where $C(\varepsilon)$ is a constant depending on ε. Similar estimates hold for $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x \, dxdy$, $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x \, dxdy$ and $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y \, dxdy$. We take ε sufficiently small to obtain

$$\begin{aligned}
&\left| 3c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x \, dxdy + c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y \, dxdy \right. \\
&\left. + c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x \, dxdy + 2c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y \, dxdy \right| \\
&\le C + \frac{1}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2((1-c^2)u_{xx}^2 + u_{xy}^2) \, dxdy \,.
\end{aligned}$$

where C is a constant.

Combining the last inequality with (3.6) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2[(1-c^2)u_{xx}^2 + u_{xy}^2] \, dxdy \\
&+ \int_{\mathbf{R}^2} \chi_n(x)x^2[(a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2] \, dxdy \\
&\le C + \int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)\left[\frac{a-bc^2}{2}u_{xx}^2 + \frac{a}{2}u_{yy}^2 + \frac{1}{2}u_y^2\right] \, dxdy \,.
\end{aligned} \tag{3.7}$$

When $n \longrightarrow \infty$ the left hand side of (3.7) tends to

$$\int_{\mathbf{R}^2} x^2\left[\frac{1-c^2}{2}u_{xx}^2 + \frac{1}{2}u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2\right] \, dxdy$$

by the monotone convergence theorem, while the right hand side tends to

$$C + \int_{\mathbf{R}^2} (a - bc^2)u_{xx}^2 + au_{yy}^2 + u_y^2 \, dxdy < \infty$$

by Lebesgue's theorem on dominated convergence. Hence

$$\int_{\mathbf{R}^2} x^2(u_{xx}^2 + u_{xy}^2 + u_{xxx}^2 + u_{xxy}^2 + u_{xyy}^2) \, dxdy < \infty \; . \tag{3.8}$$

We multiply (1.2) by $\chi_n(y)y^2u_{xx}$ and integrate over $\mathbf{R}^2$ to get, after several integrations by parts,

(3.9)

$$\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(y)y^2[(1-c^2)u_{xx}^2 + u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2] \, dxdy \\
&\quad -3c\int_{\mathbf{R}^2} \chi_n(y)y^2u_{xx}^2u_x \, dxdy + c\int_{\mathbf{R}^2} \partial_y(\chi_n(y)y^2)u_{xx}u_xu_y \, dxdy \\
&\quad -c\int_{\mathbf{R}^2} \chi_n(y)y^2(u_{xy}^2u_x + 2u_{xx}u_{xy}u_y) \, dxdy \\
&= \int_{\mathbf{R}^2} \partial_{yy}^2(\chi_n(y)y^2)\left[\frac{2a-bc^2}{2}u_{xx}^2 + 2au_{xy}^2 + u_x^2\right] \, dxdy \\
&\quad -\frac{a}{2}\int_{\mathbf{R}^2} \partial_{yyyy}^4(\chi_n(y)y^2)u_x^2 \, dxdy \; .
\end{aligned}$$

As previously, there exists a constant $C > 0$ such that the last three terms in the left side of (3.9) are dominated by

$$C + \frac{1}{2}\int_{\mathbf{R}^2} \chi_n(y)y^2((1-c^2)u_{xx}^2 + u_{xy}^2) \, dxdy \; .$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^2} \chi_n(y)y^2[(1-c^2)u_{xx}^2 + u_{xy}^2] \, dxdy \\
&+ \int_{\mathbf{R}^2} \chi_n(y)y^2[(a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2] \, dxdy \\
&\le C + \int_{\mathbf{R}^2} \partial_{yy}^2(\chi_n(y)y^2)\left[u_x^2 + \frac{2a-bc^2}{2}u_{xx}^2 + 2au_{xy}^2\right] \, dxdy \\
&\quad -\frac{a}{2}\int_{\mathbf{R}^2} \partial_{yyyy}^4(\chi_n(y)y^2)u_x^2 \, dxdy \; .
\end{aligned} \tag{3.10}$$

Passing to the limit as $n \longrightarrow \infty$ in (3.10) and using the monotone convergence theorem for the left side and Lebesgue's dominated convergence theorem for the right side one obtains:

$$\begin{aligned}
&\int_{\mathbf{R}^2} y^2\left[\frac{1-c^2}{2}u_{xx}^2 + \frac{1}{2}u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2\right] \, dxdy \\
&\quad \le C + \int_{\mathbf{R}^2} 2u_x^2 + (2a-bc^2)u_{xx}^2 + 4au_{xy}^2 \, dxdy < \infty \; .
\end{aligned}$$

Thus

$$\int_{\mathbf{R}^2} y^2(u_{xx}^2 + u_{xy}^2 + u_{xxx}^2 + u_{xxy}^2 + u_{xyy}^2)\,dxdy < \infty\,. \tag{3.11}$$

Multiplying the equation (1.2) by $\chi_n(x)x^2u_{yy}$ (respectively by $\chi_n(y)y^2u_{yy}$), integrating by parts and proceeding as above we obtain

$$\int_{\mathbf{R}^2} x^2(u_{xy}^2 + u_{yy}^2 + u_{xxy}^2 + u_{xyy}^2 + u_{yyy}^2)\,dxdy < \infty \tag{3.12}$$

and

$$\int_{\mathbf{R}^2} y^2(u_{xy}^2 + u_{yy}^2 + u_{xxy}^2 + u_{xyy}^2 + u_{yyy}^2)\,dxdy < \infty\,. \tag{3.13}$$

Theorem 3.1 follows from (3.8), (3.11), (3.12) and (3.13). □

Lemma 3.2 *We have $ru_x \in L^\infty(\mathbf{R}^2)$ and $ru_y \in L^\infty(\mathbf{R}^2)$.*

Proof. From (1.2) we deduce that

$$Q(\xi_1,\xi_2)\widehat{u_x} = ic\xi_1\mathcal{F}(3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy}) \tag{3.14}$$

and

$$Q(\xi_1,\xi_2)\widehat{u_y} = ic\xi_2\mathcal{F}(3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy}) \tag{3.15}$$

We note $h_i = \mathcal{F}^{-1}(\dfrac{\xi_i}{Q(\xi_1,\xi_2)})$ and $g = 3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy}$. The previous equations can be written as

$$u_x = ich_1 \star g \quad\text{and}\quad u_y = ich_2 \star g$$

Then

$$|ru_x| \le C|rh_1| \star |g| + C|h_1| \star |rg|\,. \tag{3.16}$$

We claim that $rg \in W^{1,p}(\mathbf{R}^2)$ for all $p \in [1,2]$. Indeed, Theorems 3.1 and 2.1 imply that $rg \in L^p(\mathbf{R}^2)$ for all $p \in [1,2]$. Moreover, since $r\nabla^3 u \in L^2(\mathbf{R}^2)$ we have (denoting by D one of the operators ∂_x or ∂_y):

$$D(rDuD^2u) = (Dr)DuD^2u + rD^2uD^2u + rDuD^3u \in L^p(\mathbf{R}^2)$$

for $1 \le p \le 2$. Thus $D(rg) \in L^p(\mathbf{R}^2)$ and so $rg \in W^{1,p}(\mathbf{R}^2)$.

It is clear now that $|rg| \in W^{1,p}(\mathbf{R}^2)$.

By Lemma A1 in Appendix we have $rh_i \in L^\infty(\mathbf{R}^2)$. Then $h_i \in L^2_w(\mathbf{R}^2)$ and using the generalized Young's theorem we deduce

$$|h_i| \star |rg| \in L^q(\mathbf{R}^2) \text{ if } 2 < q < \infty$$

and

$$D(|h_i| \star |rg|) = |h_i| \star (D|rg|) \in L^q(\mathbf{R}^2) \text{ if } 2 < q < \infty\,.$$

So $|h_i| \star |rg| \in W^{1,q}(\mathbf{R}^2)$ for $2 < q < \infty$. The Sobolev imbedding theorem gives us $|h_i| \star |rg| \in L^\infty(\mathbf{R}^2)$. But $|rh_i| \star |g|$ is also in $L^\infty(\mathbf{R}^2)$ because $rh_i \in L^\infty(\mathbf{R}^2)$ and $g \in L^1(\mathbf{R}^2)$. Using (3.16) we obtain the desired conclusion. □

We note

$$k_1 = \mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) , \quad k_2 = \mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) , \quad k_3 = \mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right) .$$

Lemma 3.3 *$\widehat{k_i} \in H^s(\mathbf{R}^2)$ for $0 \le s < 1$ and $k_i \in L^q(\mathbf{R}^2)$ if $1 < q \le 2$, $i = 1, 2, 3$.*

Proof. The proof is essentially the same as the proof of Lemma 3.4 in [15]. For the sake of completeness, we give it here.

It is easy to verify that $\widehat{k_i} \in L^2(\mathbf{R}^2)$ and

$$|\nabla \widehat{k_i}| \le C\frac{|\xi_1| + |\xi_2|}{Q(\xi_1,\xi_2)} \in L^q(\mathbf{R}^2)$$

if $1 \le q < 2$. Hence $\widehat{k_i}$ belongs to the homogeneous Sobolev space $\dot{W}^{1,q}(\mathbf{R}^2)$, $1 \le q < 2$. By Theorem 6.5.1 in [8], $\dot{W}^{1,q}(\mathbf{R}^2) \subset \dot{H}^s(\mathbf{R}^2)$ for $s = 2(1 - \frac{1}{q})$. So $\widehat{k_i} \in \dot{H}^s(\mathbf{R}^2)$ for any $s \in [0,1)$, $i = 1,2,3$. Since $\widehat{k_i} \in L^2(\mathbf{R}^2)$ we have $\widehat{k_i} \in H^s(\mathbf{R}^2)$, $s \in [0,1)$, $i = 1,2,3$.

Let $q \in (1,2]$ be given. Let $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{2}$, $\alpha \in (2,\infty]$. We choose $s \in [0,1)$ such that $s\alpha > 2$. Then we have:

$$\begin{aligned} \| k_i \|_{L^q} &\le \| (1+r^2)^{\frac{s}{2}} k_i \|_{L^2} \cdot \| \frac{1}{(1+r^2)^{\frac{s}{2}}} \|_{L^\alpha} \\ &= \| \widehat{k_i} \|_{H^s} \cdot \| \frac{1}{(1+r^2)^{\frac{s}{2}}} \|_{L^\alpha} \qquad < \infty \end{aligned}$$

Thus $k_i \in L^q(\mathbf{R}^2)$ for all $q \in (1,2]$, $i = 1,2,3$ and the lemma is proved. □

We may state now our main result.

Theorem 3.4 *Let $u \in \mathcal{V}$ be a solution of (1.2). Then*

a) $r^2 D^\alpha u \in L^\infty(\mathbf{R}^2)$ for all $\alpha \in \mathbf{N}^2$, $|\alpha| \ge 1$;

b) $ru \in L^\infty(\mathbf{R}^2)$.

In view of the remarks 2.3 and 2.4, the estimates given by Theorem 3.4 for u, u_x and u_y are optimal.

Proof. We note

$$\varphi_1 = \frac{3}{2}u_x^2 + \frac{1}{2}u_y^2 , \qquad \varphi_2 = u_x u_y .$$

The equations (3.2) and (3.3) can be written as

$$u_x = -ck_1 \star \varphi_1 - ck_2 \star \varphi_2 ,$$

$$u_y = -ck_2 \star \varphi_1 - ck_3 \star \varphi_2 .$$

Let us prove first that $r^{1+\delta}u_x$ and $r^{1+\delta}u_y$ are in $L^\infty(\mathbf{R}^2)$ if $\delta \in [0,1)$. It clearly suffices to show that $r^{1+\delta}(k_i \star \varphi_j) \in L^\infty(\mathbf{R}^2)$. We have:

$$|r^{1+\delta}(k_i \star \varphi_j)| \le C|r^{1+\delta}k_i| \star |\varphi_j| + C|k_i| \star |r^{1+\delta}\varphi_j| . \tag{3.17}$$

By Lemma A2 in Appendix (and the remark A3) we have $r^{1+\delta}k_i \in L^\infty(\mathbf{R}^2)$. But $\varphi_j \in L^1(\mathbf{R}^2)$ and so

$$|r^{1+\delta}k_i| \star |\varphi_j| \in L^\infty(\mathbf{R}^2) .$$

By Lemma 3.2 and Theorem 2.1,

$$|r^{1+\delta}\varphi_j| \leq |(1+r)^2\varphi_j| \cdot |\frac{1}{(1+r)^{1-\delta}}| \in L^p(\mathbf{R}^2)$$

for all $p > \dfrac{2}{1-\delta}$. Since $k_i \in L^q(\mathbf{R}^2)$ for $1 < q \leq 2$, we obtain (choosing $p > \dfrac{2}{1-\delta}$ and $q = \dfrac{p-1}{p}$):

$$|k_i| \star |r^{1+\delta}\varphi_j| \in L^\infty(\mathbf{R}^2) .$$

Thus the right side of (3.17) is bounded and so $r^{1+\delta}u_x$, $r^{1+\delta}u_y \in L^\infty(\mathbf{R}^2)$ for all $\delta \in [0,1)$.

We have:

$$|r^2k_i \star \varphi_j| \leq C|r^2k_i| \star |\varphi_j| + C|k_i| \star |r^2\varphi_j| .$$

Clearly, $|r^2k_i| \star |\varphi_j| \in L^\infty(\mathbf{R}^2)$ because $r^2k_i \in L^\infty(\mathbf{R}^2)$ by Lemma A2 and $\varphi_j \in L^1(\mathbf{R}^2)$.

Since $|r^{1+\delta}\nabla u| \in L^\infty(\mathbf{R}^2)$ one obtains $r^2\varphi_j \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$. But $k_i \in L^q(\mathbf{R}^2)$ for $1 < q \leq 2$ and so $|k_i|\star|r^2\varphi_j| \in L^\infty(\mathbf{R}^2)$. Thus r^2u_x, $r^2u_y \in L^\infty(\mathbf{R}^2)$.

The rest of part a) follows easily by induction. Keeping the notations of Lemma 3.2 we have $r^2g \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$. If $\alpha \in \mathbf{N}^2$ and $|\alpha| = 2$, then $Q(\xi_1,\xi_2)\widehat{D^\alpha u} = -c \cdot \xi^\alpha \widehat{g}$, so $D^\alpha u$ can be written as

$$D^\alpha u = -c \cdot k_i \star g$$

for an $i \in \{1,2,3\}$. Hence $|r^2D^\alpha u| \leq C(|r^2k_i|\star|g| + |k_i|\star|r^2g|) \in L^\infty(\mathbf{R}^2)$. Suppose now that $r^2D^\alpha u \in L^\infty(\mathbf{R}^2)$ if $1 \leq |\alpha| \leq n$. Let $\gamma \in \mathbf{N}^2$ with $|\gamma| = n+1$. Let $\beta \in \mathbf{N}^2$, $\beta \leq \gamma$ and $|\gamma - \beta| = 2$. Then $Q(\xi_1,\xi_2)\widehat{D^\gamma u} = -c \cdot \xi^{\gamma-\beta}\widehat{D^\beta g}$. Hence

$$D^\gamma u = -c \cdot k_i \star (D^\beta g)$$

for an $i \in \{1,2,3\}$. By hypothesis $r^2D^\beta g \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$ and we deduce as above that $r^2D^\gamma g \in L^\infty(\mathbf{R}^2)$.

b) We write (3.1) in the form

$$u = ich_1 \star \varphi_1 + ich_2 \star \varphi_2 . \tag{3.18}$$

As previously we prove that $r\varphi_i \in W^{1,p}(\mathbf{R}^2)$, $p \in]1,\infty]$ and so $|r\varphi_i| \in W^{1,p}(\mathbf{R}^2)$ for $p \in]1,\infty]$.

By Lemma A1 in Appendix, $h_i \in L^2_w(\mathbf{R}^2)$. The generalized Young's theorem implies

$$|h_i| \star |r\varphi_i| \in L^q(\mathbf{R}^2) \text{ if } q \in]2,\infty[$$

and

$$D(|h_i| \star |r\varphi_i|) = |h_i| \star (D|r\varphi_i|) \in L^q(\mathbf{R}^2),\ q \in]2,\infty[$$

hence $|h_i| \star |r\varphi_i| \in W^{1,q}(\mathbf{R}^2)$ for $q \in]2, \infty[$. By the Sobolev imbedding theorem, $|h_i| \star |r\varphi_i| \in L^\infty(\mathbf{R}^2)$. Clearly $|rh_i| \star |\varphi_i| \in L^\infty(\mathbf{R}^2)$ because $rh_i \in L^\infty(\mathbf{R}^2)$ and $\varphi_i \in L^1(\mathbf{R}^2)$. Thus we have

$$|ru| \leq C \sum_{i=1,2} (|rh_i| \star |\varphi_i| + |h_i| \star |r\varphi_i|) \in L^\infty(\mathbf{R}^2) .$$

This finishes the proof of Theorem 3.4. □

1.2.4 Some identities

We derive here some identities of Pohozaev type satisfied by the Benney-Luke solitary waves. If $u \in \mathcal{V}$ is a solution of (1.2), multiplying (1.2) by xu_x (respectively by yu_y) and integrating over $\mathbf{R}^2$ we obtain, after a few integrations by parts,

$$\begin{aligned} &\int_{\mathbf{R}^2} (1-c^2)u_x^2 - u_y^2 + 3(a-bc^2)u_{xx}^2 - au_{yy}^2 + (2a-bc^2)u_{xy}^2 \, dxdy \\ &\qquad + 2c \int_{\mathbf{R}^2} u_x^3 \, dxdy = 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\int_{\mathbf{R}^2} (1-c^2)u_x^2 - u_y^2 + (a-bc^2)u_{xx}^2 - 3au_{yy}^2 - (2a-bc^2)u_{xy}^2 \, dxdy \\ &\qquad + c \int_{\mathbf{R}^2} (u_x^3 - u_x u_y^2) \, dxdy = 0 . \end{aligned} \tag{4.2}$$

Multiplying (1.2) by u and integrating one obtains immediately

$$\begin{aligned} &\int_{\mathbf{R}^2} (1-c^2)u_x^2 + u_y^2 + (a-bc^2)u_{xx}^2 + au_{yy}^2 + (2a-bc^2)u_{xy}^2 \, dxdy + \\ &\qquad \frac{3c}{2} \int_{\mathbf{R}^2} u_x^3 + u_y^2 u_x \, dxdy = 0 . \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we deduce

$$\int_{\mathbf{R}^2} (1-c^2)u_x^2 + u_y^2 \, dxdy = 2 \int_{\mathbf{R}^2} (a-bc^2)u_{xx}^2 + au_{yy}^2 + (2a-bc^2)u_{xy} \, dxdy .$$

1.2.5 Appendix

We prove here some technical facts about the Fourier transform of a special kind of functions of two variables.

Lemma A1. *Let $a, b, c, d, e > 0$ and let $Q(\xi_1, \xi_2)$ be the polynomial of two variables*

$$Q(\xi_1, \xi_2) = a\xi_1^4 + b\xi_2^4 + c\xi_1^2\xi_2^2 + d\xi_1^2 + e\xi_2^2 .$$

If

$$c^2 - 4ab > 0 \tag{i}$$

then we have

a) $r\mathcal{F}^{-1}\left(\frac{\xi_1}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

b) $r\mathcal{F}^{-1}\left(\frac{\xi_2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$.

where $r = \sqrt{x^2+y^2}$ *and* $\mathcal{F}^{-1}$ *denotes the inverse Fourier transform.*

Proof. We regard separately $Q(\xi_1,\xi_2)$ as a polynomial of second degree in ξ_1^2 (respectively in ξ_2^2) and calculate its discriminant in each case.

$$Q(\xi_1,\xi_2) = a\xi_1^4 + (c\xi_2^2+d)\xi_1^2 + b\xi_2^4 + e\xi_2^2$$

$$\Delta_1(\xi_2) = (c^2-4ab)\xi_2^4 + 2(cd-2ae)\xi_2^2 + d^2$$

$$Q(\xi_1,\xi_2) = b\xi_2^4 + (c\xi_1^2+e)\xi_2^2 + a\xi_1^4 + d\xi_1^2$$

$$\Delta_2(\xi_1) = (c^2-4ab)\xi_1^4 + 2(ce-2bd)\xi_1^2 + e^2 \; .$$

Remark that we always have

$$ce - 2bd > 0 \text{ or } cd - 2ae > 0 \; .$$

Indeed, suppose that $ce - 2bd \le 0$. Then $d \ge \frac{ce}{2b}$ implies $cd - 2ae \ge \frac{c^2e}{2b} - 2ae = \frac{e}{2b}(c^2-4ab) > 0$ by (i). So we may assume without loss of generality that $ce - 2bd > 0$ and $b = 1$. In this case $Q(\xi_1,\xi_2)$ can be written as a product

$$Q(\xi_1,\xi_2) = (\xi_2^2 + A^2(\xi_1))(\xi_2^2 + B^2(\xi_1))$$

where $A(\xi)$ and $B(\xi)$ are positive and

$$A^2(\xi) = \frac{1}{2}[c\xi^2 + e - \sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}] \; ,$$

$$B^2(\xi) = \frac{1}{2}[c\xi^2 + e + \sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}] \; .$$

It is easy to check that the functions A and B have the following properties:

(1) $A \in C(\mathbf{R}) \cap C^\infty(\mathbf{R}\setminus\{0\})$, $B \in C^\infty(\mathbf{R})$, $A(-\xi) = A(\xi)$, $B(-\xi) = B(\xi)$.

(2) There exist constants $C_1, C_2 > 0$ such that

$$C_1|\xi| \le A(\xi) \le C_2|\xi| \text{ and}$$
$$C_1(1+|\xi|) \le B(\xi) \le C_2(1+|\xi|) \; , \quad \forall \xi \in \mathbf{R}.$$

(3) There are $C_1, C_2 > 0$ verifying

$$C_1 \le A'(\xi) \le C_2 \; , \quad \forall \xi > 0$$
$$C_1\xi \le B'(\xi) \le C_2\xi \; , \quad \forall \xi \in [-1,1]$$
$$C_1 \le B'(\xi) \le C_2 \; , \quad \forall \xi \in [1,\infty[\; .$$

(4) There exists $C > 0$ such that

$$|A''(\xi)| \leq \frac{C}{|\xi|}\,, \quad \forall \xi \in \mathbf{R} \setminus \{0\} \text{ and}$$
$$|B''(\xi)| \leq \frac{C}{(1+|\xi|)}\,, \quad \forall \xi \in \mathbf{R}\,.$$

Putting $h_i = \mathcal{F}^{-1}\left(\frac{\xi_i}{Q(\xi_1,\xi_2)}\right)$ we have:

$$h_1(x,y) = \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2} \frac{\xi_1}{B^2(\xi_1)-A^2(\xi_1)} \left(\frac{1}{\xi_2^2+A^2(\xi_1)} - \frac{1}{\xi_2^2+B^2(\xi_1)}\right) d\xi_1 d\xi_2\,.$$

But $\mathcal{F}^{-1}\left(\frac{1}{\xi^2+a^2}\right)(x) = -\frac{1}{2a}e^{-a|x|}$ if $Re(a) > 0$ and so we obtain

$$h_1(x,y) =$$

(5)
$$= \int_{\mathbf{R}} e^{ix\xi} \frac{\xi}{B^2(\xi)-A^2(\xi)} \left[\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right] d\xi$$
$$= \int_{\mathbf{R}} e^{ix\xi} \cdot \frac{\xi\left[\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right]}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\, d\xi\,.$$

By (2),

$$\left|e^{ix\xi}\frac{\xi}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\left[\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right]\right|$$
$$\leq Ce^{-C_1|y||\xi|}\,,$$

hence if $y \neq 0$,

(6)
$$|h_1(x,y)| \leq C\int_{\mathbf{R}} e^{-C_1|y||\xi|}\, d\xi = \frac{C}{|y|}\,.$$

To obtain an estimate of $|h_1(x,y)|$ in terms of $\frac{1}{|x|}$ we use the following elementary result:

Lemma H. *Let $I \subset \mathbf{R}$ be an interval (bounded or not) and let $f : I \longrightarrow \mathbf{R}$ be an integrable and monotone function. There exists an absolute constant $C > 0$ (we may take $C = 4\sqrt{2}\pi$) such that*

$$\left|\int_I e^{ix\xi} f(\xi)\, d\xi\right| \leq \frac{C}{|x|} \cdot \sup_{\xi \in I} |f(\xi)|\,.$$

To prove Lemma H one estimates $\int_I \sin(x\xi)f(\xi)d\xi$ and $\int_I \cos(x\xi)f(\xi)d\xi$ by splitting I into intervals on which $\sin(x\xi)$, respectively $\cos(x\xi)$ have constant sign, which gives an alternating sum of monotone terms.

Let $f_{1,y}(\xi) = \dfrac{\xi}{A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}e^{-A(\xi)|y|}$. Note that $f_{1,y}$ is differentiable on $\mathbf{R}\setminus\{0\}$. If we prove that $f'_{1,y}$ has at most N zeros where N does not depend on y, then we can decompose $\mathbf{R}$ into (at most) $N+2$ intervals where $f_{1,y}$ is monotone. Applying Lemma H on each of these intervals we finally obtain

$$\left|\int_{\mathbf{R}} e^{ix\xi} f_{1,y}(\xi)\, d\xi\right| \le \frac{C}{|x|}\cdot \sup_{\xi\in\mathbf{R}} |f_{1,y}(\xi)| \le \frac{C}{|x|}\,. \tag{7}$$

Let us count now the zeros of $f'_{1,y}$. For $\xi \neq 0$ one obtains

$$f'_{1,y}(\xi) = \frac{e^{-A(\xi)|y|}}{A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\times$$

$$\times\left[\frac{-(c^2-4a)\xi^4+e^2}{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2} - \frac{\xi A'(\xi)}{A(\xi)} - \xi|y|A'(\xi)\right]\,.$$

Thus $f'_{1,y}(\xi)=0$ clearly implies that ξ is a solution of an equation

$$P(\xi)A^2(\xi) + R(\xi)A'(\xi)A(\xi) + S(\xi,|y|)A'(\xi)A^2(\xi) = 0\,,$$

where $P(\xi)$, $R(\xi)$ are polynomials in ξ and and $S(\xi,|y|)$ is a polynomial in two variables ξ, $|y|$. Multiplying this by $\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$ we obtain

$$P_1(\xi) + R_1(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$$

$$+\left(S_1(\xi,|y|) + S_2(\xi,|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}\right)A(\xi) = 0\,,$$

where $P_1(\xi)$, $R_1(\xi)$ and $S_1(\xi,|y|)$, $S_2(\xi,|y|)$ are polynomials. Passing the last term on the right and taking the squares we deduce that ξ must satisfy

$$P_2(\xi) + R_2(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$$

$$= S_3(\xi,|y|) + S_4(\xi,|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}\,.$$

(here $P_2(\xi)$, $R_2(\xi)$, $S_3(\xi,|y|)$ and $S_4(\xi,|y|)$ are polynomials).

If we isolate $\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$ and take again the squares, we find that

$$\Phi(\xi,|y|) = 0\,,$$

where $\Phi(\xi,|y|)$ is a polynomial in two variables. Let N be the degree of Φ in the first variable. It is clear now that for a fixed y, the last equation has at most N solutions; hence for each y, $f'_{1,y}$ has at most N zeros in $\mathbf{R}\setminus\{0\}$.

Exactly the same argument applies to

$$f_{2,y}(\xi) = \frac{\xi}{B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}e^{-B(\xi)|y|}$$

and gives us the estimate

$$\left|\int_{\mathbf{R}} e^{ix\xi} f_{2,y}(\xi)\, d\xi\right| \le \frac{C}{|x|}\,. \tag{8}$$

From (6), (7) and (8) we infer that

$$|h_1(x,y)| \le \frac{C}{|x|+|y|} ,$$

that is, $rh_1 \in L^\infty(\mathbf{R}^2)$.

b) One easily checks that if $Re(a) > 0$ and $Re(b) > 0$, then

$$\mathcal{F}\left(\frac{i}{2}sgn(x)\frac{1}{b^2-a^2}(e^{-a|x|}-e^{-b|x|})\right)(\xi) = \frac{\xi}{(\xi^2+a^2)(\xi^2+b^2)}$$

or equivalently

$$\mathcal{F}^{-1}\left(\frac{\xi}{(\xi^2+a^2)(\xi^2+b^2)}\right)(x) = \frac{i}{2}sgn(x)\frac{1}{b^2-a^2}(e^{-a|x|}-e^{-b|x|}) .$$

Consequently, we have

$$h_2(x,y) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2}\frac{\xi_2}{(\xi_2^2+A^2(\xi_1))(\xi_2^2+B^2(\xi_1))}\,d\xi_1 d\xi_2$$

$$= \int_{\mathbf{R}} \frac{i}{2}sgn(y)e^{ix\xi}\frac{e^{-A(\xi)|y|}-e^{-B(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\,d\xi .$$

If $y \neq 0$,

$$\begin{aligned} |h_2(x,y)| &\le C\int_{\mathbf{R}} e^{-A(\xi)|y|}-e^{-B(\xi)|y|}\,d\xi \\ &\le C\int_{\mathbf{R}} e^{-C_1|y||\xi|} \qquad (9) \\ &= \frac{C}{|y|} . \end{aligned}$$

If $x \neq 0$, we apply Lemma H to the functions

$$g_{1,y}(\xi) = \frac{e^{-A(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}$$

and

$$g_{2,y}(\xi) = \frac{e^{-B(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}$$

and reason as in part a) to obtain

$$h_2(x,y) \le \frac{C}{|x|} . \qquad (10)$$

Inequalities (9) and (10) clearly give $h_2(x,y) \le \frac{C}{|r|}$, which is the desired conclusion.
□

Lemma A2. *With the assumptions and the notations of Lemma A1, we have:*

a) $r^2\mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

b) $r^2\mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

c) $r^2\mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$.

Proof. a) As in the proof of Lemma A1, we write

$$k_1(x,y) = \mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2}\frac{\xi_1^2}{\sqrt{(c^2-4a)\xi_1^4+2(ce-2d)\xi_1^2+e^2}}\left(\frac{1}{\xi_2^2+A^2(\xi_1)}-\frac{1}{\xi_2^2+B^2(\xi_1)}\right)d\xi_1 d\xi_2$$

$$= \int_{\mathbf{R}} e^{ix\xi}\cdot\frac{\xi^2\left(\frac{1}{2B(\xi)}e^{-B(\xi)|y|}-\frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right)}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\,d\xi\ ;$$

$$k_1(x,y) = T_B - T_A\ ,$$

where

$$T_B = \int_{\mathbf{R}} e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi^2}{2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ ,$$

$$T_A = \int_{\mathbf{R}} e^{ix\xi-A(\xi)|y|}\cdot\frac{\xi^2}{2A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ .$$

Integrating by parts we get

$$(11)\quad T_B = e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi^2}{(ix-B'(\xi)|y|)2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\Bigg|_{-\infty}^{\infty} -$$

$$-\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi}{(ix-B'(\xi)|y|)B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\,d\xi$$

$$-\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi^2\cdot B''(\xi)|y|}{(ix-B'(\xi)|y|)^2 2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\,d\xi$$

$$+\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi^2 B'(\xi)}{(ix-B'(\xi)|y|)2B^2(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\,d\xi$$

$$+\int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{2\xi^3[(c^2-4a)\xi^2+ce-2d]}{(ix-B'(\xi)|y|)2B(\xi)(\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2})^3}\, d\xi\ .$$

The first term equals 0.

Suppose $y \neq 0$. If $\xi \in [-1,1] \setminus \{0\}$ by (3) we have

$$\left|\frac{\xi}{ix - B'(\xi)|y|}\right| = \frac{1}{\left|i\frac{x}{\xi} - \frac{B'(\xi)}{\xi}|y|\right|} \leq \frac{1}{\left|\frac{B'(\xi)}{\xi}\right||y|} \leq \frac{1}{C_1|y|}\ .$$

If $\xi \in \mathbf{R} \setminus [-1,1]$, (3) gives us

$$\left|\frac{1}{ix - B'(\xi)|y|}\right| \leq \frac{1}{B'(\xi)|y|} \leq \frac{1}{C_1|y|}\ .$$

It is now easy to see that the absolute value of each of the four integrals above is less than

$$\frac{C}{|y|}\int_{-\infty}^{\infty} e^{-B(\xi)|y|} d\xi \leq \frac{C}{|y|^2}\ .$$

Hence

$$T_B \leq \frac{C}{|y|^2}\ . \tag{12}$$

Consider, for example, the first integral in (11). It can be written as

$$\int_{-\infty}^{\infty} e^{ix\xi} f_{x,y}(\xi)\, d\xi\ ,$$

where

$$f_{x,y}(\xi) = e^{-B(\xi)|y|} \cdot \frac{B'(\xi)\xi|y| + ix\xi}{(x^2 + B'(\xi)^2|y|^2)B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}$$

We argue as in the proof of Lemma A1, part a). The number of zeros of $\frac{d}{d\xi}(Re\ f_{x,y}(\xi))$ and $\frac{d}{d\xi}(Im\ f_{x,y}(\xi))$ is finite and does not depend on (x,y). Lemma H applies and we deduce that for $x \neq 0$,

$$\left|\int_{-\infty}^{\infty} e^{ix\xi} f_{x,y}(\xi)\, d\xi\right| \leq \frac{C}{|x|}\sup_{\xi\in\mathbf{R}}|f_{x,y}(\xi)| \leq \frac{C}{x^2}\ .$$

Using the same argument we obtain that the other three integrals in T_B are bounded (in absolute value) by $\frac{C}{x^2}$. Hence

$$|T_B| \leq \frac{C}{x^2}\ . \tag{13}$$

Finally,

$$|k_2(x,y)| \leq \frac{C}{x^2} + \frac{C|y|}{|x|^3} \,. \tag{16}$$

By (14) we have

$$\begin{aligned} |k_2(x,y)| &\leq C \int_{\mathbf{R}} |\xi|(e^{-A(\xi)|y|} - e^{-B(\xi)|y|})\, d\xi \\ &\leq C \int_{\mathbf{R}} |\xi| e^{-C_1|\xi|\cdot|y|}\, d\xi \qquad (17) \\ &= \frac{C'}{y^2} \,. \end{aligned}$$

From (16) and (17) we deduce that

$$\begin{aligned} |k_2(x,y)| &\leq \min\left(\frac{C}{x^2} + \frac{C|y|}{|x|^3}, \frac{C'}{y^2}\right) \\ &\leq C' \min\left(\frac{1}{x^2}, \frac{1}{y^2}\right) \\ &\leq \frac{C''}{r^2} \,. \end{aligned}$$

This proves b).

c) We have

$$k_3(x,y) = \mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right)(x,y) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1 + iy\xi_2} \cdot \frac{1}{B^2(\xi_1) - A^2(\xi_1)} \cdot \left(\frac{-A^2(\xi_1)}{\xi_2^2 + A^2(\xi_1)} + \frac{B^2(\xi_1)}{\xi_2^2 + B^2(\xi_1)}\right) d\xi_1 d\xi_2$$

$$= \frac{1}{2} \int_{\mathbf{R}} e^{ix\xi} \frac{A(\xi)e^{-A(\xi)|y|} - B(\xi)e^{-B(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}\, d\xi \,.$$

If $y \neq 0$ then clearly

$$\begin{aligned} |k_3(x,y)| &\leq C \int_{-\infty}^{\infty} A(\xi)e^{-A(\xi)|y|} + B(\xi)e^{-B(\xi)|y|}\, d\xi \\ &\leq C' \int_{-\infty}^{\infty} |\xi| e^{-C_1|\xi|\cdot|y|}\, d\xi + C \int_{-1}^{1} e^{-C|y|}\, d\xi \qquad (18) \\ &\leq \frac{C''}{y^2} \,. \end{aligned}$$

Integrating by parts we get

$$\int_{\mathbf{R}} e^{ix\xi - B(\xi)|y|} \cdot \frac{B(\xi)}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}} \, d\xi =$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{B'(\xi)}{(ix - B'(\xi)|y|)\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{B(\xi)B''(\xi)|y|}{(ix - B'(\xi)|y|)^2\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$+ \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{2B(\xi)[(c^2-4a)\xi^3 + (ce-2d)\xi]}{(ix - B'(\xi)|y|)(\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2})^3} \,.$$

We use the same argument involving Lemma H as before and conclude that the last sum of integrals is bounded by $\dfrac{C}{x^2} + \dfrac{C|y|}{|x|^3}$.

Let us estimate the term of $k_3(x,y)$ containing $A(\xi)$. Integrating by parts on $(-\infty, 0)$ and on $(0, \infty)$ we have

$$\int_{\mathbf{R}} e^{ix\xi - A(\xi)|y|} \cdot \frac{A(\xi)}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}} \, d\xi =$$

$$= \frac{2A(0)A'(0+)}{x^2 + [A'(0+)]^2} -$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{A'(\xi)}{(ix - A'(\xi)|y|)\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{A(\xi)A''(\xi)|y|}{(ix - A'(\xi)|y|)^2\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$+ \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{2A(\xi)[(c^2-4a)\xi^3 + (ce-2d)\xi]}{(ix - A'(\xi)|y|)(\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2})^3} \,.$$

Using the same method we obtain that the last sum of integrals is bounded by $\dfrac{C}{x^2}$. Finally,

$$|k_3(x,y)| \le \frac{C}{x^2} + \frac{C|y|}{|x|^3} \,. \tag{19}$$

Inequalities (18) and (19) give us $|k_3(x,y)| \le \dfrac{C}{r^2}$. Lemma A2 is proved. □

Remark A3. It is much easier to show that $|k_i(x,y)| \le \dfrac{C}{r}$. The proof is similar to that of Lemma A1 and does not use integrations by parts. Hence $|k_i(x,y)| \le \dfrac{C}{r^\alpha}$ for all $\alpha \in [1,2]$, $i = 1,2,3$.

1.3 On the existence, analyticity and decay of solitary waves to a generalized Benjamin-Ono equation

To appear in *Nonlinear Analysis: Theory, Methods and Applications*

1.3.1 Introduction

We study the solitary waves of the following generalization of the Benjamin-Ono (BO) equation

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0 \tag{1}$$

in $\mathbf{R}^2$, where $\alpha, \beta > 0$ and $(-\Delta)^{\frac{1}{2}}$ is the operator defined by $\mathcal{F}((-\Delta)^{\frac{1}{2}}u)(\xi) = |\xi|\widehat{u}(\xi)$. $\mathcal{F}$ or $\widehat{\ }$ represent the Fourier transform.

Equation (1) describes the dynamics of three-dimensional, slightly nonlinear disturbances in boundary-layer shear flows (without the assumption of a difference in their scales along and across the flow), see [1], [41].

The solitary waves of (1) are solutions of the form $A(x,y,t) = v(x-ct,y)$ where c is the speed of the solitary wave. It seems that solitary waves play an important role in the evolution of (1). Such a solution must satisfy the equation

$$-cv_x + \frac{\alpha}{2}(v^2)_x - \beta(-\Delta)^{\frac{1}{2}} v_x = 0\ . \tag{2}$$

Numerical experiments ([41]) show the existence of solitary waves (solitons). It has also been observed that the solitons decay at infinity like some power of $r = \sqrt{x^2+y^2}$.

Our aim is to give rigorous proofs of these facts.

We suppose throughout that the wave speed c is positive.

In the next section, we show that solitary waves exist and are smooth (analytic) functions. Since the techniques we use are classical, we only sketch the proofs. In the last section we prove that the solutions of some generalization of equation (2) in $\mathbf{R}^n$ decay at infinity as $\frac{1}{|x|^{n+1}}$ and this algebraic rate is nearly optimal. We hope that our results about the decay of solutions of a quite general equation should be useful elsewhere (see also Remark 8 below).

Our method to study analyticity and decay of solutions was inspired by the ideas developed by Bona and Li in [12], [26] for one-dimensional problems.

1.3.2 Existence and regularity

In order to simplify equation (2), we integrate it once in x and make the scale change $v(x,y) = au(bx,by)$, where $a = \frac{2c}{\alpha}$ and $b = \frac{c}{\beta}$. Then (2) reduces to

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \tag{3}$$

or, using the Fourier transform,

$$(1+|\xi|)\widehat{u} = \widehat{u^2}\ . \tag{4}$$

Let us introduce the functionals

$$V(u) = \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 + |u|^2 dx = \frac{1}{2(2\pi)^2}\int_{\mathbf{R}^2}(1+|\xi|)|\widehat{u}|^2 d\xi$$

and

$$I(u) = \frac{1}{3}\int_{\mathbf{R}^2} u^3 dx\ .$$

Clearly V and I are well defined and of class C^2 on the Sobolev space $H^{\frac{1}{2}}(\mathbf{R}^2)$.

For $\mu \neq 0$, we consider the minimization problem

$$(\mathcal{P}) \qquad \text{minimize } V(u) \text{ under the constraint } I(u) = \mu.$$

A minimizer of $(\mathcal{P})$ is called a ground state. If u is such a minimizer, there exists a Lagrange multiplier λ such that

$$(1 + (-\Delta)^{\frac{1}{2}})u = \lambda u^2 \, .$$

It is easy to see that $\lambda\mu$ is positive (because the above equation gives $2V(u) = 3\lambda I(u)$). Then λu is a non-trivial solution of (3). Clearly λu minimizes $V(v)$ subject to the constraint $I(v) = I(\lambda u)$.

Theorem 1. *There exists minimizers of problem $(\mathcal{P})$. Consequently, equation (3) admits non-trivial solutions.*

Proof. One may prove Theorem 1 by using the concentration-compactness principle, as it was done in [2] to show the existence of solitary waves for the ILW equation. The main difficulty is to eliminate dichotomy. To do this, one needs to estimate the L^2-norm of the commutator $(L\chi - \chi L)u$ in terms of the derivatives of order ≥ 1 of χ and the $H^{\frac{1}{2}}$-norm of u, where $Lu = \mathcal{F}^{-1}((1+|\xi|)^{\frac{1}{2}}\widehat{u})$ and $\chi \in C_0^\infty(\mathbf{R}^2)$. But this can be done and we obtain the existence of ground states.

We may also observe that problem $(\mathcal{P})$ is exactly of the type discussed by O. Lopes in a recent paper ([31]). The functionals V and I satisfy the assumptions $\mathbf{HH_1} - \mathbf{HH_6}$ of Lopes and using the Theorems 3.1 and 3.15 in [31], we infer that any minimizing sequence (u_n) of $(\mathcal{P})$ possesses a subsequence that converges strongly in $H^{\frac{1}{2}}(\mathbf{R}^2)$ (modulo translation in $\mathbf{R}^2$) to an element u which is a ground state. □

We give another variational characterization of the ground states. We consider the functionals

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 dx - \frac{1}{3}\int_{\mathbf{R}^2} u^3 dx$$

and

$$Q(u) = \frac{1}{2}\int_{\mathbf{R}^2} |u|^2 dx.$$

Proposition 2. *Let $u_\star$ be a minimizer of V under the constraint $I(u) = \mu$. Suppose that $u_\star$ satisfies the equation (3). Then $E(u_\star) = 0$ and $u_\star$ is a solution of the problem*

$$(\mathcal{P}') \qquad \textit{minimize } E(v) \textit{ under the constraint } Q(v) = Q(u_\star).$$

Proof. Multiplying (3) by u and integrating we obtain the identity

$$(5) \qquad \int_{\mathbf{R}^2} |u|^2 dx + \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 dx = \int_{\mathbf{R}^2} u^3 dx.$$

For $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ we denote $u_{a,b}(x) = bu(ax)$. Then $I(u_{a,b}) = b^3a^{-2}I(u)$, $\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u_{a,b}|^2 dx = b^2a^{-1}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 dx$ and $Q(u_{a,b}) = b^2a^{-2}Q(u)$.

We have $I(u_{\star_{a,a^{\frac{2}{3}}}}) = I(u_\star)$ and since $u_\star$ is a minimizer of the problem $(\mathcal{P})$, the function

$$f(a) = V(u_{\star_{a,a^{\frac{2}{3}}}}) = \frac{1}{2}a^{\frac{1}{3}} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \frac{1}{2}a^{-\frac{2}{3}} \int_{\mathbf{R}^2} |u_\star|^2 dx$$

has a minimum at $a = 1$. Hence $f'(1) = 0$, that is

$$\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx = 2 \int_{\mathbf{R}^2} |u_\star|^2 dx. \tag{6}$$

Combining (5) and (6) we obtain $E(u_\star) = 0$.

Let $v \in H^{\frac{1}{2}}(\mathbf{R}^2)$ such that $Q(v) = Q(u_\star)$. We want to show that $E(v) \geq E(u_\star) = 0$. This clearly holds if $I(v) \leq 0$. Suppose that $I(v) > 0$. For $a > 0$, let $b(a) = a^{\frac{2}{3}} \Big(\frac{\int u_\star^3 dx}{\int v^3 dx} \Big)^{\frac{1}{3}}$. Then $I(v_{a,b(a)}) = I(u_\star)$, hence $V(v_{a,b(a)}) \geq V(u_\star)$ and this gives

$$\begin{aligned} &\Big(\frac{\int u_\star^3 dx}{\int v^3 dx} \Big)^{\frac{2}{3}} \cdot \Big[a^{\frac{1}{3}} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx + a^{-\frac{2}{3}} \int_{\mathbf{R}^2} |v|^2 dx \Big] \\ &\geq \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \int_{\mathbf{R}^2} |u_\star|^2 dx. \end{aligned} \tag{7}$$

The minimum of the left side of (7) for $a \in (0, \infty)$ is

$$3 \Big(\frac{\int u_\star^3 dx}{\int v^3 dx} \Big)^{\frac{2}{3}} \cdot \Big(\frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx \Big)^{\frac{2}{3}} \Big(\int_{\mathbf{R}^2} |v|^2 dx \Big)^{\frac{1}{3}}.$$

Hence

$$\begin{aligned} &3 \Big(\frac{\frac{1}{2} \int |(-\Delta)^{\frac{1}{4}} v|^2 dx}{\int v^3 dx} \Big)^{\frac{2}{3}} \Big(\int_{\mathbf{R}^2} u_\star^3 dx \Big)^{\frac{2}{3}} \Big(\int_{\mathbf{R}^2} |v|^2 dx \Big)^{\frac{1}{3}} \\ &\geq \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \int_{\mathbf{R}^2} |u_\star|^2 dx. \end{aligned}$$

Since $\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx = 2 \int_{\mathbf{R}^2} |u_\star|^2 dx$ and $\int_{\mathbf{R}^2} u_\star^3 dx = 3 \int_{\mathbf{R}^2} |u_\star|^2 dx$ by (5) and (6) and $\int_{\mathbf{R}^2} |v|^2 dx = \int_{\mathbf{R}^2} |u_\star|^2 dx$ by assumption, we obtain

$$\frac{\frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx}{\int_{\mathbf{R}^2} v^3 dx} \geq \frac{1}{3},$$

that is $E(v) \geq 0$. □

Remark 3. The converse of Proposition 2 is valid modulo a scale change. More precisely, let $u_\star$ be as above and let v be a solution of problem $(\mathcal{P}')$. Then there exists $a > 0$ such that $v_{a,a}$ is a solution of problem $(\mathcal{P})$.

Indeed, $Q(v_{a,a}) = Q(v)$ and $E(v_{a,a}) = aE(v)$. Since v is a minimizer of $(\mathcal{P}')$, necessarily $E(v) = 0$. Moreover, there exists a Lagrange multiplier λ such that

$$(-\Delta)^{\frac{1}{2}} v - v^2 = -\lambda v.$$

Then $\lambda \int_{\mathbf{R}^2} |v|^2 dx = -\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx + \int_{\mathbf{R}^2} v^3 dx = \frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx$, hence $\lambda > 0$. Denote $v_\star = v_{\frac{1}{\lambda}, \frac{1}{\lambda}}$. Then $v_\star$ satisfies (3). Multiplying this by $v_\star$ and integrating, we find that $v_\star$ also satisfies (5). But $E(v_\star) = 0$ and so we deduce that $\frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v_\star|^2 dx = \frac{1}{3}\int_{\mathbf{R}^2} v_\star^3 dx = \int_{\mathbf{R}^2} |v_\star|^2 dx = \int_{\mathbf{R}^2} |u_\star|^2 dx$. Now it is clear that $I(v_\star) = I(u_\star)$ and $V(v_\star) = V(u_\star)$, i.e. $V(v_\star)$ achieves the minimum of $V(w)$ for all $w \in H^{\frac{1}{2}}(\mathbf{R}^2)$ such that $I(w) = I(u_\star)$.

Now we turn our attention to the regularity of solitary waves.

Theorem 4. *Let $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ be a solution of (3). Then $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in [1, \infty]$. In particular, u is a C^∞ function and tends to zero at infinity.*

Proof. By the Sobolev imbedding theorem, $H^{\frac{1}{2}}(\mathbf{R}^2) \subset L^4(\mathbf{R}^2)$, so that $u^2 \in L^2(\mathbf{R}^2)$. From (4) we deduce that $|\xi|\widehat{u} \in L^2(\mathbf{R}^2)$, hence $u \in H^1(\mathbf{R}^2)$. Again by Sobolev's imbedding we have $u \in L^p(\mathbf{R}^2)$ for $2 \le p < \infty$.

It is easy to check that the functions $m(\xi) = \frac{1}{1+|\xi|}$ and $m_i(\xi) = \frac{\xi_i}{1+|\xi|}$ satisfy $|\partial^\alpha m(\xi)| \le C|\xi|^{-|\alpha|}$ and $|\partial^\alpha m_i(\xi)| \le C|\xi|^{-|\alpha|}$ for $|\alpha| = 0, 1, 2$ and a classical theorem of Mikhlin implies that $m, m_i \in M_q(\mathbf{R}^2)$ for $1 < q < \infty$, i.e m, m_i are Fourier multipliers for $L^q(\mathbf{R}^2)$, $1 < q < \infty$. Equation (4) gives

$$\widehat{u}(\xi) = m(\xi)\widehat{u^2}(\xi) \text{ and } \widehat{u_{x_j}}(\xi) = im_j(\xi)\widehat{u^2}(\xi)$$

and Mikhlin's theorem implies that $u, u_{x_j} \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Hence $u \in W^{1,p}(\mathbf{R}^2)$, $\forall p \in]1, \infty[$. In particular, u is continuous and tends to zero at infinity.

It follows easily by induction that $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and $p \in]0, \infty[$. Indeed, suppose that $u \in W^{n,p}(\mathbf{R}^2)$ for all $p \in]1, \infty[$. If $\alpha_1, \alpha_2 \in \mathbf{N}$, $\alpha_1 + \alpha_2 = n$, we have for example

$$\mathcal{F}(\partial_{x_1}^{\alpha_1+1}\partial_{x_2}^{\alpha_2} u) = \frac{i\xi_1}{1+|\xi|}\mathcal{F}(\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}(u^2)).$$

The induction hypothesis implies that $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}(u^2) \in L^p(\mathbf{R}^2)$, $\forall p \in]1, \infty[$. Again by Mikhlin's theorem we obtain $\partial_{x_1}^{\alpha_1+1}\partial_{x_2}^{\alpha_2} u \in L^p(\mathbf{R}^2)$, $1 < p < \infty$ and so $u \in W^{n+1,p}(\mathbf{R}^2)$ for all $p \in]1, \infty[$.

The fact that $u \in W^{k,1}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ can be easily proved by writing (3) as a convolution equation and using Lemma 7 below (see also the proof of Theorem 11 and Remark 12). □

Theorem 5. *Let $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ be a solution of (3). Then there exists $\sigma > 0$ and an holomorphic function U of two complex variables z_1, z_2 defined in the domain*

$$\Omega_\sigma = \{(z_1, z_2) \in \mathbf{C}^2 \quad | \quad |Im(z_1)| < \sigma, \ \ |Im(z_2)| < \sigma\}$$

such that $U(x, y) = u(x, y)$ for all $(x, y) \in \mathbf{R}^2$.

Proof. By Theorem 4 we have $(1 + |\xi|^2)^{\frac{m}{2}}\widehat{u}(\xi) \in L^2(\mathbf{R}^2)$ for all $m \ge 0$. We take $m > 1$ and apply Cauchy-Schwarz' inequality to get

$$\int_{\mathbf{R}^2} |\widehat{u}|(\xi) d\xi \le \left(\int_{\mathbf{R}^2} (1 + |\xi|^2)^m |\widehat{u}|^2(\xi) d\xi\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^2} (1 + |\xi|^2)^{-m} d\xi\right)^{\frac{1}{2}} < \infty .$$

Hence $\widehat{u} \in L^1(\mathbf{R}^2)$. Equation (4) implies that

$$|\widehat{u}|(\xi) \le |\widehat{u}| \star |\widehat{u}|(\xi) \text{ and } |\xi||\widehat{u}|(\xi) \le |\widehat{u}| \star |\widehat{u}|(\xi) .$$

We note $\mathcal{C}_1|\widehat{u}| = |\widehat{u}|$ and for $n \geq 1$, $\mathcal{C}_{n+1}|\widehat{u}| = (\mathcal{C}_n|\widehat{u}|) \star |\widehat{u}|$.

Lemma 6. *We have for all $k \in \mathbf{N}$*

$$|\xi|^k|\widehat{u}|(\xi) \leq (k+1)^{k-1}\mathcal{C}_{2(k+1)}|\widehat{u}|(\xi).$$

The lemma follows easily by induction, using the identity

$$\sum_{j=0}^{k} C_k^j (1+j)^{j-1}(1+k-j)^{k-j+1} = 2(2+k)^{k-1}$$

(which is a specialization of Abel's identity).

Using Lemma 6, we have

$$|\xi|^k|\widehat{u}|(\xi) \leq (k+1)^{k-1}||\ \mathcal{C}_{2(k+1)}|\widehat{u}|\ ||_{L^\infty} \leq (k+1)^{k-1}||\ \mathcal{C}_{2k+1}|\widehat{u}|\ ||_{L^2} \cdot ||\widehat{u}||_{L^2}$$
$$\leq (k+1)^{k-1}||\widehat{u}||_{L^1}^{2k} \cdot ||\widehat{u}||_{L^2}^2 .$$

Let $a_k = \dfrac{(k+1)^{k-1}||\widehat{u}||_{L^1}^{2k} \cdot ||\widehat{u}||_{L^2}^2}{k!}$. Clearly $\dfrac{a_{k+1}}{a_k} = ||\widehat{u}||_{L^1}^2 \cdot \left(\dfrac{k+2}{k+1}\right)^k \longrightarrow$ $e||\widehat{u}||_{L^1}^2$ as $k \longrightarrow \infty$. Let $\sigma = \dfrac{1}{e||\widehat{u}||_{L^1}^2}$.

If $0 < \tau < \sigma$, the series $\displaystyle\sum_{k=0}^{\infty} \frac{(\tau|\xi|)^k}{k!}|\widehat{u}|(\xi)$ converges uniformly in L^∞-norm (because each term is dominated by $\tau^k a_k$ and the series $\displaystyle\sum_{k=0}^{\infty} \tau^k a_k$ converges absolutely).

Hence $e^{\tau|\xi|}\widehat{u}(\xi) \in L^\infty(\mathbf{R}^2)$ for $\tau < \sigma$.

We define the function

$$U(z_1, z_2) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} e^{i(z_1\xi_1+z_2\xi_2)}\widehat{u}(\xi_1, \xi_2)d\xi_1 d\xi_2.$$

By the Paley-Wiener Theorem, U is well defined and analytic in Ω_σ and by Plancherel's Theorem we have $U(x, y) = u(x, y)$ for all $(x, y) \in \mathbf{R}^2$. □

1.3.3 Decay properties

We consider a generalization of equation (3) in $\mathbf{R}^n$, namely

$$(1 + (-\Delta)^{\frac{1}{2}})u = g(u) \tag{8}$$

with the following assumptions on g:

i) $g : \mathbf{C} \longrightarrow \mathbf{C}$ is continuous and

ii) there exists $\gamma > 1$ and $C > 0$ such that $|g(z)| \leq C|z|^\gamma$, $\forall z \in \mathbf{C}$.

The aim of this paragraph is to prove that the solutions of (8) that tend to zero at infinity must decay (at least) as $\frac{1}{|x|^{n+1}}$.

Equation (8) may be written in the equivalent forms

$$\widehat{u} = \frac{1}{1+|\xi|}\widehat{g(u)} \tag{9}$$

or

$$u = k \star g(u) \ , \tag{10}$$

where $k = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|}\right)$. We begin with some estimates on the kernel k.

Lemma 7.

i) We have

$$k(x) = c_n \int_0^\infty e^{-s} \cdot \frac{s}{(|x|^2+s^2)^{\frac{n+1}{2}}}\, ds \ , \qquad \textit{where } c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \ .$$

ii) $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$ and there exist positive constants A_1^n, A_2^n such that

$$\begin{array}{lclcll} A_1^n|x|^{-n+1} & \le & k(x) & < & \frac{c_n}{n-1}|x|^{-n+1} & \textit{if } 0<|x|\le 1, n\ge 2, \quad \textit{respectively} \\ -c_1 e^{-1}\ln|x| & < & k(x) & < & -c_1\ln|x| + c_1 & \textit{if } 0<|x|\le 1, n=1 \quad \textit{and} \\ A_2^n|x|^{-n-1} & \le & k(x) & < & c_n|x|^{-n-1} & \textit{if } |x|\ge 1,\ n\ge 1. \end{array}$$

iii) $|x|^{n+1}k(x) \in L^\infty(\mathbf{R}^n)$ and for $1 \le p < \infty$ we have $|x|^\alpha k(x) \in L^p(\mathbf{R}^n)$ if and only if

$$n-1-\frac{n}{p} < \alpha < n+1-\frac{n}{p}. \tag{11}$$

In particular, $k \in L^p(\mathbf{R}^n)$ if and only if $1 \le p < \frac{n}{n-1}$.

Remark 8. From now on, we use only equation (10), the assumptions i) and ii) on g and the estimates on k given by Lemma 7, iii). Hence our result about the decay of solutions (Theorem 11 below) holds for any equation that can be written in the form (10) with a kernel k that satisfies the conclusion iii) of Lemma 7.

Proof of Lemma 7. i) For any $\phi \in \mathcal{S}$ (the Schwartz' space of rapidly decreasing functions) we have

$$\begin{aligned} < k, \phi >_{\mathcal{S}',\mathcal{S}} &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{1}{1+|\xi|}\int_{\mathbf{R}^n} e^{ix.\xi}\phi(x)dxd\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_0^\infty e^{-(1+|\xi|)s}ds \cdot \int_{\mathbf{R}^n} e^{ix.\xi}\phi(x)dxd\xi \\ &= \int_0^\infty e^{-s}\int_{\mathbf{R}^n}\left(\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix.\xi}e^{-|\xi|s}d\xi\right)\phi(x)dxds \\ &= \int_0^\infty e^{-s}\int_{\mathbf{R}^n} P_s(x)\phi(x)dx \end{aligned}$$

where $P_s(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix.\xi}e^{-|\xi|s}d\xi = \frac{c_n s}{(|x|^2+s^2)^{\frac{n+1}{2}}}$
is the Poisson kernel

$$= \int_{\mathbf{R}^n} c_n \int_0^\infty e^{-s} \cdot \frac{s}{(|x|^2+s^2)^{\frac{n+1}{2}}}\, ds\ \phi(x)dx \ .$$

This proves i).

ii) It is obvious that $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$. Using i), for $n \geq 2$ and $0 < |x| \leq 1$ we clearly have $k(x) > c_n \int_0^{|x|} \frac{se^{-1}}{(|x|^2+s^2)^{\frac{n+1}{2}}} ds = c_n \frac{e^{-1}}{n-1}\left(1 - \frac{1}{2^{\frac{n-1}{2}}}\right) \frac{1}{|x|^{n-1}}$ and $k(x) < c_n \int_0^\infty \frac{s}{(|x|^2+s^2)^{\frac{n+1}{2}}} ds = \frac{c_n}{n-1} \frac{1}{|x|^{n-1}}$.

For $n = 1$ and $0 < |x| \leq 1$, integrating by parts and using the elementary inequality $\ln(x^2 + s^2) \leq \ln(s^2 + 1) < s^2$ for $s \neq 0$ we obtain $k(x) = -\frac{1}{2}c_1 \ln x^2 + \frac{1}{2}c_1 \int_0^\infty e^{-s} \ln(x^2+s^2) ds < -c_1 \ln|x| + \frac{1}{2}c_1 \int_0^\infty e^{-s} s^2 ds = -c_1 \ln|x| + c_1$ and obviously $k(x) > c_1 \int_0^1 \frac{se^{-1}}{x^2+s^2} ds = \frac{1}{2}c_1 e^{-1}(\ln(x^2+1) - \ln x^2) > -c_1 e^{-1} \ln|x|$.

For $|x| \geq 1$ we get $k(x) > c_n \int_0^1 \frac{se^{-s}}{(2|x|^2)^{\frac{n+1}{2}}} ds = c_n(1 - \frac{2}{e})2^{-\frac{n+1}{2}} \frac{1}{|x|^{n+1}}$ and $k(x) < c_n \int_0^\infty \frac{se^{-s}}{|x|^{n+1}} ds = \frac{c_n}{|x|^{n+1}}$.

iii) is a direct consequence of ii). □

Lemma 9. *Let l and m be two constants satisfying $0 < l < m - n$. Then there exists $B > 0$ depending only on l, m and n such that for all $\varepsilon > 0$ we have*

a) $\displaystyle\int_{\mathbf{R}^n} \frac{|y|^l}{(1+\varepsilon|y|)^m(1+|x-y|)^m} dy \leq \frac{B|x|^l}{(1+\varepsilon|x|)^m}$ *for all $x \in \mathbf{R}^n$, $|x| \geq 1$ and*

b) $\displaystyle\int_{\mathbf{R}^n} \frac{1}{(1+\varepsilon|y|)^m(1+|x-y|)^m} dy \leq \frac{B}{(1+\varepsilon|x|)^m}$ *for all $x \in \mathbf{R}^n$.*

The proof of Lemma 9 is elementary and is essentially the same as the proof of Lemma 3.1.1 in [12], p. 383.

After this preparation, we may prove an integral estimate of the solutions of the convolution equation (10). This is given in the next lemma.

Lemma 10. *Suppose that $f \in L^\infty(\mathbf{R}^n)$ satisfies (10), i.e. $f = k \star g(f)$ and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.*

Then $|x|^\beta f(x) \in L^q(\mathbf{R}^n)$ for all $q \in]n, \infty[$ and for all $\beta \in [0, 1[$.

Proof. We remark first that $k \in L^1(\mathbf{R}^n)$ and $g(f) \in L^\infty(\mathbf{R}^n)$, so f is continuous. Choose $p \in]1, \frac{n}{n-1}[$. Then choose α such that

$$n - \frac{n}{p} < \alpha < n + 1 - \frac{n}{p} . \tag{12}$$

By Lemma 7 we have $k \in L^p(\mathbf{R}^n)$ and $|\cdot|^\alpha k \in L^p(\mathbf{R}^n)$. Let $K_{\alpha,p} = ||(1+|x|)^\alpha k(x)||_{L^p}$.

Now choose $l \in [0, \alpha - \frac{n(p-1)}{p}[$. For $0 < \varepsilon < 1$ we denote

$$h_\varepsilon(x) = \frac{|x|^l}{(1+\varepsilon|x|)^\alpha} f(x) .$$

Let q be the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then $h_\varepsilon \in L^q(\mathbf{R}^n)$ by the choice of l. Since

$$f(x) = (k \star g(f))(x) = \int_{\mathbf{R}^n} k(x-y)(1+|x-y|)^\alpha \cdot \frac{g(f(y))}{(1+|x-y|)^\alpha} dy ,$$

using Hölder's inequality we obtain

$$|f(x)| \le K_{\alpha,p}\left(\int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1+|x-y|)^{\alpha q}}\,dy\right)^{\frac{1}{q}}. \tag{13}$$

The assumption ii) on the function g and the fact that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ imply that for every $\delta > 0$ there exists $R_\delta > 1$ such that if $|x| \ge R_\delta$ we have

$$|g(f(x))| \le \delta |f(x)|.$$

If $0 < r < q$, by (13) and Hölder's inequality we obtain

$$\int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx = \int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^{q-r}\left(\frac{|x|^l}{(1+\varepsilon|x|)^\alpha}\right)^r |f(x)|^r dx$$

$$\le \int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^{q-r}\left(\frac{|x|^l}{(1+\varepsilon|x|)^\alpha}\right)^r \cdot K_{\alpha,p}^r\left(\int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1+|x-y|)^{\alpha q}}\,dy\right)^{\frac{r}{q}} dx$$

$$\le K_{\alpha,p}^r\left(\int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx\right)^{\frac{q-r}{q}}$$

$$\times\left[\int_{\mathbf{R}^n\setminus B(0,R_\delta)}\left(\frac{|x|^l}{(1+\varepsilon|x|)^\alpha}\right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1+|x-y|)^{\alpha q}}\,dy\,dx\right]^{\frac{r}{q}}.$$

The last sequence of inequalities gives

$$\int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx \le K_{\alpha,p}^q \int_{\mathbf{R}^n\setminus B(0,R_\delta)}\left(\frac{|x|^l}{(1+\varepsilon|x|)^\alpha}\right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1+|x-y|)^{\alpha q}}\,dy\,dx. \tag{15}$$

(since $h_\varepsilon \in L^q(\mathbf{R}^n)$, we may divide by $\int_{\mathbf{R}^n\setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx$). Observe that $lq < \alpha q - n$ by the choice of l. Using Fubini's Theorem and Lemma 9 we obtain

$$\int_{\mathbf{R}^n\setminus B(0,R_\delta)}\left[\left(\frac{|x|^l}{(1+\varepsilon|x|)^\alpha}\right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1+|x-y|)^{\alpha q}}\,dy\right] dx$$

$$= \int_{\mathbf{R}^n} |g(f(y))|^q \left[\int_{\mathbf{R}^n\setminus B(0,R_\delta)} \frac{|x|^{lq}}{(1+\varepsilon|x|)^{\alpha q}} \cdot \frac{1}{(1+|x-y|)^{\alpha q}}\,dx\right] dy$$

$$\le \int_{\mathbf{R}^n\setminus B(0,R_\delta)} |g(f(y))|^q \frac{B|y|^{lq}}{(1+\varepsilon|y|)^{\alpha q}}\,dy \tag{16}$$

$$+\int_{B(0,R_\delta)} |g(f(y))|^q \cdot \int_{\mathbf{R}^n\setminus B(0,R_\delta)} \frac{|x|^{lq}}{(1+\varepsilon|x|)^{\alpha q}} \cdot \frac{1}{(1+|x-y|)^{\alpha q}}\,dx\,dy$$

where B depends on n, l, q and α, but not on ε. The last integral is majorized by a constant C depending on f and R_δ (but not on ε).

Combining (15) and (16) and taking into account the fact that $|g(f(y))| < \delta|f(y)|$ on $\mathbf{R}^n \setminus B(0, R_\delta)$, we get

$$(17) \qquad \int_{\mathbf{R}^n \setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx \leq K_{\alpha,p}^q \left[B\delta^q \int_{\mathbf{R}^n \setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx + C \right] .$$

Choosing δ such that $K_{\alpha,p} B^{\frac{1}{q}} \delta < 1$, from (17) we deduce that

$$(18) \qquad \int_{\mathbf{R}^n \setminus B(0,R_\delta)} |h_\varepsilon(x)|^q dx \leq C'$$

where C' is a constant that does not depend on ε. We let $\varepsilon \longrightarrow 0$ in (18) and apply Fatou's Lemma to obtain

$$\int_{\mathbf{R}^n \setminus B(0,R_\delta)} |x|^{lq} |f(x)|^q dx \leq C' .$$

Hence $|x|^l f(x) \in L^q(\mathbf{R}^n)$ for $q = \frac{p}{p-1}$.

To summarize, we proved that for any $p \in]1, \frac{n}{n-1}[$, for any $\alpha \in]n - \frac{n}{p}, n+1-\frac{n}{p}[$ and for any $l \in [\, 0, \alpha - \frac{n(p-1)}{p}[$ we have $|x|^l f(x) \in L^{\frac{p}{p-1}}$.

We choose sequences (p_k), (α_k) and (l_k) such that

$$p_k \in]1, \tfrac{n}{n-1}[\,, \qquad p_k \longrightarrow \tfrac{n}{n-1} \text{ as } k \longrightarrow \infty$$

$$\alpha_k \in]n - \tfrac{n}{p_k}, n+1-\tfrac{n}{p_k}[\,, \qquad \alpha_k \longrightarrow 2 \text{ as } k \longrightarrow \infty \text{ and}$$

$$l_k \in [0, \alpha_k - \tfrac{n(p_k-1)}{p_k}[\,, \qquad l_k \longrightarrow 1 \text{ as } k \longrightarrow \infty.$$

Then $q_k = \frac{p_k}{p_k - 1} \longrightarrow n$ as $k \longrightarrow \infty$ and $|x|^{l_k} f(x) \in L^{q_k}$ for all k. This proves the lemma. □

We may now state our main result.

Theorem 11. *Suppose that f satisfies equation (10) and*
-either $f \in L^p(\mathbf{R}^n)$ for a $p \in](\gamma-1)n, \infty[$, $p \geq \gamma$,
-or $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.
Then $|x|^{n+1} f(x) \in L^\infty(\mathbf{R}^n)$.

Proof. First we show that we always have $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \longrightarrow 0$ at infinity.

Suppose that $f \in L^p(\mathbf{R}^n)$ and $p > \gamma n$. Then $g(f) \in L^{\frac{p}{\gamma}}(\mathbf{R}^n)$. Since $\frac{p}{\gamma} > n$ and $k \in L^q(\mathbf{R}^n)$ for all $q \in [1, \frac{n}{n-1}[$, it clearly follows from equation (10) that $f \in L^\infty(\mathbf{R}^n)$, f is continuous and tends to zero at infinity.

If $f \in L^{\gamma n}(\mathbf{R}^n)$, then $g(f) \in L^n(\mathbf{R}^n)$. Equation (10) and Young's theorem imply that $f \in L^q(\mathbf{R}^n)$ for all $q \in [\gamma n, \infty[$. Then the preceeding argument shows that $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Now suppose that $p \in](\gamma-1)n, \gamma n[$ and $p \geq \gamma$. Then $g(f) \in L^{\frac{p}{\gamma}}(\mathbf{R}^n)$ and by (10) and Young's theorem we obtain $f \in L^q(\mathbf{R}^n)$ for all $q \in [\frac{p}{\gamma}, \frac{pn}{\gamma n - p}[$. Iterating this argument, after a finite number of steps we get $f \in L^p(\mathbf{R}^n)$ for a $p \geq \gamma n$. As above we obtain $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \longrightarrow 0$ at infinity.

The rest of the proof is a standard bootstrap argument. We make use of the inequality

$$||x|^\delta f| \le C\left((|x|^\delta k) \star |g(f)| + k \star (|x|^\delta |g(f)|)\right) . \tag{19}$$

By Lemma 7, $|x|k \in L^q(\mathbf{R}^n)$ for $q \in]1, \frac{n}{n-2}[$ if $n \ge 3$ (respectively for $q \in]1,\infty[$ if $n = 2$ and $q \in]1,\infty]$ if $n = 1$). Lemma 10 implies that $g(f) \in L^r(\mathbf{R}^n)$ for $r \in]\frac{n}{\gamma},\infty]$, so we get $(|x|k) \star |g(f)| \in L^\infty(\mathbf{R}^n)$. Similarly, $k \in L^q(\mathbf{R}^n)$ for $q \in [1, \frac{n}{n-1}[$ and $|x|g(f) \in L^r(\mathbf{R}^n)$ for $r \in]\frac{n}{\gamma},\infty[$ by Lemma 10, hence $k \star (|x||g(f)|) \in L^\infty(\mathbf{R}^n)$. Using (19) we get $|x|f(x) \in L^\infty(\mathbf{R}^n)$.

Suppose that $|x|^\alpha f(x) \in L^\infty(\mathbf{R}^n)$ and $\alpha\gamma \le n+1$. Obviously $|x|^{\alpha\gamma}|g(f)| \in L^\infty(\mathbf{R}^n)$ and $k \in L^1(\mathbf{R}^n)$, hence $k\star(|x|^{\alpha\gamma}|g(f)|) \in L^\infty(\mathbf{R}^n)$. Observe that $|g(f)(x)| \le \frac{C}{(1+|x|)^{\alpha\gamma}}$, so $g(f) \in L^q(\mathbf{R}^n)$ for all q verifying $q > \frac{n}{\alpha\gamma}$, $q \ge 1$. Using Lemma 7 and Young's theorem, we find that $(|x|^{\alpha\gamma}k) \star |g(f)| \in L^\infty(\mathbf{R}^n)$ and from (19) it follows that $|x|^{\alpha\gamma}f(x) \in L^\infty(\mathbf{R}^n)$. Hence $|x|^\alpha f \in L^\infty(\mathbf{R}^n)$ and $\alpha\gamma \le n+1$ imply that $|x|^{\alpha\gamma} f \in L^\infty(\mathbf{R}^n)$. This clearly leads to the conclusion of the theorem. □

Remark 12. Suppose that g is C^m and $|g^{(i)}|(x) \le C_i|x|^{\gamma-i}$, $0 \le i \le m$ and f satisfies the hypothesis of Theorem 11. Then $|f(x)| \le \frac{C}{(1+|x|)^{n+1}}$, in particular $f \in L^1(\mathbf{R}^n)$. Arguing as in the proof of Theorem 4 we obtain that $f \in W^{m+1,q}(\mathbf{R}^n)$ for all $q \in [1,\infty[$. As in Theorem 11 it can be proved that the derivatives of f of order $\le m$ decay at infinity at least as $\frac{1}{|x|^{n+1}}$.

Remark 13. Suppose in addition that g is differentiable and there exists $\beta > 0$ such that $|g'(x)| \le C|x|^\beta$. If $f \in L^p(\mathbf{R}^n)$, $1 < p \le \infty$ satisfies (10) and $\int_{\mathbf{R}^n} g(f(x))\, dx \ne 0$, then the decay rate of f given by Theorem 11 is optimal. More precisely, $|x|f$ cannot belong to $L^1(\mathbf{R}^n)$.

In particular, the solutions of equation (3) in $\mathbf{R}^2$ decay at infinity as $\frac{1}{|x|^3}$ and this algebraic rate is optimal.

Indeed, $|x|f \in L^1(\mathbf{R}^n)$ would imply that x_jf, $g(f)$ and $x_jg(f)$ are L^1-functions, hence their Fourier transforms are continuous. But

$$\begin{aligned} -\widehat{ix_jf}(\xi) &= \partial_{\xi_j}\widehat{f}(\xi) = \partial_{\xi_j}\left(\frac{1}{1+|\xi|}\widehat{g(f)}\right)(\xi) \\ &= -\frac{\xi_j}{(1+|\xi|)^2|\xi|}\widehat{g(f)} + \frac{1}{1+|\xi|}\mathcal{F}(-ix_jg(f))(\xi). \end{aligned} \tag{20}$$

Take $\xi_j = s$ and $\xi_i = 0$ if $i \ne j$ in (20). For $s \downarrow 0$ we get

$$-\widehat{ix_jf}(0) = -\widehat{g(f)}(0) + \mathcal{F}(-ix_jg(f))(0) ,$$

while for $s \uparrow 0$ we obtain

$$-\widehat{ix_jf}(0) = \widehat{g(f)}(0) + \mathcal{F}(-ix_jg(f))(0) ,$$

Hence $\widehat{g(f)}(0) = 0$, i. e. $\int_{\mathbf{R}^n} g(f(x))\, dx = 0$, contrary to our assumption.

Chapter 2

Stationary solutions to a nonlinear Schrödinger equation with potential in one dimension

Sections 1-5 of this chapter will appear in
Proceedings of the Royal Society of Edinburgh: Section A.

2.1 Introduction

We consider the 1-dimensional nonlinear Schrödinger (NLS) equation with an external repulsive potential U moving at velocity $v > 0$:

$$iA_t + A_{xx} + A - |A|^2A - U(x - vt)A = 0, \quad x \in \mathbf{R}, t \in \mathbf{R}. \tag{1.1}$$

This equation arises in many physical contexts. For example, it describes the motion of an impurity (modelized by U) at constant velocity v in a NLS fluid at rest at $+\infty$. The behaviour of equation (1.1) in one dimension is similar to that in higher dimensions, vortices being replaced by propagating localized density depressions which are called gray solitons (see [23]). Equation (1.1) can be put into a hydrodynamical form using Madelung's transformation $A(x,t) = \sqrt{\rho(x,t)}e^{i\phi(x,t)}$, see [37] or [45]. This change of variables leads to the system

$$\rho_t + 2(\rho\phi_x)_x = 0, \tag{1.2}$$

$$\phi_t + |\phi_x|^2 - \frac{\rho_{xx}}{2\rho} + \frac{|\rho_x|^2}{4\rho^2} - 1 + \rho + U(x - vt) = 0. \tag{1.3}$$

Note that the Madelung transformation is singular when $A = 0$. Equation (1.2) and the derivative with respect to x of (1.3) are the equation of conservation of mass, respectively Euler's equation for a compressible inviscid fluid of density ρ and velocity $2\phi_x$. We require that the fluid be at rest at infinity with density 1. This gives the "boundary condition" $A(x) \longrightarrow 1$ at $+\infty$. Taking the derivative with respect to t of (1.3) and substituting ρ_t from (1.2) we get

$$\phi_{tt} - 2\rho\phi_{xx} - 2\rho_x\phi_x + \frac{\partial}{\partial t}\Big(|\phi_x|^2 - \frac{\rho_{xx}}{2\rho} + \frac{|\rho_x|^2}{4\rho^2} + U(x - vt)\Big) = 0. \tag{1.4}$$

For a small oscillatory motion (i.e. a sound wave), all the nonlinear terms appearing in (1.4) except $2\rho\phi_{xx}$ may be neglected and the velocity potential ϕ essentially obeys to the wave equation $\phi_{tt} - 2\rho\phi_{xx} - vU'(x - vt) = 0$. We see that sound waves propagate with velocity $\sqrt{2\rho}$ and therefore the sound velocity at infinity is $\sqrt{2}$.

Equation (1.1) can be written in the frame of the moving impurity as

$$iA_t - ivA_x + A_{xx} + A - |A|^2A - U(x)A = 0. \tag{1.5}$$

In this context, it describes the flow of a NLS fluid past a fixed obstacle when a flow of constant density is injected at velocity v at infinity. The obstacle is modelized by the localized potential U. This problem was considered by V. Hakim in [23]. In the case of a Dirac potential, he proved the existence of a critical velocity v_c such that for $v < v_c$ there exist two stationary solutions of (1.5) (i.e. solutions which do not depend on t), one of them being stable and the other unstable. Using formal asymptotic expansions and numerical experiments, he showed that a similar phenomenon takes place for small potentials and for slowly varying potentials (i.e. potentials of the form $U(\varepsilon x)$, ε small). In all these cases, the two solutions become identical at critical velocity and no stationary solution exists for $v > v_c$. The critical velocity depends on the obstacle and is less than the sound velocity. Above the critical velocity the characteristics of the time-dependent flow were studied numerically. It was found that the obstacle emitted repeatedly gray solitons propagating downstream and sound propagating upstream.

The aim of this paper is to prove rigorously that, for a general potential U, equation (1.5) admits two stationary solutions if the velocity v is reasonably small.

Since one expects, from physical considerations, that the solutions are slowly varying and have a modulus tending to 1 at $\pm\infty$, we seek for solutions of the form $A(x) = (1 + r(x))e^{i\theta(x)}$ with $r(x) \longrightarrow 0$ and $\theta'(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Substituting this expression in (1.5) one finds that the real functions r and θ must satisfy

$$-vr_x + 2r_x\theta_x + (1+r)\theta_{xx} = 0, \tag{1.6}$$

$$v(1+r)\theta_x + r_{xx} - (1+r)\theta_x^2 + (1+r) - (1+r)^3 - U(x)(1+r) = 0. \tag{1.7}$$

Multiplying equation (1.6) by $1 + r$ and integrating we find

$$\theta_x = \frac{v}{2}\Big(1 - \frac{1}{(1+r)^2}\Big). \tag{1.8}$$

This determines θ_x (half of the fluid velocity) as a function of $(1+r)^2$ (the local fluid density). Introducing (1.8) in (1.7) we find that r satisfies the equation (also derived by V. Hakim):

$$-r_{xx} - (1+r) + (1+r)^3 - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)U(x) = 0. \tag{1.9}$$

From now on, we will focus our attention on finding solutions of (1.9). Once this task accomplished, it is easy to determine the corresponding phase θ from (1.8). Then $A(x) = (1 + r(x))e^{i\theta(x)}$ will be a solution of (1.5).

Of course it is interesting to find solutions of (1.9) under the more general possible assumptions on U. In what follows, we suppose that U is a positive Borel measure with bounded total variation. A few notations are in order: by $\int_{\mathbf{R}} f(x)U(x)dx$ we denote the integral of a function f with respect to the measure U and by $||U||$ the total variation of U, i.e. $||U|| = \int_{\mathbf{R}} U(x)dx$. If $f \in L^\infty(\mathbf{R})$, then fU is also a Borel measure of bounded total variation and therefore $fU \in \mathcal{D}'(\mathbf{R})$. In particular, if $r \in L^\infty(\mathbf{R})$ and $r \neq -1$ a.e., all quantities appearing in (1.9) make sense in $\mathcal{D}'(\mathbf{R})$.

We discuss now what happens if U vanishes on some interval I. It is easily seen that equation (1.9) can be integrated explicitly on this interval. This simple observation gives an obstruction to the existence of stationary solutions of (1.5) for v greater than $\sqrt{2}$ (which is the sound velocity at infinity) in the case of a potential with compact support.

Indeed, suppose that $U \equiv 0$ on an interval I. On this interval equation (1.9) becomes

$$-r_{xx} - (1+r) + (1+r)^3 - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1+r)^3}\Big) = 0. \tag{1.10}$$

We remark that if $r > -1$ is a continuous solution, then r_{xx} is also continuous, therefore $r \in C^2(I)$. Multiplying (1.10) by $2r_x$ and integrating, it is easy to see that there exists a constant C such that

$$(1.11) \qquad -r_x^2 + \frac{1}{2}((1+r)^2 - 1)^2 - \frac{v^2}{4}\Big(1 + r - \frac{1}{1+r}\Big)^2 + C = 0.$$

If I is of the form $(-\infty, a)$ or (b, ∞), the condition $r \longrightarrow 0$ at $\pm\infty$ implies $C = 0$, that is

$$(1.12)\ r_x^2 = \frac{1}{2}((1+r)^2 - 1)^2 - \frac{v^2}{4}\Big(1 + r - \frac{1}{1+r}\Big)^2 = r^2(r+2)^2\Big(\frac{1}{2} - \frac{v^2}{4}\frac{1}{(1+r)^2}\Big).$$

Since $r^2(r+2)^2\Big(\frac{1}{2} - \frac{v^2}{4}\frac{1}{(1+r)^2}\Big) < 0$ for $r \in (-1, -1 + \frac{v}{\sqrt{2}}) \setminus \{0\}$ and $r_x^2 \geq 0$, we see that any solution r of (1.9) cannot take values in $(-1, -1+\frac{v}{\sqrt{2}}) \setminus \{0\}$. If v is greater than $\sqrt{2}$, any solution of (1.9) that tends to zero at $\pm\infty$ must be identically zero on I (since otherwise, by continuity it would take values sufficiently close to 0, but different from 0, which is impossible).

If $v \leq \sqrt{2}$, any solution r of (1.9) must be less than or equal to 0 on $\mathbf{R}$ by the maximum principle. Indeed, the function $x \longmapsto \psi_v(x) = -(1+x) + (1+x)^3 - \frac{v^2}{4}\Big(1 + x - \frac{1}{(1+x)^3}\Big)$ is strictly increasing and positive on $(0, \infty)$. Suppose that r achieves a positive maximum at x_0. Then $r''(x_0) \leq 0$. On the other hand, from (1.9) we infer that $r''(x_0) \geq \psi_v(r(x_0)) > 0$, a contradiction.

Suppose that $U \equiv 0$ on an interval I of the form $(-\infty, a)$ or (b, ∞). If $v = \sqrt{2}$, we see from (1.12) that we have also $r \geq 0$ on I, and therefore $r \equiv 0$ on I. If $v < \sqrt{2}$, we must have $-1 + \frac{v}{\sqrt{2}} \leq r \leq 0$ on I.

Suppose that $v \geq \sqrt{2}$. In the particular case $U = g\delta$ (where δ is the Dirac measure and $g \geq 0$), one has $r \equiv 0$ on $(-\infty, 0) \cup (0, \infty)$; consequently, if $g > 0$, (1.9) does not admit solutions and if $g = 0$, it admits only the trivial solution. If U has a compact support with $supp(U) \subset (a, b)$ it follows that any solution r of (1.9) that tends to zero at $\pm\infty$ must vanish on $\mathbf{R} \setminus (a, b)$. But this gives too many constraints (r and its derivatives should vanish at a and b) and so we expect that (1.9) does not possess solutions satisfying the "boundary condition" $r \longrightarrow 0$ at $\pm\infty$ if $v \geq \sqrt{2}$ and $U \neq 0$.

From now on, we will suppose throughout that $0 < v < \sqrt{2}$.

This paper is organized as follows. In the next section we give a variational formulation of equation (1.9) and we introduce our main tools. It will be seen that the solutions of (1.9) are the critical points of a functional E defined on the space $H^1(\mathbf{R})$. Section 3 is devoted to a detailed study of the particular case $U = g\delta$, where the solutions are known explicitly. It is proved that there exists a positive function $\varphi(v)$ such that if $0 < g < \varphi(v)$, there are exactly two solutions of (1.9). One of them minimizes E on an open set of $H^1(\mathbf{R})$ and the other is a critical point of E of mountain-pass type. The two solutions are the same when $g = \varphi(v)$ and no solution exists when $g > \varphi(v)$. In the general case, we show that an analogous phenomenon takes place. Our main result is:

Theorem 1.1 *a) There exists a function $\varphi_1(v) > 0$ such that if $||U|| < \varphi_1(v)$, then E admits a minimizer on an open set (which will be described later) of $H^1(\mathbf{R})$.*

b) There exists a function $\varphi_2(v) > 0$ such that if $||U|| < \varphi_2(v)$ and U has compact support, E admits a second critical point (of "mountain-pass" type).

We have $\varphi(v) > \varphi_1(v) > \varphi_2(v)$ for any $v \in [0, \sqrt{2})$. The graphs of these functions are given in Fig. 1 below. It is quite clear that the existence of nontrivial solutions for (1.9) should depend also on the shape of U, not only on its total variation. Therefore for a given potential U, we expect to have a nontrivial solution of (1.9) for values of v slightly larger than $\varphi_1^{-1}(||U||)$ and two distinct solutions for v slightly larger than $\varphi_2^{-1}(||U||)$.

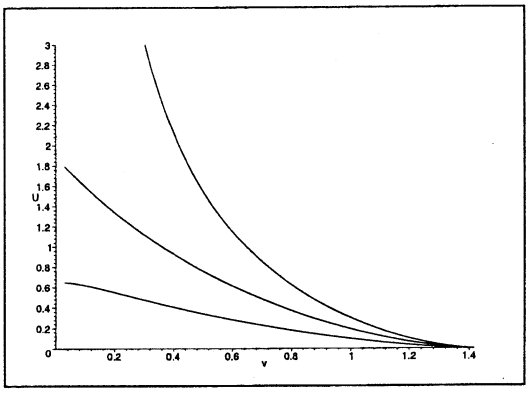

Fig. 1. The graphs of functions φ, φ_1 and φ_2.

The proof of part a) in Theorem 1.1 is rather classical and is given in Section 4. We prove part b) in Section 5. The main difficulty is that the Palais-Smale sequences of E do not converge. We use a theorem of Ghoussoub and Preiss [21] to obtain Palais-Smale sequences with a supplementary property which enables us to deduce their convergence to a solution of (1.9). We have also to impose further restriction on the total mass of U in order to be sure that this second solution is different from that one obtained in Section 4.

2.2 Variational formulation

Consider the set $V = \{u \in H^1(\mathbf{R}) | \inf_{s \in \mathbf{R}} u(s) > -1\}$. Clearly V is a not-empty open subset of $H^1(\mathbf{R})$ (recall that $H^1(\mathbf{R})$ is continuously embeded in $C_b^0(\mathbf{R})$). We introduce the following functionals:

$$G : V \longrightarrow \mathbf{R}, \quad G(u) = \int_{\mathbf{R}} |u'(x)|^2 + \frac{1}{4} u^2(x)(u(x)+2)^2 \Big(2 - \frac{v^2}{(u(x)+1)^2}\Big) dx,$$

$$H : H^1(\mathbf{R}) \longrightarrow \mathbf{R}, \quad H(u) = \int_{\mathbf{R}} u(x)(u(x)+2)U(x) dx,$$

$$E : V \longrightarrow \mathbf{R}, \qquad E(u) = G(u) + H(u).$$

It is easy to check that the functionals G and H are well defined and of class C^1 on V, respectively on $H^1(\mathbf{R})$. A function $r \in V$ satisfies (1.9) (in the distributional sense) if and only if r is a critical point of E.

We want to study the behaviour of $G(u)$ in terms of the variations of the function u. For this purpose, we use the following simple observation:

Remark 2.1 Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function such that $f(0) = 0$. Put $F(x) = \int_0^x f(s)ds$. Then for any $u \in H^1(\mathbf{R})$ and any $a, b \in \mathbf{R}$, $a < b$ we have

$$(2.1) \quad |F(u(b)) - F(u(a))| = |\int_a^b f(u(s))u'(s)ds| \leq \frac{1}{2}\int_a^b |f(u(s))|^2 + |u'(s)|^2 ds.$$

If $F(u(b)) \geq F(u(a))$, we have equality in (2.1) if and only if $u'(s) = f(u(s))$ a.e. on $[a, b]$. If $F(u(b)) < F(u(a))$, equality holds if and only if $u'(s) = -f(u(s))$ a.e. on $[a, b]$. In particular, for any $a \in \mathbf{R}$ one has $|F(u(a))| \leq \frac{1}{2}\int_{-\infty}^a |f(u(s))|^2 + |u'(s)|^2 ds$ and $|F(u(a))| \leq \frac{1}{2}\int_a^\infty |f(u(s))|^2 + |u'(s)|^2 ds$. Hence

$$(2.2) \quad 4|F(u(a))| \leq \int_{-\infty}^\infty |f(u(s))|^2 + |u'(s)|^2 ds, \quad \forall a \in \mathbf{R}.$$

Moreover, equality holds in (2.2) if and only if $u' = \sigma f(u)$ a.e. on $(-\infty, a)$ and $u' = -\sigma f(u)$ a.e. on (a, ∞), where $\sigma = sgn(F(u(a)))$.

Now take $f : [-1 + \frac{v}{\sqrt{2}}, \infty) \longrightarrow \mathbf{R}$,

$$(2.3) \quad f(x) = \frac{1}{2}x(x+2)\sqrt{2 - \frac{v^2}{(1+x)^2}}$$

and let $F(x) = \int_0^x f(s)ds$. Observe that f is negative on $(-1 + \frac{v}{\sqrt{2}}, 0)$ and positive on $(0, \infty)$, hence F is decreasing on $[-1 + \frac{v}{\sqrt{2}}, 0]$ and increasing on $[0, \infty)$, so that F is positive on $[-1 + \frac{v}{\sqrt{2}}, \infty) \setminus \{0\}$.

Let $r \in H^1(\mathbf{R})$ be so that $\inf_{x \in \mathbf{R}} r(x) = r(x_0) = a \in [-1 + \frac{v}{\sqrt{2}}, 0]$. Applying the previous remark we obtain that

$$0 \leq 4F(a) = 4F(r(x_0)) \leq G(r)$$

and equality holds if and only if $r'(x) = f(r(x))$ a.e. on $(-\infty, x_0)$ and $r'(x) = -f(r(x))$ a.e. on (x_0, ∞). Solving the Cauchy problem

$$(2.4) \quad \begin{cases} r'(x) = f(r(x)) \text{ on } (-\infty, 0] \\ r(0) = a \end{cases}$$

we find the solution

(2.5)

$$r_{1,a}(x) = -1 + \sqrt{\frac{v^2}{2} + (1 - \frac{v^2}{2})\tanh^2(\frac{1}{2}\sqrt{2 - v^2}(x + c(a)))}, \quad a \in [-1 + \frac{v}{\sqrt{2}}, 0)$$

where $c(a) = \frac{1}{\sqrt{2-v^2}} \ln \frac{\sqrt{2-v^2} - \sqrt{2(a+1)^2 - v^2}}{\sqrt{2-v^2} + \sqrt{2(a+1)^2 - v^2}}$, respectively $r_{1,0} \equiv 0$ if $a = 0$. It is obvious that the Cauchy problem $\begin{cases} r'(x) = -f(r(x)) \text{ on } [0, \infty) \\ r(0) = a \end{cases}$ has the solution

$r_{2,a}(x) = r_{1,a}(-x)$. We put

$$r_a(x) = \begin{cases} r_{1,a}(x) & \text{if } x \le 0 \\ r_{2,a}(x) & \text{if } x > 0. \end{cases}$$

The functions $(r_a)_{a\in[-1+\frac{v}{\sqrt{2}},0]}$ will be very useful in what follows. We list below some of their basic properties.

Lemma 2.2 *The following assertions hold:*

i) $r_a \in H^1(\mathbf{R})$ *and the mapping* $a \longmapsto r_a$ *is continuous from* $[-1+\frac{v}{\sqrt{2}},0]$ *to* $H^1(\mathbf{R})$.

ii) r_a *is symmetric about 0, decreasing on* $(-\infty,0]$ *and increasing on* $[0,\infty)$ *and tends exponentially to zero at* $\pm\infty$.

iii) r_a *is* C^∞ *on* $\mathbf{R}\setminus\{0\}$.

iv) $c(-1+\frac{v}{\sqrt{2}})=0$ *and* c *is strictly decreasing on* $[-1+\frac{v}{\sqrt{2}},0)$ *with* $\lim_{a\uparrow 0} c(a) = -\infty$.

v) $r_{-1+\frac{v}{\sqrt{2}}}$ *is of class* C^1 *on* $\mathbf{R}$ *with* $r'_{-1+\frac{v}{\sqrt{2}}}(0)=0$. *Moreover, for each* a *we have* $r_a(x) = r_{-1+\frac{v}{\sqrt{2}}}(x+c(a))$ *for* $x \le 0$, *respectively* $r_a(x) = r_{-1+\frac{v}{\sqrt{2}}}(x-c(a))$ *for* $x > 0$.

vi) $G(r_a)) = 4F(a)$ *and* r_a *is the unique solution of the minimization problem: "minimize* $G(r)$ *under the constraint* $r(0)=a$."

vii) If $x < y \le 0$ *or* $0 \le x < y$, *then for any function* $v \in H^1_{loc}(\mathbf{R})$ *verifying* $v(x) = r_a(x)$, $v(y) = r_a(y)$ *and* $v \ge -1+\frac{v}{\sqrt{2}}$ *on* (x,y) *we have*

$$\int_x^y |r_a'(s)|^2 + f^2(r_a(s))ds = 2|F(v(y)) - F(v(x))| \le \int_x^y |v'(s)|^2 + f^2(v(s))ds. \tag{2.6}$$

The proof is obvious.

For each $a \in [-1+\frac{v}{\sqrt{2}},0]$ define

$$h(a) = \inf\{E(u) \mid u \in H^1(\mathbf{R}),\ \inf_{x\in\mathbf{R}} u(x) = a\}.$$

Lemma 2.3 *The function* h *has the following properties:*

i) $h(a) \ge 4F(a) + a(a+2)||U||, \quad \forall a \in [-1+\frac{v}{\sqrt{2}},0]$.

ii) For all $k > 0$ *and* $a \in [-1+\frac{v}{\sqrt{2}},0]$ *we have*

$$h(a) \le 4F(a) + 2kf^2(a) + a(a+2)||\chi_{[-k,k]}U||.$$

iii) $h : [-1+\frac{v}{\sqrt{2}},0] \longrightarrow \mathbf{R}$ *is continuous,* $h(0)=0$ *and*

$$(2.7) \qquad h(-1+\frac{v}{\sqrt{2}}) = 4F(-1+\frac{v}{\sqrt{2}}) + (\frac{v^2}{2}-1)||U||.$$

Proof. *i*) is clear because for any $u \in H^1(\mathbf{R})$ such that $\inf_{x\in\mathbf{R}} u(x) = a$, we have $G(u) \geq 4F(a)$ and $H(u) \geq a(a+2)||U||$ (note that the function $y \longmapsto y(y+2)$ is increasing on $[-1,\infty)$).

ii) Define

$$(2.8) \qquad u_{a,k}(x) = \begin{cases} r_a(x+k) & \text{if} \quad x < -k \\ a & \text{if} \quad -k \leq x \leq k \\ r_a(x-k) & \text{if} \quad x > k. \end{cases}$$

Obviously $u_{a,k} \in H^1(\mathbf{R})$, $\inf_{x\in\mathbf{R}} u_{a,k}(x) = a$, $G(u_{a,k}) = 4F(a) + 2kf^2(a)$ and $H(u_{a,k}) \leq \int_{-k}^{k} u_{a,k}(u_{a,k}+2)\chi_{[-k,k]}(x)U(x)dx = a(a+2)||\chi_{[-k,k]}U||$. Since by definition $h(a) \leq E(u_{a,k}) = G(u_{a,k}) + H(u_{a,k})$, *ii*) follows.

iii) It is clear that $h(0) = 0$. Because $f(-1+\frac{v}{\sqrt{2}}) = 0$, *i*) and *ii*) give

$$4F(-1+\tfrac{v}{\sqrt{2}}) + (\tfrac{v^2}{2}-1)||U|| \leq h(-1+\tfrac{v}{\sqrt{2}})$$
$$\leq 4F(-1+\tfrac{v}{\sqrt{2}}) + (\tfrac{v^2}{2}-1)||\chi_{[-k,k]}U||$$

for all $k > 0$. Passing to the limit as $k \longrightarrow \infty$, we obtain (2.7).

Let $\varepsilon > 0$ be arbitrary, but fixed. Take k_ε sufficiently large so that $||\chi_{[-k_\varepsilon,k_\varepsilon]}U|| > ||U|| - \varepsilon$. Using *i*) and *ii*) we get

$$(2.9) \qquad 4F(a) + a(a+2)||U|| \leq h(a) \leq 4F(a) + 2k_\varepsilon f^2(a) + a(a+2)(||U||-\varepsilon).$$

Letting $a \longrightarrow -1+\frac{v}{\sqrt{2}}$ (respectively $a \longrightarrow 0$) in (2.9) we obtain

$$h\Big(-1+\frac{v}{\sqrt{2}}\Big) \leq \liminf_{a\downarrow -1+\frac{v}{\sqrt{2}}} h(a) \leq \limsup_{a\downarrow -1+\frac{v}{\sqrt{2}}} h(a) \leq h\Big(-1+\frac{v}{\sqrt{2}}\Big) + \varepsilon\Big(1-\frac{v^2}{2}\Big),$$

respectively

$$0 = h(0) \leq \liminf_{a\uparrow 0} h(a) \leq \limsup_{a\uparrow 0} h(a) \leq 0.$$

Since ε was arbitrary, we infer that h is continuous at 0 and $-1+\frac{v}{\sqrt{2}}$.

It remains to prove that h is continuous at any point $a \in (-1+\frac{v}{\sqrt{2}}, 0)$. Fix such an a and let $a_n \longrightarrow a$. All we have to do is to show that $h(a_n) \longrightarrow h(a)$.

Let $\varepsilon > 0$ be arbitrary, but fixed. Consider $u \in H^1(\mathbf{R})$ such that $\inf_{x\in\mathbf{R}} u(x) = a$ and $E(u) < h(a) + \varepsilon$. By continuity of E, $E(\frac{a_n}{a}u) \longrightarrow E(u)$ as $n \longrightarrow \infty$, so $E(\frac{a_n}{a}u) < h(a) + \varepsilon$ if n is sufficiently large. Since $\inf_{x\in\mathbf{R}} \frac{a_n}{a}u(x) = a_n$, it follows that $h(a_n) \leq E(\frac{a_n}{a}u) < h(a) + \varepsilon$ for all n sufficiently large. Thus $\limsup_{n\to\infty} h(a_n) \leq h(a) + \varepsilon$.

Now fix $\delta \in (-1+\frac{v}{\sqrt{2}}, a)$. For each n sufficiently large (so that $a_n > \delta$), choose $u_n \in H^1(\mathbf{R})$ verifying $\inf_{x\in\mathbf{R}} u_n(x) = a_n$, $a_n \leq u_n \leq 0$ and $E(u_n) < h(a_n) + \varepsilon$ (this is possible because $E(-u^-) \leq E(u)$, $\forall u \in V$, where $u^- = -\min(u,0)$). Note that f

is a Lipschitz function on $[\delta, 0]$; let L_δ be its Lipschitz constant. Observe that there exists $C_\delta > 0$ such that $f^2(x) \geq C_\delta x^2$, $\forall x \in [\delta, 0]$. It follows that

$$\int_{\mathbf{R}} |u_n'|^2 dx + C_\delta \int_{\mathbf{R}} u_n^2 dx \leq G(u_n) = E(u_n) - H(u_n) < h(a_n) + \varepsilon - a_n(a_n+2)||U||.$$

It is seen from *i)* and *ii)* that h is bounded on $[-1+\frac{v}{\sqrt{2}}, 0]$, hence (u_n) is a bounded sequence in $H^1(\mathbf{R})$. Then we have

$$\int_{\mathbf{R}} \frac{a^2}{a_n^2} |u_n'|^2 dx - \int_{\mathbf{R}} |u_n'|^2 dx \longrightarrow 0 \text{ as } n \longrightarrow \infty;$$

$$\begin{aligned} &|\int_{\mathbf{R}} f^2(\tfrac{a}{a_n} u_n) - f^2(u_n) dx| \\ &\leq L_\delta \Big(\int_{\mathbf{R}} |\tfrac{a}{a_n} u_n - u_n|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbf{R}} |f(\tfrac{a}{a_n} u_n) + f(u_n)|^2 dx\Big)^{\frac{1}{2}} \\ &\leq L_\delta |\tfrac{a}{a_n} - 1| \Big(\int_{\mathbf{R}} u_n^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbf{R}} 2f^2(\tfrac{a}{a_n} u_n) + 2f^2(u_n) dx\Big)^{\frac{1}{2}} \longrightarrow 0 \text{ as } n \longrightarrow \infty; \end{aligned}$$

$$\begin{aligned} &|\int_{\mathbf{R}} \Big(\tfrac{a}{a_n} u_n (\tfrac{a}{a_n} u_n + 2) - u_n(u_n+2)\Big) U(x) dx| \\ &\leq \Big(|\tfrac{a^2}{a_n^2} - 1|\delta^2 + 2|\tfrac{a}{a_n} - 1| \cdot |\delta|\Big) ||U|| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Consequently, $\lim_{n\to\infty} (E(\frac{a}{a_n} u_n) - E(u_n)) = 0$. But $\inf_{x\in\mathbf{R}} \frac{a}{a_n} u_n(x) = a$ and so

$$h(a) \leq E(\frac{a}{a_n} u_n) < h(a_n) + \varepsilon + \Big(E(\frac{a}{a_n} u_n) - E(u_n)\Big).$$

Thus $h(a) \leq \liminf_{n\longrightarrow\infty} h(a_n) + \varepsilon$. Therefore we proved that

$$h(a) - \varepsilon \leq \liminf_{n\longrightarrow\infty} h(a_n) \leq \limsup_{n\longrightarrow\infty} h(a_n) \leq h(a) + \varepsilon.$$

Since ε was arbitrary, it follows that $\lim_{n\longrightarrow\infty} h(a_n) = h(a)$. This proves the continuity of h at $a \in (-1+\frac{v}{\sqrt{2}}, 0)$. □

Remark 2.4 It can be proved that if U has compact support, there exists $u_a \in H^1(\mathbf{R})$ such that $\inf_{x\in\mathbf{R}} u_a(x) = a$ and $E(u_a) = h(a)$ (that is, there exists a "minimizer at level a"). We do not give here the proof because we do not make use of this result.

If u_a could be chosen in order to have a continuous map $a \longmapsto u_a$ from $[-1+\frac{v}{\sqrt{2}}, 0]$ into $H^1(\mathbf{R})$, then the proofs in Section 5 can be considerably simplified and the results slightly strengthened. We were not able to prove that a continuous path of "minimizers at level a" exists for a general U.

2.3 The case $U = g\delta$ $(g > 0)$

The case $U = g\delta$ $(g > 0)$ is very simple and one can find explicitly the solutions of (1.9) (see [23]); however, it is quite instructive and gives a good feeling of what kind of result can be expected when U is a positive Borel measure.

Consider the functions r_a, $a \in [-1+\frac{v}{\sqrt{2}}, 0]$ introduced in the previous section. On $(-\infty, 0)$ we have $r_a'' = (r_a')' = f(r_a)' = f'(r_a)r_a' = f(r_a)f'(r_a) = \frac{1}{2}(f^2)'(r_a)$, that is $r_a'' = -(1+r_a) + (1+r_a)^3 - \frac{v^2}{4}(1+r_a - \frac{1}{(1+r_a)^3})$. Obviously the same is true on $(0,\infty)$. Moreover,

$$\lim_{x\uparrow 0} r_a'(x) = \lim_{x\uparrow 0} f(r_a(x)) = f(a),$$
$$\lim_{x\downarrow 0} r_a'(x) = \lim_{x\downarrow 0} -f(r_a(x)) = -f(a).$$

We obtain that r_a satisfies (1.9) for $U = -\frac{2f(a)}{a+1}\delta$ (note that $-\frac{2f(a)}{a+1} \geq 0$).

Conversely, let $r \in H^1(\mathbf{R})$ be a solution of (1.9) for $U = g\delta$, $g \geq 0$. From the discussion in Introduction it follows that $-1+\frac{v}{\sqrt{2}} \leq r(x) \leq 0$, $\forall x \in \mathbf{R}$, $r \in C^2(\mathbf{R}\setminus\{0\})$ and (1.12) is true, i.e. $r_x^2 = f^2(r)$ on $(-\infty,0)\cup(0,\infty)$.

Observe that 0 is not a solution of (1.9) if $U \neq 0$. Let I be a maximal interval such that $I \subset \mathbf{R}\setminus\{0\}$ and $r \neq 0$, $r \neq -1+\frac{v}{\sqrt{2}}$ on I. Since r_x is continuous on I and $f(r) \neq 0$ if $r \notin \{0, -1+\frac{v}{\sqrt{2}}\}$, we have either $r_x = f(r)$ on I or $r_x = -f(r)$ on I.

Let $a = r(0)$. If $a = 0$ or $a = -1+\frac{v}{\sqrt{2}}$, it follows from (1.12) that $\lim_{x\to 0} r'(x) = 0$, hence r_x may be extended by continuity at 0. Moreover, since $\lim_{x\to 0} r''(x)$ exists, the continuous extension of r_x is differentiable at $x=0$ and consequently r satisfies (1.9) for $U = 0$, that is we must have $g = 0$. So if $g > 0$, then necessarily $a = r(0) \in (-1+\frac{v}{\sqrt{2}}, 0)$. Let

$$x_1 = \inf\{x < 0 \mid r \neq 0, r \neq -1+\tfrac{v}{\sqrt{2}} \text{ on } (x,0)\} \text{ and}$$
$$y_1 = \sup\{y > 0 \mid r \neq 0, r \neq -1+\tfrac{v}{\sqrt{2}} \text{ on } (0,y)\}.$$

Clearly $x_1 < 0$, $y_1 > 0$ and the sign of r' does not change on $(x_1, 0)$ and on $(0, y_1)$. If $r' = f(r)$ or if $r' = -f(r)$ on $(x_1,0)\cup(0,y_1)$, then r satisfies (1.9) with $U = 0$ on (x_1, y_1), a contradiction. If $r' = -f(r)$ on $(x_1, 0)$ and $r' = f(r)$ on $(0, y_1)$, then r satisfies (1.9) with $U = \frac{2f(a)}{a+1}\delta$ and $g = \frac{2f(a)}{a+1} < 0$, again a contradiction. It remains that $r' = f(r)$ on $(x_1, 0)$ and $r' = -f(r)$ on $(0, y_1)$. By a standard argument we infer that $x_1 = -\infty$, $y_1 = \infty$ and $r = r_a$. Thus we have proved that (1.9) has no other solutions than the functions r_a introduced in Section 2. Obviously we must have $g = -\frac{2f(a)}{a+1}$ if r_a is a solution.

Note that in the case $U \equiv 0$, the problem is translation invariant. Following the above discussion, one easily proves that the only solutions of (1.9) are 0 and $r_{-1+\frac{v}{\sqrt{2}}}(\cdot - z)$, $z \in \mathbf{R}$.

It is natural to ask then: for a given $g > 0$, how many solutions are there? The answer is: exactly as many as the roots of the equation

$$g = -\frac{2f(a)}{a+1} \tag{3.1}$$

are. Let $k_v(a) = -\frac{2f(a)}{a+1}$. Obviously k_v is differentiable on $(-1+\frac{v}{\sqrt{2}}, 0]$ and a straightforward computation shows that $k_v'(a) > 0$ on $(-1+\frac{v}{\sqrt{2}}, a_\star(v))$ and $k_v'(a) < 0$ on $(a_\star(v), 0)$, where $a_\star(v) = -1+\sqrt{\frac{-1+\sqrt{1+4v^2}}{2}}$. So k_v is increasing on $[-1+$

$\frac{v}{\sqrt{2}}, a_\star(v)]$, decreasing on $[a_\star(v), 0]$, $k_v(-1+\frac{v}{\sqrt{2}}) = k_v(0) = 0$ and k_v has a maximum at $a_\star(v)$. Let

$$\varphi(v) = k_v(a_\star(v)) = \frac{(1+\sqrt{1+4v^2}-2v^2)\sqrt{2-v^2}}{2v\sqrt{1+v^2+\sqrt{1+4v^2}}}. \tag{3.2}$$

Thus, if $g < \varphi(v)$, equation (3.1) has exactly two roots $a_1 \in (a_\star(v), 0)$ and $a_2 \in (-1+\frac{v}{\sqrt{2}}, a_\star(v))$. Clearly $a_1 \downarrow a_\star(v)$ and $a_2 \uparrow a_\star(v)$ as $g \uparrow \varphi(v)$. When $g = \varphi(v)$, we have the double root $a_\star(v)$. If $g > \varphi(v)$, (3.1) has no roots. Consequently, if $g < \varphi(v)$ the equation (1.9) with $U = g\delta$ has two solutions, namely r_{a_1} and r_{a_2}. These solutions are "merging" when $g = \varphi(v)$. For $g > \varphi(v)$, equation (1.9) does not admit solutions.

Note that the function φ is continuous and strictly decreasing on $(0, \sqrt{2}]$, $\lim_{v\downarrow 0} \varphi(v) = \infty$ and $\varphi(\sqrt{2}) = 0$. Therefore φ^{-1} exists, is strictly decreasing, $\varphi^{-1}(0) = \sqrt{2}$ and $\lim_{g\to\infty} \varphi^{-1}(g) = 0$. We summarize the above discussion in the following

Proposition 3.1 *Consider the equation (1.9) with the potential $U = g\delta$.*

i) For a fixed velocity $v \in (0, \sqrt{2})$, the equation has exactly two solutions if $g \in (0, \varphi(v))$, where $\varphi(v)$ is given by (3.2). If $g = \varphi(v)$, there exists only one solution. If $g > \varphi(v)$, the equation does not admit solutions.

ii) Conversely, fix $g > 0$. If $v < \varphi^{-1}(g)$, we have exactly two solutions of velocity v. There is only one solution of velocity $v = \varphi^{-1}(g)$ and there are no solutions of velocity $v > \varphi^{-1}(g)$.

Remark 3.2 It is obvious that in the case $U = g\delta$ one has

$$h(a) = E(r_a) = 4F(a) + a(a+2)g.$$

So the function h is differentiable and

$$h'(a) = 4f(a) + 2(a+1)g = 2(a+1)(g - k_v(a)).$$

If $g > \varphi(v)$, then h is strictly increasing on $[-1+\frac{v}{\sqrt{2}}, 0]$ and it does not admit critical points. If $g = \varphi(v)$, it is still strictly increasing, but it has one critical point $a_\star(v)$. Finally, if $g < \varphi(v)$, we see that the function h is increasing on $[-1+\frac{v}{\sqrt{2}}, a_2(v)]$, decreasing on $[a_2(v), a_1(v)]$ and increasing on $[a_1(v), 0]$, where $a_1(v)$ and $a_2(v)$ are the two roots of equation (3.1). We have already seen that the two solutions of (1.9) are $r_{a_1(v)}$ and $r_{a_2(v)}$. Note that $r_{a_1(v)}$ is a local minimum of E (for example, it minimizes E on the open set $\{u \in H^1(\mathbf{R}) \mid \inf_{x\in\mathbf{R}} u(x) > a_2(v)\}$). The second solution, $r_{a_2(v)}$, is a critical point of mountain-pass type of E. Indeed, for each continuous path $\gamma : [0,1] \longrightarrow H^1(\mathbf{R})$ such that $\gamma(0) = r_{-1+\frac{v}{\sqrt{2}}}$ and $\gamma(1) = r_{a_1(v)}$, there exists $t \in [0,1]$ such that $E(\gamma(t)) \geq E(r_{a_2(v)}) > \max(E(r_{-1+\frac{v}{\sqrt{2}}}), E(r_{a_1(v)}))$ (when E is suitably extended to $H^1(\mathbf{R})$).

For a general measure U, we do not know the shape of the curve $a \longmapsto h(a)$. However, it will be shown in the next two sections that quite a similar phenomenon takes place.

2.4 A local minimizer of E

We keep the notation introduced previously. The main result of this section is

Theorem 4.1 *Assume that U is a positive Borel measure and $||U||$ is finite. Then:*

i) There exists $\eta > 0$ such that $h(a) < 0$ for all $a \in (-\eta, 0)$.

ii) Suppose that there exists $a \in [-1 + \frac{v}{\sqrt{2}}, 0)$ such that $h(a) \geq 0$. Let $a_0 = \sup\{a \in [-1 + \frac{v}{\sqrt{2}}, 0) \mid h(a) \geq 0\}$. Then E has a minimum on the open set

$$V_0 = \{u \in H^1(\mathbf{R}) \mid \inf_{x \in \mathbf{R}} u(x) > a_0\}.$$

Proof. *i)* We have for any $T > 0$

$$\begin{aligned} h(a) \leq E(r_a) = 4F(a) + \int_{\mathbf{R}} r_a(r_a + 2)U(x)dx \\ \leq 4F(a) + r_a(T)(r_a(T) + 2)||\chi_{[-T,T]}U||. \end{aligned} \tag{4.1}$$

Let us denote by $\phi_T(a)$ the right hand side of the above inequality. Clearly ϕ_T is differentiable and

$$\phi_T'(a) = 4f(a) + 2(r_a(T) + 1)||\chi_{[-T,T]}U|| \frac{d}{da}(r_a(T)).$$

But $r_a(T) = r_{-1+\frac{v}{\sqrt{2}}}(T - c(a))$ and so

$$\frac{d}{da} r_a(T) = -r'_{-1+\frac{v}{\sqrt{2}}}(T - c(a))c'(a) = f(r_{-1+\frac{v}{\sqrt{2}}}(T - c(a)))c'(a). \tag{4.2}$$

Since $r_{-1+\frac{v}{\sqrt{2}}}(c(a)) = r_a(0) = a$, we get $1 = r'_{-1+\frac{v}{\sqrt{2}}}(c(a))c'(a)$. Remember that $c(a) < 0$ and $r'_{-1+\frac{v}{\sqrt{2}}}(c(a)) = f(r_{-1+\frac{v}{\sqrt{2}}}(c(a))) = f(a)$. Therefore $c'(a) = \frac{1}{f(a)}$. Combining this with (4.2), one obtains $\frac{d}{da}(r_a(T)) = \frac{f(r_{-1+\frac{v}{\sqrt{2}}}(T - c(a)))}{f(a)}$. After a straightforward computation, we get

$$\lim_{a \uparrow 0} \frac{d}{da}(r_a(T)) = e^{-\sqrt{2-v^2}T}. \tag{4.3}$$

Thus $\lim_{a \uparrow 0} \phi_T'(a) = 2e^{-\sqrt{2-v^2}T}||\chi_{[-T,T]}U||$. Now fix T such that $||\chi_{[-T,T]}U|| > 0$. Then ϕ_T is continuous, $\phi_T(0) = 0$ and $\lim_{a \uparrow 0} \phi_T'(a) > 0$, so there exists $\eta > 0$ such that $\phi_T(a) < 0$, $\forall a \in (-\eta, 0)$. This clearly implies *i)*.

ii) Obviously E is bounded from below on V_0 by $\min_{a \in [a_0, 0]} h(a)$. Let $(r_n)_{n \in \mathbf{N}}$ be a minimizing sequence for E on V_0. We may suppose that $a_0 < r_n(s) \leq 0$, $\forall s \in \mathbf{R}$ and $E(r_n) < 0$. Then we have

$$G(r_n) < -\int_{\mathbf{R}} r_n(r_n + 2)U(s)ds \leq -a_0(a_0 + 2)||U||. \tag{4.4}$$

Observe that the function $a \longmapsto 4F(a) + a(a+2)||U||$ is increasing on an interval $(-1 + \frac{v}{\sqrt{2}}, -1 + \frac{v}{\sqrt{2}} + \delta)$ for some $\delta > 0$. In view of Lemma 2.3 *i)* and *iii)*, it follows that $h(-1+\frac{v}{\sqrt{2}}) < h(a)$, $\forall a \in (-1+\frac{v}{\sqrt{2}}, -1+\frac{v}{\sqrt{2}}+\delta)$. Consequently we have $a_0 > -1+\frac{v}{\sqrt{2}}$ and there exists $C_0 > 0$ such that $f^2(x) \geq C_0 x^2$, $\forall x \in [a_0, 0]$. From (4.4) we infer that (r_n) is bounded in $H^1(\mathbf{R})$. Hence there exists a subsequence (still denoted (r_n)) and $r \in H^1(\mathbf{R})$ such that

$$r_n \rightharpoonup r \text{ weakly in } H^1(\mathbf{R}) \text{ and}$$
$$r_n \to r \text{ a.e. as } n \longrightarrow \infty.$$

By lower semicontinuity we have

$$\int_{\mathbf{R}} |r'|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbf{R}} |r_n'|^2 dx. \tag{4.5}$$

Using Fatou's lemma one has

$$\int_{\mathbf{R}} f^2(r) dx \leq \liminf_{n\to\infty} \int_{\mathbf{R}} f^2(r_n) dx. \tag{4.6}$$

Clearly $|r_n(s)(r_n(s)+2)| \leq |a_0|(2+a_0)$ for all $s \in \mathbf{R}$ and $n \in \mathbf{N}$. Since $||U|| < \infty$, Lebesgue's dominated convergence theorem can be applied and gives

$$\int_{\mathbf{R}} r(r+2)U(x) dx = \lim_{n\to\infty} \int_{\mathbf{R}} r_n(r_n+2)U(x) dx. \tag{4.7}$$

From (4.5), (4.6) and (4.7) we infer that

$$E(r) \leq \liminf_{n\to\infty} E(r_n) < 0. \tag{4.8}$$

Obviously $r \in \overline{V_0}$ since $r_n \longrightarrow r$ a.e. We cannot have $\inf_{x\in\mathbf{R}} r(x) = a_0$ because in this case we would have $E(r) \geq h(a_0) \geq 0$, which contradicts (4.8). Hence $r \in V_0$ and r is a minimizer of E on V_0. □

Remark 4.2 The assumption of Theorem 4.1, part *ii)* is clearly satisfied if, for example, $h(-1+\frac{v}{\sqrt{2}}) \geq 0$, that is if $||U|| \leq \frac{8F(-1+\frac{v}{\sqrt{2}})}{2-v^2}$. Let $\varphi_1(v) = \frac{8F(-1+\frac{v}{\sqrt{2}})}{2-v^2}$. One can see that φ_1 is smooth and positive on $[0, \sqrt{2})$ and $\varphi_1(0) = \frac{4\sqrt{2}}{3}$, $\varphi_1(0) = 0$. $\lim_{v\uparrow\sqrt{2}} \frac{\varphi_1(v)}{\sqrt{2-v^2}} = \frac{5}{12}$. If $||U|| \leq \varphi_1(v)$, then necessarily E has a critical point which is a local minimizer.

2.5 A second critical point of E

It is proved below, under certain hypothesis on U, that the functional E has a second critical point of "mountain-pass" type.

We suppose throughout this section that the assumptions of Theorem 4.1 are satisfied. Moreover, we suppose that U has compact support. Let $[x, y]$ be the smallest closed interval containing $supp(U)$.

We use the following mountain-pass theorem due to Ghoussoub and Preiss [21], based on Ekeland's variational principle:

Theorem 5.1 ([21]) *Let X be a Banach space and $\Phi : X \longrightarrow \mathbf{R}$ a C^1 functional. Let $u, v \in X$ and consider the set $\Gamma_{u,v}$ of continuous paths joining u and v, i.e.*

$$\Gamma_{u,v} = \{\gamma \in C^0([0,1], X) \mid \gamma(0) = u, \gamma(1) = v\}.$$

Define $c = \inf_{\gamma \in \Gamma_{u,v}} (\max_{s \in [0,1]} \Phi(\gamma(s)))$. Assume that there exists a closed subset M of X such that $M^c = M \cap \{x \in X \mid \Phi(x) \geq c\}$ separates u and v, i.e. u and v belong to two disjoint connected components of $X \setminus M^c$. Then there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that

i) $\lim_{n \to \infty} dist(x_n, M) = 0,$

ii) $\lim_{n \to \infty} \Phi(x_n) = c,$

iii) $\lim_{n \to \infty} ||\Phi'(x_n)||_{X^\star} = 0.$

A sequence satisfying *ii)* and *iii)* is called a Palais-Smale sequence. Note that the usual mountain-pass theorem corresponds to the case $M = X$.

In order to apply Theorem 5.1, we extend E to $H^1(\mathbf{R})$. Fix $d \in (-1, -1 + \frac{v}{\sqrt{2}})$ and consider a function $\tilde{f} : \mathbf{R} \longrightarrow \mathbf{R}$ such that $\tilde{f} \in C^1(\mathbf{R})$, $\tilde{f} \equiv f$ on $[d, \infty)$ and $\tilde{f}$ is bounded on $(-\infty, d]$. Define $\tilde{E} : H^1(\mathbf{R}) \longrightarrow \mathbf{R}$ by

$$\tilde{E}(u) = \int_{\mathbf{R}} |u'|^2 + \tilde{f}^2(u) dx + H(u).$$

Then $\tilde{E}$ is a C^1 functional on $H^1(\mathbf{R})$ and $\tilde{E} \equiv E$ on a neighbourhood of $V_\star = \{u \in H^1(\mathbf{R}) \mid \inf_{s \in \mathbf{R}} u(s) > -1 + \frac{v}{\sqrt{2}}\}$. We are going to find a critical point $r_1 \in V_\star$ of $\tilde{E}$. Clearly r_1 will be also a critical point of E.

Set

$$w(s) = \begin{cases} r_{-1+\frac{v}{\sqrt{2}}}(s - x) & \text{if} \quad s < x \\ -1 + \frac{v}{\sqrt{2}} & \text{if} \quad x \leq s \leq y \\ r_{-1+\frac{v}{\sqrt{2}}}(s - y) & \text{if} \quad s > y \end{cases} \tag{5.1}$$

so that $w \in H^1(\mathbf{R})$, $\inf_{s \in \mathbf{R}} w(s) = -1 + \frac{v}{\sqrt{2}}$ and $E(w) = h(-1 + \frac{v}{\sqrt{2}})$. Let

$$\Gamma_{r,w} = \{\gamma \in C^0([0,1], H^1(\mathbf{R})) \mid \gamma(0) = r, \gamma(1) = w\}$$

where r is a minimizer of E on V_0 (as in Section 4), and $c = \inf_{\gamma \in \Gamma_{r,w}} (\max_{s \in [0,1]} \tilde{E}(\gamma(s)))$.

We study first the convexity of f^2 on $[-1 + \frac{v}{\sqrt{2}}, 0]$. One has

$$(f^2)''(x) = 2\Big(3(x+1)^2 - 1 - \frac{v^2}{4} - \frac{3v^2}{4} \frac{1}{(x+1)^4}\Big).$$

So $(f^2)''$ is strictly increasing on $[-1 + \frac{v}{\sqrt{2}}, 0]$, $(f^2)''(-1 + \frac{v}{\sqrt{2}}) = (\frac{5}{2} + \frac{3}{v^2})(v^2 - 2) < 0$, $(f^2)''(0) = 2(2 - v^2) > 0$ and f^2 is concave on $[-1 + \frac{v}{\sqrt{2}}, -1 + \sqrt{\alpha(v)}]$ and convex on $[-1 + \sqrt{\alpha(v)}, 0]$, where $\alpha(v)$ is the unique root of the equation

$3y^3 - (1+\frac{v^2}{4})y^2 - \frac{3v^2}{4} = 0$ in the interval $[\frac{v^2}{2}, 1]$. It is also easily seen that there exists $\beta(v) \in (-1+\frac{v}{\sqrt{2}}, -1+\sqrt{\alpha(v)})$ such that $(f^2)'$ is positive (and decreasing) on $(-1+\frac{v}{\sqrt{2}}, \beta(v))$ (hence f^2 is concave, increasing and positive on $(-1+\frac{v}{\sqrt{2}}; \beta(v))$) and $(f^2)'$ is negative on $(\beta(v), 0)$. In other words, $\beta(v)$ is the maximum point of f^2 on $(-1, 0]$.

Next, we introduce the following supplementary hypothesis:

H1 $-1+\sqrt{\alpha(v)} \le a_0$ (recall that $a_0 = \sup\{a \in [-1+\frac{v}{\sqrt{2}}, 0) \mid h(a) \ge 0\}$).

H2 There exists $\varepsilon > 0$ such that for any interval $I \subset [x, y]$ we have $\int_I U(x)dx \ge \varepsilon|I|$, where $|I|$ is the length of I.

Proposition 5.2 *Assume that* **H1** *and* **H2** *are satisfied. For $\delta > 0$, δ small, consider the closed subset of $H^1(\mathbf{R})$*

$$M_\delta = \{u \in H^1(\mathbf{R}) \mid -1 + \frac{v}{\sqrt{2}} + \delta \le \inf_{s\in\mathbf{R}} u(s) \le a_0\}.$$

Then there exists $\delta > 0$ such that M_δ^c separates r and w.

Remark 5.3 We assume that **H1** holds only for technical reasons (the convexity of f^2 in a neighbourhood of $[\inf_{s\in\mathbf{R}} r(s), 0]$ is used in proofs). Using only the assumptions of Theorem 4.1, hypothesis **H2** and the fact that U has compact support, the proofs given below still work and it can be deduced, for example, that the set $\{u \in H^1(\mathbf{R}) \mid -1+\frac{v}{\sqrt{2}}+\delta \le \inf_{s\in\mathbf{R}} u(s) \le -\eta\} \cap \{u \in H^1(\mathbf{R}) \mid \tilde{E}(u) \ge c'\}$ separates 0 and w, where $c' = \inf_{\gamma\in\Gamma_{0,w}} (\max_{s\in[0,1]} \tilde{E}(\gamma(s)))$. We still get a critical point of E. However, we are not able to prove that this critical point is different from r.

In view of Lemma 2.3, *i*), a sufficient (but not necessary) condition for **H1** to be satisfied is that $4F(-1+\sqrt{\alpha(v)}) + (\alpha(v)-1)||U|| \ge 0$. Therefore, if U has compact support and $||U|| \le \varphi_2(v)$, where $\varphi_2(v) = \frac{4F(-1+\sqrt{\alpha(v)})}{1-\alpha(v)}$, then (1.9) has a second solution r_1. Moreover, it will be seen that $\inf_{s\in\mathbf{R}} r_1(s) < \inf_{s\in\mathbf{R}} r(s)$. Note that φ_2 is continuous and positive on $[0, \sqrt{2})$ and $\varphi_2(0) = 2\sqrt{2} - \frac{8}{9}\sqrt{6}$.

The proof of Proposition 5.2 is based on the following three lemmas:

Lemma 5.4 *Let $u \in H^1(\mathbf{R})$ be such that $a = \inf_{s\in\mathbf{R}} u(s) \ge -1+\frac{v}{\sqrt{2}}$. There exists a continuous path $\psi : [0,1] \longrightarrow H^1(\mathbf{R})$ with the following properties:*

i) $\psi(0) = u$;

ii) $\inf_{s\in\mathbf{R}} \psi(t)(s) \ge a$ *and* $E(\psi(t)) \le E(u)$, $\forall t \in [0,1]$;

iii) there exists $z \in [x, y]$ such that $\psi(1)(z) = a$ and $\psi(1)(s) \le r_a(s-z)$ for all $s \in \mathbf{R}$.

Proof. For $t \in [0,1]$ set $v_t = -u^- + tu^+$, where u^+ and u^- are the positive, respectively the negative part of u. Clearly the map $t \longmapsto v_t$ is continuous from

$[0,1]$ to $H^1(\mathbf{R})$, $v_1 = u$ and $v_0 = -u^-$, $a \le v_0 \le 0$. Since the functions $s \longmapsto f^2(s)$ and $s \longmapsto s(s+2)$ are increasing on $[0,\infty)$, we have $E(v_t) \le E(u)$, $\forall t \in [0,1]$.

For $t \in [0,\infty)$ define

$$u_t(s) = \begin{cases} r_{v_0(x-t)}(s-x+t) & \text{if} \quad s < x-t \\ v_0(s) & \text{if} \quad x-t \le s \le y+t \\ r_{v_0(y+t)}(s-y-t) & \text{if} \quad s > y+t \end{cases} \tag{5.2}$$

It is easy to check that $t \longmapsto u_t$ is a continuous map from $[0,\infty)$ to $H^1(\mathbf{R})$, $a \le u_t(s) \le 0$, $\forall s \in \mathbf{R}, \forall t \in [0,\infty)$ and $u_t \longrightarrow v_0$ in $H^1(\mathbf{R})$ as $t \longrightarrow \infty$.

By Lemma 2.2 *vii)* and Remark 2.1, we have for all $t \in [0,\infty)$

$$\begin{aligned} E(v_0) &= \Big(\int_{-\infty}^{x-t} + \int_{x-t}^{y+t} + \int_{y+t}^{\infty}\Big)(|v_0'|^2(s) + f^2(v_0(s)))ds \\ &\quad + \int_x^y v_0(v_0+2)U(s)ds \\ &\ge 2F(v_0(x-t)) + \int_{x-t}^{y+t} |v_0'|^2(s) + f^2(v_0(s))ds \\ &\quad + 2F(v_0(y+t)) + \int_x^y v_0(v_0+2)U(s)ds = E(u_t). \end{aligned} \tag{5.3}$$

Since u_0 is decreasing on $(-\infty, x]$ and increasing on $[y,\infty)$, there exists $z \in [x,y]$ such that $u_0(z) = b = \inf_{s\in\mathbf{R}} u_0(s)$. Clearly $b \ge a$.

If $b > a$, there exists $z_1 \in \mathbf{R} \setminus [x,y]$ such that $v_0(z_1) = a$. Suppose that $z_1 < x$. Using Remark 2.1 we have

$$\begin{aligned} E(v_0) - E(u_0) &= \Big(\int_{-\infty}^{x} + \int_{y}^{\infty}\Big)(|v_0'|^2 + f^2(v_0))ds \\ &\quad - \Big(\int_{-\infty}^{x} + \int_{y}^{\infty}\Big)(|u_0'|^2 + f^2(u_0))ds \\ &= \Big(\int_{-\infty}^{z_1} + \int_{z_1}^{x} + \int_{y}^{\infty}\Big)(|v_0'|^2 + f^2(v_0))ds - 2F(v_0(x)) - 2F(v_0(y)) \\ &\ge 2F(v_0(z_1)) + 2(F(v_0(z_1)) - F(v_0(x))) + 2F(v_0(y)) \\ &\qquad -2F(v_0(x)) - 2F(v_0(y)) \\ &= 4F(a) - 4F(v_0(x)) \ge 4F(a) - 4F(b). \end{aligned} \tag{5.4}$$

Obviously the same is true if $z_1 > y$.

For $t \in [a,0]$ set $u_t(s) = \min(u_0(s), r_t(s-z))$ (note that this definition is not ambiguous for $t=0$). Since the mapping $t \longmapsto r_t(\cdot - z)$ is continuous from $[a,0]$ to $H^1(\mathbf{R})$, we infer that the mapping $t \longmapsto u_t$ is also continuous.

Let us show that $E(u_t) \le E(v_0)$, $\forall t \in [a,0]$. Fix t. Since $a \le u_t \le u_0 \le 0$ we have

$$\int_x^y u_t(u_t+2)U(s)ds \le \int_x^y u_0(u_0+2)U(s)ds = \int_x^y v_0(v_0+2)U(s)ds. \tag{5.5}$$

The set $O_t = \{s \in \mathbf{R} \mid u_0(s) > r_t(s-z)\}$ is open, hence there exists a family at most countable of disjoint open intervals $((x_i,y_i))_{i\in I}$ such that $O_t = \cup_{i\in I}(x_i,y_i)$. For each $i \in I$ we have

- either $x_i = -\infty$ or $u_0(x_i) = r_t(x_i - z)$
- either $y_i = \infty$ or $u_0(y_i) = r_t(y_i - z)$.

Then

$$E(u_t) - E(u_0) \le \sum_{i\in I} \Big(\int_{x_i}^{y_i} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \tag{5.6}$$
$$- \int_{x_i}^{y_i} |u_0'(s)|^2 + f^2(u_0(s))ds \Big).$$

If $(x_i, y_i) \subset (-\infty, z)$ or $(x_i, y_i) \subset (z, \infty)$ then

$$\int_{x_i}^{y_i} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \le \int_{x_i}^{y_i} |u_0'(s)|^2 + f^2(u_0(s))ds \tag{5.7}$$

by Lemma 2.2, part *vii*). Note that if $t \ge b$, we have $(x_i, y_i) \subset (-\infty, z)$ or $(x_i, y_i) \subset (z, \infty)$ for all $i \in I$. If $t < b$, there exists exactly one $i_0 \in I$ such that $z \in (x_0, y_0)$ and $(x_i, y_i) \subset ((-\infty, z) \cup (z, \infty))$ for all other $i \in I$. For i_0 we have

$$\begin{aligned}
&\int_{x_{i_0}}^{y_{i_0}} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \\
&= \Big(\int_{x_{i_0}}^{z} + \int_{z}^{y_{i_0}} \Big) |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \\
&= (2F(t) - 2F(u_0(x_{i_0}))) + (2F(t) - 2F(u_0(y_{i_0})))
\end{aligned}$$

and by Remark 2.1,

$$\int_{x_{i_0}}^{y_{i_0}} |u_0'(s)|^2 + f^2(u_0(s))ds \ge 2|F(u_0(y_{i_0})) - F(u_0(x_{i_0}))|.$$

Therefore

$$\begin{aligned}
&\int_{x_{i_0}}^{y_{i_0}} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds - \int_{x_{i_0}}^{y_{i_0}} |u_0'(s)|^2 + f^2(u_0(s))ds \\
&\le 4F(t) - 4\max(F(u_0(x_{i_0})), F(u_0(y_{i_0}))) \le 4F(t) - 4F(b).
\end{aligned} \tag{5.8}$$

From (5.6), (5.7), (5.8) and (5.4) we infer that $E(u_t) - E(u_0) \le 4F(t) - 4F(b) \le 4F(a) - 4F(b) \le E(v_0) - E(u_0)$. Hence $E(u_t) \le E(v_0) \le E(u)$ for all $t \in [a, 0]$.

Finally, define $\psi : [0,1] \longrightarrow H^1(\mathbf{R})$ by

$$\psi(t) = \begin{cases} v_{1-2t} & \text{if} \quad t \in [0, \frac{1}{2}] \\ u_{a\frac{(4t-3)}{2t-1}} & \text{if} \quad t \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to check that ψ is continuous and satisfies Lemma 5.4. □

Lemma 5.5 *Suppose that the hypothesis* **H1** *and* **H2** *are satisfied. There exists* $\delta > 0$ *(depending on* ε*) such that for each* $u \in H^1(\mathbf{R})$ *verifying*

$$b = \inf_{s\in\mathbf{R}} u(s) \in [-1 + \frac{v}{\sqrt{2}}, -1 + \frac{v}{\sqrt{2}} + \delta]$$

there exists a continuous path $\lambda : [0,1] \longrightarrow \overline{V_\star}$ *such that*

i) $\lambda(0) = u$*;*

ii) $E(\lambda(t)) \le E(u)$, $\forall u \in [0,1]$;

iii) $\lambda(1)(s) \le 0$, $\forall s \in \mathbf{R}$ *and there exists* $z \in [x,y]$ *such that* $\lambda(1)(z) = -1 + \frac{v}{\sqrt{2}}$.

Proof. Fix $a \in (-1+\frac{v}{\sqrt{2}}, \beta(v))$ sufficiently close to $-1+\frac{v}{\sqrt{2}}$ so that $c(a) > x-y$. (The value of a will be chosen later). Recall that $\beta(v)$ is the maximum point of f^2 on $(-1,0]$ and f^2 is concave and increasing on $[-1+\frac{v}{\sqrt{2}}, \beta(v)]$.

Let $u \in H^1(\mathbf{R})$ be such that $b = \inf_{s\in\mathbf{R}} u(s) \in [-1+\frac{v}{\sqrt{2}}, a)$.

Consider the path ψ given by Lemma 5.4 and denote $u_1 = \psi(1)$. There exists $z \in [x,y]$ such that $u_1(z) = b$ and $u_1(s) \le r_b(s-z)$, $\forall s \in \mathbf{R}$.

Let

$$x_a = \inf\{t < z \mid u_1(s) < a \text{ on } (t,z]\},$$
$$y_a = \sup\{t > z \mid u_1(s) < a \text{ on } [z,t)\}.$$

Clearly $u_1(x_a) = u_1(y_a) = a$. Since $u_1(s) \le r_b(s-z)$ we have $x_a \le z - (c(b)-c(a))$ and $y_a \ge z + (c(b)-c(a))$. For $t \in [a,0]$ define

$$\lambda_1(t)(s) = \begin{cases} \min(u_1(s), r_t(s-x_a)) & \text{if} \quad s \in (-\infty, x_a] \\ u_1(s) & \text{if} \quad s \in (x_a, y_a) \\ \min(u_1(s), r_t(s-y_a) & \text{if} \quad s \in [y_a, \infty). \end{cases}$$

Then λ_1 is continuous from $[a,0]$ to $H^1(\mathbf{R})$, $b \le \lambda_1(t)(s) \le 0$ for all t, s and $\lambda_1(0) = u_1$. As in the proof of Lemma 5.4 one shows that $E(\lambda_1(t)) \le E(u_1)$, $\forall t \in [a,0]$. Denote $u_2 = \lambda_1(a)$. We have $u_2(s) \le r_a(s-x_a)$ on $(-\infty, x_a]$, $u_2(s) = u_1(s)$ on (x_a, y_a) and $u_2(s) \le r_a(s-y_a)$ on $[y_a, \infty)$.

For $t \in [0, b+1-\frac{v}{\sqrt{2}}]$ define

$$\lambda_2(t)(s) = \begin{cases} \min(u_2(s), r_{a-t}(s-x_a)) & \text{if} \quad s \in (-\infty, x_a] \\ u_1(s) - t & \text{if} \quad s \in (x_a, y_a) \\ \min(u_2(s), r_{a-t}(s-y_a) & \text{if} \quad s \in [y_a, \infty). \end{cases}$$

One easily checks that the map $t \longmapsto \lambda_2(t)$ is continuous for the norm of $H^1(\mathbf{R})$. As in the proof of Lemma 5.4 we obtain

$$\begin{aligned} &\int_{\mathbf{R}\setminus[x_a,y_a]} |\lambda_2(t)'(s)|^2 + f^2(\lambda_2(t)(s))ds \\ &\le \int_{\mathbf{R}\setminus[x_a,y_a]} |u_2'(s)|^2 + f^2(u_2(s))ds + 4F(a-t) - 4F(a). \end{aligned}$$

We have

$$\int_{\mathbf{R}\setminus[x_a,y_a]} \lambda_2(t)(\lambda_2(t)+2)U(s)ds - \int_{\mathbf{R}\setminus[x_a,y_a]} u_2(u_2+2)U(s)ds \le 0$$

because $-1+\frac{v}{\sqrt{2}} \le \lambda_2(t)(s) \le u_2(s)$, $\forall s \in \mathbf{R}$. Obviously $\lambda_2(t)'(s) = u_1'(s)$ for $s \in (x_a, y_a)$. Therefore

$$\begin{aligned} &E(\lambda_2(t)) - E(u_2) \\ &\le 4F(a-t) + 4F(a) + \int_{x_a}^{y_a} f^2(u_1(s)-t) - f^2(u_1(s))ds \\ &\quad + \int_{x_a}^{y_a} \Big(-2t(u_1(s)+1) + t^2 \Big) U(s)ds. \end{aligned} \tag{5.9}$$

We have $f^2(u_1(s)-t)-f^2(u_1(s))\le -2tff'(u_1(s))\le -2tff'(a)$ for $s\in[x_a,y_a]$ by the concavity of f^2. Since $u_1(s)+1\ge b+1\ge \frac{v}{\sqrt{2}}$ we obtain

$$
\begin{aligned}
&E(\lambda_2(t))-E(u_2)\\
&\le 4F(a-t)+4F(a)-2tff'(a)(y_a-x_a)+(t^2-\sqrt{2}vt)\int_{x_a}^{y_a}U(s)ds.
\end{aligned}
\tag{5.10}
$$

Using the fact that $y_a-z\ge c(b)-c(a)$, $z-x_a\ge c(b)-c(a)$, $z\in[x,y]$, $-c(a)\le y-x$ and hypothesis **H2** (note that this is the only point in the proof of Proposition 5.2 where this hypothesis is needed), we get $\int_{x_a}^{y_a}U(s)ds\ge \varepsilon|[x,y]\cap[x_a,y_a]|\ge \varepsilon(c(b)-c(a))$. Hence

$$
\begin{aligned}
&E(\lambda_2(t))-E(u_2)\le 4F(a-t)+4F(a)\\
&-4t(c(b)-c(a))ff'(a)+\varepsilon(t^2-\sqrt{2}vt)(c(b)-c(a)).
\end{aligned}
\tag{5.11}
$$

Recall that f is negative and decreasing on $[-1+\frac{v}{\sqrt{2}},a]$, so $F(a-t)-F(a)\le -tf(a)$. For $t\le\frac{\sqrt{2}v}{2}$ we have

$$
\begin{aligned}
&E(\lambda_2(t))-E(u_2)\\
&\le -4tf(a)-4t(c(b)-c(a))ff'(a)-\tfrac{\varepsilon\sqrt{2}v}{2}t(c(b)-c(a))\\
&=\Big[\Big(-4f(a)+4ff'(a)c(a)+\tfrac{\varepsilon\sqrt{2}v}{2}\cdot c(a)\Big)-\Big(4ff'(a)+\tfrac{\varepsilon\sqrt{2}v}{2}\Big)c(b)\Big]t.
\end{aligned}
\tag{5.12}
$$

By a straightforward computation one has

$$
\begin{aligned}
\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}}\frac{f(a)}{\sqrt{2(a+1)^2-v^2}}&=\frac{v^2-2}{2\sqrt{2}v};\\
\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}}\frac{c(a)}{\sqrt{2(a+1)^2-v^2}}&=-\frac{2}{2-v^2};\\
\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}}ff'(a)&=\frac{1}{v\sqrt{2}}(\frac{v^2}{2}-1)^2.
\end{aligned}
$$

Consequently, we find that

$$
\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}}\frac{-4f(a)+4ff'(a)c(a)+\frac{\varepsilon\sqrt{2}v}{2}\cdot c(a)}{\sqrt{2(a+1)^2-v^2}}=-\frac{\sqrt{2}v\varepsilon}{2-v^2}<0.
\tag{5.13}
$$

Hence $-4f(a)+4ff'(a)c(a)+\frac{\varepsilon\sqrt{2}v}{2}c(a)<0$ if a is "sufficiently close" to $-1+\frac{v}{\sqrt{2}}$.

Now choose $a\in(-1+\frac{v}{\sqrt{2}},\beta(v))$ such that $-c(a)\le y-x$ and $-4f(a')+4ff'(a')c(a')+\frac{\varepsilon\sqrt{2}v}{2}\cdot c(a')<0$ for all $a'\in[-1+\frac{v}{\sqrt{2}},a]$. In view of (5.13), this is possible.

Next, choose $\delta\in(0,\frac{v\sqrt{2}}{2})$ such that

$$
-4f(a)+4ff'(a)c(a)+\frac{\varepsilon\sqrt{2}v}{2}c(a)-\Big(4ff'(a)+\frac{\varepsilon\sqrt{2}v}{2}\Big)c(b)<0
\tag{5.14}
$$

for all $b\in[-1+\frac{v}{\sqrt{2}},-1+\frac{v}{\sqrt{2}}+\delta]$. This is also possible because $\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}}c(b)=0$.

Let $u \in H^1(\mathbf{R})$ be such that $b = \inf_{s\in\mathbf{R}} u(s) \in [-1+\frac{v}{\sqrt{2}}, -1+\frac{v}{\sqrt{2}}+\delta]$. Let ψ be the path given by Lemma 5.4 and let $u_1 = \psi(1)$. Define λ_1 as before. It is clear that $t \longmapsto \lambda_1(-t)$, $t \in [0,-a]$ is a continuous path in $\overline{V_\star}$ joining u_1 and $u_2 = \lambda_1(a)$. Next, define λ_2 as previously for $t \in J = [0, b+1-\frac{v}{\sqrt{2}}]$. Then the estimates (5.9) - (5.11) hold. We see that $b+1-\frac{v}{\sqrt{2}} < \frac{v\sqrt{2}}{2}$, hence (5.12) is true for all $t \in J$. From (5.12) and (5.14) we infer that $E(\lambda_2(t)) \leq E(u_2) \leq E(u)$, $\forall t \in J$. Let $u_3 = \lambda_2(b+1-\frac{v}{\sqrt{2}})$. It is easy to see that $\inf_{s\in\mathbf{R}} u_3(s) = u_3(z) = -1+\frac{v}{\sqrt{2}}$ and λ_2 is a continuous path in $\overline{V_\star}$ joining u_2 and u_3. It suffices to add the paths ψ, $\lambda_1(-\cdot)$ and λ_2 to obtain a continuous path $\lambda : [0,1] \longrightarrow \overline{V_\star} = \{u \in H^1(\mathbf{R}) \mid \inf_{s\in\mathbf{R}} u(s) \geq -1+\frac{v}{\sqrt{2}}\}$ such that $\lambda(0) = u$, $\lambda(1) = u_3$ and $E(\lambda(t)) \leq E(u)$, $\forall t \in [0,1]$. This proves Lemma 5.5. $\square$

Lemma 5.6 *Let $u \in H^1(\mathbf{R})$ be such that $-1+\frac{v}{\sqrt{2}} \leq u \leq 0$ and there exists $z \in [x,y]$ such that $u(z) = -1+\frac{v}{\sqrt{2}}$. Then there exists a continuous path $\mu : [0,1] \longrightarrow \overline{V_\star}$ satisfying:*

i) $\mu(0) = u$, $\mu(1) = w$, where w is given by (5.1);

ii) $\mu(t)(z) = -1+\frac{v}{\sqrt{2}}$, $\forall t \in [0,1]$;

iii) $E(\mu(t)) \leq E(u)$ for all $t \in [0,1]$.

Proof. Let $v(s) = \min(u(s), r_{-1+\frac{v}{\sqrt{2}}}(s-z))$. For $t \in [0, 1-\frac{v}{\sqrt{2}}]$ define $\mu_1(t)(s) = \min(u(s), r_{-t}(s-z))$. Then μ_1 is a continuous path joining u and v and one shows as previously that $E(\mu_1(t)) \leq E(u)$ for all t.

For $k \in [0,\infty)$ set $\mu_2^\star(k)(s) = \min(v(s), u_{-1+\frac{v}{\sqrt{2}},k}(s-z))$, where $u_{-1+\frac{v}{\sqrt{2}},k}$ was defined in (2.8). Then $\mu_2^\star$ is continuous from $[0,\infty)$ to $H^1(\mathbf{R})$ (because $k \longmapsto u_{-1+\frac{v}{\sqrt{2}},k}$ is continuous) and $\mu_2(0) = v$. As in the previous lemmas one proves that $E(\mu_2^\star(k)) \leq E(v)$, $\forall k \in [0,\infty)$. Since $v(s) \longrightarrow 0$ as $s \longrightarrow \infty$, there exists $k_0 > 0$ such that $supp(U) \subset [z-k_0, z+k_0]$ and $-1+\sqrt{\alpha(v)} < v(s) \leq 0$ for all $s \in \mathbf{R} \setminus [z-k_0, z+k_0]$. Let $v_1 = \mu_2^\star(k_0)$. Then $v_1(s) = u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z)$ if $s \in I_1 = [z-k_0+c(-1+\sqrt{\alpha(v)}), z+k_0-c(-1+\sqrt{\alpha(v)})]$ and $-1+\sqrt{\alpha(v)} < v_1(s) \leq 0$ for $s \in \mathbf{R}\setminus I_1$. Denote by μ_2 the restriction of $\mu_2^\star$ to $[0,k_0]$, so that μ_2 is a continuous path and it joins v and v_1.

Set $\mu_3(t) = (1-t)v_1 + t u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)$, $t \in [0,1]$. Obviously μ_3 is continuous and $\mu_3(t) \equiv u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)$ on I_1, for all $t \in [0,1]$.

Since $v_1(s), u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z) \in (-1+\sqrt{\alpha(v)}, 0]$ if $s \in \mathbf{R}\setminus I_1$, by the convexity of f^2 on $(-1+\sqrt{\alpha(v)}, 0]$ we get $E(\mu_3(t)) \leq (1-t)E(v_1) + tE(u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)) \leq E(u)$, for all $t \in [0,1]$ (note that $E(u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)) = h(-1+\frac{v}{\sqrt{2}}) \leq E(u)$).

For $t \in [z-k_0, x]$ set

$$\mu_4(t)(s) = \begin{cases} r_{-1+\frac{v}{\sqrt{2}}}(s-t) & \text{if} \quad s < t \\ u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z) & \text{if} \quad s \geq t. \end{cases}$$

Denote $\mu_4(x)$ by v_2. Clearly μ_4 is a continuous path joining $u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)$ and v_2.

Finally, for $t \in [0, z + k_0 - y]$ let

$$\mu_5(t)(s) = \begin{cases} v_2(s) & \text{if} \quad s \le z + k_0 - t \\ r_{-1+\frac{v}{\sqrt{2}}}(s-t) & \text{if} \quad s > z + k_0 - t. \end{cases}$$

Then μ_5 is a continuous path joining v_2 and w.

Adding the paths μ_i, $1 \le i \le 5$, we obtain a continuous path $\mu : [0,1] \longrightarrow \overline{V_\star}$ satisfying Lemma 5.6. □

Proof of Proposition 5.2. For a given path $\gamma \in \Gamma_{r,w}$, denote $l(t) = \inf_{s \in \mathbf{R}} \gamma(t)(s)$. The function l is continuous, $l(0) = \inf_{s\in\mathbf{R}} r(s) > a_0$ (as seen in Section 4) and $l(1) = -1 + \frac{v}{\sqrt{2}}$. If $l(t) \in [-1 + \frac{v}{\sqrt{2}}, 0]$ we necessarily have $\tilde{E}(\gamma(t)) = E(\gamma(t)) \ge h(l(t))$, therefore

$$\max_{t\in[0,1]} \tilde{E}(\gamma(t)) \ge \max_{a \in [-1+\frac{v}{\sqrt{2}}, a_0]} h(a).$$

Consequently, we have $c \ge \max_{a \in [-1+\frac{v}{\sqrt{2}}, a_0]} h(a)$. In particular, $c > E(r)$ and using Lemma 2.3 we infer that $c > E(w) = h(-1 + \frac{v}{\sqrt{2}})$.

Fix $\delta = \delta(\varepsilon)$ as given by Lemma 5.5. We show that Proposition 5.2 holds for this choice of δ.

We reason by contradiction. Suppose that M_δ^c does not separate r and w, i.e. there exists a continuous path $\gamma : [0,1] \longrightarrow (H^1(\mathbf{R}) \setminus M_\delta) \cup \{u \in M_\delta \mid \tilde{E}(u) < c\}$ such that $\gamma(0) = r$ and $\gamma(1) = w$. As before, set $l(t) = \inf_{s\in\mathbf{R}} \gamma(t)(s)$. Let

$$t_0 = \sup\{t \in [0,1] \mid l(t) = a_0\} \text{ and}$$
$$t_1 = \inf\{t \in [t_0, 1] \mid l(t) = -1 + \tfrac{v}{\sqrt{2}} + \delta\}.$$

Then $0 < t_0 < t_1 < 1$ and for $t \in [t_0, t_1]$ we have $-1 + \frac{v}{\sqrt{2}} + \delta \le l(t) \le a_0$, hence $\gamma(t) \in M_\delta$. By our assumption, $E(\gamma(t)) < c$ for all $t \in [t_0, t_1]$. Let $u_0 = \gamma(t_0)$, $u_1 = \gamma(t_1)$.

Using the convexity of f^2 on $[a_0, \infty)$ we have

$$\tilde{E}((1-t)r + tu_0) = E((1-t)r + tu_0) \le (1-t)E(r) + tE(u_0) < c, \quad \forall t \in [0,1].$$

Define $\gamma_1 : [0,1] \longrightarrow H^1(\mathbf{R})$, $\gamma_1(t) = (1-t)r + tu_0$.

We have $\inf_{s\in\mathbf{R}} u_1(s) = -1 + \frac{v}{\sqrt{2}} + \delta$. Therefore Lemma 5.5 can be applied for u_1 and gives us a path $\lambda : [0,1] \longrightarrow \overline{V_\star}$ such that $\inf_{s\in\mathbf{R}} \lambda(1)(s) = \min_{s\in[x,y]} \lambda(1)(s) = -1 + \frac{v}{\sqrt{2}}$ and $\lambda(1) \le 0$. Next, apply Lemma 5.6 to $\lambda(1)$ in order to obtain a continuous path μ joining $\lambda(1)$ and w. Adding the paths λ and μ we obtain a continuous path $\gamma_2 : [0,1] \longrightarrow \overline{V_\star}$ such that $\gamma_2(0) = u_1$, $\gamma_2(1) = w$ and $E(\gamma_2(t)) \le E(u_1)$, $\forall t \in [0,1]$.

We define a new path in the following way: we start from r and go to u_0 along the path γ_1; then we go from u_0 to u_1 along the path $t \longmapsto \gamma(t)$, $t \in [t_0, t_1]$; finally we go from u_1 to w along the path γ_2. It suffices to make the corresponding changes of parameter to obtain a continuous path $\gamma_\star \in \Gamma_{r,w}$. Since $\max_{t\in[0,1]} \tilde{E}(\gamma_1(t)) = E(u_0)$ and $\max_{t\in[0,1]} \tilde{E}(\gamma_2(t)) = E(u_1)$, we have

$$\max_{t\in[0,1]} \tilde{E}(\gamma_\star(t)) = \max_{t\in[t_0,t_1]} E(\gamma(t)) < c,$$

which contradicts the definition of c. This proves Proposition 5.2. □

Proposition 5.7 *Assume that the hypothesis* **H1** *and* **H2** *are satisfied. There exists a solution* r_1 *of equation (1.9) and* $z \in [x, y]$ *such that* $\inf_{s\in\mathbf{R}} r_1(s) = r_1(z) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$, *where* $\delta(\varepsilon)$ *is given by Lemma 5.5. Moreover, we have* $E(r_1) \leq c$.

Proof. From Proposition 5.2 and Theorem 5.1 it follows that there exists a sequence $(u_n) \in H^1(\mathbf{R})$ such that

$$\lim_{n\longrightarrow\infty} dist(u_n, M_{\delta(\varepsilon)}) = 0; \tag{5.15}$$

$$\lim_{n\longrightarrow\infty} \tilde{E}(u_n) = c; \tag{5.16}$$

$$\lim_{n\longrightarrow\infty} ||\tilde{E}'(u_n)||_{H^{-1}(\mathbf{R})} = 0. \tag{5.17}$$

Using (5.15) we may suppose that $\inf_{s\in\mathbf{R}} u_n(s) > -1 + \frac{v}{\sqrt{2}} + \frac{1}{2}\delta(\varepsilon)$, $\forall n \in \mathbf{N}$ and so $\tilde{E}(u_n) = E(u_n)$ and $\tilde{E}'(u_n) = E'(u_n)$. Since there exists a constant $C > 0$ such that $f^2(x) \geq Cx^2$ if $x \in [-1 + \frac{v}{\sqrt{2}} + \frac{1}{2}\delta(\varepsilon), \infty)$, (5.16) implies that the sequence (u_n) is bounded in $H^1(\mathbf{R})$.

Let $a_n = \inf_{s\in\mathbf{R}} u_n(s)$. For each n, fix a point $z_n \in \mathbf{R}$ such that $u_n(z_n) = a_n$.

The sequence $u_n(\cdot - z_n)$ is bounded in $H^1(\mathbf{R})$. Passing to a subsequence if necessary, we may suppose that there exists $u \in H^1(\mathbf{R})$ such that

$$u_n(\cdot - z_n) \rightharpoonup u \text{ weakly in } H^1(\mathbf{R}). \tag{5.18}$$

Using Arzela - Ascoli's Theorem and passing again to a subsequence, we may suppose that

$$u_n(\cdot - z_n) \longrightarrow u \text{ uniformly on each compact } K \subset \mathbf{R}. \tag{5.19}$$

It is clear that $\inf_{s\in\mathbf{R}} u(s) = u(0) \in [-1 + \dfrac{v}{\sqrt{2}} + \delta(\varepsilon), a_0]$.

Let $\phi \in \mathcal{S}(\mathbf{R})$. By (5.17), we have

$$E'(u_n)\phi(\cdot + z_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{5.20}$$

But

$$\int_{\mathbf{R}} u_n'(s)\phi'(s + z_n)ds = \int_{\mathbf{R}} u_n'(t - z_n)\phi'(t)dt \longrightarrow \int_{\mathbf{R}} u'(t)\phi'(t)dt$$

by (5.18) and

$$\int_{\mathbf{R}} ff'(u_n(s))\phi(s + z_n)ds \longrightarrow \int_{\mathbf{R}} ff'(u(t))\phi(t)dt \text{ as } n \longrightarrow \infty$$

by (5.19) and Lebesgue's dominated convergence theorem.

If there exists a subsequence (z_{n_k}) which tends to $+\infty$ or to $-\infty$ as $k \longrightarrow \infty$, we would have $\int_{\mathbf{R}} (u_{n_k}(s)+1)\phi(s+z_{n_k})U(s)ds \longrightarrow 0$ as $k \longrightarrow \infty$ by the dominated convergence theorem. From (5.20) we obtain

$$\int_{\mathbf{R}} u'(s)\phi'(s)ds + \int_{\mathbf{R}} ff'(u(s))\phi(s)ds = 0, \quad \forall \phi \in \mathcal{S}(\mathbf{R}),$$

that is u satisfies (1.9) (in the distributional sense) for $U \equiv 0$. On the other hand we have $\inf_{s\in\mathbf{R}} u(s) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$. We have seen in Section 3 that this is impossible. Therefore the sequence (z_n) is bounded.

Passing again to a subsequence, we may suppose that $\lim_{n\longrightarrow\infty} z_n = z \in \mathbf{R}$. By (5.19), $u_n(s) \longrightarrow u(s+z)$, $\forall s \in \mathbf{R}$ and

$$\int_{\mathbf{R}} (u_n(s)+1)\phi(s+z_n)U(s)ds \longrightarrow \int_{\mathbf{R}} (u(t)+1)\phi(t)U(t-z)dt.$$

From (5.20) we obtain for all $\phi \in \mathcal{S}(\mathbf{R})$

$$\int_{\mathbf{R}} u'(s)\phi'(s)ds + \int_{\mathbf{R}} ff'(u(s))\phi(s)ds + \int_{\mathbf{R}} (u(s)+1)\phi(s)U(s-z)ds = 0.$$

Therefore u satisfies the equation

$$-u''(s) + ff'(u(s)) + (u(s)+1)U(s-z) = 0$$

or equivalently, $r_1 = u(\cdot+z)$ satisfies (1.9). Furthermore, r_1 achieves its minimum at z and $r_1(z) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$. From the discussion in Introduction, it follows that $r_1 \le 0$ and r_1 satisfies (1.12) on $(-\infty, x)\cup(y,\infty)$. Let $a = r_1(x)$ and $b = r_1(y)$. By a standard argument we infer that $r_1 \equiv r_a(\cdot - x)$ on $(-\infty, x)$ and $r_1 \equiv r_b(\cdot - y)$ on (y, ∞) so that necessarily $z \in [x, y]$.

As in the proof of Theorem 4.1 one has

$$E(r_1) = E(u(\cdot+z)) \le \liminf_{n\rightarrow\infty} E(u_n) = c. \tag{5.21}$$

□

In fact, hypothesis **H2** is not necessary for the existence of a second solution of equation (1.9). It can be eliminated using Proposition 5.7 and a simple approximation procedure. This will be seen in the next theorem, which is the main result of this section.

Theorem 5.8 *Let U be a positive Borel measure with $supp(U) \subset [x,y]$. Suppose that $||U|| < \varphi_2(v)$, where φ_2 is the function introduced in Remark 5.3. Then equation (1.9) admits a solution r_1 with $\inf_{s\in\mathbf{R}} r_1(s) \in [-1+\frac{v}{\sqrt{2}}, a_0]$, where $a_0 = \sup\{a \in [-1+\frac{v}{\sqrt{2}}, 0) \mid h(a) \ge 0\}$. Furthermore, $E(r_1) \le c$.*

Proof. We have seen that if $||U|| < \varphi_2(v)$, then $h(-1+\sqrt{\alpha(v)}) > 0$.

For $\varepsilon > 0$ define $U_\varepsilon = U + \varepsilon\chi_{[x,y]}$. Denote by H_ε, E_ε, h_ε the corresponding quantities for the measure U_ε. It is easily seen that $h_\varepsilon(a) \le h(a)$, $\forall a \in [-1+\frac{v}{\sqrt{2}}, 0]$ and $h_\varepsilon(a) \longrightarrow h(a)$ as $\varepsilon \longrightarrow 0$, so $h_\varepsilon(-1+\sqrt{\alpha(v)}) > 0$ if ε is sufficiently small, say, if $\varepsilon \in (0, \varepsilon_0)$. For $\varepsilon \in (0,\varepsilon_0)$, define $a_{0,\varepsilon}$ as in Theorem 4.1. Then $a_{0,\varepsilon} \le a_0$ and E_ε

has a minimizer r_ε on the set $V_{0,\varepsilon} = \{u \in H^1(\mathbf{R}) \mid \inf_{s\in\mathbf{R}} u(s) > a_{0,\varepsilon}\}$. Define c_ε as before. It is obvious that $c_\varepsilon \le c$.

Applying Proposition 5.7 for the measure U_ε, we get a critical point $r_{1,\varepsilon}$ of E_ε and $z_\varepsilon \in [x,y]$ such that $\inf_{s\in\mathbf{R}} r_{1,\varepsilon}(z) = r_{1,\varepsilon}(z_\varepsilon) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$. Furthermore, we have $E_\varepsilon(r_{1,\varepsilon}) \le c_\varepsilon \le c$, which implies that

$$\int_{\mathbf{R}} |r'_{1,\varepsilon}|^2(s)ds \le E_\varepsilon(r_{1,\varepsilon}) - \int_{\mathbf{R}} r_{1,\varepsilon}(r_{1,\varepsilon}+2)U_\varepsilon(s)ds \le c + \Big(1-\frac{v^2}{2}\Big)||U_\varepsilon||.$$

Hence $\int_{\mathbf{R}} |r'_{1,\varepsilon}|^2(s)ds$ is uniformly bounded for $\varepsilon \in (0,\varepsilon_0)$. Let $a_\varepsilon = r_{1,\varepsilon}(x)$ and $b_\varepsilon = r_{1,\varepsilon}(y)$. We know that $r_{1,\varepsilon} = r_{a_\varepsilon}(\cdot - x)$ on $(-\infty, x)$ and $r_{1,\varepsilon} = r_{b_\varepsilon}(\cdot - y)$ on (y,∞). Since $-1+\frac{v}{\sqrt{2}} \le r_{1,\varepsilon}(s) \le 0$ for $s \in [x,y]$, we infer that $r_{1,\varepsilon}$ is uniformly bounded in $L^2(\mathbf{R})$, hence $r_{1,\varepsilon}$ is bounded in $H^1(\mathbf{R})$. Consequently, there exists a sequence $\varepsilon_n \longrightarrow 0$ and $r_1 \in H^1(\mathbf{R})$ such that $r_{1,\varepsilon_n} \rightharpoonup r_1$ weakly in $H^1(\mathbf{R})$ as $n \longrightarrow \infty$. Using Arzela - Ascoli's theorem, we may suppose that $r_{1,\varepsilon_n} \longrightarrow r_1$ uniformly on $[x,y]$. In fact, the particular form of r_{1,ε_n} implies that $r_{1,\varepsilon_n} \longrightarrow r_1$ uniformly on $\mathbf{R}$ and $r_1 = r_a(\cdot - x)$ on $(-\infty, x)$, respectively $r_1 = r_b(\cdot - y)$ on (y,∞), where $a = r_1(x)$ and $b = r_1(y)$. Clearly the minimum of r_1 on $\mathbf{R}$ is achieved at a point $z \in [x,y]$. By the uniform convergence, $r_1(z) \in [-1+\frac{v}{\sqrt{2}}, a_0]$.

For each test function ϕ one has

$$\int_{\mathbf{R}} r'_{1,\varepsilon_n}\phi' ds + \int_{\mathbf{R}} ff'(r_{1,\varepsilon_n})\phi ds + \int_{\mathbf{R}} (1+r_{1,\varepsilon_n})\phi U(s)ds + \varepsilon_n \int_x^y (1+r_{1,\varepsilon_n})\phi ds = 0.$$

Passing to the limit as $n \longrightarrow \infty$, we obtain that r_1 is a solution of (1.9).

The weak convergence of r_{1,ε_n} in $H^1(\mathbf{R})$ and the uniform convergence on $\mathbf{R}$ imply $E(r_1) \le \liminf_{n\longrightarrow\infty} E(r_{1,\varepsilon_n}) \le c$. □

Coming back to (1.8), we determine the corresponding phases θ and θ_1 for the solutions r, respectively r_1 of (1.9). If U has compact support, θ' and θ'_1 are integrable on $\mathbf{R}$ because of the particular form of r and r_1 outside *supp*(U). We impose that $\theta(x) \longrightarrow 0$, $\theta_1(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Then $\theta(x) \longrightarrow \mu$, $\theta_1(x) \longrightarrow \mu_1$ as $x \longrightarrow -\infty$ for some positive constants μ and μ_1. Thus we obtain two solutions A and A_1 of (1.5). Remark that A and A_1 tend exponentially to 1 at ∞ and to $e^{i\mu}$ (respectively to $e^{i\mu_1}$) at $-\infty$. Vortices are replaced in one dimension by a density depression around *supp*(U).

2.6 Application to a Gross-Pitaevskii-Schrödinger system

In this section we present an application of the discussion in sections 1 and 2 to the study of a system describing the motion of an uncharged impurity in a Bose condensate. In dimensionless variables, the system reads

$$(6.1)\qquad \begin{cases} 2i\dfrac{\partial \psi}{\partial t} &= -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \\ 2i\delta\dfrac{\partial \varphi}{\partial t} &= -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi. \end{cases}$$

Here ψ and φ are the wavefunctions for bosons, respectively for the impurity, $\delta = \frac{\mu}{M}$ where μ is the mass of impurity and M is the boson mass (δ is supposed to be small), $q^2 = \frac{l}{2d}$, l being the boson-impurity scattering length and d the boson diameter, k is a dimensionless measure for the single-particle impurity energy and ε is a dimensionless constant ($\varepsilon = (\frac{a\mu}{lM})^{\frac{1}{5}}$, where a is the "healing length"; in applications, $\varepsilon \doteq 0.2$). Assuming that we are in a frame in which the condensate is at rest at infinity, the solutions must satisfy the "boundary conditions"

$$(6.2)\qquad \psi \longrightarrow 1, \quad \varphi \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

This system (originally introduced by Clark and Gross) was studied by J. Grant and P. H. Roberts (see [22]). Using formal asymptotic expansions and numerical calculations, they computed the effective radius and the induced mass of the uncharged impurity.

We consider here the system (6.1) in one space dimension and we look for solitary waves, that is for solutions of the form

$$(6.3)\qquad \psi(x,t) = \tilde{\psi}(x-ct), \quad \varphi(x,t) = \tilde{\varphi}(x-ct).$$

This kind of solutions corresponds to the case where the only disturbance present in the condensate is that caused by the uniform motion of the impurity with velocity c. In view of the boundary conditions, we are looking for solutions of the form

$$(6.4)\qquad \tilde{\psi}(x) = (1+\tilde{r}(x))e^{i\psi_0(x)}, \quad \tilde{\varphi}(x) = \tilde{u}(x)e^{i\varphi_0(x)}$$

with $\tilde{r}(x) \longrightarrow 0$, $\tilde{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. By an easy computation we find that the real functions ψ_0, φ_0, $\tilde{r}$, $\tilde{u}$ must satisfy

$$(6.5)\qquad \psi_0' = c(1 - \frac{1}{(1+\tilde{r})^2}),$$

$$(6.6)\qquad \varphi_0' = c\delta,$$

$$(6.7)\qquad \tilde{r}'' = c^2\Big(\frac{1}{(1+\tilde{r})^3} - (1+\tilde{r})\Big) + \frac{1}{\varepsilon^2}\Big((1+\tilde{r})^3 - (1+\tilde{r}) + \frac{1}{\varepsilon^2}(1+\tilde{r})\tilde{u}^2\Big),$$

$$(6.8) \qquad \tilde{u}'' = \Big(\frac{q^2}{\varepsilon^2}(1+\tilde{r})^2 - c^2\delta^2 - k^2\Big)\tilde{u}.$$

From (6.6) we see that necessarily $\varphi_0(x) = c\delta x + C$. Note that the system is invariant under the transform $(\psi, \varphi) \longmapsto (e^{i\alpha}\psi, e^{i\beta}\varphi)$, so the integration constants in (6.5) and (6.6) are not important. Thus all we have to do is to solve the system (6.7)-(6.8). Thereafter it will be easy to find the corresponding phases from (6.5)-(6.6) and (6.4) will give a solitary-wave solution of (6.1).

After the scale change $\tilde{u}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})$, $\tilde{r}(x) = r(\frac{x}{\varepsilon})$, we find that the functions r and u satisfy

$$(6.9) \qquad r'' = (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2,$$

$$(6.10) \qquad u'' = (q^2(1+r)^2 - \varepsilon^2(c^2\delta^2 + k^2))u.$$

Remark that equation (6.9) is exactly equation (1.9) for $v = 2c\varepsilon$ and $U = u^2$. Equation (6.10) is linear in u; more precisely, u must be an eigenvector of the linear operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ corresponding to the eigenvalue $\varepsilon^2(c^2\delta^2 + k^2)$. According to the discussion in Introduction, we impose from now on that $c\varepsilon < \frac{1}{\sqrt{2}}$. Let

$$r(v)(x) = -1 + \sqrt{\frac{v^2}{2} + (1 - \frac{v^2}{2})\tanh^2(-\frac{\sqrt{2-v^2}}{2}|x|)}, \quad |v| < \sqrt{2}.$$

It is clear from the discussion at the beginning of section 3 that $r = r(2c\varepsilon)$, $u = 0$ satisfy (6.9)-(6.10). We call $(r(2c\varepsilon), 0)$ a trivial solution of (6.9)-(6.10).

Observe that the system (9)-(10) has a good variational formulation: its solutions are the critical points of the "energy" functional. Indeed, since $1+\tilde{r} = |\tilde{\psi}| \geq 0$, it is clear that we must have $\tilde{r} \geq -1$. Therefore we will seek for solutions r of (9) with $r > -1$. Let $V = \{r \in H^1(\mathbf{R}) \mid \inf_{x\in\mathbf{R}} r(x) > -1\}$. It is obvious that V is open in $H^1(\mathbf{R})$ because $H^1(\mathbf{R}) \subset C_b^0(\mathbf{R})$ by the Sobolev imbedding. A pair $(r, u) \in V \times H^1(\mathbf{R})$ satisfy (9)-(10) if and only if (r,u) is a critical point of the C^∞ functional $E : V \times H^1(\mathbf{R}) \longrightarrow \mathbf{R}$,

$$\begin{aligned} E(r,u) \;=\; & \int_{\mathbf{R}} |r'|^2 dx + \frac{1}{2}\int_{\mathbf{R}} \Big((1+r)^2 - 1\Big)^2\Big(1 - \frac{2c^2\varepsilon^2}{(1+r)^2}\Big)dx \\ & + \int_{\mathbf{R}} u^2(1+r)^2 dx + \frac{1}{q^2}\int_{\mathbf{R}} |u'|^2 dx - \frac{\varepsilon^2(c^2\delta^2+k^2)}{q^2}\int_{\mathbf{R}} u^2 dx. \end{aligned}$$

However, $E(r,\cdot)$ is quadratic in u for any fixed r, so it would be very difficult to find critical points of E by using a classical topological argument.

In order to show the existence of nontrivial solitary waves for the system, we follow very closely the proof of the Bifurcation from a Simple Eigenvalue Theorem (see [19]). From now on we fix c, ε, δ and q such that $\delta < q\sqrt{2}$. Denote

$$\mathbf{H} = H^2_{rad}(\mathbf{R}) = \{u \in H^2(\mathbf{R}) | u(x) = u(-x), \ \forall x \in \mathbf{R}\} \ \text{ and}$$
$$\mathbf{L} = L^2_{rad}(\mathbf{R}) = \{u \in L^2(\mathbf{R}) | u(x) = u(-x), \ \textit{a.e.} \ x \in \mathbf{R}\}.$$

Clearly $\mathbf{H} \cap V$ is an open set of $\mathbf{H}$. We define $S : (\mathbf{H} \cap V) \times \mathbf{H} \longrightarrow \mathbf{L}$, $T : \mathbf{R} \times \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{L}$,

$$(6.11)\quad S(r,u) = -r'' + (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2,$$

$$(6.12)\qquad T(k,r,u) = -u'' + (q^2(1+r)^2 - \varepsilon^2(c^2\delta^2 + k^2))u.$$

It is obvious that S and T are well defined and of class C^∞ (recall that $\mathbf{H} \subset L^\infty(\mathbf{R})$ and $\mathbf{H}$ is an algebra).

The linear operator $A = -\frac{d^2}{dx^2} + q^2(1 + r(2c\varepsilon))$ with domain $D(A) = H^2(\mathbf{R})$ is self-adjoint and strictly positive on $L^2(\mathbf{R})$ (in fact, $A \geq 2c^2\varepsilon^2q^2$). Let λ_0 be its first eigenvalue. It is well-known that λ_0 is simple, $\lambda_0 \geq 2c^2\varepsilon^2q^2$ and, denoting by u_0 a corresponding eigenvector, that u_0 is symmetric. Let $k_0 = \sqrt{\frac{\lambda_0}{\varepsilon^2} - c^2\delta^2} > 0$ and$u_0^\perp$ be the orthogonal of u_0 in $L^2(\mathbf{R})$.

We have the following result concerning the existence of non-trivial solitary waves:

Theorem 6.1 *There exists $\eta > 0$ and C^∞ functions*

$$s \longmapsto (k(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times (u_0^\perp \cap \mathbf{H})$$

defined on $(-\eta, \eta)$ such that $k(0) = k_0$, $r(0) = 0$, $u(0) = 0$ and

$$S(r(2c\varepsilon) + sr(s), s(u_0 + u(s))) = 0, \quad T(k(s), r(2c\varepsilon) + sr(s), s(u_0 + u(s))) = 0.$$

Moreover, there exists a neighbourhood U of $(k_0, r(2c\varepsilon), 0)$ in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ such that any solution of $S(r,u) = 0$, $T(k,r,u) = 0$ in U is either of the form $(k(s), r(2c\varepsilon) + sr(s), s(u_0 + u(s)))$ or of the form $(k, r(2c\varepsilon), 0)$.

That is, $r = r(2c\varepsilon) + sr(s)$, $u = s(u_0 + u(s))$ are nontrivial solutions of (6.9)-(6.10) for $k = k(s)$.

In order to prove Theorem 6.1, we need the following technical lemmas:

Lemma 6.2 *$d_rS(r(2c\varepsilon), 0) : \mathbf{H} \longrightarrow \mathbf{L}$ is invertible.*

Proof. Let $g : (-1, \infty) \longrightarrow \mathbf{R}$, $g(x) = (1+x)^3 - (1+x) - c^2\varepsilon^2\Big(1 + x - \frac{1}{(1+x)^3}\Big)$. It is esily seen that $d_rS(r(2c\varepsilon), 0) = -\frac{d^2}{dx^2} + g'(r(2c\varepsilon))$. The linear operator $B = -\frac{d^2}{dx^2} + g'(r(2c\varepsilon))$ with domain $D(B) = H^2(\mathbf{R})$ is self-adjoint in $L^2(\mathbf{R})$.

We claim that $Ker(B) = Span(\frac{d}{dx}r(2c\varepsilon))$. Indeed, we have seen at the beginning of Section 3 that

$$(6.13)\qquad \frac{d^2}{dx^2}r(2c\varepsilon) = g(r(2c\varepsilon)).$$

Differentiating with respect to x we get $\frac{d}{dx}r(2c\varepsilon)) \in Ker(B)$. Conversely, let $h \in Ker(B)$. Then $h'' = g'(r(2c\varepsilon))h$, so that

$$(h'r'(2c\varepsilon))' = h''r'(2c\varepsilon) + h'r''(2c\varepsilon) = hg'(r(2c\varepsilon))r'(2c\varepsilon) + h'g(r(2c\varepsilon)) = (hg(r(2c\varepsilon)))'.$$

Hence $h'r'(2c\varepsilon) = hg(r(2c\varepsilon)) + C$ on $\mathbf{R}$. Taking the limits as $|x| \longrightarrow \infty$, we get $C = 0$, so $h'r'(2c\varepsilon) = hg(r(2c\varepsilon)) = hr''(2c\varepsilon)$. Since $r'(2c\varepsilon) \neq 0$ on $(-\infty, 0)$

and on $(0,\infty)$, on each of these intervals we have $\left(\frac{h}{r'(2c\varepsilon)}\right)' = \frac{h'r'(2c\varepsilon)-hr''(2c\varepsilon)}{(r'(2c\varepsilon))^2} = 0$. Thus there exist constants C_1, C_2 such that $h(x) = C_1 r'(2c\varepsilon)(x)$ on $(-\infty,0)$ and $h(x) = C_2 r'(2c\varepsilon)(x)$ on $(0,\infty)$. Then $h'(x) = C_1 r''(2c\varepsilon)(x) = C_1 g(r(2c\varepsilon)(x))$ on $(-\infty,0)$ and $h'(x) = C_2 r''(2c\varepsilon)(x) = C_2 g(r(2c\varepsilon)(x))$ on $(0,\infty)$. But h' is continuous because $h \in H^2(\mathbf{R})$ and therefore $C_1 = C_2$, which proves our claim.

Since $r'(2c\varepsilon) \notin \mathbf{H}$, it is clear that the restriction of B to $\mathbf{H}$ is one-to-one and maps $\mathbf{H}$ into $\mathbf{L}$. It remains to prove that $B\mathbf{H} = \mathbf{L}$. It is well-known that $Im(B) = (r'(2c\varepsilon))^{\perp}$. We have $\mathbf{L} \subset Im(B)$ because $r'(2c\varepsilon)$ is an odd function. Let $f \in \mathbf{L}$. Clearly there exists $r \in H^2(\mathbf{R})$ such that $Br = f$. Let $\tilde{r}(x) = r(-x)$. It is easy to see that $B\tilde{r} = f$, hence there exists C such that $r - \tilde{r} = Cr'(2c\varepsilon)$. Then $r - \frac{1}{2}Cr'(2c\varepsilon) \in \mathbf{H}$ and $B(r - \frac{1}{2}Cr'(2c\varepsilon)) = f$. This completes the proof of Lemma 6.2. □

Lemma 6.3 *We have:*

i) $KerT(k_0, r(2c\varepsilon), \cdot) = Span(u_0)$;

ii) $ImT(k_0, r(2c\varepsilon), \cdot) = u_0^{\perp} \cap \mathbf{L}$.

The proof is obvious.

Proof of Theorem 6.1. Let $W = \{r \in \mathbf{H} | \sup_{x\in\mathbf{R}} |r(x)| < 1\}$ and $I = (-\sqrt{2}c\varepsilon, \sqrt{2}c\varepsilon)$. Clearly W is open in $\mathbf{H}$. We define $F : I \times \mathbf{R} \times \mathbf{H} \times (\mathbf{H} \cap u_0^{\perp}) \longrightarrow \mathbf{L} \times \mathbf{L}$,

$$F(s,k,r,u) = \begin{cases} \begin{pmatrix} \frac{1}{s} S(r(2c\varepsilon) + sr, s(u_0 + u)) \\ \\ \frac{1}{s} T(k, r(2c\varepsilon) + sr, s(u_0 + u)) \end{pmatrix} & \text{if } s \neq 0, \\ \\ \begin{pmatrix} (d_r S(r(2c\varepsilon), 0) \cdot r \\ T(k, r(2c\varepsilon), u_0 + u) \end{pmatrix} & \text{if } s = 0. \end{cases}$$

It is easily seen that F is C^∞ because

$$\begin{aligned} F_1(s,k,r,u) &= \tfrac{1}{s}\left(S(r(2c\varepsilon) + sr, s(u_0 + u)) - S(r(2c\varepsilon), 0)\right) \\ &= \tfrac{1}{s}\int_0^1 \frac{d}{dt} S(r(2c\varepsilon) + tsr, ts(u_0 + u))dt \\ &= \tfrac{1}{s}\int_0^1 d_r S(r(2c\varepsilon) + tsr, ts(u_0+u)) \cdot sr + d_u S(r(2c\varepsilon) + tsr, ts(u_0 + u)) \cdot s(u_0 + u)dt \\ &= \int_0^1 d_r S(r(2c\varepsilon) + tsr, ts(u_0+u)) \cdot r + d_u S(r(2c\varepsilon) + tsr, ts(u_0 + u)) \cdot (u_0 + u)dt \end{aligned}$$

and

$$\begin{aligned} F_2(s,k,r,u) &= \tfrac{1}{s}\left(T(k, r(2c\varepsilon) + sr, s(u_0 + u)) - T(k, r(2c\varepsilon), 0)\right) \\ &= \int_0^1 d_r T(k, r(2c\varepsilon) + tsr, ts(u_0 + u)) \cdot r + T(k, r(2c\varepsilon) + tsr, u_0 + u)dt. \end{aligned}$$

It is also clear that $F(0, k_0, 0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$d_{(k,r,u)}F(0,k_0,0,0)(\tilde{k},\tilde{r},\tilde{u})$$

$$= \tilde{k}\begin{pmatrix} 0 \\ -2\varepsilon^2 k_0 u_0 \end{pmatrix} + \begin{pmatrix} d_r S(r(2c\varepsilon, 0) \cdot \tilde{r} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T(k_0, r(2c\varepsilon), \tilde{u}) \end{pmatrix}$$

In view of Lemmas 6.2 and 6.3, $d_{(k,r,u)}F(0, k_0, 0, 0)$ is invertible. By the Implicit Function Theorem, there exist $\eta > 0$ and C^∞ functions defined on $(-\eta, \eta)$,

$$s \longmapsto (k(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times (\mathbf{H} \cap u_0^\perp)$$

such that $k(0) = k_0$, $r(0) = 0$, $u(0) = 0$ and $F(s, k(s), u(s), r(s)) = (0, 0)$. It is obvious that if $s \neq 0$, $(r(2c\varepsilon) + sr(s), s(u_0 + u(s)))$ satisfy the system (6.9)-(6.10) for $k = k(s)$. Finally, the uniqueness part in Theorem 6.1 is proved exactly in the same way as in the Bifurcation from a Simple Eigenvalue Theorem. □

Remark 6.4 Theorem 6.1 gives the existence of a branch of nontrivial solutions for (6.9)-(6.10) *locally* near $(k_0, r(2c\varepsilon), 0)$. It is an open question how long this branch of solutions exists. Note that the Global Bifurcation Theorem ([42]) and its variants do not apply in this case because the operators involved are far from being compact.

Remark 6.5 It is not hard to prove that in dimension $N = 1$, 2 or 3 the Cauchy problem for the system (6.1) is globally well-posed in $(1 + H^1(\mathbf{R}^N)) \times H^1(\mathbf{R}^N)$.

Remark 6.6 The existence of solitary waves for (6.1) in dimension greater than 1 is an open problem. It was proved by F. Bethuel and J.-C. Saut (see [11]) that in dimension 2, the Gross-Pitaevskii equation

$$2i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi, \quad |\psi| \longrightarrow 1 \text{ as } |x| \longrightarrow \infty$$

possesses travelling-waves moving with small speed.

Chapter 3

Existence of nonstationary bubbles in higher dimensions

3.1 Introduction

The aim of this work is to prove the existence of travelling "bubbles" for the nonlinear Schrödinger equation

$$(1.1) \qquad i\frac{\partial \varphi}{\partial t} + \Delta\varphi + F(|\varphi|^2)\varphi = 0 \quad \text{in } \mathbf{R}^N,$$

where the function φ is complex-valued and satisfies the "boundary condition" $|\varphi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, and r_0 is a positive real constant such that $F(r_0^2) = 0$. The case of the "$\psi^3 - \psi^5$" nonlinear Schrödinger equation

$$(1.1') \qquad i\frac{\partial \psi}{\partial t} + \Delta\psi - \alpha_1\psi + \alpha_3|\psi|^2\psi - \alpha_5|\psi|^4\psi = 0$$

with $\alpha_1, \alpha_3, \alpha_5 > 0$ and $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$ fits in this framework.

Equation (1.1) (and in particular (1.1')) appears in a large variety of physical problems, see [3]. For example, (1.1') describes the boson gas with 2-body attractive and 3-body repulsive δ-function interaction. These equations have applications to superfluidity, where the "$\psi^3 - \psi^5$" NLS equation arises on the level of the Ginzburg-Landau two-liquid theory. They also occur in the description of defectons, in the theory of one-dimensional ferromagnetic and molecular chains and in other similar problems in condensed matter. Equation (1.1') with $N = 3$ models the evolution of a monochromatic wave complex envelope in a medium with weakly saturating nonlinearity.

There is a special kind of solutions of (1.1), the "stationary bubbles". These are solutions of the form $e^{i\omega t}\psi(x)$. It was proved in [13] under general conditions on the nonlinearity F that the stationary bubbles exist and are unstable.

It was also proved (see [4]) that in space dimension one there exist some localized solutions travelling with velocity c, having the form $\varphi(t,x) = \Phi(x - ct)$ and corresponding to "nonstationary bubbles". The boundary condition is then $\lim_{x\to\pm\infty} \Phi(x) = r_0 e^{\mp i\mu}$, where μ is a real number depending on c and $\mu = 0$ when $c = 0$.

The travelling waves (or nonstationary bubbles) of (1.1) are solutions of the form $\varphi(t, x_1, \dots, x_N) = \Phi(x_1 - ct, x_2, \dots, x_N)$. In view of the boundary condition, we will seek for solutions Φ of the form $\Phi(x) = r_0 - u(x)$ with $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. The function u must satisfy

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0. \tag{1.2}$$

Now let us describe the assumptions that we make on the nonlinearity F (which are essentially the same as in [13] or [28]). We assume throughout that $F \in C^1(\mathbf{R}_+, \mathbf{R})$ and

(H1) $$F(r_0^2) = 0, \quad F'(r_0^2) < 0.$$

We will need a little bit more regularity on F only in a neighbourhood of r_0^2. We suppose that there exists $\alpha > 0$ such that $|F'(r_0^2 + s) - F'(r_0^2)| \le C|s|^\alpha$ for s small. Set

$$V(s) = \int_s^{r_0^2} F(\tau)d\tau. \tag{1.3}$$

In particular, condition (H1) implies $|F(r_0^2 + s)| \le C|s|$ and $V(r_0^2 + s) \le C's^2$ for some $C, C' > 0$ and s small.

We also have to impose some restrictions on the behaviour of F at infinity. We suppose that there exists $C > 0$ such that

(H2) $$|F'(s)| \le C|s|^{\frac{\sigma}{2}-1} \quad \text{for } s \ge 1, \text{ where } \sigma = \frac{4}{N-2}.$$

(Note that $2 + \sigma$ is the critical exponent for the embedding of $H^1(\mathbf{R}^N)$ in some $L^p(\mathbf{R}^N)$.) Of course this implies

$$|F(s)| \le C's^{\frac{\sigma}{2}} \quad \text{if } s \ge 1 \quad \text{and} \tag{1.4}$$

$$|V(s)| \le C''s^{\frac{\sigma}{2}+1} \tag{1.5}$$

for some positive constants C', C''.

We will always make the assumption

(H3) there exists $\rho_1 \in [0, r_0^2)$ such that $V(\rho_1) < 0$.

Note that assumptions (H1), (H2), (H3) are "almost" needed for the existence of stationary bubbles (see [6] and [13]). In addition, for technical reasons we impose the following condition:

(H4) there exists $M > 0$ such that $F(s) \le 0$ for $s \ge M$.

We need (H4) only in Section 5, to prove the regularity of the nonstationary bubbles.

Let $a_0 = \sup\{a > 0 \mid F(|r_0 - u|^2)(r_0 - u) > 0, \ \forall u \in (0, a)\}$. In view of (H1) and (H3), it is clear that $0 < a_0 < r_0$.

We define $J(\lambda,u) = [2u-(\lambda+2)r_0]F(|r_0-u|^2) - 2\lambda u(r_0-u)^2F'(|r_0-u|^2)$ and we suppose that the following condition is satisfied: for any $U \in (a_0, r_0)$ there exists $\lambda(U) > 0$ continuously depending on U such that

$$\textbf{(H5)} \qquad \begin{aligned} J(\lambda(U),u) \leq 0, \quad \forall u \in [0,U] \quad \text{and} \\ J(\lambda(U),u) \geq 0, \quad \forall u \in [U,r_0]. \end{aligned}$$

Note that assumption (H5) is the analogous of conditions (5)-(6) in [36] and we need it only to prove an uniqueness result in section 2 (Theorem 2.6).

A complex-valued function $u = u_1 + iu_2$ is a solution of equation (1.2) if and only if its real and imaginary parts satisfy the system

$$(1.6) \qquad -cu_{2_{x_1}} - \Delta u_1 + F((r_0-u_1)^2+u_2^2)(r_0-u_1) = 0,$$

$$(1.7) \qquad cu_{1_{x_1}} - \Delta u_2 - F((r_0-u_1)^2+u_2^2)u_2 = 0.$$

In what follows, $H^1(\mathbf{R}^N)$ always denotes the space $H^1(\mathbf{R}^N,\mathbf{R})$ and $\mathcal{D}^{1,2}(\mathbf{R}^N) = \mathcal{D}^{1,2}(\mathbf{R}^N,\mathbf{R}) = \{v \in L^{2+\sigma}(\mathbf{R}^N) \mid \nabla v \in L^2(\mathbf{R}^N)\}$, with norm $||v||^2_{\mathcal{D}^{1,2}} = \int_{\mathbf{R}^N} |\nabla v|^2 dx$. We shall identify a function $u = u_1 + iu_2$ with the pair (u_1,u_2) and we seek for solutions with $u_1 \in H^1(\mathbf{R}^N)$, $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Let $\mathbf{H} = H^1(\mathbf{R}^N) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. On $\mathbf{H}$ we consider the norm $||(u_1,u_2)||^2 = ||u_1||^2_{H^1} + ||u_2||^2_{\mathcal{D}^{1,2}}$. We identify $H^1(\mathbf{R}^N) \times \{0\}$ with $H^1(\mathbf{R}^N)$ and $\{0\} \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ with $\mathcal{D}^{1,2}(\mathbf{R}^N)$. We introduce the following functionals:

$$\begin{aligned}
T(u) &= T(u_1,u_2) &= \int_{\mathbf{R}^N} |\nabla u|^2 dx = \int_{\mathbf{R}^N} |\nabla u_1|^2 dx + \int_{\mathbf{R}^N} |\nabla u_2|^2 dx, \\
I(u) &= I(u_1,u_2) &= \int_{\mathbf{R}^N} V(|r_0-u|^2)dx = \int_{\mathbf{R}^N} V((r_0-u_1)^2+u_2^2)dx, \\
Q(u) &= Q(u_1,u_2) &= -2\int_{\mathbf{R}^N} u_1 u_{2_{x_1}} dx, \\
E(u) &= E(u_1,u_2) &= T(u) + I(u), \\
E_c(u) &= E_c(u_1,u_2) &= T(u) + I(u) + cQ(u) = E(u) + cQ(u).
\end{aligned}$$

Obviously T and Q are of class C^∞ on $\mathbf{H}$. It is easy to check that under assumptions (H1) and (H2), I is of class C^2 on $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. It will be verified at the beginning of Section 4 that I is well-defined and of class C^2 on $\mathbf{H}$ if $N \geq 4$.

Therefore E and E_c are of class C^2 on $\mathbf{H}$ if $N \geq 4$ and the $\mathbf{H}$-solutions of (1.2) are exactly the critical points of E_c, while the critical points u of E satisfy the equation

$$(1.8) \qquad -\Delta u + F(|r_0-u|^2)(r_0-u) = 0.$$

The following theorem gives the existence of a special solution of (1.8):

Theorem 1.1.([13]) *There exists a real-valued function* $u_0 \in H^1(\mathbf{R}^N)$ *which satisfies equation (1.8) and has the following properties:*

i) u_0 *is radially symmetric, i.e* $u_0(x) = u_0(|x|) = u_0(r)$*;*

ii) $0 < u_0(r) < r_0$, $\forall r \in [0, \infty)$, $u_{0r}(0) = 0$ *and* $u_{0r}(r) < 0$, $\forall r > 0$ *(i.e.* u_0 *is strictly decreasing in* r*);*

iii) $u_0 \in C^2(\mathbf{R}^N)$ *and there exist constants* $C, \delta > 0$ *such that* $|\partial_x^\alpha u_0(x)| \leq Ce^{-\delta|x|}$, $\forall x \in \mathbf{R}^N$, $\forall \alpha \in \mathbf{N}^N$ *with* $|\alpha| \leq 2$.

iv) u_0 *is a solution of the minimization problem:*

"minimize $T(u)$ *under the constraint* $I(u) = I(u_0)$*";*

v) equivalently, u_0 *is a solution of the maximization problem:*

"maximize $I(u)$ *under the constraint* $T(u) = T(u_0)$*".*

Theorem 1.1 was proved in [13] by using a general result of H. Berestycki and P.-L. Lions (see [6]). A solution having the properties listed in Theorem 1.1 will be called a *ground state* for equation (1.8).

Note that $\lim_{s \to r_0^2} \frac{V(s)}{(s-r_0)^2} = -\frac{1}{2}F'(r_0^2) > 0$, so $V(s)$ is positive on an interval $((r_0 - \eta)^2, (r_0+\eta)^2)$. Suppose that $V \geq 0$ on $[r_0^2, \infty)$ (remark that this is the case for the "$\psi^3 - \psi^5$" nonlinearity). Then $V(|r_0 - z|^2) < 0$ implies that z belongs to the ball (in $\mathbf{C}$) of center r_0 and radius $r_0 - \eta$. Let $N = \{z \in \mathbf{C} \mid V(|r_0 - z|^2) < 0\} \subset B_{\mathbf{C}}(r_0, r_0 - \eta)$. If $u \in \mathbf{H}$ and $E(u) < 0$, we have

$$\begin{aligned} E(u) &\geq \int_{\mathbf{R}^N} V(|r_0 - u|^2)dx \geq \int_{\{x|u(x)\in N\}} V(|r_0 - u|^2)dx \\ &\geq \inf_{[0,r_0^2]} V \cdot meas(\{x \mid u(x) \in N\}), \end{aligned}$$

so that $meas(\{x \mid u(x) \in N\}) \geq \frac{\int V(|r_0-u|^2)dx}{\inf_{[0,r_0^2]} V} \geq \frac{E(u)}{\inf_{[0,r_0^2]} V}$. On the other hand, by the Sobolev embedding and the fact that $dist(N, 0) \geq \eta$ we have

$$\int_{\mathbf{R}^N} |\nabla u|^2 dx \geq C_S ||u||_{L^{2^*}}^2 \geq C_S \eta^{\frac{2}{2^*}} (meas(\{x \mid u(x) \in N\}))^{\frac{2}{2^*}},$$

so that

$$E(u) \geq C_1 (meas(\{x \mid u(x) \in N\}))^{\frac{2}{2^*}} - C_2 meas(\{x \mid u(x) \in N\})$$

for some positive constants C_1, C_2. Clearly, $meas(\{x \mid u(x) \in N\})$ *does not* depend continuously on u. However, using the simple observations made above, it is possible to find a radial function $v_0 \in H^1(\mathbf{R}^N)$ such that $E(v_0) < 0$ and $\inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E(\gamma(t)) > 0$, where $\Gamma = \{\gamma \in C([0,1], \mathbf{H}) \mid \gamma(0) = 0, \gamma(1) = v_0\}$. Therefore the functional E admits a Palais-Smale sequence (nevertheless, it is not obvious at this stage that this sequence converges in $\mathbf{H}$).

Since $E_c(u) \longrightarrow E(u)$ as $c \longrightarrow 0$ uniformly on bounded sets of $\mathbf{H}$, one should expect that $\inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E_c(\gamma(t)) > 0$, at least for small values of c. However, the observations made above fail when E is replaced by E_c: it is not possible to bound $E_c(u)$ from below in terms of $meas(\{x \mid u(x) \in N\})$. There exist continuous paths connecting v_0 to functions of arbitrarily low "energy" E_c such that E_c decreases and

$meas(\{x \mid u(x) \in N\})$ is constant along these paths. To be more precise, for any $c \neq 0$ one can find a continuous path $\tilde{\gamma}_c : [0, \infty) \longrightarrow \mathbf{H}$ such that $\tilde{\gamma}_c(0) = v_0$, $\tilde{\gamma}_c(\tau)$ is of the form $r_0 - (r_0 - v_0)e^{i\varphi_\tau}$ (hence $|r_0 - \tilde{\gamma}_c(\tau)(x)| = |r_0 - v_0(x)|$) and $E_c(\tilde{\gamma}_c(\cdot))$ is strictly decreasing on $[0, \infty)$ with $\lim_{\tau \longrightarrow \infty} E_c(\tilde{\gamma}_c(\tau)) = -\infty$. We do not know whether it is possible or not to connect some $\tilde{\gamma}_c(\tau)$ for large τ (thus for $E_c(\tilde{\gamma}_c(\tau))$ very small) to zero by a continuous path in $\mathbf{H}$ such that E_c remains negative along this path. (Of course, if such a path existed, we would be able to connect zero to v_0 in the set $\{u \in \mathbf{H} \mid E_c(u) \leq 0\}$, which is not possible in the set $\{u \in \mathbf{H} \mid E(u) \leq 0\}$. Anyway, the preceeding arguments suggest that it should be extremely difficult to find Palais-Smale sequences for E_c by using a Mountain-Pass Theorem on the entire $\mathbf{H}$. Even if such a sequence is found, it should be still more difficult to prove that it converges (in some sense) to a non-trivial solution of (1.2).

We want to prove that (1.2) admits non-trivial solutions by showing that E_c possesses non-trivial critical points. But instead of searching for a change of topology of the level sets of E_c on the entire $\mathbf{H}$, we analyze what happens locally on a small neighbourhood of u_0, where u_0 is a ground state of equation (1.8) as given by Theorem 1.1.

Remark that the system (1.6)-(1.7) is of the form $\Phi_1(c, u_1, u_2) = 0$, $\Phi_2(c, u_1, u_2) = 0$ with

$$\begin{pmatrix} \dfrac{\partial \Phi_1}{\partial u_1} & \dfrac{\partial \Phi_1}{\partial u_2} \\ \\ \dfrac{\partial \Phi_2}{\partial u_1} & \dfrac{\partial \Phi_2}{\partial u_2} \end{pmatrix} (c, u_0, 0) = \begin{pmatrix} A & -c\dfrac{\partial}{\partial x_1} \\ \\ c\dfrac{\partial}{\partial x_1} & B \end{pmatrix}.$$

where A and B are linear operators in $L^2(\mathbf{R}^N)$ defined by $D(A) = D(B) = H^2(\mathbf{R}^N)$ and

$$(1.9) \qquad \begin{aligned} Au &= -\Delta u - [2F'((r_0 - u_0)^2)(r_0 - u_0)^2 + F((r_0 - u_0)^2)]u, \\ Bu &= -\Delta u - F((r_0 - u_0)^2)u, \end{aligned}$$

u_0 being the ground state. It is easy to see that A and B are self-adjoint. It follows from a classical theorem of Weyl that the essential spectrum of A is $\sigma_{ess}(A) = [-2F'(r_0^2)r_0^2, \infty)$ and the essential spectrum of B is $\sigma_{ess}(B) = [0, \infty)$. Note that $-2F'(r_0^2)r_0^2 > 0$ by (H1) and it is not hard to see that for $c < -2F'(r_0^2)r_0^2$, the essential spectrum of $(\Phi_1, \Phi_2)'(c, u_0, 0)$ is $[0, \infty)$. So even if restricted to the space orthogonal to its kernel, the linear operator $(\Phi_1, \Phi_2)'(0, u_0, 0)$ is not invertible. Therefore we cannot solve the equation $(\Phi_1, \Phi_2)(c, u_1, u_2) = (0, 0)$ for c near zero and (u_1, u_2) near $(u_0, 0)$ by an argument based on the Implicit Function Theorem (such as, for example, the Lyapunov-Schmidt method).

Our strategy is as follows: we consider the spectral decomposition

$$L^2(\mathbf{R}^N) = X \oplus Ker(A) \oplus \tilde{Y},$$

where X, $\tilde{Y}$ are the subspaces corresponding to the negative part of $\sigma(A)$, respectively to the positive part of $\sigma(A)$. It will be seen in the next section that X is one-dimensional and $X \subset H^1(\mathbf{R}^N)$. Let $Y = \tilde{Y} \cap H^1(\mathbf{R}^N)$. We consider the restrictions of the functionals E and E_c to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. We prove in Section 4 that $E(u_0 + u_1, u_2) > E(u_0, 0)$ for $u_1 \in Y$, $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $(u_1, u_2) \neq (0, 0)$, $||(u_1, u_2)||_{\mathbf{H}}$ small and $E(u_0 + v_1, 0) < E(u_0, 0)$ for $v_1 \in X$, $v_1 \neq 0$, $||v_1||_{H^1}$ small.

Therefore u_0 is a saddle-point for E restricted to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. We shall prove that for c sufficiently small, there exists an open neighbourhood Ω_c of $(0,0)$ in $Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ such that for all $(u_1, u_2) \in \Omega_c$ and (u_1, u_2) "close" to $\partial\Omega_c$ we have $E_c(u_0 + u_1, u_2) > E_c(u_0, 0)$ and $E_c(u_0 + v, 0) = E(u_0 + v, 0) < E_c(u_0, 0)$ for $v \in X$, $v \neq 0$, $||v||_{H^1}$ small. By a local Mountain-Pass type argument we infer that for c sufficiently small, there exists a critical point $(u_0 + u_1^c, u_2^c)$ of E_c restricted to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ and $||(u_1^c, u_2^c)||_{\mathbf{H}} \longrightarrow 0$ as $c \longrightarrow 0$.

It remains only to prove that $E_c'(u_0 + u_1^c, u_2^c).u = 0$ for all $u \in Ker(A)$. It is obvious that $\frac{\partial u_0}{\partial x_i} \in Ker(A)$, $i = 1, \dots, N$. It will be proved in section 2 that $Ker(A)$ is spanned by $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ and we shall get the desired conclusion thanks to the invariance of equation (1.2) by translations in $\mathbf{R}^N$. Our main result is:

Theorem. *Let $N \geq 4$. There exists $c_0 > 0$ such that for any $c \in [-c_0, c_0]$ there exists a critical point $u_c \in \mathbf{H}$ of E_c. Moreover, $u_c \longrightarrow u_0$ in $\mathbf{H}$ as $c \longrightarrow 0$ and u_c can be chosen radially symmetric in the transverse variables $(x_2, \dots, x_N)$.*

Similar results were obtained in space dimension $N = 2, 3$ by Zhiwu Lin in [28]. He used the hydrodynamical formulation of the nonlinear Schrödinger equation, searching for solitary waves of (1.1) of the form $\sqrt{\rho}e^{i\varphi}$ and he applied the Lyapunov-Schmidt method of finite-dimensional reduction to the equations in ρ and φ. He used implicitly the fact that $Ker(A) = Span\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\}$.

This paper is organized as follows: the next section is devoted to the study of the operator A introduced in (1.9). Its properties are essential for our proof of existence of nonstationary bubbles. It will be shown that A has a first negative eigenvalue, 0 is its second eigenvalue and $Ker(A) = Span\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\}$. In Section 3 we prove an abstract result in critical point theory (a local Saddle-Point Theorem). This result will be applied in Section 4 to find critical points of the functional E_c. Finally, Section 5 is devoted to the regularity of nonstationary bubbles.

3.2 Properties of the operator A

We have already defined the operator A in $L^2(\mathbf{R}^N)$ by formula (1.9). In this section we study its properties and we are particularly interested in the structure of its kernel. It turns out that the results obtained here still hold in a slightly more general framework. Therefore, consider $g \in C^1([0, \infty))$ with $g(0) = 0$, $g'(0) > 0$ and $|g'(s) - g'(0)| \leq C|s|^\alpha$ for small s and some C, $\alpha > 0$. Let $G(t) = \int_0^t g(s)ds$ and suppose that there exists $\zeta > 0$ with $G(\zeta) < 0$ (this corresponds to assumption (H3) on F). Suppose that the problem

$$(2.1) \qquad -\Delta u + g(u) = 0$$

admits a positive radial solution having the properties listed in Theorem 1.1, where I is replaced by $I(u) = \int_{\mathbf{R}^N} G(u)dx$. If $N \geq 3$ and $\limsup_{x \to \infty} \frac{g(s)}{s^{1+\sigma}} \leq 0$, it follows from a classical result of H. Berestycki and P.-L. Lions that such a solution always exists (see Theorem 1 in [6]); it is called a *ground state* for (2.1). In this section, we denote by u_0 a ground state for (2.1) and we define the operator L on $L^2(\mathbf{R}^N)$ by $D(L) = H^2(\mathbf{R}^N)$ and $Lu = -\Delta u + g'(u_0)u$. Note that in the particular case $g(s) = F((r_0 - s)^2)(r_0 - s)$, (2.1) becomes (1.8) and L equals A.

Remark that L is bounded from below. Since $g'(u_0(x))$ tends exponentially to $g'(0)$ as $|x| \longrightarrow \infty$ (at this point we use the fact that $|g'(s) - g'(0)| \le C|s|^\alpha$ for small s) it follows from a theorem of Weyl that the essential spectrum of L is the same as the essential spectrum of $-\Delta + g'(0)$, that is $\sigma_{ess}(L) = \sigma_{ess}(-\Delta + g'(0)) = [g'(0), \infty)$. Hence $\sigma(L)$ consists precisely in $\sigma_{ess}(L)$ and a finite number of discrete eigenvalues below $g'(0)$.

Lemma 2.1. *The first eigenvalue of L exists and is negative.*

Proof. It suffices to show that $\inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \dfrac{\langle Lu, u\rangle}{||u||_{L^2}^2} < 0$. We will find a function $v \in H^1(\mathbf{R}^N)$ such that $\langle Lu, u\rangle < 0$.

Because $u_0(x) = u_0(|x|) = u_0(r)$ is a solution of (2.1), u_0 (as a function of the real variable r) must satisfy

$$-u_0'' - \frac{N-1}{r} u_0' + g(u_0) = 0 \quad \text{on } (0, \infty). \tag{2.2}$$

This implies that $u_0 \in C^3(0, \infty)$; differentiating (2.2) we get

$$-u_0''' - \frac{N-1}{r} u_0'' + g'(u_0)u_0' + \frac{N-1}{r^2} u_0' = 0. \tag{2.3}$$

Let $v(x) = u_0(|x|)$. In view of Theorem 1.1 iii), $v \in H^1(\mathbf{R}^N)$ and from (2.3) we see that v satisfies $Lv + \frac{N-1}{r^2} v = 0$. Therefore $\langle Lv, v\rangle = -(N-1)\displaystyle\int_{\mathbf{R}^N} \frac{|v(x)|^2}{|x|^2} dx < 0$. This proves Lemma 2.1. □

We denote by $-\lambda_1$ the first eigenvalue of L. It is known that $-\lambda_1$ is simple and the corresponding eigenvector is radially symmetric, has constant sign and tends exponentially to zero at infinity. Denote by e_1 an eigenvector corresponding to $-\lambda_1$ with $||e_1||_{L^2} = 1$.

Differentiating equation (2.1) with respect to x_i, we get $\frac{\partial u_0}{\partial x_i} \in Ker(L)$. Therefore 0 is an eigenvalue of L. Using the fact that u_0 minimizes $T(u) = \displaystyle\int_{\mathbf{R}^N} |\nabla u|^2 dx$ subject to the constraint $I(u) = I(u_0)$, where $I(u) = \displaystyle\int_{\mathbf{R}^N} G(u)dx$, we obtain:

Lemma 2.2. *0 is the second eigenvalue of L.*

Proof. Since $-\lambda_1 < 0$ and 0 is an eigenvalue, it is clear that the second eigenvalue of L exists and is ≤ 0. In order to show that the second eigenvalue of L is ≥ 0, we will find a function $f_0 \in H^1(\mathbf{R}^N)$ such that L is positive on $f_0^\perp \cap H^1(\mathbf{R}^N)$ and we use the Min-Max Principle. We claim that for any $v \in H^1(\mathbf{R}^N)$ such that

$$I'(u_0).v = \int_{\mathbf{R}^N} g(u_0)v dx = 0 \tag{2.4}$$

we have $\langle Lv, v\rangle \ge 0$. Indeed, fix $v \in H^1(\mathbf{R}^N)$ such that $I'(u_0).v = 0$. Since $I'(u_0) \ne 0$, there exists $w \in H^1(\mathbf{R}^N)$ such that

$$I''(u_0).(v, v) + I'(u_0).w = 0. \tag{2.5}$$

Using the Implicit Function Theorem, it is not hard to see that there exists $\delta > 0$ and a C^2-curve $\psi : (-\delta, \delta) \longrightarrow H^1(\mathbf{R}^N)$ such that

$$\psi(0) = u_0, \quad \psi'(0) = v, \quad \psi''(0) = w \quad \text{and } I(\psi(t)) = I(u_0). \tag{2.6}$$

Recall that we have assumed that u_0 satisfies the conditions of Theorem 1.1, in particular u_0 minimizes $T(u)$ under the constraint $I(u) = I(u_0)$. The Euler-Lagrange equation of u_0 is exactly equation (2.1), that is $\frac{1}{2}T'(u_0) + I'(u_0) = 0$. Moreover, the real function $t \longrightarrow T(\psi(t))$ achieves a local minimum at $t = 0$, therefore $\frac{d}{dt}T(\psi(t)) \mid_{t=0} = 0$ and $\frac{d^2}{dt^2}T(\psi(t)) \mid_{t=0} \geq 0$. This gives $T'(u_0).v = 0$ and

$$T''(u_0).(v,v) + T'(u_0).w \geq 0.$$

Using the Euler-Lagrange equation and (2.5) we get

$$\frac{1}{2}T''(u_0).(v,v) \geq -\frac{1}{2}T'(u_0).w = I'(u_0).w = -I''(u_0).(v,v),$$

i.e. $\frac{1}{2}T''(u_0)(v,v) + I''(u_0)(v,v) \geq 0$, which is exactly $\langle Lv, v\rangle \geq 0$. Our claim is thus proved.

It is clear that $g(u_0) \in H^1(\mathbf{R}^N)$. By the Min-Max Principle (see, for example, [43], vol. IV, Theorem XIII.1 p. 76 and Theorem XIII.2 p. 78) the second eigenvalue of L is exactly

$$\inf_{u \in e_1^\perp \setminus \{0\}} \frac{\langle Lu, u\rangle}{||u||_{L^2}^2} = \sup_{f \in H^1(\mathbf{R}^N)} \inf_{u \in f^\perp \setminus \{0\}} \frac{\langle Lu, u\rangle}{||u||_{L^2}^2} \geq 0. \tag{2.7}$$

Therefore 0 is the second eigenvalue of L. □

Corollary 2.3. *i) For any $v \in H^1(\mathbf{R}^N) \cap e_1^\perp$ we have $\langle Lv, v\rangle \geq 0$.*
ii) For any $v \in H^1(\mathbf{R}^N) \cap g'(u_0)^\perp$ we have $\langle Lv, v\rangle \geq 0$.

Corollary 2.3 follows directly from the proof of Lemma 2.2.

Because $\sigma_{ess}(L) = [g'(0), \infty)$ and 0 is a discrete eigenvalue, we have $\beta = \inf(\sigma(L) \cap (0, \infty)) > 0$. Consider the functional calculus associated to the self-adjoint operator L. Let $L_+ = \chi_{(0,\infty)}(L)$ and $\tilde{Y} = Im(L_+)$. Then we have the orthogonal decomposition $L^2(\mathbf{R}^N) = \mathbf{R}e_1 \oplus Ker(L) \oplus \tilde{Y}$. Let $Y = \tilde{Y} \cap H^1(\mathbf{R}^N)$. We have

$$\langle Lu, u\rangle \geq \beta ||u||_{L^2}^2, \quad \forall u \in Y.$$

Lemma 2.4. *There exists $\alpha > 0$ such that*

$$\langle Lu, u\rangle \geq \alpha ||u||_{H^1}^2, \quad \forall u \in Y. \tag{2.8}$$

Proof. For any $u \in Y$ we have

$$\langle Lu, u\rangle = \int_{\mathbf{R}^N} |\nabla u|^2 + g'(u_0)|u|^2 dx \geq \beta ||u||_{L^2}^2 \geq -\beta\delta \int_{\mathbf{R}^N} g'(u_0)|u|^2 dx,$$

where $\delta = \frac{1}{||g'(u_0)||_{L^\infty}}$. It follows that $\int_{\mathbf{R}^N} |\nabla u|^2 dx + (1+\beta\delta)\int_{\mathbf{R}^N} g'(u_0)|u|^2 dx \geq 0$ (or equivalently $\frac{1}{1+\beta\delta}\int_{\mathbf{R}^N} |\nabla u|^2 dx + \int_{\mathbf{R}^N} g'(u_0)|u|^2 dx \geq 0$), which gives $\langle Lu, u\rangle \geq \frac{\beta\delta}{1+\beta\delta}\int_{\mathbf{R}^N} |\nabla u|^2 dx.$ □

Now we focus our attention on the kernel of L. First we have to introduce some notation. Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k with $dim\ \mathcal{H}_k = a_k = C^k_{N+k-1} - C^{k-2}_{N+k-3}$. For each k let $\{Y_1^{(k)}, \dots, Y_{a_k}^{(k)}\}$ be an orthonormal basis of $\mathcal{H}_k$. Let $\mathcal{P}_k$ be the space of linear combinations of the form $\sum_{i=1}^{a_k} f_i(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right)$ with $f_i \in L^2((0,\infty), r^{N-1}dr)$. Then $\mathcal{P}_k \subset L^2(\mathbf{R}^N)$, the spaces $\mathcal{P}_k$ are mutually orthogonal and invariant under the Fourier transform. More precisely, if $Y \in \mathcal{H}_k$, $f \in L^2((0,\infty), r^{N-1}dr)$ then $\mathcal{F}\left(f(|x|)Y\left(\frac{x}{|x|}\right)\right)(\xi) = g(|\xi|)Y\left(\frac{\xi}{|\xi|}\right)$ for some $g \in L^2((0,\infty), r^{N-1}dr)$. Moreover, $\sum_{k=0}^{\infty} \mathcal{P}_k = L^2(\mathbf{R}^N)$, that is any function $u \in L^2(\mathbf{R}^N)$ has an unique expansion

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{a_k} c_{k,i}(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right), \tag{2.9}$$

where $c_{k,i}(|x|) = \int_{S^{N-1}} u(|x|\theta)\overline{Y_i^{(k)}}(\theta)d\theta$. Let $p_{k,i}$ be the projection $p_{k,i}(u) = c_{k,i}(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right)$. Then $p_{k,i}$ is bounded (has norm 1) as an operator from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$, $s \geq 0$ and commutes with Δ.

After this preparation, we may prove

Theorem 2.5. *$Ker(L)$ is spanned by $\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\} \cup (Ker(L) \cap H^2_{rad}(\mathbf{R}^N))$, where $H^2_{rad}(\mathbf{R}^N) = \{u \in H^2(\mathbf{R}^N) \mid u$ is radially symmetric $\}$.*

Proof. The proof was inspired by an idea of M. Weinstein (see the proof of Proposition 2.8 b), p. 483 in [49]). Let $u \in Ker(L)$ and consider its decomposition given by (2.9). Since $u \in H^2(\mathbf{R}^N)$, we have $p_{k,i}(u) \in H^2(\mathbf{R}^N)$. Because $g'(u_0)$ is a radial function, it is clear that $p_{k,i}(g'(u_0)u) = g'(u_0)p_{k,i}(u)$. Therefore we have $L(p_{k,i}(u)) = p_{k,i}(Lu) = 0$. This implies that $c_{k,i}(r)$ satisfies

$$A_k c_{k,i} = 0 \text{ on } (0,\infty),$$

where $A_k = -\frac{d^2}{dr^2} - \frac{N-1}{r}\frac{d}{dr} + g'(u_0(r)) + \frac{k(k+N-2)}{r^2}$. Putting $u_{k,i}(x) = c_{k,i}(|x|)$ we obtain $L_k u_{k,i} = 0$, in particular $\langle L_k u_{k,i}, u_{k,i}\rangle = 0$, where $L_k = -\Delta + g'(u_0) + \frac{k(k+N-2)}{|x|^2}$ on $\mathbf{R}^N$. Taking $v(x) = u_0'(|x|)$ (as in the proof of Lemma 2.1), we see that $L_1 v = 0$, that is v is an eigenvector of L_1 corresponding to the eigenvalue 0. Moreover, v is radially symmetric and has constant sign. But it is known that L_1 possesses a first eigenvalue and the corresponding eigenvector (i.e. the ground state of L_1) is radial, does not change sign and any other eigenvector of L_1 changes sign (because it is orthogonal to the ground state). We infer that v must be the ground state of L_1, 0 its first eigenvalue and therefore $L_1 \geq 0$. Since $L_1 u_{1,i} = 0$, we have necessarily $u_{1,i} = c_i v$ for some constants c_i, so that $c_{1,i}(|x|)Y_i^{(1)}\left(\frac{x}{|x|}\right) = c_i u_0'(|x|)\frac{x_i}{|x|} = c_i\frac{\partial u_0}{\partial x_i}$. For $k \geq 2$ we have $L_k = L_1 + \frac{(k-1)(k-1+N)}{|x|^2}$, so that $\langle L_k u_{k,i}, u_{k,i}\rangle = 0$ implies $u_{k,i} = 0$, that is $c_{k,i} = 0$. Thus $u = p_{0,1}(u) + \sum_{i=1}^{N} p_{1,i}(u) = p_{0,1}(u) + \sum_{i=1}^{N} c_i\frac{\partial u_0}{\partial x_i}$ and $p_{0,1}(u) \in H^2_{rad}(\mathbf{R}^N) \cap Ker(L)$. □

Let $a_0 = \sup\{a > 0 \mid g(s) > 0,\ \forall s \in (0,a)\}$. It is clear that $G > 0$ on $(0,a]$ and (2.1) implies that u_0 satisfies the Pohozaev's identity $\int_{\mathbf{R}^N} G(u_0(x))dx = -\frac{N-2}{N}\int_{\mathbf{R}^N} |\nabla u_0|^2 dx < 0$, thus necessarily $u_0(0) > a_0$. We define

$$I(u,\lambda) = \lambda u g'(u) - (\lambda+2)g(u).$$

In the remainder of this section we will make the following assumption: there exists a continuous function $\lambda : (a_0, u_0(0)] \longrightarrow (0,\infty)$ such that for any $U \in (a_0, u_0(0)]$ we have

$$(\mathbf{H5'}) \qquad \begin{array}{ll} I(u,\lambda(U)) \leq 0, & \forall u \in [0,U] \text{ and} \\ I(u,\lambda(U)) \geq 0, & \forall u \in [U, u_0(0)]. \end{array}$$

Note that in the particular case $g(u) = F((r_0-u)^2)(r_0-u)$, we have $I(u,\lambda) = J(u,\lambda)$ and the condition (H5') is in fact assumption (H5).

Theorem 2.6. *Under assumption (H5'), we have $Ker(L) \cap H^2_{rad}(\mathbf{R}^N) = \{0\}$.* Consequently, $Ker(L) = Span\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\}$.

Proof. An easy boot-strap argument shows that any $u \in Ker(L)$ belongs to $W^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,\infty)$, so that $u \in C^{1,\alpha}(\mathbf{R}^N)$ $\forall \alpha \in [0,1)$ and u as well as $\frac{\partial u}{\partial x_i}$, $i = 1,\dots,N$ are bounded and tend to zero at infinity. Let $u(x) = \tilde{\delta}(|x|) = \tilde{\delta}(r) \in Ker(L) \cap H^2_{rad}(\mathbf{R}^N)$. Because u is C^1, necessarily $\tilde{\delta}'(0) = 0$ so $\tilde{\delta}$ must satisfy

$$(2.10) \qquad -\delta'' - \frac{N-1}{r}\delta' + g'(u_0)\delta = 0 \quad \text{on } (0,\infty)$$

together with the boundary conditions

$$(2.11) \qquad \tilde{\delta}'(0) = 0, \qquad \lim_{r\to\infty} \tilde{\delta}(r) = 0.$$

Since $\tilde{\delta} \in C^1([0,\infty))$, (2.10) implies that in fact $\tilde{\delta} \in C^3(0,\infty)$.

It is clear that the linear equation (2.10) with the condition $\delta'(0) = 0$ admits a global solution δ defined on $[0,\infty]$ and any other such solution is a multiple of δ. We may suppose without loss of generality that $\delta(0) = 1$. In order to prove Theorem 2.6, it suffices to show that the function $u_1(x) = \delta(|x|)$ does not belong to $H^2(\mathbf{R}^N)$.

Suppose by contradiction that $u_1 \in H^2_{rad}(\mathbf{R}^N)$. This implies that δ and δ' tend to zero as $r \longrightarrow \infty$. First, we prove that δ has exactly one zero in $(0,\infty)$. Since $u_1 \in L^2(\mathbf{R}^N)$, necessarily $\delta \in L^2((0,\infty), r^{N-1}dr)$. Let $w_1(r) = r^{\frac{N-1}{2}}\delta(r)$. Then $w_1 \in L^2(0,\infty)$ and satisfies

$$(2.12) \qquad Mw_1 = 0,$$

where $M = -\frac{d^2}{dr^2} + g'(u_0) + \frac{(N-1)(N+1)}{4r^2}$. Remark that $Mw = \lambda w$ if and only if $u(x) = |x|^{-\frac{N-1}{2}} w(|x|)$ satisfies $Lu = \lambda u$. Using Lemmas 2.1 and 2.2 we infer that 0 is the second eigenvalue of M, the first being $-\lambda_1$ (with corresponding eigenvector $r^{\frac{N-1}{2}} e_1(r)$). Since w_1 satisfies (2.12), a well-known result (see, for example, Theorem

XIII.8, p. 90 in [43], vol. IV) implies that the number of zeroes of w_1 in $(0,\infty)$ is exactly the number of eigenvalues of M below 0, that is one. It is obvious that $\delta(r)=0$ for $r\in(0,\infty)$ if and only if $w_1(r)=0$, thus δ has exactly one zero, say, r_1. Because δ and δ' cannot vanish simultaneously, δ must change sign at r_1. Therefore $\delta>0$ on $[0,r_1)$, $\delta<0$ on (r_1,∞) and necessarily $\delta'(r_1)<0$.

The rest of the proof was inspired by the ideas developed by K. McLeod in [36].

We show that $u_0(r_1)>a_0$. Suppose that $u_0(r_1)\le a_0$. Then $u_0(r)<a_0$ and $g(u_0(r))>0$ on (r_1,∞). Remark that equations (2.2) and (2.10) can be written as

$$(r^{N-1}u_0'(r))' = r^{N-1}g(u_0(r)), \tag{2.13}$$

respectively

$$(r^{N-1}\delta'(r))' = r^{N-1}g'(u_0(r))\delta(r). \tag{2.14}$$

We obtain from (2.13) and (2.14)

$$[(r^{N-1}u_0'(r))(r^{N-1}\delta'(r))]' = (r^{N-1}u_0'(r))'r^{N-1}\delta'(r) + r^{N-1}u_0'(r)(r^{N-1}\delta'(r))'$$

$$= r^{2N-2}[g(u_0(r))\delta'(r) + g'(u_0(r))u_0'(r)\delta(r)] = r^{2N-2}[g(u_0(r))\delta(r)]'.$$

Integrating this equality from r_1 to ∞ and then integrating by parts we get, taking into account that u_0, u_0' and $g'(u_0)$ tend exponentially to zero and δ,δ' tend to zero as $r\longrightarrow\infty$,

$$-r_1^{2N-2}u_0'(r_1)\delta'(r_1) = \int_{r_1}^{\infty} r^{2N-2}[g(u_0(r))\delta(r)]'dr$$
$$= r^{2N-2}g(u_0(r))\delta(r)\Big|_{r_1}^{\infty} - (2N-2)\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr$$
$$= -(2N-2)\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr.$$

But $r_1^{2N-2}u_0'(r_1)\delta'(r_1)>0$ and $\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr<0$ because $g(u_0)>0$ and $\delta<0$ on that interval, so we obtain a contradiction which proves that $u_0(r_1)>a_0$.

We need the following oscillation result which appears as Lemma 5 in [36] and is a special case of the Sturm comparison theorem:

Lemma 2.7.([36]) *Let Y and Z be nontrivial solutions of*

$$-Y'' - \frac{N-1}{r}Y' + H(r)Y = 0, \qquad \textit{respectively} \tag{2.15}$$

$$-Z'' - \frac{N-1}{r}Z' + h(r)Z = 0 \tag{2.16}$$

on some interval $(\mu,\nu)\subset(0,\infty)$, where H and h are continuous on (μ,ν), $H\ge h$ on (μ,ν) and $H\not\equiv h$. If either

a) $\mu>0$ and $Y(\mu)=Y(\nu)=0$, or

b) $\mu=0$, Y and Z are continuous at 0, $Y'(0)=Z'(0)=0$ and $Y(\nu)=0$, *then Z has at least one zero on (μ, ν). The same conclusion holds if $H \equiv h$ on (μ, ν), provided Y and Z are linearly independent.*

Suppose that (2.15) has at least one solution which does not vanish in some neighbourhood of ∞. We define

$$\rho = \inf\{r \in (0, \infty) \mid \text{there exists a solution of (2.15) with no zeroes in } (0, \infty)\}.$$

The interval (ρ, ∞) is called the disconjugacy interval of (2.15). It is not hard to see that any solution of (2.15) has at least one zero in $[\rho, \infty)$; in fact, it has exactly one by Lemma 2.7,a). The following result holds (for the proof, the reader may consult [36]) :

Lemma 2.8.([36]) *Assume that H is continuous on $(0, \infty)$ and $H(r) > 0$ for large r. Let the disconjugacy inteval of (2.15) be (ρ, ∞) with $\rho > 0$ and suppose that (2.15) has a solution Y_0 with $\lim_{r \to \infty} Y_0(r) = 0$. Then:*

a) $Y_0(\rho) = 0$ and if Y is a nontrivial solution of (2.15) such that $Y(\rho) = 0$, there exists c such that $Y = cY_0$.

b) If Y is a nontrivial solution of (2.15) with a zero in (ρ, ∞), then $Y(r) \longrightarrow \pm\infty$ as $r \longrightarrow \infty$.

We will also make use of the following well-known result about the ground state u_0 (for a proof, see [40]) :

Lemma 2.9.([40]) *We have* $\lim_{r \to \infty} \frac{u_0'(r)}{u_0(r)} = -\sqrt{g'(0)} < 0$.

Now let us show how assumption (H5') implies the conclusion of Theorem 2.6. For $\lambda > 0$, define

$$v_\lambda(r) = ru_0'(r) + \lambda u_0(r). \tag{2.17}$$

A simple calculation using (2.2) shows that v_λ satisfies

$$-v_\lambda'' - \frac{N-1}{r} v_\lambda' + g'(u_0)v_\lambda = \lambda g'(u_0(r))u_0(r) - (\lambda+2)g(u_0(r)) = I(u_0(r), \lambda). \tag{2.18}$$

Equivalently, v_λ is a solution of

$$-v_\lambda'' - \frac{N-1}{r} v_\lambda' + \left(g'(u_0) - \frac{I(u_0(r), \lambda)}{v_\lambda}\right) v_\lambda = 0 \tag{2.19}$$

on any interval which does not contain any zero of v_λ.

Let $\lambda_1 = \lambda(u(r_1))$ and $\lambda_2 = \lambda(u(0))$, where $\lambda(U)$ is given by assumption (H5'). Then $I(u(r), \lambda_1) \geq 0$ on $[0, r_1]$ and $I(u(r), \lambda_1) \leq 0$ on $[r_1, \infty)$, while $I(u(r), \lambda_2) \leq 0$ for all $r \in [0, \infty)$. By (2.10), (2.19) and Lemma 2.7, v_{λ_1} oscillates faster than δ on any subinterval of $[0, r_1]$ on which $v_{\lambda_1} > 0$. Since $v_{\lambda_1}(0) = \lambda_1 u_0(0) > 0$ and $(\delta(r_1)) = 0$, it follows that the first zero of v_{λ_1} occurs in $(0, r_1]$. Similarly, v_{λ_2} oscillates slower than δ as long as $v_{\lambda_2} > 0$, hence the first zero of v_{λ_2} occurs in $[r_1, \infty)$.

Lemma 2.10. *Assume that for a certain $\lambda > 0$ we have $I(u_0(r), \lambda) \leq 0$ on $[r_1, \infty)$ and there exists $r_2 \geq r_1$ such that $v_\lambda(r_2) < 0$. Then $v_\lambda < 0$ on $[r_2, \infty)$.*

Proof. Suppose by contradiction that there exists $r > r_2$ such that $v_\lambda(r) = 0$. Let $r_3 = \inf\{r > r_2 \mid v_\lambda(r) = 0\}$. Obviously $v_\lambda(r_3) = 0$ and $v'_\lambda(r_3) \geq 0$.

We claim that $v'_\lambda(r_3) > 0$. Indeed, if $v'_\lambda(r_3) = 0$, (2.18) would imply $v''_\lambda(r_3) = -I(u_0(r_3), \lambda) \geq 0$. Since $v_\lambda < 0$ on (r_2, r_3), r_3 cannot be a local minimum of v_λ; so necessarily $v''_\lambda(r_3) = 0$ and $I(u_0(r_3), \lambda) = 0$. From the equalities $v_\lambda(r_3) = v'_\lambda(r_3) = v''_\lambda(r_3) = 0$, $I(u_0(r_3), \lambda) = 0$ it can be easily deduced that $u'_0(r_3) = 0$, a contradiction. Thus $v'_\lambda(r_3) > 0$.

It follows that $v_\lambda > 0$ on an interval $(r_3, r_3 + \eta)$. On the other hand, it follows from (2.17) and Lemma 2.9 that $v_\lambda(r)$ is negative for large r, therefore v_λ must vanish after r_3. Let $r_4 = \inf\{r > r_3 \mid v_\lambda(r) = 0\}$. Then $v_\lambda > 0$ on (r_3, r_4) and comparing (2.10) and (2.19) we infer that δ oscillates faster than v_λ on (r_3, r_4), thus δ must vanish on $[r_3, r_4]$, contradicting the fact that r_1 is the unique zero of δ. This proves the lemma. □

Coming back to the proof of Theorem 2.6, we show that the first zero of v_{λ_1} occurs in $(0, r_1)$. Suppose by contradiction that it occurs exactly at r_1. Then we have $v_{\lambda_1}(r_1) = \delta(r_1) = 0$, $v_{\lambda_1} \longrightarrow 0$ exponentially and $\delta, \delta' \longrightarrow 0$ as $r \longrightarrow \infty$. Using (2.14) and (2.19) and integrating by parts we get

$$\int_{r_1}^{\infty} r^{N-1} g'(u_0)\delta v_{\lambda_1} dr = \int_{r_1}^{\infty} (r^{N-1}\delta')' v_{\lambda_1} dr = -\int_{r_1}^{\infty} r^{N-1}\delta' v'_{\lambda_1} dr$$
$$= \int_{r_1}^{\infty} (r^{N-1} v'_{\lambda_1})'\delta(r) dr = \int_{r_1}^{\infty} r^{N-1}[g'(u_0)v_{\lambda_1} - I(u_0(r), \lambda_1)]\delta(r) dr.$$

Thus $\int_{r_1}^{\infty} r^{N-1} I(u_0(r), \lambda_1)\delta(r) dr = 0$. But $I(u_0(r), \lambda_1) \leq 0$ and $\delta < 0$ on (r_1, ∞), so necessarily $I(u_0(r), \lambda_1) \equiv 0$ on $[r_1, \infty)$, that is $\lambda_1 u g'(u) - (\lambda_1 + 2)g(u) = 0$ for $u \in (0, u_0(r_1)]$, which implies $g(u) = Au^{\frac{\lambda_1+2}{\lambda_1}}$ on $(0, u_0(r_1)]$ for some constant A, contradicting the fact that $g'(0) > 0$. Hence the first zero of v_{λ_1} occurs in $(0, r_1)$.

It is clear that $v_{\lambda_1} - v_{\lambda_2} = (\lambda_1 - \lambda_2)u_0$ has the same sign as $\lambda_1 - \lambda_2$ on $[0, \infty)$. Since the first zero of v_{λ_1} occurs before the first zero of v_{λ_2}, we must have $\lambda_1 < \lambda_2$.

We infer that there exists $\lambda'_0 \in (\lambda_1, \lambda_2]$ such that the first zero of $v_{\lambda'_0}$ occurs exactly at r_1. Choose $\lambda_0 \in (\lambda_1, \lambda'_0)$ such that the first zero of v_{λ_0} occurs before r_1 and $v_{\lambda_0}(r_1) < 0$. Let $r_0^\star$ be the last zero of v_{λ_0} before r_1. Since $\lambda_1 = \lambda(u_0(r_1))$, $\lambda_2 = \lambda(u_0(0))$ and $r \longmapsto \lambda(u(r))$ is continuous, there exists $\tilde{r}_0 \in (0, r_1)$ such that $\lambda_0 = \lambda(u_0(\tilde{r}_0))$. Let $r_0 = \max(r_0^\star, \tilde{r}_0)) < r_1$. Then $I(u_0(r), \lambda_0) \leq 0$, $\forall r \in [r_0, \infty)$ and $v_{\lambda_0}(r_1) < 0$. By Lemma 2.10 we have $v_{\lambda_0} < 0$ on $[r_1, \infty)$, hence $v_{\lambda_0} < 0$ on (r_0, ∞).

Consider the solution δ_0 of (2.10) with $\delta_0(r_0) = 0$, $\delta'_0(r_0) = 1$. Then δ_0 cannot have any zero in (r_0, ∞) since if $\delta_0(r_4) = 0$ for some $r_4 \in (r_0, \infty)$ we would infer from (2.10), (2.19) and Lemma 2.7 that v_{λ_0} has a zero in (r_0, r_4), which is absurd. Consequently (r_0, ∞) is contained in the disconjugacy interval of (2.10). But δ is a solution of (2.10) which vanishes at r_1 and r_1 is an interior point of the disconjugacy interval of (2.10). Using lemma 2.8b) we infer that $\delta(r) \longrightarrow -\infty$ as $r \longrightarrow \infty$, which contradicts the assumption $u_1(x) = \delta(|x|) \in H^2(\mathbf{R}^N)$. This finishes the proof of Theorem 2.6. □

3.3 A local variant of the Saddle-Point Theorem

In this section we present a general abstract result in critical point theory which generalizes the classical Saddle-Point Theorem. The proof is based on a sharp deformation result (the Quantitative Deformation Lemma) due to M. Willem.

Theorem 3.1. *Let E be a Banach space and $\varphi : E \longrightarrow \mathbf{R}$ a C^1-functional. Let F be a finite-dimensional subspace and G a closed subspace of E such that $F+G=E$ and $F\cap G=\{0\}$. Suppose that there exist $r>0$ and an open set $\Omega \subset G$ containing 0 with the following properties:*

i) $\varphi(x)\le 0$ if $x\in B_F(0,r)$;

ii) $\varphi(x+y)\le \mu_0<0$ if $x\in F$, $r_1\le ||x||\le r$ for some $r_1<r$ and $y\in\Omega$;

iii) $\varphi(y)\ge\mu_1>\mu_0$ if $y\in\Omega$;

iv) there exists $0<\delta_0<dist(0,\partial\Omega)$ and a continuous function $h:\Omega(\delta_0)=\{y\in\Omega \mid dist(y,\partial\Omega)\le\delta_0\}\longrightarrow[0,r)$ such that for all $x\in F$ with $||x||=r$ and for all $y\in\Omega(\delta_0)$, the function $t\longmapsto\varphi(tx+y)$ is not increasing on $[\frac{h(y)}{r},1]$;

v) $\varphi(x+y)\ge 0$ if $y\in\Omega(\delta_0)$ and $||x||\le h(y)$.

Then there exists $c\in[\mu_1,0]$ and a sequence $z_n\in B_F(0,r)+\Omega$ such that:

a) $\varphi(z_n)\longrightarrow c$ and

b) $\varphi'(z_n)\longrightarrow 0$ as $n\longrightarrow\infty$.

Remark 3.2. A sequence satisfying a) and b) is called a Palais-Smale sequence for φ. The functional φ is said to have the Palais-Smale property if any Palais-Smale sequence contains a convergent subsequence. Thus if φ satisfies the assumptions of Theorem 3.1 and has the Palais-Smale property, it has a critical point in $B_F(0,r)+\Omega$.

Remark 3.3. If φ' is bounded on bounded sets of E, we may replace assumption ii) by $\varphi(x+y)\le\mu_0<0$ if $x\in F$, $||x||=r$ and $y\in\Omega$.

Proof of Theorem 3.1 We denote $\varphi^d=\varphi^{-1}((-\infty,d])$ and for a given subset $S\subset E$ and $\rho>0$ we denote $S_\rho=\{u\in E \mid dist(u,S)\le\rho\}$. We shall make use of the following Quantitative Deformation Lemma of M. Willem:

Lemma 3.4.([50]) *Let X be a Banach space, $\varphi\in C^1(X,\mathbf{R})$, $S\subset X$, $c\in\mathbf{R}$, $\varepsilon,\delta>0$ such that :*

$$||\varphi'(u)||\ge\frac{8\varepsilon}{\delta},\quad \forall u\in\varphi^{-1}([c-2\varepsilon,c+2\varepsilon]\cap S_{2\delta}). \tag{3.1}$$

Then there exists $\eta\in C([0,1]\times X,X)$ such that

i) $\eta(t,u)=u$ if $t=0$ or if $u\notin\varphi^{-1}([c-2\varepsilon,c+2\varepsilon]\cap S_{2\delta})$,

ii) $\eta(1,\varphi^{c+\varepsilon}\cap S)\subset\varphi^{c-\varepsilon}$,

iii) $\eta(t,\cdot)$ is an homeomorphism of X, $\forall t\in[0,1]$,

iv) $||\eta(t,u)-u||\le\delta$, $\forall u\in X$, $\forall t\in[0,1]$,

v) $\varphi(\eta(\cdot,u))$ is non-increasing on $[0,1]$, $\forall u\in X$,

vi) $\varphi(\eta(t,u))<c$, $\forall u\in\varphi^c\cap S_\delta$, $\forall t\in(0,1]$.

Let $\Gamma=\{\gamma\in C(B_F(0,r),B_F(0,r)+\Omega) \mid \gamma_{|\partial B_F(0,r)}=id\}$ and

$$c=\inf_{\gamma\in\Gamma}\max_{x\in B_F(0,r)}\varphi(\gamma(x)). \tag{3.2}$$

Taking $\gamma_0 = id_{B_F(0,r)} \in \Gamma$, it follows from assumption i) in Theorem 3.1 that $c \le 0$. We claim that $c \ge \mu_1$. Indeed, let p_F be the canonical projection from E onto F. For any $\gamma \in \Gamma$, $p_F \circ \gamma$ is a continuous mapping of $B_F(0,r)$ into itself and $p_F \circ \gamma_{|\partial B_F(0,r)} = id$, so that there exists $x_\gamma \in B_F(0,r)$ such that $p_F \circ \gamma(x_\gamma) = 0$, that is $\gamma(x_\gamma) \in \Omega$ (at this point we use the fact that F is finite-dimensional). From assumption iii) we have $\varphi(\gamma(x_\gamma)) \ge \mu_1$, so obviously $\max_{x \in B_F(0,r)} \varphi(\gamma(x)) \ge \mu_1$, which proves the claim.

If $c = 0$, the infimum in (3.2) is achieved for $\gamma_0 = id_{B_F(0,r)}$. We claim that in this case there exists a critical point of φ in $S = \{x \in B_F(0,r) \mid \varphi(x) = 0\}$. Indeed, suppose that this is false. Since S is compact and $S \subset Int(B_F(0,r) + \Omega)$, there exists $\varepsilon_0 > 0$ such that

$$(3.3) \qquad ||\varphi'(x)|| \ge 16\varepsilon_0, \ \forall x \in S_{\varepsilon_0} \text{ and } dist(S, \partial(B_F(0,r)+\Omega)) > 2\varepsilon_0.$$

We may apply Lemma 3.4 to φ, S, $c = 0$, $\delta = \frac{1}{2}\varepsilon_0$ and $\varepsilon = \varepsilon_0^2$ and we obtain a continuous mapping $\eta : [0,1] \times E \longrightarrow E$ with properties i)-vi) in that Lemma. Define $\gamma_1 : B_F(0,r) \longrightarrow E$ by $\gamma_1(x) = \eta(1,x)$. By (3.3) and Lemma 3.4 i) and iii) it follows that $\gamma_1 \in \Gamma$ and from Lemma 3.4 ii) and v) we infer that $\gamma_1(x) \le -\varepsilon$, $\forall x \in S$, so $\max_{x \in B_F(0,r)} \varphi(\gamma_1(x)) < 0$, contrary to the assumption that $c = 0$.

Hence Theorem 3.1 is proved in the case $c = 0$. From now on we may assume that $c < 0$. Let $S = \{x + y \mid x \in B_F(0,r_1), y \in \Omega, dist(y, \partial\Omega) \ge \frac{\delta_0}{2}\}$. Let $0 < \tilde{\delta} < \frac{1}{4} dist(S, \partial(B_F(0,r)+\Omega))$. To prove Theorem 3.1, it suffices to show that for any $\varepsilon > 0$ such that $c + 2\varepsilon < 0$ and $c - 2\varepsilon > \mu_0$, there exists $z_\varepsilon \in S_{2\tilde{\delta}}$ such that

$$(3.4) \qquad c - 2\varepsilon \le \varphi(z_\varepsilon) \le c + 2\varepsilon \quad \text{and} \quad ||\varphi'(z_\varepsilon)|| < \frac{8\varepsilon}{\tilde{\delta}}.$$

Suppose that this thesis is false. Consider h and δ_0 as given by assumption iv). Define $h_0 : \Omega(\delta_0) \longrightarrow [0,r]$ by

$$h_0(y) = \begin{cases} r & \text{if } dist(y,\partial\Omega) < \frac{\delta_0}{2} \\ \frac{2}{\delta_0}(h(y) - r) \cdot dist(y, \partial\Omega) + 2r - h(y) & \text{if } \frac{\delta_0}{2} \le dist(y,\partial\Omega) \le \delta_0. \end{cases}$$

It is clear that h_0 is continuous and $h_0(y) \ge h(y)$. Let

$$W = (B_F(0,r)+\Omega)) \setminus \{x + y \mid y \in \Omega(\delta_0), \ ||x|| < h(y)\} \quad \text{and}$$
$$W_0 = (B_F(0,r)+\Omega)) \setminus \{x + y \mid y \in \Omega(\delta_0), \ ||x|| < h_0(y)\}$$

Observe that $z \in W_0$ and $\varphi(z) \ge c - 2\varepsilon$ implies $z \in S$. Define $\psi : W \longrightarrow W_0$ by

$$\psi(x+y) = \begin{cases} h_0(y)\frac{x}{||x||} + y & \text{if } y \in \Omega \text{ and } h(y) \le ||x|| \le h_0(y) \\ x + y & \text{otherwise.} \end{cases}$$

It is easy to see that ψ is continuous and in view of assumption iv) we have $\varphi(z) \ge \varphi(\psi(z))$, $\forall z \in W$.

If ε is such that $\mu_0 < c - 2\varepsilon$ and $c + 2\varepsilon < 0$, consider $\gamma \in \Gamma$ such that $\max_{x \in B_F(0,r)} \varphi(\gamma(x)) < c + \varepsilon$. Since $\varphi(x+y) \ge 0 > c + \varepsilon$ if $y \in \Omega(\delta_0)$ and $||x|| < h(y)$, we have necessarily $\gamma(x) \in W$, $\forall x \in B_F(0,r)$. Let $\gamma_2 = \psi \circ \gamma$. Then $\gamma_2 \in \Gamma$ and $\max_{x\in B_F(0,r)} \varphi(\gamma_2(x)) \le \max_{x\in B_F(0,r)} \varphi(\gamma(x)) < c+\varepsilon$.

We apply Lemma 3.4 for the functional φ, the set S, c, ε and $\tilde{\delta}$ and we get $\eta \in C([0,1]\times E, E)$ with properties i)-vi) in that Lemma. Let $\gamma_3(x) = \eta(1,\gamma_2(x))$, $x \in B_F(0,r)$. Since $\varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \gamma_2(B_F(0,r)) \subset \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap W_0 \subset S$ and $dist(S, \partial(B_F(0,r)+\Omega)) > 2\tilde{\delta}$, we infer from Lemma 3.4 i) and iv) that $\gamma_3(x) \in B_F(0,r)+\Omega, \forall x \in B_F(0,r)$ and $g_{3|\partial B_F(0,r)} = id$, hence $\gamma_3 \in \Gamma$. From Lemma 3.4, ii) it follows that $\max_{x\in B_F(0,r)} \varphi(\gamma_3(x)) < c-\varepsilon$, contrary to (3.2). This contradiction proves Theorem 3.1. □

3.4 Application to the functional E_c

We have already introduced the functionals E and E_c in Introduction. In this section we study the behaviour of the functional E_c near the ground state u_0 of (1.8) given by Theorem 1.1 and we prove that E_c admits a nontrivial critical point if c is sufficiently small. Let us verify first that E and E_c are well-defined on $\mathbf{H}$ and of class C^2 if $N \ge 4$. It is clear that the mapping $(u_1,u_2) \longmapsto V((r_0-u_1)^2+u_2^2)$ is of class $C^2(\mathbf{R}^2)$. We have $\sigma = \frac{4}{N-2} \le 2$ because $N \ge 4$. Taking into account that for $\alpha > \beta$, $|u|^\alpha \le C|u|^\beta$ for $|u|$ small, respectively $|u|^\beta \le C|u|^\alpha$ for $|u|$ large, the following estimates hold:

$$\begin{aligned}
|V((r_0-u_1)^2+u_2^2)| &\le C|-2r_0u_1+u_1^2+u_2^2|^2\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} \\
&\quad + C|u_1^2+u_2^2|^{\frac{\sigma}{2}+1}\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'(|u_1|^2+|u_1|^{2+\sigma}+|u_2|^{2+\sigma}),
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial}{\partial u_1}V((r_0-u_1)^2+u_2^2)| &= |2F((r_0-u_1)^2+u_2^2)(r_0-u_1)| \\
&\le C|-2r_0u_1+u_1^2+u_2^2|\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} \\
&\quad + C((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}|r_0-u_1|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'(|u_1|+|u_1|^{1+\frac{\sigma}{2}}+|u_2|^{1+\frac{\sigma}{2}}) + C'(|u_1|^{1+\sigma}+|u_2|^{1+\sigma}),
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial}{\partial u_2}V((r_0-u_1)^2+u_2^2)| &= |-2F((r_0-u_1)^2+u_2^2)u_2| \\
&\le C|-2r_0u_1+u_1^2+u_2^2|\cdot|u_2|\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} \\
&\quad + C((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}|u_2|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'(|u_1|^{\frac{2\sigma}{2+\sigma}}+|u_1|^\sigma+|u_2|^\sigma)|u_2|,
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^2}{\partial u_1^2}V((r_0-u_1)^2+u_2^2)| &= |-4F'((r_0-u_1)^2+u_2^2)(r_0-u_1)^2-2F((r_0-u_1)^2+u_2^2)| \\
&\le C\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} + C(((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'(1+|u_1|^\sigma+|u_2|^\sigma)
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^2}{\partial u_1\partial u_2}V((r_0-u_1)^2+u_2^2)| &= |4F'((r_0-u_1)^2+u_2^2)(r_0-u_1)u_2| \\
&\le C|u_2|\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} \\
&\quad + C((r_0-u_1)^2+u_2^2)^{-1+\frac{\sigma}{2}}|r_0-u_1||u_2|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'|u_2|^{\frac{\sigma}{2}} + C'(|u_1|^\sigma+|u_2|^\sigma),
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^2}{\partial u_2^2}V((r_0-u_1)^2+u_2^2)| &= |-4F'((r_0-u_1)^2+u_2^2)u_2^2-2F((r_0-u_1)^2+u_2^2)| \\
&\leq C(|u_2|^2+|-2r_0u_1+u_1^2+u_2^2|)\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} \\
&\quad +C((r_0-u_1)^2+u_2^2)^\sigma\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'(|u_2|^\sigma+|u_1|)+C'(|u_1|^\sigma+|u_2|^\sigma) \\
&\leq C''(|u_1|^{\frac{2\sigma}{2+\sigma}}+|u_1|^\sigma+|u_2|^\sigma).
\end{aligned}$$

From these estimates it follows that I is a C^2-functional from $(L^2\cap L^{2+\sigma}(\mathbf{R}^N))\times L^{2+\sigma}(\mathbf{R}^N)$ to $\mathbf{R}$. In view of the Sobolev embedding, I is of class C^2 on $\mathbf{H}=H^1(\mathbf{R}^N)\times\mathcal{D}^{1,2}(\mathbf{R}^N)$ and consequently so are E and E_c.

In order to apply Theorem 3.1 to the functional E_c near u_0, we are interested in the geometry of the level sets of E and E_c in a neighbourhood of u_0. We can get some basic information about the behaviour of E and E_c near u_0 by studying the differential $E'(u_0,0)$.

We have already seen that u_0 is a critical point of E, that is $d_{u_1}E(u_0,0)=0$ and $d_{u_2}E(u_0,0)=0$. An easy calculation gives $d^2_{u_1,u_1}E(u_0,0).(v,v)=2\langle Av,v\rangle$, where A is the operator introduced in (1.9), and $d^2_{u_1,u_2}E(u_0,0)=0$. We have:

Lemma 4.1. $d^2_{u_2,u_2}E(u_0,0).(v,v)=2\displaystyle\int_{\mathbf{R}^N}(r_0-u_0)^2\left|\nabla\left(\frac{v}{r_0-u_0}\right)\right|^2dx.$

Proof. In view of Theorem 1.1, the linear mapping $v\longmapsto(r_0-u_0)v$ is a continuous isomorphism of $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and its inverse is $w\longmapsto\frac{w}{r_0-u_0}$. Using equation (1.8) satisfied by u_0 and integrating by parts we get

$$\begin{aligned}
&\int_{\mathbf{R}^N}F((r_0-u_0)^2)(r_0-u_0)^2v^2dx=\int_{\mathbf{R}^N}(\Delta u_0)(r_0-u_0)v^2dx \\
&=-\int_{\mathbf{R}^N}(r_0-u_0)\Delta(r_0-u_0)v^2dx \\
&=\int_{\mathbf{R}^N}|\nabla(r_0-u_0)|^2v^2dx+2\int_{\mathbf{R}^N}(r_0-u_0)v\nabla(r_0-u_0).\nabla v\,dx,
\end{aligned}$$

so we obtain

$$\begin{aligned}
&d^2_{u_2,u_2}E(u_0,0).((r_0-u_0)v,(r_0-u_0)v) \\
&=2\int_{\mathbf{R}^N}|\nabla((r_0-u_0)v)|^2dx+\int_{\mathbf{R}^N}\frac{\partial^2}{\partial u_2^2}(V((r_0-u_1)^2+u_2^2))_{|u_1=u_0,u_2=0}.(r_0-u_0)^2v^2dx \\
&=2\int_{\mathbf{R}^N}|\nabla((r_0-u_0)v)|^2dx-2\int_{\mathbf{R}^N}F((r_0-u_0)^2)(r_0-u_0)^2v^2dx \\
&=2\int_{\mathbf{R}^N}(r_0-u_0)^2|\nabla v|^2dx.
\end{aligned}$$

This proves Lemma 4.1. □

Let $H(v)=\displaystyle\int_{\mathbf{R}^N}(r_0-u_0)^2|\nabla\left(\frac{v}{r_0-u_0}\right)|^2dx$. Note that $H(v)^{\frac{1}{2}}$ defines a norm on $\mathcal{D}^{1,2}(\mathbf{R}^N)$ equivalent to the usual norm $||v||_{\mathcal{D}^{1,2}}=\left(\displaystyle\int_{\mathbf{R}^N}|\nabla v|^2dx\right)^{\frac{1}{2}}$.

Because we have not good estimates of $E(u_0+u_1,0)=E(u_0)$ for $u_1 \in Ker(A)$, we work for the moment only on the space $(\mathbf{R}e_1+Y)\times \mathcal{D}^{1,2}(\mathbf{R}^N)$ and we show that the restriction of E_c to this space admits a critical point near u_0 for c small. It will be seen later that this is in fact a critical point of E_c on the whole $\mathbf{H}$.

Since E is of class C^2 and $E'(u_0,0)=0$, $d^2_{u_1,u_2}E(u_0,0)=0$, using the Taylor expansion we may write for $u_1 \in Y$, $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ with $||(u_1,u_2)||_{\mathbf{H}}$ small and $t \in \mathbf{R}$, t small

$$E(u_0+u_1+te_1,u_2)=E(u_0,0)+\langle A(u_1+te_1),(u_1+te_1)\rangle+H(u_2)+h(t,u_1,u_2) \tag{4.1}$$

and

$$\begin{aligned} d_{u_1}E(u_0+u_1+te_1,u_2) &= d^2_{u_1,u_1}E(u_0,0)(u_1+te_1,\cdot) \\ &\quad +d^2_{u_1,u_2}E(u_0,0)(\cdot,u_2)+L(t,u_1,u_2) \\ &= 2A(u_1+te_1)+L(t,u_1,u_2) \end{aligned} \tag{4.2}$$

where $h:\mathbf{R}\times Y\times \mathcal{D}^{1,2}(\mathbf{R}^N)\longrightarrow \mathbf{R}$, $L:\mathbf{R}\times Y\times \mathcal{D}^{1,2}(\mathbf{R}^N)\longrightarrow H^{-1}(\mathbf{R}^N)$, $|h(t,u_1,u_2)| = o(|t|^2+||(u_1,u_2)||^2_{\mathbf{H}})$ and $||L(t,u_1,u_2) = o(|t|+||(u_1,u_2)||_{\mathbf{H}})$ as $(t,u_1,u_2)\longrightarrow (0,0,0)$.

For each $\varepsilon>0$ consider t_ε, $r_\varepsilon>0$ such that

$$\begin{aligned} &|h(t,u_1,u_2)|\le \varepsilon(|t|^2+||(u_1,u_2)||^2_{\mathbf{H}}) \quad \text{and} \\ &||L(t,u_1,u_2)||\le \varepsilon(|t|+||(u_1,u_2)||_{\mathbf{H}}) \end{aligned} \tag{4.3}$$

if $|t|\le t_\varepsilon$ and $||(u_1,u_2)||_{\mathbf{H}}\le r_\varepsilon$. For $|t|\le t_\varepsilon$ we have

$$E(u_0+te_1,0)-E(u_0,0)=t^2\langle Ae_1,e_1\rangle+h(t,0,0)\le -\lambda_1 t^2+\varepsilon t^2. \tag{4.4}$$

If $u_1\in Y$ and $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, it follows from Lemmas 2.4 and 4.1 that there exist two positive constants γ_1, γ_2 such that

$$\gamma_1||(u_1,u_2)||^2_{\mathbf{H}}\le \langle Au_1,u_1\rangle+H(u_2)\le \gamma_2||(u_1,u_2)||^2_{\mathbf{H}}. \tag{4.5}$$

Next, we show that E is "small" in a cone $\{te_1+u_1+iu_2 \in (\mathbf{R}e_1+Y)\times \mathcal{D}^{1,2}(\mathbf{R}^N) \mid ||(u_1,u_2)||_{\mathbf{H}}\le kt,\ t\in[-\tilde{t}_\varepsilon,\tilde{t}_\varepsilon]\ \}$ and is "large" in a cone $\{te_1+u_1+iu_2 \in (\mathbf{R}e_1+Y)\times \mathcal{D}^{1,2}(\mathbf{R}^N) \mid t\le l||(u_1,u_2)||_{\mathbf{H}},\ ||(u_1,u_2)||_{\mathbf{H}}\le \tilde{r}_\varepsilon\ \}$, where k and l do not depend on ε.

Let $\varepsilon \le \min(1,\frac{\lambda_1}{4},\frac{\gamma_1}{4})$. Let $k=\sqrt{\frac{\lambda_1}{4(1+\gamma_2)}}$. If $|t|\le \min(t_\varepsilon,\frac{r_\varepsilon}{k})$ and $||(u_1,u_2)||_{\mathbf{H}}\le k|t|$, by (4.1) and (4.3) we have

$$\begin{aligned} &E(u_0+u_1+te_1,u_2)-E(u_0,0) \\ &\le -\lambda_1 t^2+\gamma_2||(u_1,u_2)||^2_{\mathbf{H}}+\varepsilon(t^2+||(u_1,u_2)||^2_{\mathbf{H}}) \\ &\le -\lambda_1 t^2+\gamma_2 k^2 t^2+\varepsilon(1+k^2)t^2\le -\tfrac{\lambda_1}{2}t^2. \end{aligned} \tag{4.6}$$

Let $l=\frac{1}{4}\sqrt{\frac{\gamma_1}{1+\lambda_1}}$. If $||(u_1,u_2)||_{\mathbf{H}}\le \min(r_\varepsilon,\frac{t_\varepsilon}{l})$ and $|t|\le l||(u_1,u_2)||_{\mathbf{H}}$ we have

$$\begin{aligned} &E(u_0+u_1+te_1,u_2)-E(u_0,0) \\ &\ge -\lambda_1 t^2+\gamma_1||(u_1,u_2)||^2_{\mathbf{H}}-\varepsilon(t^2+||(u_1,u_2)||^2_{\mathbf{H}}) \\ &\ge ||(u_1,u_2)||^2_{\mathbf{H}}(\gamma_1-\lambda_1 l^2-\varepsilon l^2-\varepsilon) \\ &\ge \tfrac{\gamma_1}{2}||(u_1,u_2)||^2_{\mathbf{H}}. \end{aligned} \tag{4.7}$$

From now on, we consider throughout that $0 < \varepsilon < \min(1, \frac{\lambda_1}{4}, \frac{\gamma_1}{4}, \frac{3l\lambda_1}{4||e_1||_{H^1}})$. The next lemma says that assumption iv) in Theorem 3.1 is satisfied.

Lemma 4.2. *There exists $c_0 > 0$ such that for any $c \in [-c_0, c_0]$ and any $(u_1, u_2) \in Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ with $||(u_1, u_2)||_{\mathbf{H}} \leq \min(r_\varepsilon, \frac{t_\varepsilon}{l})$ the function*

$$t \longmapsto E_c(u_0 + u_1 + te_1, u_2)$$

is increasing on $[-t_\varepsilon, -l||(u_1, u_2)||_{\mathbf{H}}]$ and decreasing on $[l||(u_1, u_2)||_{\mathbf{H}}, t_\varepsilon]$.

Proof. Using (4.2), (4.3) and the identities $\langle Au_1, e_1\rangle = 0$, $\langle Ae_1, e_1\rangle = -\lambda_1$, we obtain on $[-t_\varepsilon, -l||(u_1, u_2)||_{\mathbf{H}}]$:

$$\frac{d}{dt}E_c(u_0 + u_1 + te_1, u_2) = d_{u_1}E(u_0 + u_1 + te_1, u_2).e_1 - 2c\int_{\mathbf{R}^N} e_1 u_{2x_1}dx$$

$$= 2\langle A(u_1 + te_1), e_1\rangle + L(t, u_1, u_2)e_1 - 2c\int_{\mathbf{R}^N} e_1 u_{2x_1}dx$$

$$\geq -2\lambda_1 t - \varepsilon(|t| + ||(u_1, u_2)||_{\mathbf{H}})||e_1||_{H^1} - 2|c| \cdot ||u_2||_{\mathcal{D}^{1,2}}$$

$$\geq (2\lambda_1 - \varepsilon)|t| - (\varepsilon||e_1||_{H^1} + 2|c|)||(u_1, u_2)||_{\mathbf{H}}$$

$$\geq [(2\lambda_1 - \varepsilon)l - (\varepsilon||e_1||_{H^1} + 2|c|)] \cdot ||(u_1, u_2)||_{\mathbf{H}}.$$

Taking $c_0 = \frac{l\lambda_1}{2}$, since $\varepsilon < \min(\frac{\lambda_1}{4}, \frac{3l\lambda_1}{4||e_1||_{\mathbf{H}}})$, it is clear that the last quantity is positive for $|c| < c_0$. A similar estimate holds on $[l||(u_1, u_2)||_{\mathbf{H}}, t_\varepsilon]$. □

Theorem 4.3. *There exists $c_1 > 0$ such that for all $c \in [-c_1, c_1]$, the functional $\varphi_c(u_1, u_2) = E_c(u_0 + u_1, u_2) - E_c(u_0, 0)$ restricted to $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ admits a critical point $(u_{1,c}, u_{2,c})$. Moreover, $(u_{1,c}, u_{2,c}) \longrightarrow (0, 0)$ as $c \longrightarrow 0$.*

Proof. Let $t_0 = \min(t_\varepsilon, \frac{r_\varepsilon}{k})$. Let $r_0 = \min(r_\varepsilon, \frac{t_\varepsilon}{l}, kt_0)$. Now fix $t \in (0, t_0]$ and let $r(t) = \min(r_0, kt)$. If c is sufficiently small, we show that the assumptions of Theorem 3.1 are satisfied for $F = \mathbf{R}e_1$, $G = Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$, $B_F(0, r) = [-t, t]e_1$, $\Omega = B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$, $\mu_0 = -\frac{\lambda_1}{4}t^2$, $\mu_1 = -\frac{\lambda_1}{8}t^2$, $\delta_0 = \frac{r(t)}{2}$ and $h(u_1, u_2) = l||(u_1, u_2)||_{\mathbf{H}}$.

If $\tau \in [-t, t]$, using (4.4) we have

$$\begin{aligned} \varphi_c(\tau e_1, 0) &= E_c(u_0 + \tau e_1, 0) - E_c(u_0, 0) \\ &= E(u_0 + \tau e_1, 0) - E(u_0, 0) \leq (-\lambda_1 + \varepsilon)\tau^2. \end{aligned} \tag{4.8}$$

Because $0 < \varepsilon < \frac{\lambda_1}{4}$, assumption i) is satisfied.

Since Q is bounded on bounded sets of $\mathbf{H}$, there exists $c(t) \in (0, c_0]$ such that for any c with $|c| \leq c(t)$,

$$|cQ(u_0 + u_1 \pm te_1, u_2)| < \frac{\lambda_1}{4}t^2 \text{ for } (u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t)) \quad \text{and} \tag{4.9}$$

$$|cQ(u_0 + u_1 + \tau e_1, u_2)| < \min(\frac{\lambda_1}{8}t^2, \frac{\gamma_1}{16}r(t)^2) \tag{4.10}$$

for $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$ and $|\tau| \le l||(u_1, u_2)||_{\mathbf{H}}$.

If $|c| \le c(t)$ and $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$, by (4.6), the choice of $r(t)$ and (4.9) we have

$$\begin{aligned} \varphi_c(\pm te_1 + u_1, u_2) &= E(u_0 + u_1 \pm te_1, u_2) - E(u_0, 0) \\ &+Q(u_0 + u_1 \pm te_1, u_2) \le -\tfrac{\lambda_1}{2}t^2 + \tfrac{\lambda_1}{4}t^2 = -\tfrac{\lambda_1}{4}t^2, \end{aligned} \tag{4.11}$$

Since φ_c' is bounded on bounded sets of $\mathbf{H}$, assumption ii) is verified (see also Remark 3.3).

Using (4.7) and (4.10) we get for $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$

$$\begin{aligned} \varphi_c(u_1, u_2) &= E(u_0 + u_1, u_2) - E(u_0, 0) + cQ(u_0 + u_1, u_2) \\ &\ge \tfrac{\gamma_1}{2}||(u_1, u_2)||_{\mathbf{H}} - \tfrac{\lambda_1}{8}t^2 \ge -\tfrac{\lambda_1}{8}t^2, \end{aligned} \tag{4.12}$$

thus assumption iii) holds. It follows from Lemma 4.2 that hypothesis iv) is verified. Also, for $|c| \le c(t)$, if $(u_1, u_2) \in Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ are such that $\frac{r(t)}{2} \le ||(u_1, u_2)||_{\mathbf{H}} \le r(t)$ and $|\tau| \le l||(u_1, u_2)||_{\mathbf{H}}$, we have by (4.7) and (4.10)

$$\begin{aligned} \varphi_c(\tau e_1 + u_1, u_2) &= E_c(u_0 + u_1 + \tau e_1, u_2) - E_c(u_0, 0) \\ &\ge \tfrac{\gamma_1}{2}||(u_1, u_2)||_{\mathbf{H}}^2 - \tfrac{\gamma_1}{16}r(t)^2 \ge \tfrac{\gamma_1}{16}r(t)^2, \end{aligned} \tag{4.13}$$

so that assumption v) is satisfied. Hence we may apply Theorem 3.1 and we obtain a Palais-Smale sequence $(u_{1,c}^n, u_{2,c}^n)$ for the functional φ_c restricted to $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. Moreover, $(u_{1,c}^n, u_{2,c}^n) \in [-t, t]e_1 + B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$ for any n. Since $(u_{1,c}^n)$, $(u_{2,c}^n)$ are bounded in $H^1(\mathbf{R}^N)$, respectively in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, we may extract a subsequence (still denoted $(u_{1,c}^n)$, $(u_{2,c}^n)$) such that

$$\begin{aligned} &u_{1,c}^n \rightharpoonup u_{1,c} && \text{weakly in } H^1(\mathbf{R}^N) \\ &u_{1,c}^n \longrightarrow u_{1,c} && \text{a.e. and in } L^p_{loc}, \quad \forall p \in [1, 2+\sigma) \\ &u_{2,c}^n \rightharpoonup u_{2,c} && \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N) \\ &u_{2,c}^n \longrightarrow u_{2,c} && \text{a.e. and in } L^p_{loc}, \quad \forall p \in [1, 2+\sigma). \end{aligned} \tag{4.14}$$

It is clear that $||u_{1,c}||_{H^1} \le t + r(t)$ and $||u_{2,c}||_{\mathcal{D}^{1,2}} \le r(t)$. Let $(v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. By weak convergence it is obvious that

$$T'(u_0 + u_{1,c}^n, u_{2,c}^n).(v_1, v_2) \longrightarrow T'(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) \text{ as } n \longrightarrow \infty, \tag{4.15}$$

$$Q'(u_0 + u_{1,c}^n, u_{2,c}^n).(v_1, v_2) \longrightarrow Q'(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) \text{ as } n \longrightarrow \infty. \tag{4.16}$$

On the other hand, it follows from the estimates at the beginning of this section that

$F((r_0 - u_0 - u_{1,c}^n)^2 + (u_{2,c}^n)^2)(r_0 - u_0 - u_{1,c}^n)$ is bounded in $L^2 + L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N)$ and
$F((r_0 - u_0 - u_{1,c}^n)^2 + (u_{2,c}^n)^2)u_{2,c}^n$ is bounded in $L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N)$.

Passing again to a subsequence, we may assume that

$$\begin{aligned} &F((r_0 - u_0 - u_{1,c}^n)^2 + (u_{2,c}^n)^2)(r_0 - u_0 - u_{1,c}^n) \rightharpoonup f_1 \text{ weakly in } L^2 + L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N) \\ &F((r_0 - u_0 - u_{1,c}^n)^2 + (u_{2,c}^n)^2)u_{2,c}^n \rightharpoonup f_2 \text{ weakly in } L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N). \end{aligned}$$

In view of the estimates at the beginning of Section 4 and of the convergence $u^n_{1,c} \longrightarrow u_{1,c}$, $u^n_{2,c} \longrightarrow u_{2,c}$ in $L^p_{loc}(\mathbf{R}^N)$, $1 \le p < 2+\sigma$, we have $F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)(r_0 - u_0 - u^n_{1,c}) \longrightarrow F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})(r_0 - u_0 - u_{1,c})$ and $F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)u^n_{2,c} \longrightarrow F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})u_{2,c}$ in $L^q_{loc}(\mathbf{R}^N)$, $1 \le q < \frac{2+\sigma}{1+\sigma}$. By the uniqueness of the limit in $\mathcal{D}'(\mathbf{R}^N)$ we infer that $f_1 = F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})(r_0 - u_0 - u_{1,c})$ and $f_2 = F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})u_{2,c}$. Now the weak convergence implies that

$$(4.17)\quad I'(u_0+u^n_{1,c}, u^n_{2,c}).(v_1, v_2) \longrightarrow 2\int_{\mathbf{R}^N} f_1v_1 - f_2v_2dx = I'(u_0+u_{1,c}, u_{2,c}).(v_1, v_2).$$

Since $\lim\limits_{n\to\infty} E'_c(u_0+u^n_{1,c}, u^n_{2,c}).(v_1, v_2) = 0$, from (4.15), (4.16) and (4.17) we infer that

$$(4.18)\quad E'_c(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) = 0 \quad \text{for all } (v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N).$$

In conclusion, we have proved that for any $t \in (0, t_0]$ there exists $c(t) > 0$ such that for $|c| \le c(t)$, the restriction of φ_c to the space $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ admits a critical point $(u_{1,c}, u_{2,c})$ and $||u_{1,c}||_{H^1} \le t + r(t)$, $||u_{2,c}||_{\mathcal{D}^{1,2}} \le r(t)$. The proof of Theorem 4.3 is completed. □

Theorem 4.4. *There exists $c_\star > 0$ such that for $|c| \le c_\star$, E_c admits a nontrivial critical point $u_c \in \mathbf{H}$. Moreover, $u_c \longrightarrow u_0$ as $c \longrightarrow 0$.*

Proof. Let $u_c = (u_0 + u_{1,c}, u_{2,c}) = u_0 + u_{1,c} + iu_{2,c}$ where $(u_{1,c}, u_{2,c})$ is given by Theorem 4.3. It follows from (4.18) that $E'_c(u_c) = 0$ on $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$, that is $d_{u_2}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $d_{u_1}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $\mathbf{R}e_1 \oplus Y = (Ker(A))^\perp \cap H^1(\mathbf{R}^N)$. All we have to do is to show that $d_{u_1}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $Ker(A)$. For small c, this will be done thanks to the invariance of E_c by translations in $\mathbf{R}^N$. (Note also that $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ are in the kernel of A just because E is translation invariant).

It will be seen in the next section that $u_{1,c}$ and $u_{2,c}$ are in $H^2(\mathbf{R}^N)$, respectively in $\mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$, where $\mathcal{D}^{2,2}(\mathbf{R}^N) = \{v \in \mathcal{D}'(\mathbf{R}^N) \ |\nabla^2 v \in L^2(\mathbf{R}^N)\ \}$. Then for each $i \in \{1, \dots, N\}$, the mapping $t \longmapsto u_c(x_1, \dots, x_i + t, \dots, x_N)$ is C^1 from $\mathbf{R}$ to $\mathbf{H}$ and

$$(4.19)\qquad E_c(u_c(x_1, \dots, x_i + t, \dots, x_N)) = E_c(u_c), \quad \forall t \in \mathbf{R}.$$

Differentiating (4.19) at $t = 0$ we get

$$(4.20)\qquad E'_c(u_c).\frac{\partial u_c}{\partial x_i} = 0.$$

Because $d_{u_2}E_c(u_c) = 0$, (4.20) gives $d_{u_1}E_c(u_c).\left(\frac{\partial u_0}{\partial x_i} + \frac{\partial u_c}{\partial x_i}\right) = 0$. By Theorem 2.6 we have $H^1(\mathbf{R}^N) = \mathbf{R}e_1 + Y + Span\{\frac{\partial u_0}{\partial x_i}, i = 1, \dots, N\}$, the sum being orthogonal in $L^2(\mathbf{R}^N)$. Note that $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ are orthogonal in $L^2(\mathbf{R}^N)$ and $\frac{\partial u_{1,c}}{\partial x_i} \longrightarrow 0$ in $L^2(\mathbf{R}^N)$ as $c \longrightarrow 0$. It follows that for c sufficiently small we also have $H^1(\mathbf{R}^N) = \mathbf{R}e_1 + Y + Span\{\left(\frac{\partial u_0}{\partial x_i} + \frac{\partial u_{1,c}}{\partial x_i}\right), i = 1, \dots, N\}$ and from (4.20) we deduce that $d_{u_1}E_c(u_c) = 0$ on $H^1(\mathbf{R}^N)$, as we need. Thus Theorem 4.4 is proved. □

Remark 4.5. Both the functional E_c and equation (1.2) are invariant by rotations in the $(x_2, \dots, x_N)$-variables. Therefore instead of working on $\mathbf{H}$, we could work on $\mathbf{H}_{1,rad} = \{u \in \mathbf{H} \mid u \text{ is radially symmetric in } (x_2, \dots, x_N)\}$. Our proofs remain valid without changes and we obtain a critical point $\tilde{u}_c$ of E_c on $\mathbf{H}_{1,rad}$ for $|c| \le c_\star$. Of course that in this case we know *à priori* that $E'_c(\tilde{u}_c).v = 0$ only for $v \in \mathbf{H}_{1,rad}$. Because the group G of rotations in $(x_2, \dots, x_N)$ acts isometrically on $\mathbf{H}$ and $Fix(G) = \mathbf{H}_{1,rad}$, from the Principle of Symmetric Criticality (see [38] or [50]) we obtain that in fact $\tilde{u}_c$ is a critical point of E_c on $\mathbf{H}$. Therefore we have the following:

Corollary 4.6. *If $|c| \le c_\star$, there exists a solution $\tilde{u}_c \in \mathbf{H}$ of (1.2) which is radially symmetric in the transverse variables $(x_2, \dots, x_N)$. Moreover, $\tilde{u}_c \longrightarrow u_0$ in $\mathbf{H}$ as $c \longrightarrow 0$.*

3.5 Regularity

In this section we show that the critical points obtained in Theorem 4.3 are in $H^2(\mathbf{R}^N) \times \mathcal{D}^{2,2}(\mathbf{R}^N)$ (thus completing the proof of Theorem 4.4) and we obtain some other regularity properties of the solutions of equation (1.2). We begin with the following simple lemma:

Lemma 5.1. Let $(u_1, u_2) \in \mathbf{H}$ satisfy $E'_c(u_1, u_2).(v_1, v_2) = 0$, $\forall (v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. Then

$$d_{u_1} E_c(u_1, u_2) \in Ker(A) \quad \text{and} \quad d_{u_2} E_c(u_1, u_2) = 0.$$

Proof. It is obvious that $d_{u_2} E_c(u_1, u_2) = 0$. Let p_1, p_2 be the orthogonal projections of $L^2(\mathbf{R}^N)$ onto $Ker(A)$, respectively onto $\mathbf{R}e_1 \oplus Y$. It is clear that $d_{u_1} E_c(u_1, u_2).p_2 v = 0$ for all $v \in H^1(\mathbf{R}^N)$. Hence for any $v \in H^1(\mathbf{R}^N)$ we have

$$\begin{aligned} |\langle d_{u_1} E_c(u_1, u_2), v\rangle_{H^{-1},H^1}| &= |\langle d_{u_1} E_c(u_1, u_2), p_1 v\rangle_{H^{-1},H^1}| \\ &\le C||p_1 v||_{H^1} \\ &\le C'||p_1 v||_{L^2} \quad \text{because } Ker(A) \text{ is finite-dimensional} \\ &\le C'||v||_{L^2}. \end{aligned}$$

By density of $H^1(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ we infer that $d_{u_1} E_c(u_1, u_2)$ has an unique extension as a bounded linear functional on $L^2(\mathbf{R}^N)$, hence $d_{u_1} E_c(u_1, u_2) \in L^2(\mathbf{R}^N)$. Observe that $\mathbf{R}e_1 \oplus Y = H^1(\mathbf{R}^N) \cap Im(A)$ is dense in $Im(A)$ and $Im(A)^\perp = Ker(A)$ because A is self-adjoint. Since $\langle d_{u_1} E_c(u_1, u_2), v\rangle = 0$, $\forall v \in \mathbf{R}e_1 \oplus Y = H^1(\mathbf{R}^N) \cap Im(A)$, by density we infer that $d_{u_1} E_c(u_1, u_2) \in Ker(A)$. □

Lemma 5.2. *Suppose that $N \ge 4$ and $F \in C^1([0, \infty))$ satisfies*

i) $F(r_0^2) = 0$ and

ii) $F(x) \le 0$ and $|F(x)| \le x^{\frac{\sigma}{2}}$ for large x.

Let $u = u_1 + iu_2$ with $u_1 \in H^1(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ be a solution of the equation

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = f_1 + if_2. \tag{5.1}$$

We have:

a) If $f_1,\ f_2 \in L^2(\mathbf{R}^N) \cap L^{2+\sigma}(\mathbf{R}^N)$, *then* $u_1 \in H^2(\mathbf{R}^N)$ *and* $u_2 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$.

b) If $f_1,\ f_2 \in L^q(\mathbf{R}^N), \forall q \in [2,\infty)$, *then* $u_1 \in W^{2,q}(\mathbf{R}^N)$, $u_2 \in \mathcal{D}^{1,q} \cap \mathcal{D}^{2,q}(\mathbf{R}^N)$ $\forall q \in [2,\infty)$ *and* $u_2 \in W^{2,q}(\mathbf{R}^N)$, $\forall q \geq 2+\sigma$.

Proof. Equation (5.1) is equivalent to the system

$$-cu_{2x_1} - \Delta u_1 + F((r_0-u_1)^2+u_2^2)(r_0-u_1) = f_1 \tag{5.2}$$

$$cu_{1x_1} - \Delta u_2 - F((r_0-u_1)^2+u_2^2)u_2 = f_2. \tag{5.3}$$

We show first that $u_1 \in L^{q_1}(\mathbf{R}^N)$ and $u_2 \in L^{q_2}(\mathbf{R}^N)$ with $q_1, q_2 \geq 2+2\sigma$. This step was inspired by the proof of Theorem 2.3 in [17]. For $i=1,2$ and $n \in \mathbf{N}$, let

$$u_i^n(x) = \begin{cases} -n & \text{if} \quad u_i(x) < -n \\ u_i(x) & \text{if} \quad -n \leq u_i(x) \leq n \\ n & \text{if} \quad u_i(x) > n. \end{cases}$$

It is clear that $u_1^n \in H^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $u_2^n \in \mathcal{D}^{1,2}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $\nabla u_i^n = \chi_{\{-n\leq u_i \leq n\}} \nabla u_i$, $i=1,2$. Let $h_p(s) = |s|^{p-2}s$, $p \geq 2$. Then $h_p(u_1^n) \in H^1(\mathbf{R}^N)$ and $h_p(u_2^n) \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Multiplying (5.3) by $h_p(u_2^n)$ and integrating we get

$$\begin{aligned} (p-1)\int_{\mathbf{R}^N} |\nabla u_2^n|^2 |u_2^n|^{p-2} dx &= \int_{\mathbf{R}^N} f_2 |u_2^n|^{p-2} u_2^n dx \\ &+ \int_{\{-n\leq u_i \leq n\}} F((r_0-u_1)^2+u_2^2)|u_2|^p dx \\ &+ \int_{\{u_2<-n\}\cup\{u_2>n\}} F((r_0-u_1)^2+u_2^2)|u_2| n^{p-1} dx \\ &- c\int_{\mathbf{R}^N} u_{1x_1} |u_2^n|^{p-2} u_2^n dx. \end{aligned} \tag{5.4}$$

Denoting by $F_{max} = \max\limits_{x\in[0,\infty)} F(x)$, we have:

$$\int_{\{-n\leq u_i \leq n\}} F((r_0-u_1)^2+u_2^2)|u_2|^p dx \leq F_{max} \int_{\mathbf{R}^N} |u_2|^p dx;$$

$$\int_{\{u_2<-n\}\cup\{u_2>n\}} F((r_0-u_1)^2+u_2^2)|u_2| n^{p-1} dx \leq 0 \text{ if } n \text{ is sufficiently large,}$$

$$\left| c\int_{\mathbf{R}^N} u_{1x_1} |u_2^n|^{p-2} u_2^n dx \right| = \left| -c(p-1)\int_{\mathbf{R}^N} u_1 |u_2^n|^{p-2} u_{2\,x_1}^n dx \right|$$

$$= \left| \tfrac{-2c(p-1)}{p^2} \int_{\mathbf{R}^N} u_1 |u_2^n|^{\frac{p}{2}-2} u_2^n \cdot \frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right) dx \right|$$

$$\leq \tfrac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right) \right|^2 dx + C(p)\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx.$$

Using the identity $\int_{\mathbf{R}^N} |\nabla u|^2 |u|^{p-2} dx = \frac{4}{p^2}\int_{\mathbf{R}^N} \left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2 dx$, (5.4) gives

$$\begin{aligned} \tfrac{4(p-1)}{p^2} \int_{\mathbf{R}^N} \left|\nabla |u_2^n|^{\frac{p}{2}}\right|^2 dx &\leq \int_{\mathbf{R}^N} |f_2||u_2|^{p-1} dx + F_{max}\int_{\mathbf{R}^N} |u_2|^p dx \\ &+ \tfrac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right) \right|^2 dx + C(p)\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx. \end{aligned} \tag{5.5}$$

Note that the right hand side of (5.5) may be infinite. Since $f_2 \in L^{2+\sigma}(\mathbf{R}^N)$ and $u_1, u_2 \in L^{2+\sigma}(\mathbf{R}^N)$ by the Sobolev embedding, taking $p = 2+\sigma$ in (5.5) we get

$$\frac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \nabla |u_2^n|^{\frac{2+\sigma}{2}} \right|^2 dx \leq K, \tag{5.6}$$

where K does not depend on n. Passing to the limit as $n \longrightarrow \infty$ in (5.6) and using Lebesgue's dominated convergence theorem we infer that $\nabla \left(|u_2|^{\frac{2+\sigma}{2}} \right) \in L^2(\mathbf{R}^N)$. By the Sobolev embedding we obtain $|u_2|^{\frac{2+\sigma}{2}} \in L^{2+\sigma}(\mathbf{R}^N)$, that is $u_2 \in L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$.

Multiplying (5.2) by $h_p(u_1^n)$ and integrating we get

$$\begin{aligned}
&(p-1) \int_{\mathbf{R}^N} |\nabla u_1^n|^2 |u_1^n|^{p-2} dx = \int_{\mathbf{R}^N} f_1 |u_1^n|^{p-2} u_1^n dx \\
&+ \int_{\{-n \leq u_1 \leq n\}} F((r_0 - u_1)^2 + u_2^2) |u_1|^p dx \\
&+ \int_{\{u_1 < -n\} \cup \{u_1 > n\}} F((r_0 - u_1)^2 + u_2^2) |u_1| n^{p-1} dx \\
&- r_0 \int_{\mathbf{R}^N} F((r_0 - u_1)^2 + u_2^2) |u_1^n|^{p-2} u_1^n dx + c \int_{\mathbf{R}^N} u_{2x_1} |u_1^n|^{p-2} u_1^n dx.
\end{aligned} \tag{5.7}$$

We have $|F((r_0 - z_1)^2 + z_2^2)| \leq C| - 2r_0 z_1 + z_1^2 + z_2^2|$ for $z_1^2 + z_2^2 \leq 4r_0^2$ and $|F((r_0 - z_1)^2 + z_2^2)| \leq C((r_0 - z_1)^2 + z_2^2)^{\frac{\sigma}{2}} \leq C'(|z_1|^\sigma + |z_2|^\sigma)$ for $z_1^2 + z_2^2 > 4r_0^2$.

If $\sigma \leq 1$ (that is, $N \geq 6$), then $|F((r_0 - z_1)^2 + z_2^2)| \leq C(|z_1| + |z_2|)$ for all z_1, z_2 and proceeding as above we infer that

$$\begin{aligned}
&\tfrac{4(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \nabla |u_1^n|^{\frac{p}{2}} \right|^2 dx \leq \int_{\mathbf{R}^N} |f_1| |u_1|^{p-1} dx + F_{max} \int_{\mathbf{R}^N} |u_1|^p dx \\
&+ C \int_{\mathbf{R}^N} (|u_1| + |u_2|) |u_1|^{p-1} dx \\
&+ \tfrac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \frac{\partial}{\partial x_1} \left(|u_1^n|^{\frac{p}{2}} \right) \right|^2 dx + C(p) \int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx.
\end{aligned} \tag{5.8}$$

Of course, the right side of (5.8) may be infinite. Because $u_1 \in L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$, it is easy to see that

$$\int_{\mathbf{R}^N} |u_1|^p dx < \infty \text{ and } \int_{\mathbf{R}^N} |f_1| |u_1|^{p-1} dx < \infty \text{ for } 2 \leq p \leq 2+\sigma,$$

$$\int_{\mathbf{R}^N} |u_2| |u_1|^{p-1} dx < \infty \text{ for } 2 + \tfrac{\sigma}{2+\sigma} \leq p \leq 2 + \sigma + \tfrac{\sigma}{2+\sigma},$$

$$\int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx < \infty \text{ for } 2 + \tfrac{2\sigma}{2+\sigma} \leq p \leq 2 + \sigma + \tfrac{2\sigma}{2+\sigma}.$$

Taking $p = 2+\sigma$ in (5.8) we obtain

$$\frac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left| \nabla |u_1^n|^{\frac{2+\sigma}{2}} \right|^2 dx \leq K < \infty, \tag{5.9}$$

where K does not depend on n. Passing to the limit as $n \longrightarrow \infty$ in (5.9) and using again Lebesgue's dominated convergence theorem we get that $\nabla\left(|u_1|^{\frac{2+\sigma}{2}}\right) \in L^2(\mathbf{R}^N)$ and therefore $u_1 \in L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$ by the Sobolev embedding.

If $2 \geq \sigma \geq 1$, we have $|F((r_0 - z_1)^2 + z_2^2)| \leq C(|z_1| + |z_1|^\sigma + |z_2|^\sigma)$ for all z_1, z_2 (note that $\sigma \leq 2$ because $N \geq 4$) so that (5.7) gives

$$\begin{aligned}
&\tfrac{4(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\nabla|u_1^n|^{\frac{p}{2}}\right|^2 dx \leq \int_{\mathbf{R}^N}|f_1||u_1|^{p-1}dx + F_{max}\int_{\mathbf{R}^N}|u_1|^p dx \\
&+C\int_{\mathbf{R}^N}(|u_1| + |u_1|^\sigma + |u_2|^\sigma)|u_1|^{p-1}dx \qquad (5.10)\\
&+\tfrac{2(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\frac{\partial}{\partial x_1}\left(|u_1^n|^{\frac{p}{2}}\right)\right|^2 dx + C(p)\int_{\mathbf{R}^N}|u_2|^2|u_1|^{p-2}dx.
\end{aligned}$$

Since $u_1 \in L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$, it is clear that

$$\int_{\mathbf{R}^N}|u_1|^p dx < \infty \text{ and } \int_{\mathbf{R}^N}|f_1||u_1|^{p-1}dx < \infty \text{ for } 2 \leq p \leq 2+\sigma,$$

$$\int_{\mathbf{R}^N}|u_1|^{p+\sigma-1}dx < \infty \text{ for } 3-\sigma \leq p \leq 3,$$

$$\int_{\mathbf{R}^N}|u_2|^\sigma|u_1|^{p-1}dx < \infty \text{ for } 2 + \tfrac{2-\sigma}{2+\sigma} \leq p \leq 2+\sigma+\tfrac{2-\sigma}{2+\sigma},$$

$$\int_{\mathbf{R}^N}|u_2|^2|u_1|^{p-2}dx < \infty \text{ for } 2 + \tfrac{2\sigma}{2+\sigma} \leq p \leq 2+\sigma+\tfrac{2\sigma}{2+\sigma}.$$

Therefore for $p \in [2 + \frac{2\sigma}{2+\sigma}, 3]$ we obtain

$$\frac{2(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\nabla|u_1^n|^{\frac{p}{2}}\right|^2 dx \leq K < \infty, \qquad (5.11)$$

with K independent of n. As previously we get that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ and $u_1 \in L^{\frac{(2+\sigma)p}{2}}(\mathbf{R}^N)$ by the Sobolev embedding. In particular, for $p = 3$ we obtain $u_1 \in L^{\frac{3}{2}(2+\sigma)}(\mathbf{R}^N)$. Thus we have proved that $u_1 \in L^{q_1}(\mathbf{R}^N)$ and $u_2 \in L^{q_2}(\mathbf{R}^N)$ with q_1, $q_2 \geq 2 + 2\sigma$.

From the above estimates it follows that

$$\begin{aligned}
&|F((r_0 - u_1)^2 + u_2^2)(r_0 - u_1)| \\
&\leq C(|u_1| + |u_2|^2)\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} + C(|u_1|^\sigma + |u_2|^\sigma)(|u_1| + |u_2|)\chi_{\{u_1^2+u_2^2> 4r_0^2\}} \\
&\leq C'(|u_1|^+|u_2|^{1+\frac{\sigma}{2}})\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} + C'(|u_1|^{1+\sigma} + |u_2|^{1+\sigma}) \in L^2(\mathbf{R}^N)
\end{aligned}$$

and similarly $F((r_0 - u_1)^2 + u_2^2)u_2 \in L^2(\mathbf{R}^N)$. From (5.2) and (5.3) we infer now that $\Delta u_1 \in L^2(\mathbf{R}^N)$ and $\Delta u_2 \in L^2(\mathbf{R}^N)$, which imply that $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{2,2}(\mathbf{R}^N)$. This proves a).

b) Suppose now that $f_1, f_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Let $r \geq \frac{(2+\sigma)^2}{2}$ if $\sigma \leq 1$, respectively $r \geq \frac{2}{3}(2+\sigma)$ if $\sigma > 1$ and $s \geq \frac{(2+\sigma)^2}{2}$ be such that $u_1 \in L^2 \cap L^r(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^s(\mathbf{R}^N)$.

It is easily seen that $\displaystyle\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx < \infty$ if $2+\sigma \leq p \leq 2 + s(1-\frac{2}{r})$.Let $p_1 = \min(s, 2+s(1-\frac{2}{r}))$. From (5.5) it follows that $\nabla\left(|u_2|^{\frac{p_1}{2}}\right) \in L^2(\mathbf{R}^N)$, thus $u_2 \in L^{\frac{2+\sigma}{2}p_1}(\mathbf{R}^N)$. We also have

$$\int_{\mathbf{R}^N} |u_1|^{p+\sigma-1} dx < \infty \text{ for } 3-\sigma \leq p \leq r+1-\sigma,$$

$$\int_{\mathbf{R}^N} |u_2|^{\sigma} |u_1|^{p-1} dx < \infty \text{ for } 2+\tfrac{2-\sigma}{2+\sigma} \leq p \leq 1 + r(1-\tfrac{\sigma}{s}),$$

$$\int_{\mathbf{R}^N} |u_2||u_1|^{p-1} dx < \infty \text{ for } 2+\tfrac{\sigma}{2+\sigma} \leq p \leq 1 + r(1-\tfrac{1}{s})$$

$$\int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx < \infty \text{ for } 2+\tfrac{2\sigma}{2+\sigma} \leq p \leq 2 + r(1-\tfrac{2}{s}).$$

In the case $\sigma \leq 1$, we obtain from (5.8) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2 + \frac{2\sigma}{2+\sigma} \leq p \leq p_2 = \min(r, 1+r(1-\frac{1}{s}), 2+r(1-\frac{2}{s}))$, while in the case $\sigma > 1$ we obtain from (5.10) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2+\frac{2\sigma}{2+\sigma} \leq p \leq p_2' = \min(r, 1+r(1-\frac{\sigma}{s}), 2+r(1-\frac{2}{s}), r+1-\sigma)$. By the Sobolev embedding, $u_1 \in L^{\frac{2+\sigma}{2}p_2}(\mathbf{R}^N)$ if $\sigma \leq 1$, respectively $u_1 \in L^{\frac{2+\sigma}{2}p_2'}(\mathbf{R}^N)$ if $\sigma > 1$. Thus we obtained that $u_1 \in L^{r'}(\mathbf{R}^N)$ and $u_2 \in L^{s'}(\mathbf{R}^N)$, where $r' = \frac{2+\sigma}{2}p_2$ if $\sigma \leq 1$, respectively $r' = \frac{2+\sigma}{2}p_2'$ if $\sigma > 1$ and $s' = \frac{2+\sigma}{2}p_1$. Repeating this argument it follows that $u_1 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$ and $u_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2+\sigma,\infty)$. Consequently $F((r_0-u_1)^2+u_2^2)(r_0-u_1), F((r_0-u_1)^2+u_2^2)u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$.

Since $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$, we have $u_{1x_1}, u_{2x_1} \in H^1(\mathbf{R}^N) \subset L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$. Using (5.2) and (5.3) we infer that $\Delta u_1, \Delta u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2, 2+\sigma]$, therefore $u_1 \in W^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,2+\sigma]$, $u_2 \in \mathcal{D}^{1,p} \cap \mathcal{D}^{2,p}(\mathbf{R}^N)$, $\forall p \in [2, 2+\sigma]$ and $u_2 \in W^{2,2+\sigma}(\mathbf{R}^N)$. Iterating this argument we obtain the conclusion in Lemma 5.2, b). □

Remark 5.3. From Lemma 5.3 b) it follows in particular that $u_1, u_1 \in C^{1,\alpha}(\mathbf{R}^N)$ for all $\alpha \in [0,1)$, u_1, u_2 are bounded and tend to zero at infinity.

Finally, suppose that F is C^k and $f_1, f_2 \in W^{k,q}(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Differentiating equation (5.2), respectively (5.3), we obtain $u_1 \in W^{k+2,q}(\mathbf{R}^N)$, $\forall q \in [2,\infty)$, $u_2 \in \mathcal{D}^{1,q} \cap \mathcal{D}^{k+2,q}(\mathbf{R}^N)$, $2 \leq q < 2+\sigma$ and $u_2 \in W^{k+2,q}(\mathbf{R}^N)$, $2+\sigma \leq q < \infty$.

b) Suppose now that $f_1, f_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Let $r \geq \frac{(2+\sigma)^2}{2}$ if $\sigma \leq 1$, respectively $r \geq \frac{2}{3}(2+\sigma)$ if $\sigma > 1$ and $s \geq \frac{(2+\sigma)^2}{2}$ be such that $u_1 \in L^2 \cap L^r(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^s(\mathbf{R}^N)$.

It is easily seen that $\displaystyle\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx < \infty$ if $2+\sigma \leq p \leq 2 + s(1-\frac{2}{r})$.Let $p_1 = \min(s, 2+s(1-\frac{2}{r}))$. From (5.5) it follows that $\nabla\left(|u_2|^{\frac{p_1}{2}}\right) \in L^2(\mathbf{R}^N)$, thus $u_2 \in L^{\frac{2+\sigma}{2}p_1}(\mathbf{R}^N)$. We also have

$$\int_{\mathbf{R}^N} |u_1|^{p+\sigma-1} dx < \infty \text{ for } 3-\sigma \leq p \leq r+1-\sigma,$$

$$\int_{\mathbf{R}^N} |u_2|^{\sigma} |u_1|^{p-1} dx < \infty \text{ for } 2 + \tfrac{2-\sigma}{2+\sigma} \leq p \leq 1 + r(1-\tfrac{\sigma}{s}),$$

$$\int_{\mathbf{R}^N} |u_2| |u_1|^{p-1} dx < \infty \text{ for } 2 + \tfrac{\sigma}{2+\sigma} \leq p \leq 1 + r(1-\tfrac{1}{s})$$

$$\int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx < \infty \text{ for } 2 + \tfrac{2\sigma}{2+\sigma} \leq p \leq 2 + r(1-\tfrac{2}{s}).$$

In the case $\sigma \leq 1$, we obtain from (5.8) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2 + \frac{2\sigma}{2+\sigma} \leq p \leq p_2 = \min(r, 1+r(1-\frac{1}{s}), 2+r(1-\frac{2}{s}))$, while in the case $\sigma > 1$ we obtain from (5.10) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2 + \frac{2\sigma}{2+\sigma} \leq p \leq p_2' = \min(r, 1+r(1-\frac{\sigma}{s}), 2+r(1-\frac{2}{s}), r+1-\sigma)$. By the Sobolev embedding, $u_1 \in L^{\frac{2+\sigma}{2}p_2}(\mathbf{R}^N)$ if $\sigma \leq 1$, respectively $u_1 \in L^{\frac{2+\sigma}{2}p_2'}(\mathbf{R}^N)$ if $\sigma > 1$. Thus we obtained that $u_1 \in L^{r'}(\mathbf{R}^N)$ and $u_2 \in L^{s'}(\mathbf{R}^N)$, where $r' = \frac{2+\sigma}{2}p_2$ if $\sigma \leq 1$, respectively $r' = \frac{2+\sigma}{2}p_2'$ if $\sigma > 1$ and $s' = \frac{2+\sigma}{2}p_1$. Repeating this argument it follows that $u_1 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$ and $u_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2+\sigma,\infty)$. Consequently $F((r_0-u_1)^2+u_2^2)(r_0-u_1), F((r_0-u_1)^2+u_2^2)u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$.

Since $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$, we have $u_{1x_1}, u_{2x_1} \in H^1(\mathbf{R}^N) \subset L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$. Using (5.2) and (5.3) we infer that $\Delta u_1, \Delta u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2, 2+\sigma]$, therefore $u_1 \in W^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,2+\sigma]$, $u_2 \in \mathcal{D}^{1,p} \cap \mathcal{D}^{2,p}(\mathbf{R}^N)$, $\forall p \in [2, 2+\sigma]$ and $u_2 \in W^{2,2+\sigma}(\mathbf{R}^N)$. Iterating this argument we obtain the conclusion in Lemma 5.2, b). □

Remark 5.3. From Lemma 5.3 b) it follows in particular that $u_1, u_1 \in C^{1,\alpha}(\mathbf{R}^N)$ for all $\alpha \in [0,1)$, u_1, u_2 are bounded and tend to zero at infinity.

Finally, suppose that F is C^k and $f_1, f_2 \in W^{k,q}(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Differentiating equation (5.2), respectively (5.3), we obtain $u_1 \in W^{k+2,q}(\mathbf{R}^N)$, $\forall q \in [2,\infty)$, $u_2 \in \mathcal{D}^{1,q} \cap \mathcal{D}^{k+2,q}(\mathbf{R}^N)$, $2 \leq q < 2+\sigma$ and $u_2 \in W^{k+2,q}(\mathbf{R}^N)$, $2+\sigma \leq q < \infty$.

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