LOUIS FUNAR

Invariants for 3-manifolds

Thèses d'Orsay, 1994
<http://www.numdam.org/item?id=BJHTUP11_1994__0371__P0_0>

ORSAY
n° d'ordre :

UNIVERSITE de PARIS-SUD

Centre d' ORSAY

THESE

présentée
Pour obtenir

Le titre de Docteur en Sciences
Spécialité : Mathématiques

par

Louis FUNAR

Sujet : **Invariants pour les variétés de dimension 3**

Soutenue le 24 février 1994 devant la Commission d'examen

MM. (Président) Larry Siebenmann
Arnaud Beauville
Valentin Poénaru
(Rapporteur) Vladimir Turaev
(Rapporteur) Pierre Vogel

INVARIANTS FOR 3-MANIFOLDS
INVARIANTS POUR LES VARIETES DE DIMENSION 3

Louis FUNAR

à ma mère Maria, avec reconnaissance

This Thesis contains six chapters:

1. A French summary.

2. The description of the abelian Witten's theory using the theory of theta functions. The 3-manifolds are presented by Heegaard splittings.

3. A semi-abelian quantization procedure is outlined for a general gauge group. This permits to follow Witten's approach using instead of the Teichmuller space the Siegel upper-space. The exact computations of the monodromy representations may be carried out via special theta functions.

4. We describe the precise relation between RCFT and TQFT which may be used further to obtain an entirely algebraic characterization of invariants (e.g. in terms of homotopy Lie algebras).

5. We show that a Markov trace which vanishes on a two-sided ideal of $\mathbf{Z}[B_k]$ it vanishes also on the ideal generated in $\mathbf{Z}[B_\infty]$.

6. A study of Markov traces on homogeneous quotients of rank 3 of cubic Hecke algebras is begun. Explicit computations are done for the quotient of maximal dimension. As a consequence the ternary Vassiliev invariants are introduced.

Keywords: Theta functions, symplectic group, Heegaard splitting, tensor representation, mapping class group, duality groupoid, Hecke algebra, Markov trace.

AMS Classification: 57 A 10, 14 K 25, 32 G 15

Remerciements

Je tiens à remercier tous ceux qui, à un instant ou l'autre, m'ont donné leur soutien et m'ont encouragé à poursuivre les mathématiques:

Les efforts de mes parents à travers toute ma vie, que je ne pourrais jamais les gratifier;

V.Boju, qui m'a introduit dans l'universe de la géométrie, pour toutes les discussions qu'on a eues, son enthousiasme et sa amitié;

S.Dimca, A.Nemethi et A.Zaharia qui m'ont initié à la théorie des singularités;

S.Papadima qui m'a guidé à travers la topologie algèbrique;

P.Ionescu qui m'a introduit à la géométrie algèbrique;

B.Berceanu qui m'a beaucoup aidé et a contribué á la realisation du dernier chapitre en acceptant de se joindre aux interminables calculs qui se trouvent dans l'appendice;

G.Masbaum et P.Vogel qui m'ont substantiellement aidé a reviser les chapitres 4, 5 et 6 pour gagner en clarté et coherence et pour corriger les nombreuses fautes que la version antérieure contenait. Ceci n'etait pas possible sans leur patience, leurs critiques et leurs suggestions;

V.Turaev et L.Guillou pour les observations qu'ils m'ont faites pour améliorer les deux premières chapitres;

Je veux exprimer ma plus profonde gratitude à mon directeur de thèse V.Poénaru qui m'a appris à faire de la recherche, m'a encouragé et soutenu tout au long de ce travail. Cette thèse est le resultat de toutes les discussions qu'on a eues pendant ce temps, qui m'ont aidé à concretiser mes idées.

J'exprime également mes sincères remerciements á:

L.Siebenmann pour l'honneur qu'il me fait de présider le jury;

A.Beauville pour avoir accepté d'être membre du jury;

V.Turaev et P.Vogel pour accepter la corvée d'être les rapporteurs de cette thèse et membres du jury.

Contents

Chapter 1

Introduction

La thése comporte cinq parties presque independantes:

1. La TQFT abélienne.

2. Une TQFT semi-abélienne.

3. TQFT et RCFT.

4. L'homogénite des traces Markov.

5. Algèbres de Hecke cubiques.

1.1 La TQFT abélienne

On suit de près la construction géométrique, dont Witten ([Wit89]) a utilisé pour définir les blocks conformes, dans le cas où le groupe de jauge est $U(1)$. Il s'agit de la quantification géométrique dependant du choix d'une structure complexe sur une surface. Il etait transparent dans les notes du seminaire d'Oxford que les fonctions thêta représentent le noyau de la théorie abélienne. Nous avons décrit explicitement la construction des invariants en termes des scindements de Heegaard.

Soit θ_m les fonctions thêta de niveau k pour les variétés abéliennes de dimension g ($m \in (Z/kZ)^g$) définies par

$$\theta_m(z,\Omega) = \sum_{l \in m+kZ^g} exp(\frac{\pi i}{k} < l, \Omega l > +2\pi i < l, z >) \quad (1.1)$$

pour $z \in C^g$, $\Omega \in \mathcal{S}_g$. $\mathcal{S}_g$ est l'espace de Siegel des matrices complexes symétriques à partie imaginaire positive définie et $<,>$ est la forme hermitienne canonique sur C^g.

On a une action naturelle du groupe sympléctique $Sp(2g, Z)$ on $C^g \times \mathcal{S}_g$ donée par la formule

$$\gamma \cdot (z, \Omega) = (((C\Omega + D)^{\top})^{-1})z, (A\Omega + B)(C\Omega + D)^{-1}). \quad (1.2)$$

où $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, A, B, C, D etant des matrices carees $g \times g$. L'équation fonctionelle satisfaite par la fonction thêta classique se généralise au cas des fonctions thêta de niveau k comme suit([Fun91]):

Théorème 1.1.1 *Le vecteur thêta $\Theta_k(z,\Omega) = (\theta_m(z,\Omega)_{m\in(Z/kZ)^g})$ satisfait l'équation fonctionelle:*

$$\begin{aligned} \Theta\ (\gamma \cdot (z, \Omega)) &= \zeta_\gamma det(C\Omega + D)^{1/2} \times \\ &\times exp(k\pi i < z, (C\Omega + D)^{-1}Cz >)\rho_k(\gamma)(\Theta_k(z, \Omega)) \end{aligned} \quad (1.3)$$

où k est pair, $\zeta_\gamma \in R_8$, (R_8 est le groupe des racines de l'unité d'ordre 8) et $\rho_k : Sp(2g, Z) \longrightarrow U(k^g)$ est une application qui devient un homomorphisme (qu'on note aussi par ρ_k) quand on passe au quotient $U(k^g)/R_8$. Cet homomorphisme est determiné par:

1.

$$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag(exp(\frac{\pi i}{k} < m, Bm >)). \quad (1.4)$$

pour $B = B^{\top}$ une matrice aux elements entiers.

2.

$$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^{\top})^{-1} \end{bmatrix} = (\delta_{A^{\top}m,n})_{m,n\in(Z/kZ)^g}. \quad (1.5)$$

pour $A \in GL(g, Z)$

3.

$$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = k^{-g/2} exp(2\pi i k^{-1} < m, l >)_{m,l \in (Z/kZ)^g}. \quad (1.6)$$

Donc on peut considerer que $Sp(2g, Z)$ agit (via ρ_k) sur un certain espace vectoriel hermitien $W_g(k)$ qui posséde une base canonique, définie à une racine de l'unité d'ordre 8 près, donnée par les fonctions thêta $\{\theta_m; m \in (Z/kZ)^g\}$. Cet espace vectoriel a aussi une interprétation géométrique remarquable: Soit θ_A le diviseur théta sur la variété abélienne principalement polarisée A de dimension g. On considère que A est le quotient de C^g par le reseaux engendré par les vecteurs colonne dans la matrice $1, \Omega$, $\Omega \in \mathcal{S}_g$. Alors $\{\theta_m(z, \Omega), m \in (Z/kZ)^g\}$ est une base de l'espace des sections holomorphes $H^0(A, \theta_A^{\otimes k})$. Mais $H^0(A, \theta_A^{\otimes k})$ est en même temps la fibre d'un fibré vectoriel $V_g(k) \longrightarrow \mathcal{A}_g$. au dessus de l'espace de modules des variétés abéliennes principalement polarisées . De plus ce fibré est muni d'une connexion projectivement plate ∇. En fait les fonctions thêta sont des solutions pour l'équation de la chaleur [Wel83, Hit90, Wit89, ADW91] et cet operateur différentiel peut être interprété comme une connexion dans $V_g(k)$. L'espace $\mathcal{A}_g$ est le quotient du démi-espace de Siegel $\mathcal{S}_g$ par l'action (libre) du groupe sympléctique $Sp(2g, Z)$. Alors la monodromie de la connexion ∇ est une représentation projective du groupe $Sp(2g, Z)$ qui est la projectivisation de ρ_k.

Maintenant on peut donner la définition des invariants: Soit M^3 une variété fermée et orientée de dimension 3. On considére une décomposition de Heegaard $M^3 = H_g \cup H_g$ de M^3 en deux tores pleins de genre g et notons par h le homéomorphisme de recollement. Le choix d'une base sympléctique dans l'homologie de la surface $\partial H_g = \Sigma_g$ permet d'identifier le morphisme induit par h en homologie avec un element $h_* \in Sp(2g, Z)$.

Theoreme 1.1.2 *On pose*

$$f_k(M^3) = c^{-g/2} < \theta_0, \rho_k(h_*)\theta_0 > (modulo R_8) \in C/R_8 \quad (1.7)$$

où $<,>$ *est la structure hermitienne naturelle dans* $W_g(k)$ *et*

$c =< \theta_0, \rho_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \theta_0 >.$

Alors $f_k(M^3)$ ne depend pas des choix faits mais seulement du type d' homéomorphisme de la variété M^3.

Il y a une variante $f_{k,p}$ des invariants qui utilise la représentation $\Lambda^p \rho_k$. Les détails se trouvent dans le texte.

Les invariants topologiques (multiplicatives) s'étendent d'une manière canonique à une TQFT. Remarquons qu'on a un analogue de décomposition de Heegaard pour les 3-variétés à bord, en utilisant des corps de compression.Soit $\Sigma_g \cong \partial H_g \subset R^3$ plongée d'une facon standard dans R^3.On choisit des cercles plongés $c_j \subset \Sigma_g, j \leq m$ qui bordent dans H_g et on met

$H_g(c_1, ..., c_s) = \Sigma_g \times [0,1] \cup_{j \leq m}$ 1-anse attachee le long du cercle c_j
On obtient ainsi tous les corps de compression [Cra91].On fixe donc le groupe $K \cong Z^g = ker(H_1(\Sigma_g, Z) \longrightarrow H_1(H_g, Z))$ et on considère le sous-groupe $L = Z < c_1, ..., c_s > \subset K$ engendré par les classes d'homologie c_j. Soit $r_k : Z^g \longrightarrow (Z/kZ)^g$ le morphisme de réduction modulo k et

$\pi_L : (Z/kZ)^g \longrightarrow (Z/kZ)^g / r_k(L) \cong (Z/kZ)^{g-r}$
la projection canonique ($g - r$ sera la somme des généra de l'autre bord de $H_g(c_1, ..., c_s)$). On définit alors

$$f_k(H_g(c_1, ..., c_s)) = \sum_{m \in (Z/kZ)^g} \theta^*_m \otimes \theta_{\pi_L(m)} \quad (1.8)$$

Maintenant pour tout cobordisme orienté M^3 on a une décomposition de Heegaard $M^3 = A \cup B$ en deux corps de compression qui sont recollés le long d'une surface commune par le homéomorphisme h. On posera alors

$$f_k(M^3) = D_h(f_k(A) \otimes f_k(B)) \quad (1.9)$$

Théorème 1.1.3 *La formule (9) définit une TQFT en dimension 3.*

On remarque que la TQFT qu'on a décrit a la même expression pour les corps de compression que celle de [MOO92]. Alors aussi les invariants déduits sont les mêmes, et on a donc une expression très simple pour eux:

Corollaire 1.1.4 *Considèrons M^3 une 3-variété fermée. Si pour toute classe $\alpha \in H^1(M, Z/kZ)$ on a le cup produit $\alpha \cup \alpha \cup \alpha = 0$ alors $f_k(M^3) = (cardH^1(M, Z/kZ))^{1/2}$ autrement $f_k(M^3) = 0$. En plus même dans le cas des variétés à bord ces invariants sont des invariants homotopiques.*

On remarque que $f_{k,p}$ ne sont pas des invariants homotopiques en general. On peut s'y echapper à l'ambiguité restant dans le groupe des racines de l'unité en ajoutant une structure spin sur M^3. Autrement on peut proceder comme dans [MOO92] quand le terme regularisant s'identifie à un invariant de Rochlin (modulo 2).

1.2 Une TQFT semi-abélienne

Dans ce chapitre on généralise l'approche décrite dans [Fun93f] pour un groupe de jauge G arbitraire mais tout en restant dans un contexte abélien. La motivation est donnée dans la version anglaise.

Soit G un groupe de Lie simple, connexe et 1-connexe de rang l ayant le tore maximale T et le groupe de Weyl W. On choisit un système de racines R dont le reseau des co-racines associé est $Q^\vee$. On va supposer que toutes les racines ont la même longueur.

Soit $\mathcal{S}_g$ l'espace de Siegel (des matrices complexes symétriques à partie imaginaire positive definie) qui paramétrise les structures holomorphes des tores de dimension g. Si $\Omega \in \mathcal{S}_g$ on a une variété abélienne principalement polarisée $Ab(\Omega) = \mathbf{C}^g/R(\Omega)$, où $R(\Omega)$ est le reseau engendré par les collonnes de la matrice $[1_g\, \Omega]$ ayant la polarisation qu'on appelle η. Maintenant $J(\Omega) = Ab(\Omega) \otimes Q^\vee$ est une variété abélienne de dimension gl et le groupe W^g agit sur $J(\Omega)$ en utilisant l'action du W sur chaque facteur du type $\mathbf{C} \otimes Q^\vee$. On a une inclusion canonique ([Fun93c, Loo76, Mum70])

$$(S^2Q^\vee)^W \otimes_{\mathbf{Z}} (H^2(Ab(\Omega), \mathbf{Z}) \cap H^{1,1}(Ab(\Omega))) \longrightarrow H^2(J(\Omega), \mathbf{Z}),$$

où $(S^2Q^\vee)^W$ est le groupe des formes bilinéaires symétriques W-invariantes et integrales sur $Q^\vee$. Soit I le générateur de $(S^2Q^\vee)^W$.

Lemme 1.2.1 *1. Il existe un fibré en droites holomorphe $\mathcal{L}$ sur $J(\Omega)$, unique jusqu'à une translation, tel que $c_1(\mathcal{L}) = I \otimes \eta$.*

2. Le fibré $\mathcal{L}$ est ample.

3. Pour tout $w \in W^g$ les fibres $w^*\mathcal{L}$ et $\mathcal{L}$ sont isomorphes.

La preuve suit du théorème d'Appell-Humbert et du théorème de Lefschetz [Mum70].
Maintenant on suit de près la construction de Witten en utilisant l'espace de Siegel à la place de l'espace de Teichmuller. On considère l'espace $N(\Sigma_g) = Hom(\pi_1(Jac(\Sigma_g)), G)/G$, où G agit par conjugaison. Alors $N(\Sigma_g)$ s'identifie a $\underbrace{T \times T ... \times T}_{2g}/W$ où W agit diagonalement. Le tore $T^{2g} = T \times T \times ... \times T$ est difféomorphe à $J(\Omega)$ donc on a une famille de structures holomorphes sur T^{2g} paramétrée par $\mathcal{S}_g$. Le fibré en droites pré-quantique sera alors $\mathcal{L}$. Soit $Th(k, g, R, \Omega) = H^0(J(\Omega), \mathcal{L}^{\otimes k})$. L'espace de Hilbert associé par quantification géométrique à $N(\Sigma_g)$ est la partie W-invariante qu'on note par $Th(k, g, R, \Omega)^W$. La famille d'espaces $Th(k, g, R, \Omega)$ est une fibré vectoriel au-dessus de $\mathcal{S}_g$ muni d'une connexion projectivement plate qui est décrite en termes des fonctions thêta (voir [Fun91, Goc92]).On va considérer k pair et $W \neq 1$ dans la suite. Alors on aura le fibré vectoriel hermitien $Th(k, g, R)^W$ ayant la fibre $Th(k, g, R, \Omega)^W$ au-dessus du point $\Omega \in \mathcal{S}_g$.

Théorème 1.2.2 *Le fibré hermitien $Th(k, g, R)^W$ admet une connexion plate, naturelle par rapport à l'action du groupe sympléctique sur $\mathcal{S}_g$.*

La preuve suit d'une description explicite des sections plates qui sont des fonctions thêta W-invariantes, comme dans [Loo76].
Soit ρ^W la monodromie de l'action du groupe $Sp(2g, \mathbf{Z})$ sur $\mathcal{S}_g$. Soit $Sp^+(2g, \mathbf{Z})$ le sous-groupe des matrices sympléctiques ayant la forme $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. Si M^3 est une variété orientée et fermée de dimension 3 et $M^3 = T_g \cup T_g$ est un scindement de Heegaard orienté en deux tores pleins de genre g on choisit $h(M) \in Sp(2g, \mathbf{Z})$ qui représente l'automorphisme induit par le homéomorphisme de recollement dans l'homologie $H_1(\partial T_g, \mathbf{Z})$ de la surface de genre g. On note par $\zeta_k = exp(2\pi i c_k)$, $c_k = kI(r, r)/h(k + h)$ etant la charge centrale en niveau k (r est la moitié de la somme des racines et h est le nombre de Coxeter du groupe G) et par $U(G, k)$ le groupe (des racines de l'unité) engendré par ζ_k.

La représentation ρ^W admet un unique vecteur unitaire (modulo un scalaire de module 1) invariant par $Sp^+(2g, \mathbf{Z})$ (modulo un caractère) en chaque genre g qu'on le note par w_g.

Proposition 1.2.3 *La classe du nombre complexe*
$I_G(M^3, k) = (k+h)^{-lg/2} < \rho^W(h(M))w_g, w_g > \in \mathbf{C}/U(G,k)$
ne depend pas des choix faits et définit donc un invariant topologique de la variété M^3.

La preuve, similaire au cas abélien est donnée dans [Fun93c] (voir aussi [Fun91]).
Cette définition de l'invariant I_G présente quelques inconvenients: nous n'avons pas de formule explicite pour la dimension du $Th(k, g, R, \Omega)^W$ et ensuite on a l'ambiguité résidant dans $U(G, k)$. Mais l'action du W s'étend à une action du W^g et la famille des W^g-invariants $Th(k, g, R, \Omega)^{W^g}$ reste un fibré hermitien au-dessus de l'espace de Siegel qu'on le note par $Th(k, g, R)^{W^g}$.

Proposition 1.2.4 *Le fibré* $Th(k, g, R)^{W^g}$ *est un sous-fibré plat du* $Th(k, g, R)^W$.

Ensuite la section plate w_g est aussi W^g-invariante. Soit ρ_W la monodromie de l'action du groupe $Sp(2g, \mathbf{Z})$ sur $Th(k, g, R)^{W^g}$. Alors w_g est un vecteur (projectivement) invariant pour $\rho_W(Sp^+(2g, \mathbf{Z}))$, et on peut exprimer l'invariant I_G de la manière suivante:
$I_G(M^3, k) = (k+h)^{-lg/2} < \rho_W(h(M))w_g, w_g >$,
$<,>$ etant la structure hermitienne canonique. Cette fois on a une base assez explicite pour les espaces $Th(k, g, R, \Omega)^{W^g}$. Soit P le reseau dual au reseau des racines et $L = I^{-1}(P)$. Alors la base canonique correspond aux représentants de $L^g/kQ^{\vee g} \bowtie W^g$. Un tel système est donné par P_k^g où on avait mis
$P_k = \{\lambda \in L \text{ tel que } 0 \leq < \lambda, \alpha > \leq k \text{ pour toute racine positive } \alpha\}$.
correspondant au représentations integrables de poids maximaux du groupe de lacets LG. Le vecteur w_g correspond à l'index $0_g = (0, 0, ..., 0) \in P_k^g$.

Theoreme 1.2.5 *Pour k pair la représentation ρ_W est determinée par les données suivantes:*

1.

$$\rho_W \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix} = diag(exp(\frac{\pi i}{k+h} < \lambda + r_g, B(\lambda + r_g) > - \frac{\pi i}{k} < r_g, Br_g >)) \quad (1.10)$$

où $B = B^{\top}$ est une matrice à coefficients entiers.

2.

$$\rho_W \begin{bmatrix} A & 0 \\ 0 & (A^{\top})^{-1} \end{bmatrix} = (\delta_{A^{\top}\lambda,\mu})_{\lambda,\mu \in L^g/kQ^{\vee g} \rtimes W^g}. \quad (1.11)$$

pour $A \in GL(g, Z)$.

3.

$$\rho_W \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} = i^{gp}(k+h)^{-lg/2} \left(\frac{vol(M)}{vol(Q^{\vee})}\right)^{g/2} \times$$
$$\left(\sum_{w \in W^g} det(w) exp(\frac{2\pi i}{k+h} < w(\lambda + r_g), \mu + r_g >) \right)_{\lambda,\mu} . \quad (1.12)$$

où p est le nombre des racines positives, vol est le volume du quotient déterminé par le reseau, le produit hermitien $<,>$ est l'extension canonique du I, $r_g = (r, r, ..., r) \in P_k^g$ et $det(w) = \Pi_{i=1}^g det(w_i)$ si $w = (w_1, w_2, ..., w_g) \in W^g$, det etant le caractère alternant sur W.

Pour k impair les mêmes formules définissent une représentation du groupe thêta $\Gamma(1, 2)$.

Le sous-groupe $Sp^+(2g, \mathbf{Z})$ agit (via ρ_W) sur w_g comme un caractère nontrivial ce qui explique l'ambiguité $U(G, k)$. La methode pour y echapper est d'ajouter une p_1-structure (comme dans [Ati90, BHMV92]). Soit X la fibre homotopique de l'application $p_1 : BO \longrightarrow K(\mathbf{Z}, 4)$ qui correspond à la première classe de Pontryaghin du fibré tautologique τ sur BO. Une p_1-structure sur une variété M est une application fibrée du fibré tangent stable τ_M du M dans le fibré $p_1^*\tau$ au-dessus de X. Si M^3 est de dimension 3 (fermée et orientée) alors c'est le bord d'une variété de dimension 4, disons Y. Pour une p_1-structure α sur M^3 soit $p_1(Y, \alpha) \in H^4(Y, M; \mathbf{Z})$ l'obstruction de l'étendre sur Y et

$\sigma(\alpha) = 3signature(Y) - < p_1(Y,\alpha), [Y] > \in \mathbf{Z}$.
Alors σ ne depend pas du choix de la variété Y d'après la formule de Hirzebruch pour la signature et c'est en fait 3 fois l'obstruction d'Atiyah ([Ati90]). L'ensemble des classes d'homotopie de p_1-structures sur M^3 est affine isomorphe à $\mathbf{Z}$ par σ. La p_1-structure canonique correspond à $\sigma(\alpha) = 0$.

Considèrons maintenant le 2-cocycle signature (où cocycle de Meyer [Ati90, Mey73]) $c : Sp(2g, \mathbf{R}) \times Sp(2g, \mathbf{R}) \longrightarrow \mathbf{R}$ qui donne une extension centrale du groupe sympléctique
$0 \longrightarrow \mathbf{R} \longrightarrow \widetilde{Sp}(2g, \mathbf{R}) \longrightarrow Sp(2g, \mathbf{R}) \longrightarrow 0$
ayant une section canonique s. La fonction de Meyer $\varphi : \widetilde{Sp}(2g, \mathbf{R}) \longrightarrow \mathbf{R}$ est l'unique (voir [Bar92]) fonction satisfaisant $c(a,b) = \varphi(s(ab)) - \varphi(s(a)) - \varphi(s(b))$. Remarque que $\varphi(\tilde{S}p(2g, \mathbf{Z})) = \mathbf{Z}$. Soit $\Phi : Sp(2g, \mathbf{Z}) \longrightarrow \mathbf{Z}$ donné par $\Phi(a) = lim_n \frac{1}{n} \varphi(s(a)^n)$. On peut alors définir l'invariant pour des variétés avec une p_1-structure α de la manière suivante: si α diffère de la structure canonique par l'entier $m \in \mathbf{Z}$ on pose
$Z_G((M^3, \alpha), k) = \zeta_k^{\Phi(h(M))+m} (k+h)^{-lg/2} < \rho_W(h(M)) w_g, w_g >$.
Le résultat principal est:

Théorème 1.2.6 *Le nombre complexe $Z_G((M^3,\alpha),k)$ est un invariant topologique des variétès orientées fermées avec p_1-structure qui se comporte multiplicativement par rapport à la somme connexe. Le changement d'orientation se traduit par la conjugaison. Si la p_1-structure est altérée par $m \in Z$ l'invariant est multiplié par ζ_k^m.*

Maintenant le principe general (voir [Fun93g]) fournit une TQFT qui étend l'invariant au cobordismes contenant des entrelacs avec une précaution: tandis que pour les variétés fermées la classe d'homotopie de la p_1-structure suffit, pour recoller les cobordismes on a besoin de p_1-structures fixes (voir [BHMV92]).

On remarque que pour M^3 qui est un espace lenticulaire ou un fibré en tores l'invariant Z_G coincide avec celui de Witten et il est calculé par Jeffrey [Jef92]. En effet les représentations du $SL(2, \mathbf{Z})$ qui correspondent aux deux invariants coincident. Ceci prouve que les invariants Z_G ne sont pas des invariants homotopiques en général. Ces resultats permetront de calculer la monodromie de la theorie de Witten pour petites valeurs de k. Remarquons que l'invariant Z_g ne distingue pas parmi les sphères d'homologie entière. D'autre part il semble que la famille des invariants $I_G(M^3, k)$ caractérise complètement le premier nombre de Betti $b_1(M)$ et la forme d'enlacement sur le sous-groupe de torsion $Tors(H_1(M, \mathbf{Z}))$ à valeurs dans $\mathbf{Q}/\mathbf{Z}$.

La TQFT associée provient d'une RCFT (voir [Fun93g, Cra91, Deg92]) dont la dimension des espaces des blocks conformes en genre g sera $d(k)^g$ ou $d(k) = card(P_k)$. Alors la RCFT est determinée par un groupe fini exactement comme la théorie de Witten avec $G = U(1)$ en niveau k provient de la RCFT associée au groupe $\mathbf{Z}/k\mathbf{Z}$ (voir l'appendice 3 du [MS89]). Ces aspects seront consideres ultérieurement.

1.3 TQFT et RCFT

Dans sa construction des invariants topologiques (pour les variétés de dimension 3) Witten [Wit89] fait refference au modèle WZW (correspondant au groupe $SU(2)$) de la théorie conforme du champs rationnelle (abbregée RCFT) bidimensionnelle. Peu après Crane [Cra91] et ensuite Degiovanni [Deg92] ont montré qu'à chaque RCFT on peut associer des invariants topologiques des variétés de dimension 3, et plus générale une théorie quantique du champs topologique (abbregée TQFT) d'après la terminologie introduite par Atiyah [Ati89]. Le propos du chapitre 3 est d'éclaircir le lien entre TQFTs et RCFTs du point de vue mathématique. Les détails se trouvent dans [Fun93a] (voir aussi [Fun94b]). Nous utiliserons la définition combinatoriale d'une RCFT due à Moore et Seiberg [MS89] qui paraitra dans la suite.

Soit $\mathcal{M}_{*,*}$ la tour des groupes des classes d'homéomorphismes (des surfaces compactes orientées à bord). On a une multiplication multivalente

$\sigma : \mathcal{M}_g \times \mathcal{M}_h \longrightarrow \mathcal{M}_{g+h}$

donnée par $\sigma(x, y) = \{\tilde{x}\sharp\tilde{y}; \pi_g(\tilde{x}) = x, \pi_h(\tilde{y}) = y\} \subset \mathcal{M}_{g+h}$. Ici $\pi_* : \mathcal{M}_{*,1} \longrightarrow \mathcal{M}_*$ est la projection canonique et $\sharp : \mathcal{M}_{g,1} \times \mathcal{M}_{h,1} \longrightarrow \mathcal{M}_{g+h}$ est le morphisme induit par le recollement des homéomorphismes. On appelle une représentation hermitienne tensorielle (abbregee r.h.t.) de la tour $\mathcal{M}_*$ (correspondant au surfaces fermées) la donnée suivante:

1. Une famille d'espaces vectoriels complexes W_* munis des formes hermitiennes non-degenerees $<,>$. Soit $U(W_g)$ le groupe des transformations linéaires du W_g qui préservent la forme hermitienne.

2. Une structure tensorielle $\otimes : W_g \times W_h \longrightarrow W_{g+h}$ (famille d'applications bilineaires) compatible avec la structure hermitienne (voir la suite).

3. Une famille de représentations $\rho_* : \mathcal{M}_* \longrightarrow U(W_*)$ telles qu'on a

$< \rho_{g+h}(c)(x \otimes y), x' \otimes y' > = < \rho_g(a)x, x' >< \rho_h(b)y, y' >$

pour toutes $x, x' \in W_g, y, y' \in W_h, a \in \mathcal{M}_g, b \in \mathcal{M}_h, c \in \sigma(a, b) \subset \mathcal{M}_{g+h}$.

La r.h.t. ρ_* est dite ponderée s'il existe un vecteur $w_g \in W_g$ (le vecteur vide) satisfaisant les conditions

4. $w_{g+h} = w_g \otimes w_h$ pour toutes g, h.

5. Soit $\mathcal{M}_g^+ \subset \mathcal{M}_g$ le sous-groupe des homéomorphismes qui s'étendent au tore solide de genre g. Alors

$\rho_g(c)w_g = w_g$ pour tout $c \in \mathcal{M}_g^+$.

6. $d = < w_1, \rho_1(\tau)w_1 > \neq 0$, ou $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2, \mathbf{Z}) = \mathcal{M}_1$.

Soit M^3 une variété fermée et orientée de dimension 3 et $M^3 = T_g \cup_\varphi \overline{T_g}$ un scindement de Heegaard orienté en deux tores pleins de genre g qui se recollent le long de la surface $\Sigma_g = \partial T_g$ en utilisant l'homéomorphisme $\varphi \in \mathcal{M}_g$. Soit ρ_* une r.h.t.p.(onderée). On pose

$f(M^3) = d^{-g} < \rho_g(\varphi)w_g, \overline{w_g} > \in \mathbf{C}$.

Proposition 1.3.1 *i) Le nombre $f(M^3)$ ne depend pas des choix faits donc c'est un invariant topologique des variétés fermées et orientées.*

ii)Reciproquement un invariant f des variétés fermées et orientées (de dimension 3) qui est

1. multiplicatif, donc $f(M \sharp N) = f(M)f(N)$

2. sensitive à l'orientation, c'est à dire $f(\overline{M}) = \overline{f(M)}$,

détermine une r.h.t.p. ρ_* telle que f est défini par la formule precedente.

On se ramène donc à l'étude des r.h.t.p.. On va considérer dans la suite que $<,>$ est positive et les espaces W_* sont de dimension finie.

Soit $c_* = \{c_1, c_2, ..., c_{3g-3}\}$ un système de courbes qui décomposent la surface Σ_g en pantalons et Γ le graph dual à la decomposition. On assume que la r.h.t.p. ρ_* est cyclique, c'est à dire que W_g est l'espace vectoriel engendré par l'orbite $\rho_g(\mathcal{M}_g)w_g$. On note par t_c le twist de Dehn autour du cercle c plongé dans Σ_g. Soit v un sommet du graphe Γ et $c_1, c_2, c_3 \in c_*$ les cercles correspondants aux arrêtes incidentes a v. On définit

$Z(\Gamma, v, (a_1, a_2, a_3)) = \{x \in W_g; \rho_g(t_{c_i})x = a_i x\} \subset W_g$.
Le choix du sommet v induit un morphisme $\mathcal{M}_{0,3} \to \mathcal{M}_g$ et $Z(\Gamma, v, (a_1, a_2, a_3))$ aura la structure de $\mathbf{C}[\mathcal{M}_{0,3}]$-module. Ce module scinde dans une somme directe de $\mathbf{C}[\mathcal{M}_{0,3}]$-modules simples (et cycliques) qui sont deux-à-deux isomorphes et dont la classe d'isomorphisme on la note par $W(\Gamma, v, (a_1, a_2, a_3))$.

Proposition 1.3.2 *Le $\mathbf{C}[\mathcal{M}_{0,3}]$-module $W(\Gamma, v, (a_1, a_2, a_3))$ ne depend pas du choix du découpage c_*, du sommet v, or du genre g.*

On le dénote donc par $W(a_1, a_2, a_3)$. L'ensemble des valeurs propres et leur inverses pour les twists de Dehn dans c_* est un sous-ensemble fini $L \subset \mathbf{C}$ qui lui aussi ne depend du découpage choisi.

Proposition 1.3.3 *Supposons que la r.h.t.p. ρ_* est cyclique, unitaire, de dimension finie et le vecteur w_g est unique en chaque genre. Alors on a la décomposition*
$W_g \stackrel{i_\Gamma}{\simeq} \oplus_{l \in \mathcal{L}} \otimes_{v \in V(\Gamma)} W(l(e_1(v)), l(e_2(v)), l(e_3(v)))$
ou $V(\Gamma)$ est l'ensemble des sommets de Γ, $\mathcal{L}$ est l'ensemble des coloriages des arrêtes de Γ (avec l'ensemble des couleurs L) et $\{e_1(v), e_2(v), e_3(v)\}$ sont les arrêtes incident au sommet v. Aussi Γ est le graphe dual d'un découpage étendu (qui peut contenir des cercles qui bordent).

On fixe les orientations du chaque cercle c_i pour distinguer parmi t_{c_i} et $t_{c_i}^{-1}$. On va spécifier ca dans la notation des blocs primaires $W(i, j, k)$ en mettant les indices comme $W^{ijk}, W_k^{ij}, W_{jk}^i, W^{ijk}$ suivant l'identification du pantalon avec le cobordisme orienté ayant deux composantes du bord: une positive et l'autre negative. On peut toujours se ramener au cas où la composante positive contient deux cercles et on trouve les symétries
$W_{jk}^i \simeq W_{kj}^i,\ W_{jk}^i \simeq W_{ji^{-1}}^{k^{-1}}$.
En cette situation ρ_* s'étend à une représentation du groupoide de dualité $\mathcal{M}_{*,*}$. On a des lois de multiplication extérieures sur $\mathcal{M}_{*,*}$ induites par le recollement des homéomorphismes (sur des surfaces recollées le long des composantes du bord). La conjecture de Grothendieck [Gro84] prouvé par Moore et Seiberg affirme que le groupoide de dualite est engendré par les elements suivantes:

1.T le twist de Dehn autour du meridien du tore avec un trou, $T \in \mathcal{M}_{1,1}$.

2. $S \in \pi_1^{-1}(\tau) \subset \mathcal{M}_{1,1}$.

3. $\Omega(-)$ qui interchange les deux cercles de la composante positive du bord du pantalon, $\Omega(-) \in \mathcal{M}_{0,3}$.

4. $\Theta(-)$ qui interchange deux cercles dans des composantes différentes du bord du pantalon, $\Theta(-) \in \mathcal{M}_{0,3}$.

5. Considerons la sphère à quatre trous $\Sigma_{0,4} = S^2 - d_1 \cup d_2 \cup d_3 \cup d_4$. Soit e un cercle plongé qui sépare $d_1 \cup d_2$ et $d_3 \cup d_4$ et f le cercle qui sépare $d_1 \cup d_4$ et $d_2 \cup d_3$. Soit F un homéomorphisme qui interchange e avec f, $F \in \mathcal{M}_{0,4}$.

Le mot engendré signifie qu'on utilise la multiplication et l'inverse dans chaque groupe et aussi les lois de multiplication extérieures entre groupes.

La Proposition 2 et les contraintes imposées sur T, S, Ω, Θ, F (pour que ce soient induites d'une représentation du $\mathcal{M}_*$) nous fournissent la donnée suivante:

i) Les espaces W_{jk}^i, avec $i, j, k \in L$, telles que $W_{1j}^i \simeq \mathbf{C}$ si $i = j$, et ce soit 0 autrement; L est muni d'une unité 1 et de l'involution $j^* = j^{-1}$.

ii) Les isomorphismes
$S(j) : \oplus_i W_{ji}^i \longrightarrow \oplus_i W_{ji}^i,\ j \in L$
$T : \oplus_i W_{ji}^i \longrightarrow \oplus_i W_{ji}^i,\ j \in L$
$\Omega_{jk}^i(-) : W_{jk}^i \longrightarrow W_{kj}^i,\ \Omega_{jk}^i(+) = \Omega_{jk}^i(-)^*$
$\Theta_{jk}^i(-) : W_{jk}^i \longrightarrow W_{ji^{-1}}^{k^{-1}},\ \Theta_{jk}^i(+) = \Theta_{jk}^i(-)^*$
$F\begin{bmatrix} i & j \\ k & l \end{bmatrix} : \oplus_{r\in L} W_{ir}^k \otimes W_{jl}^r \longrightarrow \oplus_{s\in L} W_{sl}^k \otimes W_{ij}^s.$

iii) Les équations

$$F(\Omega(\varepsilon)\otimes 1)F = (1\otimes\Omega(\varepsilon))F(1\otimes\Omega(\varepsilon)) \tag{1.13}$$

$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12} \tag{1.14}$$

$$S^2(j) = \oplus_{i\in L}\Theta_{ji}^i(-) \tag{1.15}$$

$$S(j)TS(j) = T^{-1}S(j)T^{-1} \tag{1.16}$$

$$(S\otimes 1)(F(1\otimes\Theta(-)\Theta(+)F^{-1})(S^{-1}\otimes 1) = FPF^{-1}(1\otimes\Omega(-)) \tag{1.17}$$

où P_{ij} permute les facteurs i et j d'un produit tensoriel, et F_{ij} n'agit que sur le produit des deux facteurs. On identifie W_g avec
$\oplus W^{1}_{i_1 i_1} \otimes W^{i_1}_{i_1 j_1} \otimes W^{j_1}_{i_2 k_2} \otimes ... \otimes W^{j_{g-1}}_{i_g i_g} \otimes W^{i_g}_{i_g 1}$.
Si on utilise les générateurs standard $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, \delta_2\}$ du $\mathcal{M}_g$ (voir [MS89, Bir74, Fun93a]) la representation ρ_g se retrouve comme suit:

$$\rho_g(\alpha_1) = T_1^{-1} \tag{1.18}$$

$$\rho_g(\alpha_l) = F_{j_{l-1}} \begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} T_{j_{l-1}} F^{-1}_{j_{l-1}} \begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix} \tag{1.19}$$

for $l > 1$.

$$\rho_g(\beta_l) = T_{k_l} S_{k_l i_l}(j_{l-1}, j_l) T_{k_l}. \tag{1.20}$$

$$\rho_g(\delta_2) = T_{i_2}. \tag{1.21}$$

Les indices nous indiquent sur quel facteur les transformations agissent. Remarquons que le vecteur vide w_g est le générateur de $W^1_{11} \otimes W^1_{11} \otimes ... \otimes W^1_{11} \subset W_g$. La donnée i)-iii) correspond à la definition de Moore et Seiberg d'une RCFT. Comme la représentation ρ_* obtenue a partir d'une RCFT est en général une représentation projective il suit que la charge centrale c est un entier congruent à 0 modulo 24.

Théorème 1.3.4 *Une r.h.t.p. cyclique, unitaire, de dimension finie, à vecteur vide unique est équivalente à une RCFT unitaire de charge centrale $c = 0$(modulo 24).*

Les exemples les plus interesants de telles RCFT sont celles obtenues à partir d'un groupe compact G (dont la matrice de fusion est donnée par les "6-j symboles" classiques, et le modèle de Wess-Zumino-Witten basé sur le groupe de Lie simple E_8. Un formalisme similaire permet d'etablir que les représentations projectives correspondant aux invariants des variétés framées (voir [Ati90, BHMV92]) sont determinés par une RCFT, cette fois sans des contraintes sur la charge centrale.

Soit M^3 un cobordisme orienté entre $\partial_+ M$ et $\partial_- M$. Alors $M^3 = C_+ \cup_\varphi \overline{C_-}$ se décompose en deux corps de compression (voir [Cra91, Fun93f]). La présence des automorphismes non-triviaux des composantes du bord impose l'addition d'une structure supplémentaire. Donc on fixe des identifications du bord avec une union des surfaces. D'une manière équivalente on peut rigidiser les surfaces comme suit: $(\Sigma_g, c_*, \Gamma \subset \Sigma_g)$ est une surface rigide ou on a choisit un découpage c_* dont le graphe dual est plongé dans Σ_g tel que chaque pantalon ne contient que le sommet dual et les arrêtes incidentes. Soit C un corps de compression obtenu en attachant des anses d'indice 2 sur les cercles $c_1, c_2, ..., c_s \subset \Sigma_g$ qui bordent dans T_g sur le bord du cylindre $\Sigma_g \times [0,1]$. On considère un découpage c_* qui contient les cercles d'attachement. On peut identifier Σ_g avec un voisinage tubulaire du graphe dual Γ (qui est planaire) dans $\mathbf{R}^3$. Le plongement $\Gamma \subset \Sigma_g$ est donné par le champ vectoriel perpendiculaire au plan du Γ (le "framing" normal). Alors on a une surface rigide $(\Sigma_g, c_*, \Gamma \subset \Sigma_g)$. Supposons que l'autre bord du C est connexe donc c'est la surface Σ_h avec $h = g - s$ (à part des sphères S^2). Soit

$X = \Sigma_g - \bigcup_{i=1}^{s} c_i \times [-\varepsilon, \varepsilon] \bigcup_{i=1}^{s} d_{i1} \cup d_{i2}$,

ou $c_i \times [-\varepsilon, \varepsilon] \subset \Sigma_g$ sont des voisinages réguliers disjointes des c_i's, et d_{i1}, d_{i2} sont des disques (disjoints) plongés proprement dans T_g qui bordent $c_i \times \{\varepsilon\}$ et $c_i \times \{-\varepsilon\}$ respectivement. Alors X est homéomorphe a $\Sigma_h \cup_j S^2$. On considère les cercles $\{c_i; i \in A\}$ qui restent dessines sur Σ_h (en particulier quelque unes parmi $c_1, ..., c_s$). Alors $c'_* = \{c_i; i \in A\}$ est un decoupage etendu sur Σ_h. La trace du Γ dans Σ_h est un plongement $\Gamma' \subset \Sigma_h$ du graphe dual de c'_*. On obtient ainsi le transport de la structure rigide de Σ_g sur l'autre bord du corps de compression. Un coloriage l de Γ' est admissible si $l(x) = 1$ pour $x \in \{c_1, c_2, ..., c_s\}$. Un coloriage admissible l s'étend à un coloriage l^e de Γ par 1. On trouve une application injective

$Z(C) : W_h \overset{i_{\Gamma'}}{\simeq} \oplus_{l\text{admissible}} W(\gamma', l) \simeq \oplus_l W(\Gamma, l^e) \subset \bigoplus l W(\Gamma, l) \overset{i_\Gamma}{\simeq} W_g$

ou $W(\Gamma, l) = \otimes_{v \in V(\Gamma)} W(l(e_1(v)), l(e_2(v)), l(e_3(v)))$.

Revenons au cobordisme M^3 scindé comme avant. On fixe des structures rigides $\alpha_+ = \varphi\alpha_-$ sur la surface de recollement Σ_g. Soit β_+ et β_- les structures rigides sur $\partial_+ M$ et $\partial_- M$ obtenues par transport, en utilisant les deux corps de compression C_+ et $\overline{C_-}$.

Théorème 1.3.5 *L'invariant f des variétés fermées orientées determiné par la r.h.t.p. ρ_* s'étend canoniquement à une TQFT par la formule*

$f(M^3, \beta_+, \beta_-) = d^{-g} Z(C_+)^* \circ \rho_g(\varphi) \circ Z(C_-)$.

Comme un corrolaire il suit qu'une RCFT est determinée uniquement par sa restriction au surface fermées, qui donne une reponse á une question posée par Friedan (voir [Seg88]).

Le reste du chapitre est consacré à la généralisation de la formule de Kirby et Melvin qui expriment les invariants des variétés en fonction des invariants des entrelacs coloriées dans la théorie quantique associée au groupe $SU(2)$.

En final on retrouve les RCFT abélienness qui seront définis par un groupe fini commutative.

1.4 L'homogénite des traces Markov

On veut donner une description axiomatique des invariants des entrelacs similaires à celle obtenue auparavant pour les variétés fermées. On a vu qu'un invariant pour les variétés fermées s'étend au catégorie des entrelacs coloriées (plongés dans une variété quelconque) mais le passage inverse n'est pas evident. Cette description est utilisée pour prouver le théorème d'homogénite des traces Markov:

Théorème 1.4.1 *Toute trace Markov t sur $\mathbb{C}[B_\infty]$ qui factorise par un quotient (filtré) $P(k)$ (pour un k fixé) factorise globalement par le quotient homogéne P^h associé à $P(k)$.*

Ce résultat étend la remarque elementaire suivante: les générateurs b_i du groupe de tresses sont deux-à-deux conjugués, donc une fois qu'on a une relation polynomiale $Q(b_{i_0}) = 0$ on aura $Q(b_i) = 0$ pour tout i.

1.5 Algèbres de Hecke cubiques

Un corollaire du chapitre 3 implique que tout trace Markov qui provient d'une RCFT (donc les invariants des noeuds qui provient des invariants des varietes) est en fait définie sur un quotient assez mince de l'algèbre groupale $\mathbb{C}[B_\infty]$. Ce quotient admet une filtration par de modules dimension finie. Si la RCFT est unitaire cette trace est positive et le complété de l'algebre des matrices sur le quotient d'avant est isomorphe au II_1-facteur hyperfini. D'ailleurs ceci explique pourquoi les diagrames de Brattelli de l'algèbre définie par les R-matrices dans le cas quantique (voir [TW93]) sont des restrictions de la diagrame de Brattelli de l'algèbre de Hecke standard.

Donc si on veut construire des invariants non-quantiques on essai de définir des traces de Markov qui ne factorisent pas par un quotient hyperfini.

Le but du chapitre 5 est de commencer un étude systématique de l'algébre de Hecke cubique dans l'esprit du [Jon87]. Les details des preuves se trouvent dans [Fun93d] (voir aussi [Fun94a]).

Soit B_n le groupe de tresses d'Artin en n brins ayant la présentation usuelle

$< b_1, b_2, ..., b_{n-1} \mid b_i b_j = b_j b_i, \mid i - j \mid > 1,$

$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, i = 1, n-2 >.$

On a des inclusions canoniques $B_n \subset B_{n+1}$ et on pose $B_\infty = \bigcup_{n>0} B_n$. Soit Q un polynôme cubique avec $Q(0) \neq 0$. On définit l'algèbre de Hecke cubique

$H(Q,n) = \mathbf{C}\,[B_n] \,/(Q(b_i), i = 1, n-1),\ n \in \mathbf{Z}_+ \cup \{\infty\},$

par analogie avec l'algèbre de Hecke classique (voir [Bou82]). Remarque que $H(Q,n)$ sont des espaces vectoriels de dimension infinie pour $n > 6$. Cet aspect rend l'étude des traces Markov sur $H(Q,\infty)$ plus compliqué que dans le cas quadratique, traité dans [Jon87]. On va considérer les quotients $P(\infty)$ de $H(Q,\infty)$ qui admettent une filtration naturelle induite $P(n)$, $n \in \mathbf{Z}_+$. Ce quotient est homogéne si, une fois qu'on a l'identité

$F(b_i, b_{i+1}, ..., b_j) = 0$ pour un polynome F,

dans $P(k)$, on aura aussi

$F(b_{i+k}, b_{i+k+1}, ..., b_{j+k}) = 0,\ k \in \mathbf{Z}, k \geq 1 - i.$

dans les quotients respectifs. Le rang d'un quotient homogéne est le plus petit rang oú une relation non-triviale apparait. Donc les quotients homogénes de rang 2 sont des algèbres de Hecke quadratiques. On s'interesse dans la suite aux quotients homogenes de rang 3. On a alors $P(2) = H(Q,2)$, et $P(3)$ est un quotient propre de $H(Q,3)$.

Proposition 1.5.1 *Pour tout polynôme* $Q(x) = \alpha X^2 + \beta X + \gamma$, $\gamma \neq 0$, $dim_{\mathbf{C}} H(Q,3) = 24$. *Une base de l'espace vectoriel est donnée par la famille*

$\mathcal{B} = \{b_1^i, b_1^i b_2^j b_1^k, b_1^i b_2 b_1^2 b_2, i, k = 0, 1, 2, j = 1, 2\}.$

Un système complet de relations pour l'algèbre $H(Q,3)$ *est*

$b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$
$b_2^2b_1^2b_2 = b_1b_2^2b_1^2 + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta(b_1^2b_2 - b_1b_2^2)$
$b_2b_1^2b_2^2 = b_1^2b_2^2b_1 + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta(b_2b_1^2 - b_2^2b_1)$.

Rappellons qu'une trace Markov sur $P(\infty)$ de paramétres $(z, \bar{z})$ est une fonctionnelle linéaire t qui satisfait les conditions
$t(xb_n) = zt(x)$ et $t(xb_n^{-1}) = \bar{z}t(x)$ pour $x \in P(n)$
$t(xy) = t(yx)$.
Le résultat précedent implique qu'on a une famille a 24 paramétres des relations dans $H(Q, 3)$ donc de quotients homogénes de rang 3.

Proposition 1.5.2 *Pour chaque $(z, \bar{z}) \in \mathbf{C}^*$ le quotient homogéne générique de rang 3 n'admet qu'au plus une trace Markov de paramétres $(z, \bar{z})$.*

Idée de la preuve: Soit L_n les modules définies par la récurrence
$L_{n+1} = \mathbf{C} < xb_n^iy; i = 1, 2, x, y \in L_n > \oplus L_n \subset H(Q, n+1)$,
avec $L_2 = H(Q, 2)$. On prouve que la projection de L_n sur le quotient $P(n)$ est surjective si le quotient est générique. Pour $n = 3$ c'est la généricite qui implique cette affirmation. Pour $n > 3$ on utilise une récurrence.□
En particulier le quotient générique de $H(Q, 3)$ est de dimension au plus 21. On peut supposer toujours $\gamma = 1$. On a alors

Proposition 1.5.3 *Soit $\alpha = \beta = 0$.Le quotient K_3 défini par la relation*
$b_2b_1^2b_2 + b_1b_2^2b_1 + b_1^2b_2b_1 + b_1b_2b_1^2 + b_1^2b_2^2 + b_2^2b_1^2 + b_1 + b_2 = 0$ *est de dimension 21.*

La preuve se trouve dans [Fun93d].□
Le quotient homogéne associé K_∞ admet une déformation triviale au-dessus de la droite de γ qu'on la note par $K_\infty(\gamma)$.

On va travailler dans la suite avec des réseaux dans les algèbres introduites qui contiennent plus d'information. On fait d'abord quelques notations: $P(\infty)$ est un quotient homogéne de l'algèbre groupale $A(z, \bar{z})[B_\infty]$. $A(z, \bar{z})$ est le plus petit sous-anneaux de C contenant z et $\bar{z}$ et R est un $A(z, \bar{z})$-module. Le module $MT(P(*), R)(z, \bar{z})$ des traces Markov sur $P(*)$ a valeurs dans R (ayant les paramétres $(z, \bar{z})$ est l'ensemble des morphismes $t : P(\infty) \rightarrow R$ de $A(z, \bar{z})$-modules qui satisfont les conditions d'avant. Le link module du quotient $P(\infty)$ est défini comme

$L(P(\infty))(z, \bar{z}) = P(\infty)/A(z, \bar{z}) < xy - yx, xb_n - zx, xb_n^{-1} - \bar{z}x, x, y \in P(n), n \in \mathbf{Z} >$

où le quotient considéré est un quotient de modules. On remarque que $L(\mathbf{Z}[B_\infty])(1,1)$ est le $\mathbf{Z}$-module libre engendré par les classes d'isotopie des entrelacs orientés dans S^3. Le link module d'un quotient est donc un module engendré par l'ensemble des classes d'isotopie des entrelacs orientés modulo les relations "skein" forcées par les relations qui existent dans $P(\infty)$. Ce module n'est pas toujours un module libre car on a

Théorème 1.5.4 *Le link module du quotient homogéne $K_\infty(\gamma)$ est*

$$L(K_\infty)(z, \bar{z}) = \begin{cases} A(z, \bar{z})/6z^7 A(z, \bar{z}) & \textit{si } z^3 + \gamma = 0, \bar{z} = -z^2/\gamma \\ 0 & \textit{autrement} \end{cases}$$

Idée de la preuve: On a dèja vu que le rank du link module est au plus 1. Le calcule explicite suit de la description du module des traces Markov à valeurs dans un module quelconque R. La méthode de prouver qu'une telle trace Markov existe (pour certaines modules de torsion R) est inspirée par [Ber78]. On définit un graphe géant dont les sommets sont les elements du sémi-groupe abélien engendré par le groupe libre à $n-1$ générateurs. Les arrêtes correspondent aux elements qui diffèrent par exactement une relation (dans l'ensemble des relations qui définissent $K_\infty(\gamma)$). Si on utilise les relations dans une seule direction on arrivait à orienter les arrêtes du graphe. On pourra donc se demander si les elements minimaux existent dans chaque composante connexe du graphe et si on a aussi l'unicité de ceux-ci. Ces elements minimaux seraient une base de $K_n(\gamma)$ si on ajoute suffisament des relations pour que l'unicité ait lieu. L'existence est d'habitude acquise par l'introduction de l'ordre lexicographique sur les mots. Dans notre cas se serait trop compliqué. Dans notre approche les arrêtes orientés sont definies par le remplacement d'un monôme suivant une des préscriptions:

(C0)(j) $Ab_j^3 B \rightarrow AB$
(C1)(j) $Ab_{j+1}b_j b_{j+1}B \rightarrow Ab_j b_{j+1} b_j B$
(C2)(j) $Ab_{j+1}b_j^2 b_{j+1}B \rightarrow AS_j B$
(C12)(j) $Ab_{j+1}b_j^2 b_{j+1}^2 B \rightarrow Ab_j^2 b_{j+1}^2 b_j B$

(C21)(j) $Ab_{j+1}^2 b_j^2 b_{j+1} B \rightarrow Ab_j b_{j+1}^2 b_j^2 B$.
Il faut ajouter des arretes non-orientees, qui correspondent aux changements
(P_{ij}) $Ab_i b_j B \rightarrow Ab_j b_i B$ si $\mid i-j \mid > 1$.
Observons qu'on a ajouté des relations qui rendent le processus de réduction d'un mot ambiguu. La raison est d'assurer l'existence des chemins descendants vers des sommets minimaux, même si on a des lacets (circuits fermés) orientées dans le graphe. On obtient de cette manière l'existence des elements minimaux mais pas l'unicité (qui en plus ne pourra pas etre obtenue). Ensuite on considere une tour des graphes qui modelisent non pas la structure de $K_n(\gamma)$ mais les fonctionnelles sur $K_n(\gamma)$ qui satisfont les conditions de trace Markov (a part la commutativite). L'unicité pour cette tour de graphes fournit les obstructions à l'existence de telles fonctionnelles, qui se trouvent dans $K_4(\gamma)$ donc on peut les traiter algèbriquement. La condition de commutativité correspond à une autre obstruction algèbrique, toujours dans $K_4(\gamma)$ et d'ici le résultat. □

Le calcul du link module montre l'existence d'une trace Markov t sur $K_\infty(-1)$ à valeurs dans $\mathbf{Z}/6\mathbf{Z}$. Soit $Q_\gamma = X^3 + \gamma$. On denote par $H_{\mathbf{Z}}(Q,n)$ l'algèbre de Hecke cubique définie sur $\mathbf{Z}$ qui est un réseaux dans l'algèbre $H(Q,n)$. Notre résultat principal est

Théorème 1.5.5 *1. La trace Markov t se rélève uniquement à une trace Markov F sur $H_{\mathbf{Z}}(Q_{-1},\infty)$ à valeurs dans $\mathbf{Z}$.*
2. La trace F induit un isomorphisme
$L(H_{\mathbf{Z}}(Q_{-1},\infty))(1,1) = \mathbf{Z}$.
3. $rang(L(H_{\mathbf{Z}}(Q_\gamma,\infty))(z,\bar{z})) \leq 1$ pour toutes les valeurs des paramétres.

La preuve suit du théoreme d'homogénite ([Fun93b]).□
En utilisant la méthode de Birman et Lin [BXS93] on peut verifier que F, qui est un invariant des entrelacs orientés (à deux valeurs 1 et -1) n'est pas un invariant Vassiliev de degré fini. Cet invariant s'étend à une famille qu'on appelle invariants Vassiliev ternaires. Soit SB_n le monoide des tresses singulières (voir [Bir93]) ayant les générateurs $g_i, g_i^{-1}, s_i, 1 \leq i < n$ et les relations

$[g_i, g_j] = [s_i, g_j] = [s_i, s_j] = 0$ if $\mid i-j \mid > 1$,
$[g_i, s_i] = 0$,
$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i$,

$s_{i+1}g_ig_{i+1} = g_ig_{i+1}s_i$.
Soit $\mathbf{Z}SB_n$ l'algèbre monoidale du monoide des tresses singulières. L'algèbre de Vassiliev (ternaire) W_n est définie comme le quotient de $\mathbf{Z}SB_n \otimes \mathbf{Z}[\varepsilon]$ par l'idéal engendré par les elements $g_i^2 + g_i^{-1} - \varepsilon s_i$. On note par $\mathbf{Z}(\varepsilon)$ l'algébre des polynômes Laurent en ε. L'application naturelle $i : \mathbf{Z}[B_n] \longrightarrow W_n$ induit un isomorphisme

$$\mathbf{Z}[B_n] \otimes \mathbf{Z}(\varepsilon) \longrightarrow W_n \otimes_{\mathbf{Z}[\varepsilon]} \mathbf{Z}(\varepsilon).$$

Maintenant tout invariant des entrelacs orientés s'étend aux entrelacs singulièrs (n'ayant que des points doubles transverses) par

$$\varepsilon I(L_x) = I(L_{++}) + I(L_-),$$

où L_x, L_{++}, L_- denotent les diagrames planes avec une intersection, deux twists à gauche, et un twist à droite, le reste des diagrammes etant identiques. Un invariant sera de degré d (où invariant Vassiliev ternaire) s'il s'annulle pour tout entrelac singulièr avec $d+1$, ou plus, de points doubles. On a une caractérisation simple des invariants qui sont des limites d'invariants Vassiliev (ternaires):

Proposition 1.5.6 *L'invariant I associé à la trace Markov t (sur $\mathbf{Z}[B_\infty]$) est la limite d'une suite d'invariants Vassiliev ternaires (de degré fini) si et seulement s'il existe une trace Markov*

$$\tau : W_\infty \longrightarrow A(z,\bar{z})[\varepsilon]$$

qui rend la diagrame suivante commutative:

$$\begin{array}{ccc} W_n & \xrightarrow{\tau} & A(z,\bar{z})[\varepsilon] \\ \uparrow & & \downarrow \\ \mathbf{Z}[B_\infty] & \xrightarrow{t} & A(z,\bar{z}) \end{array}$$

où le morphisme $A(z,\bar{z})[\varepsilon] \longrightarrow A(z,\bar{z})$ est donné par $\varepsilon \to 1$.

Comme corollaire on obtient que les invariants quantiques sont des limites d'invariants Vassiliev ternaires. On ne sait pas si les limites de ces invariants ternaires coincident avec les limites des invariants Vassiliev classiques ou pas.

Chapter 2

The abelian TQFT

2.1 Introduction

The aim of this paper is to give some invariants for 3-manifolds using representations of symplectic groups and the theory of Heegaard splittings. Our approach fits into the circle of ideas developped by Witten [Wit89] being the mathematical counterpart of his abelian topological quantum field theory. Also it seems to be related to Reshetikin-Turaev invariants [RT91] for $sl(2,\mathrm{C})$. The way of constructing such invariants is inspired by the work of Kohno [Koh92].

In order to get representations of $Sp(2g,\mathbf{Z})$ we generalize the functional equation of classical theta functions (see [Mum84]) to theta functions of level k. The tensor formalism gives then some invariants lying in C/R_N (R_N being the group of N^{th} roots of unity, for some N) for closed orientable 3-manifolds, depending on two parametres. Explicit computations for S^3, $S^2 \times S^1$ and lens spaces are found in the final of the paper.

2.2 Theta functions of level k

If we want to consider the dependence of the classical theta function $\theta(z,\Omega)$ on Ω the fundamental fact is a functional equation which describes its behaviour under the action of $Sp(2g,\mathbf{Z})$ on both the variables z and Ω. Let $\Gamma(1,2)$ be the so-called theta group consisting of elements

$\gamma \in Sp(2g, \mathbf{Z})$ which preserve the orthogonal form
$Q(n,m) = n^{\top} \cdot m \in \mathbf{Z}/2\mathbf{Z}$.
We represent any element $\gamma \in Sp(2g, \mathbf{Z})$ as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are $g \times g$ matrices. Then $\Gamma(1,2)$ may be alternatively described as the set of those elements γ having the property that the diagonals of $A^{\top}C$ and $B^{\top}D$ are even. Let $<,>$ denote the standard hermitian product on $\mathbf{C}^{2g}$. The functional equation, as stated in [Mum84] is:

$$\begin{aligned} &\theta((C\Omega + D)^{\top^{-1}}z, (A\Omega + B)(C\Omega + D)^{-1}) = \\ &\quad \zeta_{\gamma} det(C\Omega + D)^{1/2} exp(\pi < iz, (C\Omega + D)^{-1}Cz >)\theta(z, \Omega) \end{aligned} \tag{2.1}$$

for $\gamma \in \Gamma(1,2)$, where ζ_{γ} is a 8^{th} root of unity.
If $g = 1$ we suppose that $C > 0$ or $C = 0$ and $D > 0$ so the imaginary part $Im(C\Omega + D) \geq 0$ for Ω in the upper half plane. Then we shall choose the square root $(C\Omega + D)^{1/2}$ in the first quadrant. Now we can express the dependence of ζ_{γ} on γ as follows:

1. if C is even and D is odd $\zeta_{\gamma} = i^{(D-1)/2}(\frac{C}{|D|})$
2. if C is odd and D is even $\zeta_{\gamma} = exp(-\pi i C/4)(\frac{D}{C})$

where $(\frac{x}{y})$ is the usual Jacobi symbol ([HW79]).
For $g > 1$ it is less obvious to describe this dependence. We fix firstly the choice of the square root of $det(C\Omega + D)$ in the following manner: Let $det^{\frac{1}{2}}(\frac{Z}{i})$ be the unique holomorphic function on $\mathcal{S}_g$ satisfying
$(det^{\frac{1}{2}}(\frac{Z}{i}))^2 = det(\frac{Z}{i})$
and taking in $i1_g$ the value 1. Next define
$det^{\frac{1}{2}}(C\Omega + D) = det^{\frac{1}{2}}(D) det^{\frac{1}{2}}(\frac{\Omega}{i}) det^{\frac{1}{2}}(\frac{-\Omega^{-1} - D^{-1}C}{i})$
where the square root of $det(D)$ is taken to lie in the first quadrant. Using this convention we may express ζ_{γ} as a Gauss sum for invertible D

$$\zeta_{\gamma} = det^{\frac{-1}{2}}(D) \sum_{l \in \mathbf{Z}^g / D\mathbf{Z}^g} exp(\pi i < l, BD^{-1}l >) \tag{2.2}$$

and in particular we recover the formula from above for $g = 1$. Otherwise for $\gamma = \begin{bmatrix} A & 0 \\ 0 & (A^{\top})^{-1} \end{bmatrix}$ we have $\zeta_{\gamma} = (detA)^{-1/2}$. For $\gamma =$

$\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$ we take $\zeta_\gamma = 1$. Now for $\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we have $\zeta_\gamma = exp(\pi i g/4)$. It will became clear from below that this data determines ζ_γ.

There is also an interesting connection between the multiplier system ζ_γ and the Maslov index for lagrangian subspaces. Let $\mathbf{R}^{2g}$ be endowed with the usual symplectic structure $s = \sum_{i=1}^{g} dx_i \wedge dx_{i+g}$, and let l_i, $i = 1,3$ be lagrangian subspaces of dimension g. We may define a quadratic form on $l_1 \oplus l_2 \oplus l_3$ by

$B(x_1 + x_2 + x_3) = s(x_1, x_2) + s(x_2, x_3) + s(x_3, x_1)$.

for $x_i \in l_i$, $i = 1,3$. The signature of this quadratic form is the so called Kashiwara (or ternary) index of the triple (l_1, l_2, l_3) and is denoted by $I(l_1, l_2, l_3)$. The failure of the multiplier system ζ_γ to be a homomorphism is expressed via a 2-cocycle. Specifically set $\mu(\gamma_1, \gamma_2) = I(l, \gamma_1 l, \gamma_1\gamma_2 l)$, where l is the lagrangian space $l = \{x_{i+g} = 0$, for $i = 1, g\}$. Then μ is a 2-cocycle (called the Maslov cocycle) and we have ([LV80]):

$\zeta_{\gamma_1\gamma_2} = exp(-\frac{\pi i}{4}\mu(\gamma_1, \gamma_2))\zeta_{\gamma_1}\zeta_{\gamma_2}$.

Our aim is to give an explicit form for the functional equation satisfied by the theta functions of level k. For $m \in (\mathbf{Z}/k\mathbf{Z})^g$ these are defined by

$$\theta_m(z, \Omega) = \sum_{l \in m + k\mathbf{Z}^g} exp(\frac{\pi i}{k} < l, \Omega l > + 2\pi i < l, z >) \qquad (2.3)$$

or, equivalently, by

$\theta_m(z, \Omega) = \theta(m/k, 0)(kz, k\Omega)$

where $\theta(*, *)$ are the theta functions with rational characteristics ([Mum84]) given by

$$\theta(a, b)(z, \Omega) = \sum_{l \in \mathbf{Z}^g} exp(\frac{\pi i}{k} < l + a, \Omega(l + a) > + 2\pi i < l + a, z + b >) \qquad (2.4)$$

for $a, b \in \mathbf{Q}^g$. Obviously $\theta(0, 0)$ is the usual theta function.
Let denote by $R_8 \subset \mathbf{C}$ the group of 8^{th} roots of unity. Then R_8 becomes also a subgroup of the unitary group $U(n)$ acting by scalar multiplication.

We consider $\mathcal{S}_g$ the Siegel space of $g \times g$ symmetric matrices Ω of complex entries having the imaginary part $Im\Omega$ positive defined. There is a natural $Sp(2g, \mathbf{Z})$ action on $\mathbf{C}^g \times \mathcal{S}_g$ given by

$$\gamma \cdot (z, \Omega) = ((((C\Omega + D)^{\top})^{-1})z, (A\Omega + B)(C\Omega + D)^{-1}). \tag{2.5}$$

Consider also the theta vector of level k namely

$\Theta_k(z, \Omega) = (\theta_m(z, \Omega))_{m \in (\mathbf{Z}/k\mathbf{Z})^g}$

We can state now the main result of this section:

Theorem 2.2.1 *The theta vector satisfies the following functional equation:*

$$\begin{aligned} \Theta_k(\gamma \cdot (z, \Omega)) &= \zeta_\gamma det(C\Omega + D)^{1/2} \times \\ & exp(k\pi i < z, (C\Omega + D)^{-1}Cz >)\rho_k(\gamma)(\Theta_k(z, \Omega)) \end{aligned} \tag{2.6}$$

where

1. *γ belongs to the theta group $\Gamma(1, 2)$ if k is odd and to $Sp(2g, \mathbf{Z})$ elsewhere.*
2. *$\zeta_\gamma \in R_8$ is the mutiplier system described above.*
3. *$\rho_k : Sp(2g, \mathbf{Z}) \longrightarrow U(k^g)$ is a group homomorphism for even k ; a similar assertion holds for odd k when $Sp(2g, \mathbf{Z})$ is replaced by $\Gamma(1, 2)$.*

Remark 2.2.1 *This result is stated also in [Igu72] for some modified theta functions but in less explicit form. Our aim is to compute effectively the mapping ρ_k.*

Proof: Consider

$A(\gamma, (z, \Omega)) = \zeta_\gamma det(C\Omega + D)^{1/2} exp(k\pi i < z, (C\Omega + D)^{-1}Cz >)$

Making some messy computations we obtain the identity

$A(\gamma_1\gamma_2, (z, \Omega)) = A(\gamma_1, \gamma_2 \cdot (z, \Omega))A(\gamma_2, (z, \Omega))$.

This can also be derived from the corresponding "cocycle identity" for k equals to 1, which is known to holds via (1) (see also [Fre87] p.14). Therefore if the equation (6) holds for γ_1 and γ_2 it holds also for $\gamma_1\gamma_2$

with $\rho_k(\gamma_1\gamma_2)$ replaced by $\rho_k(\gamma_1)\rho_k(\gamma_2)$. But the theta functions of level k form a basis for the vector space $H^0(Ab_\Omega, \Theta^k)$, where $Ab_\Omega = \mathbf{C}^g/(\mathbf{Z}^g \oplus \Omega\mathbf{Z}^g)$ is the abelian variety corresponding to Ω and Θ is the theta line bundle (giving the principal polarization) over Ab_Ω (see [GH78]). Thus we obtain in fact a representation of the symplectic group and it is sufficiently to check the relation (6) for a system of generators. We know that $Sp(2g, \mathbf{Z})$ is generated by the matrices having one of the following forms ([Bir71, Mum84]):

1. $\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$ where $B = B^\top$; If we want to get the generators of $\Gamma(1,2)$ we must further impose that B has even diagonal.
2. $\begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ where $A \in GL(g, \mathbf{Z})$.
3. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We remark that the theta functions of level k can be expressed as

$$\theta_m(z,\Omega) = exp(\frac{\pi i}{k} < m, \Omega m + 2kz >)\theta(kz + \Omega m, k\Omega) \tag{2.7}$$

This relation follows immediately from [Igu72] p.50. Therefore in the first case we have

$$\begin{aligned} \theta_m(z,\Omega + B) &= exp(\frac{\pi i}{k} < m, (\Omega + B)m + 2kz >) \times \\ &\quad \theta(kz + (\Omega + B)m, k(\Omega + B)) \end{aligned}$$

But $\begin{bmatrix} 1 & kB \\ 0 & 1 \end{bmatrix}$ belongs to $\Gamma(1,2)$ if k is even and B arbitrary or k is odd and the diagonal of B is even. Therefore the relation (1) implies
$\theta(kz + (\Omega + B)m, k(\Omega + B)) = \theta(kz + (\Omega + B)m, k\Omega)$.
Since θ is periodic we obtain
$\theta(kz + (\Omega + B)m, k\Omega) = \theta(kz + \Omega m, k\Omega)$
Using (7) it follows that

$$\theta_m(z,\Omega + B) = exp(\frac{\pi i}{k} < m, Bm >)\theta_m(z,\Omega) \tag{2.8}$$

holds. Thus the equation (6) is verified if we put

$$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag(exp(\frac{\pi i}{k} < m, Bm >)). \tag{2.9}$$

In the second case we have

$$\begin{aligned} \theta_m(Az, A\Omega A^\top) &= exp(\frac{\pi i}{k} < m, A\Omega A^\top m + 2kAz >) \times \\ & \theta(kAz + A\Omega A^\top m, kA\Omega A^\top m)). \end{aligned}$$

But the equation (1) gives us
$\theta(Az, kA\Omega A^\top) = \zeta_\gamma (detA)^{-1/2} \theta(z, k\Omega)$
with $\zeta_\gamma \in R_8$. Therefore

$$\theta_m(Az, A\Omega A^\top) = \zeta_\gamma (detA)^{-1/2} \theta_{A^\top m}(z, \Omega) \tag{2.10}$$

holds. Now the equation (6) will be verified if we shall choose

$$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top m, n})_{m,n \in (\mathbf{Z}/k\mathbf{Z})^g}. \tag{2.11}$$

Above δ stands for the usual Kronecker symbol.

The last case is a little more difficult to handle. We shall recall first the well-known Poisson summation formula ([Mum84]):

Proposition 2.2.1 *Let f be a smooth function on R^g which decrease to zero faster than any rational function at infinity and*
$f^\#(\xi) = \int_{R^g} f(x) exp(2\pi i < x, \xi >) dx$
be its Fourier transform. Then the following identity

$$\sum_{n \in \mathbf{Z}^g} f(n) = \sum_{n \in \mathbf{Z}^g} f^\#(n) \tag{2.12}$$

holds.

Consider now $f(x) = exp(\pi i < kx + m, k^{-1}\Omega(kx + m) + 2z >)$
A simple computation gives us

$$\begin{aligned} f^\#(\xi) &= \zeta (det\Omega)^{1/2} k^{-g/2} exp(-2\pi i k^{-1} < m, \xi >) \times \\ & exp(-\pi < z + k^{-1}\xi, k\Omega^{-1}(z + k^{-1}\xi >). \end{aligned} \tag{2.13}$$

Here $\zeta = exp(\frac{\pi i g}{4})$. Next we are interested in computing

$$\begin{aligned}
S_l &= \sum_{\xi \in -l+k\mathbf{Z}^g} exp(-\pi i < z + k^{-1}\xi, k\Omega^{-1}(z + k^{-1}\xi) >) \\
&= \sum_{\eta \in \mathbf{Z}^g} exp(-\pi i < z - \eta - k^{-1}l, k\Omega^{-1}(z - \eta - k^{-1}l) >) \\
&= \exp(-\pi i < \eta, k\Omega^{-1}z >) exp(-\pi i < l, \Omega^{-1}l >) \times \\
&\quad exp(2\pi i < l, \Omega^{-1}z >) \times \\
&\quad (\sum_{\eta \in \mathbf{Z}^g} exp(-\pi i < \eta, k\Omega^{-1}l > + 2\pi i < \eta, -\Omega^{-1}l + k\Omega^{-1}z >) \\
&= exp(-\pi < z, k\Omega^{-1}z >) exp(-\pi < l, \Omega^{-1}l >) exp(2\pi i < l, \Omega^{-1}z) \times \\
&\quad \theta(-\Omega^{-1}l + k\Omega^{-1}z, -k\Omega^{-1}) \\
&= \exp(-\pi i < z, k\Omega^{-1}z >) \theta_l(\Omega^{-1}z, -\Omega^{-1}).
\end{aligned}$$

From the Poisson formula (12) we get

$$\begin{aligned}
\theta_m(z, \Omega) &= \sum_{n \in \mathbf{Z}^g} f(n) = \sum_{n \in \mathbf{Z}^g} f^{\#}(n) = \\
&= \zeta (det\Omega)^{-1/2} exp(-\pi i < z, k\Omega^{-1}z >) \times \\
&\quad \sum_{l \in (\mathbf{Z}/k\mathbf{Z})^g} k^{-g/2} exp(2\pi i k^{-1} < m, l >) \theta_l(\Omega^{-1}z, -\Omega^{-1}).
\end{aligned}$$

Therefore we put

$$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = k^{-g/2} exp(2\pi i k^{-1} < m, l >)_{m,l \in (\mathbf{Z}/k\mathbf{Z})^g}. \tag{2.14}$$

and the equation (6) is verified.□

2.3 Abelian Witten theory

Consider A an abelian variety with a principal polarization ω. Then for a positive line bundle $L \longrightarrow A$ with first Chern class ω we have, according to [GH78]:

$h^0(A, \mathcal{O}(L)) = 1.$

Moreover $H^0(A, \mathcal{O}(L))$ is generated by the classical theta function θ. Therefore the divisor θ is determined up to translation by (A, ω). Let now $\mathcal{S}_g$ be the Siegel space and $\Omega \in \mathcal{S}_g$. Then $(1, \Omega)$ determine a lattice in $\mathbf{C}^g$, hence an abelian variety Ab_Ω which has a natural principal polarization given by the ample line bundle L_Ω. Now it is known that L_Ω's glue together i.e. they can be viewed as the fibers of a line bundle $L \longrightarrow \mathcal{S}_g$ over the Siegel space (see [Hit90, Wel83, Wit89]). If $V_k(\Omega) = H^0(Ab_\Omega, L_\Omega^{\otimes k})$ then $V_k(\Omega)$ are also the fibers of a vector bundle V_k over $\mathcal{S}_g$. Next a local frame for V_k is provided by the theta functions of level k. Now a deep result of Welters extended to the non-abelian case by Hitchin ([Hit90, Wel83, Wit89]) asserts that V_k has a projectively flat connection. This follows from the fact that θ_m are global solutions of the heat equation: we identify the tangent space of $\mathcal{S}_g$ with the space of symmetric tensors (as any symmetric tensor give a deformation of the Kahler polarization of a torus). Therefore in this trivialization the heat operator takes the form

$$\partial_{\Omega_{st}} + \frac{i}{4\pi k}\partial^2_{z_s z_t}$$

Thus θ_m are the covariant constant sections of this connection. Now the vector bundle $V_k \longrightarrow \mathcal{S}_g$ support the action of $Sp(2g, \mathbf{Z})$. With respect to this action the above connection is not natural. If we modify θ_m as is done in [Mum84] for the case when k equals one, we can obtain a natural connection on V_k whose covariant constant sections are the modified theta functions. This is explained by the factor

$$det(C\Omega + D)^{1/2} exp(\pi < iz, (C\Omega + D)^{-1}Cz >)$$

appearing in the equation (5).

However this connection is not flat but only projectively flat. Hence if we compute the holonomy of this connection (more precisely of the induced connection on the moduli space of principally polarized abelian varieties) using the theta fuctions of level k, we shall obtain not a linear but a projective unitary representation of the symplectic group. This gives a geometric interpretation for the messy factor $\zeta_\gamma \in R_8$ and prove also that the presence of a central factor in the curvature cannot be avoided (see also [Ati90]).

2.4 Representations of symplectic groups

We begin by saying few words about the tensor formalism. A tensor group Γ_* is a collection of groups Γ_m indexed by $m \in \mathbf{Z}$ (m is called the degree) endowed with monomorphisms

$$\Gamma_m \otimes \Gamma_n \longrightarrow \Gamma_{m+n}$$

which satisfy the natural associativity law. A morphism between two tensor groups Γ_* and Γ'_* consists in a family of morphisms between the groups in same degree which are compatible with the structural arrow from above. In the same manner it could be defined tensor algebras, tensor vector spaces and so on.

Let V_* be a tensor vector space which we suppose always in the sequel has its structural arrows not only monomorphisms but isomorphisms (TVS in short). Then $GL(V_*)$ has the structure of a tensor group given by

$$GL(V_m) \otimes GL(V_n) \longrightarrow GL(V_m \otimes V_n) \longrightarrow GL(V_{m+n})$$

The same procedure yields $U(V_*)$ with a tensor group structure if V_* is a hermitian TVS. Next a linear tensor representation (and analogously an orthogonal, an unitary one etc.) of Γ_* is simply a morphism between tensor groups

$$\Gamma_* \longrightarrow GL(V_*).$$

EXAMPLE 2.4.1 *:*

Starting with the family of symplectic groups $Sp(2g, \mathbf{Z})$ we can construct a tensor group Sp_* which will be called the symplectic tensor group. This may be done as follows: We define the structural arrows

$$Sp(2g, \mathbf{Z}) \otimes Sp(2h, \mathbf{Z}) \longrightarrow Sp(2(g+h), \mathbf{Z})$$

by the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A \oplus A' & B \oplus B' \\ C \oplus C' & D \oplus D' \end{bmatrix}.$$

Therefore the theta group $\Gamma(1,2)$ is endowed with the structure of tensor subgroup of the symplectic tensor group.□

EXAMPLE 2.4.2 *:*

Consider

$$Sp_g^+ = \{\gamma \in Sp(2g, \mathbf{Z}) \text{ having the block from bottom left } C = 0\}$$

$\Gamma(1,2)^+ = \Gamma(1,2) \cap Sp_g^+$

Then $Sp_*^+, \Gamma(1,2)^+$ are tensor subgroups of the tensor symplectic group.□

Set V_g for the TVS defined bellow (which may be identified with the space of theta functions)

$V_g = \mathbf{C} < \theta_m, m \in (\mathbf{Z}/k\mathbf{Z})^g >$

with the natural hermitian product and the structural arrows induced by

$\theta_m \otimes \theta_n = \theta_{mn}$

where

$(\mathbf{Z}/k\mathbf{Z})^g \otimes (\mathbf{Z}/k\mathbf{Z})^h \longrightarrow (\mathbf{Z}/k\mathbf{Z})^{g+h}$

is the obvious concatenation map.

Proposition 2.4.1 *The mappings*

$\rho_k : Sp(2g, \mathbf{Z}) \longrightarrow U(V_g)$, *for even* k

$\rho_k : \Gamma(1,2) \longrightarrow U(V_g)$, *for odd* k

which we obtained in the first section, are unitary tensor reprentations.

Proof: We need first

Lemma 2.4.2 *1. Suppose that* γ_1 *or* γ_2 *lies in* $Sp^+(2g, \mathbf{Z})$. *Then the following relation*

$$\zeta_{\gamma_1\gamma_2} = \zeta_{\gamma_1}\zeta_{\gamma_2} \tag{2.15}$$

holds.

2. For all $\gamma_1, \gamma_2 \in Sp_*$ *with* D_i *invertible,* $(i \in 1, 2)$ *we have*

$$det^{\frac{1}{2}}(D_1 \oplus D_2)\zeta_{\gamma_1 \otimes \gamma_2} = det^{\frac{1}{2}}(D_1)det^{\frac{1}{2}}(D_2)\zeta_{\gamma_1}\zeta_{\gamma_2}. \tag{2.16}$$

Proof of lemma: If $\gamma_i \in Sp^+(2g, \mathbf{Z})$ then γ_i leaves the lagrangian subspace $l = \{x_{i+g} = 0\}$ invariant. Since the ternary index I is alternating (see [LV80],p.39) we derive that the Maslov cocycle $\mu(\gamma_1, \gamma_2)$ vanishes. Thus our first claim follows. Remark that in particular the mapping $\gamma \longrightarrow \zeta_\gamma$ is a group homomorphism on $Sp^+(2g, \mathbf{Z})$.

Next we see that ζ_γ is uniquely defined by the 2-cocycle condition and by its restriction to $Sp^+(2g, \mathbf{Z})$. The relation we claimed trivially holds if $\gamma_i \in Sp^+(2g, \mathbf{Z})$. Since the Maslov cocyle verifies

$\mu(\gamma_1 \otimes \gamma_2, \gamma_1' \otimes \gamma_2') = \mu(\gamma_1, \gamma_1') + \mu(\gamma_2, \gamma_2')$

(see [LV80]) we are done.□

We shall use now the fact that ρ_k arises as a monodromy representation.

Specifically we consider $\Omega = \Omega_1 \oplus \Omega_2$. From the definition of level k theta functions we easily find that
$\theta_{mm'}((z_1, z_2), \Omega) = \theta_m(z_1, \Omega_1)\, \theta_{m'}(z_2, \Omega_2)$
hence
$\theta_{mm'}(\gamma_1 \otimes \gamma_2((z_1, z_2), \Omega)) = \theta_m(\gamma_1(z_1, \Omega_1))\, \theta_{m'}(\gamma_2(z_2, \Omega_2))$
We can express both terms using the theorem 2.1. Since the theta functions of level k give a basis for $H^0(Ab_\Omega, \Theta^k)$ we derive that

$$det^{\frac{1}{2}}((C_1 \oplus C_2)\Omega + (D_1 \oplus D_2))\zeta_{\gamma_1\otimes\gamma_2}\rho_k(\gamma_1 \otimes \gamma_2) = \\ det^{\frac{1}{2}}(C_1\Omega_1 + D_1)\zeta_{\gamma_1}\rho_k(\gamma_1) \otimes det^{\frac{1}{2}}(C_2\Omega_2 + D_2)\zeta_{\gamma_2}\rho_k(\gamma_2).$$

On the other hand the choices of square roots of determinants of type $det^{\frac{1}{2}}(\frac{\Omega}{i})$ agree i.e.
$det^{\frac{1}{2}}(\frac{\Omega_1\oplus\Omega_2}{i}) = det^{\frac{1}{2}}(\frac{\Omega_1}{i})det^{\frac{1}{2}}(\frac{\Omega_2}{i})$
since both terms are holomorphic functions, their values on $i1_g$ coincide, and their squares are the same on $\mathcal{S}_{g_1} \times \mathcal{S}_{g_2}$. Next from the definition of $det^{\frac{1}{2}}(C\Omega + D)$ (in the case when the D_i's are invertible) we find that

$$det^{\frac{1}{2}}(D_1 \oplus D_2)\zeta_{\gamma_1\otimes\gamma_2}\rho_k(\gamma_1\otimes\gamma_2) = det^{\frac{1}{2}}(D_1)\zeta_{\gamma_1}\rho_k(\gamma_1)\otimes det^{\frac{1}{2}}(D_2)\zeta_{\gamma_2}\rho_k(\gamma_2).$$

From the lemma 4.2. we conclude that

$$\rho_k(\gamma_1 \otimes \gamma_2) = \rho_k(\gamma_1) \otimes \rho_k(\gamma_2). \tag{2.17}$$

for invertible D_i's. Suppose now that D_1 is not invertible. But any γ_1 may be written as $\lambda_1\lambda_2$ where the λ_i's have the bottom right block invertible. The relation (17) for $\gamma_1\lambda_2^{-1}$ and γ_2 reads

$$\rho_k(\gamma_1\lambda_2^{-1} \otimes \gamma_2) = \rho_k(\gamma_1\lambda_2^{-1}) \otimes \rho_k(\gamma_2).$$

Since $xy \otimes zt = (x \otimes z)(y \otimes t)$ we find that (17) is fulfilled even if D_1 is not invertible. In a similar manner we derive that (17) holds for all γ_i. Therefore ρ_k is a morphism of tensor groups and the proposition is proved.□

Now we are interested in finding invariants of the following situation: We have given Γ_* a tensor group, Γ_*^+ a tensor subgroup and $\tau \in \Gamma_1$ which induces a shift operation $\Gamma_g \longrightarrow \Gamma_{g+1}$ by the formula

$a \longrightarrow a \otimes \tau$

(for simplity the shift operation is homogeneous, i.e. does not depend on the degree; the results bellow could be extended to more general cases). Thus we want a function $f : \Gamma \longrightarrow \mathbf{C}$ which is invariant under:

(i) the action of Γ^+ on Γ by left and right multiplication

(ii) the action of shift operation on Γ (by a degree one map), and we call therefore f a $(\Gamma_*, \Gamma_*^+, \tau)$-invariant,

We outline a procedure to give invariants in terms of an unitary (tensor) representation of Γ_*, say ρ. We say that $W_* \subset V_*$ is a full TVSS if it is a tensor vector subspace and so the maps

$W_m \otimes W_n \longrightarrow W_{m+n}$

are isomorphisms. We can identify W_m with $(W_1)^{\otimes m}$. Also W_* is Γ_*^+ -invariant if

$\rho(c)W_m \subset W_m$, for all m and all $c \in \Gamma_m^+$.

Now for an unitary operator $a \in U(V)$ (V being a finite dimensional hermitian vector space) and W a (hermitian) subspace we can define the restricted determinant as follows: set $a_W : W \longrightarrow V \longrightarrow W$ where the second map is the projection; then $det_W a = det(a_W)$. Then the set $\{det_W \rho(c);$ where $c \in \Gamma_*^+\}$ is a subgroup of $U(1) \subset \mathbf{C}$(the complex numbers of module 1) which we call $U(\Gamma_*^+)$.

Proposition 2.4.3 *Consider ρ an unitary (tensor) representation of Γ_* and W be a full Γ_*^+-invariant TVSS. Suppose that*

$\det_W \rho(\tau) \neq 0.$

Then the function

$F(a) = f(a)(\ modulo\ U(\Gamma_*^+) \in \mathbf{C}/U(\Gamma_*^+),$

where

$f(a; \rho) = (det_W \rho(a))^{m(g)}(det_W \rho(\tau))^{-g}, for\ a \in \Gamma_g, k \in \mathbf{Z}$

$m(g) = (dim W_1)^{1-g},$

is a $(\Gamma_*, \Gamma_*^+, \tau)$-invariant.

Proof: We have

$det_W \rho(ca) = det_W \rho(c) \cdot det_W \rho(a)$ for all $c \in \Gamma_*^+$

if W is Γ_*^+-invariant, as a simple computation shows. This gives the first part of the claim i.e. the invariance under left (right) multiplication. Next

$det_{W_{g+1}} \rho(a \otimes \tau) = det_{W_g \otimes W_1} \rho(a) \otimes \rho(\tau)$
since W_* is a full TVSS, and
$det_{W_g \otimes W_1} \rho(a) \otimes \rho(\tau) = (det_{W_1} \rho(\tau))^{dim W_g} (det_{W_g} \rho(a))^{dim W_1}$
which proves the invariance under the shift operation. □

Consider now the data (Sp_*, Sp_*^+, τ) where $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
We wish to use the representations ρ_k (k even) of the symplectic tensor group into $U(V_*)$. We denote the target spaces $V_* = V_*(k)$ in order to distinguish them for different k's. Let p be a divisor of k and

$W_g(p, k) = \{x \in V_g(k); x_m = 0$ for all $m \in (\mathbf{Z}/k\mathbf{Z})^g$ such that p does

not divide m, where (x_m) are the coordinates in the basis

$(\theta_m)_{m \in (\mathbf{Z}/k\mathbf{Z})^g}$ from example 3.2 $\}$.

Always in the sequel the index k of the representation will be even.

Proposition 2.4.4 *$W_g(p, k)$ is a full Sp_*^+-invariant TVSS of $V_*(k)$.*

Proof: We have
$W_1(p, k) = \{x \in V_1(k); x_m = 0$ for $m \in \mathbf{Z}/k\mathbf{Z}$ not multiple of $p\}$
and we have an identification $W_g(p, k) = W_1(p, k)^{\otimes k}$. Now a system of generators for Sp_*^+ is provided by the matrices from section 2 of type (9) and (11). Matrices of type (9) are diagonal hence leave $W_g(p, k)$ invariant. Next choose $\gamma = \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ and $x \in W_g(p, k)$. Then
$x = \sum_{m \in (\mathbf{Z}/k\mathbf{Z})^g} x_{pm} \theta_{pm}$
hence
$\rho_k(\gamma) x = \sum_{m \in (\mathbf{Z}/k\mathbf{Z})^g} x_{pm} \theta_{pA^\top m}$
and therefore $\rho_k(\gamma) x \in W_g(p, k)$. □

We wish now to obtain a (Sp_*, Sp_*^+, τ)-invariant using our tensor representation ρ_k. Set $l = k/p$. We consider only the case when $gcd(l, p) = 1$. This condition is equivalent to $det_{W_1(k,p)} \rho_k(\tau) \neq 0$ (see the example 5.2). Set therefore $N(l) = 2gcd(l, 6)$.

Proposition 2.4.5 *Let p be a divisor of k, and*
$f_{p,k}(a) = f_{W_*(p,k)}(a; \rho_k)$ *(modulo $N(l)$)* $\in \mathbf{C}/R_{N(l)}$
Then $f_{p,k}$ is a (Sp_*, Sp_*^+, τ)-invariant.

Proof: We have to compute $f_{p,k}(ac)$ for $c \in Sp^+(2g, \mathbf{Z})$. The relation (17) implies

$f_{p,k}(ac) = f_{p,k}(a)det_{W_*(p,k)}\rho_k(c)$.
We may restrict ourselves to the case when c is one of the generators described above. Specifically if $c = \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix}$ then $\rho_k(c)$ is a permutation matrix. The restriction to $W_g(p,k)$ is again a permutation matrix thus having determinant in R_2. Next consider $c = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$. It will suffice then to check the case when $B = E_{st}$ where E_{st} is the matrix having only a non-zero entry which equals 1 and lies on the st position. Since $\rho_k(c)$ is diagonal and leaves therefore $W_*(p,k)$ invariant we find
$det_{W_*(p,k)}\rho_k(c) = exp(\sum_{m_i=1,k/p} \frac{\pi i}{k} p^2 < m_s, m_t >)$.
If $s \neq t$ (so $g > 1$) this equals
$exp(\pi i(\frac{k}{p})^{g-2}pl(l+1)^2/4)$,
and for $s = t$ we obtain
$exp(\pi i(\frac{k}{p})^{g-1}(l+1)(2l+1)p/6)$.
An elementary computation shows that these numbers belong to $R_{N(l)}$ for all values of g, s, t. Now the relation (17) concludes the proof as in the proposition 4.2. □

2.5 Invariants for 3-manifolds

Let M^3 be a closed connected and orientable 3-manifold. A Heegaard splitting $\mathcal{H}$ of M^3 is a pair (V_+, V_-) consisting of handlebodies of genus, say g, such that $M^3 = V_+ \cup V_-$ and $\partial V_+ = \partial V_- \cong T$, T being the orientable surface of genus g. Consider V a standard (fixed) handlebody in R^3 with boundary T and the identifications
$\varphi_{+,-} : V_{+,-} \longrightarrow V$
We can look at M^3 as being built from two standard handlebodies which are glued together using an homeomorphism between their boundaries. This gluing homeomorphism could be expressed as
$h = h(\mathcal{H}) = \varphi_-^{-1} \circ \varphi_+ : \partial V_+ \longrightarrow \partial V_-$ We identify ∂V_+ and ∂V_- with T. Observe that h is determined up to (left and right) multiplication by a homeomorphism of T which extends to a homeomorphism of V. We shall call such a homeomorphism of T a full homeomorphism. This ambiguity comes from the choice of identification maps φ_-, φ_+. Consider

now the map induced by h in the homology

$h(\mathcal{H}) : H_1(T, \mathbf{Z}) \longrightarrow H_1(T, \mathbf{Z})$

We choose a symplectic basis in $H_1(T, \mathbf{Z})$ i.e. a basis

$\{\alpha_1, \alpha_2, ..., \alpha_g, \beta_1, \beta_2, ..., \beta_g\}$

in which the intersection pairing has the form:

$\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0,\ \alpha_i \cdot \beta_j = \delta_{ij}$

Therefore we can identify $h(\mathcal{H})$ with an element of $Sp(2g, \mathbf{Z})$. Remark that the change of the symplectic basis affects $h(\mathcal{H})$ by conjugation with a homomorphism induced in homology by some full homeomorphism.

Consider next $\mathcal{M}_g$ the mapping class group of surfaces of genus g and $\mathcal{M}_g^+$ the image in $\mathcal{M}_g$ of the subgroup of full homeomorphisms. According to [Bir74] we have a surjection

$s : \mathcal{M}_g \longrightarrow Sp(2g, \mathbf{Z})$.

Proposition 2.5.1 *We have* $s(\mathcal{M}_g^+) \subseteq Sp^+(2g, \mathbf{Z})$.

Proof: Consider φ a full homeomorphism of T. Then the diagram

$$\begin{array}{ccccccc} 0 \to ker(inc_*) & \to H_1(T) & \to H_1(V) & \to H_1(V,T) \\ & \varphi_* \downarrow & \varphi_* \downarrow & \varphi_* \downarrow \\ 0 \to ker(inc_*) & \to H_1(T) & \to H_1(V) & \to H_1(V,T) \end{array}$$

commutes, where $inc : T \longrightarrow V$ denotes the inclusion. It follows that

$\varphi_*(ker(inc_*)) = ker(inc_*)$

But we have in the (standard) symplectic basis

$ker(inc_*) = \mathbf{Z} < \beta_1, \beta_2, ..., \beta_g >$

and we are done. Another proof can be given using the description of $\mathcal{M}_g^+$ from [Suz77]. In fact all the generators of $\mathcal{M}_g^+$ have their image in $Sp^+(2g, \mathbf{Z})$.

We can state now our main result:

Theorem 2.5.1 *Consider* $f_{p,k}(h(\mathcal{H}) \in \mathbf{C}/R_{N(l)}$, *for* p *dividing* k, k *even*

Then each of these numbers depends only on the homeomorphism type of M^3 , *not on the particular Heegaard splitting.*

Proof: First of way observe that, even if $h(\mathcal{H}) \in Sp(2g, \mathbf{Z})$ is not uniquely determined by the Heegaard splitting, its double coset in

$Sp^+(2g, \mathbf{Z}) \backslash Sp(2g, \mathbf{Z}) / Sp^+(2g, \mathbf{Z})$

does. Next $f_{p,k}$ is invariant under multiplication by elements from

$Sp^+(2g, \mathbf{Z})$ as Proposition 4.2 claims. Now if we have two equivalent Heegaard splittings $\mathcal{H}$ and $\mathcal{G}$ then also

$h(\mathcal{H}) = ch(\mathcal{G})d$, for some $c, d \in Sp^+(2g, \mathbf{Z})$

holds. Let denote by $\mathcal{H} \oplus \mathcal{G}$ the connected sum of the two Heegaard splittings (which is a Heegaard splitting for the connected sum of the corresponding 3-manifolds). Then the following is fulfilled

$h(\mathcal{H} \oplus \mathcal{G}) = h(\mathcal{H}) \otimes h(\mathcal{G})$.

This comes from the corresponding relation at the level of mapping class groups. Now the Reidemester-Singer theorem [Cra76, Sie80] asserts that any two Heegaard splittings $\mathcal{H}$ and $\mathcal{G}$ of a 3-manifold are stably equivalent i.e.

$\mathcal{H} \oplus m\mathcal{T} \cong \mathcal{G} \oplus n\mathcal{T}$, $m, n \in \mathbf{Z}^+$

where $\mathcal{T}$ denotes the (standard) Heegaard decomposition of genus one of the sphere S^3. But it is easy to see that

$h(\mathcal{T}) = \tau \in Sp(2, \mathbf{Z})$.

Since $f_{p,k}$ is invariant under the shift operation by τ our claim is proved. □ Next we shall outline a procedure for computing this invariants which we can denote, according to the theorem, by $f_{p,k}(M^3)$. Consider a collection of circles $\{c_j^+\}$ on $\partial V_+ \cong T$ which bounds pairwise disjoint properly embedded 2-cells in V_+ and which cut away from V_+ a 3-cell. We obtain by a similar procedure another collection $\{c_j^-\}$ starting from V_-. The pair $(\{c_j^+\}, \{c_j^-\})$ is usually called a Heegaard diagram. The action of h induces a commutative diagram:

$$\begin{array}{ccccccc} 0 \to \mathbf{Z} < c_j^+, j \leq g > & \to & H_1(T) & \to & H_1(V_+) & \to & H_1(V,T) \\ & & h \downarrow & & \downarrow & & \downarrow \\ 0 \to \mathbf{Z} < c_j^-, j \leq g > & \to & H_1(T) & \to & H_1(V_+) & \to & H_1(V,T) \end{array}$$

We derive

$h(\mathbf{Z} < c_j^+, j \leq g >) = \mathbf{Z} < c_j^-, j \leq g >$.

This relation determine $h(\mathcal{H})$ in terms of the homological information contained in the Heegaard diagram. For example we may identify V_- with the fixed V hence $\{c_j^-\} = \{\beta_j, j \leq g\}$.

Let $T = S^1 \times S^1$ be the torus. We consider c^+ as being a loop on T whose homology (or homotopy) class is $(a, b) \in \mathbf{Z} \oplus \mathbf{Z}$, with a, b pairwise prime, modulo the standard ambiguity due to the non-uniqueness of the longitude of torus. Then (c^+, β) is a Heegaard diagram for the lens space $L_{a,b}$. Set $\mathcal{H}_{a,b}$ for the corresponding Heegaard splitting. From above it follows that we may choose $h(\mathcal{H}_{a,b}) = \begin{bmatrix} b & c \\ a & d \end{bmatrix}$ with $bd - ac = 1$.

EXAMPLE 2.5.1 :

Let $a = 1, b = 0$ hence $\mathcal{H}_{a,b} = \mathcal{T}$ is the standard splitting of the sphere S^3. We derive
$f_{p,k}(S^3) = 1$.

EXAMPLE 2.5.2 :

Let $a = 0, b = 1$. We obtain $S^2 \times S^1$ and
$h(\mathcal{H}_{0,1}) = 1 \in Sp(2, \mathbf{Z})$
Therefore

$$\begin{aligned} f_{p,k}(S^2 \times S^1) &= (det_{W_1(p,k)}\rho_k(\tau))^{-1} = \\ &= k^{k/2p} det\{exp(\frac{2\pi i p^2 mn}{k})\}^{-1}_{m,n\in\{1,2,\ldots,\frac{k}{p}\}} \end{aligned}$$

Now if $gcd(l, p) > 1$ the above considered determinant vanishes since it have two equal lines. So we must consider $gcd(l, p) = 1$ (according to the first condition in 4.3). Now

$$\rho_k(\tau)^2 \mid_{W_1(k,p)} = (\delta_{pm,-pm})_{m\in\{1,2,\ldots,l\}},$$

holds, hence the absolute value of the considered determinant is $l^{l/2}$ and its phase is a 4^{th} root of unity. On the other hand this is a Vandermonde determinant hence its value is given by

$$det\{exp(\frac{2\pi i p^2 mn}{k})\}_{m,n\in\{1,2,\ldots,\frac{k}{p}\}} = \prod_{j>h}(exp(\frac{2\pi i pj}{l}) - exp(\frac{2\pi i ph}{l})).$$

But we have

$$exp(\frac{2\pi i pj}{l}) - exp(\frac{2\pi i ph}{l}) = 2i\, sin(\frac{\pi(h-j)p}{l})exp(\frac{\pi i(h+j)p}{l})$$

hence we can compute its phase modulo π (in any case the invariants are defined modulo a sign) as being

$$\frac{l(l-1)}{2} + \sum_{l \geq j > h \geq 1} \frac{\pi(j+h)p}{l} = \frac{l(l-1)}{2} + \frac{\pi p(l^2-1)}{2}.$$

Therefore

$$f_{p,k}(S^2 \times S^1) = i^{\frac{l(l-1)}{2}} p^{l/2} exp(-\frac{\pi i p(l^2-1)}{2}) \in \mathbf{C}/R_{N(l)}.$$

It is simply to check that $exp(\frac{\pi i p(l-1)}{2}) \in R_{N(l)}$, and $i^{\frac{l(l-1}{2}} \in R_{N(l)}$ if $l \neq 3(modulo\ 4)$, so we can write

$$f_{p,k}(S^2 \times S^1) = \begin{cases} ip^{l/2} \in \mathbf{C}/R_{N(l)} & \text{if } l = 3(modulo\ 4) \\ p^{l/2} \in \mathbf{C}/R_{N(l)} & \text{otherwise.} \end{cases}$$

EXAMPLE 2.5.3 *:*

We consider now the general case of lens spaces. We may restrict ourselves to the case when $0 < 2b < a$ (see [BZ85, Hem76]) since $L_{a,b}$ is homeomorphic to $L^*_{a,a-b}$. It is also known that $\pi_1(L_{a,b}) = \mathbf{Z}/a\mathbf{Z}$. Then $L_{a,b}$ is homeomorphic to $L_{a,b'}$ if and only if $b' = \varepsilon b(modulo\ a)$ or $bb' = \varepsilon(modulo\ a)$ where $\varepsilon \in \{-1, 1\}$. A homeomorphism preserves the orientation if $\varepsilon = 1$. Next there exists a homotopy equivalence between $L_{a,b}$ and $L_{a,b'}$ if and only if $b' = \varepsilon n^2 b(modulo\ a)$ for some integer n, and again, the map preserves the orientation if and only if $\varepsilon = 1$. Denote $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ by $t(a)$. Consider now the expansion in continued fraction ([HW79]):
$\frac{b}{a} = \{a_1, a_2, ..., a_m\}, a_m \geq 2, a_j > 0$ for all j.

Lemma 2.5.1 *There exist some natural numbers $c, d \in \mathbf{Z}$ satisfying the diophantine equation*
$bd - ac = 1$ *such that* $\begin{bmatrix} b & c \\ a & d \end{bmatrix}$ *may be decomposed as follows:*
$(-1)^{[\frac{m+1}{2}]} \tau t(a_1) \tau t(a_2) \tau ... \tau t(a_m) \tau.$
where the right bracket states for the integer part.

Proof: It is sufficient to prove that $\begin{bmatrix} b & c \\ a & d \end{bmatrix}$ has the following decomposition:

$$\begin{cases} s(a_1)t(a_2)s(a_3)...s(a_m) & \text{if } m \text{ is odd} \\ s(a_1)t(a_2)s(a_3)...t(a_m)\tau & \text{otherwise} \end{cases}$$

where

$$s(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = -\tau t(a)\tau.$$

Now the relation from above follows easily by recurrence. □

We consider first the case $b = 1$. Then we have to compute $f(s(a); \rho_k)$. Suppose first that $gcd(a, k) = 1$ so $gcd(a, 2k) = 1$ since k is even. Now

$$\rho_k(s(a))_{mn} = k^{-1} \sum_{r=1,k} exp(\frac{2\pi i(n-m)r}{k} + \frac{\pi i a r^2}{k}) =$$

$$= exp(-\frac{\pi i(n-m)^2 a'^2}{k}) \sum_{r=1,k} exp(\frac{\pi i a(r + (n-m)a')^2}{k}),$$

where $aa' = 1(modulo\ 2k)$. We remark that the above sum may be taken over any subset S of $\mathbf{Z}/2k\mathbf{Z}$, which has k elements and maps onto $\mathbf{Z}/k\mathbf{Z}$ under the natural morphism $\mathbf{Z}/2k\mathbf{Z} \to \mathbf{Z}/k\mathbf{Z}$. This happens because
$(r + k)^2 = r^2(modulo\ 2k)$ for even k. Furthermore

$$\sum_{r=1,k} exp(\frac{\pi i a(r + (n-m)a')^2}{k}) = \sum_{r=1,k} exp(\frac{\pi i a r^2}{k}) = \frac{1}{2} \sum_{r=1,2k} exp(\frac{\pi i a r^2}{k}).$$

We denote by $G(u, v)$ the Gauss sum $\sum_{x \in \mathbf{Z}/v\mathbf{Z}} exp(\frac{2\pi i u x^2}{v})$. We obtained

$$\rho_k(s(a))_{mn} = \frac{1}{2} k^{-1} exp(-\frac{\pi i(n-m)^2 a'}{k}) G(a, 2k).$$

We have to compute

$$det(exp(-\frac{\pi i p(n-m)^2 a'}{l})_{m,n=1,l}) =$$

$$det(exp(-\frac{\pi i p(a'n^2 + a'm^2 - 2a'mn)}{l}))_{m,n=1,l} =$$

$$\prod_{n=1,l} exp(-\frac{\pi i p a' n^2}{l}) \prod_{m=1,l} exp(-\frac{\pi i p a' m^2}{l}) det(exp(-\frac{2\pi i p a' mn}{l}))_{m,n} =$$

$$exp(-\frac{2\pi ipa'(l+1)(2l+1)}{6})det(exp(-\frac{2\pi ipa'mn}{l}))_{m,n=1,l}.$$

Now the matrix $(exp(-\frac{2\pi ipa'mn}{l}))_{m,n}$ is the same with $(exp(\frac{2\pi imn}{l}))_{m,n}$ up to a permutation of the lines. It follows that for $gcd(a,k)=1$

$$f_{p,k}(L_{a,1}) = k^{-l/2}\frac{1}{2^l}exp(\frac{2\pi ipa'(l+1)(2l+1)}{6})G(a,2k)^l.$$

but $exp(\frac{2\pi ipa'(l+1)(2l+1)}{6}) \in R_{N(l)}$ so we may write finally

$$f_{p,k}(L_{a,1}) = k^{-l/2}\frac{1}{2^l}G(a,2k)^l \in \mathbb{C}/R_{N(l)}$$

Now if $a = va_0$, $k = vk_0$, with $gcd(a_0,k_0) = 1$ and $v > 1$ then

$$\sum_{r=1,k} exp(\frac{2\pi i(n-m)r + \pi iar^2}{k}) =$$
$$\sum_{j=0,v-1}\sum_{r=1,k_0} exp(\frac{2\pi i(n-m)(r+vj) + \pi ia(r+k_0j)^2}{k}).$$

For even k_0 this expression transforms into

$$\sum_{j=0,v-1}\sum_{r=1,k_0} exp(\frac{2\pi i(n-m)(r+vj) + \pi iar^2}{k}) =$$
$$\sum_{r=1,k_0} exp(\frac{2\pi i(m-n)r}{k} + \frac{\pi ia_0r^2}{k_0}) \sum_{j=0,v-1} exp(\frac{2\pi i(n-m)j}{v}) =$$
$$\begin{cases} \frac{v}{2}exp(-\frac{\pi is^2a_0'}{k_0})G(a_0,2k_0), & \text{if } m-n=sv, \\ 0 & \text{otherwise.} \end{cases}$$

where a_0' verifies $a_0a_0' = 1(modulo\ 2k_0)$. If we set $p_0 = lcm(p,v)/v$, $l_0 = k_0/l_0$ then we find

$$f_{p,k}(L_{a,1}) = \lambda k^{-l/2}gcd(p,v)^{l/2}\frac{1}{2^l}G(a_0,2k_0)^l \tag{2.18}$$

where

$$\lambda = exp(\pi i(\frac{2p_0a_0'(l_0+1)(2l_0+1)}{6} + \frac{p_0(l_0-1)}{2})). \tag{2.19}$$

For odd k_0 a similar computation shows that the considered sum is zero so

$$f_{p,k}(L_{a,1}) = 0.$$

We consider now the lens spaces $L_{a,b}$ with $b \geq 1$. We wish to compute first $\rho_k(\begin{bmatrix} b & c \\ a & d \end{bmatrix})$. According to the lemma this is

$\rho_k((-1)^{[\frac{m+1}{2}]}\tau t(a_1)\tau ... t(a_m)\tau)$

hence

$$\rho_k(\begin{bmatrix} b & c \\ a & d \end{bmatrix})_{st} =$$

$$k^{-\frac{m+1}{2}} \sum_{r_1, r_2, ..., r_m = 1,k} exp(\frac{\pi i}{k}(Q(r_1, r_2, ..., r_m) + (-1)^{[\frac{m+1}{2}]} 2sr_1 + 2tr_m)$$

Here Q denotes the quadratic form

$Q(r_1, r_2, ..., r_m) = \sum_{j=1,m} a_j r_j^2 + 2\sum_{j=1,m-1} r_j r_{j+1}$.

Notice again that the sum may be taken over any $S_1 \times S_2 \times ... S_m$ where $S_i \subset \mathbf{Z}/2k\mathbf{Z}$ form a complete set of residues modulo k. We introduce the sequences $\Delta(a_1, a_2, ..., a_n)$ defined by the recurrence relations

$\Delta(a_1) = a_1$,

$\Delta(a_1, a_2) = a_1 a_2 - 1$,

$\Delta(a_1, a_2, ..., a_{n+1}) = a_{n+1}\Delta(a_1, a_2, ..., a_n) + \Delta(a_1, a_2, ..., a_{n-1})$.

We assume that $\Delta = \Delta(a_1, a_2, ..., a_m,)$ is invertible in $\mathbf{Z}/2k\mathbf{Z}$. Further set

$$A = A(a_1, ..., a_m) = \\ \Delta^{-2}(a_1\Delta(a_2, ..., a_m)^2 + \sum_{j=2,m} a_j \Delta(a_1, a_2, ..., a_{j-1})^2),$$

$$B = B(a_1, ..., a_m) = \\ \Delta^{-2}(a_m\Delta(a_1, ..., a_{m-1})^2 + \sum_{j=1,m-1} a_j \Delta(a_{j+1}, ..., a_m)^2),$$

$$C = C(a_1, ..., a_m) = \\ (-1)^{[\frac{m+1}{2}]+m+1}\Delta^{-2}(a_1\Delta(a_2, ..., a_n)^2 + \\ a_m\Delta(a_1, ..., a_{m-1})^2 + \sum_{j=2,m} a_j\Delta(a_1, a_2, ..., a_{j-1})\Delta(a_{j+1}, ..., a_m)),$$

Set $G(Q,n)$ for the more general Gauss sum
$G(Q,n) = \sum_{r_j=1,n} exp(\frac{2\pi i}{k} Q(r_1, r_2, ..., r_m))$.
We claim that

$$\rho_k(\begin{bmatrix} b & c \\ a & d \end{bmatrix})_{st} = k^{-\frac{m+1}{2}} \frac{1}{2^m} exp(\frac{\pi i}{k}(As^2 + Bt^2 + 2Cst))G(Q,2k).$$

holds. In fact let consider the matrix

$$L = \begin{bmatrix} a_1 & 1 & 0 & ... & 0 & & \\ 1 & a_2 & 1 & 0 & ... & 0 & \\ 0 & 1 & a_3 & 1 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & ... & & 0 & 1 & a_m \end{bmatrix}.$$

Then $\Delta = det(L)$. If x_j are the solutions of the equation
$Lx = (s, 0, 0, ...0, t)^T$ then an elementary computation shows that
$\sum_{j=1,m} a_j x_j^2 = As^2 + Byt^2 + 2Cst$.
On the other hand we have

$$Q(r_1 + x_1, r_2 + x_2, ..., r_m + x_m) = Q(r_1, r_2, ..., r_m) + (-1)^{[\frac{m+1}{2}]} 2sr_1 + 2tr_m + \sum_{j=1,m} a_j x_j^2$$

and our claim follows because

$$\sum_{r_j=1,k} exp(\frac{\pi i}{k} Q(r_1 + x_1, x_2 + x_2, ..., r_m + x_m)) = \frac{1}{2^m} G(Q, 2k). \quad (2.20)$$

It remains therefore to compute the determinant of
$(exp(\frac{\pi i}{l}(pAs^2 + pBt^2 + 2pCst)))_{s,t=1,l}$.
This may be done as in the $m = 1$ case. Specifically we have

$$det(exp(\frac{\pi i}{l}\left(pAs^2 + pBt^2 + 2pCst\right)))_{s,t=1,l} = exp(\frac{\pi ip(A+B)(l+1)(2l+1)}{6}) det(exp(\frac{2\pi ipCst}{k}))_{s,t=1,l}$$

Now if $gcd(C,l) > 1$ the last determinant vanishes. Otherwise it equals $i^{\frac{l(l-1)}{2}} l^{l/2} exp(-\frac{\pi ip(l-1)}{2})$ up to sign. Since $exp(\frac{\pi ip(A+B)(l+1)(2l+1)}{6}) \in R_{N(l)}$ we may write

$$f_{p,k}(L_{a,b}) = k^{-\frac{lm}{2}} \frac{1}{2^{ml}} G(Q, 2k)^l.$$

if Δ is invertible in $\mathbf{Z}/2k\mathbf{Z}$, and $gcd(C, l) = 1$. If $gcd(C, l) > 1$ we observed that

$$f_{p,k}(L_{a,b}) = 0.$$

Next the general Gauss sums $G(Q, n)$ may be computed as a product of usual Gauss sums if the matrix L may be diagonalized over $\mathbf{Z}/2k\mathbf{Z}$.

We shall compute explicitly some values of $f_{p,k}(L_{a,b})$ using the evaluation of the Gauss sums from [Lan70]. It is known that $L_{5,1}$ and $L_{5,2}$ are not homotopy equivalent, and $L_{7,1}$, $L_{7,2}$ and $L_{7,4}$ are homotopy equivalent but not homeomorphic, and not homotopy equivalent to $L_{7,3}$. Let $k = 2n$, with odd n. Then

$$f_{p,k}(L_{5,1}) = exp(\frac{2\pi l}{8}) \left(\frac{20}{n}\right)^l \left(\frac{-4}{5n}\right)^l k^{-l/2} \frac{1}{2^l}$$

$$f_{p,k}(L_{7,1}) = exp(\frac{2\pi l}{8}) \left(\frac{28}{n}\right)^l \left(\frac{-4}{7n}\right)^l k^{-l/2} \frac{1}{2^l}$$

On the other hand $\frac{2}{5} = \{2, 2\}$, $\frac{2}{7} = \{3, 2\}$, and $\frac{3}{7} = \{2, 3\}$. We find $\Delta(2, 2) = 3$, $C(2, 2) = 2 \cdot 9^{-1}$
where a^{-1} denotes the inverse of a modulo $2k$, so
$f_{p,k}(L_{5,2}) = 0$ for all even l.
Also $\Delta(3, 2) = \Delta(2, 3) = 5$, $C(3, 2) = C(2, 3) = 12 \cdot (25)^{-1}$. Again this implies
$f_{p,k}(L_{7,2}) = f_{p,k}(L_{7,3}) = 0$ for all l with $gcd(l, 12) > 1$.
We remark that the symmetry of the situation will imply that
$f_{p,k}(L_{7,2}) = f_{p,k}(L_{7,3})$ for all k, p.
So we see that $f_{p,k}$ is not an homotopy invariant since it distinguish $L_{7,1}$ and $L_{7,2}$. Nevertheless the topological information which is contained is not so fine because it fails to distinguish the spaces $L_{7,2}$ and $L_{7,3}$ which are not even homotopy equivalent.

Another feature of this invariants is their behaviour under connected sums:

Remark 2.5.1 *We have*
$f_{p,k}(M \cup N) = f_{p,k}(M) f_{p,k}(N)$

Proof: The proof is straightforward since ρ_k fulfills the relation (17) $W_g(p,k)$ is a full TVSS, and $h(\mathcal{H})$ behaves (tensor) multiplicatively under connected sums of Heegaard decompositions. □

We remark also that this invariants depend upon the choice of orientation. If M^* denotes the manifold M with the opposite orientation then we deduce an Heegaard splitting for M^*, namely $\mathcal{H}^* = (V_-, V_+)$. We need some precision concerning the orientation at this point: we consider oriented Heegaard splittings hence we choose a product orientation on the normal fibre bundle of ∂V_+ in M. This normal fibre bundle can be viewed as a tubular neighborhood of the two-sided surface ∂V_+ in M. Then we ask that the product orientation agrees with the induced one from M. Now there is a locally defined frame which splits in a neighborhood of ∂V_+ as $e \oplus f$ with e normal to ∂V_+. We take V_+ to be one of the regions bounded by ∂V_+ such that e points out to respect to it. Since ρ_k is unitary it follows that
$f_{p,k}(M^*) = f_{p,k}(M)^*$
where the $*$ in the right member denotes the complex conjugation.

Remark 2.5.2 *Every homology sphere could be obtained by twisting the*
homeomorphism corresponding to the standard Heegaard decomposition of genus g of S^3 by an homeomorphism lying in the Torelli group
$ker(\mathcal{M}_g \to Sp(2g, \mathbb{Z}))$
(see [Bir74]). Therefore for any homology sphere our invariants equal 1.

After this paper was written several related papers appeared ([Goc92, MOO92, Jef92]). In [MOO92] it is proved that $f_{p,p}$ are homotopy invariants which may be described in terms of the cohomology algebra of the manifold.

2.6 The extension to cobordisms

The purpose of this section is the description of the TQFT adjacent to the invariants previously introduced.

We shall prove that ([Fun93a]) any multiplicative invariant defined for closed 3-manifolds has an extension to manifolds with boundary. This extension behaves fonctorially with respect to the composition of cobordisms hence it is a TQFT. We shall explain this for the invariants which were previously introduced (see also [Fun91, Goc92]). This invariants could be obtained in several ways: using the Chern-Simons action for the gauge group Z/kZ as it is explained in [DW90], or we consider the rational conformal field theory having the fusion rules of the group Z/kZ and we apply the technic of Crane [Cra91, Deg92], or Kohno [Koh92]. Finally we can consider them as generalizations for the invariant τ_3 of Kirby et Melvin [KM91] , Reshetikin et Turaev [RT91], being in fact homotopy invariants computable in terms of the cohomology algebra. (see also [MOO92]).

So what we want now is to found a functor f_k which fulfills the following properties:

1. $f_k(\cup_{i\leq n}\Sigma_{g_i}) = \otimes_{i\leq n} V_{g_i}(k)$.

2. If Σ_g^* denotes the surface of genus g opposite orientation then $f_k(\Sigma_g^*) = f_k(\Sigma_g)^* = V_g(k)^*$.

3. Let M^3 be an oriented cobordism of dimension 3,between ∂_+M^3 and ∂_-M^3. Then $f_k(M^3)$ is a morphism defined modulo scalar multiplication by an 8^{th} root of unity

$f_k(M^3) : f_k(\partial_+M^3) \longrightarrow f_k(\partial_-M^3)$.

For two cobordisms M^3 and N^3 we have

$f_k(M^3 \cup N^3) = f_k(N^3) \circ f_k(M^3)$.

We can enlarge this condition to the case of arbitrary compositions of cobordisms. Suppose that $\partial_-M^3 \cong \Sigma_g^*$ and $\partial_+N^3 \cong \Sigma_g$ and let h be an homeomorphism of Σ_g. We choose a symplectic basis in the homology $H_1(\Sigma_g, Z)$ and we consider the map induced by h, say $h_* \in Sp(2g, Z)$. We have therefore a coupling map

$d_h : W_g(k)^* \otimes V_g(k) \longrightarrow C$,

given by:

$d_h(x^* \otimes y) = < x, \rho_k(h_*)y >$,

which can be extended to:

$D_h : Z \otimes V_g(k)^* \otimes V_g(k) \otimes X \longrightarrow Z \otimes X$

by

$D_h = 1_Z \otimes d_h \otimes 1_X$.

Therefore we can compute f_k for cobordism obtained by gluing M^3 and

N^3 along the boundary components Σ_g using the homeomorphism h by means of the following formula:

$f_k(M^3 \cup_h N^3) = D_h(f_k(M^3) \otimes f_k(N^3))$.

We identified above $Hom(V, W)$ with $V^* \otimes W$.

4.For the closed 3-manifolds f_k coincide with the invariants $f_{k,1}$ which we previously defined.

Remark that there exist an analog of the Heegaard decompositions for 3-manifolds with boundary using instead of handlebodies the compression bodies. Let $\Sigma_g \cong \partial H_g \subset R^3$ embedded in a standard way in R^3. We choose embedded circles $c_j \subset \Sigma_g, j \leq m$ which bound in H_g and we put

$H_g(c_1, ..., c_s) = \Sigma_g \times [0,1] \cup_{j \leq m}$ 1-handle attached on the circle c_j

We obtain this way all the compression bodies [Cra91]. We fix the group $K \cong Z^g = ker(H_1(\Sigma_g, Z) \longrightarrow H_1(H_g, Z))$ and we consider the subgroup $L = Z < c_1, ..., c_s > \subset K$ generated by the homology classes of the c_j's. Let $r_k : Z^g \longrightarrow (Z/kZ)^g$ be morphism of reduction modulo k and let

$\pi_L : (Z/kZ)^g \longrightarrow (Z/kZ)^g / r_k(L) \cong (Z/kZ)^{g-r}$

be the natural projection ($g - r$ will be the sum of genera of the other boundary component of $H_g(c_1, ..., c_s)$). We define then

$$f_k(H_g(c_1, ..., c_s)) = \sum_{m \in (Z/kZ)^g} \theta_m^* \otimes \theta_{\pi_L(m)} \qquad (2.21)$$

Next for any oriented cobordism M^3 we have an Heegaard splitting of $M^3 = A \cup B$ into two compression bodies which are glued along a common surface by homeomorphism h. We put therefore

$$f_k(M^3) = D_h(f_k(A) \otimes f_k(B)) \qquad (2.22)$$

Theorem 2.6.0.1 *The formula (22) define a TQFT in dimension 3.*

Proof: We shall only sketch the proof which follows the pattern of theorem 2.4.1.We consider

$Sp(H_g(c_1, ..., c_s)) = \{(\varphi_1, \varphi_2)$ such that there exists an homeomorphism φ of $H_g(c_1, ..., c_s)$ which induces φ_j in the homology of every boundary component $\} \subset Sp(2g, Z) \oplus Sp(2(g-r), Z)$.

There is a natural action of the group $Sp(2g, Z) \oplus Sp(2(g-r), Z)$ on $V_g(k)^* \otimes V_{g-r}(k)$. Then it may be checked that $Sp(H_g(c_1, ..., c_s))$ does not change $f_k(H_g(c_1, ..., c_s))$ but only with a scalar factor from R_8. Using the version Reidemester-Singer theorem for Heegaard splittings in compression bodies we obtain the independence of $f_k(M^3)$ from the various choices we have done. Let now h_M, h_N denote the gluing homeomorphisms (in some Heegaard decompositions of) M^3, N^3 and ϕ an homeomorphism of $\partial_- M^3 \cong \partial_+ N^3$. Then we found an Heegaard splitting of $M^3 \cup_\phi N^3$ having h as gluing homeomorphism and such that the following relation is fulfilled

$h_* = (h_{M*} \otimes 1)(1 \otimes \phi_* \otimes 1)(1 \otimes h_{N*})$.

at the level of maps induced in homology. Here $\otimes$ is the tensor structure on the symplectic group (the direct sum of symplectic matrices ([Fun91, Goc92]). This permits us to conclude about the functoriality of f_k. □

We remark that the TQFT which we described has the same expression for compression bodies as that from [MOO92]. Therefore also the invariants deduced are the same so we have a simple expression for them, namely

Corollary 2.6.0.2 *Consider M^3 a closed 3-manifold. If for every cohomology class $\alpha \in H^1(M, Z/kZ)$ cup product $\alpha \cup \alpha \cup \alpha = 0$ then $f_k(M^3) = (card\ H^1(M, Z/kZ))^{1/2}$ otherwise $f_k(M^3) = 0$. So f_k are homotopy invariants.*

Chapter 3

A semi-abelian TQFT

3.1 Introduction

The motivation of this chapter is the attempt of understanding a semi-abelian version of Chern-Simons-Witten invariants using representations of mapping class groups. This has been done above in the case when the gauge group G is $U(1)$ in [Fun91, Fun93f, Goc92] and for general G but only in genus 1 case in [Jef92].

The spaces $Z(\Sigma_g, k)$ associated to a genus g Riemann surface come from the quantization of M_{Σ_g} the space of representations of the fundamental group $\pi_1(\Sigma_g)$ in G, modulo conjugation. If $G = U(l)$ then a theorem of Narasimhan-Seshadri ([NS65]) identifies M_{Σ_g} with the moduli space of rank l semi-stable holomorphic bundles of degree 0 over Σ_g. The Picard group of M_{Σ_g} is generated by an ample line bundle L_{Σ_g} and it turns that $Z(\Sigma_g, k) = H^0(M_{\Sigma_g}, L^k_{\Sigma_g})$ are the fibers of a projectively flat holomorphic vector bundle over the Teichmuller space. It is clear (see[Fun93g]) that the (projective) representation of the mapping class group $\mathcal{M}_g$ arising as the monodromy of the natural action on this flat bundle will determine the topological field theory we are looking for. One way to understand $Z(\Sigma_g, k)$ was opened in [BeaSS, Bea90] where it is identified with some space of theta functions on the jacobian variety $Jac(\Sigma_g)$ in the case when $G = SU(2)$ and $k = 1, 2$.

This suggests that a first step towards the complete knowledge of the monodromy representation is provided by the quantization of the subspace N_{Σ_g} of the representations of $\pi_1(Jac(\Sigma_g))$. This space has a simple description since $\pi_1(Jac(\Sigma_g)) = \mathbf{Z}^{2g}$ but the action of $\mathcal{M}_g$ is hard to describe in terms of jacobians. So we extend the associated bundle to a projectively flat bundle over the moduli space of principally polarized abelian varieties. This way a representation of the symplectic group $Sp(2g, \mathbf{Z})$ will be obtained. In order to derive invariants for closed 3-manifolds we need a weight vector for $Sp^+(2g, \mathbf{Z})$ but we may find only a projective weight vector. So we shall modify the initial representation to one of a central extension $\widetilde{Sp}(2g, \mathbf{Z})$ of $Sp(2g, \mathbf{Z})$ and we may derive invariants for framed 3-manifolds.

3.2 The quantization of N_{Σ_g}

We choose G a compact Lie group which is assumed to be simple and connected. It has maximal torus T and Weyl group W. The usual alternating character on W is denoted by det and the rank of G (the dimension of T) is denoted by l. Let R be a reduced irreducible root system in the dual t^* of the Lie algebra t of and let $R^\vee \subset t$ denote its dual. We write Q and $Q^\vee$ for the lattices generated by R and $R^\vee$ respectively. Their dual lattices we denote by $P^\vee \subset t$ and $P \subset t^*$ and we have $Q \subset P$ and $Q^\vee \subset P^\vee$. We fix a basis $\alpha_1, \alpha_2, ..., \alpha_l$ for R and then $\alpha^\vee{}_1, \alpha^\vee{}_2, ..., \alpha^\vee{}_l$ is a basis for $R^\vee$. Let $\tilde{\alpha}$ be the highest root. We write

$\tilde{\alpha}^\vee = \sum_{i=1}^{l} s_i \alpha^\vee{}_i$

and we put $h = 1 + \sum_{i=1}^{l} s_i$. If G is simply laced (all the roots have the same length) then h will be the Coxeter number of G. We consider the positive definite, symmetric bilinear form I on t given by

$I(x, y) = (2g)^{-1} \sum_{i=1}^{l} < \alpha_i, x >< \alpha_i, y >$.

If $S^2Q^\vee$ denotes the lattice of integral symmetric bilinear forms on $Q^\vee$ then $(S^2Q^\vee)^W$ is infinite cyclic generated by I unless R is of type C_l ($l \geq 3$) in which case $\frac{1}{2}I$ is a generator. Now I determines a homomorphism $t \longrightarrow t^*$ which we also denote by I. We set $M = I^{-1}(P)$.

We return now to the moduli space of representations N_{Σ_g}. Any representation will map the whole group $\mathbf{Z}^{2g}$ into a maximal torus of G and therefore the only conjugation freedom left is the diagonal action of W. Hence

$N_{\Sigma_g} = T \times T \times ... \times T/W$
The tangent space to T^{2g} is $A = t \oplus t \oplus ... \oplus t$ and
$T^{2g} = A/(Q^{\vee})^{2g}$.
The basic symplectic form ω on A is
$\omega((\xi_1, \xi_2, ..., \xi_{2g})(\eta_1, \eta_2, ..., \eta_{2g})) = -2\pi I((\xi_1, \xi_2, ..., \xi_{2g}), S(\eta_1, \eta_2, ..., \eta_{2g}))$
where I denotes the extension of the above considered bilinear form to $t \oplus t \oplus ... \oplus t$ by direct sum, and

$$S = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} \in Sp(2g, \mathbf{Z}).$$

It is known that a connection ∇ on the trivial line bundle $A \times \mathbf{C}$ over the symplectic affine space (A, ω) with curvature $-i\omega$ is given by
$\nabla_X(a) = -\frac{i}{2}\omega(X - X_0, a)$
for any $X_0 \in A$. Our task will be the construction of a line bundle $\mathcal{L}$ on T^{2g}, the prequantum line bundle, such that $c_1(\mathcal{L}) = \frac{1}{2\pi}\omega$, which must support a lift of the W action. In order to proceed we need to introduce a holomorphic structure on T^{2g}. As in the genus 1 case ([ADW91]) a holomorphic structure on T^{2g} will be specified by a modular parameter Ω in the Siegel space $\mathcal{S}_g$ (of complex symmetric matrices of dimension g whose imaginary part is positive definite). To each such Ω there is a principally polarized abelian variety $Ab(\Omega)$ associated, namely the quotient of $\mathbf{C}^g$ by the lattice $L(\Omega)$ generated by the columns of the matrix $[1_g\Omega]$ with the Kahler polarization $\eta = \sum_{i=1}^{g} dx_i \wedge dx_{i+g}$. Here x_i are the coordinates on $\mathbf{C}^g$ duals to $L(\Omega)$. Now the product $J(\Omega) = Q^{\vee} \otimes Ab(\Omega)$ is an abelian variety of dimension gl which is diffeomorphic to T^{2g}. Also the action of the Weyl group W is naturally extended to a diagonal action on $J(\Omega)$. We set for brevity $E = Ab(\Omega)$ and $J = J(\Omega)$.

Lemma 3.2.1 *The fixed point locus J^W is a finite subgroup of J, naturally isomorphic to $P^{\vee}/Q^{\vee} \otimes H_1(E, \mathbf{Z})$.*

Proof: We know that E is isomorphic as a group with $H_1(E, \mathbf{R})/H_1(E, \mathbf{Z})$. Then $z = (z_1, z_2, ..., z_g) \in Q^{\vee} \otimes H_1(E, \mathbf{R})$ maps to J^W iff
$z - t_j z = (< \alpha_j, z_i > \alpha_j^{\vee})_{i=1,g} \in Q^{\vee} \otimes H_1(E, \mathbf{Z})$,
for all $j = 1, l$. Here t_j stands for the reflection of W which sends α_j to $-\alpha_j$. But this is equivalent to $z \in P^{\vee} \otimes H_1(E, \mathbf{Z})$, hence the lemma. □

This implies that the geometric quotient J/W is a Cohen-Macaulay variety with a finite number of singular points. We may identify therefore $Pic(J/W)$ with $(Pic(J))^W$. We regard the last group as being the set Λ of isomorphism classes of holomorphic line bundle L over J with the property that w^*L and L are isomorphic for all $w \in W$.

Proposition 3.2.2 *The exact sequence*

$$0 \longrightarrow Pic^0(J) \xrightarrow{i} Pic(J) \xrightarrow{c} H^2(J,\mathbf{Z})$$

restricts to the following exact sequence:

$$0 \longrightarrow P/Q \otimes H^1(E,\mathbf{Z}) \xrightarrow{i} \Lambda \xrightarrow{c} (S^2Q^\vee)^W \otimes H^2(E,\mathbf{Z}) \cap H^{1,1}(E).$$

Proof: The proof goes as in the genus 1 case (see [Loo76]): The theorem of Appell-Humbert [Mum70] identifies $Pic^0(J)$ in a natural way with

$$Hom(H_1(J,\mathbf{Z}),\mathbf{R}/\mathbf{Z}) \cong P \otimes H^1(E,\mathbf{R}/\mathbf{Z}).$$

On the other hand we have canonical isomorphisms

$$H^2(J,\mathbf{C}) \cong \Lambda^2 Hom_{\mathbf{R}}(H_1(J,\mathbf{R}),\mathbf{C}) \cong \Lambda^2 Hom_{\mathbf{R}}(Q^\vee \otimes H_1(E,\mathbf{R}),\mathbf{C})$$

and the last term contains

$$S^2Q^\vee \otimes \Lambda^2 Hom_{\mathbf{R}}(H_1(E,\mathbf{R}),\mathbf{C}) \cong S^2Q^\vee \otimes H^2(E,\mathbf{C})$$

as a subspace. Another application of Appell-Humbert's theorem shows that $c(Pic(J)) = S^2Q^\vee \otimes H^2(E,\mathbf{Z}) \cap H^{1,1}(E)$. Next an element $z \in P \otimes H^1(E,\mathbf{R})$ projects onto a W-invariant element of $P \otimes H^1(E,\mathbf{R}/\mathbf{Z})$ iff as in the previous lemma $z - t_j z \in P \otimes H^1(E,\mathbf{Z})$ for all $j = 1,l$. Therefore the map

$$z \longrightarrow \textstyle\sum_{i=1}^{l}(z - t_i z)$$

induces an isomorphism between $(P \otimes H^1(E,\mathbf{R}/\mathbf{Z}))^W$ and $(P/Q) \otimes H^1(E,\mathbf{Z})$. Since W acts transitively on the set of bases of the root system R and trivially on P/Q this isomorphism is canonically. We have now the exact sequence

$$0 \longrightarrow P \otimes H^1(E,\mathbf{R}/\mathbf{Z}) \xrightarrow{i} Pic(J) \xrightarrow{c} (S^2Q^\vee) \otimes H^2(E,\mathbf{Z}) \cap H^{1,1}(E).$$

which will be (noncanonically) split as an exact sequence of W-modules because W is finite. Therefore by taking the W-invariants the sequence remains exact and we are done. □

Now let $\mathcal{L}$ be a holomorphic line bundle over J whose isomorphism class belongs to Λ and $c(\mathcal{L}) = I \otimes \eta$. This will be the prequantum line bundle which we wanted.

We remark that there is also a natural product action of W^g on J. We may state:

Proposition 3.2.3 *1. The prequantum line bundle $\mathcal{L}$ is ample.*

2. For any $w \in W^g$ the line bundles $w^*\mathcal{L}$ and $\mathcal{L}$ are isomorphic.

Proof: We remark that the line bundle $\mathcal{L}$ is well defined modulo a translation in J. Now since I and η are positive definite the Lefschetz theorem on theta functions implies that $\mathcal{L}$ is ample ([GH78]).

Secondly we remark that the set Λ_g of isomorphisms classes of line bundles L over J which satisfy the condition stated at the second point may be inserted into an exact sequence similar to that appearing in Proposition 2.2., namely

$$0 \longrightarrow P/Q \otimes H^1(E, \mathbf{Z}) \xrightarrow{i} \Lambda_g \xrightarrow{c} \oplus_{i=1}^{g} (S^2 Q^{\check{}})^W \otimes \mathbf{C} < \eta_i >.$$

where η_i is the cohomology class of $dx_i \wedge dx_{i+g}$. The proof is quite similar. Since

$$I \otimes \eta \in \oplus_{i=1}^{g} ((S^2 Q^{\check{}})^W \otimes \mathbf{C} < \eta_i >).$$

and $\mathcal{L}$ is uniquely defined up to a translation the claim will follow. □

We wish to construct explicitly such a line bundle $\mathcal{L}$. Remember that $E = Ab(\Omega)$. Let $e : Q^{\check{}g} \otimes L(\Omega) \times t^g_{\mathbf{C}} \longrightarrow \mathbf{C}^*$ be defined by

$$e(u + \Omega v, z) = exp(\pi i I(2z + \Omega u, v)), \text{ where } u, v \in Q^{\check{}g}$$

and I is extended to the complexification $t^g_{\mathbf{C}}$. We have an induced action F of $Q^{\check{}g} \otimes L(\Omega)$ on $\mathbf{C} \times t^g_{\mathbf{C}}$ given by:

$$F(x)(a, z) = (a/e(x, z), z + x).$$

The orbit space of this action is in a natural way a line bundle which we call $\mathcal{L}$ over $t^g_{\mathbf{C}}/Q^{\check{}g} \otimes L(\Omega) \cong J$. Since I is W-invariant we have $F(wx)(a, wz) = (a, w(z + x))$ for any $w \in W$. Therefore the action of W on $\mathbf{C} \times t^g_{\mathbf{C}}$ defined by $w(a, z) = (a, wz)$ induces one on $\mathcal{L}$, so $\mathcal{L} \in \Lambda$. More much I is also
W^g-invariant and we see that $\mathcal{L} \in \Lambda_g$. Following the theorem of Appell-Humbert we have $c(\mathcal{L}) = I \otimes \eta$, so that $\mathcal{L}$ is the required prequantum line bundle.

Now the orbit space of the action F_k defined by

$$F_k(x)(a, z) = (ae(x, z)^{-k}, z + x))$$

determines a line bundle over J which is naturally isomorphic to $\mathcal{L}^k$. The sections of $\mathcal{L}^k$ corresponds to the level k theta functions on $J(\Omega)$ hence they are holomorphic functions θ on $t^g_{\mathbf{C}}$ satisfying

$$\theta(z + x) = e(x, z)^{-k}\theta(z).$$

We denote by $Th(k, g, R, \Omega) = H^0(J, \mathcal{L}^k)$. This space will support the W^g action coming from the action on $\mathcal{L}$, and in particular the diagonal W-action. The quantization space (in level k) for $J(\Omega)$ will be therefore the space of W-invariant sections $Th(k, g, R, \Omega)^W$.

3.3 Theta functions and Coxeter groups

The purpose of this section is to define some representations of the symplectic group arising from the study of W-invariant theta functions. We shall use the notations of the previous chapter.

The level k theta functions are defined by

$$\theta_m(z,\Omega) = \sum_{l\in m+k\mathbf{Z}^g} exp(\frac{\pi i}{k} < l, \Omega l > +2\pi i < l, z >) \qquad (3.1)$$

for $m \in (\mathbf{Z}/k\mathbf{Z})^g$. The functional equation above stated was generalized to level k theta functions (see [Fun91, Fun93f]) as follows:

Let
$\Theta_k(z,\Omega) = (\theta_m(z,\Omega))_{m\in(\mathbf{Z}/k\mathbf{Z})^g}$
be the theta vector of level k. Then the following equation is fulfilled

$$\begin{aligned}\Theta\ (\gamma\cdot(z,\Omega)) &= \zeta_\gamma det(C\Omega+D)^{1/2}\times \\ &\times exp(k\pi i < z, (C\Omega+D)^{-1}Cz >)\rho_k(\gamma)(\Theta_k(z,\Omega)) \qquad (3.2)\end{aligned}$$

where

1. γ belongs to the theta group $\Gamma(1,2)$ if k is odd and to $Sp(2g,\mathbf{Z})$ elsewhere.
2. $\zeta_\gamma \in R_8$ is the multiplier system from above.
3. $\rho_k : Sp(2g,\mathbf{Z}) \longrightarrow U(k^g)$ is a mapping which becames a group homomorphism (denoted also by ρ_k when no confusion arises) when passing to the quotient $U(k^g)/R_8$ for even k (or equivalently it gives rise to a representation of the central extension of $Sp(2g,\mathbf{Z})$ determined by the 2-cocycle $exp(-\frac{\pi i}{4}\mu(*,*))$; a similar assertion holds for odd k when $Sp(2g,\mathbf{Z})$ is replaced by $\Gamma(1,2)$.

We computed explicitly ρ_k for a system of generators:

1. $$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag(exp(\frac{\pi i}{k} < m, Bm >)). \qquad (3.3)$$

   for $B = B^\top$ a matrix with integer entries.

2.

$$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top m,n})_{m,n\in(\mathbf{Z}/k\mathbf{Z})^g}. \tag{3.4}$$

for $A \in GL(g, \mathbf{Z})$

3.

$$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = k^{-g/2} exp(2\pi i k^{-1} < m, l >)_{m,l\in(\mathbf{Z}/k\mathbf{Z})^g}. \tag{3.5}$$

where $<,>$ is the inner product on $\mathbf{R}^g$.

We wish now that a Coxeter group enter in our picture. Remember that the sections of the prequantum line bundle are theta functions which may be given explicitly. For $\Omega \in \mathcal{S}_g$ and $x \in t^g_{\mathbf{C}}$ we put $\Omega x = (\sum_{j=1}^g \Omega_{ij}x_j)_{i=1,g} \in t^g_{\mathbf{C}}$. Consider

$$\theta_\lambda(z,\Omega) = \sum_{x\in Q^g+k^{-1}\lambda} exp(k\pi i < x, \Omega x > +2k\pi i < x, z >) \tag{3.6}$$

where $\lambda \in M^g = (I^{-1}(P))^g$, $z \in t^g_{\mathbf{C}}$. It is clear that $\theta_\lambda(z,\Omega)$ lies in $Th(k,g,R,\Omega)$. We may extract moreover a C-basis of theta functions:

Proposition 3.3.1 *Consider $X \subset M^g$ be a set of representatives for M^g/kQ^g. Therefore $\{\theta_\lambda(z,\Omega); \lambda \in X\}$ is a* C*-basis for $Th(k,g,R,\Omega)$.*

The proof is analogous to the classical case (see [Kac83, Mum84, Mum70 Loo76]).□

Thus $Th(k,g,R,\Omega)$ are the fibers of a vector bundle, say $Th(k,g,R)$ over the Siegel space $\mathcal{S}_g$. We have moroever a hermitian structure on this bundle given by

$< \theta_\lambda(z,\Omega), \theta_\mu(z,\Omega) >= 2k^{-lg/2}det^{1/2}(Im(\Omega_R))\delta_{\lambda,\mu}$

where Ω_R states for the matrix with each $\Omega_{i,j}$ replaced by a block $\Omega_{i,j}1_l$. Obviously $\Omega_R \in \mathcal{S}_{lg}$ and $J(\Omega) \cong Ab(\Omega_R)$.

We can get this hermitian structure geometrically using the construction of Gocho [Goc92]. Specifically set $j_R : \mathcal{S}_g \longrightarrow \mathcal{S}_{lg}$ for the holomorphic embedding $j_R(\Omega) = \Omega_R$. Then $Th(k,g,R)$ is a subbundle of the trivial bundle $L^2\Theta$ of L^2-sections

$L^2(H^0(Ab(\Omega_R),\mathcal{L}^k)) \times \mathcal{S}_g \longrightarrow \mathcal{S}_g$.

Proposition 3.3.2 *The L^2-metric and the trivial connection on the trivial L^2-bundle induce the above hermitian structure and a projectively flat connection on $Th(k, g, R)$.*

The proof is essentially contained in [Goc92]. □

Moreover $\{\theta_\lambda(z, \Omega); \lambda \in X\}$ will be a basis of covariant constant sections with respect with the induced connection.

We remark further that the W^g-action on $Th(k, g, R, \Omega)$ takes a particularly simple form, namely:

$w\theta_\lambda(z, \Omega) = \theta_{w\lambda}(z, \Omega)$

where $w = (w_1, w_2, ..., w_g) \in W^g$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_g) \in M^g/kQ^g$ and $w\lambda = (w_1\lambda_1, w_2\lambda_2, ..., w_g\lambda_g) \in M^g/kQ^g$. The diagonal action of W is the induced one.

We shall consider now the anti-invariant theta functions with respect with this two actions, namely:

$\psi^-_{\lambda,k}(z, \Omega) = \sum_{w\in W^g} det(w)\theta_{w\lambda}(z, \Omega)$

where $det(w) = det(w_1)det(w_2)...det(w_g)$ and $det : W \longrightarrow \{-1, 1\}$ is the usual alternating character of the Coxeter group W, and

$\varphi^-_{\lambda,k}(z, \Omega) = \sum_{w\in W} det(w)\theta_\lambda(z, \Omega)$

Set

$P_k = \{\lambda \in M$, such that $0 << \lambda, \alpha >\leq k$, for all positive roots $\alpha\}$.

We may describe therefore the anti-invariant subspace $Th(k, g, R, \Omega)^{-W^g}$:

Proposition 3.3.3 *We have*

1. $Th(k, g, R, \Omega)^{-W^g} = 0$, *for* $k < h$.

2. $Th(h, g, R, \Omega)^{-W^g} = \mathbf{C} < \psi^-_{r_g,h}(z\Omega) >$, *where* $r_g = \underbrace{(r, r, ..., r)}_{g}$ *and r is determined as follows: set $f_j \in t$ such that $I(f_j, \alpha_i^\vee) = \delta_{i,j}$. Then $r = \frac{1}{h}(d_1 + d_2 + ... + d_l)$. If G is simply laced then r is the half sum of the positive roots.*

3. $Th(k + h, g, R, \Omega)^{-W^g} = \mathbf{C} < \psi^-_{\lambda+r_g,k+h}(z, \Omega); \lambda \in P_k^g >$.

Proof: The W^g-action splits into g copies of independent W-actions so

$Th(k, g, R, \Omega)^{-W^g} \cong (Th(k, 1, R, \Omega)^{-W})^{\otimes g}$,

and the case when $g = 1$ is treated in [Loo76] (see also [ADW91, Kac83]). □

Now we can deal now with the spaces of W^g-invariant theta functions $Th(k,g,R,\Omega)^{W^g}$ which will be naturally a subspace of the quantization space $Th(k,g,R,\Omega)^W$. We state

Proposition 3.3.4 *The following theta functions*
$\psi_{\lambda,k}(z,\Omega) = \psi^-_{\lambda+r_g,k+h}(z,\Omega)/\psi^-_{r_g,h}(z,\Omega)$
with $\lambda \in P^g_k$ *form a* $\mathbf{C}$*-basis of the space* $Th(k,g,R,\Omega)^{W^g}$ *of invariant theta functions.*

Proof: We have an injective homomorphism
$Th(k,g,R,\Omega)^{W^g} \longrightarrow Th(k+h,g,R,\Omega)^{-W^g}$
given by $\theta \longrightarrow \theta\psi^-_{r_g,h}$. The inverse of this homomorphism will associate to $\theta \in Th(k+h,g,R,\Omega)^{-W^g}$ the meromorphic theta function $\theta/\psi^-_{r_g,h}$. What it remains to prove is that $\theta/\psi^-_{r_g,h}$ is actually an holomorphic function.

To every root α there is an associated morphism of abelian varieties
$r_\alpha : J \cong Q^\vee \otimes E \longrightarrow \mathbf{Z} \otimes E \cong E.$
Consider Θ the theta divisor on E which pass through zero. So $c_1(\mathcal{O}(\Theta)) = \eta$. Next we consider the divisor Δ on J defined as $\Delta = \sum_{\alpha\in R^+} r^*_\alpha(\Theta)$, the sum being taken over the positive roots. Then

Lemma 3.3.5 *The divisor* $(\psi^-_{r_g,h})$ *associated to the section* $\psi^-_{r_g,h}$ *is* Δ.

Proof of lemma: Observe first that $(\psi_{r_g,h}) \geq \Delta$. Indeed if $z \in r^*_{\alpha_j}\Theta$ then the element $w_\alpha = \begin{cases} (t_j,t_j,...,t_j) \in W^g & \text{for odd } g \\ (t_j,...,t_j,1) \in W^g & \text{for even } g \end{cases}$
leaves the fiber over z fixed so that
$\psi^-_{r_g,h}(wz,\Omega) = w\psi^-_{r_g,h}(z,\Omega) = -\psi^-_{r_g,h}(z,\Omega)$
because $\psi^-_{r_g,h}$ is an anti-invariant theta function. Furthermore $\psi^-_{r_g,h}$ is a section of $\mathcal{L}^h$ hence the Chern class $c_1(\mathcal{O}(\psi^-_{r_g,h})) = hI \otimes \eta$. Next the Chern class of $r_\alpha(\Theta)$ is
$\alpha \otimes \alpha \otimes \eta \in S^2Q^\vee \otimes H^2(E),$
so that
$c_1(\Delta) = \frac{1}{2}\sum_{j=1}^{l} \alpha_j \otimes \alpha_j \otimes \eta = hI \otimes \eta.$
Therefore $(\psi^-_{r_g,h}) - \Delta$ is a nonnegative divisor of vanishing Chern class so our claim follows. □

Next we remark that the same proof as above will give

$(\psi^-_{\lambda+r_g,k+h}) \geq \Delta$
which implies that $\psi^-_{\lambda+r_g,k+h}/\psi^-_{r_g,h}(z,*)$ is a holomorphic function on z and we are done. □

Now the projectively flat connection ∇ on $Th(k,g,R)$ will induce a projectively flat connection on the subbundle of anti-invariant sections $Th(k,g,R)^{-W^g}$. We identify the vector bundle of invariant sections $Th(k,g,R)^{W^g}$ with $Th(k+h,g,R)^{-W^g}\otimes(Th(h,g,R)^{-W^g})^*$ and we shall derive an induced connection ∇^{W^g} on $Th(k,g,R)^{W^g}$. Our aim is to compute the monodromy of the symplectic action with respect to this connection. We set
$\Psi_k(z,\Omega) = (\psi_{\lambda,k}(z,\Omega))_{\lambda\in P^g_k}$
for the (k, W^g)-theta vector.

Theorem 3.3.6 *The (k, W^g)-theta vector satisfies the functional equation:*

$$\Psi_k(\gamma(z,\Omega)) = exp(lk\pi i < z,(C\Omega+D)^{-1}Cz >)\rho_k^{W^g}(\gamma)\Psi_k(z,\Omega) \quad (3.7)$$

where, for even k
$\rho_k^{W^g} : Sp(2g,\mathbf{Z}) \longrightarrow U(Th(k,g,R,\Omega)^{W^g}$
is a representation of the symplectic group given by

1.

$$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag(exp(\frac{\pi i}{k+h} < \lambda+r_g, B(\lambda+r_g > -\frac{\pi i}{k} < r_g, Br_g >)) \quad (3.8)$$

for $B = B^\top$ a matrix with integer entries.

2.

$$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top\lambda,\mu})_{\lambda,\mu\in M^g/kQ^{\vee g}\rtimes W^g}. \quad (3.9)$$

for $A \in GL(g,\mathbf{Z})$.

3.

$$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i^{gp}(k+h)^{-lg/2}(\frac{vol(M)}{vol(Q^\vee)})^{g/2} \times$$
$$\times \sum_{w\in W^g} det(w)exp(\frac{2\pi i}{k+h} < w(\lambda+r_g), \mu+r_g >). \quad (3.10)$$

where p is the number of positive roots.

where $<,>$ *is the natural extension of the inner product I on* $R^{gl} \cong t^g$. *For odd k the same formulas define a representation of the theta group* $\Gamma(1,2)$.

Proof: We consider first the symplectic action on anti-invariant theta functions:

$\psi^-_{\lambda,k}(\gamma(z,\Omega)) = \sum_{w \in W^g} \theta_{w\lambda}(\gamma(z,\Omega))$

But we may write

$\theta_{w\lambda}(\gamma(z,\Omega)) = \zeta_{\gamma_R} det(C_R + \Omega_R D_R)^{1/2} exp(lk\pi i < z, (C+\Omega D)^{-1} Cz >$

$\times$

$\times \sum_{\lambda \in M^g/kQ^{\vee g}} \rho_k(\gamma_R)^{\mu}_{w\lambda} \theta_\mu(z,\Omega)$.

since $\theta_\lambda(z,\Omega)$ are theta functions for $Ab(\Omega_R)$. Because the inner product $<,>$ is W^g-invariant it can be checked on the generators that

$\rho_k(\gamma_R)^{w\mu}_{w\lambda} = \rho_k(\gamma_R)^{\mu}_{\lambda}$.

It follows

$\psi^-_{\lambda,k}(\gamma(z,\Omega)) = \zeta_{\gamma_R} det(C_R + \Omega_R D_R)^{1/2} exp(lk\pi i < z, (C+\Omega D)^{-1} Cz >$

$\times$

$\times \sum_{\mu \in P^g_{k-h}} (\sum_{w \in W^g} det(w) \rho_k(\gamma_R)^{\mu}_{w\lambda}) \psi^-_{\mu,k}(z,\Omega)$

We derive

$\rho^{W^g}_k(\gamma)^{\mu}_{\lambda} = (\sum_{w \in W^g} det(w) \rho_h(\gamma_R)^{r_g}_{wr_g})^{-1} (\sum_{w \in W^g} det(w) \rho_{k+h}(\gamma_R)^{\mu+r_g}_{w(\lambda+r_g)})$

Using the calculations performed in [Jef92, Kac83] for the transformation rules of $\psi^-_{r_g,h}$ we get our claim for the generators considered. Finally we remark that the map

$\gamma \longrightarrow exp(lk\pi i < z, (C + \Omega D)^{-1} Cz >)$ is a character for $Sp(2g, \mathbf{Z})$ which implies that $\rho^{W^g}_k$ is a group representation and our claim follows. □

Remark 3.3.7 *It is interesting to note that in the non-abelian case* $W \neq 1$ *the messy factor* ζ_γ *is cancelled out. This comes from the fact that the connection* ∇^{W^g} *is actually flat not only projectively flat.*

We come back now to the invariant theta functions arising from the diagonal W-action. This time we don't have such an explicit description for the space $Th(k,g,R,\Omega)^W$. However we can state:

Proposition 3.3.8 *1. Consider $B^0_{k,g}$ be a set of representatives for $M^g/kQ^{\vee G} \bowtie W$. Set $B_{k,g} \subset B^0_{k,g}$ be the subset of those λ having an even isotropy group $Stab(\lambda) = \{w \in W; w\lambda = \lambda\}$ (i.e. the character det on $Stab(\lambda)$ is identically one for $\lambda \in B_{k,g}$). Therefore we have*

$Th(k,g,R,\Omega)^{-W} = \mathbf{C}C < \varphi^-_{\lambda,k}(z,\Omega); \lambda \in B_{k,g} >$.

2. The W-invariant theta functions

$\{\varphi_{\lambda-r_g,k}(z,\Omega) = \varphi^-_{\lambda,k+h}(z,\Omega)/\varphi_{r_g,h}(z,\Omega)\}$, *with* $\lambda \in B_{k+h,g}$ *form a* **C**-*basis for the space* $Th(k,g,R,\Omega)^W$.

Proof: It is clear that $\varphi^-_{\lambda,k}$ are W-anti-invariant. These theta functions will generate the space $Th(k,g,R,\Omega)$ from the general theory of invariants of finite group actions. It remains to prove the linear independence. We make first a little digression on formal theta functions (see [Loo76]). Let F denote the lattice of affine linear functions on $V = t^g_{\mathbf{C}}$ which takes integral values on $Q^{\vee g}$ and let $e(F)$ denote the subgroup of $\mathbf{Z}^F$ whose elements are of the form

$\xi = \sum_{f(\Omega r_g)\geq n} c_f e(f)$

for some real number n.Here $e(f)$ stands for the element of $\mathbf{Z}^F$ which is one on f and zero on $F-\{f\}$. The order of ξ is $o(\xi) = inf\{f(\Omega r_g); c_f \neq 0\}$ and the initial part of ξ is by definition

$in(\xi) = \sum_{f(\Omega r_g)=o(\xi)} c_f e(f)$.

Now V acts on F by translation and hence on $\mathbf{Z}^F$. We call $\xi \in e(F)$ a formal theta function of level k if for any $v \in Q^{\vee g} \otimes L(\Omega)$ we have

$(u+\Omega v)^*\xi = e(-kI(v) - \frac{1}{2}I(\Omega v, v))\xi$. The set of theta functions of level k will be denoted by Th^k. Any element of Th^0 has the form

$\sum_{n\geq n_0} c_n e(n)$, with $n, n_0 \in \mathbf{Z}$.

where $e(n)$ is the constant function n. We put for any $\lambda \in M^g$

$\theta_\lambda = \sum_{\mu\in k^{-1}\lambda+Q^{\vee g}} e(-kI(v) + \frac{1}{2}k(I(\Omega v, v) - I(\Omega\lambda,\lambda))$.

It follows that $\{\theta_\lambda; \lambda \in S\}$, for S a system of representatives for $M^g/kQ^{\vee g}$ is a Th^0-basis for Th^k. Next we take into account the diagonal W-action which is given by

$w\theta_\lambda = \theta_{w\lambda}$.

Define the anti-invariant (formal) theta functions by

$\theta^-_\lambda = \sum_{w\in W} det(w)\theta_{w\lambda}$.

To any $\lambda \in B_{k,g}$ we associate some $\tilde{\lambda} \in (Q^{\vee g} \bowtie W)\lambda$ with the property that the (convex) function $I(\Omega(x-r_g), x-r_g)$ for $x \in (Q^{\vee g} \bowtie W)\lambda$ has a minimum in $x = \tilde{\lambda}$. Therefore it will follow that, for real and positive definite Ω

$in(\theta^-_{\tilde{\lambda}}) = card(Stab(\lambda))e(-kI(\tilde{\lambda}))$.

Indeed we have for $w \in Q^{\vee g} \bowtie W$, and $m = \tilde{\lambda}$ the following relations

$$-kI(wm, \Omega r_g) + \tfrac{1}{2}k(I(\Omega wm, wm) - I(\Omega m, m)) =$$
$$\tfrac{1}{2}k(I(\Omega(wm - r_g), wm - r_g) - I(\Omega(m - r_g), m - r_g)) - kI(m, \Omega r_g) \geq$$
$$\geq -kI(m, \Omega r_g)$$

But now for generic Ω the convex function $I(\Omega(x - r_g), x - r_g)$ has exactly one minimum on the orbit of λ under the affine Weyl group. Therefore equality can hold before only if $wm = m$. If $\lambda \in B_{k,g}$ then our claim follows. Otherwise there exists some $w \in Stab(\lambda)$ with $det(w) = -1$. Then

$$\theta^-_{\lambda} = -\theta^-_{w\lambda} = -\theta^-_{\lambda}$$

hence $\theta^-_{\lambda} = 0$.

Now we remark that the initial parts we obtained $in(\theta^-\tilde{\lambda})$ will be linear independent over Th^0 since the family $e(-kI(\lambda))$ fulfills this property. This will prove the linear independence of the corresponding family of formal theta functions. The same proof will work if we take $e(if)$ in the place of $e(f)$ and $i\Omega$ in place of Ω. But if we replace $e(f)$ by $exp(2\pi if)$ and Ω by $i\Omega$ we derive some multiples of the usual theta functions. Therefore for generic and purely imaginary $\Omega \in \mathcal{S}_g$ the usual anti-invariant theta functions which we considered will be linear independent over **C**. Since the independence is an open condition this will be true for Ω in a Zariski open subset of $\mathcal{S}_g$. Since $Th(k, g, R, \Omega)^{-W} \subset Th(k, g, R, \Omega)$ and the second family of spaces is a vector bundle endowed with a W-invariant projectively flat connection we obtain that the dimension of $Th(k, g, R, \Omega)$ is constant. This will prove our first claim.

We consider first the case of odd g. Then $\psi^-_{r_g,h}(z, \Omega)$ is a W-anti-invariant theta function. Then for any $k \geq 0$ we have

$$(\varphi^-_{\tilde{\lambda},k+h}) \geq (\Delta)$$

as in the proof of Proposition 3.6. It will follow that

$$\{\varphi^-_{\tilde{\lambda},k+h}(z, \Omega)/\psi^-_{r_g,h}(z, \Omega); \lambda \in B_{k,g}\}$$

is a basis for $Th(k, g, R, \Omega)^W$. The proof is similar. We may consider the induced $Sp(2g, \mathbf{Z})$-action on the associated vector bundle $Th(k, g, R)^W$. Essentially the same computation as in 3.6. (remark that $\mathbf{C} < \psi^-_{r_g,h} >$

is $Sp(2g,\mathbf{Z})$-invariant !) will give that

$\gamma(\varphi^-_{r_g,k+h}/\psi^-_{r_g,h}) = \chi(\gamma)\varphi^-_{r_g,k+h}/\psi^-_{r_g,h}$

where $\gamma \in Sp^+(2g,\mathbf{Z})$ and χ is a character for $Sp^+(2g,\mathbf{Z})$. Moreover this vector is the only (projectively) invariant vector of $Sp^+(2g,\mathbf{Z})$. On the other hand $\psi_{0_g,k+h} \in Th(k,g,R,\Omega)^W$ and has the same property. We derive that

$\varphi^-_{r_g,k+h}(z,\Omega) = s(\Omega)\psi^-_{r_g,k+h}(z,\Omega)$

where $s : S_g \longrightarrow \mathbf{C}$ is a holomorphic $Sp(2g,\mathbf{Z})$-invariant function. This will prove the claim in case of odd g.

Further we have

$\varphi^-_{r_g,h}(z,\Omega) = \varphi^-_{r_{g+1},h}((z,0),(\Omega \oplus i1)).$

so $\varphi^-_{r_g,h}(z,\Omega)$ is $Sp(2g,\mathbf{Z})$ invariant also for even g. Next the proof proceeds as in 3.6. and we are done. □

Denote now by $\Phi_k(z,\Omega) = (\varphi_{\lambda,k}(z,\Omega))_{\lambda\in B_{k,g}}$ the (k,W)-theta vector. Then we may compute the monodromy of the symplectic action actually using the connection ∇^W on the vector bundle $Th(k,g,R)^W$ which comes from its identification with $Th(k+h,g,R)^{-W} \otimes \mathbf{C} < \varphi^-_{r_g,h} >^*$.

Theorem 3.3.9 *The (k,W)-theta vector satisfies the functional equation*

$\Phi_k(\gamma(z,\Omega)) = exp(lk\pi i < z,(C\Omega+D)^{-1}Cz > \rho_k^W(\gamma)\Phi_k(z,\Omega)$

where for even k

$\rho_k^W : Sp(2g,\mathbf{Z}) \longrightarrow U(Th(k,g,R,\Omega)^W$

is a group representation determined by

1.

$$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag(exp(\frac{\pi i}{k+h} < \lambda+r_g, B(\lambda+r_g > -\frac{\pi i}{k} < r_g, Br_g >)) \tag{3.11}$$

for $B = B^\top$ a matrix with integer entries.

2.

$$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top\lambda,\mu})_{\lambda,\mu\in B_{k+h,g}}. \tag{3.12}$$

for $A \in GL(g,\mathbf{Z})$

3.

$$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i^{gp}(k+h)^{-lg/2}(\frac{vol(M)}{vol(Q^{\vee})})^{g/2} \times$$
$$\times \sum_{w \in W} det(w) exp(\frac{2\pi i}{k+h} < w(\lambda + r_g), \mu + r_g >). \qquad (3.13)$$

where p is the number of positive roots.

For odd k the same formulas define a representation of the theta group $\Gamma(1,2)$.

Proof: Since $\varphi^{-}_{r_g,h}$ is $Sp(2g,\mathbf{Z})$-invariant the proof goes as in the previous theorem.□
We remark that the natural map induced by $A \in Gl(g,\mathbf{Z})$
$A : M^g/Q^{\vee g} \bowtie W \longrightarrow M^g/Q^{\vee g} \bowtie W$
maps $B_{k,g}$ onto itself so that the formula 2. makes sense.

3.4 Invariants for framed 3-manifolds

We wish to define some invariants for closed orientable 3-manifolds using the method of [Fun91] for the representations ρ_k^W.

We start with the $\rho_k^{W^g}$ which parallels the $W=1$ case. We identify $Th(k, g_1+g_2, R, \Omega_1 \oplus \Omega_2)^{W^g}$ with $Th(k, g_1, R, \Omega_1)^{W^{g_1}} \otimes Th(k, g_2, R, \Omega_2)^{W^{g_2}}$ via the map
$\psi_{\lambda_1,k} \otimes \psi_{\lambda_2,k} \longrightarrow \psi_{(\lambda_1,\lambda_2),k}$
Set $c_k = k < r, r > /h(k+h)$ for the central charge in level k, and $\zeta_k = exp(2\pi i c_k)$. We define the symplectic sum of two matrices
$\oplus_c : Sp(2g,\mathbf{Z}) \times Sp(2h,\mathbf{Z}) \longrightarrow Sp(2(g+h),\mathbf{Z})$
by the formula
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A \oplus A' & B \oplus B' \\ C \oplus C' & D \oplus D' \end{bmatrix}.$$
Therefore we can state:

Proposition 3.4.1 *1. The representation $\rho_k^{W^g}$ is a tensor representation i.e.*
$\rho_k^{W^g}(\gamma_1 \oplus_c \gamma_2) = \rho_k^{W^g}(\gamma_1) \otimes \rho_k^{W^g}(\gamma_2)$
holds.

2. If $Sp^+(2g, \mathbf{Z})$ denotes the subgroup of symplectic matrices of the form $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ *then $\phi_{0_g,k}$ is a projective weight vector for $Sp^+(2g,\mathbf{Z})$ i.e.*

$\rho_k^{W^g}(\gamma)\psi_{0_g,k} = \chi(\gamma)\psi_{0_g,k}$

for $\gamma \in Sp^+(2g,\mathbf{Z})$, where $\chi : Sp^+(2g,\mathbf{Z}) \longrightarrow U_W$ is the character taking values in the group of roots of unity generated by ζ_k. This character is determined by

$\chi(\begin{bmatrix} A & 0 \\ 0 & (A^{\perp})^{-1} \end{bmatrix} = 1$

$\chi(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = (\zeta_k)^{(\sum_{i,j} B_{i,j})}$

Proof: It is known that ρ_k is tensorial. Now the W^g-action being split we can pass to the W^g-anti-invariant part and we are done. Otherwise this property can be checked directly on the generators. The second part is a corollary of Theorem 3.6. □

So we obtained a tensor representation of $(Sp(2g,\mathbf{Z}), Sp^+(2g,\mathbf{Z}))$ in the terminology of [Fun91, Fun93f]. Hence there is a standard way to derive invariants for closed 3-manifolds: Let M^3 be a closed orientable 3-manifold and $M^3 = T_g \cup \bar{T}_g$ be a Heegaard splitting into two handlebodies of genus g. The gluing homeomorphism induces an automorphism in homology $H_1(\partial T_g)$ which we may identify with an element $h(M) \in Sp(2g,\mathbf{Z})$. This identification corresponds to the choice of a canonical basis in the homology of a genus g surface. We set

$I_W(M^3, k) = (k+h)^{-lg/2} < \rho_k^{W_g}(h(M^3))\psi_{0_g,k}, \psi_{0_g,k} >$

We have then

Proposition 3.4.2 *1. The class of equivalence $I_W(M^3,k) \in \mathbf{C}/U_W$ does not depend upon the various choices made and defines therefore a topological invariant of M^3.*

2. The invariant $I_W(*,k)$ behaves multiplicatively under connected sums.

The proof is standard (see [Fun91]). □

We wish now to pass to the representation ρ_k^W.The only point here is that $Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W$ is a proper subspace of $Th(k, g_1 + g_2, R\Omega)$. Also there is no canonical inclusion mapping

$$Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W \longrightarrow Th(k, g_1 + g_2, R, \Omega)$$

as for the expansive tensor structures (see [Fun93g]) but a surjective mapping:

$$\pi : Th(k, g_1 + g_2, R\Omega) \longrightarrow Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W$$

which is defined as follows: we take

$$\pi_i : M^{g_1+g_2}/kQ^{\vee g_1+g_2} \bowtie W \longrightarrow M^{g_i}/kQ^{\vee g_i} \bowtie W$$

be the canonical projections and set also π_i for the induced maps

$$\tilde{\pi}_i : B_{k,g_1+g_2} \longrightarrow B_{k,g_i} \cup \{\phi\}$$

which are given by

$$\tilde{\pi}_i(x) = \begin{cases} \pi_i(x) & \text{if } \pi_i(x) \in B_{k,g_i} \\ \phi & \text{otherwise} \end{cases}$$

We put formally $\theta_\phi = 0$. Therefore the mapping π is given by

$$\pi(\varphi_\lambda) = \varphi_{\pi_1(\lambda)} \otimes \varphi_{\pi_2(\lambda)}.$$

Furthermore we have

$$\theta_{(\lambda_1,\lambda_2)}((z_1, z_2), \Omega_1 \oplus \Omega_2)) = \theta_{\lambda_1}(z_1, \Omega_1) \otimes \theta_{\lambda_2}(z_2, \Omega_2).$$

If ρ denotes the symplectic action on $Th^-(k, g, R, \Omega)$ then it follows that

$$\rho(\gamma_1 \oplus_s \gamma_2)\varphi_\lambda^- = \sum_{\mu_1=\pi_1(\mu),\mu_2=\pi_2(\mu)} \rho(\gamma_1)^{\pi_1(\mu)}_{\pi_1(\lambda)}\rho(\gamma_2)^{\pi_2(\mu)}_{\pi_2(\lambda)}\varphi_\mu^-$$

where $\oplus_s$ denotes the symplectic direct sum of matrices and the coefficients of the matrices on the righthand are zero if some index is ϕ. This implies that

$$< \rho_k^W(\gamma_1 \oplus \gamma_2)\varphi_{k,\lambda}, \varphi_{k,\mu} >=$$
$$< \rho_k^W(\gamma_1)\varphi_{\pi_1(\lambda)}, \varphi_{k,\pi_1(\mu)} >< \rho_k^W(\gamma_2)\varphi_{k,\pi_2(\lambda)}, \varphi_{k,\pi_1(\mu)} >.$$

Then if we define

$$I'_W(M^3, k) = (k + h)^{-lg/2} < \rho_k^W(h(M^3))\varphi_{0_g,k}, \varphi_{0_g,k} >$$

it will follow that I'_W is a topological invariant as above. Since $\varphi_{0_g,k}$ is the only projective weight vector of the same character χ and $Th(k, g, R, \Omega)^{W^g}$ is a $Sp(2g, \mathbb{Z})$-submodule of $Th(k, g, R, \Omega)^W$ we find that in fact

$$I'_W(M^3, k) = I_W(M^3, k),$$

so nothing new appears. This is a particular case of the following more general principle which is used in [Fun93g] for mapping class groups: if we have a tensor representation of $Sp(2g, \mathbb{Z})$ in the unitary automorphisms of the hermitian vector space V_g which define topological invariants for 3-manifolds then we may restrict to the sub-representations on $V' = Span(Sp(2g, \mathbb{Z})v_g)$ where v_g is the projective $Sp^+(2g, \mathbb{Z})$ weight vector. This implies that we may restrict ourselves to the full symplectic submodule i.e. of type $V_g = V_1^{\otimes g}$.

We want now to remove the ambiguity U_W in the definition of our invariants. This will be done by adding some structure on the manifold M^3, namely a framing. For technical reasons we shall consider a p_1-structure on M^3 (see [BHMV92]) which is a notion equivalent to Atiyah's 2-framings ([Ati90]).Let X denote the homotopy fiber of the map $p_1 : BO \longrightarrow K(\mathbf{Z}, 4)$ corresponding to the first Pontryaghin class of the tautological bundle τ of BO. Then a p_1-structure on a manifold M is fiber map from τ_M the stable tangent bundle of M^3 to $p_1^*\tau$ the pull-back of τ over X. Actually we shall consider only homotopy classes of p_1-structures. If M^3 is an oriented closed 3-manifold then M^3 bounds a 4-manifold Y. If α is a p_1-structure on M^3 then let $p_1(Y, \alpha) \in H^4(Y, M, \mathbf{Z})$ denote the obstruction to extending it to Y. Set

$\sigma(\alpha) = 3signature(Y) - < p_1(Y, \alpha), [Y] > \in \mathbf{Z}$

which does not depend on Y according to Hirzebruch's signature theorem and is equal to 3 times Atiyah's σ. It is known that the set of homotopy classes of p_1-structures on M^3 is affine isomorphic to $\mathbf{Z}$, the isomorphism being given by σ. A similar statement holds for the set of homotopy classes of p_1-structures on an oriented, compact, connected 3-manifold with boundary which restrict to a given p_1-structure on the boundary. We shall be concerned only with homotopy classes of p_1-structures bellow. The canonical p_1-structure on M^3 is that on which σ vanishes.

We come back to our representation $\rho_k^{W^g}$. The ambiguity comes from the fact that $\psi_{0_g,k}$ is only a projective weight vector for $Sp^+(2g, \mathbf{Z})$. Now we consider the central extension of $Sp(2g, \mathbf{Z})$ corresponding to the 2-cocycle signature (or cocycle de Meyer [Bar92]) $c : Sp(2g, \mathbf{Z}) \times Sp(2g, \mathbf{Z}) \longrightarrow \mathbf{Z}$. This may be constructed as follows ([Ati90]). Let Γ_g be the mapping class group of genus g surfaces and set $\tilde{\Gamma}_g$ for the set of isomorphism classes of fibrations $Y \longrightarrow S^1$ with fibre a surface of genus g, which are endowed with a p_1-structure. There is a natural group law on $\tilde{\Gamma}_g$. For $f, g \in \Gamma_g$ we construct a 4-manifold T which is fibred (with fibre the genus g surface) over the pants $D^2 - D_1^2 - D_2^2$ and has the monodromies fg, f, g on the circles $\partial D^2, \partial D_1^2, \partial D_2^2$ respectively. Set X_f for the boundary component which fibres over ∂D_1^2.Given two p_1-structures α, β on X_f, X_g respectively, then there is a unique p_1-structure γ on X_{fg} which extends the p_1-structure on boundary to T.

Since $\tilde{\Gamma}_g$ is essentially the set of pairs (f, α) with α a p_1-structure on X_f we may define the group law on $\tilde{\Gamma}_g$ by:

$(f,\alpha)(g,\beta) = (fg,\gamma)$

We obtain this way a central extension of Γ_g

$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{\Gamma}_g \longrightarrow \Gamma_g \longrightarrow 0$

with a canonical section s given by

$s(f) = (f,\alpha)$ where $\sigma(\alpha) = 0$.

The canonical 2-cocycle for this extension will be therefore

$c(f,g) = s(f)s(g)s(fg)^{-1} = signature(T)$.

Now the cohomology of T depends only on the elements f_*, g_* in $Sp(2g,\mathbf{Z})$ induced by the action of f, g in the homology of the fibre (see [Ati90, Mey73]). therefore we have an induced central extension of the symplectic group

$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{Sp}(2g,\mathbf{Z}) \longrightarrow Sp(2g,\mathbf{Z}) \longrightarrow 0$

which is also endowed with a canonical section denoted also by s. Then Meyer's function $\Phi : \widetilde{Sp}(2g,\mathbf{Z}) \longrightarrow \mathbf{Z}$ which lifts to $\widetilde{Sp}(2g,R)$ is the quasi-morphisme defined by the equation $c(f,g) = \Phi(s(fg)) - \Phi(s(f)) - \Phi(s(g))$.

There exists exactly one quasi-morphism on $\widetilde{Sp}(2g,\mathbf{Z})$ which satisfies the previous relation (see [Bar92]). We shall consider now the homogeneous quasi-morphisme associated, namely

$\Psi(f) = lim_{n\to\infty}\frac{1}{n}\Phi((s(f))^n)$

on $Sp(2g,\mathbf{Z})$ and similar

$\Psi(f) = lim_{n\to\infty}\frac{1}{n}\Phi((f)^n)$

for $f \in \widetilde{Sp}(2g,\mathbf{Z})$. Then set

$\rho_{k,W} : \widetilde{Sp}(2g,\mathbf{Z}) \longrightarrow U(Th(k,g,R,\Omega)^{W^g}$

defined by

$\rho_{k,W}(s(f)+m) = (\zeta_k)^{(\Psi(f)+m)}\rho_k^{W^g}(f)$.

Here $m \in \mathbf{Z}$ makes sense since we may alter a p_1-structure with an integer. It is clear that $\rho_{k,W}$ is a projective representation of $\widetilde{Sp}(2g,\mathbf{Z})$.

Consider now an oriented closed 3-manifold M^3 presented by a Heegaard splitting $M^3 = T_g \cup \bar{T}_g$ with gluing homeomorphism $h(M^3)$. Set $h_*(M)$ for the corresponding element of $Sp(2g,\mathbf{Z})$. Suppose that a p_1-structure α is chosen on M^3. Then α differs from the canonical p_1-structure by an integer m. We define

$Z_W((M^3,\alpha),k) = < \rho_{k,W}(s(h_*(M)) + m)\psi_{0_g,k}, \psi_{0_g,k} >$

Our main result is

Theorem 3.4.3 *The complex number $Z_W(*, k)$ is a topological invariant for closed 3-manifolds with p_1-structure which behaves multiplicatively under connected sums and pass to the conjugate when the orientation is changed. If the p_1-structure is altered by an integer m then the invariant is multiplied by ζ_k^m.*

Proof: We remark that it is sufficient to prove the following:

Lemma 3.4.4 *We have :*
1. The 2-cocycle $\tilde{c}$ associated to Ψ satisfies
$\tilde{c}(\gamma_1, \gamma_2) = 0$ *if* $\gamma_1 \in Sp^+(2g, \mathbf{Z})$.
2. $\chi(f) = exp(2\pi i c_k \Psi(f))$ *if* $f \in Sp^+(2g, \mathbf{Z})$.

In fact $\tilde{c}$ could be obtained as follows:
$\tilde{c}(f_1, f_2) = lim_{n\to\infty}\frac{1}{n}\mu(f_1^n, f_2^n)$
where μ is the Maslov 2-cocycle from the introduction (see [Bar92]). Therefore the first claim follows since the Maslov cocycle verifies the required relation.

In particular Ψ is a character on $Sp^+(2g, \mathbf{Z})$. Also Ψ is constant on conjugation classes hence
$\Psi(\begin{bmatrix} A & 0 \\ 0 & (A^{\perp})^{-1} \end{bmatrix}) = 0$
On the other hand
$\Psi(ab) = \Psi(a) + \Psi(b)$
if a and b commute according to [Bar92]. Thus it remains to compute
$\Psi(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix})$
in the case when B has only one nonzero entry. But
$\Psi(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) = 1$
and Ψ is constant under direct sum with identity. Every concerned element is conjugate to a stabilization of $\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$ and therefore
$\Psi(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}) = \sum_{i,j} B_{ij}$.
This proves our lemma. □

Since Ψ takes integer values on $Sp(2g, \mathbf{Z})$ the theorem follows.□
We can do something also in the case when $G = U(1)$ so $W = 1$ by taking into account the spin structures. Say M^3 has a spin structure α. Then the Heegaard splitting will be one in the context of spin manifolds. But the spin structure on the surface ∂T_g induces a quadratic form

$$q_\alpha : H_1(\partial T_g, \mathbf{Z}) \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

defined as follows. Let $x \in H_1(\partial T_g, \mathbf{Z})$ and x' be a circle representing the homology class x. If the spin structure induced on x' is the bounding spin structure of the circle then we set $q_\alpha(x) = 0$ otherwise q_α equals one (see [Kir89]). Now the gluing homeomorphism $h(M)$ will be compatible with q_α so $h_*(M)$ may be identified with an element of $\Gamma(1,2)$. It follows that $I_W(M, \alpha) \in \mathbf{C}$ is well defined.

Chapter 4

TQFT and RCFT

After Witten [Wit89] introduced his invariants for 3-manifolds many work has been done for understanding them from the mathematical point of view. A counterpart to the Feynmann path integral formalism in the Chern-Simons theory has been given via quantum groups by Reshetykin and Turaev [RT91]. The $SU(2)$-theory has been extensively studied in [RT91, BHMV92, KM91, Koh92]. Recently the quantum group construction of invariants has been extended to the simple Lie groups in the series A, B, C, D by Turaev and Wenzl [TW93]. Several generalizations were given by Crane [Cra91] and Degiovanni [Deg92] which started from Rational Conformal Field Theories (abbrev. RCFT) in dimension 2 and derived Topological Quantum Field Theories (abbrev. TQFT) in dimension 2+1. Also Kohno [Koh92] computed the mapping class group representation arising in the SU(2)- WZW model and show how we can construct topological invariants from this data, by pointing out that these ideas work more generally for any RCFT. The case of $\mathbf{Z}/k\mathbf{Z}$-fusion rules which turns out to be the same as the abelian Witten's theory, has been discussed in [Koh92], and from a different point of view in [Fun91, Fun93f, Goc92, MOO92]. In fact the Dijgraaf-Witten's approach ([DW90]) in the case of abelian groups provides the same system of homotopy invariants. The TQFT based on a finite group was completely described by Freed and Quinn [FQ93].

In this paper we wish to give an axiomatic treatment of the topological invariants. In the first part we introduce the tensor representations of the mapping class group. It turns out that all multiplicative invariants for closed 3-manifolds come from such representations. Therefore if we restrict to the finite dimensional case, we outline a splitting procedure which permits to decompose the target spaces of tensor representations using the rules of sewing conformal blocks. Next we extend the representation to the duality groupoid [MS89] and show that also the representation may be split into some pieces of data which, with minor modifications corresponds to the data needed for a RCFT as axiomatized by Moore and Seiberg [MS89].

Remark that a result of Ocneanu [Ocn92] describes all RCFTs (and also the TQFTs which may be described in terms of triangulations) in terms of systems of bimodules over II_1-factors. Therefore we have an algebraic description of unitary rational invariants of 3-manifolds even if, upon now, it is only a theoretical one and the encoded topological information is not so transparent.

4.1 Invariants for closed 3-manifolds

We shall consider in this paper only the case of orientable 3-manifolds. We choose an oriented Heegaard splitting of the closed 3-manifold $M = H_g \cup_\varphi \overline{H}_g$ into two genus g handlebodies, where $\varphi \in Homeo(\Sigma_g)$ states for the gluing homeomorphism and Σ_g is the surface of genus g. The Reidemester-Singer stabilization theorem ([Sie80]) states that the homeomorphism type of M is uniquely determined by the Heegaard splitting modulo the following (elementary) operations:

1. replacing an Heegaard splitting by an isomorphic one.
2. taking the connected sum with the standard Heegaard splitting of the sphere S^3 into two genus one handlebodies.

So two 3-manifolds are homeomorphic iff any two Heegaard splittings of them are stably isomorphic. But the Heegaard splitting consists in the data (g, φ) where $\varphi \in \mathcal{M}_g$ is the class of φ in the mapping class group of Σ_g. We wish to translate the Reidemester-Singer criterion into a purely algebraic one.

Remark firstly that φ is not uniquely defined. In fact different identifications of ∂H_g with the genus g surface Σ_g may give distinct classes in $\mathcal{M}_g$. On the other hand $\mathcal{M}_g$ itself is $Out^+(\pi_1\Sigma_g)$ and there are as many self-identifications as generator systems for $\pi_1\Sigma_g$. All these choices correspond to the first type operation: two Heegaard splittings determined by the pairs (g,φ) and (g,φ') are isomorphic if and only if

$\varphi' = c\varphi d$ where $c, d \in \mathcal{M}_g^+$,

where $\mathcal{M}_g^+$ is the subgroup of $\mathcal{M}_g$ of the classes of homeomorphisms $\psi : \Sigma_g \longrightarrow \Sigma_g$ which extend to homeomorphisms of the handlebody H_g. A system of generators for $\mathcal{M}_g^+$ was given by Suzuki in [Suz77].

We wish now to obtain the algebraic counterpart of the connected sum of Heegaard splittings. The interesting feature of the tower of groups $\mathcal{M}_g$ is that no group homomorphisms $\mathcal{M}_g \times \mathcal{M}_h \to \mathcal{M}_{g+h}$ actually exist away from the trivial one (as was pointed to us by F.Laudenbach). Nevertheless we dispose of a multivalued mapping

$\sigma : \mathcal{M}_g \times \mathcal{M}_h \longrightarrow \mathcal{M}_{g+h}$

which makes the following diagram commutative

$$\begin{array}{ccccc} \mathcal{M}_{g,1} & \times & \mathcal{M}_{h,1} & & \\ & & & \overset{\otimes}{\searrow} & \\ \pi_g \downarrow & & \pi_h \downarrow & & \\ \mathcal{M}_g & \times & \mathcal{M}_h & \xrightarrow{\sigma} & \mathcal{M}_{g+h} \end{array}$$

where we denoted by $\mathcal{M}_{g,1}$ the mapping class group of the genus g surface with a disk removed, π_g is the usual projection and $\otimes$ is the group morphism induced by composition of homeomorphisms. Specifically

$\sigma(x,y) = \{a \otimes b; a \in \pi_g^{-1}(x), b \in \pi_h^{-1}(y)\} \subset \mathcal{M}_{g+h}$.

We can identify $\mathcal{M}_1$ with $SL(2,\mathbf{Z})$. Then the standard Heegaard decomposition of the sphere S^3 has the gluing morphism $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, up to right and left multiplication by an element from

$$\mathcal{M}_1^+ = SL^+(2,\mathbf{Z}) = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, n \in \mathbf{Z} \right\} \subset SL(2,\mathbf{Z}).$$

Set now $M(\varphi) = H_g \cup_\varphi \overline{H}_g$ for $\varphi \in \mathcal{M}_g$. We can rephrase the Reidemester-Singer criterion as follows:

Proposition 4.1.1 *The manifolds $M(\varphi)$ and $M(\varphi')$ are homeomorphic if and only if $\varphi, \varphi' \in \mathcal{M}_\infty = \cup_{g>0}\mathcal{M}_g$ are equivalent under the equivalence relation generated by the following elementary moves*

1. $\varphi \sim c\varphi d$, *for* $\varphi \in \mathcal{M}_g$, $c, d \in \mathcal{M}_g^+$,
2. $\varphi \sim \psi$ *for any* $\varphi \in \mathcal{M}_g, \psi \in \sigma(\varphi,\tau) \subset \mathcal{M}_{g+1}$.

It should be interesting if we can replace σ by an univalent mapping such that the conclusion of the proposition remains valid. If we look only at the homological information carried by φ (i.e. we consider its image in the symplectic group $Sp(2g, \mathbf{Z})$) then a similar problem has an affirmative answer (see [Fun91, Fun93f, Fun93e, Fun93g]).

Consider K a field and $\mathcal{I}_K$ the K-algebra of (3-manifold) invariants i.e. the set of graded functions $f_* : \mathcal{M}_* \longrightarrow K$, which fulfill

$f_g(cxd) = f_g(x)$, for all $x \in \mathcal{M}_g, c, d \in \mathcal{M}_g^+, g \in N$.

$f_{g+1}(x') = f_g(x)$,for all $x \in \mathcal{M}_g$, $x' \in \sigma(x, \tau)$.

We shall say that a set of invariants $R \subset \mathcal{I}_K$ is complete if

$\varphi_1 \cong \varphi_2$ if and only if $f_*(\varphi_1) = f_*(\varphi_2)$ for all $f_* \in R$

holds.

Proposition 4.1.2 *If K is infinite then the whole K-algebra of invariants $\mathcal{I}_K$ is complete.*

Proof: Consider the (multivalued) map $x \rightarrow \sigma(x, \tau)$ which enables us to identify $\mathcal{M}_g$ with a subset of $\mathcal{M}_{g+1}$. This map induces another map between cosets

$\mathcal{M}_g^+ \backslash \mathcal{M}_g / \mathcal{M}_g^+ \longrightarrow \mathcal{M}_{g+1}^+ \backslash \mathcal{M}_{g+1} / \mathcal{M}_{g+1}^+$.

Then the set of (closed oriented) 3-manifolds (modulo a homeomorphism) may be identified with the direct limit of the system

$Manif = lim_{\rightarrow} \mathcal{M}_g^+ \backslash \mathcal{M}_g / \mathcal{M}_g^+$.

But $\mathcal{M}_g^+ \backslash \mathcal{M}_g / \mathcal{M}_g^+$ has cardinal at most $\aleph_0$ hence $Manif$ is countable. Thus the direct limit will admit an injective map into K, from which we can recover an invariant, and we are done.□

This says that there is an universal invariant which classifies 3-manifolds, but we don't know if it is algorithmically computable.

We say now that $f_* \in \mathcal{I}_K$ is an multiplicative invariant if the following condition is fulfilled

$f_{g+h}(z) = f_g(x) f_h(y)$ for all $x \in \mathcal{M}_g, y \in \mathcal{M}_h$, $z \in \sigma(x, y)$.

The 3-manifold invariant associated to f_* is obviously given by

$f(M(\varphi)) = f_g(\varphi)$, if $\varphi \in \mathcal{M}_g$.

Then f_* is a multiplicative invariant iff

$f(M \sharp N) = f(M) f(N)$

holds for all closed manifolds M, N, where $\sharp$ denotes the connected sum of the manifolds. We denote by $\mathcal{MI}_K$ the set of multiplicative invariants.

Proposition 4.1.3 *For an infinite K the set $\mathcal{MI}_K$ is a complete set of invariants.*

Proof: Set $P_g = \{\varphi \in \mathcal{M}_g$ such that $M(\varphi)$ is prime $\}$. The set P_g has not a subgroup structure. Let $x \in \sigma(P_g, \mathcal{M}_1)$, so $x \in \sigma(y, \lambda)$, and $M(x) = M(y)\sharp M(\lambda)$. Suppose $x \in P_{g+1}$. If we agree that S^3 will be not prime then $\lambda \cong \tau$. Therefore if λ is not equivalent to τ then the map
$\sigma(*, \lambda) : P_g \longrightarrow \mathcal{M}_{g+1}$ has image disjoint from P_{g+1}. Now the direct limit

$$PManif = lim_{\rightarrow} \mathcal{M}_g^+ \backslash P_g / \mathcal{M}_g^+.$$

may be injectively mapped into K. This gives us a collection of maps $f_g : P_g \longrightarrow K$ fulfilling the conditions stated in Proposition 2.1., and which classifies prime 3-manifolds. Using the multiplicativity one may extend it to all 3-manifolds. Now a well-known theorem of Milnor(see[Hem76]) asserts the uniqueness of the decomposition of 3-manifolds into prime ones (modulo connected sums with S^3), and we are done. □

From now one we shall consider $K = \mathbf{C}$, and that the multiplicative invariants are sensitive to the change of orientation i.e.

$$f(\overline{M}) = \overline{f(M)},$$

$\overline{M}$ being the manifold M with opposite orientation, and the bar on the right hand being the complex conjugation. We may restrict without loss of generality to these multiplicative invariants.

We define next the hermitian tensor representations (abbrev. h.t.r.) of $\mathcal{M}_*$: consider an indexed family of complex vector spaces W_g endowed with non-degenerate hermitian forms $<,>$. Set

$$U(W_g) = \{A \in GL(W_g); < Ax, Ay >=< x, y > \text{ for all } x, y \in W_g\}.$$

We assume that W_* has a tensor structure i.e. a multiplication map

$$\otimes : W_g \times W_h \longrightarrow W_{g+h}$$

which is compatible with the hermitian structures, hence

$$< x \otimes x', y \otimes y' >=< x, y >< x', y' >, \text{ if } x, x' \in W_g, y, y' \in W_h.$$

We have a family of ("unitary") group representations

$$\rho_g : \mathcal{M}_g \longrightarrow U(W_g)$$

such that

$$< \rho_{g+h}(c)(x \otimes y), x' \otimes y' >=< \rho_g(a)x, x' >< \rho_h(b)y, y' >,$$

for all $x, x' \in W_g, y, y' \in W_h$, and $a \in \mathcal{M}_g$, $b \in \mathcal{M}_h$, and $c \in \sigma(a, b) \subset \mathcal{M}_{g+h}$.

The h.t.r. is a weight one (or a h.t.r. of $(\mathcal{M}_*, \mathcal{M}_*^+)$) if we have a weight (or vacuum) vector $w_g \in W_g$ in every level g satisfying

$$w_{g+h} = w_g \otimes w_h,$$

$\rho_g(c)(w_g) = w_g$, for all $c \in \mathcal{M}_g^+$,
and
$d =< w_1, \rho_1(\tau)(w_1) > \neq 0$.
Denote by WHTR the set of weight h.t.r. of $\mathcal{M}_*$. We associates to every element $(\rho_*, W_*) \in$ WHTR a function $f_* = f(\rho_*, W_*)$ by the formula:
$f_g(x) = d^{-g} < \rho_g(x)w_g, \overline{w}_g >$ if $x \in \mathcal{M}_g$.

Proposition 4.1.4 *The functions $f(\rho_*, W_*)$ define a multiplicative* **C**-*invariant.*

Proof: Let $a, b \in \mathcal{M}_g^+$. Then
$f_g(axb) = d^{-g} < \rho_g(axb)w_g, \overline{w}_g >= d^{-g} < \rho_g(ax)w_g, \overline{w}_g >$
since $\rho_g(b)w_g = w_g$. Also
$\rho_g(a)(\mathbf{C} < w_g >^{\perp}) = \mathbf{C} < w_g >^{\perp}$
holds from the unitarity condition. Therefore $f_g(axb) = f_g(x)$.

Further we have
$f_{g+h}(z) = d^{-g-h} < \rho_{g+h}(z)w_{g+h}, \overline{w}_{g+h} >=$
$= d^{-g-h} < \rho_{g+h}(z)w_g \otimes w_h, \overline{w}_g \otimes \overline{w}_h >= f_g(x)f_h(y)$
for all $z \in \sigma(x, y)$. But $f_1(\tau) = 1$, hence $f_{g+1}(x \otimes \tau) = f_g(x)$. The unitarity implies now that f_* is sensitive to the orientation, which ends the proof.□
We obtained a map f :WHTR$\longrightarrow \mathcal{MI}_{\mathbf{C}}$. We can state now the main result of this section:

Theorem 4.1.5 *The map f is surjective hence any multiplicative* **C**-*invariant (always sensitive to the orientation) arise from a weight hermitian tensor representation.*

Proof: Consider the set of (compact orientable) 3-manifolds with marked boundary:
$MB_g = \{(M, \varphi); M$ is a 3-manifold with boundary and $\varphi : \partial M \longrightarrow \Sigma_g$
is an orientation preserving homeomorphism$\}$/modulo homeomorphisms
compatible with the markings φ's on the boundary.
Here Σ_g denote the standard genus g surface. More precisely Σ_g is a tubular neighborhood of the graph shown in figure 1 hence it inherits

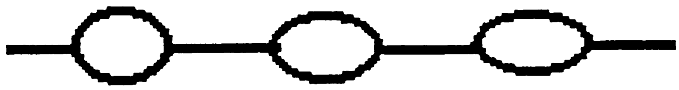

Figure 4.1: The spine of the standard H_g

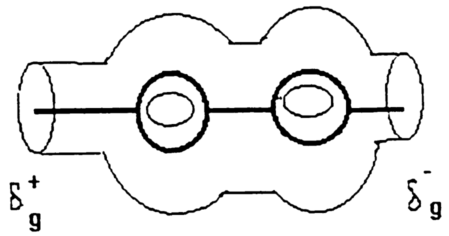

Figure 4.2: The standard Σ_g

a natural cut system $c_{0,*}$, it bounds the standard handlebody H_g and there are two disks δ_g^+, δ_g^- embedded in Σ_g (see figure 2). There is a canonical way to fix the marking φ in terms of a combinatorial data on ∂M. We choose a cut system c_* on ∂M having the dual graph Γ isomorphic to the dual graph of $c_{0,*}$. We can see the graph Γ as the spine of the surface ∂M. A framing of the graph Γ will be an embedding into ∂M. We suppose that the intersection of the framing with each trinion be the neighborhood of a vertex in the graph (a star configuration). The surface with the cut system and the framing of the dual graph satisfying the above written condition we shall call a rigid surface. The reason is very simple: once we have two rigid surfaces with an identification of the dual graphs there is an unique homeomorphism ψ (up to an isotopy) between the rigid surfaces extending the combinatorial isomorphism at the graph level. In fact if a trinion is cutted along the framing we obtain a disk which implies our claim. So instead of marking the boundary we can add a rigid structure on the boundary. This will be useful in the further sections.

Consider now F_* a non-trivial multiplicative invariant. There is a induced map

$B_F^{(1)} : \mathbf{C} < MB_g > \times \mathbf{C} < MB_g > \longrightarrow \mathbf{C}$

defined on generators by

$B_F^{(1)}((M,\varphi),(N,\psi)) = F(M \cup_{\varphi\psi^{-1}} N)$

where the manifold on the right is obtained by gluing the boundaries according to the prescribed homeomorphism. Then $B_F^{(2)}(x,y) = B_F^{(1)}(x,\overline{y})$ where the bar denotes the complex conjugation of the coordinates (in the canonical basis) is a hermitian bilinear form on the huge space $\mathbf{C} < MB_g >$. Set

$W_g^1 = \mathbf{C} < MB_g > /\ker B_F^{(2)}$.

Now we may assume that $\overline{(M,\varphi)} = (\overline{M},\varphi)$, where the first bar is the complex conjugation on coordinates (the complex strucure), and the second denotes the reversal of the orientation. This may be achieved by passing to a quotient which we shall call W_g. Next $B_F^{(2)}$ induce a nondegenerate hermitian form

$B_F : W_g \times W_g \longrightarrow \mathbf{C}$.

We have a mapping $\rho_g : \mathcal{M}_g \longrightarrow GL(W_g)$ given by

$\rho_g(x)\,[M,\varphi] = [M,\tilde{x}\varphi]$

where $[,]$ denotes the class of the corresponding element in W_g and

$\tilde{x} \in Homeo(\Sigma_g, \delta_g^+, \delta_g^-)$

is an arbitrary lift of $x \in \mathcal{M}_g$.

Lemma 4.1.6 *The mapping ρ_g is a well-defined group representation.*

Proof: It suffices to prove that whenever $h \in Homeo(\Sigma_g, \delta_g^+, \delta_g^-)$ is isotopic to identity on Σ_g (by an isotopy not necessary trivial on the two disks) the following identity $[M,\varphi] = [M,h\varphi]$ holds in W_g. Let h_t be an isotopy with $h_0 = h$ and $h_1 = id$. Consider the pseudo-isotopy

$H : \partial M \times [0,1] \longrightarrow \partial M \times [0,1]$

given by $H(x,t) = (\varphi^{-1}h_t^{-1}\varphi(x), t)$.

We identify $\partial M \times [0,1]$ with a collar V of ∂M in M. Define further $\phi : M \longrightarrow M$ by

$$\phi(x) = \begin{cases} H(x) & \text{if } x \in V \\ x & \text{elsewhere} \end{cases}$$

Then ϕ is a homeomorphism of M and the following diagram is commutative

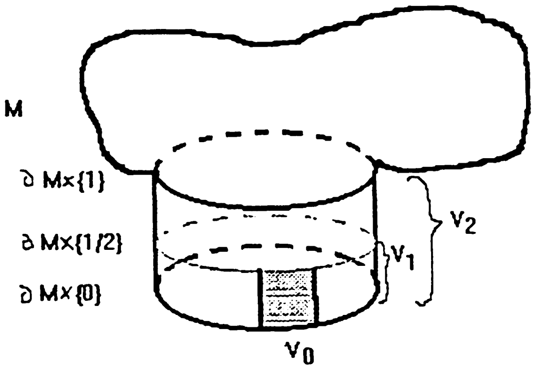

Figure 4.3: The tubular neighborhhoods V_0, V_1, V_2

$$\begin{array}{ccccc} M & \supset & \partial M & & \\ \phi \downarrow & & \varphi^{-1}h^{-1}\varphi \downarrow & \overset{\varphi}{\searrow} & \\ M & \supset & \partial M & \overset{h\varphi}{\to} & (\Sigma_g, \delta^+ g, \delta_g^-) \end{array}$$

which implies that $[M, \varphi] = [M, h\varphi]$ getting our claim. Finally every $x \in \mathcal{M}_g$ has a lift $\hat{x} \in Homeo(\Sigma_g)$, which may be isotoped on Σ_g to $\tilde{x} \in Homeo(\Sigma_g, \delta^+ g, \delta_g^-)$. Since $Homeo^0(\Sigma_g)$ acts trivially on W_g it follows that ρ_g is a representation of $\mathcal{M}_g$. □

It is clear that $\rho_g(x)$ is an isometry with respect to the bilinear form B_F. We shall define now the tensor structure on W_*. Let $[M, \varphi] \in W_g$ and $[N, \psi] \in W_h$. Consider the tubular neighborhoods (see figure 3) in M and N which satisfy:

$$\begin{array}{ccccccc} V_0 & \subset & V_1 & \subset & V & \subset M \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\ \varphi^{-1}(\delta_g^+) \times \left[0, \frac{1}{2}\right] & \subset & \partial M \times \left[0, \frac{1}{2}\right] & \subset & \partial M \times [0,1] & \end{array}$$

and respectively

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T & \subset N \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\ \psi^{-1}(\delta_h^-) \times \left[0, \frac{1}{2}\right] & \subset & \partial N \times \left[0, \frac{1}{2}\right] & \subset & \partial N \times [0,1] & \end{array}$$

We construct the 3-manifold

$$X = M - int(V_0) \bigcup_{\partial V_0 \overset{id}{\simeq} \partial T_0} N - int(T_0),$$

which has the boundary

$\partial X = \partial M - int(\varphi^{-1}(\delta_g^+)) \bigcup_{\partial\varphi^{-1}(\delta_g^+)) \simeq \partial\psi^{-1}(\delta_h^-)} \partial N - int(\psi^{-1}(\delta_h^-))$.
Now we have an homeomorphism

$\partial X \xrightarrow{\varphi\sharp\psi} \Sigma_g - int(\delta_g^+)) \bigcup_{\partial\delta_g^+ \simeq \partial\delta_h^-} \Sigma_h - int(\delta_h^-) = \Sigma_{g+h}$.
of ∂X in the standard surface of genus $g+h$ by simply taking the connected sum of the homeomorphism φ and ψ (on the respective subsets). The uniqueness of the tubular neighborhhood implies that $(X, \varphi\sharp\psi) \in MB_g$ does not depend on the various choices we made but only on (M, φ) and (N, ψ). We put therefore

$[M, \varphi] \otimes [N, \psi] = [X, \varphi\sharp\psi]$,
which may be extended to a tensor structure on W_* by linearity. Suppose now that for another two elements $(M', \varphi') \in MB_g$, $(N', \psi') \in MB_h$ the same construction yields the marked manifold $(X', \varphi'\sharp\psi') \in MB_{g+h}$. Then we have an homeomorphism between the closed 3-manifolds

$X \cup_{\varphi\sharp\psi\circ(\varphi'\sharp\psi')^{-1}} X'$ and

$M \cup_{\varphi\varphi'^{-1}} M'\sharp N \cup_{\psi\psi'^{-1}} N'$.
Since F is a multiplicative invariant we derive the compatibility of the tensor structure on W_* and the bilinear forms B_F.

Set now $w_g = [H_g, id] \in W_g$. We show that w_g is the vacuum vector. Obviously $w_{g+h} = w_g \otimes w_h$. Let $a \in Homeo(\Sigma_g, \delta_g^+, \delta_g^-)$ representing a class in $\mathcal{M}_g^+$. Then

$\rho_g([H_g, id]) = [H_g, a]$.
and

$B_F([H_g, id], [M, \psi]) = F(H_g \cup_{\psi^{-1}} M)$

$B_F([H_g, a], [M, \psi]) = F(H_g \cup_{a\psi^{-1}} M)$.
But a extends to a homeomorphism $A : H_g \longrightarrow H_g$. Therefore we have an homeomorphism between $H_g \cup_{\psi^{-1}} M$ and $H_g \cup_{a\psi^{-1}} M$ obtained by gluing A and id_M and taking a quotient. This gives $F(H_g \cup_{\psi^{-1}} M) = F(H_g \cup_{a\psi^{-1}} M)$. Since B_F is non-degenerate we derive $[H_g, 1] = [H_g, a]$, hence w_g is
$\rho_g(\mathcal{M}_g^+)$-invariant. Now

$F(M(\varphi)) = B_F([H_g, \varphi], \left[\overline{H_g}, 1\right]) = < \rho_g(\varphi)w_g, \overline{w}_g >$.
Since f is non-trivial and multiplicative $F(S^3) = 1$ so $d = 1$. This ends the proof of our theorem. □
Consider now the set of cobordisms M with boundary $\partial M = \partial_1 M \cup \partial_2 M$, where we suppose for simplicity that $\partial_j M$ are connected. Denote

$MB_{g_1,g_2} = \{(M,\varphi_1,\varphi_2),\varphi_j : \partial_j \longrightarrow \Sigma_{g_j}\}$/modulo homeomorphisms compatibles with the markings φ_j.

We have a multilinear mapping induced by the invariant F

$u_F^1 : \mathbf{C} < MB_g > \times \mathbf{C} < MB_{g,h} > \times \mathbf{C} < MB_h > \longrightarrow \mathbf{C}$

defined on the generators by

$u_F^1((M,\varphi),(N,\varphi_1,\varphi_2),(P,\psi)) = F(M \cup_{\varphi\varphi_1^{-1}} N \cup_{\varphi_2\psi_1^{-1}} P)$.

Suppose that $[M_1,\varphi] = [M_2,\psi]$. Then

$F(M_1 \cup_{\varphi\theta^{-1}} Q) = F(M_2 \cup_{\psi\theta^{-1}} Q)$

for any $(Q,\theta) \in MB_g$. So u_F^1 induces a map

$u_F : \mathbf{C} < MB_{g,h} > \longrightarrow W_g \otimes W_h$.

We can identify W_h and its dual W_h^* by means of the form B_F, so we think u_F as having image in $Hom(W_g, W_h)$. We have also a twist composition

$\mathcal{M}_h \times \mathbf{C} < MB_{g,h} > \times \mathbf{C} < MB_{h,k} > \longrightarrow \mathbf{C} < MB_{g,k} >$

extending linearly the composition

$(\varphi,(M,\psi_1,\psi_2),(N,\mu_1,\mu_2)) \rightarrow (M \cup_{\psi_2\varphi\mu_1^{-1}} N,\psi_1,\mu_2)$.

We denote $\xi_1 = (M,\psi_1,\psi_2), \xi_2 = (N,\mu_1,\mu_2)$ and their twist composition by $\xi_1 \times_\varphi \xi_2$. Observe that u_F has a simple expression as element of $Hom(W_g, W_h)$, given by

$u_F(\xi_1)([Q,\theta]) = \left[Q \cup_{\theta\psi_1^{-1}} M, \psi_2\right]$.

We have also a twisted version for the composition of morphisms:

$\mathcal{M}_h \times Hom(W_g, W_h) \times Hom(W_h, W_k) \longrightarrow Hom(W_g, W_k)$,

given by

$(\varphi, a, b) \longrightarrow b \times_\varphi a = b \circ \rho_h(\varphi) \circ a$.

Proposition 4.1.7 *We have* $u_F(\xi_1 \times_\varphi \xi_2) = u_F(\xi_1) \times_\varphi u_F(\xi_2)$.

Proof: Consider (Q,θ) arbitrary. Then

$u_F(\xi_1 \times_\varphi \xi_2)([Q,\theta]) = \left[Q \cup_{\theta\psi_1^{-1}} M \cup_{\psi_2\varphi\mu_1^{-1}} N, \mu_2\right]$

and

$u_F(\xi_1) \times_\varphi u_F(\xi_2)([Q,\theta]) = u_F(\xi_2) \circ \rho_h(\varphi)(\left[Q \cup_{\theta\psi_1^{-1}} M, \psi_2\right]) =$

$= u_F(\xi_2) \circ (\left[Q \cup_{\theta\psi_1^{-1}} M, \psi_2\varphi\right]) = \left[Q \cup_{\theta\psi_1^{-1}} M \cup_{\psi_2\varphi\mu_1^{-1}} N, \mu_2\right]$

and we are done. □

Remark 4.1.8 *We observe that when we write the morphism u_F the dependence on F is not explicit. In fact the aspect of W_g and ρ_g recover all the information on F. In the same manner we can get a functor* *from the category of all cobordisms (so with not necessarily connected boundaries) into that of hermitian vector spaces. This is usually called a topological quantum field theory TQFT (in dimension 2+1) ([Ati89, Wit89]).*

Corollary 4.1.9 *i) Any multiplicative invariant extends canonically to a TQFT.*
ii) The invariants coming from TQFTs form a complete set of invariants.

Remark that the computation of the TQFT extending an invariant is not always obvious. An example is given in [Fun93f]. The general case will be discussed in section 5, once we obtain the structure of a WTHR, following the same pattern.

We can make further some easy simplifications. First of way we consider the orbit of the weight vector $O_g = \rho_g(\mathcal{M}_g(w_g))$ and set $\widetilde{W_g} = Span(O_g) \subset W_g$.

Lemma 4.1.10 *ρ_* restricts to a tensor representation on $\widetilde{W_*}$.*

Proof: It suffices to prove that $\widetilde{W_*}$ has a tensor structure, so
$\widetilde{W_g} \otimes \widetilde{W_h} \subset \widetilde{W_{g+h}}$.
Let consider
$x = \sum_i a_i \rho_g(g_i) w_g \in \widetilde{W_g}$ and $y = \sum_i b_i \rho_g(g_i') w_g \in \widetilde{W_h}, a_i, b_i \in \mathbf{C}$.
Then we can write
$x \otimes y = \sum_{i,j} a_i b_j \rho_g(g_i) w_g \otimes \rho_h(g_j') w_h$.
Now
$\rho_g(x) w_g = [H_g, \tilde{x}]$ and $\rho_h(y) w_h = [H_h, \tilde{y}]$
for two lifts $\tilde{x}, \tilde{y}$ in the appropriated homeomorphisms groups. The construction of the tensor structure enables us to obtain
$\rho_g(x) w_g \otimes \rho_h(y) w_h = [H_{g+h}, \tilde{x} \sharp \tilde{y}] = \rho_{g+h}(\tilde{x} \sharp \tilde{y}) w_{g+h}$.
from which we derive our claim. □
Thus we may restrict ourselves to the case when W_g is spanned by O_g since $f(\rho_*, W_*) = f(\rho_*, \widetilde{W_*})$. In this case the h.t.r. will be called a cyclic h.t.r..

We shall make some remarks concerning the irreducibility of h.t.r.. The tensor subspace $H_* \subset W_*$ is an invariant tensor subspace if H_g is an invariant subspace of W_g for all g and $H_g \otimes H_h \subset H_{g+h}$ (it is a tensor vector subspace). If equality holds above we say that H_* is fully invariant. An h.t.r. is (weakly) irreducible if it does not contains proper fully invariant tensor subspaces. Set

$H_g^\perp = \{z \in W_g; < z, v >= 0 \text{ for all } v \in H_g\}$.

Suppose we have a cyclic but not irreducible h.t.r. and H_* is an invariant tensor subspace. Set

$\pi_1 : H_g \longrightarrow Z_g = H_g / H_g \cap H_g^\perp$

$\pi_2 : H_g^\perp \longrightarrow V_g = H_g^\perp / H_g \cap H_g^\perp$

for the canonical projections. We shall decompose

$w_g = z'_g + v'_g$ with $z'_g \in H_g, v'_g \in H_g^\perp$.

This decomposition is not necessary unique. We have induced hermitian forms $<,>$ on Z_g and V_g. Since ρ_* is unitary we find that $H_g^\perp$ is an invariant subspace, henceforth $H_g \cap H_g^\perp$ is also invariant. Thus we have two induced representations of $\mathcal{M}_g$ into $U(Z_g)$ and $U(V_g)$ respectively. Set $z_g = \pi_1(z'_g)$ and $v_g = \pi_2(v'_g)$. Since the h.t.r. is cyclic the vectors z_g, v_g are nonzero.

Proposition 4.1.11 *Suppose that H_* is a fully invariant tensor subspace. Then the induced representations $\rho_{g,Z}$ and $\rho_{g,V}$ are in WHTR, with vacuum vectors z_g and v_g respectively. The invariants associated satisfy*

$f(\rho_*, W_*) = f(\rho_{*,Z}, Z_*) + f(\rho_{*,V}, V_*)$.

Proof: Take $a \in \mathcal{M}_g^+$. Then $\rho_g(a)w_g = w_g$, hence

$\rho_g(a)z'_g - z'g = \rho_g(v'_g) - v'g \in H_g \cap H_g^\perp$.

Therefore $\rho_g(a)z_g = z_g, \rho_g(a)v_g = v_g$. Now we claim that $H_*^\perp$ is a tensor vector space, with the induced structure. In fact $x \in H_g^\perp, y \in H_h^\perp$ implies $< x \otimes y, z >= 0$ for all $z \in H_g \otimes H_h = H_{g+h}$. Thus $x \otimes y \in H_{g+h}^\perp$. This implies that $H_* \cap H_*^\perp$ is a tensor (vector) subspace, hence V_g and Z_g will be tensor vector spaces. The compatibility between the hermitian and the tensor structures is immediate. Finally

$< \rho_g(x)(w_g), \overline{w}_g >=< \rho_g(x)(z'_g), \overline{z'}_g > + < \rho_g(x)(v'_g), \overline{v'}_g >=$

$=< \rho_g(x)(z_g), \overline{z}_g > + < \rho_g(x)(v_g), \overline{v}_g >$

and this ends the proof of the proposition.□

Remark 4.1.12 *In the case when the hermitian form $<,>$ is positive we may complete W_g to a tensor structure of Hilbert spaces. This will be called the geometric (or unitary) situation. Then $H_g \cap H_g^\perp = \phi$ and* *we may found that we have an induced h.t.r. on H_* for any invariant (not necessary fully invariant) tensor subspace H_*.*

4.2 Representations of the mapping class group

We shall restrict now to the geometric situations, and also, we assume that the representations ρ_g are finite dimensional. The invariants which are derived are called rational.

Let consider $c_* = \{c_1, c_2, ..., c_{3g-3}\}$ be a cut system (see [HT82]) on Σ_g. The Dehn twists around the curves in the cut system generate an abelian subgroup $\mathbf{Z}^{3g-3}$ of $\mathcal{M}_g$. Now we know that a finite family of pairwise commuting unitary operators on W_g could be simultaneously diagonalized. We shall carry out this diagonalization procedure in all genera by taking into account the tensor structure of W_*. Then the decomposition of W_g into the sum of eigenspaces of a fixed operator will be iterated and we shall obtain the sewing rules of conformal blocks in a RCFT.

We wish to derive firstly a comparison result for the blocks W_g in different genera. Consider some curve c lying in the cut system c_* on Σ_{g+h}.

We suppose that c is a separating curve so $\Sigma_{g+h} - c = \Sigma_{g,1} \cup \Sigma_{h,1}$. Set $\mathcal{M}_{g+h}(c)$ for the subgroup of $\mathcal{M}_{g+h}$ generated by the homeomorphisms φ having the property that $\varphi(c)$ is isotopic to c. We put then

$W_{g+h|1,0} = Span < \rho_{g+h}(x) w_{g+h}; x \in \mathcal{M}_{g+h}(c) >$.

Let d_c denote the Dehn twist around c and $t_c = \rho_{g+h}(d_c)$. We consider the eigenspaces of t_c, namely

$W_{g+h|\lambda} =< x \in W_{g+h}; t_c x = \lambda x >$.

Remark that all these subspaces are $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. In fact $\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1} \subset \mathcal{M}_{g+h}$ is a subgroup contained in the centralizer of d_c so

$t_c \rho_{g+h}(u) w_{g+h} = \rho_{g+h}(u d_c) w_{g+h} = \lambda \rho_{g+h}(u) w_{g+h}$

for all $u \in \mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}$ having the property that $\rho_{g+h}(u) w_{g+h} \in W_{g+h|\lambda}$. The algebra $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$ is an integral algebra hence $W_{g+h|\lambda}$, for $\lambda \neq 1$ splits into simple modules:

$W_{g+h|\lambda} = \bigoplus_i W_{g+h|\lambda,i}$
where $W_{g+h|\lambda,i}$ are simple cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. When $\lambda = 1$ we observe that $W_{g+h|1,0} \subset W_{g+h|1}$ and the above decomposition takes the form

$W_{g+h|1} = W_{g+h|1,0} \bigoplus_i W_{g+h|1,i}$.
Here $W_{g+h|1,0}$ is not necessary simple but all the rest are simple cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. Consider now

$W_{g|1} = Span < \rho_{g+h}(z)w_{g+h}; z \in \sigma(\mathcal{M}_g, 1) = \mathcal{M}_{g,1} \otimes 1 \subset \mathcal{M}_{g+h} >$

$W_{h|1} = Span < \rho_{g+h}(z)w_{g+h}; z \in \sigma(1, \mathcal{M}_h) = 1 \otimes \mathcal{M}_{g,1} \subset \mathcal{M}_{g+h} >$.
We have natural isomorphisms $W_g \simeq W_{g|1}$ and $W_h \simeq W_{h|1}$ given respectively by

$x \to x \otimes w_h$ and $x \to w_g \otimes x$.
Denote for instance by $\tilde{\otimes}$ the tensor structure on W_* which a priori has nothing to do with the natural tensor product of vector spaces.

Lemma 4.2.1 *The natural map*

$\theta : W_{g|1} \otimes W_{h|1} \simeq W_g \otimes W_h \xrightarrow{\tilde{\otimes}} W_{g+h|1,0}$
is an isomorphism.

Proof: Since $W_{g+h|1,0}$ is a cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module, $w_g \tilde{\otimes} w_h = w_{g+h}$, and $\theta(W_{g|1} \otimes W_{h|1})$ is also a $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module it follows that θ is onto. It remains to prove that θ is injective. Consider

$z = \sum_{i,j} a_{ij} z_i \otimes t_j \in ker(\theta)$
where $z_i = \rho_{g+h}(\sigma(x_i, 1))w_{g+h}$ and $t_j = \rho_{g+h}(\sigma(1, y_j))w_{g+h}$, with $x_i \in \mathcal{M}_g, y_j \in \mathcal{M}_h$. We can compute now

$\theta(z) = \sum_{i,j} \rho_{g+h}(\sigma(x_i, y_j))w_{g+h} = 0$.
Therefore

$< \theta(z), u\tilde{\otimes}v >= 0$
holds for all $u \in W_g$ and $v \in W_h$. This implies that

$\sum_{i,j} a_{ij} < \rho_g(x_i)w_g, u >< \rho_h(y_j)w_h, v >= 0$
for all u and v. Since the hermitian product $<,>$ is nondegenerate we derive $a_{ij} = 0$ hence $z = 0$ and our claim follows. □
As a consequence we derive that the map $\tilde{\otimes} : W_g \otimes W_h \longrightarrow W_{g+h}$ is injective hence

$dim(W_{g+h}) \geq dim(W_g)dim(W_h)$.
Suppose now that $(\lambda, i) \neq (1, 0)$. We consider the generators $w_{g+h}(c; \lambda, i)$ for the $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules $W_{g+h|\lambda,i}$. We set

$Z_{g|\lambda,i} = Span < \rho_{g+h}(z)w_{g+h}(c;\lambda,i); z \in \sigma(\mathcal{M}_g,1) >$.

$Z_{h|\lambda,i} = Span < \rho_{g+h}(z)w_{g+h}(c;\lambda,i); z \in \sigma(1,\mathcal{M}_g) >$.

In an obvious manner $Z_{g,|\lambda,i}$ is a $\mathbf{C}[\mathcal{M}_{g,1}]$-module which decompose further into simple (and cyclic) $\mathbf{C}[\mathcal{M}_{g,1}]$-modules:

$Z_{g|\lambda,i} = \sum_{j=1}^{s^+(i)} W_{g|\lambda,i,j}$,

and in a similar manner

$Z_{h|\lambda,i} = \sum_{j=1}^{s^-(i)} W_{h|\lambda,i,j}$.

We wish to construct a natural mapping

$\theta_{i,j,k} : W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k} \longrightarrow W_{g+h|\lambda,i}$

similar to θ. We choose the generators $w^+_{g+h}(c;\lambda,i,j)$ for the cyclic $\mathbf{C}[\mathcal{M}_{g,1}]$-modules $W_{g|\lambda,i,j}$ and the generators $w^-_{g+h}(c;\lambda,i,k)$ for the $\mathbf{C}[\mathcal{M}_{h,1}]$-modules $W_{h|\lambda,i,k}$. Observe that $w^+_{g+h}(c;\lambda,i,j), w^-_{g+h}(c;\lambda,i,k) \in W_{g+h|\lambda,i}$. Consider $z = x \otimes 1 \in \mathcal{M}_{g,1} \otimes 1 \subset \mathcal{M}_{g+h}$ and $t = 1 \otimes y \in 1 \otimes \mathcal{M}_{g,1} \subset \mathcal{M}_{g+h}$. We set

$\theta_{i,j,k}(\rho_{g+h}(z)w^+_{g+h}(c;\lambda,i,j) \otimes \rho_{g+h}(t)w^-_{g+h}(c;\lambda;i,k)) = \rho_{g+h}(x \otimes y)w_{g+h}(c;\lambda,i)$.

which extends by linearity to $W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k}$. This map is well-defined. Indeed suppose that

$v_0 = \sum_u a_u \rho_{g+h}(z_u) w^+_{g+h}(c;\lambda,i,j) = 0$.

Since $w^+_{g+h}(c;\lambda,i,j) \in W_{g+h,\lambda,i}$ we find that

$\rho_{g+h}(s)v_0 = 0$

for all $s \in \mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}$. But such s and z commute with each other. On the other hand the module L defined by

$0 \subset L = Span < \rho_{g+h}(\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1})w^+_{g+h}(c;\lambda,i,j) > \subset W_{g+h|\lambda,i}$.

is a nontrivial $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module so we derive

$L = W_{g+h|\lambda,i}$.

Thus $w_{g+h}(c;\lambda,i) \in L$ so

$\sum_u a_u \rho_{g+h}(z_u) w_{g+h}(c;\lambda,i) = 0$

which implies $\theta_{i,j,k}(v_0 \otimes w) = 0$ for all w so $\theta_{i,j,k}$ is well-defined. The same argument based on simplicity implies that $\theta_{i,j,k}$ is onto.

Lemma 4.2.2 *The map $\theta_{i,j,k}$ is injective.*

Proof: Consider $s_0 = \sum_{u,v} a_{uv} X_u \otimes Y_v \in ker(\theta_{i,j,k})$, where

$X_u = \rho_{g+h}(z_u)w^+_{g+h}(c;\lambda,i,j)$

$Y_v = \rho_{g+h}(t_v)w^-_{g+h}(c;\lambda,i,k)$

are chosen so that $\{X_u; u\}$ and $\{Y_v; v\}$ are bases of $W_{g|\lambda,i,j}$ and $W_{h|\lambda,i,k}$ respectively. We suppose that a_{uv} are not all zero and let

$L = Span < X_u; u$ is such that $a_{uv} \neq 0$ for some $v >$.
Therefore $\rho_{g+h}(t)s_0 = 0$ for all $t \in \mathcal{M}_{g,1} \otimes 1$ hence $0 \subset L \subset W_{g|\lambda,i,j}$ is a nontrivial $\mathbf{C}[\mathcal{M}_{g,1}]$-module. The simplicity hypothesis implies that $L = W_{g|\lambda,i,j}$. Therefore we have some unitary matrices L_u acting on $W_{g|\lambda,i,j}$ such that:

i) For any $X \in W_{g|\lambda,i,j}$ the elements $\{L_u(X); u\}$ form a basis of $W_{g|\lambda,i,j}$.

ii) We have $\sum_{u,v} a_{uv} L_u(X) \otimes Y_v = 0$ for all X.

A similar reasoning on the Y_v's yields the existence of the unitary matrices S_v satisfying the analog of condition (i) and

$\sum_{u,v} a_{uv} L_u \otimes S_v(X \otimes Y) = 0$

for all X, Y. But the matrices $\{L_u \otimes S_v; u, v\}$ are linearly independent in $End(W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k})$ so $a_{uv} = 0$. Thus our claim follows. □

As an immediate consequence the spaces $W_{g|\lambda,i,j}$ for arbitrary j are all isomorphic. Let denote by $W_{g|\lambda,i}$ this isomorphism class if $(\lambda, i) \neq (1,0)$ and $W_{g|1,0} = W_g$ elsewhere. The above two lemmas permit to conclude

Proposition 4.2.3 *To a separating curve c in the cut system there is associated the following splitting of the target space of an unitary weight h.t.r.:*

$W_{g+h} \simeq \bigoplus_{(\lambda,i)} W_{g|\lambda,i} \otimes W_{h|\lambda,i}$.

It is clear that for a non-separating curve c the space W_g splits into the eigenspaces of t_c which are also $\mathbf{C}[\mathcal{M}_{g,2}]$-modules. We consider

$W_{g+1|1;1} =< x \in W_{g+1}; t_e x = x >$

where e is the edge associated to the non-separating curve c. Denote also by

$S_{g+1} = Span < \rho_{g+1}(\mathcal{M}_{g,2} \otimes 1) w_{g+1} > \subset W_{g+1|1;1}$.

Both S_{g+1} and $W_{g+1|1;1}$ are $\mathbf{C}[\mathcal{M}_{g,2}]$-modules. Now the tensor product with w_1 establishes an isomorphism between W_g and S_{g+1}. which will be useful further.

So we obtained upon now some natural inclusions

$W_g \otimes W_h \longrightarrow W_{g+h|1,0} \hookrightarrow W_{g+h}$

and

$W_g \longrightarrow S_{g+1} \hookrightarrow W_{g+1}$

depending on the choice of some curve in the cut system. On the other hand we have the splitting of proposition 3.3. of the block W_g. There

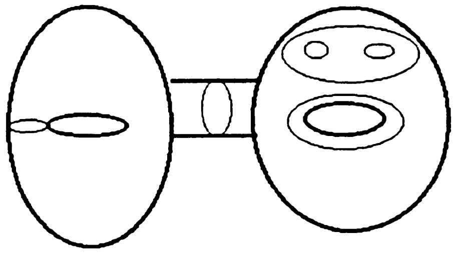

Figure 4.4: An extended cut system

is an obvious one in the non-separating case. We wish to iterate this procedure until all the curves of the cut system are cut off.

A cut system c_* defines a dual 3-valent graph Γ of genus g which is usually called by physicists a ϕ^3-diagram. Its vertices are in one-to-one correspondence with the connected components of $\Sigma_g - c_1 \cup c_2 \cup ... \cup c_{3g-3}$, which are all isomorphic to a sphere with 3 holes ($g > 1$). Two vertices are adjacent if the boundaries of the closures of the corresponding components contain the same curve c_j. It is convenient to enlarge the notion of cut system such that the case $g = 1$ fits also in this description. An extended cut system $c_* = \{c_1, c_2, ..., c_{3g-3+2h}\}$ (on Σ_g) is given by a collection $\{c_{3g-2+h}, c_{3g-1+h}, ..., c_{3g-3+h}\}$ of h disjoint embedded circles in Σ_g which bound the 2-disks $\delta_1, \delta_2, ..., \delta_h \subset \Sigma_g$ together with the cut system on the h-holed surface $\Sigma_{g,h} = \Sigma_g - \cup_{i=1}^{h} \delta_i$. The associated graph $\Gamma = \Gamma(c_*)$ has $2g - 2 + h$ vertices of valence 3 and h vertices of valence 1 which we call leaves. Let $V(\Gamma)$ denote the set of 3-valent vertices of Γ, $\partial\Gamma$ be the set of leaves, $E(\Gamma)$ be the set of edges and $F(\Gamma)$ be the subset of edges incident to the leaves. The graph Γ is planar. Once we have chosen an orientation of the plane, say the clockwise one, we have a cyclic order on the set of edges incident to a vertex. If $v \in V(\Gamma)$ let $\{e_1(v), e_2(v), e_3(v)\}$ be the set of the edges incident to v which are clockwise ordered. We shall write e also for the curve of the cut system associated to the edge e when no confusion arises.

Define

$Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) = < x; t_{e_i(v)} x = \lambda_i x > \subset W_g$.

Further the choice of some $v \in V(\Gamma)$ determines an embedding $\Sigma_{0,3}(v) \subset \Sigma_g$, hence a morphism $\mathcal{M}_{0,3} \simeq \mathcal{M}_{0,3} \otimes 1 \rightarrow \mathcal{M}_g$, (which is an injection if the vertex is 3-valent) corresponding to take the connected sum with the identity outside $\Sigma_{0,3}(v)$. This induces on $Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$ a structure of a $\mathbf{C}[\mathcal{M}_{0,3}]$-module since any $\varphi \in \mathcal{M}_{0,3} \otimes 1$ commutes with $d_{e_i(v)}$.

We deduce a splitting

$Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) = \bigoplus_j W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j)$

into simple and cyclic $\mathbf{C}[\mathcal{M}_{0,3}]$-modules, each of them generated by some $w(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j) \in W_g$. This means that

$W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j) = Span < \rho_g(\mathcal{M}_{0,3}) w(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j) >$.

On the other hand suppose that a labelling $l : E(\Gamma) \longrightarrow \mathbf{C}$ is chosen. It will be always supposed that $l(F(\Gamma)) = 1$. We set

$W_g(l) = < x; t_e x = l(e)x; e \in E(\Gamma) > \subset W_g$

for the eigenspace corresponding to l. It follows that $W_g(l)$ is a $\mathbf{C}[\mathcal{M}_{0,3} \otimes \mathcal{M}_{0,3} \otimes ... \otimes \mathcal{M}_{0,3}]$-module. This structure is induced from the map

$\mathcal{M}_{0,3} \otimes \mathcal{M}_{0,3} \otimes ... \otimes \mathcal{M}_{0,3} \longrightarrow \mathcal{M}_g$

which represents the connected sum of homeomorphisms defined on the various components $\Sigma_{0,3}$ using the graph Γ. Therefore $W_g(l)$ splits into simple and cyclic submodules

$W_g(l) = \oplus_j W_g(l)(j)$

which are respectively generated by $w(l, j)$. Set also

$W(\Gamma, l) = \bigotimes_{v \in V(\Gamma)} W(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v))))(j_v)$.

We claim that we have an isomorphism of $\mathbf{C}[\mathcal{M}_{0,3} \otimes \mathcal{M}_{0,3} \otimes ... \otimes \mathcal{M}_{0,3}]$-modules given by

$\bigotimes_v \sum_i a_{iv} \rho_g(x_{iv} w(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v))))(j_v)) \rightarrow$

$\sum_{i_1, ..., i_r} \left(\prod_{s \in V(\Gamma)} a_{i_s s} \right) \rho_g(x_{i_1 1} \otimes x_{i_2 2} \otimes ... \otimes x_{i_r r}) w(l, j)$

where r is the cardinal of $V(\Gamma)$. The fact that this application is well-defined follows as in lemma 3.2. Also as a morphism between simple modules it is an isomorphism. We derive that $W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j)$ are isomorphic for all j, $W_g(l)(j)$ are also isomorphic for all j, and we denote by $W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$ and respectively by $W(\Gamma, l)$ these isomorphism classes.

Set

$L = \{\lambda \in \mathbf{C}^*$; such that λ or λ^{-1} is an eigenvalue for some $t_e, e \in$

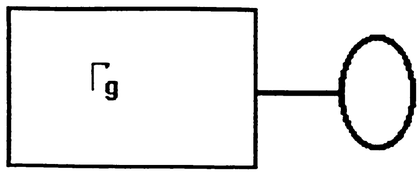

Figure 4.5: The inclusion $\Gamma_g \subset \Gamma_{g+1}$

$E(\Gamma)\}$.
Then we may restrict ourselves to the set of labellings $\mathcal{L}$ taking values in L.

We obtained the following splitting

$W_g = \bigoplus_{l\in\mathcal{L}} \bigoplus_{j=1}^{s(l)} \otimes_{v\in V(\Gamma)} W(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v))))$
into the primary blocks $W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$. A priori these primary blocks may depend upon the extended cut system c_*, the choice of $v \in \Gamma$ and of the ordered set $e_1(v), e_2(v), e_3(v)$.

Extension Lemma 4.2.4 *The primary blocks do not depend upon the extension $\tilde{c_*}$ of the cut system c_*.*

Proof: This is clear since c_i are bounding for $i > 3g-3$ so $t_{c_i} = 1$. □.

First Stabilization Lemma 4.2.5 *Assume that there is only one vector $w_1 \in W_1$ which is $SL^+(2,\mathbf{Z})$-invariant. Consider $c_{*,g} \subset \Sigma_g$ and $c_{*,g+1} \subset \Sigma_{g+1}$ having the properties:*

1. if we identify Σ_{g+1} as $\Sigma_g \sharp S^1 \times S^1$ then $c_{*,g+1}|_{\Sigma_g} = c_{*,g}$.

2. If Γ_g and Γ_{g+1} are the dual graphs then these are positioned as in figure 5. Let $v \in \Gamma_g \subset \Gamma_{g+1}$. Therefore we have an isomorphism $W(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3))$.

Proof: We choose the labels of the additional edges to be 1. These outer labels are irrelevant in the definition of $W(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3))$. We claim that

$W_g(l) \simeq W_{g+1}(l')$
where l' is the extension of the labelling l by 1. Consider that e is the new separating edge (see the figure 5). Then we have

$W_{g+1}(l') = < x \in W_{g+1|1}; t_f x = l'(f)x$ for all $f \neq e >=$

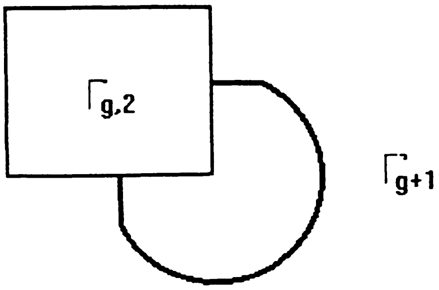

Figure 4.6: The inclusion $\Gamma_{g,2} \subset \Gamma_{g+1}$

$=< x \in W_{g+1|1,0}; t_f x = l'(f)x > \oplus_{j>0} < x \in W_{g+1|1,j}; t_f(x) = l'f)x >$.

Further we know from lemma 3.1. that the first space decompose as a tensor product

$< x \in W_{g+1|1,0}; t_f x = l'(f)x >=< x \in W_g; t_f x = l(f)x > \otimes W_1^{SL^+(2,\mathbf{Z})}$.

On the other hand each space from the second term decompose also in a tensor product according to proposition 3.3.

$< x \in W_{g+1|1,j}; t_f x = l'(f)x >=$

$< x \in W_{g|1,j}; t_f(x) = l(f)x > \otimes < x \in W_{1,j}; t_a x = x >$,

where a is the meridian of the torus. We know that $W_{1|1} = \oplus_{j\geq 0} W_{1|1,j}$ and the assumption of lemma implies

$< x \in W_{1|1,j}; t_a x = x >= 0$ if $j > 0$.

This will establishes our claim. But now we find that

$Z(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq Z(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3))$

as $\mathbf{C}[\mathcal{M}_{0,3}]$-modules so the lemma follows.□

Second Stabilization Lemma 4.2.6 *Assume that there is only one vector $w_1 \in W_1$ which is $SL^+(2,\mathbf{Z})$-invariant. Consider $c_{*,g,2} \subset \Sigma_{g,2}$ and $c_{*,g+1} \subset \Sigma_{g+1}$ having the properties:*

1. if we identify Σ_{g+1} as $\Sigma_g \sharp S^1 \times [0,1]$ then $c_{*,g+1} |_{\Sigma_{g,2}} = c_{*,g,2}$.

2. If $\Gamma_{g,2}$ and Γ_{g+1} are the dual graphs then these are positioned as in figure 6. Let $v \in \Gamma_{g,2} \subset \Gamma_{g+1}$ and suppose the leaves of $\Gamma_{g,2}$ are labelled by 1. Consider a simple path p in $\Gamma_{g,2}$ between the endpoints of the new atached edge e and a vertex v not incident to the path p. Therefore we have an isomorphism $W(\Gamma_{g,2}, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3))$.

Proof: We use the same method as above but we look this time at the

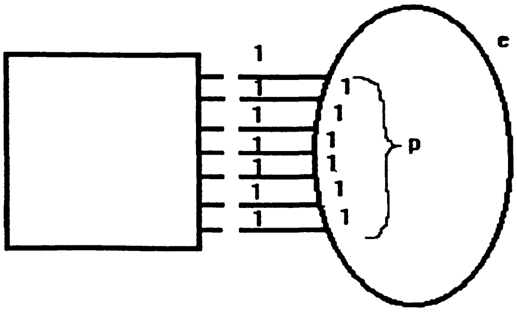

Figure 4.7: The attached great torus

non-separating curve corresponding to the edge e. We shall use now only the labellings l which take the value 1 on the edges of the path p, and denote by l' the extension by 1 on e. We claim that

$W_g(l) \simeq W_{g+1}(l')$

holds. Remark that

$W_{g+1}(l') = W_{g+1|1;1} \cap < x \in W_{g+1}; t_f x = l(f)x >$.

and

$W_{g+1|1;1} \supset S_{g+1}$.

Then we have an isomorphism

$S_{g+1} \cap < x \in W_{g+1}; t_f x = l(f)x > \simeq W_g(l)$,

coming from the identification of S_{g+1} and W_g. Consider the circuit $p \cup e$ which from geometric viewpoint represents a great (holed) torus which is attached to a surface of genus h with s holes for obtaining the surface of genus $g+1$. Remark that this torus is attached in s places depending on the combinatorics of the path p (see the figure 7)

Now a decomposition principle holds also in the non-separating case as

$W_{g+1|1;1} = S_{g+1} \oplus_r W_{g;r}$

where $W_{g;r}$ are isomorphic simple $\mathbf{C}[\mathcal{M}_{g,2}]$-submodules of W_g. The great torus has the attaching edges $f_1, f_2, ..., f_s$ all labelled by 1. We wish now to change the splitting procedure as follows: we cut firstly all the edges $f_1, f_2, ..., f_s$ and in final the edge e. This does not matter for the primary blocks we considered. The first $s-1$ edges now are non-separating and the last one is separating. A recurrence on s permits to obtain

$< x \in W_{g+1}; t_{f_i} x = x; i = 1, s-1 > \simeq W_{g-s+1} \oplus_r W_{g-s+1;r}$, where $W_{g-s+1;r}$ are simple cyclic $\mathbf{C}[\mathcal{M}_{g-s+1,s}]$-submodules of W_{g-s}. But the last one move will separate the genus $g+1$-surface into a genus $g-s$ surface with s holes and the great torus. Following the Extension Lemma the space associated to this torus does not depend upon the number of leaves, being in fact isomorphic to W_1. We have according to the splitting principle

$$< x \in W_{g+1}; t_{f_i}x = x; i = 1, s >\simeq W_{g-s} \otimes W_1 \oplus_r W_{g-s|r} \otimes W_{1|r},$$

hence

$$< x \in W_{g+1}; t_{f_i}x = x; i = 1, s \text{ and } t_e x = x >\simeq$$
$$W_{g-s} \otimes W_1^{SL^+(2,\mathbf{Z})} \oplus W_{g-s|1,j} \otimes W_{1|1,j}^{SL^+(2,\mathbf{Z})}.$$

As in the previous lemma we conclude that

$$< x \in W_{g+1}; t_{f_i}x = x; i = 1, s \text{ and } t_e x = x >\simeq W_{g-s}.$$

This implies our claim and we are done. □

Homogeneity Lemma 4.2.7 *Let $\sigma \in Aut(\Gamma)$ be a combinatorial isomorphism preserving the cyclic order on edges incident to a vertex. Then*

$$W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma, \sigma(v), (\lambda_1, \lambda_2, \lambda_3))$$

holds.

Proof: Any such σ admits a lift $\varphi \in Homeo(\Sigma_g, c_*)$. Therefore $\rho_g(\varphi)$ induces the wanted isomorphism. □

Lemma 4.2.8 *The primary blocks do not depend upon the choice of the vertex $v \in \Gamma$.*

Proof: We claim that for every pair of vertices $v_1, v_2 \in \Gamma$ we may use extensions and stabilizations of $\Gamma \subset \Gamma'$ such that the images of v_1 and v_2 become equivalent under $Aut(\Gamma')$. Then the homogeneity lemma will conclude.

Also it suffices to check our claim for pairs of adjacent vertices because we may use a recurrence on the length of the shortest path between them (Γ is arcwise connected).

We may enlarge the stabilization procedure to include also the transformation from figure 8. The conclusion of the stabilization lemmas remains valid for this type of transformations on the cut system level because we may use a recurrence. Here A and B stands for 3-valent graphs eventually with leaves.

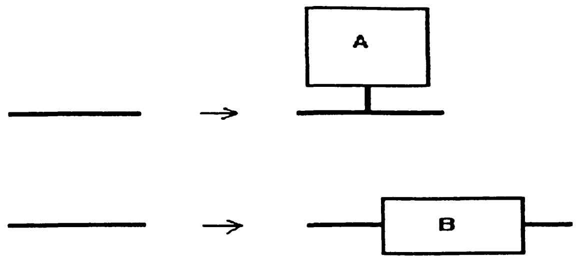

Figure 4.8: The stabilization procedure

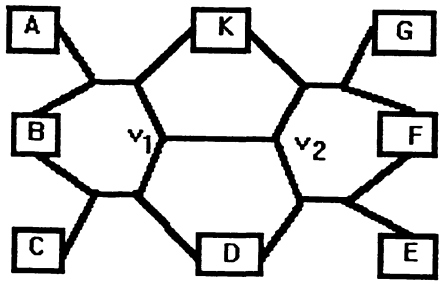

Figure 4.9: The relative position of vertices

Now the general situation of v_1 and v_2 in Γ is depicted in figure 9, where some of the graphs $A, B, ..., H$ may be void and B, D, G, E may be disconnected. We stabilize this graph using the pattern from figure 10. Now v_1 and v_2 are equivalent under the rotation of angle π of the plane. □
So we can drop the index v from the indices of a primary block.

Lemma 4.2.9 *The cyclic permutations on the labels don't change the isomorphism class of primary blocks.*

Proof: We use the same method as above. The general position of a 3-valent vertex in Γ is described in figure 11. We stabilize Γ as in figure 12. Then we may perform the cyclic permutations of the edges e_1, e_2, e_3 using the automorphism of the stabilized graph.The homogeneity lemma proves our claim. □

Lemma 4.2.10 *The label set L and the spaces $W(\Gamma, (\lambda_1, \lambda_2, \lambda_3))$ do not depend on the cut system.*

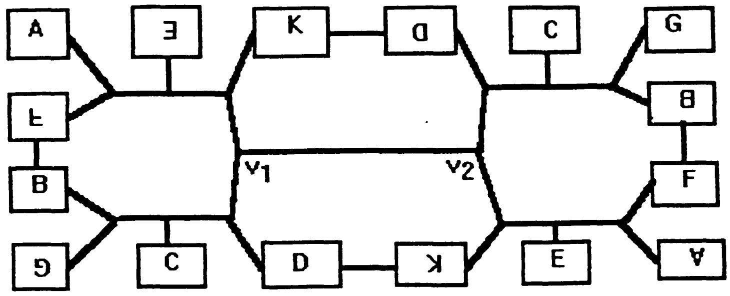

Figure 4.10: The stabilized graph

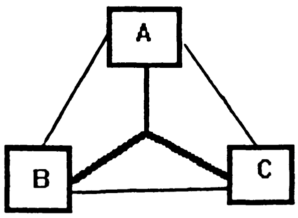

Figure 4.11: The position of a vertex

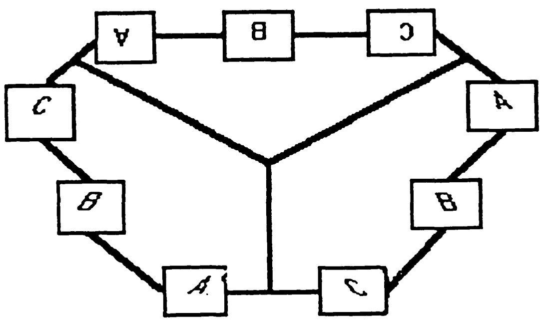

Figure 4.12: The stabilized graph

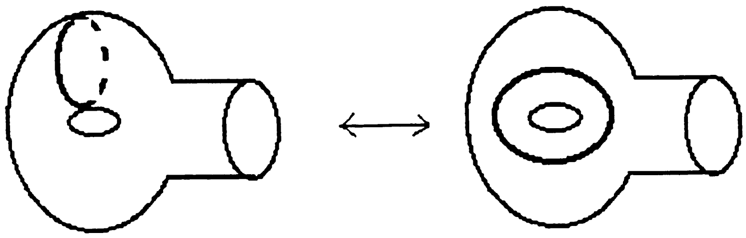

Figure 4.13: The C operation

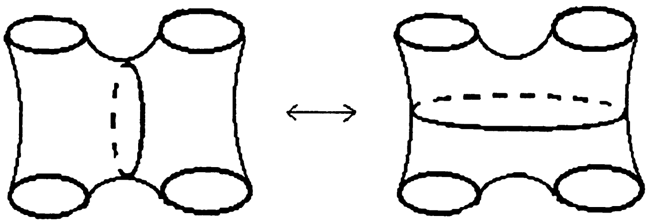

Figure 4.14: The F operation

Proof: A theorem of Hatcher and Thurston ([HT82]) states that two cut systems $c_{*,0}$ and $c_{*,1}$ on a surface are obtained one from the other by a sequence of operations C and F and their inverses. The operations C and F are described in the figures 13 and 14.

The move C does not affect the graph Γ and replace α by β. Now the following relation

$\alpha\beta\alpha = \beta\alpha\beta$

holds in $\mathcal{M}_{1,1}$. Therefore $\beta = \alpha\beta\alpha(\alpha\beta)^{-1}$ is conjugate to α so the eigenvalues of t_α and t_β coincide. Further the map

$\rho_g(d_\alpha d_\beta \otimes 1) : W(\Gamma), (\lambda_1, \lambda_2, \lambda_3)) \longrightarrow W(C\Gamma, (\lambda_1, \lambda_2, \lambda_3))$

is an isomorphism if the vertex v considered is incident to α in Γ.

The move F changes the graph according to the picture 15.

Consider now ω_i the class of the homeomorphism which interchanges e_i and e_{i+1} in the mapping class group $\mathcal{M}_{0,4}$. It is well-known that $\omega_i, i = 1,2,3$ and $d_{e_i}, i = 1,2,3,4$ generate $\mathcal{M}_{0,4}$. We have further

$d_{c_1} = d_{e_2}^{-1} d_{e_3}^{-1} d_{\omega_2}^2$

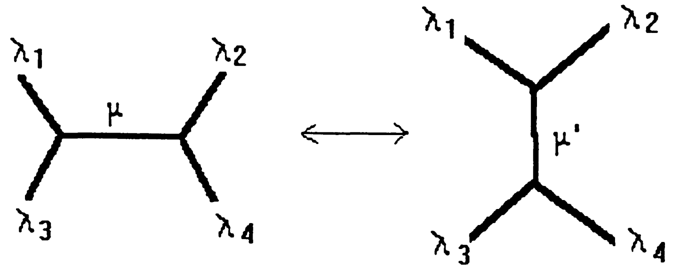

Figure 4.15: The move F on the graph level

$d_{c_2} = d_{e_1}^{-1} d_{e_2}^{-1} d_{\omega_1}^2$
so $d_{c_2} = d_{e_1}^{-1} d_{e_3} \omega_1 \omega_2 d_{c_1} (\omega_1\omega_2)^{-1}$. We wish to realize the primary block $W(\Gamma, (\lambda_1, \lambda_3, \mu)$ in both labelled graphs Γ and $F\Gamma$. So in the two labelled graphs from picture 16 we must find a pair of vertices having the same circular labels. From the previous two lemmas it suffices to check only one case, namely $\mu = \lambda_1, \mu' = \lambda_3$. Thus $\mu' = \lambda_1^{-1}\lambda_3\mu$. So from the relation $t_{c_1} x = \mu x$ we shall derive $t_{c_2}\rho_g(\omega_1\omega_2 \otimes 1)x = \mu'\rho_g(\omega_1\omega_2 \otimes 1)x$. Hence the map
$\rho_g(\omega_1\omega_2 \otimes 1)$ gives an isomorphism between the primary blocks $W(\Gamma, (\lambda_1, \lambda_2\lambda_3))$ and $W(F\Gamma, (\lambda_1, \lambda_2, \lambda_3))$ corresponding to the fixed vertices. This proves also that the label set is invariant.□
As an immediate consequence of these lemmas we derive

Theorem 4.2.11 *Assume that the cyclic vector generating the h.t.r. is the unique vector $SL^+(2, \mathbf{Z})$-invariant in genus $g = 1$. Then the target spaces of a cyclic geometric h.t.r. of $\mathcal{M}_*$ have the following decomposition*
$W_g \overset{i_\Gamma}{\simeq} \bigoplus_{l\in\mathcal{L}} \bigoplus_{j=1}^{s(l)} \otimes_{v\in V(\Gamma)} W(l(e_1), l(e_2), l(e_3))$
into primary blocks $W(i, j, k)$.

Remark now that the tensor structure $W_g \otimes W_h \to W_{g+h}$ is given by the usual tensor product of vector spaces according to lemma 3.1. Now once we have chosen an embedding of graphs $\Gamma_g \cup \Gamma_h \hookrightarrow \Gamma_{g+h}$ we have a corresponding multiplication rule for labellings $\mathcal{L}_g \times \mathcal{L}_h \hookrightarrow \mathcal{L}_{g+h}$ by extending the product labelling by 1 on the new edge and preserving the labels of an edge after we introduced a new vertex on it (so defining two adjacent edges). This induces the tensor structure on the decomposed blocks in an obvious manner.

Remark 4.2.12 *In the infinite dimensional unitary context the h.t.r. of $\mathcal{M}_*$ into $U(W_*)$ has a Hilbert completion to a h.t.r. into $U(\overline{W_*})$. Then the set of labels may be infinite and the direct sum replaced by an integral but the same decomposition principle holds for the completed blocks. The proof is essentially the same.*

Observe finally that we have chosen an orientation of each circle of the cut system, without any restriction because we must distinguish between d_e and d_e^{-1}. The change of the orientation of a curve corresponds to change the eigenvalues by their inverses. But we may restrict to some almost canonical choices. We look at the standard surface of genus g without the two disks bounded by δ_g^+, δ_g^- as being an oriented cobordism between the two circles. Each trinion lying will be therefore an oriented cobordism between its positive boundary and its negative boundary. Suppose we have 2 circles labelled j and k in the positive boundary and one circle labelled by i as the negative boundary. Therefore we specify in the primary block associated to the vertex-trinion $W(i,j,k)$ by putting the indices differently as W^i_{jk}. So we shall encounter 4 types of (oriented) primary blocks $W^{ijk}, W^i_{j^*k^*}, W^{ij}_{k^*}, W_{i^*j^*k^*}$ which are all isomorphic. But when we write the decomposition of the block W_g this notational convention specifies the orientation of all circles in the cut system.

Lemma 4.2.13 *We have the symmetries*
$W^i_{jk} \simeq W^i_{kj}$
$W^i_{jk} \simeq W^{k^{-1}}_{ji^{-1}}$

Proof: The proof is similar to that of the invariance of the primary blocks to cyclic permutations. Specifically we stabilize the graph from figure 11 to arrive at the graph depicted in figure 16. In the first case, when the edges e_1 and e_2 correspond to oriented circles lying on the positive boundary of the trinion, we can interchanges e_1 and e_2 using a homeomorphism $\varphi \in Homeo(\Sigma_g, c_*)$ preserving the orientation. In the second case the homeomorphism φ interchanges e_1 with e_2^{-1} hence the change of the labelling.□

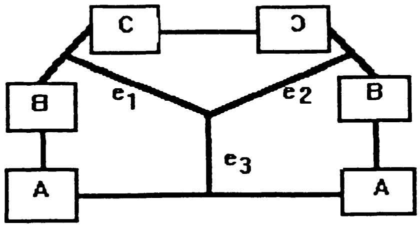

Figure 4.16: Another stabilization

This permits to start with a cut system and to obtain the decomposition specifying the orientation of each circle.

4.3 The structure of invariants

Our aim now is to get a similar decomposition for the representation ρ_* which follows the decomposition of target spaces.

We shall consider a groupoid which is closely related to the mapping class group having a tensor structure itself, and which is called the Teichmuller groupoid in [Dri91] or the duality groupoid in physical literature [MS89]. If T_g denotes the Teichmuller space [Abi77, Gro84] then $\mathcal{M}_g$ acts properly discontinuous on T_g and the quotient $M_g = T_g/\mathcal{M}_g$ is the moduli space of genus g non-singular algebraic curves. Due to the presence of curves with automorphisms M_g is not smooth but a V-manifold (see [Sat75, Wol83]) or a Q-manifold [Mat72, Mum74]. The set of its non-singular points M_g^{ns} is an open manifold, and we shall consider its (fundamental) path groupoid $\Pi_1(M_g^{ns})$. This is the duality groupoid D_g. It will became clear that it has a tensor structure when we derive another description of D_g.

We remember that an alternative description of M_g is as the moduli space of hyperbolic structures on Σ_g (or conformal structures). For $c \in c_*$ we set $l(c)$ for the hyperbolic length of the geodesic lying in the isotopy class of c, for an hyperbolic structure on Σ_g. But now the hyperbolic trinions up to conformal or anticonformal equivalence are determined by the lengths of boundary circles (which we suppose to be geodesic). Consider now the geodesic connecting two boundary circles and which are orthogonal to them. Fix the order of the loops in the cut system. There are two orthogonal geodesic which are intersect a boundary circle c. Set Δl for the oriented distance between their endpoints and consider the angles $\theta(c) = Arcsin(\Delta l/l(c)) \in [0, 2\pi)$ Now the $(3g-3)$ pairs $(l(c), \theta(c))$ give a function $f_{c_*} : T_g \longrightarrow R^{3g-3} \times (S^1)^{3g-3}$. It is a result of Bers which say that f_{c_*} is a Z^{3g-3} -covering and the Galois group acts as follows:

$Z^{3g-3} \xrightarrow{i(c_*)} \mathcal{M}_g,$
where $i(c_*)$ identifies Z^{3g-3} with the subgroup of $\mathcal{M}_g$, generated by the Dehn twists around the cut circles. These are the so-called Fenchel-Nielsen coordinates on Teichmuller space. There are real analytic ones (see [Abi77]).

To an unitary representation $\rho_g : \mathcal{M}_g \longrightarrow U(W_g)$ there is associated an holomorphic flat hermitian and $\mathcal{M}_g$-invariant vector bundle over T_g, such that the monodromy of the mapping class group is precisely ρ_g. Further this bundle descends to a flat holomorphic V-bundle E_g on M_g. Equivalently the pull-back of E_g on a smooth finite covering of M_g is a flat holomorphic bundle. Such a smooth covering is well-known to be the moduli space of algebraic curves with a level l structure.

Now there is a canonical identification of W_g with the space of flat sections of the V-bundle $E_g |_{M_g}$. The set $f_{c_*}^{-1}((0,\varepsilon)^{3g-3} \times (0,\pi)^{3g-3}))$ is a disjoint union of contractible open sets in T_g (for little ε) on which Z^{3g-3} acts freely. The flat and $\mathcal{M}_g$-invariant sections over one such contractible set U_{c_*} may be analytically continued at all of T_g (modulo the path groupoid action). The monodromy representation we get this way is nothing but the initial ρ_g from the beginning. Taking another cut system c'_* or another coordinates chart, (i.e. we consider $f_{c_*}^{-1}(\Pi_{j=1,3g-3}(l_j, l_j+\varepsilon) \times \Pi_{j=1,3g-3}(v_j, v_j+\pi)))$, we shall get a matrix which relates the two basis of flat sections $\mathcal{M}_g$-invariant obtained by analytic continuation. Therefore we have a representation of the groupoid G_g acting on the set of cut systems, so in particular on labelled 3-valent graphs (with leaves). In our case the particular labellings are the Fenchel-Nielsen coordinates and some extra marking from the identification of Z^{3g-3} as a subgroup of $\mathcal{M}_g$. We can get a covering for T_g by taking a sufficiently large family of points (l_j, v_j). Now

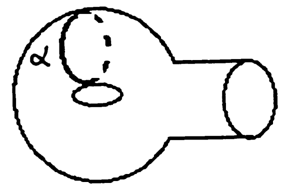

Figure 4.17: T=Dehn twist around α

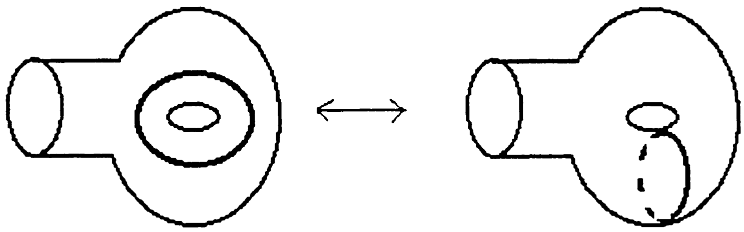

Figure 4.18: S corresponding to the C-move on cut systems

we project on M_g and we find that we can extract an open covering with contractible sets of M_g^{ns} – { a neighborhood of the variety of singular points}. Since the path groupoid is an homotopic invariant and the singular divisor is triangulable we derive $G_g \simeq D_g$. Hence we may describe D_g by looking only at its action on labelled 3-valent graphs. It is a result of Moore and Seiberg [MS89] (which in particular settles a question raised by Grothendieck) which asserts that D_g is generated by finitely many moves and relations among them. Specifically the five duality moves can be described geometrically as in the pictures 17-21.

There is another operation called braiding which can be described as the composition of F and Ω moves (see the picture 22) or alternatively, by a change in the pants decomposition (the cut system is changed but the dual graph remains the same), as in the picture 23.

The fact that these five moves suffice to generate D_g is easy to prove: in fact S and F act transitively on the set of 3-valent graphs (with a

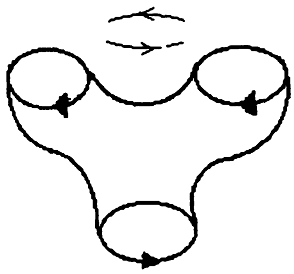

Figure 4.19: Ω interchanges two boundary circles

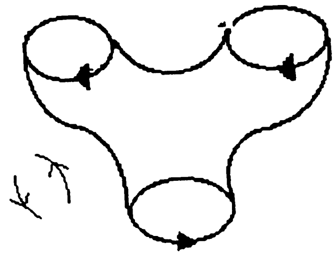

Figure 4.20: Θ interchanges two boundary circles differently oriented

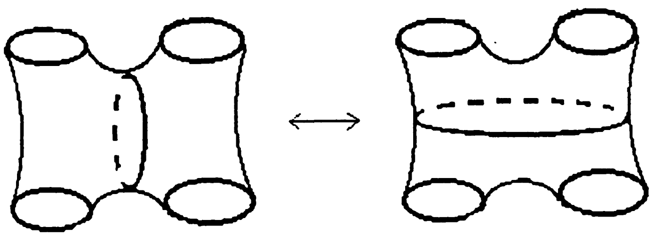

Figure 4.21: F coming from the F move on cut systems

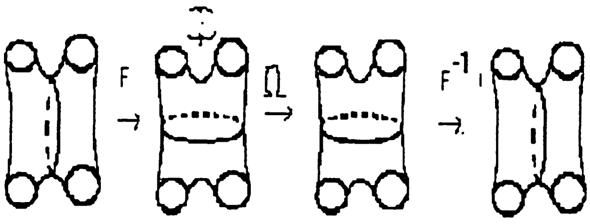

Figure 4.22: The braiding move B

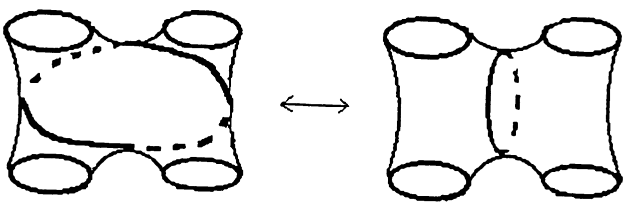

Figure 4.23: Changing the cut system for braiding

fixed number of leaves) with fixed labels, T ensures the $\mathbf{Z}^{3g-3}$ marking, and Ω, Θ acts transitively on the set of Fenchel-Nielsen labels.

Another way to look at these moves is the following: observe that S and T are classes in $\mathcal{M}_{1,1}$, Ω and Θ are lying in $\mathcal{M}_{0,3}$ and $F \in \mathcal{M}_{0,4}$. The original statement of Grothendieck conjecture states that $\mathcal{M}_{1,1}, \mathcal{M}_{0,3}, \mathcal{M}_{0,4}$ generate the whole tower of groups $\mathcal{M}_{*,*}$. What means to generate it is clear: to every decomposition of a h-holed surface $\Sigma_{g,h}$ into pieces homeomorphic to a 1-holed torus, a trinion or a 4-holed sphere we get a subgroup of $\mathcal{M}_{g,h}$ using the gluing of homeomorphisms defined on each piece. When we carry out this procedure for all possible decomposition we obtain a family of subgroups which generate $\mathcal{M}_{g,h}$.

Proposition 4.3.1 *The representation ρ_* extends naturally to a h.t.r. of the whole duality groupoid D_*.*

We have seen that ρ_g extends naturally to a representation of D_g. Consider $\Sigma_{1,1}$ a 1-holed torus embedded in Σ_g. Then there exists a cut system c_* on Σ_g containing the boundary of $\Sigma_{1,1}$. Actually when looking at W_g as being identified to $i_{\Gamma(c_*)}(W_g)$ we see that $\mathcal{M}_{1,1} \otimes 1 \subset \mathcal{M}_g$ acts only on the primary blocks corresponding to the vertex associated to $\Sigma_{1,1}$. So we have a family of transformations

$S(j) : \oplus_i W_{ji}^i \longrightarrow \oplus_i W_{ji}^i,\ j \in L$

$T(j) : \oplus_i W_{ji}^i \longrightarrow \oplus_i W_{ji}^i,\ j \in L$

which together give a representation of $\mathcal{M}_{1,1}$ for each j. But the map $T(j)$ acts by multiplication by i on W_{ji}^i hence $T(j) = T$ is a diagonal matrix which does not depend upon the external index j. A priori all these representations depend upon the choice of the particular embedding of the 1-holed torus in Σ_g. Fortunately this is not the case due to

Lemma 4.3.2 *The primary blocks $W(i,j,k)$ are $\mathbf{C}[\mathcal{M}_{0,3}]$-modules, not only vector spaces, which depend only on the labels not on the particular choices we made in the previous section.*

Proof: All the isomorphism we get in the lemmas 3.1-3.9 are module isomorphisms. □

Since $\mathcal{M}_{1,1} \hookrightarrow \mathcal{M}_{0,3}$ we derive that $S(j)$ and T are independent on the particular embedding chosen.

For the moves Θ and Ω we obtain in the same manner the family of isomorphisms

$\Omega^i_{jk}(-) : W^i_{jk} \longrightarrow W^i_{kj}, \Omega^i_{jk}(+) = \Omega^i_{jk}(-)^*$

$\Theta^i_{jk}(-) : W^i_{jk} \longrightarrow W^{k^{-1}}_{ji^{-1}}, \Theta^i_{jk}(+) = \Theta^i_{jk}(-)^*$

Geometrically these arise as follows: we identify the trinion with a domain in the plane $D - D_1 \cup D_2$ where $D_i \subset D$ are equal 2-disks. Consider another disk $D_0 \subset D$ containing D_i and an homeomorphism of D which is identity outside D_0, and the rotation by π which interchanges the disks D_1 and D_2 on a smaller disk contained in D_0.

This time it is not a representation of $\mathcal{M}_{0,3}$ which is obtained but of an object related to it. Let $\varepsilon : \{1,2,3\} \longrightarrow \mathbf{Z}/2\mathbf{Z}$ be the signature of the boundary where the circle numbered $j = 1, 2, 3$ lies on. Here we adopt the previous convention by looking at the 3-holed sphere as to an oriented cobordism. A homeomorphism h of $\Sigma_{0,3}$ which preserves globally the boundary but not pointwise induces a permutation of the boundary circles leading to another marking $h^*(\varepsilon) \in (\mathbf{Z}/2\mathbf{Z})^3$. We consider the triples $(\varepsilon, h(\textit{modulo isotopy}), h^*(\varepsilon))$. Their set is the mapping class groupoid $\mathcal{M}_{0,3}(2)$ of the 2-colored (or oriented) 3-holed sphere. In the same manner it could be defined the mapping class groupoid of c-colored h-holed surface of genus g. So actually the mappings Θ and Ω (together with S and T) define a representation of this groupoid $\mathcal{M}_{0,3}(2)$. Again this structure is uniquely defined from the previous considerations and lemma 3.13.

Finally the move F (called also the fusion move) define the isomorphisms

$F\begin{bmatrix} i & j \\ k & l \end{bmatrix} : \oplus_{r\in L} W^k_{ir} \otimes W^r_{jl} \longrightarrow \oplus_{s\in L} W^k_{sl} \otimes W^s_{ij}.$

Its action is induced from that of $d_{\omega_1} d_{\omega_2}$. But ω_i are both lying in a $\mathcal{M}_{0,3}$-factor (for two different cut systems). So each of them is canonically defined henceforth the mappings F do not depend on the particular 4-holed sphere used. Otherwise it is simply to check that the spaces on which F acts are $\mathcal{M}_{0,4}$-modules intrinsically defined.

On the other hand these isomorphisms must define a representation of the mapping class group. Using the identities from [MS89] we derive that the following conditions must be verified:

$$F(\Omega(\varepsilon) \otimes 1)F = (1 \otimes \Omega(\varepsilon))F(1 \otimes \Omega(\varepsilon)) \quad (4.1)$$

$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12} \quad (4.2)$$

$$S^2(j) = \oplus_{i\in L}\Theta^i_{ji}(-) \quad (4.3)$$

$$S(j)TS(j) = T^{-1}S(j)T^{-1} \quad (4.4)$$

$$(S\otimes 1)(F(1\otimes\Theta(-)\Theta(+)F^{-1})(S^{-1}\otimes 1) = FPF^{-1}(1\otimes\Omega(-)) \quad (4.5)$$

with the usual convention: F_{ij} acts on the i^{th} and j^{th} factors of a tensor product, and P_{ij} interchanges these factors.

But once these conditions are satisfied we know from [MS89] that the five moves define a representation of whole duality groupoid D_g which respect the tensor structure. □

Several comments are necessary now. We know that the h.t.r. admits also a weight vector w_g, which is uniquely defined by the weight condition at level 1. We say that the vacuum is irreducible if this condition is fulfilled in each genus. We have the splitting

$W_g \simeq \bigoplus_{l\in\mathcal{L}}\bigoplus_{j=1}^{s(l)} W(\Gamma_g, l)$,

where we denoted

$W(\Gamma_g, l) = \bigotimes_{v\in V(\Gamma)} W(\Gamma_g, v, l(e_1), l(e_2), l(e_3))$.

Since w_1 is uniquely determined we derive

$W^1_{11} \simeq \mathbf{C}w_0$

and $w_1 = w_0 \otimes w_0$. In particular

$w_g = w_0^{\otimes 2g} \in W^1_{11}\otimes ... \otimes W^1_{11} = W(\Gamma_g, \mathbf{1})$

where we used for Γ_g the simplest 3-valent graph of genus g with 2 leaves. Above $\mathbf{1}$ stands for the labelling identical 1. In particular if the vacuum is irreducible it follows that $s(\mathbf{1}) = 1$. Because the theory is a cyclic one generated by w_g and the representation of D_g is defined on the primary blocks directly (and not on sums of primary blocks) we obtain $s(l) = 1$ for all labellings l. So the splitting principles has the canonical form

$W_g \simeq \bigoplus_{l\in\mathcal{L}}\bigotimes_{v\in\Gamma_g} W(l(e_1), l(e_2), l(e_3))$.

We use now this expression to compute W_1 in the case of two graphs which may be seen in figure 24. Suppose all the representations ρ_g are finite dimensional and denote by $n_g = dim_{\mathbf{C}} W_g$ and $n^i_{jk} = dim_{\mathbf{C}} V^i_{jk}$. It follows $n_1 = \sum_i n^i_{1i} = \sum_{i,j}(n^i_{1j})^2$. Therefore

$W^i_{1j} \simeq \delta_{ij}\mathbf{C}$

where δ_{ij} states for the Kronecker symbol.

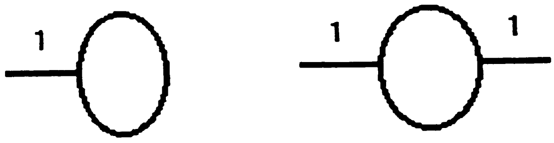

Figure 4.24: Two graphs of genus 1

Remark also that $w_g = w_0^{2g}$ is in fact a vacuum vector for our representation. For the group of Dehn twists around curves which bound in the handlebody this is already clear. But from the description given by Suzuki (see also [Cra91, Koh92]) we derive that w_g is in fact $\mathcal{M}_g^+$-invariant.

As a notational convenience we denote by $exp(2\pi\sqrt{-1}\Delta_j)$ the eigenvalue corresponding to j, this time j being a natural number. This is possible since all the matrices are unitary.

Using the relations in $\mathcal{M}_{0,3}$ we derive that $\Omega(\varepsilon)^2$ can be expressed in terms of the Dehn twists around the boundary circles as

$\Omega(-)^2 = t_1^{-1}t_2t_3$.

This implies that

$\Omega^i_{jk}(-)^2 = exp(2\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))\mathbf{1}_{n^i_{jk}}$,

$\Theta^i_{jk}(-)^2 = exp(2\pi\sqrt{-1}(\Delta_i + \Delta_k - \Delta_j))\mathbf{1}_{n^i_{jk}}$,

where $\mathbf{1}_n$ stands for the identity matrix of rank n. From the geometric interpretation we shall have natural identifications of the bases on the spaces W^i_{jk}, W^i_{kj} and $W^{k^*}_{ij^*}$ which we call σ_{12}, σ_{13} and σ_{23} respectively. These will produce a representation of the symmetric group S_3 and we are able to get the following form for the matrices Ω and Θ (in this bases)

$\Omega^i_{jk}(-) = exp(\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))(\delta_{a,\sigma_{12}a})$,

$\Theta^i_{jk}(-) = exp(\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))(\delta_{a,\sigma_{13}a})$,

where the indices a run in a basis for W^i_{jk}.

Now from this data we can recover the representation ρ_* as follows: Suppose we take Γ_g be again the simplest 3-valent graph with 2-leaves (see the figure 25). Then W_g is identified with

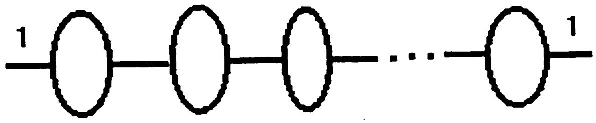

Figure 4.25: The graph Γ_g

$\oplus W^{1}_{i_1 i_1} \otimes W^{i_1}_{i_1 j_1} \otimes W^{j_1}_{i_2 k_2} \otimes ... \otimes W^{j_{g-1}}_{i_g i_g} \otimes W^{i_g}_{i_g 1}$.
We consider as generators of the mapping class group the Dehn twists around the curves $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, \delta_2\}$ as in the picture 26 (see [MS89, Bir74]). Then

$$\rho_g(\alpha_1) = T_1^{-1} \tag{4.6}$$

$$\begin{aligned}
&\rho_g(\alpha_l) = \\
&= T_{i_{l-1}}^{-1}(B^{-}_{j_{l-1}} \begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} B^{-}_{j_{l-1}} \begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix})T_{i_l}^{-1} = \\
&= F_{j_{l-1}} \begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} T_{j_{l-1}} F^{-1}_{j_{l-1}} \begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix}
\end{aligned}$$
for $l > 1$.

$$\rho_g(\beta_l) = T_{k_l} S_{k_l i_l}(j_{l-1}, j_l) T_{k_l}. \tag{4.7}$$

$$\rho_g(\delta_2) = T_{i_2}. \tag{4.8}$$

Above we used the braiding matrix B given by
$B = F^{-1}(1 \otimes \Omega(-))F$.
Also the indices on the linear transformations tell us on which of the subspaces it acts on. Remark that what we have obtained as data for the h.t.r. is exactly the axiomatic definition of a unitary RCFT (see [MS89, Deg92]) having the central charge $c = 0$(modulo 24). This is due to the fact that we have a representation of the mapping class group not of a central extension of it.

Theorem 4.3.3 *A geometric h.t.r. of $\mathcal{M}_*$ with irreducible vacuum is equivalent to a RCFT of central charge 0 (modulo 24).*

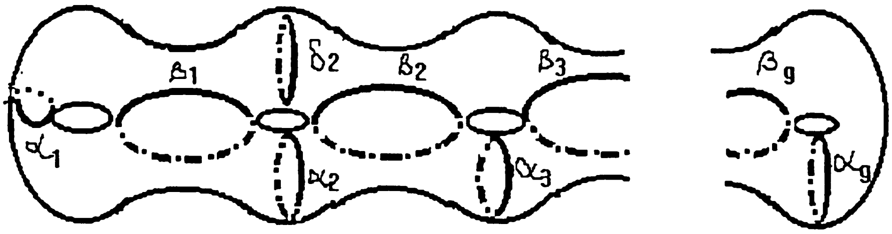

Figure 4.26: Generators for $\mathcal{M}_g$

We suppose from now one that we are working with rational unitary invariants given by a h.t.r. with irreducible vacuum so the theorem above applies.

4.4 TQFT for cobordisms

We obtained in the previous section the combinatorial data of a RCFT having the central charge $c = 0$(modulo 24). This allowed us to reobtain the initial representation ρ_g in terms of $(F, S, T, \Omega, \Theta)$. Our invariant is therefore given by the formula

$F(M(\varphi)) = d^{-g} < \rho_g(\varphi)w_g, \overline{w_g} >$

where

$d = S(0)_{11}$.

We wish to obtain a similar description for the TQFT extending the invariant F. We start with an oriented cobordism M^3 having the positive boundary $\partial_+ M^3$ and the negative boundary $\partial_- M^3$. We have an analog of the Heegaard splitting for cobordisms by using instead of handlebodies the compression bodies (see [Cra91]). A compression body C may be obtained as follows: consider $c_1, c_2, ..., c_s \subset \Sigma_g$ be disjointly embedded circles (which we suppose to be pairwise non-isotopic) which bound in H_g. Then consider

$C_1 = \Sigma_g \times [0,1] \cup$ 2-handles attached on $c_1, c_2, ..., c_s$.

A general compression body has the form

$C = C_1 \cup$ 3-handles,

permitting thus to capping off the S^2 components of the boundary. We assume that $\partial_+ C = \Sigma_g$. Now a Heegaard splitting of M is a decomposition into compression bodies

$M^3 = C_+ \cup \overline{C_-}$,

where the boundaries are identified as

$\partial_- C_+ \simeq \partial_- M^3$,

$\partial_- C_- \simeq \partial_+ \overline{C_-} \simeq \partial_+ M^3$.

The compression bodies C_+ and C_- are glued together along their boundary components $\partial_+ C_+$ and $\partial_- \overline{C_-} \simeq \partial_+ C_-$ using some homeomorphism of Σ_g whose class in the mapping class group is φ. In order to find $F(M^3)$ it suffices to know the value of invariant on compression bodies (see [Ati89]).

We construct first the functor F on surfaces. Set

$F(\Sigma_g) = W_g$ with its hermitian structure,

$F(\overline{\Sigma_g}) = W_g^*$ if the orientation changes,

$F(\phi) = \mathbf{C}$.

Further for a disjoint union of surfaces we have

$F(\bigcup_{i=1}^r \Sigma_{g_i}) = \bigotimes_{i=1}^r F(\Sigma_{g_i})$.

Next we have the morphisms

$F(C_+) : F(\partial_+ C_+) \longrightarrow F(\partial_- C_+) = F(\partial_- M^3)$,

$F(\overline{C_-}) : F(\partial_- \overline{C_+}) \longrightarrow F(\partial_+ \overline{C_+})$.

The second morphism is the transposed of $F(C_-)$. Using proposition 2.7 we derive that

$F(M^3) = F(C_+) \circ \rho_g(\varphi) \circ F(\overline{C_-})$.

On the other hand F is defined for cobordisms with marked boundaries, i.e. some fixed homeomorphisms

$\varphi_+ : \partial_+ M^3 \longrightarrow \cup_i \Sigma_{g_i}$,

$\varphi_- : \partial_- M^3 \longrightarrow \cup_i \Sigma_{h_i}$.

Suppose we choose once for all the cut systems $c_*^0 \subset \Sigma_g$. For the compression body C we have $\partial_+ C = \Sigma_g$, $\partial_- C \simeq \cup_i \Sigma_{h_i}$ where $\sum_i h_i = g - s$. Once we have chosen a cut system $\gamma_*^+ \subset \partial_+ C$ we have the natural marking

$\varphi^0(\gamma_*^+) : \partial_+ C \longrightarrow \Sigma_g$.

Let Γ^+ be the dual graph of γ_*. The surface Σ_g could be identified with the boundary of a tubular neighborhood of the 3-valent graph Γ embedded trivially into $\mathbf{R}^3$. Since the graph is actually planar the blackboard framing f provides the surface of a rigid structure. Then

$\varphi^0(\gamma_*^+)$ is the homeomorphism of Σ_g respecting the rigid structure. A similar canonical marking may be defined on $\partial_- C$ if a cut system γ_*^- and a framing are chosen. Set $\Gamma+, \Gamma^-$ for the corresponding dual graphs of γ_*^+ and γ_*^- respectively. We may suppose, for simplicity, that $\partial_- C$ is connected so it is a surface of genus $h = g - s$. We start with the (eventually extended) cut system $\gamma_* \subset \Sigma_g$ which contains the attaching circles of the 2-handles, hence $\gamma_i = c_i$ for $i = 1, s$. Each curve c_i has a natural framing given by $c_i \times [-\varepsilon, \varepsilon] \subset \Sigma_g$. For small ε these tubular neighborhoods remain disjoint. Consider

$X = \Sigma_g - \bigcup_{i=1}^s c_i \times [-\varepsilon, \varepsilon] \cup_{i=1}^s d_{i1} \cup d_{i2}$,

where d_{i1} are 2-disks (disjointly embedded in H_g) bounding $c_i \times \{-\varepsilon\}$, and respectively d_{i2} are 2-disks capping off $c_i \times \{\varepsilon\}$. Therefore

$X = \Sigma_h \cup_j S^2$.

We shall identify the negative boundary of C with the surface Σ_h which is a boundary component of X. Consider the curves γ_i, for $i > s$, which remain drawn on this surface Σ_h. We add those curves $c_i \times \{+ - \varepsilon\}$ which also lie on Σ_h. Their set represents an extended cut system on $\Sigma_h = \partial_- C$ which we denote by $[C]\gamma_*$ and we call the transport by C of γ_*. The pieces of the framing which remain on Σ_h give the transport of the framing , hence a rigid structure on the negative boundary. Let Γ^- be its dual graph. A labelling l of $[C]\gamma_*$ is admissible if

$l(x) = 1$ if x is not in $\{\gamma_i, i > s\}$.

Any such labelling extends to a labelling l^e of γ_* (or, equivalently Γ^+) by 1. Further we have a canonical isomorphism (by the stabilization lemmas) between $W(\Gamma^-, l)$ and $W(\Gamma^+, l^e)$. We obtain a natural injective mapping

$W_h \overset{i_{\Gamma^-}}{\simeq} \oplus_l W(\Gamma^-, l) \simeq \oplus_l W(\Gamma^+, l^e) \subset \oplus_l W(\Gamma^+, l) \overset{i_{\Gamma^+}}{\simeq} W_g$,

where in the first two direct sum the l's run over all admissible labellings of Γ^-, while the third sum is taken over all labellings of Γ^+.

Now we can get the expression of $F(C)$ for some special markings of the boundary. This is sufficient since $\mathcal{M}_g \times \mathcal{M}_h$ acts transitively on the markings. Namely we choose $\varphi^+ = \varphi^0(\gamma_*)$, and $\varphi^- = \varphi^0([C]\gamma_*)$. We can state now

Proposition 4.4.1 *The morphism*

$F(C, \varphi^+, \varphi^-) : W_g \longrightarrow W_h$

is the projection dual to the above described inclusion mapping.

Proof: Observe first that for a handlebody F has the wanted description because $F(H_g, id) = w_g$. This equality follows from the proof of theorem 2.5.

On the other hand it suffices to check the result for a particular cut system since $\mathcal{M}_g$ acts transitively on the set of cut systems, and in a compatible manner on $F(C, \varphi^+, \varphi^-)$ as given above. So we consider

$X_{g,h} = \Sigma_h \times [0,1] \bigcup_{\Sigma_h \times 1 \supset b_i} H_{g-h}$

where after we take the union we identify the 2-disks b_i leaving in $\Sigma_h \times 1$ and ∂H_{g-h}. Let consider some $\varphi \in \mathcal{M}_g$, $\varphi = \varphi_1 \sharp id$, with $\varphi_1 \in Homeo(\Sigma_h, b_1, ..., b_r)$. Therefore

$X_{g,h} \cup_\varphi \overline{H_g} = \Sigma_h \times [0,1] \cup_{\varphi_1} \overline{H_h \sharp_{b_i} H_{g-h} \cup H_{g-h}}$.

Next for any $\psi \in \mathcal{M}_h$ we have

$F(H_h \cup_\psi X_{g,h} \cup_\varphi \overline{H_g}) = F(H_h \cup_{\varphi_1^{-1}\psi} \overline{H_h \sharp_{b_i} H_{g-h} \cup H_{g-h}})$

since the two considered manifolds are homeomorphic. We wish to replace the quotient space on the right by an usual connected sum. Choose a null homotopic curve which pass trough the centers of the 2-disks b_i in both manifolds. Then Dehn's lemma gives us two embedded disks (in $M(\varphi_1^{-1}\psi)$ and S^3 respectively) D_1 and D_2. The usual connected sum may be carried out by identifying some collars of these two disks. This says that replacing the quotient space with the connected sum has the effect of a connected sum with the S^3. Thus the homeomorphism type does not change. It follows from the multiplicativity of F that

$F(H_h \cup_\psi X_{g,h} \cup_\varphi \overline{H_g}) = F(M(\varphi_1^{-1}\psi))F(S^3) = F(M(\varphi_1^{-1}\psi))$.

Let $Z = Span < \rho_g(\mathcal{M}_h \otimes 1) w_g > \subset W_g$. The above formula reads

$F(X_{g,h}, id, id) \mid_Z = 1$.

On the other hand $Z \simeq W_h$ which implies that we have a cross section of $F : W_g \longrightarrow W_h$ given by $x \rightarrow x \otimes w_{g-h}$. Then the position of Z in W_g is that arising from the inclusion of graphs $\Gamma^- \subset \Gamma^+$. This establishes our claim. □

Remark 4.4.2 *The value of F on compression bodies is universal because it not depends upon the particular invariant chosen but only on the primary blocks. As a direct consequence this value (for compression bodies only) is the same in the classical RCFT associated to a compact group and for the quantum RCFT obtained from the associated quantum group (for a parameter value not a root of unity).*

In the abelian TQFT (the gauge group $U(1)$) coming from the Chern-Simons-Witten theory the extension to cobordisms was described in [Fun93f].

Remark that

$F(M \cup_\varphi N) = F(M) \circ \rho_g(\varphi) \circ F(N)$

from proposition 1.4, so the twist factor from the middle does not depend upon the choice of the splitting (not necessary a Heegaard splitting).

We shall give an example. If $V \xrightarrow{\pi} S^1$ is a Σ_g-bundle over the circle having the monodromy mapping $\varphi \in \mathcal{M}_g$ we decompose

$V = \pi^{-1}([0,1/2]) \cup \pi^{-1}([1/2,1])$.

Both components in the right are two cylinders over Σ_g. But the positive boundary of $\pi^{-1}([0,1/2])$ consists into two copies of Σ_g and the other one is void. The marking may be chosen to be $(1 \otimes 1)$. The negative boundary of $\pi^{-1}([1/2,1])$ consists also into two copies of Σ_g and we can consider the marking $(1 \otimes \varphi)$. Since

$F(\Sigma_g \times [0,1], 1, 1) = 1$

we derive

$F(\Sigma_g \times [0,1], 1 \otimes 1) = \sum_{i=1,k} e_i \otimes e_i^*$

where $\{e_1, e_2, ..., ek\}$ is a basis for W_g. Thus

$F(\Sigma_g \times [0,1], 1 \otimes \varphi) = \sum_{i=1,k} e_i \otimes \rho_g(\varphi)(e_i)^*$,

and we can compute

$F(V) = \sum_{i,j=1,k} < e_i \otimes e_i^*, e_j \otimes \rho_g(\varphi)(e_j)^* >= \sum_{i=1,k} < e_i, \rho_g(\varphi)(e_i) >= tr(\rho_g(\varphi))$,

which agrees with Atiyah's formula (see [Ati89]).

Corollary 4.4.3 *Suppose we have a Hilbert h.t.r. yielding unitary invariants for 3-manifolds. Then $\rho_g(\mathcal{M}_g)$ consists in trace class operators on W_g.*

4.5 Colored link invariants

Consider $K \subset M^3$ be a link with k components having the framing f. The framing is equivalent to the choice of k longitudes on the tori bounding the tubular neighborhood $T(K) \subset M^3$. Choose some circle on each torus which bounds a small 2-disk embedded in $S^1 \times S^1$ disjoint from the framing. This gives an extended cut system $c_*(f)$ on ∂M^3. We have further canonical identifications

$F(\partial(M - T(K)) = W_1^{\otimes k}$

$F(M - T(K), f) = F(M - T(K), \varphi^0(c_*(f))) = v \in W_1^{\otimes k}$.

The second one comes from the choice of the rigid structure on $\partial T(K)$ given by the framing f. Also we know that

$W_1 = \oplus_i W_{1i}^i$, and $W_{1i}^i \simeq \mathbf{C}e_i$

with fixed unitary e_i (defined up to a modulus 1 scalar).

Suppose we have a coloring on the components of the link K , say $c : \{1, 2, ..., k\} \longrightarrow L$. We have then a naturally associated invariant for framed colored links given by

$F(M^3, K, f, c) = < v, e_{c(1)} \otimes e_{c(k)} \otimes ... \otimes e_{c(k)} > \in \mathbf{C}$.

Proposition 4.5.1 *Consider M^3 obtained by Dehn surgery on the framed link $(K, f) \subset S^3$. Then the following formula*

$F(M) = \sum_{c \ coloring} S(0)_{c(1)1} S(0)_{c(2)1} ... S(0)_{c(k)1} F(S^3, K, f, c)$

holds.

Proof: We may decompose $M^3 = S^3 - T(K) \cup_\varphi T(K)$ where $\varphi = \tau \oplus \tau \oplus ... \oplus \tau \in SL(2, \mathbf{Z})^k$, under the framing identification. On the other hand $T(K)$ is a union of solid tori (with their canonical markings of their boundaries $\partial H_1 = \Sigma_1$) hence

$F(T(K)) = w_1^{\otimes k} \in W_1^{\otimes k}$

if we use the standard marking of the boundary. Therefore

$F(M) = < F(S^3 - T(K)), (\rho_1(\tau) w_1)^{\otimes k} > =$

$\sum_{c \text{ coloring}} S(0)_{c(1)1} S(0)_{c(2)1} ... S(0)_{c(k)1} F(S^3, K, f, c)$. □

This formula permits to recover the invariant for closed 3-manifolds once we know its values for colored links. This way was used in [Deg92, KM91, KT93] to define 3-manifold invariants.

There is another approach to obtain link invariants directly from the data of RCFT. Start with a braid representative for the link K having the strands colored (this coloring is induced from a coloring of the link components) (see the figure 27).

Define now the spaces $W_{0,n}(c)$ where c is a strand coloration compatible with respect to the Artin's closure. Consider $\Sigma_{0,n}$ be the sphere with n-holes, having the boundary circles c_i, $i = 1, n$. Extend the set of c_i's to an extended cut system c_* on $\Sigma_{0,n}$ having the dual graph $\Gamma_{0,n}$.

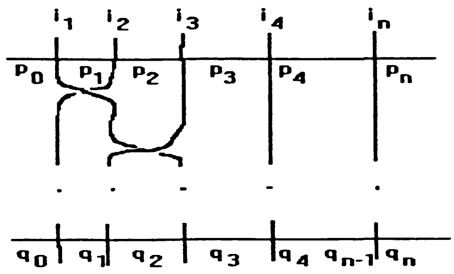

Figure 4.27: A colored braid representative

A labelling $l : E(\Gamma_{0,n}) \longrightarrow L$ is admissible if $l(c_i) = c(i)$, where $c(i)$ is the color of the i^{th} strand. We set further

$W_{0,n}(c) = \oplus_{l \text{ admissible}} W(\Gamma_{0,n}, l)$.

This definition may be done more generally for a h-holed surface $\Sigma_{g,h}$ of genus g having a fixed coloring c of the boundary components. The corresponding spaces are

$W_{g,h}(c) = \oplus_l W(\Gamma_{g,h}, l)$

the sum being taken over all the labellings extending the boundary one. Remark that whenever $\Sigma_{g,h} \cup \Sigma_{g',h'} = \Sigma_{g+g',h+h'-2s}$ under the identification of s boundary circles we have a splitting

$W_{g+g',h+h'-2s}(c) = \oplus_d W_{g,h}(c_0 d) \otimes W_{g',h'}(c_1 d)$

where c_0 is the coloring of the $h - s$ circles of $\Sigma_{g,h}$ induced by c and $c_0 d$ is the extension of c_0 by an arbitrary labelling d of the remaining circles (and similar for c_1 and $c_1 d$).

Observe that

$W_{0,n}(i_1, ..., i_n) = \oplus_{p_1,...p_{n-1}} W^0_{i_1 i_1} \otimes W^{i_1}_{i_2 p_1} \otimes ... \otimes W^{i_n}_{i_n 0} \hookrightarrow$
$\oplus_{p_1,...,p_{n-1},j} W^0_{i_1 i_1} \otimes W^{i_1}_{i_2 p_1} \otimes ... \otimes W^{i_n}_{i_n j} \simeq \oplus_j W_{0,n+1}(i_1, ..., i_n, j)$.

We have a natural representation of the groupoid of c-colored braids $B_n(c)$ (see [Fun93b]) on $W_{0,n+1}(i_1, i_2, ..., i_n, j)$ given by

$$\rho_{0,n,j}(b_s) = 1 \otimes B_{p_s} \begin{bmatrix} i_s & i_{s+1} \\ p_s & p_{s+1} \end{bmatrix} \otimes 1, \text{ with } p_n = j.$$

We set

$\rho_{0,n} = \oplus_j \rho_{0,n,j}$.

We can compute $\rho_{0,n}(x)$ using (a recurrence on) the graphical resolution of crossings from figure 28. Finally we obtain an identity as in figure 29.

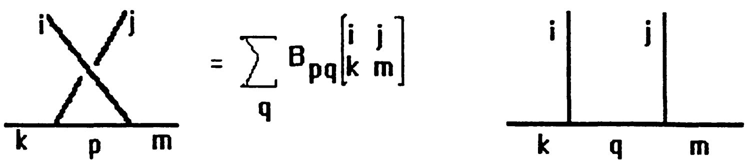

Figure 4.28: The resolution of a crossing

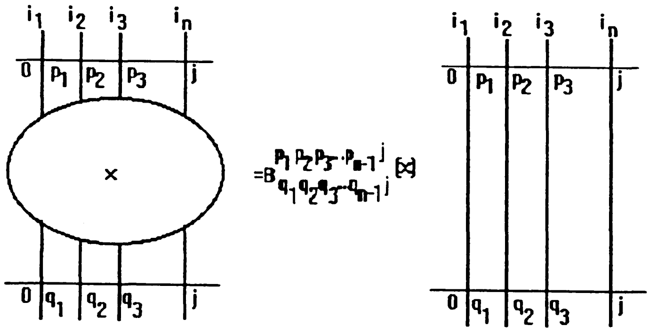

Figure 4.29: The expression for $\rho_{0,n}(x)$

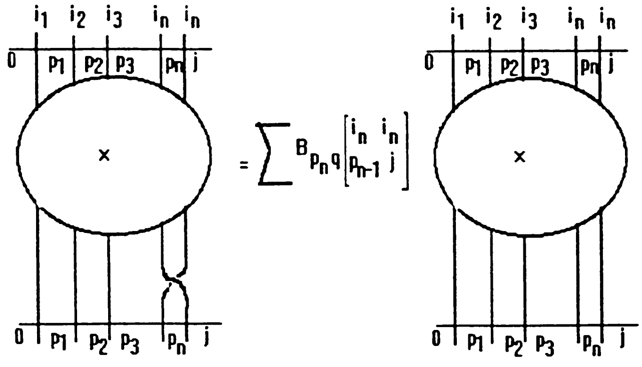

Figure 4.30: The resolution for xb_n

We derive that the trace of the representation
$tr(\rho_{0,n}(x) \mid_{W_{0,n+1}(i_1,\ldots,i_n,j)}) = \sum_{p_1,\ldots,p_n} B^{p_1 p_2 \ldots p_{n-1} j}_{p_1 p_2 \ldots p_{n-1} j}(x)$.
where the B^*_* are certain products of braiding matrices, depending on x. Define
$J(x,c) = \left(\prod_{s=1}^{n} \frac{S(0)_{11}}{S(0)_{i_s 1}}\right) \sum_{j=1}^{n} \frac{S(0)_{j1}}{S(0)_{11}} tr(\rho_{0,n}(x) \mid_{W_{0,n+1}(i_1,\ldots,i_n,j)})$.

Proposition 4.5.2 *Let x be a braid representative for the colored link (K,c). Then $J(K,c) = J(x,c)$ defines an invariant for colored links.*

Proof: It is clear that $J(x,c)$ is constant on conjugation classes. It remains to compute $J(xb_n, c)$ for $x \in B_n(c)$. Since the last two strands belong to the same component of the link (after the closure) the induced color i_{n+1} of the $n+1$-strand is i_n. We have the graphical identity from figure 30. We derive that
$tr(\rho_{0,n+1}(xb_n) \mid_{W_{0,n+2}}(i_1,\ldots,i_n,i_n,j) =$
$\sum_{p_1,\ldots,p_n} B_{p_s} \begin{bmatrix} i_s & i_{s+1} \\ p_s & p_{s+1} \end{bmatrix} B^{p_1\ldots p_{n-1}p_n j}_{p_1\ldots p_{n-1}p_n j}(x)$.
Observe that
$B^{p_1\ldots p_{n-1}p_n j}_{p_1\ldots p_{n-1}p_n j}(x) = B^{p_1\ldots p_{n-1}p_n}_{p_1\ldots p_{n-1}p_n}(x)$
because the last strand is not touched in the resolution process. From the Moore-Seiberg equation we derive the identity
$\sum_j \frac{S(0)_{j1}}{S(0)_{11}} B_{pp} \begin{bmatrix} i & i \\ q & j \end{bmatrix} =$
$= \frac{S(0)_{p1}}{S(0)_{11}} \sum_j B^{-1}_{1j} \begin{bmatrix} p & 1 \\ p & 1 \end{bmatrix} B^{-1}_{j1} \begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} B_{pp} \begin{bmatrix} 1 & 1 \\ q & j \end{bmatrix} \times$

$exp(2\pi\sqrt{-1}(\Delta_j - \Delta_p)) = \frac{S(0)_{p1}}{S(0)_{11}} exp(2\pi\sqrt{-1}(\Delta_p - \Delta_q))\times$
$\sum_j B_{1j}^{-1} \begin{bmatrix} p & 1 \\ p & 1 \end{bmatrix} B_{pp} \begin{bmatrix} 1 & 1 \\ q & j \end{bmatrix} B_{j1}^{-1} \begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} =$
$= \frac{S(0)_{p1}}{S(0)_{11}} exp(2\pi\sqrt{-1}(\Delta_p - \Delta_q)\Omega_{p1}^q(-)\Omega_{p1}^q(-)B_{11}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} =$
$= \frac{S(0)_{p1}}{S(0)_{11}} \delta_{pq^*}.$
This implies that
$J(x,c) = J(xb_n, c)$
proving that J is in fact an invariant for colored links. □

Let f_0 be the blackboard framing of K induced from a braid representative of it. We set
$J(K, f_0, c) = J(K, c).$
An arbitrary framing f differs from f_0 by a sequence of integers $r_1, r_2, ..., r_k$. We define then
$J(K, f, c) = \prod_{j=1}^k exp(2\pi\sqrt{-1}\Delta_{c(j)}r_j)J(K, f_0, c).$
Observe that if we alter the framing f_0 by the same sequence of integers in the first definition of the link invariant then $\rho_1(\tau)$ changes to $\rho_1(\tau)T^{r_j}$, hence
$F(K, f, c) = \prod_{j=1}^k exp(2\pi\sqrt{-1}\Delta_{c(j)}r_j)F(K, f_0, c).$
This says that the framing dependence is the same in the two approaches. Now we can stay the main result of this section

Theorem 4.5.3 *The invariants $F(K, f, c)$ and $J(K, f, c)$ coincide.*

Proof: There is perhaps an explicit description which allows us to pass from $x \in B_n$ to its Artin closure $\hat{x}$, to change the Dehn surgery presentation on $\hat{x}$ into a Heegaard splitting and to recover some $x' \in \mathcal{M}_g$ but it seems to be a complicated one. Our strategy is simpler: we show that these invariants extend to invariants of colored framed 3-valent graphs. Further an analog of Dehn surgery could be defined for such framed graphs. The analog Kirby moves may be described and we derive that the formula of proposition 6.1 gives actually 3-manifold invariants in both cases. Now the corresponding h.t.r. corresponding to the two TQFT are coming from the same RCFT hence the 3-manifold invariants must be the same and our claim will follow.

First step: Let Γ be a connected 3-valent framed graph of genus g embedded in the manifold M^3. A tubular neighborhood $T(\Gamma) \subset M^3$ of

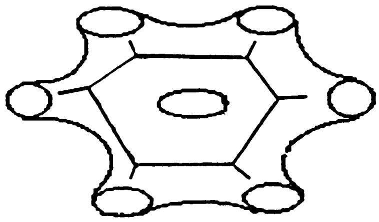

Figure 4.31: The s-holed torus

Γ bounds a genus g surface $\partial T(\Gamma)$. We have a natural cut system on $\partial T(\Gamma)$ obtained in the following manner: over each edge e of the graph there is a cylinder sitting in $T(\Gamma)$ which is a trivial S^1-bundle over e. We consider the meridian $\gamma(e)$ of this cylinders. Their set give a cut system γ_* on $\partial T(\Gamma)$.

Now a coloring of Γ consists in

i) a coloring of its edges $c : E(\Gamma) \longrightarrow L$,

ii) a labelling c of its vertices: assume we have chosen once for all the basis B_{ijk} for the primary block W_{ijk}. Then a vertex $v \in V(\Gamma)$ has three incident edges e_i.We consider that $c(v) \in B_{c(e_1)c(e_2)c(e_3)}$.

Consider now the colored graph Γ having r connected components Γ_i, $i = 1, r$. Assume that the framing gives a rigid structure on $\partial T(\Gamma)$. Then

$F(M - T(\Gamma), \varphi^0(\gamma_*, f)) = v \in \otimes_{i=1}^r F(\partial T(\Gamma_i))$

and

$F(\partial T(\Gamma)) = \oplus_{l\text{labelling}} W(\Gamma, l) = \oplus_i \otimes_{l_i} W(\Gamma_i, l_i)$.

We define

$F(M, \Gamma, f, c) = < v, \otimes_{v \in V(\Gamma)} c(v) > \in \mathbf{C}$.

We wish to define now the Dehn surgery on a framed graph $(\Gamma, f) \subset S^3$. As in the classical case we remove a tubular neighborhood of Γ and glue it back differently

$D(\Gamma, f) = S^3 - T(\Gamma) \cup_{\varphi(f)} T(\Gamma)$,

where $\varphi(f)$ is a homeomorphism depending on the framing f. We have the cut system γ_* on $\partial T(\Gamma)$. Consider an irreducible cycle z (of length s) in the graph Γ. The part of $\partial T(\Gamma)$ sitting over z is a s-holed torus $T(z)$ (see the figure 30). The framing of the loop z describes a longitude

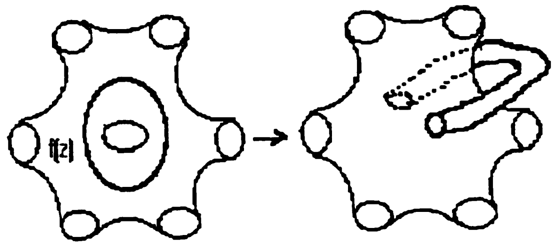

Figure 4.32: Reidentification of the s-holed torus

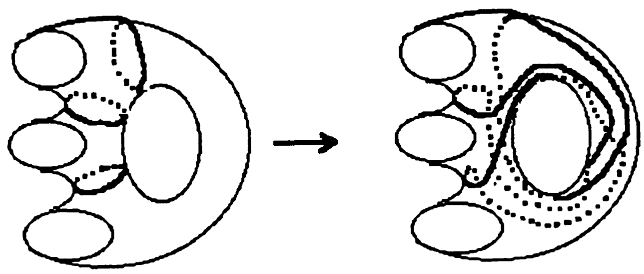

Figure 4.33: The change of the cut system

$f(z)$ of the torus $T(z)$ (avoiding the holes). If we cut the holed torus along $f(z)$ we get a $s+2$-holed torus. We identify again the two new circles but changing the orientation of one of them. We obtain again a s-holed torus (see figure 31). This transformation may be described on a fixed (holed) torus by a change in the cut system preserving the dual graph. Each curve $\gamma(e)$ with e an edge in z is sliding over the 1-handle (see the figure 32). This change on the cut system (see the picture 33) , once it was done for all irreducible cycles, define a homeomorhism $\varphi(f)$ of $\partial T(\Gamma)$. In fact it corresponds to the homeomorphism between the two adjacent rigid surfaces determined by the framings.

We obtain as in 6.1. a decomposition

$F(D(\Gamma, f)) = \sum_{c \text{ coloring}} [c, \Gamma]\, F(S^3, \Gamma, f, c)$

where $[c, \Gamma]$ are certain universal constants. The computation of these constants is done as follows: At the graph level we perform a transformation $S(z)$ for each irreducible cycle which preserves the dual graph

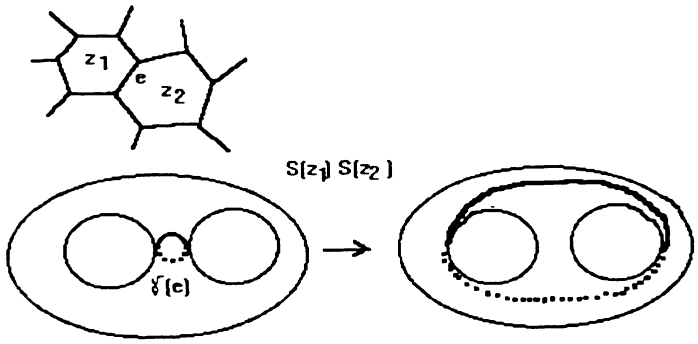

Figure 4.34: Commutativity of cycle transformations

hence we have a mapping

$1 \otimes S(z) \otimes 1 : W(\Gamma, l) \longrightarrow \oplus_{l(e), e \subset z} W(\Gamma, l)$.

If we have two disjoint cycles the associated transformations commute in an obvious manner. But even if the cycles z_1, z_2 are not disjoint the associated transformations commute. It suffices to look at the images of each curve in the cut system. If e is not a common edge of z_1 and z_2 then only one of the transformations $S(z_i)$ changes $\gamma(e)$. If e is a common edge then $S(z_1)S(z_2)\gamma(e)$ is the curve surrounding both 1-handles of the holed genus 2 surface sitting over $z_1 \cup z_2$ (the cycles are irreducible) as can be seen on the figure 34. Further we restrict to a cycle z and look for the expression of $S(z)$. We may perform s fusion moves to change the initial cut system into a cut system having the dual graph with a length 1 loop as in figure 35. Therefore we perform an usual S-move on the 1-loop and we come back using the inverses of the s fusion moves used above. We obtained

$S(z) = \prod_{i=1}^{s-1} F(e_i) S(e_s) \prod_{i=1}^{s} F^{-1}(e_i)$

where $F(e_i)$ is the fusion moves which contracts the edge e_i.

However there is not a local formula for

$[c, \Gamma] = < \prod_z S(z) w_g, \otimes_{v \in V(\Gamma)} c(v) >$

because the labellings change at each cycle transform.

Second step: Also $J(K, f, c)$ extends to 3-valent graphs using the RCFT data. We represent Γ as Artin's closure of a singular braid (as Birman described in the case of 4-valent graphs). A singular braid is the composition of

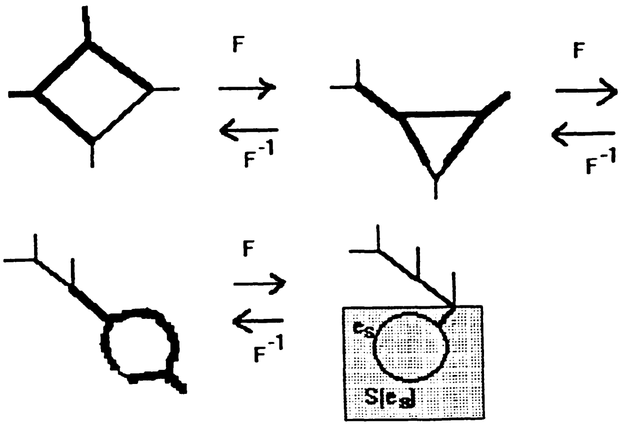

Figure 4.35: Getting $S(z)$ from elementary moves

Figure 4.36: The vertex elements

1) usual braid elements giving a crossing in a generic plane projection,

2) vertex elements as in figure 36.

Now the resolution of crossings must take into account the vertices. The two graphical rules from figure 37 give the resolution of vertices. One caution is need: when we pass from the space associated to the upper line (indexed by $p_1, \dots, p_{n-1}$) to the bottom line, when we encounter a vertex the vector space changes at this level. The change consists into a tensor product with W^i_{jk} (i, j, k are the labels of the three edges incident to the vertex v). We shall identify then the element $x \in W^k_{lm}$ with $x \otimes c(v) \in W^k_{lm} \otimes W^i_{jk}$.

After all singularities are inductively solved we obtain a matrix $\tilde{B}^{p_1 \dots p_{n-1} j}_{q_1 \dots q_{n-1} j}(x)$ analog to $B^{p_1 \dots p_{n-1} j}_{q_1 \dots q_{n-1} j}(x)$. The formula

$J(\Gamma, f_0, c) = \left(\prod_{s=1}^n \frac{S(0)_{11}}{S(0)_{i_s 1}}\right) \sum_{j=1}^n \frac{S(0)_{j1}}{S(0)_{11}} \sum_{p_1, \dots, p_n} \tilde{B}^{p_1 p_2 \dots p_{n-1} j}_{p_1 p_2 \dots p_{n-1} j}(x).$

gives a topological invariant for the colored graph γ (the closure of

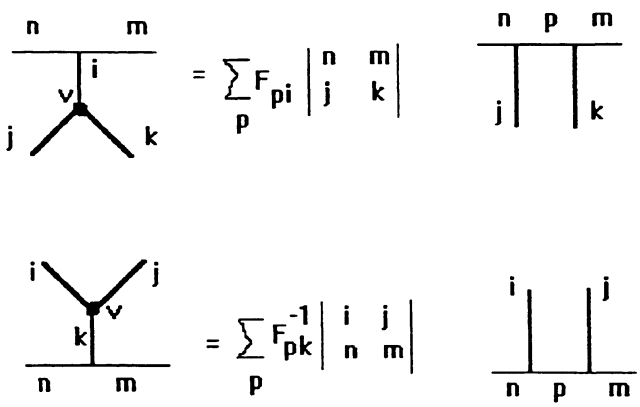

Figure 4.37: The resolution of vertices

the singular braid x). This can be derived immediately from the Reidemester's moves for 3-valent graphs. Alternatively the method of Degiovanni ([Deg92] appendix B.1) gives essentially the same invariant. The change of the framing is the same as in the case of links. In fact it will be clear from below that we may always replace a graph surgery by a link surgery.

Set now

$J(D(\Gamma, f)) = \sum_{c \text{ coloring}} [c, \Gamma]\, J(\Gamma, f, c)$

We claim that J defines a topological invariant for 3-manifolds. We need the analog of Kirby moves for graph surgery. Away from the usual K-move given in figure 38 we have another move which permits the reduction of the number of loops in the graph. We choose an irreducible cycle z in Γ having the length $s > 1$. If we have an edge between two distinct vertices we can push one vertex along e in order to get an unknotted edge e in S^3. This may be done for all but one the edges of the cycle z. Let e_0 be the edge which remains knotted. Eventually changing the cut system (hence the framing) we perform fusion moves at the graph level which kill the unknotted edges one by one. We arrive at a graph with the cycle z replaced by a single loop e_0. Moreover this loop is disjoint from the rest of the graph.

Consider now (Γ, f) a 3-valent (framed) graph and (K, f_K) a disjoint framed link. Choose two points $x \in \Gamma$, and $y \in K$ and an unknotted

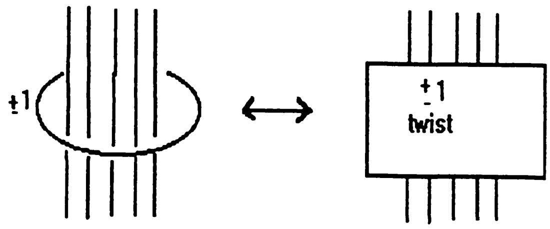

Figure 4.38: The K-move

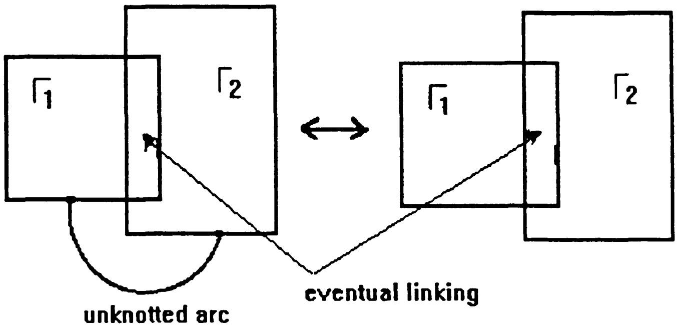

Figure 4.39: Stabilization-Destruction move

arc a between x and y. Then $\Gamma' = \Gamma \cup K \cup a$ is again a 3-valent graph with a natural framing $f' = f \cup f_K$. We claim that

$D(\Gamma', f') \simeq D(\Gamma \cup K, f \cup f_K)$,

where on the right hand side we have a disjoint union. This is clear from the definition of the graph surgery. So when we try to kill all the loops in the graph Γ we arrive at a g-component link. So the second allowed move is that from picture 39. If we apply directly the theorem of Kirby [Kir78] it follows that two surgery presentations are equivalent under the equivalence relation given by these two moves, because we may restrict to the link presentations.

Now the same reasoning as in [Deg92] permits to obtain the invariance of J under this generalized Kirby moves.

Third step: We prove that $F(M) = J(M)$ for closed 3-manifolds M. Both are multiplicative invariants which are therefore determined by some h.t.r. ρ_* and $\rho_{J,*}$ respectively.

1) In the definition of $J(M)$ the conformal blocks $W_{J,g} = W(\partial T(\Gamma))$

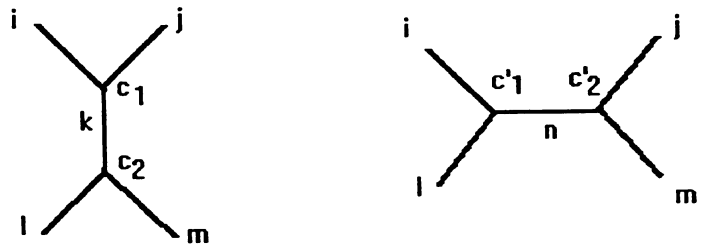

Figure 4.40: The fusion move on the surface

for a genus g graph may be identified with $\oplus_l W(\Gamma, l)$ hence with W_g, or eventually with a quotient if the representation ρ_J splits. Also the label set coincide with L and the primary blocks must be the same.

2) The change of framing is given in both cases by some function on the colors, Γ and the conformal weights. Therefore Δ_j are the same.

3)The fusion matrix corresponds to a change on the cut system of $\partial(M - T(\Gamma))$ as in figure 40. This is the same to allow a move on the graph level $\Gamma \to \Gamma'$ where we assumed that the edge labelled k is unknotted. From the definition of $J(\Gamma, f_0, c)$ we derive that

$$J(\Gamma, f_0, c) = \sum < F_{kn} \begin{bmatrix} i & j \\ l & m \end{bmatrix} (c_1 \otimes c_2), c_1' \otimes c_2' > J(F\Gamma, f, c').$$

This proves that the fusion matrices are equal in both approaches.

4) The S-matrix comes from the constants $[c, \Gamma]$ hence it must coincide.

5) Finally the vacuum vector is unique.

Therefore $F(M) = J(M)$ for closed 3-manifolds. But F extends canonically to manifolds with boundary hence $F(M - T(\Gamma)) = J(M - T(\Gamma))$ and our claim follows. □

Corollary 4.5.4 *A TQFT is determined by the matrices S, T and the braid matrices S. Equivalently the associated invariants for colored links determine uniquely the TQFT.*

Remark that if the primary blocks W_{jk}^i have dimension 0 or 1 for all labels then we can drop the coloring of vertices. This is the case for example in the quantum (or classical) $SU(2)$-theory. In particular the invariant

$$F_{SU(2)}(S^3 - T(K)) = \sum_{c \text{ coloring}} J_{SU(2)}(K, c) \otimes_{j=1}^{k} e_{c(j)}$$

where the terms on the right hand side are the Jones polynomials for colored links. These are expressed in terms of cablings (see [KM91]) of the link K..

We wish finally to derive a general property fulfilled by the unitary link invariants coming from the RCFT. When all colors to the components are the same $j \in L$ we get an usual Markov trace $t_j : \mathbf{C}[B_\infty] \longrightarrow \mathbf{C}$. But this Markov trace factors through a filtered quotient P_k of the group algebra of B_k (for each k) which is finite dimensional matrix algebra. In fact we can take for P_k the endomorphism algebra $End(\oplus_{i\in L} W_{0,k+1}(j,j,...,j,i))$. Now this P_k is a $\mathbf{C}^*$-algebra since the representation $\rho_{0,n}$ is unitary. We claim that

$tr(xx^*) \geq 0,$

so the trace is positive. This may be proved directly, but the simplest way is to use the formalism of [Fun93b]. Any link invariant is expressed as

$t(x) =< w_{0,2n}, \tilde{\rho}_{0,2n}(\tilde{x}) w_{0,2n} >$

for a plat representative $\tilde{x}$ of the Artin closure of x. However the representation $\tilde{\rho}_{0,2n}$ is the same as that described above and the vacuum vector $w_{0,2n}$ is $e_j^{\otimes 2n}$. It corresponds to the standard semi-link with 2n endpoints (see [Fun93b] for details). Now the positivity of the trace is straightforward.

Define generally the definition quotient $D(t)$ of a Markov trace t be the smallest nontrivial homogeneous quotient on which t factors.

Proposition 4.5.5 *Let t be a Markov trace coming from an unitary RCFT. Since t is positive it defines an hermitian product on $D(t)$. Let $\overline{D(t)}$ be the completion of $D(t)$ with respect to the hermitian product. Then $\overline{D(t)}$ is isomorphic to the hyperfinite II_1-factor.*

The proof is straightforward: $\overline{D(t)}$ is a von Neumann algebra by construction which has a Markov trace (unique). Since it is an hyperfinite factor (a quotient of P_∞) it is the hyperfinite II_1-factor. □

Consider now that the orientation of each component of the link L is reversed. This is the same thing to consider that the colors $c(j)$ change into $c(j)^*$. When changing all the labels in L to their $*$-conjugated the data F,Ω,Θ, S,T pass to the dual matrices and the primary blocks $W_{j^*k^*}^{i^*}$ pass to their duals $(W_{jk}^i)^*$. This together with the unitarity imply that

$J(\overline{K}, c) = \overline{J(K, c)}.$

This means that

Remark 4.5.6 *The unitary RCFT invariants for colored links do not detect the non-invertibility of links.*

Therefore in order to have a family of complete invariants for links we must allow non-unitary RCFT. For example the RCFT derived from a quantum super-group yield indefinite nondegenerate forms.

We end this section by

Conjecture 4.5.7 *The Markov trace I defined in [Fun93d] for the cubic Hecke algebra* $\mathbf{C}[B_\infty]/(b_i^3+1=0, i>0)$ *is positive and the completion of* $D(t)$ *in this hermitian product is a* II_1 *factor which is not isomorphic to the hyperfinite factor.*

4.6 Abelian RCFTs

We say that a RCFT is abelian if we have the isomorphisms of vector spaces

$W_g \simeq W_1^{\otimes g}$.

Proposition 4.6.1 *An abelian RCFT is determined by a finite abelian group structure on L such that* $W_{hk}^g = \mathbf{C}$ *iff* $g = h + k$, *and otherwise it vanishes. The unity is 0 and the involution * corresponds to taking the inverse.*

Proof: In genus one we have $n_1 = card(L)$. Further $n_2 = \sum_{i,j,k} n_{ji}^i n_{jk}^k$. This implies $n_{ji}^i = 0$ if $j \neq 0$. This proves also that 0 is a unit. From the expression of n_3 we derive $n_{jk}^i \in \{0,1\}$ and for fixed j,k there is an unique i with $n_{jk}^i = 1$. We denote it by $j+k$. Since $n_{jk}^i = n_{kj}^i$ this law is commutative. The associativity follows from the fact that F is an isomorphism. Also $n_{jk}^i = n_{ji^*}^{k^*}$ so $k^* = -k$. □

The RCFT determined by a finite group were treated by Dijgraaf and Witten in [DW90], the abelian theories were classified by Moore and Seiberg in the appendix 3 of [MS89], and the general case was settled by Freed and Quinn [FQ93].

In particular it follows that the h.t.r. associated factors through the symplectic groups

$\rho_g : \mathcal{M}_g \longrightarrow Sp(2g, \mathbf{Z}) \longrightarrow U(W_g)$.

The basic data is $(S(0), T)$ since $S(j) = 0$ for $j > 0$ and the fusion matrix is 1. In particular the conformal weights Δ_j vanish. We can compute the h.t.r. in terms of generators of the symplectic group as

$$\rho_g \begin{pmatrix} A & 0 \\ 0 & {}^T A^{-1} \end{pmatrix} = (\delta_{{}^T A\lambda,\mu})_{\lambda,\mu \in L^g},\ a \in GL(g, \mathbf{Z})$$

$$\rho_g \begin{pmatrix} 1_g & B \\ 0 & 1_g \end{pmatrix} = 1$$

for a symmetric matrix B with integer entries,

$$\rho_g \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = S(0)^{\otimes g}.$$

If we allow projective unitary representations, or unitary invariants for framed 3-manifolds there are more interesting examples. In particular the conformal weights may be nonzero. The general form of these representations (we normalize them to be true representations but allowing that the vacuum vector be invariant up to a character) is

$$\rho_g \begin{pmatrix} A & 0 \\ 0 & {}^T A^{-1} \end{pmatrix} = (\delta_{{}^T A\lambda,\mu})_{\lambda,\mu \in L^g},\ a \in GL(g, \mathbf{Z})$$

$$\rho_g \begin{pmatrix} 1_g & B \\ 0 & 1_g \end{pmatrix} = diag(exp(2\pi\sqrt{-1}\Delta_x < Bx, x >))_{x \in L^g}$$

for a symmetric matrix B with integer entries, with $\Delta_x = \sum_{i=1}^g \Delta_{x_i}$, the scalar product being the natural one on L^g and $\Delta_j card(L) \in \mathbf{Z}$ for all j.

$$\rho_g \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = S^{\otimes g},$$

where S and $T = diag(exp(2\pi\sqrt{-1}\Delta_x))_{x \in L}$ give a $SL(2, \mathbf{Z})$ representation.

In particular we get the abelian Witten's theory for the gauge group $G = U(1)$ (or equivalently the $\mathbf{Z}/k\mathbf{Z}$-theory) and the family of theories obtained by the semi-abelian quantization in [Fun93e, Fun93g].

The invariants for 3-manifolds we get in the canonical framing are not longer multiplicative invariants. Their modulus corresponds to a multiplicative invariant and therefore is an homotopic invariant determined by the first Betti number and the torsion pairing on $Tors(H_1(M, \mathbf{Z}))$. When also the phase factor is taken into account we obtain in particular the Witten's invariants for torus bundles and lens spaces (see [Fun93e, Jef92] hence even in the abelian setting we can obtain non-homotopic invariants.

It seems that any h.t.r. ρ_* which factors through the tower of symplectic groups has a fully invariant tensor subspace hence it reduces to an abelian RCFT.

4.7 Open 3-manifolds

We shall give a brief insight of what an extension of the TQFT to open 3-manifolds may be. Actually the functor F which associates to each surface a vector space may be extended to some category of closed sets having Haussdorff dimension at most 2. This direction will be detalied elsewhere.

Let W^3 be an (oriented) open 3-manifold without boundary. Consider an exhaustion of W^3 by compact submanifolds

$... \subset K_n \subset int(K_{n+1}) \subset K_{n+1} \subset int(K_{n+2}) \subset ...,$

$W^3 = \bigcup_n K_n.$

The closures $V_i = cl(K_{i+1} - K_i)$ are cobordisms between ∂K_i and ∂K_{i+1}. Consider Z be an arbitrary TQFT having the properties

$F(S^2) = \mathbf{C}.$

All TQFTs with irreducible vacua satisfy this condition. Then we have an inductive system of vector spaces $(Z(\partial K_i), Z(V_i))$. We set

$Z_\infty(W) = lim_{\rightarrow}(Z(\partial K_i), Z(V_i)).$

Definition-Lemma 4.7.1 *The vector space $Z_\infty(W)$ is a topological invariant for open 3-manifolds.*

Remark that this invariant depends only on the structure of W at infinity. If W' is another open 3-manifold, such that $cl(W - K)$ is homeomorphic to $cl(W' - K')$ for some compact submanifolds $K \subset W$ and $K' \subset W'$, then $Z_\infty(W) = Z_\infty(W')$. We may prove that it is naturally associated to the space of ends of W.

Definition 4.7.2 *The open manifold W is homologically 1-connected at infinity if any compact $K \subset W$ can be engulfed into a compact submanifold Y such that $K \subset int(Y)$ and $H_1(Y) = 0$.*

For this class of manifolds the invariants at infinity Z_∞ are easy to compute. In fact we may state:

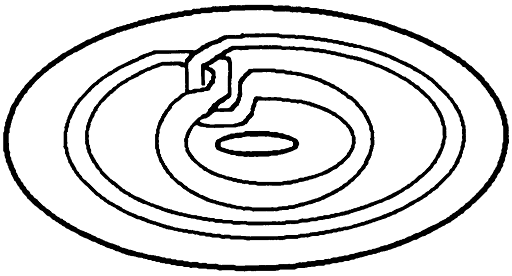

Figure 4.41: $T_1 \hookrightarrow T_0$

Proposition 4.7.3 *For an open 3-manifold W homologically 1-connected at infinity we have* $dim(Z_\infty(W)) \leq 1$.

Proof: If Y is a compact 3-manifold with boundary and $H_1(Y) = 0$ then its boundary components are spheres. In fact the Euler-Poincare characteristic of the double of Y is two times the sum of genera of the boundary components of Y. So all the genera vanish.

Let now K_n be an exhaustion of W as above. For each n we can include K_n into some compact submanifold Y_n having the boundary a disjoint union of spheres. Further each Y_n is contained into $K_{r(n)}$ for large $r(n)$. We know that

$Z(\partial Y_n) = \bigotimes Z(S^2) = \mathbf{C}$.

Since $Z(cl(K_{r(n)} - K_n)$ factors through $\mathbf{C}$ for all n our claim follows. $\square$.

We are interested now in the case of Whitehead manifold Wh. Let $T_1 \subset T_0$ be the embedding of solid tori from figure 41. There exists an homeomorphism h of the sphere S^3 such that $h(T_0) = T_1$. Therefore the Whitehead manifold Wh is defined as

$S^3 - \bigcup_{n \geq 0} h^n(T_0)$.

It is the typical example of a contractible open manifold which is not simply connected at infinity. Our aim is to compute $Z_\infty(Wh)$. Set $V = cl(T_0 - T_1)$ which is a cobordism between two tori. By definition

$Z_\infty(Wh) = lim_\rightarrow(Z(\Sigma_1), Z(V))$.

We can compute $Z(V)$ using the method from section 5. As a cobordism

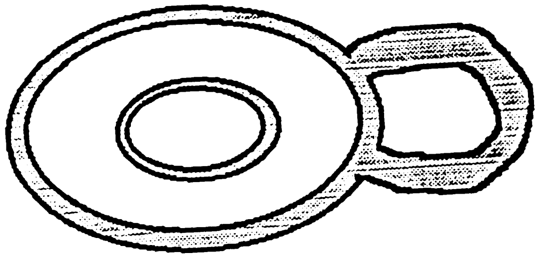

Figure 4.42: R_1

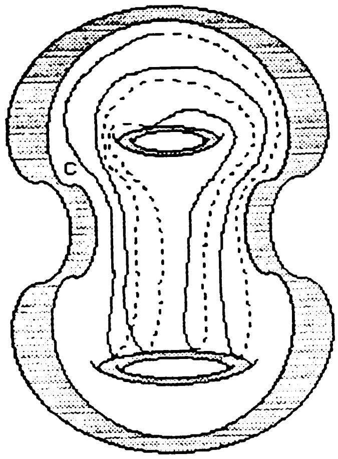

Figure 4.43: R_2 and the attaching curve c

V is the composition of two simple cobordisms $R_1 \cup R_2$ where $R_1 = \Sigma_1 \times [0,1] \cup$2-handle,

$R_2 = \Sigma_2 \times [0,1] \cup$1-handle attached on the curve c,
where the curve $c \hookrightarrow \Sigma_2 \times \{1\}$ is depicted in figure 43. Equivalently $\bar{R}_1$ is

$R_1 = \Sigma_2 \times [0,1] \cup$1-handle attached on the curve l,
where $l \hookrightarrow \Sigma_2 \times \{0\}$ is drawn in figure 44. The cut systems change according to the graphical rules from figure 45. Then $Z(V)$ is the following composition:

$Z(\Sigma_1) \cong \bigoplus W_{1i}^i \otimes W_{ij}^j \hookrightarrow \bigoplus W_{kj}^i \otimes W_{ij}^j \rightarrow$

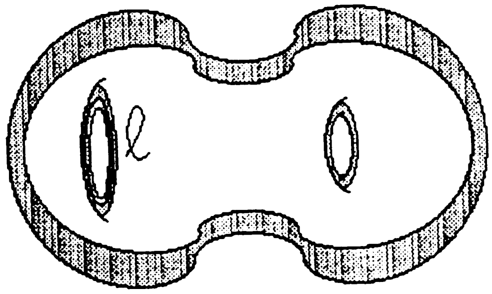

Figure 4.44: $\bar{R}_1$ and l

$$\xrightarrow{1\otimes S(k)} \bigoplus W_{il}^i \otimes W_{ij}^j \xrightarrow{\oplus F_{im}\begin{bmatrix} j & j \\ l & l \end{bmatrix}} \bigoplus W_{jl}^m \otimes W_{jl}^m \rightarrow$$

$$\xrightarrow{\oplus B_{mn}^2 \begin{bmatrix} j & j \\ l & l \end{bmatrix}} \bigoplus W_{jl}^n \otimes W_{jl}^n \xrightarrow{\pi} \bigoplus W_{jl}^1 \otimes W_{jl}^1 \cong Z(\Sigma_1)$$

where B is the braiding matrix and π is the canonical projection.

We apply this for Z being TQFT associated to the group $SU(2)$ in level $k = 2$ described in [Koh92]. We have $L = \{0, \frac{1}{2}, 1\}$, the identity is 0, and

$W_{00}^0 = \mathbf{C}f_0, W_{\frac{1}{2}\frac{1}{2}}^0 = \mathbf{C}f_{\frac{1}{2}}, W_{11}^0 = \mathbf{C}f_1, W_{1\frac{1}{2}}^{\frac{1}{2}} = \mathbf{C}e.$

The permutations of indices correspond to isomorphisms and the other vector spaces W_{jk}^i are 0. Fix the decompositions having the dual graphs from figure 46.

Therefore:

$Z(\Sigma_1, \delta_1) = \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1 \overset{not}{=} V,$

$Z(\Sigma_2, \delta_2) = \oplus_{i,j}\mathbf{C}f_i \otimes f_j \oplus \mathbf{C}e \otimes e \cong V \otimes V \oplus \mathbf{C}e \otimes e.$

The only nontrivial fusion matrices are:

$F_{00}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} : \mathbf{C}f_1 \otimes f_1 \longrightarrow \mathbf{C}f_1 \otimes f_1,$

$F_{01}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} : \mathbf{C}f_1 \otimes f_1 \longrightarrow \mathbf{C}e \otimes e,$

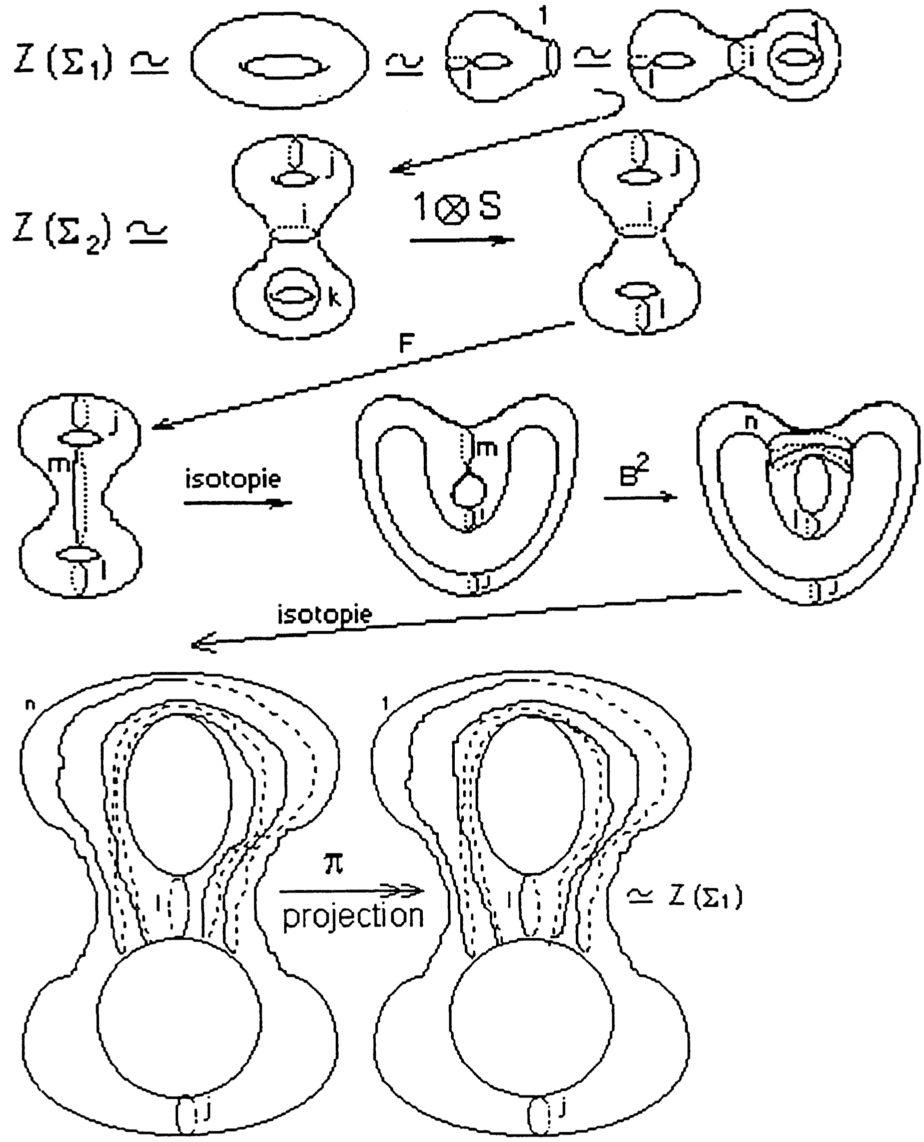

Figure 4.45: The cut system change

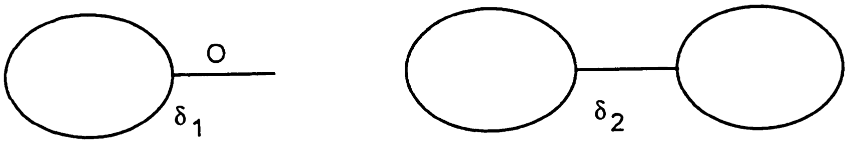

Figure 4.46: δ_1 and δ_2

$F_{10}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} : \mathbf{C}e \otimes e \longrightarrow \mathbf{C}f_1 \otimes f_1,$

$F_{11}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} : \mathbf{C}e \otimes e \longrightarrow \mathbf{C}e \otimes e$

where (in terms of the chosen bases)

$\left(F_{ij}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$

The matrix $S(0) : \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1 \longrightarrow \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1$ is given by

$S(0) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

and $S(1) : \mathbf{C}e \otimes e \longrightarrow \mathbf{C}e \otimes e$ is the scalar multiplication by $exp\left(\frac{-3\pi\sqrt{-1}}{4}\right)$.

From the results of Kohno we derive that

$\left(B^2_{ij}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}.$

At the (dual) graph level the transformations are described in figure 47. Observe that the projection π identifies $f_i \otimes f_i$ with f_i. Therefore $Z(F)f_0$ may be computed as follows:

$f_0 \to f_0 \otimes f_0 \to \frac{1}{2}(f_0 \otimes f_0 + f_0 \otimes f_{\frac{1}{2}} + f_0 \otimes f_1) \to$

$\to \frac{1}{2}(f_0 \otimes f_0 + f_0 \otimes f_{\frac{1}{2}} + f_0 \otimes f_1) \xrightarrow{\pi} \frac{1}{2}f_0 = Z(F)f_0,$

because the fusing and braiding matrices act trivially on the third vector. In a similar manner we get

$Z(F)f_{\frac{1}{2}}) = \frac{1}{\sqrt{2}}f_{\frac{1}{2}}.$

Finally we have

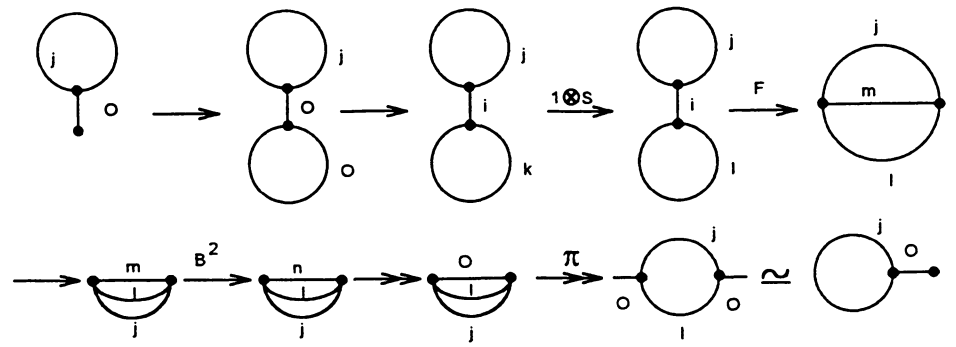

Figure 4.47: The graph transformations

$f_1 \to f_1 \otimes f_0 \to \frac{1}{2}(f_1 \otimes f_0 + \sqrt{2} f_1 \otimes f_{\frac{1}{2}} + f_1 \otimes f_1) \to$
$\xrightarrow{F} z - \frac{1}{2\sqrt{2}}(f_1 \otimes f_1 + e \otimes e) \xrightarrow{B^2} z + \frac{1}{4}(-1 + \sqrt{-1}) f_1 \otimes f_1 + \frac{1}{4}(1 - \sqrt{-1}) e \otimes e \to$
$\xrightarrow{\pi} \frac{1}{4}(-1 + \sqrt{-1}) f_1 = Z(F) f_0,$
where z is a vector lying in the kernel of π. Remark that π vanish on all (basis) vectors which are not corresponding to the marked graphs from figure 48. Also the graph associated to $e \otimes e$ is given in figure 49 hence $e \otimes e$ lies in the kernel of π. We obtain $Z(V) : \mathbf{C}^3 \longrightarrow \mathbf{C}^3$ be

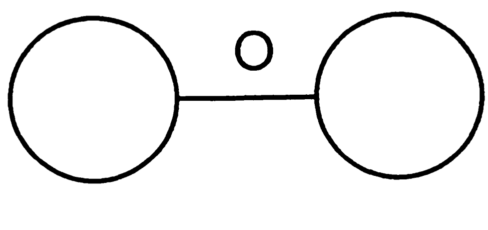

Figure 4.48: The graphs preserved by π

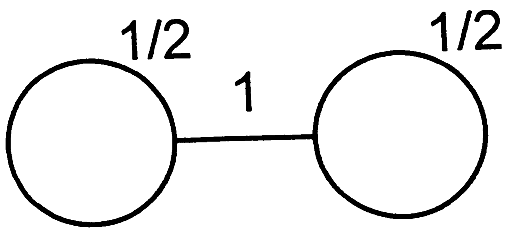

Figure 4.49: The graph associated to $e \otimes e$

given by the formula:

$Z(F) = diag(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{4}(-1+\sqrt{-1}))$.

It is easy to see that $Z_\infty(Wh) \simeq \mathbf{C}^3$ hence the extension at infinity of a TQFT is not trivial.

Remark that the end of Wh is a Cantor set. So the complete extension of the TQFT may associate even to a Cantor set a finite dimensional vector space.

Chapter 5

The homogeneity of Markov traces

5.1 Introduction

There are several ways, essentially equivalent, to look at a link $L \subset S^3$, all of them dealing with objects related to the braid group B_*. The most common description is the Artin's closure of a braid $\alpha \in B_n$ denoted by $\hat{\alpha}$ (see the figure 1) We remark that the group B_* has a natural and nonambiguous tensor group structure. The morphism $\otimes : B_n \otimes B_k \longrightarrow B_{n+k}$ is simply given by putting together the strands corresponding to the two geometric braids, as follows from picture 2.

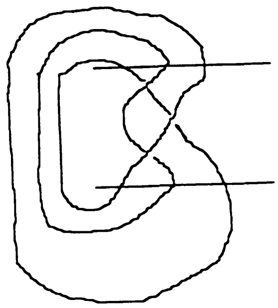

Figure 5.1: The Artin closure of braids

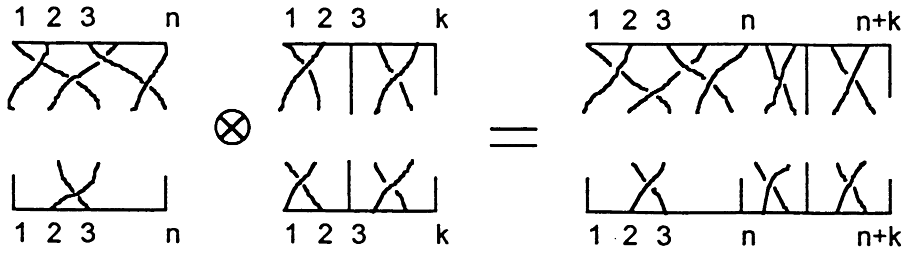

Figure 5.2: The tensor structure on B_*

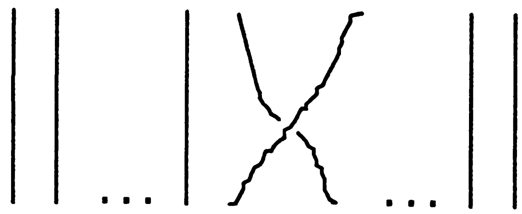

Figure 5.3: Geometric view of b_i

Proposition 5.1.1 *B_* is a (noncommutative) tensor group with one generator, namely*

$t = b_1 \in B_2$

and one relation helding in B_3:

$$t(1 \otimes t)t = (1 \otimes t)t(1 \otimes t). \tag{5.1}$$

Proof: We precise that $1 \in B_1$ consists in one strand. Next remember that B_n has a finite group presentation

$B_n =< b_1, b_2, ..., b_{n-1} \mid b_i b_j = b_j b_i$, if $\mid i - j \mid \geq 2, b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, i = 1, 2, ..., n-2 >$.

where b_i is represented by the element from figure 3. But we observe that

$b_i = \overbrace{1 \otimes 1 \otimes ... \otimes 1 \otimes t}^{i} \otimes 1 \otimes ... \otimes 1$

and therefore the commutativity relations $b_i b_j = b_j b_i$, (if $\mid i - j \mid \geq 2$) are consequences of the compatibility between the tensor structure and the group structure i.e.$(x \otimes y)(z \otimes t) = (xz \otimes yt)$. Also the other one relation comes from (3.1).□

Consequence 5.1.2 *Let W_* be a tensor vector space.There is a bijection between*
$\{B_* \longrightarrow U(W_*)\} \leftrightarrow \{B_2 \longrightarrow U(W_2)$ *which lift to* $B_3 \longrightarrow U(W_3)\}$

We have a result of Markov saying that two braids have isotopic Artin closures if and only if they are equivalent under the equivalence relation generated by the following elementary moves:

$$x \cong cxc^{-1}, \text{ where } x, c \in B_n. \tag{5.2}$$

$$x \cong (x \otimes 1)b_n \cong (x \otimes 1)b_n^{-1}, \text{with } x \in B_n. \tag{5.3}$$

We can consider the K-algebra of link invariants consisting in the maps $f_* : B_* \longrightarrow K$ satisfying:
$f_n(x) = f_n(cxc^{-1}), x, c \in B_n$ for all n.
$f_n(x) = f_{n+1}((x \otimes 1)b_n^{\varepsilon}), \varepsilon \in \{-1, 1\}, x \in B_n$
We may identify B_n with a subgroup of B_{n+1} via the map $x \longrightarrow x \otimes 1$ and thereafter we can think f_* as a single map
$f : B_\infty = \bigcup_{n>0} B_n \longrightarrow K$
We shall that f_* is a multiplicative invariant if the following condition
$f_{n+k}(x \otimes y) = f_n(x) f_k(y)$
is fulfilled for any $x \in B_n, y \in B_k$.

Proposition 5.1.3 *If K is an uncountable field the K-algebra of multiplicative invariants for links ML_K is complete.*

Proof: We remember that a link L is called split if $L = L_1 \cup L_2$ such that there exists a sphere S^2 embedded in S^3 which separates L_1 from L_2. This is equivalent to saying that we may write $L = \hat{\alpha}$ where $\alpha = x \otimes y$. In fact if the above condition is satisfied we choose an intermediate strand separating x from y, which may be lifted to a plane in R^3 separating $\hat{x}$ from $\hat{y}$, so L is split. From now one since any split link may be uniquely splitted in nonsplit components the proof is immediate. □

Definition 5.1.4 *A Markov trace on* $\mathbf{C}[B_\infty]$ *is a linear functional, say* tr*, satisfying the conditions bellow:*

$$tr(1) = 1. \tag{5.4}$$

$$tr(b_j) = z, tr(b_j^{-1}) = \bar{z}, \textit{ with } z, \bar{z} \in C^*. \tag{5.5}$$

$$tr(x \otimes y) = tr(x)tr(y). \tag{5.6}$$

$$tr(ab_n) = z\,tr(a), tr(ab_n^{-1}) = \bar{z}\,tr(a), \textit{for } a \in B_n. \tag{5.7}$$

$$tr(ab) = tr(ba), \textit{for } a, b \in B_n. \tag{5.8}$$

The trace is called a strong trace if the parameters $z, \bar{z}$ *equal* 1.

We observe that the C-algebra ML_C could be identified with the set of strong Markov traces. However there is a simple procedure to convert any Markov trace into a strong one by the formula:

$$\tilde{tr}(x) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} tr(x), \tag{5.9}$$

where $x \in B_n$, and $e(x)$ is the sum of the exponents of the b_i's appearing in the word x. The same definition works for a quotient of $\mathbf{C}[B_\infty]$. Remark that a Markov trace on a quotient is converted into a link invariant but it may happens that this invariant be not a linear functional on the quotient in order to get a strong Markov trace on the last one.

Remark 5.1.5 *The usual Markov traces on Hecke algebras (of quadratic type) [Jon87] may be interpreted as Markov traces in our setting using the projection*

$\mathbf{C}[B_\infty] \longrightarrow \lim_{n\to\infty} H(q,n)$

Recall also that $tr(ab) = tr(a)tr(b)$ *if* $a \in B_n$ *and* b *belongs to the algebra generated by* $1, b_n, b_{n+1}, ..., b_{n+k-1}$. *This follows from the fact that any* $x \in H(q,n)$ *may be written as a sum of words each one containing at most one* b_{n-1}. *This property is specific to quadratic Hecke algebras.*

Set $Q \in \mathbf{C}[X]$ for a monic polynomial having roots different from zero and set $H(Q,n)$ for the (generalized Hecke) algebra presented by

$< 1, b_1, b_2, ..., b_{n-1} \mid b_i b_j = b_j b_i, \mid i-j \mid > 1;$

$b_{i+1}b_ibi+1 = b_ib_{i+1}b_i, i = 1,2,...,n-2; Q(b_i) = 0, i = 1,2,...,n-2 >$

There is an induced tower of inclusions

$\subset H(Q,n) \subset H(Q,n+1) \subset ...$

whose union (or inductive limit) we denote by $H(Q,\infty)$. If we should have a Markov trace t on $\mathbf{C}[B_\infty]$ which, when regarded only on the first stage $\mathbf{C}[B_2]$ it factors through $H(Q,2)$ then t factors through all of the tower of Hecke algebras hence on $H(Q,\infty)$. The reason is that all the generators b_i are conjugated with each other. Our main question is whenever this property can be extended to other quotients of the group algebra. A (filtered)quotient of $\mathbf{C}[B_\infty]$ will be in the sequel a sequence of quotients which makes the following diagram commutative:

$$\begin{array}{ccccccc} \mathbf{C}[B_2] & \hookrightarrow & \mathbf{C}[B_3] & \hookrightarrow & \mathbf{C}[B_4] & ... \\ \downarrow & & \downarrow & & \downarrow & .... \\ P(2) & \rightarrow & P(3) & \rightarrow & P(4) & ... \end{array}$$

Notice that the bottom arrows are not necessary injective. We set $P(\infty)$ also for the inductive limit of the system $P(n)$.

The (filtered) quotient $P(\infty)$ of $\mathbf{C}[B_\infty]$ is homogeneous if any identity

$F(b_i, b_{i+1}, ..., b_j) = 0,\ F \in \mathbf{C}[X_0, X_1, ..., X_{j-i}]$

which holds in $P(\infty)$ remains valid under the translation of indices i.e. also

$F(b_{i+k}, b_{i+k+1}, ..., b_{j+k}) = 0$, for $k \in \mathbf{Z}, k \geq 1-i$.

The rank of the quotient is the first rank where a nontrivial relation appears.

Consider now $P(2) \rightarrow P(3) \rightarrow ... \rightarrow P(k)$ be a (filtered) quotient of $\mathbf{C}[B_k]$. Then $P(k)$ defines a filtered homogeneous quotient $P^h(*)$ of $\mathbf{C}[B_\infty]$ by adding all missing relations in every rank. We call it the homogeneous quotient defined by $P(k)$.

The main result of this paper is

Homogeneity Theorem 5.1.1 *If a Markov trace, defined on $\mathbf{C}[B_\infty]$, factors through the quotient $P(k)$ of $\mathbf{C}[B_k]$ (for a fixed k) then it factors through all the homogeneous quotient $P^h(\infty)$.*

Some nontrivial applications of the result are obtained in [Fun93d].

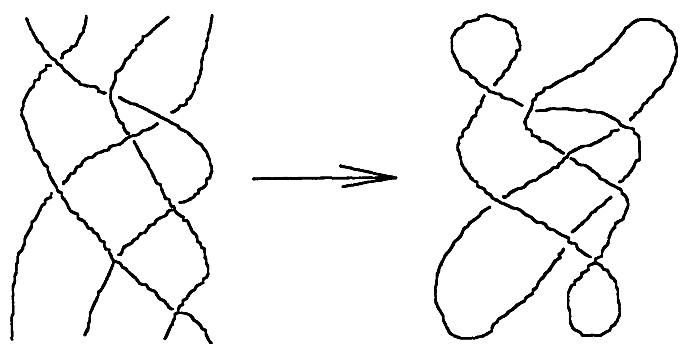

Figure 5.4: The plat closure of α

5.2 Plats and concentric links

We shall continue by describing the other approach of obtaining links, very similar to that for closed 3-manifolds, by means of plats. We begin with the nonoriented case: a plat will be a closed, unoriented link $\tilde{\alpha}$ that is formed when a braid $\alpha \in B_{2n}$ is closed according the pattern from figure 4. Then any unoriented link is of the form $\tilde{\alpha}$ for some $\alpha \in B_{2n}$. We have an analog of Markov's theorem which is close to Reidemester's description of Heegaard splittings (see[Bir74]):

Proposition 5.2.1 *For each m let B_{2m}^{+} be the subgroup of B_{2m} generated by*

$b_{2i-1}, i = 1, m; b_{2i}b_{2i-1}b_{2i+1}b_{2i}, i = 1, m-1; b_{2i}b_{2i-1}b_{2i+1}^{-1}b_{2i}^{-1}, i = 1, m-1$;.

Let also the map $i : B_{2m} \longrightarrow B_{2m+2}$ be defined by $i(\alpha) = \alpha b_{2m}$. Then two plats $\tilde{\alpha}$ and $\tilde{\beta}$ with $\alpha, \beta \in \cup_m B_{2m}$ are isotopic (as unoriented links) if and only if α and β are equivalent under the equivalence relation on $\cup_m B_{2m}$ generated by the following elementary moves:

$\alpha \cong x\alpha y, \alpha \in B_{2k}, x, y \in B_{2k}^{+}$.

$\alpha \cong i(\alpha)$.

It is known that $\tilde{\alpha}$ only depends on the image of α in the mapping class group $\mathcal{M}_{0,2m}$ of the pointed sphere $S^2 - \{p_1, p_2, ..., p_{2m}\}$. Now we can exhibit a (little bit more general) construction of unoriented links which also parallels Heegaard's splittings. If we consider $c_i, i = 1, m$ being disjoint, unlinked and unknotted arcs in the ball B^3 and having endpoints $\partial c_i = \{p_{2i-1}, p_{2i}\} \subset S^2$ we think the pair $L_m = (B^3, \cup_i c_i)$ as an analog of the standard handlebody H_m, and call him a standard semi-link. Now the pair (S^3, K) , with K a link, may be obtained by gluing two such standard semi-links according to a homeomorphism which identifies the boundaries, hence from $Homeo(S^2, \{p_1, p_2, ..., p_{2k}\})$. Of course only its image in $\mathcal{M}_{0,2k}$ does matter.

Observe now that B_{2*} is a tensor subgroup of B_*. Notice that the unit $1_2 \in B_2$ is a 2-string and not a 1-string braid.

Proposition 5.2.2 *The tensor group B_{2*} has two generators $t = b_1 \in B_2$, and $b = b_2 \in B_4$ and is defined by two relations helding in B_4*

$(t \otimes 1_2)b(t \otimes 1_2) = b(t \otimes 1_2)b$.

$(1_2 \otimes t)b(1_2 \otimes t) = b(1_2 \otimes t)b$.

In fact for any k, $b_j \in B_{2k}$ can be expressed as $1_2^i \otimes t \otimes 1_2^{k-i-1}$ if $j = 2i+1$ and as $1_2^{i-1} \otimes b \otimes 1_2^{k-i-1}$ if $j = 2i$. Next the usual braid relations follows from above using the tensor strucure.□

From Proposition 2.1 it follows that we have an equivalence between

{tensor representations of B_*} and {Yang-Baxter operators}.

So it would be nice to see how closed to Yang-Baxter operators is the category of tensor representations of B_{2*} since the invariants of unoriented links actually come from the last ones. In fact consider the category of scalar tensor representations (abbrev. s.t.r.) of (B_{2*}, B_{2*}^+), denoted by $STR(B_{2*}, B_{2*}^+)$. We say that such a s.t.r.ρ is Markov if the following additional condition is fulfilled:

$$< x \otimes w_1, \rho(\beta \otimes 1_2)\rho(1_2^{k-1} \otimes b)(y \otimes w_1) >= d < x, \rho(\beta)y > \quad (5.10)$$

where $x, y \in W_k, \beta \in B_{2k}$ and $d =< w_1, w_1 >\neq 0$. Because the scalar product is nondegenerated this is equivalent to ask that the composite map

$W_g \overset{x \to x \otimes w_1}{\longrightarrow} W_{g+1} \overset{1_2 \otimes \rho(b)}{\longrightarrow} W_{g+1} \overset{proj}{\longrightarrow} W_g$

where $proj$ is the canonical projection, be the identity.

Therefore we can state the following characterization of invariants:

Theorem 5.2.3 *Any multiplicative* **R**-*invariant for unoriented links can be obtained from a Markov s.t.r. of* (B_{2*}, B_{2*}^+) *by the following formula:*

$I(\tilde{\alpha}) = d^{-g} < w_g, \rho(\alpha) w_g >$, *for* $\alpha \in B_{2g}$,

where $\tilde{\alpha}$ *is the plat closure of the braid* α.

Proof: Consider I a multiplicative R-invariant for links, and set

$F_g = \{(B^3, L), L$ being a 1-dimensional manifold with $\partial L \subset S^2$,

and a marking $(S^2, \partial L) \longrightarrow (S^2, \{p_1, p_2, ..., p_2g\})\}$

modulo isotopy relative to boundary

Now I extends to a bilinear map

$B_I : \mathbf{R} < F_g > \times \mathbf{R} < F_g > \longrightarrow \mathbf{R}$.

We define $W_g = \mathbf{R} < F_g > / ker B_I$ so B_I descends to a nondegenerate bilinear form on W_g. We have then a representation of $\mathcal{M}_{0,2g}$ in $U(W_g)$ which induces a tensor representation of B_{2*}. The tensor structure on W_g is given by connected sum, and the B_{2*}^+ invariant vector is simply the class of (L_*, id), L_g being the standard semi-link. The rest of the proof is standard (see also the previous chapter and [BHMV92]).□

If we wish to obtain C-invariants we start also from scalar representations since the lack of orientation implies that no hermiticity assumptions on the bilinear form could naturally be imposed.

We shall consider now a slight variation in the manner in which the braid is closed for obtaining a link. This is of course relevant only in the 2-dimensional picture. Consider an equipartition of $\{1, 2, ..., 2n\} = A_1 \cup A_2 \cup ... \cup A_n$ every set A_i containing two elements a_i, and b_i. Then we can use this pattern (by joining a_i and b_i when regarded as bottom or top endpoints of the braid, by a simple arc in a ball) for closing the braids in B_{2n}. In this way nothing changes but the identification of B_{2n} since the morphism $B_{2n} \longrightarrow \mathcal{M}_{0,2n}$ is modified. The partition with $A_i = \{i, 2n - i\}$ gives what we shall call the concentric closure x^c of the braid x (see the figure 5). We mention that if we wish that the link $(x \otimes_c y)^c$ be isotopic to $x^c \cup y^c$ we must change accordingly the tensor structure on B_{2*}. Schematically the new tensor product $\otimes_c$ is given in the picture 6. When we refer to $(B_{2*}, \otimes_c)$ we often put an upperscript B_{2*}^c. Remark first that any unoriented link may be obtained as the

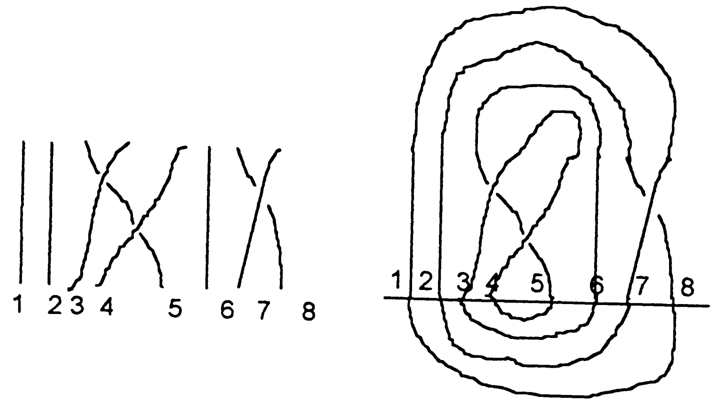

Figure 5.5: The concentric closure α^c of α

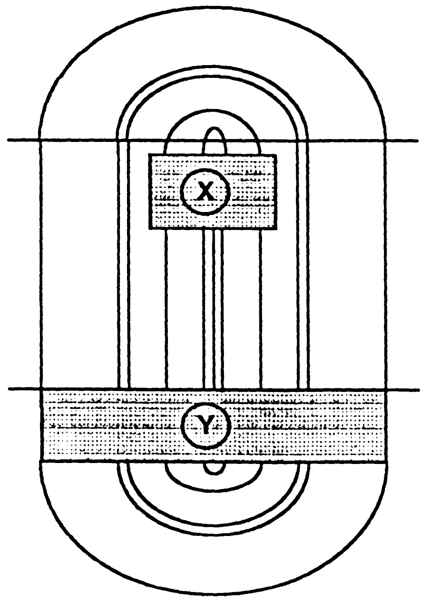

Figure 5.6: The $\otimes_c$ product of x and y

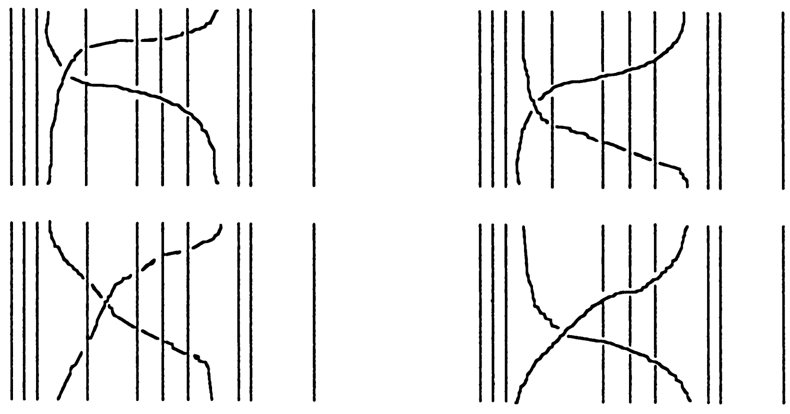

Figure 5.7: Generators for B_{2m}^{c+}

concentric closure of some braid, and we have the following Markov's type theorem:

Proposition 5.2.4 *Let B_{2m}^{c+} be the subgroup of B_{2m} generated by the elements drawn in figure 7. Let the maps $i_c^\varepsilon : B_{2m} \longrightarrow B_{2m+2}$, be defined by*

$i_c^\varepsilon(x) = 1 \otimes x b_{2m}^\varepsilon, \varepsilon \in \{-1, 1\}$.

Then two concentric closures α^c and β^c are isotopic as unoriented links if and only if $\alpha, \beta \in \cup_m B_{2m}$ are equivalents under the equivalence relation generated by the following elementary moves:

$\alpha \cong x\alpha y, x, y \in B_{2m}^{c+}, \alpha \in B_{2m}$.

$\alpha \cong i_c^\varepsilon(\alpha)$.

Proof: Remark that the mapping $\pi_c : B_{2m} \longrightarrow \mathcal{M}_{0,2m}$ obtained using a different numerotation of endpoints of braids is given by the composition of the usual one π by conjugation with an automorphism of $\mathcal{M}_{0,2m}$. Set $\mathcal{M}_{0,2m}^+$ for the image of the group of homeomorphisms of $(S^2, \{p_1, p_2, ..., p_{2m}\}$ which extend to the semi-link L_m. Then $\pi(B_{2m}^+) = \mathcal{M}_{0,2m}^+$. But we may pass from an identification of L_m with itself in the plat model, to L_m viewed in the concentric model by moving its endpoints as in picture 8. We may draw now the image of $b_i \in B_{2m}$ under $\pi_c^{-1}\pi$ as in figure 9., where $i = 2j - 1$, or $i = 2n - 2j$. Of course the strands between $j + 1$ and $2n - j - 1$ can pass over if we choose the

Figure 5.8: Moves of L_m's endpoints

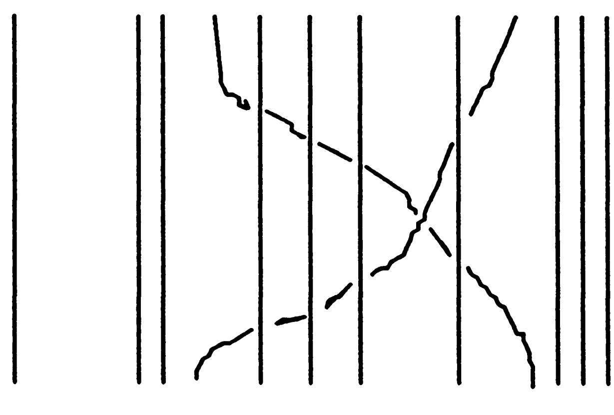

Figure 5.9: The image of b_i

mirrored identification. Now the last proposition gives our claim. Also a long a detailed proof may be given on the lines of that of Birman for the plat closure (see[Bir74]).□
We have an analog of theorem 2.3, namely:

Theorem 5.2.5 *Any multiplicative R-invariant for unoriented links can be obtained from a Markov s.t.r. ρ of $(B_{2*}, B_{2*}^{c+}, \otimes_c)$ by the formula*
$I(\alpha^c) = d^{-g} < w_g, \rho_g(\alpha) w_g >, \alpha \in B_{2g}$.

The proof is essentially the same as for 2.3. This result gives nothing new and is of purely theoretical interest since the tensor structure $\otimes_c$ is much more complicated than the usual one. However this characterization will be used later due because there is a close connection between the concentric and Artin's closure which cannot be expected for the plat closure.

5.3 Oriented plats and concentric links

The next step will be a similar description for oriented links. The point is that there is no canonical way to associate an orientation to the plat (or the concentric) closure of a braid. Trying to impose somewhat artificially an orientation we have to enlarge the braid group to a groupoid structure.

Consider $\tau \in S_{2n}$ be a fixed involution without fixed points, and denote by
$M(\tau) = \{\varepsilon : \{1, 2, ..., 2n\} \longrightarrow Z/2Z \text{ such that } \varepsilon(i) = -\varepsilon(\tau(i))\}$
$M(\tau) \subset (Z/2Z)^{2n}$
the set of τ-anti-invariant signatures. We have a natural projection morphism, $p : B_{2n} \longrightarrow S_{2n}$ inducing an action of B_{2n} on $(Z/2Z)^{2n}$ by
$(b, (x_1, x_2, ..., x_{2n})) = (x_{p(b)1}, x_{p(b)2}, ..., x_{p(b)2n}) = b^* x$.
We set $O(b; \tau) = M(\tau) \cap b^* M(\tau) \subset (Z/2Z)^{2n}$.

Lemma 5.3.1 *For any $b \in B_{2n}$ there exists some $m > 0$ so that $O(b; \tau) \cong (Z/2Z)^m$.*

Proof: For $x \in \{1, 2, ..., 2n\}, \sigma \in S_{2n}$ define the sequence $x_1 = x, x_{2k} = \sigma^{-1}\tau\sigma x_{2k-1}, x_{2k+1} = \tau x_{2k}$. For some p we shall have $x_{2p+1} = x_1$. We put $A_x = \{x_1, ..., x_{2p}\}$. We have then a partition of $\{1, 2, ..., 2n\} = A_1 \cup A_2 \cup ... \cup A_k$ where the A_i's are subsets of type A_x and moreover $A_i = B_i \cup \tau B_i, B_i \cap \tau B_i = \phi$. Then $B_i, \tau B_i$ are supports for the maximal cycles of $\sigma^{-1}\tau\sigma$. If we change $\sigma^{-1}\tau\sigma$ with $\sigma\tau\sigma^{-1}$ we get the similar partition with A_i replaced by $\bar{A}_i = \sigma^{-1}A_i$. Take now $\sigma = p(b)^{-1}$. Since the sequence x_j has not cycles of even length it follows that $\varepsilon \in O(b; \tau)$ iff

$$\varepsilon = \begin{cases} \varepsilon_i & \text{on } B_i \\ -\varepsilon_i & \text{on } \tau B_i \end{cases}$$

where ε_i are arbitrary. This proves our claim.□

We remark that if we connect the pairs of points labelled x and τx for all x in both upper and bottom lines of the geometric braid b then the k obtained above is exactly the number of components of the resulting link.

Consider now $\mathcal{B}^o_{2n}(\tau)$ be the set of pairs $\{(b, \varepsilon), b \in B_{2n}, \varepsilon \in O(b; \tau)\}$. Then $\mathcal{B}^o_{2n}(\tau)$ is a groupoid, called the groupoid of τ-braids. In fact we may compose (b, ε) and (b', ε') iff $b'^*\varepsilon' = \varepsilon$, and if this is so the composition $(b, \varepsilon)(b', \varepsilon')$ equals (bb', ε'). If the involutions τ for different values of n are related as in the case of plats ($\tau(2i-1) = 2i$) or concentric links ($\tau(i) = 2n - i$) we can put also some tensor structure on $\mathcal{B}^o_{2*}(\tau)$ as follows:

$(b, \varepsilon) \otimes (b', \varepsilon') = (b \otimes b', \varepsilon \otimes \varepsilon')$ where for the plat closure:

$$\varepsilon \otimes \varepsilon'(j) = \begin{cases} \varepsilon(j) & \text{if } j \leq 2n, b \in \mathcal{B}^o_{2n}(\tau) \\ \varepsilon'(j-2n) & \text{otherwise} \end{cases}$$

and for concentric closure:

$$\varepsilon \otimes \varepsilon'(j) = \begin{cases} \varepsilon(j) & \text{if } j \leq n, b \in \mathcal{B}^o_{2n}(\tau) \\ \varepsilon'(j-n) & \text{if } n < j \leq n+m, b' \in \mathcal{B}^o_{2m}(\tau) \\ \varepsilon(j-n-m) & \text{if } n+m < j \leq 2n+m \\ \varepsilon'(j-2n-m) & \text{otherwise} \end{cases}$$

and the tensor product on braids is the appropriate one.

Now for any element in $\mathcal{B}^o_{2n}(\tau)$ we can attach a natural orientation to its closure as follows: represent $b \in B_{2n}$ as a geometric braid lying in $R^2 \times [0, 1]$ and having endpoints in $R^2 \times \{0, 1\}$ and which has some markings given by ε and $b^*\varepsilon$ respectively. For a string of b having upper endpoint x and bottom endpoint y we know that $\varepsilon(x) = b^*\varepsilon(y)$; if $\varepsilon(x) = 1$ we orient the string from y to x, otherwise we take the reverse orientation. When we close the braid b we glue another $2n$ arcs,

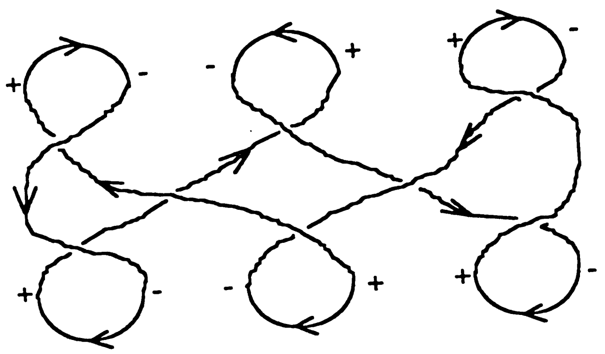

Figure 5.10: Orienting a plat

n of them lying in $R^2 \times [-1,0]$ (the lower ones) and n of them lying in $R^2 \times [1,2]$ (the upper arcs). They are joining points related by τ hence of different signatures. We orient these arcs as follows: the upper arcs are oriented from the point marked with $+1$ to the other, and the lower arcs in a opposite manner. It is easy to see that all these local orientations agree and give a global orientation for the link closure. We can state now an analog of Birman's theorem in the framework of oriented plats and concentric links:

Proposition 5.3.2 *For each m let $\mathcal{B}_{2n}^{+o}(\tau) = \{(b,\varepsilon), b \in B_{2n}^{+}(\tau)\}$ be the subgroupoid of $\mathcal{B}_{2n}^{o}(\tau)$ generated by the images of the elements from $B_{2n}^{+}(\tau)$. Then two oriented plat (or concentric) closures $\tilde{\alpha}, \tilde{\beta}$ are isotopic if and only if $\alpha, \beta \in \bigcup_n \mathcal{B}_{2n}^{o}(\tau)$ are equivalent under the equivalence relation generated by the following elementary moves:*

$\alpha \cong x\alpha y, \alpha \in \mathcal{B}_{2n}^{+o}(\tau), x, y \in \mathcal{B}_{2n}^{+o}(\tau)$

$\alpha \cong i(\alpha)$

where, if $\alpha = (b,\varepsilon), b \in B_{2k}$ we met

$i(\alpha) = (bb_{2k}, \tilde{\varepsilon})$

$$\tilde{\varepsilon}(j) = \begin{cases} \varepsilon(j), & \text{if } j \leq 2n \\ \varepsilon(2n), & \text{if } j = 2n+1 \\ -\varepsilon(2n), & \text{if } j = 2n+2 \end{cases}$$

for plats, and

$i(\alpha) = (1 \otimes bb_{2k}^{\delta}, \varepsilon^c)$ *,where* $\delta \in \{-1, 1\}$

$$\varepsilon^c(j) = \begin{cases} \varepsilon(j-1), & \textit{if } 1 < j \leq 2n+1 \\ \varepsilon(2n), & \textit{if } j = 2n+2 \\ -\varepsilon(2n), & \textit{if } j = 1 \end{cases}$$

for concentric links.

This follows directly from 2.1 and 2.4. by taking into account the orientations.□

We wish now to exhibit a description of C-invariants for links using the representations of the groupoid $\mathcal{B}^o_{2*}(\tau)$. We shall restrict ourselves to those invariants sensitive to the change of orientation i.e. satisfying $\overline{I(L)} = I(\bar{L})$ where $\bar{L}$ is the link L with the orientations of all its components reversed.

First of way we must explain what means exactly a (tensor) representation of a groupoid G. There are two applications defined , usually called source and target $S, T : G \longrightarrow A$, where A is a fixed set (the category of final objects). Now for two objects $x, y \in G$ their composite xy is defined if and only if $S(x) = T(y)$. Then G has a partition as $G = \cup_{a,b \in G} G(a,b)$ where $G(a,b) = \{x \in G; S(x) = a, T(y) = b\}$; the subset $G(a,a)$ will be therefore a group which acts by left multiplication on $G(b,a)$. This action is usually (e.g.in our case of oriented braid groupoid) a transitive one. If $G = \mathcal{B}^o_{2n}(\tau)$ the source and target maps are $S(b,\varepsilon) = \varepsilon, T(b,\varepsilon) = b^*\varepsilon$ and $A = (\mathbf{Z}/2\mathbf{Z})^{2n}$. Then a representation of G will be one in the categorical acception: we have an indexed set of vector spaces (eventually carrying some extra structure as hermitian, scalar etc) $W_a, a \in A$ and mappings $\rho_{a,b} : G(a,b) \longrightarrow Hom(W_a, W_b)$ compatible with the composition rules. If G is complete i.e. each element has an inverse and $G(a,b) = (G(b,a))^{-1}$ then the maps $\rho_{a,b}$ are supposed to take values in the group of isomorphisms $Isom(W_a, W_b)$, and W_a will be isomorphic. If we have a tensor groupoid then the categories of final objects will inherit a tensor structure and then the compatibility between this and the maps $\rho_{a,b}$ is required.

Theorem 5.3.3 *Any C-invariant sensible to orientation of oriented links may be obtained from an hermitian tensor representation $\rho_{*,*}$ of* $(\mathcal{B}^o_{2*}(\tau), \mathcal{B}^{+o}_{2*}(\tau))$ *having the following properties:*

1. $\rho_{*,*}$ *is hermitian i.e. the vector spaces* $W_{k,\varepsilon}, k \in N, \varepsilon \in (Z/2Z)^{2k}$

are endowed with non-degenerate hermitian forms $<,>$ *compatible with* $\otimes$ *and*

$\rho_{(k,\varepsilon),(k,\delta)}(\mathcal{B}^o_{2k}(\tau)(\varepsilon,\delta) \in Isom(W_{k,\varepsilon}, W_{k,\delta})$
(here $Isom$ *represents the group of isometries between the respective hermitian spaces).*

2. $\rho_{(k,\varepsilon),(k,\varepsilon)}(\mathcal{B}^o_{2k}(\tau)(\varepsilon,\varepsilon))$ *acts transitively on* $\rho_{(k,\varepsilon),(k,\delta)}(\mathcal{B}^o_{2k}(\tau)(\varepsilon,\delta))$ *by left multiplication.*

3. *We have the invariant (tensor) vectors* $w_{k,\varepsilon} \in W_{k,\varepsilon}$ *so that*

   $\rho_{(k,\varepsilon),(k,\delta)}(\mathcal{B}^{+o}_{2k}(\tau)(\varepsilon,\delta)(w_{k,\varepsilon}) = w_{k,\delta}$;
   $w_{k+n,\varepsilon\otimes\delta} = w_{k,\varepsilon} \otimes w_{n,\delta}$

4. *The representation is Markov i.e. the following composition is identity:*

   $W_{k,\varepsilon} \xrightarrow{sh} W_{k+1,\varepsilon\otimes\delta} \xrightarrow{F} W_{k+1,\varepsilon\otimes\delta} \xrightarrow{proj} W_{k,\varepsilon}$
   where

   $sh(x) = x \otimes w_{1,\delta}$ *is the shift map.*

   $$F = \begin{cases} (1_2^{k-1}, \lambda) \otimes \rho(b,\mu) & \text{for plats}, b = b_2 \in B_4 \\ (1^{k-1}, \lambda) \otimes_c \rho(b_1,\mu) & \text{for concentric links} \end{cases}$$

   $\varepsilon \otimes \delta = \lambda \otimes \mu$.

   proj is the projection.

5. $d = < w_{1,\varepsilon}, w_{1,\varepsilon} > \neq 0$

and the invariant could be expressed as
$I(\tilde{\alpha}) = d^{-k} < w_{k,\varepsilon}, \rho_{(k,\varepsilon),(k,\delta)}(\alpha) w_{k,\delta} >$, *for* $\alpha \in \mathcal{B}^o_{2k}(\tau)(\varepsilon,\delta)$.

Proof: We have to define
$M_{k,\varepsilon} = \{$marked oriented semi-links (B^3, L) with some identification $(\partial(B^3,L),\varepsilon) \cong (S^2, \{p_1,...,p_2k), \varepsilon)\}$/modulo isotopy
and take the image of the standard semi-link in $W_{k,\varepsilon} = C < M_{k,\varepsilon} > /kerB_I$, (B_I being the bilinear form associated to the invariant I) to be $w_{k,\varepsilon}$. We observe that we may suppose that the orbit of $w_{k,\varepsilon}$ spans $W_{k,\varepsilon}$. Now the Markov condition reduces to the usual relation encountered in 2.3 and the proof is straightforward.□

We can generalize this approach for including colored links; if G is a finitely generated group we replace $Z/2Z$ by G in the above definitions. Remark that Lemma 3.1 holds in this setting and gives $O(b,\tau,G) \cong G^k$. Also G^{2*} has a natural tensor structure and 3.2,3.3 can be rephrased for G-colored braids (and links). However if we wish to include framings (so taking a Z factor in G specifying the framing) we must review the definition of $\tilde{\varepsilon}$ in 3.2 as follows:

$$\tilde{\varepsilon_Z}(j) = \begin{cases} \varepsilon_Z(j), & j \leq 2n \\ \varepsilon_Z(2n)+1, & j = 2n+1 \\ \varepsilon_Z(2n)+1, & j = 2n+2 \end{cases}$$

for plats, and similarly for concentric links.

Proposition 5.3.4 *The tensor group $(B^c_{2*}, \otimes_c)$ is a finitely generated tensor group; in fact it is generated by $1_2, b_1^{(2)}, b_1^{(4)}, b_3^{(4)}$ (where the superscript denotes the braid group where the element lies), with the relations derived from the usual ones in B_4 and B_6.*

Proof: We have firstly $b_2^{(4)} = 1_2 \otimes_c b_1^{(2)}$ and

$b_1^{(6)} = b_1^{(4)} \otimes_c 1_2$

$b_2^{(6)} = 1_2 \otimes_c b_1^{(4)}$

$b_3^{(6)} = 1_2 \otimes_c 1_2 \otimes_c b_1^{(2)}$

$b_4^{(6)} = 1_2 \otimes_c b_3^{(4)}$

$b_5^{(6)} = b_3^{(4)} \otimes_c 1_2$

Now we can proceed by induction: if $1 < i < 2n-1$ then $b_i^{(2n)} = 1_2^{i-1} \otimes_c x$, where $x \in B_{2n-2i+2}$; also $n \geq 3$ hence we can write $b_{2n-1}^{(2n)} = b_3^{(4)} \otimes_c 1_2^{n-2}$ and $b_1^{(2n)} = b_1^{(4)} \otimes_c 1^{n-2}$. All braid relations will follow now by functoriality of $\otimes_c$ from those of B_4 and B_6. □

Now if we denote by $\mathcal{B}^G_{2*}(\tau)$ the groupoid of G-colored braids it seems that $\mathcal{B}^G_{2*}(\tau)$ is not a finitely generated tensor groupoid. Consider $R = \{1, z_1, z_2, ..., z_p\}$ a reduced system of generators for G, and the following elements:

$e_k = b_{2k}b_{2k-1}...b_1b_{4k-1}b_{4k-2}...b_{2k} \in B_{4k}$

$f_k = b_{2k}b_{2k-1}...b_2b_{4k-2}b_{4k-3}...b_{2k} \in B_{4k}$

Therefore it is easy to compute the number of components of their plat closures as $c(e_k) = c(f_k) = 2$. Let also R_G denote

$\{(1_2, v), (b_1^{(2)}, v), ((b_2^{(4)})^2, v_1 \otimes v_2), (e_k, v_1 \otimes v_2), (f_k, v_1 \otimes v_2)$ with $v, v_1, v_2 \in R, k \in N\}$

We formulate the following

Conjecture 5.3.5 *1) The tensor groupoid of G-colored braids is not finitely generated for any nontrivial groups G, in both plat or concentric closure settings.*

2) In the case of plats R_G is a reduced system of generators for the tensor groupoid of G-colored braids.

5.4 The proof of the homogeneity theorem

A Markov trace on $C[B_\infty]$ is said to be unoriented if the invariant deduced from it does not depend upon the choice of the orientation of the components of link.

Proposition 5.4.1 *Let $t : C[B_\infty] \longrightarrow C$ be an unoriented strong Markov trace which factors in rank k through $P(k)$. Then t factors through the homogeneous quotient $P^h(\infty)$.*

Proof: We remark first that the tensor representations of B_{2*} which have been discussed in sections 2 and 3 could be extended to tensor representations of algebras $C[B_{2*}]$ in the corresponding endomorphisms algebras, because at universal level this can be done.

Observe that the pull-back of relations by a negative translation of indices is always trivial since the trace must be constant on braids having isotopic closures. The really problem is to push the relations by a positive translation of indices since the ideal on which the trace must vanish widens once the rank increases. Assume that for some polynomial Q of k variables

$Q(b_1, b_2, ..., b_{k-1}) = 0$,

is a relation fulfilled in $P(k)$. This leads to

$t(aQ(b_1, ..., b_{k-1})b) = 0$ for all $a, b \in B_k$

We remark that for $x \in B_n$ the Artin's closure $\hat{x}$ is isotopic to the concentric closure $(1^n \otimes x)^c$, as unoriented links. Since t is an unoriented strong Markov trace we have a $\otimes_c$ representation of (B_{2*}, B_{2*}^{c+}) such that

$t(x) = < w_n, \rho_n(1^n \otimes x) w_n >$

Therefore

$< w_k, \rho_(1^k \otimes a)(1^k \otimes Q(b_1, b_2, ..., b_{k-1})(1^k \otimes b) w_k >= 0$

Observe next that as in the first part of the paper we may restrict ourselves to the case when the B_{2k}-orbit of w_k spans W_k, as we shall suppose from now one.

Consider now $V_n \subset W_n$ to be the span of $\rho_n(1^n \otimes B_n)w_n$. The last equality implies

$< x, Dy >= 0$ for all $x, y \in V$

where $D = 1^k \otimes Q(b_1, ..., b_{k-1})$. But D as an element of $End(W_k)$ has the property $D(V_k) \subset V_k$. We derive $D \mid_{V_k} = 0$. Since the representation ρ is tensorial we derive

$V_k \otimes W_1 \supset V_{k+1}$.

On the other hand

$D \otimes 1 \mid_{V_k \otimes W_1} = 0$,

from functoriality which implies that

$D \otimes 1 \mid_{V_{k+1}} = 0$.

We obtained

$Q(b_{k+1}^{(2k)}, b_{k+2}^{(2k)}, ..., b_{2k-1}^{(2k)}) = 0$, in $End(V_{k+1})$, since

$b_{k+i}^{(2k)} \otimes_c 1^2 = b_{k+i+2}^{(2k+2)} = 1^{k+1} \otimes b_{i+1}^{(k+1)}$.

This implies that

$t(aQ(b_2, b_2, ..., b_k)b) = 0$ for all $a, b \in B_{k+1}$

so the relation is pushed by the translation by one of the indices. □

Proof of Homogeneity Theorem:Suppose that the trace is a strong Markov trace. We have as above $t(aQ(b_1, ..., b_{k-1})b) = 0$ for all $a, b \in B_k$.

Let $\varepsilon_n = (\underbrace{+1, +1, ... +1}_{n} -1, ..., -1) \in (\mathrm{Z}/2\mathrm{Z})^{2n}$. Then the Artin's closure of the braid $x \in B_n$ is isotopic (as oriented links!) with the concentric closure of $(1^n \otimes x, \varepsilon_n)^c$. Therefore for some $\otimes_c$-representation of the groupoid $\mathcal{B}_{2*}^o$ we may write

$t(x) =< w_{n\varepsilon_n}, \rho_{n,\varepsilon_n}(1^n \otimes x, \varepsilon_n) w_{n,\varepsilon} >$.

We may suppose that $\mathcal{B}_{2n}^o(\varepsilon_n, \varepsilon_n) w_{n,\varepsilon_n}$ span W_{n,ε_n}. In lemma 4.2 we can add the marking ε_n without complications, hence

$(1^n \otimes B_n, \varepsilon_n)\mathcal{B}_{2n}^{o+}(\varepsilon_n, \varepsilon_n) = \mathcal{B}_{2n}(\varepsilon_n, \varepsilon_n)$.

holds. Reasoning as in the previous proposition and using $\varepsilon_{n+k} = \varepsilon_n \otimes_c \varepsilon_k$) we find that

$Q((1^k \otimes b_1^{(k)}, \varepsilon_k), ..., (1^k \otimes b_1^{(k)}, \varepsilon_k)) = 0$ in $End(V_k, \varepsilon_k)$.
Therefore we derive
$Q((1^{k+1} \otimes b_2^{(k+1)}, \varepsilon_{k+1}), ..., (1^{k+1} \otimes b_1^{(k+1)}, \varepsilon_{k+1})) = 0$ in $End(V_{k+1}, \varepsilon_{k+1})$.
so
$t(aQ(b_2, b_2, ..., b_k)b) = 0$ for all $a, b \in B_{k+1}$.
This means that t factors through $P^h(\infty)$. The same proof works for an arbitrary Markov trace not necessary a strong one using the associated invariant. □

Consider now a filtered ideal $J \subset C[B_\infty]$, so that F^jJ is an ideal in $F^jC[B_\infty]$. Then J is called homogeneous if it is closed under the natural shift operator
$S : C[B_\infty] \longrightarrow C[B_\infty]$
defined by $S(x) = 1 \otimes x$.

Then as a direct consequence of the theorem we have:

Corollary 5.4.2 *If J is a filtered ideal annihilated by a strong Markov trace (i.e. $J \subset ker\ t$) then J is homogeneous.*

Chapter 6

Cubic Hecke algebras

6.1 Introduction

The aim of this paper is to begin a systematic study of cubic Hecke algebras. The generalized Hecke algebras were introduced by analogy with the classical case as the quotients

$H(Q,n) = \mathbf{C}[B_n]/(Q(b_j); j = 1, n-1)$

of the group algebra of the braid group by the ideal generated by $Q(b_j)$ where Q is a polynomial having $Q(0) \neq 0$.

The structure of these algebras is well-known in the quadratic case (see [Bou82]). They are finite dimensional semi-simple modules of dimension $n!$. In the general case we notice that some new features arise. In particular

$dim_{\mathbf{C}} H(Q,n) = \infty$ if $deg(Q) > 6$, and $n \geq 3$.

Also even in the cubic case

$dim_{\mathbf{C}} H(Q,n) = \infty$ if $n > 6$, $deg(Q) = 6$.

Since our main interest is the study of Markov traces on the cubic Hecke algebras we shall be mainly concerned with its homogeneous quotients.

The (filtered) quotient $P(\infty)$ of $H(Q,\infty)$ is homogeneous if any identity

$F(b_i, b_{i+1}, ..., b_j) = 0,\ F \in \mathbf{C}[X_0, X_1, ..., X_{j-i}]$

which holds in $P(\infty)$ remains valid under the translation of indices i.e. also

$F(b_{i+k}, b_{i+k+1}, ..., b_{j+k}) = 0$, for $k \in \mathbf{Z}, k \geq 1 - i$.

The rank of the quotient is the first rank where a nontrivial relation appears. So the homogeneous quotients (of cubic Hecke algebras) of rank 2 are trivial or quadratic Hecke algebras. We should be interested in the study of homogeneous quotients of rank 3. Therefore the first relation lies in $H(Q,3)$. We prove (Proposition 2.1) that $dim_{\mathbf{C}} H(Q,3) = 24$. So we have a 24-dimensional space parameterizing the main stratum of quotients of $H(Q,3)$.

A Markov trace t on a quotient P_∞ of $\mathbf{C}[B_\infty]$ is said to have parameters $(z,\bar{z})$ if $t(xb_n) = zt(x)$ and $t(xb_n^{-1}) = \bar{z}t(x)$ hold for all $x \in P_n$. To a Markov trace we can always associate a link invariant. Notice that the link invariant is not a Markov trace on P_∞ with parameters $(1,1)$ unless the defining relations of the quotient are homogeneous polynomials in each rank.

We derive therefore that the generic rank 3 homogeneous quotient has at most one Markov trace when the parameters are fixed. We are looking now to the quotients of maximal dimension, which in rank 3 is 21. After some computations we find that there is a such quotient e.g. that given by the following relation

$$b_2b_1^2b_2 + b_1b_2^2b_1 + b_1^2b_2b_1 + b_1b_2b_1^2 + b_1^2b_2^2 + b_2^2b_1^2 + \gamma b_1 + \gamma b_2 = 0$$

for $Q = X^3 - \gamma$.

We focus in section 3 on the study of Markov traces on the homogeneous quotient, $K_n(\gamma)$, determined by this relation. Dually we compute the link group associated to $K_\infty(\gamma)$ and the two parameters $(z,\bar{z})$. It is (Theorem 3.4) a cyclic torsion group of order 6 or it vanishes. Roughly speaking the link group of a quotient is the group generated by the isotopy classes of oriented links modulo the skein relations dictated by the quotient itself. So it turns out that these various link groups (which must not be confounded with the fundamental group of the link's complement, but is really a group of links!) are not always torsion free. What we have to prove is that a certain functional on $K_\infty(\gamma)$ really exists, since its uniqueness may be easily obtained. The method of proof is greatly inspired from [Ber78]. We define a huge graph whose vertices are the elements of the abelian semigroup associated to the free group in $n-1$ letters (in first instance) and whose edges correspond to elements which differ by exactly one relation (from the set of relations defining $K_n(\gamma)$).

If we should use our relations in only one direction (i.e. we may replace a by b but not b by a) we should arrive to orient the edges of this graph and we may ask whenever the minimal elements of each connected component of the graph exist and there are unique. This will provide a basis for $K_n(\gamma)$ if sufficiently many relations are added in order to obtain the unicity. For the existence of minimal elements the usual procedure is to use the lexicographic order on the free group on $n-1$ letters and to replace always a word by smaller ones. We have carried out this algorithm for $H(Q,3)$ in appendix and we can see the technical difficulties which may be encountered. So having yet in mind a certain reduction process we define the oriented edges as follows: exactly one monomial may be changed using one of the rules

(C0)(j) $Ab_j^3B \to AB$
(C1)(j) $Ab_{j+1}b_jb_{j+1}B \to Ab_jb_{j+1}b_jB$
(C2)(j) $Ab_{j+1}b_j^2b_{j+1}B \to AS_jB$
(C12)(j) $Ab_{j+1}b_j^2b_{j+1}^2B \to Ab_j^2b_{j+1}^2b_jB$
(C21)(j) $Ab_{j+1}^2b_j^2b_{j+1}B \to Ab_jb_{j+1}^2b_j^2B$.

Also some unoriented edges must be added. They correspond to a change in a monomial of type

(P_{ij}) $Ab_ib_jB \to Ab_jb_iB$ whenever $\mid i-j \mid > 1$.

Remark that we was forced to add some relations (knowing that they hold already in $H(Q,n)$) which additionally make the reduction process ambiguous. The reason is to assure the existence of descending paths to some minimal points even if closed oriented loops may be found in the graph. And we shall check the existence and unicity of minimal elements up to unoriented paths in this semi-oriented graph by means of so-called Pentagon Lemma 3.7. When this approach will be not successful we shall widen our graph to a tower of graphs modelling not $K_n(\gamma)$ but the functionals on $K_n(\gamma)$ satisfying a recurrent condition which permits to reduce further the minimal elements. Here the Colored Pentagon Lemma 3.8. (in fact a variant of 3.7.) can be applied and reduces all the problem to some algebraic computations. This shows that the main obstructions lie in $K_4(\gamma)$ not in $K_3(\gamma)$ as it could be expected from the study of quadratic Hecke algebras. When we wish to check the commutativity condition for the functional be actually a Markov trace another one obstruction appears in $K_4(\gamma)$. This explain why torsion arise in the link group.

In the fourth section we prove that the Markov trace on $K_\infty(\gamma)$

and taking values in $\mathbf{Z}/6\mathbf{Z}$ actually has an unique lift to $H(Q_{-1},\infty)$ which is integer valued, where $Q_{-1} = x^3 + 1$. Thus we derive our main result (Theorem 4.5) which computes the link group of $H(Q_{-1},*)$ with parameters (1,1). A link invariant F is derived which is not a Vassiliev invariant of finite degree. Finally using a method due to Baez [Bae92] for looking at Vassiliev invariants we arrive to a whole sequence of 3-rd order Vassiliev invariants which in degree 0 correspond to F. They are always algorithmically computable from theoretic point of view. Whenever some of them are really new (so they are not limits of classical Vassiliev invariants) we don't know at this moment.

6.2 The rank 3 quotients

The generalized Hecke algebras were introduced by analogy with the classical case as the quotients

$H(Q,n) = \mathbf{C}[B_n]/(Q(b_j); j = 1, n-1)$

of the group algebra of the braid group by the ideal generated by $Q(b_j)$ where Q is a polynomial having $Q(0) \neq 0$. Since our main interest is the study of Markov traces on the cubic Hecke algebras we shall be mainly concerned with its homogeneous quotients. The rank of the quotient is the first rank where a nontrivial relation appears. So the homogeneous quotients (of cubic Hecke algebras) of rank 2 are trivial or quadratic Hecke algebras. We should be interested in the study of homogeneous quotients of rank 3. Therefore the first relation lies in $H(Q,3)$.

Proposition 6.2.1 *For all cubic polynomials Q with $Q(0) \neq 0$*
$dim_{\mathbf{C}} H(Q,3) = 24$.

Proof: Since $Q(0) \neq 0$ we may restrict to the case when the exponents of the b_i's are 0,1 or 2. Set $w_{n,k} = b_n b_{n-1} \ldots b_{k+1} b_k^2 b_{k+1} \ldots b_n \in H(Q, n+1)$ and $Q = X^3 - \alpha x^2 - \beta x - \gamma$.

Lemma 6.2.2 *Set $r_{n,j} = b_j b_{j+1} \ldots b_{n-1}$. The following commutation rules hold in $H(Q,n+1)$:*

$b_i w_{n,j} = w_{n,j} b_i$ *if* $i \neq j-1$ *and* $i < n$.

$b_n w_{n,j} = \alpha w_{n,j} + \beta w_{n-1,j} b_n + \gamma r_{n,j} w_{n,j+1} r_{n,j}^{-1}$.

$w_{n,j} b_n = \alpha w_{n,j} + \beta b_n w_{n-1,j} + \gamma r_{n,j}^{-1} w_{n,j+1} r_{n,j}$.

where $w_{n,n} = b_n^2$.

Proof of lemma: If $i < j-1$ we obtain the first relation since the b_k's involved in $w_{n,j}$ have $| k-i | \geq 2$. Now

$$b_{n-1}w_{n,j} = (b_{n-1}b_nb_{n-1})...b_j^2...b_{n-1}b_n = b_nb_{n-1}b_n...b_j^2...b_{n-1}b_n =$$
$$= b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}(b_nb_{n-1}b_n) = b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}b_nb_{n-1} =$$
$$= w_{n,j}b_{n-1}$$

proving the first relation for $i = n-1$. Similarly for all $i \geq j$ we have

$$b_iw_{n,j} = b_ib_nb_{n-1}...b_{i+2}b_{i+1}b_ib_{i-1}...b_j^2...b_n =$$
$$= b_nb_{n-1}...b_{i+2}(b_ib_{i+1}b_i)b_{i-1}...b_j^2...b_n =$$
$$= b_nb_{n-1}...b_{i+2}b_{i+1}b_ib_{i+1}b_{i-1}...b_j^2...b_{i-1}b_ib_{i+1}...b_n =$$
$$= b_nb_{n-1}...b_{i+2}b_{i+1}b_ib_{i-1}...b_j^2...b_{i-1}(b_{i+1}b_ib_{i+1})...b_n =$$
$$= b_nb_{n-1}...b_j^2...b_{i-1}b_ib_{i+1}b_i...b_n = w_{n,j}b_i.$$

So the first commutation relations are proved.

Sublemma 6.2.3 $w_{n,j}b_n = b_{n-1}^{-1}b_nw_{n-1,j}b_{n-1}b_nb_{n-1}$ *for* $j = 1, n-1$.

Set first $j < n-1$. We shall use $b_{n-1}b_n = b_nb_{n-1}b_nb_{n-1}^{-1}$ in what follows:

$$w_{n,j}b_n = b_nb_{n-1}...b_j^2...b_{n-1}b_n^2 = b_nb_{n-1}...b_j^2...b_{n-2}b_nb_{n-1}b_nb_{n-1}^{-1}b_n =$$
$$= (b_nb_{n-1}b_n)b_{n-2}...b_j^2...b_{n-2}b_{n-1}b_nb_{n-1}^{-1}b_n =$$
$$= b_{n-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}b_n(\gamma^{-1}b_{n-1}^2 - \alpha\gamma^{-1}b_{n-1} - \beta\gamma^{-1})b_n =$$
$$= \gamma^{-1}b_{n-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}(b_{n-1}b_nb_{n-1})b_{n-1}b_n-$$
$$-\alpha\gamma^{-1}b_{n-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}b_nb_{n-1}b_n-$$
$$-\beta\gamma^{-1}b_{n-1}(b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}b_n)b_n.$$

Using also the previous commutation rules for $i = n-1$ we obtain

$$w_{n,j}b_n = \gamma^{-1}b_{n-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_nb_{n-1}(b_nb_{n-1}b_n)-$$
$$-\alpha\gamma^{-1}b_{n-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}^2b_nb_{n-1}-$$
$$-\beta\gamma^{-1}b_nb_{n-1}b_{n-2}...b_j^2...b_{n-2}b_{n-1}(b_nb_{n-1}b_n) =$$
$$= \gamma^{-1}b_{n-1}(b_nb_{n-1}b_n)b_{n-2}...b_j^2...b_{n-2}b_{n-1}^2b_nb_{n-1}-$$
$$-\alpha\gamma^{-1}b_{n-1}\left[b_nw_{n-1,j}b_{n-1}\right]b_nb_{n-1}-$$
$$-\beta\gamma^{-1}\left[b_nw_{n-1,j}b_{n-1}\right]b_nb_{n-1} =$$
$$= (\gamma^{-1}b_{n-1}^2 - \alpha\gamma^{-1}b_{n-1}-$$
$$= \beta\gamma^{-1})\left[b_nw_{n-1,j}b_{n-1}\right]b_nb_{n-1} =$$
$$= b_{n-1}^{-1}b_nw_{n-1,j}b_{n-1}b_nb_{n-1}.$$

For $j = n-1$ we may write in the same manner:

$$b_{j+1}b_j^2b_{j+1}^2 = b_{j+1}b_j(b_{j+1}b_jb_{j+1}b_j^{-1})b_{j+1} = b_jb_{j+1}b_j^2b_{j+1}b_j^{-1}b_{j+1} =$$
$$= \gamma^{-1}b_jb_{j+1}b_j^2b_{j+1}b_j^2b_{j+1} - \alpha\gamma^{-1}b_jb_{j+1}b_j^2b_{j+1}b_jb_{j+1} - \beta\gamma^{-1}b_jb_{j+1}b_j^2b_{j+1}^2 =$$
$$= \gamma^{-1}b_jb_{j+1}b_jb_{j+1}b_jb_{j+1}b_jb_{j+1} - \alpha\gamma^{-1}b_jb_{j+1}b_j^3b_{j+1}b_j - \beta\gamma^{-1}b_{j+1}b_jb_{j+1}b_jb_{j+1}^2$$
$$= \gamma^{-1}b_j^2b_{j+1}b_j^3b_{j+1}b_j - \alpha\gamma^{-1}b_jb_{j+1}b_j^3b_{j+1}b_j - \beta\gamma^{-1}b_{j+1}b_j^2b_{j+1}b_jb_{j+1} =$$

$= (\gamma^{-1}b_j^2 - \alpha\gamma^{-1}b_j - \beta\gamma^{-1})b_{j+1}b_j^3b_{j+1}b_j = b_j^{-1}b_{j+1}b_j^3b_{j+1}b_j.$
which ends the proof of the sublemma.□
We are able now to prove our lemma. From above we have:
$w_{n,j}b_n = b_{n-1}^{-1}b_n(w_{n-1,j}b_{n-1})b_nb_{n-1} =$
$= b_{n-1}^{-1}b_nb_{n-2}^{-1}b_{n-1}(w_{n-2,j}b_{n-2})b_{n-1}b_{n-2}b_nb_{n-1} =$
$= ... = r_{n,j+1}^{-1}\left[b_nb_{n-1}...b_{j+2}(w_{j+1,j}b_{j+1})b_{j+2}...b_n\right]r_{n,j+1}.$
We denote $b_n...b_{j+2} = s$ and $b_{j+2}...b_n = t$ for simplicity. Thus
$w_{n,j}b_n = r_{n,j+1}^{-1}s(b_j^{-1}b_{j+1}b_j^3b_{j+1}b_j)tr_{n,j+1} =$
$\alpha r_{n,j+1}^{-1}s(b_j^{-1}b_{j+1}b_j^2b_{j+1}b_j)tr_{n,j+1} + \beta r_{n,j+1}^{-1}s(b_j^{-1}b_{j+1}b_jb_{j+1}b_j)tr_{n,j+1} +$
$+\gamma r_{n,j+1}^{-1}s(b_j^{-1}b_{j+1}^2b_j)tr_{n,j+1} = \alpha r_{n,j}^{-1}sb_{j+1}b_j^2b_{j+1}tr_{n,j}+\beta y+\gamma r_{n,j}^{-1}w_{n,j+1}r_{n,j},$
where we met
$y = r_{n,j}^{-1}(b_n...b_{j+1}b_jb_{j+1}...b_n)r_{n,j} = r_{n,j}^{-1}(b_n...b_jb_{j+1}b_j...b_n)r_{n,j} =$
$= r_{n,j}^{-1}b_j(b_n...b_{j+2}b_{j+1}b_{j+2}...b_n)b_jr_{n,j} =$
$= ... = r_{n,j}^{-1}(b_jb_{j+1}...b_{n-2}(b_nb_{n-1}b_n)b_{n-2}...b_j)r_{n,j} = b_nw_{n-1,j}.$
and this proves that
$w_{n,j}b_n = \alpha r_{n,j}^{-1}w_{n,j}r_{n,j} + \beta b_nw_{n-1,j} + \gamma r_{n,j}^{-1}w_{n,j+1}r_{n,j}.$
But $r_{n,j}w_{n,j} = w_{n,j}r_{n,j}$ according to the first commutation rule so we are done. □.
For $n = 2$ the relations of lemma read
$b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$
$b_2^2b_1^2b_2 = b_1b_2^2b_1^2 + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta(b_1^2b_2 - b_1b_2^2)$
$b_2b_1^2b_2^2 = b_1^2b_2^2b_1 + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta(b_2b_1^2 - b_2^2b_1).$

We claim that using these relations any word w in b_1 and b_2 is equivalent (as element of $H(Q,3)$) to a sum of words having the degree in b_2 at most 2. In fact if the degree in b_2 is at least 3 then it contains one of the monomials $b_2^ab_1^bb_2^c$ with $a+c \geq 3$ or $b_2b_1^bb_2b_1^cb_2$. We prove that in both situations the degree may be reduced.

In the first case if $b = 1$ then we replace $b_2b_1b_2$ by $b_1b_2b_1$. If $b = 2$ then a or c equals 2 so we can apply one of the above written relations.

In the second case if b or c equals 1 again we may replace $b_2b_1b_2$ by $b_1b_2b_1$. If $b = c = 2$ then
$b_2b_1^2b_2b_1^2b_2 = b_1b_2b_1^2b_1b_2 = b_1^2b_2b_1^2b_2^2$
so the third relation may be used to reduce the degree of w thus proving our claim. It follows that the following elements generates the vector space $H(Q,3)$:
$e_1 = 1, e_2 = b_1, e_3 = b_1^2, e_4 = b_2, e_5 = b_2^2, e_6 = b_1b_2, e_7 = b_2b_1, e_8 = b_1^2b_2,$

$e_9 = b_2b_1^2, e_{10} = b_1b_2^2, e_{11} = b_2^2b_1, e_{12} = b_1^2b_2^2, e_{13} = b_2^2b_1^2, e_{14} = b_1b_2b_1, e_{15} = b_1^2b_2b_1, e_{16} = b_1b_2b_1^2, e_{17} = b_1b_2^2b_1^2, e_{18} = b_1^2b_2b_1^2, e_{19} = b_1^2b_2^2b_1, e_{20} = b_1b_2^2b_1, e_{21} = b_1^2b_2^2b_1^2, e_{22} = b_2b_1^2b_2, e_{23} = b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2, e_{24} = b_2b_1^2b_2b_1^2 = b_1b_2b_1^2b_2b_1 = b_1^2b_2b_1^2b_2$.

Remark that for $\alpha = \beta = 0, \gamma = 1$, so $Q_0 = X^3 - 1$, the algebra $H(Q_0, 3)$ is the group algebra of a group of order 24. In fact $\{e_1, e_2, ..., e_{24}\}$ becomes a group in which the multiplication law is induced by the following identities

$b_2^2b_1^2b_2 = b_1b_2^2b_1^2$; $b_2b_1^2b_2^2 = b_1^2b_2^2b_1$; $b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$.

It follows that $H(Q_0, 3)$ is a semi-simple algebra hence for Q generic and sufficiently close to Q_0 the algebra $H(Q, 3)$ will be also a semi-simple algebra of the same dimension. This ends the proof of the proposition for generic Q close to Q_0. The complete proof for all Q is given in appendix. □

Remember that the Markov trace on the quadratic Hecke algebras (which is unique [Jon87]) has the following multiplicative property:

$tr(xb_n) = tr(x)tr(b_n)$, when $x \in H(Q, n)$,

which implies that:

$tr(xy) = tr(x)tr(y)$ when $x \in H(Q, n), y \in <1, b_n, b_{n+1}, ..., b_{n+k}>$.

However we cannot expect that this property will extend to higher level algebras and Markov traces on them. We say that a Markov trace t is multiplicative if

$t(xb_n^k) = t(x)t(b_n^k)$

holds when $x \in H(Q, n), k \in Z$. Also the trace will be called ideal if $t^{-1}(0) \subset H(Q, n)$ is an ideal, not only a vector subspace.

Remark 6.2.4 *In the case of cubic Hecke algebras the Markov traces are multiplicative.*

In fact we have $b_n^2 = \alpha b_n + \beta + \gamma b_n^{-1}$. We derive then the multiplicativity for $k = 2$, since for $k \in \{-1, 0, 1\}$ is already contained in the definition of the Markov traces. This will imply the multiplicative property for all k.□

Set then B for the base of $H(Q, 3)$ considered above. A general relation yielding a rank 3 quotient takes therefore the form:

$R(\mu) : \sum_{x \in B} \mu_x x = 0$, where $(\mu_x)_{x \in B} \in \mathbf{C}^{24}$.

Set $\omega = (\mu_{b_2b_1^2b_2}, \mu_{b_1b_2b_1^2b_2}, \mu_{b_1^2b_2b_1^2b_2}) \in \mathbf{C}^3$ and $M_Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & \beta & \alpha \end{bmatrix}$.
Let λ_i, $i = 1,3$ be the eigenvalues of M_Q and $E_Q = \{(1, \lambda_i, \lambda_i^2), i = 1,3\}$ be the eigenvectors of M_Q. Observe that $\lambda_i \neq 0$ since $\gamma \neq 0$. Consider now a rank 3 homogeneous quotient $P(\infty)$ of $H(Q,\infty)$. Its third term $P(3)$ is thus a proper quotient of $H(Q,3)$, hence several nontrivial relations $R(\mu)$ are fulfilled in $P(3)$.

Proposition 6.2.5 *Suppose that for some relation holding in $P(3)$ the following (generic) condition:*
(*) (the degree 2 polynomial having the vector of coefficients ω has no common roots with Q)
is fulfilled. Then for fixed $(z,t) \in \mathbf{C}^{*2}$ there exists at most one Markov trace on $P(\infty)$ with parameters (z,t).

Proof: Define recursively the modules L_n by
$L_2 = H(Q,2)$,
$L_3 = \mathbf{C} < b_1^i b_2^j b_1^k;\ i,j,k \in \{0,1,2\} >$,
$L_{n+1} = \mathbf{C} < ab_n^\varepsilon b;\ \varepsilon \in \{1,2\} > \oplus L_n$.
We claim that, under the natural projection π on $P(n)$, L_n surjects onto $P(n)$. For $n = 2$ it is trivial.
For $n = 3$ remark that
$\sum_{x\in B'} \omega_x x \in \pi(L_3)$
where we met $B' = \{b_2b_1^2b_2, b_1b_2b_1^2b_2, b_1^2b_2b_1^2b_2\}$. But L_3 is b_1-invariant so also
$\sum_{x\in B'}(M_Q\omega)_x x = \sum_{x\in B'} b_1 x \in \pi(L_3)$.
The hypothesis implies $b_2b_1^2b_2 \in \pi(L_3)$ and we are done.

Consider now $w \in P(n+1)$ represented by a word in the b_i's having only positive exponents. We assume the degree of the word in the variable b_n be minimal among all words (with positive exponents) representing w.

If the degree is less or equal to 1 there is nothing to prove.

If the degree is 2 then $w = ub_n^2v$, $u, v \in P(n)$ so using the induction hypothesis we are done, or else $w = ub_nzb_nv$, and $u,z,v \in P(n)$. So $z = xb_{n-1}^\varepsilon y$ with $x, y \in P(n-1)$ by the induction and $\varepsilon \in \{0,1,2\}$. If $\varepsilon = 0$ then w may be reduced to uzb_n^2v. If $\varepsilon = 1$ then $w = ub_nxb_{n-1}yb_nv = uxb_{n-1}b_nb_{n-1}yv$ hence the degree of w may be reduced by 1, contradiction. If $\varepsilon = 2$ then $w = uxb_nb_{n-1}^2b_nyv$. Because $P(n+1)$ is homogeneous we derive

$b_nb_{n-1}^2b_n \in \mathbf{C} < b_{n-1}^i b_n^j b_{n-1}^k; i,j,k = 0,2 >$,

hence we reduced w to a word of type $u'b_n^2v'$.

If the degree of w is at least 3 we shall contradict the minimality. In fact w contains a subword $w' = b_n^a u b_n^b$, $u \in P(n)$ and $a+b \geq 3$, or else a subword $w'' = b_nub_nvb_n$, $u,v \in P(n)$.

In the first case using the induction we can write $u = xb_{n-1}^\varepsilon y$, $x,y \in P(n-2)$.

If $\varepsilon = 0$ then

$w' = b_n^{a+b}xy = \alpha b_n^{a+b-1}xy + \beta b_n^{a+b-2}xy + \gamma b_n^{a+b-3}xy$,

hence its degree reduces by 1.

If $\varepsilon = 1$ then

$w' = b_n^{a-1}xb_nb_{n-1}b_nyb_n^{b-1} = b_n^{a-1}xb_{n-1}b_nb_{n-1}yb_n^{b-1}$,

again its degree being reduced by 1.

If $\varepsilon = 2$ then a or b equals 2. Say $a = 2$. We can write

$w' = xb_n^2b_{n-1}^2b_nyb_n^{b-1} = xb_1b_2^2b_1^2yb_n^{b-1} + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1)yb_n^{b-1} +$
$\beta(b_1^2b_2 - b_1b_2^2)yb_n^{b-1}$.

still contradicting the minimality of the degree of w.

In the second case we write also $u = xb_{n-1}^\varepsilon y$, $v = rb_{n-1}^\delta s$ with $x,y,r,s \in P(n-1)$.

If ε or δ equals 1 then, after some obvious commutation the word w'' contains a factor $b_2b_1b_2$ which, when replaced by $b_1b_2b_1$ reduces the degree by 1.

If $\varepsilon = \delta = 2$ then

$w'' = xb_nb_{n-1}^2b_nyrb_{n-1}^2b_ns$.

We use the homogeneity to replace $b_nb_{n-1}^2b_n$ by a sum of elements of type $b_{n-1}^ib_n^jb_{n-1}^k$. Each term of the expression of w'' which comes from a factor having $j < 2$ has the degree less than it had before. The remaining terms are

$xb_{n-1}^ib_n^2b_{n-1}^kyrb_{n-1}^2b_ns$,

so they contains a subword $b_n^2ub_n$ whose degree we already know that may be reduced as above. This proves our claim.

Now the Markov traces t on $H(Q,\infty)$ are multiplicative hence

$t(xb_n^\varepsilon y) = t(b_n^\varepsilon)t(yx)$,

and $P(n)$ it is an algebra hence $yx \in P(n)$. Therefore the extension of

t, by recursion, from $P(n)$ to $P(n+1)$ if ever exists it is unique. This ends the proof of our proposition. □

Remark 6.2.6 *We know that on $H(Q,\infty)$ there are at least 3 families of Markov traces whose parameters satisfy some linear conditions: $T = s_i Z + r_i$, $i = 1,3$, coming from the Markov traces on quadratic Hecke algebras. In fact we have natural projections $H(Q,\infty) \to H(Q_i,\infty)$, if Q_i is a degree 2 factor of Q.*

We shall begin therefore to investigate the case when $dim_{\mathbf{C}} P(3)$ is maximal in order to have more chance to find a quotient admitting a Markov trace. According to the above stated claim $dim_{\mathbf{C}} P(3) \leq 21$ for a generic quotient. Remark that we may always suppose $\gamma = 1$ because we have an isomorphism of algebras $H(Q,\infty) \cong H(\gamma^{-1}Q,\infty)$. Set then

$D = \{(\alpha,\beta) \in \mathbf{C}^2;\ H(Q,3)$ has a quotient with $dim_{\mathbf{C}} P(3) = 21\}$.

We shall suppose that the hypothesis of proposition 2.6 is fulfilled in what follows. We may state therefore:

Proposition 6.2.7 *For $\alpha = \beta = 0$ there is only one quotient (satisfying (*)) of dimension 21, say K_3, which is determined by the relation*

$b_2 b_1^2 b_2 + b_1 b_2^2 b_1 + b_1^2 b_2 b_1 + b_1 b_2 b_1^2 + b_1^2 b_2^2 + b_2^2 b_1^2 + b_1 + b_2 = 0$

Proof: Since the dimension of $P(3) = H(Q,3)/R(\mu)$ is 21 the following relations will become identities in $P(3)$:

$$b_1 b_2 b_1^2 b_2 = b_2 b_1^2 b_2 b_1, \tag{6.1}$$

$$b_2 R(\mu) = 0. \tag{6.2}$$

These equation are written in implicit form. We must express all factors in terms of basis elements from $B - B'$. We have seen that $R(\mu)$ may be reduced to a simpler relation

$b_2 b_1^2 b_2 = \sum_{i,j} \lambda_{ij} b_1^i b_2 b_1^j + \mu_{ij} b_1^i b_2^2 b_1^j + \sum_i \nu_i b_1^i$.

Lemma 6.2.8 *The equation (1) has the following solutions:*

$\mu_{ij} = \mu_{ji}$ and $\lambda_{ij} = \lambda_{ji}$ for all i,j.

$\mu_{00} = \mu_{12} - \beta\mu_{02};\ \mu_{22} = \mu_{01} + \alpha\mu_{02};\ \mu_{11} = (\alpha\beta+1)\mu_{02} + \beta\mu_{01} - \alpha\mu_{12}$

$\lambda_{00} = \lambda_{12} - \beta\lambda_{02};\ \lambda_{22} = \lambda_{01} + \alpha\lambda_{02};\ \lambda_{11} = (\alpha\beta+1)\lambda_{02} + \beta\lambda_{01} - \alpha\lambda_{12}$

Proof: We have

$b_1 b_2 b_1^2 b_2 = \sum_{ij} \lambda_{ij} b_1^{i+1} b_2 b_1^k + \mu_{ij} b_1^{i+1} b_2^2 b_1^k + \sum_i \nu_i b_1^{i+1} =$
$= \gamma\lambda_{02} b_2 + (\lambda_{00} + \beta\lambda_{02}) b_1 b_2 + \gamma\lambda_{21} b_2 b_1 + (\lambda_{01} + \beta\lambda_{12}) b_1 b_2 b_1 + (\lambda_{02} + \beta\lambda_{22}) b_1 b_2 b_1^2 + (\lambda_{01} + \alpha\lambda_{02}) b_1^2 b_2 + \gamma\lambda_{22} b_2 b_1^2 + (\lambda_{11} + \alpha\lambda_{12}) b_1^2 b_2 b_1 + (\lambda_{12} + \alpha\lambda_{22}) b_1^2 b_2 b_1^2 + \gamma\mu_{02} b_2^2 + (\mu_{00} + \beta\mu_{02}) b_1 b_2^2 + \gamma\mu_{21} b_2^2 b_1 + (\mu_{01} + \beta\mu_{12}) b_1 b_2^2 b_1 + (\mu_{02} + \beta\mu_{22}) b_1 b_2^2 b_1^2 + (\mu_{01} + \alpha\mu_{02}) b_1^2 b_2^2 + \gamma\mu_{22} b_2^2 b_1^2 + (\mu_{11} + \alpha\mu_{12}) b_1^2 b_2^2 b_1 + (\mu_{12} + \alpha\mu_{22}) b_1^2 b_2^2 b_1^2 + \gamma\nu_2 + (\nu_0 + \beta\nu_2) b_1 + (\nu_1 + \alpha\nu_2) b_1^2.$

and a similar expression (the symmetric one) may be written for $b_2 b_1^2 b_2 b_1$. By comparison of these expressions and using the hypothesis (that the basis of L_3 descends to one of $P(3)$) we find the relations stated above. □

In the same manner we can write the system of equations derived from (2). After we used the reductions from lemma 2.9 we obtain the following quadratic system (S): Set $c_1 = \lambda_{02} + \alpha\mu_{02}$; $c_2 = \lambda_{12} + \alpha\mu_{12}$; $c_3 = \lambda_{22} + \alpha\mu_{22}$

$$\begin{aligned}
0 &= \nu_0 + c_3\lambda_{01} + (c_2 + \alpha c_3 - \beta c_1 + \alpha\beta)\lambda_{02} + (c_1 - \alpha)\lambda_{12} + \alpha\beta\mu_{01} + \\
&\quad \beta(\alpha^2 - \beta)\mu_{02} + 2\beta\mu_{12} && (6.3)\\
0 &= (c_1 - \alpha + \beta c_3)\lambda_{01} + (\alpha\beta + 1)c_3\lambda_{02} + (c_2 + 1)\lambda_{12} && (6.4)\\
0 &= \nu_1 + (c_1 - \alpha + \beta c_3)\lambda_{01} + (\alpha\beta + 1)c_3\lambda_{02} + c_2\lambda_{12} + \beta(\alpha\beta + 2)\mu_{01} + \\
&\quad \alpha\beta(\alpha\beta + 1)\mu_{02} + \beta^2\mu_{12} && (6.5)\\
0 &= (1 - \alpha\beta + \beta c_1 + c_2 + \beta^2 c_3)\lambda_{01} + (\alpha\beta + 1)(c_1 + \beta c_3 - \alpha)\lambda_{02} + \\
&\quad (\beta + \alpha^2 - \alpha c_1 + \beta c_2 - \alpha\beta c_3)\lambda_{12} && (6.6)\\
0 &= (\beta + (\beta^2 + \alpha\beta + 1)c_3)\lambda_{01} + \\
&\quad (\alpha\beta + 1 + c_2 + (\alpha + \beta)(\alpha\beta + 1)c_3)\lambda_{02} + \\
&\quad (c_1 + \beta c_2 - \alpha\beta c_3 - \alpha)\lambda_{12} && (6.7)\\
0 &= (c_2 + \alpha c_3)\lambda_{01} + (c_1 + \alpha c_2 + \alpha^2 c_3 - \alpha)\lambda_{02} + c_3\lambda_{12} + \mu_{12} - \beta && (6.8)\\
0 &= \nu_2 + ((c_2 + \alpha c_3)\lambda_{01} + (c_1 + \alpha c_2 + \alpha^2 c_3 - \alpha)\lambda_{02} + c_3\lambda_{12} + \\
&\quad \beta(\alpha^2 + \beta)\mu_{01} + (\alpha\beta(\alpha^2 + \beta) + 2\beta)\mu_{02} + \alpha\beta\mu_{12} && (6.9)\\
0 &= (\beta c_2 + (\alpha\beta + 1)c_3)\lambda_{01} + ((\alpha\beta + 1)c_2 + \alpha(\alpha\beta + 1 - \alpha)c_3)\lambda_{02} + \\
&\quad (c_1 + (\alpha^2 + \beta)c_3 - \alpha)\lambda_{12} + \mu_{01} + \beta\mu_{12} && (6.10)\\
0 &= (c_1 + \alpha c_2 + (\alpha^2 + \beta)c_3 - \alpha\beta)\lambda_{01} + \\
&\quad (\alpha c_1 + \alpha^2 c_2 + (\alpha(\alpha^2 + \beta) + 1))c_3 - \alpha(\alpha\beta + 1))\lambda_{02} +
\end{aligned}$$

$$
\begin{aligned}
& (c_2+\alpha c_3+\alpha^2)\lambda_{12}+(\alpha\beta+1)\mu_{02}+\beta\mu_{01} && (6.11)\\
0 &= \lambda_{12}-\beta\lambda_{02}+(c_3-\beta)\mu_{01}+(c_2-\beta c_1+\alpha c_3-\alpha\beta)\mu_{02}+c_1\mu_{12} && (6.12)\\
-\beta &= (c_1+\beta c_3-2\alpha)\mu_{01}+((\alpha\beta+1)c_3-\alpha^2)\mu_{02}+c_2\mu_{12} && (6.13)\\
0 &= \lambda_{01}+(c_1+\beta c_3-\beta^2)\mu_{01}+(\alpha\beta+1)(c_3-\beta)\mu_{02}+c_2\mu_{12} && (6.14)\\
-\alpha &= (\beta c_1+c_2+\beta^2 c_3-2\alpha\beta)\mu_{01}+(\alpha\beta+1)(c_1+\beta c_3-2\alpha)\mu_{02}+ \\
& (\beta c_2-\alpha c_1+c_3+\alpha^2)\mu_{12} && (6.15)\\
1 &= (\beta c_2+(\alpha\beta+1)c_3-\alpha^2)\mu_{01}+ \\
& ((\alpha\beta+1)c_2+\alpha(\alpha\beta+1)c_3-\alpha^3)\mu_{02}+(c_1+\beta c_3-2\alpha)\mu_{12} && (6.16)\\
0 &= (c_2+\alpha c_3+1)\mu_{01}+(c_1+\alpha c_2+\alpha^2 c_3)\mu_{02}+c_3\mu_{12} && (6.17)\\
0 &= \lambda_{12}+(c_2+\alpha c_3-\alpha\beta)\mu_{01}+(c_1+\alpha c_2+\alpha^2 c_3-\alpha^2\beta)\mu_{02}+ \\
& (c_3-\beta)\mu_{12} && (6.18)\\
0 &= (\beta c_2+(\alpha\beta+1)c_3+\beta)\mu_{01}+(\alpha\beta+1)(c_2+\alpha c_3+1)\mu_{02}+ \\
& (c_1+(\beta-\alpha)c_3)\mu_{12} && (6.19)\\
0 &= (c_1+\alpha c_2+(\alpha^2+\beta)c_3)\mu_{01}+\alpha(c_1+\alpha c_2+(\alpha^2+\beta)c_3)\mu_{02}+ \\
& (c_2+\alpha c_3+1)\mu_{12} && (6.20)\\
0 &= c_3\nu_0+(c_2+\alpha c_3+1)\nu_1+(c_1+\alpha c_2+(\alpha^2+\beta)c_3-\alpha)\nu_2+\mu_{02} && (6.21)\\
0 &= c_2\nu_0+(c_1+\beta c_3-\alpha)\nu_1+(\beta c_2+(\alpha\beta+1)c_3)\nu_2+\mu_{01} && (6.22)\\
0 &= (c_1-\alpha)\nu_0+c_3\nu_1+(c_2+\alpha c_3)\nu_2+\mu_{12}-\beta\mu_{02}=0 && (6.23)
\end{aligned}
$$

We are able now to prove the first part of the proposition. Remark that, for $\alpha=\beta=0$ we obtain an unique solution for (S) namely:
$\mu^0_{01}=\mu^0_{12}=0; \mu^0_{02}=-1; \lambda^0_{01}=\lambda^0_{02}=0; \lambda^0_{12}=-1,$
$\nu^0_0=\nu^0_2=0; \nu^0_1=-1.$
What it remains to show is that this solution gives indeed a quotient of dimension 21. So let Q be X^3-1 from now on. Let I be the ideal generated by the relation
$R_0=b_2b_1^2b_2+b_1b_2^2b_1+b_1^2b_2b_1+b_1b_2b_1^2+b_1^2b_2^2+b_2^2b_1^2+b_1+b_2.$
Set also
$R_1=b_1b_2b_1^2b_2+b_1^2b_2b_1^2+b_1b_2^2b_1^2+b_1^2b_2^2b_1+B_1b_2+b_2b_1+b_1^2+b_2^2$
$R_2=b_1^2b_2b_1^2b_2+b_1^2b_2^2b_1^2+b_1^2b_2+b_2b_1^2+b_1b_2^2+b_2^2b_1+b_1b_2b_1+1=0,$
and denote by $R\subset H(Q,3)$ the span of R_0, R_1, R_2.

Lemma 6.2.9 *We have an isomorphism of vector spaces $I\cong R$.*

Proof: Remark that

$b_1R_0 = R_0b_1 = R_1; b_1R_1 = R_1b_1 = R_2; b_1R_2 = R_2b_1 = R_0$.

Since $b_j^3 = 1$ we derive $b_1b_2^2b_1^2 = b_2^2b_1^2b_2$. This implies also

$b_2R_0 = R_0b_2 = R_1; b_2R_1 = R_1b_2 = R_2; b_2R_2 = R_2b_2 = R_0$.

From these relation we find $xR_0y \in R$ for all $x, y \in H(Q,3)$, hence $I \subset R$. The other one inclusion is trivial. □

Finally $K_3 = H(Q,3)/I$ will have dimension 21 as wanted since it is easy to see that $dim_{\mathbf{C}}R = 3$. This ends the proof of the proposition. □

In order to study the Markov traces on rank 3 quotients which begins with K_3 it is sufficiently to look at the homogeneous quotients, namely

$K_n =< 1, b_1, ..., b_{n-1} \mid b_{i+1}b_ib_{i+1} = b_ib_{i+1}b_i$ for $i = 1, n-1$;
$b_ib_j = b_jb_i$; for $\mid i-j \mid > 1$; $b_i^3 = 1$ for all i
$b_{i+1}b_i^2b_{i+1} = -b_ib_{i+1}^2b_i - b_i^2b_{i+1}b_i - b_ib_{i+1}b_i^2 - b_i^2b_{i+1}^2 - b_{i+1}^2b_i^2 - b_i - b_{i+1} >$.

Remark first that K_n has an obvious deformation over $\mathbf{C}^*$ given by

$K_n(\gamma) =< 1, b_1, ..., b_{n-1} \mid b_{i+1}b_ib_{i+1} = b_ib_{i+1}b_i$ for $i = 1, n-1$;
$b_ib_j = b_jb_i$; for $\mid i-j \mid > 1$; $b_i^3 = \gamma$ for all i
$b_{i+1}b_i^2b_{i+1} = -b_ib_{i+1}^2b_i - b_i^2b_{i+1}b_i - b_ib_{i+1}b_i^2 - b_i^2b_{i+1}^2 - b_{i+1}^2b_i^2 - \gamma b_i - \gamma b_{i+1} >$.

Remark 6.2.10 *There is in fact exactly one solution for the system (S) for general α and β which is polynomial in this parameters. This was pointed to me by P. Vogel.*

6.3 Markov traces on $K_\infty(\gamma)$

Let now work with the algebra $\mathbf{Z}[B_\infty]$ instead of $\mathbf{C}[B_\infty]$. Let P_* be a quotient of $\mathbf{Z}[B_\infty]$. Consider $A(z,\bar{z})$ be the smallest subring of $\mathbf{C}$ containing $z, \bar{z} \in \mathbf{C}^*$.

Definition 6.3.1 *i) Let R be a $A(z,\bar{z})$-module. The module $AF(P_*, R)(z,\bar{z})$ of admissible functionals on P_* taking values in R is the set of those*
$t \in Hom_{A(z,\bar{z})}(P_\infty, R)$ satisfying
$t(xb_ny) = zt(xy)$ for $x, y \in P_n$,
$t(xb_n^{-1}y) = \bar{z}t(xy)$ for $x, y \in P_n$.
ii) The module of Markov traces with values in R is
$MT(P_*, R)(z,\bar{z}) = AF(P_*^{ab}, R)(z,\bar{z})$,
where $P_k^{ab} = P_k/[P_k, P_k]$ with the induced inductive system structure. Observe that P_k^{ab} are only modules not algebras.

iii) We define the link module of P_* with parameters $(z, \bar{z})$ as
$L(P_*)(z,\bar{z}) = P^{ab}_\infty / << xb_n - zx, xb_n^{-1} - \bar{z}x; x \in P_n >>$
$L(P_*) = L(P_*)(1,1)$,
where <<>> stands for the module spanned by the considered elements.

If P_* is defined by homogeneous relations in each rank then $L(P_*)(z,\bar{z})$, as abelian group, is isomorphic to $L(P_*)$ via the map
$x \in P_n \rightarrow (\frac{1}{z\bar{z}})^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} x$,
where $e(x)$ is the exponent sum for words. Observe also that the Markov traces descends to $L(P_*)(z,\bar{z})$ and we have
$MT(P_*,R)(z,\bar{z}) = Hom_{A(z,\bar{z})}(L(P_*)(z,\bar{z}),R)$,
so the knowledge of Markov traces is enlightening when computing $L(P_*)$. We have natural morphisms
$L(\mathbf{Z}[B_\infty])(z,\bar{z}) \longrightarrow L(P_*)(z,\bar{z})$
and their duals
$MT(P_*,R)(z,\bar{z}) \longrightarrow MT(\mathbf{Z}[B_\infty])(z,\bar{z})$.
Let $\mathcal{L}$ be the set of isotopy classes of oriented links and $\mathbf{Z} << \mathcal{L} >>$ be the free abelian group generated by $\mathcal{L}$.

Lemma 6.3.2 *The mapping $\mathbf{Z} << \mathcal{L} >> \longrightarrow L(\mathbf{Z}[B_\infty])$ defined by*
$L \rightarrow$ *class of x in $L(\mathbf{Z}[B_\infty])$,*
where x is some braid word representing L, is an isomorphism.

The proof follows from Markov's theorem in a straightforward manner.
□

Example 6.3.3 *If $P_* = H(q,*)$ is the usual quadratic Hecke algebra then from Proposition 2.4. we derive*

$$L(H(q,*))(z,\bar{z}) = \begin{cases} A(z,\bar{z}) & \text{if } z = q - 1 + q\bar{z} \\ 0 & \text{elsewhere} \end{cases}$$

We can state the main result of this section:

Theorem 6.3.4 *We have*

$$L(K_*(\gamma))(z,\bar{z}) = \begin{cases} A(z,\bar{z})/6z^7A(z,\bar{z}) & \text{if } z^3+\gamma = 0, \bar{z} = -z^2/\gamma \\ 0 & \text{elsewhere} \end{cases}$$

Proof: In order to get the result we need the description of Markov traces on $K_*(\gamma)$. Firstly we wish to deal with the module of admissible functionals. We shall use the following type of presentation of a module:

$M = A < x_1, x_2, ..., x_p \mid r_1, r_2, ..., r_q \mid\mid w_1, w_2, ..., w_s >$

which have to be read as follows: $x_1, ..., x_p$ generates the A-algebra $\bar{A}$ whose defining relations are $r_1, ..., r_q$. Therefore M is the quotient of $\bar{A}$ by the submodule spanned by the images of $w_1, ..., w_s$ in $\bar{A}$.

Consider now the following sets of words in the b_i's:

$W_1 = \{1\}$,

$W_{n+1} = W_n \cup W_n b_n Z_n \cup W_n b_n^2 Z_n$.

$Z_n = \{b_{n-1}^{i_1} b_{n-2}^{i_2} ... b_{n-p}^{i_p}; i_1, i_2, ..., i_p \in \{1, 2\}, p = 1, n-1\}$. First of way

Lemma 6.3.5 *We have a surjection of (K_n, K_n)-bimodules*

$K_n \oplus K_n \otimes_{K_{n-1}} K_n \oplus K_n \otimes K_{n-1} K_n \longrightarrow K_{n+1}$

given by

$x \oplus y \otimes z \oplus u \otimes v \rightarrow x + y b_n z + u b_n^2 v.$

The proof follows from that of proposition 2.6.□

As a corollary we derive that $AF(K_n, R)(z, \bar{z})$ is $R \otimes_{A(z,\bar{z})} M$ where M has the module presentation

$$\left\langle \begin{array}{l|l||l} & b_{i+1}b_ib_{i+1} = b_ib_{i+1}b_i & \\ & b_i^3 = \gamma & ab_ib = zab \\ b_1, b_2, ..., b_{n-1} & b_ib_j = b_jb_i, \mid i-j \mid > 1 & ab_i^2b = tab \\ & b_{i+1}b_i^2b_{i+1} = S_i & a, b \in W_i, i = 1, n-1. \end{array} \right\rangle$$

where $t = \gamma\bar{z}$,

$S_i = -(b_ib_{i+1}^2b_i + b_i^2b_{i+1}^2 + b_{i+1}^2b_i^2 + b_i^2b_{i+1}b_i + b_ib_{i+1}b_i^2 + \gamma b_i + \gamma b_{i+1})$

and the algebra $A = A(z, \bar{z})$. We shall use this presentation for proving first

Proposition 6.3.6 *The module of admissible functionals is*

$AF(K_*(\gamma), R)(z, \bar{z}) = R/(12\gamma^3 z + 8\gamma^2 z^2 \bar{z} - 4\gamma^3 \bar{z})R.$

Proof: Most of this section will be concerned with the proof of this proposition. Firstly from lemma 3.5 we derive that an admissible functional, if ever exists, is unique up to the choice of $t(1) \in R$. Look now at the algebra $K_n(\gamma)$. We wish to use the following transforms on the words

$b_{i+1}b_ib_{i+1} \rightarrow b_ib_{i+1}b_i$
$b_i^3 \rightarrow \gamma$
$b_{i+1}b_i^2b_{i+1} \rightarrow S_i$
and only in this direction, in order to reduce the degree of b_{n-1} as much as possible. According to lemma 3.5 every word is equivalent to a sum of words of type $\sum_i x_i b_{n-1}^{\epsilon_i} y_i$. Unfortunately we are forced to use the relations
$b_ib_j \leftrightarrow b_jb_i$ for $\mid i-j \mid > 1$,
in both directions. Assume this is the process we shall carry out. So we obtain finally a sum $\sum_i x_i b_{n-1}^{\epsilon_i} y_i$ with $x_i, y_i \in K_{n-1}(\gamma)$. Of course this "normal form " for the word we started with is not unique since we may perform again permutations of its letters in each term. But if any two such normal forms would be equivalent under eventual permutations of its letters (of b_ib_j with $\mid i-j \mid > 1$ always!) we should have an almost canonical description of the basis of $K_n(\gamma)$. Indeed the last assumption is equivalent to say that the surjection of lemma 3.5 is actually an isomorphism. However this is not the case. We can at least to obtain the obstructions to the unicity of this almost canonical form. We return now to the module of admissible functionals. The last group of relations enables us to make a further reduction, namely
$ab_{n-1}b \rightarrow z\, ab$
$ab_{n-1}b \rightarrow t\, ab.$
This way we may reduce finally a word to a sum of words lying in $K_{n-2}(\gamma)$. Assume that we are using a recurrence on n. Then each element of $K_{n-2}(\gamma)$ may be uniquely reduced to an element of R (the value of the functional on the element). So it suffices to check the obstructions directly on the values in order to obtain that the functional is well-defined. This is what we shall formalize now.

Let Γ be a semi-oriented graph. This means that the edges have two types: oriented edges and unoriented ones. A path $v_1v_2...v_n$ is a semi-oriented path if for all j or $v_j \rightarrow v_{j+1}$ or else v_jv_{j+1} is unoriented. If all edges are unoriented we say that its endpoints are unoriented equivalents. We states first the pentagon condition for semi-oriented graphs:

(PC) If $v_2 \rightarrow v_1$, $v_2v_3...v_{n-1}$ is an unoriented path, $v_{n-1} \rightarrow v_n$ then there exists semi-oriented paths $v_1x_1x_2...x_me$ and $v_ny_1y_2...y_pe$ having the same endpoint (see figure 1).

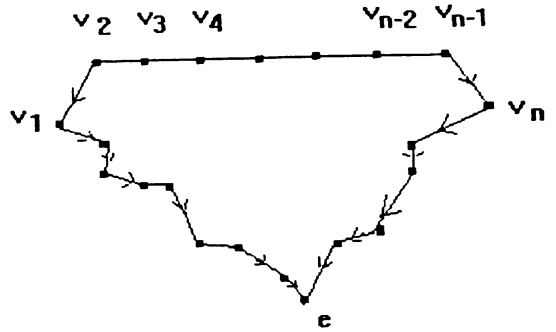

Figure 6.1: The Pentagon Condition

We say now that $x \leq y$ if there exists an semi-oriented path from y to x in Γ. Of course $\leq$ is not always a partial order relation. It is necessary and sufficient that no closed semi-oriented loops exists in Γ. However we can say that x is minimal if $y \leq x$ implies that y is unoriented equivalent to x.

Pentagon lemma 6.3.7 *Suppose that the (PC) holds. If a connected component C of the graph Γ has a minimal element m_C then it is unique up to unoriented equivalence.*

Proof of lemma: Consider two minimal elements x and y which lie in C. Then there exists some path $xx_0x_1...x_ny$ joining them. Since x is minimal the closest oriented edge (if ever exists) is ingoing, and the same is true for y. If this path is not unoriented and again from minimality there are at least two oriented edges. Therefore open pentagon configurations (i.e. those configurations where (PC) applies) exist. We apply then (PC) iteratively whenever such configurations exist or has appeared. When this process stops we find two semi-oriented $xz_1z_2...z_pe$ and $yu_1u_2...u_se$ having the same endpoint e. So $e \leq x$ and $x \leq y$. Again from minimality these paths must be unoriented so x and y are unoriented equivalent. (see figure 2) □

Remark that a priori we can nothing say about the existence of such minimal elements. If $\leq$ would be a partial order with descending chain condition then the existence of minimal elements is standard. However in the case which we shall work out even if $\leq$ is not a partial order the existence of minimal elements can be established.

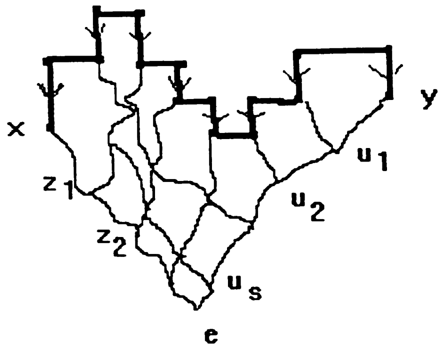

Figure 6.2: Proof of Pentagon Lemma

Suppose now we have a more complicated structure: a sequence of disjoint graphs Γ_n. In every Γ_n there exists a distinguished subset of vertices V_n^0 which are minimal elements in their connected components.Suppose that each connected component admits at least one minimal element. Each such vertex from V_n^0 has exactly one outgoing edge going to a vertex of Γ_{n-1}. We color these new edges in red. Set Γ_n^* for the union of all Γ_j, $j \leq n$ and with the red edges added in each rank j. We say that Γ_n^* is coherent if any connected component of Γ_n has an unique minimal element (with respect to Γ_n^*) in Γ_0 up to unoriented equivalence. We shall state now the colored version of Pentagon lemma for this type of graphs.

(CPC) If $v_1 v_2 ... v_n$ is an open pentagon configuration in Γ_n then there exists bicolored semi-oriented paths (in Γ_n^*) from v_1 and v_2 having the same endpoint. In addition if xy is an unoriented edge in Γ_n with $x, y \in V_n^0$ then there exists semi-oriented paths in Γ_n^* starting with red edges and having the same endpoint (see the figure 3).

Colored Pentagon Lemma 6.3.8 *Suppose that Γ_{n-1}^* is coherent and the (CPC) condition is fulfilled. Then Γ_n^* is coherent.*

The proof is similar to that of Pentagon Lemma. □.
We are ready define now our graph Γ_n. Its vertices are the elements of

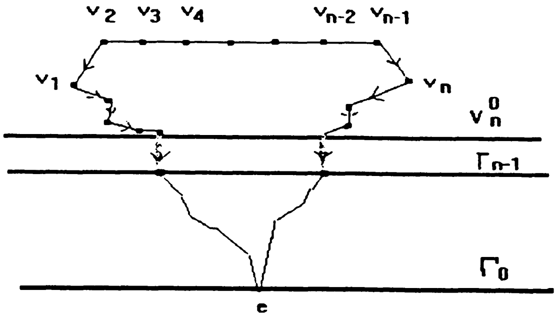

Figure 6.3: The colored pentagon condition

the group algebra of the free monoid F_n in $(n-1)$ letters $\{b_1, b_2, ..., b_{n-1}\}$. Two vertices $v = \sum_i \alpha_i x_i$ and $w = \sum_i \beta_i y_i$, $\alpha_i, \beta_i \in A(z, \bar{z})$ are related by an oriented edge if exactly one monomial of v is changed following one of the rules

(C0)(j) $Ab_j^3 B \rightarrow AB$
(C1)(j) $Ab_{j+1}b_jb_{j+1}B \rightarrow Ab_jb_{j+1}b_jB$
(C2)(j) $Ab_{j+1}b_j^2b_{j+1}B \rightarrow AS_jB$
(C12)(j) $Ab_{j+1}b_j^2b_{j+1}^2B \rightarrow Ab_j^2b_{j+1}^2b_jB$
(C21)(j) $Ab_{j+1}^2b_j^2b_{j+1}B \rightarrow Ab_jb_{j+1}^2b_j^2B$.

Also an unoriented edge between v and w correspond to a change in a monomial of v of type

(P_{ij}) $Ab_ib_jB \rightarrow Ab_jb_iB$ whenever $| i-j |> 1$.

Remark that the use of (C12) and (C21) is somewhat ambiguous since we may always use (C2) for a subword. Their role is to break in some sense the closed oriented loops in Γ_n. In fact consider V_n^0 be the set of vertices corresponding to elements of the free abelian $A = A(z, \bar{z})$-module generated by W_n.

Lemma 6.3.9 *Each connected component of Γ_n has a minimal element in V_n^0, not necessarily unique.*

Proof of lemma: We prove our claim by induction on n. For $n = 1$ it is trivially. Say now w is a word in the b_i's having only positive exponents. If its degree in b_{n-1} is zero or one we apply the induction hypothesis and we are done. If the degree is 2 and only one b_{n-1}^2 we are already in position to apply the induction hypothesis. Also we may suppose that no exponents greater than 2 occur by using (C0) several times. If the degree is 2 then $w = xb_{n-1}yb_{n-1}z$ with $x, y, z \in F_{n-1}$. The induction applied to y implies that $w \geq xb_{n-1}ab_{n-2}^{\varepsilon}bz$ with $a, b \in F_{n-2}$. Then several transforms of type (P_{n-1j}) and $(C\varepsilon)$ will do the job. Consider now that the degree is strictly greater than 2. So we have a subword of type

$b_{n-1}^{\alpha}xb_{n-1}^{\beta}$ with $3 \leq \alpha + \beta \leq 4$
or else one of the type

$b_{n-1}xb_{n-1}yb_{n-1}$. The second case reduce to the first one as above. Next say that $x \geq ab_{n-2}^{\varepsilon}b$, $a, b \in F_{n-2}$. Several applications of (P_{n-1j}) leads us to consider the word $b_{n-1}^{\alpha}b_{n-2}^{\varepsilon}b_{n-1}^{\beta}$. If $\varepsilon = 1$ we apply two times (C1) and we are done. Otherwise we shall apply $(C\alpha\beta)$ if $\alpha \neq \beta$ or both (C12) and (C21) if $\alpha = \beta = 2$. This proves that every vertex descends to V_n^0. But these vertices have not outgoing edges as can be easily seen. When we use the unoriented edges some new vertices have to be added. But it is easy to see that these also does not have outgoing edges. Since any vertex has a semi-oriented path ending in V_n^0 we are done. □

We remark that the moves (C12) and (C21) are really necessary for the conclusion of the above remain valid. In fact from $b_2b_1^2b_2^2$ only (C2) may be applied. We obtain a factor $b_2^2b_1^2b_2$. If we continue, at each stage we shall find one of these two monomials. When all reductions are used at the second stage we recover $b_2b_1^2b_2^2$ so we have a closed oriented loop in the graph. Its connected component should not have a minimal element without the use of (C12) or (C21).

We are able now to define the bicolored graph $\Gamma_n^*(z, t)$. The red edges are added as follows: Each minimal vertex $v = \sum_i \alpha_i x_i b_n^{\varepsilon_i} y_i$, is related to $w = \sum_i \alpha_i z_i x_i y_i$ which this time is a vertex of Γ_{n-1}, where we met $z_0 = 1, z_1 = z, z_2 = t$. Since $b_j^3 = \gamma$ we have $t = \gamma\bar{z}$ and we prefer working with (z, t) instead with the couple of parameters $(z, \bar{z})$. Finally we define $\Gamma_0(H)$ as the graph having the vertices corresponding to the module R. Two vertices are connected by an unoriented edge iff the corresponding elements lie in the same coset of R/H, H being a certain submodule of R. Let also $H_0 = (12\gamma^3 z - 4\gamma^3\bar{z} + 8\gamma^2\bar{z}^2z^2)R$ be a fixed cyclic submodule of R. We specify the choice of H in the tower bicolored graph by denoting it $\Gamma_n^*(H)$, even if only the first stage depends upon H. With these notations we can state:

Proposition 6.3.10 *The bicolored graph $\Gamma^*_\infty(H)$ is coherent if and only if $H_0 \subset H$.*

Before we proceed remark that this result has as an immediate corollary the proposition 3.6. Indeed for $H = H_0$ we know how associate to each element of $K_n(\gamma)$ an element of R. The only problem which we can encounter is that moving on different descending paths we should obtain different elements. But the previous proposition states the unicity of the endpoint viewed as an element of R/H_0. So our claim follows.
Proof: We shall prove the coherence of each $\Gamma^*_n(H)$ by recurrence on n. For $n = 1, 2$ this may be easily established without any condition on H. We wish to make use the Colored Pentagon Lemma. For instance we shall look only at the Pentagon Condition in Γ_n. For those configurations that we cannot prove the (PC) directly we shall check that the (CPC) (which is weaker since it regards all of $\Gamma^*_n(H)$) is still verified. Of course this implies that we may apply the Colored Pentagon Lemma.

Consider an open pentagon configuration (abbrev. o.p.c.) $[w_0, w_1, ..., w_n]$. So $w_1 \to w_0, w_1, ..., w_{n-1}$ are unoriented equivalent and $w_{n-1} \to w_n$. We say that this o.p.c. is irreducible if none of the vertices $w_1, w_2, ..., w_{n-1}$ has an outgoing edge.

Reduction Lemma 6.3.11 *i) In order to verify (PC) it suffices to restrict to irreducible configurations.*

ii) It suffices to verify (PC) only for monomials from F_n.

iii) Suppose $w'_j = Aw_jB$, for $j = 0, n$ (so A, B are not touched by any transform) in the o.p.c.. If (PC) holds for $[w_0, w_1, ..., w_n]$ it also holds for $[w'_0, w'_1, ..., w'_n]$.

iv) Suppose that (PC) holds for $[w_0, w_1, ..., w_n]$ and for $[y_0, y_1, ..., y_m]$. Then for all A, B, C the (PC) is valid also for

$[Aw_0By_1C, Aw_1By_1C, ..., Aw_{n-1}By_1C, Aw_{n-1}By_2C, Aw_{n-1}By_3C, ...$
$.., Aw_{n-1}By_{m-1}C, Aw_{n-1}By_mC]$.

In fact when we fix the endpoints of the o.p.c. we can mixed the unoriented edges of each subjacent o.p.c. in any order we want. Let $(i_k, j_k) \in \{0, 1, ..., n\} \times \{0, 1, ..., m\}, k = 1, p$ such that $i_0 = 0 < i_1 \le i_2 \le ... \le i_p, j_p = m > j_{p-1} \ge ... \ge 0$, and $i_{k+1} - i_k + j_{k+1} - j_k = 1$ for all k. Then the o.p.c.

$\left[Aw_{i_0}By_{j_0}C, Aw_{i_1}By_{j_1}C, ..., Aw_{i_p}By_{j_p}C\right]$

fulfills the (PC).

Proof: i) We may always decompose a configuration into irreducible ones and iterate the construction.

ii) The reduction transforms on different monomials commute with each other so we are done.

iii) Obvious.

iv) The reductions of x_{n-1} and y_1 commute again with each other. □

Thus the top line of a o.p.c. corresponds to a word w_1 and a sequence of permutations of its letters giving in order $w_2, w_3, ..., w_{n-1}$. We may suppose that $w = w_1$ has no proper subwords w_1' which fulfill the following two conditions:

i) Say $w = Aw'B$. Then each of the considered permutations acts on the letters of A, of B or w'. So there is an equivalent of w' say w".

ii) The reduction transforms performed at w_1 and w_2 acts actually on w' and w".

In other words only those letters may be not permuted which enter in a block which is reduced. Also it follows that or $n = 2$ so the top line is trivial and exactly two outgoing edges are incident in w_1 or else the corners w_1 and w_{n-1} have each exactly one outgoing edge, except for the case when a move (Cij) may be applied. When such an edge (Cij) exists then of course also (C2) exists so there are two outgoing edges. We shall choose always the edge (Cij) if ever exists and we shall say that the corresponding word has unique reduction. If ever in our o.p.c. the outgoing edge from w is an (C2) even if a (Cij) may be performed then we know that a semi-oriented loop exists permitting to come back in w. So the (PC) is trivially satisfied.

Now the top line is determined by the sequence of transpositions of the letters of w. Let l be the length of w. Otherwise this is the same to giving a permutation $\sigma \in S_l$ with a prescribed decomposition into transpositions. Set T_l for the transposition which interchanges the letters on the positions l and $l+1$. Notice that for a fixed w not all σ are suitable. In fact only a subset of the group of permutations, which we call permitted may work. Say $P(w)$ is the set of permitted permutations. If $e_w : \{1, 2, ..., l\} \longrightarrow \{1, 2, ..., n-1\}$ is the evaluation map

$e_w(j)$ =index of the letter lying in position j on w

then $T_j\sigma$ is permitted (where $\sigma \in P(w)$) iff

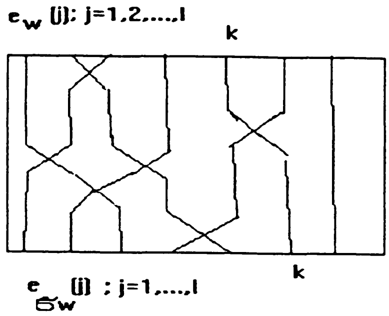

Figure 6.4: The complete diagram associated to an o.p.c.

$| e_{\sigma(w)}(j) - e_{\sigma(w)}(j+1) |> 1$.
Say that two permitted permutations σ and σ' are equivalent if for the o.p.c. corresponding to σ and σ' the (PC) is valid or not for both in same time.

Lemma 6.3.12 *i) Suppose that $\sigma_1 T_j T_i \sigma_2 \in P(w)$, $| i - j |> 1$. Then $\sigma_1 T_i T_j \sigma_2 \in P(w)$ and these two permutations are equivalent.*

ii) Suppose that $\sigma_1 T_{i+1} T_i T_{i+1} \sigma_2 \in P(w)$. Then $\sigma_1 T_i T_{i+1} T_i \sigma_2 \in P(w)$ and these two permutations are equivalent. The converse is still true.

iii) If $\sigma_1 T_i^2 \sigma_2 \in P(w)$ then $\sigma_1\sigma_2$ is permitted and equivalent to the previous one.

Proof: The existence in the first case is equivalent to
$| e_{\sigma_2(w)}(j) - e_{\sigma_2(w)}(j+1) |> 1$
and
$| e_{\sigma(w)}(i) - e_{\sigma(w)}(i+1) |> 1$,
so it is symmetric. In the second case also it is equivalent to
$| e_{\sigma_2(w)}(j+\varepsilon_1) - e_{\sigma_2(w)}(j+\varepsilon_2) |> 1$ for all $\varepsilon_j \in \{0,1,2\}$,
so it is again symmetric. The equivalence is trivial. □
We shall use a graphical representation for the decomposition of σ into transpositions similar to the braid pictures (see picture 4), were we specify on the top and bottom lines of the diagram the values of the evaluation maps. This picture encodes all information about the o.p.c. because the two words w and $\sigma(w)$ have unique reduction. Initially we are interested only in drawing the trajectories of the six (to ten) elements which enter in the two blocks which reduces. Suppose for instance that the two reduction moves are two (C0). So $w = xiiiy$ and

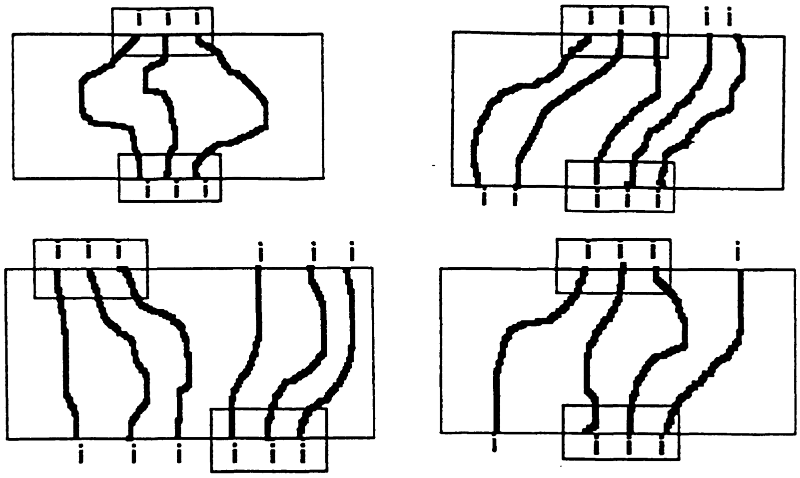

Figure 6.5: The essential trajectories for (CO)(i)-(C0)(i)

$\sigma(w) = x'jjjy'$. Say that $i = j$. The trajectories of the i's may be disjointed since the transposition acting on the couple ii is trivial in fact. So the possible trajectories fit into 4 cases which may be seen in picture 5.a,b,c,d.

Suppose now we have two trajectories of i and $j \neq i$ which intersects. First of way we derive that $| i - j | > 1$. Orient all the arcs from the top to the bottom.

Lemma 6.3.13 *i) Suppose that the arcs labelled i and j have algebraic intersection number 0. Then we can replace the diagram by an equivalent one where the arcs are disjoint.*

ii) Suppose that the arcs labelled i and j have algebraic intersection number 1. Then we can replace the diagram by an equivalent one where the arcs have exactly one intersection point.

Proof: We consider the diagram is that from figure 6.

We can assume that the biangle in the middle is minimal, hence it does not contain any other biangle. In fact we can apply repeatedly the disjointedness procedure only for minimal biangles. Such biangle have two walls: one coming from i and the other from j. From minimality no other arc cross twice the same wall (see picture 7).

Let consider the region L and R such that: the set of arcs labelled by something not commuting with j is contained in L, and those labelled

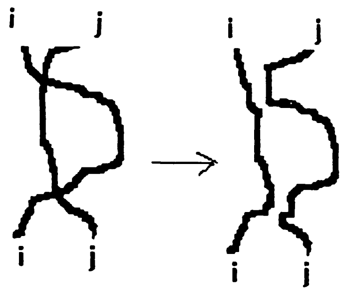

Figure 6.6: Disjointing trajectories

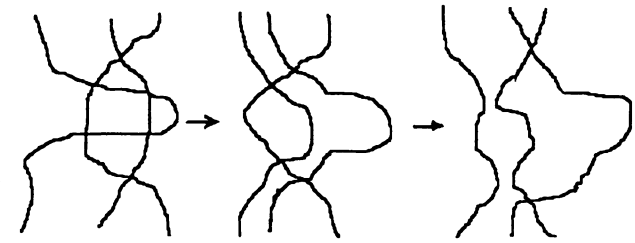

Figure 6.7: Non minimal biangle's procedure

by some k not commuting with i are contained in R. Then the situation is that from picture 8.

Thus all arcs which cross the biangle are labelled by some k which commutes with both i and j. The same commutation transforms may be performed whenever we make the arcs i and j disjoint. □

A similar reasoning permits to say that the diagrams from picture 9 are equivalent. When the triangle in the middle is not touched by any arc then it is a simple consequence of lemma 3.12 ii). If it is minimal, any arc which cross it is labelled by something which commutes with

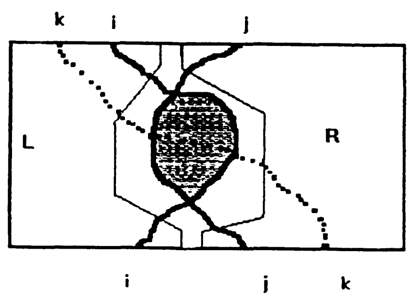

Figure 6.8: The regions R and L

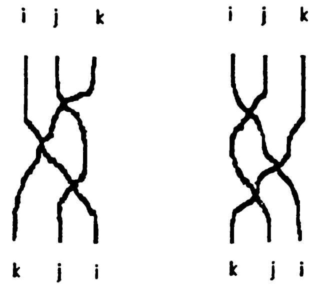

Figure 6.9: Equivalent diagrams

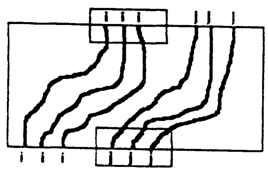

Figure 6.10: The diagram for (C0)(i)-(C0)(j) when $\mid i-j\mid = 1$

j.

Remark now the similitude of pictures 6 and 9 with the Reidemester's moves on link diagrams. So we can actually isotopy our arcs leaving the endpoints fixed and keeping the tangent (in a C^1-approximation of arcs) away from the horizontal.

Now we can continue our discussion on the trajectories of i's and j's. If $\mid i-j\mid= 1$ the trajectories are disjoint so there are as in picture 10.

If i and j commutes there are essentially sixteen diagrams (up to an isotopy) which can be seen in picture 11.

In order to represent graphically the possible diagrams for the (C1), (C2), (C12), (C21) moves we shall picture the trajectories of a couple of neighbor points having the same label as a single thicker trajectory. This may be done since every arc crossing the dashed region (see figure 12) between the trajectories of the two i's has a label commuting with i. In addition the trajectories of i and $i+1$ are disjoint.

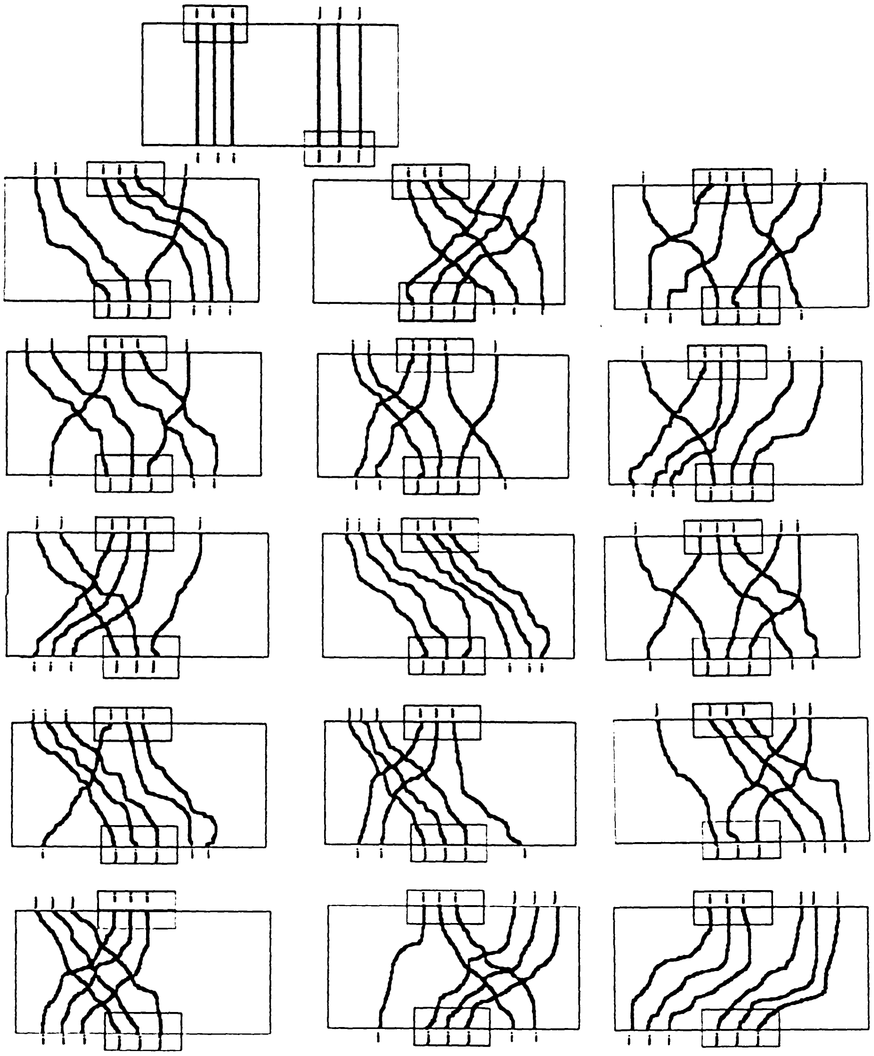

Figure 6.11: The 16 diagrams for (C0)(i) -(C0)(j) in the commuting case

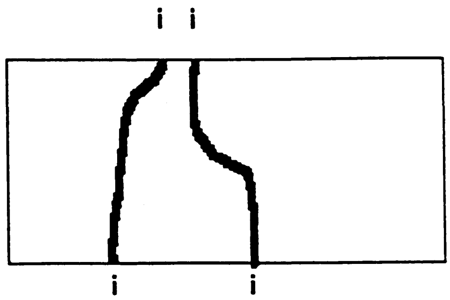

Figure 6.12: The grahical representation of the dashed region

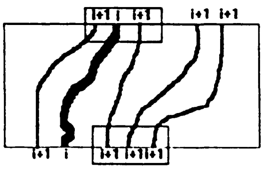

Figure 6.13: The new diagram for (C1)(i)-(C0)(i+1)

Suppose we are in the case (C1)(i)-(C0)(j). For $j \neq i-1, i, i+1, i+2$ the sixteen diagrams from above appear appropriately labelled. For $j = i-1, i, i+2$ some diagrams are not realized because the arcs labelled by $i-1$ and i does not intersect, so several cases have to be left. For $j = i+1$ another diagram have to be considered, that from figure 13.

The same situation we encounter when we describe the possible trajectories for the couple of reduction transforms (C2)-(C0), (C12)-(C0), (C21)-(C0). A simple analysis shows that in the remaining cases the only new diagrams are those from figure 14.

The other ones are obtained from the previous twelve using the suitable labelling, and taking into account the constraints of disjointeness imposed by the labels. We say now that a diagram is interactive if there is some marked arc relating the top and bottom blocks where the reduction transforms act. Our task will be to eliminate the non-interactive diagrams where the (PC) trivially holds.

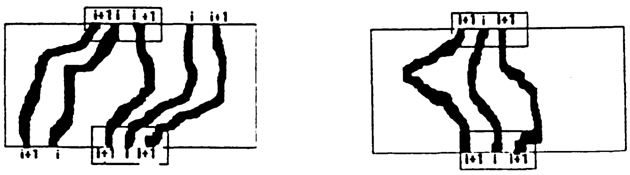

Figure 6.14: The new diagrams for (Cx)(i)-(Cy)(i) $x, y \neq 0$

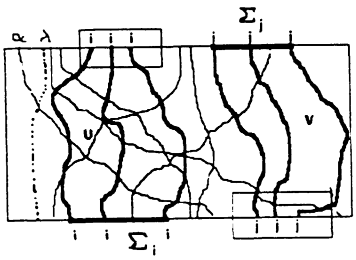

Figure 6.15: The whole picture of a non-interactive diagram without crossings

Lemma 6.3.14 *The usual (PC) is valid in Γ_n for non-interactive diagrams.*

Proof: We consider first the case where no crossings of the essential arcs exist. The typical case is that from picture 9. We draw now all trajectories as in figure 15. We have the dashed regions U and V which are bounded by the i's arcs and respectively j's arcs.

Everything crossing the regions U and V commutes with i and j respectively. We claim first that U and V are tangent to the end lines from left and right respectively. If not there exists some arc labelled λ lying to the left of U. Assume that this arc is the first from the left having this property. In particular λ commutes with every label α which stands to the left of λ. Thus we may perform these commutation transforms at any moment, to get λ on the first position. Since λ does not cross U we may leave it on the first position replacing the o.p.c. by

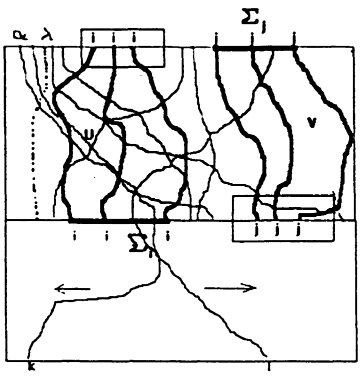

Figure 6.16: The simplification of a non-interactive diagram without crossings of essential arcs

an equivalent one. Thus the new configuration corresponds to a word which is not minimal with respect to the reduction procedure (see the reduction lemma and the subsequent comments).

Let now Σ_i be the convex hull of the three points labelled i coming from essential arcs and lying on the bottom line. Similarly set Σ_j for the convex hull of the j's on the top line. Every arc which arrive on Σ_i must cross U hence is labelled by some k commuting with i. We can move these endpoints using the commutation rules from the left or the right according to the following principle: if the startpoint of the arc labelled k is in the left of the block of i''s on the top line, then we move to the left. Otherwise we move to the right. The only problem which we can have is in the following case: the startpoint of some k is in the left of the arc labelled l, both arrive on Σ_i, but this time the endpoint of l is in the left of k. A topological argument shows that these two arcs cross each other. Therefore k and l are commuting and we can perform our transforms as it was said (see figure 16).

Finally we shall recover a diagram which this time has crossings but is equivalent to the standard one of picture 17.

Suppose now that the reduction transforms $AiiiB \to AB$ and $CjjjD \to CD$ are also performed. We may use the simplification transforms (commutations which are still valid even if the i or the j are collapsed)

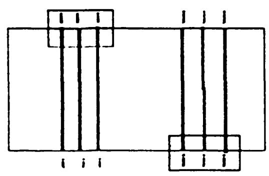

Figure 6.17: The standard non-interactive diagram

for above for each word: to AB in the part of j's and to CD in the part of i's. Due to the particular form of the standard diagram we shall get (see the picture 16) the words $UjjjV$ and $U'iiiV'$ respectively, with $UV = U'V'$. So again the use of a reduction transform will get the same word. Thus the (PC) is satisfied for these configurations. It is almost the same reasoning for the other non-interactive diagrams without crossings.

It remains the case when crossings of essential arcs appear. But the commutation transforms may be also be performed in such way that the starting points of j's on the top line will be all on the same part with respect to the iii block. In other words we make Σ_j and the block iii disjoints. The same is true for the bottom line. The worst case is again when iii is in the left of Σ_j on the top line and down the situation is reversed. But again i and j commutes with everything starts or arrive on the convex hulls of $iii \cup \Sigma_j$ and $jjj \cup \Sigma_i$. So we can rearrange them to obtain the same order in the top and bottom lines. This ends the proof of the lemma. □

So it remains to look at the interactive configuration. The same reasoning as in the above permits us to restrict to the normal forms draw in figure 18.a-f. Some of the trajectories may be thick trajectories.

The cases a,b,c,d,f are trivially verified because only the consistency of relations defining $K_3(\gamma)$ is involved.

Let do a subcase of d , corresponding to (Cε)-C(0): The monomial has the form $w = b_{i+1}b_i^{\varepsilon}b_{i+1}xb_{i+1}^2$ which is unoriented equivalent to $w' = b_{i+1}b_i\varepsilon b_{i+1}^3$. Here x commutes with b_{i+1}. so we may suppose it lies in F_i. Therefore $x \rightarrow x_0 b_{i-1}^{j_1} b_{i-2}^{j_2} ... b_{i-p}^{j_p}$, with $x_0 \in F_{i-1}$. So again we can

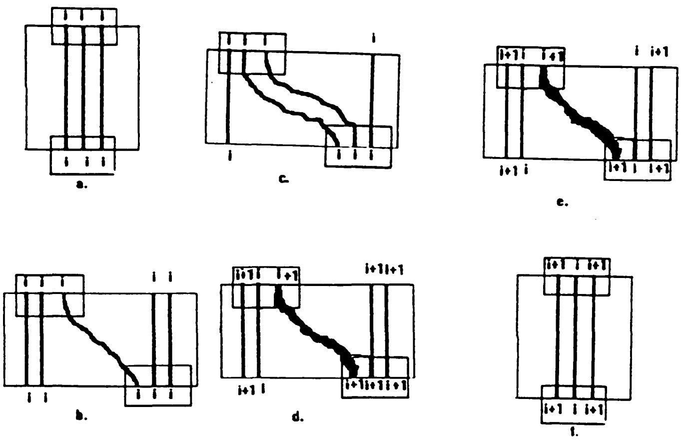

Figure 6.18: The normal forms of interactive configurations

restrict to the case $x_0 = 1$. Now w is reduced to $Sb_{i-1}^{j_1}b_{i-2}^{j_2}...b_{i-p}^{j_p}b_{i+1}^2$. This is equivalent to $Sb_{i+1}^2 b_{i-1}^{j_1}b_{i-2}^{j_2}...b_{i-p}^{j_p}$. Since Sb_{i+1}^2 and $b_{i+1}b_i^\varepsilon$ may be related by semi-oriented path to a common endpoint (here we use the induction hypothesis for n=2) we are done. All other cases are similar.

In the case e the situation is different. Using the commutation rules, as above we must preserve the term $b_{i-1}^{j_1}$. So we must check the configurations

$w = xb_{i+1}^\alpha b_i^\varepsilon b_{i+1}^\beta b_{i-1}^\mu b_i^\delta b_{i+1}^\gamma b_{i-2}^{j_2}...b_{i-p}^{j_p}$,

where $x \in F_{i-1}$ and α, β, γ equal 1 if $\varepsilon = \delta = 1$. It is here where we cannot prove that the (PC) holds. In fact it does not hold since the surjection of Proposition 3.5 has a nontrivial kernel in rank n=3. Fortunately we may prove that the (CPC) condition is still verified.

Lemma 6.3.15 *For the above family of o.p.c. the (CPC) is satisfied in $\Gamma_n^*(H)$, $n \geq 3$ iff $H_0 \subset H$.*

Proof: We claim first that it suffices to restrict to $x = 1, p = 1$. We observe that any admissible functional t on $K_\infty(\gamma)$ satisfies

$t(xuv) = t(u)t(xv)$ for $x, v \in H(Q, m)$ and $u \in < 1, b_m, b_{m+1}, ..., b_{m+k} >$.

If $k = 0$ this is nothing but the multiplicative property for Markov traces. If $k > 0$ then in the process which compute the value of the functional we arrive to replace u by αb_m^ε where $t(u) = \alpha t(b_n^\varepsilon)$. Another one step of the reduction and we found $t(xuv) = \alpha t(b_m^\varepsilon)t(xv)$.

But the (CPC) is equivalent to the existence of the admissible functional. This proves our claim which says that in fact we can consider $i = 2$. Consider first $\alpha = \beta = \gamma = 1$. We have to check the o.p.c. corresponding to

$b_3b_2^\varepsilon b_3b_1^\mu b_2^\delta b_3$.

In order to shorten the computations we observe that for symmetric words the (CPC) is satisfied. For a word $w = w_1....w_l$ its symmetric is $w^* = w_lw_{l-1}...w_1$. This has a simple proof by induction. In fact we perform in both situations the same reduction transforms (the words have unique reduction) which leads us to two sums of minimal elements $\sum_i x_i$ and $\sum_i x_i^*$. Now we apply the induction hypothesis and we are done. Roughly speaking

$t(x) = t(x^*)$

for any admissible functional t. So it suffices to check the case when

$\varepsilon = 1$ and $\delta = 2$.

1)Say $\mu = 1$. Then

$w = b_3b_2b_3b_1b_2^2b_3 \to b_2b_3b_1^2b_2b_1b_3 \sim b_2b_1^2b_3b_2b_1 \to zb_2b_1^2b_2^2b_1 \to zb_1^2b_2^2b_1^2 \to \gamma tz^2$.

$w' = b_3b_2b_1b_3b_2^2b_3 \to -b_3b_2b_1b_2b_3^2b_2 - b_3b_2b_1b_2^2b_3b_2 - b_3b_2b_1b_2b_3b_2^2 - -b_3b_2b_1b_2^2b_3^2 - b_3b_2b_1b_3^2b_2^2 - \gamma b_3b_2b_1b_3 - \gamma b_3b_2b_1b_2$.

Now

$b_3b_2b_1b_2b_3^2b_2 \to b_3b_1b_2b_1b_3^2b_2 \sim b_1b_3b_2b_3^2b_1b_2 \to b_1b_2^2b_3b_2b_1b_2 \to zb_1b_2^3b_1b_2 \to \gamma tz^2$.

$b_3b_2b_1b_2^2b_3b_2 \to b_3b_1^2b_2b_1b_3b_2 \to b_1^2b_2b_3b_2b_1b_2 \to zb_1^2b_2^2b_1b_2 \to zb_1^3b_2b_1^2 \to \gamma tz^2$.

$b_3b_2b_1b_2b_3b_2^2 \to b_3b_1b_2b_1b_3b_2^2 \to b_1b_2b_3b_2b_1b_2^2 \to zb_1b_2^2b_1b_2^2 \to$
$\to zb_1^2b_2b_1^2b_2 \to -6\gamma z^2t - \gamma^2 z$.

$b_3b_2b_1b_2^2b_3^2 \to b_3b_1^2b_2b_1b_3^2 \to b_1^2b_3b_2b_3^2b_1 \to b_1^2b_2^2b_3b_2b_1 \to \gamma^2 z$.

$b_3b_2b_1b_3^2b_2^2 \to b_3b_2b_3^2b_1b_2^2 \to b_2^2b_3b_2b_1b_2^2 \to \gamma t^2 z$.

$b_3b_2b_1b_3 \to b_2b_3b_2b_1 \to \gamma z^2 t$.

$b_3b_2b_1b_2 \to zb_1b_2b_1 \to \gamma tz^2$.

We conclude

$w' = b_3b_2b_1b_3b_2^2b_3 \to \gamma z^2 t$ thus the (CPC) is verified.

2) Consider now $\mu = 2$. We shall write all details up to the use of red edges where we arrive in Γ_2^* and the computation becames canonical.

$w = b_3b_2b_3b_1^2b_2^2b_3 \sim w' = b_3b_2b_1^2b_3b_2^2b_3.$

$w \to$
$-b_2b_1^2b_2b_3^3b_2b_1 - b_2b_1^2b_2^2b_3b_2b_1 - b_2b_1^2b_2b_3b_2^2b_1 - b_2b_1^2b_2^2b_3^3b_1 - b_2b_1^2b_3^2b_2^2b_1 - \gamma b_2b_1^2b_2b_1 - \gamma b_2b_1^2b_3b_1 \to$
$\to -b_1^2b_2^2b_1b_3b_2b_1 - b_1^2b_2^2b_1b_3^2b_1 - b_2b_1^2b_3^2b_2^2b_1 - \gamma b_2b_1^2b_3b_1 + b_1b_2^2b_1b_3^2b_2b_1 + b_1^2b_2b_1b_3^2b_2b_1 + b_1b_2b_1^2b_3^2b_2b_1 + b_1^2b_2^2b_3^2b_2b_1 + b_2^2b_1^2b_3^2b_2b_1 + \gamma b_2b_3^2b_2b_1 + \gamma b_1b_3^2b_2b_1 + b_1b_2^2b_1b_3b_2^2b_1 + b_1^2b_2b_1b_3b_2^2b_1 + b_1b_2b_1^2b_3b_2^2b_1 + b_1^2b_2^2b_3b_2^2b_1 + b_2^2b_1^2b_3b_2^2b_1 + \gamma b_2b_3b_2^2b_1 + \gamma b_1b_3b_2^2b_1 - \gamma b_2b_1^2b_2b_1 \to -4\gamma^2t - 3\gamma t^2z.$

On the other hand
$w' \to -b_3b_2b_1^2b_2b_3^2b_2 - b_3b_2b_1^2b_2^2b_3b_2 - b_3b_2b_1^2b_2b_3b_2^2 - b_3b_2b_1^2b_2^2b_3^2 - b_3b_2b_1^2b_3^2b_2^2 - \gamma b_3b_2b_1^2b_3 - \gamma b_3b_2b_1^2b_2.$

The last four terms are easy to compute:

$b_3b_2b_1^2b_2 \to -(3\gamma t^2z + 4\gamma^2z^2).$
$b_3b_2b_1^2b_3 \to t^2z.$
$b_3b_2b_1^2b_3^2b_2^2 \to b_2^2b_3b_2b_1^2b_2^2 \to \gamma t^2z.$
$b_3b_2b_1^2b_2b_3^2 \to b_1b_2^2b_3^2b_2b_1 \to \gamma^2t.$

For the other ones the computation becames more complicated. We shall write this time as an array where the reduction of each term is specified.

$b_3b_2b_1^2b_2b_3^2b_2 \to$

$$\begin{array}{cccc}
-b_3b_1b_2^2b_1b_3^2b_2- & b_3b_1^2b_2b_1b_3^2b_2 & -b_3b_2b_1b_2b_1^2b_3^2b_2 & -b_3b_1^2b_2^2b_3^2b_2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
-b_1b_2^2b_3^2b_2b_1b_2 & -b_1^2b_2^2b_3b_2b_1b_2 & -b_1b_2^2b_3b_2b_1^2b_2 & -b_1^2b_2^2b_3^2b_2^2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
-\gamma t^2z & -\gamma^2z^2 & -\gamma^2z^2 & -\gamma t^2z
\end{array}$$

$$\begin{array}{ccc}
-b_3b_2^2b_1^2b_3^2b_2 & -\gamma b_3b_1b_3^2b_2 & -\gamma b_3b_2b_3^2b_2 \\
\downarrow & \downarrow & \downarrow \\
-b_2^2b_3^2b_2b_1^2b_2 & -\gamma^2b_1b_2 & -\gamma b_2^2b_3b_2^2 \\
\downarrow & \downarrow & \downarrow \\
-\gamma t^2z & -\gamma^2z^2 & -\gamma^2z^2
\end{array}$$

so we obtain

$b_3b_2b_1^2b_2b_3^2b_2 \to -(3\gamma t^2z + 4\gamma^2z^2).$

Also

$b_3b_2b_1^2b_2^2b_3b_2 \to b_1^2b_3b_2^2b_3b_1b_2 \to -b_1^2b_2b_3^2b_2b_1b_2 - b_1^2b_2^2b_3b_2b_1b_2 - b_1^2b_2b_3b_2^2b_1b_2 - b_1^2b_2^2b_3^2b_1b_2 - b_1^2b_3^2b_2^2b_1b_2 - \gamma b_1^2b_3b_1b_2 - \gamma b_1^2b_2b_1b_2 \to -(3\gamma t^2z + 4\gamma^2z^2).$

The last term is given by

$$-b_3b_2b_1^2b_2b_3b_2^2 \longrightarrow$$

$$\begin{array}{cccc}
b_3b_1b_2^2b_1b_3b_2^2 & +b_3b_1^2b_2b_1b_3b_2^2 & +b_3b_1b_2b_1^2b_3b_2^2 & +b_3b_1^2b_2^2b_3b_2^2 \\
 & \downarrow & \downarrow & \downarrow \\
 & b_1^2b_2b_3b_2b_1b_2^2 & b_1b_2b_3b_2b_1^2b_2^2 & -(6\gamma t^2z+\gamma^2t) \\
 & \downarrow & \downarrow & \\
 & -(3\gamma t^2z+4\gamma^2z^2) & \gamma t^2z &
\end{array}$$

$$\begin{array}{ccc}
+b_3b_2^2b_1^2b_3b_2^2 & +\gamma b_3b_2b_3b_2^2 & +\gamma b_3b_1b_3b_2^2 \\
\downarrow & \downarrow & \downarrow \\
-(6\gamma t^2z+\gamma^2t) & \gamma^2z^2 & \gamma t^2z
\end{array}$$

and

$$b_3b_1b_2^2b_1b_3b_2^2 \longrightarrow$$

$$\begin{array}{cccc}
-b_1b_2b_3^2b_2b_1b_2^2 & -b_1b_2^2b_3b_2b_1b_2^2 & -b_1b_2b_3b_2^2b_1b_2^2 & -b_1b_2^2b_3^2b_1b_2^2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
6\gamma t^2z+\gamma^2t & -\gamma t^2z & -\gamma t^2z & 6\gamma zt^2+\gamma^2t
\end{array}$$

$$\begin{array}{ccc}
-b_1b_3^2b_2^2b_1b_2^2 & -\gamma b_1b_2b_1b_2^2 & -\gamma b_1b_3b_1b_2^2 \\
\downarrow & \downarrow & \downarrow \\
6\gamma zt^2+\gamma^2t & -\gamma^2z^2 & -\gamma t^2z
\end{array}$$

hence

$$b_3b_1b_2^2b_1b_3b_2^2 \longrightarrow 15\gamma t^2z+3\gamma^2t-\gamma^2z^2.$$

We introduce in the previous expression and we obtain for the last term

$$b_3b_2b_1^2b_2b_3b_2^2 \longrightarrow \gamma^2t+2\gamma t^2z-4\gamma^2z^2,$$

so the final result is

$$w' \longrightarrow 9\gamma t^2z+8\gamma^2z^2.$$

So a necessary condition for the coherence is $H_0 \subset H$. The other configurations we must check are with α, β, γ greater than 1. We left these cases for the reader since nothing new appears. Thus the (CPC) holds for the considered o.p.c.. Remark that the second part of the (CPC) is trivially satisfied when using an induction. This ends the proof of Lemma 3.15. □

It follows now we may apply the Colored Pentagon Lemma to get the coherence of $\Gamma_n^*(H_0)$ so the proof of Proposition 3.10 which we observed that implies the result of Proposition 3.6.□

We are ready now to prove our theorem 3.4. In fact it suffices to describe the module of Markov traces taking values in R for fixed parameters $(z, \bar{z})$. There is essentially only one admissible functional on $K_\infty(\gamma)$

from above. It suffices to check the commutativity condition

$t(ab) = t(ba)$ for all x, y.

At the first stage $K_3(\gamma)$ we derive

$t(b_2 b_1^2 b_2) = t(b_1^2 b_2^2)$

$t(b_1 b_2 b_1^2 b_2) = t(b_2 b_1 b_2 b_1^2) = \gamma t(b_1 b_2)$.

But t is a functional on $H(Q,3)/I_3$ hence

$t(R_0) = t(R_1) = 0$.

These conditions imply $z^3 + \gamma = 0$ and $t = -z^2$. So we conclude

$L(K_*(\gamma)(z, \bar{z}) = 0$ if z, t does not satisfy the previous stated conditions.

Suppose now that the parameters satisfy these conditions from now one. Therefore we see that $H_0 = 0$.

We shall prove the commutativity by induction on n. It suffices now to check the commutativity conditions for $b \in \{b_1, ..., b_n\}$ and a lying in a system of generators of $K_{n+1}(\gamma)$, say W_{n+1}. For $b = b_i$, $i < n$ it is obvious. It remains to check whenever

$t(ab_n) = t(b_n a)$.

We have three cases

i) $a \in K_n(\gamma)$.

ii) $a = xb_n y$, $x, y \in K_n(\gamma)$.

iii) $a = xb_n^2 y$, $x, y \in K_n(\gamma)$.

which will be discussed in combination with any of the six subcases

1) $x \in K_{n-1}(\gamma)$, and $y \in K_{n-1}(\gamma)$.

2) $x \in K_{n-1}(\gamma)$, and $y = ub_{n-1}v$, $u, v \in K_{n-1}(\gamma)$.

3) $x \in K_{n-1}(\gamma)$, and $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\gamma)$.

4) $x = rb_{n-1}s$, $r, s \in K_{n-1}(\gamma)$, $y = ub_{n-1}v$, $u, v \in K_{n-1}(\gamma)$.

5) $x = rb_{n-1}s$, $r, s \in K_{n-1}(\gamma)$, $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\gamma)$.

6) $x = rb_{n-1}^2 s$, $r, s \in K_{n-1}(\gamma)$, $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\gamma)$.

Now (*,i), (1,ii) and (1,iii) are trivial.

(2,ii) $t(b_n x b_n^2 u b_{n-1} v) = tzt(xuv) = t(xb_n u b_{n-1} v b_n)$.

(2,iii) $t(b_n x b_n^2 u b_{n-1} v) = \gamma t(xub_{n-1}v) = \gamma zt(xuv)$.

$t(xb_n^2 u b_{n-1} v b_n) = t(xub_n^2 b_{n-1} b_n v) = t(xub_{n-1} b_n b_{n-1}^2 v) = \gamma zt(xuv)$.

(3,ii) $t(b_n x b_n u b_{n-1}^2 v) = t^2 t(xuv)$

$t(xb_n u b_{n-1}^2 v b_n) = t(xub_n b_{n-1}^2 b_n v) = t(b_n b_{n-1}^2 b_n) t(xuv) = t^2 t(xuv)$.

(3,iii) $t(b_n x b_n^2 u b_{n-1}^2 v) = \gamma t(xub_{n-1}^2 v) = \gamma tt(xuv)$

$t(xb_n^2 u b_{n-1}^2 v b_n) = t(xuvb_{n-1} b_n^2 b_{n-1}^2 v) = \gamma tt(xuv)$.

(4,ii) $t(b_n r b_{n-1} s b_n u b_{n-1} v) = zt(rb_{n-1}^2 s u b_{n-1} v)$.

$t(rb_{n-1}sb_nub_{n-1}vb_n) = zt(rb_{n-1}sub_{n-1}^2v)$

Let $su = pb_{n-2}^{\varepsilon}w$ with $p, w \in K_{n-2}(\gamma)$. If $\varepsilon = 0$ it is trivial. If $\varepsilon = 1$ then both terms equal $\gamma zt(rpwv)$ and if $\varepsilon = 2$ again both terms equal $\gamma tt(rpwv)$ so we are done.

(4,iii) $t(b_nrb_{n-1}sb_n^2ub_{n-1}v) = t(rb_{n-1}^2b_nb_{n-1}sub_{n-1}v) = \gamma zt(rsub_{n-1}v) =$
$= \gamma z^2t(rsuv)$

and it is easy to check that also $t(rb_{n-1}sb_n^2ub_{n-1}vb_n) = \gamma z^2t(rsuv)$.

(5,iii) $t(b_nrb_{n-1}sb_n^2ub_{n-1}^2v) = \gamma zt(rsub_{n-1}^2v) = \gamma ztt(rsuv)$.
$t(rb_{n-1}sb_n^2ub_{n-1}^2vb_n) = t(rb_{n-1}sub_{n-1}b_n^2b_{n-1}^2v) = \gamma ztt(rsuv)$.

(6,ii) $t(b_nrb_{n-1}^2sb_nub_{n-1}^2v) = -3tt(rb_{n-1}^2sub_{n-1}^2v) - 3\gamma zt(rsub_{n-1}^2v) -$
$-\gamma t(rb_{n-1}sub_{n-1}^2v)$ and

$t(rb_{n-1}^2sb_nub_{n-1}^2vb_n) = -3tt(rb_{n-1}^2sub_{n-1}^2v) - 3\gamma zt(rb_{n-1}^2suv) - \gamma t(rb_{n-1}^2sub_{n-1}v)$.

and as in (4,ii) we conclude that the two terms are equal.

(6,iii) $t(b_nrb_{n-1}^2sb_n^2ub_{n-1}^2v) = t(rb_{n-1}^2b_n^2b_{n-1}sub_{n-1}^2v) = \gamma t^2t(rsuv)$.
$t(rb_{n-1}^2sb_n^2ub_{n-1}^2vb_n) = t(rb_{n-1}^2sub_{n-1}b_n^2b_{n-1}^2v) = \gamma t^2t(rsuv)$.

(5,ii) The last case!

$t(b_nrb_{n-1}sb_nub_{n-1}^2v) = zt(rb_{n-1}^2sub_{n-1}^2v)$
$t(rb_{n-1}sb_nub_{n-1}^2vb_n) = -3tt(rb_{n-1}sub_{n-1}^2v) - 3\gamma zt(rb_{n-1}suv) - \gamma t(rb_{n-1}sub_{n-1})$

Let consider again $su = pb_{n-2}^{\varepsilon}q$ with $p, q \in K_{n-2}(\gamma)$. If $\varepsilon = 0$ it is clear. If $\varepsilon = 1$ then both terms are equal to

$-3\gamma z^2t(rpb_{n-2}qv) - \gamma zt(rpb_{n-2}^2qv) - 3\gamma ztt(rpqv)$.

If $\varepsilon = 2$ then the first term equals

$\gamma^2t(rpb_{n-2}qv)$.

The second one it turns to be

$\gamma^2t(rpb_{n-2}qv) + 6\gamma^2zt(rpqv) - 6\gamma z^2t(rpb_{n-2}^2qv)$.

But r and v are arbitrary in $K_{n-1}(\gamma)$. We derive that

$MT(K_*(\gamma), R)(z, \bar{z}) = R/(6\gamma z^2)R$.

When we pass to the dual we recover the result as stated in Theorem 3.4.□

6.4 Link groups and invariants

In the last section we obtained a Markov trace

$t : K_\infty(\gamma)(z, \bar{z}) \longrightarrow A(z, \bar{z})/6z^7A(z, \bar{z})$.

The natural way to get an invariant is to consider the function

$f(x) = (\frac{1}{z\bar{z}})^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} t(x).$

Since $\gamma = -z^3, \bar{z} = 1/z$ we find that $t(x)$ is an homogeneous polynomial in z, hence $f(x) \in \mathbf{Z}$ and does not depend upon γ and the choice of z. Now the class of $f(x)$ modulo 6 is well defined and represents a Markov trace on $K_\infty(-1)$ with values in $\mathbf{Z}/6\mathbf{Z}$. By composition with the natural projection $H(Q_{-1}, \infty) \to K_\infty(-1)$, where $Q_{-1} = X^3 + 1$, we get an element $f \in MT(H(Q_{-1}, *), \mathbf{Z}/6\mathbf{Z})(1,1)$.

Proposition 6.4.1 *There exists an unique lift F of f in $MT(H(Q_{-1}, *), \mathbf{Z})(1,1)$ determined by*
$F(1) = 1$ *and* $F(b_1^2 b_2 b_1^2 b_2) = -1.$

Proof: We need first

Lemma 6.4.2 *If $Tors(A)$ denotes the torsion subgroup of the abelian group A then*
$2Tors(L(H(Q_{-1}, *))(1,1)) = 0.$

Proof: Since $b_i^3 = -1$ all the relations defining the module $L(H(Q_{-1}, *))(1,1)$ have the following form

$w_1 = \varepsilon w'$, $\varepsilon \in \{-1, 1\}$, where w and w' are words in the b_i's. The only possibility that torsion appears will be that $w = -w$ holds, hence the torsion elements have order 2. □.

Assume now that f is normalized by $f(1) = 1$. Due to the form of the relations R_0, R_1, R_2 we obtain $f(x) = \varepsilon$ (modulo 6), $\varepsilon \in \{-1, 1\}$ if x is a word. Then the previous lemma enables us to get a lift

$f_0 \in MT(H(Q_{-1}, *), \mathbf{Z})(1,1).$

whose reduction modulo 6 is f. Remark that f is a Markov trace so its values on $e_1, e_2, ..., e_{23}$ are uniquely determined from $f(1) = 1, z = 1, t = -1$. The only freedom degree in the definition of f (on $H(Q_{-1}, 3)$) is the choice of

$f(b_1^2 b_2 b_1^2 b_2) = -7 + 6k.$

Lemma 6.4.3 *For fixed k there exists at most one Markov trace in $MT(H(Q_{-1}, *), \mathbf{Z})(1,1)$ satisfying*
$f(1) = 1$ *and* $f(b_1^2 b_2 b_1^2 b_2) = -7 + 6k.$

Proof: Suppose there exists two such Markov traces , f and f'. Then $f - f' \in MT(H(Q_{-1}, *), \mathbf{Z})(1,1)$ and it vanishes on $H(Q,3)$. In particular $f - f'$ factors through $K_3(-1)$ in rank 3. According to the homogeneity of Markov traces (see [Fun93b]) $f - f'$ factors through the homogeneous quotient extending $K_3(-1)$ i.e. $K_\infty(-1)$. But on $K_\infty(-1)$ any Markov trace which vanishes on 1 is identically zero. Thus our claim follows. □

Therefore we may denote by $f(k)$ the Markov trace determined by the above conditions.

Lemma 6.4.4 *There exists at most one value of k for which $f(k)$ exists.*

Proof: We observe first that $f(0)$ does not exist. If $f(0)$ should exist therefore $f(0)(R_0) = f(0)(R_1) = f(0)(R_2) = 0$, so again $f(0)$ factors through $K_3(-1)$ and further it factors through $K_\infty(-1)$. Since $f(0)(1) = 1$ $f(0)$ must coincide with f_0. The last one is defined only modulo 6 which leads us to a contradiction.

Consider the subgroup $S \subset \mathbf{Z}$ of those m for which a Markov trace t on $H(Q_{-1}, \infty)$ which fulfills $t(1) = 0$, and $t(b_1^2 b_2 b_1^2 b_2) = m$ exists. We claim that $S = 0$ or $S = \mathbf{Z}$. Let $m \neq 0$ be an element of S. Then t reduces modulo m to a Markov trace t_{red} in $MT(H(Q_{-1}, *), \mathbf{Z}/m\mathbf{Z})(1,1)$ which vanishes on $H(Q_{-1}, 3)$. In particular it factors through $K_3(-1)$ hence through $K_\infty(-1)$. But on $K_\infty(-1)$ the Markov trace is unique so t_{red} vanishes identically. This means that

$t(x) = 0$ (modulo m) for all $x \in H(Q_{-1}, \infty)$.

Therefore $\frac{1}{m}t$ is a Markov trace taking values in $\mathbf{Z}$ hence $1 \in S$. This proves our claim.

Suppose now that $f(k_1)$ and $f(k_2)$ exist. Their difference is a Markov trace t satisfying $t(1) = 0$, $t(b_1^2 b_2 b_1^2 b_2) = 6(k_1 - k_2)$. Therefore $6(k_1 - k_2) \in S$. This implies that $S = \mathbf{Z}$. Let t_1 be the Markov trace satisfying $t_1(1) = 0$ and $t_1(b_1^2 b_2 b_1^2 b_2) = 1$. Therefore $f(k_0) - 6k_0 t_1$ is again a Markov trace, and in fact it must be $f(0)$. We arrive at a contradiction since we have seen that $f(0)$ does not exist.

It follows $S = 0$ and k is unique. □

It remains to determine the value of k.

If the class of $b_1^2 b_2 b_1^2 b_2$ in $L(H(Q_{-1}, *)(1,1)$ would be a torsion element then a $\mathbf{Z}$-valued Markov trace will vanish on it. So $7 = 6k$ which is false since k is an integer.

Otherwise $b_1^2 b_2 b_1^2 b_2$ lies in the free abelian part of $L(H(Q_{-1},*)(1,1)$. We claim that the rank of this free abelian subgroup is exactly 1. It is clear that its rank is at least 1 since a nontrivial Markov trace exists. If $b_1^2 b_2 b_1^2 b_2$ does not lie in $\mathbf{Z}1 \subset L(H(Q_{-1},*)(1,1)$ then a Markov trace which separates them exists. So for some $m \neq 0$ we obtain a Markov trace satisfying $t(1) = 0$ and $t(b_1^2 b_2 b_1^2 b_2) = m$, contradiction. Next the uniqueness of a Markov trace with given values on 1 and $b_1^2 b_2 b_1^2 b_2$ follows as in lemma 4.3. so getting our claim. We derive that

$b_1^2 b_2 b_1^2 b_2 = \lambda 1$ in $L(H(Q_{-1},*))(1,1)$.

But we already remarked in the proof of lemma 4.2. that the only scalars which could appear in a relation between words are 1 and -1. So $-7+6k$ equals 1 or -1. The only convenient solution is $k=1$. This ends the proof of the proposition. □

We can state now our main result

Theorem 6.4.5 *$L(H(Q_{-1},*))(1,1) = \mathbf{Z}$, the isomorphism being given by F.*

Proof: We already know that the free abelian part is $\mathbf{Z}$. If the torsion subgroup is not zero then there is some torsion module R and some Markov trace taking values in R such that $t(1) = 0$, $t(b_1^2 b_2 b_1^2 b_2) = 0$ which is not identically zero. From the previous considerations this is impossible so actually $L(H(Q_{-1},*))(1,1)$ is free abelian. It remains to prove that F is the only one trace with $t(1) = 1$ which follows from $S = 0$. □.

We may derive a partial result concerning the Markov traces with other parameters $(z,\bar{z})$.

Proposition 6.4.6 *$rank L(H(Q_\gamma,*))(z,\bar{z}) \leq 1$.*

Proof: First a Markov trace in $MT(H(Q_\gamma,*),R)(z,\bar{z})$ is determined by its values on 1 and $b_1^2 b_2 b_1^2 b_2$. In fact any Markov trace which vanishes on both elements will factor through K_∞ so it is identically zero. So the rank of the considered group is less than 2. Say now $R = A(z,\bar{z})$. A Markov trace t having $t(1) = 0$ and $t(b_1^2 b_2 b_1^2 b_2) = m \neq 0$ induces a Markov traces taking values in $\mathbf{Z}/m\mathbf{Z}$ which will factor through $K_\infty(\gamma)$. So again we obtain $m = 1$ so we should have two factors $A(z,\bar{z})$

in the link module. We know that $b_1^2b_2b_1^2b_2 \sim -1$ in the link group $L(H(Q_{-1},*)(1,1)$. The relation in this group have the form $w = \varepsilon w'$ with w and w' words. But also in $L(H(Q_\gamma,*)(1,1)$ the relations have a similar form $w = \lambda w'$ with w, w' words and λ some scalars depending on $\gamma, z, \bar{z}, w, w'$. Therefore using the same reduction procedure we must arrive at $b_1^2b_2b_1^2b_2 \sim \lambda 1$ in $L(H(Q_\gamma))(z,\bar{z})$. This contradicts the presence of two factors $A(z,\bar{z})$ in the link module and proves also that $b_1^2b_2b_1^2b_2$ cannot be a torsion element. □
We know that equality holds for $\bar{z} = \gamma^{-2/3}(z+\gamma^{1/3})$ when a Markov trace with the prescribed parameters could be obtained by lifting that on the corresponding quadratic quotient of $H(Q_\gamma)$.

We have yet remarked that the trace F when restricted to words takes only two values -1 and 1. We derive then a link invariant, say F, which associes to a link L the number $F(x) \in \{-1,1\}$ for any braid word x representing L. Hopefully we may compute algorithmically $F(x)$ since its reduction modulo 6 (which is $F(x)$ itself!) is f_0 so we can use the algorithm described in the previous section.

Proposition 6.4.7 *The invariant F is not a Vassiliev invariant of finite degree.*

Proof: We shall consider K the classical torus knot of type $(1,12k)$ and set $K^{(12k)}$ for the singular knot having all crossings identified. We remark that $F(b_1^j) = \sigma(j)$ where σ has period 6 and $\sigma(0) = \sigma(1) = \sigma(5) = 1, \sigma(2) = \sigma(3) = \sigma(4) = -1$. So $\sigma(j) = \sigma(-j)$. Let F denotes also the extension of F to singular knots. According to [BXS93] we may write

$F(K^{(12k)} = \sum_{p=0}^{12k}\binom{12k}{p}(-1)^p\sigma(2p-12k) = \sum_{p=0}^{12k}\binom{12k}{p}(-1)^p\sigma(2p)$.

Let $\zeta = exp(\frac{2\pi i}{3})$ and

$a_j = \sum_{p=0;p=j(3)}^{12k}\binom{12k}{p}(-1)^p,\ j \in \{0,1,2\}$.

Then

$F(K^{(12k)}) = a_0 - a_1 - a_2$

so from elementary combinatorics we derive

$F(K^{(12k)} = \frac{4}{3}27^{2k}$,

which proves our claim. □
We don't know however if F is not the limit of a sequence of Vassiliev invariants. On the other hand F generates a whole sequence of Vassiliev

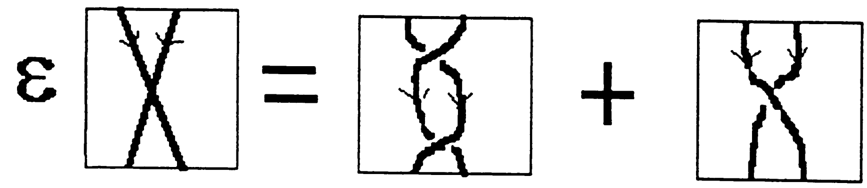

Figure 6.19: The resolution of a self-intersection singularity

type invariants, as follows. Let SB_n be the monoid of singular braids (see [BXS93, Bir93]) with generators $g_i, g_i^{-1}, s_i, 1 \leq i < n$ and relations

$[g_i, g_j] = [s_i, g_j] = [s_i, s_j] = 0$ if $\mid i - j \mid > 1$,

$[g_i, s_i] = 0$,

$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i$,

$s_{i+1} g_i g_{i+1} = g_i g_{i+1} s_i$.

Let $\mathbf{Z}SB_n$ be the monoid algebra of the singular braid monoid. The (3-order) Vassiliev algebra W_n is defined as the quotient of $\mathbf{Z}SB_n \otimes \mathbf{Z}[\varepsilon]$ by the ideal generated by the following elements

$g_i^2 + g_i^{-1} = \varepsilon s_i$.

If $\mathbf{Z}(\varepsilon)$ denotes the algebra of Laurent polynomials in ε then it is clear that the natural map $i : \mathbf{Z}[B_n] \longrightarrow W_n$ produces an isomorphism

$\mathbf{Z}[B_n] \otimes \mathbf{Z}(\varepsilon) \longrightarrow W_n \otimes_{\mathbf{Z}[\varepsilon]} \mathbf{Z}(\varepsilon)$.

Now any link invariant I will extends to singular links admitting transverse double points by means of the following resolution rule for the singularities

$\varepsilon I(L_x) = I(L_{++}) + I(L_-)$,

where L_x, L_{++}, L_- denotes the link diagrams with an intersection, two left-handed crossings and one right-handed crossing respectively all the rest of the diagrams being the same (see picture 20). A link invariant is of degree d if it vanishes on all singular links with $d + 1$ or more self intersections. A Markov trace on W_∞ taking values in the $A(z, \bar{z})[\varepsilon]$-module E is a map $t \in Hom_{A(z,\bar{z})[\varepsilon]}(W_\infty, E)$ satisfying

$t(xy) = t(yx)$ for all x, y.

$t(xg_n) = zt(x), t(xg_n^{-1}) = \bar{z}t(x)$ if $x \in W_n$.

We say that a Markov trace on W_∞ is homogeneous of degree d if for every $x \in \mathbf{Z}[B_\infty]$, $t(i(x))$ is a homogeneous polynomial in ε of degree d.

Proposition 6.4.8 *There is a one-to-one correspondence between the Markov traces $t \in MT(\mathbf{Z}[B_\infty], A(z,\bar{z})(z,\bar{z})$ whose associated invariant I is of degree d and Markov traces $\tau : W_\infty \longrightarrow A(z,\bar{z})[\varepsilon]$ that are homogeneous of degree d, given by*

$\tau(i(x)) = \varepsilon^d t(x)$.

Proof: Suppose that I is of degree d. The formula from above defines a Markov trace on W_∞ with values in $A(z,\bar{z})(\varepsilon)$. Let $x \in W_n$ which is a product of l elements of the form s_j and arbitrarily many of the form g_j. Then

$\tau(x) = \varepsilon^{d-l}(z\bar{z})^{\frac{n-1}{2}}\left(\frac{z}{\bar{z}}\right)^{\frac{e(x)}{2}} I(\hat{x})$

where $\hat{x}$ denotes the closure of the singular braid x. Remark that $\hat{x}$ has exactly l self-intersections. So if $l > d$, $I(\hat{x})$ vanishes hence τ takes values in $A(z,\bar{z})[\varepsilon]$.

Conversely assume τ is given and satisfies the hypothesis. Then t is a Markov trace. Let L be a link with l self-intersections. Then L is isotopic to some $\hat{x}$, where $x \in SB_n$ may be written

$x = w_1 s_{i_1} w_2 s_{i_2} w_3 \ldots w_l s_{i_l} w_{l+1}$,

with w_i words in the g_i's. Set

$x' = w_1(g_{i_1}^2 + g_{i_1}^{-1})w_2(g_{i_2}^2 + g_{i_2}^{-1})w_3 \ldots w_l(g_{i_l}^2 + g_{i_l}^{-1})w_{l+1}$.

Therefore

$\varepsilon^d t(x) = \tau(i(x)) = \varepsilon^l \tau(x')$.

Assume that $l > d$. Since $\tau(x') \in \mathbf{C}[\varepsilon]$ and $t(x) \in \mathbf{C}$ we derive that $t(x) = 0$ hence $I(L) = 0$. □

We can obtain therefore a necessary and sufficient condition that an invariant be the limit of (3-rd order) Vassiliev invariants of finite degree.

Proposition 6.4.9 *The invariant I associated to the Markov trace t is the limit of a Vassiliev invariants of finite degree iff there exists a Markov trace*

$\tau : W_\infty \longrightarrow A(z,\bar{z})[\varepsilon]$

which makes the following diagram commutative

$$\begin{array}{ccc} W_n & \xrightarrow{\tau} & A(z,\bar{z})\,[\varepsilon] \\ \uparrow & & \downarrow \\ \mathbf{Z}\,[B_\infty] & \xrightarrow{t} & A(z,\bar{z}) \end{array}$$

where the morphism $A(z,\bar{z})\,[\varepsilon] \longrightarrow A(z,\bar{z})$ *is given by* $\varepsilon \rightarrow 1$.

Proof: If $t = \sum_d t_d$, where t_d are of degree d then take $\tau = \sum_d \varepsilon^d \tilde{t}_d$ where $\tilde{t}$ is the natural extension of t_d to W_∞ and conversely. The rest of the proof is similar to that of the previous proposition. □.

Now we observe that in the quadratic Hecke algebra the following relation

$b_j^2 + b_j^{-1} = (q^2 - q + 1)q^{-1}(b_i - 1)$

holds. This implies that the Markov trace with parameter z, say $t_{q,z}$ fulfills the following condition

$t_{q,z}(w_1(g_{i_1}^2 + g_{i_1}^{-1})w_2(g_{i_2}^2 + g_{i_2}^{-1})w_3...w_l(g_{i_l}^2 + g_{i_l}^{-1})w_{l+1}) = (q-\zeta)^l c$

where $\zeta^3 = -1$ and $c \in \mathbf{C}$.

Corollary 6.4.10 *If* $H(q,z)(*)$ *is the Homfly polynomial then*

$H(\zeta exp(q), z)(L) = \sum_{j=1}^{\infty} w_i(L,z)q^i$

where $w_i(*,z)$ *are Vassiliev invariants of degree* i.

The proof follows from the previous proposition, when we take $\varepsilon = q - \zeta$.
□

In particular nontrivial Vassiliev invariants of every degree exist. In the same manner we can prove that all quantum invariants of Turaev are obtained as the limit of 3-rd order Vassiliev invariants.

Remark that all constructions we made could actually be performed by replacing $x^2 + x^{-1}$ by any polynomial Q. If Q is quadratic then only classical Vassiliev invariants are obtained. Consider $Q = x + 1$ and call the induced invariants 1-st order Vassiliev invariants. The previous results may be restated word-by-word also for this case. Let $V(i)^d$ denote the space of Vassiliev invariants of order i and degree d and $\bar{V}(i)^\infty$ the space of invariants which are limits of finite degree invariants. We remark that the quantum invariants are already contained in $\bar{V}(1)^\infty$. Consider the case of Homfly polynomial. We make first a change of variable in the Homfly polynomial by setting $h = \frac{1}{q}$. Then in the quadratic Hecke algebra $H(q,\infty)$ we have

$b_i + 1 = h(h-1)^{-1}(b_i^2 + 1)$.

Therefore the Homfly polynomial $H(h,z)(k)$, when developping in series the factor $\bar{z}^{-1}$ which normalizes the trace, has the coefficients 1-st order Vassiliev invariants. It is simply to check that

$V(2)^d \cap V(3)^f \subset V(1)^{d+f}$,

which implies that

$\bar{V}(2)^\infty \cap \bar{V}(3)^\infty = \bar{V}(1)^\infty$.

Whenever these three spaces are distinct we don't know. However the space of 1-st order invariants seems to be more treatable from algebraic viewpoint.

6.5 Appendix

This part of the paper is joint work with Barbu Berceanu.
We shall get in this section the complete proof of Proposition 2.1. for all cubic polynomials $Q = x^3 - \alpha x^2 - \beta x - \gamma$ with $\gamma \neq 0$. The method of the proof is due to Bergmann [Ber78] and even if simple it is powerful (see [Ber93] for many interesting examples coming basically from rational homotopy theory).

Briefly, instead of having a semi-oriented graph as in section 3 we shall work with an oriented graph and the relation $\leq$ will be a total order namely the lexicographic one. So we try to solve inductively all the ambiguities eventually adding new relations. Remark that this way all ambiguities are interactive ones so they could be listed. It remains to find the patience of checking all.

We proceed with 3 relations :

(1) $b_2b_1b_2 = b_1b_2b_1$

(2) $b_1^3 = \alpha b_1^2 + \beta b_1 + \gamma$

(3) $b_2^3 = \alpha b_2^2 + \beta b_2 + \gamma$

and the system of generators S containing all words in b_1 and b_2 without subwords appearing in the left hand of some relation, i.e. upon now without containing a $b_2b_1b_2$, b_1^3, b_2^3. Now we shall develop each ambiguity word (i.e. which has two resolutions) by underlining the subword replaced in each case. Away from the starting point the computations, even messy, became canonically , so the words have unique reduction, and we shall write only the final result. Also if an ambiguity is solvable, so no new relation appear we mark by a □ in the final.

The interactions (2-2), (3-3), (1-2), (1-3) give only identities. Further

(1-1) $b_2b_1\underline{b_2b_1b_2} = b_2b_1^2b_2b_1$ and $\underline{b_2b_1b_2}b_1b_2 = b_1b_2b_1^2b_2$ so we obtain a new relation

(4) $b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$.

Next we have an interaction

(1-4) $\underline{b_2b_1^2b_2b_1}b_2 = b_1b_2b_1^2b_2^2$ and

$b_2b_1^2\underline{b_2b_1b_2} = \alpha a b_1b_2b_1^2b_2 + \beta b_1b_2b_1^2 + \gamma b_2^2b_1$

and a new relation is obtained

(5) $b_1b_2b_1^2b_2^2 = \alpha b_1b_2b_1^2b_2 + \beta b_1b_2b_1^2 + \gamma b_2^2b_1$.

(4-2) $\underline{b_2b_1^2b_2b_1}b_1^2 = \alpha b_1^2b_2b_1^2b_2 + \beta b_1b_2b_1^2b_2 + \gamma b_2b_1^2b_2$

$b_2b_1^2b_2\underline{b_1^3} = \alpha b_1^2b_2b_1^2b_2 + \beta b_1b_2b_1^2b_2 + \gamma b_2b_1^2b_2$. □

(1-4) $\overline{b_2b_1b_2}b_1^2b_2b_1 = \alpha b_1^2b_2b_1^2b_2 + \beta b_1^2b_2b_1^2 + \gamma b_1b_2^2b_1$

$b_2b_1\overline{b_2b_1^2b_2b_1} = \alpha b_1^2b_2b_1^2b_2 + \beta b_1^2b_2b_1^2 + \gamma b_1b_2^2b_1$. □

(3-4) $\overline{b_2^3b_1^2b_2b_1} = (\alpha^2+\beta)b_1b_2b_1^2b_2 + (\alpha\beta+\gamma)b_1^2b_2b_1 + \alpha\gamma b_1b_2$

$\underline{b_2^2b_2b_1^2}\overline{b_2b_1} = (\alpha^2+\beta)b_1b_2b_1^2b_2 + (\alpha\beta+\gamma)b_1^2b_2b_1 + \alpha\gamma b_1b_2$. □

(2-5) $\overline{\underline{b_1^3}b_2}b_1^2b_2^2 = \alpha^2b_1^2b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1^2 + \alpha\gamma b_1b_2^2b_1 + \alpha\beta b_1b_2b_1^2b_2 + \beta^2b_1b_2b_1^2 + \beta\gamma b_2^2b_1 + \gamma b_2b_1^2b_2^2$

$b_1^2\underline{b_1b_2b_1^2b_2^2} = \alpha^2b_1^2b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1^2 + \alpha\gamma b_2b_1^2b_2 + \alpha\beta b_1b_2b_1^2b_2 + \beta^2b_1b_2b_1^2 + \beta\gamma b_2b_1^2 + \gamma b_1^2b_2^2b_2$.

Since we supposed $\gamma \neq 0$ we derive a new relation:

(6) $b_2b_1^2b_2^2 = b_1^2b_2^2b_1 + \alpha(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta(b_2b_1^2 - b_2^2b_1)$

Now the left hand of (6) is included in the left hand of (5) and

(5-6) $b_1\underline{b_2b_1^2b_2^2} = \alpha b_1b_2b_1^2b_2 + \beta b_1b_2b_1^2 + \gamma b_2^2b_1$. □

so the relation (5) is cancelled out when introducing (6).

(3-6) $\underline{b_2b_1^2\overline{b_2^2}b_2} = \alpha b_2b_1^2b_2^2 + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_2b_1^2 - b_2^2b_1) + \beta b_2b_1^2b_2 + \gamma b_2b_1^2$

$b_2b_1\underline{b_2^3} = \alpha b_2b_1^2b_2^2 + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_2b_1^2 - b_2^2b_1) + \beta b_2b_1^2b_2 + \gamma b_2b_1^2$. □

(6-1) $\underline{b_2b_1^2b_2^2}b_1b_2 = b_1^2b_2^2b_1^2b_2 - \alpha b_1b_2^2b_1^2b_2 - \beta b_2^2b_1^2b_2 + \alpha\beta b_2b_1^2b_2 + \beta^2b_1b_2b_1 + \beta\gamma b_2^2 + \alpha^2b_1b_2b_1^2b_2 + \alpha\beta b_1b_2b_1^2 + \alpha\gamma b_2^2b_1$

$b_2b_1^2b_2\underline{b_2b_1b_2} = \alpha b_1^2b_2b_1^2b_2 + \alpha\beta b_1b_2b_1^2 + \beta^2b_1b_2b_1 + \beta\gamma b_1b_2 + \gamma b_2^2b_1^2$

so another relation must be added:

(7) $b_1^2b_2^2b_1^2b_2 = \alpha(b_1^2b_2b_1^2b_2 + b_1b_2^2b_1^2b_2) + \beta b_2^2b_1^2b_2 - \alpha^2b_1b_2b_1^2b_2 + \gamma b_2^2b_1^2 - \alpha\beta b_2b_1^2b_2 - \alpha\gamma b_2^2b_1 + \beta\gamma(b_1b_2 - b_2^2)$

(1-6) $\underline{b_2b_1b_2}b_1^2b_2^2 = \alpha^2b_1b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1 + \alpha\gamma b_1b_2^2 + \alpha\beta b_1b_2b_1^2 + \beta^2b_1b_2b_1 + \beta\gamma b_1b_2 + \alpha\gamma b_2^2b_1 + \beta\gamma b_2b_1 + \gamma^2b_1$

$b_2b_1\underline{b_2b_1^2b_2^2} = \alpha^2 b_1b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1 + \alpha\gamma b_1b_2^2 + \alpha\beta b_1b_2b_1^2 + \beta^2 b_1b_2b_1 + \beta\gamma b_1b_2 + \alpha\gamma b_2^2b_1 + \beta\gamma b_2b_1 + \gamma^2 b_1$. □

(3-6) $\underline{b_2}^3b_1^2b_2^2 = \gamma b_1^2b_2^2 + \alpha^2 b_2^2b_1^2b_2 + \beta b_1^2b_2^2b_1 + \alpha\beta(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta^2(b_2b_1^2 - b_2^2b_1) + \alpha b_1^2b_2^2b_1^2 + \alpha^2(b_1b_2b_1^2b_2 - b_1b_2^2b_1^2) - \alpha^2 b_1^2b_2b_1^2 - \alpha^2\beta b_2^2b_1 - \alpha\beta\gamma b_1 + \alpha^2\beta b_2b_1^2 + \alpha\beta\gamma b_2$

$b_2^2\underline{b_2b_1^2b_2^2} = \gamma b_1^2b_2^2 + \alpha^2 b_2^2b_1^2b_2 + \beta b_1^2b_2^2b_1 + \alpha\beta(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta^2(b_2b_1^2 - b_2^2b_1) + \alpha b_1^2b_2^2b_1^2 + \alpha^2(b_1b_2b_1^2b_2 - b_1b_2^2b_1^2) - \alpha^2 b_1^2b_2b_1^2 - \alpha^2\beta b_2^2b_1 - \alpha\beta\gamma b_1 + \alpha^2\beta b_2b_1^2 + \alpha\beta\gamma b_2$. □

(2-6) $\underline{b_1^3}b_2^2b_1^2b_2 = \alpha b_1^2b_2^2b_1^2b_2 + \beta b_1b_2^2b_1^2b_2 + \gamma b_2^2b_1^2b_2$

$b_1\underline{b_1^2b_2^2b_1^2b_2} = \alpha b_1^2b_2^2b_1^2b_2 + \beta b_1b_2^2b_1^2b_2 + \gamma b_1b_2^2b_1^2 + \alpha\gamma(b_2b_1^2b_2 - b_1b_2^2b_1) + \beta\gamma(b_1^2b_2 - b_1b_2^2)$

hence we get the last relation we used in section 2, namely

(8) $b_2^2b_1^2b_2 = b_1b_2^2b_1^2 + \alpha(b_2b_1^2b_1 - b_1b_2^2b_1) + \beta(b_1^2b_2 - b_1b_2^2)$

Now using these eight relations the system of generators S reduces to the 24 elements $e_1, e_2, ..., e_{24}$. So it remains to check that all ambiguities are solvable.

Before to proceed remark that the left hand of (8) is included in that of (7) and

(7-8) $b_1^2\underline{b_2^2b_1^2b_2} = \alpha b_1^2b_2b_1^2b_2 + \alpha b_1^2b_2^2b_1^2 + \beta b_1b_2^2b_1^2 + \gamma b_2^2b_1^2 - \alpha^2 b_1^2b_2^2b_1 - \alpha\beta b_1b_2^2b_1 - \alpha\gamma b_2^2b_1 + (\alpha^2\beta + \beta^2)b_1^2b_2 + (\alpha\beta^2 + \beta\gamma)b_1b_2 + \alpha\beta\gamma b_2 - \alpha\beta b_1^2b_2^2 - \beta^2 b_1b_2^2 - \beta\gamma b_2^2$

so (7) is cancelled out.

(1-8) $\underline{b_2b_1b_2}b_2b_1^2b_2 = \alpha\beta b_1^2b_2b_1 + \beta^2 b_1b_2b_1 + \beta\gamma b_2b_1$

$b_2b_1\underline{b_2^2b_1^2b_2} = \alpha\beta b_1^2b_2b_1 + \beta^2 b_1b_2b_1 + \beta\gamma b_2b_1$. □

(3-8) $\underline{b_2}^3b_1^2b_2 = \alpha b_1b_2^2b_1^2 + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_1^2b_2 - b_1b_2^2)$

$b_2\underline{b_2^2b_1^2b_2} = \alpha b_1b_2^2b_1^2 + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_1^2b_2 - b_1b_2^2)$. □

$(6\text{-}8)_1$ $b_2^2b_1^2b_2b_2 = b_1^2b_2^2b_1^2 + \alpha(b_1b_2b_1^2b_2 - b_1^2b_2b_1^2) + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_1^2b_2 + b_2b_1^2 - b_2^2b_1 - b_1b_2^2) + \beta\gamma(b_2 - b_1)$

$b_2\underline{b_2b_1^2b_2^2} = b_1^2b_2^2b_1^2 + \alpha(b_1b_2b_1^2b_2 - b_1^2b_2b_1^2) + \alpha^2(b_2b_1^2b_2 - b_1b_2^2b_1) + \alpha\beta(b_1^2b_2 + b_2b_1^2 - b_2^2b_1 - b_1b_2^2) + \beta\gamma(b_2 - b_1)$. □

$(6\text{-}8)_2$ $b_2b_1^2b_2b_1^2b_2^2 = 2\alpha^2 b_1^2b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1^2 - \alpha^3 b_1b_2b_1^2b_2 + \alpha\gamma(b_1^2b_2^2 + b_2^2b_1^2 + b_1b_2^2b_1) + \beta^2 b_2b_1^2b_2 + \beta\gamma(b_1^2b_2 + b_2b_1^2) - \alpha^2\gamma(b_1b_2^2 + b_2^2b_1) + (\alpha\beta^2 + \beta\gamma)b_1b_2b_1 + \gamma^2 b_1^2 - \alpha\beta\gamma b_2^2 - \alpha\gamma^2 b_1 - \beta^2\gamma b_2 - \beta\gamma^2$

$b_2b_1^2\underline{b_2^2b_1^2b_2} = 2\alpha^2 b_1^2b_2b_1^2b_2 + \alpha\beta b_1^2b_2b_1^2 - \alpha^3 b_1b_2b_1^2b_2 + \alpha\gamma(b_1^2b_2^2 + b_2^2b_1^2 + b_1b_2^2b_1) + \beta^2 b_2b_1^2b_2 + \beta\gamma(b_1^2b_2 + b_2b_1^2) - \alpha^2\gamma(b_1b_2^2 + b_2^2b_1) + (\alpha\beta^2 + \beta\gamma)b_1b_2b_1 + \gamma^2 b_1^2 - \alpha\beta\gamma b_2^2 - \alpha\gamma^2 b_1 - \beta^2\gamma b_2 - \beta\gamma^2$. □

(4-4) $b_2b_1^2b_2b_1b_1b_2b_1 = \alpha^2 b_1^2b_2b_1^2b_2+\alpha\beta b_1b_2b_1^2b_2+\alpha\beta b_1^2b_2b_1^2+\gamma b_1b_2^2b_1^2+\beta^2b_1^2b_2b_1+\alpha\gamma b_2b_1^2b_2+\beta\gamma b_1^2b_2$

$b_2b_1^2b_2b_1^2b_2b_1 = \alpha^2 b_1^2b_2b_1^2b_2+\alpha\beta b_1b_2b_1^2b_2+\alpha\beta b_1^2b_2b_1^2+\gamma b_1b_2^2b_1^2+\beta^2b_1^2b_2b_1+\alpha\gamma b_2b_1^2b_2+\beta\gamma b_1^2b_2$. □

(4-6) $b_2b_1^2b_2b_1b_1b_2^2 = (\alpha^2+\beta)b_1^2b_2b_1^2b_2+(\alpha\beta+\gamma)b_1^2b_2b_1^2+\alpha\gamma b_1b_2^2b_1$

$b_2b_1^2b_2b_1^2b_2^2 = (\alpha^2+\beta)b_1^2b_2b_1^2b_2+(\alpha\beta+\gamma)b_1^2b_2b_1^2+\alpha\gamma b_1b_2^2b_1$. □

(6-4) $b_2b_1^2b_2b_2b_1^2b_2b_1 = \alpha^3b_1^2b_2b_1^2b_2+\alpha^2\beta b_1^2b_2b_1^2+(2\alpha^2\beta+\beta^2)b_1b_2b_1^2b_2+\alpha\gamma(b_1^2b_2^2b_1+b_1b_2^2b_1^2)-\alpha^2\gamma b_1b_2^2b_1+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\beta\gamma+\alpha\beta^2)b_1b_2b_1^2+2\alpha^2\gamma b_2b_1^2b_2+\alpha\beta\gamma(b_1^2b_2+b_2b_1^2)+\alpha\gamma^2b_2^2+\beta\gamma^2b_2+\gamma^3$

$b_2b_1^2b_2b_2b_1^2b_2b_1 = \alpha^3b_1^2b_2b_1^2b_2+\alpha^2\beta b_1^2b_2b_1^2+(2\alpha^2\beta+\beta^2)b_1b_2b_1^2b_2+\alpha\gamma(b_1^2b_2^2b_1+b_1b_2^2b_1^2)-\alpha^2\gamma b_1b_2^2b_1+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\beta\gamma+\alpha\beta^2)b_1b_2b_1^2+2\alpha^2\gamma b_2b_1^2b_2+\alpha\beta\gamma(b_1^2b_2+b_2b_1^2)+\alpha\gamma^2b_2^2+\beta\gamma^2b_2+\gamma^3$. □

(6-6) $b_2b_1^2b_2^2b_1^2b_2^2 = (2\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2+(2\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2+\beta^2b_1^2b_2^2b_1-\alpha^4b_1b_2b_1^2b_2+\alpha\gamma b_1b_2^2b_1^2+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\alpha^2\gamma+\beta\gamma)b_1^2b_2^2-(\alpha^3\beta+\alpha^2\gamma)b_1b_2b_1^2+(\alpha^2\gamma-\alpha\beta^2)b_1b_2^2b_1+(\alpha\beta^2+\alpha^2\gamma+\beta\gamma)b_2b_1^2b_2+(2\alpha\beta\gamma+\gamma^2)b_1^2b_2-(\alpha^3\gamma+\alpha\beta\gamma)b_1b_2^2+\beta^3b_2b_1^2-(\alpha^3\gamma+\beta^3)b_2^2b_1+\alpha\gamma^2b_1^2-(\alpha^2\beta\gamma+\alpha\gamma^2)b_1b_2-(\alpha^2\beta\gamma+\beta^2\gamma)b_2^2-\alpha^2\gamma^2b_1+(\alpha\beta^2\gamma+\beta\gamma^2)b_2-\alpha\beta\gamma^2$

$b_2b_1^2b_2b_2b_1^2b_2^2 = (2\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2+(2\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2+\beta^2b_1^2b_2^2b_1-\alpha^4b_1b_2b_1^2b_2+\alpha\gamma b_1b_2^2b_1^2+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\alpha^2\gamma+\beta\gamma)b_1^2b_2^2-(\alpha^3\beta+\alpha^2\gamma)b_1b_2b_1^2+(\alpha^2\gamma-\alpha\beta^2)b_1b_2^2b_1+(\alpha\beta^2+\alpha^2\gamma+\beta\gamma)b_2b_1^2b_2+(2\alpha\beta\gamma+\gamma^2)b_1^2b_2-(\alpha^3\gamma+\alpha\beta\gamma)b_1b_2^2+\beta^3b_2b_1^2-(\alpha^3\gamma+\beta^3)b_2^2b_1+\alpha\gamma^2b_1^2-(\alpha^2\beta\gamma+\alpha\gamma^2)b_1b_2-(\alpha^2\beta\gamma+\beta^2\gamma)b_2^2-\alpha^2\gamma^2b_1+(\alpha\beta^2\gamma+\beta\gamma^2)b_2-\alpha\beta\gamma^2$. □

(8-1) $b_2^2b_1^2b_2b_1b_2 = \alpha\gamma b_1b_2^2+\beta\gamma b_1b_2+\gamma^2b_1$

$b_2^2b_1^2b_2b_1b_2 = \alpha\gamma b_1b_2^2+\beta\gamma b_1b_2+\gamma^2b_1$. □

(8-3) $b_2^2b_1^2b_2b_2^2 = \alpha b_1^2b_2^2b_1^2+\alpha^2(b_1b_2b_1^2-b_1^2b_2b_1^2)+(\alpha^3+\alpha\beta)(b_2b_1^2b_2-b_1b_2^2b_1)+\alpha^2\beta(b_2b_1^2-b_2^2b_1)+(\alpha^2\beta+\beta^2)(b_1^2b_2-b_1b_2^2)+\beta b_1b_2^2b_1^2+\alpha\beta\gamma(b_2-b_1)$

$b_2^2b_1^2b_2{}^3 = \alpha b_1^2b_2^2b_1^2+\alpha^2(b_1b_2b_1^2-b_1^2b_2b_1^2)+(\alpha^3+\alpha\beta)(b_2b_1^2b_2-b_1b_2^2b_1)+\alpha^2\beta(b_2b_1^2-b_2^2b_1)+(\alpha^2\beta+\beta^2)(b_1^2b_2-b_1b_2^2)+\beta b_1b_2^2b_1^2+\alpha\beta\gamma(b_2-b_1)$. □

(8-4)$_1$ $b_2^2b_1^2b_2b_1 = \alpha b_1b_2b_1^2b_2+\beta b_1^2b_2b_1+\gamma b_1b_2^2$

$b_2b_2b_1^2b_2b_1 = \alpha b_1b_2b_1^2b_2+\beta b_1^2b_2b_1+\gamma b_1b_2^2$. □

(8-4)$_2$ $b_2^2b_1^2b_2b_1^2b_2b_1 = (\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2+(\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2+(\alpha^2\beta+\beta^2)b_1b_2b_1^2b_2+\alpha\gamma b_1b_2^2b_1^2+(\alpha^2\gamma+\beta\gamma)b_2b_1^2b_2+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\alpha\beta\gamma+\gamma^2)b_1^2b_2$

$b_2^2b_1^2b_2b_1^2b_2b_1 = (\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2+(\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2+(\alpha^2\beta+\beta^2)b_1b_2b_1^2b_2+\alpha\gamma b_1b_2^2b_1^2+(\alpha^2\gamma+\beta\gamma)b_2b_1^2b_2+(\alpha\beta^2+\beta\gamma)b_1^2b_2b_1+(\alpha\beta\gamma+\gamma^2)b_1^2b_2$. □

(8-6) $b_2^2b_1^2b_2b_1^2b_2^2 = (\alpha^3+2\alpha\beta+\gamma)b_1^2b_2b_1^2b_2+(\alpha^2\beta+\alpha\gamma+\beta^2)b_1^2b_2b_1^2+(\alpha^2\gamma+\beta\gamma)b_1b_2^2b_1$

$b_2^2b_1^2\underline{b_2b_1^2b_2^2} = (\alpha^3+2\alpha\beta+\gamma)b_1^2b_2b_1^2b_2 + (\alpha^2\beta+\alpha\gamma+\beta^2)b_1^2b_2b_1^2 + (\alpha^2\gamma+\beta\gamma)b_1b_2^2b_1$. □

(8-8) $\underline{b_2^2b_1^2b_2}b_2b_1^2b_2 = (2\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2 + (2\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2 + \alpha\gamma b_1^2b_2^2b_1 + \beta^2b_1b_2^2b_1^2 - \alpha^4b_1b_2b_1^2b_2 - (\alpha^3\beta+\alpha^2\gamma)b_1^2b_2b_1 + (\alpha\beta^2+\beta\gamma)b_1b_2b_1^2 + (\alpha^2\gamma-\alpha\beta^2)b_1b_2^2b_1 + (\alpha\beta^2+\beta\gamma+\alpha^2\gamma)b_2b_1^2b_2 + (\alpha^2\gamma+\beta\gamma)b_2^2b_1^2 + \beta^3b_1^2b_2 - (\alpha^3\gamma+\beta^3)b_1b_2^2 + (2\alpha\beta\gamma+\gamma^2)b_2b_1^2 - (\alpha\beta\gamma+\alpha^3\gamma)b_2^2b_1 + \alpha\gamma^2b_1^2 - (\alpha^2\beta\gamma+\alpha\gamma^2)b_2b_1 - (\alpha^2\beta\gamma+\beta^2\gamma)b_2^2 - \alpha^2\gamma^2b_1 - (\alpha\beta^2\gamma+\beta\gamma^2)b_2 - \alpha\beta\gamma^2$

$b_2^2b_1^2\underline{b_2^2b_1^2b_2} = (2\alpha^3+\alpha\beta)b_1^2b_2b_1^2b_2 + (2\alpha^2\beta+\alpha\gamma)b_1^2b_2b_1^2 + \alpha\gamma b_1^2b_2^2b_1 + \beta^2b_1b_2^2b_1^2 - \alpha^4b_1b_2b_1^2b_2 - (\alpha^3\beta+\alpha^2\gamma)b_1^2b_2b_1 + (\alpha\beta^2+\beta\gamma)b_1b_2b_1^2 + (\alpha^2\gamma-\alpha\beta^2)b_1b_2^2b_1 + (\alpha\beta^2+\beta\gamma+\alpha^2\gamma)b_2b_1^2b_2 + (\alpha^2\gamma+\beta\gamma)b_2^2b_1^2 + \beta^3b_1^2b_2 - (\alpha^3\gamma+\beta^3)b_1b_2^2 + (2\alpha\beta\gamma+\gamma^2)b_2b_1^2 - (\alpha\beta\gamma+\alpha^3\gamma)b_2^2b_1 + \alpha\gamma^2b_1^2 - (\alpha^2\beta\gamma+\alpha\gamma^2)b_2b_1 - (\alpha^2\beta\gamma+\beta^2\gamma)b_2^2 - \alpha^2\gamma^2b_1 - (\alpha\beta^2\gamma+\beta\gamma^2)b_2 - \alpha\beta\gamma^2$. □.

This ends the proof of Proposition 2.1.

Bibliography

[Abi77] H. Abikoff. *The geometry of Teichmuller space.* Lect.Notes Math., 1977.

[ADW91] S. Axelrod, S. Dellapietra, and E. Witten. Geometric quantization of Chern-Simons gauge theory. *J.Diff.Geometry*, 33:787–902, 1991.

[Ati89] M.F. Atiyah. Topological quantum field theory. *Publ.Math.Inst.Hautes Etudes Sci.*, 68:175–186, 1989.

[Ati90] M. Atiyah. On framings of 3-manifolds. *Topology*, 29:1–7, 1990.

[Bae92] J.C. Baez. Link invariants of finite type and perturbation theory. *Letters in Math.Phys.*, 26:43–51, 1992.

[Bar92] E. Barge, J.and Ghys. Cocycles d'Euler et de Maslov. *Math.Ann.*, 294:235–265, 1992.

[Bea88] A Beauville. Fibres de rang 2 sur une courbe, fibre determinant et fonctions theta. *Bull.Soc.Math.France*, 116:431–448, 1988.

[Bea90] A Beauville. Fibres de rang 2 sur une courbe, fibre determinant et fonctions theta ii. *Preprint*, 1990.

[Ber78] G.M. Bergmann. The diamond lemma for ring theory. *Advances in Math.*, 29:178–218, 1978.

[Ber93] B. Berceanu. Local homotopy of low dimensional varieties. *thesis*, 1993.

[BHMV92] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *preprint*, 1992.

[Bir71] J. Birman. On Siegel's modular group. *Math.Ann.*, 191:59–68, 1971.

[Bir74] J. Birman. *Braids, links and mappings class groups.* Princeton Univ. Press, 1974.

[Bir93] J. Birman. New points of view in knot theory. *Bull.A.M.S.*, 28:253–287, 1993.

[Bou82] N. Bourbaki. *Groupes et algebres de Lie IV, V, VI.* Masson, Paris, 1982.

[BXS93] J. Birman and Lin X.-S. Knot polynomials and Vassiliev's invariants. *Inventiones Math.*, 111:225–270, 1993.

[BZ85] G. Burde and H. Zieschang. *Knots.* Walter de Gruyter, 1985.

[Cra76] R. Craggs. A new proof of Reidemester-Singer theorem on stable equivalence of Heegaard splittings. *Proc.A.M.S.*, 57:143–147, 1976.

[Cra91] L. Crane. 2-d physics and 3-d topology. *Commun.Math.Phys.*, 135:615–640, 1991.

[Deg92] P. Degiovanni. Moore and Seiberg's equations and 3-d topological quantum field theory. *Commun.Math.Phys.*, 145:459–505, 1992.

[Dri91] V.G. Drinfeld. On quasi-triangular quasi-Hopf algebras and a group closely connected with Gal$(\bar{Q}, Q)$. *Leningrad Math.J.*, 2:829–861, 1991.

[DW90] R. Dijgraaf and E. Witten. Topological gauge theories and group cohomology. *Commun.Math.Phys.*, 129:393–429, 1990.

[FQ93] D. Freed and F. Quinn. Chern-Simons theory with finite gauge group. *Commun.Math.Phys.*, 156:435–472, 1993.

[Fre87] J. Freitag. *Theta functions.* Lect.Notes in Math., 1987.

[Fun91] L. Funar. Invariants for closed orientable 3-manifolds and theta functions. *Preprint Orsay*, 91-28, 1991.

[Fun93a] L. Funar. 2+1-D Toplogical Quantum Field Theory and 2-D Conformal Field Theory. *Preprint Orsay*, 93-80, 1993.

[Fun93b] L. Funar. Homogeneity of Markov traces. *Preprint Orsay*, 93-69, 1993.

[Fun93c] L. Funar. Invariants for 3-manifolds II: Conformal field theory and Hecke algebras. *Preprint Orsay*, 93-01, 1993.

[Fun93d] L. Funar. On the quotients of cubic Hecke algebras. *Preprint Orsay*, 93-70, 1993.

[Fun93e] L. Funar. Quantification des tores et varietes de dimension 3. *C.R.Acad.Sci.Paris*, 317:1179–1184, 1993.

[Fun93f] L. Funar. Representations du groupe symplectique et varietes de dimension 3. *C.R.Acad.Sci.Paris*, 316:1067–1072, 1993.

[Fun93g] L. Funar. Theta functions, root systems and 3-manifold invariants. *Preprint Orsay*, 93-54, 1993.

[Fun94a] L. Funar. Sur un quotient de l'algèbre de Hecke cubique. *soumis à C.R.Acad.Sci.Paris*, 1994.

[Fun94b] L. Funar. Théories quantiques du champs topologiques en dimension 3 et théories conformes du champs en dimension 2. *C.R.Acad.Sci.Paris (to appear)*, 1994.

[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry.* John Wiley and Sons, 1978.

[Goc92] T. Gocho. The topological invariant of three manifolds based on the U(1) gauge theory. *J.Fac.Sci.Univ.Tokyo,Sect.1A,Math.*, 39:169–184, 1992.

[Gro84] A. Grothendieck. Esquisse d'un programme. *Preprint*, 1984.

[Hem76] J. Hempel. *3-manifolds.* Princeton Univ. Press, 1976.

[Hit90] N. Hitchin. Flat connections and geometric quantization. *Comm.Math.Phys.*, 131:347–380, 1990.

[HT82] S. Hatcher and W. Thurston. A set of generators of mapping class group. *Topology*, 14:36–59, 1982.

[HW79] G. Hardy and E. Wright. *An introduction to the theory of numbers.* Oxford at the Claredon Press, 1979.

[Igu72] J.-i. Igusa. *Theta functions.* Springer Verlag, Band 194, 1972.

[Jef92] L. Jeffrey. Chren-Simons-Witten invariants of lens spaces and torus bundles and the semi-classical approximation. *Commun.Math.Phys.*, 147:563–604, 1992.

[Jon87] V. Jones. Hecke algebras and a polynomial invariant for knots. *Ann.of Math.*, 126:335–388, 1987.

[Kac83] V. Kac. *Infinite dimensional Lie algebras.* Progress in Math., 1983.

[Kir78] R. Kirby. A calculus for framed links in S^3. *Invent.Math.*, 45:35–56, 1978.

[Kir89] R. Kirby. *Topology of 4-manifolds.* Lect.Notes Math. 1374, 1989.

[KM91] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C). *Invent.Math.*, 105:473–545, 1991.

[Koh92] T. Kohno. Topological invariants of 3-manifolds using representations of mapping class group. *Topology*, 31:203–230, 1992.

[KT93] T. Kohno and T. Takata. Symmetry of Witten's 3-manifold invariants for sl(n,C). *Journal of Knot theory and its Ramifications*, 2:149–169, 1993.

[Lan70] S. Lang. *Algebraic number theory.* Addison-Wesley Comp., 1970.

[Loo76] E. Looijenga. Root systems and elliptic curves. *Invent.Math.*, 38:17–32, 1976.

[LV80] G. Lion and M. Vergne. *The Weil representation, Maslov indiex and theta series.* Progress in Math., 1980.

[Mat72] H. Matsusaka. *The theory of Q-varieties.* Japan Acad. of Sciences, 1972.

[Mey73] W. Meyer. Die Signatur von Flachenbundeln. *Math.Ann.*, 201:239–264, 1973.

[MOO92] H. Murakami, T. Ohtsuki, and M. Okada. Invariants of three-manifolds derived from linking matrices of framed links. *Osaka J.Math.*, 29:545–572, 1992.

[MS89] G. Moore and N. Seiberg. Classical and quantum field theory. *Commun.Math.Phys.*, 123:177–254, 1989.

[Mum70] D. Mumford. *Abelian varieties.* Oxford Univ.Press, 1970.

[Mum74] D. Mumford. *Geometric invariant theory.* Grund.der Math., 1974.

[Mum84] D. Mumford. *Tata lectures on theta I,II.* Progress in Math., 1984.

[NS65] M. Narasimhan and C. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann.of Math.*, 61:540–567, 1965.

[Ocn92] A. Ocneanu. Topological invariants of 3-manifolds and von Neumann algebras. *Lectures at College de France*, 1992.

[RT91] N. Reshetykin and V. Turaev. Invariants of three-manifolds via link polynomials and quantum groups. *Invent.Math.*, 103:547–597, 1991.

[Sat75] I. Satake. The theory of algebraic V-manifolds. *Amer.J.Math.*, 73:36–59, 1975.

[Seg88] G. Segal. The definition of conformal field theory. *preprint Oxford*, 1988.

[Sie80] L. Siebenmann. Les bisections expliquent le theoreme de Reidemester-Singer. *Preprint Orsay*, 1980.

[Suz77] S. Suzuki. On homeomorphisms of 3-dimensional handlebodies. *Canadian J.Math*, 1:111–124, 1977.

[TW93] V. Turaev and H. Wenzl. Quantum invariants of 3-manifolds associated with classical simple Lie algebras. *International Journal of Math.*, 4:323–358, 1993.

[Wel83] G. Welters. Polarized abelian varieties and the heat equation. *Compositio.Math.*, 49:173–194, 1983.

[Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Commun.Math.Phys.*, 121:351–399, 1989.

[Wol83] S. Wolpert. On the dynamical behaviour of Teichmuller space. *Ann.of Math.*, 96:36–59, 1983.