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Application des groupes de cohomologie non ramifiée en degré 3 ou 4 à la construction de corps unirationnels non rationnels

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par
Emmanuel PEYRE

SUJET: **Application des groupes de cohomologie non ramifiée en degré 3 ou 4 à la construction de corps unirationnels non rationnels**

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Title: Unramified cohomology in degree three or four and rationality problems.

Abstract: The aim of this work is to construct unirational function fields K over an algebraically closed field k of characteristic 0 such that the unramified cohomology group $H^i_{nr}(K, \mu_p^{\otimes i})$ is not trivial for $i = 2, 3$ or 4. This implies that the field K is not stably rational over k. For this purpose, let us consider a function field K over k and a finite subgroup U of K^*/K^{*p}. We get a morphism Φ from the exterior algebra of this subgroup to $\bigoplus_{j \in \mathbf{N}} H^j(K, \mu_p^{\otimes j})$. We give then a sufficient condition for an element in this exterior algebra to have a non-ramified image. This condition is based on the kernel of Φ. Thus the problem reduces essentially to determining this kernel. This is made possible in particular cases by theorems proved by Amitsur in degree two, by Arason and Suslin in degree three and by Jacob and Rost in degree four.

Key words: rationality problems, Galois cohomology, unramified cohomology.

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Table des Matières

Résumé de la thèse

1 Cohomologie non ramifiée

Un des premiers exemples de variété X unirationnelle non rationnelle sur $\mathbf{C}$ a été construit par Artin et Mumford en utilisant le sous-groupe de torsion de $H^3(X, \mathbf{Z})$. Pour une variété X projective, lisse et unirationnelle, ce groupe est isomorphe au groupe de Brauer non ramifié du corps de fonctions de X. Cet invariant fut ensuite utilisé sous cette dernière forme par Saltman puis par Bogomolov pour donner des exemples dans le cas du problème de Noether. Le théorème 2 ci-dessous s'inspire des méthodes utilisées dans leur construction. Colliot-Thélène et Ojanguren ont été les premiers à donner un exemple basé sur les groupes de cohomologie non ramifiée en degré 3.

On rappelle les définitions suivantes:

Définition

(i) Un corps L est un ***corps de fonctions*** sur K s'il est engendré par un nombre fini d'éléments en tant que corps au-dessus de K.

(ii) Un corps de fonctions L sur K est dit *rationnel* s'il existe des indéterminées $T_1, \dots, T_n$ et un isomorphisme

$$L \xrightarrow{\sim} K(T_1, \dots, T_n)$$

au-dessus de K.

(iii) Deux corps de fonctions L et M sur K sont dits ***stablement isomorphes*** s'il existe des indéterminées $(T_1, \dots, T_l)$ et $(U_1, \dots, U_m)$ et un isomorphisme

$$L(T_1, \dots, T_l) \xrightarrow{\sim} M(U_1, \dots U_m)$$

au-dessus de K. On dit que L est ***stablement rationnel*** sur K s'il est stablement isomorphe à K.

(iv) Un corps de fonctions L sur K est ***unirationnel*** sur K s'il existe des indéterminées $T_1, \dots, T_m$ et une injection

$$L \longrightarrow K(T_1, \dots, T_m)$$

au-dessus de K.

Si le corps L est rationnel sur K, alors il est stablement rationnel et L stablement rationnel sur K implique L unirationnel sur K.

Si L est un corps de caractéristique première à n, on note L^s une clôture séparable de L, μ_n le groupe des racines n-ièmes de l'unité et pour tout $\mathrm{Gal}(L^s/L)$-module discret M, on note $H^i(L, M)$ le groupe $H^i(\mathrm{Gal}(L^s/L), M)$. On fixe k un corps algébriquement clos et de caractéristique 0. Si K est un corps de fonctions sur k, on note $\mathcal{P}(K)$ l'ensemble des anneaux de valuation discrète A tels que $k \subset A \subset K$ et $\mathrm{Fr}(A) = K$. Pour tout $A \in \mathcal{P}(K)$, κ_A désigne le corps résiduel de A et

$$\partial_A : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

le morphisme résidu.

Définition Les groupes de cohomologie non ramifiée sont les groupes

$$H^i_{nr}(K, \mu_n^{\otimes j}) = \bigcap_{A \in \mathcal{P}(K)} \mathrm{Ker}(H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})).$$

Ces groupes généralisent le groupe de Brauer non ramifié. En effet celui-ci est isomorphe à $\varinjlim H^2_{nr}(K, \mu_n)$.

Proposition 1 (Colliot-Thélène, Ojanguren) *Si K est stablement rationnel alors $H^i_{nr}(K, \mu_n^{\otimes j}) = \{0\}$ pour $i \geq 1$.*

L'objectif de la thèse est de construire des corps unirationnels qui ne sont pas stablement rationnels en appliquant cette proposition avec $i = 2$, 3 ou 4.

2 Caractérisation d'éléments non ramifiés

On part des données suivantes: un corps de fonctions K sur k, un nombre premier p, un entier strictement positif i, un $\mathbf{F}_p$-espace vectoriel de dimension finie U dont le dual est noté $U^\vee$ et un morphisme

$$\Phi^1 : U^\vee \to K^*/K^{*p} \xrightarrow{\sim} H^1(K, \mu_p).$$

Le morphisme Φ^1 induit un morphisme d'algèbres graduées

$$\Phi : \Lambda^* U^\vee \to H^*(K, \mu_p^{\otimes *}).$$

et donc une application $\Lambda^i(U^\vee) \to H^i(K, \mu_p^{\otimes i})$. Or on a un isomorphisme canonique $(\Lambda^i U)^\vee \xrightarrow{\sim} \Lambda^i(U^\vee)$. Par conséquent on obtient une application

$$\Phi^i : (\Lambda^i U)^\vee \to H^i(K, \mu_p^{\otimes i}).$$

On note $S^i = (\mathrm{Ker}\,\Phi^i)^\perp$ et on note $\hat{\Phi}^i$ l'unique application qui rende commutatif le diagramme:

$$\begin{array}{ccc} \Lambda^i(U^\vee) & \longrightarrow & H^i(K,\mu_p^{\otimes i}) \\ \uparrow & \overset{\Phi^i}{\nearrow} & \uparrow \hat{\Phi}^i \\ (\Lambda^i U)^\vee & \xrightarrow{\mathrm{Res}} & (S^i)^\vee \end{array}$$

Par définition de S^i, cette application $\hat{\Phi}^i$ est injective. Soit $S^i_{dec} \subset S^i$ le sous-groupe de S^i engendré par les éléments de S^i de la forme $u \wedge v$ avec $u \in U$ et $v \in \Lambda^{i-1}U$.

Théorème 2 *Avec les notations introduites ci-dessus, si $f \in (S^i)^\vee$ vérifie $f_{|S^i_{dec}} = 0$ alors $\hat{\Phi}^i \in H^i_{nr}(K,\mu_p^{\otimes i})$ et on obtient ainsi une injection*

$$(S^i/S^i_{dec})^\vee \hookrightarrow H^i_{nr}(K,\mu_p^{\otimes i}).$$

En particulier, par la proposition 1, si $S^i \neq S^i_{dec}$, le corps K n'est pas stablement rationnel.

3 Construction d'exemples de corps unirationnels non rationnels

Soient n et i deux entiers strictement positifs et p un nombre premier. On choisit une racine primitive p-ième de l'unité. On pose $F' = k(T_1,\ldots,T_n)$, $X_i = T_i^p$, $F = k(X_1,\ldots,X_n)$. Soit U un $\mathbf{F}_p$-espace vectoriel de dimension n avec une base $u_1,\ldots,u_n$. On note Φ^1_F l'application $U^\vee \to H^1(F,\mu_p)$ qui envoie u_j sur (X_j) pour $1 \leq j \leq n$. Si K est un corps de fonctions sur k contenant F, on note Φ^1_K la composée

$$U^\vee \to H^1(F,\mu_p) \to H^1(K,\mu_p).$$

Elle induit comme ci-dessus des morphismes

$$\Phi^i_K : (\Lambda^i U)^\vee \to H^i(K,\mu_p^{\otimes i}).$$

On est donc ramené à trouver $S \subset \Lambda^i U$ et un corps de fonctions K sur F tels que

(i) K est unirationnel sur k

(ii) $S^\perp = \mathrm{Ker}\,\Phi^i_K$

(iii) $S \neq S_{dec}$.

Le problème se réduit donc essentiellement à déterminer le noyau de $\Phi^i_{F(Y)}$ pour des variétés Y particulières.

3.1 Exemples avec $H^2_{nr}(K, \mu_p^{\otimes 2})$ non nul

Lorsque $i = 2$, on peut appliquer le théorème suivant d'Amitsur:

Théorème 3 (Amitsur) *Si A est une algèbre simple centrale sur un corps L et Y la variété de Severi-Brauer associée, alors*

$$\mathrm{Ker}(\mathrm{Br}(L) \to \mathrm{Br}(L(Y))) = < [A] >$$

où $[A]$ désigne la classe de A dans $\mathrm{Br}(L)$.

Soit S un sous-espace quelconque de $\Lambda^2 U$, soient $s_1, \ldots, s_m$ une famille génératrice de $S^\perp$ et $S_1, \ldots, S_m$ des algèbres simples centrales représentant les images de $s_1, \ldots, s_m$ dans $H^2(F, \mu_p)$. On note Y_j les variétés de Severi-Brauer correspondantes et on pose $K = F(Y_1 \times \cdots \times Y_m)$. On déduit alors du théorème d'Amitsur et du théorème 2 la proposition suivante:

Proposition 4 *Le corps K est unirationnel sur k. Mais si $S \neq S_{dec}$, alors $H^2_{nr}(K, \mu_p^{\otimes 2}) \neq 0$ et K n'est pas stablement rationnel.*

Exemple On considère le cas $n = 4$. Les sous-espaces S tels que $S \neq S_{dec}$ ont alors été décrits par Bogomolov et on peut prendre, par exemple,

$$S = < u_1 \wedge u_2, u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_4 > .$$

3.2 Exemples avec $H^3_{nr}(K, \mu_p^{\otimes 3})$ non nul

On utilise ici le théorème suivant de Suslin:

Théorème 5 (Suslin) *Soit L un corps de caractéristique première à n contenant μ_n. Soient A une algèbre simple centrale cyclique d'indice n sur L et c un élément de L^*. On note Y la variété de norme associée définie par $\mathrm{Nrd}_A(x) = c$ et alors*

$$\mathrm{Ker}(H^3(L, \mu_n^{\otimes 2}) \to H^3(L(Y), \mu_n^{\otimes 2})) = < [A] \cup (c) > .$$

Soit S un sous-espace de $\Lambda^3 U$. On suppose que $S^\perp$ possède une base $(s_1, \ldots, s_m)$ telle que l'on puisse écrire

$$s_j = v_j \wedge w_j \wedge y_j \text{ pour } 1 \leq j \leq m$$

avec $v_j, w_j, y_j \in U^\vee$ tels que pour tout $k \in \{1, \ldots, m\}$ et tout $j \in \{1, \ldots, n\}$ au plus un des éléments v_k, w_k, y_k a une valeur non nulle sur u_j. On considère alors V_j, W_j, Y_j des représentants des images de v_j, w_j, y_j dans $H^1(K, \mu_p)$. On note $A_\xi(V_j, W_j)$ l'algèbre simple centrale engendrée par des éléments I et J avec les relations:

$$I^p = V_j, \quad J^p = W_j, \quad IJ = \xi JI.$$

On note Z_j la variété de norme associée à $A_\xi(V_j, W_j)$ et Y_j et on pose $K = F(Z_1 \times \cdots Z_m)$. On montre alors en utilisant notamment le théorème de Suslin et le théorème 2:

Proposition 6 *Le corps de fonctions K est unirationnel,*

$$\mathrm{Br}_{nr}(K) = \{0\}$$

mais si $S \neq S_{dec}$ alors

$$H^3_{nr}(K, \mu_p^{\otimes 3}) \neq \{0\}$$

et K n'est pas stablement rationnel.

Exemple On peut prendre $n = 6$ et

$$S =< u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 >$$

3.3 Exemples avec $H^4_{nr}(K, \mathbf{Z}/2\mathbf{Z})$ non nul

On pose ici $p = 2$. On peut alors appliquer le théorème de Jacob et Rost:

Théorème 7 (Jacob et Rost) *Soient L un corps de caractéristique différente de 2 et a_1, a_2, a_3, a_4 des éléments de L^*. La forme de Pfister associée est notée*

$$<< a_1, a_2, a_3, a_4 >>=< 1, -a_1 > \otimes < 1, -a_2 > \otimes < 1, -a_3 > \otimes < 1, -a_4 >$$

et Q désigne la quadrique correspondante. Alors

$$\mathrm{Ker}(H^4(L, \mathbf{Z}/2\mathbf{Z}) \to H^4(L(Q), \mathbf{Z}/2\mathbf{Z})) =< (a_1, a_2, a_3, a_4) > .$$

Soit $S \subset \Lambda^4 U$. On suppose que $S^\perp$ possède une base $(s_1, \ldots, s_m)$ telle que

$$s_j = u_{1,j} \wedge u_{2,j} \wedge u_{3,j} \wedge u_{4,j}$$

avec $u_{k,j} \in U^\vee$ pour $1 \leq k \leq 4$ et $1 \leq j \leq m$. On considère $U_{k,j}$ des représentants des images de $u_{k,j}$ et $q_j =<< U_{1,j}, U_{2,j}, U_{3,j}, U_{4,j} >>$ la forme de Pfister associée. On note Q_j la quadrique correspondante et $K = F(Q_1 \times \cdots \times Q_m)$. On déduit du théorème de Jacob et Rost la proposition suivante:

Proposition 8 *K est unirationnel sur k mais si $S \neq S_{dec}$ alors $H^4_{nr}(K, \mathbf{Z}/2\mathbf{Z}) \neq \{0\}$ et K n'est pas stablement rationnel.*

Exemple L'entier $n = 8$ et

$$S = < u_1 \wedge u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 \wedge u_8 >$$

conviennent.

Unramified cohomology and rationality problems

Among the first examples of smooth projective varieties X over $\mathbf{C}$ which are unirational but not rational was the example constructed by Artin and Mumford using the torsion part of $H^3(X, \mathbf{Z})$. When X is unirational, this group may also be described as the unramified Brauer group of the function field of X. From this point of view, Saltman [Sa] and Bogomolov [Bo] gave examples related to Noether's problem. Colliot-Thélène and Ojanguren [C-T,O] were the first to use the unramified cohomology groups in degree 3 to prove the non-rationality of a unirational field.

The plan of this paper is the following: first we recall some basic facts about unramified cohomology. In the second section, we state the main result, Theorem 2, which enables one to characterize unramified elements by calculations in the exterior algebra. This generalizes some of the methods used in [Sa] and [Bo]. In the next section, we prove Theorem 2. In this proof, we show how one can lift the residue map in the exterior algebra of a subgroup of $H^1(K, \mu_p)$ of finite dimension. The fourth section applies the main result to the construction of several unirational non-rational fields. In this part, to prove the non-triviality of elements in $H^3_{nr}(K, \mu_p^{\otimes 3})$, we use a recent result by Suslin [Su] and to have a similar result for $H^4_{nr}(K, \mu_2^{\otimes 4})$, we apply a theorem of Jacob and Rost [J,R]. For the examples with non-trivial $H^3_{nr}(K, \mu_p^{\otimes 3})$, we prove also that the unramified Brauer group is trivial.

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1 Unramified cohomology: definition and basic properties

Let us first give a few definitions about fields. These definitions are used throughout this paper.

Definition

(i) A field L is a *function field* over a field K if it is generated by a finite number of elements as a field over K.

(ii) A function field L over K is *rational* over K if there exist indeterminates $T_1, \ldots, T_m$ and an isomorphism $L \xrightarrow{\sim} K(T_1, \ldots, T_m)$ over K.

(iii) Two function fields L and M over K are *stably isomorphic* if there exist indeterminates $U_1, \ldots, U_l$, $T_1, \ldots, T_m$ and an isomorphism $L(U_1, \ldots, U_l) \xrightarrow{\sim} M(T_1, \ldots, T_m)$ over K. A function field L is *stably rational* over K if L is stably isomorphic to K.

(iv) A function field L over K is *unirational* over K if there exist indeterminates $T_1, \ldots, T_m$ and an injection $L \hookrightarrow K(T_1, \ldots, T_n)$ over K.

We have the following relations between the various kind of rationalities: L rational over K implies L stably rational over K and L stably rational over K implies L unirational over K.

From now on we shall omit "over K" when K is clear from the context.

Notation Let k be an algebraically closed field of characteristic 0. If L is a field, let us denote by L^s a separable closure of L, and for any $\mathrm{Gal}(L^s/L)$-module M, $H^i(L, M) = H^i(\mathrm{Gal}(L^s/L), M)$. In particular, the Brauer group is defined by $\mathrm{Br}(L) = H^2(L, L^{s*})$. If L is of characteristic prime to n, we use μ_n to denote the group of n-th roots of unity in L^s and, when the characteristic of L is 0, μ_∞ to denote the union of the groups μ_n. Let K be a function field over k. We denote by $\mathcal{P}(K)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and the fraction field $\mathrm{Fr}(A)$ of A is K. If $A \in \mathcal{P}(K)$ then κ_A denotes the residue field. For any $i \in \mathbf{N} - \{0\}$ and $j \in \mathbf{Z}$

$$\partial_A : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

denotes the residue map. We also denote by ∂_A the residue map

$$\partial_A : \mathrm{Br}(K) \to H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}).$$

We recall that the residue maps may be defined as follows: let $\hat{K}$ be the completion of K for A, $\hat{K}_{alg}$ an algebraic closure of $\hat{K}$ and $\hat{K}_{nr}$ the maximal unramified extension of $\hat{K}$ in $\hat{K}_{alg}$. Since there exists an isomorphism $\hat{K} \xrightarrow{\sim} \kappa_A((T))$, we have an isomorphism from $\hat{K}_{nr}$ to the algebraic closure $\kappa_A^s((T))_{alg}$ of $\kappa_A((T))$ in $\kappa_A^s((T))$ and

$$\hat{K}_{alg} \xrightarrow{\sim} \varinjlim \kappa_A^s((T^{1/n}))_{alg}.$$

Therefore we get an isomorphism

$$\mathrm{Gal}(\hat{K}_{alg}/\hat{K}_{nr}) \xrightarrow{\sim} \varprojlim \mu_n.$$

But the cohomological dimension of $\hat{\mathbf{Z}}$ is one (see [Se], example 1 on page I-19). Therefore $H^q(\hat{K}_{nr}, \mu_n^{\otimes j}) = 0$ if $q \geq 2$ and the Hochschild-Serre spectral sequence

$$H^p(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), H^q(\hat{K}_{nr}, \mu_n^{\otimes j})) \Rightarrow H^{p+q}(\hat{K}, \mu_n^{\otimes j})$$

gives rise to morphisms

$$H^i(\hat{K}, \mu_n^{\otimes j}) \to H^{i-1}(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), H^1(\hat{K}_{nr}, \mu_n^{\otimes j})).$$

But

$$H^{i-1}(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), H^1(\hat{K}_{nr}, \mu_n^{\otimes j})) \xrightarrow{\sim} H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

and ∂_A is the composed map

$$H^i(K, \mu_n^{\otimes j}) \to H^i(\hat{K}, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1}).$$

For the Brauer group, the residue map is defined as follows: By Hilbert's theorem 90, we have an exact sequence:

$$0 \to H^2(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), \hat{K}_{nr}^*) \to \mathrm{Br}(\hat{K}) \to \mathrm{Br}(\hat{K}_{nr}).$$

But the cohomological dimension of $\mathrm{Gal}(\hat{K}_{alg}/\hat{K}_{nr})$ is one and $\hat{K}_{alg}^*$ is divisible. Thus $\mathrm{Br}(\hat{K}_{nr}) = 0$ and we get a canonical isomorphism

$$\mathrm{Br}(\hat{K}) \xrightarrow{\sim} H^2(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), \hat{K}_{nr}^*).$$

Let ν_A be the valuation on $\hat{K}_{nr}$ extending the one defined by A on K. The morphism ∂_A is then the composed map

$$\begin{aligned}\mathrm{Br}(K) &\to \mathrm{Br}(\hat{K}) \xrightarrow{\sim} H^2(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), \hat{K}_{nr}^*)\\ &\xrightarrow{\nu_A} H^2(\mathrm{Gal}(\hat{K}_{nr}/\hat{K}), \mathbf{Z}) \xrightarrow{\sim} H^2(\kappa_A, \mathbf{Z}) \xrightarrow{\sim} H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}).\end{aligned}$$

Definition The *unramified cohomology groups* are the groups

$$H^i_{nr}(K, \mu_n^{\otimes j}) = \bigcap_{A \in \mathcal{P}(K)} \mathrm{Ker}(H^i(K, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})).$$

Similarly the *unramified Brauer group* is

$$\mathrm{Br}_{nr}(K) = \bigcap_{A \in \mathcal{P}(K)} \mathrm{Ker}(\mathrm{Br}(K) \xrightarrow{\partial_A} H^1(\kappa_A, \mathbf{Q}/\mathbf{Z})).$$

The unramified cohomology groups were denoted by $F_n^{i,j}(K/k)$ in [C-T,O], but, here, the ground field k is fixed. Therefore we do not include it in the notation.

Proposition 1 (Colliot-Thélène and Ojanguren [C-T,O]) *If the function fields K and L are stably isomorphic over k then*

$$H^i_{nr}(K, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{nr}(L, \mu_n^{\otimes j}).$$

In particular, if K is stably rational then $H^i_{nr}(K, \mu_n^{\otimes j}) = \{0\}$.

Remark One can also show that the unramified Brauer group depends only on the stable rationality class of the field. This is the invariant which was used by Artin and Mumford in [A,M]. The unramified cohomology groups may be considered as generalizations of the unramified Brauer group. Indeed, if $i = 2$, the unramified cohomology groups are isomorphic to the n-torsion part of the unramified Brauer group:

$$\mathrm{Br}_{nr}(K)_{(n)} \xrightarrow{\sim} H^2_{nr}(K, \mu_n).$$

2 Characterization of unramified elements using the exterior algebra

Let p be a prime number and k an algebraically closed field of characteristic 0. Throughout this paper, we shall start from data of the following type: a function field K, an $\mathbf{F}_p$ vector space U of finite dimension, whose dual is denoted by $U^\vee$ and a morphism $\phi^1 : U^\vee \to H^1(K, \mu_p)$.

Since $\mu_p \subset k$, we can choose a primitive p-th root of unity. Thus, if ϕ^1 is an injection, the group U may also be considered as a quotient of the absolute Galois group of K. In the examples we have in mind, the field K will be of the form L^U, where L is a rational extension of k endowed with an action of U.

Then we take n to be a strictly positive integer. For a ring B, we denote by $H^i_{ét}(B, \mu_n^{\otimes i})$ the group $H^i_{ét}(\mathrm{Spec}(B), \mu_n^{\otimes i})$. Kummer theory then yields a canonical morphism $B^* \longrightarrow H^1_{ét}(B, \mu_n)$. The image of $x \in B^*$ under this map will be denoted by (x). We shall consider the group $H(B)_n = \sum_{i \in \mathbf{N}} H^i_{ét}(B, \mu_n^{\otimes i})$, (mainly when B is a field or a local ring). Cup-product makes $H(B)_n$ into a ring. We know that this ring is anticommutative ([Mi], Chapter V, §1) but we shall use the following result:

Lemma 1 *Let B be an integral ring of characteristic prime to $2n$ such that n is invertible in B and which contains the $2n$-th roots of unity. Then, for any $x \in B^*$, $(x) \cup (x) = 0$.*

In other words, the subalgebra of $H(B)_n$ generated by the symbols (x) for $x \in B^*$ is strictly anticommutative.

Proof Let $B' = B[T]/(T^n - x)$. Since n and x belong to B^*, the map $\pi : \mathrm{Spec}\, B' \longrightarrow \mathrm{Spec}\, B$ is étale. Moreover it is finite and of constant degree n. Therefore, for any sheaf F of n-torsion on $\mathrm{Spec}\, B$, one can define the transfer map $tr : \pi_*\pi^* F \longrightarrow F$ ([SGA4], exposé XVIII, théorème 2.9) which yields morphisms $tr : H^i_{ét}(B', \mu_n^{\otimes i}) \longrightarrow H^i_{ét}(B, \mu_n^{\otimes i})$ and if ξ is a primitive n-th root of unity,

$$tr\big((-T)\big) = \big(N_{B'/B}(-T)\big) = (\xi^{\frac{n(n-1)}{2}}(-1)^n x) = (-x)$$

and we have the formula ([Mi], Chapter V, §1)

$$\begin{aligned}
(x) \cup (-x) &= (x) \cup tr\big((-T)\big) \\
&= tr\Big(\pi^*\big((x)\big) \cup (-T)\Big) \\
&= tr\big((x) \cup (-T)\big) \\
&= tr\big((T^n) \cup (-T)\big) \\
&= 0.
\end{aligned}$$

But $(-1) = 0$ since $\mu_{2n} \subset B^*$ and we get $(x) \cup (x) = (x) \cup (-x) = 0$. ■

Thanks to this lemma, we get a morphism of graded $\mathbf{F}_p$-algebras $\phi : \Lambda^* U^\vee \longrightarrow H(K)_p$. Thus for a fixed strictly positive integer i, we have a natural morphism $\Lambda^i(U^\vee) \longrightarrow H^i(K, \mu_p^{\otimes i})$. We may identify $\Lambda^i(U^\vee)$ and $(\Lambda^i U)^\vee$ by the map:

$$\begin{array}{ccc}
\Lambda^i(U^\vee) & \longrightarrow & (\Lambda^i U)^\vee \\
f_1 \wedge \ldots \wedge f_i & \longmapsto & \left(\begin{array}{ccc} \Lambda^i U & \longrightarrow & \mathbf{F}_p \\ v_1 \wedge \ldots \wedge v_i & \longmapsto & \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma) f_1(v_{\sigma(1)}) \ldots f_i(v_{\sigma(i)}) \end{array}\right)
\end{array}$$

With this identification, for any basis $(u_1,\ldots,u_n)$ of U, the dual basis of $(u_{j_1}\wedge\ldots\wedge u_{j_i})_{j_1<\ldots<j_i}$ is the basis $(u_{j_1}^\vee\wedge\ldots\wedge u_{j_i}^\vee)_{j_1<\ldots<j_i}$, where $(u_1^\vee,\ldots,u_n^\vee)$ denotes the dual basis of $(u_1,\ldots,u_n)$.

Notation In this way we get a morphism

$$\phi^i : (\Lambda^i U)^\vee \longrightarrow H^i(K,\mu_p^{\otimes i}).$$

Let $S^i = (\mathrm{Ker}\,\phi^i)^\perp \subset \Lambda^i U$. We obtain an injection

$$\hat{\phi}^i : \mathrm{Hom}(S^i,\mathbf{F}_p) \hookrightarrow H^i(K,\mu_p^{\otimes i}).$$

Let $S^i_{dec}\subset S^i$ be the subgroup of S^i generated by the elements of the form $u\wedge v\in S^i$ with $u\in U$ and $v\in\Lambda^{i-1}U$.

I am thankful to Bruno Kahn who pointed out to me that this construction also applies to the case $p=2$.

Theorem 2 *With notation as above, if f is an element of* $\mathrm{Hom}(S^i,\mathbf{F}_p)$ *such that $f_{|S^i_{dec}}$ is zero, then*

$$\hat{\phi}^i(f)\in H^i_{nr}(K,\mu_p^{\otimes i}).$$

Since $\hat{\phi}^i$ is injective, by proposition 1, this theorem implies the following result:

Corollary 3 *If $S^i_{dec}\neq S^i$ then $H^i_{nr}(K,\mu_p^{\otimes i})\neq\{0\}$ and K is not stably rational.*

Remark If the ground field k is not algebraically closed but is of characteristic prime to $2n$ and contains μ_{2n}, it is possible to prove a generalization of this result. Namely, let $S^i=(\Phi^{i-1}(H^i(k,\mu_p^{\otimes i})))^\perp$ and S^i_{dec} be the subgroup of S^i generated by the elements of the form $u\wedge v$ with $u\in U$ and $v\in\Lambda^{i-1}U$. Then we get an injection

$$(S^i/S^i_{dec})^\vee \hookrightarrow H^i_{nr/k}(K,\mu_p^{\otimes i})/H^i(k,\mu_p^{\otimes i})$$

3 Proof of Theorem 2

Let A be an element of $\mathcal{P}(K)$ and ν_A be the corresponding valuation. ν_A defines an element of $(K^*/K^{*p})^\vee \xrightarrow{\sim} H^1(K,\mu_p)^\vee$ and therefore an element of $U^{\vee\vee}$. But there is a natural isomorphism $\rho: U\longrightarrow U^{\vee\vee}$ and we obtain a vector $\tau_A \in U$. In other words, we have a commutative diagram:

$$\begin{array}{ccc} H^1(K,\mu_p) & \tilde{\rightarrow} & K^*/K^{*p} \\ \phi^1 \uparrow & & \nu_A \downarrow \\ U^\vee & \xrightarrow{\rho(\tau_A)} & \mathbf{Z}/p\mathbf{Z} \end{array}$$

Let us denote by $\tilde{\tau}_A$ the transpose of: $\begin{array}{ccc} \Lambda^{i-1}U & \rightarrow & \Lambda^i U \\ u & \mapsto & \tau_A \wedge u \end{array}$

Main lemma 2 *For any $\lambda \in (\Lambda^i U)^\vee$, if $\tilde{\tau}_A(\lambda) = 0$ then*

$$\partial_A(\phi^i(\lambda)) = 0.$$

This lemma implies the theorem:

Proof of Theorem 2 Let f be an element of $\mathrm{Hom}(S^i, \mathbf{F}_p)$ such that $f_{|S^i_{dec}} = 0$. As $f_{|S^i_{dec}} = 0$, $f_{|(\tau_A \wedge \Lambda^{i-1}U)\cap S^i} = 0$. Let $T_1 \subset \tau_A \wedge \Lambda^{i-1}U$ be such that $(\tau_A \wedge \Lambda^{i-1}U \cap S^i) \oplus T_1 = \tau_A \wedge \Lambda^{i-1}U$. Let $T_2 \subset \Lambda^i U$ be such that $(S^i + \tau_A \wedge \Lambda^{i-1}U) \oplus T_2 = \Lambda^i U$ and let $T = T_1 \oplus T_2$. Then we have $S^i \oplus T = \Lambda^i U$. Let $\lambda \in (\Lambda^i U)^\vee$ be defined by $\lambda_{|S^i} = f$ and $\lambda_{|T} = 0$. Then the following relation holds:

$$(\lambda)_{|\tau_A \wedge \Lambda^{i-1}U} = 0.$$

So by the lemma $\partial_A(\phi^i(\lambda)) = 0$. But, by definition of $\hat{\phi}^i$, since $\lambda_{|S_i} = f$, we have $\phi^i(\lambda) = \hat{\phi}^i(f)$. Finally we get $\partial_A(\hat{\phi}^i(f)) = 0$, as wanted. ■

The main lemma will be deduced from a series of lemmata:

Lemma 3 (Colliot-Thélène and Ojanguren) *Let L be a field over k, $B \in \mathcal{P}(L)$ and ν_B the corresponding valuation. Let $a \in L^*$, $b \in H^{i-1}_{\acute{e}t}(B, \mu_n^{\otimes j})$, a' the image of a in $H^1(L, \mu_n)$, b' the image of b in $H^{i-1}(L, \mu_n^{\otimes j})$ and β the image of b in $H^{i-1}(\kappa_B, \mu_n^{\otimes j})$. Then the image of $a' \cup b' \in H^i(L, \mu_n^{\otimes j+1})$ by ∂_A verifies:*

$$\partial_A(a' \cup b') = \nu_B(a)\beta.$$

Proof The proof we give here for self-completeness is similar to the one given by J.-P. Serre in his course at the Collège de France in 1991-92. With a notation similar to the one used in the definition of ∂_A, we consider the spectral sequence described in [H,S]

$$H^p(\mathrm{Gal}(\hat{L}_{nr}/\hat{L}), H^q(\hat{L}_{nr}, \mu_n^{\otimes j})) \Rightarrow H^{p+q}(\hat{L}, \mu_n^{\otimes j}).$$

Let $G = \text{Gal}(\hat{L}_{alg}/\hat{L})$, $N = \text{Gal}(\hat{L}_{alg}/\hat{L}_{nr})$. The spectral sequence may be defined in the following way. Let $A^m(j)$, also denoted by $C^m(G, \mu_n^{\otimes j})$, be the group of normalized m-cochains for G with coefficients in $\mu_n^{\otimes j}$. Let $A(j) = \sum_{m \in \mathbf{N}} A^m(j)$. The filtration is defined as follows: if $k < 0$, $A_k(j) = A(j)$ and if $k \geq 0$, $A_k(j)$ is given by $A_k(j) = \sum_{m \in \mathbf{N}} A_k(j) \bigcap A^m(j)$, where $A^m(j) \bigcap A_k(j) = 0$ if $k > m$ and is otherwise the subgroup of $A^m(j)$ of the cochains $\gamma : G^m \to \mu_n^{\otimes j}$ such that $\gamma(g_1, \ldots, g_m)$ depends only on $g_1, \ldots, g_{m-k}$ and $g_{m-k+1}N, \ldots, g_m N$. We denote by $E_r^{p,q}(j)$ the groups of the spectral sequence corresponding to $A(j)$ with the graduation $A^m(j)$ and the filtration $A_k(j)$. It is proved in [H,S] that the natural map

$$A^{m+k}(j) \bigcap A_k(j) \to C^k(G/N, C^m(N, \mu_n^{\otimes j}))$$

yields an isomorphism

$$E_2^{p,q}(j) \xrightarrow{\sim} H^p(\text{Gal}(\hat{L}_{nr}/\hat{L}), H^q(\hat{L}_{nr}, \mu_n^{\otimes j})).$$

Let $\bar{\alpha}'$ be defined by

$$\forall g \in G, \quad \bar{\alpha}'(g) = \frac{g(a^{1/n})}{a^{1/n}} \in \mu_n$$

for any n-th root $a^{1/n}$ of a. The cocycle $\bar{\alpha}' \in A^1(1)$ represents the image α' of a in $H^1(G, \mu_n)$. Let β' be the image of b in $H^{i-1}(G, \mu_n^{\otimes j})$. We have a commutative diagram

$$\begin{array}{ccccc} H^{i-1}_{\acute{e}t}(B, \mu_n^{\otimes j}) & & \longrightarrow & & H^{i-1}(L, \mu_n^{\otimes j}) \\ \downarrow & & & & \downarrow \\ H^{i-1}(\kappa_B, \mu_n^{\otimes j}) & \xrightarrow{\sim} & H^{i-1}(\text{Gal}(\hat{L}_{nr}/\hat{L}), \mu_n^{\otimes j}) & \to & H^{i-1}(\hat{L}, \mu_n^{\otimes j}) \end{array}$$

therefore β' may be represented by a cocycle

$$\bar{\beta}' \in A^{i-1}(j) \bigcap A_{i-1}(j).$$

By definition of the cup-product, the cocycle $\bar{\alpha}' \cup \bar{\beta}'$ represents $\alpha' \cup \beta'$, the image of $a' \cup b'$ in $H^i(G, \mu_n^{\otimes j+1})$. However

$$(\bar{\alpha}' \cup \bar{\beta}')(g_1, g_2, \ldots, g_i) = \bar{\alpha}'(g_1) \otimes \bar{\beta}'(g_2, \ldots, g_i).$$

Thus $\bar{\alpha}' \cup \bar{\beta}'$ belongs in fact to $A^i(j+1) \bigcap A_{i-1}(j+1)$ and its image in $C^{i-1}(G/N, C^1(N, \mu_n^{\otimes j+1}))$ is the cocycle

$$\bar{\gamma} : \bar{g}_1, \ldots, \bar{g}_{i-1} \mapsto (m \mapsto \bar{\alpha}'(m) \otimes \bar{\beta}'(g_1, \ldots, g_{i-1})).$$

And, through the maps $C^1(N, \mu_n^{\otimes j+1}) \to H^1(N, \mu_n^{\otimes j+1}) \to \mu_n^{\otimes j}$, the image of $m \mapsto \bar{\alpha}'(m) \otimes \bar{\beta}'(g_1, \ldots, g_{i-1})$ is $\nu_A(a)\bar{\beta}'(g_1, \ldots, g_{i-1})$. Therefore the image of $\bar{\alpha}' \cup \bar{\beta}'$ in $H^{i-1}(\kappa_B, \mu_n^{\otimes j}) \xrightarrow{\sim} E_2^{i-1,1}(j+1)$, which is, by definition, $\partial_A(a' \cup b')$, is the product of $\nu_A(a)$ by β. ■

Lemma 4 *With notation as above, there exists a morphism ϕ_A^1 which fits into the commutative diagram:*

$$\begin{array}{ccc} \tau_A^\perp \hookrightarrow U^\vee & \xrightarrow{\phi^1} & H^1(K,\mu_p) \\ \searrow \phi_A^1 & \nearrow & \\ H^1_{\acute{e}t}(A,\mu_p) & & \end{array}$$

Proof Let $x \in \tau_A^\perp$. Then $\phi^1(x) \in H^1(K,\mu_p) \tilde{\rightarrow} K^*/K^{*p}$. Let y be an element of K^* which represents $\phi^1(x)$. By the very definition of τ_A, we have $\nu_A(y) \equiv 0(p)$. Let π_A be a uniformizing element of K for ν_A. We may write y in the form $y = \pi_A^{kp} z$ for a $k \in \mathbf{Z}$ and a $z \in A^*$.Thus $\phi^1(x)$ is the image of $\bar{z} \in A^*/A^{*p}$ by

$$A^*/A^{*p} \hookrightarrow K^*/K^{*p}$$

which is an embedding. We define $\phi_A^1(x)$ as the image of $\bar{z}$ in $H^1_{\acute{e}t}(A,\mu_p)$. Then the commutativity of the diagram

$$\begin{array}{ccc} A^*/A^{*p} & \hookrightarrow & K^*/K^{*p} \\ \downarrow & & \downarrow \\ H^1_{\acute{e}t}(A,\mu_p) & \rightarrow & H^1(K,\mu_p) \end{array}$$

yields the lemma. ■

If $\tau_A = 0$, the main lemma follows from lemma 1 and lemma 4, since ∂_A is zero on the image of $H^i_{\acute{e}t}(A,\mu_p^{\otimes i})$. Let us assume that $\tau_A \neq 0$.

Lemma 5 *With notation as above, there is a map $\hat{\tau}_A$ making the following diagram commutative,*

$$\begin{array}{ccc} (\Lambda^i U)^\vee & \xrightarrow{\tilde{\tau}_A} & (\Lambda^{i-1} U)^\vee \\ \downarrow \hat{\tau}_A & \nearrow & \\ \Lambda^{i-1}(\tau_A^\perp) & & \end{array}$$

Here the map $\Lambda^{i-1}(\tau_A^\perp) \rightarrow (\Lambda^{i-1}U)^\vee$ is the natural injection.

Proof Let us choose a basis $v_1, \ldots, v_n$ of U with $v_1 = \tau_A$. Then $\tilde{\tau}_A$ is given by the formula: if $j_1 < \ldots < j_i$ then

$$\tilde{\tau}_A(v_{j_1}^\vee \wedge \ldots \wedge v_{j_i}^\vee) = 0 \text{ if } j_1 \neq 1$$

$$\tilde{\tau}_A(v_{j_1}^\vee \wedge \ldots \wedge v_{j_i}^\vee) = v_{j_2}^\vee \wedge \ldots \wedge v_{j_i}^\vee \text{ if } j_1 = 1$$

■

Thanks to lemma 1 and lemma 4 we get a morphism of graded $\mathbf{F}_p$-algebras

$$\Lambda^*(\tau_A^\perp) \to H(A)_p$$

and in particular a morphism $\Lambda^{i-1}(\tau_A^\perp) \xrightarrow{\phi_A^{i-1}} H_{\acute{e}t}^{i-1}(A, \mu_p^{\otimes i-1})$

Lemma 6 *With notation as above, the diagram*

$$\begin{array}{ccc} (\Lambda^i U)^\vee & \xrightarrow{\hat{\tau}_A} & \Lambda^{i-1}(\tau_A^\perp) \\ \downarrow \phi^i & & \downarrow \\ H^i(K, \mu_p^{\otimes i}) & & H_{\acute{e}t}^{i-1}(A, \mu_p^{\otimes i-1}) \\ \searrow \partial_A & & \swarrow \\ & H^{i-1}(\kappa_A, \mu_p^{\otimes i-1}) & \end{array}$$

is commutative.

Proof The computation of the preceding proof implies that for any $\lambda \in (\Lambda^i U)^\vee \xrightarrow{\sim} \Lambda^i(U^\vee)$

$$\lambda - v_1^\vee \wedge \tilde{\tau}_A(\lambda) \in \Lambda^i(\tau_A^\perp).$$

Thus $\phi^i(\lambda - v_1^\vee \wedge \tilde{\tau}_A(\lambda))$ comes from $H_{\acute{e}t}^i(A, \mu_p^{\otimes i})$ and its image by ∂_A is 0. Therefore, using lemma 3 with $a' = \phi^1(v_1^\vee)$ and $b = \phi_A^{i-1}(\hat{\tau}_A(\lambda))$,

$$\begin{aligned} \partial_A \phi^i(\lambda) &= \partial_A(\phi^1(v_1^\vee) \cup \phi^{i-1}(\tilde{\tau}_A(\lambda))) \\ &= \nu_A(\phi^1(v_1^\vee))\phi_{\kappa_A}^{i-1}(\hat{\tau}_A(\lambda)) \\ &= \phi_{\kappa_A}^{i-1}\hat{\tau}_A(\lambda) \end{aligned}$$

where $\phi_{\kappa_A}^{i-1}$ is the composite map

$$\Lambda^{i-1}\tau_A^\perp \xrightarrow{\phi_A^{i-1}} H_{\acute{e}t}^{i-1}(A, \mu_p^{\otimes i-1}) \to H^{i-1}(\kappa_A, \mu_p^{\otimes i-1}).$$

∎

Proof of the main lemma The case $\tau_A = 0$ has already been settled. If $\tau_A \neq 0$ and $\lambda \in (\Lambda^i U)^\vee$ verifies $\tilde{\tau}_A(\lambda) = 0$ then by lemma 5 $\hat{\tau}_A(\lambda) = 0$. Lemma 6 then implies that $\partial_A(\phi^i(\lambda)) = 0$. ∎

Remark In fact, the lemmata of this section also apply to any field over k and any discrete valuation ring $A \subset K$ such that $\mathrm{Fr}(A) = K$.

4 Construction of non-rational fields

Notation Let k be an algebraically closed field of characteristic 0, p a prime number, i a strictly positive integer, n an integer. Let us fix a primitive p-th root of unity ξ, and let $F' = k(T_1, \ldots, T_n)$, $X_j = T_j^p$ for $j = 1, \ldots, n$ and $F = k(X_1, \ldots, X_n) \subset F'$. Let U denote an $\mathbf{F}_p$-vector space of dimension n with a chosen basis $(u_1, \ldots, u_n)$. This yields an isomorphism $U \xrightarrow{\sim} \mathrm{Gal}(F'/F)$ and an injection $U^\vee \overset{\phi_F^1}{\hookrightarrow} H^1(F, \mu_p)$ (which sends $u_j^\vee$ to the class of X_j).

This notation will be used throughout section 4.

Lemma 7 *The morphism $\Lambda^i(U^\vee) \to H^i(F, \mu_p^{\otimes i})$ is an injection.*

Proof Let $\hat{F}_m$ be the field $k((X_1))\ldots((X_m))$ for $m \leq n$. Let us prove by induction on m that

$$\Lambda^j(U_m^\vee) \xrightarrow{\sim} H^j(\hat{F}_m, \mu_p^{\otimes j})$$

where U_m is the subgroup of U generated by $u_1 \ldots u_m$. The result is true for $m = 0$. We assume that this is true for $m-1$. Let us consider the valuation associated to (X_m). The residue field is isomorphic to $\hat{F}_{m-1}$. Since $\hat{F}_m$ is complete, the inertia group is isomorphic to $\varprojlim \mu_n$. We get an exact sequence

$$0 \to \varprojlim \mu_n \to \mathrm{Gal}(\hat{F}_m) \to \mathrm{Gal}(\hat{F}_{m-1}) \to 0$$

which is split. The Hochschild-Serre spectral sequence yields short exact sequences

$$0 \to H^j(\hat{F}_{m-1}, \mu_p^{\otimes j}) \to H^j(\hat{F}_m, \mu_p^{\otimes j}) \to H^{j-1}(\hat{F}_{m-1}, \mu_p^{\otimes j-1}) \to 0.$$

Let $A = \hat{F}_{m-1}[[X_m]]$ be the valuation ring corresponding to (X_m). Using the notation of section 3, we have that $\tau_A = u_m$. Therefore lemma 6 implies the commutativity of the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & H^j(\hat{F}_{m-1}, \mu_p^{\otimes j}) & \to & H^j(\hat{F}_m, \mu_p^{\otimes j}) & \to & H^{j-1}(\hat{F}_{m-1}, \mu_p^{\otimes j-1}) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \Lambda^j(U_{m-1}^\vee) & \to & \Lambda^j(U_m^\vee) & \overset{\hat{u}_m}{\to} & \Lambda^{j-1}(U_{m-1}^\vee) & \to & 0 \end{array}$$

where the lines are exact and, by induction hypothesis, the left and right vertical maps are isomorphisms, the morphism $\hat{u}_m$ being defined in the same way as $\hat{\tau}_A$. The exactness of the bottom line comes from the decomposition $U_m = U_{m-1} \oplus \mathbf{F}_p u_m$. Thus the central vertical map is also an isomorphism and the result for m is proved. ∎

If K is a function field over k which contains F, we define ϕ^1_K as the composed map

$$U^\vee \to H^1(F,\mu_p) \to H^1(K,\mu_p)$$

and by lemma 1 we get a morphism $\phi^i_K : (\Lambda^i U)^\vee \to H^i(K,\mu_p^{\otimes i})$

Now the problem of finding a unirational field which is not rational reduces to producing a subspace $S \subset \Lambda^i U$ and an extension K/F of function fields satisfying the following three conditions:

(i) K is unirational over k

(ii) the kernel of the map $\phi^i_K : (\Lambda^i U)^\vee \to H^i(K,\mu_p^{\otimes i})$ is $S^\perp$.

(iii) $S \neq S_{dec}$

We can then apply Corollary 3 to the map

$$\phi^1_K : U^\vee \to H^1(K,\mu_p)$$

Indeed the orthogonal in $\Lambda^i U$ of the kernel of the induced map $(\Lambda^i U)^\vee \to H^i(K,\mu_p^{\otimes i})$ is S by (ii). Therefore using corollary 3, we see that assumption (iii) implies that K is not stably rational.

For each of the following examples, the road map is as follows. First we give conditions on a subspace V of $(\Lambda^i U)^\vee$ which imply the following two properties: there exists an extension K/F such that K is unirational and $\mathrm{Ker}(H^i(F,\mu_p^{\otimes i}) \to H^i(K,\mu_p^{\otimes i}))$ is exactly the image of V by the injection ϕ^i_F. Then we produce $S \subset \Lambda^i U$ such that $S^\perp \subset \Lambda^i U^\vee$ verifies these conditions and such that $S \neq S_{dec}$.

4.1 Examples with non-trivial $H^2_{nr}(K,\mu_p^{\otimes 2})$

Notation If A is a central simple algebra over an arbitrary field L, we shall denote by $[A]$ its class in the Brauer group $\mathrm{Br}(L)$ and Y_A the corresponding Severi-Brauer variety.

Theorem 4 (Amitsur [Am]) *The kernel of the morphism*

$$\mathrm{Br}(L) \to \mathrm{Br}(L(Y_A))$$

is the finite subgroup of $\mathrm{Br}(L)$ *generated by* $[A]$.

Let us now consider a field L of characteristic prime to p and $[A_1],\ldots,[A_m]$ in $(\mathrm{Br}\,L)_{(p)} \xrightarrow{\sim} H^2(L,\mu_p)$, then we deduce from the theorem the following lemma:

Lemma 8

$$\mathrm{Ker}(H^2(L,\mu_p) \to H^2(L(Y_{A_1}\times\cdots\times Y_{A_m}),\mu_p)) = <[A_1],\ldots,[A_m]> .$$

Proof We shall prove the lemma by induction on m. When $m=0$ the lemma is trivial. Assume that the result is true for $m-1$. Let us denote by L_m the field $L(Y_{A_1}\times\cdots\times Y_{A_m})$ and by L_{m-1} the field $L(Y_{A_1}\times\cdots\times Y_{A_{m-1}})$. Let $\rho_j : H^2(L,\mu_p) \to H^2(L_j,\mu_p)$ for $j=m,m-1$ and $\rho : H^2(L_{m-1},\mu_p) \to H^2(L_m,\mu_p)$ be the canonical maps. Let λ be an element of $H^2(L,\mu_p)$ such that $\rho_m(\lambda)=0$. Then $\rho(\rho_{m-1}(\lambda))=0$ and by theorem 4, $\rho_{m-1}(\lambda)=k[A_m]$ for some $k\in\mathbf{Z}/p\mathbf{Z}$. therefore $\rho_{m-1}(\lambda-k[A_m])=0$ and by the induction hypothesis

$$\lambda - k[A_m] \in <[A_1],\ldots,[A_{m-1}]> .$$

Thus $\lambda \in <[A_1],\ldots,[A_m]>$. ■

We shall now apply this lemma to the construction described at the beginning of section 4.

Let S be a subspace of $\Lambda^2 U$. Let $s_1,\ldots,s_m$ be a family generating $S^\perp \subset \Lambda^2 U^\vee$, and $S_1,\ldots,S_m$ be central simple algebras representing the images of $s_1,\ldots,s_m$ in $H^2(F,\mu_p)$ which our choice of $\xi\in\mu_p$ enables us to identify with $H^2(F,\mu_p^{\otimes 2})$. Let $K=F(Y_{S_1}\times\cdots\times Y_{S_m})$.

Proposition 5 *With notation as above, the function field K is unirational. However if $S\neq S_{dec}$ then*

$$H^2_{nr}(K,\mu_p^{\otimes 2}) \neq \{0\}$$

and K is not stably rational.

Proof By their very definition, the images of $s_1,\ldots,s_m$ in the group $H^2(F,\mu_p^{\otimes 2})$ come from $H^2(\mathrm{Gal}(F'/F),\mu_p^{\otimes 2})$. Therefore they become zero when lifted to F'. So the Severi-Brauer varieties corresponding to the S_j are split by F' and the composite $F'K$ is rational over F' and thus over k. So K is unirational. We deduce from lemma 8 that the extension K/F satisfies condition (ii) above and by the principle above, $S\neq S_{dec}$ implies $H^2_{nr}(K,\mu_p^{\otimes 2})\neq\{0\}$. ■

Example For $n\leq 3$ any element of $\Lambda^2 U$ may be written as $u\wedge v$ with u and v in U. We shall therefore consider the case $n=4$. The subspaces S of $\Lambda^2 U$ such that $S_{dec}\neq S$ are described by Bogomolov in [Bo] when $p \neq 2$. This description is the following: the elements of the form $u \wedge v$ with $u, v \in U$ are, in this case, the isotropic vectors for the quadratic form

$$\begin{array}{rccc} q: & \Lambda^2 U & \longrightarrow & \Lambda^4 U \\ & u & \longmapsto & u \wedge u \end{array}$$

and $S \neq S_{dec}$ if and only if $S = \mathrm{Ker}(q_{|S}) \oplus T$ where $T \neq \{0\}$ and $q_{|T}$ is anisotropic. The following cases are possible:

case	dim S	dim S_{dec}
(a)	1	0
(b)	2	0
(c)	2	1
(d)	3	1
(e)	3	2

Case (a) was studied by Saltman in [Sa]. For an example of (e) we may choose

$$S = < u_1 \wedge u_2, u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_4 > .$$

Indeed $q_{|S}$ is represented by the matrix: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Then

$$S^{\perp} = < u_3^{\vee} \wedge u_4^{\vee}, u_2^{\vee} \wedge u_3^{\vee}, u_1^{\vee} \wedge u_3^{\vee} - u_2^{\vee} \wedge u_4^{\vee} >,$$

and

$$K = F(Y_{A_\xi(X_3,X_4)} \times Y_{A_\xi(X_2,X_3)} \times Y_{A_\xi(X_1,X_3) \otimes A_\xi(X_4,X_2)}),$$

where $A_\xi(a,b)$ is the algebra over F generated by two elements I and J with the relations

$$I^p = a, J^p = b, IJ = \xi JI.$$

4.2 Examples with non-trivial $H^3_{nr}(K, \mu_p^{\otimes 3})$

Notation If L is a field of characteristic prime to p which contains the p-th roots of unity, ξ a primitive p-th root of unity and A a cyclic central simple algebra of the form $A_\xi(a,b)$ for $a, b \in L^*$ then, for any $c \in L^*$, I denote by $Z_{A,c}$ the *norm variety* defined by $\mathrm{Nrd}(x) = c$.

Theorem 6 (Suslin [Su]) *With this notation, the kernel of the map*

$$H^3(L,\mu_p^{\otimes 2}) \to H^3(L(Z_{A,c}),\mu_p^{\otimes 2})$$

is the subgroup generated by $[A] \cup c$.

The case $p = 2$ is due to Arason [Ar].

Let us apply this theorem to our problem.

Let S be a subspace of $\Lambda^3 U$. We make the following hypotheses:

(H1) we can choose a basis $(s_1, \ldots, s_m)$ of $S^\perp$ such that each s_j is of the form $v_j \wedge w_j \wedge y_j$ for $v_j, w_j, y_j \in U^\vee$.

(H2) For each $k \in \{1,\ldots,m\}$ and each $j \in \{1,\ldots,n\}$, at most one of the elements v_k, w_k, y_k has a non zero value on u_j.

Notation Let $Z_j = Z_{A_\xi(V_j,W_j),Y_j}$ where V_j, W_j, Y_j are the images of v_j, w_j, y_j in $H^1(F,\mu_p)$. Let $K = F(Z_1 \times \cdots \times Z_m)$.

(H1) enables us to apply theorem 6 whereas **(H2)** is used to prove that $\mathrm{Br}_{nr}(K)$ is trivial.

Proposition 7 *With notation as above, the function field K is unirational and the group* $\mathrm{Br}_{nr}(K)$ *is trivial. However, if* $S \neq S_{dec}$, *then*

$$H^3_{nr}(K,\mu_p^{\otimes 3}) \neq \{0\}$$

and K is not stably rational.

Proof

• Let us first prove the last claim: Using theorem 6, since $K = F(Z_1)\ldots(Z_m)$, we may prove as in section 4.1, Lemma 8 that

$$\mathrm{Ker}(H^3(F,\mu_p^{\otimes 3}) \to H^3(K,\mu_p^{\otimes 3})) =< (V_j,W_j,Y_j), 1 \leq j \leq m >$$

and therefore

$$\mathrm{Ker}((\Lambda^3 U)^\vee \to H^3(K,\mu_p^{\otimes 3})) = S^\perp.$$

As above, if $S \neq S_{dec}$ then K is not stably rational.

• The images of v_j, w_j, y_j in $H^1(F,\mu_p)$ come from the group $H^1(\mathrm{Gal}(F'/F),\mu_p)$ and have therefore trivial images in $H^1(F',\mu_p)$. Thus

$$A_\xi(V_j,W_j)\otimes F' \xrightarrow{\sim} M_p(F')$$

and $Z_{A_\xi(V_j,W_j),Y_j} \xrightarrow{\sim} SL_{p,F'}$. This shows that the composite field KF' is rational over F' hence also over k. So K is unirational over k.

• We shall now prove that $\mathrm{Br}_{nr}(K) = \{0\}$. For this purpose, we shall use the following lemmata:

Lemma 9 *Let X be a non-singular geometrically integral variety over a field M of characteristic 0. Let $\overline{M}$ be an algebraic closure of M. Let $\mathcal{G} = \mathrm{Gal}(\overline{M}/M)$ and $\overline{X} = X \times_M \overline{M}$. Let $\overline{M}[X]$ be the ring $\Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ and $\overline{M}(X)$ the function field of $\overline{X}$.*

If $\overline{M}[X]^* = \overline{M}^*$, then there is an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{\mathcal{G}} \to \mathrm{Br}(M) \to \\ \to \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{M}(X)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{X}))\right) \to H^1(\mathcal{G}, \mathrm{Pic}(\overline{X})).$$

Proof We have the following exact sequence:

$$0 \to H^1(\mathcal{G}, \overline{M}[X]^*) \to \mathrm{Pic}(X) \to (\mathrm{Pic}\,\overline{X})^{\mathcal{G}} \to H^2(\mathcal{G}, \overline{M}[X]^*) \to \\ \to \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{M}(X)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{X}))\right) \to H^1(\mathcal{G}, \mathrm{Pic}\,\overline{X}).$$

This is the exact sequence (1.5.0) in [C-T,S]. We then use the fact that $\overline{M}[X]^* = \overline{M}^*$ and Hilbert's theorem 90 to get the lemma. ■

The exact sequence of the lemma can also be obtained using the following exact sequence of $\mathcal{G}$-modules

$$1 \to \overline{M}^* \to \overline{M}(X)^* \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0$$

and the similar one for X.

Lemma 10 *Let L be as in theorem 6, let A_j for $j = 1, \ldots, m$ be central simple algebras over L and let c_j belong to L^* for $j = 1, \ldots, m$, we denote by Z_{A_j,c_j} the norm variety for A_j and c_j and $Z = \prod_{1 \le j \le m} Z_{A_j,c_j}$. If $\lambda \in \mathrm{Br}_{nr}(L(Z))$ then*

$$\lambda \in \mathrm{Im}\left(\mathrm{Br}(L) \to \mathrm{Br}(L(Z))\right).$$

Proof We denote by $\overline{L}$ the algebraic closure of L, $\overline{Z} = Z \times_L \overline{L}$ and $\mathcal{G} = \mathrm{Gal}(\overline{L}/L)$. Let $\lambda \in \mathrm{Br}_{nr}(L(Z))$. By Hilbert's theorem 90 we have an exact sequence:

$$0 \to H^2(\mathcal{G}, \overline{L}(Z)^*) \to \mathrm{Br}(L(Z)) \xrightarrow{\rho} \mathrm{Br}(\overline{L}(Z)).$$

But $\overline{Z} \xrightarrow{\sim} \prod_{j=1}^{m} SL_{p,\overline{L}}$ is $\overline{L}$-rational and therefore $\mathrm{Br}_{nr}(\overline{L}(Z)) = \{0\}$. As

$$\rho\left(\mathrm{Br}_{nr}(L(Z))\right) \subset \mathrm{Br}_{nr}(\overline{L}(Z))$$

$\rho(\lambda) = 0$ and λ comes from $\lambda' \in H^2(\mathcal{G}, \overline{L}(Z)^*)$.

Let us prove that $\lambda' \in \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{L}(Z)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{Z}))\right)$ For any $x \in Z^{(1)}$, the set of points of codimension 1 in Z, let us choose $x' \in \overline{Z}^{(1)}$ above x. Then x' defines $B \in \mathcal{P}(\overline{L}(Z))$ whereas x corresponds to $A = L(Z) \bigcap B$. Let $\mathcal{H}_x$ be the stabilizer of B. Let $\widehat{L(Z)}$ (respectively $\widehat{\overline{L}(Z)}$) be the completion of $L(Z)$ (respectively $\overline{L}(Z)$) for A (respectively B), $\widehat{L(Z)}_{nr}$ (respectively $\widehat{\overline{L}(Z)}_{nr}$) the corresponding maximal unramified extensions. Let $\widetilde{\overline{L}(Z)}$ (respectively $\widetilde{\overline{L}(Z)}_{nr}$) be the algebraic closure of $\widehat{L(Z)}$ (respectively $\widehat{L(Z)}_{nr}$) in $\widehat{\overline{L}(Z)}$ (respectively $\widehat{\overline{L}(Z)}_{nr}$). Since the ramification index is one, the fields $\widetilde{\overline{L}(Z)}_{nr}$ and $\widehat{L(Z)}_{nr}$ are actually equal. Therefore we have the following diagram of fields:

$$\begin{array}{ccccc}
 & & \widetilde{\overline{L}(Z)}_{nr} & & \\
 & / & & \| & \\
 & \widetilde{\overline{L}(Z)} & \mathcal{H}_x & \widehat{L(Z)}_{nr} & \\
 / & & \diagdown & & / \\
\overline{L}(Z) & & \mathcal{G} & \widehat{L(Z)} & \\
 & \diagdown & & / & \\
 & & L(Z) & &
\end{array}$$

Let us define ν_x as the valuation associated to B and i_x as the injection $\mathcal{H}_x \hookrightarrow \mathcal{G}$. The definition of ∂_A for the Brauer group and the diagram of fields yields the following commutative diagram

$$\begin{array}{ccc}
H^2(\mathcal{G}, \overline{L}(Z)^*) & \hookrightarrow & \mathrm{Br}(L(Z)) \\
\downarrow (i_x, \nu_x)^* & & \downarrow \partial_A \\
H^2(\mathcal{H}_x, \mathbf{Z}) & & H^1(\mathrm{Gal}(\overline{\kappa_A}/\kappa_A), \mathbf{Q}/\mathbf{Z})) \\
\downarrow & & \downarrow \\
H^1(\mathcal{H}_x, \mathbf{Q}/\mathbf{Z}) & & H^1(\mathrm{Gal}(\widehat{L(Z)}_{nr}/\widehat{L(Z)}), \mathbf{Q}/\mathbf{Z}) \\
\searrow & & \swarrow \\
 & H^1(\mathrm{Gal}(\widetilde{\overline{L}(Z)}_{nr}/\widehat{L(Z)}), \mathbf{Q}/\mathbf{Z}). &
\end{array}$$

Here the isomorphisms

$$H^2(\mathcal{H}_x, \mathbf{Z}) \xrightarrow{\sim} H^1(\mathcal{H}_x, \mathbf{Q}/\mathbf{Z})$$

and

$$H^1(\mathrm{Gal}(\overline{\kappa_A}/\kappa_A), \mathbf{Q}/\mathbf{Z})) \xrightarrow{\sim} H^1(\mathrm{Gal}(\widehat{L(Z)}_{nr}/\widehat{L(Z)}), \mathbf{Q}/\mathbf{Z})$$

are the inverses of the natural maps. Besides the map

$$H^2(\mathcal{H}_x, \mathbf{Z}) \hookrightarrow H^1(\mathrm{Gal}(\widetilde{\overline{L}(Z)}_{nr}/\widehat{L(Z)}), \mathbf{Q}/\mathbf{Z})$$

is injective. But $\mathrm{Div}(\overline{Z}) \xrightarrow{\sim} \underset{x \in Z^{(1)}}{\oplus} \mathbf{Z}[\mathcal{G}/\mathcal{H}_x]$ as a $\mathcal{G}$-module and by Shapiro's lemma

$$H^2(\mathcal{G}, \mathrm{Div}(\overline{Z})) \xrightarrow{\sim} \underset{x \in Z^{(1)}}{\oplus} H^2(\mathcal{H}_x, \mathbf{Z}).$$

By the diagram, for any $x \in Z^{(1)}$ we have $(i_x, \nu_x)^*(\lambda') = 0$ and

$$\lambda' \in \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{L}(Z)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{Z}))\right).$$

Let us show that $\overline{L}[Z]^* = \overline{L}^*$ and that $\mathrm{Pic}(\overline{Z}) = 0$. These facts may be proved in the following elementary way: let $U = \mathbf{A}^1_{\overline{L}} - \{0\}$ and $P = \overline{Z} \times U^m$ Then $P \xrightarrow{\sim} (GL_{p,\overline{L}})^m$ Let $\pi : P \to \overline{Z}$ be the natural projection and $j : \overline{Z} \to P$ the immersion corresponding to $(1, \dots, 1)$. If $f \in \overline{L}[Z]^*$ then $\pi^*(f)$ is an inversible element in the ring of function of $(GL_{p,\bar{L}})^m$ which has the following form:

$$\overline{L}[X_{i,j,k}, 1 \leq k \leq m, 1 \leq i,j \leq p]\left[\frac{1}{\mathrm{Det}\,(X_{i,j,k})_{1 \leq i,j \leq p}}, 1 \leq k \leq m\right].$$

Therefore $\pi^*(f)$ can be written in the form $c \prod_{k=1}^{m} (\mathrm{Det}_k)^{n_k}$ where $c \in \overline{L}^*$, $n_k \in \mathbf{Z}$ and $\mathrm{Det}_k = \mathrm{Det}\,(X_{i,j,k})_{1 \leq i,j \leq p}$ for $1 \leq k \leq m$. But, by definition of j, we have the relation $j^*(\mathrm{Det}_q) = 1$. Therefore

$$f = j^*(\pi^*(f)) \in \overline{L}^*.$$

Moreover we have an injection $\mathrm{Pic}(\overline{Z}) \hookrightarrow \mathrm{Pic}(P)$. And P is an open set in $(M_{n,\overline{L}})^m$. We hence have a surjection $\mathrm{Pic}((M_{n,\overline{L}})^m) \twoheadrightarrow \mathrm{Pic}(P)$. But the Picard group of $(M_{n,\overline{L}})^m$ is trivial. Therefore $\mathrm{Pic}(\overline{Z}) = \{0\}$. Since $\overline{L}[X]^* = \overline{L}^*$, by lemma 9, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Pic}(Z) \to \mathrm{Pic}(\overline{Z})^{\mathcal{G}} \to \mathrm{Br}(L) \to \\ \to \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{L}(Z)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{Z}))\right) \to H^1(\mathcal{G}, \mathrm{Pic}(\overline{Z})). \end{aligned}$$

We get an isomorphism

$$\mathrm{Br}(L) \xrightarrow{\sim} \mathrm{Ker}\left(H^2(\mathcal{G}, \overline{L}(Z)^*) \to H^2(\mathcal{G}, \mathrm{Div}(\overline{Z}))\right).$$

Therefore λ comes from $\mathrm{Br}(L)$. ■

Lemma 11 *With notation as in proposition 7, if A is an element of $\mathcal{P}(F)$ corresponding to a point of codimension one of $\mathbf{A}^n_k$ then there exists $B \in \mathcal{P}(K)$ such that we have $B \bigcap F = A$, the map*

$$H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(\kappa_B, \mathbf{Q}/\mathbf{Z})$$

is injective and the ramification index $e_{B/A} = \nu_B(\pi_A) = 1$ (for ν_B the valuation corresponding to B, and π_A an uniformizing element of A).

The proof of lemma 11, which uses **(H2)** is based on the following lemma

Lemma 12 *Let L be a field over k, we denote by ξ a primitive p-th root of unity. Let $V, W, X \in L^*$, $A = A_\xi(V, W)$ and $Z = Z_{A,X}$. Let $B \in \mathcal{P}(L)$ such that V, W, X belong to B and at most one of the V, W, X belongs to $\mathcal{M}_B$, the maximal ideal of B. Then there exists $B' \in \mathcal{P}(L(Z))$ above B such that the morphism $H^1(\kappa_B, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(\kappa_{B'}, \mathbf{Q}/\mathbf{Z})$ is injective and such that the ramification index of B' over B is* 1.

Proof The algebra A is generated by two generators I and J with the relations $I^p = V$,$J^p = W$ and $IJ = \xi JI$ and has therefore the basis $(I^k J^k)_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}}$. If $V \notin (L^*)^p$, let $L' = L[T]/(T^p - V)$, otherwise let $L' = L$ and T be a p-th root of V. The field L' is a splitting field for A. We define an isomorphism $A \otimes L' \xrightarrow{\sim} M_p(L')$ by sending I to the diagonal matrix $D(T, \xi T, \ldots, \xi^{p-1}T)$ and J to

$$\begin{pmatrix} 0 & \ldots\ldots\ldots & & 0 & W \\ 1 & 0 & \ldots\ldots\ldots & & 0 \\ 0 & 1 & 0 & .. & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & .. & 0 & 1 & 0 \end{pmatrix}$$

Therefore, if $y \in A \otimes L(y_{j,k})_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}}$ is given by $y = \sum_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}} y_{j,k} I^j J^k$, the image of y in the ring $M_p(L'(y_{j,k})_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}})$ is $M_y = (m_{j,k})_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}}$ where

$$\begin{aligned} m_{j,k} &= y_{0,j-k} + y_{1,j-k}\xi^j T + \cdots + y_{p-1,j-k}\xi^{j(p-1)}T^{p-1} && \text{if } j \ge k \\ m_{j,k} &= W(y_{0,p+j-k} + \cdots + y_{p-1,p+j-k}\xi^{j(p-1)}T^{p-1}) && \text{otherwise} \end{aligned}$$

$\mathrm{Det}(M_y)$ is then a polynomial defined over L, which, by definition, gives the reduced norm on A. Therefore the equation of Z is given by $\mathrm{Det}(M_y) - X = 0$ and

$$L(Y) = \mathrm{Fr}\left(L[y_{j,k}]_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}} / (\mathrm{Det}(M_y) - X)\right).$$

Let

$$B'_0 = B[y_{j,k}]_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}} / (\mathrm{Det}(M_y) - X).$$

This is well defined: the coefficients of the polynomial $\mathrm{Det}(M_y) - X$ are in B, since $V, W, X \in B$. Let π_B be a uniformizing element for B. We have

$$B'_0/(\pi_B) = \kappa_B[y_{j,k}]_{\substack{0 \le j \le p-1 \\ 0 \le k \le p-1}} / (\overline{\mathrm{Det}(M_y) - X}).$$

The polynomial $\overline{\mathrm{Det}(M_y)}$ is given by the same computation over the residue field κ_B. And by hypothesis the only possible cases are the following ones:

(a) None of V, W, X is in $\mathcal{M}_B$, then $(\overline{\mathrm{Det}(M_y) - X})$ is the equation of the norm variety corresponding to the central simple algebra $A_\xi(\bar{V}, \bar{W})$ and to $\bar{X}$ where $\bar{V}, \bar{W}, \bar{X}$ are the images of V, W, X in κ_B

(b) If $W \in \mathcal{M}_B$ but neither of V, X is in $\mathcal{M}_B$ then $\overline{\mathrm{Det}(M_y)}$ becomes equal to the determinant of a lower triangular matrix over an extension of κ_B which splits the polynomial $T^p - \overline{V}$ and we get the following equality in $\kappa_B[y_{j,k}]$

$$\overline{\mathrm{Det}(M_y) - X} = \prod_{j=0}^{p-1} (y_{0,0} + y_{1,0}\xi^j T + \cdots + y_{p-1,0}\xi^{j(p-1)} T^{p-1}) - \bar{X}.$$

We obtain the equation of $Y \times \mathbf{A}_{\kappa_B}^{p(p-1)}$ where Y is geometrically integral. Y is, in fact, birationally isomorphic to the Severi-Brauer variety corresponding to $A_\xi(\overline{V}, \overline{X})$

(c) We assume that $V \in \mathcal{M}_B$ but neither of W, X is in $\mathcal{M}_B$. We may exchange V and W in the definition of Z and this case reduces to the preceding one.

(d) If $X \in \mathcal{M}_B$ but neither of V, W is in $\mathcal{M}_B$, we have

$$\overline{\mathrm{Det}(M_y) - X} = \overline{\mathrm{Det}(M_y)}.$$

We get a variety which becomes isomorphic over $\overline{\kappa_B}$, the algebraic closure of κ_B, to the subvariety of $M_p(\overline{\kappa_B})$ defined by $\mathrm{Det}(M) = 0$. This subvariety is integral.

Therefore in each case $B'_0/(\pi_B)$ is the ring of functions of a geometrically integral variety over κ_B. Thus $B'_0/(\pi_B)$ is integral and (π_B) is a prime ideal of B'_0. Let $B' = B'_{0(\pi_B)}$. B' is a local ring and, since $\mathcal{M}_{B'} = (\pi_B)$, B' is a discrete valuation ring of rank one. Moreover $B' \cap L = B$, $\mathrm{Fr}(B') = L(Z)$, $e_{B'/B} = 1$ and κ_B is algebraically closed in $\kappa_{B'}$, the residue field of B' therefore

$$H^1(\kappa_B, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(\kappa_{B'}, \mathbf{Q}/\mathbf{Z}).$$

■

Proof of lemma 11 Since A corresponds to a point of codimension 1 of $\mathbf{A}_k^n$, at most one of $X_1, \ldots, X_n$ is in $\mathcal{M}_A$. Since **(H2)** is satisfied, we see that, by removing if necessary terms of the form π^{kp}, we can reduce to the case where, for each j, at most one of the V_j, W_j, Y_j is in $\mathcal{M}_A$. We shall prove by induction on $k \leq m$ that there exists $B_k \in \mathcal{P}(F(Z_1)\ldots(Z_k))$ such that $B_k \bigcap F = A$, $e_{B_k/A} = 1$ and the map

$$H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(\kappa_{B_k}, \mathbf{Q}/\mathbf{Z})$$

is an injection. For $k = 0$, A verifies the conditions. If it is true for $k < m$, we may use the construction of the preceding lemma to obtain B_{k+1}, because $V_{k+1}, W_{k+1}, Y_{k+1} \in A = B_k \bigcap F$ and at most one of them belong to $\mathcal{M}_{B_k}$.

The ring $B = B_m$ satisfies the conditions we wanted. ■

Completion of the proof of proposition 7 Let $\lambda \in \mathrm{Br}_{nr}(K)$. By lemma 10, we know that $\lambda \in \mathrm{Im}(\mathrm{Br}(F) \to \mathrm{Br}(K))$. Let λ' be an element of $\mathrm{Br}(F)$ whose image is λ. Let $A \in \mathcal{P}(F)$ corresponding to an irreducible divisor of $\mathbf{A}_k^n$. By lemma 11, there exists $B \in \mathcal{P}_K$ above A such that $e_{B/A} = 1$ and

$$H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(\kappa_B, \mathbf{Q}/\mathbf{Z})$$

is injective. Therefore we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(K) \\ \downarrow \partial_A & & \downarrow \partial_B \\ H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) & \hookrightarrow & H^1(\kappa_B, \mathbf{Q}/\mathbf{Z}) \end{array}$$

where the bottom line is injective. Since $\partial_B(\lambda) = 0$, we get that $\partial_A(\lambda') = 0$. Thus

$$\lambda' \in \bigcap_{A \in (\mathbf{A}_k^n)^{(1)}} \mathrm{Ker}\, \partial_A.$$

But, as in [C-T], using an induction on n one can check that, since k is algebraically closed of characteristic 0,

$$\bigcap_{A \in (\mathbf{A}_k^n)^{(1)}} \mathrm{Ker}\, \partial_A = \{0\}.$$

Therefore $\mathrm{Br}_{nr}(K) = \{0\}$. ■

Example 1 To get an example with $S \neq S_{dec}$, we need to take $n \geq 6$. If $n = 6$, $S =< u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 >$ verifies $S \neq S_{dec}$ and we have

$$\begin{aligned} S^\perp &= < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{(1,2,3),(4,5,6)\}, \end{cases} \\ &\quad u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee > \\ &= < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{(1,2,3),(4,5,6)\}, \end{cases} \\ &\quad (u_1^\vee - u_4^\vee) \wedge (u_2^\vee + u_5^\vee) \wedge (u_3^\vee + u_6^\vee) > . \end{aligned}$$

Therefore **(H1)** and **(H2)** are satisfied.

Example 2 We shall now give other examples with $n = 6$.

$$\begin{aligned} \text{Let } \; g_1 &= u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1 \\ g_2 &= u_2 \wedge u_3 \wedge u_4 + u_4 \wedge u_5 \wedge u_6 + u_6 \wedge u_1 \wedge u_2 \\ h_1 &= u_1 \wedge u_3 \wedge u_5 \\ h_2 &= u_2 \wedge u_4 \wedge u_6. \end{aligned}$$

Let us prove that, if $s = \{g_1\}$, $\{g_1, g_2\}$, $\{g_1, h_1\}$ or $\{g_1, h_1, h_2\}$, the subspace S generated by s verifies **(H1)**, **(H2)** and $S \neq S_{dec}$

• We first prove that S verifies **(H1)** and **(H2)**. If $s_0 = \{g_1, h_1, h_2\}$ and S_0 is the subspace generated by s_0 then

$$\begin{aligned} S_0^\perp &= < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{ \; (1,2,3),(3,4,5),(1,5,6), \\ \qquad\qquad\quad (1,3,5),(2,4,6)\}, \end{cases} \\ &\quad u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee, \\ &\quad u_3^\vee \wedge u_4^\vee \wedge u_5^\vee - u_5^\vee \wedge u_6^\vee \wedge u_1^\vee > \\ &= < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{ \; (1,2,3),(3,4,5),(1,5,6), \\ \qquad\qquad\quad (1,3,5),(2,4,6)\}, \end{cases} \\ &\quad u_3^\vee \wedge (u_1^\vee + u_5^\vee) \wedge (u_2^\vee + u_4^\vee), \\ &\quad u_5^\vee \wedge (u_1^\vee + u_3^\vee) \wedge (u_4^\vee + u_6^\vee) > . \end{aligned}$$

Therefore S_0 verifies **(H1)** and **(H2)**. This implies **(H1)** and **(H2)** in the cases $s = \{g_1\}$ and $s = \{g_1, h_1\}$. Indeed, for $s = \{g_1\}$, we have

$$S^\perp =< S_0^\perp, u_1^\vee \wedge u_3^\vee \wedge u_5^\vee, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee >$$

and for $s = \{g_1, h_1\}$, we get

$$S^\perp =< S_0^\perp, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee > .$$

For $s = \{g_1, g_2\}$, we have:

$$S^\perp = < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{ \quad (1,2,3),(3,4,5),(1,5,6), \\ \qquad\qquad\qquad (2,3,4),(4,5,6),(1,2,6)\}, \end{cases}$$

$$u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee,$$
$$u_3^\vee \wedge u_4^\vee \wedge u_5^\vee - u_5^\vee \wedge u_6^\vee \wedge u_1^\vee,$$
$$u_2^\vee \wedge u_3^\vee \wedge u_4^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee,$$
$$u_4^\vee \wedge u_5^\vee \wedge u_6^\vee - u_6^\vee \wedge u_1^\vee \wedge u_2^\vee >$$

$$= < u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } \begin{cases} 1 \leq j < l < m \leq 6 \\ (j,l,m) \notin \{ \quad (1,2,3),(3,4,5),(1,5,6), \\ \qquad\qquad\qquad (2,3,4),(4,5,6),(1,2,6)\}, \end{cases}$$

$$u_3^\vee \wedge (u_1^\vee + u_5^\vee) \wedge (u_2^\vee + u_4^\vee),$$
$$u_5^\vee \wedge (u_1^\vee + u_3^\vee) \wedge (u_4^\vee + u_6^\vee),$$
$$u_4^\vee \wedge (u_2^\vee + u_6^\vee) \wedge (u_3^\vee + u_5^\vee),$$
$$u_6^\vee \wedge (u_2^\vee + u_4^\vee) \wedge (u_1^\vee + u_5^\vee) > .$$

• We shall now prove that $S \neq S_{dec}$. First we remark that an element $g \in \Lambda^k U$ may be written in the form $g = u \wedge v$ with $u \in U$ and $v \in \Lambda^{k-1} U$ if and only if there exists $u \in U - \{0\}$ such that $g \wedge u = 0$. Let $\alpha, \beta, \gamma, \delta \in \mathbf{F}_p$, $g = \alpha g_1 + \beta g_2 + \gamma h_1 + \delta h_2$, $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbf{F}_p$ and $u = \sum_{1 \leq k \leq 6} a_i u_i$ We are interested in the equation:

$$(\star) \qquad u \wedge g = 0.$$

Let us compute $u_k \wedge g$ for $1 \leq k \leq 6$

$$\begin{aligned}
u_1 \wedge g &= \alpha u_1 \wedge u_3 \wedge u_4 \wedge u_5 + \beta u_1 \wedge u_2 \wedge u_3 \wedge u_4 \\
&\quad + \beta u_1 \wedge u_4 \wedge u_5 \wedge u_6 + \delta u_1 \wedge u_2 \wedge u_4 \wedge u_6 \\
u_2 \wedge g &= \alpha u_2 \wedge u_3 \wedge u_4 \wedge u_5 - \alpha u_1 \wedge u_2 \wedge u_5 \wedge u_6 \\
&\quad + \beta u_2 \wedge u_4 \wedge u_5 \wedge u_6 - \gamma u_1 \wedge u_2 \wedge u_3 \wedge u_5 \\
u_3 \wedge g &= -\alpha u_1 \wedge u_3 \wedge u_5 \wedge u_6 + \beta u_3 \wedge u_4 \wedge u_5 \wedge u_6 \\
&\quad + \beta u_1 \wedge u_2 \wedge u_3 \wedge u_6 - \delta u_2 \wedge u_3 \wedge u_4 \wedge u_6 \\
u_4 \wedge g &= -\alpha u_1 \wedge u_2 \wedge u_3 \wedge u_4 - \alpha u_1 \wedge u_4 \wedge u_5 \wedge u_6 \\
&\quad + \beta u_1 \wedge u_2 \wedge u_4 \wedge u_6 + \gamma u_1 \wedge u_3 \wedge u_4 \wedge u_5 \\
u_5 \wedge g &= -\alpha u_1 \wedge u_2 \wedge u_3 \wedge u_5 - \beta u_2 \wedge u_3 \wedge u_4 \wedge u_5 \\
&\quad + \beta u_1 \wedge u_2 \wedge u_5 \wedge u_6 + \delta u_2 \wedge u_4 \wedge u_5 \wedge u_6 \\
u_6 \wedge g &= -\alpha u_1 \wedge u_2 \wedge u_3 \wedge u_6 - \alpha u_3 \wedge u_4 \wedge u_5 \wedge u_6 \\
&\quad - \beta u_2 \wedge u_3 \wedge u_4 \wedge u_6 - \gamma u_1 \wedge u_3 \wedge u_5 \wedge u_6.
\end{aligned}$$

Therefore $(\star)$ is equivalent to the following system of equations:

$$\begin{array}{rclcrcl} a_1\beta - a_4\alpha & = & 0 & \qquad & a_1\alpha + a_4\gamma & = & 0 \\ -a_2\gamma - a_5\alpha & = & 0 & & -a_3\alpha - a_6\gamma & = & 0 \\ a_3\beta - a_6\alpha & = & 0 & & -a_3\delta - a_6\beta & = & 0 \\ a_1\delta + a_4\beta & = & 0 & & a_2\beta + a_5\delta & = & 0 \\ -a_2\alpha + a_5\beta & = & 0. & & & & \end{array}$$

Let us first consider the case $s = \{g_1, h_1, h_2\}$ and $S =< s >$. If $S = S_{dec}$ then there exist $\alpha, \gamma, \delta \in \mathbf{F}_p$ with $\alpha \neq 0$ and $u \in U - \{0\}$ such that

$$(\alpha g_1 + \gamma h_1 + \delta h_2) \wedge u = 0$$

We may assume $\alpha = 1$. Then, solving the system of equations with $\beta = 0$ and $\alpha = 1$, we find $u = 0$ and get a contradiction. Thus $S \neq S_{dec}$. From this we also deduce that $\dim(S/S_{dec}) = 1$ if $s = \{g_1, h_1\}$ or $s = \{g_1\}$. For the case $s = \{g_1, g_2\}$, we resolve the system with $\gamma = \delta = 0$ and find that $(\alpha g_1 + \beta g_2) \wedge u = 0$ implies $\alpha g_1 + \beta g_2 = 0$ or $u = 0$. Therefore $S_{dec} = \{0\}$. To sum up, we have found the following examples:

s	dim S	dim S_{dec}
g_1	1	0
g_1, h_1	2	1
g_1, h_1, h_2	3	2
g_1, g_2	2	0

Moreover one can show that

$$S =< u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1 >$$

is not in the same orbit under the action of $GL_6(\mathbf{F}_p)$ as the subgroup used in example 1. See [Re] for details.

4.3 Examples with non-trivial $H^4_{nr}(K, \mu_2^{\otimes 4})$

In this case we assume $p = 2$ and use the following result of Jacob and Rost on the quadratic forms [J,R]. We recall that the n-fold Pfister form $<< a_1, \ldots, a_n >>$ is the quadratic form

$$< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > .$$

Theorem 8 (Jacob and Rost) *Let L be a field of characteristic prime to 2, let Φ be a 4-fold Pfister form $<< a_1, a_2, a_3, a_4 >>$ and let $L(\Phi)$ be the field of functions of the quadric associated to Φ. Then we have*

$$\mathrm{Ker}(H^4(L, \mathbf{Z}/2\mathbf{Z}) \to H^4(L(\Phi), \mathbf{Z}/2\mathbf{Z})) = \{0, (a_1) \cup (a_2) \cup (a_3) \cup (a_4)\}.$$

As in section 4.2 we assume the following:

(H1) We may choose a basis $s_1, \ldots, s_m$ of $S^\perp$ such that each s_j may be written as $u_{1,j} \wedge u_{2,j} \wedge u_{3,j} \wedge u_{4,j}$ with $u_{k,j} \in U^\vee$ for $1 \leq k \leq 4$ and $1 \leq j \leq m$.

We then let $U_{k,j}$ represent the image of $u_{k,j}$ in F^*/F^{*2}. Set

$$\Phi_j = << U_{1,j}, U_{2,j}, U_{3,j}, U_{4,j} >>$$

and let $K = F(\Phi_1)(\Phi_2)\ldots(\Phi_m)$, the field of functions of a product of quadrics.

Proposition 9 *K is unirational over k, but if $S \neq S_{dec}$ then $H^4_{nr}(K, \mu_2^{\otimes 4}) \neq \{0\}$ and K is not stably rational.*

The proof is similar to those of the other cases.

Example We may take $n = 8$ and

$$S =< u_1 \wedge u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 \wedge u_8 > .$$

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