THÈSES D'ORSAY

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Convergence faible de la statistique linéaire de rang pour des variables aléatoires faiblement dépendantes et non stationnaires

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THESE

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par

Michel HAREL

Sujet : Convergence faible de la statistique linéaire de rang pour des variables aléatoires faiblement dépendantes et non sationnaires.

Soutenue le 30 Novembre 1989 devant le Jury composé de

M.M.	PURI	Madan	Président
	RAOULT	Jean Pierre	
	DACUNHA-CASTELLE	Didier	
	DEHEUVELS	Paul	
	DOLECKI	Szymon	
	HALLIN	Marc	
	MASSART	Pascal	
	RUYMGAART	Fritz	

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Abstract : There is three parts in this work. In the first part we study the weak convergence of the truncated and weighted empirical process for sequences of φ mixing or strong mixing and non stationary random variables. Then we extend this results to the weighted empirical process indexed by rectangles. After the definition of a new process called split process, we prove the weak convergence of this process weighted by a function. In the second part, we deduce the weak convergence of the multi dimensional linear rank statistic as well as the serial linear rank statistic always under mixing and non stationary conditions. Then, we establish the weak convergence of a two sample linear rank statistic. At last, in the third part, we prove the weak invariance of the U-statistic as well as a signed rank statistic under absolutely regularity and non stationary conditions.

Key words : weak convergence, weighed and truncated empirical process, rank process, split process, linear rank statistic, serial linear rank statistic, U-statistic, φ mixing, strong mixing, absolute regularity, Skorohod topology.

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TABLE DES MATIERES

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INTRODUCTION

Ce travail a pour objet essentiel d'obtenir la convergence faible de la statistique linéaire de rang dans des conditions trés générales (faible dépendance, scores non bornés et non stationnarité).

Il a été mené à bien sous la direction du Professeur Jean - Pierre Raoult.

Mon souci a été de rechercher quelles étaient les hypothèses les plus larges de non stationnarité qui permettaient malgré tout d'obtenir des théorèmes de convergence faible. En effet, les processus issus de la réalité physique sont rarement stationnaires. Or, la plupart des théorèmes de convergence faible ont été démontrés sous des hypothèses de stationnarité.

Après avoir rappelé les bases sur les théorèmes de convergence faible, nous énonçons les avancées déjà effectuées par d'autres auteurs tout en expliquant la portée de notre travail qui se présente sous forme de trois chapitres subdivisés en treize articles dont certains, écrits en collaboration avec le Professeur Madan Puri.

Les théorèmes de convergence faible

Soient $X_1, X_2, \ldots$ une suite de variables aléatoires (V.A) réelles indépendantes et identiquement distribuées de fonction de répartition continue F.

La fonction de répartition aléatoire

$$\tilde{F}_n(x) = n^{-1} \sum_{i=1}^{n} I_{[X_i \leq x]} \qquad -\infty < x < +\infty$$

($I_{[\,]}$ désigne la fonction indicatrice) qui affecte la probabilité n^{-1} à chaque valeur aléatoire X_i est appelée la fonction de répartition empirique de $X_1, \ldots, X_n$. La normalisation naturelle de $\tilde{F}_n$ conduit au processus empirique défini par

$$U_n(x) = n^{1/2} [\tilde{F}_n(x) - F(x)] \qquad -\infty < x < +\infty$$

On note F^{-1} la fonction inverse de F.

On part du résultat de convergence fonctionnelle suivant

$$W_n = U_n \circ F^{-1} \Rightarrow W_0 \text{ quand } n \rightarrow \infty$$

où W_0 désigne un pont Brownien et $\Rightarrow$ désigne la convergence faible pour la topologie de Skorohod sur [0, 1].

Si W_n est affecté d'une fonction correctrice, Shorack et Wellner (1982) ont montré que le processus empirique corrigé (par r) W_n/r vérifie

$$W_n/r \Rightarrow W_o/r \text{ si et seulement si}$$
$$r^2(t) / [t \log \log (1/t)] \to \infty \text{ quand } t \to 0$$

où r est une fonction continue, positive ou nulle, croissante sur [0, 1/2] et symétrique au point 1/2.

Si par contre, les V.A. ne sont plus supposées ni indépendantes, ni identiquement distribuées, notons

$$S_n = \sum_{i=1}^{n} X_i$$

Mc Leish (1975) a montré que si les V.A. sont faiblement stationnaires φ mélangeantes avec le taux $\varphi(n) = 0[1/n\ (\log n)^{2+\varepsilon}]$, $\varepsilon > 0$ sous la condition $E(\frac{S_n^2}{n}) \to \sigma^2$ (σ^2 constante positive) quand $n \to \infty$, la fonction aléatoire $W_n(t) = \frac{S_{[nt]}}{n^{1/2}\sigma}$ ([x] désigne la partie entière de x) appelée somme partielle empirique converge en loi (pour l'extension de Stone (1963) sur $[0, +\infty[$ de la topologie de Skorohod) vers un processus Brownien.

Par contre, Billingsley (1968) a montré qu'avec la condition plus forte $\sum_{n=1}^{+\infty} (\varphi(n))^{1/2} < +\infty$, alors la condition $E(\frac{S_n^2}{n}) \to \sigma^2$ est satisfaite.

Les résultats sont similaires lorsque les V.A. sont seulement fortement mélangeantes.

Reprenons le cas de V.A. indépendantes et identiquement distribuées. Du processus empirique, on a dérivé le processus de rang L_n défini par

$$L_n(t) = n^{-1/2} \sum_{i=1}^{n} (I_{[\tilde{F}_n(X_i) \le t]} - t) \qquad 0 \le t \le 1$$

On considére la statistique linéaire de rang $\mathcal{S}_n$ définie par

$$\mathcal{S}_n = \sum_{i=1}^{n} c_{ni}\, a_n(R_i)$$

où les c_{ni} sont des constantes connues appelées constantes de régression, $a_n(i)$ sont les scores connus et R_i est le rang de X_i parmi $X_1, \ldots, X_n$. Si on note J_n les fonctions de score définies par

$$J_n(t) = a_n(i) \qquad \text{si } \frac{i-1}{n} < t \leq \frac{i}{n} \text{ et } 1 \leq i \leq n$$

la statistique de rang $\mathcal{S}_n$ peut s'écrire

$$\mathcal{S}_n = n^{1/2} \int_0^1 h_n(t)\, dL_n(t) = -\, n^{1/2} \int_0^1 L_n(t)\, dh_n(t)$$

En utilisant la propriété de martingale de la suite $\{L_n(i/n+1) / (1 - i/_n) : 0 \leq i \leq n-1\}$, Shorack et Wellner (1986) en ont déduit que $n^{-1/2}\, \mathcal{S}_n$ converge en loi vers une loi Gaussienne.

Enfin, on définit une U-statistique.

$$U_n = \binom{n}{k}^{-1} \sum_{(i)}^{(n)} g(X_{i_1}, \dots, X_{i_k}) \qquad k \leq n$$

où la sommation $\sum_{(i)}^{(n)}$ désigne toutes les suites possibles $1 \leq i_1 < \dots < i_k \leq n$ et g est une fonction mesurable symétrique dans ses k coordonnées.

Soit $Z_n = \{Z_n(t), 0 \leq t \leq 1\}$

un processus aléatoire défini par :

$$Z_n(t) = \begin{cases} 0 & \text{si } 0 \leq t \leq (k-1)/n \\ (kn^{1/2})^{-1} i\, U_i & \text{si } t = i/n \qquad k \leq i \leq n \\ \text{linéairement interpolé si } t \in [i/n, (i+1)/n], & k-1 \leq i \leq n-1 \end{cases}$$

Miller et Sen (1972) ont montré la convergence faible du processus $Z_n(t)$ vers un processus de mouvement Brownien. Ce résultat est appelé principe d'invariance faible de la U-statistique.

Nous espérons que ce bref rappel permettra de mieux saisir nos objectifs.

Nos objectifs

Einmahl, Ruymgaart et Wellner (1984) ont montré, pour des V.A. indépendantes, la convergence faible du processus empirique corrigé multidimensionnel.

Une généralisation possible de ce processus est le processus empirique indexé par ensembles.

$$W_n(B) = n^{-1/2} \sum_{i=1}^{n} (I_{[X_i \in B]} - \mu_i(B)) \qquad B \in \mathsf{B}$$

où $\mathcal{B}$ est un ensemble de parties de $\mathbb{R}^k$ et μ_i est la mesure de probabilité associée à X_i.

Ils ont ensuite établi la convergence du processus empirique corrigé indexé par rectangles.

La convergence du processus empirique indexé par des ensembles très généraux a été étudiée avec correction pour des V.A. indépendantes par Alexander (1982) et sans correction pour des V.A. mélangeantes par Massart (1987).

Notre premier objectif (chapitre 1) a été de généraliser les résultats d'Einmahl, Ruymgaart et Wellner (1984) dans un cadre faiblement dépendant de type mélangeant non stationnaire.

Dans le chapitre 2, toujours dans un cadre mélangeant non stationnaire, on utilise la convergence du processus empirique corrigé pour montrer la convergence de la statistique linéaire de rang via la convergence du processus de rang corrigé.

La correction des processus est utilisée pour montrer la convergence des statistiques de rang à scores non bornés. En effet, selon une idée de Rüschendorf (1976), on peut exprimer la statistique de rang $\mathcal{S}_n$ sous la forme

$$\mathcal{S}_n = n^{1/2} \int_{[0,1]^k} L_n(t) . \frac{1}{r(t)} . r(t) \, d\lambda_n(t)$$

où L_n est le processus de rang, r la fonction corrective et λ_n une mesure donnée par les scores.

La difficulté rencontré pour des V.A. multidimensionnelles est que le processus empirique et le processus de rang s'annulent sur la frontière inférieure de $[0, 1]^k$ (une seule des coordonnées de t est égale à 0) mais non sur la frontière supérieure (une seule des coordonnées de t est égale à 1) et ceci nous réduit considérablement la classe des fonctions de score.

Pour résoudre cette difficulté, on a été amené à définir un nouveau processus appelé processus éclaté qui présente l'avantage de s'annuler aussi sur la frontière supérieure de $[0, 1]^k$.

Bien que son intérêt se justifie dans le cas multidimensionnel, définissons ici le processus empirique éclaté unidimensionnel pour illustrer cette nouvelle notion.

Il est donné par

$$\hat{W}_n(t) = \begin{cases} n^{-1/2} \sum_{i=1}^{n} (I_{[X_i \le F^{-1}(t)]} - t) & \text{si } t < 1/2 \\ n^{-1/2} \sum_{i=1}^{n} (I_{[X_i \ge F^{-1}(t)]} - (1 - t)) & \text{si } t \ge 1/2 \end{cases}$$

lorsque les V.A. sont identiquement distribuées.

La première partie de chapitre 2 étudie la convergence de la statistique linéaire de rang multidimensionnelle. Ruymgaart et Van Zuylen (1978) l'on démontrée pour des V.A.

indépendantes et non identiquement distribuées. Mais leurs techniques du type poissonisation ne sont plus utilisables dans notre contexte mélangeant.

La deuxième partie du chapitre 2 étudie la convergence de la statistique sérielle linéaire de rang. Cette statistique avait été étudiée par Hallin, Ingenbleek et Puri (1985), mais ils obtenaient la convergence uniquement lorsqu'il y avait contiguïté à l'indépendance, en utilisant un lemme classique de Lecam. Avec nos techniques, nous généralisons leurs résultats.

La troisième partie démontre la convergence faible d'une statistique linéaire de rang ayant une plus large classe de fonctions de score pour une suite de V.A. uni-dimensionnelle à deux échantillons. Pour cela, on étend un résultat de Denker et Rössler (1985) du cas stationnaire au cas non stationnaire en utilisant un théorème central limite de Withers (1975) pour des V.A. non stationnaires.

Le chapitre 3 établit l'invariance faible de la U-statistique pour des V.A. absolument régulières (mélange intermédiaire entre le φ mélange et le mélange fort) non stationnaires, puis on déduit l'invariance faible d'une statistique de rang signée utilisée dans la pratique pour tester la symétrie. Ces résultats avaient été établi par Yoshihara (1976, 1978) pour des V.A. absolument régulières mais stationnaires. Pour généraliser, on utilise le théorème central limite non stationnaire de Withers (1975).

Nous montrons à présent comment nous avons pu réaliser ces objectifs.

Présentation de nos résultats.

On considère une suite triangulaire $X_{ni} = (X_{ni}^{(1)}, \dots, X_{ni}^{(k)})$ $1 \leq i \leq n$, $n \geq 1$ de variables aléatoires à valeurs dans $\mathbb{R}^k$ avec pour fonctions de répartition F_{ni} et fonctions de répartition marginales $F_{ni}^{(j)}$ supposées continues pour tout $j \in \{1, \dots, k\}$.

Pour tout ce qui suit, nous travaillerons avec des suites de variables aléatoires qui seront soit φ mélangeantes, soit fortement mélangeantes, soit encore absolument régulières.

On rappelle que la suite non stationnaire $\{X_{ni}\}$ est dite φ mélangeante si

$$\sup_{n\geq 1} \max_{j\leq n-m} \left\{ \sup_{A,B} | P(A \mid B) - P(A) | \,;\, B \in \sigma(X_{ni}, 1 \leq i \leq j), A \in \sigma(X_{ni}, i \geq j+m) \right\} = \varphi(m) \downarrow 0$$

où $\sigma(X_{ni}, i \leq j)$ et $\sigma(X_{ni}, i \geq j + m)$ sont les tribus engendrées par $(X_{n1}, \dots, X_{nj})$ et $(X_{n,j+m}, X_{n,j+m+1}, \dots, X_{n,n})$ respectivement.

Elle est dite fortement mélangeante si

$$\sup_{n\geq 1} \max_{j\leq n-m} \left\{ \sup_{A,B} | P(A \cap B) - P(A)P(B) | \,;\, A \in \sigma(X_{ni}, 1 \leq i \leq j), B \in \sigma(X_{ni}, i \geq j+m) \right\}$$
$$= \alpha(m) \downarrow 0$$

et enfin absolument régulière si

$$\sup_{n\geq 1} \max_{j\leq n-m} E\left\{ \sup_{A\in\sigma(X_{ni},\, i\geq j+m)} | P(A \mid \sigma(X_{ni}, 1\leq i\leq j)) - P(A)| \right\} = \beta(m) \downarrow 0$$

On a toujours $\alpha(m) \leq \beta(m) \leq \varphi(m)$.

Pour une meilleure compréhension des résultats, certaines définitions et conditions techniques seront, dans ce qui suit, seulement évoquées et non données en détail.

Analyse du chapitre 1.

Soit H_{ni} la fonction de répartition définie par

$$H_{ni}(t_1, \ldots, t_k) = F_{ni}(F_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)^{-1}}(t_k))$$

pour tout $(t_1, \ldots, t_k) \in [0,1]^k$ où $F_n^{(j)} = n^{-1} \sum_{i=1}^{n} F_{ni}^{(j)}$

On considère maintenant le processus empirique tronqué W_n défini par

$$W_n(t_o, \mathbf{t}) = n^{-1/2} \sum_{i=1}^{[nt_o]} \left\{ \prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)}) \leq t_j]} - H_{ni}(\mathbf{t}) \right\}$$

pour tout $t_o \in [0, 1]$ et $\mathbf{t} = (t_1, \ldots, t_k) \in [0, 1]^k$ où $[n\, t_o]$ désigne la partie entière de $n\, t_o$.

Soit $f : [0, 1]^{k+1} \longrightarrow \mathbb{R}$. Pour $\rho \in \{0, 1\}^{k+1}$, on définit

$$f_\rho(t) = \lim_{\substack{s_i \uparrow t_i\ \rho(i) = 1 \\ s_i \downarrow t_i\ \rho(i) = 0}} f(s) \quad (s, t) \in [0, 1]^{k+1}$$

si la limite existe et on appelle $f_\rho(t)$ la ρ limite en t. On note D_{k+1} l'espace de toutes les applications $f : [0, 1]^{k+1} \longrightarrow \mathbb{R}$ telles que , pour tout $\rho \in \{0, 1\}^{k+1}$, f_ρ existe et $f_\rho = f$ pour $\rho = (0, \ldots, 0)$. L'espace D_{k+1} généralise l'espace des fonctions c.a.d. l.a.g. Parfois appelé espace des fonctions admettant une limite dans les 2^{k+1} quadrants et continues dans le premier quadrant, il fut introduit par Neuhaus (1971).

On dit que la suite $\{H_{ni}\}$ est μ-bornée s'il existe une mesure finie et positive μ sur $[0, 1]^k$ avec ses fonctions de répartition marginales continues telle que pour tout $n \geq 1$ et $1 \leq i \leq n$, $\mu_{ni}(B) \leq \mu(B)$ pour tout rectangle B dans $[0, 1]^k$ où μ_{ni} est la mesure sur $[0, 1]^k$ associée à H_{ni}.

On appelle fonction correctrice une fonction continue $r : [0, 1]^{k+1} \longrightarrow \mathbb{R}^+$

On a obtenu le résultat suivant

THÉOREME 1

On suppose que la suite $\{X_{ni}\}$ *est*

(a) φ *mélangeante avec* $\varphi(m) = 0(m^{-1-\varepsilon})$, $\quad \varepsilon > 0$

ou

(b) *fortement mélangeante avec* $\sum_{n=1}^{\infty} m^{2(k+1)} \alpha^{\varepsilon}(m) < \infty$, $\varepsilon \in]0, 1/(2k+4)[$

la suite $\{H_{ni}\}$ *est*

(c) μ *bornée où* μ *est absolument continue avec une densité bornée*

ou

(d) $\{H_{ni}\}$ *a des marginales uniformes pour tout* $n \geq 1$ *et* $1 \leq i \leq n$.

De plus, on suppose que

(e) *la fonction de covariance* $C_n(\cdot,\cdot)$ *du processus empirique* W_n *converge vers une fonction* $C(\cdot,\cdot)$. *Alors, pour toute fonction correctrice* r *telle que* $r(t) = 0$ *s'il existe au moins un* $j \in \{0, \ldots, k\}$ *tel que* $t_j = 0$ *ou si* $\mathbf{t} = (1, \ldots, 1)$ *et satisfaisant*

$$r(t) \geq A\left[\prod_{j=0}^{k} t_j \left(1 - \prod_{j=1}^{k} t_j\right)\right]^{1/2-\delta}, t \in [0, 1]^{k+1}, A > 0 \quad 0 < 1/2 - \delta < \frac{1}{2k+4}$$

W_n/r (*à valeurs p.s. dans* D_{k+1}) *converge faiblement pour la topologie de Skorohod vers un processus Gaussien* W_0/r *à trajectoires p.s. continues* (W_0 *est le processus limite de* W_n).

Les résultats d'Einmahl, Ruymgaart et Wellner (1984) étaient vérifiés pour une plus large classe de fonctions correctrices (satisfaisant les conditions de Shorack et Wellner (1982)) en raison de leur technique de démonstration différente, non utilisable pour nos conditions de mélange.

Soit maintenant le processus empirique indexé par rectangle $\tilde{W}_n$ défini par

$$\tilde{W}_n(B) = n^{-1/2} \sum_{i=1}^{n} \left(\prod_{j=1}^{k} I_{[a_j < F_n^{(j)}(X_{ni}^{(j)}) \leq b_j]} - \mu_{ni}(B) \right)$$

pour tout rectangle $\qquad B = \prod_{j=1}^{k}]a_j, b_j[\subset [0, 1]^k$

Si $\mathcal{F}(k)$ est l'ensemble des rectangles semi ouverts de $[0, 1]^k$, on généralise l'espace D_k en un espace noté $\tilde{D}_k$ (trop long à définir ici) qui est un sous-espace de l'ensemble des fonctions $f : \mathcal{F}(k) \longrightarrow \mathbb{R}$. Sur $\tilde{D}_k$, on est amené également à définir une topologie de Skorohod adaptée.

On appelle <u>processus empirique modifié</u> le processus $\hat{W}_n$ défini par

$$\hat{W}_n(B) = \begin{cases} \tilde{W}_n(B) & \text{si } |B| \geq n^{-1} \text{ et } |B| \leq 1 - n^{-1} \\ 0 & \text{sinon} \end{cases}$$

où $|B|$ est la mesure de Lebesgue de B.

Pour toute fonction correctrice $r : [0, 1] \to \mathbb{R}^+$, on introduit le processus empirique modifié corrigé défini par

$$(\hat{W}_n/r)(B) = \begin{cases} \hat{W}_n(B) / r(|B|) & \text{si } |B| \neq 0 \\ 0 & \text{sinon} \end{cases}$$

THÉOREME 2

Supposons que les conditions (a) *ou* (b) *et* (c) *ou* (d) *ainsi que* (e) *du théorème 1 soient satisfaites, alors pour toute fonction correctrice* r *s'annulant en* 0 *ou* 1 *et satisfaisant*

$$r(u) \geq A[u(1-u)]^{1/2-\delta} \quad A > 0,\ 0 < 1/2 - \delta < 1/2\, k(k+1)$$

$\hat{W}_n/r$ *converge faiblement pour la topologie de Skorohod vers un processus Gaussien.*

Les résultats d'Einmahl, Ruymgaart et Wellner (1984) furent établis dans le cas indépendant en recourant à la bien connue construction de Skorohod, mais uniquement pour des processus équivalents puisqu'ils n'ont pas envisagé les notions d'espaces $\tilde{D}_k$ et de topologie de Skorohod sur $\tilde{D}_k$.

Pour tout $\rho = (\rho(i))_{0 \leq i \leq k} \in \{0, 1\}^{k+1}$ donné, et tout $n \geq 1$ on définit une application $\psi_\rho^{(n)} : (\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ comme suit

$$\psi_\rho^{(n)}(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \text{ où}$$
$$x_i = (x_i^{(1)}, \ldots, x_i^{(k)}),\ y_i = (y_i^{(1)}, \ldots, y_i^{(k)}) \text{ et où}$$
$$y_i^{(j)} = \begin{cases} x_i^{(j)} & \text{si } \rho(0) = 0 \text{ et } \rho(j) = 0 \\ -x_i^{(j)} & \text{si } \rho(0) = 0 \text{ et } \rho(j) = 1 \\ x_{n+1-i}^{(j)} & \text{si } \rho(0) = 1 \text{ et } \rho(j) = 0 \\ -x_{n+1-i}^{(j)} & \text{si } \rho(0) = 1 \text{ et } \rho(j) = 1 \end{cases} \qquad 1 \leq j \leq k$$

On définit aussi

$$\Psi'_\rho : [0, 1]^{k+1} \longrightarrow [0, 1]^{k+1} \text{ comme suit}$$

$$\Psi'_\rho(t_0, t_1, \dots, t_k) = (t'_0, \dots, t'_k) \text{ où}$$

$$t'_j = \begin{cases} t_j & \text{si } \rho(j) = 0 \\ 1-t_j & \text{si } \rho(j) = 1 \end{cases} \qquad 0 \le j \le k$$

Pour tout $\rho \in \{0, 1\}^{k+1}$, on note W_n^ρ le processus $W_n(\Psi'_\rho(t))$ associé à la variable aléatoire $\Psi_\rho^{(n)}(X_{n1}, \dots, X_{nn})$.

Puis, si on note $I(\rho) = I_{\rho(o)} \times \dots \times I_{\rho(k)}$ où

$$I_\ell = \begin{cases} [0, 1/2[& \text{si } \ell = 0 \\ [1/2, 1] & \text{si } \ell = 1 \end{cases}$$

on définit enfin le processus empirique tronqué éclaté par

$$W_n^*(t) = W_n^\rho(t) \text{ si } t \in I(\rho)$$

Le processus W_n^* n'est plus défini sur l'espace D_{k+1} mais sur un espace que nous notons D_{k+1}^* où les fonctions ont leur continuité dans un quadrant dépendant de ρ.

Sur l'espace D_{k+1}^*, nous définissons également une topologie adaptée dite de Skorohod éclatée.

THÉOREME 3

Supposons que les conditions (a) *ou* (b) *et* (c) *ou* (d) *ainsi que* (e) *du Théorème 1 soient satisfaites, alors pour toute fonction correctrice* $r : [0, 1]^{k+1} \longrightarrow \mathbb{R}$ *s'annulant s'il existe au moins un* $j \in \{0, \dots, k\}$ *tel que* $t_j = 0$ *ou* $t_j = 1$ *et satisfaisant*

$$r(t) \ge A \left[\prod_{j=0}^{k} t_j \prod_{j=0}^{k} (1-t_j) \right]^{1/2 - \delta}, \quad A > 0, \ 0 < 1/2 - \delta < \frac{1}{2k+4}$$

$W_n^* \frac{1}{r}$ *(à valeurs p.s. dans* D_{k+1}^* *) converge faiblement pour la topologie de Skorohod vers un processus Gaussien.*

Cette notion de processus éclatés est nouvelle et permettra dans le chapitre 2 d'obtenir la convergence de la statistique linéaire de rang multidimensionnelle et de la statistique sérielle linéaire de rang avec des constantes de régression et des scores non bornés.

Analyse du chapitre 2.

Deux types de statistique sont envisagées

- la statistique linéaire de rang où les observations sont k-dimensionnelles que l'on écrit

$$\mathcal{S}_n = \sum_{i=1}^{n} c_{ni}\, a_n\, (R_{ni}^{(1)}, \ldots, R_{ni}^{(k)})$$

où k désigne une dimension, $R_{ni}^{(j)}$ est donc le rang de la coordonnée $X_{ni}^{(j)}$ parmi $(X_{n1}^{(j)}, \ldots, X_{nn}^{(j)})$, c_{ni} est la constante de régression et a_n le score.

- la statistique sérielle linéaire de rang où les observations sont 1-dimensionnelles que l'on écrit

$$\mathcal{S}_n^{*} = \sum_{i=1}^{n} c_{ni}\, a_n\, (R_{n,i-k+1}, \ldots, R_{ni})$$

où k est une longueur de regroupement, R_{ni} est donc le rang de X_{ni} parmi $(X_{n1}, \ldots, X_{nn})$.

Dans toute la suite (pour des raisons de lisibilité des résultats), a_n sera supposée définie à partir d'une fonction de score J avec telle propriété (sauf mention explicite).

$J = J_d + J_c$ où J_d est une fonction en escalier ayant seulement un nombre fini de points de discontinuité et J_c admet des dérivées partielles continues $\dfrac{\partial^I J_c}{(\partial t_j)_{j\in I}}$ sur $]0, 1[^{k+1}$ pour tout $I \subset \{0, \ldots, k\}$.

On considére le processus de rang tronqué L_n

$$L_n(t_o, \mathbf{t}) = n^{-1/2} \sum_{i=1}^{[n\, t_o]} \left\{ \prod_{j=1}^{k} I_{[\tilde{F}_n^{(j)} (X_{ni}^{(j)}) \le t_j]} - H_{ni}(\mathbf{t}) \right\}$$

pour tout $(t_o, \mathbf{t}) \in [0, 1]^{k+1}$ où $\tilde{F}_n^{(j)}$ est la fonction de répartition empirique de la suite $\{X_{ni}^{(j)}\}_{1\le i\le n}$

On définit également un processus de rang tronqué éclaté $\hat{L}_n$.

Après avoir rajouté une condition technique appelée condition de différentiabilité, nous avons pu démontrer la convergence faible du processus $\hat{L}_n$ muni d'une fonction correctrice vers un processus Gaussien pour la topologie uniforme afin d'en déduire le résultat suivant.

THÉOREME 4.

Supposons que les conditions (a) *ou* (b) *et* (c) *ou* (d) *ainsi que* (e) *du théorème 1 soient satisfaites, avec en plus la condition de différentiabilité et qu'il existe pour tout* $j \in \{1, \ldots k\}$ *une fonction de répartition* $F^{(j)}$ *telle que*

$$\sup_{x \in \mathbb{R}} | F_n^{(j)}(x) - F^{(j)}(x) | = 0(n^{-\alpha}), \ \alpha > 0$$

alors si la fonction de score J *vérifie*

$$\left| \frac{\partial^I J_c(t)}{(\partial t_j)_{j \in I}} \right| \leq A \left[\prod_{j \in I} t_j(1 - t_j) \right]^{-3/2+\delta}$$
$$A > 0, \ 0 < 1/2 - \delta < \frac{1}{2k+4}$$

$n^{-1/2} \mathcal{S}_n$ *converge en loi vers une distribution normale de variance finie.*

Etudions à présent la convergence de la statistique $\mathcal{S}_n^*$, on obtient deux théorèmes selon que la fonction de score J est bornée ou non.

On affaiblit alors la condition de non stationnarité en supposant $F_{ni} = F_n$ pour tout $i \in \{1, \ldots, n\}$. On note f_n la densité de probabilité de F_n et g_n la densité de probabilité de la variable aléatoire $(X_{n1}, \ldots, X_{nk})$.

La condition de différentiabilité est remplacée par une condition un peu plus forte que nous appellerons ici <u>condition de pseudo-différentiabilité.</u>

THÉOREME 5.

Supposons que les conditions (a) *ou* (b) *et* (d) *et* (e) *du théorème 1 soient satisfaites avec la condition de pseudo-différentiabilité, alors si la fonction de score* J *est bornée,* $n^{-1/2} \mathcal{S}_n^*$ *converge en loi vers un distribution normale de variance finie.*

Nous avons alors trouvé des applications aux processus de Markov et processus ARMA.

THÉOREME 6.

Supposons que les conditions (a) *ou* (b) *et* (d) *et* (e) *du théorème 1 soient satisfaites ainsi que la condition de pseudo-différentiabilité. On suppose de plus qu'il existe une fonction de répartition* F *telle que*

$$\sup_{x \in \mathbb{R}} | F_n(x) - F(x) | = 0\,(n^{-\alpha}) \quad \alpha > 0$$

et que

$$g_n(x_1, \ldots, x_k) \leq A \left[\prod_{j=1}^{k} F_n(x_j)\,(1 - F_n(x_j)) \right]^{-1/2+\delta} \prod_{j=1}^{k} f_n(x_j)$$
$$A > 0, \ \ 0 < 1/2 - \delta < \frac{1}{2\,k+4}$$

alors si la fonction de score J *vérifie*

$$\left| \frac{\partial^I J_c(t)}{(\partial t_j)_{j\in I}} \right| \leq A \left[\prod_{j\in I} t_j(1 - t_j) \right]^{-3/2 + \delta'} \qquad \delta' > \delta$$

$n^{-1/2}\, \mathcal{S}_n^*$ *converge en loi vers une distribution normale de variance finie.*

Enfin, nous envisageons le cas où la suite $\{X_{ni}\}$ est 1-dimensionnelle et la réunion de deux suites indépendantes de variables aléatoires faiblement dépendantes, c'est-à-dire

$$X_{ni} = \begin{cases} Y_{n_1 i} & 1 \leq i \leq n_1 \\ Z_{n_2 j} & j = i - n_1 \qquad n_1 + 1 \leq i \leq n \end{cases}$$

où $n = n_1 + n_2$ et $\lim_{n\to\infty} n_1 n^{-1} = \lambda_o \in \,]0, 1[$

les deux suites $\{Y_{n_1 i}\}_{1\leq i\leq n_1}$ et $\{Z_{n_2 j}\}_{1\leq j\leq n_2}$ sont supposées indépendantes.

Nous montrons alors le comportement asymptotique de la statistique $\mathcal{S}_n$ lorsque la fonction de score J s'écrit $J = J_o \times J_1$ où J_o est défini à partir des constantes de régression.

Pour tout $\delta \geq 0$ on pose $\eta = \delta(4 + \delta)^{-1}$ et on note μ_δ la mesure sur [0, 1] donnée par la densité $t(1-t)^{--1/2-\eta}$ par rapport à la mesure de Lebesgue. On note $\mathfrak{G}_\delta$ l'espace des fonctions J_1 à variations bornées, μ_δ intégrables et pour lesquelles les mesures η définies par $J_1 = \int d\eta$ sont absolument continues par rapport à la mesure de Lebesgue.

Si on note $F^{(1)}_{n_1,i,\ell}$ et $F^{(2)}_{n_2,j,k}$ les fonctions de répartition respectives des variables aléatoires $(Y_{n_1 i}, Y_{n_1 \ell})$ et $(Z_{n_2 j}, Z_{n_2 k})$, on obtient

THÉOREME 7.

Supposons que les suites $(Y_{n_1 i})$ et $(Z_{n_2 j})$ *soient* (a') φ *mélangeantes avec* $\sum_{m=1}^{+\infty} m(\varphi(m))^{(2+3\delta)/(4+2\delta)} < +\infty$ *pour* $2 > \delta \geq 0$, *ou* (b') *fortement mélangeantes avec* $\sum_{m=1}^{+\infty} m^2 (\alpha(m))^{\delta/(2+\delta)} < +\infty$ *pour* $\delta > 0$, que (c') *la fonction* J_o *admet une dérivée à variations bornées et* J_1 *appartient à* $\mathfrak{G}_\delta$ *et que de plus* (d') *pour tout* $m \in \mathbb{N}^*$ *et tout* $\ell \in \{1, 2\}$ *il existe une fonction de répartition continue* $G^{(\ell)}_m$ *sur* $\mathbb{R}^2$ *de marges* $F^{(\ell)}$ *telle que*

$$\lim_{n\to\infty} \max_{1\leq i<j\leq n} \left| F^{(\ell)}_{n_\ell,i,j} (F_n^{-1}(t_1), F_n^{-1}(t_2)) - G^{(\ell)}_{j-i}(H^{-1}(t_1), H^{-1}(t_2)) \right| = 0$$

pour tout $(t_1, t_2) \in [0, 1]^2$ *où*

$$F_n = n^{-1} \sum_{i=1}^{n} F_{ni} \quad \text{et} \quad H = \lambda_o F^{(1)} + (1 - \lambda_o) F^{(2)}$$

alors $n^{-1/2} \mathcal{S}_n$ *converge en loi vers une distribution normale de variance finie.*

Analyse du chapitre III.

La U-statistique est définie par

$$U(F_n) = \binom{n}{p}^{-1} \sum_{(i)}^{(n)} g(X_{ni_1}, \ldots, X_{ni_p}) \quad n \geq p \geq 1$$

où la sommation $\sum_{(i)}^{(n)}$ recouvre toutes les inégalités $1 \leq i_1 < \ldots < i_p \leq n$ et $g : \mathbb{R}^p \longrightarrow \mathbb{R}$ est une fonction Borel mesurable qui est symétrique dans ses p coordonnées. Si on note $F_{n,i,j}$ la fonction de répartition de (X_{ni}, X_{nj}), on obtient alors le résultat suivant :

THÉOREME 8.

On suppose que la suite $\{X_{ni}\}$ *est absolument régulière avec* $\beta(m) = 0(m^{-(2+\delta)/\delta})$, *que la fonction* g *satisfait certaines conditions techniques d'intégrabilité (non données ici) et appartient à l'espace* D_p *et que de plus pour tout* $m \in \mathbb{N}^*$, *il existe une fonction de répartition continue* G_m sur $\mathbb{R}^2$ *de marges continues* F *telle que*

$$\lim_{n \to \infty} \max_{1 \leq i < j \leq n} | F_{n,i,j}(x_1, x_2) - G_{j-i}(x_1, x_2)| = 0$$

pour tout $(x_1, x_2) \in \mathbb{R}^2$,
alors $n^{1/2} U(F_n)$ *converge en loi vers une distribution normale de variance finie.*

Des applications aux processus de Markov et processus ARMA sont données.

Remarque : Les U-statistiques ne convergent pratiquement jamais en condition de mélange fort.

Pour terminer, nous définissons la notion de U-statistique généralisée.

Soit un entier $c \geq 2$ et pour tout j $(1 \leq j \leq c)$ un entier $m_j \geq 1$; soit g une application de $\prod_{j=1}^{c} (\mathbb{R}^k)^{m_j}$ dans $\mathbb{R}$, symétrique pour tout j (c'est à dire que $g(x_1, \ldots, x_c)$ est invariant pour toute permutation sur les m_j indices de x_j).

On considère c tableaux triangulaires indépendants de variables aléatoires à valeurs dans $\mathbb{R}^k$, notées $X_{j,n,i}$ $(1 \leq j \leq c,\ 1 \leq n,\ 1 \leq i \leq n)$ à fonctions de répartition continues $F_{j,n,i}$.

Pour tout $\mathbf{n}(= (n_1, \ldots, n_c))$, tel que, pour tout j, $n_j \geq m_j$ soit $S_\mathbf{n} = \prod_{j=1}^{c} S_{j,n_j}$, où S_{j,n_j} est l'ensemble de toutes les suites strictement croissantes à m_j éléments dans $\{1, \ldots, n_j\}$ $(1 \leq i_{j,1} < i_{j,2} < \ldots < i_{j,m_j} \leq n_j)$. La U-statistique généralisée de degré $\mathbf{m} = (m_1, \ldots m_c)$ est définie par

$$U(\mathbf{n}) = \left[\prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \right] \sum_{S_\mathbf{n}} g(\mathbf{X}_{j,n_j,i_{j,\ell}},\ 1 \leq \ell \leq m_j,\ 1 \leq j \leq c)$$

On suppose que, pour tout j, le tableau triangulaire des variables aléatoires $\mathbf{X}_{j,n,i}$ $(1 \leq n, 1 \leq i \leq n)$ est absolument régulier de taux noté β_j.

On étudie le comportement asymptotique de $U(\mathbf{n})$ quand les n_j tendent vers l'infini de telle sorte que, pour tout j, $n_j / n_1 + \ldots + n_c$ tende vers $\lambda_j (0 < \lambda_j < 1)$; les λ_j seront fixés dans toute la suite et on note $\lim_{\mathbf{n}}$ ce type de convergence.

Pour tout $\mathbf{t} = (t_1, \ldots, t_c) \in [0, 1]^c$ et $\mathbf{n} = (n_1, \ldots, n_c)$ on note $[\mathbf{n}\ \mathbf{t}] = ([n_1\ t_1], \ldots, [n_c\ t_c])$.

Soit $Z(\mathbf{n}) = \{Z(\mathbf{t}, \mathbf{n}), \mathbf{t} \in [0, 1]^c$ le processus défini par

$$Z(\mathbf{t}\ ;\mathbf{n}) = \begin{cases} U([\mathbf{n}\ \mathbf{t}]) - \theta(F_\mathbf{n}) & \text{pour tout } [\mathbf{n}\ \mathbf{t}] \geq \mathbf{m} \\ 0 & \text{autrement} \end{cases}$$

où $\theta(F_\mathbf{n})$ est un coefficient de centrage

($\mathbf{a} \leq \mathbf{b}$ signifie $a_j \leq b_j$ pour tout $j = 1, \ldots, c$)

Si on note $G_{j,n,i,\ell}$ la fonction de répartition de la variable aléatoire $(\mathbf{X}_{j,n,i}, \mathbf{X}_{j,n,\ell})$, on obtient

THÉOREME 9.

On suppose que les variables aléatoires $\{\mathbf{X}_{j,n,i}\}$ *vérifient*

$$\max_{1 \leq j \leq c} \beta_j(m) = 0(m^{-6-\delta})$$

que la fonction g *appartient à* D_{km_0} *où* $m_0 = \sum_{j=1}^{c} m_j$ *et satisfait certaines conditions d'intégrabilité (non données ici) et que de plus il existe une famille de fonctions de répartition sur* $\mathbb{R}^{2k}$, $G_{j,\ell}$ *de marges* $F_j (1 \leq j \leq c, \ell > 1)$ *telle que pour tout* $(x_1, x_2) \in \mathbb{R}^{2k}$

$$\lim_{\mathbf{n}} \max_{1 \leq j \leq c} \max_{1 \leq i < \ell \leq n_j} | G_{j,n_j,i,\ell}(\mathbf{x}_1, \mathbf{x}_2) - G_{j,\ell-i}(\mathbf{x}_1, \mathbf{x}_2)| = 0$$

alors $Z(\mathbf{t}\ ;\mathbf{n})$ *converge en loi pour la topologie de Skorohod sur* D_c *vers le processus Gaussien* $W = \{W(\mathbf{t}), \mathbf{t} \in [0, 1]^c\}$ *où*

$$W(t) = \begin{cases} \sum_{j=1}^{c} \gamma_j W_j(t_j) & t \geq 0 \\ 0 \textit{ avec la probabilité } 1 \textit{ si } t_j = 0 \textit{ pour au moins un } j, 1 \leq j \leq c \end{cases}$$

où les γ_j *sont des constantes et les* W_j *sont* c *copies indépendantes d'un mouvement brownien sur* [0, 1].

En utilisant des techniques similaires à la démonstration des théorèmes 8 et 9, on montre également un théorème d'invariance faible pour une certaine statistique de rang où on prend les rangs des valeurs absolues des variables aléatoires. Cette statistique est utilisée pour tester la symétrie.

Actuellement, nous nous attaquons toujours sous des hypothèses de non stationnarité à d'autres types de problèmes de convergence, en particulier à la loi du logarithme itéré. Nous espérons ici avoir fourni un outil plus adéquat de travail aux statisticiens appliqués qui utilisent les théorèmes de convergence faible.

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Chapitre 1

Convergence faible du processus empirique en condition de mélange.

CONVERGENCE EN LOI POUR LA TOPOLOGIE DE SKOROHOD DU PROCESSUS EMPIRIQUE MULTIDIMENSIONNEL NORMALISE TRONQUE ET SEMI-CORRIGE (ETUDE AU VOISINAGE DE LA FRONTIERE INFERIEURE DE $[0,1]^{1+k}$).

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Travail mené en liaison avec J.P. RAOULT, S. BALACHEFF et G. DUPONT (Laboratoire de Mathématiques, Université de ROUEN).

I PRESENTATION DE L'ETUDE.

I-a) Rappels.

En [6], dans ce même fascicule, S. BALACHEFF et G. DUPONT établissent des conditions de convergence pour le processus de rang normalisé tronqué. Cette étude repose sur la considération préalable du processus empirique normalisé tronqué; voir aussi [5]pour plus de détails.

Ce processus, noté ici W_n, est défini de la manière suivante (nous utilisons dans ce texte, sauf avis contraire les mêmes notations que S. BALACHEFF et G. DUPONT dans [6]).

Pour tout t ($\in [0,1]^{1+k}$), on convient de noter :

$t = (t_0,\dots t_k)$ et $\tilde{t} = (t_1,\dots, t_k)$ ($\in [0,1]^k$)

(alors que dans [6], il est noté en général :

$t = (s, t_1,\dots, t_k)$)

On note en particulier $\tilde{1} = (1,\dots, 1)$ ($\in [0,1]^k$)

Etant observé x $(= (x^1,\dots, x^n))$, suite de n observations dans $\mathbb{R}^k$, on note :

$$[W_n(t)]\ (x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt_0]} I\,[\bar{F}_n\ (x^j) \leq \tilde{t}] - H_n^j\ (\tilde{t})$$

(les notations [ ], I [ ], $\leq$ (ordre dans $\mathbb{R}^k$), $\bar{F}_n$, H_n^j sont introduites dans [6], III-a, III-b et III-c);

Rappelons en particulier que :

$$I\,[\bar{F}_n(x^j) \leq \tilde{t}\,] = \prod_{i=1}^{k} [\bar{F}_{n,i}\ (x_i^j) \leq t_i]$$

$$\text{et } H_n^j\ (t_1,\dots, t_k) = F_n^j\ (\bar{F}_{n,1}^{-1}(t_1),\dots,\bar{F}_{n,k}^{-1}(t_k))$$

(H_n^j est la fonction de répartition de la probabilité notée μ_n^j).

Il résulte alors de [5] (Chap. 7, Th. 1) (et, à quelques détails de normalisation près [6], Th. 5) que la suite des processus W_n converge en loi, pour la topologie de SKOROHOD sur $[0,1]^{1+k}$ (voir [13] et [6]), vers un processus gaussien à trajectoires p.s. continues dès que sont satisfaites les hypothèses suivantes :

H_1 - Les marges $P_{n,i}$ ($1 \leq i \leq k$) de la probabilité P_n régissant l'observation x dans $(\mathbb{R}^k)^n$ sont supposées diffuses sur $\mathbb{R}^n$.

H_2 - La suite $(C_n \; ; \; n \in \mathbb{N}^*)$ des fonctions de covariance des processus W_n converge simplement vers une fonction limite c.

H_3 - Il existe une application décroissante $\phi : \mathbb{N} \to [0,1]$ vérifiant $\phi(0) = 1$, $\sum_{n \in \mathbb{N}} n \; \phi^{1/2}(n) < \infty$, et pour laquelle la suite $(P_n, n \in \mathbb{N}^*)$ est ϕ- mélangeante.

H_4 - Il existe une mesure μ sur $[0,1]^k$, finie, positive, à marges diffuses et vérifiant :

$(\forall n \in \mathbb{N}^*)$ $(\forall j \in \{1, 2, \ldots, n\})$ $(\forall B$ bloc de $[0,1]^k)$

$\mu_n^j \; (B) \leqslant \mu(B)$

(où comme en [5], si T est une partie de $[0,1]^k$ de la forme $\prod_{i=1}^{k} T_i$, on appelle bloc de T, toute partie de T de la forme $\prod_{i=1}^{k}] \, \underline{t_i}, \overline{t_i} \,] \cap T$ où, pour tout i, $\underline{t_i}$ et $\overline{t_i}$ sont deux éléments de T_i vérifiant $\underline{t_i} < \overline{t_i}$.

I-b) Cadre de cet article.

L'usage proposé par RÜSCHENDORF [2], des processus de rang pour la convergence des statistiques de rang multidimensionnelles repose sur l'étude (qui va faire l'objet du présent article) de la convergence en loi, pour la topologie de SKOROHOD de la suite des processus du type $W_n \cdot \frac{1}{r}$, où r est une application continue de $[0,1]^{1+k}$ dans $\mathbb{R}_+$ et où, par convention, on note :

$$\frac{1}{r} \; (t) = \frac{1}{r(t)} \quad \text{si } r \; (t) \neq 0$$

$$\frac{1}{r} \; (t) = 0 \qquad \text{si } r \; (t) = 0$$

L'étude de tels processus est effectuée par FEARS et MEHRA ([3]) (dans le cas unidimensionnel), par MEHRA et RAO ([8]) (dans le cas multidimensionnel, mais pour un processus stationnaire non tronqué); ils la présentent comme l'étude de la convergence de la suite des processus W_n pour la topologie déduite de la topologie de

SKOROHOD en adoptant, pour distance entre deux éléments f et g de D_k, d $(\frac{f}{r}, \frac{g}{r})$ où d est la distance de SKOROHOD (les auteurs notent $d_r(f.g) = d\ (\frac{f}{r}, \frac{g}{r})$ et parlent de "d_r metrics" ; remarquons que cette nouvelle topologie n'est, en toute rigueur, définie que sur l'ensemble des éléments f de D_k tels que $\frac{f}{r}$ soit lui-même un élément de D_k ; nous traiterons cette difficulté en posant ci-dessous une hypothèse assurant que, presque sûrement, la trajectoire des processus corrigés $W_n.\frac{1}{r}$ appartient à D_k.

Comme le remarque RÜSCHENDORF, les fonctions r doivent, pour permettre d'obtenir les résultats de convergence des statistiques de rang qui constituent son but, s'annuler en tout point de $[0,1]^{1+k}$ en lequel le processus W_n prend presque sûrement la valeur 0, c'est-à-dire en tout point t vérifiant l'une des conditions suivantes :

(i) = $t_o = 0$

(ii) Il existe i ($1 \leq i \leq k$) tel que $t_i = 0$

(on dit alors que $\tilde{t}$ appartient à la frontière inférieure de $[0,1]^k$, notée $\overleftarrow{[0,1]}^k$.

(iii) $\tilde{t} = \tilde{1}$

Nous nous limitons en fait au cas où r est une fonction correctrice (et même, dans une première étape, semi-correctrice) aux sens donnés par la définition suivante :

<u>Définition 1.</u>

On appelle fonction correctrice (respectivement semi-correctrice) toute application r de $[0,1]^{1+k}$ dans $\mathbb{R}_+$, vérifiant :

$$(\forall t)\ r(t) = r_o\ (t_o)\ \tilde{r}\ (\tilde{t})$$

où

(i) r_o est une application continue de [0,1] dans $\mathbb{R}_+$, s'annulant en 0 et en 0 seulement.

(ii) $\tilde{r}$ est une application continue de $[0,1]^k$ dans $\mathbb{R}_+$ dont l'ensemble des points d'annulation est $\overleftarrow{[0,1]}^k \cup \{\tilde{1}\}$ (resp. $\overleftarrow{[0,1]}^k$).

Si r est une fonction correctrice (resp. semi-correctrice) on dit que le processus $\frac{W_n}{r}$ est le r-corrigé (resp. r-semi-corrigé) de W_n (ou corrigé (resp. semi-corrigé) si il n'y a pas de risque de confusion sur r).

Sont en particulier correctrices les fonctions dites de WELLNER (par référence à [14]) de la forme :

$$r(t) = [\, t_0 \prod_{i=1}^{k} t_i \,(1 - \prod_{i=1}^{k} t_i)\,]^{1/2-\varepsilon}$$

(où $0 < \varepsilon < \frac{1}{2}$).

I-c) Objet de cet article.

En [2], RÜSCHENDORF présente un théorème (Th. 2.2) dans l'énoncé duquel la convergence (en loi pour la topologie de SKOROHOD) d'un processus (non nécessairement empirique) corrigé est déduite uniquement de la convergence du processus lui-même ("avant correction") et d'une condition sur la probabilité limite de ce processus.

Ce théorème n'est pas exact (pour un contre exemple, voir [12]). En fait, nous pensons même qu'il est vain d'espérer obtenir un théorème "général" de ce type.

Le but de notre travail est donc de donner des conditions sur la fonction correctrice (ou semi-correctrice) r, conditions liées intimement aux éléments intervenant dans la définition du processus empirique non tronqué, et assurant la convergence du processus empirique normalisé tronqué corrigé.

La première étape de cette étude consiste à établir la proposition suivante (qui est, pour l'essentiel, le lemme 3.2 de [8] et dont nous ne redonnerons pas ici la démonstration).

Proposition.

Soit W_n ($n \in \mathbb{N}^*$), une suite de processus, à valeurs dans D_{k+1} et convergeant en loi vers W_0, processus gaussien à trajectoires presque sûrement continues.

Soit r une application continue de $[0,1]^{k+1}$ dans $\mathbb{R}_+$.

On pose, pour tout $\theta > 0$

$$R_\theta = \{v \in \mathbb{R}^{1+k} \; ; \; (\exists\, w \in \mathbb{R}^{1+k})\; r(w) = 0 \text{ et } \sup_{0 \leq i \leq k} |v_i - w_i| \leq \theta\}$$

On suppose vérifiées les deux conditions suivantes :

(A) Pour tout n, le processus $W_n \cdot \frac{1}{r}$ est à trajectoires presque sûrement dans D_k.

(B) $(\forall\, \delta > 0)$ $(\forall\, \varepsilon > 0)$ $(\exists\, \theta > 0)$ $(\exists\, N_0)$ $(\forall\, n \geq N_0)$

$$P_n \, [\sup_{v \in R_\theta} |(W_n \cdot \frac{1}{r}) \; (v)| > \delta] < \varepsilon$$

Alors la suite des processus $W_n \cdot \frac{1}{r}$ converge aussi, en loi vers le processus corrigé $W_o \cdot \frac{1}{r}$ (qui est évidemment lui-même gaussien et à trajectoires p.s. continues).

Dans le cas particulier, où W_n est le processus empirique normalisé tronqué, la condition (A) est évidemment assurée par l'hypothèse suivante :

H_5 - r est une fonction correctrice (resp. semi-correctrice) telle que, pour tout n et tout j ($1 \leq j \leq n$), soient satisfaites les conditions (1) et (2) (resp. soit satisfaite la condition (1)) ci-dessous :

(1) En tout $\tilde{t}$ appartenant à la frontière inférieure, on a :

$$\lim_{\tilde{u} \to \tilde{t}} (H_n^j \frac{1}{\tilde{r}}) \; (\tilde{u}) = 0$$

(2) $$\lim_{\tilde{u} \to \tilde{1}} [(1 - H_n^j) \frac{1}{\tilde{r}}] \; (\tilde{u}) = 0$$

Notre tâche essentielle consiste donc à assurer la réalisation de la condition (B) de la proposition.

On remarque que, par définition même des fonctions correctrices (resp. semi-correctrices), on a, pour tout $\theta < \frac{1}{2}$,

$$R_\theta = \bigcup_{i=o}^{k} C_{\theta,i} \cup C'_\theta \quad (\text{resp. } R_\theta = \bigcup_{i=o}^{k} C_{\theta,i})$$

où, pour tout i ($0 \leq i \leq k$)

$$C_{\theta,i} = \{t; \; o \leq t_i \leq \theta\}$$

et $$C'_\theta = \{t \; ; \; (\forall i \in \{1,\ldots, k\}) \; : \; 1 - \theta \leq t_i \leq 1\}$$

Nous nous limiterons désormais aux fonctions correctrices et semi-correctrices régulières au sens suivant :

<u>Définition 2.</u>

La fonction correctrice r (resp. semi-correctrice) est dite régulière si et seulement si sont satisfaites les conditions (1) (2) et (3) (resp. (1) et (2)) ci-dessous.

(1) - Il existe θ_o (> 0) tel que r_o est strictement croissante sur $[0, \theta_o]$

(2) - Il existe θ' (> 0) tel que $\tilde{r}$ est strictement croissante sur $\bigcup_{i=1}^{k} C_{\theta',i}$

(3) - Il existe θ'' (> 0) tel que $\tilde{r}$ est strictement décroissante sur $C'_{\theta''}$

L'étude des processus régulièrement corrigés conduit à décomposer R_θ en l'union des parties des types suivants :

(1) $C_{\theta,i}$ $(1 \leq i \leq k)$

(2) $C_{\theta,o} \cap \bigcup_{i=1}^{k} C_{\theta',i}$

(3) $C_{\theta,o} \cap [[0,1]^{1+k} - \bigcup_{i=1}^{k} C_{\theta',i} - C'_{\theta''}]$

(4) $C_{\theta,\delta} \cap C'_{\theta''}$

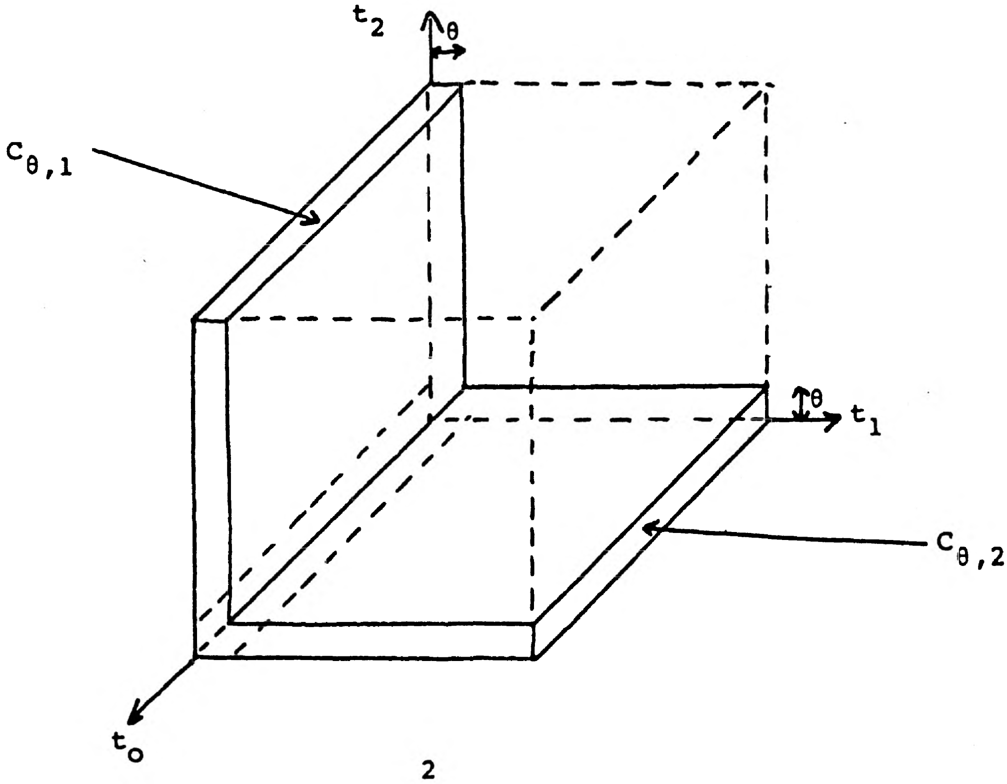
(5) $C'_{\theta''}$

et à démontrer que, pour chacune (soit R) de ces parties, on a :

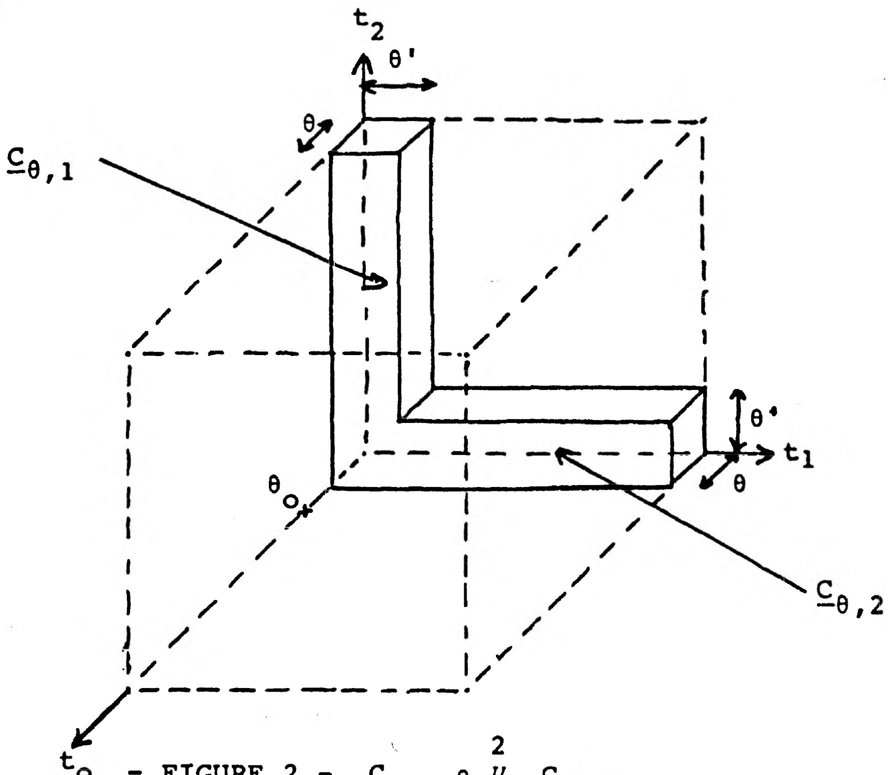
(B_R) $(\forall\delta > 0)$ $(\forall\varepsilon > 0)$ $(\exists\, \theta > 0)$ $(\exists\, N_o)$ $(\forall n \geq N_o)$

$$P_n\, [\sup_{v \in R} |(W_n . \frac{1}{r})\ (v)| > \delta] < \varepsilon$$

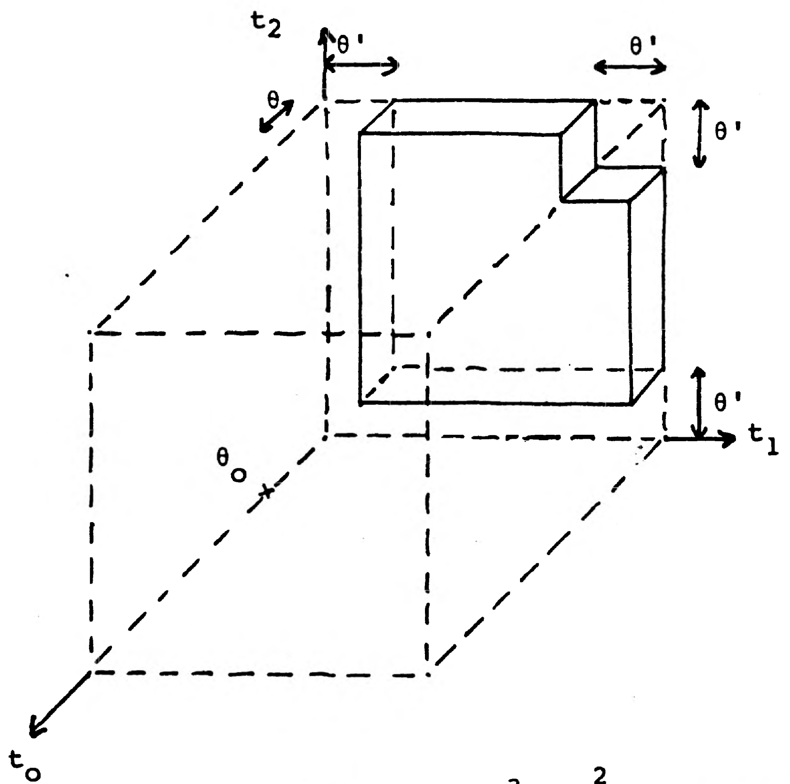
Pour k = 2, on a les schémas suivants :



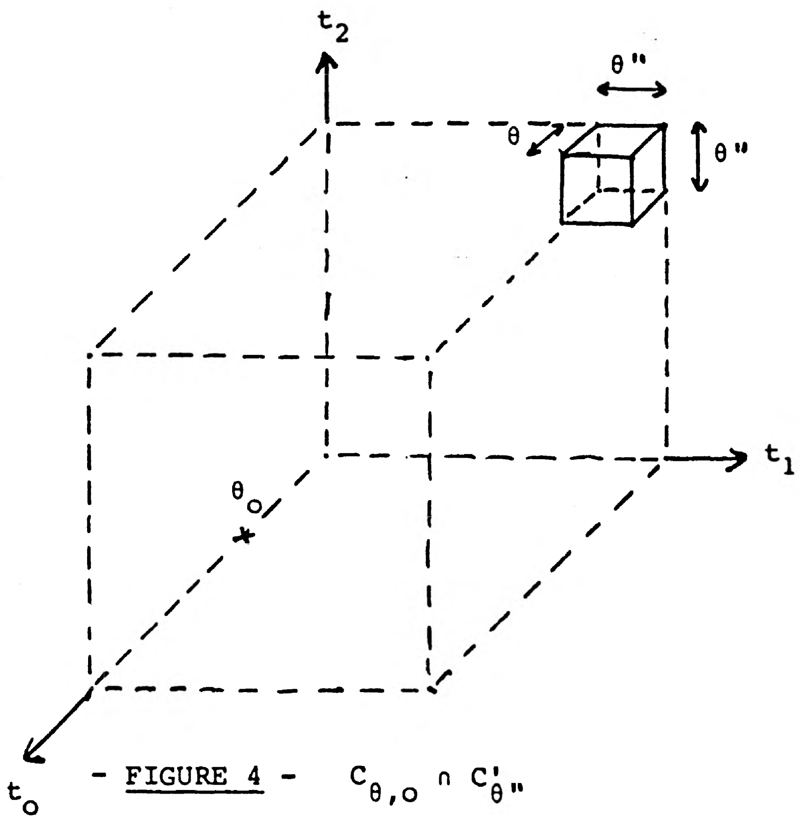
- <u>FIGURE 1</u> - $\bigcup_{i=1}^{2} C_{\theta,i}$



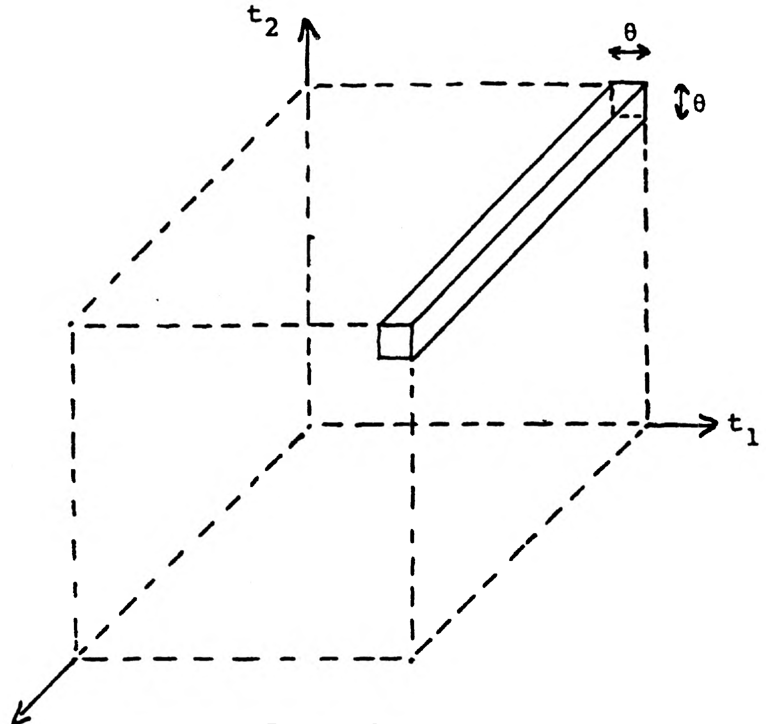
- <u>FIGURE 2</u> - $C_{\theta,o} \cap \bigcup_{i=1}^{2} C_{\theta',i}$



- **FIGURE 3** - $C_{\theta,o} \cap [[0,1]^3 - \bigcup_{i=1}^{2} C_{\theta',i} - C'_{\theta'}]$



- **FIGURE 4** - $C_{\theta,o} \cap C'_{\theta''}$



- FIGURE 5 - C'_θ

Pour les processus régulièrement semi-corrigés, il suffit de procéder ainsi en décomposant R_θ en l'union des parties suivantes:

(1) $C_{\theta,i}$ $(1 \leq i \leq k)$

(2) $C_{\theta,o} \cap \bigcup_{i=1}^{k} C_{\theta',i}$

(3') $C_{\theta,o} \cap \complement\left(\bigcup_{i=1}^{k} C_{\theta',i}\right)$

Nous allons nous limiter dans le présent article à l'étude des processus régulièrement semi-corrigés ; ceci implique évidemment que, pour un processus corrigé par une fonction régulière r, la condition B_R est satisfaite sur les parties (1), (2) et (3) pour lesquelles la condition B_R coïncide avec celle, relative respectivement à (1), (2) et (3'),pour tout fonction correctrice régulière r' coïncidant avec r sur $[0,1] \times C'_{\theta''}$ (il est clair qu'il en existe).

II - NOTATIONS ET LEMMES TECHNIQUES.

II-a) - Notations.

0. On notera $K = \{1,\dots, k\}$ et $K' = \{0, 1,\dots, k\}$

1 - Soit $(x_i \; ; \; 1 \leq i \leq k)$ une suite d'éléments d'un ensemble X, de longueur k.

1°) Soit $I \subset K$; on note $(I : x_i)$ la suite partielle $(x_i \; ; \; i \in I)$ (élément de $[0,1]^I$).

(en particulier $(K : x_i) = (x_i \; ; \; 1 \leq i \leq k)$

Soit $\complement I = K - I$; on note $\boxed{I : x_i}$ l'application de $X^{\complement I}$ dans X^K, qui à tout $(\complement I : y_i)$ associe l'élément $(z_i \; ; \; 1 \leq i \leq k)$ défini par :

si $i \in I$, $z_i = x_i$

si $i \notin I$, $z_i = y_i$

Si f est une application d'ensemble de définition X, $f \circ \boxed{I : x_i}$ désigne alors bien sûr l'application déduite de f par "fixation aux valeurs x_i des coordonnées d'indices appartenant à I"

2°) Soit $(y_i \; ; \; 1 \leq i \leq k)$ une autre suite, de longueur k et soient I et J deux parties disjointes de K; alors

$$(I : x_i \; ; \; J : y_i) = (I \cup J : z_i)$$

où si $i \in I$, $z_i = x_i$

si $i \notin J$, $z_i = y_i$

Ce type de notation peut bien sûr s'étendre à plus de trois parties disjointes de $\{1, \ldots, k\}$

(<u>Exemple</u> : $(I_1 : x_i^1 \; ; \; I_2 : x_i^2 \; ; \; I_3 : x_i^3)$ où I_1, I_2, I_3 sont trois parties disjointes de K.

3°) Soit $X \subset \mathbb{R}$ et $I \subset J \subset K$ et soit f une application de X^J dans $\mathbb{R}$.

On note, si cela a un sens, $\partial_I f$ la dérivée de f selon les coordonnées appartenant à I.

(si $I = \{i_1, \ldots, i_\ell\}$ où $\ell \leq k$

$$\partial_I f = \frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} f) \; ;$$

On remarque que, si f est une application de X^k dans $\mathbb{R}$, on a :

$$(\partial_I f)\ (K : x_i) = ((\partial_I f) \circ \boxed{\complement I : x_i})\ \ (I : x_i)$$

autrement dit :

$$(\partial_I f) \circ \boxed{\complement I : x_i} = \partial_I\ (f \circ \boxed{\complement I : x_i})$$

2 - Soit $((a_i, b_i)\ ;\ 1 \leq i \leq k)$ une suite d'éléments de $[0,1]^2$, de longueur k et soient I et J deux parties disjointes de K; alors

$$[I : a_i, b_i\ ;\ J : c_i, d_i] = \prod_{i \in I \cup J} [x_i, y_i]$$

où
| si $i \in I$, $[x_i, y_i] = [a_i, b_i]$
| si $i \in J$, $[x_i, y_i] = [c_i, d_i]$

Ce type de notation peut bien sûr s' étendre à plus de deux parties disjointes de $\{1, \ldots, k\}$

(<u>Exemple</u> : $[I_1 : a_i^1, b_i^1\ ;\ I_2 : a_i^2, b_i^2\ ;\ I_3 : a_i^3, b_i^3]$ où I_1, I_2, I_3 sont trois parties disjointes de K.

3 - "L'opérateur différence" $\Delta_B f$ (où B est un bloc de $[0,1]^k$ et f une application de $[0,1]^k$ dans $\mathbb{R}$) s'exprime naturellement avec les notations ci-dessous :

Si $B = [K : a_i, b_i]$,

$$\Delta_B f = \sum_{I \subset K} (-1)^{\text{card } I}\ f(I : b_i, \complement I : a_i)$$

II-b) - <u>Lemmes techniques</u>.

Dans les démonstrations de la partie III, nous serons amenés à utiliser certaines propriétés sur les notations que nous venons de définir ; nous allons les énoncer ici sous forme de lemmes dont on trouvera les démonstrations (fastidieuses mais élémentaires) dans [7].

<u>Lemme 1</u>.

Soit f une application de $[0,1]^k$ dans $\mathbb{R}$ nulle sur $\overleftarrow{[0,1]}^k$; alors on a :

$$\Delta_{[I\ :\ a_i,\ b_i\ ;\ \complement I\ :\ 0,\ c_i]}\ f = \Delta_{[I\ :\ a_i,\ b_i]} (f \circ \boxed{\complement I : c_i})$$

Lemme 2.

Soient f et g deux applications de $[0,1]^k$ dans $\mathbb{R}$, nulles sur $\overline{[0,1]}^k$; soit pour tout i ($1 \leq i \leq k$), $0 \leq a_i \leq b_i \leq 1$; alors on a :

$$\Delta_{[K\ :\ a_i,\ b_i]}\, f.g = \sum_{I \subset J} \Delta_{[I\ :\ a_i,\ b_i\ ;\ J-I\ :\ 0,\ a_i\ ;\ \complement J\ :\ 0,\ b_i]} f.$$

$$\Delta_{[I\ :\ 0,\ b_i\ ;\ J-I\ :\ a_i,\ b_i\ ;\ \complement J\ :\ 0,\ b_i]} g$$

Lemme 3.

Soit f une application de $[0,1]^k$ dans $\mathbb{R}$ nulle sur $\overline{[0,1]}^k$; soit, pour tout i ($1 \leq i \leq k$), $0 \leq a_i \leq b_i \leq 1$; alors on a :

$$f(K) : a_i) = -\sum_{\substack{I \subset K \\ I \neq \phi}} \Delta_{[I\ :\ a_i,\ b_i\ ;\ \complement I\ :\ 0,\ b_i]} f \quad + f(K : b_i)$$

Enfin, nous énoncerons une propriété du processus empirique W_n qui est en réalité une des deux seules que nous utiliserons dans la démonstration du théorème, découlant de la définition même du processus empirique tronqué (l'autre propriété sera utilisée dans le lemme 9).

Soit H la fonction de répartition de la mesure μ citée dans l'hypothèse (H_4).

Soit H' la fonction de répartition de la mesure μ' sur $[0,1]^{1+k}$ produit de la mesure μ et de la mesure uniforme sur $[0,1]$

Lemme 4.

Soient $B = \prod_{i=0}^{k} [a_i,\ b_i]$ et $B' = \prod_{i=0}^{k} [c_i,\ d_i]$ deux blocs de $[0,1]^{1+k}$ tels que : $B' \subset B$ (c'est-à-dire :

$a_i \leq c_i \leq d_i \leq b_i$, $0 \leq i \leq k$), $a_o = c_o$, $b_o = d_o$ et a_o, b_o, c_o, d_o de la forme :

$a_o = c_o = \frac{\ell}{n}$, $b_o = d_o = \frac{\ell'}{n}$ où ℓ et ℓ' sont des entiers compris entre 0 et n.

Alors on a :

$$|\Delta_{B'}\, W_n| \leq |\Delta_B\, W_n| + n^{1/2}\, \Delta_B\, H'$$

Démonstration.

On va majorer séparément $\Delta_{B'} W_n$ et $- \Delta_{B'} W_n$

a) $\Delta_{B'} W_n = \Delta_{[K' : c_i, d_i]} W_n$

$$= n^{-1/2} \sum_{j=\ell+1}^{\ell'} \left(\prod_{i=1}^{k} I [c_i \leq \overline{F}_{n,i} (x_i^j) \leq d_i] - \Delta_{[K : c_i, d_i]} H_n^j \right) \leq$$

$$= n^{-1/2} \sum_{j=\ell+1}^{\ell'} \left(\prod_{i=1}^{k} I [a_i \leq \overline{F}_{n,i} (x_i^j) \leq b_i] - \Delta_{[K : a_i, b_i]} H_n^j \right) +$$

$$n^{-1/2} \sum_{j=\ell+1}^{\ell'} (\Delta_{[K : a_i, b_i]} H_n^j - \Delta_{[K : c_i, d_i]} H_n^j)$$

$$\leq \Delta_{[K' : a_i, b_i]} W_n + n^{-1/2} \sum_{j=\ell+1}^{\ell'} \Delta_{[K : a_i, b_i]} H_n^j$$

$$\leq |\Delta_B W_n| + n^{-1/2} (\ell' - \ell) \Delta_{[K : a_i, b_i]} H$$ (d'après l'hypothèse (H_4))

$$= |\Delta_B W_n| + n^{1/2} \Delta_{[K' : a_i, b_i]} H'$$

b) D'autre part :

$$- \Delta_{[K' : c_i, d_i]} W_n = n^{-1/2} \sum_{j=\ell+1}^{\ell'} \left(\Delta_{[K : c_i, d_i]} H_n^j - \prod_{i=1}^{k} I[c_i \leq \overline{F}_{n,i}(x_i^j) \leq d_i]\right)$$

$$\leq n^{-1/2} \sum_{j=\ell+1}^{\ell'} \Delta_{[K : c_i, d_i]} H_n^j$$

$$\leq n^{1/2} \Delta_{[K' : a_i, b_i]} H'$$

$$\leq n^{1/2} \Delta_{[K' : a_i, b_i]} H' + |\Delta_B W_n|$$

III - CONVERGENCE EN LOI DANS UN VOISINAGE DE LA FRONTIERE INFERIEURE.

III-a) - On note :

$\tilde{C}_{\theta,i} = \{\tilde{t}\ ;\ 0 \leq t_i \leq \theta\}$ $(1 \leq i \leq k)$ $(C_{\theta,i} = [0,1] \times \tilde{C}_{\theta,i})$

$F_1, F_2, \ldots, F_k$ les fonctions de répartition des marges de μ,
F_o la mesure uniforme sur $[0,1]$
H'' la fonction de répartition de la mesure μ'' sur $[0,1]^{1+k}$, mesure produit des marges de μ et de la mesure uniforme sur $[0,1]$ (c'est-à-dire : $H'' = \prod_{i=0}^{k} F_i$).

Théorème.

Supposons que :

1. (1-1) Les conditions (H_1), (H_2), (H_3), (H_4) (H_5) sont vérifiées

(1-2) La fonction r est semi-correctrice

2. Il existe θ_o et θ' strictement positifs tels que :

(2-1) Sur $]0,\theta_o]$, $\frac{1}{r_o}$ est de classe $\mathcal{C}_1$ et $\partial(\frac{1}{r_o}) < 0$

(2-2) Sur $\bigcup_{i=1}^{k} \tilde{C}_{\theta',i} - \bigcup_{i=1}^{k} \tilde{C}_{o,i}$, $\frac{1}{\tilde{r}}$ est de classe $\mathcal{C}_k$ et

- si card I est pair, $\partial_I(\frac{1}{\tilde{r}})$ prend des valeurs positives
- si card I est impair, $\partial_I(\frac{1}{\tilde{r}})$ prend des valeurs négatives.

(2-3) Il existe $\beta > \frac{3k+2}{k+2}$

tel que :

a) $a_o.\frac{1}{r_o^\beta(a_o)}$ est fonction croissante, sur $[0,\theta_o]$, de a_o

b) pour tout $J \subset K$

$\Delta_{[I\ :\ a_i,b_i\ ;\ \complement I\ :\ 0,a_i]^H} \cdot \frac{1}{\tilde{r}^\beta}(I : b_i, \complement I : a_i)$

et $\Delta_{[I\ :\ a_i, b_i\ ;\ \complement I\ :\ 0, a_i]}(\prod_{i=1}^{k} F_i) \cdot \frac{1}{\tilde{r}^{\beta}}\ (I : b_i, \complement I : a_i)$

sont, sur $\bigcup_{i=1}^{k} \tilde{C}_{\theta',i}$ fonctions croissantes de ($\complement I : a_i$)

(2-4)

$$\int_{[0,\theta_o]} \frac{1}{r_o^{1+\beta}}\, dt_o < + \infty$$

$$\int_{[0,\theta_o]} \left| \frac{1}{r_o^{\beta}}\, \partial \left(\frac{1}{r_o}\right) t_o \right| dt_o < + \infty$$

(2-5) Pour tout $I \subset K$

$$\int_{\bigcup_{i=1}^{k} \tilde{C}_{\theta',i}} \left| \frac{1}{\tilde{r}^{\beta}}\, \partial_I H\, \partial_{\complement I} \left(\frac{1}{\tilde{r}}\right) \right| \lambda^k < + \infty$$

$$\int_{\bigcup_{i=1}^{k} \tilde{C}_{\theta',i}} \left| \frac{1}{\tilde{r}^{\beta}}\, \partial_I \left(\prod_{i=1}^{k} F_i \right) \partial_{\complement I} \left(\frac{1}{\tilde{r}}\right) \right| d\lambda^k < + \infty$$

Alors la suite des processus $W_n.\frac{1}{r}$ converge en loi vers le processus $W_o.\frac{1}{r}$ gaussien et à trajectoires p.s. continues (où W_o est le processus limite de W_n).

Remarques.

1 - Il résulte évidemment des hypothèses 2-1 et 2-2 que la fonction semi-correctrice r est régulière.

2 - Il sera pratique d'introduire

H^* la fonction de répartition de la mesure μ^* sur $[0,1]^{1+k}$ somme des mesures μ' et μ'' (c'est-à-dire : $H^* = H' + H''$).

$\tilde{H}^*$ la fonction de répartition de la mesure $\tilde{\mu}^*$, marginale de μ^* sur $[0,1]^k$ $(\tilde{H}^* = H + \prod_{i=1}^{k} F_i)$

Alors l'hypothèse 2-3 b exprime une propriété de croissance de :

$$\Delta_{[I\ :\ a_i,b_i\ ;\ \complement I\ :\ 0,a_i]}\ \tilde{H}^*\ \ \frac{1}{\tilde{r}^\beta}(I : b_i,\ \complement I : a_i)$$

et l'hypothèse 2-5 exprime que :

$$\int_{\bigcup_{i=1}^{k} \tilde{C}_{\theta',i}} \left| \frac{1}{\tilde{r}^\beta}\ \partial_I\ \tilde{H}^*\ \partial_{\complement I}\ \left(\frac{1}{\tilde{r}}\right) \right|\ d\lambda^k < +\infty$$

3 - Un processus W_n vérifiant la condition (B) de la proposition avec une certaine fonction correctrice r, vérifiera aussi la condition (B) pour toute fonction correctrice égale à r à une constante multiplicative près ; c'est pourquoi par la suite, nous supposerons toujours que :

$r_o \leq 1$ sur $[0,1]$

et $\tilde{r} \leq 1$ sur $[0,1]^k$

Démonstration du théorème.

Nous montrerons que la conditions B_R est vérifiée dans les trois cas suivants :

$R = \bigcup_{i=1}^{k} C_{\theta,i}$, $R = C_{\theta,o} \cap \bigcup_{i=1}^{k} C_{\theta',i}$ (noté par la suite $\underline{C}_{\theta,o}$)

et $R = C_{\theta,o} \cap [[0,1]^{1+k} - \bigcup_{i=1}^{k} C_{\theta',i}]$ (notée par la suite $\overline{C}_{\theta,o}$)

On remarque que $\underline{C}_{\theta,o} = \bigcup_{i=1}^{k} \underline{C}_{\theta,i}$ (où pour tout i,

$\underline{C}_{\theta,i} = \underline{C}_{\theta,o} \cap C_{\theta',i}$)

L'idée directrice de la démonstration va consister à généraliser dans chacun des trois cas une idée donnée, dans le cas unidimensionnel, par FEARS et MEHRA [3].

Ainsi dans le premier cas ($R = \bigcup_{i=1}^{k} C_{\theta,i}$), pour tout i ($1 \leq i \leq k$) et pour tout $n \in \mathbb{N}^*$, on note $C^*_{\theta,i}(n)$ le sous ensemble de $C_{\theta,i}$ constitué par les valeurs de t telles que

$F_o(t_o) \leq \frac{1}{n}$ et pour tout $j \geq 1$, $F_j(t_j) \geq (\frac{1}{n})^{1+\alpha}$

On démontre alors la propriété suivante (dans l'énoncé de laquelle ν est une mesure diffuse et γ une constante positive qui seront précisées par la suite).

1 - Pour tout i ($1 \leq i \leq k$), il existe une constante $K''_i > 0$ telle que pour tout θ ($< \theta'$), pour tout ε (> 0), il existe $N^i_o(\varepsilon, \theta)$ tel que pour tout $n \geq N^i_o(\varepsilon, \theta)$ on ait

$$P_n [\sup_{t \in C^*_{\theta,i}(n)} |(W_n \cdot \frac{1}{r})(t)| > \varepsilon] < \frac{K''_i}{\varepsilon^4} \nu^{1+\gamma}(C_{\theta,i})$$

Pour tout i ($1 \leq i \leq k$), pour tout θ ($< \theta'$) pour tout $\delta_i > 0$ et pour tout $\varepsilon_i > 0$, il existe $N^i_1(\varepsilon_i, \delta_i, \theta)$ tel que pour tout $n \geq N^i_1(\varepsilon_i, \delta_i, \theta)$ on ait

$$P_n [\sup_{t \in C_{\theta,i} - C^*_{\theta,i}(n)} |(W_n \cdot \frac{1}{r})(t)| > \delta_i] < \varepsilon_i$$

Cette propriété (1) assure la réalisation de B_R dans le premier cas, démontrons le :

En effet, on a :

$$\underset{t\in\bigcup_{i=1}^{k} C_{\theta,i}}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| \leq \sum_{i=1}^{k} \Big(\underset{t\in C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| + \underset{t\in C_{\theta,i}-C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| \Big),$$

c'est-à-dire :

$$P_n\Big[\underset{t\in\bigcup_{i=1}^{k} C_{\theta,i}}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| > \delta\Big] \leq \sum_{i=1}^{k} \Big(P_n\Big[\underset{t\in C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| > \frac{\delta}{2k}\Big] +$$

$$P_n\Big[\underset{t\in C_{\theta,i}-C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| > \frac{\delta}{2k}\Big]\Big) .$$

Pour tout i $(1 \leq i \leq k)$, soit $\theta_i > 0$ tel que l'on ait, si $\theta<\theta_i$

$$\nu^{1+\gamma}(C_{\theta,i}) < \Big[K''_i \Big(\frac{2k}{\delta}\Big)^4\Big]^{-1} \Big(\frac{\varepsilon}{2k}\Big) .$$

Pour tout $\theta < \theta_i$, soit alors $N^i_o(\frac{\delta}{2k},\theta)$ tel que pour tout $n \geq N^i_o(\frac{\delta}{2k},\theta)$, on ait :

$$P_n\Big[\underset{t\in C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| > \frac{\delta}{2k}\Big] < K''_i \Big(\frac{2k}{\delta}\Big)^4 \nu^{1+\gamma}(C_{\theta,i})$$

$$< \frac{\varepsilon}{2k} .$$

D'autre part, pour tout i, et pour tout $\theta \leq \theta_i$, il existe egalement $N^i_1(\frac{\varepsilon}{2k}, \frac{\delta}{2k}, \theta)$ tel que pour tout $n \geq N^i_1(\frac{\varepsilon}{2k}, \frac{\delta}{2k},\theta)$, on ait :

$$P_n\Big[\underset{t\in C_{\theta,i}-C^*_{\theta,i}(n)}{\operatorname{Sup}} |(W_n\cdot\frac{1}{r})(t)| > \frac{\delta}{2k}\Big] < \frac{\varepsilon}{2k}$$

Alors, si on prend $\theta < \operatorname{Inf}_{1 \leq i \leq k} \{\theta_i\}$

et $N_o = \operatorname{Sup}_{1 \leq i \leq k} \{\operatorname{Sup} (N_o^i (\frac{\delta}{2k}, \theta), N_1^i (\frac{\varepsilon}{2k}, \frac{\delta}{2k}, \theta))\}$,

on aura pour tout $n \geq N_o$,

$$P_n [\operatorname{Sup}_{t \in \bigcup_{i=1}^{k} C_{\theta,i}} |(W_n . \frac{1}{\tilde{r}})(t)| > \delta] = k (\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k}) = \varepsilon ,$$

qui est la propriété B_R (pour $R = \bigcup_{i=1}^{k} C_{\theta,i}$) annoncée.

De même, dans le second cas ($R = \underline{C}_{\theta,o}$), on définit pour tout i ($1 \leq i \leq k$) $\underline{C}^*_{\theta,i}(n)$ comme le sous ensemble de $\underline{C}_{\theta,i}$ constitué par les valeurs de t telles que $F_o(t_o) \leq \frac{1}{n}$ et pour tout j ($1 \leq j \leq k$), $F_j(t_j) \geq (\frac{1}{n})^{1+\alpha}$, et on démontre la propriété (2) formellement identique à (1) ci-dessus (en y remplaçant θ' par θ_o et en y conservant les mêmes mesure diffuse ν et constante γ) ; le même raisonnement que précédemment permet d'en déduire la propriété B_R pour $R = \underline{C}_{\theta,o}$

Enfin, dans le troisième cas ($R = \overline{C}_{\theta,o}$) il suffit, $\tilde{r}$ étant minoré sur R par une constante strictement positive, de définir $\overline{C}^*_{\theta,o}(n)$, sous ensemble de $\overline{C}_{\theta,o}$ constitué par les t tels que $t_o \geq \frac{1}{n}$; on démontre alors la propriété suivante (dans l'énoncé de laquelle la mesure ν' diffère de celle utilisée dans les propriétés (1) et (2)):

3 - Il existe une constante $K'' > 0$ telle que pour tout $\theta < \theta_o/2$, pour tout ε (> 0), il existe N_o (ε,θ) tel que pour tout $n \geq N_o$ (ε,θ) on ait

$$P_n [\operatorname{Sup}_{t \in \overline{C}_{\theta,o}} |W_n(t) . \frac{1}{r_o(t_o)}| > \varepsilon] < \frac{K''_o}{\varepsilon^4} \nu'^{1+\gamma}(C_{2\theta,o})$$

la condition (B_R) sera alors réalisée en choisissant θ tel que $\nu'^{1+\gamma} (C_{2\theta,o}) \leq (\frac{K''_o}{(m\delta)^4})^{-1}\varepsilon$ car pour tout $n \geq N_o$ (ε,θ) on aura :

$$P_n [\operatorname{Sup}_{t \in \overline{C}_{\theta,o}} |(W_n . \frac{1}{\tilde{r}})(t)| > \delta] = P_n [\operatorname{Sup} |W_n(t) \frac{1}{r_o(t_o)}| > m\delta]$$

$$\leq \left(\frac{K''_o}{(m\delta)^4}\right) \left(\frac{K''_o}{(m\delta)^4}\right)^{-1} \varepsilon = \varepsilon$$

III-b) - <u>Démonstration des propriétés 1 et 2</u>.

Nous allons décomposer la démonstration en quatre lemmes dont les deux derniers énoncerons les propriétés 1 et 2. Sans perte de généralité et pour la commodité de la démonstration, nous supposerons dans cette partie que :

- r_o est de classe $\mathcal{C}_1$, $\partial(\frac{1}{r_o}) < 0$ sur $[0,1]$, ce que nous appellerons l'hypothèse (2-1)';

- pour tout $a_o \in [0,1]$,

$$a_o . \frac{1}{r^\beta\ (a_o)}$$ est une fonction croissante sur $[0,1]$ de a_o,

ce que nous appellerons l'hypothèse (2-3) a').

En effet, r_o étant continue et strictement positive sur $[\theta_o,1]$, il existe m tel que $0 < m \leq r_o\ (t_o)$ si $t_o \in [\theta_o,1]$;

soit r'_o une fonction définie par :

$$r'_o\ (t_o) = r_o\ (t_o) \text{ si } t_o \in [0,\theta_o]$$

$$r'_o\ (t_o) = (At_o)^\delta \text{ si } t_o \in [\theta_o,1]$$

avec $0 < \delta < 1$ et $\delta\ \beta < 1$ et $A = \dfrac{(r_o(\theta_o))^{1/2}}{\theta_o}$

(donc : $r'_o\ (\theta_o) = r_o\ (\theta_o)$)

r'_o vérifie bien toutes les hypothèses du théorème 1, ainsi que (2-1)' et (2-3) a') et on a :

$$\sup_{\substack{t_o\in[\theta_o,1]\\ \tilde{t}\in[0,1]^k}} \left|W_n(t)\,\frac{1}{r_o}(t)\,.\,\frac{1}{\tilde{r}}(t)\right| \leq \sup_{\substack{t_o\in[\theta_o,1]\\ \tilde{t}\in[0,1]^k}} \left|\frac{W_n(t)}{m}\,\frac{1}{\tilde{r}}(t)\right| =$$

$$\sup_{\substack{t_o\in[\theta_o,1]\\ \tilde{t}\in[0,1]^k}} \left|\frac{W_n(t)}{r_o'(t_o)}\,\frac{r_o'(t_o)}{m}\,\frac{1}{\tilde{r}}(\tilde{t})\right| \leq \frac{A^\delta}{m} \sup_{\substack{t_o\in[\theta_o,1]\\ \tilde{t}\in[0,1]^k}} \left|W_n(t)\,\frac{1}{r_o}\,r(t_o)\frac{1}{\tilde{r}}(\tilde{t})\right| ,$$

où $\frac{A^\delta}{m}$ est un nombre fini positif.

On définit sur $[0,1]^{k+1}$ la mesure ν par :

$$\nu(A) = \sum_{I \subset K} \int_A \left|\frac{1}{r^\beta}\,\partial_I\, H^*\partial\, \complement_I(\frac{1}{r})\right| d\,\lambda^{k+1}$$

où $\beta > \frac{3k+2}{k+2}$

Remarquons que la mesure ν est la mesure produit des mesures ν_o et $\tilde{\nu}$, ou ν_o est la mesure définie sur $[0,1]$ par :

$$\nu_o(A_o) = \int_{A_o} \frac{1}{r_o^{\beta+1}}\,dt_o + \int_{A_o} \left|\frac{1}{r_o^{\beta}}\,\partial\,(\frac{1}{r_o})\,F_o\right| dt_o \quad ,$$

et $\tilde{\nu}$ est la mesure définie sur $[0,1]^k$ par :

$$\tilde{\nu}(\tilde{A}) = \sum_{I \subset K} \int_{\tilde{A}} \left|\frac{1}{\tilde{r}^{\,\beta}}\,\partial_I\,\tilde{H}^*\,\partial\,\complement_I(\frac{1}{\tilde{r}})\right| d\,\lambda^k \quad .$$

Par la suite β étant fixé, il sera pratique de l'écrire sous la forme :

$$\beta = \frac{3\,k\,(1+\alpha)+2}{k\,(1+\alpha)+2} \quad (\text{où } \alpha > 0)$$

et on définira :

$$\gamma = \frac{1}{k\,(1+\alpha)+1} \quad (\,\gamma \text{ vérifie } (1+\gamma)\,(1+\beta) = 4) \quad .$$

Pour tout n ($\in \mathbb{N}^*$) on définit $\mathcal{B}(n)$, ensemble des blocs

$$B = \prod_{i=o}^{k} [\underline{t}_i, \overline{t}_i] \quad \text{tels que}$$

$$(\underline{t}_o, \overline{t}_o) \in (\{0, \frac{1}{r}, \ldots, \frac{n-1}{n}, 1\})^2 .$$

Pour tout bloc $B = \prod_{i=o}^{k} [\underline{t}_i, \overline{t}_i]$ on pose

$$m_H(B) = \inf_{1 \leq i \leq k} (F_i(\overline{t}_i) - F_i(\underline{t}_i))$$

Lemme 5.

Sous les conditions du théorème , il existe K(> 0) tel que pour tout n et tout bloc de $\mathcal{B}(n)$ contenu dans $(\bigcup_{i=1}^{k} C_{\theta',i}) \cup \underline{C}_{\theta_o,o}$, vérifiant $m_H(B) \geq (\frac{1}{n})^{1+\alpha}$ on ait :

$$E(\Delta_B(W_n \cdot \frac{1}{r}))^4 \leq K\nu^{1+\gamma}(B)$$

Démonstration.

On commence par montrer qu'il existe une constante K' > 0 telle que pour tout bloc $B = (\prod_{i=o}^{k} [\underline{t}_i, \overline{t}_i])$ appartenant à $\mathcal{B}(n)$ et vérifiant $m_H(B) \geq (\frac{1}{n})^{1+\alpha}$ on ait :

$$E(\Delta_B W_n)^4 \leq K' (\Delta_B H^*)^{1+\gamma} \quad ;$$

d'après un lemme technique, dans [4] p. 133, faisant intervenir l'hypothèse de mélange H_3, il existe une constante K_1(> 0) telle que :

$$E(\Delta_B W_n)^4 \leq K_1 (\overline{t}_o - \underline{t}_o)^2 (\Delta_{[K : \underline{t}_i, \overline{t}_i]} H)^2 + 2 \frac{(\overline{t}_o - \underline{t}_o)}{n} \Delta_{[K : \underline{t}_i, \overline{t}_i]} H$$

de l'hypothèse $m_H(B) \geq (\frac{1}{n})^{1+\alpha}$ on en déduit :

$$(\underline{t}_o - \overline{t}_o) \prod_{i=1}^{k} (F_i(\overline{t}_i) - F_i(\underline{t}_i)) \geq (\frac{1}{n})^{k(1+\alpha)+1}$$

$$(\Delta_B H^*)^{\gamma} \geq \frac{1}{n} \ ;$$

d'où

$$E\ (\Delta_B\ W_n)^4 \leq 2K_1 \left[(\bar{t}_o-\underline{t}_o)^2(\Delta_{[K:\underline{t}_i,\bar{t}_i]}H)^2+(\Delta_B H^*)^\gamma(\bar{t}_o-\underline{t}_o)\Delta_{[K:\underline{t}_i,\bar{t}_i]}H\right]$$

$$\leq 2K_1\ [(\Delta_B H^*)^2 + (\Delta_B H^*)^{1+\gamma}]$$

$$\leq K'\ (\Delta_B H^*)^{1+\gamma} \text{ avec } K' = 2K_1\ (H^*(1,\ldots,1)^{1-\gamma}+1)$$

Soit maintenant $J = \{i\in K' : \underline{t}_i > 0\}$.

On a d'après le lemme 2,

$$\Delta_B(W_n.\tfrac{1}{r}) = \sum_{I\subset J} \Delta_{[I:\ \underline{t}_i,\bar{t}_i\ ;\ J-I\ :\ 0,\underline{t}_i\ ;\ \complement J:\ 0,\bar{t}_i]}W_n \ .$$

$$\Delta_{[I:\ 0,\bar{t}_i\ ;\ J-I\ :\ \underline{t}_i,\bar{t}_i\ ;\ \complement J:\ 0,\bar{t}_i]}\tfrac{1}{r}$$

Utilisant l'inégalité $(\sum_{e\in E} |x_e|)^4 \leq (2^3)^{\text{card } E} \sum_{e\in E} x_e^4$,

et posant : $K^{(3)} = (2^3)^{2^{k+1}}.K'$, il vient

$$E\ (\Delta_B\ (W_n.\tfrac{1}{r}))^4 \leq K^{(3)} \sum_{I\subset J} Y_I$$

avec :

$$Y_I = (\Delta_{[I:\ \underline{t}_i,\bar{t}_i\ ;\ J-I\ :\ 0,\underline{t}_i\ ;\ \complement J:\ 0,\bar{t}_i]}H^*)^{1+\gamma}\ .$$

$$(\Delta_{[I:\ 0,\bar{t}_i\ ;\ J-I\ :\ \underline{t}_i,\bar{t}_i\ ;\ \complement J:\ 0,\bar{t}_i]}\tfrac{1}{r})^4$$

Or :

$$|\Delta_{[I:\ 0,\bar{t}_i\ ;\ J-I\ :\ \underline{t}_i,\bar{t}_i\ ;\ \complement I:\ 0,\bar{t}_i]}\tfrac{1}{r}|^{1+\beta} =$$

$$|\Delta_{[J-I\ :\ \underline{t}_i,\bar{t}_i]}\ (\tfrac{1}{r} \circ \boxed{I\cup\complement J\ :\ \bar{t}_i})|^{1+\beta} \leq$$

$$|\Delta_{[J-I\ :\ \underline{t}_i,\bar{t}_i]}\ (\tfrac{1}{r} \circ \boxed{I\cup\complement J\ :\ \bar{t}_i})|\ (\sum_{L\subset J-I} \tfrac{1}{r}(L:\underline{t}_i;\complement L:\bar{t}_i))^\beta \leq$$

$$2^{(k+1)\beta}|\Delta_{[J-I:\ \underline{t}_i,\bar{t}_i]}\ (\tfrac{1}{r} \circ \boxed{I\cup\complement J:\ \bar{t}_i})\ |(\sup_{L\subset J-I} \tfrac{1}{r}(L:\underline{t}_i:\complement L:\bar{t}_i))^\beta,$$

d'où, par croissance de r sur $(\bigcup_{i=1}^{k} C_{\theta',i}) \cup \underline{C}_{\theta,o}$

$$|\Delta_{[IU \complement J\ :\ 0,\overline{t}_i\ ;J\ -\ I:\underline{t}_i,\overline{t}_i]}\tfrac{1}{r}|^{1+\beta} \leq$$

$$2^{(k+1)\beta}|\Delta_{[IU \complement J:\ 0,\overline{t}_i;\ J\ -\ I:\ \underline{t}_i,\overline{t}_i]}\tfrac{1}{r}|\ \tfrac{1}{r^\beta}\ (J-I:\ \underline{t}_i;\ IU\ \complement J:\overline{t}_i)\ .$$

Comme $(1+\beta)\ (1+\gamma) = 4$, on a alors

$Y_I \leq 2^{(k+1)}\beta\ \tilde{Y}_I$ avec

$$\tilde{Y}_I = (\Delta_{[I:\underline{t}_i,\overline{t}_i;\ J-I:0,\underline{t}_i;\ \complement J:0,\overline{t}_i]}H^*\ \tfrac{1}{r^\beta}\ (J-I:\underline{t}_i;\ IU\ \complement J:\overline{t}_i).$$

$$|\Delta_{[IU \complement J:\ 0,\overline{t}_i\ ;\ J\ -\ I:\ \underline{t}_i,\overline{t}_i]}\tfrac{1}{r}\ |)^{1+\gamma} =$$

$$(\Delta_{[I:\ \underline{t}_i,\overline{t}_i\ ;\ \complement J:\ 0,\overline{t}_i]}H^*\ \circ\ \boxed{J\ -\ I:\ \underline{t}_i}\cdot\tfrac{1}{r^\beta}(J-I:\ \underline{t}_i\ ;IU\ \complement J:\ \overline{t}_i).$$

$$|\Delta_{[IU \complement J:\ \underline{t}_i,\overline{t}_i]}\tfrac{1}{r}\ \circ\ \boxed{IU\ \complement J:\ \overline{t}_i}\ |)^{1+\gamma}$$

D'après les hypothèses (2-1)' et (2-2)

$\partial_{J-I}(\frac{1}{r})$ est de signe constant sur $(\bigcup_{i=1}^{k} C_{\theta',i})\cup(\underline{C}_{\theta,o})$; on a :

$$|\Delta_{[J-I:\ \underline{t}_i,\overline{t}_i]}\tfrac{1}{r}\ \circ\ \boxed{IU\ \complement J:\ \overline{t}_i}| = \int_{[J-I:\underline{t}_i,\overline{t}_i]} |\partial_{J-I}(\tfrac{1}{r})\ (IU\complement J:\overline{t}_i;J-I:v_i)\,|\lambda^{J-I}(dv_i)$$

d'où en vertu des hypothèses (2-3)a') et (2-3) b)

$$\tilde{Y}_I \leq (\int_{[J-I:\underline{t}_i,\overline{t}_i]} \left[\Delta_{[I:\underline{t}_i,\overline{t}_i;\ \complement I:0,\overline{t}_i]}H^*\ \circ\ \boxed{J-I:\ v_i}\right]\cdot\tfrac{1}{r^\beta}(J-I:v_i;\ IU\ \complement J:\ \overline{t}_i)\ \cdot$$

$$|\partial_{J-I}\ \tfrac{1}{r}\ (IU\ \complement J:\ \overline{t}_i\ ;\ J-I:\ v_i)|\ \lambda^{J-I}(dv_i))^{1+\gamma}$$

et en vertu des hypothèses (2-1)' et (2-2) on a ,
pour tout $(K'\ :\ v_i)\ \epsilon[K'\ :\ \underline{t}_i,\overline{t}_i]$,

$$|(\partial_{J-I}\ \tfrac{1}{r})(K':\ v_i)|\ \geq |(\partial_{J-I}\ \tfrac{1}{r})\ (IU\ \complement J:\ \overline{t}_i\ ;\ J-I:\ v_i)|\ ;$$

en particulier

$$\tfrac{1}{r}\ (K':\ v_i)\ \geq \tfrac{1}{r}\ (IU\ \complement J\ :\ \overline{t}_i\ ;\ J-I:\ v_i)$$

donc :

$$\left(\int_{[IU\,(J-I):\underline{t}_i,\overline{t}_i\ ;\ \complement J:0,\overline{t}_i]} \frac{1}{r^\beta}|\partial_{J-I}(\frac{1}{r})|\cdot\partial_{IU\,\complement J}H^* \; d\lambda^{k+1}\right)^{1+\gamma} =$$

$$\left(\int_{[I:\underline{t}_i,\overline{t}_i:\ \complement J:0,\overline{t}_i]}\int_{[J-I:\underline{t}_i,\overline{t}_i]} \frac{1}{r^\beta}(K':v_i)\cdot\partial_{IU\complement J}H^*(K':v_i)\,|\partial_{J-I}\frac{1}{r}\,(K':v_i)|\; d\lambda^{J-I}\, d\lambda^{IU\,\complement J}\right)^{1+\gamma}$$

$$\geq \tilde{Y}_I$$

donc si on pose $K = K^{(3)}\ 2^{(k+1)\beta}$, on a ,en raison de l'inégalité (où a >1):

$$\left(\sum_{e\in E} |x_e|^a\right) \leq \left(\sum_{e\in E}|x_e|\right)^a ,$$

$$E(\Delta_B(W_n\cdot\tfrac{1}{r}))^4 \leq K\sum_{I\subset K'}\left(\int_B \frac{1}{r^\beta}\cdot|\ \partial_{J-I}\ \frac{1}{r}|\cdot\partial_{IU\,\complement J}H^*\ d\lambda^{k+1}\right)^{1+\gamma}$$

$$\leq\ K\ \nu^{1+\gamma}\ (B)$$ (fin de la démonstration du lemme) .

— Pour tout i ($1 \leq i \leq k$), on définit un quadrillage de $C_{\theta,i}$ ($\theta < \theta'$), de base $B_{\theta,i}(n) = \{(t_j^\ell\ (n,\theta)\ ;\ 0 \leq \ell \leq L_j^i : 0 \leq i \leq k\}$,

vérifiant

$$t_j^o\ (n,\theta) = 0 \qquad (0 \leq j \leq k)$$

$$t_o^\ell\ (n,\theta) = \frac{\ell}{n}\ ,\ L_o^i = n$$

$$t_j^{L_j^i}\ (n,\theta) = 1 \text{ si } j \neq i$$

$$t_i^{L_i^i}\ (n,\theta) = \theta$$

$F_j\ (t_j^{\ell+1}(n,\theta)) - F_j(t_j^\ell(n,\theta))$ (où $j \neq o$) est indépendant de ℓ et vérifie : $(\frac{1}{n})^{1+\alpha} < F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^{\ell}(n,\theta)) < (\frac{1}{n})^{1+\alpha'}$ (où $0 \leq \alpha' \leq \alpha$).

Autrement dit, si on note $\frac{F_j(1)}{L_j^i} = D_j^i(n,\theta)$ quand $j \neq i$,

et $\frac{F_i(\theta)}{L_i^i} = D_i^i(n,\theta)$, on a $F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^{\ell}(n,\theta)) = D_j^i(n,\theta)$

et la mesure (pour la mesure μ") de chaque bloc élémentaire de notre quadrillage vaut $\frac{1}{n} \prod_{j=1}^{k} D_j^i(n,\theta)$

(où $(\frac{1}{n})^{1+\alpha} < D_j^i(n,\theta) < (\frac{1}{n})^{1+\alpha'}$)

On définit également pour tout i $(1 \leq i \leq k)$ un quadrillage de $\underline{C}_{\theta,i}$ $(\theta < \theta_o)$ de base $\underline{B}_{\theta,i}(n) =$

$((t_j^{\ell}(n,\theta)\ ;\ 0 \leq 0 \leq L_j^i : 0 \leq j \leq k)$, vérifiant,

$t_j^o(n,\theta) = 0 \quad (0 \leq j \leq k)$

$t_o^{\ell}(n,\theta) = \frac{\ell}{n},\ L_o^i = [n\theta] + 1$

$t_i^{L_i^i}(n,\theta) = \theta'$

$t_j^{L_j^i}(n,\theta) = 1$ si j est différent de 0 et de i

$F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^{\ell}(n,\theta))$ (où $j \neq 0$) est indépendant de ℓ

et vérifie $(\frac{1}{n})^{1+\alpha} < F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^{\ell}(n,\theta)) < (\frac{1}{n})^{1+\alpha'}$

(où $0 < \alpha' < \alpha$)

(On introduit comme ci-dessus, $D_j^i(n,\theta)$ qui est la valeur commune des $F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^{\ell}(n,\theta))$.

<u>Lemme 6.</u>

Sous les conditions du théorème , pour tout i $(1 \leq i \leq k)$, il existe $K_i' > 0$ tel que pour tout θ $(< \theta')$ et tout ε (> 0), on ait :

$$P_n[\underbrace{\sup_{t\in B_{\theta,i}(n)}}|(W_n.\frac{1}{r})(t)| > \varepsilon] \lesssim \frac{K_i'}{\varepsilon^4} \nu^{1+\gamma} (C_{\theta,i}) .$$

Il existe également $\underline{K}_i' > 0$ tel que pour tout θ ($< \theta_o$) et tout $\varepsilon(> 0)$ on ait :

$$P_n [\underbrace{\sup_{t\in \underline{B}_{\theta,i}(n)}}|(W_n.\frac{1}{r})(t)| > \varepsilon] \leq \frac{\underline{K}_i'}{\varepsilon^4} \nu^{1+\gamma} (\underline{C}_{2\theta,i})$$

Démonstration.

C'est une conséquence directe de l'inégalité de MARKOV, du lemme 5 et du théorème 1 de BICKEL et WICHURA [10](sous la forme affaiblie qui en est donnée par BALACHEFF et DUPONT en [5] p. 82 en [6] lemme 1).

Pour tout bloc $B = \prod_{i=o}^{k} [a_i,b_i]$ de $[0,1]^{k+1}$,

si $J = \{i\in K' : a_i > 0\}$ on note ,

$$X(B) = \prod_{i\in J} \frac{F_i(b_i)}{F_i(a_i)}$$

$(= 1$ si $J = \emptyset)$

Lemme 7.

Sous les conditions du théorème , pour tout $i(1 \leq i \leq k)$, il existe $K_i'' > 0$ tel que pour tout $\theta(< \theta')$ et tout ε (> 0), il existe N_o^i $(\varepsilon, \theta) \in N^*$ tel que, pour tout $n \geq N_o^i$ (ε, θ) on ait

$$P_n [\underbrace{\sup_{t\in C^*_{\theta,i}(n)}}|(W_n.\frac{1}{r})(t)| < \varepsilon] \leq \frac{K_i''}{\varepsilon^4} \nu^{1+\gamma} (C_{\theta,i}) .$$

Il existe également $\underline{K}_i'' > 0$ tel que pour tout $\theta(< \theta_o)$ et tout ε (> 0) il existe $\underline{N}_o^i$ $(\varepsilon, \theta)\in N^*$ tel que pour tout $n \geq \underline{N}_o^i$ (ε, θ) on ait :

$$P_n [\underbrace{\sup_{t\in \underline{C}^*_{\theta,i}(n)}}|(W_n.\frac{1}{r})(t)| > \varepsilon] \leq \frac{\underline{K}_i''}{\varepsilon^4} \nu^{1+\gamma} (\underline{C}_{2\theta,i})$$

<u>Démonstration</u>.

Pour ε et θ donnés, fixons également n et notons D_j^i pour D_j^i (n, θ).

Soit $t \in [K' : \underline{t}_j, \overline{t}_j]$ avec $\underline{t}_j = t_j^{\ell_j}$ $\overline{t}_j = t_j^{\ell_j+1}$ $(1 \leq \ell_j \leq L_j^i, 0 \leq j \leq k)$;

alors on a :

$|(W_n.\frac{1}{r})(t)| = |(W_n.\frac{1}{r})(\{0\} : t_o ; K : t_j)| \leq$

$|(W_n.\frac{1}{r})\ (\{0\}) : t_o\ ;\ K : t_j)| \leq$

$$\sum_{I \subset K} |\ \Delta_{[\{0\}:0,\underline{t}_o ; I:\ t_j,\overline{t}_j\ ;\ \complement I:\ 0,\overline{t}_j]} W_n | \frac{1}{r}\ (\{0\}:\ \underline{t}_o\ ;\ K:\ t_j)$$

(d'après le lemme 3).

Posons :

$$Y_I' = |\Delta_{[\{0\}:0,\underline{t}_o\ ;\ I:\ t_j,\overline{t}_j\ ;\ \complement I:\ 0,\overline{t}_j]} W_n | \frac{1}{r}\ (\{0\}:\ 0,\underline{t}_o\ ;\ K:\ t_j)$$

$(K' : t_j)$ appartenant au bloc élémentaire : $[K' : t_j^{\ell_j}, t_j^{\ell_j+1}]$,

on a pour tout j $(0 \leq j \leq k)$

$$\frac{F_j\ (t_j^{\ell_j+1})}{F_j\ (t_j)} \leq 2$$

donc $X\ ([K' : t_j, \overline{t}_j]) \leq 2^{k+1}$.

D'autre part, les hypothèses de croissance (2-3) a') et (2-3) b) entraînant également la croissance de la fonction

$\prod_{i=o}^{k} F_i.\frac{1}{r}$ (puisque $\beta > 1$), on en déduit pour tout $J \subset K$:

$$\frac{1}{r}\ (K' : t_j) \leq 2^{k+1}\ \frac{1}{r}\ (J : \overline{t}_j, \complement J : t_j)\ .$$

Si $I = \emptyset$, on a :

$$Y_I' = |\Delta_{[\{0\}:\ 0,t_o ;\ K:\ 0,\overline{t}_j]} W_n | . \frac{1}{r}\ (\{0\}:\ \underline{t}_o ;\ K:\ t_j) \leq$$

$2^{k+1} \; |(W_n \cdot \frac{1}{r})\ (\{0\}: \underline{t}_o;\ \ K: \bar{t}_j)|$.

Si $I \neq \emptyset$

$$Y'_I \leq |\Delta_{[\{0\}:\ 0,\underline{t}_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]} W_n|\frac{1}{r}\ (\{0\}: \underline{t}_o;\ K:\ t_j) +$$

$$n^{1/2}\ \Delta_{[\{0\}:\ 0,\underline{t}_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]} H' \cdot \frac{1}{r}\ (\{0\}: \underline{t}_o;\ K:\ t_j)$$

(d'après le lemme 4)

Posons $K'^{*}(I) = \{\underline{t}_o\} \times \prod_{j\in I} \{\underline{t}_j,\bar{t}_j\} \times \prod_{j\notin I} \{\bar{t}_j\}$

et soit $(K': a(j))$ l'élément générique de $K'^{*}(I)$

(c'est-à-dire $a(0)= t_o$, $a(j)= \underline{t}_j$ ou $\bar{t}_j$ si $j\in I$, et $a(j)= \bar{t}_j$ si $j\notin I$).

On a alors :

$$|\Delta_{[\{0\}:\underline{t}_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]}\ W_n \cdot |\frac{1}{r}\ (\{0\}: \underline{t}_o;\ K:\ t_j) \leq$$

$$2^{k+1} \sum_{(K':\ a(j))\in K'^{*}(I)} |(W_n \cdot \frac{1}{r})\ (K':\ a(j))| .$$

D'autre part, la mesure d'un bloc pour la mesure μ' étant majorée par la mesure marginale de toute face, on a pour tout j $(\in I)$,

$$\Delta_{[\{0\}:\ 0,t_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]} H' \leq D^{i}_{j}$$

d'où

$$\Delta_{[\{0\}:\ \underline{t}_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]} H' \leq \left(\prod_{j\in I} D^{i}_{j}\right)^{\frac{1}{\text{card } I}} .$$

On en déduit (en utilisant le fait que $\beta > 1$ et que r est majorée par 1)

$$n^{1/2}\ \Delta_{[\{0\}:\ \underline{t}_o;\ I:\ \underline{t}_j,\bar{t}_j;\ \complement I:\ 0,\bar{t}_j]} H'\ \frac{1}{r}\ (\{0\}: \underline{t}_o;\ K:\ t_j) \leq$$

$$2^{k+1}\left(\left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}}\right)^{1/2}\left(\Delta_{[\{0\}:\underline{t}_o;\ I:\ \underline{t}_j,\overline{t}_j;\ \complement I:\ 0,\overline{t}_j]}H' \cdot \frac{1}{r^2}(\{0\}:\ \underline{t}_o;\ K:\ \overline{t}_j)\right)^{1/2} \leq$$

$$2^{K+1}\left(\left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}}\right)^{1/2}\left(\Delta_{[\{0\}:\underline{t}_o;\ I:\ \underline{t}_j,\overline{t}_j;\ \complement I:\ 0,\overline{t}_j]}H' \cdot \frac{1}{r^{1+\beta}}(\{0\}:\ \underline{t}_o;\ K:\ \overline{t}_j)\right)^{1/2} \leq$$

$$\leq 2^{k+1}\left(\left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}}\right)^{1/2} \nu^{1/2}([\{0\}:0,\underline{t}_o;\ I:\underline{t}_j,\overline{t}_j;\complement I:0,\overline{t}_j])$$

$$\leq 2^{k+1}\left(\left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}}\right)^{1/2} \nu^{1/2}\left(\bigcup_{i=1}^{k} C_{\theta',i}\right)$$

On a donc :

$$Y'_I \leq 2^{k+1}\left(\sum_{(K':a(j))\in K'^*(I)} \left|W_n\cdot\tfrac{1}{r})(K':\ a(j))\right| + \left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{2\,\mathrm{card}\, I}} \cdot \nu^{1/2}\left(\bigcup_{i=1}^{k} C_{\theta',i}\right)\right)$$

Soit finalement

$$\sup_{t\in C^*_{\theta,i}(n)}\left|(W_n\cdot\tfrac{1}{r})(t)\right| \leq 2^{k+1}\ 3^k \sup_{t\in B_{\theta,i}(n)}\left|(W_n\cdot\tfrac{1}{r})\ (t)\right| +$$

$$2^{k+1}(2^k-1)\sup_{\substack{I\subset K\\ I\neq\emptyset}}\left(n^{\mathrm{card}\, I}\prod_{j\in I} D_j^i\right)^{\frac{1}{2\,\mathrm{card}\, I}} \nu^{1/2}\left(\bigcup_{i=1}^{k} C_{\theta',i}\right)$$

(et la même inégalité en remplaçant $C^*_{\theta,i}(n)$ par $\underline{C}^*_{\theta,i}(n)$ et $B_{\theta,i}(n)$ par $\underline{B}_{\theta,i}(n)$)

D'autre part, par hypothèse, pour tout j, $D_j^i\ (n,\theta) = o\ (\frac{1}{n})^{1+\alpha'}$

Donc $\left(n^{\mathrm{card}\, I} \prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}} = o\,(1/n)^{\alpha'}$

On pourra donc trouver un $N_o^i\,(\varepsilon,\theta)$ tel que, si $n \geq N_o^i\,(\varepsilon,\theta)$,

on ait

$$\underset{\substack{I\subset K\\ I\neq\emptyset}}{\mathrm{Sup}} \left(\left[n^{\mathrm{card}\, I} \prod_{j\in I} D_j^i\right)^{\frac{1}{\mathrm{card}\, I}} \leq \left[\frac{\varepsilon}{2}\,(2^{k+1}(2^k-1)\,\nu^{1/2}\,\left(\bigcup_{i=1}^{k} C_{\theta',i}\right)^{[illegible]}\right]$$

Donc, si on pose

$$K_i'' = (2^{k+2}.3^k)^4\; K_i' \;,$$

on aura pour tout $n \geq N_o^i\,(\varepsilon,\theta)$

$$P_n\,[\underset{t\in C^*_{\theta,i}(n)}{\mathrm{Sup}} |(W_n.\tfrac{1}{r})\;(t)| > \varepsilon] \leq P_n\,[\underset{t\in B_{\theta,i}(n)}{\mathrm{Sup}}\, 2^{k+1}\; 3^k|(W_n.\tfrac{1}{r})\,(t)| > \tfrac{\varepsilon}{2}]$$

$$\leq \frac{K_i''}{\varepsilon^4}\,\nu^{1+\gamma}\;(C_{\theta,i})$$ (d'après le lemme 7)

(et de même, si $n \geq \underline{N}_o^i\,(\varepsilon,\theta)$

$$P_n\,[\underset{t\in \underline{C}^*_{\theta,i}(n)}{\mathrm{Sup}} |(W_n.\tfrac{1}{r})\,(t)| > \varepsilon] \leq \frac{K_i''}{\varepsilon^4}\;\nu^{1+\gamma}(\underline{C}_{2\theta,i})$$

<u>Lemme 8</u>.

Sous les conditions du théorème _, pour tout i $(1 \leq i \leq k)$, pour tout θ $(< \theta')$, pour tout $\delta_i > 0$ et tout $\varepsilon_i > 0$, il existe $N_1^i\,(\varepsilon_i,\delta_i,\theta) \in \mathbb{N}^*$ tel que pour tout $n \geq N_1^i\,(\varepsilon_i,\delta_i,\theta)$ on ait :

$$P_n\,[\underset{t\in C_{\theta,i}-C^*_{\theta,i}(n)}{\mathrm{Sup}} |(W_n.\tfrac{1}{r})\,(t)| > \delta_i] < \varepsilon_i$$

De même pour tout $\theta(<\theta_o)$ pour tout $\underline{\delta}_i > 0$ et tout $\underline{\varepsilon}_i > 0$ il existe $\underline{N}_1^i\,(\underline{\varepsilon}_i,\,\underline{\delta}_i,\theta) \in \mathbb{N}^*$ tel que pour tout $n \geq \underline{N}_1^i\,(\underline{\varepsilon}_i,\underline{\delta}_i,\theta)$ on ait :

$$P_n\,[\underset{t\in \underline{C}_{\theta,i}-C^*_{\theta,i}(n)}{\mathrm{Sup}} |(W_n.\tfrac{1}{r})\,(t)| > \delta_i] < \varepsilon_i$$

Démonstration.

Soit, pour tout j, $\hat{C}_j = \{t \in C_{\theta,i} - C^*_{\theta,i}(n) \; ; \; t_j \leq t^1_j \; (n,\theta)\}$.

On a pour tout i $(0 \leq i \leq k)$

$$P_n \Big[\underbrace{\operatorname{Sup}}_{t \in C_{\theta,i} - C^*_{\theta,i}(n)} |(W_n \cdot \tfrac{1}{r})(t)| > \delta_i\Big] \leq \sum_{i=o}^{k} P_n \Big[\underbrace{\operatorname{Sup}}_{t \in \hat{C}_j} |(W_n \cdot \tfrac{1}{r})(t)| > \delta_j / k+1\Big] \; ;$$

le terme d'indice $j = 0$ est nul, car pour tout t tel que $t_o < \frac{1}{n}$, on a $(W_n \cdot \frac{1}{r})(t) = 0$;

si $j \neq 0$, il vient (en posant provisoirement t^ℓ_j pour t^ℓ_j (n,θ) et en utilisant la croissance de $\frac{H'}{r}$)

$$P_n \Big[\underbrace{\operatorname{Sup}}_{t \in \hat{C}_j} |(W_n \cdot \tfrac{1}{r})(t)| \leq \frac{\delta_i}{k+1}\Big]$$

$$\geq P_n\{[\underbrace{\operatorname{Sup}}_{t \in \hat{C}_j} |(W_n \cdot \tfrac{1}{r})(t)| \leq \frac{\delta_i}{k+1}] \cap [\tilde{F}_{n,j}(\bar{F}^{-1}_{n,j}(t^1_j)) = 0]\}$$

$$\geq P_n\{[n^{1/2} \frac{H'}{r} (\{j\} : t^1_j \, ; \, \complement\{j\} : t_q^{L^i_q}) \leq \frac{\delta_i}{k+1}] \cap [\tilde{F}_{n,j}(\bar{F}^{-1}_{n,j}(t^1_j)) = 0]\}$$

On majore $\sqrt{n}\, \frac{H'}{r}$ par $(n\, H')^{1/2} \left(\frac{H'}{r^{1+\beta}}\right)^{1/2}$;

Or on a, d'une part,

$$H' (\{j\} : t^1_j \, ; \, \complement\{j\} : t_q^{L^i_q}) \leq F_j(t^1_j)$$

et d'autre part,

$$\frac{H'}{r^{1+\beta}} (\{j\} : t^1_j \, ; \, \complement\{j\} : t_q^{L^i_q}) \leq \int_{[\{j\}:0,t^1_j \, ; \, \complement\{j\} : 0,\, t_q^{L^i_q}]} \frac{1}{r^{1+\beta}} \cdot \partial_{K'} H \, d\lambda^{1+k}$$

$$\leq \nu (\{j\}:0,t^1_j \, ; \, \complement\{j\}:0,t_q^{L^i_q})$$

$$\leq \nu (C_{\theta,i}).$$

Il en résulte que :

$$P_n\,[\underset{t\in \hat{C}_j}{\mathrm{Sup}}\ |(W_n.\tfrac{1}{r})(t)| \leq \frac{\delta_i}{k+1}] \geq$$

$$P_n\{[(nF_j(t_j^1(n,\theta))^{1/2}\nu^{1/2}(C_{\theta,i}) \leq \frac{\delta_i}{k+1}] \cap [\tilde{F}_{n,j}(\overline{F}_{n,j}^{-1}(t_j^1(n,\theta)) = 0]\}$$

Or $n\,F_j(t_j^1(n,\theta)) = o\,(\frac{1}{n})^{\alpha'}$; il existe donc $N'^{i}_{j}\,(\delta_i,\theta)$ tel que, pour tout $n \geq N'^{i}_{1}\,(\delta_i,\theta)$ on ait

$$n\,F_j\ (t_j^1(n,\theta)) \leq (\frac{\delta_i}{k+1})^2\ (\nu(C_{\theta,i}))^{-1}\ ,$$

d'où

$$P_n\,[\underset{t\in \hat{C}_j}{\mathrm{Sup}}\ |(W_n.\tfrac{1}{r})(t)| \leq \frac{\delta_i}{k+1}] > P_n[\tilde{F}_{n,j}(\overline{F}_{n,j}^{-1}(t_j^1\ (n,\theta)) = 0]$$

$$\geq 1 - \sum_{\ell=1}^{n} F_{n,j}^{\ell}\ (\overline{F}_{n,j}^{-1}(t_j^1\ (n,\theta))$$

$$\geq 1 - n\,F_j(t_j^1\ (n,\theta))$$

$$\geq 1 - n\,D_j^i\ (n,\theta)\ .$$

Comme $n\,D_j^i\ (n,\theta) = o\ (\frac{1}{n})^{\alpha'}$, il existe un $N''^{i}_{1}\ (\varepsilon_i,\theta)$ tel que si $n \geq N''^{i}_{1}\ (\varepsilon_i,\theta)$

$$n\,D_j^i\ (n,\theta) < \frac{\varepsilon_i}{k+1} \quad (1 \leq j \leq k)\ ;$$

l'inégalité cherchée s'en déduit avec ;

$$N_1^i\ (\varepsilon_i,\delta_i,\theta) = \mathrm{Sup}\ \{N'^{i}_{1}\ (\delta_i,\theta),\ N''^{i}_{1}\ (\varepsilon_i,\theta)\}\ .$$

On fait un raisonnement similaire pour avoir :

$$P_n\,[\underset{t\in \underline{C}_{\theta,i}-\underline{C}^*_{\theta,i}(n)}{\mathrm{Sup}}\ |(W_n.\tfrac{1}{r})(t)| > \underline{\delta}_i] < \underline{\varepsilon}_i$$

III-c) - Démonstration de la propriété 3.

Soit ν' la mesure définie sur $[0,1]^{1+k}$ par

$$\nu'(A) = \int_A \frac{1}{r_o^{1+\beta}} \cdot \partial_K \tilde{H}^* d\lambda^{k+1} + \int_A \left| \frac{1}{r_o^{\beta}} \cdot \partial\left(\frac{1}{r_o}\right) F_o \cdot \partial_K \tilde{H}^* \right| d\lambda^{k+1}$$

(ou β est définie comme dans III-b).

On remarque, que ν est la mesure produit de la mesure ν_o' et de $\tilde{\mu}^*$, où $\tilde{\mu}^*$ est la mesure de fonction de répartition $\tilde{H}^*$ (voir remarque 2 après l'énoncé du théorème)et où ν_o' est la mesure sur $[0,1]$ définie par :

$$\nu_o'(A_o) = \int_{A_o} \frac{1}{r_o^{1+\beta}} dt_o + \int_{A_o} \left| \frac{1}{r_o^{\beta}} \partial\left(\frac{1}{r_o}\right) F_o \right| dt_o \ ;$$

On définit aussi, comme dans III-b) γ, $\mathcal{B}(n)$ et $m_H(B)$.

On convient de noter $\frac{W_n}{r_o}$ l'application définie sur $[0,1]^{1+k}$ par $\frac{W_n}{r_o}(t_o,t_1,\dots,t_k) = W_n(t_o,\dots,t_k) \cdot \frac{1}{r_o(t_o)}$

Lemme 9.

Sous les conditions du théorème , il existe K (> 0) tel que pour tout n et tout bloc de $\mathcal{B}(n)$ contenu dans $C_{\theta,o}$, vérifiant $m_H(B) \geq \left(\frac{1}{n}\right)^{1+\alpha}$, on ait :

$$E\left(\Delta_B\left(W_n \cdot \frac{1}{r_o}\right)\right)^4 \leq \nu'^{1+\gamma}(B)$$

Démonstration.

Dans le lemme 5, on a déjà montré qu'il existe K' > 0 tel que $E(\Delta_B W_n)^4 \leq K'(\Delta_B H^*)^{1+\gamma}$ avec $B = \prod_{i=0}^{k} [\underline{t}_i, \overline{t}_i]$ vérifiant les conditions figurant dans l'énoncé du lemme.

Si $\underline{t}_o = 0$ on a :

$$\Delta_B\left(W_n \cdot \frac{1}{r_o}\right) = (\Delta_B W_n) \frac{1}{r_o}(\overline{t}_o)$$

d'où (utilisant en particulier l'égalité $(1+\beta)(1+\gamma) = 4$)

$$E\ (\Delta_B(W_n \cdot \frac{1}{r_o}))^4 \leq K'(\Delta_B H^*)^{1+\gamma} (\frac{1}{r_o}(\bar{t}_o))^4$$

$$= K' \ (\Delta_B H^* \cdot \frac{1}{r_o^{1+\beta}} (\bar{t}_o))^{1+\gamma}$$

$$\leq K' (\left| \int_{\underline{t}_o}^{\bar{t}_o} \frac{du_o}{r_o^{1+\beta}(u_o)} \right| \int_{[K:\underline{t}_i,\bar{t}_i]} \partial_K \tilde{H}^* \, d\lambda^k)^{1+\gamma}$$

$$\leq K' \ \nu'^{1+\gamma}(B)$$

Si $\underline{t}_o > 0$, il vient :

$$\Delta_B\ (W_n \cdot \frac{1}{r_o}) = (\Delta_{[K:\underline{t}_i,\bar{t}_i;\ \{0\}:0,\bar{t}_o]} W_n)(\frac{1}{r_o}(\underline{t}_o) - \frac{1}{r_o}(\bar{t}_o)) +$$

$$(\Delta_B\ W_n)\ \frac{1}{r_o}\ (\bar{t}_o)$$

d'où :

$$E(\Delta_B(W_n \cdot \frac{1}{r_o}))^4 \leq 2^3\ (\Delta_{[K:\underline{t}_i,\bar{t}_i;\ \{0\}:0,\bar{t}_o]} H^*)^{1+\gamma} (\frac{1}{r_o}(\underline{t}_o) - \frac{1}{r_o}(\bar{t}_o))^4 +$$

$$2^3\ (\Delta_B H^*)^{1+\gamma}\ (\frac{1}{r_o}(\bar{t}_o))^4.$$

Il résulte de la croissance de r_o sur $[0,\theta_o]$ que

$$\left|\frac{1}{r_o}(\underline{t}_o) - \frac{1}{r_o}(\bar{t}_o)\right|^{1+\beta} \leq \left|\frac{1}{r_o}(\underline{t}_o) - \frac{1}{r_o}(\bar{t}_o)\right| \frac{1}{r_o^{\beta}}(\underline{t}_o)$$

d'où

$$E\ (\Delta_B(W_n \cdot \frac{1}{r_o}))^4 \leq 2^3 (\frac{\Delta_{[K:\underline{t}_i,\bar{t}_i;\ \{0\}:0,\bar{t}_o]} H^*}{r_o^{\beta}\ (\underline{t}_o)} \left|\frac{1}{r_o}(\bar{t}_o) - \frac{1}{r_o}(\underline{t}_o)\right|)^{1+\gamma} +$$

$$2^3\ (\Delta_B H^*\ \frac{1}{r_o^{1+\beta}} (\bar{t}_o))^{1+\gamma}\ .$$

En vertu de l'hypothèse (2-3)a) on aura alors :

$$\frac{\Delta_{[K:\underline{t}_i,\bar{t}_i;\ \{0\}:0,\underline{t}_o]} H^*}{r_o^{\beta}\ (\underline{t}_o)} \left| \frac{1}{r_o}\ (\bar{t}_o) - \frac{1}{r_o}\ (\underline{t}_o) \right|$$

$$\leq \int_{\underline{t}_o}^{\overline{t}_o} |\Delta_{[K:\underline{t}_i,\overline{t}_i]}\tilde{H}^* \frac{u_o}{r_o^{\beta}(u_o)} \partial(\frac{1}{r_o})(u_o)| \, du_o$$

$$= \left[\int_{\underline{t}_o}^{\overline{t}_o} |\frac{1}{r_o^{\beta}} \partial(\frac{1}{r_o})| du_o \right] \cdot \Delta_{[K:\underline{t}_i,\overline{t}_i]}\tilde{H}^*$$

$$= \int_{\underline{t}_o}^{\overline{t}_o} |\frac{1}{r_o^{\beta}} \partial(\frac{1}{r_o}) \, du_o \cdot \int_{[K:\underline{t}_i,\overline{t}_i]} (\partial_K \tilde{H}^*) d\lambda^k$$

On a de même :

$$\Delta_B H^* \frac{1}{r_o^{1+\beta}} (\overline{t}_o)$$

$$\leq (\int_{\underline{t}_o}^{\overline{t}_o} (\frac{1}{r_o^{1+\beta}}) \, du_o) \cdot \int_{[K:\underline{t}_i,\overline{t}_i]} (\partial_K \tilde{H}^*) d\lambda^k$$

d'où :

$$E(\Delta_B(W_n \cdot \frac{1}{r_o}))^4 \leq 2^3 (\int_{\underline{t}_o}^{\overline{t}_o} |\frac{1}{r_o^{\beta}} \partial(\frac{1}{r_o})| du_o \int_{[K:\underline{t}_i,\overline{t}_i]} (\partial_K \tilde{H}^*) \, d\lambda^k)^{1+\gamma} +$$

$$2^3 (\int_{\underline{t}_o}^{\overline{t}_o} \frac{1}{r_o^{1+\beta}} \, du_o \cdot \int_{[K:\underline{t}_i,\overline{t}_i]} (\partial_K \tilde{H}^*) \, d\lambda^k)^{1+\gamma}$$

$$\leq K \, \nu'^{1+\gamma}(B)$$

(avec $K = 2^3$) (fin de la démonstration du lemme)

— On définit un quadrillage de $C_{\theta,o}$ $(\theta < \theta_o)$ de base :

$B_{\theta_o,o}(n) = ((t_j^{o} \, (n,\theta) \; ; \; 0 \leq \ell \leq L_j \; : \; 0 \leq j \leq k)$ telle que :

$t_j^o\ (n,\theta) = 0\ , \theta \leq j \leq k$

$t_o^\ell\ (n,\theta) = \frac{\ell}{n}\ , L_o = [n\theta] + 1$

$t_j^{L_j}(n,\theta) = 1$ si $j \neq 0$

$F_j(t_j^{\ell+1}(n,\theta)) - F_j(t_j^\ell(n,\theta))$ (où $j \neq 0$) est indépendant de ℓ et vérifie $(\frac{1}{n})^{1+\alpha} < F_j(t_j^{\ell+1}\ (n,\theta)) - F_j(t_j^\ell(n,\theta)) < (\frac{1}{n})^{1+\alpha'}$

(où $0 \leq \alpha' < \alpha$)

Lemme 10.

Sous les conditions du théorème _, il existe K' > 0 tel que pour tout $\theta (< \theta_o)$ et tout $\varepsilon > 0$ on ait :

$$P_n\ [\underset{t\in B_{\theta,o}(n)}{\mathrm{Sup}}\ |(W_n.\frac{1}{r_o})(t)|\ > \varepsilon]\ \frac{K'}{\varepsilon^4}\ \nu'^{1+\gamma}(C_{2\theta,o})$$

Ce lemme se déduit du lemme 9 comme, ci-dessus, le lemme 6 du lemme 5.

On déduit alors du lemme 10, par une méthode tout à fait analogue à celle permettant d'établir ci-dessus le lemme 7 et en remarquant que, si $t_o < \frac{1}{n}$, $W_n(t) = 0$, la propriété (3) énoncée.

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WEAK CONVERGENCE OF WEIGHTED MULTIVARIATE EMPIRICAL PROCESSES UNDER MIXING CONDITIONS

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ABSTRACT. Harel (1980) established the weak convergence of the multivariate truncated empirical processes under φ-mixing conditions for weight functions which vanish on the lower boundary of $[0,1]^{k+1}$. In this paper we extend the results under strong mixing conditions and also when the weight functions not only vanish on the lower boundary of $[0,1]^{k+1}$ but also on the upper corner of $[0,1]^{k+1}$.

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1. INTRODUCTION

Let $\underset{\sim}{X}_{ni} = (X_{ni}^{(1)},\ldots,X_{ni}^{(k)})$, $1\leq i\leq n$, $n\geq 1$ be $\mathbb{R}^k$-valued random variables with continuous distribution functions $F_{ni}(\underset{\sim}{x})$, $\underset{\sim}{x}\in\mathbb{R}^k$ and continuous marginal distribution functions $F_{ni}^{(j)}(x)$, $x\in\mathbb{R}$, $1\leq i\leq n$, $n\geq 1$, $1\leq j\leq k$. We are interested in the asymptotic behavior of the truncated empirical process W_n defined by

$$(1.1)\qquad W_n(t_o,\underset{\sim}{t}) = n^{-\frac{1}{2}} \sum_{i=1}^{[nt_o]} \left\{ \prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)})\leq t_i]} - H_{ni}(\underset{\sim}{t}) \right\}$$

for all $t_o\in[0,1]$ and $\underset{\sim}{t} = (t_1,\ldots,t_k) \in [0,1]^k$ where $[nt_o]$ is the largest integer $\leq nt_o$, $F_n^{(j)}(x) = n^{-1}\sum_{i=1}^{n} F_{ni}^{(j)}(x)$, and H_{ni} is the measure on $[0,1]^k$ defined by

$$(1.2)\qquad H_{ni}(\underset{\sim}{t}) = F_{ni}(F_n^{(1)^{-1}}(t_1),\ldots,F_n^{(k)^{-1}}(t_k)) .$$

In section 2, we introduce spaces D_{k+1} and C_{k+1} and suitable weight function r . Section 3 deals with some basic tools and the weak convergence of the truncated empirical process W_n defined in (1.1). In section 4, we study the weak convergence of the weighted truncated process $W_n \cdot \frac{1}{r}$ with respect to the Skorohod topology where r is a continuous function from $[0,1]^{k+1}$ into $\mathbb{R}^+$ (called the weight function). The convergence properties are studied when the sequence $\{\underset{\sim}{X}_{ni}\}$ is

$$(1.3)\qquad \varphi\text{-mixing with rates } \varphi(m) = O(m^{-1-\epsilon}),\ \epsilon>0$$

or

(1.4) strong-mixing with rates $\sum_{m=1}^{\infty} m^{2(k+1)}\alpha^{\epsilon}(m) < \infty$ for some $\epsilon \in (0,1/(2k+4))$.

Recall that $\{X_{ni}\}$ is φ-mixing if $\sup\{|P(B|A)-P(B)|;\ A \in \sigma(X_{ni},1\le i\le j),\ B \in \sigma(X_{ni},i\ge j+m)\} = \varphi(m)\downarrow 0$ for positive integers j and m ; and it is strong mixing if $\sup\{|P(A\cap B)-P(A).P(B)|;\ A \in \sigma(X_{ni},1\le i\le j),\ B \in\sigma(X_{ni},i\ge j+m)\} = \alpha(m)\downarrow 0$ for positive integers j and m . Here $\sigma(X_{ni},i\le j)$ and $\sigma(X_{ni},i\ge j+m)$ are the σ-fields generated by $(X_{n1},\dots,X_{nj})$ and $(X_{n,j+m},X_{n,j+m+1},\dots)$ respectively .

Our results are the extensions of the results of Fears and Mehra (1974) and Ahmad and Lin (1978) who considered the stationary univariate empirical processes under φ-mixing. Einmahl, Ruymgaart and Wellner (1984) studied the weak convergence of the weighted multivariate empirical processes when the underlying random variables are independent and the class of weight functions is sufficiently broad. However, their approach based on exploiting the representation of an empirical process as a conditioned Poisson process cannot be used in the context of this paper. Our results carry the approach of Fears and Mehra (1974) to the nonstationary as well as multivariate case.

2. PRELIMINARIES

2.1a. The D_{k+1} and C_{k+1} spaces

Let $f : [0,1]^{k+1} \to \mathbb{R}$. For $\rho \in \{0,1\}^{k+1}$, define

$$f_\rho(t) = \lim_{\substack{s_i\uparrow t_i,\rho(i)=1 \\ s_i\downarrow t_i,\rho(i)=0}} f(s) \quad ((s,t)\in[0,1]^{k+1})$$

, if it exists; in which case, call $f_\rho(t)$ the ρ-limit of f at t . Denote by D_{k+1} , the space of all maps $f : [0,1]^{k+1} \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^{k+1}$, f_ρ exists and $f_\rho = f$ for $\rho = (0,\dots,0)$. Denote by C_{k+1} , the space of all continuous maps $f : [0,1]^{k+1} \to \mathbb{R}$, and note that for any bounded function f, $f \in C_{k+1}$ if and only if $\lim_{\delta\to 0} \omega(f,\delta) = 0$ where

(2.1) $\omega(f,\delta) = \sup\{|f(t)-f(t')|;(t,t')\in([0,1]^{k+1})^2, \|t-t'\| \le \delta\}$ and $\|t\| = \sup\{|t_j|, 0\le j\le k\}$.

2.1b. Weight function

A function $r : [0,1]^{k+1} \to \mathbb{R}^+$ is called a weight function if it satisfies the following conditions:
(i) There exists an $r_o : [0,1] \to \mathbb{R}^+$ and $\underset{\sim}{r} : [0,1]^k \to \mathbb{R}^+$ such that $r(t) = r_o(t_o)\underset{\sim}{r}(\underset{\sim}{t})$ for all $t = (t_o,\underset{\sim}{t})\in[0,1]^{k+1}$,
(ii) r belongs to C_{k+1} ,
(iii) $r_o=0$ if $t_o=0$; $\underset{\sim}{r}=0$ if there exists at least one

$$j\in\{1,\ldots,k\} \text{ such that } t_j=0 \text{ or } \underset{\sim}{t} = (1,\ldots,1) .$$

2.1c. Grids accompanying a sequence of probability measures

A grid T of $[0,1]^{k+1}$ is a subset of $[0,1]^{k+1}$ such that $T = \prod_{j=0}^{k} T^{(j)}$ where $T^{(j)}$ is a finite subset of $[0,1]$ which includes 0 and 1 .

A pace τ of a grid $T = \prod_{j=0}^{k} T^{(j)}$ is the number $\tau = \max_{0\le j\le k} \tau_j$ where $\tau_j = \max\{|t_j'-t_j|,\ t_j'$ and t_j are successive elements in $T^{(j)}\}$.

A subpace $\tilde{\tau}$ of a grid $T = \prod_{j=0}^{k} T^{(j)}$ is the number $\tilde{\tau} = \inf_{0\le j\le k} \tilde{\tau}_j$ where $\tilde{\tau}_j = \inf\{|t_j'-t_j|;\ t_j'$ and t_j are successive elements in $T^{(j)}\}$.

We denote the lower boundary of T by $\underset{\leftarrow}{T}$ where

$$\underset{\leftarrow}{T} = \bigcup_{j=0}^{k} \Big[\prod_{\ell=0}^{j-1} T^{(\ell)} \times \{0\} \times \prod_{\ell=j+1}^{k} T^{(\ell)}\Big]$$

We call block B of T any rectangle of $[0,1]^{k+1}$ which is of the form

$$B = \prod_{j=0}^{k} \{[t_j, t'_j] \text{ where } t_j \text{ and } t'_j \text{ belong to } T^{(j)}, \text{ and } t_j < t'_j\}.$$

We call <u>evaluation</u> $e^{(B)}$ of B, the operator $e^{(B)}$: $D_{k+1} \to \mathbb{R}^+$ such that

$$e^{(B)}(f) = \sum_{(\epsilon_0,\dots,\epsilon_k)\in\{0,1\}^{k+1}} (-1)^{\sum_{i=0}^{k}\epsilon_i} f[(1-\epsilon_0)t_0+\epsilon_0 t'_0, \dots$$

$$\dots, (1-\epsilon_k)t_k + \epsilon_k t'_k] .$$

For any $\delta>0$, set

$\omega_T(f,\delta) = \sup\{|f(t)-f(t')|; (t,t')\in T^2, \|t-t'\|<\delta\}$.

We say that a sequence $\{T_n\}_{n\in\mathbb{N}^*}$ of grids is asymptotically dense in $[0,1]^{k+1}$ if the pace τ_n of T_n satisfies $\lim_{n\to\infty} \tau_n = 0$; $\mathbb{N}^* = \mathbb{N}-\{0\}$, $\mathbb{N} = \{0,1,2,\dots\}$.

Let P_n, $n\in\mathbb{N}^*$ be a sequence of probability measures on $(D_{k+1}, \mathscr{D}_{k+1})$ where $\mathscr{D}_{k+1}$ is the σ-field generated by the Skorohod topology (on D_{k+1}). we say that the sequence $\{T_n\}$ of grids accompanies the measure P_n if and only if $\forall\ \epsilon>0$, $\exists\ \epsilon'>0$ and $\forall\ \delta\in[0,1/2)$, $\exists\ N_0\in\mathbb{N}^*$ such that

$P_n[\{f\in D_{k+1}; \omega(f,\delta)\geq\epsilon \text{ and } \omega_{T_n}(f,2\delta)<\epsilon'\}] = 0 \quad \forall\ n\geq N_0$.

For the ease of convenience, we state the following Propositions due to Balacheff and Dupont (1980) which will be used in the sequel. Proposition 2.1 is, however, a slight modification of a result of Neuhaus (1971).

<u>Proposition</u> 2.1. Let P_n, $n\in\mathbb{N}^*$ be probability measures on $(D_{k+1}, \mathscr{D}_{k+1})$ such that the following conditions are satisfied:

(2.2) $\Phi_U(P_n) \to$ weakly to some probability measure P_U on $\mathbb{R}^U$ where U is a finite subset of $[0,1]^{k+1}$. (Φ_U is the projection of D_{k+1} on $\mathbb{R}^U$).

(2.3) $\lim\limits_{\delta\to 0} \limsup\limits_{n\to\infty} P_n[f\in D_{k+1}; \omega(f,\delta)\geq\epsilon] = 0 \quad \forall\ \epsilon>0$

Then P_n converges weakly with respect to the Skorohod topology to some probability measure P, and $P(C_{k+1}) = 1$.

<u>Proposition</u> 2.2. Let υ be a positive finite measure on $[0,1]^{k+1}$ with continuous marginals. Let P_n, $n\in\mathbb{N}^*$ be a sequence of probability measures on $(D_{k+1},\mathcal{D}_{k+1})$ such that $\forall\ n\in\mathbb{N}^*$, $P_n[f\in D_{k+1}, f|\underline{[0,1]}^{k+1}=0] = 1$. Let T_n, $n\in\mathbb{N}^*$ be a sequence of grids asymptotically dense in $[0,1]^{k+1}$ and accompanying P_n. Furthermore suppose that for any block B_n of T_n,

(2.4) $P_n[f\in D_{k+1}; |e^{(B_n)}(f)|>\lambda] \leq \lambda^{-\gamma}(\upsilon(B_n))^{\beta}$, $\beta>1$ and $\gamma>0$.

Then, $\forall\ \epsilon>0$, $\exists$ a $\delta\in(0,1)$ and $N_o\in\mathbb{N}$ such that

(2.5) $P_n[f\in D_{k+1}; \omega(f,\delta)\geq\epsilon] < \epsilon \quad \forall\ n\geq N_o$.

3. CONVERGENCE OF THE TRUNCATED EMPIRICAL PROCESS

3.1. Some basic tools

To investigate the weak convergence of the truncated multivariate empirical process, we use some of the ideas of Balacheff and Dupont (1980) who studied the convergence of the unweighted empirical processes. We need two lemmas.

<u>Lemma</u> 3.1. Let the sequence $\{X_{ni}\}$ of $\mathbb{R}$-valued random variables centered at their expectations be φ-mixing with rates $\sum_{m\geq 1} m^{-1}\varphi^{1/2q}(m) < \infty$, $q\in\mathbb{N}^*$. Let N_n be the number of

indexes i $(1 \le i \le n)$ for which X_{ni} is not identically zero. Set $S_n = \sum_{i=1}^{n} X_{ni}$, and $\|X_{ni}\|_\ell = (\int |X_{ni}|^{2\ell} dP_n)^{1/2\ell}$. Then, there exists a constant $C_q(\varphi)$ depending only on q and φ such that

$$(3.1) \qquad E(S_n^{2q}) \le C_q(\varphi) \sum_{\ell=1}^{q} N_n^{q/\ell} (\sup_{1 \le i \le n} \|X_{ni}\|_\ell)^{2q} .$$

<u>Proof</u>. The proof is a slight modification of Theorem 2.1 of Neumann (1982) and is therefore omitted.

<u>Lemma</u> 3.2. Let the sequence $\{X_{ni}\}$ of $\mathbb{R}$-valued random variables centered at their expectations be strong mixing with rates $\sum_{m \ge 1} m^{2q-2} \alpha^{\varepsilon}(m) < \infty$, $q \ge 1$, $\varepsilon \in (0,1/2q)$, and $|X_{ni}| \le 1, 1 \le i \le n, n \ge 1$. Let N_n be the number of indexes i $(1 \le i \le n)$ for which X_{ni} is not identically zero. Set $S_n = \sum_{i=1}^{n} X_{ni}$ and $\|X_{ni}\|_\varepsilon = (\int |X_{ni}|^{2/(1-\varepsilon)})^{1-\varepsilon}$. Then, there exists a constant $C_q(\alpha)$ depending only on q and α such that

$$(3.2) \qquad E(S_n^{2q}) \le C_q(\alpha) \sum_{\ell=1}^{q} N_n^{\ell} (\sup_{1 \le i \le n} \|X_{ni}\|_\varepsilon^{\ell}) .$$

<u>Proof</u>. The proof is essentially the same as in Doukhan and Portal (1983) and is therefore omitted.

We shall say that the sequence $\{H_{ni}\}$ is μ-bounded if there exists a finite and positive measure μ on $[0,1]^k$ with continuous marginal distributions such that for every $n \ge 1$ and $1 \le i \le n$, $H_{ni}(B) \le \mu(B)$ for all rectangles B in $[0,1]^k$.

<u>Theorem</u> 3.1. Assume that the sequence $\{\underset{\sim}{X}_{ni}\}$ is (a) φ-mixing with rates (1.3) <u>or</u> (b) strong mixing with rates (1.4); the sequence $\{H_{ni}\}$ <u>is</u> (c) μ-bounded where μ is absolutely continuous with bounded density f_μ or (d) $\{H_{ni}\}$ has uniform marginals for all $n \ge 1$ and $1 \le i \le n$. Furthermore

assume that (e) the covariance function C_n of the empirical process W_n converge to a function C . Then, W_n converges weakly in the Skorohod topology to a Gaussian process W_o with trajectories a.s. in C_{k+1} .

Proof. To prove this theorem, we have to verify (2.2) and (2.3) for the probability measure Q_n on $(D_{k+1},\mathscr{D}_{k+1})$ defined by W_n . To verify (2.2) we have to show that the finite projections of the process W_n converge in law to a normal law; equivalently, we have to show that $\sum_{\ell=1}^{p} \lambda_\ell W_n(t^\ell)$ converges in law to a normal law for every $(\lambda_1,\ldots,\lambda_p) \in \mathbb{R}^p$, for every $(t^1,\ldots,t^p) \in [0,1]^{k+1}$ where $t^\ell = (t_0^\ell,\ldots,t_k^\ell)$, $1 \leq \ell \leq p$, $p \in N^*$. Without loss of generality, we assume that $t_0^1 <\ldots< t_0^p$. Now set

$$g_\ell^i = \prod_{j=1}^{k} I_{[F_{ni}^{(j)}(X_{ni}^{(j)}) \leq t_i^\ell} - H_{ni}(t_1^\ell,\ldots,t_k^\ell) \text{ , and}$$

$$Y_{ni} = \begin{cases} \sum_{\ell=1}^{p} \lambda_\ell \, g_\ell^i & \text{if } i \leq [nt_0^1] . \\ \sum_{\ell=m}^{p} \lambda_\ell \, g_\ell^i & \text{if } [nt_0^{m-1}] \leq i \leq [nt_0^m],\ 1 < m \leq p . \\ 0 & \text{if } i > [nt_0^p] . \end{cases}$$

Then, we can write

$$(3.3) \qquad \sum_{\ell=1}^{p} \lambda_\ell W_n(t^\ell) = n^{-1/2} \sum_{i=1}^{n} Y_{ni} .$$

Following Withers (1975, Cor 1) we have to show that

(i) $E\left[\sum_{i=1}^{n} Y_{ni}\right]^2/n$ converges to some constant,

(ii) $\sum_{n\geq 1} \alpha(n) < \infty$ and (iii) $n^{1-a}\alpha([n]^b) \longrightarrow 0$ (as $n \longrightarrow \infty$) where $0 < 2b < a < 1 - b$. Now in our situation (i) holds by (3.3) and assumption (e); (ii) follows from (1.3) and (1.4), and (iii) from (1.3) and (1.4) by taking $a = 3/4 - \epsilon/8$, $b = 1/4$ and ϵ sufficiently small (since taking $\alpha(n) = n^{-1-\epsilon}$, $n^{1-a}\alpha([n]^b) \leq An^{-\epsilon/8}$ where $A > 0$ is some constant). Thus (2.2) is proved.

To prove (2.3), we use Proposition 2.2 and verify (2.4) which will imply (2.5).

Let $T_n = \{i/n; 0\leq i\leq n\}^{k+1}$ be a sequence of grids asymptotically dense in $[0,1]^{k+1}$, and we prove that T_n accompanies Q_n. Now for every $\underset{\sim}{t} \in [0,1]^k$, let $(\underset{\approx}{t},\bar{\underset{\sim}{t}})$ be the points of T_n^P (projection of T_n from $[0,1]^{k+1}$ to $[0,1]^k$) such that $\underset{\approx}{t} \leq \underset{\sim}{t} \leq \bar{\underset{\sim}{t}}$ and $\|\bar{\underset{\sim}{t}}-\underset{\approx}{t}\| \leq 1/n$, and denote $\underline{t}_o = [nt_o]/n$ for every $t_o \in [0,1]$. Then, with the conditions (c) or (d) of Theorem 3.1, we obtain, after some computations

$|W_n(t_o,\underset{\sim}{t}) - W_n(t_o',\underset{\sim}{t}')| \leq \frac{2kK(\mu)}{\sqrt{n}} + |W_n(\underline{t}_o,\bar{\underset{\sim}{t}}) - W_n(\underline{t}_o',\underset{\approx}{t}')|$ for every $(t_o,\underset{\sim}{t})$ and $(t_o',\underset{\sim}{t}') \in [0,1]^{k+1}$ where $K(\mu) = \sup_{\underset{\sim}{t}\in[0,1]^k} f_\mu(\underset{\sim}{t})$ if we have (c), and $K(\mu)=1$ if we have (d). Consequently, for every $\delta\in(0,1/2]$, we have $\omega(W_n,\delta) \leq \frac{2kK(\mu)}{\sqrt{n}} + \omega_{T_n}(W_n,2\delta)$. It follows that T_n accompanies Q_n. It remains to show that Q_n satisfies (2.4).

Suppose we have condition (c). Let $\sum_{m=1}^{\infty} m^{-1}\varphi^{1/4}(m) < \infty$ (implied by (1.3)), and let B_n be a block of T_n as defined in 2.1c. Then, using Lemma 3.1, with $q=2$, we obtain

$$E[e^{(B_n)}(W_n)]^4 \le n^{-2}C_2(\varphi)[(nt_o-nt_o')^2K(\mu))^2(\prod_{j=1}^{k}(t_j-t_j')^2 + (nt_o-nt_o')K(\mu)\prod_{j=1}^{k}(t_j-t_j')] .$$

Let $\upsilon = (C_2(\varphi)(K(\mu)+K^2(\mu)))^{\beta^{-1}}U^{k+1}$ where U^{k+1} is the uniform probability measure on $[0,1]^{k+1}$ and $\beta = (k+2)/(k+1)$. Then, by the Markov inequality,

$$Q_n[f\in D_{k+1}; |e^{(B_n)}(f)| > \lambda] \le \lambda^{-4}(\upsilon(B_n))^{\beta}$$

which implies (2.4) for the φ-mixing case with rates (1.3). (2.5) follows.

For the strong mixing case with rates (1.4), we use Lemma 3.2 for $q=2$ and $\epsilon < 1/(2k+4)$, and obtain

$$E[e^{(B_n)}(W_n)]^4$$
$$\le \frac{1}{n^2} C_2(\alpha)[(nt_o-nt_o')^2\{K(\mu)\prod_{j=1}^{k}(t_j-t_j')\}^{2(1-\epsilon)} + + (nt_o-nt_o')\{K(\mu)\prod_{j=1}^{k}(t_j-t_j')\}^{1-\epsilon}]$$

which (with $\beta = \frac{(k+1)(1-\epsilon)+1}{k+1}$) implies (2.3) and hence (2.4) by proceeding as above.

Now let us suppose we have condition (d). Then, for the φ-mixing case with rates $\sum_{m=1}^{\infty} m^{-1}\varphi^{1/2(k+2)}(m) < \infty$ (implied by (1.3)), we use Lemma 3.1 with $q = k+2$, and obtain

$$E[e^{(B_n)}(W_n)]^{2(k+2)}$$
$$\le C_{k+2}(\varphi)\sum_{\ell=1}^{k+2} n^{-(k+2)}[(nt_o-nt_o')^{(k+2)/\ell} \times$$
$$\times (\prod_{j=1}^{k}(t_j-t_j'))]^{(k+2)/k\ell}$$

and now proceeding as in the φ-mixing case dealt with above, we get the desired result.

For the strong mixing case with rates (1.3), we use Lemma 3.2 with $q=k+2$ and $\epsilon < 1/2(k+2)$, and obtain

$$E[e^{(B_n)}(W_n)]^{2(k+2)} \le C_{k+2}(\alpha) \sum_{\ell=1}^{k+2} n^{-(k+2-\ell)}(t_o-t_o')^{\ell} \times \times \left(\prod_{j=1}^{k}(t_j-t_j')\right)^{\ell(1-\epsilon)/k} .$$

Now proceeding precisely as in the first φ-mixing case, we get the desired result. The proof follows.

4. CONVERGENCE OF THE WEIGHTED EMPIRICAL PROCESS

We start with a basic proposition given in Mehra and Rao (1975), and Harel (1980).

<u>Proposition</u> 4.1. Let Y_n, $n \in \mathbb{N}^*$ be a process with values in D_{k+1}, and suppose that Y_n converges in law (in Skorohod topology) to a Gaussian process Y_o with trajectories a.s. in C_{k+1}. Let the weight function r be such that $Y_n \cdot \frac{1}{r}$, $n \in \mathbb{N}$ has a.s. trajectories in D_{k+1}. Furthermore assume that $\forall\ \eta>0$, $\exists\ \theta>0$ and N_o such that $\forall\ n \ge N_o$.

$$(4.1) \qquad P_n[\sup\{|Y_n(t)\cdot \frac{1}{r(t)}| \ge \eta\}] \le \eta$$

where P_n is the law of Y_n, and sup is taken over $t = (t_o,t_1,\ldots,t_k)$ with the condition that there is at least one j, $0 \le j \le k$ such that $t_j \le \theta$ or $\prod_{j=1}^{k} t_j \ge 1-\theta$. <u>Then</u> $Y_n \cdot \frac{1}{r}$ converges weakly in Skorohod topology to the Gaussian process $Y_o \cdot \frac{1}{r}$ with trajectories a.s. in C_{k+1}.

We now prove our main theorem.

<u>Theorem</u> 4.1. <u>Let</u> $\{X_{ni}\}$ satisfy the assumptions of Theorem 3.1. Then, for any weight function $r : [0,1]^{k+1} \to \mathbb{R}^+$ satisfying

$$(4.2) \qquad r(t) \geq A\left[\prod_{j=0}^{k} t_j\left(1 - \prod_{j=1}^{k} t_j\right)\right]^{\frac{1}{2}-\delta}, \quad t\in[0,1]^{k+1}, A>0,$$
$$0 < \frac{1}{2} - \delta < \frac{1}{2(k+2)}$$

$W_n \cdot \frac{1}{r}$ converges weakly in Skorohod topology to $W_o \cdot \frac{1}{r}$ with trajectories a.s. in C_{k+1}.

<u>Proof</u>. From Proposition 4.1, it is clear that the proof will follow if we prove (4.1).

Denote

$$(4.3) \qquad C_\theta^{(1)} = \{t; t\in[0,1]^{k+1}, \exists \text{ at least one } j\in\{0,1,\dots,k\} \text{ such that } t_j \leq \theta\}$$

and

$$(4.4) \qquad C_\theta^{(2)} = \{t; t\in[0,1]^{k+1}. \prod_{j=1}^{k} t_j \geq 1-\theta\}.$$

Then (4.2) will follow if we show that

$$(4.5) \qquad P_n\left[\sup_{t\in C_\theta^{(i)}} \{|W_n(t) \cdot \frac{1}{r(t)}| \geq \eta\}\right] \leq \eta, \quad i = 1,2.$$

(By convention $\frac{1}{r(t)} = 0$ if $r(t) = 0$).

We first prove (4.5) for $i=1$. Without loss of generality, take $r(t) = (\prod_{j=1}^{k} t_j)^{1/2-\delta}$. Define a measure ν on $[0,1]^{k+1}$ by

$$(4.6) \qquad \nu(A) = \sum_{I\subset\{0,1,\dots,k\}} \int_A \left| \frac{1}{r^{\beta}(t)} \frac{\partial^{I}}{(\partial t_j)_{j\in I}} U^{k+1}(t) \frac{\partial^{I^c}}{(\partial t_j)_{j\in I^c}} \left(\frac{1}{r}\right)(t) \right| \times dt_o \dots dt_{k+1}$$

where A is any Borel subset of $[0,1]^{k+1}$, U^{k+1} is the distribution function of the uniform probability measure on $[0,1]^{k+1}$, $\beta>0$ is a constant, and I^c is the complement of I.

Let $\{T_n\}_{n\in\mathbb{N}^*}$ be a sequence of grids of $[0,1]^{k+1}$ with pace τ_n such that $\tau_n \le (\frac{1}{n})^{1+\alpha'}$; $\tilde{\tau}_n$ the subpace of the projection T_n^P of T_n such that $\tilde{\tau}_n \ge (\frac{1}{n})^{1+\alpha}$; $0<\alpha'<\alpha$, and $T_n^{(0)} = \{0,\frac{1}{n},\dots,\frac{n-1}{n},1\}$.

(a) We first consider the case when $\{X_{ni}\}$ is φ-mixing with rates (1.3) and $\{H_{ni}\}$ is μ-bounded (viz. conditions (a) and (c) of Theorem 3.1).

Then the proof of (4.5) will follow if we prove the following Lemmas.

<u>Lemma</u> 4.1. For any block B_n of T_n,

$$(4.7) \qquad E\left[e^{B_n}\left(\frac{W_n}{r}\right)\right]^4 \le K[\nu(B_n)]^{1+\gamma},$$

where $K>0$ is some constant, $\gamma = [k(1+\alpha)+1]^{-1}$ and β in (4.6) is chosen such that $(1+\beta)(1+\gamma) = 4$.

<u>Lemma</u> 4.2. $\exists$ $C>0$ such that $\forall$ $\theta>0$ and $\forall$ $\eta>0$,

$$(4.8) \qquad P_n\left[\sup_{t\in C_{\theta_n}^{(1)}\cap T_n} \{|W_n(t)\cdot\frac{1}{r(t)}| > \eta\}\right] < C\eta^{-4}\nu(C_{2\theta}^{(1)})$$

where $\theta_n = \theta + n^{-1-\alpha'}$.

Let $C_\theta^{(1)}(n)$ be a subset of $C_\theta^{(1)}$ defined by

(4.9) $C_\theta^{(1)}(n)$

$$= \{t; t\in C_\theta^{(1)}; \forall j\in\{0,\dots,k\}, t_j \geq \inf\{t_j'; t_j'\in T_n^{(j)}, t_j'\neq 0\}\} .$$

<u>Lemma</u> 4.3. $\exists\ C_1$ such that $\forall\ \theta>0$ and $\forall\ \eta>0\ \exists\ N_o(\theta,\eta)$ such that

$$(4.10)\qquad P_n\left[\sup_{t\in C_\theta^{(1)}(n)}\{|W_n(t)\cdot \tfrac{1}{r}(t)|>\eta\}\right] < C_1\eta^{-4}\nu(C_{2\theta}^{(1)})\ \ \forall\ n \geq N_o(\theta,\eta)$$

<u>Lemma</u> 4.4. $\forall\ \theta>0$ and $\forall\ \eta>0$, $\exists\ N_1(\theta,\eta)$ such that

$$(4.11)\qquad P_n\left[\sup_{t\in C_\theta^{(1)}-C_\theta^{(1)}(n)}\{|W_n(t)\cdot \tfrac{1}{r}(t)| > \eta\}\right] < \eta\quad \forall\ n \geq N_1(\theta,\eta) .$$

<u>Proof</u> <u>of</u> <u>Lemma</u> 4.1. Using Lemma 3.1. with $q=2$, we obtain

$$(4.12)\qquad E[e^{B_n}(W_n)]^4 \leq K_1[e^{B_n}(U^{k+1})]^{1+\gamma}$$

where $K_1>0$ is some constant. Let $J = \{j\in\{0,\dots,k\}; t_j>0\}$, and for any $I \subset J$, associate to B_n two other blocks $B_n(I)$ and $B_n'(I)$ defined as

$$B_n(I) = \prod_{j\in I}[t_j,t_j']\prod_{j\in J-I}[0,t_j]\prod_{j\notin J}[0,t_j']$$

and

$$B_n'(I) = \prod_{j\in I}[0,t_j']\prod_{j\in J-I}[t_j,t_j']\prod_{j\notin J}[0,t_j'] .$$

Then we have

$$e^{B_n}(W_n\cdot \tfrac{1}{r}) = \sum_{I\subset J} e^{B_n(I)}(W_n)\ e^{B_n'(I)}(\tfrac{1}{r}) .$$

Consequently,

(4.13) $$E[e^{B_n}(W_n \cdot \tfrac{1}{r})]^4 \leq K_2 \sum_{I \subseteq J} Y_I$$ where K_2 is some constant, where

(4.14) $$Y_I = \left[[e^{B_n(I)}(U^{k+1})] \, |e^{B'_n(I)}(\tfrac{1}{r})|^{1+\beta} \right]^{1+\gamma} .$$

As U^{k+1} and r are nondecreasing, we obtain after some computations (along the lines of Harel (1980, Lemma 5)) that

(4.15) $$(e^{B_n(I)}(U^{k+1}))\,|e^{B'_n(I)}(\tfrac{1}{r})|^{1+\beta}$$
$$\leq K_3 \int_{B_n} \Big| \frac{1}{r^{\beta}(t)} \frac{\partial^{I\cup J^c}}{(\partial t_j)_{j\in I\cup J^c}} U^{k+1}(t) \times$$
$$\times \frac{\partial^{J-I}}{(\partial t_j)_{j\in J-I}} (\tfrac{1}{r}(t)) \Big| \, dt_o \ldots dt_k .$$

where $K_3 > 0$ is some constant. Using (4.15) in (4.14) we obtain

(4.16) $$Y_I \leq$$
$$\left[K_3 \int_{B_n} \Big| \frac{1}{r^{\beta}(t)} \frac{\partial^{I\cup J^c}}{(\partial t_j)_{j\in I\cup J^c}} U^{k+1}(t) \, \frac{\partial^{J-I}}{(\partial t_j)_{j\in J-I}} (\tfrac{1}{r})(t)) \Big| \times dt_o \ldots dt_k \right]^{1+\gamma}$$

Substituting (4.16) in (4.13) we obtain (4.7).

Proof of Lemma 4.2. This follows from the Markov inequality, Lemma 1 of Balacheff and Dupont (1980) and (4.7).

Proof of Lemma 4.3. Let η, θ and n be fixed, and let $t \in C_\theta^{(1)}(n)$. For any $j \in \{0,\ldots,k\}$, let $\underline{t}_j$ and $\overline{t}_j$ be elements of T_n^j such that $\underline{t}_j \leq t_j \leq \overline{t}_j$ where $\underline{t}_j$ and $\overline{t}_j$ are successive elements of T_n . For $I \subset \{1,\ldots,k\}$, let $B_n''(I)$ be the rectangle in $C_\theta^{(1)}$ defined by

$$B''(I) = [0,t_o] \prod_{j\in I} [t_j,\overline{t}_j] \prod_{\substack{j\notin I \\ j\neq 0}} [0,\overline{t}_j] \ .$$

Then, we have an inequality

$$(4.17) \qquad |(W_n\cdot\tfrac{1}{r})(t)| \leq \sum_{I\subset\{1,\ldots,k\}} |e^{B_n''(I)}(W_n)|\,|\tfrac{1}{r}(\underline{t}_o,\underset{\sim}{\underline{t}}| \ .$$

If $I=\phi$, then

$$(4.18) \qquad |e^{B_n''(I)}(W_n)|\ \tfrac{1}{r}(\underline{t}_o,\underset{\sim}{\underline{t}}) \leq 2^{k+1}|W_n\cdot\tfrac{1}{r}(\underline{t}_o,\underset{\sim}{\overline{t}}|$$

where $\underset{\sim}{\overline{t}} = (\overline{t}_1,\ldots,\overline{t}_k)$, and $(\underline{t}_o,\underset{\sim}{\overline{t}}) \in T_n \cap C_{\theta_n}^{(1)}$.

If $I\neq\phi$, then we show that

$$(4.19) \qquad |e^{B_n''(I)}(W_n)|\cdot\tfrac{1}{r}(\underline{t}_o,\underset{\sim}{\underline{t}}) \leq K_4 \sup_{r\in C_{\theta_n}^{(1)}\cap T_n} |(W_n\cdot\tfrac{1}{r})(s)| + $$
$$+ o(n^{-\alpha'})$$

where $K_4>0$ is some constant. (4.19) follows because of the following fact:

For any rectangles B_1 and B_2 in $[0,1]^{k+1}$ where $B_2 \subset B_1$, we have

$$(4.20) \qquad |e^{B_2}(W_n)| \leq |e^{B_1}(W_n)| + n^{1/2}\ e^{B_1}(UH)$$

where U is the uniform distribution over $[0,1]$ and H is the distribution corresponding to measure μ .

From (4.15),(4.16) and (4.17), we obtain

(4.21)

$$\sup_{t\in C_\theta^{(1)}(n)} \{|W_n(t)\cdot\tfrac{1}{r}(t)|\} \le K_5 \sup_{s\in C_{\theta_n}^{(1)}\cap T_n} \{|W_n(s)\cdot\tfrac{1}{r}(s)|\} + $$
$$+ o(n^{-\alpha'}) ,$$

where K_5 is some constant. Using Lemma 4.2 and (4.21) we get (4.10).

Proof of Lemma 4.4. Set

$C(n,j) = \{t; t\in[0,1]^{k+1},\ t_j < \text{Inf}\ [t_j'; t_j'\in T_n^{(j)}, t_j'\neq 0]\}$,

$j=0,1,\ldots,k$. We have

$$P_n\left[\sup_{t\in C_\theta^{(1)}-C_\theta^{(1)}(n)} \{|W_n(t)\cdot\tfrac{1}{r}(t)| > \eta\}\right]$$
$$\le \sum_{j=0}^{k} P_n\left[\sup_{t\in C(n,j)} \{|W_n(t)\cdot\tfrac{1}{r}(t)| > \tfrac{\eta}{k+1}\}\right] .$$

If $j=0$, then $\forall\ t\in C(n,0)$, we have $W_n(t) = 0$. If $j\neq 0$, then

$$P_n\left[\sup_{t\in C(n,j)} \{|W_n(t)\cdot\tfrac{1}{r}(t)| \le \tfrac{\eta}{k+1}\right]$$
$$\ge P_n\left[\left[\sup_{t\in C(n,j)} \{|W_n(t)\cdot\tfrac{1}{r}(t)| \le \tfrac{\eta}{k+1}\}\right]\right.$$
$$\left.\cap \left[(F_n^{(1)}(X_{ni}^{(1)}),\ldots,F_n^{(k)}(X_{ni}^{(k)}))\in C^c(n,j),\ \forall\ i\in\{1,\ldots,n\}\right]\right]$$
$$= P_n\left[\left[\sup_{t\in C(n,j)} \{n^{-1/2}\sum_{i=1}^{[nt_o]} \frac{H_{ni}(\underset{\sim}{t})}{r(t)} \le \tfrac{\eta}{k+1}\}\right]\right.$$
$$\left.\cap \left[F_n^{(1)}(X_{ni}^{(1)}),\ldots,F_n^{(k)}(X_{ni}^{(k)}))\in C^c(n,j)\ \forall\ i\in\{1,\ldots,n\}\right]\right]$$

$$= P_n\left[(F_n^{(1)}(X_{ni}^{(1)}),\ldots,(F_n^k(X_{ni}^{(k)}))\in C^c(n,j),\ \forall\ i\in\{1,\ldots,n\}\right]$$

for sufficiently large n

$$\geq 1 - \sum_{i=1}^{n} F_{ni}^{(j)} \circ F_n^{(j)^{-1}}(n^{-1-\alpha'}) \geq 1 - n^{-\alpha'} .$$

It follows that (for sufficiently large n)

$$P_n\left[\sup_{C_\theta^{(1)}-C_\theta^{(1)}(n)} \{|W_n(t)\cdot\tfrac{1}{r}(t)| > \eta\}\right] < \eta .$$

This proves Lemma 4.4.

Now from (4.10) and (4.11), we obtain (4.5) for $i=1$ when conditions (a) and (c) of Theorem 3.1 are satisfied.

(b) We now consider the case when $\{X_{ni}\}$ is strong mixing with rates (1.4) and $\{H_{ni}\}$ is μ-bounded (viz. conditions (b) and (c) of Theorem 3.1).

To prove the result for this case, we use Lemma 3.2 with $q=2$ and obtain the inequality (4.7) for $\gamma = [k(1+\alpha)+1]^{-1} - \epsilon$, $\epsilon < [2(k+2)]^{-1}$ and β satisfying $(1+\beta)(1+\gamma) = 4$. (Note for α sufficiently small, ν is a finite measure, and $\gamma>0$). Proceeding precisely as in (a), we obtain for this case also the results of Lemmas 4.2, 4.3 and 4.4. Consequently, we get (4.5) with $i=1$.

(c) Let now $\{X_{ni}\}$ be φ-mixing with rates (1.3) and let $\{H_{ni}\}$ have uniform marginals for all $n\geq 1$ and $1\leq i\leq n$ (viz. conditions (a) and (d) of Theorem 3.1).

To prove the result for this case, we use Lemma 3.1 with $q=k+2$ and we obtain the following variant of the inequality (4.7).

$$E[e^{B_n}(W_n\cdot\tfrac{1}{r})]^{2(k+2)} \leq K_6[\nu(B_n)]^{1+\gamma} \tag{4.22}$$

where $K_6>0$ is some constant, $\gamma = [k(1+\alpha)+1]^{-1}$, β is such that $(1+\beta)(1+\gamma) = 2(k+2)$. (For α sufficiently small, ν is a finite measure). The proofs follow proceeding as in (a).

(d) Finally, let $\{X_{ni}\}$ be strong mixing with rates (1.4) and let $\{H_{ni}\}$ have uniform marginals for all $n\geq 1$ and $1\leq i\leq n$ (viz. conditions (b) and (d) of Theorem 3.1).

Here we use Lemma 3.2 with $q=k+2$ and get the inequality (4.25) where $\gamma = [k(1+\alpha)+1]^{-1} - \epsilon$, $\epsilon < [2(k+2)]^{-1}$ and β is such that $(1+\beta)(1+\gamma) = 2(k+2)$. The proof follows proceeding as in (a).

We now prove (4.5) for $i=2$.

Without loss of generality, let us take

$$(4.23) \qquad r(t) = [t_o(1 - \prod_{j=1}^{k} t_j)]^{1/2 - \delta} .$$

Following the ideas of Einmahl, Ruymgaart and Wellner (1984), we start with the equality

$$n^{-1/2} \sum_{i=1}^{[nt_o]} (I_{[(F_n^{(1)}(X_{ni}^{(1)}),\ldots,F_n^{(k)}(X_{ni}^{(k)}))\in B]} - H_{ni}(B))$$

$$= -n^{-1/2} \sum_{i=1}^{[nt_o]} (I_{[(F_n^{(1)}(X_{ni}^{(1)}),\ldots,F_n^{(k)}(X_{ni}^{(k)}))\in B^c]} - H_{ni}(B^c))$$

and then using the union-intersection principle (see Einmahl, Ruymgaart and Wellner (1984)), we obtain

$$(4.24) \qquad |W_n(t)| \leq \sum_{\ell\in L} |e^{\bar{R}_\ell(t)}(W_n)| , \quad t\in[0,1]^{k+1}$$

where L is a finite index set, $R_\ell(t)$ one semi-open rectangle and $\bar{R}_\ell(t)$ is the closure of $R_\ell(t)$. (By convention $e^{\bar{R}_\ell(t)}(W_n) = 0$ if $R_\ell(t) = \phi$). Using (4.23) and (4.24), we get

$$(4.25) \qquad \sup_{t\in C_\theta^{(2)}} \frac{|W_n(t)|}{r(t)} \leq \sum_{\ell\in L} \sup_{t\in C_\theta^{(2)}} \frac{|e^{\bar{R}_\ell(t)} W_n|}{(U^{k+1}(R_\ell(t)))^{1/2 - \delta}} .$$

Now let ℓ be fixed. Then there exists a $J \subset \{1,\dots,k\}$, $J \neq \phi$ such that $R_\ell(t) = (0,t_o] \times \prod_{j \in J} (t_j,1] \times (0,1]^{k-q}$ where q is the cardinal of J. For the convenience of simplicity, take $J = \{1,\dots,q\}$. Set $X_{ni}^{*(j)} = -X_{ni}^{(j)}$, and let F_{ni}^* be the distribution function of $(X_{ni}^{*(1)},\dots,X_{ni}^{*(q)})$, $F_{ni}^{*(j)}$ the marginal distribution function of $X_{ni}^{*(j)}$, and let $F_n^{*(j)} = n^{-1} \sum_{i=1}^n F_{ni}^{*(j)}$. Let H_{ni}^* be the measure on $[0,1]^q$ defined as $H_{ni}^*(t_1,\dots,t_q) = F_{ni}^*(F_{ni}^{*(1)^{-1}}(t_1),\dots,F_{ni}^{*(q)^{-1}}(t_q))$, and let $R_\ell^P(t)$ denote the projection of $R_\ell(t)$ from $[0,1]^{k+1}$ for $[0,1]^k$. Then, if $R_\ell(t) \neq \phi$, we get

$$\frac{e^{\overline{R}_\ell(t)}(W_n)}{(U^{k+1}(R_\ell(t)))^{\frac{1}{2}-\delta}}$$

$$= \left[n^{-1/2} \sum_{i=1}^{[nt_o]} \left(\sum_{j=1}^{q} I_{\{F_n^{(j)}(X_{ni}^{(j)})>t_j]} - H_{ni}(R_\ell^P(t)) \right] \times$$

$$\times \left(t_o \prod_{j=1}^{q} (1-t_j)\right)^{-1/2+\delta}$$

$$= \left[n^{-1/2} \sum_{i=1}^{[nt_o]} \left(\prod_{j=1}^{q} I_{[F_n^{(j)}(X_{ni}^{(j)}) \le 1-t_j]} - H_{ni}^*(1-t_1,\dots,1-t_q) \right) \right] \left(t_o \prod_{j=1}^{q} (1-t_j)\right)^{-1/2+\delta}$$

which is the empirical process (1.1) weighted by $(t_o \prod_{j=1}^{q} (1-t_j))^{1/2-\delta}$ for $k=q$, and associated with the random variable $(X_{ni}^{*(1)},\dots,X_{ni}^{*(q)})$, $1 \le i \le n;\ n \ge 1$, and t_j replaced by $1-t_j$, $j \neq 0$. Now proceeding exactly as in the proof of (4.5) for $i=1$ we get (4.5) for $i=2$.

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Statistique/*Statistics*

Convergence faible du processus empirique multidimensionnel corrigé en condition de mélange

Michel HAREL et Madan PURI

Résumé — On a établi dans [6] la convergence faible du processus empirique tronqué multidimensionnel en condition de φ mélange pour des fonctions correctrices s'annulant sur la frontière inférieure de $[0,1]^{k+1}$. Dans cette Note, nous étendons les résultats en mélange fort et aussi quand les fonctions correctrices s'annulent non seulement sur la frontière inférieure de $[0,1]^{k+1}$ mais aussi sur le coin supérieur de $[0,1]^{k+1}$.

Weak convergence of weighted multivariate empirical processes under mixing conditions

Abstract — *We established in* [6] *the weak convergence of the multivariate truncated empirical processes under* φ*-mixing conditions for weight functions which vanish on the lower boundary of* $[0,1]^{k+1}$. *In this paper we extend the results under strong mixing conditions and also when the weight functions not only vanish on the lower boundary of* $[0,1]^{k+1}$ *but also on the upper corner of* $[0,1]^{k+1}$.

1. INTRODUCTION. — Soient $\mathbf{X}_{ni}=(X_{ni}^{(1)}, \ldots, X_{ni}^{(k)})$, $1\leqq i\leqq n$, $n\geqq 1$ des variables aléatoires à valeurs dans $\mathbb{R}^k$ avec des fonctions de répartition continues $F_{ni}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^k$ et des fonctions de répartition marginales continues $F_{ni}^{(j)}(x)$, $x\in\mathbb{R}$, $1\leqq i\leqq n$, $1\leqq j\leqq k$. Nous nous intéressons au comportement asymptotique du processus empirique tronqué $\mathbf{W}_n$ défini par :

$$\mathbf{W}_n(t_0, \mathbf{t})=n^{-1/2}\sum_{i=1}^{[nt_0]}\left\{\sum_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)})\leqq t_j]}-H_{ni}(\mathbf{t})\right\} \tag{1}$$

pour tout $t_0\in[0,1]$ et $t=(t_1, \ldots, t_k)\in[0,1]^k$ où $[nt_0]$ est le plus grand entier $\leqq nt_0$, $F_n^{(j)}(x)=n^{-1}\sum_{j=1}^{n} F_{ni}^{(j)}(x)$ et H_{ni} est la mesure définie sur $[0,1]^k$ par

$$H_{ni}(\mathbf{t})=F_{ni}(F_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)^{-1}}(t_k)). \tag{2}$$

Les propriétés de convergence sont étudiées quand la suite $\{\mathbf{X}_{ni}\}$ est

(3) φ mélangeante avec le taux $\varphi(m)=O(m^{-1-\varepsilon})$, $\varepsilon>0$

ou

(4) fortement mélangeante avec le taux $\sum_{m=1}^{\infty} m^{2(k+1)}\alpha^{\varepsilon}(m)<\infty$ pour un certain $\varepsilon\in]0, 1/(2k+4)[$.

Ces résultats sont une extension des résultats de Fears et Mehra [5] et Ahmad et Lin [1] qui considéraient le processus empirique unidimensionnel stationnaire et φ-mélangeant. Einmahl, Ruymgaart et Wellner [4] ont étudié la convergence faible du processus empirique multidimensionnel corrigé lorsque les variables aléatoires sont indépendantes.

2. DÉFINITIONS ET NOTATIONS. — 2.1. *Les espaces* D_{k+1} *et* C_{k+1}.

Soit f: $[0,1]^{k+1}\to\mathbb{R}$. Pour $\rho\in\{0,1\}^{k+1}$, on définit

$$f_\rho(t)=\lim_{\substack{s_i\uparrow t_i\ \rho(i)=1\\ s_i\downarrow t_i\ \rho(i)=0}} f(s)\ ((s, t)\in[0, 1]^{k+1}),$$

Note présentée par Robert FORTET.

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si la limite existe et dans ce cas on appelle $f_\rho(t)$ la ρ limite de f en t.

On note D_{k+1} l'espace de toutes les applications f: $[0, 1]^{k+1} \to \mathbb{R}$ telles que pour tout $\rho \in \{0,1\}^{k+1}$, f_ρ existe et $f_\rho = f$ si $\rho = (0, \ldots, 0)$.

Finalement on note par C_{k+1} l'espace de toutes les applications continues f: $[0, 1]^{k+1} \to \mathbb{R}$.

2.2. *Fonction correctrice.* — Une fonction r : $[0, 1]^{k+1} \to \mathbb{R}^+$ est appelée fonction correctrice si elle satisfait les conditions suivantes :

(i) il existe $r_0 : [0,1] \to \mathbb{R}^+$ et $\mathbf{r} : [0,1]^k \to \mathbb{R}_+$ tels que $r(t) = r_0(t_0)\mathbf{r}(\mathbf{t})$ pour tout $t = (t_0, \mathbf{t}) \in [0,1]^{k+1}$;

(ii) r appartient à C_{k+1} ;

(iii) $r_0 = 0$ si $t_0 = 0$; $\mathbf{r} = 0$ s'il existe au moins un $j \in \{1, \ldots, k\}$ tel que $t_j = 0$ (frontière inférieure) ou si $\mathbf{t} = (1, \ldots, 1)$ (coin supérieur).

3. Convergence du processus empirique tronqué. — On dit que la suite $\{\mathbf{X}_{ni}\}$ est μ bornée s'il existe une mesure finie et positive μ sur $[0,1]^k$ avec des fonctions de répartition marginales continues telle que, pour chaque $n \geqq 1$ et $1 \leqq i \leqq n$, $H_{ni}(B) \leqq \mu(B)$ pour tout rectangle B dans $[0,1]^k$.

Théorème 1. — *On suppose que la suite* $\{\mathbf{X}_{ni}\}$ *est*

(*a*) φ *mélangeante avec la taux* (3)

ou

(*b*) *fortement mélangeante avec le taux* (4);

la suite $\{H_{ni}\}$ *est*

(*c*) μ *bornée où* μ *est absolument continue avec une densité bornée*

ou

(*d*) $\{H_{ni}\}$ *a des marginales uniformes pour tout* $n \geqq 1$ *et* $1 \leqq i \leqq n$.

De plus on suppose que

(*e*) *la fonction de covariance du processus empirique* W_n *converge vers une fonction.*

Alors W_n (*à valeurs p. s. dans* D_{k+1}) *converge faiblement pour la topologie de Skorohod vers un processus Gaussien* W_0 *avec ses trajectoires p. s. dans* C_{k+1}.

Preuve. — On fait une démonstration similaire à celle du théorème 5 de [2] où avec nos hypothèses on utilise d'abord un corollaire de Withers [8] puis une légère modification du théorème 2.1 de [7] dans le cas φ mélangeant et du théorème 10 de [3] dans le cas fortement mélangeant.

4. Convergence du processus empirique corrigé.

Théorème 2. — *On suppose que la suite* $\{\mathbf{X}_{ni}\}$ *satisfait les hypothèses du théorème* 1. *Alors, pour toute fonction correctrice* r : $[0, 1]^{k+1} \to \mathbb{R}^+$ *satisfaisant*

$$r(t) \geqq A\left[\prod_{j=0}^{k} t_j\left(1 - \prod_{j=1}^{k} t_j\right)\right]^{(1/2)-\delta}, \qquad t \in [0, 1]^{k+1}, \quad A > 0$$

$$0 < \frac{1}{2} - \delta < \frac{1}{2k+4}.$$

$W_n\, 1/r$ (*à valeurs p. s. dans* D_{k+1}) *converge faiblement pour la topologie de Skorohod vers* $W_0\, 1/r$ *à trajectoires p. s. dans* C_{k+1}.

Preuve. — Quand la fonction correctrice s'annule sur la frontière inférieure, on reprend la ligne directrice de la démonstration du théorème de [6] et quand la fonction correctrice s'annule sur le coin supérieur, on reprend une idée utilisée dans [4] pour le cas indépendant.

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THE SPACE $\mathfrak{D}_k$ AND WEAK CONVERGENCE FOR THE RECTANGLE–INDEXED PROCESSES UNDER MIXING

RUNNING HEAD: SPACE $\mathfrak{D}_k$ AND WEAK CONVERGENCE UNDER MIXING

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Abstract

In this paper we study the weak convergence of the weighted empirical processes indexed by rectangles of $[0,1]^k$ under both weak and strong mixing conditions. This is accomplished by generalizing the Skorohod topology on a space of functions defined on a set of rectangles.

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1. Introduction. Let $X_{ni} = (X_{ni}^{(1)},\dots,X_{ni}^{(k)})$, $1 \le i \le n$, $n \ge 1$ be $\mathbb{R}^k$–valued random variables with continuous d.f.s (distribution functions) F_{ni} and continuous marginal d.f.s $F_{ni}^{(j)}$ of $X_{ni}^{(j)}$, $1 \le j \le k$, $1 \le i \le n$. Let $F_n^{(j)} = n^{-1} \sum_{i=1}^{n} F_{ni}^{(j)}$, and let $\{H_{ni}\}$ be a sequence of measures on $[0,1]^k$ defined by

$$H_{ni}(t_1,\dots,t_k) = F_{ni}(F_n^{(1)^{-1}}(t_1),\dots,F_n^{(k)^{-1}}(t_k)),\ 1 \le i \le n\ . \tag{1.1}$$

Let $\hat{W}_n$ be an empirical process defined by

$$\hat{W}_n(B) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \Big(\prod_{j=1}^{k} I_{[a_j < F_n^{(j)}(X_{ni}^{(j)}) \le b_j]} - H_{ni}(B) \Big) \tag{1.2}$$

where $B = \prod_{j=1}^{k} (a_j,b_j] \subset [0,1]^k$ when $I_{[\,]}$ denotes the indicator of $[\,]$.

Our aim is to study the asymptotic behavior of $\hat{W}_n$ for a certain Skorohod topology (to be defined in Section 3) when the sequence $\{X_{ni}\}$ is

$$\varphi\text{–mixing with rates } \varphi(m) = 0(m^{-1-\epsilon}) \tag{1.3}$$

or

$$\text{strong mixing with rates } \sum_{m=1}^{\infty} m^{2(k+1)} \alpha^{\epsilon}(m) < \infty \text{ for some } \epsilon \in (0,\tfrac{1}{2(k+2)})\ . \tag{1.4}$$

Recall that $\{X_{ni}\}$ is φ–mixing if $\sup \{|P(B|A)-P(B)|;\ A \in \sigma(X_{ni}, 1 \le i \le j)$, $B \in \sigma(X_{ni}, i \ge j+m)\} = \varphi(m) \downarrow 0$ for positive integers j and m; and it is strong mixing if $\sup \{|P(A \cap B)-P(A).P(B)|;\ A \in \sigma(X_{ni}, 1 \le i \le j),\ B \in \sigma(X_{ni}, i \ge j+m)\} = \alpha(m) \downarrow 0$ for positive integers j and m. Here $\sigma(X_{ni}, 1 \le i \le j)$ and $\sigma(X_{ni}; i \ge j+m)$ are the σ–fields generated by $(X_{n1},\dots,X_{nj})$ and $(X_{n,j+m}, X_{n,j+m+1},\dots)$ respectively.

Bass and Pyke (1985) extended the M_2 Skorohod (1956) topology for functions on $[0,1]$ to set functions that are outer continuous with inner limits, but the empirical processes indexed by rectangles that we are considering have trajectories concentrated on the set of functions for which an extension of the Skorohod (1956) J_1 topology is more appropriate. Most of the work dealing with the empirical processes indexed by points is concentrated on the Skorohod's J_1 topology. In their basic paper, Bass and Pyke (1985) posed an open problem of determining a general class of sets on which an extension of Skorohod's J_1 topology is possible. In this paper we provide an extension of the Skorohod J_1 topology to functions indexed by rectangles and thus answer a part of the problem posed by Bass and Pyke (loc. cit.). We may also mention an interesting paper of Straf (1972) which deals with the extension of J_1 topology to very general index sets. This requires the existence of a group Λ of homeomorphisms on an arbitrary space and does not seem to give real answers perhaps because of its very theoretical nature. Also Straf's methods seem to be applicable on the space of functions defined only on $[0,1)^k$ and not on $[0,1]^k$, and they do not seem to be applicable to empirical processes indexed by rectangles.

The weak convergence of the weighted univariate empirical process indexed by points was established for the independent case by Pyke and Shorack (1968), and for the φ–mixing case by Fears and Mehra (1974) and later by Ahmad and Lin (1980). The generalization to the φ–mixing multivariate nonstationary case was carried on by Harel (1980). Shorack and Wellner (1982) established the weak convergence of the weighted univariate empirical process indexed by intervals when the underlying random variables are independent. Their results were later generalized by Einmahl, Ruymgaart and Wellner (1984) to the multivariate case by using directly the well known Skorohod construction when the underlying random variables are independent, identical and uniformly distributed over $[0,1]^k$. Later Ruymgaart and Wellner (1984) considered the case when the random variables are not uniformly distributed, but left open the problem of the convergence of the weighted empirical processes (see Ruymgaart and Wellner (1984), remark p. 221). Our method of constructing Skorohod topology on the space of functions indexed by rectangles leads also to the extension of the results of Ruymgaart and Wellner (loc. cit.) for more general classes of distributions (not necessarily uniform distributions).

We may also mention for reference the work of Alexander (1982) on weighted empirical processes indexed by Vapnik–Cervonenkis classes of sets for the independent case. For the weak convergence of (nonweighted) empirical processes the reader is referred to the interesting papers of Neuhaus (1971, 1975) for the independent case, Rüschendorf (1974) and Balacheff and Dupont (1980) for the φ–mixing cases.

2. The $\mathfrak{D}_k$ and $\mathfrak{C}_k$ spaces and preliminaries.

We write $t = (t_1,\dots,t_k)$, and half–open rectangles $R(t,t') = \prod_{j=1}^{k} (t_j,t'_j]$. By convention, any point $t_j \in [0,1]$ will be called a half–open interval and will be written as $(t_j,t_j]$. Note that $R(t,t) = t$. For $(t,t') \in ([0,1]^k)^2$, $t \le t'$ will mean $t_j \le t'_j \ \forall\, j=1,\dots,k$, and $t<t'$ will mean $t_j<t_j \ \forall\, j=1,\dots,k$. For $(t,t') \in ([0,1]^k)^2$ with $t<t'$ or $t \le t'$ we can associate a rectangle $R(t,t')$ defined as before.

Let $\mathcal{J}(k) = \{R(t,t');R(t,t') \subset [0,1]^k\}$, and associate with the space $\mathcal{J}(k)$ the Hausdorff metric d_H where $d_H(R(t,t'),R(s,s')) = \max_{1\le j\le k} \max\{|s_j-t_j|,|s'_j-t'_j|\}$.

Consider a family of k strictly increasing finite sequences of elements of $[0,1]$ such that 0 is the first element of each sequence. For example, let $B = \{t_{ji}\}$, $1\le i\le n_j$, $1\le j\le k$ be k sequences such that $t_{j1} = 0$ and $t_{ji} < t_{ji+1}$ $\forall\, i \in \{1,\dots,n_j-1\}$ and $j \in \{1,\dots,k\}$. Now associate with B a set $I^{(j)}(i,\ell)$ defined as

$$I^{(j)}(i,\ell) = \begin{cases} \{t_j,t'_j]\subset[0,1],\, t_{ji}\le t_j<t_{ji+1},\, t_{j\ell}\le t'_j<t_{j\ell+1}\} & \text{if } 1\le i,\ell\le n_j-1 \\ \{(t_j,t'_j]\subset[0,1],\, t_{ji}\le t_j<t_{ji+1},\, t_{j\ell}\le t'_j\le 1\} & \text{if } 1\le i\le n_j-1,\ \ell=n_j \end{cases}$$

and

$$I^{(j)}(n_j,n_j) = \{(t_j,t'_j]\subset[0,1];t_{jn_j}\le t_j\le t'_j\le 1\}\ .$$

(Note that $I^{(j)}(i,\ell) = \phi$ if $t_{j\ell}\le t_{ji}$).

Then a partition G of $\mathcal{J}(k)$ defined as

$$G = \left\{\prod_{j=1}^{k} I^{(j)}(i_j,\ell_j), 1\le i_j\le n_j, 1\le \ell_j\le n_j, \text{ and } \prod_{j=1}^{k} I^{(j)}(i_j,\ell_j)\ne\phi\right\} \tag{2.2}$$

will be called a grid of $\mathcal{J}(k)$ with base B.

Let S^* be a finite subset of $\mathcal{J}(k)$ where $S^* = \{R(a^{(1)},b^{(1)}),\dots,R(a^{(p)},b^{(p)})\}$. Then the base generated by S^* is the smallest base of the grid B such that

$\{a_j^{(1)},\ldots,a_j^{(p)}\} \cup \{b_j^{(1)},\ldots,b_j^{(p)}\} \subset \{t_{j1},\ldots,t_{jn_j}\}\ \forall\, j = 1,\ldots,k$.

Let $R(t,t') \in \mathcal{X}(k)$ and $(\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$. Then the (ρ,ϵ) quadrant of $\mathcal{X}(k)$ with top $R(t,t')$ is a subset $Q(R(t,t'),\rho,\epsilon)$ of $\mathcal{X}(k)$ defined as

$$Q(R(t,t'),\rho,\epsilon) = \prod_{j=1}^{k} Q_j(t_j,t'_j,\rho_j,\epsilon_j) \quad \text{where}$$

$$Q_j(t_j,t'_j,\rho_j,\epsilon_j) = \begin{cases} \{(s_j,s'_j]\subset[0,1],\, s_j<t_j,\, s'_j<t'_j\} & \text{if } \rho_j=\epsilon_j=0 \\ \{(s_j,s'_j]\subset[0,1],\, s_j\geq t_j,\, s'_j<t'_j\} & \text{if } \rho_j=1,\ \epsilon_j=0 \\ \{(s_j,s'_j]\subset[0,1],\, s_j<t_j,\, s'_j\geq t'_j\} & \text{if } \rho_j=0,\ \epsilon_j=1 \\ \{(s_j,s'_j]\subset[0,1],\, s_j\geq t_j,\, s'_j\geq t'_j\} & \text{if } \rho_j=\epsilon_j=1 \end{cases} .$$

Let $R(t,t') \in \mathcal{X}(k)$ and $(\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(t,t'),\rho,\epsilon) \neq \phi$. Then, it is easy to see that if G is a grid, there exists an $S \in G$ such that $S \cap Q(R(t,t'),\rho,\epsilon)$ is a neighborhood of $R(t,t')$ in $Q(R(t,t'),\rho,\epsilon)$ in the topology induced on $Q(R(t,t'),\rho,\epsilon)$ by the Hausdorff metric.

Let S be a nonempty subset of $\mathcal{X}(k)$. S is called a <u>pavement</u> of $\mathcal{X}(k)$ if S is of the form $S = \prod_{j=1}^{k} S_j$ where for any $1\leq j\leq k$, $\exists(a_j^{(1)},b_j^{(1)},a_j^{(2)},b_j^{(2)}) \in [0,1]^4$ such that $S_j = \{(t_j,t'_j],\ a_j^{(1)}\leq t_j\leq a_j^{(2)},\ b_j^{(1)}\leq t'_j\leq b_j^{(2)}\}$.

For any $R(s,s') \in S$, we call the pair (J,L) the <u>indicator</u> of $R(s,s')$ into S if $(J,L) \subset \{1,\ldots,k\} \times \{1,\ldots,k\}$ is such that

$$s_j \in \{a_j^{(1)},a_j^{(2)}\}\ \forall\, j\in J \text{, and } s_j \in (a_j^{(1)},a_j^{(2)})\ \forall\, j\notin J$$
$$s'_\ell \in \{b_\ell^{(1)},b_\ell^{(2)}\}\ \forall\, \ell\in L \text{, and } s'_\ell \in (b_\ell^{(1)},b_\ell^{(2)})\ \forall\, \ell\notin L .$$

We say F , a subset of S , is the <u>face</u> of $R(s,s')$ in S with indicator (J,L) if F is of the form

$$F = \{R(u,u') \in S;\ u_j=s_j\ \forall\, j\in J \text{, and } u'_\ell=s'_\ell\ \forall\, \ell\in L\} .$$

Note that if $(J,L) = (\phi,\phi)$, then $F=S$, and if $(J,L) = \{1,\ldots,k\}\times\{1,\ldots,k\}$, then $F = R(s,s')$.

We say that $f : \mathcal{X}(k) \to \mathbb{R}$ admits a (ρ,ϵ) limit in $R(t,t')$ if and only if the restriction $f|Q(R(t,t'),\rho,\epsilon)$ admits a limit in $R(t,t')$ with respect to the metric d_H and the usual metric on $\mathbb{R}$. We shall denote the (ρ,ϵ) limit of f in R(t,t') by $f(R(t,t')+0(\rho,\epsilon))$. If $(\rho,\epsilon)=((1,\dots,1),(1,\dots,1))$, then the (ρ,ϵ) limit of f is $f(R(t,t'))$.

Denote by $\tilde{D}_k$, the set of maps $f: \mathcal{J}(k) \to \mathbb{R}$ such that for any $R(t,t') \in \mathcal{J}(k)$ and any $(\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$ for which $Q(R(t,t'),\rho,\epsilon) \neq \phi$, f admits a (ρ,ϵ) limit in R(t,t').

Finally denote by $\tilde{C}_k$, the set of maps $f: \mathcal{J}(k) \to \mathbb{R}$ which are continuous in d_H and the usual metric in $\mathbb{R}$.

2.2 Properties of the spaces $\tilde{D}_k$ and $\tilde{C}_k$.

Let $f \in \tilde{D}_k$ and $R(t,t') \in \mathcal{J}(k)$, and for any $(J,L) \subset \{1,\dots,k\} \times \{1,\dots,k\}$, set

(2.2.1) $$H(f,R(t,t'),J,L) = \max \left\{ \begin{array}{l} |f(R(t,t')+0(\rho,\epsilon)) - f(R(t,t')+0(\rho',\epsilon'))|, \\ Q(R(t,t'),\rho,\epsilon) \neq \phi,\ Q(R(t,t'),\rho',\epsilon') \neq \phi \\ \text{where } \forall\ j \in J,\ \rho_j = \rho'_j \text{ and } \forall\ \ell \in L,\ \epsilon_\ell = \epsilon'_\ell \end{array} \right\}$$

Note that if $(J',L') \subset (J,L)$, then $H(f,R(t,t'),J,L) \le H(f,R(t,t'),J',L')$ and if $(J,L) = \{1,\dots,k\} \times \{1,\dots,k\}$, then $H(f,R(t,t'),J,L) = 0$. If $(J,L) = (\phi,\phi)$, then we shall denote $H(f,R(t,t'),J,L)$ by $H(f,R(t,t'))$.

Lemma 2.2.1. Let $f \in \tilde{D}_k$, $R(t,t') \in \mathcal{J}(k)$, $(J,L) \subset \{1,\dots,k\} \times \{1,\dots,k\}$ and $\eta > 0$. Let $B(R(t,t'),\alpha)$ be an open ball with center R(t,t') and radius α such that $\forall\ (\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$, $\forall\ (s,s') \in B(R(t,t'),\alpha) \cap Q(R(t,t'),\rho,\epsilon)$

(2.2.2) $$|f(R(s,s')) - f(R(t,t') + 0(\rho,\epsilon))| < \eta .$$

Then, for any $R(u,u') \in B(R(t,t'),\alpha)$ for which $t_j = u_j\ \forall\ j \in J$, $t_j \neq u_j\ \forall\ j \notin J$; $t'_\ell = u'_\ell\ \forall\ \ell \in L$ and $t'_\ell \neq u'_\ell\ \forall\ \ell \notin L$, we have

(2.2.3) $$H(f,R(u,u'),J,L) \le 2\eta .$$

Proof: For any $(\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(u,u'),\rho,\epsilon) \neq \phi$, we denote by $(\bar{\rho},\bar{\epsilon})$, an element of $\{0,1\}^k \times \{0,1\}^k$ defined as follows:

if $j \in J$, $\bar{\rho}_j = \rho_j$	if $\ell \in L$, $\bar{\epsilon}_\ell = \epsilon_\ell$
if $j \notin J$, $\bar{\rho}_j = \begin{cases} 0 & \text{if } u_j < t_j \\ 1 & \text{if } u_j > t_j \end{cases}$	if $\ell \notin L$, $\bar{\epsilon}_\ell \begin{cases} 0 & \text{if } u'_\ell > t_\ell \\ 1 & \text{if } u'_\ell < t_\ell \end{cases}$

then $Q(R(u,u'),\rho,\epsilon) \cap Q(R(t,t'),\bar{\rho},\bar{\epsilon}) \neq \phi$.

Let $\eta_1>0$ and $\alpha'>0$ be such that

$$|f(R(s,s'))-f(R(u,u')+0(\rho,\epsilon))|<\eta_1$$

$\forall\ (\rho,\epsilon)\in\{0,1\}^k\times\{0,1\}^k$ and $\forall\ R(s,s')\in B(R(u,u'),\alpha')\cap Q(R(u,u'),\rho,\epsilon)$.

As $B(R(u,u'),\alpha')\cap Q((R(u,u'),\rho,\epsilon)\cap B(R(t,t'),\alpha)\cap Q(R(t,t'),\overline{\rho},\overline{\epsilon})\neq\phi$ we can find $R(s,s')$ such that (using (2.2.2))

$$|f(R(s,s'))-f(R(u,u')+0(\rho,\epsilon))|<\eta_1$$

and

$$|f(R(s,s'))-f(R(t,t')+0(\overline{\rho},\overline{\epsilon}))|<\eta\,.$$

From the above inequalities, we deduce

$$|f(R(t,t')+0(\overline{\rho},\overline{\epsilon}))-f(R(u,u')+0(\rho,\epsilon))|<\eta+\eta_1\,,$$

Since η_1 is arbitrary, the left side of the above inequality $\leq\eta$.

Let (ρ',ϵ') be another element of $\{0,1\}^k\times\{0,1\}^k$ such that $Q(R(u,u'),\rho',\epsilon')\neq\phi$. Since $(\overline{\rho}',\overline{\epsilon}')=(\overline{\rho},\overline{\epsilon})$, we deduce, proceeding analogously,

$$|f(R(u,u')+0(\rho',\epsilon'))-f(R(t,t')+0(\overline{\rho},\overline{\epsilon}))|\leq\eta$$

and

$$|f(R(u,u')+0(\rho',\epsilon'))-f(R(u,u')+0(\rho,\epsilon))|\leq 2\eta\,.$$

Now using (2.2.1), we conclude (2.2.3).

Lemma 2.2.2. Let $f\in\tilde{D}_k$ and $(J,L)\subset\{1,\ldots,k\}\times\{1,\ldots,k\}$. Set $T\subset\mathcal{J}(k)$ be such that $\forall\ (R(t,t'),R(s,s'))\in T\times T$ with $R(t,t')\neq R(s,s')$, we have $t_j=s_j\ \forall\ j\in J$, $t_j\neq s_j\ \forall\ j\notin J$, $t'_\ell=s'_\ell\ \forall\ \ell\in L$, and $t'_\ell\neq s'_\ell\ \forall\ \ell\notin L$. Then, for any $\eta>0$, the set of elements of T such that $H(f,R(t,t'),J,L)\geq\eta$ is finite.

Proof. Let $\eta>0$ be fixed, and let T_η be the set of elements of T such that $H(R(t,t'),J,L)\geq\eta$. If T_η is infinite, then it has a limiting point $R(t,t')$, and we can find a sequence $R(t^{(n)},t'^{(n)})$ of points of T_η which admits $R(t,t')$ as limit, and is such that $\forall\ n\in\mathbb{N}$, $t_j^{(n)}\neq t_j$ for $\forall\ j\notin J$ and $t_\ell'^{(n)}\neq t'_\ell\ \forall\ \ell\notin L$.

Let $\eta'<\eta/2$ and $\alpha>0$ be such that $B(R(t,t'),\alpha)$ satisfies $|f(R(s,s'))-f(R(t,t'),\rho,\epsilon)|<\eta'\ \forall\ (\rho,\epsilon)\in\{0,1\}^k\times\{0,1\}^k$ and $\forall\ R(s,s')\in B(R(t,t'),\alpha)\cap Q(R(t,t'),\rho,\epsilon)$.

Also let $n_0 \in \mathbb{N}$ be such that $R(t^{(n)},t'^{(n)}) \in B(R(t,t'),\alpha)\ \forall\, n \geq n_0$. Then, using Lemma 2.2.1, it follows that $H(f,R(t^{(n)},t'^{(n)}),J,L) \leq 2\eta' < \eta$ which contradicts the hypothesis.

Theorem 2.2.1. Let R be the set of grids of $\mathcal{J}(k)$ and let $f : \mathcal{J}(k) \to \mathbb{R}$. Then $f \in \check{D}_k$ if and only if $\forall\, \eta>0,\ \exists\, \delta>0,\ \exists\, G \in R$ such that $\forall\, S \in G$ and $\forall\, (R(t,t'),R(s,s')) \in S \times S$ we have

(2.2.4) $$d_H(R(t,t'),R(s,s')) \leq \delta \Rightarrow |f(R(t,t') - f(R(s,s'))| < \eta .$$

Proof. (Sufficiency). Let $f : \mathcal{J}(k) \to \mathbb{R}$, and let $f \notin \check{D}_k$. Then we show that (2.2.4) is not true, that is we show that $\forall\, \delta>0,\ \forall\, G \in R,\ \exists\, S \in G,\ \exists\, (R(t,t'),R(s,s') \in S \times S$ such that $d_H(R(t,t'),R(s,s')) \leq \delta$ and $|f(R(t,t')-f(R(s,s'))| \geq \eta$. Since $f \notin \check{D}_k$, it follows that $\exists$ $R(u,u') \in \mathcal{J}(k)$ and $\exists\, (\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(u,u'),\rho,\epsilon) \neq \phi$ and f has no (ρ,ϵ) limit in $R(u,u')$. This means that $\forall\, \eta_1>0$, $\exists\, (R(t,t'),R(s,s')) \in Q^2(R(u,u'),\rho,\epsilon)$ such that $d_H(R(t,t'),R(s,s')) < \eta_1$ and $|f(R(t,t'))-f(R(s,s'))| \geq \eta$.

Let $\delta>0$ and $G \in R$ be fixed. Then we can find $S \in G$ such that $Q(R(u,u'),\rho,\epsilon) \cap S$ is a neighborhood of $R(u,u')$ in $Q(R(u,u'),\rho,\epsilon)$. Then, there exists an $\alpha>0$ such that

$$Q(R(u,u'),\rho,\epsilon) \cap B(R(u\ u'),\alpha) \subset Q(R(u,u'),\rho,\epsilon) \cap S .$$

Now choose $\eta_1 = \min(\delta,\alpha/2)$. Then $\forall\, (R(t,t'),R(s,s')) \in (Q(R(u,u'),\rho,\epsilon) \cap B(R(u,u'),\alpha))^2$ such that $d_H(R(t,t'),R(s,s')) < \eta_1$. We see that $(R(t,t'),R(s,s')) \in S \times S$ and $d_H(R(t,t'),R(s,s') \leq \delta$.

We can now choose $(R(t,t'),R(s,s'))$, such that $|f(R(t,t')-f(R(s,s'))| \geq \eta$ and this leads to contradiction. Sufficiency is established.

(Necessary Part). Let $f \in \check{D}_k$. Choose an $\eta>0$, and let $\mathcal{H}$ be a class of subsets T of $\mathcal{J}(k)$ such that

(a) $\forall\, R(t,t') \in T,\ H(f,R(t,t')) \geq \eta$

(b) $\forall\, (R(t,t'),R(s,s')) \in T \times T,\ R(t,t') \neq R(s,s') \Rightarrow t_j \neq s_j\ \ \forall\, j \in \{1,\ldots,k\}$ and $t'_\ell \neq s'_\ell\ \forall\, \ell \in \{1,\ldots,k\}$.

Note that T is finite $\forall\, T \in \mathcal{H}$. Let S_1^* $(\in \mathcal{H})$ be a maximal element (in $\mathcal{H}$).

By iteration, we construct a sequence $S_1^*,\ldots,S_{2k}^*$ of sets in $\mathcal{J}(k)$ as follows.

For any $R(s^{(1)},s'^{(1)}) \in S_1^*$ and any $(J,L) \in \{1,\ldots,k\} \times \{1,\ldots,k\}$ such that the Card J + Card $L = 1$, let $\mathcal{H}(R(s^{(1)},s'^{(1)}),J,L)$ be a class of subsets T^* of $\mathcal{J}(k)$ such that

(c) $\forall\, R(t,t') \in T^*$, $H(f,R(t,t'),J,L) \geq \eta$

(d) $\forall\, R(t,t') \in T^*$, $t_j = s_j^{(1)}\ \forall\, j \in J$ and $t'_j = s'^{(1)}_j\ \forall\, \ell \in L$ and

(e) $\forall\, (R(t,t'),R(s,s')) \in T \times T$, $R(t,t') \neq R(s,s') \Rightarrow t_j \neq s_j\ \forall\, j \notin J$ and $t'_\ell \neq s'_\ell\ \forall\, \ell \notin L$.

Now let $S^*(R(s^{(1)},s'^{(1)}),J,L)$ be a maximal element in $\mathscr{H}(R(s^{(1)},s'^{(1)}),J,L)$. We set $S_2^* = S_1^* \cup \bigcup_{(1)} \bigcup_{(2)} S^*(R(s^{(1)},s'^{(1)}),J,L)$ where $\bigcup_{(1)}$ in the union over $R(s^{(1)},s'^{(1)}) \in S_1^*$ and $\bigcup_{(2)}$ is the union over $(J,L) \in \mathscr{P}_1$ where $\mathscr{P}_1$ is the class of subsets of $J \subset \{1,\ldots,k\}$ and $L \subset \{1,\ldots,k\}$ such that $\text{Card } J + \text{Card } L = 1$.

Proceeding this way we get a sequence $S_1^* \subset \ldots \subset S_{2k}^*$ of sets in $\mathscr{X}(k)$. Let G be a grid generated by S_{2k}^*. Now denote

$$J(R(t,t')) = \{j \in \{1,\ldots,k\} : \exists\, R(s,s') \in S_{2k}^* \text{ and } s_j = t_j\}$$

and

$$L(R(t,t')) = \{\ell \in \{1,\ldots,k\} : \exists\, R(s,s') \in S_{2k}^* \text{ and } s'_\ell = t'_\ell\}.$$

Then , we first prove the following Lemma.

<u>Lemma</u> 2.2.3. <u>Let</u> $R(t,t') \in \mathscr{X}(k)$. <u>Then for every</u> (J,L) <u>such that</u> $(J(R(t,t'),L(R(t,t')) \subset (J,L)$, <u>we have</u>

(2.2.4) $$H(f,R(t,t'),J,L) < \eta .$$

<u>Proof</u>. Let $\mathscr{S}$ be a set of sequences $(R(s^{(1)},s'^{(1)}),j_1),\ldots,(R(s^{(h)},s'^{(h)}),j_h)$ where $1 \leq h \leq 2k$, $j_\ell \in \{1,\ldots,2k\}$ and $1 \leq \ell \leq h$ such that

$$s_{j_\ell}^{(\ell)} = t_{j_\ell} \text{ if } j_\ell \leq k,\ s'^{(\ell)}_{j_\ell - k} = t'_{j_\ell - k} \text{ if } j_\ell > k \quad (1 \leq \ell \leq h)$$

(2.2.5) $$R(s^{(1)},s'^{(1)}) \in S_1^*, R(s^{(\ell)},s'^{(\ell)}) \in S^*(R(s^{(\ell-1)},s'^{(\ell-1)}),J_1,L_1)$$

where $J_1 = \bigcup_{p \leq \ell-1, j_p \leq k} \{j_p\}$, $L_1 = \bigcup_{p \leq \ell-1, j_p > k} \{j_p - k\}$, and $j_\ell \notin \{j_1,\ldots,j_{\ell-1}\}$.

Note that $\mathscr{S}$ has at least one maximal sequence and it is easy to check that

(2.2.6) $$\forall\, \ell \in \{1,\ldots,h\},\ s_{j_\ell}^{(h)} = t_{j_\ell} \text{ if } j_\ell \leq k \text{ and } s'^{(h)}_{j_\ell - k} = t'_{j_\ell - k} \text{ if } j_\ell > k$$

Since $J_1 \subset J(R(t,t'))$ and $L_1 \subset L(R(t,t'))$, it suffices to prove that any maximal sequence satisfies

(2.2.7) $\quad H(f,R(t,t'),J_2,L_2) < \eta$ where $(J_2,L_2) = (J_1,L_1)$ with $h=\ell-1$

There are three cases to be disposed of.

Case 1. Let $h = 2k$. Then from (2.2.6), $R(t,t') = R(s^{(2k)},s'^{(2k)})$ (from (2.2.6)), $J(R(t,t')) = L(R(t,t')) = \{1,\ldots,k\}$ and so $H(f,R(t,t'),J(R(t,t')),L(R(t,t'))) = 0\ (<\eta)$.

Case 2. Let $\mathscr{F} = \phi$. Then $H(f,R(t,t')) < \eta$, and also for any $(J,L) \subset \{1,\ldots,k\} \times \{1,\ldots,k\}$, $H(f,R(t,t'),J,L) < \eta$.

Case 3. Let $1 \leq h < 2k$. Then, for any $R(s,s') \in S^*(R(s^{(h)},s'^{(h)}),J_2,L_2)$ and $j \notin \{j_1,\ldots,j_h\}$, we have $s_j \neq t_j$ if $j \leq k$, $s'_{j-k} \neq t'_{j-k}$ if $j>k$. Now if $H(f,R(t,t'),J_2,L_2) \geq \eta$, then the sequence would not be maximal, and so we have the contradiction. This proves the Lemma.

Now let G be a grid generated by S^*_{2k} , and let $S \in G$.

Let $\bar{S}$ denote the closure of S under d_H . (Note that $\bar{S}$ is a pavement). If $R(t,t') \in \bar{S}$, and (J,L) is the indicator of $R(t,t')$ in $\bar{S}$, then we have $H(f,R(t,t'),J,L) < \eta$.

Now we prove that for any $S \in G$ $\exists\ \delta>0$ such that $\forall\ (R(t,t'),R(s,s')) \in S \times S$,

(2.2.8) $$d_H(R(t,t'),R(s,s')) \leq \eta \Rightarrow |f(R(t,t'))-f(R(s,s'))| < \eta .$$

Suppose (2.2.8) is not true. Then, we can find an $S \in G$ such $\forall\ \delta>0$, $\exists$ $(R(t,t'),R(s,s') \in S \times S$, such that

(2.2.9) $$d_H(R(t,t'),R(s,s')) \leq \delta \text{ and } |f(R(t,t'))-f(R(s,s'))| > \eta .$$

In this case, we can extract two sequences $R(t^{(n)},t'^{(n)})$ and $R(s^{(n)},s'^{(n)})$ in S such that they converge to the same limit $R(t,t')$ in $\bar{S}$; furthermore $\exists\ (\rho,\epsilon) \in \{0,1\}^k \times \{0,1\}^k$ and $(\rho',\epsilon') \in \{0,1\}^k \times \{0,1\}^k$ with $R(t^{(n)},t'^{(n)}) \in Q(R(t,t'),\rho,\epsilon)$ and $R(s^{(n)},s'^{(n)})$ in $Q(R(s,s'),\rho',\epsilon')$ for any $n \geq 1$ such that $|f(R(t^{(n)},t'^{(n)})) - f(R(s^{(n)},s'^{(n)})| > \eta$.

As $f \in \breve{D}_k$, $\lim_{n\to\infty} f(R(t^{(n)},t'^{(n)})) = f(R(t,t') + 0(\rho,\epsilon))$ and $\lim_{n\to\infty} f(R(s^{(n)},s'^{(n)})) = f(R(t,t') + 0(\rho',\epsilon'))$.

Consequently,

(2.2.10) $$|f(R(t,t') + 0(\rho,\epsilon)) - f(R(t,t') + 0(\rho',\epsilon'))| \geq \eta .$$

Let (J,L) be the indicator of R(t,t') in $\overline{S}$. If (J,L) = (ϕ,ϕ) , J(R(t,t')) = ϕ and L(R(t,t')) = ϕ , then $\mathcal{I}$ is ϕ and so H(f,R(t,t'))<η which is not compatible with (2.2.10).

If (J,L) $\neq$ (ϕ,ϕ) , then $\rho_j=\rho'_j$ $\forall$ j$\in$J , and $\epsilon_\ell=\epsilon'_\ell$ $\forall$ $\ell\in$L , and so

$$|f(R(t,t') + 0(\rho,\epsilon)) - f(R(t,t') + 0(\rho',\epsilon'))| \leq H(f(R(t,t'),J,L)$$
$$\leq H(f(R(t,t'),J(R(t,t')),L(R(t,t')) < \eta$$

(from Lemma 2.2.3) and this is not compatible with (2.2.10). This proves the theorem.

Definition (Balacheff and Dupont (1980)). A grid G' is finer than a grid G if and only if $\forall$ S'$\in$G' , $\exists$ S$\in$G such that S'$\subset$S .

Property of a finer grid. Given any $\delta>0$ and any grid G , we can find another grid G' finer than G such that the diameter of each element S' of G' is less than or equal to δ.

For any grid G with base B = $\{\{t_{ji}\},1\leq i\leq n_j,1\leq j\leq k\}$, we associate the number m(G) , called permeability m(G) of G , defined as

$$m(G) = \inf_{1\leq j\leq k} \inf_{1\leq i\leq n_j} \{|t_{ji}-t_{ji+1}|,t_{ji}\neq t_{ji+1}\} , t_{jn_j+1}=1$$

by convention. (Note that this concept of permeability is an extension of the similar concept given by Neuhaus (1971) for a rather specialized situation connected with the space D_k). Now denote

(2.2.11) $$R_\eta = \{G;m(G) > \eta \text{ for any } \eta>0\} .$$

Corollary 2.3.1. Let f : $\mathcal{J}$(k)$\to\mathbb{R}$. Then

(a) f$\in\tilde{D}_k$ if and only if $\forall$ $\eta>0$, $\exists$ a grid G$\in$R such that $\forall$ S$\in$G and $\forall$ (R(t,t'),R(s,s')) $\in$ S$\times$S , we have |f(R(t,t'))–f(R(s,s'))| < η .

(b) f$\in\tilde{D}_k$ if and only if $\forall$ $\eta>0$, $\exists$ a $\delta>0$ and G$\in R_\delta$ such that $\forall$ S$\in$G and $\forall$ (R(t,t'),R(s,s')) $\in$ S$\times$S , we have |f(R(t,t'))–f(R(s,s'))| < η .

Proof. (a) is a consequence of Theorem 2.3.1 and the property of the finer grid, and (b) is a consequence of (a).

For any function f : $\mathcal{J}$(k)$\to\mathbb{R}$, and any $\delta>0$, we define the 'modulus of continuity'

$\omega'(f,\delta)$ in $\check{D}_k$ and $\tilde{\omega}(f,\delta)$ in $\check{C}_k$ as

(2.2.12) $$\omega'(f,\delta) = \inf_{G\in R_\delta} \max_{S\in G} \sup_{(R(t,t'),R(S,s'))\in S\times S} |f(R(t,t'))-f(R(s,s'))|$$

and

(2.2.13) $$\tilde{\omega}(f,\delta) = \sup\{|f(R(t,t'))-f(R(s,s'))|;\ (R(t,t'),R(s,s'))\in\mathcal{J}(k)\times\mathcal{J}(k), d_H(R(t,t'),R(s,s'))\leq\delta\}$$

Note that a function $g: \delta\mapsto\omega'(f,\delta)$ where $\delta\in(0,1]$ is nondecreasing.

Corollary 2.3.2. Let $f: \mathcal{J}(k)\to\mathbb{R}$. Then

(a) $f\in\check{D}_k$ if and only if $\lim_{\delta\to 0} \omega'(f,\delta)=0$

(b) $f\in\check{C}_k$ if and only if $\lim_{\delta\to 0} \tilde{\omega}(f,\delta)=0$.

Proof. (a) is a consequence of Corollary 2.3.1(b), and (b) follows by definition.

Note that for any bounded function $f: \mathcal{J}(k)\to\mathbb{R}$, and any $\delta\in(0,\frac{1}{2}),\omega'(f,\delta) \leq \tilde{\omega}(f,2\delta)$.

3. Skorohod topology on $\check{D}_k$.

3.1 Preliminaries. In what follows, Λ denotes the space of maps $h: [0,1]\to[0,1]$ which are nondecreasing, continuous and bijective. $\lambda^{(k)}$ denotes the space of maps $\lambda: [0,1]^k\to[0,1]^k$ where $\lambda(t_1,\ldots,t_k) = (\lambda_1(t_1),\ldots,\lambda_k(t_k))$, $\lambda_j\in\Lambda$, $1\leq j\leq k$.

$I_{(k)}$ denotes the identity map on $[0,1]^k$.

$$|||\lambda||| = \max_{1\leq j\leq k} \sup_{0\leq t_j<s_j\leq 1} \left|\log \frac{\lambda_j(t_j)-\lambda_j(s_j)}{t_j-s_j}\right|$$

for any $R(t,t')\in\mathcal{J}(k)$, $\lambda(R(t,t'))$ denotes an element $R(s,s')$ of $\mathcal{J}(k)$ defined by $\lambda_j(t_j) = s_j$ and $\lambda_j(t_j') = s_j'$, $1\leq j\leq k$. For any bounded maps $f,g: \mathcal{J}(k)\to\mathbb{R}$, we denote

$$d(f,g) = \inf_{\lambda\in\Lambda^{(k)}} \max\{\|f-g\circ\lambda\|,\|\lambda-I_{(k)}\|\};$$

$$d_o(f,g) = \inf_{\lambda\in\Lambda^{(k)}} \max\{\|f-g\circ\lambda\|,|||\lambda|||\}.$$

where $\|f-g\circ\lambda\| = \sup_{R(t,t')\in\mathcal{J}(k)} \{|f(R(t,t'))-g\circ\lambda(R(t,t'))|\}$ and $\|\lambda-I_{(k)}\| = \sup_{t\in[0,1]^k} \{|\lambda(t)-I_{(k)}(t)|\}$. We shall call the topologies associated with d and d_o as Skorohod and modified Skorohod topologies respectively.

Lemma 3.1. *Let* $f_n : \check{D}_k \to \mathbb{R}$ *be a sequence of maps such that* $f_n \to f \in \check{D}_k$ *in Skorohod topology. Let* $R(t,t') \in \mathcal{X}(k)$ *be such that the restriction of* f *to the face of* $R(t,t')$ *in* $\mathcal{X}(k)$ *is continuous in* $R(t,t')$. *Then* $\lim_{n\to\infty} f_n(R(t,t')) = f(R(t,t'))$.

Proof. Proof follows by using the inequality

$$|f_n(R(t,t')) - f(R(t,t'))| \le |f_n(R(t,t')) - f(\lambda^{-1}(R(t,t')))| + |f(\lambda^{-1}(R(t,t'))) - f(R(t,t'))|$$

where λ^{-1} is the inverse function of λ.

Remark 3.1. Following Billingsley (1968) and Neuhaus (1971), the following facts can easily be established.

(i) Skorohod topology as well as modified Skorohod topology implies uniform topology.

(ii) Uniform topology is finer than the Skorohod topology.

(iii) The modified Skorohod topology is finer than the Skorohod topology.

(iv) The Skorohod topology and the modified Skorohod topology are equivalent in $\check{D}_k$.

(v) The space $(\check{D}_k, d)$ is separable.

(vi) The space $(\check{D}_k, d_0)$ is complete.

(vii) $\forall\ \delta>0$, $\omega'(f,\delta)$ is upper semicontinuous in $f \in \check{D}_k$ with respect to the Skorohod topology.

Lemma 3.2. *For any* $R(t,t') \in \mathcal{X}(k)$; *let* $\varphi_{R(t,t')} : \check{D}_k \to \mathbb{R}$ *be a map defined by* $\varphi_{R(t,t')} = f(R(t,t'))$. *If the restriction of the face of* $R(t,t')$ *into* $\mathcal{X}(k)$ *is continuous, then* $\varphi_{R(t,t')}$ *is continuous with respect to the Skorohod topology*.

Proof. Consequence of Lemma 3.1.

Theorem 3.1. *Let* $K \subset \check{D}_k$. *Then the closure of* $\overline{K}$ *of* K *(with respect to the Skorohod topology) is compact if and only if*

$$\sup_{f\in K} \|f\| < \infty \tag{3.1}$$

and

$$\lim_{\delta\to\infty} \sup_{f\in K} \omega'(f,\delta) = 0 . \tag{3.2}$$

Proof. Let $\overline{K}$ be compact. Then (3.1) and (3.2) follow from Remark 3.1(vii). We now prove the sufficiency part.

Let $K\subset \tilde{D}_k$, and let (3.1) and (3.2) hold. Since $(\tilde{D}_k,d_o)$ is complete, we have to show that $\forall\ \eta>0$, $\exists$ a finite subset $K(\eta)$ of $\tilde{D}_k$ which is η–net in K with respect to d_o.

Choose an $\eta>0$, and an integer m such that $\eta > 1/m$, and $\omega'(f,1/m) < \eta\ \forall\ f\in K$.

Let $\mathscr{L}_m = \{j/m;0\le j\le m\}$ and let $R(\mathscr{L}_m)$ be the set of grids G of $\mathscr{I}(k)$, with each grid having the base of the form $\{t_{ji};1\le i\le n_j,1\le j\le k\}$ where $t_{ji}\in\mathscr{L}_m\ \forall i$ and $\forall j$.

Let H be an η–net set of $[-\sup\|f\|,\sup\|f\|]$ and let $K(\eta)$ be the space of step functions of the form $\sum_{S\in G}\alpha_S I_S$ where $G\in R(\mathscr{L}_m)$ and for any $S\in G$, α_S belongs to H. Then it is easy to check that $K(\eta)$ is 2η–net in K with respect to d, and hence $K(\eta)$ is η–net in K with respect to d_o.

4. <u>Weak convergence of probability measures in</u> $\tilde{D}_k$.

4.1 <u>Measurability on</u> $\tilde{D}_k$.

For any $T\subset\mathscr{I}(k)$, let φ_T denote the projection of $\tilde{D}_k$ in $\mathbb{R}^T$, and let $\mathscr{D}_k$ be the Borel σ–field generated by the Skorohod topology in $\tilde{D}_k$. For any $(J,L)\subset\{1,\dots,k\}\times\{1,\dots,k\}$, let $F_{J,L} = \{R(t,t');t_j=1\ \forall\ j\in J$ and $t'_\ell=1\ \forall\ \ell\in L\}$. Then, we have

<u>Theorem</u> 4.1. <u>$\mathscr{D}_k$ is the restriction to $\tilde{D}_k$ of the σ–field $\mathscr{B}^{\mathscr{I}(k)}$ on $\mathbb{R}^{\mathscr{I}(k)}$ where $\mathscr{B}$ is the usual Borel σ–field on $\mathbb{R}$.</u>

<u>Proof</u>. First we show that $\mathscr{B}^{\mathscr{I}(k)}\subset\mathscr{D}_k$. To prove this we have to show that $\forall\ R(t,t')\in\mathscr{I}(k)$, the map $\varphi_{R(t,t')}:\tilde{D}_k\to\mathbb{R}$ is measurable.

If $t=t'=(1,\dots,1)$, then the measurability of $\varphi_{R(t,t')}$ follows as a consequence of Lemma 3.1. If t or $t'\neq(1,\dots,1)$, then setting $J=\{j;j\in\{1,\dots,k\},t_j\neq1\}$ and $L=\{\ell;\ell\in\{1,\dots,k\},t'_\ell\neq1\}$, we notice that $L\subset J$. Then, from the continuity of the right with respect to t_j and t'_j in $R(t,t')$, it follows that $\varphi_{R(t,t')} = \lim_{\epsilon\to0} h_\epsilon$ where

$$h_\epsilon(f) = \frac{1}{\epsilon^{(\mathrm{Card}\ J\ +\ \mathrm{Card}\ L)}}\int_{\prod_{j\in J}[t_j,t_j+\epsilon)\prod_{\ell\in L}[t'_\ell,t_\ell+\epsilon)} (f(R(u,u'))(du_j)_{j\in J}(du'_\ell)_{\ell\in L}$$

and

$$\epsilon < \inf\,(\inf_{j\in J}\,(1-t_j), \inf_{\ell\in L}\,(1-t'_\ell), \inf_{\ell\in L}\,(t'_\ell-t_\ell))$$

where $u_j = 1$ if $j\notin J$, and $u'_\ell = 1$ if $\ell\notin L$.

We prove that h_ϵ is continuous with respect to the Skorohod topology.

For any $f\in\breve{D}_k$, denote

(4.1) $C(f) = \{R(s,s');R(s,s')\in\mathcal{J}(k)$ and the restriction of f to the face of $R(s,s')$ into $\mathcal{J}(k)$ is continuous in $R(s,s')\}$,

and the map

(4.2) $$\Pi_{J,L} : \mathcal{J}(k) \to [0,1]^J\times[0,1]^L$$

defined by

$$\Pi_{J,L}(R(s,s')) = ((s_j)_{j\in J},(s'_\ell)_{\ell\in L}) .$$

If $f_n\to f_0$ in the Skorohod topology, then $f_n(R(s,s')) \to f_0(R(S,s'))\ \forall\ R(s,s')\in C(f_0)$, and from Lemma 2.2.2 we deduce that $\Pi_{J,L}(C^*(f_0))$ where, C^* denotes the complement of C , has the Lebesque measure 0 in $[0,1]^J\times[0,1]^L$. It follows (from the Lebesgue Theorem) that $h_\epsilon(f_n)\to h_\epsilon(f_0)$, and so h_ϵ is continuous in the Skorohod topology. This implies that h_ϵ is measurable and hence $\varphi_{R(t,t')}$ is measurable. Thus $\mathcal{B}^{\mathcal{J}(k)}\subset\mathcal{D}_k$. Now we prove that $\mathcal{D}_k\subset\mathcal{B}^{\mathcal{J}(k)}$.

Let $T\subset\mathcal{J}(k)$, and suppose T is dense in $\mathcal{J}(k)$ and also for $(J,L)\subset\{1,\ldots,k\}\times\{1,\ldots,k\}$, $F_{J,L}\cap T$ is dense in $F_{J,L}$. Then, we prove that $\mathcal{D}_k$ is generated by $\{\varphi_{R(t,t')};R(t,t')\in T\}$.

Following Billingsley (1968, p. 121), because d_0 is separable, it suffices to show that for any $f\in\breve{D}_k$ and $r>0$, the open ball $B(f,r) = \{g\in\breve{D}_k;d_0(f,g)<r\}$ belongs to the σ–field generated by $\{\varphi_{R(t,t')};R(t,t')\in T\}$. Take a sequence $\{R(t^{(n)},t'^{(n)}\}$ of T , and suppose that it is dense in $\mathcal{J}(k)$, and the intersection of this sequence with $F_{J,L}$ is also dense in $F_{J,L}$. For any $\eta<r$ and any $N^*\in\mathbb{N}$, denote

(4.3) $A_N{}^*(\eta) = \{g\in\breve{D}_k, |||\lambda|||<r-\eta, |g(R(t^{(i)},t'^{(i)}))-f(\lambda(R(t^{(i)},t'^{(i)}))| < r-\eta, 0\le i\le N^*,$ for some $\lambda\in\Lambda^{(k)}\}$.

Then, it follows that

(4.4) $$A_{N^*}(\eta) = \varphi^{-1}_{\{R(t^{(o)},t'^{(o)}),\dots,R(t^{(N^*)},t'^{(N^*)})\}}(H_{N^*}(\eta))$$

where

(4.5) $$H_{N^*}(\eta) = \{(a_o,\dots,a_{N^*}) \in \mathbb{R}^{N^*+1}; |||\lambda||| < r-\eta \text{ and } |a_i - f(\lambda(R(t^{(i)},t'^{(i)})))| < r-\eta, 0 \le i \le N^* \text{ for some } \lambda \in \Lambda^{(k)}\}$$

which implies that $A_{N^*}(\eta)$ belongs to the σ–field generated by $\{\varphi_{R(t,t')}; R(t,t') \in T\}$.

Next, we show that $B(f,r) = \bigcup_{\eta \in Q\cap(0,r)} (\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta))$ where Q is the set of rational numbers.

It is clear that $B(f,r) \subset \bigcup_{\eta \in Q\cap(0,r)} (\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta))$. It remains to show that for each $\eta \in Q\cap(0,r)$

(4.6) $$\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta) \subset B(f,r) .$$

Let $g \in \bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta)$. Choose N^* and $\lambda^{(N^*)} \in \Lambda^{(k)}$ such that

$$|||\lambda^{(N^*)}||| < r-\eta \text{ and } |g(R(t^{(i)},t'^{(i)})) - f(\lambda^{(N^*)}(R(t^{(i)},t'^{(i)})))| < r-\eta \quad \forall\, 0 \le i \le N^* .$$

Following Billingsley (1968, p.122), we can find a subsequence $\{\lambda^{(i_{N^*})}; N^* \in \mathbb{N}\}$ of $\{\lambda^{(N^*)}, N^* \in \mathbb{N}\}$ such that $\lambda^{(i_{N^*})} \to \lambda \in \Lambda^{(k)}$, and $|||\lambda||| < r-\eta$.

For any $R(t,t') \in T$, we can find an i such that $t_j \le t_j^{(i)}$ and $t'_j \le t_j'^{(i)}$, and then we have

$$|g(R(t^{(i)},t'^{(i)})) - f(\lambda^{(i_{N^*})}(R(t^{(i)},t'^{(i)})))| < r-\eta$$

$\forall\, N^* \ge i, i \in \{0,\dots,i_{N^*}\}$. It follows that there exists a $(\rho(R(t,t')),\epsilon(R(t,t'))) = ((\rho_j)_{1\le j\le k},(\epsilon_\ell)_{1\le \ell\le k}) \in \{0,1\}^k \times \{0,1\}^k$ such that

(4.7) $$t_j = 1 \Rightarrow \rho_j = 1 \ \forall\, 1 \le j \le k \text{ and } t'_\ell = 1 \Rightarrow \epsilon_\ell = 1 \ \forall\, 1 \le \ell \le k$$

(4.8) $$Q(R(t,t'),\rho,\epsilon) \ne \phi$$

and

(4.9) $$|g(R(t,t')) - f(\lambda(R(t,t')) + 0(\rho,\epsilon))| \le r-\eta .$$

We now prove that $\|g-f\circ\lambda\| \leq r-\eta$. To prove this, suppose there exists $R(u,u')\in\mathcal{X}k)$ such that $|g(R(u,u'))-f(\lambda(R(u,u'))| > r-\eta$. Then we can find an $R(t,t')\in T$ such that for any $(\rho,\epsilon)\in\{0,1\}^k\times\{0,1\}^k$ such that (4.9) is not satisfied. This leads to contradiction. Thus $\|g-f\circ\lambda\| \leq r-\eta$. Now since $|||\lambda|||<r-\eta$, it follows that $g\in B(f,r)$. (4.6) holds.

Corollary 4.1. Let T be denumerable and dense in $\mathcal{X}k)$. Also let $F_{J,L}\cap T$ be dense in $F_{J,L}$ $\forall$ $(J,L)\subset\{1,...,k\}\times\{1,..,k\}$. Let $\mathcal{D}_T\subset\mathcal{D}_k$ where $\mathcal{D}_T = \{\varphi_U^{-1}(H_U)$, U is a finite subset of T, and H_U a Borel–subset of $\mathbb{R}^U\}$, and φ_U is the projection of D_{k+1} on $\mathbb{R}^U$. If P and Q are probability measures on $\mathcal{D}_k$, and if $P=Q$ on $\mathcal{D}_T$, then $P=Q$ on $\mathcal{D}_k$.

Proof. Consequence of Theorem 4.1.

For any probability measures on $(\breve{D}_k,\mathcal{D}_k)$, denote

$T_P=\{R(t,t');R(t,t')\in\mathcal{X}k);P(\{f;\varphi_{R(t,t')}$ is discontinuous on $f\})=0\}$

$T_P'=\{R(t,t');R(t,t')\in\mathcal{X}k);P(\{f;$ the restriction of f to the face of $R(t,t')$ into $\mathcal{X}k)$ is discontinuous in $R(t,t')\})=0\}$.

Then, it is clear that $T_P'\subset T_P$.

Theorem 4.2. $T_P\cap F_{J,L}$ is dense in $F_{J,L}$. $\forall$ $(J,L)\subset\{1,...,k\}\times\{1,...,k\}$.

Proof. It suffices to prove that $T_P'\cap F_{J,L}$ is dense in $F_{J,L}$. To prove this it suffices to show that $T_P'\cap F_{\phi,\phi}$ is dense in $F_{\phi,\phi}$.

For any $R(t,t')\in F_{\phi,\phi}$, let

$$J_{R(t,t')} = \{f;f \text{ is discontinuous in } R(t,t')\}$$
$$C = \{R(t,t');R(t,t')\in F_{\phi,\phi};P(J_{R(t,t')})\neq 0\} ,$$

and

$$J_{\{R(t,t'),\eta\}} = \{f;H(f,R(t,t')) \geq \eta\} .$$

Obviously,

$$C = \bigcup_{\ell\geq 1}\ \bigcup_{m\geq 1}\{R(t,t')\in F_{\phi,\phi};P(J_{\{R(t,t'),1/m\}}) \geq \tfrac{1}{\ell}\} .$$

Now using Lemma 2.2.2 and proceeding as in Billingsley (1968, p. 124) we find that

$\{R(t,t');R(t,t')\in F_{\phi,\phi};P(J_{\{R(t,t')\}})\geq\eta_1\}$ has a Lebesque measure zero. This implies that C has a Lebesgue measure zero. Consequently $T'_P \cap F_{\phi,\phi}$ is dense in $F_{\phi,\phi}$.

Theorem 4.3. The sequence $\{P_n\}_{n\geq 1}$ of probability measures on $(\breve{D}_k,\mathcal{D}_k)$ converges weakly to a probability measure P on $(\breve{D}_k,\mathcal{D}_k)$ if and only if

(4.10) $\varphi_U(P_n)$ converges weakly to $\varphi_U(P)$ for all finite subsets U of T_P.

(4.11) the sequence is tight.

Proof. The necessary part is obvious. To prove the sufficiency part, note that since the sequence $\{P_n\}$ is tight, it is weakly relatively compact, we have to prove (cf. Billingsley (1968), Theorem 2.3) that any subsequence of $\{P_n\}$ which converges admits P as limit. For this we use the same line of argument as in Billingsley (1968, Theorem 15.1), the fact that for any probability measure Q and any finite subset U of T_Q, φ_U is a.s. Q–continuous, and the Corollary 4.1.

Corollary 4.3.1. Let $\{P_n\}_{n\geq 1}$ be a sequence of probability measures on $(\breve{D}_k,\mathcal{D}_k)$ such that

(4.12) $$\lim_{\delta\to 0}\limsup_{n\to\infty} P_n(f;\omega'(f,\delta)\geq\epsilon) = 0 \quad \forall\ \epsilon>0$$

and there exists a probability measure P on $(\breve{D}_k,\mathcal{D}_k)$ such that

(4.13) $$\varphi_U(P_n) \to \varphi_U(P) \quad \text{weakly for any finite subset U of } T_P.$$

Then,

(4.14) $$P_n \to P \quad \text{weakly with respect to the Skorohod topology.}$$

Proof. Follows from Theorems 3.1 and 4.3.

Corollary 4.3.2. Let $\{P_n\}_{n\geq 1}$ be a sequence of probability measures on $(\breve{D}_k,\mathcal{D}_k)$ such that

(4.15) $$\lim_{\delta\to 0}\limsup_{n\to\infty} P_n(f;\tilde{\omega}(f,\delta)\geq\epsilon) = 0 \quad \forall\ \epsilon>0$$

and

(4.16) $\varphi_U(P_n)$ converges weakly to some probability measure P_U on R^U for every finite subset U of $\mathcal{J}(k)$. Then, (4.17) P_n converges weakly to some probability measure P in Skorohod topology with $P(\check{C}_k) = 1$.

Proof. Follows from Corollary 4.3.1 and the inequality $\omega'(f,\delta) \leq \tilde{\omega}(f,\delta)$.

5. Convergence of the process $\hat{W}_n$.

5.1 Preliminaries and some basic tools. In this section we study the asymptotic behavior of the empirical process $\hat{W}_n$ defined in (1.2). Before we do that we introduce the space D_k .

Let $f : [0,1]^k \to \mathbb{R}$. For $\rho \in \{0,1\}^k$, define $f_\rho(t) = \lim_{\substack{s_i \uparrow t_i, \rho(i)=1 \\ s_i \downarrow t_i, \rho(i)=0}} f(s) \quad ((s,t) \in [0,1]^k)$, if it exists; in which case, call $f_\rho(t)$ the ρ–limit of f at t . Denote by D_k , the space of all maps $f : [0,1]^k \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^k$, f_ρ exists and $f_\rho = f$ for $\rho = (0,\ldots,0)$.

For any map $f : [0,1]^k \to \mathbb{R}$, and any rectangle $B = \prod_{j=1}^{k} (a_j,b_j]$, we denote a difference operator $\Delta_B f$ by

(5.1) $$\Delta_B f = \Sigma(-1)^{\text{card } I} f((b_i)_{i \in I}, (a_i)_{i \notin I})$$

where card I is the cardinal of I , and Σ is over all the 2^k subsets $I \subset \{1,\ldots,k\}$.

We will study the asymptotic behavior of $\hat{W}_n$ via the empirical process W_n defined as

(5.2) $$W_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \left[\prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)}) \leq t_j]} - H_{ni}(t) \right]$$

The process W_n has been studied by Balacheff and Dupont (1980), among others. (See the references in Balacheff and Dupont (1980)).

It is well known that the process W_n has a.s. (almost sure) trajectories in D_k . Set $B = \prod_{j=1}^{k} (a_j,b_j] \in \mathcal{J}(k)$, and by convention, we shall put $\hat{W}_n(B)=0$ if there exists at least one j for which $b_j=a_j$. Then the process $\hat{W}_n$ has a.s. trajectories in the space $\check{D}_k$.

Our results of section 5 as well as section 6 are based on the following two lemmas.

Lemma 5.1. Let the sequence $\{X_{ni}\}$ of real–valued random variables centered at their expectations be φ–mixing with rates $\sum_{m\geq 1} m^{-1}\varphi^{1/2q}(m) < \infty$. Let N_n be the number of indexes i $(1\leq i\leq n)$ for which X_{ni} is not identically zero. Set $S_n = \sum_{i=1}^{n} X_{ni}$, and $\|X_{ni}\|_\ell = (\int |X_{ni}|^{2\ell} dP_n)^{1/2\ell}$. Then, there exists a constant $C_q(\varphi)$ depending only on $q\in N^* = \{1,2,\ldots\}$ and φ such that

$$E(S_n^{2q}) \leq C_q(\varphi) \sum_{\ell=1}^{q} N_n^{q/\ell} (\sup_{1\leq i\leq n} \|X_{ni}\|_\ell)^{2q} . \tag{5.3}$$

Proof. The proof is a slight modification of Theorem 2.1 of Neumann (1982) and is therefore omitted.

Lemma 5.2. Let the sequence $\{X_{ni}\}$ of real–valued random variables centered at their expectations be strong mixing with rates $\sum_{m>1} m^{2q-2}\alpha^{\epsilon}(m) > \infty$, $q\geq 1$, $\epsilon \in (0,1/2q)$, and $|X_{ni}|\leq 1$, $1\leq i\leq n$, $n\geq 1$. Let N_n be the number of indexes i $(1\leq i\leq n)$ for which X_{ni} is not identically zero. Set $S_n = \sum_{i=1}^{n} X_{ni}$ and $\|X_{ni}\|_\epsilon = (\int |X_{ni}|^{2/(1-\epsilon)})^{1-\epsilon}$. Then, there exists a constant $C_q(\alpha)$ depending only on q and α such that

$$E(S_n^{2q}) \leq C_q(\alpha) \sum_{\ell=1}^{q} N^{\ell} (\sup_{1\leq i\leq n} \|X_{ni}\|_\epsilon^{\ell}) . \tag{5.4}$$

Proof. The proof is essentially the same as in Doukhan and Portal (1983) and is therefore omitted.

Now for any grid G with base $B = \{t_{ji}, 1\leq i\leq n_j, 1\leq j\leq k\}$, the number $\tau = \max_{1\leq j\leq k} \max_{1\leq j\leq n_j-1} \{|t_{ji}-t_{ji+1}|\}$ is called the pace of the grid G.

Let $\{G_n\}_{n\geq 1}$ be a sequence of grids with paces $\{\tau_n\}_{n\geq 1}$. $\{G_n\}_{n\geq 1}$ is called asymptotically dense in $[0,1]^k$ if $\tau_n \to 0$ as $n\to\infty$. We denote the lower boundary of a subset C of $[0,1]^k$ by $\underset{\leftarrow}{C}$ where $\underset{\leftarrow}{C} = \{t; t\in C, t_j = 0 \text{ for at least one j}, 1\leq j\leq k\}$. For any base $B = \{t_{ji}, 1 \le i \le n_j, 1 \le j \le k\}$, we say that a subset $\tilde{B}$ is tied to B if $\tilde{B} = \{t; t_j \in \{t_{j1}, \ldots, t_{jn_j}\}, 1 \le j \le k\}$ and $R(t,t')$ is <u>tied</u> to B if $t_j \in \{t_{j1}, \ldots, t_{jn_j}\}$ and $t'_j \in \{t_{j1}, \ldots, t_{jn_j}\}$ $\forall\ 1 \le j \le k$ and $|R(t,t')| \ne 0$ where $|R(t,t')|$ is the Lebesgue measure of $R(t,t')$, i.e. $|R(t,t')| = \prod_{j=1}^{k} (t'_j - t_j)$. Note that $\tilde{B}$ is unique whereas $R(t,t')$ is not unique.

For any base B of a grid G and the corresponding subset $\tilde{B}$ tied to B , denote

(5.5) $$\omega_B(f,\delta) = \sup\{|f(t)-f(t')|; (t,t') \in \tilde{B} \times \tilde{B}, \|t-t'\| \le \delta\} \text{ for any } \delta > 0$$

where $\|t\| = \sup\{|t_j|, 1 \le j \le k\}$.

Now for any bounded function $f: [0,1]^k \to \mathbb{R}$, denote

(5.6) $$\omega(f,\delta) = \sup\{|f(t)-f(t')|, (t,t') \in [0,1]^k \times [0,1]^k, \|t-t'\| \le \delta\} .$$

Let P_n, $n \ge 1$ be a sequence of probability measures on $(D_k, \mathcal{D}_k)$ where $\mathcal{D}_k$ is the σ-field generated by the Skorohod topology (on D_k) . We say that the sequence $\{G_n\}$ of grids with bases $\{B_n\}$ <u>accompanies</u> the measure P_n if and only if $\forall\ \epsilon > 0$, $\exists\ \epsilon' > 0$ and $\forall\ \delta \in [0,1/2)$, $\exists\ N_0 \ge 1$ such that $P_n[\{f \in D_k; \omega(f,\delta) \ge \epsilon$ and $\omega_{B_n}(f,2\delta) < \epsilon'\}] = 0\ \forall\ n \ge N_0$.

For the ease of convenience we state the following lemma due to Balacheff and Dupont (1980) which will be used in the sequel.

<u>Lemma</u> 5.3. <u>Let</u> ν <u>be a positive finite measure on</u> $[0,1]^k$ <u>with continuous marginals. Let</u> $\{P_n\}_{n \ge 1}$ <u>be a sequence of probability measures on</u> $(D_k, \mathcal{D}_k)$ <u>such that</u> $P_n[f \in D_k; f|\partial[0,1]^k = 0] = 1\ \forall\ n \ge 1$. <u>Suppose</u> $\{G_n\}_{n \ge 1}$, <u>a sequence of grids with bases</u> $\{B_n\}_{n \ge 1}$, <u>is asymptotically dense in</u> $[0,1]^k$, <u>and accompanies</u> P_n . <u>Furthermore suppose that for any</u> $R(t^{(n)}, t'^{(n)}) \in \mathcal{J}(k)$ <u>tied to</u> B_n ,

(5.7) $$P_n[f \in D_k; |\Delta_{R(t^{(n)}, t'^{(n)})} f| > \lambda] \le \lambda^{-\gamma} [\nu(R(t^{(n)}, t'^{(n)}))]^{\beta}, \quad \beta > 1, \gamma > 0 .$$

<u>Then,</u> $\forall\ \epsilon > 0$, $\exists\ \delta \in (0,1)$ <u>and</u> $N_0 \ge 1$ <u>such that</u>

(5.8) $$P_n[f \in D_k; \omega(f,\delta) \ge \epsilon] \le \epsilon\ \forall\ n \ge N_0 .$$

Finally, we need the notion of μ-boundedness.

We shall say that the sequence $\{H_{ni}\}$ is μ–bounded if there exists a finite and positive measure μ on $[0,1]^k$ with continuous marginal distributions such that for every $1\le i\le n$, $n\ge 1$, $H_{ni}(R(t,t')) \le \mu(R(t,t'))$ for any rectangle $R(t,t')$ in $\mathcal{J}(k)$.

5.2 Convergence of $\check{W}_n$.

Theorem 5.2.1. Assume that the sequence $\{X_{ni}\}$ is (a) φ–mixing with rates (1.3) or (b) strong mixing with rates (1.4); the sequence $\{H_{ni}\}$ is (c) μ–bounded where μ is absolutely continuous with bounded density f_μ or (d) $\{H_{ni}\}$ has uniform marginals for all $n\ge 1$ and $1\le i\le n$. Furthermore assume that (e) the covariance function C_n of the empirical process W_n defined in (5.2) converges to a function C. Then, $\check{W}_n$ converges weakly in the Skorohod topology to a Gaussian process $\check{W}_0$ with trajectories a.s. in $\check{C}_k$.

Proof. Let the probability measure $\check{Q}_n$ (resp. Q_n) on $(\check{D}_k,\check{\mathcal{D}}_k)$ (resp. $(D_k,\mathcal{D}_k)$) be associated with $\check{W}_n$ (resp. W_n). To prove this theorem, we have to verify (4.12) and (4.13). Then following Withers (1975) $\varphi_U(\check{Q}_n) \to$ weakly to a Gaussian measure $\check{Q}_U$ if (i) $C_n \to$ some function C, (ii) $\sum_{m\ge 1} \alpha(m) < \infty$, and (iii) $m^{1-a}\alpha([m^b]) \to 0$ (as $m\to\infty$) where $0<2b<a<1-b$. Now in our situation (i) holds by assumption (e); (ii) follows from (1.3) and (1.4); and (iii) from (1.3) and (1.4) by taking $a = 3/4 - \epsilon/8$, $b = 1/4$ and ϵ sufficiently small (since taking $\alpha(m) = m^{-1-\epsilon}$, $m^{1-a}\alpha([m^b]) \le Am^{-\epsilon/8}$ where $A>0$ is some constant). Thus (4.13) is proved. We now prove (4.12). Since $|\check{W}_n(B)| = |\Delta_B W_n|$ where Δ_B is defined in (5.1), we have

$$(5.9)\qquad \sup\{|\check{W}_n(R(t,t'))-\check{W}_n(R(s,s'))|;d_H(R(t,t'),R(s,s'))\le\delta\}$$
$$\le \sup\{2k|W_n(t)-W_n(s)|;\|t-s\|\le\delta\},$$

to prove (4.12), it suffices to prove (5.8) for W_n. (5.8) will follow if we prove (5.7).

Let $B_n = \{i/n;0\le i\le n\}^k$, $n\ge 1$ be a sequence of bases of grids G_n, $n\ge 1$. Note that G_n is asymptotically dense in $[0,1]^k$, and we prove that G_n accompanies Q_n. Now for every $t\in[0,1]^k$, let $(\underline{t},\bar{t})$ be the points of $\check{B}_n$ where $\check{B}_n$ is tied to B_n such that $\underline{t}\le t\le \bar{t}$ and $\|\bar{t}-\underline{t}\|\le 1/n$. Then, with the conditions (c) or (d) we obtain, after some computations that $|W_n(t)-W_n(t')| \le \frac{2kK(\mu)}{\sqrt{n}} + |W_n(\bar{t}) - W_n(\underline{t}')|$ for every $t\in[0,1]^k$ and $\forall\, t'\in[0,1]^k$ where $K(\mu) = \sup_{t\in[0,1]^k} f_\mu(t)$ if we have (c), and $K(\mu)=1$ if we have (d). Consequently, for every $\delta\in(0,1/2]$, we have $\omega(W_n,\delta) \le \frac{2kK(\mu)}{\sqrt{n}} + \omega_{B_n}(W_n,2\delta)$. It follows that G_n accompanies Q_n. It remains to show that Q_n satisfies (5.7).

Suppose we have condition (c). Let $\sum_{m=1}^{\infty} m^{-1}\varphi^{\frac{1}{4}}(m) < \infty$ (implied by (1.3)) and let $R(t^{(n)},t'^{(n)})$ be tied to B_n. Using Lemma 5.1, with $q=2$, we obtain

$$(5.10)\quad E[\hat{W}_n(R(t^{(n)},t'^{(n)}))]^4 \le C_2(\varphi)[(K(\mu)\prod_{j=1}^{k}(t_j'-t_j))^2 + n^{-1}(K(\mu)\prod_{j=1}^{k}(t_j'-t_j))] .$$

Let $\nu = (C_2(\varphi)(K(\mu)+K^2(\mu)))^{\beta^{-1}} U^k$ where U^k is the uniform probability measure on $[0,1]^k$ and $\beta = 1+k^{-1}$. Then, by the Markov inequality, we obtain

$$Q_n[f\in D_k; |\Delta_{R(t^{(n)},t'^{(n)})} f| > \lambda] \le \lambda^{-4}[\nu(R(t^{(n)},t'^{(n)}))]^{\beta}$$

which implies (5.7) for the φ–mixing case with rates (1.3). (5.8) follows.

For the strong mixing case with rates (1.4), we use Lemma 5.2 for $q=2$ and $\epsilon < (2k+4)^{-1}$, and obtain

$$E[\hat{W}_n(R(t^{(n)},t'^{(n)}))]^4 \le C_2(\alpha)[K(\mu)\prod_{j=1}^{k}(t_j-t_j'))^{2(1-\epsilon)} + n^{-1}(K(\mu)\prod_{j=1}^{k}(t_j-t_j'))^{1-\epsilon}]$$

which (with $\beta=(1-\epsilon)+k^{-1}$) implies (5.7) and hence (5.8) by proceeding as above.

Now let us suppose we have condition (d). Then, for the φ–mixing case with rates $\sum_{m=1}^{\infty} m^{-1}\varphi^{1/2(k+1)}(m) < \infty$ (implied by (1.3)), we use Lemma 5.1 with $q=k+1$, and obtain

$$E[\hat{W}_n(R(t^{(n)},t'^{(n)}))]^{2(k+1)} \le C_{k+1}(\varphi) \sum_{\ell=1}^{k+1} n^{-(k+1)}[n^{(k+1)/\ell}\prod_{j=1}^{k}(t_j'-t_j)^{(k+1)/k\ell}]$$

and proceeding as in the φ–mixing case dealt with above, we get the desired result.

For the strong mixing case with rates (1.4), we use Lemma 5.2 with $q=k+1$ and $\epsilon < 1/2(k+2)$, and obtain

$$E[\hat{W}_n(R(t^{(n)},t'^{(n)}))]^{2(k+1)} \le C_{k+1}(\alpha) \sum_{\ell=1}^{k+1} n^{-(k+1-\ell)} \prod_{j=1}^{k} (t_j'-t_j)^{\ell(1-\epsilon)/k}$$

and then proceed as above for the first φ-mixing case.

The proof follows.

6. Convergence of the weighted empirical process.

We start with the definition of the weight function.

Definition 6.1. A function $r : [0,1] \to \mathbb{R}^+$ is called a weight function if it satisfies the following conditions:

(i) r is continuous.

(ii) $r(u)=0$ if $u=0$ or $u=1$.

We will consider a modified empirical process $\hat{W}_n$ defined as

(6.1) $$\hat{W}_n(R(t,t')) = \begin{cases} \tilde{W}_n(R(t,t')) & \text{if } |R(t,t')| \ge n^{-1} \text{ and } |R(t,t')| \le 1-n^{-1} \\ 0 & \text{otherwise} \end{cases}$$

where $|R(t,t')|$ is defined in section 5.

For any weight function r , we introduce a weighted modified empirical process $\hat{W}_n/r$ defined as

(6.2) $$\frac{\hat{W}_n}{r}(R(t,t')) = \begin{cases} \hat{W}_n(R(t,t'))/r(|R(t,t')|) & \text{if } |R(t,t')| \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the following lemma is a slight variation of a proposition in Harel (1980).

Lemma 6.1. For any $n \ge 1$, let Y_n be a process with values in $\tilde{D}_k$, and measurable with respect to $\mathcal{D}_k$. Suppose $Y_n \to Y_o$ in law where Y_o is a Gaussian process with trajectories a.s. continuous. Let P_n be the probability measure associated with Y_n . Let r be a weight function such that

(6.3) $$Y_n \cdot \frac{1}{r} \text{ has trajectories a.s. in } \tilde{D}_k , \ (n \ge 1)$$

(6.4) $\forall\, \epsilon>0, \exists\, \theta>0, \exists\, N_o \ge 1$ such that $P_n[\sup \{|Y_n(R(t,t'))\frac{1}{r}(|R(t,t')|)|\} \ge \epsilon] \le \epsilon \quad \forall\, n \ge N_o$ where sup is over $R(t,t')$ with the condition that $|R(t,t')| \le \theta$ or $1-|R(t,t')| \le \theta$.

Then $Y_n \cdot \frac{1}{r}$ converges weakly in Skorohod topology to the Gaussian process $Y_o \cdot \frac{1}{r}$ with trajectories a.s. in $\tilde{C}_k$.

Theorem 6.1. If the sequence $\{X_{ni}\}$ satisfies the assumptions of Theorem 5.2, then for any weight function r satisfying

(6.5) $$r(u) \geq A[u(1-u)]^{\frac{1}{2}-\delta}, \quad A>0$$

where

(6.6) $$0 < \tfrac{1}{2}-\delta < 1/8k \text{ if the condition (c) of Theorem 5.2. is satisfied}$$

or

(6.7) $$0 < \tfrac{1}{2}-\delta < \frac{1}{2k(k+1)} \text{ if the condition (d) of Theorem 5.2 is satisfied.}$$

$\hat{W}_n/r$ converges weakly in the Skorohod topology to the Gaussian process $\hat{W}_o/r$ with trajectories a.s. in $\tilde{C}_k$ (where $\hat{W}_o$ is the same as in Theorem 5.2).

Proof. Convergence of $\hat{W}_n$ to $\hat{W}_o$ follows from the definition of $\hat{W}_n$ and Theorem 5.2. The theorem will follow if (6.3) and (6.4) are satisfied. (6.3) follows from the definition of $\hat{W}_n$. We now prove (6.4).

For any $\theta\in[0,1]$, let $C_\theta^{(1)}$ and $C_\theta^{(2)}$ be two subsets of $\mathcal{X}(k)$ where

$C_\theta^{(1)} = \{R(t,t'); |R(t,t')| \leq \theta\}$

$C_\theta^{(2)} = \{R(t,t'); |R(t,t')| \geq 1-\theta\}$.

(6.4) will follow if we show that $\forall\ \eta>0, \exists\ \theta>0, \exists\ N_o \geq 1$ such that

(6.8) $$P_n\Big[\sup_{R(t,t')\in C_\theta^{(i)}} \Big|\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}\Big| \geq \eta\Big] \leq \eta \quad \forall\ n\geq N_o,\ i=1,2\ .,$$

First let us take $i=1$.

Without any loss of generality, let $r(u) = u^{\frac{1}{2}-\delta}$. Set, (for any $n\geq 1$),

(6.9) $$m_n = \max\{m; \tfrac{1}{n} \leq \tfrac{1}{2^m}, m\geq 0\}$$

and

(6.10) $$m(\theta) = \max\{m; \theta \leq 1/2^m, m\geq 0\}\ .$$

For any $p\geq 0$, consider a base B_p of some grid G_p where

(6.11) $$B_p = \{t_{ji}; 1\leq i\leq n_j^{(p)}, 1\leq j\leq k,\ t_{ji} = i/2^p,\ n_j^{(p)} = 2^p\}\ .$$

We need a few lemmas.

Lemma 6.1. Let $\theta \in [0,1]$ and suppose a function $f: [0,1] \to \mathbb{R}$ is given. Then, for any two points $(u,v) \in \{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, \frac{2^m-1}{2^m}, 1\} \times \{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, \frac{2^m-1}{2^m}, 1\}$ with $|u-v| \le 2^{-m(\theta)}$ where m is an integer $> m(\theta)$, the following inequality holds:

$$|f(u)-f(v)| \le 4 \sum_{r=m(\theta)}^{m} \sup |f(u_1+2^{-r}) - f(u_1)|$$

where the sup is taken for all $u_1 \in \{0, \frac{1}{2^r}, \frac{2}{2^r}, \ldots, \frac{2^r-1}{2^r}, 1\}$ and $u_1+2^{-r} \in [0,1]$.

Proof. Follows from Neuhaus (1971, Lemma 5.1).

Lemma 6.2. Let $\theta \in [0,1]$ and suppose a function $f: [0,1]^k \to \mathbb{R}$ be given. Then, for any $(t,t') \in B_m \times B_m$ with $t<t'$ and $|R(t,t')| \le 2^{-m(\theta)}$, the following inequality holds:

$$\frac{|\Delta_{R(t,t')} f|}{r(|R(t,t')|)} \le 4^k \sum_{\substack{r_1 \le m \\ r_1+\ldots+r_k \ge m(\theta)}} \cdots \sum_{r_k \le m} \sup \left\{ \frac{\left| \Delta_{\prod_{j=1}^{k}(u_j, u_j+2^{-r_j}]} f \right|}{\left(\prod_{j=1}^{k} 2^{-r_j}\right)^{1/2-\delta}} \right\} \tag{6.13}$$

where the sup is taken over all $u_j \in \{0, \frac{1}{2^{r_j}}, \ldots, \frac{2^{r_j}-1}{2^{r_j}}\}$ and $u_j+2^{-r_j} \in [0,1]$, $1 \le j \le k$, and where m is an integer $> m(\theta)$.

Proof. From Lemma 6.1, we deduce (by iteration)

$$|\Delta_{R(t,t')} f| \le 4^k \sum_{\substack{r_1 \le m \\ r_1+\ldots+r_k \ge m(\theta)}} \cdots \sum_{r_k \le m} \sup \{ |\Delta_{\prod_{j=1}^{k}(u_j, u_j+2^{-r_j}]} f| . \tag{6.14}$$

Since each term on the left side of (6.14) is less than or equal to a finite number of terms on the right side of (6.14) for which the Lebesgue measure $|R(t,t')| \ge \prod_{j=1}^{k} 2^{-r_j}$, and since r is a nondecreasing function, we obtain (6.13).

Lemma 6.3. $\forall \eta>0, \exists \theta>0, \exists N \ge 1$ such that

$$P_n \left[\sup_{\substack{R(t,t') \in \mathbb{B}_m \cap C_\theta^{(1)} \\ |R(t,t')| \ne 0^n}} \left\{ |\mathbb{W}_n(R(t,t')) \cdot \frac{1}{r(|R(t,t')|)}| \ge \eta \right\} \right] \le \eta \;\; \forall n \ge N . \tag{6.15}$$

Proof. See Appendix.

Now we prove (6.8) for $i=1$. We assume that the condition (c) of Theorem 5.2 holds. Let $\theta\in[0,1]$ and n be fixed.

For any $R(t,t')\in\mathcal{X}(k)$ for which $|R(t,t')|\geq n^{-1}$, let $\underline{t},\bar{t},\underline{t}'$ and $\bar{t}'$ be points of B_{m_n} such that

$\underline{t}_j\leq t_j\leq\bar{t}_j$, $\underline{t}'_j\leq t'_j\leq\bar{t}'_j$, $\bar{t}_j-\underline{t}_j\leq 2^{-m_n}$ and $\bar{t}'_j-\underline{t}'_j\leq 2^{-m_n}$, $1\leq j\leq k$.

For any $(J,L)\subset\{1,\ldots,k\}\times\{1,\ldots,k\}$ for which $J\cap L=\phi$, we define an element $R(t,t')(J,L)$ of $\mathcal{X}(k)$ by

$$R(t,t')(J,L)=\prod_{j\in J}(\underline{t}_j,t_j]\prod_{j\in L}(t'_j,\bar{t}'_j]\prod_{j\notin J\cup L}(\underline{t}_j,\bar{t}'_j].$$

Then, we have the inequality:

$$(6.16)\qquad |\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}|\leq\sum_{\substack{(J,L)\subset\{1,\ldots,k\}\times\{1,\ldots,k\}\\ J\cap L=\phi}}|\hat{W}_n(R(t,t')(J,L))\frac{1}{r(|R(t,t')|)}|.$$

If $J=L=\phi$, then we have the inequality

$$(6.17)\qquad |\hat{W}_n(R(t,t')(\phi,\phi))\frac{1}{r(|R(t,t')|)}|\leq 3^k|\hat{W}_n(R(\underline{t},\bar{t}'))\frac{1}{r(|R(\underline{t},\bar{t}')|)}|$$

If $J\cup L\neq\phi$, then we have the inequality

$$(6.18)\qquad |\hat{W}_n(R(t,t')(J,L))\frac{1}{r(|R(t,t')|)}|$$
$$\leq\hat{W}_n(G_k)\times\frac{1}{r(|R(t,t')|)}|+n^{\frac{1}{2}}K(\mu)U^k(G_k)\cdot\frac{1}{r(|R(t,t')|)}$$

where

$$G_k=\prod_{j\in J}(\underline{t}_j,\bar{t}_j]\prod_{j\in L}(\underline{t}'_j,\bar{t}'_j]\prod_{j\notin J\cup L}(\underline{t}_j,\bar{t}'_j].$$

(6.18) follows due to the fact that for any $(B_1,B_2)\in\mathcal{X}(k)\times\mathcal{X}(k)$ where $B_2\subset B_1$,

$$(6.19)\qquad |\hat{W}_n(B_2)|\leq|\hat{W}_n(B_1)|+n^{\frac{1}{2}}\mu(B_1).$$

As r is non–decreasing, we deduce

$$(6.20)\qquad |\hat{W}_n(R(t,t')(J,L))\frac{1}{r(|R(t,t')|)}|$$
$$\leq 3^k|\hat{W}_n(G_k)\times(r(|G_k|))^{-1}+n^{\frac{1}{2}}(K(\mu)\prod_{j\in J}(\bar{t}_j-\underline{t}_j)\prod_{j\in L}(\bar{t}'_j-\underline{t}'_j))^{(\frac{1}{2}+\delta)}\}.$$

Now

(6.21) $$n^{\frac{1}{2}}\Big(\prod_{j\in J}(\bar{t}_j-\underline{t}_j)\prod_{j\in L}(\bar{t}'_j-\underline{t}'_j)\Big)^{(\delta+\frac{1}{2})}=n^{\frac{1}{2}}(2^{-m_n})^{(\delta+\frac{1}{2})\ell}\to 0$$

as $n\to\infty$, where $\ell=\text{Card } J\cup L$.

Using (6.16), (6.17), (6.20) and (6.21), we obtain

(6.22) $$\sup_{R(t,t')\in C_\theta^{(1)}}|\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}|\le$$

$$\le K\sup_{\substack{R(t,t')\in\mathcal{B}_m\cap C_\theta^{(1)}\\|R(t,t')|\ne 0}}|\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}|+0(n^{\frac{1}{2}}(2^{-m_n})^{(\delta+\frac{1}{2})\ell})$$

where $K>0$ is some constant.

Using (6.22) along with Lemma (6.3) we get (6.8) for $i=1$ when the condition (c) of Theorem 5.2 is satisfied. The proof when the condition (d) of Theorem 5.2 is satisfied is essentially similar, and is therefore omitted.

Now we prove (6.8) for $i=2$.

Without loss of generality, we take $r(u)=(1-u)^{\frac{1}{2}-\delta}$. Following the ideas of Einmahl, Ruymgaart and Wellner (1984), we start with the equality $\hat{W}_n(B)=-\hat{W}_n(B^*)$ where B^* is the complement of B and $\hat{W}_n(B^*)=n^{-\frac{1}{2}}\sum_{i=1}^{n}[I_{[F_n(X_{ni})\in B^*]}-H_{ni}(B^*)]$, where $F_n=(F_n^{(1)},\dots,F_n^{(k)})$. For any $R(t,t')\in\mathcal{X}(k)$ with $|R(t,t')|\le 1-n^{-1}$ and using the union–intersection principle (see Einmahl, Ruymgaart and Wellner (1984)), we obtain

(6.23) $$|\hat{W}_n(R(t,t'))|\le\sum_{\ell\in L}|\hat{W}_n(R_\ell(t,t'))|$$

where L is a finite index set, $R_\ell(t,t')$ is an element of $\mathcal{X}(k)$ with the condition that there exists $J\subset\{1,\dots,k\}$ and $M\subset\{1,\dots,k\}$ with $J\cap M=\phi$ and $J\cup M\ne\phi$ such that

$$R_\ell(t,t')=\prod_{j\in J}(0,t_j]\cdot\prod_{j\in M}(t'_j,1]\cdot\prod_{j\notin J\cup M}(0,1].$$

From (6.23), we get

(6.24) $$\sup_{R(t,t')\in C_\theta^{(2)}}|\hat{W}_n(R(t,t'))\frac{1}{r(|R(t,t')|)}|\le\sum_{\ell\in L}|\hat{W}_n(R_\ell(t,t'))\frac{1}{(|R_\ell(t,t')|\vee n^{-1})^{\frac{1}{2}-\delta}}$$

where $a\vee b=\max(a,b)$.

If $|R_\ell(t,t')|\ge n^{-1}$, then $R_\ell(t,t')\in C_\theta^{(1)}$ and from (6.8) for $i=1$, we get the desired result for $\hat{W}_n(R_\ell(t,t'))\cdot(r(|R(t,t')|))^{-1}$.

If $|R_\ell(t,t')| < n^{-1}$, then using the inequality (6.19), we get

$$(6.25)\qquad \hat{W}_n(R_\ell(t,t'))\cdot\frac{1}{(n^{-1})^{\frac{1}{2}-\delta}}| \le |\hat{W}_n(R_\ell^{(n)}(t^{(n)},t'^{(n)}))\,\frac{1}{(n^{-1})^{\frac{1}{2}-\delta}}| + n^{\frac{1}{2}}\,\mu(R_\ell^{(n)}(t^{(n)},t'^{(n)}))/(n^{-1})^{\frac{1}{2}-\delta}$$

where $R_\ell^{(n)}(t^{(n)},t'^{(n)})\in\mathcal{J}(k)$, $|R_\ell^{(n)}(t^{(n)},t'^{(n)})| = n^{-1}$ and $R_\ell(t,t') \subset R_\ell^{(n)}(t^{(n)},t'^{(n)})$.

Since $R_\ell^{(n)}(t^{(n)},t'^{(n)})\in C_\theta^{(1)}$, we get the desired result from (6.8) for i=1 for $\hat{W}_n(R_\ell^{(n)}(t^{(n)},t'^{(n)}))\cdot\frac{1}{r(|R_\ell^{(n)}(t^{(n)},t'^{(n)})|)}$. Since $|R_\ell^{(n)}(t,t')| = n^{-1}$,

$$n^{\frac{1}{2}}\,\frac{\mu(R_\ell^{(n)}(t^{(n)},t'^{(n)})}{(n^{-1})^{\frac{1}{2}-\delta}} \le K(\mu)\,\frac{n^{\frac{1}{2}}\cdot n^{-1}}{(n^{-1})^{\frac{1}{2}-\delta}} = K(\mu)\,n^{-\delta} \to 0$$

as $n\to\infty$.

Using (6.24), and the properties of $\hat{W}_n(R_\ell(t,t'))\cdot\frac{1}{(|R_\ell(t,t')|\vee n^{-1})^{\frac{1}{2}-\delta}}$ for each $\ell\in L$ obtained in the discussion following (6.24), we get the desired result, viz. (6.8) for i=2. The proof follows.

7. Appendix.

Proof of Lemma 6.3. (a) Let us suppose that $\{X_{ni}\}$ is φ-mixing with rates (1.3) and $\{H_{ni}\}$ is μ-bounded (viz. conditions (a) and (c) of Theorem 5.2.1).

From Lemma 6.2, we obtain if $m(\theta) \le m_n$ that

$$(7.1)\qquad \sup_{\substack{R(t,t')\in\mathcal{B}_{m_n}\cap C_\theta^{(1)}\\ |R(t,t')|\neq 0^n}} |\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}|$$
$$\le 4^k \sum_{\substack{r_1\le m_n\\ r_1+\dots+r_k\ge m(\theta)}}\cdots\sum_{r_k\le m_n} \sup|\hat{W}_n(\prod_{j=1}^{k}(u_j,u_j+2^{-r_j}])\,\frac{1}{(\prod_{j=1}^{k}2^{-r_j})^{\frac{1}{2}-\delta}}|$$

where the sup is taken as in (6.13). Let a be a real number such that

$$(7.2)\qquad 0<a<1 \ \text{ and } \ a^4\cdot 2^{\frac{1}{k}-4(\frac{1}{2}-\delta)} > 1\,.$$

If the left side of (7.1) exceeds η, then $\exists\,(r_1,\dots,r_k)\in\{1,\dots,m_n\}\times\dots\times\{1,\dots,m_n\}$ with

$r_1+\ldots+r_k \ge m(\theta)$ such that

$$\sup \{|\hat{W}_n(A_k) \cdot \frac{1}{(\prod_{j=1}^{k} 2^{-r_j})^{\frac{1}{2}-\delta}}|\} > B_k \tag{7.3}$$

where $A_k = \prod_{j=1}^{k} (u_j, u_j+2^{-r_j}]$, $B_k = \frac{a^{\sum_{i=1}^{k} r_i - m(\theta)}}{4^k B_k^{(1)}} \eta$ and

$$B_k^{(1)} = \sum_{r_1+\ldots+r_k \ge m(\theta)} a^{\sum_{i=1}^{k} r_i - m(\theta)} = 0(m^{k-1}(\theta)) .$$

From (7.3), we deduce

$$P_n\left[\sup_{\substack{|R(t,t')|\neq 0 \\ R(t,t')\in B_{m_n}\cap C_\theta^{(1)}}} \{|\hat{W}_n(R(t,t')) \frac{1}{r(|R(t,t')|)}| \ge \eta\}\right] \le$$

$$\le \sum_{\substack{r_1\le m_n \\ r_1+\ldots+r_k\ge m(\theta)}} \cdots \sum_{r_k\le m_n} \sum_{u_1=0}^{2^{r_1}-1} \cdots \sum_{u_k=0}^{2^{r_k}-1} P_n[|\hat{W}_n(A_k) \frac{1}{(\prod_{j=1}^{k} 2^{-r_j})^{\frac{1}{2}-\delta}}| > B_k] \tag{7.4}$$

$$\le \sum_{\substack{r_1\le m_n \\ r_1+\ldots+r_k\ge m(\theta)}} \cdots \sum_{r_k\le m_n} \sum_{u_1=0}^{2^{r_1}-1} \cdots \sum_{u_k=0}^{2^{r_k}-1} (B_k^{-4}) E[\hat{W}_n(A_k) (\prod_{j=1}^{k} 2^{-r_j})^{\delta-\frac{1}{2}}]^4$$

by the Markov inequality.

Using Lemma 5.1 for $q=2$, the right side of (7.4)

$$\leq \sum_{\substack{r_1\leq m_n \\ r_1+..+r_k\geq m(\theta)}} \cdots \sum_{r_k\leq m_n} K_1(B_k)^{-4}\left(2^{-\sum_{j=1}^{k} r_j}\right)^{\frac{1}{k}-4(\frac{1}{2}-\delta)}, \quad K_1>0 \text{ a constant}$$

$$\leq \left(\frac{4^k}{\eta} B_k^{(1)}\right)^4 \sum_{\substack{r_1\leq m_n \\ r_1+...+r_k\geq m(\theta)}} \cdots \sum_{r_k\leq m_n} b^{-m(\theta)} \cdot (ba^4)^{m(\theta)-\sum_{j=1}^{k} r_j}, \quad b=2^{\frac{1}{k}-4(\frac{1}{2}-\delta)}>1$$

$$(7.5) \leq K_1\left(\frac{4^k}{\eta}B_k^{(1)}\right)^4 B_k^{(2)} b^{-m(\theta)} \quad \text{where } B_k^{(2)} = \sum_{r_1+...+r_k\geq m(\theta)} (ba^4)^{-\sum_{i=1}^{k} r_i+m(\theta)} .$$

Using (7.2), we note that $B_k^{(2)} = 0(m^{k-1}(\theta))$ and since $(m^{k-1}(\theta))^5 \cdot b^{-m(\theta)} \to \infty$ as $\theta\to 0$ $(m(\theta)\to\infty$ as $\theta\to 0)$ it follows that the extreme side of (7.5) $\to 0$ as $\theta\to 0$ which implies that $n\to\infty$. This proves (6.15) with conditions (a) and (c) of Theorem 5.2.1.

(b) Let $\{X_{ni}\}$ be strong mixing with rates (1.4), and let $\{H_{ni}\}$ be μ–bounded (viz. the conditions (b) and (c) of Theorem 5.2.1). Choose a real number a such that

$$(7.6) \qquad 0<a<1 \text{ and } a^4 \cdot 2^{\frac{1}{k}-\epsilon-4(\frac{1}{2}-\delta)} > 1 .$$

Now using Lemma 5.2 for $q=2$, and proceeding as above, we obtain

$$(7.7)\ P_n\left[\sup_{\substack{R(t,t')\in \mathcal{B}_m \cap C_\theta^{(1)} \\ |R(t,t')|\neq 0^n}} \left|\mathcal{W}_n(R(t,t')\cdot\frac{1}{r(|R(t,t')|)}\right|\geq\eta\right] \leq K_2\left(\frac{4^k}{\eta} B_k^{(1)}\right)^4 B_k^{(2)} b^{-m(\theta)}$$

where $b = 2^{\frac{1}{k}-\epsilon-4(\frac{1}{2}-\delta)}$ and $K_2>0$ is a constant. Using (7.7) and arguing as in (7.5), we find that the right side of (7.7) $\to 0$ as $\theta\to 0$ (and therefore $n\to\infty$). This proves (6.15) with conditions (b) and (c) of Theorem 5.2.1.

(c) Let $\{X_{ni}\}$ be φ–mixing with rates (1.3) and let $\{H_{ni}\}$ have uniform marginals (viz. conditions (a) and (d) of Theorem 5.2.1).

Choose a real number a such that

(7.8) $$1<a<1 \text{ and } a^{2(k+1)}\, 2^{\frac{1}{k}-2(k+1)(\frac{1}{2}-\delta)} > 1$$

(This is always possible since from (6.7), $\frac{1}{k} - 2(k+1)(\frac{1}{2}-\delta) > 0$, and so we choose $a \in (0,1)$ such that (7.8) is satisfied). Using Lemma 5.1 for $q=k+1$, and proceeding as above, we note that the left side of (7.7) $\le K_3\left(\frac{4^k}{\eta} B_k^{(1)}\right)^{2k+2} B_k^{(3)} b^{-m(\theta)}$ where

$$B_k^{(3)} = \sum_{r_1+\ldots+r_k \ge m(\theta)} (ba^{2k+2})^{-\sum_{i=1}^{k} r_i + m(\theta)} = 0(m^{k-1}(\theta)) \text{ , } b = 2^{\frac{1}{k}-2(k+1)(\frac{1}{2}-\delta)}$$

and $K_3>0$ is some constant. Using (7.8) and arguing as in (7.5), we prove (6.15) with conditions (a) and (d) of Theorem 5.2.1.

(d) Finally, let $\{X_{ni}\}$ be strong mixing with rates (1.4) and let $\{H_{ni}\}$ have uniform marginals (viz. conditions (b) and (d) of Theorem 5.2.1). Here choosing $0<a<1$ and $a^{2(k+2)} 2^{(2-\epsilon(k+1))k^{-1}-2(k+2)(\frac{1}{2}-\delta)} > 1$ using Lemma 5.2 for $q=k+2$, and proceeding as in (b), we get (6.15). This proves Lemma 6.3.

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Statistique/*Statistics*

L'espace $\tilde{D}_k$ et la convergence faible du processus empirique indexé par rectangles en condition de mélange

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Résumé — On a établi dans [5] la convergence faible du processus empirique corrigé multidimensionnel indexé par points en condition de mélange. Dans cette Note, nous étendons les résultats au processus empirique indexé par les rectangles de $[0, 1]^k$ après avoir au préalable généralisé la topologie de Skorohod sur un espace de fonctions indexées par rectangles.

The space $\tilde{D}_k$ and weak convergence for rectangles-indexed processes under mixing conditions

Abstract — *We established in* [5] *the weak convergence of the weighted multivariate empirical process indexed by points under mixing conditions. In this paper we extend the results for the empirical process indexed by rectangles of* $[0, 1]^k$ *after a generalization of the Skorohod topology on a space of functions indexed by rectangles.*

1. INTRODUCTION. — Soient $\underset{\sim}{X}_{ni}=(X_{ni}^{(1)}, \ldots, X_{ni}^{(k)})$, $1 \leqq i \leqq n$, $n \geqq 1$ des variables aléatoires à valeurs dans $\mathbb{R}^k$ avec des fonctions de répartition continues $F_{ni}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^k$ et des fonctions de répartition marginales continues $F_{ni}^{(j)}(x)$, $x \in \mathbb{R}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$. Nous nous intéressons au comportement asymptotique du processus empirique $\tilde{W}_n$ défini par :

$$\tilde{W}_n(B)=n^{-1/2} \sum_{i=1}^{n} \left(\prod_{j=1}^{k} I_{[a_j < F_n^{(j)}(X_{ni}^{(j)}) \leqq b_j]} - \mu_{ni}(B) \right) \tag{1}$$

pour tout rectangle $B=\prod_{j=1}^{k}]a_j, b_j] \subset [0, 1]^k$ où

$$F_n^{(j)}(x)=n^{-1} \sum_{i=1}^{n} F_{ni}^{(j)}(x)$$

et μ_{ni} est la mesure, sur $[0, 1]^k$, admettant pour fonction de répartition la fonction définie par

$$H_{ni}(t_1, \ldots, t_k)=F_{ni}(F_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)^{-1}}(t_k)), \qquad 1 \leqq i \leqq n. \tag{2}$$

Les propriétés de convergence sont étudiées quand la suite $\{X_{ni}\}$ est

(3) φ mélangeante avec le taux $\varphi(m)=O(m^{-1-\varepsilon})$, $\quad \varepsilon>0$

ou

(4) fortement mélangeante avec le taux $\sum_{m=1}^{\infty} m^{2(k+1)} \alpha^{\varepsilon}(m) < \infty$ pour un certain $\varepsilon \in]0, 1/(2k+4)[$.

Ces résultats sont une extension du cas indépendant au cas mélangeant des résultats de Einmahl, Ruymgaart et Wellner [3]. L'introduction d'une topologie de Skorohod permet une extension de leurs résultats pour des classes plus générales de variables aléatoires (non nécessairement uniformément distribuées).

Note présentée par Robert FORTET.

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2. DÉFINITIONS ET NOTATIONS. — 2.1. — *Les espaces* $\tilde{D}_k$ *et* $\tilde{C}_k$. — Nous écrivons $t=(t_1, \ldots, t_k)\in[0, 1]^k$ s'il est souhaité de détailler les coordonnées de t. On note les rectangles semi ouverts de $[0, 1]^k$ par $R(t, t')=\prod_{j=1}^{k}]t_j, t'_j]$.

Soit $\mathscr{I}(k)=\{R(t, t');\ R(t, t')\subset[0, 1]^k\}$; on associe à l'espace $\mathscr{I}(k)$ la métrique de Hausdorff d_H où

$$d_H(R(t, t'), R(s, s'))=\max_{1\leqq j\leqq k}\max\{|s_j-t_j|, |s'_j-t'_j|\}.$$

Soient $R(t, t')\in\mathscr{I}(k)$ et $(\rho, \varepsilon)\in\{0, 1\}^k\times\{0, 1\}^k$. Alors le (ρ, ε)-quadrant de $\mathscr{I}(k)$ de sommet $R(t, t')$ est le sous-ensemble $Q(R(t, t'), \rho, \varepsilon)$ de $\mathscr{I}(k)$ défini par

$$Q(R(t, t), \rho, \varepsilon)=\prod_{j=1}^{k} Q_j(t_j, t'_j, \rho_j, \varepsilon_j)$$

où

$$Q_j(t_j, t'_j, \rho_j, \varepsilon_j)=\begin{cases}\{]s_j, s'_j]\subset[0, 1],\ s_j<t_j,\ s'_j<t'_j\} & \text{si}\quad \rho_j=\varepsilon_j=0\\ \{]s_j, s'_j]\subset[0, 1],\ s_j\geqq t_j,\ s'_j<t'_j\} & \text{si}\quad \rho_j=1\ \ et\ \ \varepsilon_j=0\\ \{]s_j, s'_j]\subset[0, 1],\ s_j<t_j,\ s'_j\geqq t'_j\} & \text{si}\quad \rho_j=0\ \ \text{et}\ \ \varepsilon_j=1\\ \{]s_j, s'_j]\subset[0, 1],\ s_j\geqq t_j,\ s'_j\geqq t'_j\} & \text{si}\quad \rho_j=\varepsilon_j=1.\end{cases}$$

On dit que $f:\mathscr{I}(k)\to\mathbb{R}$ admet une (ρ, ε)-limite en $R(t, t')$ si et seulement si la restriction de f à $Q(R(t, t'), \rho, \varepsilon)$ admet une limite en $R(t, t')$ par rapport à la métrique d_H et la métrique usuelle sur $\mathbb{R}$.

On note $\tilde{D}_k$ l'ensemble des applications $f:\mathscr{I}(k)\to\mathbb{R}$ telles que pour tout $R(t, t')\in\mathscr{I}(k)$ et tout $(\rho, \varepsilon)\in\{0, 1\}^k\times\{0, 1\}^k$ pour lesquels $Q(R(t, t'), \rho, \varepsilon)\neq\varnothing$, f admet une (ρ, ε)-limite en $R(t, t')$.

Enfin on note $\tilde{C}_k$, l'ensemble des applications $f:\mathscr{I}(k)\to\mathbb{R}$ qui sont continues pour d_H et la métrique usuelle sur $\mathbb{R}$.

2.2. *Topologie de Skorohod sur* $\tilde{D}_k$. — Soit Λ l'espace des applications $h:[0, 1]\to[0, 1]$ qui sont continues, croissantes et bijectives. On note $\Lambda^{(k)}$ l'espace des applications $\lambda:[0, 1]^k\to[0, 1]^k$ où

$$\lambda(t_1, \ldots, t_k)=(\lambda_1(t_1), \ldots, \lambda_k(t_k)),\qquad \lambda_j\in\Lambda,\quad 1\leqq j\leqq k.$$

$I_{(k)}$ désigne l'application identité sur $[0, 1]^k$.

Pour tout $R(t, t')\in\mathscr{I}(k)$, $\lambda(R(t, t'))$ désigne l'élément $R(s, s')$ de $\mathscr{I}(k)$ défini par $\lambda_j(t_j)=s_j$ et $\lambda_j(t'_j)=s'_j$, $1\leqq j\leqq k$.

Pour tout couple d'applications bornées $f, g:\mathscr{I}(k)\to\mathbb{R}$, on note

$$d(f, g)=\inf_{\lambda\in\Lambda^{(k)}}\max\{\|f-g\circ\lambda\|, \|\lambda-I_{(k)}\|\}$$

où

$$\|f-g\circ\lambda\|=\sup_{R(t, t')\in\mathscr{I}(k)}\{|f(R(t, t'))-(g\circ\lambda)(R(t, t'))|\}$$

et

$$\|\lambda-I_{(k)}\|=\sup_{t\in[0, 1]^k}\{|\lambda(t)-I_{(k)}(t)|\}.$$

La topologie associée à d est dite de Skorohod.

2.3. *Fonction correctrice.* — Une fonction $r : [0, 1] \to \mathbb{R}^+$ est appelée une fonction correctrice si elle satisfait les conditions suivantes :

(i) r est continue

(ii) $r(u)=0$ si $u=0$ ou $u=1$.

3. CONVERGENCE DU PROCESSUS $\tilde{W}_n$. — On dit que la suite $\{\mu_{ni}\}$ est μ-bornée s'il existe une mesure finie et positive μ sur $[0, 1]^k$ avec des fonctions de répartition marginales continues telles que, pour chaque $n \geqq 1$ et $1 \leqq i \leqq n$, $\mu_{ni}(R(t, t')) \leqq \mu(R(t, t'))$ pour tout rectangle $R(t, t')$ de $\mathscr{I}(k)$.

On note W_n le processus empirique indexé par points défini par

$$W_n(t) = n^{1/2} \sum_{i=1}^{n} \left\{ \prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)}) \leq t_j]} - H_{ni}(t) \right\} \tag{5}$$

pour tout $t=(t_1, \ldots, t_k) \in [0, 1]^k$.

Soit $B = \prod_{j=1}^{k}]a_j, b_j] \in \mathscr{I}(k)$; par convention on pose $\tilde{W}_n(B)=0$ s'il existe au moins un j tel que $a_j = b_j$.

THÉORÈME 1. — *On suppose que la suite* $\{\underset{\sim}{X}_{ni}\}$ *est soit* φ *mélangeante avec le taux* (3), *soit fortement mélangeante avec le taux* (4); *la suite* $\{\mu_{ni}\}$ *vérifie l'une des deux conditions suivantes :*

(*a*) *elle est* μ*-bornée,* μ *étant absoluement continue avec une densité bornée;*

(*b*) *pour tout* $n \geqq 1$ *et tout* $1 \leqq i \leqq n$, μ_{ni} *a des marges uniformes.*

De plus, on suppose que la fonction de covariance du processus empirique W_n *converge vers une fonction. Alors* $\tilde{W}_n$ (*à valeurs p. s. dans* $\tilde{D}_k$) *converge faiblement pour la topologie de Skorohod vers un processus gaussien* $\tilde{W}_0$ *avec ses trajectoires p. s. dans* $\tilde{C}_k$.

Preuve. — Dans un premeier temps, on démontre que l'espace $\tilde{D}_k$ possède des propriétés identiques à celles du très connu espace D_k, *voir* Balacheff et Dupont [1] et Billingsley [2] pour $k=1$. Puis, on montre que la topologie de Skorohod sur $\tilde{D}_k$ possède également les principales propriétés de la topologie de Skorohod sur D_k. Enfin on termine en utilisant des techniques déjà utilisées dans [5] pour la convergence du processus W_n.

4. CONVERGENCE DU PROCESSUS EMPIRIQUE INDEXÉ PAR RECTANGLES ET CORRIGÉ. — On considère maintenant un processus empirique modifié $\hat{W}_n$ défini par

$$\hat{W}_n(R(t, t')) = \begin{cases} \tilde{W}_n(R(t, t')) & \text{si } |R(t, t')| \geqq n^{-1} \text{ et } |R(t, t')| \leqq 1-n^{-1} \\ 0 & \text{sinon} \end{cases} \tag{6}$$

où $|R(t, t')|$ est la mesure de Lebesgue de $R(t, t')$ c'est-à-dire

$$|R(t, t')| = \prod_{j=1}^{k} (t'_j - t_j).$$

Pour toute fonction correctrice r, on introduit le processus empirique modifié et corrigé défini par

$$(\hat{W}_n/r)(R(t, t')) = \begin{cases} \hat{W}_n(R(t, t'))/r(|R(t, t')|) & \text{si } |R(t, t')| \neq 0 \\ 0 & \text{sinon.} \end{cases}$$

THÉORÈME 2. — *On suppose que la suite* $\{\underset{\sim}{X}_{ni}\}$ *satisfait les hypothèses du théorème* 1. *Alors* $\hat{W}_n/r$ *converge faiblement pour la topologie de Skorohod vers le processus gaussien* $\tilde{W}_0/r$ *à trajectoires p. s. dans* $\tilde{C}_k$, *pour toute fonction correctrice* $r : [0, 1] \to \mathbb{R}^+$ *satisfaisant*

$$r(u) \geqq A[u(1-u)]^{1/2-\delta}, \qquad A > 0$$

où $0<1/2-\delta<1/8k$ *si la condition (a) du théorème* 1 *est satisfaite, ou bien* $0<1/2-\delta<1/2k(k+1)$ *si la condition (b) du théorème* 1 *est satisfaite.*

Preuve. — On généralise une idée déjà utilisée par Neuhaus dand [6].

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CONVERGENCE EN LOI POUR LA TOPOLOGIE DE SKOROHOD ECLATEE DU PROCESSUS EMPIRIQUE MULTIDIMENSIONNEL NORMALISE TRONQUE ECLATE ET CORRIGE

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Résumé

Nous avons établi dans (6) la convergence du processus empirique multidimensionnel en condition de ϕ mélange avec des fonctions correctrices égales à zéro seulement sur la frontière inférieure et dans cet article nous établissons la convergence d'un nouveau processus : le processus empirique éclaté avec des fonctions correctrices égales à zéro sur toute la frontière. C'est une première étape pour obtenir la convergence des statistiques de rang.

Abstract

We established in (6) the convergence of the multidimensional empirical process in ϕ mixing condition with weighted functions equal to zero only on the lower boundary and in this paper we establish the convergence of a new process : the split empirical process with weighted functions equal to zero on all the boundary. It is a first step to obtain the convergence of the rank statistics.

Mots clés : Processus empiriques multidimensionnels éclatés, espaces de Skorohod multidimensionnels éclatés, fonctions correctrices éclatées, ϕ mixing.

0. INTRODUCTION

Parmi les différentes méthodes envisagées pour établir des théorèmes de convergence pour des statistiques de rang relatives à des suites d'expériences ϕ-mélangeantes et à valeurs dans $\mathbb{R}^k$, L. Rüschendorf propose dans (7) d'exprimer ces statistiques sous la forme :

$$(\forall n \in N^*)\ (\forall x \in (\mathbb{R}^k)^n)$$

$$T_n(x) = \int_{[0,1]\times[0,1]^k} M_{n,x}(s,u)\, d\mu_n(s,u)$$

où μ_n est une mesure signée sur $[0,1]^{1+k}$, et où M_n est le processus de rang défini par

$$(\forall x \in (R^k)^n\ (\forall s \in [0,1])\ (\forall u \in [0,1]^k)$$

$$M_{n,x}(s,u) = \frac{1}{n} \sum_{j=1}^{[ns]} \prod_{i=1}^{k} I_{[\tilde{F}_{n,x_i}(x_i^j) \le u_i]}$$

où $[ns]$ désigne la partie entière du nombre réel ns, I la fonction indicatrice, et $\tilde{F}_{n,x_i}$ la fonction de répartition empirique associée à la suite $x_i = (x_i^1,\ldots\ldots,x_i^n)$ des $i^{\text{èmes}}$ composantes des observations.

Pour obtenir des applications au comportement asymptotique des tests de rang dans les cas les plus généraux avec des fonctions de scores prenant des valeurs infiniment grandes, il est indispensable d'envisager des mesures signées μ_n qui ne convergent pas faiblement (en particulier de variations totales non bornées) on est alors conduit à exprimer $T_n(x)$, une fois convenablement centré et normalisé, sous la forme :

$$\int_{[0,1]\times[0,1]^k} L_{n,x}(s,u)\ \frac{1}{r}(s,u)\ r(s,u)\ d\mu_n(s,u)$$

où L_n est de la forme $\frac{M_n-K_n}{\sqrt{n}}$ (où K_n est une fonction de centrage convenable) et où r est une application continue de $[0,1]^{1+k}$ dans $\mathbb{R}_+$ (et on note par convention $\frac{1}{r}(t) = 0$ si $r(t) = 0$), on doit alors vérifier la convergence faible des mesures signées $r(s,u)d\mu_n(s,u)$ et la convergence au sens de la topologie de Skorohod des processus $L_n.\frac{1}{r}$ (autrement dit la convergence des processus M_n au sens de la topologie de Skorohod corrigée par r, en adoptant pour distance entre deux éléments f et g de D_{1+k}, $d(\frac{f}{r}, \frac{g}{r})$ où d est la distance de Skorohod).

Il est clair que cette nouvelle topologie (Skorohod corrigée) n'est, en toute rigueur, définie que sur l'ensemble des éléments f de D_{1+k} tels que $\frac{f}{r}$ soit lui-même un élément de D_{1+k} ; par conséquent, on ne peut obtenir des résultats de convergence de la suite L_n que si la fonction r s'annule en tout t tel que $L_n(t) \overset{p.s}{=} 0$, c'est-à-dire en tout $t(=(t_o,\tilde{t})) \in [0,1]\times[0,1]^k$ vérifiant l'une des conditions suivantes :

(i) $t_o = 0$

(ii) l'une au moins des coordonnées t_i (ou $1 \le i \le k$) est nulle

(iii) $\tilde{t} = \tilde{1} = (1,\ldots,1)$.

Une étude préalable à celle de la convergence de la suite L_n est celle de la suite des processus W_n empiriques normalisés tronqués définis par :

$$W_{n,x}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nto]} \sum_{i=1}^{k} I_{[F_{n,i}(x_i^j) \le t_i]} - K_n(\tilde{t})$$

où

$$\bar{F}_{n,i} = \frac{1}{n} \sum_{j=1}^{n} F_{n,i}^j \quad (1 \le i \le k)$$

et $F_{n,i}^j$ est la fonction de répartition de la marge régissant l'observation x_i^j $(1 \le i \le k,\ 1 \le j \le n)$.

Le passage de la convergence des (W_n) à celle des (M_n) est traité par Balacheff et Dupont (1) et (2) dans le cas de la topologie usuelle. Dans le présent travail, nous abordons seulement l'étude du processus W_n pour la topologie de Skorohod corrigée.

Cette étude a déjà été partiellement effectuée par nous en (6) pour des fonctions r vérifiant (i) et (ii) ; par contre en imposant la condition (iii), nous avons été astreints à des conditions excessivement fortes sur la fonction de mélange, ce qui nous a conduit à penser qu'en exprimant les statistiques de rang sous la forme proposée par Rüschendorf, on ne pourra obtenir les résultats souhaités. Il semble préférable d'exprimer ces statistiques de rang au moyen de processus s'annulant à la fois sur la frontière inférieure et sur la frontière supérieure de $[0,1]^k$.

Les nouveaux processus que nous allons définir et que nous appellerons "processus éclatés" répondront à ces critères.

L'idée directrice est la suivante (exprimée ici pour k = 1) on considère le processus W_n^* déduit de W_n par :

- si $(t_o,t_1) \in [0,1/2[^2 \quad W_n^*(t_o,t_1) = W_n(t_o,t_1)$

- si $(t_o,t_1) \in [1/2,1] \times [0,1/2[\; W_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(1-t_o)]} I_{[\bar{F}_{n,1}(x_1^{*j}) \le t_1]} - K_n(t_1)$

- si $(t_o,t_1) \in [0,1/2[\times [1/2,1]\; W_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt_o]} I_{[\bar{F}_{n,1}(x_1^j) \ge t_1]} - (1-K_n(t_1))$

- si $(t_o,t_1) \in [1/2,1]\; W_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(1-t_o)]} I_{[\bar{F}_{n,1}(x_1^{*j}) \ge t_1]} - (1-K_n(t_1))$

où $x_1^{*j} = x_1^{n+1-j}$ $(1 \le j \le n)$.

L'étape suivante de cette étude consistera à déduire de la convergence de W_n^* au sens de la topologie de Skorohod corrigée, celle de L_n pour la topologie de la convergence uniforme (où L_n^* se déduit de L_n par éclatement comme W_n^* de W_n) puis à exprimer T_n (centré et normalisé) sous la forme $\int L_n^*.\frac{1}{r}.r\, d\mu_n^*$, où μ_n^* est une mesure sur $[0,1]^{1+k}$, que l'on aura construite à partir des fonctions de scores d'une manière presque similaire à celle que l'on a déjà utilisée pour la construction de μ_n.

1. L'ESPACE D^*_{1+k} ET LA TOPOLOGIE DE SKOROHOD ASSOCIEE

1.1 L'espace D^*_{1+k}

Définitions et notations

On reprend les notations de (2) sur les espaces D_k, on note $I_0 = [0,1/2[$ et $I_1 = [1/2,1]$; on pose pour tout $\rho = (\rho_i)_{0\le i\le k} \in \{0,1\}^{1+k}$ $I_\rho = \prod_{i=0}^{k} I_{\rho_i}$ pour toute application $f \in D_{1+k}$, tout $\rho \in \{0,1\}^{1+k}$ et tout $(x_0,\dots,x_k) \subset I_\rho$ on pose

$$f^*(x_0,\dots,x_k) = f((x_0,\dots,x_k)+0_\rho)$$

(qui est la limite de $f(x'_0,\dots,x'_k)$ quand $(x'_0,\dots,x'_k)$ tend vers $(x_0,\dots,x_k)$ par valeurs supérieures pour les x'_i tels que $\rho_i = 1$ par valeurs inférieures pour les x'_i tels que $\rho_i = 0$ (on dit que c'est la limite de f en $(x_0,\dots,x_k)$ dans la direction ρ)) ; aussi, pour $k = 1$, on a :

$$f^*(x_0,x_1) = f(x_0^+,x_1^+) \quad \text{si} \quad (x_0,x_1) \in [0,1/2[^2 \ ;$$

$$f^*(x_0,x_1) = f(x_0^+,x_1^-) \quad \text{si} \quad (x_0,x_1) \in [0,1/2[\times[1/2,1] \ ;$$

$$f^*(x_0,x_1) = f(x_0^-,x_1^+) \quad \text{si} \quad (x_0,x_1) \in [1/2,1]\times[0,1/2[\ ;$$

$$f^*(x_0,x_1) = f(x_0^-,x_1^-) \quad \text{si} \quad (x_0,x_1) \in [1/2,1]^2$$

Pour toute application $f \in D_{1+k}$, tout $J \subset \{0,\dots,k\}$ et tout pavé $\prod_{i=0}^{k} [a_i,b_i]$ inclus dans $[0,1]^{1+k}$, on note

$$\Delta_{1+k}\,(J,f,\prod_{i=0}^{k}[a_i,b_i])$$

défini par récurrence avec

$$\Delta_{1+k}(J,f,\prod_{i=0}^{k}[a_i,b_i]) = \Delta_k(J\cap(\{0,\dots,k-1\}),f(.,b_k),\prod_{i=0}^{k-1}[a_i,b_i]) -$$

$$\Delta_k(J\cap(\{0,\dots,k-1\}),\ f(.,a_k)\ \prod_{i=0}^{k-1}[a_i,b_i]) \text{ si } k\in J$$

$$= \Delta_k(J\cap(\{0,\dots,k-1\}),\ f(.,b_k),\ \prod_{i=0}^{k-1}[a_i,b_i]) -$$

$$\Delta_k(J\cap(\{0,\dots,k-1\}),\ f^*(.,a_k),\ \prod_{i=0}^{k-1}[a_i,b_i]) \text{ si } k\notin J$$

Pour tout $\rho \in \{0,1\}^{1+k}$, on note

$$J(\rho) = \{0 \le i \le k \ ; \ \rho_i = 0\}$$

On définit l'application γ sur D_{1+k} par

$$(\forall\rho \in \{0,1\}^{1+k})(\forall(x_0,\dots,x_k) \in I_\rho) \quad \gamma(f)(x_0,\dots,x_k) = D_{1+k}(J(\rho),f,\ \prod_{i=1}^{k} I_{x_i})$$

où I_{x_i} est égal à $[0,x_i]$ où à $[x_i,1]$ selon que $x_i < 1/2$ ou $x_i \ge 1/2$.

Soit l'application γ' définie sur D_{1+k} par

$$\gamma'(f) = (\Delta_{1+k}(J(\rho),\ f|I_\rho,\ \prod_{i=1}^{k} I_{x_i})_{\rho\in\{0,1\}^{1+k}}$$

Définition 1

Pour tout $\rho \in \{0,1\}^{1+k}$, on note D_ρ l'ensemble des applications f_ρ de I_ρ dans $\mathbb{R}$ prolongeables à $\bar{I}_\rho$ en une application admettant des limites dans les 2^{1+k} directions en tout point, et continue dans la direction ρ ; on note D^*_{1+k} l'espace des fonctions f telles que pour tout $\rho \in \{0,1\}^{1+k}$ la restriction de f à I_ρ appartienne à D_ρ.

Par une surjection évidente de D_{1+k} sur D_ρ, on retrouve pour D_ρ toutes les propriétés énoncées pour D_{1+k} dans (2).

On a aussi une bijection évidente γ'' entre $\prod_{\rho \in \{0,1\}^{1+k}} D_\rho$ et D^*_{1+k}.

Lemme 1

On a $\gamma(D_{1+k}) = D^*_{1+k}$

$$\gamma = \gamma'' \circ \gamma'$$

Caractérisation de l'espace D^*_{1+k}

On appelle base de quadrillage éclaté de $[0,1]^{1+k}$ toute famille de k+1 suites finies d'éléments réels telle que chaque élément de chaque suite appartienne à $[0,1[$, que le premier élément de chaque suite soit égal à 0 et que 1/2 appartienne à chaque suite.

Soit par exemple, $B = \{(t_i^j \; ; \; 1 \leq j \leq L_i, \; 0 \leq i \leq k)\}$ une telle base.
Posons pour tout i $(0 \leq i < k)$

$$M_i^j = [t_i^j, t_i^{j+1}[\text{ si } t_i^{j+1} \leq 1/2 \qquad j \in \{1,\dots,L_i\}$$

$$M_i^j = \{1/2\} \text{ si } t_i^j = 1/2 \qquad j \in \{1,\dots,L_i\}$$

$$M_i^{j+1} =]t_i^j, t_i^{j+1}] \text{ si } t_i^j \geq 1/2 \qquad j \in \{1,\dots,L_i+1\}$$

(où par convention $t_i^{L_i+1} = 1$).

Alors l'ensemble des parties non vides de $[0,1]^{1+k}$ de la forme $\prod_{i=0}^{k} M_i^{s_i}$ où pour tout i, $s_i \in \{1,\dots,L_i+1\}$ est appelé quadrillage éclaté de $[0,1]^{1+k}$ de base B.

On note R^* l'ensemble des quadrillages éclatés de $[0,1]^{1+k}$.

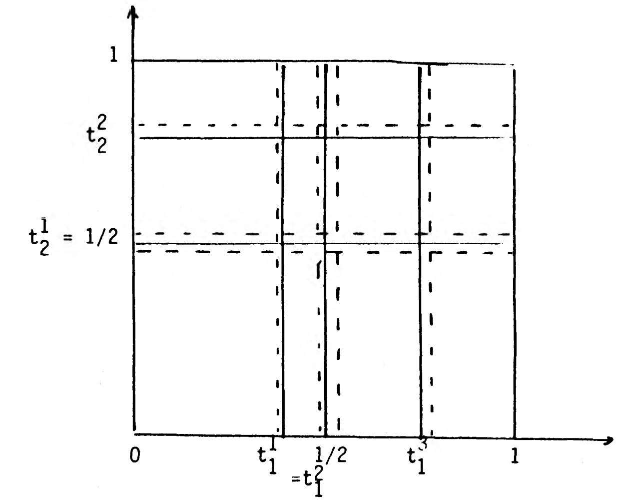
A tout quadrillage R appartenant à R^* de base $B = \{(t_i^j) \; ; \; 1 \leq j \leq L_i, \; 0 \leq i \leq k)\}$, on associe le nombre réel strictement positif

$$m(R) = \operatorname*{Inf}_{0 \leq i \leq k} \; \inf_{0 \leq j \leq L_i} \{t_i^{j+1} - t_i^j\}$$

$m(R)$ est appelé la perméabilité de R.

On note R^*_δ le sous-ensemble des quadrillages de R^* de perméabilité strictement plus grande que δ.

Exemple de schéma pour k = 1



On a $B = \{0, t_1^1, t_1^2, t_1^3\} \times \{0, t_2^1, t_2^2\}$;

la perméabilité du quadrillage de base B est ici $1/2 - t_1^1$.

Définition 2

Pour toute application f bornée de $[0,1]^{1+k}$ dans $\mathbb{R}$, on pose :

$$w'_*(f,\delta) = \underset{\mathcal{R}\in R^*_\delta}{\text{Inf}} \ \underset{R\in\mathcal{R}}{\text{Max}} \ \sup_{(t,t')\in R^2} |f(t)-f(t')|$$

Proposition 1

Soit f une application de $[0,1]^{1+k}$ dans $\mathbb{R}$; pour que f appartienne à D^*_{1+k} il faut il suffit que :

$$\lim_{\delta\to 0} w'_*(f,\delta) = 0$$

Démonstration :

Cette caractérisation des éléments de D^*_{1+k} se déduit élémentairement de la caractérisation classique des éléments de D_{1+k} par la convergence vers 0 de $w'(f,\delta)$ où

$$w'(f,\delta) = \underset{\mathcal{R}\in R_\delta}{\text{Inf}} \ \underset{R\in\mathcal{R}}{\text{max.}} \ \sup_{(t,t')\in R^2} |f(t)-f(t')|.$$

où R_δ est l'ensemble des quadrillages usuels composés des pavés de la forme $\prod_{i=0}^{k} [t_i,t'_i]$ et de perméabilité plus grande que δ et de la bijection γ ".

L'espace C^*_{1+k}

On note

$$C^*_{1+k} = \{f \ ; (\forall\rho \in \{0,1\}^{1+k}) \ f|I_\rho \text{ admet un prolongement continu à } \bar{I}_\rho\}$$

$$w_*(f,\delta) = \max_{\rho\in\{0,1\}^{1+k}} \ \sup_{(t,t')\in I^2_\rho} |f(t)-f(t')|$$

Proposition 1

Soit f une application de $[0,1]^{1+k}$ dans $\mathbb{R}$; pour que f appartienne à C^*_{1+k} il faut et il suffit que

$$\lim_{\delta\to 0} w_*(f,\delta) = 0$$

le principe de démonstration est le même que pour la proposition 1.

1.2 - Topologie de Skorohod

Topologie de Skorohod associée à D^*_{1+k}

Soit Λ l'ensemble des applications λ continues, bijectives et [0,1] dans [0,1] vérifiant $\lambda(0) = 0 \quad \lambda(1/2) = 1/2 \quad \lambda(1) = 1$.

Si $\lambda = (\lambda_o,\ldots,\lambda_k)$ on note :

$$(f \circ \lambda)(x_o,\ldots,x_k) = f(\lambda_o(x_o),\ldots,\lambda_k(x_k))$$

$$d^*(f,g) = \underset{\lambda\in\Lambda^{1+k}}{\text{Inf}} \ \max.\{\|f-g\circ\lambda\|, \|\lambda-i_{1+k}\|\}$$

(où i_{1+k} est l'application identique de $[0,1]^{1+k}$).

La distance d^* et la topologie qu'elle définit sur l'ensemble des applications bornées de $[0,1]^{1+k}$ dans $\mathbb{R}$ sont dites de "Skorohod éclatée".

Il est clair que l'application γ" définie dans 1.1 ci-dessus établit un homéomorphisme de $\prod_{\rho\in\{0,1\}^{1+k}} D_\rho$ (muni de la topologie produit des topologies de Skorohod sur chacun des D_ρ) sur D^*_{1+k} (muni de la topologie de Skorohod éclatée).

Caractérisation des compacts de D^*_{1+k}

On peut montrer l'équivalent d'un théorème connu de Billingsley (généralisé à D_k par Balacheff et Dupont en (1)).

Proposition 2

Soit K une partie de D^*_{1+k} ; la fermeture de K est compacte si, et seulement si les deux propriétés suivantes sont vérifiées :

P_1 - $\sup_{f\in K} \|f\| < +\infty$

P_2 - $\lim_{\delta\to 0} \sup_{f\in K} w'_*(f,\delta) = 0$

Démonstration

Elle découle immédiatement de l'homéomorphisme γ'' de la caractérisation des ensembles compacts de D_{1+k}.

1.3 - Convergence faible de probabilités sur D^*_{1+k}

On retrouve élémentairement des résultats équivalents à ceux énoncés dans (1). (I.1-b proposition 1 et I.1-c théorèmes 1 et 2) pour les espaces D_k.

Structure mesurable sur D^*_{1+k}

On note $\mathcal{D}^*_{1+k}$ la tribu borélienne associée à la topologie de Skorohod éclatée sur D^*_{1+k}.

Pour tout $T \subset [0,1]^{1+k}$, on note ϕ_T la projection de D^*_{1+k} sur $\mathbb{R}^T$.

Proposition 3

$\mathcal{D}^*_{1+k}$ est la restriction à D^*_{1+k} de la tribu puissance sur $\mathbb{R}^{[0,1]^{1+k}}$ ($\mathbb{R}$ étant muni de la tribu borélienne).

Démonstration

Pour tout $\rho \in \{0,1\}^{1+k}$, notons $\mathcal{D}_\rho$ la tribu borélienne associée à la topologie de Skorohod usuelle sur D_ρ ; on sait, comme pour D_{1+k}, que $\mathcal{D}_\rho$ est un espace séparable ; la démonstration va donc résulter du lemme suivant :

Lemme 2

Soit T une partie dense dénombrable dans $[0,1]^{1+k}$ telle que de plus pour tout $J \subset \{0,\dots,k\}$ si on note

$$F_J = \{(t_0,\dots,t_k) \in [0,1]^{1+k} ; \forall j \in J\ t_j = 1/2\}.$$

$F_J \cap T$ soit dense dans F_J ;

alors la tribu $\mathcal{D}^*_{1+k}$ est engendrée par $\{\phi_{\{t\}} ; t \in T\}$.

Démonstration

On note $\sigma(\ )$ pour "tribu engendrée par ( )".

On a d'abord $\sigma(\phi_{\{t\}} ; t \in T) \subset \mathcal{D}^*_{1+k}$; en effet, si pour tout $\rho(\in\{0,1\}^{1+k})$ et tout $T_\rho(\subset I_\rho)$, on note $\phi^\rho_{T_\rho}$ la projection de D_ρ sur $\mathbb{R}^{T_\rho}$ on a $\sigma(\phi^\rho_{\{t\}} ; t \in T\cap I_\rho) = \mathcal{D}_\rho$ et $\mathcal{D}^*_{1+k} = \prod_{\rho\in\{0,1\}^{1+k}} \mathcal{D}_\rho$

Donc, pour tout t, $\phi_{\{t\}}$ est mesurable.

Montrons maintenant la réciproque ; on note :

$$\pi_\rho : D^*_{1+k} \to D_\rho \quad ; \text{ soit } \mathcal{D}_{T\cap I_\rho}$$
$$f \quad f|I_\rho$$

l'ensemble des parties de D_ρ de la forme $\phi^{\rho\,-1}_U (H^U)$, où U est une partie finie de $T\cap I_\rho$ et H^U une partie borélienne de $\mathbb{R}^U$ ($\mathcal{D}_{T\cap I}$ est une algèbre qui engendre $\mathcal{D}_\rho$).

Soit enfin $\mathcal{D}^*_T$ l'ensemble des parties de D^*_{1+k} de la forme $\phi^{-1}_{U'}(H^{U'})$ où U' est une partie finie de T et H^U une partie borélienne de $\mathbb{R}^{U'}$.

On a alors :

$$\pi_\rho^{-1}(\mathcal{D}_{T\cap I_\rho}) \subset \mathcal{D}_T^* \quad \text{et} \quad \sigma(\pi_\rho^{-1}(\mathcal{D}_{T\cap I_\rho})) = \pi_\rho^{-1}(\sigma(\mathcal{D}_{T\cap I_\rho}))$$

donc

$$\pi_\rho^{-1}(\sigma(\mathcal{D}_{T\cap I_\rho})) \subset \sigma(\mathcal{D}_T^*)$$

ou encore

$$\pi_\rho^{-1}(\mathcal{D}_\rho) \subset \sigma(\mathcal{D}_T^*)$$

soit

$$C = \prod_{\rho\in\{0,1\}^{1+k}} C_\rho (\in \prod_{\rho\in\{0,1\}^{1+k}} \mathcal{D}_\rho) \text{ ; on a}$$

$$\bigcap_{\rho\in\{0,1\}^{1+k}} C_\rho = \bigcap_{\rho\in\{0,1\}^{1+k}} \pi_\rho^{-1}(C_\rho) \in \sigma(\mathcal{D}_T^*)$$

d'où

$$\prod_{\rho\in\{0,1\}^{1+k}} \mathcal{D}_\rho \subset \sigma(\mathcal{D}_T^*).$$

soit encore

$$\mathcal{D}_{1+k}^* \subset \sigma(\mathcal{D}_T^*) = \sigma(\phi_{\{t\}} \text{ ; } t \in T)$$

Convergence faible

Une suite $(P_n \text{ ; } n \in \mathbb{N})$ de probabilités sur $(D_{1+k}^*, \mathcal{D}_{1+k}^*)$ est dite faiblement convergente si elle est faiblement convergente pour la topologie de Skorohod éclatée.

Pour toute probabilité P sur $(D_{1+k}^*, \mathcal{D}_{1+k}^*)$, on note T_P l'ensemble des points t (dans $[0,1]^{1+k}$) en lesquels $\phi_{\{t\}}$ est P presque surement continue (on rappelle que $\phi_{\{t\}}(f) = f(t)$, et que $f \in D_{1+k}^*$).

Proposition 4

Soit $(P_n \text{ ; } n \in \mathbb{N})$ une suite de probabilités sur $(D_{1+k}^*, \mathcal{D}_{1+k}^*)$; elle admet une probabilité P sur $(D_{1+k}^*, \mathcal{D}_{1+k}^*)$ pour limite faible si et seulement si sont vérifiées les conditions suivantes :

1. $(\forall\varepsilon > 0) \lim_{\delta\to 0} \limsup_{n\to\infty} P_n(\{f \text{ ; } w'_*(f,\delta) \geq \varepsilon\}) = 0$;
2. Pour toute partie finie U de T_P, $\phi_U(P_n)$ converge faiblement vers $\phi_U(P)$ quand n tend vers l'infini.

Démonstration

Conséquence des propositions 2 et 3.

Corollaire 1

Soit $(P_n \text{ ; } n \in \mathbb{N})$ une suite de probabilités sur $(D_{1+k}^*, \mathcal{D}_{1+k}^*)$ vérifiant :

1'. $(\forall\varepsilon > 0) \lim_{\delta\to 0} \limsup_{n\to\infty} P_n(\{f \text{ ; } w_*(f,\delta) \geq \varepsilon\}) = 0$;

2'. La condition 2 de la proposition 4

Alors la suite (P_n) admet une limite faible P vérifiant $P(C_{1+k}^*) = 1$

Enfin on énonce un résultat qui est une variante d'un théorème de Dudley (5) et que nous utiliserons pour la convergence des processus L_n^*.

Corollaire 2

Soit $(P_n \text{ ; } n \in \mathbb{N})$ une suite de probabilités sur $(D_{1+k}^*, \mathcal{U}_{1+k}^*)$ (où $\mathcal{U}_{1+k}^*$ est la tribu engendrée par la topologie de la convergence uniforme sur D_{1+k}^*) ; alors il existe une probabilité P avec $P(C_{1+k}^*) = 1$ pour laquelle la suite (P_n) converge faiblement pour la topologie de la convergence uniforme si et seulement si sont vérifiées les conditions 1' et 2 du corollaire 1.

Démonstration

La condition suffisante est une conséquence du corollaire 1 comme cela a été fait par Billingsley (4) chapitre 3, p. 151, pour D_1.

Montrons la condition nécessaire; on suppose donc que P_n converge faiblement vers P avec $P(C^*_{k+1}) = 1$.

Soit $\tilde{U}_{1+k}$ la topologie de la convergence uniforme sur C^*_{1+k} ; comme P est concentré sur un espace séparable, il résulte du théorème 1, p. 284 dans (8) qu'il existe un espace de probabilité (Ω, a, μ) et des variables aléatoires $(X_n ; n \in IN)$ et X telles que $\mu(X_n) = P_n$, $\mu(X) = P$ et que X_n converge vers X p.s. pour μ.

Pour tout $\delta(>0)$, on considère l'application de D^*_{1+k} dans IR définie par :

$$Y_\delta : D^*_{1+k} \to \mathbb{R}$$

$$Y_\delta(f) = \max_{\rho \in \{0,1\}^{1+k}} \sup \{|(f)-f(t')| ; (t,t') \in I^2_\rho, |t-t'| \le \delta\}$$

alors Y_δ est une application continue pour la topologie de la convergence uniforme.

Soient $(Z_{n,\delta} ; n \in \mathbb{N}^*)$ et Z_δ les variables aléatoires définies par

$$Z_{n,\delta} = Y_\delta \circ X_n \qquad Z_\delta = Y_\delta \circ X$$

alors on a

$$\forall \varepsilon > 0 \quad \exists n_o \in IN \quad \forall n \ge n_o \quad \mu\{|Z_{n,\delta}-Z| > \varepsilon/2\} < \varepsilon/2$$

comme X est concentré sur C^*_{1+k}, il existe, pour tout $\varepsilon > 0$, $\delta > 0$ tel que $\mu\{Z_\delta > \varepsilon/2\} < \varepsilon/2$, et ceci entraîne $\mu\{Z_{n,\delta} > \varepsilon\} < \varepsilon$, soit encore

$$P_n(\{f ; w(f,\delta) > \varepsilon\}) = \mu\{Z_{n,\delta} > \varepsilon\} < \varepsilon$$

on a ainsi établi la condition 1' ; la vérification de la condition 2 est immédiate.

II - CONVERGENCE EN LOI DU PROCESSUS EMPIRIQUE NORMALISE TRONQUE ECLATE CORRIGE PAR LA TOPOLOGIE DE SKOROHOD

2.1 - Nature des observations

Soit $x = (x^1,\dots,x^n)$ une suite de n observations dans IR^k ; pour tout $i \in \{1,\dots,k\}$, notons $x_i = (x^1_i,\dots,x^n_i)$ la suite des ièmes composantes des observations, $\tilde{F}_{n,x_i}$ la fonction de répartition empirique associée à la suite x_i, F^j_n $(1 \le j \le n)$ la fonction de répartition de la marge Q^j_n de la probabilité Q_n régissant l'observation x dans $(IR^k)^n$, et $F^j_{n,i}$ $(1 \le i \le k)$ la fonction de répartition de la marge $Q^j_{n,i}$.

On note également :

$$\bar{F}_{n,i} = \frac{1}{n} \sum_{j=1}^{n} F^j_{n,i} \qquad 1 \le i \le k$$

(pour plus de détails voir (6)).

Nous nous plaçons sur l'hypothèse de continuité suivante :

H_1- les marges $Q_{n,i}$ $(1 \le i \le k)$ de Q_n sont supposées diffuses sur $\mathbb{R}^n$

le processus empirique W_n sur lequel nous travaillerons sera défini en notant pour tout $t = (t_o,\tilde{t}) = (t_o,t_1,\dots,t_k)$ $(\in [0,1]^{1+k})$,

$$(W_n(t))(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt_o]} I_{(\bar{F}_n(x^j) \le \tilde{t})} - H^j_n(\tilde{t}) ,$$

avec

$$H^j_n(t_1,\dots,t_k) = F^j_n(\bar{F}^{-1}_{n,1}(t_1),\dots,\bar{F}^{-1}_{n,k}(t_k))$$

On suppose également que sont satisfaites les hypothèses suivantes :

H2 - la suite $(c_n ; n \in IN^*)$ des fonctions de covariance des processus W_n converge simplement vers une fonction c.

H3 - Il existe une application décroissante $\Phi : \mathbb{N}^* \to [0,1]$ vérifiant $\Phi(1) = 1$, $\sum_{n \in N^*} n\,\Phi^{1/2}(n) < +\infty$ et pour laquelle la suite $(Q_n : n \in \mathbb{N}^*)$ est Φ mélangeante.

H4 - Il existe une mesure μ sur $[0,1]^k$, finie positive à marges diffuses et vérifiant :
$(\forall n \in \mathbb{N}^*)\ (\forall j \in \{1,\dots,n\})\ (\forall$ B bloc de $[0,1]^k)$ $\mu_n^j(B) \le \mu(B)$
(μ_n^j est la probabilité ayant H_n^j pour fonction de répartition).
On note H la fonction de répartition de la mesure μ.

2.1 - Convergence en loi

Notations et définitions

Définition 3

On appelle fonction correctrice éclatée toute application r de $[0,1]^{1+k}$ dans $\mathbb{R}_+$ vérifiant

(i) Il existe r_o et $\tilde{r}$ applications de $[0,1]$ et de $[0,1]^k$ dans $\mathbb{R}_+$ telle que pour tout $t\ (=(t_o,\tilde{t})$, $r(t) = r_o(t_o)\,\tilde{r}(\tilde{t})$;

(ii) pour tout $\rho \in \{0,1\}^{1+k}$, $r|I_\rho$ admet un prolongement continu sur $\bar{I}_\rho$.

(iii) r est nul sur toute la frontière de $[0,1]^{1+k}$.

Résultat préliminaire

Nous énonçons d'abord une généralisation immédiate de la proposition équivalente énoncée en (6) p. 50.

Proposition 5

Soit pour tout n, un processus Y_n défini sur $([0,1]^k)^n$ à valeur dans D^*_{1+k} ; on suppose que la suite des Y_n converge en loi vers Y_o gaussien, à trajectoire presque surement dans C^*_{1+k} pour la topologie de Skorohod éclatée ; on note P_n la loi de Y_n (c'est la probabilité sur D^*_{1+k}) et on suppose que les conditions 1' et 2 du corollaire 1 de la proposition 4 sont vérifiées.

Soit r une application de C^*_{1+k} positive ou nulle.
Soit également pour tout $\alpha (> 0)$

$$R_\alpha = \{(v_o,\dots,v_k) \in [0,1]^{1+k}\ ;\ (\exists\, i \in \{0,\dots,k\})\ \sup(v_i, 1-v_i) \le \alpha\}$$

On suppose vérifiées les deux conditions suivantes :

(A) pour tout n, le processus $Y_n.\frac{1}{r}$ est à trajectoires p.s. dans D^*_{1+k}.

(B) $(\forall \delta > 0)\ (\forall \varepsilon > 0)\ (\exists \alpha > 0)\ (\exists n_o)\ (\forall n \ge n_o)\ P_n(\sup_{v \in R_\alpha} |Y_n.\frac{1}{r}| > \delta) < \varepsilon$.

Alors la suite des processus $Y_n.\frac{1}{r}$ converge aussi en loi vers le processus corrigé $Y_o.\frac{1}{r}$ (qui est lui-même gaussien et à trajectoire p.s. dans C^*_{1+k}) pour la topologie de Skorohod éclatée.

Nous devons maintenant étudier dans quelles conditions on peut appliquer la proposition 5 au processus éclaté, W_n^* défini par :

$$W_n^*(t_o,\tilde{t}) \begin{cases} = \gamma(W_n) & \text{si } 0 \le t_o \le \frac{1}{2} \\ = \gamma(W_n)\ (t_o + \frac{1}{n},\tilde{t}) & \text{si } \frac{1}{2} \le t_o \le \frac{n-1}{n} \\ = 0 & \text{si } \frac{n-1}{n} < t_o \le 1 \end{cases}$$

Vérifions tout d'abord la convergence en loi du processus W_n^*.

Par la continuité de l'application γ, on sait déjà que $\gamma(W_n)$ converge en loi vers $\gamma(W_o)$ (où W_o est le processus limite de W_n) pour la topologie de Skorohod éclatée.

$\gamma(W_o)$ est p.s. à trajectoires dans C_{1+k}^* et de plus nul p.s. sur toute la frontière de $[0,1]^{1+k}$.

Considérons maintenant pour tout $n \in N^*$, l'application ψ_n, de D_{1+k}^* dans lui-même défini par :

$$\psi_n(f)(t_o,\tilde{t}) \begin{cases} = f(t_o,\tilde{t}) & \text{si } 0 \le t_o \le \frac{1}{2} \\ = f(t_o,1/n+\tilde{t}) & \text{si } \frac{1}{2} \le t_o \le \frac{n-1}{n} \\ = 0 & \text{si } \frac{n-1}{n} < t_o \le 1 \end{cases}$$

Soit $f_o \in C_{1+k}^*$, nul sur toute la frontière de $[0,1]^{1+k}$ et soit f_n une suite de fonctions de D_{1+k}^* tels que f_n converge vers f_o pour la topologie de Skorohod éclatée ; alors $\psi_n(f_n)$ converge aussi vers f_o pour la topologie de Skorohod éclatée, donc, d'après le lemme 3 de (2), W_n^* convergera en loi vers $\gamma(W_o)$ pour la topologie éclatée.

Les conditions 1' et 2 du corollaire 1 sont élémentairement vérifiées pour le processus W_n^*.

Il résulte de l'expression même de W_n^* (qui est identiquement nul pour t suffisamment proche de 0 ou suffisamment proche de 1) que la condition A est vérifiée pour W_n^* si la fonction correctrice éclatée r et les fonctions H_n^j sont liées par la condition (H_1') ci-dessous :

H_1' pour tout $\tilde{t}$ appartenant à la frontière de $[0,1]^k$, on a :

$$\lim_{\tilde{u} \to \tilde{t}} (\tilde{\gamma}(H_n^j).\frac{1}{\tilde{r}}(\tilde{u}) = 0$$

où pour tout $\tilde{\rho} = (\rho_i)_{1\le i\le k}$ $(\in\{0,1\}^k)$ et tout $(v_1,\dots,v_k) \in \tilde{I}_{\tilde{\rho}} = \prod_{i=1}^{k} I_{\rho_i}$ on a

$$\tilde{\gamma}(H_n^j)(v_1,\dots,v_k) = \Delta_k(\tilde{J}(\tilde{\rho}),\ H_n^j|\tilde{I}_{\tilde{\rho}}\ ,\ \prod_{i=1}^{k} J_{v_i})$$

avec

$$\tilde{J}(\tilde{\rho}) = \{1 \le i \le k\ ;\ \rho_i = 0\}$$

Il nous reste à assurer la réalisation de la condition (B).

Pour cela, on va décomposer R_α en 2^{1+k} sous-ensembles

$$(R_\alpha^\rho)_{\rho \in \{0,1\}^{1+k}} \quad \text{définis par } R_\alpha^\rho = R_\alpha \cap I_\rho$$

et montrer que (B) est vérifiée sur chacun des sous ensembles R_α^ρ.

Notons $\overset{0}{\le} = \le$ et $\overset{1}{\le} = \ge$

<u>Définition 4</u>

Soit $J \subset \{0,\dots,k\}$ et soit f une application d'une partie de $[0,1]^J$ dans $\mathbb{R}$; f est dite <u>monotone éclatée</u> si, pour tout ρ et tout couple $((t_i)_{i\in J}, (t_i')_{i\in J})$ de points de la projection de R_α^ρ sur $[0,1]^J$ vérifiant, pour tout i appartenant à J, l'inégalité $t_i \overset{\rho}{\le} t_i'$, on a $f((t_i)_{i\in J}) \le f((t_i')_{i\in J})$.

On note, $F_1, F_2,\dots,F_k$ les fonctions de répartitions des marges de μ, F_o celle de la mesure uniforme sur $[0,1]$, et H' la fonction de répartition de la mesure produit de la mesure uniforme sur $[0,1]$ et de μ.

Pour tout $f \in C_{1+k}^*$ et pour tout $\rho \in \{0,1\}^{1+k}$, on note β^ρ l'application qui à $f|I_\rho$ associe son prolongement continu $\tilde{f}^\rho$ sur $\bar{I}_\rho$.

Enfin, pour tout $L \subset \{0,\dots,k\}$ et toute fonction f de classe C_{1+k}, on note $\partial_L f$ sa dérivée partielle d'ordre card(L), par rapport aux coordonnées appartenant à L.

On suppose alors qu'il existe $\alpha(>0)$ tel que, pour tout $\rho \in \{0,1\}^{1+k}$, les conditions suivantes (dans l'énoncé desquelles J désigne $\{0 \le j \le k$ et $\rho_j = 0\}$) soient satisfaites.

H_5 - sur l'intérieur de R_α^ρ, $\frac{1}{r}$ est de classe C_{k+1} et, pour tout $L \subset \{0,\dots,k\}$, $\partial_L(\frac{1}{r})$ prend des valeurs positives ou négatives selon que card $(L \cap J)$ est pair ou impair.

H_6 - Il existe $c > \frac{3\ k+2}{k+2}$ tel que, pour tout $L \subset \{0,\dots,k\}$, toutes les fonctions, notées f_1 ou f_2, définies ci-dessous soient $\complement L$ monotones éclatées sur R_α^ρ : on fixe tout d'abord, pour tout $i \in L$, t_i et t_i' tels que $t_i \le t_i' \le \frac{1}{2}$ si $i \in L \cap J$, et $\frac{1}{2} \le t_i' \le t_i$ si $i \in L \cap \complement J$; on définit alors, pour toute famille $((t_i)_{i \in \complement L \cap J}, (t_i')_{i \in \complement L \cap \complement J})$ telle que $t_i \le \frac{1}{2}$ si $i \in \complement L \cap J$ et $t_i' > \frac{1}{2}$ si $i \in \complement L \cap \complement J$,

$$f_1((t_i)_{i\in\complement L\cap J}, (t_i')_{i\in\complement L\cap\complement J}) = \Delta_{\prod_{i=0}^{k} [a_i,b_i]} (\beta^\rho(\gamma(H'))\frac{1}{(\beta^\rho(r))^c}((c_i)_{0\le i\le k})$$

et

$$f_2((t_i)_{i\in\complement L\cap J}, (t_i')_{i\in\complement L\cap J}) = \Delta_{\prod_{i=0}^{k} [a_i,b_i]} (\beta^\rho(\gamma(\prod_{i=0}^{k} F_i))\frac{1}{(\beta^\rho(r))^c} ((c_i))_{0\le i\le k})$$

où $[a_i,b_i]$ est égal à $[t_i,t_i']$, $[t_i',t_i]$, $[0,t_i]$ ou $[t_i',1]$

selon que i appartient à $L\cap J$, $L\cap\complement J$, $\complement L\cap J$, $\complement L\cap\complement J$, et où c_i est égal à t_i ou t_i' selon que i appartient à $(\complement L\cap J) \cup (L\cap\complement J)$ ou à $(J\cap L) \cup (\complement L\cap\complement J)$.

H_7 - Pour tout $L \subset \{0,\dots,k\}$

$$\int_{R_\alpha^\rho} |\frac{1}{r^c} \partial_L\ (\gamma(H'))\ \partial_{\complement L}\ (\frac{1}{r})|\ d\ \lambda^{1+k} < +\infty \quad \text{et}$$

$$\int_{R_\alpha^\rho} |\frac{1}{r^c} \partial_L\ (\gamma(\prod_{i=0}^{k} F_i))\ \partial_{\complement L}\ (\frac{1}{r})|\ d\ \lambda^{1+k} < +\infty$$

Pour établir que les propriétés (H_5), (H_6) et (H_7) impliquent la propriété (B) sur tout R_α^ρ, nous remarquons que, en (2), nous avons établi cette implication pour $\rho^0 = (0,0,\dots,0)$.

Pour la généraliser à tout R_α^ρ, nous considérons ρ étant fixé, l'application ψ_ρ de $\mathbb{R}^k$ dans lui-même, telle que $\psi_\rho\ (x_1,\dots,x_k) = (x_1',\dots,x_k')$ où

si $\rho_0 = 0$ $\quad x_i' = x_i$ si $\rho_i = 0$

$\qquad\qquad\quad x_i' = -x_i$ si $\rho_i = 1$

si $\rho_0 = 1$ $\quad x_i' = x_{n+1-i}$ si $\rho_i = 0$

$\qquad\qquad\quad x_i' = -x_{n+1-i}$ si $\rho_i = 1$

et l'application ψ_ρ', de $[0,1]^{1+k}$ dans lui-même, telle que $\psi_\rho'\ (t_0,t_1,\dots,t_k) = (t_0',t_1',\dots,t_k')$ où

$t_i' = t_i$ si $\rho_i = 0$

$t_i' = 1 - t_i$ si $\rho_i = 1$

On note $F_n^{\rho,j}$ la fonction de répartition de la marge $Q_n^{\rho,j}$ de la probabilité Q_n^ρ régissant l'observation $\psi_\rho(x)$ dans $(\mathbb{R}^k)^n$ et $F_{n,i}^{\rho,j}$ $(1 \le i \le k)$ la fonction de la marge $Q_{n,i}^{\rho,j}$.

On note également $\bar{F}_{n,i}^\rho = \frac{1}{n} \sum_{j=1}^{n} F_{n,i}^{\rho,j}$ $(1 \le i \le k)$

et $\quad H_n^{\rho,j}(t_1,\dots,t_k) = F_n^{\rho,j}(\bar{F}_{n,1}^{\rho-1}\ (t_1),\dots,\bar{F}_{n,k}^{\rho-1}(t_k))$

Si $\quad \psi_\rho'(t_0,\tilde{t})) = (t_0',\tilde{t}')$, on a $H_n^{\rho,j}(\tilde{t}') = \tilde{\gamma}\ (H_n^j)(\tilde{t})$

on en déduit que pour tout $(t_0,\tilde{t})$ appartenant à I_ρ,

$$W_n^*(t_0,\tilde{t})(x) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt_0']} I_{[\bar{F}_n(\psi_\rho(x^j) \le \tilde{t}']} - H_n^{\rho,j}(\tilde{t}')$$

c'est-à-dire

$$\frac{W_n^*(t)(x)}{r(t)} = \frac{W_n(\psi'_\rho(t))\psi_\rho(x))}{r(\psi'_\rho(t))}$$

où r_ρ est défini sur I_ρ par $r_\rho(t) = r(\psi_\rho'^{-1}(t))$

donc si on pose $t' = \psi_\rho(t)$ et $x' = \psi_\rho(x)$ le processus $\frac{w_n(t')(x')}{r_\rho(t')}$ vérifie (B) pour $\rho = \rho°$.

Théorème fondamental

Il résulte de la proposition 5 et de l'étude menée au paragraphe précédent qu'on a le théorème suivant :

Théorème

On suppose que le processus des observations vérifie les conditions (H_1) à (H_4) et (H_1') et que la fonction correctrice r est monotone éclatée et qu'il existe $\alpha(>0)$ tel que, pour tout $\rho(\in\{0,1\}^{1+k})$ les conditions (H_5) à (H_7) soient satisfaites.

Alors la suite des processus $W_n^*.\frac{1}{r}$ converge en loi pour la topologie de Skorohod éclatée vers le processus $\gamma(W_0).\frac{1}{r}$ gaussien et à trajectoire presque surement dans C_{1+k}^*.

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Chapitre 2

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WEAK CONVERGENCE OF MULTIDIMENSIONAL RANK STATISTICS UNDER φ-MIXING CONDITIONS

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Abstract: We consider arrays of φ-mixing multidimensional random variables. The weak convergence of the associated weighted empirical process was established in Harel (1980). In this paper we continue this research and prove weak convergence results for weighted rank processes and certain rank statistics.

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Key words and phrases: D^*_{k+1} space; weak convergence; weighted empirical process; weighted rank process, rank statistic; φ-mixing; Skorohod topology; Gaussian process.

1. Introduction

Let $X_{ni}=(X_{ni}^{(1)},\dots,X_{ni}^{(k)})$, $1\le i\le n$, $n\ge 1$, be $\mathbb{R}^k$-valued r.v.'s (random variables) with continuous d.f.'s (distribution-functions) $F_{ni}(x)$, $x\in\mathbb{R}^k$, and continuous marginal distribution functions $F_{ni}^{(j)}(x)$, $x\in\mathbb{R}$, $1\le i\le n$, $n\ge 1$, $1\le j\le k$. Our aim is to study the asymptotic behavior of the rank statistics

$$\mathscr{S}_n=\sum_{i=1}^{n} c_{ni}a_n(R_{ni}^{(1)},\dots,R_{ni}^{(k)}) \tag{1.1}$$

where c_{ni} are the regression constants, $a_n(\cdots)$ are the scores, and $R_{ni}^{(j)}$ is the rank of $X_{ni}^{(j)}$ among $\{X_{nl}^{(j)},\ 1\le l\le n\}$. We assume that the underlying r.v.'s are φ-mixing with rates

$$\varphi(m)=O(m^{-1-\varepsilon})\quad\text{for some } \varepsilon>0\ (m\ge 1) \tag{1.2}$$

or

$$\sum_{m=1}^{\infty} m^{-1}\varphi^{1/4}(m)<\infty. \tag{1.3}$$

Recall that the sequence $\{X_{ni}\}$ is φ-mixing if

$$\sup\{|P(B\mid A)-P(B)|;\ A\in\sigma(X_{ni},\ 1\le i\le j),\ B\in\sigma(X_{ni},\ i\ge j+m)\}=\varphi(m)\downarrow 0$$

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for positive integers j and m. Here $\sigma(X_{ni}, 1 \le i \le j)$ and $\sigma(X_{ni}, i \ge j+m)$ are the σ-fields generated by $(X_{n1}, \dots, X_{nj})$ and $(X_{n,j+m}, X_{n,j+m+1}, \dots)$ respectively. Our methods of proofs are based on some of the ideas of Fears and Mehra (1974), Mehra and Rao (1975), Rüschendorf (1976), and Ahmad and Lin (1980). For the case of independent observations X_{ni}, the asymptotic behavior of $\mathcal{S}_n$ is studied by Ruymgaart and Van Zuijlen (1978).

Denote by $\tilde{F}_n^{(j)}$ the right continuous empirical d.f. of $X_{ni}^{(j)}$, $i=1,\dots,n$; i.e. let

$$\tilde{F}_n^{(j)}(x) = n^{-1} \sum_{i=1}^{n} I_{[X_{ni}^{(j)} \le x]}$$

where $I_{[\,]}$ denotes the indicator function, and set $F_n^{(j)}(x) = E(\tilde{F}_n^{(j)}(x)) = n^{-1} \sum_{i=1}^{n} F_{ni}^{(j)}(x)$. Let $\{H_{ni}\}$ be a sequence of measures on $[0,1]^k$ defined by

$$H_{ni}(t) = F_{ni}(F_n^{(j)^{-1}}(t_1), \dots, F_n^{(k)^{-1}}(t_k)), \quad t=(t_1,\dots,t_k) \in [0,1]^k. \tag{1.4}$$

We now define the *truncated rank process* L_n as

$$L_n(t_0, t) = n^{-1/2} \sum_{i=1}^{[nt_0]} \left\{ \prod_{j=1}^{k} I_{[\tilde{F}_n^{(j)}(X_{ni}^{(j)}) \le t_j]} - H_{ni}(t) \right\}, \quad t_0 \in [0,1], \tag{1.5}$$

where $[nt_0]$ is the largest integer $\le nt_0$. Then, we can rewrite $\mathcal{S}_n$ equivalently as

$$\mathcal{S}_n = n^{-1/2} \int L_n(\mathbf{t}) \cdot \frac{1}{r}(\mathbf{t}) \cdot r(\mathbf{t}) \cdot \mathrm{d}\mu_n(\mathbf{t}) + d_n \tag{1.6}$$

where $\mathbf{t} = (t_0, t)$, μ_n is a signed measure on $[0,1]^{k+1}$, r is a continuous function from $[0,1]^{k+1}$ into $\mathbb{R}^+$ (called a weight function, see definition in Section 2.1b) and d_n is a centering constant. (By convention, $(1/r)(\mathbf{t}) = 0$ if $r(\mathbf{t}) = 0$.) The weight function is introduced to deal with the situations when the regression constants and/or score functions are unbounded. We shall first establish the weak convergence of the process $L_n \cdot (1/r)$ with respect to the topology of the uniform convergence (i.e. uniform topology), and then the convergence of $\mathcal{S}_n$ will follow. Since the process $L_n \cdot (1/r)$ does not necessarily vanish on the upper boundary of $[0,1]^{k+1}$, we introduce (in Section 3) a new process $\hat{L}_n$ called the split rank process which vanishes on the upper boundary and which admits the representation of $\mathcal{S}_n$ in the form

$$\mathcal{S}_n = n^{-1/2} \int_{[0,1]^{k+1}} \hat{L}_n(\mathbf{t}) \frac{1}{r}(\mathbf{t}) \cdot r(\mathbf{t}) \, \mathrm{d}\lambda_n(\mathbf{t}) + b_n \tag{1.7}$$

where λ_n (a certain measure) and b_n (a centering constant) are defined in Section 4. In Section 3, we shall establish the convergence of $\hat{L}_n \cdot (1/r)$ via the convergence of $W_n \cdot (1/r)$ where W_n is the *truncated empirical process* W_n defined as

$$W_n(t_0, t) = n^{-1/2} \sum_{i=1}^{[nt_0]} \left\{ \prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)}) \le t_j]} - H_{ni}(t) \right\}, \quad t_0 \in [0,1]. \tag{1.8}$$

Our methods are the extensions of those of Fears and Mehra (1974) and Ahmad and Lin (1980) who studied the problem for the one-dimensional stationary case, and of Mehra and Rao (1975) who considered the multidimensional stationary case. These authors study the convergence of the processes without truncation and their findings are limited to the case where the regression constants are bounded. The results of our paper take care of these limitations, and provide not only the extensions of the results of Fears and Mehra, and of Ahmad and Lin, but also improve upon the results of Mehra and Rao (loc. cit.). For example, in the k-dimensional case with stationary observations having $\prod_{j=1}^{k} t_j$ as their common d.f., we obtain the convergence of empirical process with weight functions like $r(t_1, \ldots, t_k) = (\prod_{j=1}^{k} t_j)^{\alpha}$ if $\alpha < (k+1)/4k$ ($\alpha < (k+2)/4(k+1)$ with truncation), where Mehra and Rao provide results for $\alpha < 1/2k$. Moreover the main theorem (Theorem 3.1) of Mehra and Rao (loc. cit.) can be used only for weight functions which vanish on the lower boundary and not on the upper boundary. Here the results hold for weight functions which vanish on the lower as well as on the upper boundary. We may also mention that in the one-dimensional case, Neumann (1982) uses a much slower mixing rate for φ-mixing than in Harel (1980), but his results can only be applied for stationary processes and in untruncated situations. Here we also obtain a slight generalization of his results by considering intermediate φ-mixing and for truncated and nonstationary processes.

We would also like to draw the attention of the readers to work of Ruymgaart (1974) who studied the multidimensional i.i.d. case, and of Ruymgaart and Van Zuijlen (1978) who investigated a similar problem for the case of independent but not identically distributed random vectors.

2. Preliminaries

2.1a. The D_{k+1}^* and C_{k+1}^* spaces

Let $f: [0,1]^{k+1} \to \mathbb{R}$. For $\varrho = (\varrho(0), \ldots, \varrho(k)) \in \{0,1\}^{k+1}$, define

$$f_\varrho(t) = \lim_{\substack{s_i \uparrow t_i,\, \varrho(i)=1 \\ s_i \downarrow t_i,\, \varrho(i)=0}} f(s) \quad ((s,t) \in [0,1]^{k+1}),$$

if it exists; in which case, call $f_\varrho(t)$ the ϱ-limit of f at t. Denote by D_{k+1}, the space of all maps $f: [0,1]^{k+1} \to \mathbb{R}$ such that for all $\varrho \in \{0,1\}^{k+1}$, f_ϱ exists and $f_\varrho = f$ for $\varrho = (0, \ldots, 0)$. More generally, for a closed rectangle R in $[0,1]^{k+1}$. Let $D_\varrho(R) = \{f; f: R \to \mathbb{R};\ f_{\varrho'} \text{ exists } \forall \varrho' \in \{0,1\}^{k+1} \text{ and } f_\varrho = f\}$.

Let $D_{k+1}^* = \{f; f: [0,1]^{k+1} \to \mathbb{R};\ \forall \varrho$, restriction $f \mid I(\varrho)$ has an extension $\bar{f}_\varrho$ on $\bar{I}(\varrho)$ with $\bar{f}_\varrho \in D_\varrho(\bar{I}(\varrho))\}$ where $I(\varrho) = I_{\varrho(0)} \times \cdots \times I_{\varrho(k)}$,

$$I_l = \begin{cases} [0, \frac{1}{2}) & \text{if } l = 0, \\ [\frac{1}{2}, 1] & \text{if } l = 1, \end{cases}$$

where $\bar{A}$ denotes the closure of the set A. Denote by C_{k+1} the space of all continuous maps $f:[0,1]^{k+1}\to\mathbb{R}$, and note that for any bounded function f, $f\in C_{k+1}$ if and only if $\lim_{\delta\to 0}\omega(f,\delta)=0$ where

$$\omega(f,\delta)=\sup\{|f(t)-f(t')|;\ (t,t')\in([0,1]^{k+1})^2,\ \|t-t'\|\le\delta\}\tag{2.1}$$

and $\|t\|=\sup\{|t_j|,\ 0\le j\le k\}$.

Now put $C^*_{k+1}=\{f;\ f:[0,1]^{k+1}\to\mathbb{R};\ \forall\varrho,\ f|I(\varrho)$ has a continuous extension to $\bar{I}(\varrho)\}$. We define a modulus of continuity for any bounded function $f:[0,1]^{k+1}\to\mathbb{R}$, to be denoted by $\omega_*(f,\delta)$, $(\delta>0)$ by setting

$$\omega_*(f,\delta)=\max_{\varrho\in\{0,1\}^{k+1}}\ \sup_{(t,t')\in I^2(\varrho)}|f(t)-f(t')|,\quad \|t-t'\|<\delta.\tag{2.2}$$

Consider an operator $\gamma:D_{k+1}\to D^*_{k+1}$ defined as

$$\gamma(f)(t)=\sum_{I\subset\{0,\dots,k\}}(-1)^{\operatorname{Card} I}f((b_i)_{i\in I},(a_i)_{i\notin I}),\quad t\in I(\varrho),\tag{2.3}$$

where $a_i=0$ and $b_i=t_i$ if $\varrho(i)=0$, $a_i=t_i-$ and $b_i=1$ if $\varrho(i)=1$ and Card I denotes the cardinality of I.

For example, suppose $k=1$. Then

$$\begin{aligned}
\gamma(f)(t)&=f(t_0,t_1)-f(t_0,0)-f(0,t_1)+f(0,0) &&\text{if } (t_0,t_1)\in[0,\tfrac12)^2,\\
\gamma(f)(t)&=f(1,t_1)-f(t_0-,t_1)-f(1,0)+f(t_0-,0) &&\text{if } (t_0,t_1)\in[\tfrac12,1]\times[0,\tfrac12),\\
\gamma(f)(t)&=f(t_0,1)-f(t_0,t_1-)-f(0,1)+f(0,t_1-) &&\text{if } (t_0,t_1)\in[0,\tfrac12)\times[\tfrac12,1],\\
\gamma(f)(t)&=f(1,1)-f(t_0-,1)-f(1,t_1-)+f(t_0-,t_1-) &&\text{if } (t_0,t_1)\in[\tfrac12,1]^2.
\end{aligned}$$

2.1b. Weight function

A function $r:[0,1]^{k+1}\to\mathbb{R}^+$ is called a split weight function if it satisfies the following conditions:

(i) there exists an $r_0:[0,1]\to\mathbb{R}^+$ and $\bar{r}:[0,1]^k\to\mathbb{R}^+$ such that $r(t)=r_0(t_0)\bar{r}(t)$ for all $t=(t_0,t)\in[0,1]^{k+1}$;

(ii) r belongs to C^*_{k+1};

(iii) $r=0$ if there exists at least one $j\in\{0,\dots,k\}$ such that $t_j=0$ or $t_j=1$.

r is simply called a weight function if (i), (ii) and (iii) are replaced by

(i)′ r belongs to C_{k+1};

(ii)′ $r=0$ if there exists at least one $j\in\{0,\dots,k\}$ such that $t_j=0$ or $t=(1,\dots,1)$.

2.1c. Grids accompanying a sequence of probability measures

A *grid* T of $[0,1]^{k+1}$ is a subset of $[0,1]^{k+1}$ such that $T=\prod_{j=0}^k T^{(j)}$ where $T^{(j)}$ is a finite subset of $[0,1]$ which includes 0 and 1.

A *pace* τ of a grid $T=\prod_{j=0}^k T^{(j)}$ is the number $\tau=\max_{0\le j\le k}\tau_j$ where $\tau_j=\max\{|t'_j-t_j|,\ t'_j$ and t_j are successive elements in $T^{(j)}\}$.

We denote the lower boundary of T by $T_\leftarrow$ where

$$T_{\leftarrow}=\bigcup_{j=0}^{k}\left[\prod_{l=0}^{j-1}T^{(l)}\times\{0\}\times\prod_{l=j+1}^{k}T^{(l)}\right].$$

We call block B of T any rectangle of $[0,1]^{k+1}$ which is of the form

$$B=\prod_{j=0}^{k}\{[t_j,t_j'] \text{ where } t_j \text{ and } t_j' \text{ belong to } T^{(j)}, \text{ and } t_j<t_j'\}.$$

We call *evaluation* $e^{(B)}$ of B, the operator $e^{(B)}:D_{k+1}\to\mathbb{R}^+$ such that

$$e^{(B)}(f)=\sum_{(\varepsilon_0,\dots,\varepsilon_k)\in\{0,1\}^{k+1}}(-1)^{\sum_{i=0}^{k}\varepsilon_i}f[(1-\varepsilon_0)t_0+\varepsilon_0t_0',\dots,(1-\varepsilon_k)t_k+\varepsilon_kt_k'].$$

For any $\delta>0$, set

$$\omega_T(f,\delta)=\sup\{|f(t)-f(t')|;\ (t,t')\in T^2,\ \|t-t'\|<\delta\}.$$

We say that a sequence $\{T_n\}_{n\in\mathbb{N}^*}$ of grids is asymptotically dense in $[0,1]^{k+1}$ if the pace τ_n of T_n satisfies $\lim_{n\to\infty}\tau_n=0$ ($\mathbb{N}^*=\mathbb{N}-\{0\}$, $\mathbb{N}=\{0,1,2,\dots\}$).

Let P_n, $n\in\mathbb{N}^*$, be a sequence of probability measures on $(D_{k+1},\mathscr{D}_{k+1})$ where $\mathscr{D}_{k+1}$ is the σ-field generated by the Skorohod topology (on D_{k+1}). We say that the sequence $\{T_n\}$ of grids accompanies the measure P_n if and only if $\forall\varepsilon>0$, $\exists\varepsilon'>0$ and $\forall\delta\in[0,\frac{1}{2})$, $\exists N_0\in\mathbb{N}^*$ such that $P_n[\{f\in D_{k+1};\ \omega(f,\delta)\geq\varepsilon \text{ and } \omega_{T_n}(f,2\delta)<\varepsilon'\}]=0$ $\forall n\geq N_0$.

For the ease of convenience, we state the following results due to Balacheff and Dupont (1980) which will be used in the sequel.

Proposition 2.1. *Let P_n, $n\in\mathbb{N}^*$, be probability measures on $(D_{k+1},\mathscr{D}_{k+1})$ such that the following conditions are satisfied:*

$$\Phi_U(P_n)\to \text{weakly to some probability measure } P_U \text{ on } \mathbb{R}^U \tag{2.4}$$

where U is a finite subset of $[0,1]^{k+1}$ (Φ_U is the projection of D_{k+1} on $\mathbb{R}^U$); and

$$\lim_{\delta\to 0}\limsup_{n\to\infty}P_n[f\in D_{k+1};\ \omega(f,\delta)\geq\varepsilon]=0\quad\forall\varepsilon>0. \tag{2.5}$$

Then P_n converges weakly with respect to the Skorohod topology to some probability measure P, and $P(C_{k+1})=1$.

Proposition 2.2. *Let ν be a positive finite measure on $[0,1]^{k+1}$ with continuous marginals. Let P_n, $n\in\mathbb{N}^*$, be a sequence of probability measures on $(D_{k+1},\mathscr{D}_{k+1})$ such that $\forall n\in\mathbb{N}^*$, $P_n[f\in D_{k+1}, f\,|\,[0,1]_{\leftarrow}^{k+1}=0]=1$. Let T_n, $n\in\mathbb{N}^*$, be a sequence of grids asymptotically dense in $[0,1]^{k+1}$ and accompanying P_n. Furthermore suppose that for any block B_n of T_n,*

$$P_n[f\in D_{k+1};\ |e^{(B_n)}(f)|>\lambda]\leq\lambda^{-\gamma}(\nu(B_n))^{\beta},\quad \beta>1,\ \gamma>0. \tag{2.6}$$

Then, $\forall\varepsilon>0$, $\exists\delta\in(0,1)$ and $N_0\in\mathbb{N}$ such that

$$P_n[f\in D_{k+1};\ \omega(f,\delta)\geq\varepsilon]<\varepsilon\quad\forall n\geq N_0. \tag{2.7}$$

3. Weak convergence of the weighted and split rank process

In this section we shall study the weak convergence of the weighted rank process as well as that of the split rank process. We start with preliminaries:

3.1. Split rank process

For a given $\varrho=(\varrho(0),\bar{\varrho})\in\{0,1\}^{k+1}$, we define a map

$$\psi_\varrho^{(n)}:(\mathbb{R}^k)^n\to(\mathbb{R}^k)^n \tag{3.1}$$

as follows:

$$\psi_\varrho^{(n)}(x_1,\dots,x_n)=(z_1,\dots,z_n)$$

where $x_i=(x_i^{(1)},\dots,x_i^{(k)})$, $z_i=(z_i^{(1)},\dots,z_i^{(k)})$ and where

$$z_i^{(j)}=\begin{cases} x_i^{(j)} & \text{if } \varrho(0)=0 \text{ and } \varrho(j)=0,\\ -x_i^{(j)} & \text{if } \varrho(0)=0 \text{ and } \varrho(j)=1,\\ x_{n+1-i}^{(j)} & \text{if } \varrho(0)=1 \text{ and } \varrho(j)=0,\\ -x_{n+1-i}^{(j)} & \text{if } \varrho(0)=1 \text{ and } \varrho(j)=1.\end{cases}$$

Also define

$$\psi'_\varrho:[0,1]^{k+1}\to[0,1]^{k+1},\quad \tilde{\psi}_{\bar{\varrho}}:[0,1]^k\to[0,1]^k,\quad \psi'_{0,\varrho(0)}:[0,1]\to[0,1] \tag{3.2}$$

as follows:

$$\psi'_\varrho(t_0,\dots,t_k)=(\psi'_{0,\varrho(0)}(t_0),\tilde{\psi}_{\bar{\varrho}}(t))=(t'_0,\dots,t'_k)$$

where

$$t'_j=\begin{cases} t_j & \text{if } \varrho(j)=0,\\ 1-t_j & \text{if } \varrho(j)=1.\end{cases}$$

For any $\varrho\in\{0,1\}^{k+1}$, denote by L_n^ϱ the process $L_n\circ\psi'_\varrho$ associated with the r.v. $\psi_\varrho^{(n)}(X_{n1},\dots,X_{nn})$ on $(\mathbb{R}^k)^n$, and consider the process L_n^* defined as

$$L_n^*(t)=\begin{cases} 0 & \text{if } t\notin\left[\dfrac{1}{n},\dfrac{n-1}{n}\right]^{k+1},\\ L_n^\varrho(t) & \text{if } t\in\left[\dfrac{1}{n},\dfrac{n-1}{n}\right]^{k+1}\cap I(\varrho).\end{cases} \tag{3.3}$$

Then, we define the split rank process $\hat{L}_n$ as

$$\hat{L}_n(t)=\begin{cases} 0 & \text{if } t\notin\left[\dfrac{1}{n+1},\dfrac{n}{n+1}\right]^{k+1},\\ L_n^\varrho(t') & \text{if } t\in\left[\dfrac{1}{n+1},\dfrac{n}{n+1}\right]^{k+1}\cap I(\varrho),\end{cases} \tag{3.4}$$

where

$$t'_j = \begin{cases} \dfrac{n+1}{n} t_j & \text{if } \varrho(j)=0, \\ \dfrac{n+1}{n} t_j - \dfrac{1}{n} & \text{if } \varrho(j)=1. \end{cases} \tag{3.5}$$

Definition. We say that the sequence $\{H_{ni}\}$ of measures on $[0,1]^k$ is μ-bounded if there exists a finite and positive measure μ on $[0,1]^k$ with continuous marginal distributions such that for every $n \geq 1$ and $1 \leq i \leq n$, $H_{ni}(B) \leq \mu(B)$ for all rectangles B in $[0,1]^k$.

$\{H_{ni}\}$ is said to satisfy the differentiability condition if:

(j) $(\partial/\partial t_j)H_{ni}$ exists and is continuous; and

(jj) $l_{nj} = n^{-1} \sum_{i=1}^{[nt_0]} (\partial/\partial t_j) H_{ni}(t) \to l_j(t_0, t)$ in uniform topology as $n \to \infty$; $l_j \in C_{k+1}$ and $(\partial^{k-1}/(\partial t_l)_{l \neq j}) l_j$ exists and is bounded.

Theorem 3.1. *Assume that* (a) *the sequence* $\{X_{ni}\}$ *is φ-mixing with rates* (1.2); (b) *the sequence* $\{H_{ni}\}$ *is μ-bounded where μ is absolutely continuous with bounded density* f_μ; (c) *the sequence* $\{H_{ni}\}$ *satisfies the differentiability condition*; (d) *the covariance function* C_n *of the empirical process* W_n *defined in* (1.8) *converges to a function C; and* (e) *there exists a d.f.* $F^{(j)}(x)$ *such that*

$$\sup_{x \in \mathbb{R}} |F_n^{(j)}(x) - F^{(j)}(x)| = O(n^{-\alpha}), \quad \alpha > 0. \tag{3.6}$$

Then, for any split weight function r satisfying

$$r(t) \geq A \left[\prod_{j \in J(\varrho)} t_j \prod_{j \notin J(\varrho)} (1 - t_j) \right]^{1/2 - \delta} \quad \forall t \in I(\varrho) \tag{3.7}$$

where $A > 0$ *is a constant,* $\frac{1}{2} - (k+2)/(4k+4) < \delta < \frac{1}{2}$ *and* $J(\varrho) = \{j;\ j \in \{0, \ldots, k\}, \varrho(j) = 0\}$, $\tilde{L}_n \cdot (1/r)$ *converges weakly in uniform topology to a Gaussian process* $\gamma(L_0) \cdot (1/r)$ *with trajectories a.s. in* C^*_{k+1} *where* L_0 *is defined in Proposition* 5.1.3.

Proof of this theorem is given in Section 5.

Let us now consider the *untruncated* process $\tilde{L}_n$ defined as

$$\tilde{L}_n(t) = \begin{cases} 0 & \text{if } t \notin \left[\dfrac{1}{n+1}, \dfrac{n}{n+1} \right]^k, \\ L_n^{(1,\tilde{\varrho})}(t') & \text{if } t \in \left[\dfrac{1}{n+1}, \dfrac{n}{n+1} \right]^k \cap \tilde{I}(\tilde{\varrho}), \end{cases} \tag{3.8}$$

where

$$t'_j = \begin{cases} \dfrac{n+1}{n} t_j & \text{if } \varrho(j)=0, \\ \dfrac{n+1}{n} t_j - \dfrac{1}{n} & \text{if } \varrho(j)=1, \end{cases}$$

and where $\tilde{I}(\tilde{\varrho}) = I_{\varrho(1)} \times \cdots \times I_{\varrho(k)}$. Then, we have:

Corollary 3.1. *Assume that* (a′) *the sequence* $\{X_{ni}\}$ *is stationary, and φ-mixing with rates* (1.3); (b′) *the common d.f.* H_{ni} *has a continuous density*; and (c′) H_{ni} *satisfies the differentiability condition. Then, for any split weight function* $\tilde{r}: [0,1]^k \to \mathbb{R}^+$ *satisfying*

$$\tilde{r}(t) \geq \tilde{A}\left[\prod_{j \in \tilde{J}(\tilde{\varrho})} t_j \prod_{j \notin \tilde{J}(\tilde{\varrho})} (1-t_j)\right]^{1/2-\delta} \quad \text{for all } t \in \tilde{I}(\tilde{\varrho}) \tag{3.9}$$

where $\tilde{A}>0$ *is a constant,* $\frac{1}{2}-(k+1)/4k<\delta<\frac{1}{2}$, *and* $\tilde{J}(\tilde{\varrho}) = \{j;\, j \in \{1,\dots,k\},\ \varrho(j)=0\}$. $\tilde{L}_n \cdot (1/r)$ *converges weakly in uniform topology to a Gaussian process* $\tilde{\gamma}(\tilde{L}_0)\cdot(1/r)$ *with trajectories a.s. in* C_k^*. *Here* $\tilde{L}_0 = L_0(1,t)$ *and* $\tilde{\gamma}: D_k \to D_k^*$ *is defined as in* (2.3).

The proof is given in Section 5.

Example 3.1. Let the d.f. F_{ni} be given by

$$F_{ni} = \prod_{j=1}^{k} F^{(j)}\left[1+\beta_{ni}\prod_{j=1}^{k}(1-F^{(j)})\right]$$

where $F^{(j)}$ is a continuous d.f. and $-1 \leq \beta_{ni} \leq 1$. Suppose there exists a $\bar{\beta} \in [-1,+1]$ and an $\alpha>0$ such that $|\beta_{ni}-\bar{\beta}| = O(n^{-\alpha})$. Then, it is easy to check that the conditions of Theorem 3.1 are satisfied. (Note that here $H_{ni}(t) = \prod_{j=1}^{k} t_j[1+\beta_{ni}\prod_{j=1}^{k}(1-t_j)]$.)

Example 3.2. Let F be a continuous d.f. with marginals $F^{(j)}$, $1 \leq j \leq k$, such that

$$F_{ni}(u_1,\dots,u_k) = F(u_1+1/n,\dots,u_k+1/n)$$

for all $(u_1,\dots,u_k) \in \mathbb{R}^k$ and $F(F^{(1)^{-1}}(\ ),\dots,\ F^{(k)^{-1}}(\ ))$ has a continuous density. Then the conditions of Theorem 3.1 are satisfied.

Example 3.3 (Dupont). Assume that $F_{ni}^{(j)} = \alpha_{ni}^{(j)} F^{(j)} + \beta_{ni}^{(j)} G^{(j)}$, $j=1,\dots,k$, where $(\alpha_{ni}^{(j)}, \beta_{ni}^{(j)}) \in [0,1]^2$ and $\alpha_{ni}^{(j)} + \beta_{ni}^{(j)} = 1$; and let $F^{(j)}$ and $G^{(j)}$ admit continuous density functions which are strictly positive on some bounded interval, and zero outside. Let $\alpha^{(j)} \in [0,1]$ and $\gamma>0$ be such that $|\alpha_{ni}^{(j)} - \alpha^{(j)}| = O(n^{-\gamma})$, $1 \leq j \leq k$. Then, the conditions of Theorem 3.1 are satisfied (since $l_{ni}(t)$ converges uniformly to $t_0 \prod_{l \neq j} t_l$).

4. Asymptotic normality of linear rank statistics

A measure λ on $[0,1]^{k+1}$ is called a pseudo-measure of order $I\subset\{0,\dots,k\}$ if for any $f\in C^*_{k+1}$,

$$\int_{[0,1]^{k+1}} f(t)\lambda(\mathrm{d}t)=\int_{\{(t_0,\dots,t_k):\, t_j=1/2-,\ \forall j\in I\}} f(t)\lambda(\mathrm{d}t).$$

λ will be called a general measure if it is the finite sum of pseudo measures.

For any n, we define a measure λ_n concentrated on $\{1/(n+1),\dots,n/(n+1)\}^{k+1}$ by setting

$$\lambda_n\left(\prod_{j\in J(\varrho)}\left[\frac{l_j}{n+1},\frac{1}{2}\right)\prod_{j\notin J(\varrho)}\left[\frac{1}{2},\frac{l_j}{n+1}\right]\right)=c_{nl_0}a_n(l_1,\dots,l_k)$$

for all $(l_0/(n+1),\dots,l_k/(n+1))\in I(\varrho)\cap\{1/(n+1),\dots,n/(n+1)\}^{k+1}$. Denote by F^{ϱ}_{ni} the d.f. of $X^{\varrho}_{ni}=(X^{(1)\varrho}_{ni},\dots,X^{(k)\varrho}_{ni})$ and $F^{(j)\varrho}_{ni}$ the marginal d.f. of $X^{(j)\varrho}_{ni}$ where $\psi^{\varrho}_n(X_{n1},\dots,X_{nn})=(X^{\varrho}_{n1},\dots,X^{\varrho}_{nn})$ and set

$$F^{(j)\varrho}_n=n^{-1}\sum_{i=1}^{n}F^{(j)\varrho}_{ni},$$

$$H^{\varrho}_{ni}=F^{\varrho}_{ni}(F^{(1)\varrho^{-1}}_n(\ ),\dots,F^{(k)\varrho^{-1}}_n(\ ))\quad\text{for any } \varrho\in\{0,1\}^{k+1}.$$

For any n, we also define a centering coefficient b_n by

$$b_n=\sum_{\varrho\in\{0,1\}^{k+1}}\int_{I(\varrho)}H^{\varrho}_n(t)\lambda_n(\mathrm{d}t)$$

where H^{ϱ}_n is a map: $[0,1]^{k+1}\to\mathbb{R}^+$ such that

$$H^{\varrho}_n(t)=\sum_{i=1}^{[n\psi'_{0,\varrho(0)}(t'_0)]}H^{\varrho}_{ni}\circ\bar{\psi}_{\bar{\varrho}}(t')$$

where

$$t'_j=\begin{cases}\dfrac{n+1}{n}t_j & \text{if } \varrho(j)=0,\\[2ex] \dfrac{n+1}{n}t_j-\dfrac{1}{n} & \text{if } \varrho(j)=1.\end{cases}$$

We now state the following theorem, the proof of which is given in Section 6.

Theorem 4.1. *Let r be a split weight function such that for some general measure λ_0 we have*

$$\lim_{n\to\infty}\int_{[0,1]^{k+1}} fr\,\mathrm{d}\lambda_n=\int_{[0,1]^{k+1}} fr\,\mathrm{d}\lambda_0\quad \textit{for all } f\in C^*_{k+1},\tag{4.1}$$

$$\sup_{n\in\mathbb{N}}\int_{[0,1]^{k+1}} fr\,\mathrm{d}|\lambda_n|<\infty,\tag{4.2}$$

where $|\lambda_n|$ denotes the measure of total variation. If the sequence $\{X_{ni}\}$ and r satisfy the assumptions of Theorem 3.1 *then $n^{-1/2}(\mathscr{S}_n - b_n)$ converges in law to the normal distribution with mean* 0 *and variance T^2 where*

$$T^2 = \int_{[0,1]^{k+1}} \int_{[0,1]^{k+1}} E[\gamma(L_0)(t)\gamma(L_0)(t')]\lambda_0(\mathrm{d}t)\lambda_0'(\mathrm{d}t'). \tag{4.3}$$

Remark that the above theorem is proved under the assumption that the sequence $\{X_{ni}\}$ is φ-mixing with rates (1.2). The theorem does not hold with the φ-mixing rates (1.3) unless one assumes stationarity and the special case when $c_{ni}=1$ for all i.

Let $\tilde{\mathscr{S}}_n$ denote the statistic $\mathscr{S}_n$ when $c_{ni}=1$ for all i, i.e. let

$$\tilde{\mathscr{S}}_n = \sum_{i=1}^{n} a_n(R_{ni}^{(1)}, \dots, R_{ni}^{(k)})$$

and let $\tilde{b}_n$ denote the corresponding centering coefficient i.e.

$$\tilde{b}_n = \sum_{\tilde{\varrho} \in \{0,1\}^k} \int_{\tilde{I}(\tilde{\varrho})} \tilde{H}_n^{\tilde{\varrho}}(t)\tilde{\lambda}_n(\mathrm{d}t)$$

where

$$\tilde{H}_n^{\tilde{\varrho}}(t) = \sum_{i=1}^{n} H_{ni}^{(0,\tilde{\varrho})} \circ \tilde{\psi}_{\tilde{\varrho}}(t'), \quad t_j' = \begin{cases} \dfrac{n+1}{n} t_j & \text{if } \varrho(j)=0, \\[2ex] \dfrac{n+1}{n} t_j - \dfrac{1}{n} & \text{if } \varrho(j)=1. \end{cases}$$

and

$$\tilde{\lambda}_n\left(\prod_{j \in \tilde{J}(\tilde{\varrho})} \left[\frac{l_j}{n+1}, \frac{1}{2} \right) \prod_{j \notin \tilde{J}(\tilde{\varrho})} \left[\frac{1}{2}, \frac{l_j}{n+1} \right] \right) = a_n(l_1, \dots, l_k).$$

Corollary 4.1. *Let $\tilde{r}$ be a split weight function such that for some general measure $\tilde{\lambda}_0$ on $[0,1]^k$ we have*

$$\lim_{n\to\infty} \int_{[0,1]^k} \tilde{f}\tilde{r}\,\mathrm{d}\tilde{\lambda}_n = \int_{[0,1]^k} \tilde{f}\tilde{r}\,\mathrm{d}\tilde{\lambda}_0 \quad \textit{for all } f \in C_k^*, \tag{4.4}$$

$$\sup_{n\in\mathbb{N}} \int \tilde{f} \cdot \tilde{r}\,\mathrm{d}|\tilde{\lambda}_n| < \infty. \tag{4.5}$$

If the sequence $\{X_{ni}\}$ and $\tilde{r}$ satisfy the assumptions of Corollary 3.1, *then $n^{-1/2}(\tilde{\mathscr{S}}_n - \tilde{b}_n)$ converges to the normal distribution with mean* 0 *and variance $\tilde{T}^2$ where*

$$\tilde{T}^2 = \int_{[0,1]^k} \int_{[0,1]^k} E[\tilde{\gamma}(\tilde{L}_0(t))\tilde{\gamma}(\tilde{L}_0(t'))]\tilde{\lambda}_0(\mathrm{d}t)\tilde{\lambda}_0(\mathrm{d}t').$$

The proof of Corollary 4.1 can be deduced from the proof of Theorem 4.1. Let $\hat{I}(\varrho)=\prod_{j=0}^{k}\hat{I}_{\varrho(j)}$ where

$$\hat{I}_l=\begin{cases}(0,\frac{1}{2}] & \text{if } l=0,\\ [\frac{1}{2},1) & \text{if } l=1.\end{cases}$$

The following corollary gives sufficient conditions under which the conditions (4.1) and (4.2) are satisfied.

Corollary 4.2. *Let J be a function on $[0,1]^{k+1}$ such that*

$$J\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right)=c_{nl_0}a_n(l_0,\dots,l_k)$$

for all $(l_0,\dots,l_k)\in\{1,\dots,n\}^{k+1}$, $J=J_d+J_c$ where J_d is a step function taking only a finite number of jumps, and where for any $I\subset\{0,\dots,k\}$, J_c has a derivate $\partial^I J_c/(\partial t_j)_{j\in I}$ which admits a continuous extension on $\hat{I}(\varrho)$ and satisfies (on $\hat{I}(\varrho)$)

$$\left|\frac{\partial^I}{(\partial t_j)_{j\in I}}J_c\right|\le A\left[\prod_{j\in I\cap J(\varrho)}(t_j)\prod_{j\in I\cap J^c(\varrho)}(1-t_j)\right]^{-3/2+\delta'}$$

for all $\tilde{t}\in\hat{I}(\varrho)$, with $A>0$ and $\frac{1}{2}-(k+2)/(k+4)<\delta'<\frac{1}{2}$. Then the conditions (4.1) and (4.2) are satisfied for any split weight function r which satisfies (3.7) with $\delta<\delta'$.

Proof. It is sufficient to prove the above corollary in the case when J_d has only one jump, say at $a=(a_0,\dots,a_k)\in[0,\frac{1}{2})^{k+1}$ let λ_n' and λ_n'' be measures on $[0,1]^{k+1}$ defined by

$$\lambda_n'\left(\prod_{j\in J(\varrho)}\left[\frac{l_j}{n+1},\frac{1}{2}\right)\prod_{j\notin J(\varrho)}\left[\frac{1}{2},\frac{l_j}{n+1}\right]\right)=J_c\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right),$$

$$\lambda_n''\left(\prod_{j\in J(\varrho)}\left[\frac{l_j}{n+1},\frac{1}{2}\right)\prod_{j\notin J(\varrho)}\left[\frac{1}{2},\frac{l_j}{n+1}\right]\right)=J_d\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right),$$

for all $(l_0/(n+1),\dots,l_k/(n+1))\in I(\varrho)\cap\{1/(n+1),\dots,n/(n+1)\}^{k+1}$.

It is easy to check that

$$\lim_{n\to\infty}\int_{I(\varrho)}f\cdot r\,\mathrm{d}\lambda_n'$$

$$=\sum_{I\subset\{0,\dots,k\}}\int_{[0,\frac{1}{2})^i}\left(f\cdot r\cdot\frac{\partial^I}{(\partial t_j)_{j\in I}}J_c\right)((t_j)_{j\in I'}(\tfrac{1}{2}-)^{k+1-i})(\mathrm{d}t_j)_{j\in I}$$

for all $f\in C_{k+1}^*$ where $\varrho=(0,\dots,0)$ and $i=\operatorname{Card}I$ and similar limits exist if $\varrho\neq(0,\dots,0)$.

Thus we obtain a general measure λ_0' satisfying

$$\lim_{n\to\infty}\int_{[0,1]^{k+1}}f\cdot r\,\mathrm{d}\lambda_n'=\int_{[0,1]^{k+1}}f\cdot r\,\mathrm{d}\lambda_0'.$$

Analogously, we obtain

$$\lim_{n\to\infty}\int_{[0,1]^{k+1}} f\cdot r\,\mathrm{d}\lambda_n''=(f\cdot r)(a)\sum_{I\subset\{0,\dots,k\}}(-1)^i J_d((a_j-)_{j\in I}(a_j+))_{J\notin I})$$

for all $f\in C_{k+1}^*$ where $i=\operatorname{Card} I$.

5. Proof of Theorem 3.2 and Corollary 3.2

A previous work is to obtain the weak convergence of the process W_n defined in (1.8) then the weak convergence of the process L_n defined in (1.5) and finally the weak convergence of the weighted process $W_n\cdot(1/r)$.

5.1. Convergence of the empirical process and the rank process

Proposition 5.1.1. *Assume that the sequence $\{X_{ni}\}$ satisfies the assumptions* (a), (b) *and* (d) *of Theorem* 3.1. *Then W_n converges weakly in the Skorohod topology to a Gaussian process W_0 with trajectories a.s. in C_{k+1}.*

The proof is based on the ideas of Balacheff and Dupont (1980) who considered the asymptotic normality of the truncated empirical processes under φ-mixing with rates $\sum_{m=1}^{\infty} m\varphi^{1/2}(m)<\infty$.

Further in this paper, we consider the rates (1.2) or (1.3) which are slower than the one considered by them. To establish their result, Balacheff and Dupont (1980) used a slight modification of an inequality due to Rüschendorf (1974) which is not applicable in our situation. Our result is based on the following lemma.

Lemma 5.1.1. *Let the sequence $\{X_{ni}\}$ of $\mathbb{R}$-valued random variables centered at their expectations be φ-mixing with rates $\sum_{m\geq 1} m^{-1}\varphi^{1/2q}(m)<\infty$. Let N_n be the number of indexes i $(1\leq i\leq n)$ for which X_{ni} is not identically zero. Set $S_n=\sum_{i=1}^{n} X_{ni}$, and $\|X_{ni}\|_l=(\int|X_{ni}|^{2l}\,\mathrm{d}P_n)^{1/2l}$. Then, there exists a constant $C_q(\varphi)$ depending only on q and φ such that*

$$E(S_n^{2q})\leq C_q(\varphi)\sum_{l=1}^{q} N_n^{q/l}\Big(\sup_{1\leq i\leq n}\|X_{ni}\|_l\Big)^{2q}. \tag{5.1}$$

Proof. The proof is a slight modification of Theorem 2.1 of Neumann (1982) and is therefore omitted.

Proof of Proposition 5.1.1. We have to verify (2.4) and (2.5). Following Withers (1975, Cor. 1) we have to show that if Q_n is the probability measure on $(D_{k+1},\mathcal{D}_{k+1})$ defined by W_n, then $\Phi_U(Q_n)$ converges weakly to a Gaussian measure Q_n if (i) C_n converges to some function C, (ii) $\sum_{m\geq 1}\alpha(m)<\infty$, and (iii) $m^{1-a}\alpha([m]^b)\to 0$ (as

$m \to \infty$) where $0<2b<a<1-b$. Now in our situation (i) holds by assumption (d); (ii) follows from (1.2), and (iii) from (1.2) by taking $a=\frac{3}{4}-\frac{1}{8}\varepsilon$, $b=\frac{1}{4}$ and ε sufficiently small (since taking $\alpha(m)=m^{-1-\varepsilon}$, $m^{1-a}\alpha([m]^b)\le Am^{-\varepsilon/8}$ where $A>0$ in some constant). Thus (2.4) is proved.

To prove (2.5), we use Proposition 2.2 and verify (2.6) which will imply (2.7).

Let $T_n=\{i/n; 0\le i\le n\}^{k+1}$ be a sequence of grids asymptotically dense in $[0,1]^{k+1}$; we prove that T_n accompanies Q_n. Now for every $t\in[0,1]^k$, let $(t^{\mathrm{L}},t^{\mathrm{U}})$ be the points of T_n^P (projection of T_n from $[0,1]^{k+1}$ to $[0,1]^k$) such that $t^{\mathrm{L}}\le t\le t^{\mathrm{U}}$ and $\|t^{\mathrm{L}}-t^{\mathrm{U}}\|\le 1/n$, and denote $t_0^{\mathrm{L}}=[nt_0]/n$ for every $t_0\in[0,1]$. Then, with condition (b) of Theorem 3.1, we obtain after some computations

$$|W_n(t_0,t)-W_n(t_0',t')|\le\frac{2kK(\mu)}{\sqrt{n}}+|W_n(t_0^{\mathrm{L}},t^{\mathrm{U}})-W_n(t_0'^{\mathrm{L}},t'^{\mathrm{L}})|$$

for every (t_0,t) and $(t_0',t')\in[0,1]^{k+1}$ where $K(\mu)=\sup_{t\in[0,1]^k}f_\mu(t)$. Consequently, for every $\delta\in(0,\frac{1}{2}]$, we have $\omega(W_n,\delta)\le 2kK(\mu)/\sqrt{n}+\psi_{T_n}(W_n,2\delta)$. It follows that T_n accompanies Q_n. It remains to show that Q_n satisfies (2.6).

Let $\sum_{m=1}^{\infty}m^{-1}\varphi^{1/4}(m)<\infty$ (implied by (1.2)), and let B_n be a block of T_n as defined in 2.1c. Then, using Lemma 5.1.1, with $q=2$, we obtain

$$E[e^{(B_n)}(W_n)]^4\le n^{-2}C_2(\varphi)\left[(nt_0-nt_0')^2K(\mu)^2\prod_{j=1}^{k}(t_j-t_j')^2 + (nt_0-nt_0')K(\mu)\prod_{j=1}^{k}(t_j-t_j')\right].$$

Let $\nu=(C_2(\varphi)(K(\mu)+K^2(\mu)))^{\beta^{-1}}U^{k+1}$ where U^{k+1} is the uniform probability measure on $[0,1]^{k+1}$ and $\beta=(k+2)/(k+1)$. Then, by the Markov inequality,

$$Q_n[f\in D_{k+1}; |e^{(B_n)}(f)|>\lambda]\le\lambda^{-4}(\nu(B_n))^\beta$$

which implies (2.6) for the φ-mixing case with rates (1.3) and therefore (1.2). (2.5) follows.

Proposition 5.1.2. *Assume that the sequence* $\{X_{ni}\}$ *satisfies the assumptions* (a′) *and* (b′) *of Corollary* 3.1. *Then* $W_n(1,t)$ *converges weakly in the Skorohod topology to a Gaussian process* $W_0(1,t)$ *with trajectories a.s. in* C_k.

Proof. From the proof of Proposition 5.1.1 it is sufficient to prove that the finite-dimensional distribution of the nontruncated process $\tilde{W}_n(t)$ $(=W_n(1,t))$ converges in law to a normal distribution; equivalently, that $\sum_{l=1}^{p}\lambda_l\tilde{W}_n(t^{(l)})$ converges in law to a normal distribution for any $p\in\mathbb{N}^*$, any $t^{(l)}=(t_1^{(l)},\ldots,t_k^{(l)})\in[0,1]^k$ and any $\lambda_l\in\mathbb{R}$ $(1\le l\le p)$.

Let $F^{(j)}$ be the common distribution function of the $F_n^{(j)}$ $(n\in\mathbb{N}^*)$ and let $g_i^{(l)}(X_i)$ and $g_i(X_i)$ be the random variables defined by $g_i^{(l)}(X_i)=\prod_{j=1}^{k}I_{[F^j(X_i^{(j)})\le t_j^{(l)}]}-\mu(t^{(l)})$ and $g_i(X_i)=\sum_{l=1}^{p}\lambda_l g_i^{(l)}(X_i)$.

We have

$$\sum_{l=1}^{p} \lambda_l \tilde{W}_n(t^{(l)}) = n^{-1/2} \sum_{i=1}^{n} g_i(X_i)$$

where $g_i(X_i)$ is a sequence of strictly stationary φ-mixing random variables. We deduce from Theorem 2.3 of Neumann (1982) and Theorem 18.5.1 of Ibragimov and Linnik (1971) that $\sum_{l=1}^{p} \lambda_l \tilde{W}_n(t^{(l)})$ converge weakly to a normal law.

Proposition 5.1.3. *Assume that the sequence $\{X_{ni}\}$ satisfies the assumptions* (a) *to* (d) *of Theorem* 3.1. *Then L_n converges weakly in the uniform topology to a Gaussian process L_0 with trajectories a.s. in C_{k+1} where*

$$L_0(t_0, t) = W_0(t_0, t) - \sum_{j=1}^{k} l_j(t_0, t) \times W_0(1, \ldots, t_j, \ldots, 1).$$

Proof. It is proved by Balacheff and Dupont (1980) in their Theorem 6 under φ-mixing rates $\sum_{m=1}^{\infty} m\varphi^{1/2}(m) < \infty$ but the φ-mixing rate is not used in the proof of this theorem because the rate is used only for the convergence of W_n.

Proposition 5.1.4. *Assume that the sequence $\{X_{ni}\}$ satisfies the assumptions* (a′) *to* (c′) *of Corollary* 3.1. *Then $L_n(1, t)$ converges weakly in the uniform topology to $L_0(1, t)$ with trajectories a.s. in C_k.*

Proof. Consequence of Proposition 5.1.3.

5.2. Convergence of the weighted empirical process

We start with a basic proposition which is given in Mehra and Rao (1975) and Harel (1980).

Proposition 5.2.1. *Let Y_n, $n \in \mathbb{N}^*$, be a process with values in D_{k+1}, and suppose that Y_n converges in law (in Skorohod topology) to a Gaussian process Y_0 with trajectories a.s. in C_{k+1}. Let the weight function r be such that $Y_n \cdot (1/r)$, $n \geq \mathbb{N}$, has a.s. trajectories in D_{k+1}. Furthermore assume that $\forall \eta > 0$, $\exists \theta > 0$ and N_0 such that $\forall n \geq N_0$,*

$$P_n\left[\sup\left\{\left| Y_n(t) \cdot \frac{1}{r(t)} \right| \geq \eta \right\}\right] \leq \eta \tag{5.2}$$

where P_n is the law of Y_n, and sup *is taken over $t = (t_0, t_1, \ldots, t_k)$ with the condition that there is at least one j, $0 \leq j \leq k$, such that $t_j \leq \theta$. Then $Y_n \cdot (1/r)$ converges weakly in Skorohod topology to the Gaussian process $Y_0 \cdot (1/r)$ with trajectories a.s. in C_{k+1}.*

We now state the proposition for the convergence of $W_n \cdot (1/r)$.

Proposition 5.2.2. *Assume that the sequence* $\{X_{ni}\}$ *satisfies the assumptions* (a), (b) *and* (d) *of Theorem* 3.1. *Then for any weight function r satisfying*

$$r(t)\geq A\left(\sum_{j=0}^{k} t_j\right)^{1/2-\delta}, \quad t\in[0,1]^{k+1},\ A>0,\ \frac{1}{2}-\frac{k+2}{4k+4}<\delta<\frac{1}{2}. \qquad (5.3)$$

$W_n\cdot(1/r)$ *converges weakly with respect to the Skorohod topology to the Gaussian process* $W_0\cdot(1/r)$ *with trajectories a.s. in* C_{k+1}.

Proof. See Harel and Puri (1987).

5.3. Some preliminary lemmas

Before the formal proof we give five lemmas.

Lemma 5.3.1. *Let* Y_n *be a process with values in* D^*_{k+1} *and measurable with respect to* $\mathcal{U}^*_{k+1}$, *the* σ*-field generated by the uniform topology (on* D^*_{k+1}*). Let* P_n *denote the law of* Y_n. *Then, there exists a probability measure P with* $P(C^*_{k+1})=1$ *for which* P_n *converges weakly with respect to the uniform topology if and only if:*

(i) *for all finite subsets U of* $[0,1]^{k+1}$, $\Phi_U(P_n)$ *converges weakly to* $\Phi_U(P)$;

(ii) $\lim_{\delta\to 0}\limsup_{n\to\infty} P_n[\{f;\omega_*(f,\delta)\geq\varepsilon\}]=0,\ \forall\varepsilon>0.$

Lemma 5.3.1 is a variant of Theorem (12) of Dudley (1978) and the proof will be omitted.

Lemma 5.3.2. *For any* $n\geq 1$, *let* Y_n *be a process with values in* D^*_{k+1} *which converges in law with respect to the uniform topology to the Gaussian process* Y_0 *with the trajectories a.s. in* C^*_{k+1}. *Let* P_n *denote the law of* Y_n, $n\geq 0$, *(it is a probability measure on* $(D^*_{k+1},\mathcal{U}^*_{k+1})$*). Let r be a function on* C^*_{k+1} *with values in* $\mathbb{R}_+$ *such that*

$$Y_n\cdot\frac{1}{r}\ \text{has a.s. trajectories in}\ D^*_{k+1}, \qquad (5.4)$$

$$\forall\delta>0\ \exists\theta>0\ \exists N_0\ \text{such that}\ \forall n\geq N_0,$$

$$P_n\left(\sup\left\{\left|Y_n(t)\frac{1}{r}(t)\right| : \exists 0\leq j\leq k\ \min(t_j,1-t_j)\leq\theta\right\}\geq\delta\right)<\delta. \qquad (5.5)$$

Then the sequence of processes $Y_n\cdot(1/r)$ *converges weakly with respect to the uniform process* $Y_0\cdot(1/r)$ *(with trajectories a.s. in* C^*_{k+1}*).*

Remark 5.3. This lemma improves slightly the results of Pyke and Shorack (1968) and of Fears and Mehra (1974) (in the one-dimensional case).

Proof. According to Lemma 5.3.1 it is sufficient to verify the following conditions. For any $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P_n\left(\left\{f;\ \omega_*\left(f\cdot\frac{1}{r},\delta\right)>\varepsilon\right\}\right)=0. \tag{5.6}$$

For all finite subsets U of $[0,1]^{k+1}$,

$$\Phi_U\left(Y_n\cdot\frac{1}{r}\right) \text{ converges in law to } \Phi_U\left(Y_0\cdot\frac{1}{r}\right). \tag{5.7}$$

it suffices to show (5.6) because (5.7) is an immediate consequence of the hypothesis. From Lemma 5.3.1 and the hypothesis above we deduce that

$$\forall\varepsilon>0\quad \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{f;\ \omega_*(f,\delta)\geq\varepsilon\})=0$$

where $\omega_*(f,\delta)$ is defined in (2.1).

For any $\theta>0$ set $C_\theta^*=\{t;\ t\in[0,1]^{k+1}, \exists$ at least one $j\in\{0,\dots,k\}$ such that $t_j\leq\theta$ or $t_j\geq 1-\theta\}$. We can write

$$\begin{aligned}\omega_*\left(f\cdot\frac{1}{r},\delta\right) &= \max_{\varrho\in\{0,1\}^{k+1}} \sup_{\substack{(t,t')\in I^2(\varrho)\\ |t-t'|<\delta}} \left|f(t)\frac{1}{r}(t)-f(t')\frac{1}{r}(t')\right| \\ &\leq \max_{\varrho\in\{0,1\}^{k+1}} \sup_{t\in I(\varrho)\cap C_{\theta-\delta}^*} \sup_{|t-t'|<\delta} \left|f(t)\frac{1}{r}(t)-f(t')\frac{1}{r}(t')\right| \\ &\quad + \max_{\varrho\in\{0,1\}^{k+1}} \sup_{t\in I(\varrho)\cap C_{\theta-\delta}^*} \sup_{|t-t'|<\delta} \left|(f(t)-f(t'))\frac{1}{r}(t)+f(t')\left(\frac{1}{r}(t)-\frac{1}{r}(t')\right)\right| \\ &\leq 2\sup_{t\in C_\theta^*}\left|f(t)\frac{1}{r}(t)\right| + \frac{\omega_*(f,\delta)}{m} + \frac{\omega_*(r,\delta)}{m^2}\sup_{t\in[0,1]^{k+1}}|f(t)|,\end{aligned}$$

where $m=\min_{t\in I(\varrho)-C_{\theta-2\delta}^*} r(t)$ and we obtain (5.6) easily.

Lemma 5.3.3. *Let $\{Y_{ni},\ 1\leq i\leq n,\ n>1\}$ be real-valued random variables with continuous distribution functions F_{ni}, $n\geq 1$. Assume that the Y_{ni}'s are φ-mixing with rates $\sum_{m\geq 1} m^{-1}\varphi^{1/2}(m)<\infty$. Furthermore, assume that there exists $\alpha_0>0$ and a continuous distribution function F such that*

$$\sup_{x\in\mathbb{R}}|F_n(x)-F(x)|\leq A(n^{-\alpha_0}),\quad A>0,\ n\geq N_0\in\mathbb{N}, \tag{5.8}$$

where $F_n=n^{-1}\sum_{i=1}^n F_{ni}$. Then, $\forall(\alpha,\alpha')$ where $0<\alpha<\alpha'\leq(\alpha_0\wedge\frac{1}{3})$, $\exists K>0$ and $N(\alpha,\alpha')$ such that

$$P_n\left[\sup_{x\in[0,1]}|F_n\circ\tilde{F}_n^{-1}(x)-x|\geq K(n^{-\alpha})\right]\leq K(n)^{-1+2\alpha+\alpha'} \tag{5.9}$$

$\forall n\geq N(\alpha,\alpha')$ where $\tilde{F}_n$ is the right continuous empirical distribution function of $\{Y_{ni},\ i=1,\dots,n\}$, and $\tilde{F}_n^{-1}$ is the inverse function defined as

$$\tilde{F}_n^{-1}(t) = \begin{cases} -\infty & \text{if } t \in [0, 1/n), \\ Y_{(i)} & \text{if } t \in \left[\frac{i}{n}, \frac{i+1}{n}\right), \\ Y_{(n)} & \text{if } t = 1, \end{cases}$$

where $Y_{(1)} < \cdots < Y_{(n)}$ *is the ordered sequence* $Y_{n1}, \ldots, Y_{nn}$.

Proof. The proof follows by showing that

$$P_n\left[\sup_{u \in \mathbb{R}} |\tilde{F}_n(u) - F(u)| \geq Kn^{-\alpha}\right] \leq K' n^{-1+2\alpha+\alpha'}$$

and then making standard transformation (see Harel (1986)).

Lemma 5.3.4. *Assume that the random variables* $\{Y_{ni}\}$ *satisfy the assumptions of Lemma* 5.3.3 *and condition* (b) *of Theorem* 3.1. *Then* $\forall \varepsilon > 0$ *and* $\forall \tau$ $(0 < \tau < \frac{1}{2})$, $\exists \beta > 2$ *such that*

$$P_n[F_n \circ \tilde{F}_n^{-1}(x) \leq \beta x^{1-\tau},\ x \geq n^{-1}] > 1 - \varepsilon \quad \forall n \geq N_0 \in \mathbb{N}^*. \tag{5.10}$$

Proof. Follows from Theorem 3.1 of Fears and Mehra (1974).

Lemma 5.3.5. *Under the conditions of Proposition* 5.1.3 *the process* $\hat{L}_n$ *converges in law to* $\gamma(L_0)$.

Proof. For any $\varrho \in \{0, 1\}^{k+1}$, we put

$$l_{nj}^{\varrho} = n^{-1} \sum_{i=1}^{[n\psi'_{0,\varrho(0)}(t_0)]} \frac{\partial}{\partial t_j} H_{ni}^{\varrho} \circ \tilde{\psi}_{\bar{\varrho}}(t).$$

If we denote by L'_n the process defined by

$$L'_n(t) = L_n\left(\frac{n+1}{n} t \wedge 1\right)$$

where $1 = (1, \ldots, 1)$ and $a \wedge b = (a_0 \wedge b_0, \ldots, a_k \wedge b_k)$, we have

$$\hat{L}_n(t) = \begin{cases} \gamma(L'_n)(t) + \varepsilon_n(t) & \text{if } t \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right]^{1+k}, \\ 0 & \text{if } t \notin \left[\frac{1}{n+1}, \frac{n}{n+1}\right]^{1+k}, \end{cases}$$

where

$$\varepsilon_n(t) = n^{-1/2} \sum_{i=1}^{[n\psi'_{0,\varrho(0)}(t_0)]} H_{ni}^{\varrho}(\tilde{\psi}_{\bar{\varrho}}(t')) - H_{ni}^{\varrho}(\tilde{\psi}_{\bar{\varrho}}(t)), \quad t \in I(\varrho),$$

with

$$t'_j = \begin{cases} t_j & \text{if } \varrho(j)=0, \\ \dfrac{n}{n+1} t_j & \text{if } \varrho(j)=1. \end{cases}$$

From Proposition 5.1.3, L_n converges in law to L_0 therefore L'_n converges weakly to L_0 and consequently $\gamma(L'_n)$ to $\gamma(L_0)$.

It remains to show that ε_n is uniformly convergent to 0.

Using the differentiability of H^{ϱ}_{ni} we obtain

$$\varepsilon_n(t) = n^{-1/2} \sum_{i=1}^{k} (t'_j - t_j) \sum_{i=1}^{[n\psi'_{0,\varrho(0)}(t_0)]} \frac{\partial}{\partial t_j} H^{\varrho}_{ni} \circ \bar{\psi}_{\bar{\varrho}}(c^{(j)})$$

where

$$c^{(j)} = (c_1^{(j)}, \dots, c_k^{(j)}), \quad c_l^{(j)} \begin{cases} = t_l & \text{if } l > j, \\ \in [t_j, t'_j] & \text{if } l = j, \\ = t_l & \text{if } l < j; \end{cases}$$

equivalently, $\varepsilon_n(t) = n^{1/2} \sum_{j=1}^{k} (t'_j - t_j) l^{\varrho}_{nj}(c^j)$. We deduce from the differentiability condition that there exists an $A>0$ and an $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$ and $j=1,\dots,k$,

$$|l^{\varrho}_{nj}| \leq A.$$

Therefore $|\varepsilon_n| \leq n^{1/2} k (n+1)^{-1} A < n^{-1/2} kA$, which converges to 0 as $n \to \infty$.

5.4. Proof of Theorem 3.1 and Corollary 3.1

It is sufficient to prove Theorem 3.1 because in the formal proof the rate of φ-mixing is not used.

Once it has been shown that (5.5) is satisfied for $\hat{L}_n$, the theorem follows from Lemma 5.3.2 and Lemma 5.3.5. However, it is sufficient to show it for L^*_n defined in (3.3) and even for $L^*_n \mid I(\varrho)$ for $\varrho = (0,\dots,0)$; the case for $\varrho \neq (0,\dots,0)$ will be deduced by symmetrization.

Let $\tilde{F}_n^{(j)^{-1}}$ be the right continuous inverse of $\tilde{F}^j_n$, i.e.

$$\tilde{F}_n^{(j)^{-1}} = \begin{cases} -\infty & \text{for } t_j \in \left[0, \dfrac{1}{n}\right), \\ X^j_{i:n} & \text{for } t_j \in \left[\dfrac{i}{n}, \dfrac{i+1}{n}\right), \\ X^j_{n:n} & \text{for } t_j = 1, \end{cases}$$

where $X^{(j)}_{i:n}$ denotes the i-th order statistic of $(X^{(j)}_{n1}, \dots, X^{(j)}_{nn})$.

We have

$$L_n^* \mid I(\varrho) = \hat{W}_n \mid I(\varrho) + Z_n \mid I(\varrho)$$

where

$$\hat{W}_n(t) = \begin{cases} W_n(t_0, F_n^{(1)} \circ \tilde{F}_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)} \circ \tilde{F}_n^{(k)^{-1}}(t_k)) & \text{for } t \in \left[\frac{1}{n}, \frac{n-1}{n}\right]^{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_n(t) = \begin{cases} n^{-1/2} \sum_{i=1}^{[nt_0]} H_{ni}(F_n^{(1)} \circ \tilde{F}_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)} \circ \tilde{F}_n^{(k)^{-1}}(t_k)) - H_{ni}(t) \\ \quad \text{if } t \in \left[\frac{1}{n}, \frac{n-1}{n}\right]^{k+1}, \\ 0 \quad \text{otherwise.} \end{cases}$$

We first show the condition (5.5) for the process $\hat{W}_n \cdot (1/r)$.

Let δ, δ' be such that

$$r(t) \geq A\left[\prod_{j=0}^{k} t_j\right]^{1/2-\delta}, \quad A > 0, \quad t \in [0,1]^{k+1},$$

and such that $\frac{1}{2} - (k+2)/(4k+4) < \delta' < \delta < \frac{1}{2}$, and define r' to be a function by setting

$$r'(t) = \left[\prod_{j=0}^{k} t_j\right]^{1/2-\delta'}, \quad t \in [0,1]^{k+1}.$$

Using Proposition 5.2.2, we have $\forall \varepsilon > 0$, $\exists \theta^{(1)} > 0$, $\exists N_0^{(1)} \in \mathbb{N}^*$, $\forall n > N_0^{(1)}$,

$$P\left[\sup_{t \in C_{\theta^{(1)}} \cap I(\varrho)} \left| W_n(t) \cdot \frac{1}{r'(t)} \right| > \varepsilon\right] < \varepsilon$$

where $C_\theta = \{t;\ t \in [0,1]^{k+1},\ \exists$ at least one $j \in \{0, 1, \ldots, k\}$ such that $t_j \leq \theta\}$.

From Lemmas 5.3.3 and 5.3.4 and the preceding inequality we deduce that $\forall \varepsilon > 0$, $\exists \theta^{(2)} > 0$, $\exists N_0^{(2)} \in \mathbb{N}$, $\forall n > N_0^{(2)}$,

$$P\left[\sup_{t \in C_{\theta^{(2)}} \cap I(\varrho)} \left| \hat{W}_n(t) \cdot \frac{1}{r'(F_n^{(1)} \circ \tilde{F}_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)} \circ \tilde{F}_n^{(k)^{-1}}(t_k))} \right| > \varepsilon\right] < \tfrac{1}{2}\varepsilon$$

and $\forall \varepsilon > 0$, $\exists \theta^{(3)} > 0$, $\exists N_0^{(3)} \in \mathbb{N}^*$, $\forall n > N_0^{(3)}$,

$$P\left[\sup_{t \in C_{\theta^{(3)}} \cap I(\varrho)} \frac{r'(F_n^{(1)} \circ \tilde{F}_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)} \circ \tilde{F}_n^{(k)^{-1}}(t_k))}{r(t)} > 1\right] < \tfrac{1}{2}\varepsilon$$

(take τ with $1 - \tau > (\frac{1}{2} - \delta)/(\frac{1}{2} - \delta')$) setting $\theta = \theta^{(2)} \wedge \theta^{(3)}$ and $N_0 = N_0^{(2)} \vee N_0^{(3)}$. The condition (5.5) follows.

Now we show the condition (5.5) for $Z_n\cdot(1/r)$. We define the function r' as before. The process Z_n can be written in the following form:

$$Z_n(t)=n^{-1/2}\sum_{i=1}^{[nt_0]}\sum_{j=1}^{k} H_{ni}(a^{(j)})-H_{ni}(b^{(j)})$$

where

$$a^{(j)}=(a_1^{(j)},\dots,a_k^{(j)}),\quad a_l^{(j)}=\begin{cases} t_l & \text{if } l>j,\\ F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_l) & \text{if } l\le j,\end{cases}$$

$$b^{(j)}=(b_1^{(j)},\dots,b_k^{(j)}),\quad b_l^{(j)}=\begin{cases} t_l & \text{if } l\ge j,\\ F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_l) & \text{if } l< j.\end{cases}$$

As $\{H_{ni}\}$ is bounded and the measure μ admits a bounded density, we have

$$|Z_n(t)|\le n^{-1/2}A\sum_{i=1}^{[nt_0]}\sum_{j=1}^{k}|F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)-t_j|\prod_{l<j}F_n^{(l)}\circ\bar F_n^{(l)^{-1}}(t_l)\prod_{p>j}t_p$$

where $A>0$. From the equalities

$$\begin{aligned}F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)-t_j&=F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)-\bar F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)+\bar F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)-t_j\\ &=n^{-1/2}W_n(1,\dots,1,F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j),1,\dots,1)+o(n^{-1}).\end{aligned}$$

We deduce that

$$\left|\frac{Z_n(t)}{r(t)}\right|\le A\frac{[nt_0]}{nt_0^{1/2-\delta}}\sum_{j=1}^{k}\left|\frac{W_n(1,\dots,F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j),\dots,1)+o(n^{-1/2})}{t_j^{1/2-\delta}}\right|$$
$$\cdot\prod_{l<j}\frac{F_n^{(l)}\circ\bar F_n^{(l)^{-1}}(t_l)}{t_l^{1/2-\delta}}\cdot\prod_{p>j}\frac{t_p}{t_p^{1/2-\delta}}\,.$$

We apply the previous estimate to $W_n(1,\dots,F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j),\dots,1)/t_j^{1/2-\delta}$ in the non-truncated one-dimensional case, and Lemma 5.3.4 to $F_n^{(j)}\circ\bar F_n^{(j)^{-1}}(t_j)/t_l^{1/2-\delta}$ with $1-\tau>\frac{1}{2}-\delta$. Since $t_j\ge 1/n$ we deduce $o(n^{-1/2})/t_j^{1/2-\delta}<t_j^{\delta}$ and the conclusion follows immediately.

6. Proof of Theorem 4.1

First we give the integral representation of $\mathscr{S}_n$ in the form of (1.7). By definition,

$$\begin{aligned}n^{-1/2}(\mathscr{S}_n-b_n)&=n^{-1/2}\left[\sum_{i=1}^{n}c_{ni}a_{ni}(R_{ni}^{(1)},\dots,R_{ni}^{(k)})-b_n\right]\\ &=n^{-1/2}\left[\sum_{i=1}^{n}c_{ni}\sum_{\substack{1\le l_j\le n\\ 1\le j\le k}}a_n(l_1,\dots,l_k)\prod_{j=1}^{k}I_{[R_{ni}^{(j)}=l_j]}-b_n\right]\end{aligned}$$

$$= n^{-1/2} \sum_{I \subset \{1,\dots,k\}} \sum_{A} \lambda_n\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right) \sum_{i=1}^{l_0} \prod_{j\in I} I_{[R_{ni}^{(j)} \le l_j]} \prod_{j \notin I} I_{[R_{ni}^{(j)} \ge l_j]}$$

$$- \lambda_n\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right) H_n^{(0,\bar{\varrho}(I))}\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right)$$

$$+ n^{-1/2} \sum_{I \subset \{1,\dots,k\}} \sum_{B} \lambda_n\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right) \sum_{i=l_0+1}^{n} \prod_{j\in I} I_{[R_{ni}^{(j)} \le l_j]} \prod_{j \notin I} I_{[R_{ni}^{(j)} \ge l_j]}$$

$$- \lambda_n\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right) H_n^{(1,\bar{\varrho}(I))}\left(\frac{l_0}{n+1},\dots,\frac{l_k}{n+1}\right)$$

$$= \int_{[0,1]^{k+1}} \hat{L}_n(t)\lambda_n(\mathrm{d}t)$$

where $\sum_A$ is the sum over all $1 \le l_j < \frac{1}{2}[1+n]$, $j \in I$, $\frac{1}{2}[1+n] \le l_j \le n$, $j \notin I$, $1 \le l_0 < \frac{1}{2}[1+n]$; and $\sum_B$ is the sum over all $1 \le l_j < \frac{1}{2}[1+n]$, $j \in I$, $\frac{1}{2}[1+n] \le l_j \le n$, $j \notin I$, $\frac{1}{2}[1+n] \le l_0 \le n$ and

$$\bar{\varrho}(I) = (\varrho(1),\dots,\varrho(k)), \quad \varrho(j) = \begin{cases} 0 & \text{if } j \in I, \\ 1 & \text{if } j \notin I. \end{cases}$$

We now prove that

$$\int_{[0,1]^{k+1}} \hat{L}_n \cdot \frac{1}{r} r\lambda_n(\mathrm{d}t) \to \int_{[0,1]^{k+1}} \frac{\gamma(L_0)}{r} r\lambda_0(\mathrm{d}t). \tag{6.1}$$

Let $h_n : D^*_{k+1} \to \mathbb{R}$ be defined as

$$h_n(f) = \int_{[0,1]^{k+1}} f \cdot r \cdot \lambda_n(\mathrm{d}t), \quad n \ge 0,$$

let $\{f_n: n \ge 1\}$ be a sequence of functions in D^*_{k+1}, and suppose that $f_n/r \to f_0/r$ in uniform topology where $f_0 \in C^*_{k+1}$ and $f_0 \cdot (1/r) \in C^*_{k+1}$. We show that $h_n(f_n \cdot (1/r)) \to h_0(f_0 \cdot (1/r))$ as $n \to \infty$:

$$\left| \int_{[0,1]^{k+1}} f_n \lambda_n(\mathrm{d}t) - \int_{[0,1]^{k+1}} f_0 \lambda_0(\mathrm{d}t) \right|$$

$$\le \left| \int_{[0,1]^{k+1}} \left(f_n \cdot \frac{1}{r} - f_0 \cdot \frac{1}{r}\right) r\lambda_n(\mathrm{d}t) \right| + \left| \int_{[0,1]^{k+1}} f_0 \cdot \frac{1}{r} r(\lambda_n - \lambda_0)(\mathrm{d}t) \right|$$

$$\le \sup_{t \in [0,1]^{k+1}} \left| f_n \cdot \frac{1}{r}(t) - f_0 \cdot \frac{1}{r}(t) \right| \left| \int_{[0,1]^{k+1}} r\lambda_n(\mathrm{d}t) \right| + \left| \int_{[0,1]^{k+1}} \frac{f_0}{r} r(\lambda_n - \lambda_0)(\mathrm{d}t) \right|.$$

From (4.1) and (4.2), we obtain $h_n(f_n \cdot (1/r)) \to h_0(f_n \cdot (1/r))$, $n \to \infty$, and by Theorem 5.5 of Billingsley, (6.1) follows.

It remains to show that $\sigma^2 < \infty$. For this, we have to prove

$$\int_{[0,1]^{k+1}} \int_{[0,1]^{k+1}} E[\gamma(L_0)(t)\gamma(L_0)(t')]\lambda_0(\mathrm{d}t)\lambda_0(\mathrm{d}t') < \infty. \tag{6.2}$$

By symmetrization it is sufficient to show

$$\int_{[0,\frac{1}{2})^{k+1}}\int_{[0,\frac{1}{2})^{k+1}} E[(L_0)(t)(L_0)(t')]\lambda_0(\mathrm{d}t)\lambda_0(\mathrm{d}t')<\infty.$$

We have by Proposition 5.1.3,

$$L_0(t)=W_0(t)-\sum_{j=1}^{k} l_j(t)W_0(1,\ldots,t_j,\ldots,1).$$

By assumption (d) of Theorem 3.1, if $(t,t')\in[0,\frac{1}{2})^2$,

$$\lim_{n\to\infty}\left|E\left[\left\{W_n(t)-\sum_{j=1}^{k} l_j(t)W_n(1,\ldots,t_j,\ldots,1)\right\}\cdot\left\{W_n(t')-\sum_{j=1}^{k} l_j(t')W_n(1,\ldots,t_j',\ldots,1)\right\}\right]\right|$$

$$=|E[L_0(t)L_0(t')]|$$

$$\le\lim_{n\to\infty}\left|E\left[\left\{W_n(t)-\sum_{j=1}^{k} l_j(t)W_n(1,\ldots,t_j,\ldots,1)\right\}^2\right]^{1/2}\cdot E\left[\left\{W_n(t')-\sum_{j=1}^{k} l_j(t')W_n(1,\ldots,t_j',\ldots,1)\right\}^2\right]^{1/2}\right|$$

$$=\lim[A_nB_n]$$

by the Schwarz inequality.

From Lemma 5.1.1 for $q=1$, and condition (c) of Theorem 3.1, we obtain

$$A_n\le\left[E\left\{W_n^2(t)+2|W_n(t)|\sum_{j=1}^{k}|W_n(1,\ldots,t_j,\ldots,1)|l_j(t)+\sum_{s=1}\sum_{j=1} l_j(t)l_s(t)|W_n(1,\ldots,t_j,\ldots,1)W_n(1,\ldots,t_s,\ldots,1)|\right\}\right]^{1/2}$$

$$\le C_1\left[\prod_{j=0}^{k} t_j\right]^{1/2}$$

where C_1 is some constant. Similarly $B_n\le C_1[\prod_{j=0}^{k} t_j']^{1/2}$.

Thus $|E[L_0(t)L_0(t')]|$ is bounded by a function which is $\lambda_0\times\lambda_0$ integrable, and so $|E[L_0(t)L_0(t')]|$ is also $\lambda_0\times\lambda_0$ integrable.

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Statistique/*Statistics*

Convergence faible de la statistique de rang multidimensionnelle en condition de φ mélange

Michel HAREL

Résumé — On considère des suites de variables aléatoires multidimensionnelles et φ mélangeantes. La convergence faible des processus empiriques corrigés associés fut établie par Harel [3]. Dans cette Note on continue cette recherche et on prouve des résultats de convergence pour les processus de rang corrigés et certaines statistiques de rang.

Weak convergence of multidimensional rank statistics under φ mixing condition

Abstract — *We consider arrays of φ mixing multidimensional random variables. The weak convergence of the associated weighted empirical processes was established in Harel* [3]. *In this paper we continue this research and prove weak convergence results for weighted rank processes and certain rank statistics.*

1. INTRODUCTION. — Soient $\mathbf{X}_{ni}=(X_{ni}^{(1)}, \ldots, X_{ni}^{(k)})$, $1 \leqq i \leqq n$, $n \geqq 1$ des variables aléatoires à valeurs dans $\mathbb{R}^k$ avec pour fonctions de répartition $F_{ni}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^k$, supposées continues et pour fonctions de répartition marginales $F_{ni}^{(j)}(x)$, $x \in \mathbb{R}$, $1 \leqq i \leqq n$, $n \geqq 1$, $1 \leqq j \leqq k$, supposées continues également. Notre but est d'étudier le comportement asymptotique de la statistique de rang $\mathfrak{S}_n=\sum_{i=1}^{n} c_{ni}\, a_n(R_{ni}^{(1)}, \ldots, R_{ni}^{(k)})$ où les c_{ni} sont les constantes de régression, $a_n(\ldots)$ sont les fonctions de score, et $R_{ni}^{(j)}$ est le rang de $X_{ni}^{(j)}$ parmi $\{X_{nl}^{(j)}, 1 \leqq l \leqq n\}$. On suppose que les variables aléatoires sous jacentes sont φ mélangeantes où le coefficient de mélange vérifie :

$$\varphi(m)=O(m^{-1-\varepsilon}) \quad \text{où} \quad \varepsilon>0 \quad (m \geqq 1) \tag{1}$$

ou

$$\sum_{m=1}^{\infty} m^{-1}\, \varphi^{1/4}(m)<\infty. \tag{2}$$

On note $\tilde{F}_n^{(j)}$ la fonction de répartition continue à droite de $X_{ni}^{(j)}$, $i=1, \ldots, n$; c'est-à-dire $\tilde{F}_n^{(j)}(x)=n^{-1}\sum_{i=1}^{n} I_{[X_{ni}^{(j)} \leqq x]}$ où $I_{[X_{ni}^{(j)} \leqq x]}$ est la fonction indicatrice, et on pose $F_n^{(j)}(x)=E(\tilde{F}_n^{(j)}(x))=n^{-1}\sum_{i=1}^{n} F_{ni}^{(j)}(x)$. Soit $\{H_{ni}\}$ une suite de mesures sur $[0, 1]^k$ définie par

$$H_{ni}(\mathbf{t})=F_{ni}(F_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)^{-1}}(t_k)), \qquad \mathbf{t}=(t_1, \ldots, t_k) \in [0, 1]^k.$$

On considère maintenant le processus empirique tronqué W_n défini par

$$W_n(t_0, \mathbf{t})=n^{-1/2}\sum_{i=1}^{[nt_0]}\left\{\prod_{j=1}^{k} I_{[F_n^{(j)}(X_{ni}^{(j)}) \leqq t_j]}-H_{ni}(\mathbf{t})\right\}, \qquad t_0 \in [0, 1]$$

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où $[nt_0]$ désigne la partie entière de nt_0 et on considère le processus de rang tronqué L_n défini par

$$L_n(t_0, \mathbf{t}) = n^{-1/2} \sum_{i=1}^{[nt_0]} \left\{ \prod_{j=1}^{k} I_{[\tilde{F}_n^{(j)}(X_{ni}^{(j)}) \leq t_j]} - H_{ni}(\mathbf{t}) \right\}$$

pour tout $(t_0, \mathbf{t}) \in [0, 1]^{k+1}$.

2. DÉFINITIONS ET NOTATIONS. — 2.1. — *Les espaces* D_{k+1}^* *et* C_{k+1}^*. — Soit $f: [0, 1]^{k+1} \to \mathbb{R}$. Pour $\rho \in \{0, 1\}^{k+1}$, on définit $f_\rho(t) = \lim_{\substack{s_i \uparrow t_i\ \rho(i)=1 \\ s_i \downarrow t_i\ \rho(i)=0}} f(s)$ $((s, t) \in [0, 1]^{k+1})$, si la limite existe et on appelle $f_\rho(t)$ la ρ limite de f en t.

On note D_{k+1} l'espace de toutes les applications $f: [0, 1]^{k+1} \to \mathbb{R}$ telles que, pour tout $\rho \in \{0, 1\}^{k+1}$, f_ρ existe et $f_\rho = f$ pour $\rho = (0, \ldots, 0)$. Plus généralement, pour un rectangle fermé R dans $[0, 1]^{k+1}$, soit $D_\rho(R) = \{f; f: R \to \mathbb{R}, f_{\rho'}$ existe $\forall \rho' \in \{0, 1\}^{k+1}$ et $f_\rho = f\}$. Soit $D_{k+1}^* = \{f; f: [0, 1]^{k+1} \to \mathbb{R}; \forall \rho$, la restriction $f \mid I(\rho)$ a une extension $\bar{f}_\rho$ sur $\bar{I}(\rho)$ avec $\bar{f}_\rho \in D_\rho(I(\rho))\}$ où $I(\rho) = I_{\rho(0)} \times \ldots \times I_{\rho(k)}$,

$$I_l = \begin{cases} [0, 1/2[& \text{si } l = 0, \\ [1/2, 1[& \text{si } l = 1 \end{cases}$$ et $\bar{A}$ désigne la fermeture de l'ensemble A.

On note C_{k+1} l'espace de toutes les applications continues $f: [0, 1]^{k+1} \to \mathbb{R}$ et $C_{k+1}^* = \{f; f: [0, 1]^{k+1} \to \mathbb{R}; \forall \rho, f \mid I(\rho)$ a une extension continue sur $\bar{I}(\rho)\}$.

On considère un opérateur $\gamma: D_{k+1} \to D_{k+1}^*$ défini par

(3) $$\gamma(f)(t) = \sum_{I \subset \{0, \ldots, k\}} (-1)^{\operatorname{card} I} f((b_i)_{i \in I}, (a_i)_{i \notin I}), \qquad t \in I(\rho)$$

où $a_i = 0$ et $b_i = t_i$ si $\rho(i) = 0$, $a_i = t_i-$ et $b_i = 1$ si $\rho(i) = 1$ et card I désigne le cardinal de I.

2.2. *Fonction correctrice.* — Une fonction $r: [0, 1]^{k+1} \to \mathbb{R}^+$ est appelée fonction correctrice si elle satisfait les conditions suivantes :

(i) il existe $r_0: [0, 1] \to \mathbb{R}^+$ et $\tilde{r}: [0, 1]^k \to \mathbb{R}^+$ tel que $r(t) = r_0(t_0)\,\tilde{r}(\mathbf{t})$ pour tout $t = (t_0, \mathbf{t}) \in [0, 1]^{k+1}$,

(ii) r appartient à C_{k+1}^*,

(iii) il existe au moins un $j \in \{0, \ldots, k\}$ tel que $t_j = 0$ ou $t_j = 1$.

3. CONVERGENCE DU PROCESSUS DE RANG ÉCLATÉ. — Pour $\rho = (\rho(0), \tilde{\rho}) \in \{0, 1\}^{k+1}$ donné, on définit une application $\psi_\rho^{(n)}: (\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ comme suit :

$$\psi_\rho^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$$

où $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(k)})$, $\mathbf{y}_i = (y_i^{(1)}, \ldots, y_i^{(k)})$ et où

$$y_i^{(j)} = \begin{cases} x_i^{(j)} & \text{si } \rho(0) = 0 \text{ et } \rho(j) = 0 \\ -x_i^{(j)} & \text{si } \rho(0) = 0 \text{ et } \rho(j) = 1 \\ x_{n+1-i}^{(j)} & \text{si } \rho(0) = 1 \text{ et } \rho(j) = 0 \\ -x_{n+1-i}^{(j)} & \text{si } \rho(0) = 1 \text{ et } \rho(j) = 1. \end{cases}$$

On définit aussi

$$\psi'_\rho: [0, 1]^{k+1} \to [0, 1]^{k+1}, \qquad \tilde{\psi}_{\tilde{\rho}}: [0, 1]^k \to [0, 1]^k, \qquad \psi'_{0, \rho(0)}: [0, 1] \to [0, 1]$$

comme suit :

$$\psi'_\rho(t_0, \ldots, t_k) = (\psi'_{0, \rho(0)}(t_0), \tilde{\psi}_{\tilde{\rho}}(\mathbf{t})) = (t'_0, \ldots, t'_k)$$

où

$$t'_j = \begin{cases} t_j & \text{si } \rho(j) = 0 \\ 1 - t_j & \text{si } \rho(j) = 1. \end{cases}$$

Pour tout $\rho \in \{0, 1\}^{k+1}$, on note L_n^ρ le processus $L_n \circ \psi'_\rho$ associé à la variable aléatoire $\psi_\rho^{(n)}(X_{n1}, \ldots, X_{nn})$ sur $(\mathbb{R}^k)^n$, puis on définit le processus de rang $\hat{L}_n$ par

$$\hat{L}_n(t) = \begin{cases} 0 & \text{si } t \notin \left[\dfrac{1}{n+1}, \dfrac{n}{n+1}\right]^{k+1} \\ L_n^\rho(t') & \text{si } t \in \left[\dfrac{1}{n+1}, \dfrac{n}{n+1}\right]^{k+1} \cap I(\rho) \end{cases}$$

où

$$(4) \qquad t'_j = \begin{cases} \dfrac{n+1}{n} t_j & \text{si } \rho(j) = 0 \\ \dfrac{n+1}{n} t_j - \dfrac{1}{n} & \text{si } \rho(j) = 1 \end{cases}$$

et enfin on définit le processus de rang non tronqué $\tilde{L}_n$ par

$$\tilde{L}_n(\mathbf{t}) = \begin{cases} 0 & \text{si } \mathbf{t} \notin \left[\dfrac{1}{n+1}, \dfrac{n}{n+1}\right]^{k} \\ L_n^{(1, \tilde{\rho})}(\mathbf{t}') & \text{si } \mathbf{t} \in \left[\dfrac{1}{n+1}, \dfrac{n}{n+1}\right]^{k} \cap \tilde{I}(\tilde{\rho}); \end{cases}$$

$(\mathbf{t}')$ est défini comme dans (4), et $\tilde{I}(\tilde{\rho}) = I_{\rho(1)} \times \ldots \times I_{\rho(k)}$.

On dit que la suite $\{H_{ni}\}$ de mesures sur $[0, 1]^k$ est μ-bornée s'il existe une mesure μ finie et positive sur $[0, 1]^k$ avec ses marginales continues telles que pour chaque $n \geqq 1$ et $1 \leqq i \leqq n$, $H_{ni}(B) \leqq \mu(B)$ pour tout rectangle B de $[0, 1]^k$. $\{H_{ni}\}$ est dit satisfaire la condition de différentiabilité si

(j) $(\partial/\partial t_j) H_{ni}$ existe et est continue;

(jj) $l_{nj} = n^{-1} \sum_{i=1}^{[nt_0]} \dfrac{\partial}{\partial t_j} H_{ni}(\mathbf{t}) \to l_j(t_0, \mathbf{t})$ pour la topologie uniforme quand $n \to \infty$; $l_j \in C_{k+1}$ et $(\partial^{k-1}/(\partial t_l)_{l \neq j}) l_j$ existe et est continue.

THÉORÈME 1. — *Supposons que la suite* $\{X_{ni}\}$ *est* (*a*) φ *mélangeante avec le taux* (1), *ou* (*b*) *stationnaire et* φ *mélangeante avec le taux* (2); (*c*) *la suite* $\{H_{ni}\}$ *est* μ-*bornée où* μ *est absolument continue avec une densité bornée, et* (*d*) *la suite* $\{H_{ni}\}$ *satisfait la condition de différentiabilité,* (*e*) *la fonction de covariance du processus empirique* W_n *converge vers une fonction et* (*f*) *il existe une fonction de répartition* $F^{(j)}(x)$ *telle que*

$$\sup_{x \in \mathbb{R}} |F_n^{(j)}(x) - F^{(j)}(x)| = O(n^{-\alpha}), \qquad \alpha > 0.$$

Alors, sous les hypothèses (*a*) *et* (*c*) *à* (*f*), *pour toute fonction correctrice r satisfaisant*

$$r(t) \geqq A \left[\prod_{j \in J(\rho)} t_j \prod_{j \notin J(\rho)} (1 - t_j)\right]^{(1/2) - \delta}, \qquad \forall t \in I(\rho)$$

où $A > 0$, $(1/2) - ((k+2)/(4k+4)) < \delta < 1/2$ *et* $J(\rho) = \{j \in \{0, \ldots, k\}, \rho(j) = 0\}$, $\hat{L}_n . 1/r$ *converge faiblement pour la topologie uniforme vers un processus gaussien* $\gamma(L_0) . 1/r$ *à trajectoires p.s. dans* C_{k+1}^*, (L_0 *processus limite de* L_n) *et, sous les hypothèses* (*b*) *à* (*e*), *pour toute fonction correctrice* $\tilde{r}$: $[0, 1]^k \to \mathbb{R}^+$ *satisfaisant*

$$\tilde{r}(\mathbf{t}) \geqq \tilde{A} \left[\prod_{j \in \tilde{J}(\tilde{\rho})} t_j \prod_{j \notin \tilde{J}(\tilde{\rho})} (1 - t_j)\right]^{1/2 - \delta} \quad \text{pour tout } \mathbf{t} \in \tilde{I}(\tilde{\rho})$$

où $\tilde{A} > 0$, $(1/2) - ((k+1)/4k) < \delta < 1/2$, *et* $\tilde{J}(\tilde{\rho}) = \{j;\ j \in \{1, \ldots, k\}, \rho(j) = 0\}$, $\tilde{L}_n . 1/\tilde{r}$ *converge faiblement pour la topologie uniforme vers un processus gaussien* $\tilde{\gamma}(\tilde{L}_0) . 1/\tilde{r}$ *à trajectoires p.s. dans* C_k^*. [$\tilde{L}_0 = L_0(1 . \mathbf{t})$ *et* $\tilde{\gamma}: D_k \to D_k^*$ *est défini comme dans* (3)].

4. NORMALITÉ ASYMPTOTIQUE DE LA STATISTIQUE DE RANG. — Une mesure λ sur $[0, 1]^{k+1}$ est appelée une pseudo mesure d'ordre $I \subset \{0, \ldots, k\}$ si pour tout $f \in C^*_{k+1}$,

$$\int_{[0,1]^{k+1}} f(t)\lambda(dt) = \int_{\{(t_0, \ldots, t_k) : t_j = 1/2-, \ \forall j \in I\}} f(t)\,\lambda(dt);$$

λ sera appelée une mesure générale si c'est une somme finie de pseudo mesures.

Pour tout n, on définit une mesure λ_n concentrée sur $\{1/n+1), \ldots, n/(n+1)\}^{k+1}$ par

$$\lambda_n \left[\prod_{j \in J(\rho)} \left[\frac{l_j}{n+1}, \frac{1}{2}\right[\prod_{j \notin J(\rho)} \left[\frac{1}{2}, \frac{l_j}{n+1}\right]\right] = c_{nl_0} a_n(l_1, \ldots, l_k)$$

pour tout $(l_0/(n+1), \ldots, l_k/(n+1)) \in I(\rho) \cap \{1/(n+1), \ldots, n/(n+1)\}^{k+1}$. On note F^{ρ}_{ni} la fonction de répartition de $X^{\rho}_{ni} = (X^{(1)\rho}_{ni}, \ldots, X^{(k)\rho}_{ni})$ et $F^{(j)\rho}_{ni}$ la fonction de répartition de $X^{(j)\rho}_{ni}$ où $\psi^{(n)}_{\rho}(X_{n1}, \ldots, X_{nn}) = (X^{\rho}_{n1}, \ldots, X^{\rho}_{nn})$ et soit $F^{(j)\rho}_n = n^{-1} \sum_{i=1}^{n} F^{(j)\rho}_{ni}$, $H^{\rho}_{ni} = F^{\rho}_{ni}(F^{(1)\rho^{-1}}_n(\ ), \ldots, F^{(k)\rho^{-1}}_n(\ ))$. Pour tout n, on définit aussi un coefficient de centrage b_n par

$$b_n = \sum_{\rho \in \{0, 1\}^{k+1}} \int_{I(\rho)} H^{\rho}_n(t)\,\lambda_n(dt)$$

où H^{ρ}_n est une application : $[0, 1]^{k+1} \to \mathbb{R}_+$ telle que

$$H^{\rho}_n(t) = \sum_{i=1}^{[n\psi'_{0,\rho(0)}(t'_0)]} H^{\rho}_{ni}(\mathbf{t}')$$

où $(t'_0, \mathbf{t})$ est défini comme dans (4).

THÉORÈME 2. — *Soit r une fonction correctrice telle que pour une certaine mesure générale λ_0 on a :*

$$\lim \int_{[0,1]^{k+1}} f.r\,d\lambda_n = \int_{[0,1]^{k+1}} f.r\,d\lambda_0 \quad \textit{pour tout } f \in C^*_{k+1}$$

$$\sup_{n \in \mathbb{N}} \int_{[0,1]^{k+1}} f.r\,d|\lambda_n| < \infty$$

où $|\lambda_n|$ est la mesure de variation totale. Si la suite $\{X_{ni}\}$ et r (ou $\tilde{r}$) satisfont les hypothèses (a) et (c) à (f) [resp. (b) à (e)] du théorème 1, alors $n^{-1/2}(\mathfrak{S}_n - b_n)$ (resp. avec $c_{ni} = 1$ pour tout i) converge en loi vers la distribution normale de moyenne nulle et variance T^2 où

$$T^2 = \int_{[0,1]^{k+1}} \int_{[0,1]^{k+1}} E[\gamma(L_0)(t)\,\gamma(L_0)(t')]\,\lambda_0(dt)\,\lambda_0(dt').$$

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CONVERGENCE FAIBLE DE LA STATISTIQUE DE RANG MULTIDIMENSIONNELLE EN CONDITION DE ϕ MELANGE OU DE MELANGE FORT

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RESUME

Nous établissons la convergence des statistiques de rang multidimensionnelles munies de fonctions de scores et de constantes de régression non bornées sous des conditions de ϕ mélange ou de mélange fort. Les démonstrations seront pour la plupart dérivées de M. Harel (1980, 1984, 1985).

Mots clés : processus empiriques et processus de rang multidimensionnels éclatés et corrigés; statistiques de rang multidimensionnelles; ϕ mélange, mélange fort; fonctions correctrices éclatées.

1. *INTRODUCTION*

Après les résultats de F.M. Ruymgaart (1974, 1978) dans le cas indépendant, une méthode plus adaptée au cas mélangeant est celle qui fut utilisée par R. Pyke et G. Shorack (1968) dans le cas unidimensionnel indépendant puis reprise ensuite par T.R. Fears et K. Mehra (1974) dans le cas unidimensionnel ϕ mélangeant.

Cette méthode consiste à écrire la statistique T_n sous la forme

$\forall n \in \mathbb{N}^* \qquad \forall x \in (\mathbb{R}^k)^n$

$$T_n(x) = \int_{[0,1]\times[0,1]^k} L_n(t_o,\tilde{t})(x) \cdot \frac{1}{r}(t_o,t)r(t_o,\tilde{t})d\mu_n(t_o,\tilde{t})$$

où μ_n est une mesure signée définie sur $[0,1]^{1+k}$ et L_n le processus de rang normalisé tronqué et centré défini par

$\forall x \in (\mathbb{R}^k)^n \qquad \forall t_o \in [0,1] \qquad \forall \tilde{t} = (t_1, \ldots, t_k) \in [0,1]^k$

$$L_n(t_o,\tilde{t})(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\,t_o]} \prod_{i=1}^{k} I_{[\tilde{F}_{n,x_i}(x_i^j \leq t_i]} - H_n^j(\tilde{t})$$

où $[n\,t_o]$ est la partie entière de $n\,t_o$, $\tilde{F}_{n,x_i}$ est la fonction de répartition empirique continue à droite associée à la suite $(x_i^1, \ldots, x_i^n)$ de la $i^{\text{ème}}$ composante des observations, H_n^j une fonction de centralisation et r une fonction continue de $[0,1]^{1+k}$ dans $\mathbb{R}^+$ (on note par convention $\frac{1}{r}(t) = 0$ si $r(t) = 0$).

La fonction r est appelée fonction correctrice et fut introduite pour obtenir des applications au comportement asymptotique des tests de rang dans les cas les plus généraux, c'est-à-dire avec des fonctions de scores et des constantes de régression non bornées. Nous devons vérifier tout d'abord la convergence faible du processus $L_n(t_o,\tilde{t}) \cdot \frac{1}{r}(t_o,\tilde{t})$ par rapport à la topologie de la convergence uniforme et ensuite la convergence de T_n.

Nous n'obtiendrons la convergence de la suite $L_n \cdot \frac{1}{r}$ que si la fonction r s'annule pour tout t tel que $L_n(t) \xrightarrow{p.s} 0$ c'est-à-dire en tout $t = (t_o,\tilde{t}) \in [0,1] \times [0,1]^k$ vérifiant une des conditions suivantes :

(i) $t_o = 0$

(ii) au moins une des coordonnées t_i $(1 \leq 1 \leq k)$ est égale à zéro

(iii) $\tilde{t} = \tilde{1} = (1, \ldots, 1)$.

Pour obtenir le résultat souhaité, l'utilisation du processus L_n n'est pas suffisant car il ne s'annule pas sur la frontière supérieure (l'ensemble de tous les $t = (t_o, \ldots, t_k)$ pour lesquels au moins une des coordonnées est égale à 1).

Nous sommes conduits à écrire T_n sous la forme suivante :

$$T_n = \int \hat{L}_n \cdot \frac{1}{r} \cdot r \, d\lambda_n$$

où $\hat{L}_n$ est un nouveau processus possédant la même limite qu'un autre nouveau processus L_n^* appelé processus de rang éclaté (ces deux nouveaux processus s'annulent sur la frontière inférieure et sur la frontière supérieure de $[0,1]^{1+k}$)(voir M. Harel (1985)).

λ_n est une mesure générale définie sur $[0,1]^{1+k}$.

La principale idée est la suivante : (pour k = 1) on déduit de L_n^* le processus L_n, en posant :

$$L_n^*(t_o,t_1) = L_n(t_o,t_1) \qquad \text{si} \quad (t_o,t_1) \in [0,\tfrac{1}{2}[^2$$

$$L_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(1-t_o)]} I_{[\tilde{F}_{n,x_1}(x_1^{*j}) \leq t_1]} - H_n^j(t_1) \quad \text{si} \quad (t_o,t_1) \in [\tfrac{1}{2},1] \times [0,\tfrac{1}{2}[$$

$$L_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\, t_o]} I_{[\tilde{F}^*_{n,x_1}(x_1^j) \leq 1-t_1]} - (1-H_n^j(t_1)) \quad \text{si} \quad (t_o,t_1) \in [0,\tfrac{1}{2}[\times [\tfrac{1}{2},1]$$

$$L_n^*(t_o,t_1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(1-t_o)]} I_{[\tilde{F}^*_{n,x_1^*}(x_1^{*j}) \leq 1-t_1]} - (1-H_n^j(t_1)) \quad \text{si} \quad (t_o,t_1) \in [\tfrac{1}{2},1]^2$$

où $x_1^{*j} = x_1^{n+1-j}$ $(1 \leq j \leq n)$ et $\tilde{F}^*_{n,x_1}(u) = \frac{1}{n} \sum_{j=1}^{n} I_{[x_1^j \geq u]}$ $(u \in \mathbb{R})$.

Une étape préliminaire est la convergence du processus empirique tronqué éclaté et corrigé $W_n^* \cdot \frac{1}{r}$ déduit de W_n, comme L_n^* est déduit de L_n, où W_n est le processus empirique tronqué normalisé défini par

$$W_n(t)(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\,t_0]} \prod_{i=1}^{k} I_{[\bar{F}_{n,i}(x_i^j) \leq t_i]} - H_n^j(\tilde{t})$$

avec $\bar{F}_{n,i} = \frac{1}{n} \sum_{j=1}^{n} F_{n,i}^j$ $(1 \leq i \leq k)$ et $F_{n,i}^j$ la fonction de répartition de la marge régissant l'observation x_i^j $(1 \leq i \leq k,\ 1 \leq j \leq n)$.

Nous avons établi dans M. Harel (1980) la convergence du processus W_n corrigé et dans M. Harel (1983, 1984) la convergence du processus W_n^* corrigé par rapport à la topologie de Skorohod, puis dans M. Harel (1985) nous montrons la convergence du processus L_n^* corrigé par rapport à la topologie de la convergence uniforme et en déduisons la convergence de la statistique de rang T_n sous des conditions de ϕ mélange. Nous montrerons ici que la convergence de T_n reste vraie pour des observations ayant des lois de probabilités un peu plus générales et aussi sous des conditions de mélange fort.

2. DEFINITIONS ET NOTATIONS

2.1. *L'espace* D_{1+k}^*

On note $I_0 = [0, \frac{1}{2}[$ $[I_1 = \frac{1}{2}, 1]$

soit $k \in \mathbb{N}^*$ fixé pour tout

$$\rho = (\rho_0, \tilde{\rho}) = (\rho_i)_{i \in \{0,\dots,k\}} \in \{0,1\}^{1+k}$$

on pose

$$I_\rho = \prod_{i=0}^{k} I_{\rho_i} \qquad \tilde{I}_{\tilde{\rho}} = \prod_{i=1}^{k} I_{\rho_i}$$

$$J(\rho) = \{i \in \{0, \dots, k\};\ \rho_i = 0\}$$

$$\tilde{J}(\tilde{\rho}) = \{i \in \{1, \dots, k\};\ \rho_i = 0\}.$$

<u>*Définition 2.1.*</u>

Pour tout $\rho \in \{0,1\}^{1+k}$, *on note* D_ρ *l'espace des fonctions* f_ρ *de* I_ρ *dans* $\mathbb{R}$ *qui admettent une prolongation sur* $\bar{I}_\rho$ *(fermeture de* I_ρ*), ont une limite dans les* 2^{1+k} *directions en tout point et sont continues dans la direction* ρ*; on note* D_{1+k}^* *l'espace des fonctions* f *telles que pour tout* $\rho \in \{0,1\}^{1+k}$ *la restriction de* f *à* I_ρ *appartienne à* D_ρ.

2.1. *L'espace* C_{1+k}^*

On note $C_{1+k}^* = \{f$ définie sur $[0,1]^{1+k}$; $\forall\rho \in \{0,1\}^{1+k}$ f/I_ρ admet une prolongation continue sur $\bar{I}_\rho\}$ et par

$C_k(i)$, $1 \leqslant i \leqslant k$ l'espace de toutes les applications continues et bornées $f : A(i) \longrightarrow \mathbb{R}$ où $A(i) = [0,1]^{i-1} \times]0,1[\times[0,1]^{k-i}$

2.3. La nature des observations

Soit $x = (x^1, \ldots, x^n)$ une suite de n observations dans $\mathbb{R}^k$. Notons pour tout $i \in \{1, \ldots, k\}$, $x_i = (x_i^1, \ldots, x_i^n)$ la suite des $i^{\text{ièmes}}$ composantes des observations, $\tilde{F}_{n,x_i}$ la fonction de répartition empirique continue à droite, pour tout $j \in \{1,\ldots,n\}$ F_n^j la fonction de répartition de la marge Q_n^j de la probabilité Q_n régissant l'observation x dans $(\mathbb{R}^k)^n$ et $F_{n,i}^j$ la fonction de répartition de la marge $Q_{n,i}^j$ de Q_n^j.

On note aussi :

$$\bar{F}_{n,i} = \frac{1}{n} \sum_{j=1}^{n} F_{n,i}^j \qquad i \in \{1, \ldots, k\}$$

H_n^j la fonction de répartition définie sur $[0,1]^k$ par $\forall t = (t_1, \ldots, t_k) \in [0,1]^k$

$$H_n^j(t_1, \ldots, t_k) = F_n^j(\bar{F}_{n,1}^{-1}(t_1), \ldots, \bar{F}_{n,k}^{-1}(t_k)).$$

2.4. Les processus empiriques

Les processus empirique tronqué normalisé W_n est défini par

$$\forall x \in (\mathbb{R}^k)^n \quad \forall t_o \in [0,1] \qquad \forall \tilde{t} = (t_1, \ldots, t_k) \in [0,1]^k$$

$$W_n(t_o,\tilde{t})(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\,t_o]} \prod_{i=1}^{k} I_{[\bar{F}_{n,i}(x_i^j) \leq t_i]} - H_n^j(\tilde{t}).$$

Le processus de rang tronqué normalisé L_n est défini par :

$$\forall x \in (\mathbb{R}^k)^n \quad \forall t_o \in [0,1] \qquad \forall \tilde{t} = (t_1, \ldots, t_k) \in [0,1]^k$$

$$L_n(t_o,\tilde{t})(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\,t_o]} \prod_{i=1}^{k} I_{[\tilde{F}_{n,x_i}(x_i^j) \leq t_i]} - H_n^j(\tilde{t}).$$

Pour tout $\rho = (\rho_o,\tilde{\rho}) \in \{0,1\}^{1+k}$, on définit ψ_ρ l'application de $(\mathbb{R}^k)^n$ dans lui-même et pour tout $i \in \{0, \ldots, k\}$. ψ_ρ^i l'application de $\mathbb{R}^n$ dans lui-même tels que

$$\psi_\rho(x_1, \ldots, x_k) = (\psi_\rho^1(x_1), \ldots, \psi_\rho^k(x_k)) = (x_1', \ldots, x_k')$$

où

$$x_i = (x_i^1, \ldots, x_i^n),\ x_i' = (x_i'^1, \ldots, x_i'^n) \qquad i \in \{1, \ldots, k\} \qquad j \in \{1, \ldots, n\}$$

$$x_i'^j = x_i^j \quad \text{si} \quad \rho_o = 0 \quad \text{et} \quad \rho_i = 0$$

$$x_i'^j = x_i^j \quad \text{si} \quad \rho_o = 0 \quad \text{et} \quad \rho_i = 1$$

$$x_i'^j = x_i^{n+1-j} \quad \text{si} \quad \rho_o = 1 \quad \text{et} \quad \rho_i = 0$$

$$x_i'^j = -x_i^{n+1-j} \quad \text{si} \quad \rho_o = 1 \quad \text{et} \quad \rho_i = 1$$

On définit aussi l'application ψ_ρ' de $[0,1]^{1+k}$ dans lui-même, pour tout $i \in \{0,\ldots,k\}$ ψ_{i,ρ_i}'

l'application de [0,1] dans lui-même, pour tout $i \in \{1, \ldots, k\}$ $\psi'^{i}_{\rho_i}$ l'application de $[0,1]^{\{0,\ldots,k\}-\{i\}}$ dans lui-même et $\widetilde{\psi}_{\widetilde{\rho}}$ l'application de $[0,1]^k$ dans lui-même tels que :

$$\psi'_\rho(t_o,t_1, \ldots, t_k) = (\psi'_{0,\rho_0}(t_o), \ldots, \psi'_{k,\rho_k}(t_k)) = (\psi'_{i,\rho_i}(t_i), \psi'^{i}_{\rho}(t^i))$$

$$= (\psi'_{0,\rho_0}(t_o), \widetilde{\psi}_{\widetilde{\rho}}(\widetilde{t})) = (t'_1, t'_1, \ldots, t'_k)$$

où

$t'_i = t_i$ si $\rho_i = 0$

$t'_i = 1 - t_i$ si $\rho_i = 1$

Le processus empirique éclaté tronqué et normalisé W^*_n est défini par

$\forall \rho \in \{0,1\}^{1+k}$ $\forall t \in I_\rho$ $\forall x \in (\mathbb{R}^k)^n$

$$W^*_n(t)(x) = 0 \quad \text{si} \quad t \notin \prod_{i=0}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

$$W^*_n(t)(x) = W_n(\psi'_\rho(t))(\psi_\rho(x)) \quad \text{si} \quad t \in \prod_{i=0}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

Le processus de rang éclaté tronqué et normalisé est défini par

$\forall \rho \in \{0,1\}^{1+k}$ $\forall t \in I_\rho$ $\forall x \in (\mathbb{R}^k)^n$

$$L^*_n(t)(x) = 0 \quad \text{si} \quad t \notin \prod_{i=0}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

$$L^*_n(t)(x) = L_n(\psi'_\rho(t))(\psi_\rho(x)) \quad \text{si} \quad t \in \prod_{i=0}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

Le processus de rang éclaté non tronqué $\widetilde{L}_n$ est défini par

$\forall \rho \in \{0,1\}^k$ $\forall \widetilde{t} \in \widetilde{I}_{\widetilde{\rho}}$ $\forall x \in (\mathbb{R}^k)^n$

$$\widetilde{L}_n(\widetilde{t})(x) = 0 \quad \text{si} \quad \widetilde{t} \notin \prod_{i=1}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

$$\widetilde{L}_n(\widetilde{t})(x) = L_n(1,\widetilde{\psi}_{\widetilde{\rho}}(\widetilde{t}))(\psi_{(0,\widetilde{\rho})}(x)) \quad \text{si} \quad t \in \prod_{i=1}^{k} \left[\frac{1}{n}, \frac{n-1}{n}\right]$$

On note également pour tout $\rho = (\rho_0,\widetilde{\rho}) \in \{0,1\}^{1+k}$ $F^{\rho,j}_n$ la fonction de répartition de la marge $Q^{\rho,j}_n$ de la probabilité Q^ρ_n régissant l'observation $\psi_\rho(x)$ dans $(\mathbb{R}^k)^n$ et $F^{\rho,j}_{n,i}$ la fonction de répartition de la marge $Q^{\rho,j}_{n,i}$.

On note enfin

$$\bar{F}^\rho_{n,i} = \frac{1}{n} \sum_{j=1}^{n} F^{\rho,j}_{n,i} \qquad i \in \{1, \ldots, k\}$$

$$H^{\rho,j}_{n,i}(t_1, \ldots, t_k) = F^{\rho,j}_n(\bar{F}^{\rho-1}_{n,1}(t_1), \ldots, \bar{F}^{\rho-1}_{n,k}(t_k))$$

$$\ell_{n,i}^{\rho,j} = \frac{\partial}{\partial t_i} H_n^{\rho,j} \circ \widetilde{\psi}_{\rho}$$

$$\ell_{n,i}^{\rho}(t_o,\widetilde{t}) = (\frac{1}{n}) \mathop{\quad}_{j=1}^{[n \psi'_{o,\rho_o}(t_o)]} \ell_{n,i}^{\rho,j}(\widetilde{t})$$

2.5. *Mesure générale*

Pour tout $I \subset \{0, \ldots, k\}$ on appelle pseudo-mesure d'ordre I une mesure λ sur $[0,1]^k$ concentrée sur $\{\frac{1}{2}\}^{\text{card } I} \times \; 0,1^{\;k-\text{card } I}$ telle que pour tout $f \in C^*_{1+k}$ l'intégrale $\int f \, d\lambda$ est définie par

$$\int_{[0,1]^{1+k}} f \, d\lambda = \int_{[0,1]^{k-\text{card } I}} f((x_i)_{i \in \{0, \ldots, k\} - I}, (\tfrac{1}{2})^{-\text{card } I}) \lambda((\tfrac{1}{2})^{\text{card } I}, d(x_i)_{i \in \{0, \ldots, k\} - I}).$$

λ est appelée pseudo-mesure, s'il existe $I \subset \{0, \ldots, k\}$ tel que λ est une pseudo-mesure d'ordre I.

On appelle mesure générale une somme finie de pseudo-mesures.

Remarque 2.5.

Une pseudo-mesure d'ordre $\emptyset$ est une mesure.

3. *LA NATURE DES OBSERVATIONS*

Nous allons supposer que les observations vérifient deux types d'hypothèses :

H.1. *Les marges* $Q_{n,i}$ $i \in \{1,\ldots,k\}$ *sont diffuses sur* $\mathbb{R}^n$.

H.2. *La suite* $(c_n ; n \in \mathbb{N})$ *des fonctions de covariance du processus empirique* W_n *converge simplement vers une fonction* c.

H.3.a. *Il existe une fonction décroissante* $\phi : \mathbb{N}^* \to [0,1]$ *vérifiant* $\phi(0) = 1$, $\phi(n) = O(n^{-1-\varepsilon})$ *où* $\varepsilon > 0$ *et pour laquelle la suite* Q_n *est* ϕ *mélangeante.*

ou

H.3.b. *Il existe une fonction décroissante* $\alpha : \mathbb{N}^* \to [0,1]$ *vérifiant* $\alpha(0) = 1$, $\sum_{n \geq 1} (n+1)^2 \alpha^{\delta}(n) < +\infty$ avec $\delta \in]0, \frac{1}{2(k+1)}[$ *et pour laquelle la suite* Q_n *est fortement mélangeante de taux* $\alpha(n)$.

H.4. *Il existe une mesure* μ *sur* $[0,1]^k$ *qui est positive, de marges diffuses et qui vérifie*

$\forall n \in \mathbb{N}^* \quad \forall j \in \{1,\ldots,n\} \quad \forall B$ (*Bloc dans* $[0,1]^k$ $\mu_n^j(B) \leqslant \mu(B)$

(μ_n^j est la probabilité qui a H_n^j pour fonction de répartition et on note H la fonction de répartition de la mesure μ).

H.5.a. $\forall n \in \mathbb{N}^*$ $\forall i \in \{1, \ldots, k\}$ $\forall j \in \{1, \ldots, n\}$

$\ell^j_{n,i} = \frac{\partial}{\partial t_i} H^j_n$ *existe* sur $A(i)$ et appartient à $C_k(i)$

Pour tout $n \in \mathbb{N}^*$ et $i \in \{1, \ldots, k\}$, on note $\ell_{n,i}$ l'élément de D_{1+k} défini par

$$(t_o, \tilde{t}) \in [0,1]^k \qquad \ell_{n,i}(t_o,\tilde{t}) = \frac{1}{n} \sum_{j=1}^{[n\, t_o]} \ell^j_{n,i}(\tilde{t})$$

H.5.b. $\forall i \in \{1, \ldots, k\}$ $\exists \ell_i \in D_{1+k}$ *tel que* $\ell_{n,i}$ *converge vers* ℓ_i *pour la topologie uniforme* sur tout compact de $A(i)$.

H.6. $\forall i \in \{1, \ldots, k\}$ $\forall \tilde{t} \in [0,1]^k$

$\frac{1}{n} \ell^n_{n,i}(\tilde{t})$ *converge vers* 0.

$\forall \rho \in \{0,1\}^{1+k}$ on note H^ρ la fonction de répartition de la mesure $\psi'_\rho(\mu)$.

$\forall i \in \{1, \ldots, k\}$ on note F^ρ_i la marge H^ρ $(1, \ldots, 1, t_i, 1, \ldots, 1)$ et ℓ^ρ_i la limite de $\ell^\rho_{n,i}$

On note F_o la fonction de répartition uniforme sur $[0,1]$.

H.7. *Il existe une fonction continue* r *définie sur* $[0,1]^{1+k}$ *telle que*

$\forall \beta > \frac{3k+2}{k+2}$ $\forall \rho \in \{0,1\}^{1+k}$ ($\beta > \frac{6k+3}{2k+3}$ *si* H.3.B.).

H.7.a. $\forall I \subset \{0, \ldots, k\}$ $\forall\ [a_i,b_i]^{1+k} \subset [0, \frac{1}{2}]^{1+k}$

$$\Delta \prod_{i\in I} [a_i,b_i] \prod_{i\notin I} [0,a_i]\ (F_o\ H^\rho)\ \frac{1}{(r\circ\psi'_\rho)^\beta}\ ((b_i)_{i\in I},(a_i)_{i\notin I})$$

$$\Delta \prod_{i\in I} [a_i,b_i] \prod_{i\notin I} [0,a_i]\ (\prod_{i=0}^{k} F^\rho_i)\ \frac{1}{(r\circ\psi'_\rho)^\beta}((b_i)_{i\in I},(a_i)_{i\notin I})$$

sont fonctions croissantes de $(a_i)_{i\notin I}$ (Δ *est l'opérateur de différence multidimensionnel*).

H.7.b. $\forall I \subset \{0, \ldots, k\}$

$$\int_{[0,\frac{1}{2}]^{1+k}} \left|\frac{1}{(r\circ\psi'_\rho)^\beta}\ \partial_I(F_o\ H^\rho)\ \partial_{CI}\left(\frac{1}{r\circ\psi'_\rho}\right)\right| d\lambda^{1+k} + \infty$$

$$\int_{[0,\frac{1}{2}]^{1+k}} \left|\frac{1}{(r\circ\psi'_\rho)^\beta}\ \partial_I(\prod_{i=0}^{k} F^\rho_i)\ \partial_{CI}\ \frac{1}{r\circ\psi'_\rho}\right|\ d\lambda^{1+k} < + \infty$$

H.7.c. $\forall i \in \{1, \ldots, k\}$, $\forall u \in R_i = \{t \in [0,1]^{1+k};\ \exists j \neq i \quad t_j = 0\}$

$$\lim_{t = (t_o,\tilde{t}) \to u} \frac{t_i^{\frac{1}{2}}\ \ell^\rho_i(t)}{r\circ\psi'_\rho(t)} = 0$$ *et* ℓ^ρ_i appartient à $C_k(i)$

H.7.d. $\forall n \in \mathbb{N}$ $\forall j \in \{1, \ldots, n\}$ $\forall \tilde{u} \in \tilde{R} = \{\tilde{t} \in [0,1]^k;\ \exists \ell\ t_\ell = 0\}$ $\forall t_0 \in [0,1]$

$$\lim_{\tilde{t} \to \tilde{u}} H_n^j(\tilde{t}) \cdot \frac{1}{r \circ \psi'_\rho(t_0, \tilde{t})} = 0$$

H.8. *Pour tout* $i \in \{1, \ldots, k\}$, *il existe une fonction de répartition continue* F_i^*, $\alpha > 0$, $A > 0$ et $N_0 \in \mathbb{N}^*$ *tel que*

$$\forall n \geq N_0 \qquad \sup_{u \in \mathbb{R}} |\bar{F}_{n,i}(u) - F_i^*(u)| < A\left(\frac{1}{n}\right)^\alpha .$$

Le deuxième type d'hypothèse est :

H^*.1. = H.1.

H^*.2. *Les observations sont stationnaires*

H^*.3.a. *Il existe une fonction décroissante* $\phi : \mathbb{N}^* \to [0,1]$ *vérifiant* $\phi(0) = 1$, $\sum_{n\,1} n^{-1} \phi^{1/4}(n) < +\infty$ *et pour laquelle* Q_n *est* ϕ *mélangeante*

ou

H .3.b.= H.3.B.

H .4. *Si on note la fonction de répartition commune aux fonctions* H_n^j ($H_n^j = H$ $\forall n \in \mathbb{N}^*$ $\forall j \in \{1, \ldots, n\}$) *elle vérifie* H.5.a. *et* H.7.

4. CONVERGENCE DE LA STATISTIQUE DE RANG

L'objet de cette section est de montrer la convergence des statistiques de rang de la forme

$$T_n = \sum_{j=1}^{n} C_n^j\, a_n(R_{n,1}^j, \ldots, R_{n,k}^j)$$

où $a_n(\ldots)$ est une fonction de score, C_n^j une constante de régression et $R_{n,i}^j$ ($1 \leq j \leq n$, $i \in \{1, \ldots, k\}$) est le rang de $X_{n,i}^j$ dans la suite $(X_{n,i}^1, \ldots, X_{n,i}^n)$.

4.1. *Définitions et notations*

Pour tout n, on définit une mesure λ_n concentrée sur $\left\{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\right\}^{1+k}$ définie par

$$\forall \rho \in \{0,1\}^{1+k} \qquad \forall \left(\frac{\ell_i}{n}\right)_{i \in \{0,\ldots,k\}} \in I_\rho \cap \left\{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\right\}^{1+k}$$

$$\lambda_n\Big(\prod_{i \in J(\rho)} \left[\frac{\ell_i}{1+n}, \frac{1}{2}\right[\prod_{i \notin J(\rho)} \left[\frac{1}{2}, \frac{\ell_i}{1+n}\right]\Big) = C_n^{\ell_i}\, a_n((\ell_i)_{i \in \{1,\ldots,k\}}).$$

Pour tout $n \in \mathbb{N}^*$ et tout $\rho \in \{0,1\}^{1+k}$, on définit une fonction $H_n^\rho(t)$ définie sur $[0,1]^{1+k}$

$$H_n^\rho(t) = \sum_{j=1}^{[n\psi'_{0,\rho_0}(t_0)]} H_n^{\rho,j}(\underset{\sim}{\tilde{\psi}'_\rho}(\tilde{t}'))$$

avec $t'_i = \frac{n+1}{n} t_i$ si $\rho_i = 0$ $\forall i \in \{0, \ldots, k\}$

$t'_i = \frac{n+1}{n} t_i - \frac{1}{n}$ si $\rho_i = 1$

et pour n, on définit un coefficient b_n appelé coefficient de centrage par

$$b_n = \sum_{\rho\in\{0,1\}^{1+k}} \int_{I_\rho} H_n^\rho(t)\ \lambda_n(dt).$$

Lorsque toutes les constantes de régression sont toutes égales à 1, on note alors pour tout n

$\tilde{T}_n$ la statistique de rang

$\tilde{\lambda}_n$ la mesure concentrée sur $\{\frac{1}{1+n}, \ldots, \frac{n}{n+1}\}^k$ définie par

$$\forall\tilde{\rho} \in \{0,1\}^k \quad \forall(\frac{\ell_i}{n})_{i \in \{1,\ldots,k\}} \subset \tilde{I}_{\tilde{\rho}} \cap \{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\}^k$$

$$\tilde{\lambda}_n \left(\prod_{i\in J(\tilde{\rho})} \left[\frac{\ell_i}{1+n}, \frac{1}{2}\right[\prod_{i\notin J(\tilde{\rho})} \left[\frac{1}{2}, \frac{\ell_i}{1+n}\right]\right) = a_n((\ell_i)_{i \in \{1,\ldots,k\}})$$

$$\tilde{H}^{\tilde{\rho}}(\tilde{t}) = \sum_{j=1}^{n} H_n^{(o,\tilde{\rho}),j}\ (\tilde{\Psi}'_{\tilde{\rho}}(\tilde{t}'))$$

avec

$t'_i = \frac{n+1}{n} t_i$ si $\rho_i = 0$

$t'_i = \frac{n+1}{n} t_i - \frac{1}{n}$ si $\rho_i = 1$

$\tilde{b}_n$ le coefficient de centrage par

$$\tilde{b}_n = \sum_{\tilde{\rho}\in\{0,1\}^k} \int_{\tilde{I}_{\tilde{\rho}}} \tilde{H}_n^{\tilde{\rho}}(\tilde{t})\ \tilde{\lambda}_n(d\tilde{t})$$

4.2. *Convergence de T_n et $\tilde{T}_n$*

Théorème 4.2.

On suppose que les observations satisfont les conditions H.1. *à* H.6. *et* H.8. *ainsi que les conditions suivantes.*

Il existe une fonction r positive ou nulle et appartenant à C^*_{1+k} *vérifiant* H.7. *et*

H.9. $\exists B > 0 \quad \forall\rho \in \{0,1\}^{1+k} \quad \forall t \in I_\rho$

$$r(t) \geq B \left[\prod_{i\in J(\rho)} t_i \prod_{i\notin J(\rho)} (1 - t_i)\right]^{\frac{1}{2} - \delta} \quad \text{avec } 0 < \delta < \frac{1}{2}.$$

Il existe une mesure générale λ_0 *définie sur* $[0,1]^{1+k}$ *telle que*

H.10. *pour tout* $f \in C^*_{1+k}$

$$\lim_{n\to\infty} \int f\cdot r\ d\lambda_n = \int f\cdot r\ d\lambda_0$$

$$\sup_{n\in\mathbb{N}} \int r\ d|\lambda_n| \leq M < +\infty$$

où $|\lambda_n|$ *est la mesure de variation totale. Alors* $n^{-1/2}(T_n - b_n)$ *converge en loi vers une loi gaussienne* $N(0,\sigma^2)$ *avec*

$$\sigma^2 = \int_{[0,1]^{1+k}} \int_{[0,1]^{1+k}} E(L_o^*(t)L_o^*(t'))d\lambda_o(t)d\lambda_o(t') < +\infty$$

où L_o^* *est le processus limite du processus* L_n^*.

Remarque 4.1.

Si H admet une densité bornée, on a

$$\frac{1}{2} - \frac{k+2}{4k+4} < \delta < \frac{1}{2} \quad \text{avec H.3.a.}$$

et

$$\frac{1}{2} - \frac{2k+3}{8k+6} < \delta < \frac{1}{2} \quad \text{avec H.3.b.}$$

Corollaire 4.2.

On suppose que les observations satisfont les conditions $H^*.1.$ *à* $H^*.3.$ *ainsi que les conditions suivantes.*

Il existe une fonction $\tilde{r}$ *positive ou nulle et appartenant à* C_k^* *vérifiant* $H^*.4.$ *et*

$H^*.5.$ $\exists \tilde{B} > 0 \qquad \forall \tilde{\rho} \in \{0,1\}^k \qquad \forall \tilde{t} \in \tilde{I}_{\tilde{\rho}}$

$$\tilde{r}(\tilde{t}) \geq B \left[\prod_{i \in \mathcal{J}(\tilde{\rho})} t_i \prod_{i \notin \mathcal{J}(\tilde{\rho})} (1 - t_i) \right]^{\frac{1}{2} - \delta} \quad \text{avec} \quad 0 < \delta < \frac{1}{2}.$$

Il existe une mesure générale $\tilde{\lambda}_o$ *définie sur* $[0,1]^k$ *telle que*

$H^*.6.$ *pour tout* $f \in C_k^*$,

$$\lim_{n \to \infty} \int \tilde{f} \cdot \tilde{r} \, d\tilde{\lambda}_n = \int \tilde{f} \cdot \tilde{r} \, d\tilde{\lambda}$$

$$\sup_{n \in \mathbb{R}} \int \tilde{r} \, d|\tilde{\lambda}_n| \leq M < +\infty.$$

Alors $n^{-1/2}(\tilde{T}_n - \tilde{b}_n)$ *converge en loi vers une loi gaussienne* $N(0,\tilde{\sigma}^2)$ *avec*

$$\tilde{\sigma}^2 = \int_{[0,1]^k} \int_{[0,1]^k} E(\tilde{L}_o(t)\tilde{L}_o(t'))d\tilde{\lambda}_o(t) \, d\tilde{\lambda}_o(t') < +\infty$$

où $\tilde{L}_o(\tilde{t})$ *est le processus limite du processus* $L_n^*(1,\tilde{t})$.

Remarque 4.2.

Si H admet une densité bornée, on a

$$\frac{1}{2} - \frac{k+1}{4k} < \delta < \frac{1}{2} \quad \text{avec } H^*.3.a.$$

et

$$\frac{1}{2} - \frac{2k+1}{8k-2} < \delta < \frac{1}{2} \quad \text{avec } H^*.3.b.$$

Démonstration

Nous ne regarderons que les points de démonstration où les nouvelles hypothèses par rapport à M. Harel (1985) entraînent une modification.

L'hypothèse H.3.b. ou H^*.3.b. est nouvelle et intervient dans le Lemme 5 de M. Harel (1980).

Dans cet article, au bas de la page 67, on prend comme nouvelle valeur de β

$$\beta = \frac{6k(1+\alpha) + 3}{2k(1+\alpha) + 3}$$

on pose à nouveau

$$\gamma = \frac{1}{k(1+\alpha) + 1}$$

alors si

$$\delta = \frac{1}{2k(1+\alpha) + 2}$$

on aura

$$(1+\gamma)(1+\beta) = 4$$

et

$$1 - \delta + \gamma > 1.$$

On aura alors dans les conditions du Lemme 5

$\exists K\ (>0)$ tel que pour tout n et tout bloc de B(n) contenu dans $(\bigcup_{i=1}^{k} C_{\theta',i}) \cup \underline{C}_{\theta_0,0}$ vérifiant $m_H(B) \geq (\frac{1}{n})^{1+\alpha}$ on ait

$$E(\Delta_B(W_n \cdot \frac{1}{r}))^4 \leq K\, \nu^{1+\gamma-\delta}\ (B)$$

(voir M. Harel (1980) pour la définition des notations) et la démonstration est identique à celle du Lemme 5 en utilisant le Théorème 10 de P. Doukhan et F. Portal (1983). L'hypothèse H.3.b. intervient aussi dans la Proposition 2 de Balacheff et Dupont, mais la proposition reste vraie comme conséquence immédiate du Corollaire 1 de Withers (1975).

L'hypothèse H.7. est plus générale que H.7. de M. Harel (1985). Les hypothèses H.7.a., H.7.b. et H.7.d. n'interviennent que dans la convergence du processus $W_n \cdot \frac{1}{r}$ donc dans les démonstration de M. Harel (1980), mais H.7.a. et H.7.b. sont moins générales que les hypothèses (2.3) à (2.5) de cet article et H.7.d. c'est l'hypothèse H.5. (1) de ce même article.

L'hypothèse H.7.c. intervient dans la démonstration de la Proposition 4.4. de M. Harel(1985) et on vérifie facilement que la proposition reste vraie avec cette nouvelle hypothèse.

Exemple 4.2. Si k = 2 et si on a :

$\forall n \in \mathbb{N} \quad \forall j \in \{1, \ldots, n\}$ F_n^j est la fonction de répartition d'une loi gaussienne de marginales centrées réduites et de coefficient de corrélation ρ avec $0 < \rho < 1$, dans ce cas l'hypothèse H.7. est vérifiée alors que ceci n'est pas vrai pour l'hypothèse H.7. de M. Harel (1985).

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Weak Convergence of Serial Rank Statistics under Dependence with Applications in Time Series and Markov Processes

Abbreviated Title: Serial Rank Statistics under Dependence

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The asymptotic normality of linear serial rank statistics introduced by Hallin, Ingenbleek and Puri (1985) for the problem of testing white noise against ARMA alternatives is established for φ–mixing as well as strong mixing sequences of random variables using Rüschendorf's (1976) approach. Applications in Markov processes and ARMA processes in time series are provided.

AMS 1980 subject classification. 60F05, 60J05, 62M10.

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1. Introduction. Let $\{X_{n,i}, 1\le i\le n, n\ge 1\}$ be real–valued random variables with continuous distribution functions $F_n(x) = P(X_{n,i}\le x)$, $1\le i\le n$, $n\ge 1$. Consider the statistics

$$\mathscr{S}_n = (n-k)^{-1} \sum_{i=k+1}^{n} c_{n,i}\, a_n(R_{n,i-k},\dots,R_{n,i}) \tag{1.1}$$

where the $c_{n,i}$ are known constants, $a_n(.,\dots,.)$ are the scores, $R_{n,i}$ denotes the rank of $X_{n,i}$ among $(X_{n,1},\dots,X_{n,n})$, and $k\ge 1$ is a fixed integer ($< n$). Our aim is to study the asymptotic behavior of $\mathscr{S}_n$ when the sequence $\{X_{n,i}\}$ is φ–mixing with rates

$$\varphi(m) = O(m^{-1-\epsilon}) \quad \text{for some } \epsilon > 0 \ \ (m\ge 1) \tag{1.2}$$

or

$$\sum_{m=1}^{\infty} m^{-1}\, \varphi^{1/2(3+k)}(m) < \infty \tag{1.3}$$

or strong–mixing with rates

$$\sum_{m=1}^{\infty} m^{2(k+2)}\, \alpha^{\epsilon}(m) < \infty \quad \text{for some } \epsilon \in (0,1/2(3+k)) \tag{1.4}$$

Recall that the array $\{X_{n,i}, 1\le i\le n, n\ge 1\}$ is φ–mixing if $\sup_{m\le n} \sup_{1\le j\le n-m} \{|P(A|B)-P(A)| :$ $B \in \sigma(X_{n,i}, 1\le i\le j),\ A \in \sigma(X_{n,i}, i\ge j+m)\} = \varphi(m)\downarrow 0$ as $m\uparrow\infty$ for positive integers j and m , and it is strong-mixing if

$\sup_{m\le n} \sup_{1\le j\le n-m} \{|P(A\cap B)-P(A)P(B)|, A\in\sigma(X_{n,i},1\le i\le j), B\in\sigma(X_{n,i},i\ge j+m)\} = \alpha(m)\downarrow 0$ as $m\uparrow\infty$, for positive integers j and m . Here $\sigma(X_{n,i},i\le j)$ and $\sigma(X_{n,i},i\ge j+m)$ are the σ–fields generated by $(X_{n,1},\dots,X_{n,j})$ and $(X_{n,j+m},X_{n,j+m+1},\dots)$ respectively. The asymptotic behavior of the statistic $\mathscr{S}_n$ under strong–mixing conditions leads to interesting applications in ARMA processes in time series as well as in Markov processes (section 6). In passing we may mention that Hallin, Ingenbleek and Puri (1985) established the asymptotic normality of linear serial rank statistics $\mathscr{S}_n$ defined in (1.1) for an ARMA process contiguous to white noise. We show (in section 2) that contiguity is not necessary for the derivation of the asymptotic distribution theory derived in Hallin et al. (1985) and our results also lead to applications in some Markov processes which are either geometrically ergodic or Doeblin recurrent, and to some ARMA processes. For a related problem dealing with the applications of U–statistics (see Harel and Puri (1989a), (1989c)) to some Markov processes and ARMA models, the reader is referred to Harel and Puri (1989b).

2. Asymptotic normality. We start with a few preliminaries.

Denote by $\hat{F}_n(x)$, the right continuous empirical distribution function of $X_{n,i}$, $i=1,\dots,n$; i.e. let $\hat{F}_n(x) = n^{-1} \sum_{i=1}^{n} I_{\{X_{n,i} \le x\}}$ where $I_{\{.\}}$ denotes the indicator function. Denote by G_n the distribution function of the $k+1$ of the successive random variables $X_{n,1},\dots,X_{n,n}$. Let H_n (for each $n \ge k+1$) be a sequence of continuous distribution functions on $(0,1)^{k+1}$, defined by

(2.1) $$H_n(\underset{\sim}{t}) = G_n(F_n^{-1}(t_1),\dots,F_n^{-1}(t_{k+1})) \text{ for all } \underset{\sim}{t} = (t_1,\dots,t_{k+1}) \in (0,1)^{k+1}$$

where $F_n^{-1}(u) = \inf\{t : F_n(t) \ge u\}$, $0<u<1$. (Since H_n is continuous, it is actually well defined on $[0,1]^{k+1}$). Though G_n, H_n and $\underset{\sim}{t}$ depend on k, we have suppressed this fact for notational convenience.

Denote by C_{k+2}, the space of all continuous maps $f : [0,1]^{k+2} \to \mathbb{R}$, and by $C_{k+1}(j)$, $(1 \le j \le k+1)$, the space of all continuous and bounded maps $f : A(j) \to \mathbb{R}$ where $A(j) = [0,1]^{j-1} \times (0,1) \times [0,1]^{k+1-j}$.

Definition. We say that the sequence $\{H_n\}$ satisfies the differentiability condition if (i) $\frac{\partial}{\partial t_j} H_n$ exists on $A(j)$ and belongs to $C_{k+1}(j)$, $1 \le j \le k+1$, and (ii) $\frac{\partial H_n}{\partial t_j} \to \ell_j$ in the uniform topology on any compact subset of $A(j)$ as $n \to \infty$, and ℓ_j belongs to $C_{k+1}(j)$.

We define the graduate empirical process (also called the copula process, see e.g. Gaenssler and Stute (1987, Chapter V)) W_n as

(2.2) $$W_n(t) = (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{[nt_0]} \left\{ \prod_{j=1}^{k+1} I_{\{F_n(X_{n,i+j-k-1}) \le t_j\}} - H_n(\underset{\sim}{t}) \right\}$$

for all $t = (t_0, \underset{\sim}{t}) = (t_0, t_1, \dots, t_{k+1}) \in (0,1)^{k+2}$, where $[nt_0]$ denotes the integral part of the real number nt_0.

We also consider the rank process L_n (called the graduate rank process) defined as

(2.3) $$L_n(t) = (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{[nt_0]} \left\{ \prod_{j=1}^{k+1} I_{\{\hat{F}_n(X_{n,i+j-k-1}) \le t_j\}} - H_n(\underset{\sim}{t}) \right\} .$$

For any $n \geq 1$, we define a signed measure λ_n concentrated on $\{\frac{1}{n},\ldots,\frac{n-1}{n},1\}^{k+2}$ by setting

$$\lambda_n\Big(\prod_{j=0}^{k+1} [\tfrac{i_j}{n},1]\Big) = c_{n,i_0}\, a_n(i_1,\ldots,i_{k+1})$$

for all $(i_0,\ldots,i_{k+1}) \in \{1,\ldots,n\}^{k+2}$. (By convention $c_{n,i_0} = 0$ if $i_0 < k+1$).

We also define a centering coefficient b_n by

(2.4) $$b_n = \int_{[0,1]^{k+2}} \hat{H}_n(t)\lambda_n(dt)$$

where $\hat{H}_n$ is the function: $[0,1]^{k+2} \to \mathbb{R}^+$ such that $\hat{H}_n(t) = ([nt_0]-k)H_n(\underset{\sim}{t})$.

We now state the following theorem the proof of which is given in section 5.

THEOREM 2.1. *Assume that there exists a Radon measure* λ_0 *on* $[0,1]^{k+2}$ *such that*

(2.5) $$\lim_{n\to\infty} \int f d\lambda_n = \int f d\lambda_0 \quad \text{for all } f \in C_{k+2}$$

and

(2.6) $$\sup_{n\in\mathbb{N}} \int f d|\lambda_n| < \infty \; ; \; \mathbb{N} = \{0,1,2,\ldots\}$$

where $|\lambda_n|$ *denotes the measure of total variation.*

Assume that the sequence $\{X_{n,i}\}$ *is (a)* φ*–mixing with rates* (1.2) *or (b) strong-mixing with rates* (1.4) *. Furthermore, assume that (c) the covariance functions* $\{K_n, n\geq 1\}$ *of the empirical processes* $\{W_n, n\geq 1\}$ *defined in* (2.2) *converge to a function* $K(\cdot,\cdot)$ *in pointwise topology as* $n\to\infty$ *and (d)* $\{H_n\}$ *satisfies the differentiability conditions, then* L_n *converges weakly in uniform topology to a Gaussian process* L_∞ *with trajectories a.s. in* C_{k+2} *, and* $(n-k)^{\frac{1}{2}}(\mathscr{S}_n - b_n)$ *converges in law to the normal distribution with mean* 0 *and variance* σ^2 *where*

(2.7) $$\sigma^2 = \int_{[0,1]^{k+2}} \cdots \int_{[0,1]^{k+2}} E[L_\infty(t),L_\infty(t')]d\lambda_0(t)d\lambda_0(t') \qquad (<\infty)\ .$$

REMARK 2.1. The above theorem is proved under the assumption that the sequence $\{X_{n,i}\}$ is nonstationary and, either φ–mixing with rates (1.2) or strong–mixing with rates (1.4). The theorem does not hold with the φ–mixing rates (1.3) unless one assumes stationarity (which implies that the distribution functions F_n, G_n and H_n are equal to unique distribution functions F,G and H respectively) and the special case when $c_{n,i}=1$ for all i .

Let $\tilde{\mathscr{S}}_n$ denote the statistics $\mathscr{S}_n$ when $c_{n,i} = 1$ for all i , i.e. let

(2.8) $$\tilde{\mathscr{S}}_n = \sum_{i=k+1}^{n} a_n(R_{n,i-k},\dots,R_{n,i})$$

and let $\tilde{\mathfrak{b}}_n$ denote the corresponding centering constant, i.e.

(2.9) $$\tilde{\mathfrak{b}}_n = \int_{[0,1]^{k+1}} H_n(\underset{\sim}{t})\lambda_n(d\underset{\sim}{t})$$

where λ_n is a measure concentrated on $\{\frac{1}{n},\dots,\frac{n-1}{n},1\}^{k+1}$ and

$$\lambda_n\left[\prod_{j=1}^{k+1} [\tfrac{i_j}{n}, 1]\right] = a_n(i_1,\dots,i_{k+1})\ .$$

Then, we have the following theorem.

THEOREM 2.2. *Assume there exists a Radon measure* λ_0 *on* $[0,1]^{k+1}$ *such that*

(2.10) $$\lim_{n\to\infty} \int \mathfrak{f}\, d\lambda_n = \int \mathfrak{f}\, d\lambda_0$$

and

(2.11) $$\sup_{n\in\mathbb{N}} \int \mathfrak{f} d|\lambda_n| < \infty$$

where $|\tilde{\lambda}_n|$ *denotes the measure of total variation.*

Assume that the sequence $\{X_{n,i}\}$ *is* (*a'*) φ*–mixing with rates* (1.3) *and* (*b'*) H *satisfies the differentiability condition, then* $L_n(1,\underset{\sim}{t})$ *converges weakly in uniform topology to a Gaussian process* $\mathfrak{L}_\infty$ *with trajectories a.s. in* C_{k+1} , *and* $(n-k)^{\frac{1}{2}}(\tilde{\mathscr{S}}_n - \tilde{\mathfrak{b}}_n)$ *converges in law to the normal distribution with mean* 0 *and variance* $\tilde{\sigma}^2$ *where*

(2.12) $$\tilde{\sigma}^2 = \int_{[0,1]^{k+1}} \cdots \int_{[0,1]^{k+1}} E[\tilde{L}_0(\underset{\sim}{t}),\tilde{L}_0(\underset{\sim}{t}')]d\tilde{\lambda}_0(\underset{\sim}{t})d\tilde{\lambda}_0(\underset{\sim}{t}') \quad (<\infty) ,$$

The proof follows from Theorem 2.1 by putting $t_0=1$ for the processes W_n and L_n, and showing that the finite projections of W_n converge to a normal law (the proof of which is given in Proposition 3.4).

The following corollary gives sufficient conditions under which the conditions (2.5) and (2.6) are satisfied.

COROLLARY 2.2. *Let* J *be a function on* $[0,1]^{k+2}$ *such that* $J(\frac{i_0}{n},\ldots,\frac{i_{k+1}}{n}) = c_{n,i_0} a_n(i_1,\ldots,i_{k+1})$ *for all* $(i_0,\ldots,i_{k+1}) \in \{1,\ldots,n\}^{k+2}$, $J = J_d+J_c$ *where* J_d *is a step function taking only a finite number of jumps, and where for any* $I\subset\{0,\ldots,k+1\}$, J_c *has a continuous derivative* $\frac{\partial^I J_c}{(\partial t_j)_{j\in I}}$, *then the conditions* (2.5) *and* (2.6) *are satisfied.*

PROOF. It suffices to prove the above corollary in the case when J_d has only one jump, say at $a=(a_0,\ldots,a_{k+1}) \in [0,1]^{k+2}$. Let λ_n' and λ_n'' be measures on $[0,1]^{k+2}$ defined by

$$\lambda_n'(\prod_{j=0}^{k+1} [\frac{i_j}{n},1]) = J_c(\frac{i_0}{n},\ldots,\frac{i_{k+1}}{n})$$

and

$$\lambda_n''(\prod_{j=0}^{k+1} [\frac{i_j}{n},1]) = J_d(\frac{i_0}{n},\ldots,\frac{i_{k+1}}{n})$$

for all $(i_0,\ldots,i_{k+1}) \in \{\frac{1}{n},\ldots,\frac{n-1}{n},1\}^{k+2}$.

It is easy to check that

$$\lim_{n\to\infty} \int_{[0,1]^{k+2}} f d\lambda_n' = \sum_{I\subset\{0,\ldots,k+1\}} \int_{[0,1]^{k+2}} f \frac{\partial^I J_c}{(\partial t_j)_{j\in I}}((t_j)_{j\in I},(1)^{k+2-i})$$

for all $f \in C_{k+2}$ whose $i = \text{card } I$.

Thus, we obtain a measure λ_0' satisfying

$$\lim_{n\to\infty} \int_{[0,1]^{k+2}} f d\lambda_n'(t) = \int_{[0,1]^{k+2}} f d\lambda_0'(t) .$$

Analogously, we obtain

$$\lim_{n\to\infty} \int_{[0,1]^{k+2}} f d\lambda_n''(t) = f(a) \sum_{I\subset\{0,\ldots,k+1\}} (-1)^i J_d((a_i-)_{i\in I},(a_i+)_{i\notin I})$$

for all $f \in C_{k+2}$ where $i = \text{card } I$.

3. Weak convergence of the graduate empirical process and the graduate rank process.

We start with preliminaries.

3.1a The spaces D_{k+2} and C_{k+2}.

Let $f : [0,1]^{k+2} \to \mathbb{R}$. For $\rho \in \{0,1\}^{k+2}$, define

$$f_\rho(t) = \lim_{\substack{s_i \uparrow t_i,\ \rho(i)=1 \\ s_i \downarrow t_i,\ \rho(i)=0}} f(s),\ (s,t) \in ([0,1]^{k+2})^2 \ (i=0,1,\ldots,k+1),$$ if it exists, in which case,

call $f_\rho(t)$ the ρ–limit of f at t. Denote by D_{k+2}, the space of all maps $f : [0,1]^{k+2} \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^{k+2}$, f_ρ exists and $f_\rho = f$ for $\rho = (0,\ldots,0)$.

We say that we have special Skorohod topology on D_{k+2} if we have the uniform topology for the first coordinate and the J_1–Skorohod topology for the other coordinates. (For definition of Skorohod topology, cf. Skorohod (1956) and Billingsley (1968)).

We define a modulus of continuity for any bounded function $f : [0,1]^{k+2} \to \mathbb{R}^+$ to be denoted by $\omega(f,\delta),(\delta>0)$ by setting

(3.1) $$\omega(f,\delta) = \sup_{(t,t')\in([0,1]^{k+2})^2} |f(t)-f(t')|\ ,\quad \|t-t'\| < \delta\ ,$$

where $\|t\| = \sup\{|t_j|, 0\le j\le k+1\}$.

Note that f belongs to C_{k+2} if and only if $\lim_{\delta\to 0} \omega(f,\delta) = 0$.

The following proposition which is a variant of Theorem 1.2 of Dudley (1978), will be used to prove Proposition 3.4.

PROPOSITION 3.1. *Let* Y_n *be a process with values in* D_{k+2} *and measurable with respect to* $\mathcal{U}_{k+2}$, *the σ–field generated by the uniform topology (on* D_{k+2}*). Let* P_n *denote the law of* Y_n. *Then, there exists a probability measure* P *with* $P(C_{k+2}) = 1$ *for which* P_n *converges weakly with respect to the uniform topology if and only if*

(i) *for all finite subsets* U *of* $[0,1]^{k+2}$, $\phi_U(P_n)$ *converges weakly to* $\phi_U(P)$. (ϕ_U *is the projection of* D_{k+2} *on* $\mathbb{R}^U$),

(ii) $\forall \epsilon > 0$, $\lim_{\delta\to 0} \limsup_{n\to\infty} P_n[\{f;\omega(f,\delta)\geq\epsilon\}] = 0$.

The proof is given in the Appendix.

3.1b Grid accompanying a sequence of probability measures:

We call a grid T of $[0,1]^{k+2}$ a subset of $[0,1]^{k+2}$ such that $T = \prod_{j=0}^{k+1} T^{(j)}$ where $T^{(j)}$ is a finite subset of [0,1] which includes 0 and 1.

We call a pace τ of a grid $T = \prod_{j=0}^{k+1} T^{(j)}$ the number $\tau = \max_{0\leq j\leq k+1} \tau_j$ where $\tau_j = \max\{|t_j'-t_j|, t_j' \text{ and } t_j \text{ are successive elements in } T^{(j)}\}$.

We denote the lower boundary of T by $\underset{\leftarrow}{T}$ where

$$\underset{\leftarrow}{T} = \bigcup_{j=0}^{k+1} \left[\prod_{\ell=0}^{j-1} T^{(\ell)} \times \{0\} \times \prod_{\ell=j+1}^{k+1} T^{(\ell)}\right]$$

We call block B of T any part of T in the form

$$B = \prod_{j=0}^{k+1} \{(t_j,t_j'] \cap T^{(j)} \text{ where } t_j \text{ and } t_j' \text{ belong to } T^{(j)} \text{ and } t_j<t_j'\}.$$

We call evaluation $e_T^{(B)}$ of B into T, the operator $e_T^{(B)} : D_{k+2}\to\mathbb{R}^+$ such that

$$(3.2)\quad e_T^{(B)}(f) = \sum_{(\epsilon_0,\dots,\epsilon_{k+1})\in\{0,1\}^{k+2}} (-1)^{\sum_{i=0}^{k+1}\epsilon_i} f[(1-\epsilon_0)t_0+\epsilon_0 t_0',\dots,(1-\epsilon_{k+1})t_{k+1}+\epsilon_{k+1}t_{k+1}'].$$

Let ν be a finite measure on $[0,1]^{k+2}$ and let T be a grid of $[0,1]^{k+2}$. We call reduction $\tilde{\nu}$ of ν on T the measure on T defined by

$$\tilde{\nu}(\{t\}) = \begin{cases} 0 & \text{if } t \in \underset{\leftarrow}{T} \\ \nu(\prod_{j=0}^{k+1} (t_j',t_j]) & \text{if } t \notin \underset{\leftarrow}{T} \end{cases}$$

where

$$t_j' = \max\{x; x\in T^{(j)}; x<t_j, t_j\in T^{(j)}\}.$$

For any $\delta>0$, we set

$$\omega_T(f,\delta) = \sup\{|f(t)-f(t')|;(t,t') \in T^2, \|t-t'\| \le \delta\} .$$

We say that a sequence $\{T_n\}_{n\in\mathbb{N}^*}$ of grids is asymptotically dense in $[0,1]^{k+2}$ if the pace τ_n of T_n satisfies $\lim_{n\to\infty} \tau_n = 0$ $(\mathbb{N}^* = \mathbb{N}-\{0\}, \mathbb{N}=0,1,2,\ldots)$.

Let P_n , $n \in \mathbb{N}^*$ be a sequence of probability measures on $(D_{k+2}, \mathscr{D}_{k+2})$ where $\mathscr{D}_{k+2}$ is the σ–field generated by the Skorohod topology (on D_{k+2}). We say that the sequence $\{T_n\}$ of grids accompanies the sequence $\{P_n\}$ if and only if $\forall\ \epsilon>0$, $\exists\ \epsilon'>0$ and $\forall\ \delta \in [0,1/2)$, $\exists\ N_0 \in N^*$, we have

$$P_n[\{f \in D_{k+2}, \omega(f,\delta) \ge \epsilon \text{ and } \omega_{T_n}(f,2\delta) < \epsilon'\}] = 0 \quad \forall\ n \ge N_0 .$$

The following propositions (3.2 and 3.3) are variants of a result of Neuhaus (1971) (see e.g. Theorems 2 and Theorem 4 in Balacheff and Dupont (1980)) and will be used in section 4.

PROPOSITION 3.2. *Let* $P_n, n\in\mathbb{N}$ *be probability measures on* $(D_{k+2}, \mathscr{D}_{k+2})$ *such that the following conditions are satisfied:*

(3.3) $\Phi_U(P_n)$ *converges weakly to some probability measure* P_U *on* $\mathbb{R}^U$ *for any finite subset* U *of* $[0,1]^{k+2}$

and

(3.4) $$\forall\ \epsilon > 0 ,\ \lim_{\delta\to 0} \limsup_{n\to\infty} P_n[f\in D_{k+2}; \omega(f,\delta,) \ge \epsilon] = 0 ,$$

then, P_n *converges weakly with respect to the Skorohod topology to some probability measure* P *and* $P(C_{k+2}) = 1$.

PROPOSITION 3.3. *Let* ν *be a positive finite measure on* $[0,1]^{k+2}$ *with continuous marginals. Let* P_n *be a sequence of probability measures on* $(D_{k+2}, \mathscr{D}_{k+2})$ *such that* $\forall\ n \in \mathbb{N}$, $P_n[f\in D_{k+2}; f|[0,1]^{k+2} = 0] = 1$. *Let* T_n *be a sequence of grids asymptotically dense in* $[0,1]^{k+2}$ *and accompanying* P_n . *Furthermore suppose that for any block* B_n *of* T_n ,

(3.5) $$P_n[f\epsilon D_{2+k}; |e_{T_n}^{(B_n)}(f)| \ge \lambda] \le \lambda^{-\gamma}(\tilde{\nu}_n(B_n))^{\beta}$$

where $\tilde{\nu}_n$ *is the reduction of* ν *on* T_n, *and* $\beta>1$ *and* $\gamma>0$. *Then, we have* $\forall\ \epsilon>0$, $\exists$ $\delta\in(0,1)$ *and* $N_0\in\mathbb{N}$ *such that*

(3.6) $$P_n[f\in D_{k+2};\omega(f,\delta)\geq\epsilon] \leq \epsilon \quad \forall\ n \geq N_0 .$$

3.2a <u>Convergence of the graduate empirical processes.</u>

PROPOSITION 3.4. *Under the conditions (a) and (c) or (b) and (c) of Theorem* 2.1, W_n *converges weakly in the special Skorohod topology to a Gaussian process* W_∞ *with trajectories a.s. in* C_{k+2}. *Under the conditions (a') of Theorem* 2.2, $\mathfrak{W}_n = W_n(1,\underset{\sim}{t})$ *converges in the Skorohod topology to a Gaussian process* $\mathfrak{W}_\infty$ *with trajectories a.s. in* C_{k+1}.

3.2b <u>Convergence of the graduate rank process.</u>

PROPOSITION 3.5. *Under the conditions (a), (c) and (d) or (b), (c) and (d) of Theorem* 2.1, L_n *converges weakly in uniform topology to a Gaussian process* L_∞ *with trajectories a.s. in* C_{k+2}. *Under the conditions (a') and (b') of Theorem* 2.2, $\mathfrak{L}_n = L_n(1,\underset{\sim}{t})$ *converges weakly in uniform topology to a Gaussian process* $\mathfrak{L}_\infty$ *with trajectories a.s. in* C_{k+1}.

4. <u>Proofs of Propositions</u> 3.4 <u>and</u> 3.5.

Our proofs of the Propositions 3.4 and 3.5 are based on the ideas of Balacheff and Dupont (1980) who considered the asymptotic normality of the truncated empirical processes under φ–mixing with rates $\Sigma_{m=1}^{\infty} m\varphi^{1/2}(m) < \infty$. Here in this paper, we consider the rates (1.2) and (1.3) which are slower than the one considered by them. In addition we also derive results under strong mixing (1.4) which have not been considered in the literature. To establish their result Balacheff and Dupont (1980) used a slight modification of an inequality due to Rüschendorf (1974) which is not applicable in our situation. Our proofs are based on the following two Lemmas.

LEMMA 4.1. *Let the sequence* $\{X_{n,i}\}$ *of real–valued random variables (centered at its expectation) be* φ*–mixing with rates* $\Sigma_{m=1}^{\infty} m^{-1}\varphi^{1/2q}(m) < \infty$, *where* q *is an integer. Denote by* N_n *the number of indices* i $(1\leq i\leq n)$ *for which* $X_{n,i}$ *is not identical to zero. Set*

$S_n = \sum_{i=1}^{n} X_{n,i}$ *and* $\|X_{n,i}\|_\ell = (\int |X_{n,i}|^{2\ell} dP_n)^{1/2\ell}$. *Then, for any* $q \geq 1$, *there exists a constant* $C_q(\varphi)$ *depending only on* q *and* φ *such that*

$$\text{(4.1)} \qquad E(S_n^{2q}) \leq C_q(\varphi) \sum_{\ell=1}^{q} N_n^{q/\ell} (\sup_{1 \leq j \leq n} \|X_{n,j}\|_\ell)^{2q} .$$

The proof is a slight modification of Theorem 2.1 of Neumann (1982) and is sketched briefly in the Appendix.

LEMMA 4.2. *Let the sequence* $\{X_{n,i}\}$ *of real–valued random variables (centered at its expectation) be strong mixing with rates* $\Sigma_{m=1}^{\infty} m^{2q-2} \alpha^\epsilon(m) < \infty$, $\epsilon \in (0, \frac{1}{2q})$ *and* $|X_{n,i}| \leq 1$, $1 \leq i \leq n$, $n \geq 1$ *where* q *is an integer. Let* N_n *be the number of indexes* i $(1 \leq i \leq n)$ *for which* $X_{n,i}$ *is not identical to zero. Set* $S_n = \sum_{i=1}^{n} X_{n,i}$ *and* $\|X_{n,i}\|_\epsilon = (\int |X_{n,i}|^{2/(1-\epsilon)} dP_n)^{1-\epsilon}$. *Then, for any* $q \geq 1$, *there exists a constant* $C_q(\alpha)$ *depending only on* q *and* α *such that*

$$\text{(4.2)} \qquad E(S_n^{2q}) \leq C_q(\alpha) \sum_{\ell=1}^{q} N_n^{\ell} (\sup_{1 \leq i \leq n} \|X_{n,i}\|_\epsilon)^{\ell} .$$

The proof is similar to Theorem II.10 of Doukhan and Portal (1987), and is therefore omitted.

LEMMA 4.3 (Neumann, 1982). *Let* $\{Y_i, i \geq 1\}$ *be a stationary sequence of real–valued random variables centered at its expectation and with finite second moment. Assume that the sequence is* φ*–mixing with rates* $\sum_{m=1}^{\infty} m^{-1} \varphi^{\frac{1}{2}}(m) < \infty$. *Then, there exists a positive constant* K *such that* $n^{-1} E(\sum_{i=1}^{n} Y_i)^2 \longrightarrow K^2$ *as* $n \to \infty$.

Since the reference Neumann (1982) is not readily available, we have (at the suggestion of one of the referees) given the proof in the Appendix.

Proof of Proposition 3.4. Consider a sequence $Z_{m,i}$, $1 \leq i \leq m$, $m \geq 1$ of $\mathbb{R}^{k+1}$–valued random variables defined by $Z_{m,i} = (X_{m+k,i}, \ldots, X_{m+k,i+k}) = (Z_{m,i}^{(1)}, \ldots, Z_{m,i}^{(k+1)})$, $1 \leq i \leq m$, $m \geq 1$. Then, the $(k+1)$–variate truncated empirical process $\tilde{W}_m$ associated with this sequence is given by

$$(4.3) \qquad \tilde{W}_m(t_0,\underset{\sim}{t}) = m^{-\frac{1}{2}} \sum_{i=1}^{[(m+k)t_0]-k} \left[\prod_{j=1}^{k+1} I_{\{F_{m+k}(Z^{(j)}_{m,i}) \le t_j\}} - H_{m+k}(\underset{\sim}{t}) \right]$$

and this is the same as the graduate process W_n defined in (2.2). Now the process W_n defines a probability measure Q_n on $(D_{k+2},\mathcal{D}_{k+2})$.

To prove this proposition we have to verify (3.3) and (3.4). Following Withers (1975, Corollary 1) it can be shown that $\phi_U(Q_n)$ converges weakly to a Gaussian measure Q_U if (i) $K_n \to$ some function K, (ii) $\Sigma_{m\geq 1}\ \alpha(m) < \infty$, and (iii) $m^{1-a}\ \alpha([m^b]) \to 0$ as $m \to \infty$ where $0<2b<a<1-b$. Now in our situation (i) holds by assumption (c), (ii) follows from (1.2) or 1.4), and (iii) follows from (1.2) or (1.4) by taking $a = 3/4 - \epsilon/8$, $b = 1/4$ and ϵ sufficiently small. (Since taking $\alpha(m) = m^{-1-\epsilon}$, $m^{1-a}\alpha([m^b]) \leq Am^{-\epsilon/8} \to 0$ as $m \to \infty$). Thus (3.3) holds whenever conditions (a) and (c) or (b) and (c) of Theorem 2.1 are satisfied.

Now suppose that the condition (a′) of Theorem 2.2 holds with $X_{n,i} \equiv X_i$. Then, for any $p \in \mathbb{N}^*$, any $\underset{\sim}{t}^{(\ell)} \in [0,1]^k$ and any $\lambda_\ell \in \mathbb{R}$, $(1 \leq \ell \leq p)$, let $g_i^{(\ell)}(X_i)$ and $g_i(X_i)$ be the random variables defined by

$$g_i^{(\ell)}(X_i) = \prod_{j=1}^{k+1} [I_{\{F(X_{i+j-k-1}) \leq t_j\}} - H(\underset{\sim}{t}^{(\ell)})] \text{ and } g_i(X_i) = \sum_{\ell=1}^{p} \lambda_\ell\, g_i^{(\ell)}(X_i)$$

where $X_i = (X_{i-k},X_{i-k+1},\ldots,X_i)$. Then, we have $\sum_{\ell=1}^{p} \lambda_\ell \tilde{W}_n(\underset{\sim}{t}^{(\ell)}) = (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{n} g_i(X_i)$, and so (3.3) also holds by Lemma 4.3 and the central limit theorem for the stationary and φ–mixing case (cf. Ibragimov and Linnik (1971, Theorem 18.5.1 and Lemma 4.3)). Now to prove (3.4), we shall use Proposition 3.3 and verify (3.5) (which will imply (3.4)).

Let $T_n = \{i/m; 0 \leq i \leq m\}^{k+2}$ be a sequence of grids with $n = m+k$. T_n is asymptotically dense in $[0,1]^{k+2}$ and we prove that T_n accompanies Q_n. Now for every $\underset{\sim}{t} \in (0,1)^{k+1}$, let $(\underline{\underset{\sim}{t}},\overline{\underset{\sim}{t}})$ be the points of $\pi(T_n)$ where π is the projection defined by $\pi(t) = \underset{\sim}{t}$ such that $\underline{\underset{\sim}{t}} \leq \underset{\sim}{t} \leq \overline{\underset{\sim}{t}}$, and $\|\overline{\underset{\sim}{t}} - \underline{\underset{\sim}{t}}\| \leq 1/m$. Let us write $\underline{t}_0 = [nt_0]/n$ for every

$t_0 \in [0,1]$. As the marginals of H_n are uniform, we obtain (after some computations) that

$$|W_n(t_0,\underset{\sim}{t}) - W_n(t'_0,\underset{\sim}{t}')| \le \frac{2k}{\sqrt{m}} + |W_n(\underset{-}{t}_0,\bar{t}) - W_n(t'_0,\underset{\simeq}{t}')|$$

$\forall (t_0,\underset{\sim}{t}) \in [0,1]^{k+2}$ and $\forall (t'_0,\underset{\sim}{t}') \in [0,1]^{k+2}$.

Consequently, $\forall\ \delta \in (0,1/2]$, we have $\omega(W_n,\delta) \le \frac{2k}{\sqrt{m}} + \omega_{T_n}(W_n,2\delta)$. It follows that T_n accompanies Q_n. It remains to show that Q_n satisfies (3.5).

Let $\sum_{m=1}^{\infty} m^{-1}\varphi^{1/2(3+k)}(m) < \infty$ (see (1.3)), and let B_n be a block of T_n defined in section 3.1.b. Using Lemma 4.1 with $q = k+3$, we obtain (see (3.2))

$$E[e_{T_n}^{(B_n)}(W_n)]^{2(k+3)} \le$$

$$\le C_{k+3}(\varphi) \sum_{\ell=1}^{k+3} m^{-(k+3)}[(m+k)(t_0-t'_0)]^{(k+3)/\ell} \times [\prod_{j=1}^{k+1}(t_j-t'_j)]^{(k+3)/(k+1)\ell}$$

$$\le C_{k+3}(\varphi) \sum_{\ell=1}^{k+3} m^{-(k+3)} \times (m+k)^{(k+3)/\ell} \times [\prod_{j=0}^{k+1}(t_j-t'_j)]^{(k+3)/(k+2)\ell}$$

$$\le C_{k+3}(\varphi)(k+3)[\prod_{j=0}^{k+1}(t_j-t'_j)]^{(k+3)/(k+2)}.$$

Now let $\nu = (C_{k+3}(\varphi)(k+3))^{\frac{1}{\beta}} U$ where U is a uniform measure on $[0,1]^{k+2}$ and $\beta = (k+3)/(k+2)$. Then, by the Markov inequality, we obtain (see (3.5))

$$Q_n[f \in D_{k+2}; |e_{T_n}^{(B_n)}(f)| \ge \lambda] \le \lambda^{-2(k+3)}(\tilde{\nu}_n(B_n))^{\beta}$$

which implies (3.6) for the φ–mixing rates (1.3) (and so also for (1.2)). For the strong mixing case with rates (1.4), we use Lemma 4.2 with $\epsilon < (2(k+3))^{-1}$ and obtain from (4.2)

$$E[e_{T_n}^{(B_n)}(W_n)]^{2(k+3)} \le C_{k+3}(\alpha) \sum_{\ell=1}^{k+3} m^{-(k+3-\ell)}(t_0-t'_0)^{\ell}(\prod_{j=1}^{k+1}(t_j-t'_j))^{\ell(1-\epsilon)/(k+1)}$$

which (with $\beta = \frac{(k+2)(1-\epsilon)+1}{k+2}$) implies (3.5) and hence (3.6). We derive the convergence with respect to the special Skorohod topology because W_n is measurable with respect to this topology and we use Proposition 3.1 to the first coordinate (of W_n).

Proof of Proposition 3.5. The main line of proof is as follows:

We consider a map $G_n : \mathscr{V}_n \to D_{k+2}$ where $\mathscr{V}_n$ is a subset of D_{k+2} and is such that $L_n = G_n \circ W_n$, $n \geq 1$. We show that $G_n : (\mathscr{V}_n, d) \to (D_{k+2}, \rho)$ is a continuous map where d is the special Skorohod metric and ρ is the uniform metric.

Let $\mathscr{V}$ be a subset of D_{k+2} such that for any $v \in \mathscr{V}$, v equals zero on the lower boundary of $[0,1]^{k+2}$ and also for $\underset{\sim}{t} = (1,\dots,1)$. It will be noted that $\mathscr{V}_n \subset \mathscr{V}$ for $\forall\, n \geq 1$.

Let $G : \mathscr{V} \to D_{k+2}$ be a map defined by

(4.4)
$$G(v)(t) = v(t) - t_0 \sum_{j=1}^{k+1} [v(1,\dots,t_j,\dots,1) \times \ell_j(t_1,\dots,t_{k+1})]$$

where ℓ_j is the limit of $\frac{\partial}{\partial t_j} H_n$ as $n \to \infty$. We will show that $\forall\, (v_n)_{n \in \mathbb{N}^*} \in (\Pi_{n \in \mathbb{N}^*} \mathscr{V}_n)$ and $\forall\, v \in \mathscr{V} \cap C_{k+2}$, $v_n \xrightarrow{d} v \Longrightarrow G_n(v_n) \xrightarrow{\rho} G(v)$ as $n \to \infty$. Now using Lemma 3 of Balacheff and Dupont (1980), we get the desired convergence.

Let $\mathscr{Y}_n = \{y \in [0,1]^n : (y^{(1)},\dots,y^{(n)})$ are distinct points of $(0,1)\}$. We define $Y_n : [0,1]^n \to D_{k+2}$ by setting $Y_n(y)(t) = (n-k)^{-1/2} \sum_{i=k+1}^{[n t_0]} \left[\prod_{j=1}^{k+1} I_{\{y^{(i+j-k-1)} \leq t_j\}} - H_n(\underset{\sim}{t}) \right]$ for all $y = (y^{(1)},\dots,y^{(n)}) \in \mathscr{Y}_n$ and $t = (t_0, \underset{\sim}{t}) \in [0,1]^{k+2}$.

We define the space $\mathscr{V}_n$ by $\mathscr{V}_n = Y_n(\mathscr{Y}_n)$. For any $j \in \{1,\dots,k+1\}$ we define an operator $\tau_j : \mathscr{V}_n \to D_1$ as follows:

Let $y_{(1)} < \dots < y_{(n)}$ be the order values of $(y^{(1)},\dots,y^{(n)})$. (By convention, $y_{(0)} = 0$, $y_{(n+1)} = 1$), and let $v_n = Y_n(y)$. Then

(4.5)
$$\tau_j(v_n)(t_j) = \begin{cases} y^{(\ell)} \text{ where } y^{(\ell)} = \max\{y^{(m)} ; m \in \{j,\dots,j+n-k-1\}\} & \text{if } t_j = 1 \\ y^{(q)} \text{ where } y^{(q)} = \max\{y^{(m)} \leq y_{(i)} ; m \in \{0,j,\dots,j+n-k-1\}\} & \text{if } t_j \in [\frac{i}{n}, \frac{i+1}{n}) \end{cases}$$

where $i = \{0,1,\dots,n-1\}$.

Now the map $G_n : \mathscr{V}_n \to D_{k+2}$ is given by

$$(4.6)\qquad G_n(v_n)(t) = v_n(t_0,\tau_1(v_n)(t_1),\dots,\tau_{k+1}(v_n)(t_{k+1}))$$
$$+ (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{[n t_0]} \left[H_n(\tau_1(v_n)(t_1),\dots,\tau_{k+1}(v_n)(t_{k+1}) - H_n(t_1,\dots,t_{k+1})\right]$$

We now give the formal proof.

The first thing we have to show is that G_n is continuous for every n.

Let $\{v_{n,\ell}\}$ $n\geq 1$, $\ell\geq 1$ be a sequence of functions in $\mathcal{V}_n$ and let $v_{n,\ell} \to v_n$ $(\in \mathcal{V}_n)$ with respect to special Skorohod topology. We show that $G_n(v_{n,\ell}) \to G_n(v_n)$ in uniform topology. From the definition of the special Skorohod topology, we have a sequence $\{\lambda_{j,\ell}\}_{1\leq j\leq k+1,\ell\geq 1} \in \Lambda^{k+1}$ such that $\forall\, \epsilon>0$, $\exists \ell_\epsilon \in \mathbb{N}$ such that $\max_{1\leq j\leq k+1} |\lambda_{j,\ell}(t_j)-t_j| \leq \epsilon$ and

$$(4.7)\qquad |v_{n,\ell}(t)-v_n(t_0,\lambda_{1,\ell}(t_1),\dots,\lambda_{k+1,\ell}(t_{k+1}))| < \epsilon \;\forall\, \ell\geq \ell_\epsilon$$
$$\text{and } \forall\, t \in [0,1]^{k+2}$$

where Λ denotes the space of maps $h : [0,1]\to[0,1]$ which are nondecreasing, continuous and bijective, and Λ^{k+1} denotes the space of maps $\lambda : [0,1]^{k+1}\to[0,1]^{k+1}$ where $\lambda(t_1,\dots,t_{k+1}) = (\lambda_1(t_1),\dots,\lambda_{k+1}(t_{k+1}))$, $\lambda_j\in\Lambda$, $1\leq j\leq k+1$. Then, we have

LEMMA 4.4. $\exists\, \ell_0>0$ *such that* $\forall\, \ell\geq\ell_0$, $\forall j \in \{1,\dots,k+1\}$ *and* $\forall\, t_j \in [0,1]$,

$$\lambda_{j,\ell}(\tau_j(v_{n,\ell})(t_j)) = \tau_j(v_n)(t_j)$$

PROOF. For fixed j, let $(y^{\ell,1},\dots,y^{\ell,n-k})$ be a nondecreasing sequence of discontinuity points of $\tau_j(v_{n,\ell})$, and let $(y^{0,1},\dots,y^{0,n-k})$ be a (nondecreasing) sequence of discontinuity points of $\tau_j(v_n)$. (By convention, $y^{\ell,0} = y^{0,0} = 0$, $y^{\ell,n-k+1} = y^{0,n-k+1} = 1$). For $i \in \{0,1,\dots,n-k+1\}$, let $t_j \in [y^{\ell,i},y^{\ell,i+1})$. Then

$$(4.8)\qquad (n-k)^{-\frac{1}{2}}[v_{n,\ell}(1,\dots,t_j,\dots,1)+H_n(1,\dots,t_j,\dots,1)] = i(n-k)^{-1}.$$

Let $h \in \{0,1,\dots,n-k+1\} \to \lambda_{j,\ell}(t_j) \in [y^{0,h},y^{0,h+1})$. Then, we have

$$(4.9)\qquad (n-k)^{-\frac{1}{2}}[v_n(1,\dots,\lambda_{j,\ell}(t_j),\dots,1)+H_n(1,\dots,\lambda_{j,\ell}(t_j),\dots,1)] = h(n-k)^{-1}$$

From (4.8) and (4.9), we deduce

(4.10) $\left|\frac{h}{n-k} - \frac{i}{n-k}\right| \le |t_j - \lambda_{j,\ell}(t_j)| + (n-k)^{-1/2}|v_{n,\ell}(1,\ldots,t_j,\ldots,1) - v_n(1,\ldots,\lambda_{j,\ell}(t_j),\ldots,1)|$
$< \frac{1}{n-k}$ $\forall\, \ell \ge$ some ℓ_j .

Thus $\left|\frac{h}{n-k} - \frac{i}{n-k}\right| < \frac{1}{n-k}$ and this implies that $h=i$. Now let $\ell_0 = \max_{1\le j\le k} \ell_j$. Then, $\forall \ell \ge \ell_0$ and $\forall\, t_j \in [y^{\ell,i}, y^{\ell,i+1})$, we have $\lambda_{j,\ell}(t_j) \in [y^{0,i}, y^{0,i+1})$. Since the functions $\lambda_{j,\ell}$ are continuous and strictly nondecreasing, the proof follows.

We now decompose G_n defined in (4.7) as $G_n = \gamma_n + \delta_n$ where $\gamma_n(v_n)(t) = v_n(t_0, \tau_1(v_n)(t_1), \ldots, \tau_{k+1}(v_n)(t_{k+1}))$ and $\delta_n = G_n - \gamma_n$.

LEMMA 4.5.

(a) $\gamma_n : (\mathscr{V}_n, d) \to (D_{k+2}, \rho)$ *is continuous.*

(b) $\delta_n : (\mathscr{V}_n, d) \to (D_{k+2}, \rho)$ *is continuous.*

PROOF. For $t \in [0,1]^{k+2}$, for $\forall\, \epsilon > 0$, $\exists\, \ell_\epsilon \ni \forall \ell \ge \ell_\epsilon$, we have (using Lemma 4.4)

$$|v_{n,\ell}(t_0, \tau_1(v_{n,\ell})(t_1), \ldots, \tau_{k+1}(v_{n,\ell})(t_{k+1})) - v_n(t_0, \tau_1(v_n)(t_1), \ldots, \tau_{k+1}(v_n)(t_{k+1}))| < \epsilon .$$

The proof follows. Part (b) follows analogously noting that H_n has uniform marginals.

We now prove the convergence of the sequence $\{G_n\}$.

Let $v_n \in \mathscr{V}_n$, $n \in \mathbb{N}^*$ and suppose that $v_n \xrightarrow{d} v \in C_{k+2}$ and $v=0$ on the lower boundary of $[0,1]^{k+2}$ and also when $\underset{\sim}{t} = (1,\ldots,1)$. We have to prove that $G_n(v_n) \xrightarrow{\rho} G(v)$. The proof is based on the following Lemmas.

LEMMA 4.6. $\forall\, j \in \{1,\ldots,k+1\}$

(a) $\tau_j(v_n) \to \mathrm{id}_{[0,1]}$ *in uniform topology.*

(b) $(n-k)^{1/2}(\tau_j(v_n) - \mathrm{id}_{[0,1]}) \to -\, v(1,\ldots,\mathrm{id}_{[0,1]},\ldots,1)$ *in uniform topology. Where* $\mathrm{id}_{[0,1]}$ *is an identity function on* $[0,1]$.

PROOF. Note that $\forall\, v_n$, $\exists\, y_n = (y_n^{(1)}, \ldots, y_n^{(n)})$ such that $v_n = Y_n(y_n)$. Now for fixed j , and for each $n \ge k+1$, define $v_n^{(j)}(t_j)$ as $v_n^{(j)}(t_j) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \{I_{\{y_n^{(i)} \le t_j\}} - t_j\}$ and note that $v_n^{(j)}(t_j)$ can also be written as

$$v_n^{(j)}(t_j) = ((n-k)/n)^{\frac{1}{2}} v_n(1,\ldots,t_j,\ldots,1) + n^{-\frac{1}{2}} \sum_{i=1}^{k-j} [I_{\{y_n^{(i)} \le t_j\}} - t_j] + n^{-\frac{1}{2}} \sum_{i=n-j}^{n} [I_{\{y_n^{(i)} \le t_j\}} - t_j].$$

Since $v_n \xrightarrow{d} v$ (which $\Rightarrow v_n \xrightarrow{\rho} v$), it follows that $v_n^{(j)}(t_j) \xrightarrow{\rho} v(1,\ldots,t_j,\ldots,1)$. Thus, we can write

$$|\tau_j(v_n)(t_j)-t_j| = |n^{-1}\sum_{i=1}^{n}\{I_{\{y_n^{(i)}\le\tau_j(v_n)t_j)\}}-t_j\} - n^{-\frac{1}{2}}v_n^{(j)}(\tau_j(v_n)(t_j))|$$
$$\le \frac{k}{n} + n^{-1/2}[\rho(v_n^{(j)},v(1,\ldots,\cdot,\ldots,1)) + \rho(v(1,\ldots,\cdot,\ldots,1),g)] \to 0 \text{ as } n\to\infty \text{ where } g \equiv 0 .$$

This proves part (a). The proof of part (b) is similar.

LEMMA 4.7. $\gamma_n(v_n) \to v$ *in uniform topology.*

PROOF. Follows by definition and Lemma 4.6(a).

LEMMA 4.8. $\delta_n(v_n) \to \delta(v) = G(v) - v$ *in uniform topology.*

PROOF. For $t \in [0,1]^{k+2}$, we have

$$\delta_n(v_n)(t) = \frac{[nt_0]-k}{(n-k)^{1/2}}\left\{H_n(\tau_1(v_n)(t_1),\ldots,\tau_{k+1}(v_n)(t_{k+1})) - H_n(t_1,\ldots,t_{k+1})\right\}$$

if there exists a $j \in \{1,\ldots,k+1\}$, $t_j < n^{-1}$, then

$$\delta_n(v_n)(t) = \frac{k-[nt_0]}{(n-k)^{1/2}} H_n(t_1,\ldots,t_{k+1}) \le \frac{k-[nt_0]}{(n-k)^{1/2}} \cdot \frac{1}{n}$$

and so $\delta_n(v_n) \to 0$ as $n\to\infty$.

If $\forall j \in \{1,\ldots,k+1\}$, $t_j \ge n^{-1}$, then by the Taylor expansion

$$\delta_n(v_n)(t) = \frac{[nt_0]-k}{(n-k)^{1/2}}\sum_{j=1}^{k+1}\{\tau_j(v_n)(t_j) - t_j\} \times \frac{\partial}{\partial t_j}H_n(\theta_{n,1}(t_1),\ldots,\theta_{n,k+1}(t_{k+1}))$$

where $\theta_{n,j}(t_j) \in [t_j \wedge \tau_j(v_n)(t_j), t_j \vee \tau_j(v_n)(t_j)]$.

Since $\{H_n\}$ satisfies the differentiability condition, we deduce from Lemma 4.6, the desired result.

Now since $G_n(v_n) = \gamma_n(v_n) + \delta_n(v_n)$, we obtain (using Lemmas 4.7 and 4.8) that $G_n(v_n) \to G(v) = v + \delta(v)$. The proof of Proposition 3.5 follows.

5. Proof of Theorem 2.1.

First, we show $\mathscr{S}_n$ can be written as

$$\mathscr{S}_n = (n-k)^{-\frac{1}{2}}\left[\int_{[0,1]^{k+2}} L_n(t)\,\lambda_n(dt)\right] + b_n \tag{5.1}$$

where λ_n is a signed measure on $[0,1]^{k+2}$ and b_n is the centering constant defined in section 2.

$$(n-k)^{\frac{1}{2}}(\mathscr{S}_n-b_n) = (n-k)^{-\frac{1}{2}}\left[\sum_{i=k+1}^{n} c_{n,i}a_n(R_{n,i-k},\ldots,R_{n,i}) - b_n\right]$$

$$= (n-k)^{-\frac{1}{2}}\left[\left[\sum_{i=k+1}^{n} c_{n,i}\left[\sum_A a_n(i_1,\ldots,i_{k+1}) \prod_{j=1}^{k+1} I_{[R_{n,i+j-k-1}=i_j]}\right]\right] - \right.$$
$$\left. - \sum_B \lambda_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n})\hat{H}_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n})\right]$$

$$= (n-k)^{-\frac{1}{2}}\left[\sum_B \lambda_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n})\left[\sum_{i=k+1}^{i_0}\left(\prod_{j=1}^{k+1} I_{[R_{n,i+j-k-1}\le i_j]} - \hat{H}_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n})\right)\right]\right]$$

$$= \sum_B \lambda_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n})L_n(\tfrac{i_0}{n},\ldots,\tfrac{i_{k+1}}{n}) = \int_{[0,1]^{k+2}} L_n(t)\lambda_n(dt)$$

where $\sum_A$ is the sum over all $(i_1,\ldots,i_{k+1})$ in $\{1,\ldots,n\}^{k+1}$ and $\sum_B$ is the sum over all $(i_0,\ldots,i_{k+1})$ in $\{1,\ldots,n\}^{k+2}$, where λ_n is defined in Section 2, and L_n is given by (2.3). We now prove that

(5.2) $$\int_{[0,1]^{k+2}} L_n(t)\lambda_n(dt) \to \int_{[0,1]^{k+2}} L_\infty(t)\lambda_0(dt) \quad \text{as } n\to\infty .$$

Let $h_n : D_{k+2} \to \mathbb{R}$ be defined as

(5.3) $$h_n(f) = \int_{[0,1]^{k+2}} f\cdot\lambda_n(dt) \text{ , } n\ge 0 .$$

Let $\{f_n, n\ge 1\}$ be a sequence of functions in D_{k+2}, and suppose that $f_n \to f_0$ in uniform topology where $f_0 \in C_{k+2}$. We show that

(5.4) $$h_n(f_n) \to h_0(f_0) .$$

We have

$$\left|\int_{[0,1]^{k+2}} f_n\lambda_n(dt) - \int_{[0,1]^{k+2}} f_0\lambda_0(dt)\right| \le$$

$$\le \left| \int_{[0,1]^{k+2}} |f_n - f_0| \lambda_n(dt) \right| + \left| \int_{[0,1]^{k+2}} f_0(\lambda_n - \lambda_0)(dt) \right|$$

$$\le \sup_{t \in [0,1]^{k+2}} |f_n(t) - f(t)| \left| \int_{[0,1]^{k+2}} \lambda_n(dt) \right| + \left| \int_{[0,1]^{k+2}} f_0(\lambda_n - \lambda_0)(dt) \right|$$

(5.4) follows using (2.5), (2.6) and (5.3), and (5.2) follows using Billingsley (1968, Theorem 5.5) and Proposition 3.5.

Now we prove that condition (2.6) of Theorem 2.1 is satisfied. By using (5.1) and (5.2) we deduce that

$$\sigma^2 = \int_{[0,1]^{k+2}} \int_{[0,1]^{k+2}} E[L_\infty(t) L_\infty(t')] \lambda_0(dt) \lambda_0(dt') . \tag{5.5}$$

We have (see (4.4))

$$L_\infty(t) = W_\infty(t) - \sum_{j=1}^{k+1} t_0 W_\infty(1,\dots,t_j,\dots,1) \ell_j(\underset{\sim}{t}) \tag{5.6}$$

From (5.5) and (5.6) the equality in (2.6) holds.

It remains to show that $\sigma^2 < \infty$.

By assumption (d) of Proposition 3.4,

$$\text{(5.7)} \lim_{n\to\infty} |E[\{W_n(t) - \sum_{j=1}^{k+1} t_0 W_n(1,\dots,t_j,\dots,1)\ell_j(\underset{\sim}{t})\}\{W_n(t') - \sum_{j=1}^{k+1} t'_0 W_n(1,\dots,t'_j,\dots,1)\ell_j(\underset{\sim}{t})\}]|$$

$$= |E[L_\infty(t) L_\infty(t')]|$$

$$\le \lim_{n\to\infty} [E\{W_n(t) - \sum_{j=1}^{k+1} t_0 W_n(1,\dots,t_j,\dots,1)\ell_j(\underset{\sim}{t})\}^2]^{\frac{1}{2}} \times [E\{W_n(t')$$

$$- \sum_{j=1}^{k+1} t'_0 W_n(1,\dots,t'_j,\dots,1)\ell_j(\underset{\sim}{t'})\}^2]^{\frac{1}{2}}$$

$$= \lim_{n\to\infty} [A_n B_n]$$

by the Schwarz inequality.

Let now $\{X_{n,i}\}$ be φ–mixing with rates (1.2) or (1.3). Then from Lemma 4.1. with $q=1$, we obtain

$$(5.8)\ A_n \le [E\{W_n^2(t)+2|W_n(t)|t_0 \sum_{j=1}^{k+1} |W_n(1,\dots,t_j,\dots,1)|\ell_j(\underset{\sim}{t})$$

$$+ t_0^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} \ell_j(\underset{\sim}{t})\ell_s(\underset{\sim}{t})|W_n(1,\dots,t_j,\dots,1)W_n(1,\dots,t_s,\dots,1)|\}]^{\frac{1}{2}}$$

$$\le C_1[(\prod_{m=1}^{k+1} t_m)^{\frac{1}{k+2}} + 2t_0(\prod_{m=1}^{k+1} t_m)^{\frac{1}{2(k+1)}} \ell_j(\underset{\sim}{t}) + t_0^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} \ell_j(\underset{\sim}{t})\ell_s(\underset{\sim}{t}) t_j^{\frac{1}{2}} t_s^{\frac{1}{2}}]^{1/2}$$

where $C_1 > 0$ is some constant.

Similarly $B_n \le$ some inequality with t's changed to t''s . Thus $|E[L_0(t)L_0(t')]|$ is bounded by a function which is $\lambda_0 \times \lambda_0$ integrable, and so $|E[L_0(t)L_0(t')]|$ is also $\lambda_0 \times \lambda_0$ integrable.

Let now $\{X_{n,i}\}$ be strong mixing with rates (1.4). Then, using Lemma 4.2 with $q=2$, we obtain

$$A_n \le C_2\left[\left[\prod_{m=0}^{k+1} t_m\right]^{\frac{1-\epsilon}{k+2}} + 2t_0\left[\prod_{m=1}^{k+1} t_m\right]^{\frac{1-\epsilon}{2(k+1)}} \ell_j(\underset{\sim}{t}) + t_0^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} \ell_j(\underset{\sim}{t})\ell_s(\underset{\sim}{t}) t_j^{\frac{1-\epsilon}{2}} t_s^{\frac{1-\epsilon}{2}}\right]^{1/2}$$

and a similar inequality for B_n , and the result follows as in the case of φ–mixing.

The proof of Theorem 2.2 follows analogously.

6. Applications to Markov processes and ARMA processes.

6.1 Markov processes. Consider a sequence $\{X_{t,n}; n\in\mathbb{Z}\}$, of $\mathbb{R}$–valued processes such that for all $t\in\mathbb{N}^*$, $\{X_{t,n}\}$ is a k–Markov process with stationary transition probabilities $P_t(x_1,\dots,x_k;A)$ where $A\in\mathscr{B}$, $\mathscr{B}$ is the Borel σ–field of $\mathbb{R}$, and $(x_1,\dots,x_k) \in \mathbb{R}^k$.

We say that the Markov process is ergodic, if there exists a unique probability measure μ_t on $\mathbb{R}^k$ with marginals Π_t on $\mathbb{R}$ such that

$$\Pi_t(A) = \int_{\mathbb{R}^k} P_t(x_1,\dots,x_k;A)\mu_t(dx_1,\dots,dx_k) \quad \text{for all } A\in\mathscr{B}.$$

We denote by P_t^m , the m–step transition probability defined by

$$P_t^{m+\ell}(x_1,\dots,x_k;A) = \int_{\mathbb{R}} P_t^{\ell}(x_2,\dots,x_k,y;A)P_t^m(x_1,\dots,x_k;dy)$$

for all $A\in\mathscr{B}$ and $(x_1,\dots,x_k) \in \mathbb{R}^k$.

For a transition probability $P_t(\dots;.)$ and invariant measure μ_t and marginal Π_t, we denote by $P_t^*(.,.)$ the transition probability defined by

$$\int_{\prod_{i=1}^{k}(-\infty,y_i]} P_t(u_1,\dots,u_k;(-\infty,y_{k+1}])\mu_t(du_1,\dots,du_k) = \int_{-\infty}^{y_{k+1}} P_t^*(u_{k+1};\prod_{i=1}^{k}(-\infty,y_i])\Pi_t(du_{k+1}) .$$

We say that the Markov process is geometrically ergodic if it is ergodic and if there exists $0<\rho_t<1$ such that

$$\|P_t^m(x_1,\dots,x_k;.) - \Pi_t(.)\| = O(\rho_t^m) \text{ for all a.s. } (x_1,\dots,x_k)\in\mathbb{R}^k,$$

where $\|.\|$ denotes the norm of total variation. (ρ_t is called the rate).

The Markov process is Harris recurrent if there exists a σ–finite measure ν_t on $\mathbb{R}$ with $\nu_t(\mathbb{R})>0$ such that $\nu_t(A)>0$ implies $P_t(x_1,\dots,x_k;X_{t,n}\in A \text{ i.o.}) = 1$ for all $(x_1,\dots,x_k)\in\mathbb{R}^k$.

Finally, the Markov process is Doeblin recurrent if it is ergodic and there exists a finite measure ν_t on $\mathbb{R}$ with $\nu_t(\mathbb{R})>0$, an $m\geq1$ and $\epsilon>0$ such that $P_t^m(x_1,\dots,x_k;A) \leq 1-\epsilon$ if $\nu_t(A) \leq \epsilon$ for all $(x_1,\dots,x_k) \in \mathbb{R}^k$ and $A\in\mathcal{B}$.

Let us denote $\forall\, j \in \{1,\dots,k+1\}$ and $\forall\, M > 0$,

$$R_j(M) = (-\infty,+\infty)^{j-1} \times [-M,M] \times (-\infty,+\infty)^{k-j+1} .$$

Then we have the following theorem:

THEOREM 6.1. *Let* $\{X_{t,n},n\in\mathbb{Z}\}$ *be a Markov process such that for every* $t\in N^*$, $\{X_{t,n}\}$ *is either (a) aperiodic, Harris recurrent and geometrically ergodic with rates* $0<\rho_t<\rho_0$, $\rho_0 \in (0,1)$ *or (b) aperiodic and Doeblin recurrent. Suppose there exists a probability* μ_0 *on* $\mathbb{R}^k$ *and a transition probability* $P_0(\dots;.)$ *such that*

(6.1) $$\sup_{A\in\mathcal{B}^k} |\mu_t(A)-\mu_0(A)| = O(t^{-\alpha}) ,\ \alpha>0 ,$$

(6.2) $$\sup |P_t(x_1,\dots,x_k;A) - P_0(x_1,\dots,x_k;A)| \to 0 \text{ as } t\to\infty,$$

where sup *is over* $A\in\mathcal{B}$ *and* $(x_1,\dots,x_k)\in R_j(M)$ *for every* $j \in \{1,\dots,k+1\}$, $\forall\, M > 0$, *and*

(6.3) $$\sup |P_t^*(x_{k+1};A_k) - P_0^*(x_{k+1};A_k)| \to 0 \text{ as } t\to\infty ,$$

where sup *is over* $|x_{k+1}|\leq M$ *and* $A_k \in \mathscr{B}^k$.

Then, under the assumptions (3.2) *and* (3.3) $(n-k)^{1/2}(\mathscr{S}_n - b_n)$ *converges in law to a normal distribution with mean* 0 *and variance* σ^2 *where* b_n *and* σ^2 *are given by* (2.4) *and* (2.7) *respectively.* (It is assumed that P_t, P_0, P_t^*, P_0^* have densities continuous in x's and, μ_t and μ_0 have densities).

PROOF. (i) Suppose (a) holds. First, we show that the process is geometrically strong mixing. It is well known (see Nummelin and Tuominen (1982)) that if a Markov chain is aperiodic, Harris recurrent and geometrically ergodic with rate ρ_t, then

$$\int \| P_t^m(x_1,\dots,x_k;.) - \Pi_t(.)\| \mu_t(dx_1,\dots,dx_k) = O(\rho_t^m)$$

and this property is equivalent to strong mixing with rate ρ_t^m (see Rosenblatt (1971, p.199)). Next, we show that the covariance functions of the associated graduate empirical process (2.2), converge to a function K, but this is a consequence of Lemma 6 of Rüschendorf (1974) which remains true for strong mixing conditions with a geometric rate.

Let G_t be the distribution function of the k+1 successive random variables of $\{X_{tn}\}$ and let H_t be the measure on $[0,1]^{k+1}$ defined by $H_t(y_1,\dots,y_{k+1}) = G_t(\Pi_t^{-1}(y_1),\dots,\Pi_t^{-1}(y_{k+1}))$ where H_t is the marginal of μ_t for all $(y_1,\dots,y_{k+1}) \in [0,1]^{k+1}$ and $t\geq 0$ (note that we also denote by Π_t the distribution function associated with the measure Π_t).

We have to show that $\{H_t\}_{t>0}$ satisfies the differentiability condition (given in section 2).

Set $\ell_t^{(j)} = \frac{\partial}{\partial t_j} H_t$, and let $F_t^{(j)}$ be the conditional distribution function defined as

$$F_t^{(j)}(u_j;y_1,\dots,y_{j-1},y_{j+1},\dots,y_{k+1}) = \int_A P_t(u_1,\dots,u_k;(-\infty,y_{k+1}])\mu_t^j(du_1,\dots,du_{j-1},du_{j+1},du_k)$$

where $A = \prod_{\ell=1}^{j-1}(-\infty,y_\ell]\prod_{\ell=j+1}^{k}(-\infty,y_\ell]$, μ_t^j is the measure associated with the distribution function $H_t(u_1,\dots,u_{j-1},1,u_{j+1},\dots,u_k,1)$ if $j\leq k$, and $F_t^{(k+1)}$ is the conditional distribution function associated with P_t^*. We have

$$\ell_t^{(j)}(y_1,\dots,y_{k+1}) = F_t^{(j)}(\Pi_t^{-1}(y_j);\Pi_t^{-1}(y_1),\dots,\Pi_t^{-1}(y_{j-1}),\Pi_t^{-1}(y_{j+1}),\dots,\Pi_t^{-1}(y_{k+1}))\ .$$

Also

$$\begin{aligned}
&\ell_t^{(j)}(y_1,\dots,y_k) - \ell_0^{(j)}(y_1,\dots,y_k) = \\
&= [F_t^{(j)}(\Pi_t^{-1}(y_j);\Pi_t^{-1}(y_1),\dots,\Pi_t^{-1}(y_{j-1}),\Pi_t^{-1}(y_{j+1}),\dots,\Pi_t^{-1}(y_{k+1})) \\
&\qquad - F_0^{(j)}(\Pi_t^{-1}(y_j);\Pi_t^{-1}(y_1),\dots,\Pi_t^{-1}(y_{j-1}),\Pi_t^{-1}(y_{j+1}),\dots,\Pi_t^{-1}(y_{k+1}))] \\
&\qquad + [F_0^{(j)}(\Pi_t^{-1}(y_j);\Pi_t^{-1}(y_1),\dots,\Pi_t^{-1}(y_{j-1}),\Pi_t^{-1}(y_{j+1}),\dots,\Pi_t^{-1}(y_{k+1})) \\
&\qquad - F_0^{(j)}(\Pi_0^{-1}(y_j);\Pi_0^{-1}(y_1),\dots,\Pi_0^{-1}(y_{j-1}),\Pi_0^{-1}(y_{j+1}),\dots,\Pi_0^{-1}(y_{k+1}))] = A + B\ .
\end{aligned}$$

To simplify the notation take $j=1$, (for $j\neq 1$, the method is exactly the same).

Then, we have

$$\begin{aligned}
A &= F_t^{(1)}(\Pi_t^{-1}(y_1);\Pi_t^{-1}(y_2),\dots,\Pi_t^{-1}(y_{k+1})) - F_0^{(1)}(\Pi_t^{-1}(y_1);\Pi_t^{-1}(y_2),\dots,\Pi_t^{-1}(y_{k+1})) \\
&= \int_{\prod_{\ell=2}^{k}(-\infty,x_\ell]} P_t(x_1;u_2,\dots,u_k;(-\infty,x_{k+1}])\mu_t^1(du_2,\dots,du_k) \\
&\quad - \int_{\prod_{\ell=2}^{k}(-\infty,x_\ell]} P_0(x_1,u_2,\dots,u_k;(-\infty,x_{k+1}])\mu_0^1(du_2,\dots,du_k) \text{ for } \Pi_t^{-1}(y_\ell)=x_\ell\,,\ \ell\in\{1,\dots,k+1\}\ . \\
&\le \epsilon + \left| \int_{\prod_{\ell=2}^{k}(-\infty,x_\ell]} \mu_t^1(du_2,\dots,u_k) - \mu_0^1(du_2,\dots,du_k) \right| \le 2\epsilon
\end{aligned}$$

for all $(x_1,\dots,x_{k+1}) \in R_1(M)$.

We also have

$$\begin{aligned}
B &= F_0^{(1)}(\Pi_t^{-1}(y_1);\Pi_t^{-1}(y_2),\dots,\Pi_t^{-1}(y_{k+1})) - F_0^{(1)}(\Pi_0^{-1}(y_1);\Pi_0^{-1}(y_2),\dots,\Pi_0^{-1}(y_{k+1})) \\
&= F_0^{(1)}(\Pi_0^{-1}\circ\Pi_0\circ\Pi_t^{-1}(y_1);\Pi_0^{-1}\circ\Pi_0\circ\Pi_t^{-1}(y_2),\dots,\Pi_0^{-1}\circ\Pi_0\circ\Pi_t^{-1}(y_{k+1})) \\
&\quad - F_0^{(1)}(\Pi_0^{-1}\circ\Pi_0\circ\Pi_0^{-1}(y_1);\Pi_0^{-1}\circ\Pi_0\circ\Pi_0^{-1}(y_2),\dots,\Pi_0^{-1}\circ\Pi_0\circ\Pi_0^{-1}(y_{k+1})\ .
\end{aligned}$$

Noting that

$$\begin{aligned}
\sup_{y_1\in[0,1]} |\Pi_0\circ\Pi_t^{-1}(y_1)-y_1| &= \sup_{y_1\in[0,1]} |y_1-\Pi_t\circ\Pi_0^{-1}(y_1)| \\
&= \sup_{y_1\in[0,1]} |\Pi_0\circ\Pi_0^{-1}(y_1)-\Pi_t\circ\Pi_0^{-1}(y_1)|
\end{aligned}$$

we find (using (6.1)) that $B<\epsilon$ for sufficiently large t. Thus $\ell_t^{(j)}(y_1,\ldots,y_{k+1}) \to \ell_0^{(j)}(y_1,\ldots,y_{k+1})$ as $t\to\infty$ uniformly in $(y_1,\ldots,y_k)\in R_j(M)$ for any $M>0$ and so $\{H_t\}_{t>0}$ satisfies the differentiability condition.

(ii) Suppose (b) is satisfied. Then the proof follows from Davydov (1973) who proved that a Markov process which is Doeblin recurrent and aperiodic is geometrically φ–mixing.

Example 6.1. Consider the process $\{X_n, n\in\mathbb{Z}\}$ where $X_{n+1} = a_1X_n + a_2X_n\epsilon_{n+1} + a_3\epsilon_{n+1} + a_4\epsilon_{n+1}^2 + a_5$ where the a's are real numbers and $\{\epsilon_n, n\in\mathbb{Z}\}$ is a white noise with strictly positive density. Then Mokkadem (1985) has shown that if $a_1^2 + a_2^2E(\epsilon_1^2) < 1$ and $E(\epsilon_1^4) < \infty$, then the process $\{X_n, n\in\mathbb{Z}\}$ is geometrically ergodic and geometrically strong mixing. Thus the asymptotic normality of the statistic $\mathscr{S}_n$ based on the ranks of $\{X_n\}$ follows.

Example 6.2. Consider the process $\{X_n, n\in\mathbb{Z}\}$ where $X_{n+1} = f(X_n) + \epsilon_{n+1}$ where the ϵ's are independent and identically distributed random variables with strictly positive density, and $f: \mathbb{R} \to \mathbb{R}$ is bounded, nondecreasing and continuous. (This model was studied by Collomb and Doukhan (1983)). It is easy to check that this model is Doeblin recurrent and aperiodic, and we deduce that $\{X_n\}$ is geometrically φ–mixing and we can apply Theorem 6.1.

6.2 ARMA processes. Consider a sequence of ARMA (k_1,k_2) processes

(6.4)
$$\prod_{j=1}^{k_1} (1 - a_j^{(n)}U)X_{n,i} = Q_{k_2}^{(n)}(U)\epsilon_i\ ,\ i\in\mathbb{Z},\ n\in\mathbb{N}^*$$

where $U\lambda_i = \lambda_{i-1}$, $Q_{k_2}^{(n)}(U) = \sum_{\ell=0}^{k_2} b_\ell^{(n)}U^\ell$, $b_0^{(n)} = 1$ and $\{\epsilon_i, i\in\mathbb{Z}\}$ is a sequence of independent random variables such that $E(\epsilon_i) = 0$, and ϵ_i has a density $g_i(x)$, $i\in\mathbb{Z}$.

Then

LEMMA 6.2. (Gorodetskii (1977), Withers (1981)). *Let the sequence* $\{X_{n,i}, i\in\mathbb{Z}\}$ *satisfy the following conditions:*

(6.5)
$$\sup_{i\in\mathbb{Z}} \int_{-\infty}^{\infty} |g_i(x+\beta) - g_i(x)|\,dx \le c_1|\beta|\ ,\ \forall\,\beta \text{ and some } c_1>0$$

(6.6) $$\sup_{i\in\mathbb{Z}} E|\epsilon_i| < c_2 < \infty \text{ and } \sup_{n\in\mathbb{N}} \sup_{1\le j\le k_1} |a_j^{(n)}| < \rho < 1$$

where c_2 *and* ρ *are some constants. Then for any* $n\in\mathbb{N}^*$, *the process* $\{X_{n,i};i\in\mathbb{Z}\}$ *is strong mixing with rate* $\alpha(m) = O(\rho_0^{m/2})$ *for each* $\rho_0 > \rho$.

THEOREM 6.2. *Let the sequence* $\{X_{n,i},i\in\mathbb{Z}\}$ *of ARMA* (k_1,k_2) *process given by* (6.4) *satisfy the following conditions:*

(6.7) $\{\epsilon_i,i\in\mathbb{Z}\}$ *is a sequence of independent and identically distributed random variables, each having* $\mathcal{N}(0,\sigma^{*2})$ *distribution.*

(6.8) $\forall\, j\in\{1,\dots,k_1\}$, $\exists\ \alpha>0$ *and* $a_j\in(-1,1)$, $a_j\neq 0$ *such that* $|a_j^{(n)}-a_j| = O(n^{-\alpha})$, *and* $\forall\, \ell\in\{1,\dots,k_2\}$, $\exists\ \beta>0$, *and* $b_\ell\in\mathbb{R}$ *such that* $|b_\ell^{(n)}-b_\ell| = O(n^{-\beta})$.

Then for the rank statistic $\mathcal{S}_n$ *associated with the sequence* $\{X_{n,1},\dots,X_{n,n}\}$ *and the score functions satisfying the assumptions of Theorem* 2.1, $(n-k)^{1/2}(\mathcal{S}_n-b_n)$ *converges in law to the* $\mathcal{N}(0,\sigma^2)$ *distribution where* b_n *and* σ^2 *are given by* (2.4) *and* (2.7) *respectively.*

PROOF. To prove this theorem, we first note using Lemma 6.2 that the sequence $\{X_{n,i}\}$ is geometrically strong mixing. Let now, F_n be the distribution function of $X_{n,i}$, and F_0 the distribution function of a stationary random variable X_{0i} defined by an ARMA (k_1,k_2) process with coefficients a_j, $1\le j\le k_1$ and b_ℓ, $1\le\ell\le k_2$. Now we prove the differentiability condition for $H_n(\underset{\sim}{t})$ defined in (2.1) by verifying (6.1), (6.2) and (6.3).

Let P_n^j be the transition distribution function of $X_{n,1},\dots,X_{n,j-1},X_{n,j+1},\dots,X_{n,k_1}$, and G_n^j the distribution function of $(X_{n,1},\dots,X_{n,k_1-1})$, $n\ge 0$. Then (6.1), (6.2) and (6.3) are satisfied in view of the following well known result.

LEMMA 6.3. *Let* $\{G_n,n\ge 0\}$ *be a sequence of* k*–dimensional normal distribution functions each with mean vector* $\underset{\sim}{0}$. *Let the covariance matrices of* G_n *and* G_0 *be* $\underset{\sim}{\Sigma}_n = ((\sigma_{i\ell}^{(n)}))$ *and* $\underset{\sim}{\Sigma}_0 = ((\sigma_{i\ell}^{*}))$ *and assume that* $|\sigma_{i\ell}^{(n)}-\sigma_{i\ell}^{*}| = O(n^{-\alpha})$ *for each* $i,\ell = 1,\dots,k$. *Then* G_n *converges uniformly to* G_0.

7. Appendix

7.A Proof of Proposition 3.1.

(i) and (ii) are sufficient conditions; They follow immediately from Proposition 3.2 by using a result from of Billingsley ((1968), p. 151, ℓ. 15).

We have only to prove that (i) and (ii) are necessary conditions.

Let $\mathcal{U}^*_{k+2}$ be the σ–field generated by the uniform topology on C_{k+2}.

As P is concentrated on a separable space $(C_{k+2}, \mathcal{U}^*_{k+2})$, it follows from Wichura ((1970), Theorem 1) that there exists a probability space $(\Omega, \mathcal{A}, \mu)$ and a sequence of random variables $\{Y^*_n\}$, $n \in \mathbb{N}^*$ and a random variable Y^* such that $\mu(Y^*_n) = P_n$, $\mu(Y^*) = P$ and $Y^*_n \longrightarrow Y^*$ a.s. μ. For any δ (>0), we consider the map $T_\delta : D_{k+2} \longrightarrow \mathbb{R}$ defined by

$$T_\delta(f) = \sup \{ |f(t) - f(t'); \|t-t'\| \leq \delta\} .$$

Then, T_δ is a continuous map for the uniform topology on $\mathcal{U}_{k+2}$.

Now, consider a sequence of random variables $\{Z_{n,\delta}\}$, $n \in \mathbb{N}^*$ and a random variable Z_δ defined as

$$Z_{n,\delta} = T_\delta \circ Y^*_n , \quad Z_\delta = T_\delta \circ Y^* .$$

As Y^*_n converges a.s. to Y^*, it follows that $\forall \epsilon>0$, $\exists N_0 \in \mathbb{N}$ such that

(7.1) $$\mu\{|Z_{n,\delta} - Z| > \epsilon/2\} < \epsilon/2 , \quad \forall n \geq N_0$$

As Y^* is concentrated on C_{k+2}, we have also $\forall \epsilon>0$, $\exists \delta>0$ such that

(7.2) $$\mu\{|Z_\delta| > \epsilon/2\} < \epsilon/2 .$$

(7.1) and (7.2) imply

$$\mu\{|Z_{n,\delta}| > \epsilon\} < \epsilon$$

or

(7.3) $$P_n[f; \omega(f,\delta) \geq \epsilon] = \mu\{|Z_{n,\delta}| > \epsilon\} < \epsilon$$

and from (7.3) we obtain condition (ii). Condition (i) is immediate. Proposition 3.1. is proved.

7.B Proof of Lemma 4.1.

Without loss of generality, we can take $N_n = n$. First, we prove that for any p $(1 \le p \le n)$

$$(7.4) \qquad E\Big[\sum_{i=1}^{p} X_{n,i}\Big]^{2q} \le C_q \sum_{\ell=1}^{q} p^{q/\ell}\big(\sup_{1\le j\le n} \|X_{n,j}\|_\ell\big)^{2q} (h(p,\ell))^{2q}$$

where C_q is a constant depending only on q and φ and

$$h(p,\ell) = \exp\Big\{\sum_{\ell=1}^{s} \varphi^{1/2q}([2^{1/2(\ell+1)}])\Big\} \quad \text{for } 2^s \le p < 2^{s+1} .$$

For any (ℓ,p) $(1\le \ell \le p \le n)$, we define $S(\ell,p)$ by

$$S(\ell,p) = \sum_{i=\ell\wedge(n+1)}^{(\ell+p-1)\wedge(n+1)} X_{n,i} \quad \text{where by convention } X_{n,n+1} \equiv 0 .$$

Denote $S = S(\ell,p)$, $S' = S(\ell+p+r,p)$, $R = S(\ell+p,r) - S(2p+\ell,r)$ for $r \ge 1$,

$$a(p,q) = \sup_{\ell\ge 1} \Big[E\Big[\sum_{i=\ell\wedge(n+1)}^{(\ell+p-1)\wedge(n+1)} X_{n,i}\Big]^{2q}\Big]^{1/2q}, \text{ and } m_\ell = \sup_{1\le j\le n} \|X_{n,j}\|_\ell .$$

Then, after some computations, we obtain the following inequality

$$(7.5) \qquad E(S+S')^{2q} \le 2(a(p,q))^{2q} \exp\{(2q\,\varphi(r))^{1/2q}\} + (2a(p,q-1))^{2q} .$$

From the Minkowski inequality, it follows that

$$\Big\|\sum_{\ell=1}^{(\ell+2p-1)\wedge(n+1)} X_{n,i}\Big\|_{2q} = \|S+S'+R\|_{2q} \le \|S+S'\|_{2q} + 2rm_q$$

$$\le 2^{1/2q} a(p,q) \exp\{(\varphi(r))^{1/2q}\} + 2a(p,q-1) + 2rm_q .$$

Now take $p = 2^s$, $s \ge 1$ and put $r = r(s,q) = [2^{s/2(q+1)}]$, $\psi(s,q) = (\varphi(r))^{1/2q}$.

Then, from (7.5), we can write

$$a(2^s,q) \le 2^{1/2q} a(2^{s-1},q) \exp\{\psi(s,q)\} + 2a(2^{s-1},q-1) + 2r(s,q)m_q ,$$

(7.6) $$a(2^s,q) \le 2^{s/2q}(1 + 2 \sum_{i=1}^{s} 2^{-i/2q}\, r(i,q))\, m_q \exp\{ \sum_{j=1}^{s} \psi(j,q)\}$$
$$+ 2 \sum_{i=1}^{s} 2^{(s-i)/2q}\, a(2^{i-1},q-1) \exp\{ \sum_{j=i+1}^{s} \psi(j,q)\}$$

where by convention $\sum_{j=s+1}^{s} \psi(j,q) = 0$.

For $q=1$, we have

$$a(2^s,1) \le K_1\, 2^{s/2}\, h(2^s,1)m_1$$

where K_1 is a positive constant.

We give a proof by recurrence on q . Suppose that for all $q \ge 2$ and $p = 2^s$, we have

$$a(2^s,q-1) \le K_{q-1} \sum_{\ell=1}^{q-1} 2^{s/2\ell}\, h(2^s,\ell)m_\ell$$

where K_{q-1} is a positive constant. From (7.6), we deduce

$$a(2^s,q) \le 2^{s/2q} (\exp\{ \sum_{j=1}^{s} \psi(j,q)\})(1 + 2 \sum_{j=1}^{s} 2^{-\frac{1}{2}(\frac{j}{q} - \frac{j}{q+1})})m_q + 2A_q K_{q-1} \sum_{\ell=1}^{q-1} 2^{s/2\ell} h(2^s,\ell)m_\ell$$

where A_q is a constant depending only on q and φ . That is,

(7.7) $$a(2^s,q) \le K_q \sum_{\ell=1}^{q} 2^{s/2\ell}\, h(2^s,\ell)m_\ell .$$

Finally, for each $p \le n$, we can write the binary decomposition as

$$p = \sum_{i=0}^{s} v_i 2^i , \quad v_i \in \{0,1\} .$$

From the equality $h(p,\ell) = h(2^s,\ell)$ for $2^s \le p < 2^{s+1}$ and (7.7), it follows that

$$a(p,q) \le \sum_{i=0}^{s} v_i\, a(2^i,q) \le \sum_{i=0}^{s} a(2^i,q)$$

$$a(p,q) \le K_q \sum_{i=0}^{s} \sum_{\ell=1}^{q} 2^{i/2\ell}\, h(2^i,\ell)m_\ell$$

$$(a(p,q))^{2q} \le C_q \sum_{\ell=1}^{q} p^{q/\ell}(h(p,\ell)m_\ell)^{2q} \tag{7.8}$$

and (7.4) is proved.

(4.1) now follows by putting $p=n=N_n$ and by using the following relation

$$\sum_{\ell=0}^{\infty} (\varphi(2^\ell))^{1/2q} < +\infty \Longleftrightarrow \sum_{m=1}^{\infty} m^{-1}(\varphi(m))^{1/2q} < +\infty . \tag{7.9}$$

Lemma 4.1 is proved.

7.C Proof of Lemma 4.3.

For every $p,N,r \in \mathbb{N}$, we define $S_N = \sum_{i=1}^{N} Y_i$, $T_{N,j} = \sum_{i=1}^{N} Y_{j(N+r)+i}$,

$R_{N,j} = \sum_{i=1}^{r} (Y_{j(N+r)+N+i} - Y_{pN+jr+i})$ for $j=0,\dots,p-1$. For every $\ell \in \mathbb{N}$, we denote $K_\ell^2 = E(\sum_{i=1}^{\ell} Y_i)^2$.

From the property of stationarity, we have

$$K_N^2 = E(S_N^2) = E(T_{N,j})^2 \quad \text{for} \quad j=0,\dots,p-1$$

and

$$|E(S_N)^2 - \tfrac{1}{p} E(\sum_{j=0}^{p-1} T_{N,j})^2| \le p(\varphi(r))^{\frac{1}{2}} K_N^2 . \tag{7.10}$$

We have

$$S_{pN} = \sum_{i=1}^{pN} Y_i = \sum_{j=0}^{p-1} T_{N,j} + \sum_{j=0}^{p-1} R_{N,j} .$$

We deduce from (7.10)

$$|E(S_N)^2 - \tfrac{1}{p} E(S_{pN})^2| \le p(\varphi(r))^{\frac{1}{2}} K_N^2 + 4pK_rK_N . \tag{7.11}$$

From Lemma 4.1, there exists a constant C depending only on φ such that

$$K_r^2 \le C\, r\, K_1^2 \quad \text{and} \quad K_N^2 \le C\, N\, K_1^2 .$$

Now taking $r = r(N) = [N^{\frac{1}{2}}]$ and using (7.10) for $p=2$, we obtain

$$\left|\frac{1}{2^s} E(S_{2^s})^2 - \frac{1}{2^{s+m}} E(S_{2^{s+m}})^2\right| \le \frac{1}{2^s} \sum_{k=0}^{m-1} \frac{1}{2^k} \left|E(S_{2^{s+k}})^2 - \frac{1}{2} E(S_{2^{s+k+1}})^2\right|$$

$$\le 8\, C\, K_1 \sum_{k=s}^{2^{s+m-1}} ((\varphi([2^{k/2}]) + 2^{-k/4})$$

It follows that $\frac{1}{2^s} E(S_{2^s})^2$ is a Cauchy sequence. Hence there exists a constant K such that

$$\frac{1}{2^s} E(S_{2^s})^2 \longrightarrow K^2 \text{ as } s\to\infty .$$

We deduce that for every $p_0 \in \mathbb{N}$

$$\sup_{p\ge p_0} \left|K^2 - \frac{1}{p2^s} E(S_{p2^s})^2\right| \longrightarrow 0 \text{ as } s\to\infty .$$

Let ℓ be such that $pq \le \ell \le (p+1)q$. Then

$$\left| \frac{1}{pq} E(S_{pq})^2 - \frac{1}{\ell} E(S_\ell)^2\right| \le 2\, C\, K_1^2 (\frac{1}{p} + \frac{1}{p^{\frac{1}{2}}}) \text{ for all } q\ge 1 .$$

Consequently $\forall\, \epsilon>0$, $\exists\, p_0 \in \mathbb{N}$, such that

$$\sup_{\ell\ge p} \left| \frac{1}{p[1/p]} E(S_{p[1/p]})^2 - \frac{1}{\ell} E(S_\ell)^2\right| < \epsilon/2 \;,\; \forall\, p\ge p_0$$

Now if we choose s_0 such that $2^{s_0} \ge p_0$ and

$$\sup_{p\le 2p_0} \left|K^2 - \frac{1}{p2^s} E(S_{p2^s})^2\right| < \epsilon/2 \text{ for all } s\ge s_0 ,$$

then there exists for every $n \ge n_0 = p_0 2^{s_0}$, an $s\ge s_0$ and a p for which $p_0 \le p < 2p_0$ and $p2^s \le n < (p+1)2^s$.

We deduce that $\forall\, n\ge n_0$

$$\left|K^2 - \frac{1}{n} E(S_n)^2\right| \le \left|K^2 - \frac{1}{p2^s} E(S_{p2^s})^2\right| + \left|\frac{1}{p2^s} E(S_{p2^s})^2 - \frac{1}{n} E(S_n)^2\right| < \epsilon/2 + \epsilon/2 = \epsilon$$

which implies

$$\frac{1}{n} E(S_n)^2 \longrightarrow K^2 \text{ as } n\to\infty$$

and Lemma 4.3 is proved.

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STATISTIQUE. — *Convergence faible de la statistique sérielle linéaire de rang en condition de dépendance avec applications aux séries chronologiques et processus de Markov.* Note de **Michel Harel** et **Madan Puri**, présentée par Robert Fortet.

La normalité asymptotique de la statistique sérielle linéaire de rang introduite par Hallin, Ingenbleek et Puri [2] pour le problème d'éprouver un bruit blanc contre des alternatives ARMA est établie pour une suite de variables aléatoires φ mélangeantes ainsi que fortement mélangeantes en utilisant l'approche de Rüschendorf [5]. Des applications sont données pour les processus de Markov et les processus ARMA en séries chronologiques.

STATISTICS. — Weak convergence of serial rank statistics under dependence with applications in time series and Markov processes.

The asymptotic normality of linear serial rank statistics introduced by Hallin, Ingengleek and Puri [2] *for the problem of testing white noise against ARMA alternatives is established for φ-mixing as well as strong mixing sequences of random variables using Rüschendorf's* [5] *approach. Applications in Markov processes and ARMA processes in time series are provided.*

1. Introduction, définitions et hypothèses. — Soient $X_{n1}, \ldots, X_{nn}$, $n \geqq 1$ des variables aléatoires réelles avec $F_n(x)$ $n \geqq 1$ la fonction de répartition supposée continue. On considère les statistiques :

$$\mathfrak{S}_n = (n-k)^{-1} \sum_{i=k+1}^{n} c_{ni}\, a_n(R_{n,\,i-k}, \ldots, R_{n,\,i})$$

où les c_{ni} sont des constantes connues, $a_n(\ldots)$ sont les fonctions de scores et $R_{n,\,i}$ le rang de X_{ni} parmi $(X_{n1}, \ldots, X_{nn})$. Notre but est d'étudier le comportement asymptotique de $\mathfrak{S}_n$ quand la suite $\{X_{ni}\}$ est φ mélangeante où le coefficient de mélange vérifie :

(1) $$\varphi(m) = O(m^{-1-\varepsilon}) \qquad \text{où} \quad \varepsilon > 0 \quad (m \geqq 1)$$

ou

(2) $$\sum_{m=1}^{\infty} m^{-1} \varphi^{1/2(3+k)}(m) < \infty$$

ou fortement mélangeante de coefficient de mélange α vérifiant

(3) $$\sum_{m=1}^{\infty} m^{2(k+2)} \alpha^{\varepsilon}(m) < \infty \qquad \text{où} \quad \varepsilon \in]0,\ 1/2(3+k)[$$

On note $\tilde{F}_n(x)$ la fonction de répartition empirique continue de X_{ni}, $i = 1, \ldots, n$; c'est-à-dire $\tilde{F}_n(x) = n^{-1} \sum_{i=1}^{n} I_{[X_{ni} \leq x]}$ où $I_{[\ ]}$ est la fonction indicatrice. On note G_n la fonction de répartition de $k+1$ variables aléatoires successives parmi $(X_{n1}, \ldots, X_{nn})$. Soit $\{H_n\}$ une suite de mesures sur $[0,1]^{k+1}$ définie par :

$$H_n(\mathbf{t}) = G_n(F_n^{-1}(t_1), \ldots, F_n^{-1}(t_{k+1}))$$

pour tout $\mathbf{t} = (t_1, \ldots, t_{k+1}) \in [0,1]^{k+1}$.

On considère maintenant le processus empirique gradué W_n défini par :

$$W_n(t) = (n-k)^{-1/2} \sum_{i=k+1}^{[nt_0]} \left\{ \sum_{j=1}^{k+1} I_{[F_n(X_{n,\,i+j-k-1}) \leq t_j]} - H_n(\mathbf{t}) \right\}$$

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pour tout $t=(t_0, \mathbf{t})=(t_0, t_1, \ldots, t_{k+1})\in[0,1]^{k+2}$, où $[nt_0]$ désigne la partie entière de nt_0 et on considère le processus de rang gradué L_n défini par

$$L_n(t)=(n-k)^{-1/2}\sum_{i=k+1}^{[nt_0]}\left\{\sum_{j=1}^{k+1} I_{[\tilde{F}_n(X_{n,\,j-k-1})\leqq t_j]}-H_n(\mathbf{t})\right\}$$

pour tout $t=(t_0, \mathbf{t})\in[0,1]^{k+2}$.

La convergence de $\mathfrak{S}_n$ sera déduite de la convergence de L_n qui sera elle-même déduite de la convergence de W_n.

Finalement, on note par C_{k+2}, l'espace de toutes les applications continues $f: [0,1]^{k+2}\to\mathbb{R}$ et par $C_{k+1}(j)$, $1\leqq j\leqq k+1$, l'espace de toutes les applications continues et bornées $f: A(j)\to\mathbb{R}$ où $A(j)=[0,1]^{j-1}\times]0,1[\times[0,1]^{k+1-j}$.

2. Résultat préliminaire. — On dit que la suite $\{H_n\}$ satisfait la condition de différentiabilité si $\partial H_n/\partial t_j$ existe sur $A(j)$ et appartient à $C_{k+1}(j)$, $1\leqq j\leqq k+1$ et $\partial H_n/\partial t_j\to l_j$ pour la topologie uniforme sur tout compact de $A(j)$ quand $n\to\infty$, et l_j appartient à $C_{k+1}(j)$.

Proposition. — *Supposons que la suite* $\{X_{ni}\}$ *soit* (a) φ *mélangeante avec le taux* (1) *ou* (*b*) *stationnaire et* φ *mélangeante avec le taux* (2) *ou encore* (*c*) *fortement mélangeante avec le taux* (3). *Supposons de plus que* (*d*) *les fonctions de covariance des processus empiriques* W_n *convergent vers une fonction et que* (*e*) $\{H_n\}$ *satisfait la condition de différentiabilité.*

Alors sous (*a*) (*d*) *et* (*e*) *ou* (*c*) (*d*) *et* (*e*), L_n *converge faiblement pour la topologie uniforme vers un processus gaussien* L_0 *avec des trajectoires p. s. dans* C_{k+2}. *Sous* (*b*) *et* (*e*) $L_n(1, \mathbf{t})$ *converge pour la topologie uniforme vers un processus gaussien* $L_0(1, \mathbf{t})$ *avec ses trajectoires p. s. dans* C_{k+1}.

3. Normalité asymptotique de la statistique sérielle linéaire de rang $\mathfrak{S}_n$. — Pour tout n, on définit une mesure signée λ_n concentrée sur $\{1/n, \ldots, (n-1)/n, 1\}^{k+2}$ en posant :

$$\lambda_n\left(\prod_{j=0}^{k+1}\left[\frac{l_j}{n}, 1\right]\right)=c_{nl_0}\,a_n(l_1, \ldots, l_{k+1})$$

pour tout $(l_1, \ldots, l_{k+1})\in\{1, \ldots, n\}^{k+2}$ (par convention $c_{nl_0}=0$ si $l_0<k+1$).

On définit un coefficient de centrage b_n par :

$$b_n=\int_{[0,1]^{k+2}}\hat{H}_n(t)\,\lambda_n(dt),$$

où $\hat{H}_n$ est une fonction : $[0,1]^{k+2}\to\mathbb{R}_+$ telle que $\hat{H}_n(t)=([nt_0]-k)\,H_n(\mathbf{t})$.

Théorème 1. — *Soit* λ_0 *une mesure telle que :*

$$\lim_{n\to\infty}\int f.\,d\lambda_n=\int f\,d\lambda_0 \qquad \textit{pour tout} \quad f\in C_{k+2} \tag{4}$$

et

$$\sup_{n\in\mathbb{N}}\int f.\,d\,|\lambda_n|<\infty; \qquad \mathbb{N}=\{1, 1, 2, \ldots\} \tag{5}$$

où $|\lambda_n|$ *est la mesure de variation totale.*

Si la suite $\{X_{ni}\}$ *satisfait les hypothèses* (*a*) (*d*) *et* (*e*) *ou* (*c*) (*d*) *et* (*e*) *ou encore* (*b*) *et* (*e*) *avec* $c_{ni}=1$ *pour tout* i, *alors* $(n-k)^{1/2}$ $(\mathfrak{S}_n-b_n)$ *converge en loi vers une distribution* *normale avec moyenne nulle et écart type* σ *où*

$$\sigma^2=\int_{[0,1]^{k+2}}\int_{[0,1]^{k+2}} E[L_0(t)L_0(t')]\,d\lambda_0(t)\,d\lambda_0(t')\,(<\infty)$$

Preuve. — Après avoir montré que $\mathfrak{S}_n$ peut se mettre sous la forme :

$$\mathfrak{S}_n=(n-k)^{-1/2}\left[\int_{[0,1]^{k+1}} L_n(t)\,\lambda_n(dt)\right]+b_n$$

on utilise la proposition.

COROLLAIRE. — *Soit* J *une fonction sur* $[0,1]^{k+2}$ *telle que*

$$J\left(\frac{l_0}{n},\ldots,\frac{l_{k+1}}{n}\right)=c_{nl_0}\,a_n(l_1,\ldots,l_{k+1})$$

pour tout $(l_0,\ldots,l_{k+1})\in\{1,\ldots,n\}^{k+2}$, $J=J_d+J_c$ *où* J_d *est une fonction en escalier prenant seulement un nombre fini de sauts et où pour tout* $I\subset\{0,\ldots,k+1\}$, J_c *a une dérivée continue* $\partial^I J_c/(\partial t_j)_{j\in I}$, *alors les conditions* (4) *et* (5) *sont satisfaites.*

4. APPLICATIONS AUX PROCESSUS DE MARKOV ET PROCESSUS ARMA. — 4.1. On considère une suite $\{X_{tn};\,n\in\mathbb{Z}\}$ de processus à valeurs réelles telle que pour tout $t\in\mathbb{N}^*$, $\{X_{tn}\}$ est un k processus de Markov avec pour probabilité de transition $P_t(x_1,\ldots,x_k;\,A)$ où $A\in\mathscr{B}$ la tribu borélienne de $\mathbb{R}$ et $(x_1,\ldots,x_k)\in\mathbb{R}^k$.

On suppose que le processus de Markov est ergodique avec pour probabilité invariante μ_t sur $\mathbb{R}^k$ et marginale π_t sur $\mathbb{R}$.

On note $P_t^*(.;\,.)$ la transition de probabilité de X_{tn} vers $X_{t-k,\,n},\ldots,X_{t-1,\,n}$ définie par $\mu_t P_t=\pi_t P_t^*$. $\forall j\in\{1,\ldots,k+1\}$, $\forall M>0$, on note

$$R_j(M)=]-\infty,\,+\infty[^{j-1}\times[MM]\times]-\infty,\,+\infty[^{k-j+1}$$

THÉORÈME 2. — *Soit* $\{X_{tn};\,n\in\mathbb{Z}\}$ *un processus de Markov tel que pour chaque* $t\in\mathbb{N}^*$, $\{X_{tn}\}$ *est soit apériodique, Harris récurrent et géométriquement ergodique avec pour taux* $0<\rho_t<\rho_0$, $\rho_0\in]0,1[$ *soit apériodique et Doeblin récurrent. On suppose qu'il existe une probabilité* μ_0 *sur* $\mathbb{R}^k$ *et une transition de probabilité* $P_0(\ldots;\,.)$ *telle que :*

$$\sup_{A\in\mathscr{B}^k}|\mu_t(A)-\mu_0(A)|=0(t^{-\alpha}),\qquad \alpha>0$$

$$\sup_{\substack{A\in\mathscr{B}\\(x_1,\ldots,x_k)\in R_j(M)}}|P_t(x_1,\ldots,x_k;\,A)-P_0(x_1,\ldots,x_k;\,A)|\xrightarrow[t\to\infty]{}0$$

pour chaque $j\in\{1,\ldots,k\}$, $\forall M>0$ *et*

$$\sup_{\substack{|x_{k+1}|<M\\A_k\in\mathscr{B}^k}}|P_t^*(x_{k+1};\,A_k)-P_0^*(x_{k+1};\,A_k)|\xrightarrow[t\to\infty]{}0$$

alors, sous les hypothèses (4) *et* (5) $(n-k)^{1/2}$ $(\mathfrak{S}_n-b_n)$ *converge en loi vers la distribution normale de moyenne nulle et écart type* σ *où* b_n *et* σ *sont donnés dans le théorème* 1. (*On a supposé que* P_t, P_0, P_t^*, P_0^* *ont des densités continues en* x' *s et* μ_t *et* μ_0 *ont des densités.*)

Exemple 1. — On considère le processus $\{X_n;\,n\in\mathbb{Z}\}$ où

$$X_{n+1}=a_1X_n+a_2X_n\varepsilon_{n+1}+a_3\varepsilon_{n+1}+a_4\varepsilon_{n+1}^2+a_5$$

où les coefficients $a_1,\ldots,a_5$ sont des nombres réels et $\{\varepsilon_n,\,n\in\mathbb{Z}\}$ est un bruit blanc avec une densité strictement positive. Alors si $a_1^2+a_2^2E(\varepsilon_1^2)<1$ et $E(\varepsilon_1^4)<\infty$, le processus

$\{X_n, n \in \mathbb{Z}\}$ est géométriquement fortement mélangeant [4] et la normalité asymptotique de $\mathfrak{S}_n$ suit du théorème 1.

Exemple 2. — On considère le processus $\{X_n, n \in \mathbb{Z}\}$ où $X_{n+1} = f(X_n) + \varepsilon_{n+1}$ où les ε_n sont indépendants et identiquement distribués avec une densité strictement positive et $f: \mathbb{R} \to \mathbb{R}$ est bornée, croissante et continue alors $\{X_n\}$ est géométriquement φ mélangeant [1] et on peut appliquer le théorème 1.

4.2. On considère une suite de processus ARMA d'ordre (k_1, k_2)

$$\sum_{j=1}^{k_1} (1 - a_j^{(n)} U) X_{ni} = Q_{k_2}^{(n)} U \varepsilon_i, \qquad i \in \mathbb{Z}, \quad n \in \mathbb{N}^*$$

où

$$U \lambda_i = \lambda_{i-1}, \qquad Q_{k_2}^{(n)}(U) = \sum_{l=0}^{k_2} b_l^{(n)} U^l, \qquad b_0^{(n)} = 1$$

et $\{\varepsilon_i; i \in \mathbb{Z}\}$ est une suite de variables aléatoires indépendantes et identiquement distribuées ayant une $\mathfrak{N}(0, \sigma^{*2})$ distribution.

THÉORÈME 3. — *Soit la suite $\{X_{ni}, i \in \mathbb{Z}\}$ de processus ARMA d'ordre (k_1, k_2) satisfaisant les conditions suivantes :* $\forall j \in \{1, \ldots, k_1\}$, $\exists \alpha > 0$ *et* $a_j \in]-1, 1[$, $a_j \neq 0$ *tel que* $|a_j^{(n)} - a_j| = O(n^{-\alpha})$ *et* $\forall l \in \{1, \ldots, k_2\}$, $\exists \beta > 0$ *et* $b_l \in \mathbb{R}$ *tel que* $|b_l^{(n)} - b_l| = O(n^{-\beta})$ *alors pour la statistique de rang $\mathfrak{S}_n$ associée à la suite $X_{n1}, \ldots, X_{nn}$ et les fonctions de scores satisfaisant les hypothèses du théorème* 1, $(n-k)^{1/2}$ $(\mathfrak{S}_n - b_n)$ *converge en loi vers la* $\mathfrak{N}(0, \sigma^2)$ *distribution où b_n et σ^2 sont donnés par le théorème* 1.

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WEAK CONVERGENCE OF THE SERIAL LINEAR RANK STATISTIC WITH UNBOUNDED SCORES AND REGRESSION CONSTANTS UNDER MIXING CONDITIONS

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Harel and Puri (1987b) established the weak convergence of a class of the serial linear rank statistics with bounded score functions and regression constants when the random variables are φ–mixing or strong mixing. This paper extends these results to the case when the score functions as well as the regression constants are not necessarily bounded.

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1. Introduction. Let $X_{n1},\dots,X_{nn}$, $n\geq 1$ be real–valued random variables with continuous distribution functions $F_n(x)$, $n\geq 1$.

Consider the statistics

$$(1.1) \qquad \mathscr{S}_n = \sum_{i=k+1}^{n} c_{ni}\, a_n(R_{n,i-k},\dots,R_{n,i})$$

where the c_{ni} are known constants, $a_n(\dots)$ are the scores, and R_{ni} denotes the rank of X_{ni} among $(X_{n1},\dots,X_{nn})$.

Our aim is to study the asymptotic behavior of $\mathscr{S}_n$ when the sequence $\{X_{ni}\}$ is φ–mixing with rates

$$(1.2) \qquad \varphi(m) = 0(m^{-1-\epsilon}) \text{ for some } \epsilon>0 \ (m\geq 1)$$

or

$$(1.3) \qquad \sum_{m=1}^{\infty} m^{-1}\varphi^{\frac{1}{2}(3+k)}(m) < \infty$$

or strong mixing with rates

$$(1.4) \qquad \sum_{m=1}^{\infty} m^{2(k+2)}\alpha^{\epsilon}(m) < \infty \text{ for some } \epsilon\in(0,1/2(3+k)) .$$

Recall that the sequence $\{X_{ni}\}$ is φ–mixing if $\sup\{|P(A|B)-P(A)| : B\in\sigma(X_{ni},1\leq i\leq j),\ A\in\sigma(X_{ni},\ i\geq j+m)\} = \varphi(m)\downarrow 0$ for positive integers j and m, and it is strong mixing if $\sup\{|P(A\cap B)-P(A)P(B)|,\ A\in\sigma(X_{ni},1\leq i\leq j),\ B\in\sigma(X_{ni},i\geq j+m)\} = \alpha(m)\downarrow 0$ for positive integers j and m. Here $\sigma(X_{ni},1\leq i\leq j)$ and $\sigma(X_{ni},i\geq j+m)$ are the σ–fields generated by $(X_{n1},\dots,X_{nj})$ and $(X_{n,j+m},X_{n,j+m+1},\dots,X_{nn})$ respectively.

The asymptotic normality of the serial linear rank statistic $\mathscr{S}_n$ in (1.1) was established by Hallin, Ingenbleek and Puri (1985) for an ARMA process under the null hypothesis of white noise, and under alternatives contiguous to white noise. In this paper we show that contiguity is not necessary for the derivation of asymptotic normality, and we derive the results under φ–mixing as well as strong mixing sequences of non–stationary random variables using Rüschendorf (1976)'s approach. In passing, we may also mention that Harel and Puri (1987b) derived the weak convergence of $\mathcal{S}_n$ when both the score function and the regression constants c_{Ni} , $1 \le i \le n$, $n \ge 1$ are bounded. The present paper also extends the results of Harel and Puri (loc. cit.) to the case of unbounded score functions as well as the regression constants. In section 6, we provide some examples.

We start with some preliminaries.

Denote by $\tilde{F}_n(x)$, the right continuous empirical distribution function of X_{ni}, $i=1,...,n$; i.e. let $\tilde{F}_n(x) = n^{-1} \sum_{i=1}^{n} I_{[X_{ni} \le x]}$ where $I_{[.]}$ denotes the indicator function. Denote by G_n the distribution function of the $k+1$ of the successive random variables among $X_{n1},...,X_{nn}$. Let g_n denote the probability density function of G_n , and let f_n be the marginal density of G_n (equivalently, the density of F_n). Let $\{H_n\}$ be a sequence of measures on $[0,1]^{k+1}$ defined by

$$H_n(\underset{\sim}{t}) = G_n(F_n^{-1}(t_1),...,F_n^{-1}(t_{k+1})) \tag{1.5}$$

for all $\underset{\sim}{t} = (t_1,...,t_{k+1}) \in [0,1]^{k+1}$, and consider the rank process L_n (called the <u>graduate</u> <u>rank</u> <u>process</u>) defined by

$$L_n(t) = (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{[nt_o]} \left\{ \prod_{j=1}^{k+1} I_{[\tilde{F}_n(X_{n,i+j-k-1}) \le t_j]} - H_n(\underset{\sim}{t}) \right\} \tag{1.6}$$

for all $t = (t_o,\underset{\sim}{t}) = (t_o,t_1,...,t_{k+1}) \in [0,1]^{k+2}$, where $[nt_o]$ denotes the integral part of the real number nt_o .

With the above notations, we can rewrite $\mathcal{S}_n$ as

$$\mathcal{S}_n = (n-k)^{\frac{1}{2}} \int_{[0,1]^{k+2}} L_n(t) \frac{1}{r(t)} r(t) \, d\mu_n(t) + d_n \tag{1.7}$$

where μ_n is a signed measure on $[0,1]^{k+2}$ and where $r(.)$ is a continuous function from $[0,1]^{k+2}$ into $\mathbb{R}^+$, called a weight function, and d_n is a centering constant. (By convention $\frac{1}{r}(t) = 0$ if $r(t) = 0$). The weight function is introduced to deal with the case when the score functions $a_n(...)$ are unbounded. We shall verify the weak convergence of the process $L_n \cdot \frac{1}{r}$ with respect to the topology of the uniform convergence (i.e. uniform topology). Since the process $L_n \cdot \frac{1}{r}$ does not vanish on the upper boundary of $[0,1]^{k+2}$, we shall introduce in section 2 a new process $\hat{L}_n$ called the split graduate rank process which vanishes on the boundary of $[0,1]^{k+2}$ and which permits a representation of $\mathscr{S}_n$ in the form

$$\mathscr{S}_n = (n-k)^{\frac{1}{2}} \int_{[0,1]^{k+2}} \hat{L}_n(t) \tfrac{1}{r}(t) r(t) d\lambda_n(t) + b_n \tag{1.8}$$

where λ_n (a certain measure) and b_n (a centering constant) are defined in section 3.

The graduate empirical process W_n defined by

$$W_n(t) = (n-k)^{-\frac{1}{2}} \sum_{i=k+1}^{[nt_0]} \left\{ \prod_{j=1}^{k+1} I_{[F_n(X_{n,i+j-k-1}) \le t_j]} - H_n(\underset{\sim}{t}) \right\} \tag{1.9}$$

is a variant of the multivariate empirical process studied by Harel and Puri (1987a). (Applications of $\mathscr{S}_n$ in ARMA processes in time series as well as in Markov processes are provided in Harel and Puri (1987b)).

2. Weak convergence of the graduate empirical process, the split graduate rank process and the weighted split and graduate rank process. We start with preliminaries.

2.1. The spaces D^*_{k+2} and C^*_{k+2}. Let $f : [0,1]^{k+2} \to \mathbb{R}$. For $\rho=(\rho(0),\dots,\rho(k+1)) \in \{0,1\}^{k+2}$, define $f_\rho(t) = \lim\limits_{\substack{s_i \uparrow t_i \; \rho(i)=1 \\ s_i \downarrow t_i \; \rho(i)=0}} f(s)$ $(s,t) \in ([0,1]^{k+2})^2$ if it exists, in which case, call $f_\rho(t)$ the ρ–limit of f at t. Denote by D_{k+2} or $D_{k+2}(0)$ the space of all maps $f : [0,1]^{k+2} \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^{k+2}$, f_ρ exists and $f_\rho = f$ for $\rho=(0,\dots,0)$. Denote by $D_{k+2}(1)$, the space of all maps $f : [0,1]^{k+2} \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^{k+2}$, f_ρ exists and $f_\rho = f$ for $\rho = (1,0,\dots,0)$.

More generally, for a closed rectangle R in $[0,1]^{k+2}$, let $D_\rho(R)$ be the set of all maps $f : R \to \mathbb{R}$ such that f_{ρ^*} exists for all $\rho^* \in \{0,1\}^{k+2}$ and $f_\rho = f$.

Put $D^*_{k+2} = \{f : [0,1]^{k+2} \to \mathbb{R}$, for all ρ, restriction $f|I(\rho)$ has an extension f_ρ on $\overline{I(\rho)}$ with $f_\rho \in D_\rho(\overline{I(\rho)})\}$ where $I(\rho) = I_{\rho(0)} \times \dots \times I_{\rho(k+1)}$, $I_\ell = \begin{cases} [0,\frac{1}{2}) & \text{if } \ell=0 \\ [\frac{1}{2},1] & \text{if } \ell=1 \end{cases}$ and $\overline{A}$ denotes the closure of the set A.

In what follows, Λ denotes the space of maps $h:[0,1]\to[0,1]$ which are nondecreasing, continuous and bijective and $\Lambda^{(k+2)}$ denotes the space of maps $\lambda:[0,1]^{k+2}\to[0,1]^{k+2}$ where $\lambda(t_o,\dots,t_{k+1})=(\lambda_o(t_o),\dots,\lambda_{k+1}(t_{k+1}))$, $\lambda_j\in\Lambda$, $0\leq j\leq k+1$. For any bounded maps $f,g:[0,1]^{k+2}\to\mathbb{R}$, we denote

$$d(f,g) = \inf_{\lambda\in\Lambda^{(k+2)}} \max\{\|f-g\circ\lambda\|,\|\lambda-I_{(k+2)}\|\}$$

where $I_{(k+2)}$ denotes the identity map on $[0,1]^{k+2}$

$$\|f-g\circ\lambda\| = \sup_{t\in[0,1]^{k+2}}\{|f(t)-(g\circ\lambda)(t)|\}$$

$$\|\lambda-I_{(k+2)}\| = \sup_{t\in[0,1]^{k+2}}\{|\lambda(t)-I_{(k+2)}(t)|\} .$$

We shall call the topology associated with the metric d Skorohod topology.

We say that we have special Skorohod topology on D_{k+2} if we have uniform topology for the first coordinate and Skorohod topology for the other coordinates (for more details on the Skorohod topology cf. Balacheff and Dupont (1980)).

We define an operator $\gamma: D_{k+2}\to D^*_{k+2}$ by setting

(2.1) $$\gamma(f)(t) = \sum_{I\subset\{0,\dots,k+1\}} (-1)^{\text{card } I} f((b_i)_{i\in I},(a_i)_{i\notin I}))$$

for all $t\in I(\rho)$ where

$$\begin{cases} a_i = 0 \quad\text{and}\quad b_i = t_i & \text{if } \rho(i)=0 \\ a_i = t_i- \text{ and } b_i = 1 & \text{if } \rho(i)=1 \end{cases}$$

and card I means the cardinal of I.

Finally, denote by C_{k+2}, the space of all continuous maps $f:[0,1]^{k+2}\to\mathbb{R}$, and by $C_{k+1}(j)$, $(1\leq j\leq k+1)$, the space of all continuous and bounded maps $f: A(j)\to\mathbb{R}$ where $A(j) = [0,1]^{j-1}\times(0,1)\times[0,1]^{k+1-j}$.

Now put $C^*_{k+2} = \{f:[0,1]^{k+2}\to\mathbb{R} :$ for all ρ, $f|I(\rho)$ has a continuous extension to $\bar{I}(\rho)\}$. We define a modulus of continuity for any bounded function $f:[0,1]^{k+2}\to\mathbb{R}^+$ to be denoted by $\omega_*(f,\delta)$, $(\delta>0)$ by setting

(2.2) $$\omega_*(f,\delta) = \max_{\rho\in\{0,1\}^{k+2}} \sup_{(t,t')\in I(\rho)} |f(t)-f(t')| , \ \|t-t'\|\leq\delta .$$

Note that f belongs to C^*_{k+2} if and only if $\lim_{\delta\to 0} \omega_*(f,\delta) = 0$.

The following proposition, which is a variant of a theorem of Dudley (1978) will be used in section 2.2.a.

PROPOSITION 2.1. Let Y_n be a process with values in D^*_{k+2} and measurable with respect to $\mathscr{U}^*_{k+2}$, the σ–field generated by the uniform topology on D^*_{k+2}. Let P_n denote the law of Y_n. Then there exists a probability measure P with $P(C^*_{k+2}) = 1$ for which P_n converges weakly with respect to the uniform topology if and only if

(i) for all finite subsets U of $[0,1]^{k+2}$, $\varphi_U(P_n)$ converges weakly to $\varphi_U(P)$. ($\varphi_U(P)$ is the projection of D^*_{k+2} on $\mathbb{R}^U$),

(ii) $\lim_{\delta\to 0} \limsup_{n\to\infty} P_n[\{f;\omega_*(f,\delta)\geq\epsilon\}] = 0$, $\forall\ \epsilon>0$.

2.2.a. Convergence of the graduate empirical process.

PROPOSITION 2.2. Assume that the sequence $\{X_{ni}\}$ is (a) φ–mixing with rates (1.2), (b) stationary with rates (1.3), or (c) strong mixing with rates (1.4). Furthermore assume that (d) the covariance functions $\{(C_n;\ n\geq 1\}$ of the empirical process $\{W_n;\ n\geq 1\}$ defined in (1.9) converge to a function C. Then under (a) and (d) or (c) and (d), W_n converges weakly in the Skorohod topology to a Gaussian process W_0 with trajectories a.s. in C_{k+2}. Under (b), $W_n(1,\underset{\sim}{t})$ converges in the Skorohod topology to a Gaussian process $W_0(1,\underset{\sim}{t})$ with trajectories a.s. in C_{k+1}.

PROOF. Consider a sequence Z_{mi}, $1\leq i\leq m$, $m\geq 1$ of $\mathbb{R}^{k+1}$ valued random variables defined by $(X_{m+k,i},\ldots,X_{m+k,i+k}) = (Z^{(1)}_{m,i},\ldots,Z^{(k+1)}_{m,i})$, $1\leq i\leq m$, $m\geq 1$. Then the (k+1)–variate truncated empirical process $\tilde{W}_m$ associated with this sequence is given by

$$\text{(2.3)}\qquad \tilde{W}_m(t_0,\underset{\sim}{t}) = m^{-\frac{1}{2}} \sum_{i=1}^{[(m+k)t_0]-k} \Big[\prod_{j=1}^{k+1} I_{[F_{m+k}(Z^{(j)}_{mi})\leq t_j]} - H_{m+k}(\underset{\sim}{t})\Big]$$

and this is the same as the graduate process W_n defined in (1.9) for $m+k = n$.

For the process $\tilde{W}_m$, the proof follows from Theorem 3.1 in Harel and Puri (1987a). We derive the convergence with respect to the special Skorohod topology by using Proposition 2.1 to the first coordinate of W_n .

2.2b. Convergence of the split graduate rank process. To consider the convergence of the split graduate rank processes, we need some preliminaries.

DEFINITION. We say that the sequence $\{H_n\}$ satisfies the differentiability condition if (i) $\frac{\partial}{\partial t_j} H_n$ exists on $A(j)$ and belongs to $C_{k+1}(j)$, $1 \le j \le k+1$, and (ii) $\frac{\partial H_n}{\partial t_j} \to \ell_j$ in the uniform topology on any compact subset of $A(j)$ as $n \to \infty$ and ℓ_j belongs to $C_{k+1}(j)$.

For given $\epsilon \in \{0,1\}$, we define $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ by $\psi_\epsilon = (-1)^\epsilon I_d$ where I_d is an identity function. We also define $\psi'_\epsilon : [0,1] \to [0,1]$ by $\psi'_\epsilon = \epsilon + (-1)^\epsilon I_d$.

For any $\epsilon \in \{0,1\}$, denote by $F_{n,\epsilon}$ the distribution function of $\psi_\epsilon(X_{n1})$, and set $\tilde{F}_{n,\epsilon}(x) = n^{-1} \sum_{i=1}^{n} I_{[\psi_\epsilon(X_{ni}) \le x]}$. For any $\tilde{\rho} \in \{0,1\}^{k+1}$, we denote by $G_n^{\tilde{\rho}}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^{k+1}$, the distribution function of $(\psi_{\rho(1)}(X_{n,1}), \dots, \psi_{\rho(k+1)}(X_{n,k+1}))$ and by $H_n^{\tilde{\rho}}$ the sequence of measures on $[0,1]^{k+1}$ defined by

$$H_n^{\tilde{\rho}}(t_1, \dots, t_{k+1}) = G_n^{\tilde{\rho}}(F_{n,\rho(1)}^{-1}(t_1), \dots, F_{n,\rho(k+1)}^{-1}(t_{k+1}))$$

for any $\rho = (\rho(0), \tilde{\rho}) \in \{0,1\}^{k+2}$ the process L_n^ρ defined by

$$(2.4)\quad L_n^\rho(t) = (n-k)^{-\frac{1}{2}} \sum_{i=a}^{b} \Big\{ \prod_{j=1}^{k+1} I_{[\tilde{F}_{n,\rho(j)}(\psi_{\rho(j)})(X_{n,i+j-k-1}) \le \psi'_{\rho(j)}(t_j)]} - H_n^{\tilde{\rho}}(\psi'_{\rho(1)}(t_1), \dots, \psi'_{\rho(k+1)}(t_{k+1})) \Big\}$$

where $a = k+1$, $b = [nt_o]$ if $\rho(0) = 0$ and $a = \sup\{[nt_o]^* + 1, k+1\}$, $b = n$ if $\rho(0) = 1$, and

$$[nt_o]^* = \begin{cases} nt_o & \text{if } nt_o \text{ is an integer} \\ [nt_o]+1 & \text{otherwise.} \end{cases}$$

The split rank process $\hat{L}_n$ is defined by

$$(2.5)\quad \hat{L}_n(t) = \begin{cases} 0 & \text{if } t \notin [1/n+1, n/n+1]^{k+2} \\ L_n^\rho(t') & \text{if } t \in [1/n+1, n/n+1]^{k+2} \cap I(\rho) \end{cases}$$

where $t_j' = \frac{n+1}{n} t_j$ if $\rho(j)=0$ and $t_j' = \frac{n+1}{n} t_j - \frac{1}{n}$ if $\rho(j)=1$ then we have

PROPOSITION 2.3. If the sequence $\{X_{ni}\}$ satisfies the assumption (a) and (d) or (c) and (d) in Proposition 2.2, and furthermore if $\{H_n\}$ satisfies the differentiability condition, then $\hat{L}_n$ converges weakly in uniform topology to a Gaussian process $\hat{L}_0$ with trajectories a.s. in C^*_{k+2}.

The proof is given in section 4.

We now define the nontruncated split rank process $\tilde{L}_n$ as follows

(2.6)
$$\tilde{L}_n(\underset{\sim}{t}) = \begin{cases} 0 & \text{if } t \notin [1/n+1,\ n/n+1]^{k+1} \\ L_n^{1,\tilde{\rho}}(1,\underset{\sim}{t}') & \text{if } t \in [1/n+1,\ n/n+1]^{k+1} \cap \tilde{I}(\tilde{\rho}) \end{cases}$$

where $t_j' = \frac{n+1}{n} t_j$ if $\rho(j)=0$, $t_j' = \frac{n+1}{n} t_j - \frac{1}{n}$ if $\rho(j)=1$ and $\tilde{I}(\tilde{\rho}) = I_{\rho(1)} \times \dots \times I_{\rho(k+1)}$.

PROPOSITION 2.4. If the sequence $\{X_{ni}\}$ satisfies the assumption (b) of Proposition 2.2, and $\{H_{ni}\}$ satisfies the differentiability condition, then $\tilde{L}_n$ converges weakly in the uniform topology to a Gaussian process $\tilde{L}_0$ with trajectories a.s. in C^*_{k+1}.

The proof is an easy consequence of Proposition 2.3 because the mixing rate is not used in the proof.

2.2.c. Convergence of the weighted rank processes. First we define the split weight function.

DEFINITION. A function $r : [0,1]^{k+2} \to \mathbb{R}^+$ is called a split weight function if it satisfies the following conditions:

(i) there exists an $r_0 : [0,1] \to \mathbb{R}^+$ and an $\underset{\sim}{r} : [0,1]^{k+1} \to \mathbb{R}^+$ such that $r(t) = r_0(t_0)\underset{\sim}{r}(\underset{\sim}{t})$ for all $(t_0,\underset{\sim}{t}) \in [0,1]^{k+2}$,

(ii) r belongs to C^*_{k+2},

(iii) $r=0$ on the boundary of $[0,1]^{k+2}$.

PROPOSITION 2.5. If (i) the sequence $\{X_{ni}\}$ satisfies the assumptions of Proposition 2.3, (ii) if there exists a distribution function F such that $\sup_{x\in\mathbb{R}} |F_n(x)-F(x)| = 0(n^{-\alpha})$, $\alpha>0$, and (iii)

(2.7) $$g_n(x_1,\dots,x_{k+1}) \le A_1\left[\prod_{j=1}^{k+1} F_n(x_j)(1-F_n(x_j))\right]^{-\frac12+\delta}\prod_{j=1}^{k+1} f_n(x_j)$$

where $A_1>0$, then, for any split weight function r satisfying

(2.8) $$r(t) \ge A_2\left[\prod_{j\in J(\rho)} t_j \times \prod_{j\notin J(\rho)} (1-t_j)\right]^{\frac12-\delta} \quad \text{for all } t\in I(\rho)$$

where $A_2>0$, $0 < \frac12-\delta < \frac{1}{4(k+2)}$, and $J(\rho) = \{j\in\{0,\dots,k+1\}:\rho(j)=0\}$. $\hat{L}_n\cdot\frac{1}{r}$ converges weakly in uniform topology to the Gaussian process $\hat{L}_o\cdot\frac{1}{r}$ with trajectories a.s. in C^*_{k+2}.

The proof is given in section 4.

PROPOSITION 2.6. If the sequence $\{X_{ni}\}$ satisfies the assumptions of Proposition 2.4 and (2.7), then for any split weight function $\underset{\sim}{r} : [0,1]^{k+1}\to\mathbb{R}^+$ satisfying

(2.9) $$\underset{\sim}{r}(\underset{\sim}{t}) \ge \tilde{A}\left[\prod_{j\in \tilde{J}(\tilde{\rho})} t_j \times \prod_{j\notin \tilde{J}(\tilde{\rho})} (1-t_j)\right]^{\frac12-\delta}$$

for all $\underset{\sim}{t}\in\tilde{I}(\tilde{\rho})$ where $\tilde{A}>0$, $0 < \frac12-\delta < \frac{1}{4(k+2)}$, and $\tilde{J}(\tilde{\rho}) = \{j\in\{1,\dots,k+1\};\rho(j)=0\}$, $\tilde{L}_n\cdot\frac{1}{\underset{\sim}{r}}$ converges weakly in uniform topology to the Gaussian process $\tilde{L}_o\cdot\frac{1}{\underset{\sim}{r}}$ with trajectories a.s. in C^*_{k+1}.

The proof follows also by Proposition 2.5.

3. Asymptotic normality of linear serial rank statistic $\mathscr{S}_n$. For any n, we define a signed measure λ_n concentrated on $\{1/n+1,\dots,n/n+1\}^{k+2}$ by setting

$$\lambda_n\Big(\prod_{j\in J(\rho)} \big[\tfrac{\ell_j}{n+1},\tfrac12\big) \prod_{j\notin J(\rho)} \big[\tfrac12,\tfrac{\ell_j}{n+1}\big]\Big) = c_{n\ell_o}\, a_n(\ell_1,\dots,\ell_{k+1})$$

for all $(\frac{\ell_o}{n+1},\dots,\frac{\ell_{k+1}}{n+1}) \in I(\rho)\cap\{\frac{1}{n+1},\dots,\frac{n}{n+1}\}^{k+2}$ (by convention $c_{n\ell_o}=0$ if $\ell_o<k+1$).

We also define a centering coefficient b_n by

(3.1) $$b_n = \sum_{\rho\in\{0,1\}^{k+1}} \int_{I(\rho)} H_n^{\rho}(t)\lambda_n(dt)$$

where H_n^{ρ} is a function: $[0,1]^{k+2}\to\mathbb{R}$ such that

(3.2) $$H_n^{\rho}(t) = (b-a+1)(H_n^{\tilde{\rho}}(\psi'_{\rho(1)}(t'_1),\dots,\psi'_{\rho(k+1)}(t'_{k+1})))$$

$a=k+1$, $b=[(n+1)t_o]$ if $\rho(0)=0$; $a=[(n+1)t_o-1]^*+1$, $b=n$ if $\rho(0)=1$, $t'_j = \frac{n+1}{n}t_j$ if $\rho(j)=0$, $t'_j = \frac{n+1}{n}t_j - \frac{1}{n}$ if $\rho(j)=1$, $j\ge1$.

In the sequel, we will use another measure, called a pseudo measure.

DEFINITION. A measure λ on $[0,1]^{k+2}$ is called a pseudo measure of order $I\subset\{0,1,\ldots,k+1\}$ if for any $f\in C^*_{k+2}$,

(3.3)
$$\int_{[0,1]^{k+2}} f d\lambda = \int_{\{(t_0,\ldots,t_{k+1});t_i=\frac{1}{2}-\forall i\in I\}} f(t)\lambda(dt) .$$

λ will be called a general measure if it is the finite sum of pseudo measures.

THEOREM 3.1. Let r be a split weight function such that for some general measure λ_0 we have

(3.4)
$$\lim \int f.r\, d\lambda_n = \int f.r\, d\lambda_0 \quad \text{for all } f\in C^*_{k+2}$$

(3.5)
$$\sup_{n\in\mathbb{N}} \int f.r\, d|\lambda_n| < \infty \; ; \mathbb{N} = \{0,1,2,\ldots\}$$

where $|\lambda_n|$ denotes the measure of total variation.

If the sequence $\{X_{ni}\}$ and r satisfy the assumptions of Proposition 2.5, then $(n-k)^{-\frac{1}{2}}(\mathscr{S}_n-b_n)$ converges in law to the normal distribution with mean 0 and variance σ^2 where

(3.6)
$$\sigma^2 = \int_{[0,1]^{k+2}} \cdots \int_{[0,1]^{k+2}} E[\hat{L}_0(t)\hat{L}_0(t')]d\lambda_0(t)d\lambda_0(t') < \infty .$$

REMARK. The above theorem is proved under the assumption that the sequence $\{X_{ni}\}$ is nonstationary and either φ–mixing with rates (1.2) or strong mixing with rates (1.4). The theorem does not hold with the φ–mixing rates (1.3) unless one assumes stationarity and the special case when $c_{ni}=1$ for all i .

Let $\tilde{\mathscr{S}}_n$ denote the statistics $\mathscr{S}_n$ when $c_{ni}=1$ for all i , i.e. let

(3.7)
$$\tilde{\mathscr{S}}_n = \sum_{i=k+1}^{n} a_n(R_{n,i-k},\ldots,R_{n,i})$$

and let $\tilde{b}_n$ denote the corresponding centering constant i.e.

(3.8)
$$\tilde{b}_n = \sum_{\tilde{\rho}\in\{0,1\}^{k+1}} \int_{\tilde{I}(\tilde{\rho})} \tilde{H}^{\tilde{\rho}}_n(\underset{\sim}{t})\tilde{\lambda}_n(d\underset{\sim}{t})$$

where $\tilde{H}^{\tilde{\rho}}_n(\underset{\sim}{t}) = (n-k)H^{\tilde{\rho}}_n(\psi'_{\rho(1)}(t'_1),\ldots,\psi'_{\rho(k+1)}(t'_{k+1}))$, $t'_j = (\frac{n+1}{n})\, t_j$ if $\rho(j)=0$,

$t'_j = (\frac{n+1}{n})t_j - \frac{1}{n}$ if $\rho(j)=1$ and

$$\tilde{\lambda}_n\left[\prod_{j\in\tilde{J}(\tilde{\rho})}[\tfrac{\ell_j}{n+1},\tfrac{1}{2})\prod_{j\notin\tilde{J}(\tilde{\rho})}[\tfrac{1}{2},\tfrac{\ell_j}{n+1}]\right] = a_n(\ell_1,\dots,\ell_{k+1}) ,$$

then we have the following proposition.

THEOREM 3.2. *Let* $\underset{\sim}{r}$ *be a split weight function such that for some general measure* $\tilde{\lambda}_o$ *on* $[0,1]^{k+1}$ *we have*

$$\lim \int \tilde{f}\underset{\sim}{r}d\tilde{\lambda}_n = \int \tilde{f}\underset{\sim}{r}d\tilde{\lambda}_o \tag{3.9}$$

$$\sup_{n\in\mathbb{N}} \int \tilde{f}\underset{\sim}{r}d|\tilde{\lambda}_n| < \infty . \tag{3.10}$$

If the sequence $\{X_{ni}\}$ *and* $\underset{\sim}{r}$ *satisfy the assumptions of Proposition* 2.6, *then* $(n-k)^{-\frac{1}{2}}(\tilde{\mathscr{S}}_n-\tilde{b}_n)$ *converges to the normal distribution with mean* 0 *and variance* $\tilde{\sigma}^2$ *where*

$$\tilde{\sigma}^2 = \int_{[0,1]^{k+1}} \cdots \int_{[0,1]^{k+1}} E[\tilde{L}_o(\underset{\sim}{t})\tilde{L}_o(\underset{\sim}{t}')]d\tilde{\lambda}_o(d\underset{\sim}{t})d\tilde{\lambda}_o(\underset{\sim}{t}') < \infty . \tag{3.11}$$

The proof is similar to that of Theorem 3.1 and is therefore omitted.

Let $\hat{I}(\rho) = \prod_{j=0}^{k+1} \hat{I}_{\rho(j)}$ where $\hat{I}_\ell = \begin{cases}(0,\frac{1}{2}] & \text{if } \ell=0\\ [\frac{1}{2},1) & \text{if } \ell=1\end{cases}$

The following corollary gives sufficient conditions under which the conditions (3.4) and (3.5) are satisfied.

COROLLARY 3.1. *Let* J *be a function on* $[0,1]^{k+2}$ *such that* $J(\frac{\ell_o}{n+1},\dots,\frac{\ell_{k+1}}{n+1}) = c_{n\ell_o}a_n(\ell_1,\dots,\ell_{k+1})$ *for all* $(\ell_o,\dots,\ell_{k+1})\in\{1,\dots,n\}^{k+2}$, $J = J_d+J_c$ *where* J_d *is a step function taking only a finite number of jumps, and where for any* $I\subset\{0,\dots,k+1\}$, J_c *has a continuous derivate* $\frac{\partial^I J_c}{(\partial t_j)_{j\in I}}$ *which admits a continuous extension on* $\hat{I}(\rho)$ *and satisfies (on* $\hat{I}(\rho)$*)*

$$\left|\frac{\partial^I J_c}{(\partial t_j)}\right|_{j\in I} \le A\left[\prod_{j\in I\cap J(\rho)} t_j \prod_{j\in I\cap J^c(\rho)}(1-t_j)\right]^{-\frac{3}{2}+\delta'}\left[\prod_{j\in I^c\cap J(\rho)} t_j \prod_{j\in I^c\cap J^c(\rho)}(1-t_j)\right]^{-\frac{1}{2}+\delta'} \tag{3.12}$$

for all $t\in\hat{I}(\rho)$ *where* $A>0$, c *denotes the complement and* $0 < \frac{1}{2}-\delta' < \frac{1}{4(k+2)}$, *then the conditions* (3.4) *and* (3.5) *are satisfied for any split weight function* r *which satisfies* (2.7) *with* $\delta<\delta'$.

PROOF. It suffices to prove the above corollary in the case when J_d has only one jump, say at $a=(a_0,\dots,a_{k+1})\in[0,\frac{1}{2})^{k+2}$.

Let λ_n' and λ_n'' be measures on $[0,1]^{k+2}$ defined by

$$\lambda_n'\left(\prod_{j\in J(\rho)}\left[\frac{\ell_j}{n+1},\frac{1}{2}\right)\prod_{j\notin J(\rho)}\left[\frac{1}{2},\frac{\ell_j}{n+1}\right]\right)=J_c\left(\frac{\ell_0}{n+1},\dots,\frac{\ell_{k+1}}{n+1}\right)$$

and

$$\lambda_n''\left(\prod_{j\in J(\rho)}\left[\frac{\ell_j}{n+1},\frac{1}{2}\right)\prod_{j\notin J(\rho)}\left[\frac{1}{2},\frac{\ell_j}{n+1}\right]\right)=J_d\left(\frac{\ell_0}{n+1},\dots,\frac{\ell_{k+1}}{n+1}\right)$$

for all $\left(\frac{\ell_0}{n+1},\dots,\frac{\ell_{k+1}}{n+1}\right)\in I(\rho)\cap\left\{\frac{1}{n+1},\dots,\frac{n}{n+1}\right\}^{k+2}$.

It is easy to check that

$$\lim_{n\to\infty}\int_{I(\rho)} f.r.d\lambda_n' = \sum_{I\subset\{0,\dots,k+1\}}\int_{[0,\frac{1}{2})^i}\left(f.r\frac{\partial^I J_c}{(\partial t_j)_{j\in I}}\right)\left((t_j)_{j\in I},\left(\tfrac{1}{2}-\right)^{k+2-i}\right)(dt_j)_{j\in I}$$

for all $f\in C_{k+2}^*$ where $\rho=(0,\dots,0)$ and $i=\text{card } I$ and similar limits exists if $\rho\neq(0,\dots,0)$.

Thus we obtain a general measure λ_0' satisfying

$$\lim_{n\to\infty}\int_{[0,1]^{k+2}} f.rd\lambda_n'(t)=\int_{[0,1]^{k+2}} f.rd\lambda_0'(t)\,.$$

Analogously, we obtain

$$\lim_{n\to\infty}\int_{[0,1]^{k+2}} f.rd\lambda_n''(t) = (f.r)(a)\sum_{I\subset\{0,\dots,k+1\}}(-1)^i J_d\left((a_i-)_{i\in i},(a_i+)_{i\notin I}\right)$$

for all $f\in C_{k+2}^*$ where $i=\text{card } I$.

4. Proof of Propositions 2.3 to 2.6.

PROOF OF PROPOSITION 2.3. The main line of proof is as follows: We consider a map $G_n;\mathscr{V}_n\to D_{k+2}^*$ where $\mathscr{V}_n$ is a subset of D_{k+2} and is such that $\hat{L}_n=G_n\circ W_n$, $n\geq1$. We show that $G_n:(\mathscr{V}_n,d)\to(D_{k+2}^*,d^*)$ is a continuous map where d is the special Skorohod metric and d^* is the uniform metric.

Let $\mathscr{V}$ be the subset of D_{k+2} such that for any $v\in\mathscr{V}$, v equals zero on the lower boundary of $[0,1]^{k+2}$. It will be noted that $\mathscr{V}_n\subset\mathscr{V}$ for any $n\geq1$.

Let $G:\mathscr{V}\to D_{k+2}^*$ be a map defined by

(4.1) $$G(v)(t) = \gamma(v)(t)-\psi'_{\rho(0)}(t_o)\sum_{j=1}^{k+1}\gamma(v)(1,\dots,t_j,\dots,1)\times \ell_j^{\tilde\rho}(\psi'_{\rho(1)}(t_1),\dots,\psi'_{\rho(k+1)}(t_{k+1}))$$

for any $\rho = (\rho(0),\tilde\rho)\in\{0,1\}^{k+2}$ and $t\in I(\rho)$ where $\ell_j^{\tilde\rho}$ is the limit of $\frac{\partial}{\partial t_j} H_n^{\tilde\rho}$ as $n\to\infty$.

We will show that $\forall\, (v_n)_{n\in\mathbb{N}^*} \in \prod_{n\in\mathbb{N}^*}\mathscr{V}_n$ and $\forall\, v \in \mathscr{V}\cap C_{k+2}$, $v_n \overset{d}{\to} v \Rightarrow G_n(v_n) \overset{d^*}{\Longrightarrow} G(v)$ as $n\to\infty$.

Now using Lemma 3 of Balacheff and Dupont (1983), we get $\hat{L}_n$ converges in law to $G(Q)$.

We now give a formal proof.

Let $\mathscr{Y}_n = \{y\in[0,1]^n;(y^{(1)},\dots,y^{(n)})$ are distinct points of $(0,1)\}$. For $\forall\, t\in\{0,1\}$, we define $Y_n^\epsilon : [0,1]^n\to D_{k+2}(\epsilon)$ by setting

$$Y_n^\epsilon(y)(t) = (n-k)^{-\frac{1}{2}} \sum_{i=a}^{b}\Big[\prod_{j=1}^{k+1} I_{[y^{(i+j-k-1)}\le t_j]} - H_n(\underset{\sim}{t})\Big]$$ where $a=k+1$, $b=[nt_o]$ if $\epsilon=0$ and $a=\sup\{[nt_o]^*+1,k+1\}$, $b=n$ if $\epsilon=1$ for all $y = (y^{(1)},\dots,y^{(n)})\in\mathscr{Y}_n$, $t = (t_o,\underset{\sim}{t})\in[0,1]^{k+2}$.

We define the space $\mathscr{V}_n$ by $\mathscr{V}_n = Y_n^o(\mathscr{Y}_n)$. For any $j\in\{1,\dots,k+1\}$ and $\epsilon\in\{0,1\}$ we define an operator $\tau_j^\epsilon : \mathscr{V}_n\to D_1$ as follows.

Let $y_{(1)}<\dots<y_{(n)}$ be the order values of $(y^{(1)},\dots,y^{(n)})$ (by convention, $y_{(0)}=0$, $y_{(n+1)}=1$), and let $v_n^\epsilon = Y_n^\epsilon(y)$. Then

(4.2) $$\tau_j^o(v_n^o)(t_j) = \begin{cases} y^{(\ell)} \text{ where } y^{(\ell)}=\max\{y^{(m)};m\in\{j,\dots,j+n-k-1\}\} & \text{if } t_j=1 \\ y^{(q)} \text{ where } y^{(q)}=\max\{y^{(m)}\le y_{(i)};m\in\{0,j,\dots,j+n-k-1\}\} & \text{if } t_j\in[i/n,i+1/n) \text{ where } i\in\{0,1,\dots,n-1\} \end{cases}$$

(4.3) $\tau_j^1(v_n^o)(t_j) =$

$$\begin{cases} 1-y^{(\ell)} \text{ where } 1-y^{(\ell)}=\max\{1-y^{(m)};m\in\{j,\ldots,j+n-k-1\}\} & \text{if } t_j=1 \\ 1-y^{(q)} \text{ where } 1-y^{(q)}=\max\{1-y^{(m)}\le 1-y_{(n+1-i)};m\in\{0,j,\ldots,j+n-k-1\}\} & \text{if } t_j\in[i/n,i+1/n)\ ,\ i\in\{0,1,..,n-1\} \end{cases}$$

Now the map $G_n : \mathscr{V}_n \to D^*_{k+2}$ is given by

(4.4) $$G_n(v_n^o)(t) = v_n^{\rho(0)}(t'_o,\tau_1^{\rho(1)}(v_n^o)(\psi'_{\rho(1)}(t'_1)),\ldots,\tau_{k+1}^{\rho(k+1)}(v_n^o)(\psi'_{\rho(k+1)}(t'_{k+1})) + (n-k)^{-\frac{1}{2}}(b-a+1)[H_n^{\tilde\rho}(\tau_1^{\rho(1)}(v_n^o)(\psi'_{\rho(1)}(t'_1)),\ldots,\tau_{k+1}^{\rho(k+1)}(v_n^o)(\psi'_{\rho(k+1)}(t'_{k+1})) - H_n^{\tilde\rho}(\psi'_{\rho(1)}(t'_1),\ldots,\psi'_{\rho(k+1)}(t'_{k+1}))]$$

where $a=k+1$, $b=[nt'_o]$ if $\rho(0)=0$; $a=[nt'_o]^*+1$, $b=n$ if $\rho(0)=1$; $t'_j = \frac{n+1}{n}t_j$ if $\rho(j)=0$, $0\le j\le k+1$, $t'_j = \frac{n+1}{n}t_j - \frac{1}{n}$ if $\rho(j)=1$, $0\le j\le k+1$ for all $\rho = (\rho(0),\tilde\rho)\in\{0,1\}^{k+2}$ and $t\in I(\rho)$.

The first thing we have to show is that G_n is continuous for every n . Let $\{v_{n\ell}^o\}$, $n\ge1$, $\ell\ge1$ be a sequence of functions in $\mathscr{V}_n$ and let $v_{n\ell}^o \to v_n^o$ $(\in\mathscr{V}_n)$ with respect to the special Skorohod topology. We show that $G_n(v_{n\ell}^o)\to G_n(v_n^o)$ in uniform topology.

From the definition of the special Skorohod topology we have a sequence $\{\lambda_{j\ell}\}_{1\le j\le k+1,\ell\ge1} \in \Lambda^{(k+1)}$ such that $\forall\ \epsilon>0$, $\exists\ \ell_\epsilon\in\mathbb{N}$ such that $\max|\lambda_{j\ell}(t_j)-t_j| \le \epsilon$ and

$$|v_{n\ell}^o(t)-v_n^o(t_o,\lambda_{1\ell}(t_1),\ldots,\lambda_{k+1,\ell}(t_{k+1}))| < \epsilon\ \ \forall\ \ell\ge\ell_\epsilon$$

and $\forall\ t\in[0,1]^{k+2}$.

Let $\lambda'_{j\ell}$ be the bijection on $[0,1]$ defined by $\lambda'_{j\ell}(t_j) = 1-\lambda_{j\ell}(1-t_j)$, $1\le j\le k+1$, $\ell\in\mathbb{N}^*$ then we have

LEMMA 4.1. $\exists\ \ell_o>0$ <u>such that</u> $\forall\ \ell\ge\ell_o$, $\forall\ j\in\{1,\ldots,k+1\}$ <u>and</u> $\forall\ t_j\in[0,1]$

(a) $\lambda_{j\ell}(\tau_j^o(v_{n\ell}^o)(t_j) = \tau_j^o(v_n^o(t_j)$

(b) $\lambda'_{j\ell}(\tau_j^1(v_{n\ell}^o)(t_j) = \tau_j^1(v_n^o(t_j)$

PROOF. It was proved in Harel and Puri (1987b) for (a), it is similar for (b) and the proof is therefore omitted.

We now decompose G_n defined in (4.4) as $G_n = \gamma_n+\delta_n$ where $\gamma_n(v_n(t)) =$ $= v_n^{\rho(0)}(t'_o,\tau_1^{\rho(1)}(v_n^o)(\psi'_{\rho(1)}(t'_1)),\ldots,\tau_{k+1}^{\rho(k+1)}(v_n^o)(\psi'_{\rho(k+1)}(t'_{k+1}))$ and $\delta_n = G_n-\gamma_n$.

LEMMA 4.2. (a) $\gamma_n : (\mathcal{V}_n,d)\to(D^*_{k+2},d^*)$ <u>is continuous</u>

(b) $\delta_n : (\mathcal{V}_n,d)\to(D^*_{k+2},d^*)$ <u>is continuous</u>.

PROOF. It was established in Harel and Puri (1987b) for $t\in I(\rho)$ for $\rho=(0,\dots,0)$, the proof is similar for $t\in I(\rho)$ with $\rho\neq(0,\dots,0)$ and therefore omitted.

We now prove the convergence of the sequence $\{G_n\}$. Let $v_n\in\mathcal{V}_n$, $n\in\mathbb{N}^*$ and suppose that $v_n \xrightarrow{d^*} v\in c_{k+2}$ and $v=0$ on the lower boundary of $[0,1]^{k+2}$. We have to prove that $G_n(v_n)\to G(v)$. The proof is based on the following lemmas.

LEMMA 4.3. $\forall\, j\in\{1,\dots,k+1\}$

(a) $\tau^0_j(v^0_n)\to id_{[0,1]}$ <u>in uniform topology</u>

(b) $\tau^1_j(v^0_n)\to 1-id_{[0,1]}$ <u>in uniform topology</u>

(c) $(n-k)^{\frac{1}{2}}(\tau^0_j(v_n)-id_{[0,1]})\to -v(1,\dots,id_{[0,1]},\dots,1)$

(d) $(n-k)^{\frac{1}{2}}(\tau^1_j(v_n)-(1-id_{[0,1]}))\to v(1,\dots,id_{[0,1]},\dots,1)-v(1,\dots,1)$ <u>where</u> $id_{[0,1]}$ <u>is an identity function</u>

PROOF. It was established in Harel and Puri (1987b) for (a) and (c). The proof for (b) and (d) is similar and therefore omitted.

LEMMA 4.4.

(a) $\gamma_n(v_n)\to\gamma(v)$ <u>in uniform topology</u>

(b) $\delta_n(v_n)\to-\sum_{j=1}^{k+1}\psi'_{\rho(0)}(\cdot)\gamma(v)(1,\dots,\cdot,\dots,1)\times\ell^{\tilde{\rho}}_j(\psi'_{\rho(1)}(\cdot),\dots,\psi'_{\rho(k+1)}(\cdot))=\delta(v)$ <u>in uniform topology on the restriction to</u> $I(\rho)$ <u>for any</u> $\rho\in\{0,1\}^{k+2}$.

PROOF. It was established in Harel and Puri (1987b) for the particular case $\rho=(0,\dots,0)$ and the proof is similar for $\rho\neq(0,\dots,0)$ by using the following fact

$$\gamma_n(v_n)(t)=\begin{cases}\gamma(v_n)\left(\frac{n+1}{n}t\right) & \text{if } t\le\frac{n}{n+1}\\ 0 & \text{otherwise}\end{cases}$$

and is therefore omitted.

Now since $G_n(v_n)=\gamma_n(v_n)+\delta_n(v_n)$, we obtain (using Lemma 4.4) that $G_n(v_n)\to G(v)=\gamma(v)+\delta(v)$. The proof of Proposition 2.3 follows.

PROOF OF PROPOSITION 2.5. We first give three lemmas.

LEMMA 4.5. Assume that the sequence $\{X_{ni}\}$ satisfies the assumptions (a) or (c) of Proposition 2.2, then for any weight function r satisfying

$$r(t) \geq A[\prod_{j=0}^{k+1} t_j]^{\frac{1}{2}-\delta} \quad t \in [0,1]^{k+2}, \quad A>0 \tag{4.5}$$

and $0 < \frac{1}{2}-\delta < \frac{1}{4(k+2)}$, $\forall\, \epsilon>0$, $\exists\, \theta>0$, $\exists\, N_0 \geq 1$ such that $\forall\, n \geq N_0$

$$P_n[\sup_{t \in C_\theta} |W_n(t) \cdot \frac{1}{r(t)}| > \epsilon] < \epsilon$$

where $C_\theta = \{t \in [0,1]^{k+2};\ \exists$ at least one $j \in \{0,\ldots,k+1\}$ such that $t_j \leq \theta\}$.

PROOF. The proof of this lemma is given in Harel and Puri (1987a) noting that W_n is the same as $\tilde{W}_m$ defined in (2.3).

LEMMA 4.6. Let $\{Y_{ni}; 1 \leq i \leq n, n \geq 1\}$ be real valued random variables with continuous distribution functions F_n, $n \geq 1$. Assume that the Y_{ni}'s are φ-mixing with rates $\sum_{m \geq 1} m^{-1} \varphi^{\frac{1}{2}}(m) < \infty$ or strong mixing with rates (1.4).

Furthermore, assume that there exists $\alpha_0 > 0$ and a continuous distribution F such that

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq A(n^{-\alpha_0}) \quad A>0,\ n \geq N_0 \in \mathbb{N} \tag{4.6}$$

then $\forall (\alpha,\alpha')$ where $0<\alpha<\alpha' \leq (\alpha_0 \wedge 1/3)$ $\exists\, K>0$ and $N(\alpha,\alpha')$ such that

$$P_n[\sup_{x \in \mathbb{R}} |F_n \circ \tilde{F}_n^{-1}(x) - x| \geq K(n^{-\alpha})] \leq K(n^{-1+2\alpha+\alpha'}) \tag{4.7}$$

$\forall\, n \geq N(\alpha,\alpha')$ where $\tilde{F}_n$ is the right continuous empirical distribution function of $(Y_{n1},\ldots,Y_{nn})$ and $\tilde{F}_n^{-1}$ is the inverse function defined as

$$\tilde{F}_n^{-1}(s) = \begin{cases} -\infty & \text{if } s \in [0,1/n) \\ Y_{(i)} & \text{if } s \in [i/n, i+1/n) \\ Y_{(n)} & \text{if } s=1 \end{cases}$$

where $Y_{(1)} < \ldots < Y_{(n)}$ is the ordered sequence $Y_{n1},\ldots,Y_{nn}$.

PROOF. See Appendix.

LEMMA 4.7. Assume that the random variables $\{Y_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfy the assumptions of Lemma 4.2. Then $\forall\ \epsilon>0$ and $\forall\ \tau(0<\tau<1/2)\ \exists$ a $\beta>2$ such that

(4.8) $$P_n[F_n\circ\tilde{F}_n^{-1}(s) \le \beta s^{1-\tau}, s\ge n^{-1}] > 1-\epsilon \quad \forall\ n\ge N_0\in\mathbb{N}^* .$$

PROOF. See Appendix.

Proposition 2.5 will be proved if we establish the following lemma.

LEMMA 4.8. Assume that the sequence $\{X_{ni}\}$ satisfies the assumptions of Proposition 2.5, then for any weight $r(t)$ satisfying (2.8) we have $\forall\ \epsilon>0$, $\exists\ \theta>0$, $\exists\ N_0\ge 1$ such that $\forall\ n\ge N_0$

(4.9) $$P_n[\sup_{t\in C_\theta} |\hat{L}_n(t)\cdot\frac{1}{r(t)}|>\epsilon] < \epsilon$$

where $C_\theta^* = \{t\in[0,1]^{k+2}$; $\exists$ at least one $j\in\{0,\dots,k+1\}$ such that $t_j\le\theta$ or $1-t_j\le\theta\}$.

PROOF. It is sufficient to prove (4.9) for $L_n^\rho|I(\rho)$ with $\rho=(0,\dots,0)$, the case for $\rho\ne(0,\dots,0)$ will be deduced by symmetrization, and for $\hat{L}_n$ follows immediately. We now suppose $\rho=(0,\dots,0)$. We have $L_n(t) = \hat{W}_n(t)+Z_n(t)$ where

(4.10) $$\hat{W}_n(t) = \begin{cases} W_n(t_0,F_n\circ\tilde{F}_n^{-1}(t_1),\dots,F_n\circ\tilde{F}_n^{-1}(t_{k+1})) & \text{for } t\in[1/n,n-1/n]^{k+2} \\ 0 & \text{otherwise.}\end{cases}$$

(4.11) $$Z_n(t) = \begin{cases} (n-k)^{-\frac{1}{2}}([nt_0]-k)(H_n(F_n\circ\tilde{F}_n^{-1}(t_1),\dots,F_n\circ\tilde{F}_n^{-1}(t_{k+1}))-H_n(\underset{\sim}{t})) & \text{for } t\in[1/n,n-1/n]^{k+2} \\ 0 & \text{otherwise.}\end{cases}$$

Now we show (4.9) for $\hat{W}_n$ and Z_n on $I(\rho)$.

Let δ,δ' be such that

$$r(t) \ge A[\prod_{j=0}^{k+1} t_j]^{\frac{1}{2}-\delta}, \quad t\in[0,1]^{k+2}$$

and such that $1 < \frac{1}{2}-\delta < \frac{1}{2}-\delta' < \frac{1}{4(k+2)}$ and define r' to be a function by setting

$$r'(t) \ge A[\prod_{j=0}^{k+1} t_j]^{\frac{1}{2}-\delta'}, \quad t\in[0,1]^{k+2} .$$

Using Lemma 4.5, we have $\forall\ \epsilon>0$, $\exists\ \theta^{(1)}>0$, $\exists\ N_0^{(1)}\in\mathbb{N}^*$, $\forall\ n\ge N_0$

$$P_n[\sup_{t\in C^*_{\theta^{(1)}}\cap I(\rho)} |W_n(t)\cdot\frac{1}{r'(t)}|>\epsilon] < \epsilon .$$

From Lemmas 4.6 and 4.7 and the preceding inequality we deduce that $\forall\ \epsilon>0$, $\exists\ \theta^{(2)}>0$,

$\exists\ N_o^{(2)} \in \mathbb{N}^*$, $\forall\ n \geq N_o^{(2)}$

$$P[\sup_{t\in C^*_{\theta^{(1)}}\cap I(\rho)} |\hat{W}_n(t) \cdot \frac{1}{r'(F_n\circ\tilde{F}_n^{-1}(t_1),\ldots,F_n\circ\tilde{F}_n^{-1}(t_{k+1})}|>\epsilon] < \epsilon/2$$

and $\forall\ \epsilon>0$, $\exists\ \theta^{(3)}>0$, $\exists\ N_o^{(3)}\in\mathbb{N}$, $\forall\ n>N_o^{(3)}$

$$P[\sup_{t\in C^*_{\theta^{(3)}}\cap I(\rho)} \frac{r'(F_n\circ\tilde{F}_n^{-1}(t_1),\ldots,F_n\circ\tilde{F}_n^{-1}(t_{k+1})}{r(t)} > 1] < \frac{\epsilon}{2}$$

(take τ with $1-\tau > \frac{\frac{1}{2}-\delta}{\frac{1}{2}-\delta'}$) setting $\theta=\theta^{(2)}\wedge\theta^{(3)}$ and $N_o=N_o^{(2)}\vee N_o^{(3)}$ the condition (4.9) follows for $\hat{W}_n$ on $I(\rho)$.

It remains to show (4.9) for Z_n .

We define r' as before and note that Z_n can be written as

$$Z_n(t) = (n-k)^{-\frac{1}{2}}([nt_o]-k) \sum_{j=1}^{k+1} H_n(a^{(j)}-H_n(b^{(j)})$$

where

$a^{(j)} = (a_1^{(j)},\ldots,a_{k+1}^{(j)})$, $\qquad b^{(j)} = (b_1^{(j)},\ldots,b_{k+1}^{(j)})$

$a_\ell^{(j)} = t_\ell$ if $\ell>j$, $\qquad b_\ell^{(j)} = t_\ell$ if $\ell\geq j$

$a_\ell^{(j)} = F_n\circ\tilde{F}_n^{-1}(t_\ell)$ if $\ell\leq j$, $\qquad b_\ell^{(j)} = F_n\circ\tilde{F}_n^{-1}(t_\ell)$ if $\ell<j$.

We can also write

$$Z_n(t) = (n-k)^{-\frac{1}{2}}([nt_o]-k) \sum_{j=1}^{k+1} (F_n\circ\tilde{F}_n^{-1}(t_j)-t_j)\frac{\partial H_n}{\partial t_j}(F_n\circ\tilde{F}_n^{-1}(t_1),\ldots$$
$$\ldots,F_n\circ\tilde{F}_n^{-1}(t_{j-1}),\mu_{j,n},t_{j+1},\ldots,t_{k+1})$$

where $\mu_{jn} \in [\min\{F_n\circ\tilde{F}_n^{-1}(t_j),t_j\}, \max\{F_n\circ\tilde{F}_n^{-1}(t_j),t_j\}]$ and we remark that

$$F_n\circ\tilde{F}_n^{-1}(t_j)-t_j = (n-k)^{-\frac{1}{2}}W_n(1,\ldots,1,F_n\circ\tilde{F}_n^{-1}(t_j),1,\ldots,1)+o(n^{-1}) .$$

We deduce that

$$\left|\frac{Z_n(t)}{r(t)}\right| \le A\frac{([nt_o]-k)}{(n-k)t_o^{\frac12-\delta}}\sum_{j=1}^{k+1}\left|\frac{W_n(1,\dots,1,F_n\circ\tilde F_n^{-1}(t_j),1,\dots,1)+o(n^{-\frac12})}{t_j^{\frac12-\delta}}\right|$$

$$\frac{\partial H_n}{\partial t_j}(F_n\circ\tilde F_n^{-1}(t_1),\dots,F_n\circ\tilde F_n^{-1}(t_{j-1}),\mu_{jn},t_{j+1},\dots,t_{k+1})\times\frac{1}{(\prod_{\substack{\ell\ne j\\ \ell\ne 0}}t_\ell)^{\frac12-\delta}}$$

We have

$$\frac{\partial H_n}{\partial t_j}(F_n\circ\tilde F_n^{-1}(t_1),\dots,F_n\circ\tilde F_n^{-1}(t_{j-1}),\mu_{jn},t_{j+1},\dots,t_{k+1})$$

$$=\int_0^{F_n\circ\tilde F_n^{-1}(t_1)}\cdots\int_0^{F_n\circ\tilde F_n^{-1}(t_{j-1})}\int_0^{t_{j+1}}\cdots\int_0^{t_{k+1}}\frac{\partial^{k+1}H_n}{(\partial t_\ell)_{1\le\ell\le k+1}}(u_1,\dots,u_{j-1},\mu_{jn},u_{j+1},\dots,u_{k+1})$$

$$du_1\dots du_{j-1},du_{j+1}\dots du_{k+1}\,.$$

From condition (2.7), we deduce that

$$\frac{\partial H_n}{\partial t_j}(F_n\circ\tilde F_n^{-1}(t_1),\dots,F_n\circ\tilde F_n^{-1}(t_{j-1}),\mu_{jn},t_{j+1},\dots,t_{k+1})\frac{1}{(\prod_{\ell\ne j}t_\ell)^{\frac12-\delta}}\le$$

$$\le A_1(\mu_{jn})^{-\frac12+\delta}\prod_{\ell<j}\frac{F_n\circ\tilde F_n^{-1}(t_\ell))^{\frac12-\delta'}}{t_\ell^{\frac12-\delta}}\left|\int_0^{F_n\circ\tilde F_n^{-1}(t_1)}\cdots\int_0^{F_n\circ\tilde F_n^{-1}(t_{j-1})}\int_0^{t_{j+1}}\cdots\right.$$

$$\cdots\int_0^{t_{k+1}}(\prod_{\ell<j}u_\ell)^{-1+\delta+\delta'}(\prod_{\ell>j}u_\ell)^{-1+2\delta}du_1\dots du_{j-1}du_{j+1}\dots du_{k+1}\,.$$

If we apply Lemma 4.7 we have $\forall\,\epsilon>0$, $\exists\;\theta^{(1)}>0$, $\exists\;N_0^{(1)}\in\mathbb{N}$, $\forall\;n\ge N_0^{(1)}$

(4.12) $$P[\sup_{t_\ell\le\theta^{(1)}}\frac{F_n\circ\tilde F_n^{-1}(t_\ell))^{\frac12-\delta'}}{t_\ell^{\frac12-\delta}}>1]<\frac{\epsilon}{4}\quad\forall\;\ell\in\{1,\dots,k+1\}\,.$$

We also have $\forall\,\epsilon>0$, $\exists\;\theta^{(2)}>0$, $\exists\;N_0^{(2)}\in\mathbb{N}$, $\forall\;n\ge N_0^{(2)}$

$$\frac{[nt_o]-k}{(n-k)t_o^{\frac12-\delta}}<(\epsilon/(k+1))^{1/(k+2)}\,,\quad\forall\;t_o\le\theta^{(2)}$$

$$\int_0^{t_\ell}(u_\ell)^{-1+2\delta}du_\ell<(\epsilon/k+1)^{1/k+2}\quad\forall\;t_\ell\le\theta^{(2)}\,,\;\forall\;\ell\in\{1,\dots,k+1\}\,.$$

(4.13) $$P[\sup_{t_\ell\le\theta^{(2)}}\int_0^{F_n\circ\tilde F_n^{-1}(t_\ell)}(u_\ell)^{-1+\delta+\delta'}du_\ell<(\tfrac{\epsilon}{k+1})^{1/(k+2)}]>1-\frac{\epsilon}{4}\quad\forall\;\ell\in\{1,\dots,k+1\}$$

If we apply Lemma 4.5 in the one dimensional case and Lemmas 4.6 and 4.7 we obtain $\forall\ \epsilon>0\ ,\ \exists\ \theta^{(3)}>0\ ,\ \exists\ N_0^{(3)}\in\mathbb{N}\ ,\ \forall\ n\geq N_0^{(3)}$

$$(4.14)\quad P[\sup_{t_j\leq\theta^{(3)}} |\frac{W_n(1,\ldots,1,F_n\circ\tilde{F}_n^{-1}(t_j),1,\ldots,1)}{(t_j)^{\frac{1}{2}-\delta}(\mu_{jn})^{\frac{1}{2}-\delta}}| < (\frac{1}{2(k+1)})^{1/(2k+2)}] > 1-\epsilon/4$$

$\forall\ j\in\{1,\ldots,k+1\}$.

We also have $\forall\ \alpha>0$,

$$P[\inf_{t_j\geq 1/n} F_n\circ\tilde{F}_n^{-1}(t_j)\geq(1/n)^{1+\alpha}] = P[F_n(X_{ni})\geq(1/n)^{1+\alpha},\quad \forall\ i\in\{2,\ldots,n\}]$$

and

$$P[\bigcup_{i=1}^{n}\{F_n(X_{ni})<(1/n)^{1+\alpha}\}] \leq \sum_{i=1}^{n} P[F_n(X_{ni})<(1/n)^{1+\alpha}] = n\, n^{-1-\alpha} = n^{-\alpha}$$

which implies $\forall\ \epsilon>0\ ,\ \exists\ \theta^{(4)}>0\ ,\ \forall\ n\geq N_0^{(4)}$

$$(4.15)\qquad P[\sup_{1/n\leq t_j\leq\theta^{(4)}} \frac{o(n^{-\frac{1}{2}})}{t_j^{\frac{1}{2}-\delta}\mu_{jn}^{\frac{1}{2}-\delta}} < (\frac{\epsilon}{2(k+1)})^{1/(k+2)}] > 1-\epsilon/4$$

(for α sufficiently small).

Letting $\theta_0 = \bigwedge_{1\leq p\leq 4}\theta^{(p)}$ and $N_0 = \bigvee_{1\leq p\leq 4} N^{(p)}$. From (4.12) to (4.15) we obtain $\forall\ \theta\leq\theta_0$, $\forall\ n\geq N_0$

$$P_n[\sup_{t\in C^*_{\theta_0}\cap I(\rho)} |\frac{Z_n(t)}{r(t)}| > \epsilon] < \epsilon$$

i.e. condition (4.9) for Z_n on $I(\rho)$.

It remains to prove Proposition 2.5. From Proposition 2.3 we deduce that $\hat{L}_n$ verifies the conditions (i) and (ii) of Proposition 2.1. From Lemma 4.4 we deduce easily that $\frac{\hat{L}_n}{r}$ verifies conditions (i) and (ii) of Proposition 2.1 which inducts the convergence of $\frac{\hat{L}_n}{r}$ which is Proposition 2.5.

5. Proof of Theorem 3.1. First we give the integral representation of $\mathscr{S}_n$ in the form of (1.8).

By definition

$$(n-k)^{\frac{1}{2}}(\mathscr{S}_n - b_n) = (n-k)^{-\frac{1}{2}}\left(\sum_{i=k+1}^{n} c_{ni} a_n(R_{n,i-k},\dots,R_{ni}) - b_n\right)$$

$$= (n-k)^{\frac{1}{2}}\left(\sum_{i=k+1}^{n} c_{ni} \sum_{\substack{1\le \ell_j \le n \\ 1\le j\le k+1}} a_n(\ell_1,\dots,\ell_{k+1}) \prod_{j=1}^{k+1} I_{[R_{n,i+j-k-1}=\ell_j]} - b_n\right)$$

$$= (n-k)^{\frac{1}{2}}\Big(\sum_{I\subset\{1,\dots,k+1\}} \sum_A \lambda_n\left(\tfrac{\ell_o}{1+n},\dots,\tfrac{\ell_{k+1}}{1+n}\right)\Big(\sum_{i=k+1}^{\ell_o} \prod_{j\in I} I_{[R_{n,i+j-k-1}\le \ell_j]}$$

$$\times \prod_{j\notin I} I_{[R_{n,i+j-k-1}\ge \ell_j]} - H_n^{(0,\tilde\rho(I))}\left(\tfrac{\ell_o}{1+n},\dots,\tfrac{\ell_{k+1}}{1+n}\right)\Big)\Big)$$

$$+ (n-k)^{\frac{1}{2}}\Big(\sum_{I\subset\{1,\dots,k+1\}} \sum_B \lambda_n\left(\tfrac{\ell_o}{1+n},\dots,\tfrac{\ell_{k+1}}{1+n}\right)\Big(\sum_{i=\ell_o+1}^{n} \prod_{j\in I} I_{[R_{n,i+j-k-1}\le \ell_j]}$$

$$\times \prod_{j\notin I} I_{[R_{n,i+j-k-1}\ge \ell_j]} - H_n^{(1,\tilde\rho(I))}\left(\tfrac{\ell_o}{1+n},\dots,\tfrac{\ell_{k+1}}{1+n}\right)\Big)\Big)$$

$$= \int_{[0,1]^{k+2}} \hat L_n(t)\lambda_n(dt)$$

where $\sum_A$ is the sum over $1 \le \ell_j < \frac{[1+n]}{2}$ $j\in I$, $\frac{[1+n]}{2} \le \ell \le n$ $j\in I$, $1 \le \ell_o < \frac{[1+n]}{2}$; and $\sum_B$ is the sum over $1 \le \ell_j < \frac{[1+n]}{2}$ $j\in I$, $\frac{[1+n]}{2} \le \ell \le n$ $j\notin I$, $\frac{[n+1]}{2} \le \ell_o \le n$ and

$$\tilde\rho(I) = (\rho(1),\dots,\rho(k+1))$$

$$\rho(j) = 0 \text{ if } j\in I$$

$$\rho(j) = 1 \text{ if } j\notin I$$

From Lemma 2.5 we easily deduce

$$(n-k)^{-\frac{1}{2}}(\mathscr{S}_n - b_n) = \int_{[0,1]^{k+2}} \hat L_n \cdot \tfrac{1}{r}\, r\, d\lambda_n \to \int_{[0,1]^{k+2}} \hat L_o \cdot \tfrac{1}{r}\, r\, d\lambda_o .$$

It remains to show that $\sigma^2 < \infty$.

For this, we have to prove

$$\int_{[0,1]^{k+2}} \int_{[0,1]^{k+2}} E[\hat L_o(t)\hat L_o(t')]\lambda_o(dt)\lambda_o(dt') < \infty \tag{5.1}$$

by symmetrization it is sufficient to show

(5.2)
$$\int_{[0,\frac{1}{2}]^{k+2}} \int_{[0,\frac{1}{2}]^{k+2}} E[\hat{L}_o(t)\hat{L}_o(t')]\lambda_o(dt)\lambda_o(dt') < \infty .$$

We have by proof of Proposition 2.3

$$\hat{L}_o(t) = W_o(t) - \sum_{j=1}^{k+1} t_o W_o(1,\dots,t_j,\dots,1)\ell_j(\underset{\sim}{t}) \quad \forall\, t\in[0,\tfrac{1}{2})^{k+2}$$

by assumption (d) of Proposition 2.2

$$\lim |E[\{W_n(t) - \sum_{j=1}^{k+1} t_o W_n(1,\dots,t_j,\dots,1)\ell_j(\underset{\sim}{t})\}\{W_n(t') - \sum_{j=1}^{k+1} t_o' W_n(1,\dots,t_j',\dots,1)\ell_j(\underset{\sim}{t}')\}]|$$

$$= |E[\hat{L}_o(t)\hat{L}_o(t')]|$$

$$\le \lim [E\{W_n(t) - \sum_{j=1}^{k+1} t_o W_n(1,\dots,t_j,\dots,1)\ell_j(\underset{\sim}{t})\}^2]^{\frac{1}{2}}$$

$$\times [E\{W_n(t') - \sum_{j=1}^{k+1} t_o' W_n(1,\dots,t_j',\dots,1)\ell_j(\underset{\sim}{t}')\}^2]^{\frac{1}{2}}$$

$$= \lim_{n\to\infty} A_n^* B_n^*$$

by the Schwarz inequality.

Let now $\{X_{ni}\}$ be φ–mixing with rates (1.2) or (1.3), then from Lemma 4.1 with $q=1$ of Harel and Puri (1987b) we obtain

$$A_n^* \le [[E\{W_n^2(t) + 2|W_n(t)|t_o \sum_{j=1}^{k+1} |W_n(1,\dots,t_j,\dots,1)|\ell_j(\underset{\sim}{t})$$

$$+ t_o^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} \ell_j(\underset{\sim}{t})\ell_s(\underset{\sim}{t})|W_n(1,\dots,t_j,\dots,1)W_n(1,\dots,t_s,\dots,1)|\}]^{\frac{1}{2}}$$

$$\le C_1[(\prod_{m=0}^{k+1} t_m)^{1/(k+2)} + 2t_o (\prod_{m=1}^{k+1} t_m)^{1/2(k+1)}\ell_j(\underset{\sim}{t})$$

$$+ t_o^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} \ell_j(\underset{\sim}{t})\ell_s(\underset{\sim}{t}) t_j^{\frac{1}{2}} t_s^{\frac{1}{2}}]^{\frac{1}{2}} \quad \text{(where } C_1 \text{ is some constant)}$$

$$\le C_1[(\prod_{m=0}^{k+1} t_m)^{1/(k+2)} + 2A_1 t_o (\prod_{m=1}^{k+1} t_m)^{1/2(k+1)} t_j^{-\frac{1}{2}+\delta} (\prod_{\ell\neq j} t_\ell)^{\frac{1}{2}+\delta}$$
$$+ A_1 t_o^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} t_j^\delta t_s^\delta (\prod_{\ell\neq j} t_\ell)^{\frac{1}{2}+\delta} (\prod_{p\neq s} t_p)^{\frac{1}{2}+\delta}]^{\frac{1}{2}}$$

by condition (2.7).

Similarly $B_n^* \le$ some inequality with t's changed to t''s. Thus $|E[\hat{L}_o(t)\hat{L}_o(t')]|$ is bounded by a function which is $\lambda_o \times \lambda_o$ integrable and so $|E[\hat{L}_o(t)\hat{L}_o(t')]|$ is $\lambda_o \times \lambda_o$ integrable.

Let now $\{X_{ni}\}$ be strong mixing with rates (1.4), then, using Lemma 4.2 with $q=2$ of Harel of Puri (1987b), we obtain

$$A_n^* \le C_1[(\prod_{m=0}^{k+1} t_m)^{(1-\epsilon)/(k+2)} + 2A_1 t_o (\prod_{m=1}^{k+1})^{(1-\epsilon)/2(k+1)} t_j^{-\frac{1}{2}+\delta} (\prod_{\ell\neq j} t_\ell)^{\frac{1}{2}+\delta}$$
$$+ A_1 t_o^2 \sum_{s=1}^{k+1} \sum_{j=1}^{k+1} t_j^{\delta-\epsilon/2} t_s^{\delta-\epsilon/2} (\prod_{\ell\neq j} t_\ell)^{\frac{1}{2}+\delta} (\prod_{p\neq s} t_p)^{\frac{1}{2}+\delta}]^{\frac{1}{2}}$$

and a similar inequality for B_n^*, and the result follows as in the case of φ-mixing.

6. Examples. For convenience, let us take $k=1$.

6.1 Consider a sequence of autoregressive processes

$$X_{n,i} = a_n X_{n,i-1} + \epsilon_i, \quad i\in\mathbb{Z},\ n\in\mathbb{N},\ a_n\in\mathbb{R}$$

where $\{\epsilon_i;\ i\in\mathbb{Z}\}$ is a sequence of independent and identically distributed random variables each having a $\mathcal{N}(0,1)$ distribution.

If there exists an a such that $|a_n - a| = 0(n^{-\alpha})$, $\alpha>0$ and $\frac{|a|}{1-a^2} < \frac{1-2\delta}{1+2\delta}$, $0 < \frac{1}{2}-\delta < \frac{1}{12}$, then for the rank statistic $\mathcal{S}_n$ associated with the sequence $\{X_{n,1},\dots,X_{n,n}\}$ and the score functions satisfying the assumptions of Theorem 3.1 $(n-k)^{-\frac{1}{2}}(\mathcal{S}_n - b_n)$ converges in law to the $\mathcal{N}(0,\sigma^2)$ distribution where b_n and σ^2 are given by (3.1) and (3.6).

Indeed, from Theorem 6.2 in Harel and Puri (1987b) we have only to satisfy the condition (2.7). The bivariate random variable $(X_{n,i-1}, X_{n,i})$ is gaussian with correlation coefficient $\frac{a_n}{1-a_n^2}$. From Ruymgaart, Shorack and Van Zwet (1977), the condition (2.7) is satisfied if

$$\frac{|a_n|}{1-a_n^2} < \frac{2(\frac{1}{2}-\delta)}{2-2(\frac{1}{2}-\delta)} = \frac{1-2\delta}{1+2\delta}$$

and (2.7) follows for n sufficiently large.

6.2 Consider a sequence of moving average processes

$$X_{n,i} = \epsilon_i + b_n\epsilon_{i-1}, \quad i\in\mathbb{Z}, \quad n\in\mathbb{N} < b_n\in\mathbb{R}$$

where $\{\epsilon_i;\ i\in\mathbb{Z}\}$ is a sequence of independent and identically distributed random variables, each having $\mathscr{N}(0,1)$ distribution.

If there exists a constant b such that $|b_n-b| = 0(n^{-\beta})$, $\beta>0$, and $\frac{|b|}{1+b^2} < \frac{1-2\delta}{1+2\delta}$, $0 < \frac{1}{2}-\delta < \frac{1}{12}$, then for the rank statistic $\mathscr{S}_n$ associated with the sequence $\{X_{n,1},\dots,X_{n,n}\}$ and the score functions satisfying the assumptions of Theorem 3.1, $(n-k)^{-\frac{1}{2}}(\mathscr{S}_n-b_n)$ converges in law to the $\mathscr{N}(0,\sigma^2)$ distribution where b_n and σ^2 are given by (3.1) and (3.7)

7. Appendix. Proofs of Lemma 4.6 and Lemma 4.7.

The proofs of Lemmas 4.6 and 4.7 are based on the following lemmas.

LEMMA 7.1. Let $\{Y_{ni}, 1\le i\le n, n\ge 1\}$ be real valued random variables which are φ–mixing with rates

(7.1) $$\sum_{m=1}^{\infty} m^{-1}\varphi^{\frac{1}{2}}(m) < \infty$$

or strong mixing with rates

(7.2) $$\sum_{m=1}^{\infty} m^2\alpha^{\epsilon}(m) < \infty, \ \epsilon\in(0,\tfrac{1}{2}) \ \text{ and } \ |Y_{ni}|\le 1,\ 1\le i\le n, n\ge 1 .$$

Set $S_n = \sum_{i=1}^{n} Y_{ni}$, $\|Y_{ni}\| = (\int|Y_{ni}-E(Y_{ni})|^2 dP_n)^2$ and

$\|Y_{ni}\|_\epsilon = (\int |Y_{ni}-E(Y_{ni})|^{2/(1-\epsilon)} dP_n)^{1-\epsilon}$.

Furthermore, assume that there exists $\alpha_0>0$, $N_0 \in \mathbb{N}$ and $\mu(|\mu|<\infty)$ such that

(7.3) $$|n^{-1} \sum_{i=1}^{n} E(Y_{ni})-\mu| \le A(n^{-\alpha_0}), \quad A>0 , n\ge N_0$$

then $\forall\ \alpha \le (\alpha_0 \wedge 1/3)$, $\exists\ K>0$ and $N(\alpha)$ such that

(7.4) $$P_n[|n^{-1} \sum_{i=1}^{n} Y_{ni}-\mu| \ge K(n^{-\alpha})] \le K(n^{-1+2\alpha})M , \forall\ n\ge N(\alpha)$$

where $M = \sup_n \max_{1\le i\le n} \|Y_{ni}\|$ if the sequence of r.v.'s is φ-mixing and $M = \sup_n \max_{1\le i\le n} \|Y_{ni}\|_\epsilon$ if the sequence of r.v.'s is strong mixing.

PROOF. Suppose the sequence of r.v.'s is φ-mixing. Using Lemma 4.1 in Harel and Puri (1987b) with $q=1$ we obtain

$$E[(n^{-1} \sum_{i=1}^{n} (Y_{ni}-E(Y_{ni}))]^2 \le C_1(\varphi)Mn^{-1}$$

where $C_1(\varphi)$ is a constant depending only on φ . We deduce from the Markov inequality

(7.5) $$P_n[|n^{-1} \sum_{i=1}^{n} (Y_{ni}-E(Y_{ni}))| > K_0 n^{-\alpha}] \le (K_0)^2 n^{2\alpha} n^{-1} C_1(\varphi)M .$$

From (7.3) and (7.5) we obtain (7.4) for $K = \max\{K_0+A,(K_0)^2 C_1(\varphi)M\}$.

If the sequence of r.v.'s are strong mixing, we use Lemma 4.2 in Harel and Puri (1987b) for $q=1$ and obtain $E[n^{-1} \sum_{i=1}^{n} (Y_{ni}-E(Y_{ni}))]^2 \le C_1(\alpha)Mn^{-1}$ where $C_1(\alpha)$ is a constant depending only on α , the method is similar and therefore omitted.

LEMMA 7.2. Assume that the random variables $\{Y_{ni}, 1\le i\le n, n\ge 1\}$ with continuous distribution functions F_n , $n\ge 1$ satisfy condition (7.1) or (7.2) of Lemma 7.1.

Furthermore, assume that there exists $\alpha_0>0$, $N_0 \in \mathbb{N}$ and a continuous distribution function F such that

(7.6) $$\sup_{x\in\mathbb{R}} |F_n(x)-F(x)| \le A(n^{-\alpha_0}) \quad A>0, n\ge N_0$$

then $\forall\,(\alpha,\alpha')$ where $0<\alpha<\alpha' \le (\alpha_0 \wedge 1/3) < \exists\, K>0$ and $N(\alpha,\alpha')\in\mathbb{N}$ such that

$$P_n\,[\sup_{x\in\mathbb{R}} |\tilde{F}_n(x)-F(x)| \ge K(n^{-\alpha})] \le K(n^{-1+2\alpha+\alpha'}) \tag{7.7}$$

$\forall\, n\ge N(\alpha,\alpha')$ where $\tilde{F}_n$ is the empirical distribution function of $\{Y_{n1},\dots,Y_{nn}\}$.

PROOF. For any $j\in\mathbb{N}$, $k\in\mathbb{N}_*$ such that $j\le k$ we denote by $u_{j;k}\in\mathbb{R}\cup\{-\infty,+\infty\}$ the element defined by

$$F(u_{j;k}) = j/k\ .$$

From (7.4) of Lemma 7.1 we deduce that $K_0>0$ exists (not depending on j and k) and $N(\alpha)\in\mathbb{N}$ such that

$$P_n[|\tilde{F}_n(u_{j;k})-F(u_{j;k})| \ge K_0(n^{-\alpha})] \le K_0 n^{-1+2\alpha} \quad \forall\, n\ge N(\alpha)\ . \tag{7.8}$$

For j and k fixed, let u be so that $u_{j-1;k}\le u\le u_{j;k}$. We have

$$F(u_{j-1;k}) \le F(u) \le F(u_{j;k})$$
$$\tilde{F}_n(u_{j-1;k}) \le \tilde{F}_n(u) \le \tilde{F}_n(u_{j;k})$$
$$F(u_{j;k})-F(u_{j-1;k}) = 1/k$$

which inducts

$$\begin{aligned}\tilde{F}_n(u)-F(u) &\le \tilde{F}_n(u_{j;k})-F(u_{j-1;k})\\ &= \tilde{F}_n(u_{j;k})-F(u_{j;k})+F(u_{j;k})-F(u_{j-1;k})\\ &= \tilde{F}_n(u_{j;k})-F(u_{j;k}) + 1/k\ .\end{aligned}$$

By the same argument we obtain

$$\tilde{F}_n(u)-F(u) \ge \tilde{F}_n(u_{j-1;k})-F(u_{j-1;k}) + 1/k$$

and we deduce

$$\sup_{u\in\mathbb{R}} |\tilde{F}_n(u)-F(u)| \le 1/k + \max_{1\le j\le k} |\tilde{F}_n(u_{j;k})-F(u_{j;k})|$$

we choose $N_0(\alpha,\alpha')$ sufficiently large such that for any $n\ge N_0(\alpha,\alpha')$ we can find k verifying

$$K_0 n^{\alpha} \le k \le K_0 n^{\alpha'}\ .$$

For this number k we have

$$\sup_{u\in\mathbb{R}} |\tilde{F}_n(u)-F(u))| \le K_0 n^{-\alpha} + \max_{1\le j\le k} |\tilde{F}_n(u_{j;k})-F(u_{j;k})|\ . \tag{7.9}$$

Then from (7.8) and (7.9) we obtain

$$\forall\, n\geq N(\alpha)\vee N_0(\alpha,\alpha') = N(\alpha,\alpha')$$

$$P_n[\sup_{u\in\mathbb{R}} |\tilde{F}_n(u)-F(u)| \geq 2K_0 n^{-\alpha}] \leq kK_0 n^{1-2\alpha} \leq K_0^2 n^{1-2\alpha-\alpha'}$$

which is (7.7) for $K = \max\{2K_0, K_0^2\}$.

PROOF OF LEMMA 4.6. It comes from (7.6), (7.7) and the following inequality

$$\sup_{u\in[0,1]} |F_n\circ\tilde{F}_n^{-1}(u)-u| \leq \sup_{u\in[0,1]} |F_n\circ\tilde{F}_n^{-1}(u)-F\circ\tilde{F}_n^{-1}(u)|$$
$$+ \sup_{u\in[0,1]} |F\circ\tilde{F}_n^{-1}(u)-\tilde{F}_n\circ\tilde{F}_n^{-1}(u)| + \sup_{u\in[0,1]} |\tilde{F}_n\circ\tilde{F}_n^{-1}(u)-u|$$

where by definition $|\tilde{F}_n\circ\tilde{F}_n^{-1}(u)-u| \leq 1/n$.

PROOF OF LEMMA 4.7. Let U_n be the empirical process associated with the sequence $\{Y_{ni}\}$ that is

$$U_n(s) = n^{-\frac{1}{2}} \sum_{i=1}^{n} I_{[F_n(Y_{ni})\leq s]} - s \quad \text{for all} \quad s\in[0,1] . \tag{7.10}$$

U_n is the particular case of the process W_n defined in (1.9) for which $k=0$ and which is nontruncated.

From Lemma 4.5 we deduce that $\forall\, \epsilon>0$, $\exists\, \theta>0$, $\exists\, N_0\geq 1$ such that $\forall\, n\geq N_0$. $P_n(A_n) > 1 - \epsilon/2$ where
$A_n = \{|U_n(F_n\circ\tilde{F}_n^{-1}(s)| \leq (F_n\circ\tilde{F}_n^{-1}(s))^{\frac{1}{2}-\delta}, 0\leq s\leq\theta\}$ where $0<\frac{1}{2}-\delta<\frac{1}{2}$. From (4.7) of Lemma 4.6 we have $\forall\, \epsilon>0$, $\exists\, N_1\geq 1$ such that $\forall\, n\geq N_1$, $P_n(A_n') > 1 - \epsilon/2$ where

$$A_n' = \{F_n\circ\tilde{F}_n^{-1}(s)) \leq \beta s^{1-\tau}, \theta\leq s\leq 1\}$$

where $\beta>1$ and $0<\tau<1$.

Now we can write

$$n^{-\frac{1}{2}}U_n(F_n\circ\tilde{F}_n^{-1}(s)) = n^{-1} \sum_{i=1}^{n} I_{[F_n(Y_{ni})\leq F_n\circ\tilde{F}_n^{-1}(s)]} - F_n\circ\tilde{F}_n^{-1}(s)$$
$$= \tilde{F}_n\circ\tilde{F}_n^{-1}(s)-F_n\circ\tilde{F}_n^{-1}(s)$$

if $n^{-1}\leq s\leq\theta$ we obtain on A_n

$$F_n\circ\tilde{F}_n^{-1}(s) \leq \tilde{F}_n\circ\tilde{F}_n^{-1}(s) + n^{-\frac{1}{2}}(F_n\circ\tilde{F}_n^{-1}(s))^{\frac{1}{2}-\delta} \leq s + s(F_n\circ\tilde{F}_n^{-1}(s))^{\frac{1}{2}-\delta} . \tag{7.11}$$

Put $u=s^{\frac{1}{2}}$ and $z = F_n \circ \tilde{F}_n^{-1}(s)$, from (7.11) we deduce

$$z \le u^2 + uz^{\frac{1}{2}-\delta}$$
$$z^{\frac{1}{2}+\delta} \le zu$$
$$z \le 2^{2/(1+2\delta)} u^{1/(1+2\delta)}$$

or $F_n \circ \tilde{F}_n^{-1}(s) \le \beta s^{1-\tau}$ where $\beta = 2^{2/(1+2\delta)}$ and $1-\tau = 1/(1+2\delta)$. At last we have $\forall\, n \ge N_0 \vee N_1$, $F_n \circ \tilde{F}_n^{-1}(s) \le \beta s^{1-\tau}$ on $A_n \cap A_n'$ if $s \ge 1/n$ and $P(A_n \cap A_n') > 1-\epsilon$ which is (4.8) and Lemma 4.7 is proved.

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Statistique/*Statistics*

Convergence faible de la statistique sérielle linéaire de rang avec des fonctions de scores et des constantes de régression non bornées en condition de mélange

Michel HAREL et Madan PURI

Résumé — Nous établissions dans [1] la convergence faible d'une classe de statistiques de rang sérielles linéaires avec des fonctions de scores et des constantes de régression bornées lorsque les variables sont φ mélangeantes ou fortement mélangeantes. Cette Note généralise ces résultats au cas où les fonctions des scores ainsi que les constantes de régression ne sont pas nécessairement bornées.

Weak convergence of the serial linear rank statistic with unbounded scores and regression constants under mixing conditions

Abstract — *We established in* [1] *the weak convergence of a class of the serial linear rank statistics with bounded score functions and regression constants when the random variables are* φ*-mixing or strong mixing. This paper extends these results to the case when the score functions as well as the regression constants are not necessarily bounded.*

1. INTRODUCTION, DÉFINITIONS ET HYPOTHÈSES. — Soient $X_{n1}, \ldots, X_{nn}$, $n \geqq 1$ des variables aléatoires réelles avec des fonctions de répartition continues $F_n(x)$, $n \geqq 1$. On considère les statistiques

$$S_n = \sum_{i=k+1}^{n} c_{ni}\, a_n(R_{n,\,i-k}, \ldots, R_{n,\,i}) \tag{1}$$

où les c_{ni} sont des constantes connues, $a_n(\ldots)$ sont les fonctions de score et $R_{n,\,i}$ le rang de X_{ni} parmi $(X_{n1}, \ldots, X_{nn})$. Notre but est d'étudier le comportement asymptotique de S_n quand la suite $\{X_{ni}\}$ est φ-mélangeante où le coefficient de mélange vérifie :

$$\varphi(m) = O(m^{-1-\varepsilon}) \quad \text{où} \quad \varepsilon > 0 \quad (m \geqq 1) \tag{2}$$

ou

$$\sum_{m=1}^{\infty} m^{-1}\, \varphi^{1/2(3+k)}(m) < \infty \tag{3}$$

ou fortement mélangeante de coefficient de mélange α vérifiant

$$\sum_{m=1}^{\infty} m^{2(k+2)}\, \alpha^{\varepsilon}(m) < \infty \qquad \text{où} \quad \varepsilon \in]0,\ 1/2(3+k)[. \tag{4}$$

On note $\tilde{F}_n(x)$ la fonction de répartition empirique continue à droite de X_{ni}, $i = 1, \ldots, n$; c'est-à-dire $\tilde{F}_n(x) = n^{-1} \sum_{i=1}^{n} I_{[X_{ni} \leqq x]}$ où $I_{[\ ]}$ est la fonction indicatrice. On note G_n la fonction de répartition de $k+1$ variables aléatoires successives parmi $(X_{n1}, \ldots, X_{nn})$. Soient g_n la densité de probabilité de G_n et f_n la densité marginale de G_n (c'est-à-dire la densité de F_n). Soit encore μ_n la mesure sur $[0, 1]^{k+1}$ admettant pour fonction de répartition la fonction définie par

$$H_n(\mathbf{t}) = G_n(F_n^{-1}(t_1), \ldots, F_n^{-1}(t_{k+1})) \quad \text{pour tout} \quad \mathbf{t} = (t_1, \ldots, t_{k+1}) \in [0, 1]^{k+1}. \tag{5}$$

On considère également le processus empirique gradué W_n défini par

$$W_n(t) = (n-k)^{-1/2} \sum_{i=k+1}^{[nt_0]} \left\{ \prod_{j=1}^{k+1} I_{[F_n(X_{n,\,i+j-k-1}) \leqq t_j]} - H_n(\mathbf{t}) \right\} \tag{6}$$

Note présentée par Robert FORTET.

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pour tout $t=(t_0, \mathbf{t})=(t_0, t_1, \ldots, t_{k+1})\in[0, 1]^{k+2}$, où $[nt_0]$ désigne la partie entière de nt_0. Ce processus est une variante du processus empirique multivarié étudié par Harel et Puri [2].

1.1. *Les espaces* D^*_{k+2} *et* C^*_{k+2}. — Soit $f: [0, 1]^{k+2}\to\mathbb{R}$. Pour tout

$$\rho=(\rho(0), \ldots, \rho(k+1)\in\{0, 1\}^{k+2},$$

on définit

$$f_\rho(t)=\lim_{\substack{s_i\uparrow t_i\ \rho(i)=1\\ s_i\downarrow t_i\ \rho(i)=0}} f(s)\,((s, t)\in[0, 1]^{k+2})$$

si la limite existe et dans ce cas on appelle $f_\rho(t)$ la ρ limite de f en t.

Pour tout rectangle fermé R de $[0, 1]^{k+2}$, soit $D_\rho(R)$ l'ensemble de toutes les applications $f: R\to\mathbb{R}$ telles que f_{ρ^*} existe pour tout $\rho^*\in\{0, 1\}^{k+2}$ et $f_\rho=f$.

On pose $D^*_{k+2}=\{f: [0, 1]^{k+2}\to\mathbb{R}$, pour tout ρ, la restriction de f à $I(\rho)$ a une extension f_ρ sur $\bar{I}(\rho)$ telle que $f_\rho\in D_\rho(\bar{I}(\rho))\}$ où $\bar{A}$ désigne la fermeture de A et

$$I(\rho)=I_{\rho(0)}\times\ldots\times I_{\rho(k+1)},\qquad I_l=\begin{cases}[0, 1/2[\text{ si } l=0\\ [1/2, 1] \text{ si } l=1\end{cases}.$$

Finalement on note $C_{k+1}(j)$, $(1\leq j\leq k+1)$ l'espace de toutes les applications continues et bornées $f: A(j)\to\mathbb{R}$ où $A(j)=[0, 1]^{j-1}\times]0, 1[\times[0, 1]^{k+1-j}$. On pose $C^*_{k+2}=\{f=[0, 1]^{k+2}\to\mathbb{R}$: pour tout ρ, la restriction de f à $I(\rho)$ a une extension continue sur $\bar{I}(\rho)\}$.

1.2. *Fonction correctrice.* — Une fonction $r: [0, 1]^{k+2}\to\mathbb{R}^+$ est appelée fonction correctrice éclatée si elle satisfait les conditions suivantes :

(i) il existe $r_0: [0, 1]\to\mathbb{R}^+$ et $\tilde{r}: [0, 1]^{k+1}\to\mathbb{R}^+$ telles que

$$r(t)=r_0(t_0)\tilde{r}(\mathbf{t})\quad\text{pour tout } (t_0, \mathbf{t})\in[0, 1]^{k+2};$$

(ii) r appartient à C^*_{k+2};

(iii) $r=0$ sur la borne de $[0, 1]^{k+2}$.

2. Résultats préliminaires. — On dit que la suite $\{H_n\}$ satisfait à la condition de différentiabilité si :

(i) $\partial H_n/\partial t_j$ existe sur $A(j)$ et appartient à $C_{k+1}(j)$, $1\leq j\leq k+1$ et

(ii) $\partial H_n/\partial t_j\to l_j$ pour la topologie uniforme pour tout compact de $A(j)$ quand $n\to\infty$ et l_j appartient à $C_{k+1}(j)$.

Pour $\varepsilon\in\{0, 1\}$ donné, on définit $\psi_\varepsilon: \mathbb{R}\to\mathbb{R}$ par $\psi_\varepsilon=(-1)^\varepsilon I_d$, où I_d est la fonction identité. On définit aussi $\psi'_\varepsilon: [0, 1]\to[0, 1]$ par $\psi'_\varepsilon=\varepsilon+(-1)^\varepsilon I_d$. Pour tout $\varepsilon\in\{0, 1\}$, on note $F_{n,\varepsilon}$ la fonction de répartition de $\psi_\varepsilon(X_{n,1})$ et soit $\tilde{F}_{n,\varepsilon}(x)=n^{-1}\sum_{i=1}^{n} I_{[\psi_\varepsilon(X_{ni})\leq x]}$.

Pour tout $\tilde{\rho}\in\{0, 1\}^{k+1}$ on note $G^{\tilde{\rho}}_n(\mathbf{x}), \mathbf{x}\in\mathbb{R}^{k+1}$, la fonction de répartition de $(\psi_{\rho(1)}(X_{n,1}), \ldots, \psi_{\rho(k+1)}(X_{n,k+1}))$ et $H^{\tilde{\rho}}_n$ la suite de fonctions de répartition définie par

$$H^{\tilde{\rho}}_n(t_1, \ldots, t_{k+1})=G^{\tilde{\rho}}_n(F^{-1}_{n,\rho(1)}(t_1), \ldots, F^{-1}_{n,\rho(k+1)}(t_{k+1}))$$

Pour tout $\rho=(\rho(0), \tilde{\rho})\in\{0, 1\}^{k+2}$ le processus L^ρ_n est défini par

$$L^\rho_n(t)=(n-k)^{-1/2}\sum_{i=a}^{b}\left\{\prod_{j=1}^{k=1} I_{[\tilde{F}_{n,\rho(j)}(\psi_{\rho(j)})(X_{n,i+j-k-1})\leq\psi'_{\rho(j)}(t_j)]}\right.$$

$$\left.-H^{\tilde{\rho}}_n(\psi'_{\rho(1)}(t_1), \ldots, \psi'_{\rho(k+1)}(t_{k+1}))\right\}$$

où $a=k+1$, $b=[nt_0]$ si $\rho(0)=0$, et $a=\sup\{[nt_0]^*+1, k+1\}$, $b=n$ si $\rho(0)=1$

$$[nt_0]^* = \begin{cases} nt_0 & \text{si } nt_0 \text{ est entier} \\ 1[nt_0]+1 & \text{autrement.} \end{cases}$$

Le processus de rang éclaté $\tilde{L}_n$ est défini par

$$\hat{L}_n(t) = \begin{cases} 0 & \text{si} \quad t \notin [1/(n+1), n/(n+1)]^{k+2} \\ L_n^\rho(t') & \text{si} \quad t \in [1/(n+1), n/(n+1)]^{k+2} \cap I(\rho) \end{cases} \tag{7}$$

où t et t' sont liés par la relation

$$t'_j = (n+1/n)\,t_j \quad \text{si } \rho(j)=0 \qquad \text{et} \qquad t'_j = (n+1/n)\,t_j - 1/n \quad \text{si } \rho(j)=1. \tag{8}$$

2.1. *Convergence du processus de rang gradué éclaté.*

PROPOSITION 1. — *On suppose que la suite* $\{X_{ni}\}$ *est soit* φ*-mélangeante avec le taux* (2), *soit fortement mélangeante avec le taux* (4); *la suite* $\{H_n\}$ *satisfait à la condition de différentiabilité. De plus, on suppose que la fonction de covariance du processus empirique gradué* W_n *défini en* (6) *converge vers une fonction. Alors* $\hat{L}_n$ (*à valeurs p. s. dans* D_{k+2}^*) *converge faiblement pour la topologie uniforme vers un processus gaussien* $\hat{L}_0$ *à trajectoire p. s. dans* C_{k+2}^*.

Si on définit le processus de rang éclaté gradué non tronqué comme suit

$$\tilde{L}_n(\mathbf{t}) = \begin{cases} 0 & \text{si} \quad t \in [1/(n+1),\ n/(n+1)]^{k+1} \\ L_n^{1,\tilde{\rho}}(1, \mathbf{t}') & \text{si} \quad t \in [1/(n+1),\ n/(n+1)]^{k+1} \cap \tilde{I}(\tilde{\rho}) \end{cases} \tag{9}$$

où $\mathbf{t}$ et $\mathbf{t}'$ sont liés par la relation (8) et $\tilde{I}(\tilde{\rho}) = I_{\rho(1)} \times \ldots \times I_{\rho(k+1)}$, alors on a :

PROPOSITION 2. — *On suppose que la suite* $\{X_{ni}\}$ *est stationnaire et* φ*-mélangeante avec le taux* (3) *et que la suite* $\{H_n\}$ *satisfait la condition de différentiabilité. Alors* $\tilde{L}_n$ *converge faiblement pour la topologie de la convergence uniforme vers un processus gaussien* $\tilde{L}_0$ *à trajectoire p. s. dans* C_{k+1}^*.

2.2. *Convergence du processus de rang gradué éclaté et corrigé.*

PROPOSITION 3. — *Si la suite* $\{X_{ni}\}$ *satisfait les conditions de la proposition* 1, *s'il existe une fonction de répartition* F *telle que*

$$\sup |F_n(x) - F(x)| = O(n^{-\alpha}), \qquad \alpha > 0 \tag{10}$$

et si de plus

$$g_n(x_1, \ldots, x_{k+1}) \leqq A_1 \left[\prod_{j=1}^{k+1} F_n(x_j)(1-F_n(x_j))\right]^{-1/2+\delta} \prod_{j=1}^{k+1} f_n(x_j) \tag{11}$$

où $A_1 > 0$, *alors, pour toute fonction correctrice* r *satisfaisant*

$$r(t) \geqq A_2 \left[\prod_{j \in J(\rho)} t_j \prod_{j \notin J(\rho)} (1-t_j)\right]^{1/2-\delta} \quad \textit{pour tout} \quad t \in I(\rho), \tag{12}$$

où $A_2 \geqq 0$, $0 < 1/2 - \delta < 1/4(k+2)$ *et* $J(\rho) = \{j \in \{0, \ldots, k+1\} : \rho(j) = 0\}$, $\hat{L}_n \cdot 1/r$ *converge faiblement pour la topologie uniforme vers le processus gaussien* $\hat{L}_0 \cdot 1/r$ *à trajectoires p. s. dans* C_{k+2}^*.

PROPOSITION 4. — *Si la suite* $\{X_{ni}\}$ *satisfait les conditions de la proposition* 2 *et* (11), *alors, pour toute fonction correctrice* $\tilde{r} : [0, 1]^{k+1} \to \mathbb{R}^+$ *satisfaisant*

$$\tilde{r}(\mathbf{t}) \geqq \tilde{A} \left[\prod_{j \in \tilde{J}(\tilde{\rho})} t_j \prod_{j \notin \tilde{J}(\tilde{\rho})} (1-t_j)\right]^{1/2-\delta} \quad \textit{pour tout} \quad \mathbf{t} \in \tilde{I}(\tilde{\rho}),$$

où $\tilde{A}>0$, $0<1/2-\delta<1/4(k+2)$ *et* $\tilde{J}(\tilde{\rho})=\{j\in\{1, \ldots, k+1\};\ \rho(j)=0\}$, $\tilde{L}_n.1/\mathbf{r}$ *converge faiblement pour la topologie de la convergence uniforme vers le processus gaussien* $\tilde{L}_0.1/\mathbf{r}$ *à trajectoires p. s. dans* C^*_{k+1}.

3. Normalité asymptotique de la statistique linéaire de rang S_n. — Pour tout n, on définit une mesure signée λ_n concentrée sur $\{1/n+1, \ldots, n/n+1\}^{k+2}$ en posant

$$\lambda_n\left(\prod_{j\in J(\rho)}[l_j/n+1, 1/2[\prod_{j\notin J(\rho)}[1/2, l_j/n+1]\right]=c_{nl_0}a_n(l_1, \ldots, l_{k+1})$$

pour tout $(l_0/n+1, \ldots, l_{k+1}/n+1)\in I(\rho)\cap\{1/n+1, \ldots, n/n+1\}^{k+2}$ (par convention $c_{nl_0}=0$ si $l_0<k+1$).

On définit aussi un coefficient de centrage b_n par

$$b_n=\sum_{\rho\in\{0,\ 1\}^{k+2}}\int_{I(\rho)}H^{\rho}_n(t)\lambda_n(dt) \tag{13}$$

où H^{ρ}_n est une fonction : $[0, 1]^{k+2}\to\mathbb{R}$ telle que

$$H^{\rho}_n(t)=(b-a+1)(H^{\tilde{\rho}}_n(\psi'_{\rho(1)}(t'_1), \ldots, \psi'_{\rho(k+1)}(t'_{k+1}))) \tag{14}$$

où $a=k+1$, $b=[(n+1)t_0]$ si $\rho(0)=0$, $a=[(n+1)t_0-1]^*+1$, $b=n$ si $\rho(0)=1$, et les t_j et t'_j sont liés par (8).

Une mesure λ sur $[0, 1]^{k+2}$ est appelée pseudo-mesure d'ordre $I\subset\{0, 1, \ldots, k+1\}$ si pour tout $f\in C^*_{k+2}$

$$\int_{[0,\ 1]^{k+2}}f\,d\lambda=\int_{\{(t_0, \ldots, t_{k+1});\ t_i=1/2-,\ \forall i\in I\}}f(t)\lambda(dt).$$

λ sera appelé une mesure générale si elle est une somme finie de pseudo-mesures.

Théorème. — *Soit* r (*ou* $\tilde{r}$) *une fonction correctrice telle que pour une certaine mesure générale* λ_0 *on a*

$$\lim\int f.r.d\lambda_n=\int f.r.d\lambda_0\quad pour\ tout\ f\in C^*_{k+2} \tag{15}$$

$$\sup_{n\in\mathbb{N}}\int f.r.d|\lambda_n|<\infty\quad où\ |\lambda_n|\ désigne\ la\ mesure\ de\ variation\ totale. \tag{16}$$

Si la suite $\{X_{ni}\}$ *et* r (*ou* $\tilde{r}$) *satisfont aux conditions de la proposition* 3 (*resp. proposition* 4), *alors* $(n-k)^{-1/2}(S_n-b_n)$ (*resp. avec* $c_{ni}=1$ *pour tout i*) *converge en loi vers la loi normale de moyenne nulle et variance* σ^2 *où*

$$\sigma^2=\int_{[0,\ 1]^{k+2}}\cdots\int_{[0,\ 1]^{k+2}}E[\hat{L}_0(t)\hat{L}_0(t')]\,d\lambda_0(t)\,d\lambda_0(t')<\infty.$$

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WEAK CONVERGENCE OF THE SIMPLE LINEAR RANK STATISTIC UNDER MIXING CONDITIONS IN THE NONSTATIONARY CASE

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Abstract. The asymptotic distribution theory of simple linear rank statistics for the case when the underlying random variables are nonstationary is studied both for the φ–mixing and strong mixing cases.

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1. Introduction. Let X_{ni} , $1\le i\le n$, $n\ge 1$ be real–valued r.v.'s (random variables) with continuous d.f.'s (distribution functions) $F_{ni}(x)$, $x\in\mathbb{R}$ and let c_{ni} ($1\le i\le n$, $n\ge 1$) be an array of regression constants defined by a function g on $[0,1]$ as

$$c_{ni} = g(i/n) , \quad 1\le i\le n , n\ge 1 . \tag{1.1}$$

Denote by $\hat{H}_n(x) = n^{-1}\sum_{i=1}^{n} c_{ni} I_{[X_{ni}\le x]}$ the weighted empirical process where $I_{[\ ]}$ denotes the indicator function and by $\hat{F}_n(x) = n^{-1}\sum_{i=1}^{n} I_{[X_{ni}\le x]}$ the usual empirical process.

The corresponding expectations are denoted by

$$H_n = E(\hat{H}_n) \text{ and } F_n = E(\hat{F}_n) .$$

We will study the asymptotic behavior of the simple linear rank statistic of the form

$$\mathscr{S}_n(J) = n^{\frac{1}{2}} \int_{-\infty}^{+\infty} J(\tfrac{n}{n+1}\hat{F}_n(x))d\hat{H}_n(x) - n^{\frac{1}{2}} \int_{-\infty}^{+\infty} J(F_n(x))dH_n(x) \tag{1.2}$$

where J is a score function defined on the open unit interval.

The problem of finding a sufficiently large class of score functions for which the linear rank statistic is asymptotically normal was first considered by Chernoff and Savage (1958). Their results were later on strengthened considerably by several authors, mainly by Govindarajulu et al. (1966), Pyke and Shorack (1968), Hájek (1968) and Dupač and Hájek (1969) for the independent case, by Fears and Mehra (1974) for the φ–mixing case with stationary random variables, and by Denker and Rösler (1985) for the φ–mixing as well as strong mixing case but under a stationary set–up. In this paper we investigate the asymptotic distribution theory of the simple linear rank statistics (1.2) for the case when the underlying random variables are nonstationary.

2. Preliminaries. In this section, we give some propositions which are minor variations of Denker and Rösler (1985) and so their proofs will be either omitted or briefly outlined.

For $\delta\ge 0$, set $\eta = \delta(4+2\delta)^{-1}$.

Let μ_δ denote the measure on $[0,1]$ given by its density $(z(1-z))^{-\frac{1}{2}-\eta}$ relative to the Lebesgue measure. For a monotone function J, let $\|J\|_\delta$ be the L_1–norm of J in $L_1(\mu_\delta)$. By the Jordan decomposition of any right continuous function, J has a unique decomposition: $J = J_1 - J_2$ where J_1 and J_2 are monotone functions and $J_1(1/2) = 0$. For such a function, we set

$$\|J\|_\delta = \|J_1\|_\delta + \|J_2\|_\delta$$

where J_1 and J_2 belong to $L_1(\mu_\delta)$.

Denote by $\mathscr{H}_\delta$ the space of all right continuous functions J with $\|J\|_\delta < \infty$ and $J(1/2) = 0$ and let $\mathscr{G}_\delta$ be the set of all $J \in \mathscr{H}_\delta$ for which the measure ν defined by $J = \int d\nu$ is absolutely continuous with respect to Lebesgue measure. It is well known that $\mathscr{G}_\delta$ is the $\| \ \|_\delta$ closure of $C_{2,b}$: the space of functions with bounded second derivatives. The à priori assumption of having the space $\mathscr{H}_\delta$ of right continuous functions J with $J(1/2) = 0$ is no restriction because $\bar{J}$ defined by $\bar{J}(x) = J(x+) = \lim_{y \downarrow x} J(y)$ is a well defined right continuous function and if $\bar{J}$ belongs to $\mathscr{H}_\delta$, $\mathscr{S}_n(J)$ and $\mathscr{S}_n(\bar{J})$ are asymptotically equivalent (see Denker and Rösler (1985)).

We consider the array G_{ni} $(1 \le i \le n,\ n \ge 1)$ of d.f.'s on $[0,1]$ defined by

(2.1) $$G_{ni} = F_{ni} \circ F_n^{-1}.$$

Denote by $\hat{G}_n$ the empirical process in $[0,1]$ derived from $\hat{H}_n$ and defined by

(2.2) $$\hat{G}_n(t) = n^{-1} \sum_{i=1}^{n} c_{ni} I_{[F_n(X_{ni}) \le t]}, \quad t \in [0,1]$$

and $\hat{I}_n$ the empirical process on $[0,1]$ derived from $\hat{F}_n$ and defined by

(2.3) $$\hat{I}_n(t) = n^{-1} \sum_{i=1}^{n} I_{[F_n(X_{ni}) \le t]}, \quad t \in [0,1].$$

We also denote $G_n = E(\hat{G}_n)$ and $I_n = E(\hat{I}_n)$. The linear rank statistic defined in (1.2) can then be written as

(2.4) $$\mathscr{S}_n(J) = n^{\frac{1}{2}} \int_0^1 J(\tfrac{n}{n+1} \hat{I}_n(t)) d\hat{G}_n(t) - n^{\frac{1}{2}} \int_0^1 J(I_n(t)) dG_n(t).$$

The connection between the dependence structure of the processes and the class of functions for which asymptotic normality holds is expressed by the following condition

(2.5) $$\begin{cases} nE(\hat{G}_n(t)-G_n(t))^2 \le C\Lambda_n^2(t(1-t))^{1-2\eta} \\ nE(\hat{I}_n(t)-I_n(t))^2 \le C(t(1-t))^{1-2\eta} \end{cases}$$

for all $t \in (0,1)$ and $n\geq 1$ where $\eta = \delta(4+2\delta)^{-1}$, $\Lambda_n = \sup_{1\leq i\leq n} |g(i/n)|$ for g defined in (1.1) and C is some constant > 0.

PROPOSITION 2.1. Let $K>0$ and $\delta\geq 0$ be given. Then there exists a constant C_1 such that the following holds: If $\{X_{ni}\}$ is an array of r.v.'s satisfying (2.5) and if $\{c_{ni}\}$ are regression constants defined by (1.1), we have

(2.6) $$nE\{\int_{-\infty}^{+\infty} J(\tfrac{n}{n+1}\hat{F}_n(x))d\hat{H}_n(x) - \int_{-\infty}^{+\infty} J(\tfrac{n}{n+1}F_n(x))d\hat{H}_n(x)\}^2 \le C_1\Lambda_n^2\|J\|_\delta^2$$

(2.7) $$nE\{\int_{-\infty}^{+\infty} J(\tfrac{n}{n+1}F_n(x))d(\hat{H}_n-H_n)(x)\}^2 \le C_1\Lambda_n^2\|J\|_\delta^2$$

(2.8) $$n\{\int_{-\infty}^{+\infty} (J(F_n(x)) - J(\tfrac{n}{n+1}F_n(x)))dH_n(x)\}^2 \le C_1\Lambda_n^2\|J\|_\delta^2$$

(2.9) $$nE(\mathscr{S}_n(J))^2 \le C_1\Lambda_n^2\|J\|_\delta^2 .$$

PROOF. We only prove (2.6) because our method of proof is similar to that of Proposition 2 in Denker and Rösler (1985). It suffices to consider an increasing function $J\in\mathscr{H}_\delta$. Define

$$\varphi(x,t) = \begin{cases} 1 & \text{if } \frac{n}{n+1}F_n(x) \le t < \frac{n}{n+1}\hat{F}_n(x) \\ -1 & \text{if } \frac{n}{n+1}\hat{F}_n(x) \le t < \frac{n}{n+1}F_n(x) \\ 0 & \text{otherwise}. \end{cases}$$

and denote by $\hat{F}_n^{-1}(t) = \inf\{x\in\mathbb{R} : \hat{F}_n(x)\geq t\}$ the left continuous inverse of $\hat{F}_n$. Since for $t \le \frac{n}{n+1}$,

$$\varphi(x,t) = \begin{cases} 1 & \text{if } \hat{F}_n^{-1}(\frac{n+1}{n}t) \le x < F_n^{-1}(\frac{n+1}{n}t) \\ -1 & \text{if } F_n^{-1}(\frac{n+1}{n}t) \le x < \hat{F}_n^{-1}(\frac{n+1}{n}t) \\ 0 & \text{otherwise}. \end{cases}$$

it follows that for fixed $t \le \frac{n}{n+1}$,

$$|\int_{-\infty}^{\infty}\varphi(x,t)d\hat{H}_n(x)| \le \Lambda_n \int_{-\infty}^{\infty}|\varphi(x,t)|d\hat{F}_n(x) = \Lambda_n \mid \langle \tfrac{n+1}{n} t \rangle - \hat{F}_n(F_n^{-1}(\tfrac{n+1}{n} t)-)|$$

$$\le \Lambda_n(|\hat{F}_n(F_n^{-1}(\tfrac{n+1}{n} t)-) - \tfrac{n+1}{n} t| + 2 \inf \{\tfrac{1}{n},t\})$$

where $\langle t\rangle = \frac{k-1}{n}$ if $\frac{k-1}{n} < t \le \frac{k}{n}$.

From the assumption (2.5) on the sequence $\{X_{ni}\}$, it follows for $t \le \frac{n}{n+1}$ that

$$(\int_{-\infty}^{+\infty}\varphi(x,t)d\hat{H}_n(x))^2 \le 8\, \Lambda_n^2(\tfrac{1}{n}(1 \wedge nt))^2 + 2\, E(\hat{I}_n(\tfrac{n+1}{n} t) - I_n(\tfrac{n+1}{n} t))^2$$

$$\le 8\, \Lambda_n^2(\tfrac{1}{n}(1 \wedge nt))^2 + 2 \tfrac{C}{n}\, \Lambda_n^2(t(1-t))^{1-2\eta}.$$

Finally, interchanging the order of integration, and using the Cauchy–Schwarz inequality, we obtain

$$nE(\int_{-\infty}^{+\infty}J(\tfrac{n}{n+1}\hat{F}_n(x))d\hat{H}_n(x) - \int_{-\infty}^{+\infty}J(\tfrac{n}{n+1}F_n(x))d\hat{H}_n(x))^2$$

$$= nE(\int_0^{\frac{n}{n+1}}\int_{-\infty}^{+\infty}\varphi(x,t)d\hat{H}_n(x)dJ(t))^2$$

$$\le n(\int_0^{\frac{n}{n+1}}(E(\int_{-\infty}^{+\infty}\varphi(x,t)d\hat{H}(x))^2)^{\frac{1}{2}}dJ(t))^2$$

$$\le n\, C_1\Lambda_n^2(\int_0^{\frac{n}{n+1}}\tfrac{1}{n}(1 \wedge nt)dJ(t) + \int_0^{\frac{n}{n+1}} n^{-\frac{1}{2}}(t(1-t))^{\frac{1}{2}-\eta}\, dJ(t))^2$$

$$\le C_1\Lambda_n^2\|J\|_\delta^2$$

where C_1 is some positive constant, since $n^{\frac{1}{2}} \int_0^{\frac{n}{n+1}} \frac{1}{n}(1 \wedge nt)dJ(t) \le C_1 \int_0^1(t(1-t))^{\frac{1}{2}}dJ(t)$.

(2.6) is proved.

PROPOSITION 2.2. Let $\{X_{ni}\}$ satisfy condition (2.5) for some $\delta>0$, and let the regression constants c_{ni} satisfy (1.1) and $\sup_{n\in\mathbb{N}} \Lambda_n < +\infty$. Assume that $K\subset\mathcal{H}_\delta$ is a subset possessing the following property: for every $J\in K$, there exists a normal distribution

$\mathcal{N}(0,\sigma^2)$ where $0<\sigma<+\infty$ such that $\mathcal{S}_n(J)$ converges in law to $\mathcal{N}(0,\sigma^2)$, then the $\| \|_\delta$-norm closure of K has the same property.

PROOF. Let $J_1 \in \bar{K}$ where $\bar{K}$ is the closure of K, and $J \in K$. By Proposition 2.1 and the fact that $\mathcal{S}_n(J_1) = \mathcal{S}_n(J) + \mathcal{S}_n(J_1 - J)$, the distributions of $\mathcal{S}_n(J_1)$ and $\mathcal{S}_n(J)$ are closed, uniformly in n, in the weak topology for sufficiently small $\|J-J_1\|_\delta$. This proves the proposition.

For two probability measures P and Q on $\mathbb{R}$, denote by $D_2(P,Q) = \operatorname{Inf}\,(E(X-Y)^2)^{\frac{1}{2}}$ where the infinitum extends over all r.v.'s X and Y defined on the same probability space and having distributions P and Q respectively.

Let $\mathcal{L}(Z)$ denote the distribution of the r.v. Z.

PROPOSITION 2.3. Let $\{X_{ni}\}$ satisfy condition (2.5) for some $\delta>0$ and let c_{ni} satisfy (1.1) and $\sup_{n\in\mathbb{N}}^* \Lambda_n<+\infty$. Assume that there exists an operator $\sigma: \mathcal{H}_\delta \to \mathbb{R}$ which is uniformly bounded and satisfies the Lipschitz condition for the $\| \|_\delta$ norm.

If for every $J \in K \subset \mathcal{H}_\delta$, $\mathcal{S}_n(J)$ converges in law to a normal distribution $\mathcal{N}(0,\sigma^2(J))$, then the $\| \|_\delta$-norm closure of K has the same property.

PROOF. Let $J_1 \in \bar{K}$ and $J \in K$. Then, we have

$$D_2(\mathcal{L}(\mathcal{S}_n(J_1)),\mathcal{N}(0,\sigma^2(J_1)))$$
$$\leq D_2(\mathcal{L}(\mathcal{S}_n(J_1)),(\mathcal{L}(\mathcal{S}_n(J)))+D_2(\mathcal{L}(\mathcal{S}_n(J)),\mathcal{N}(0,\sigma^2(J)))+D_2(\mathcal{N}(0,\sigma^2(J)),\mathcal{N}(0,\sigma^2(J_1)))$$
$$\leq D_2(\mathcal{L}(\mathcal{S}_n(J)),\mathcal{N}(0,\sigma^2(J))) + C'\|J-J_1\|_\delta$$
for some $C'>0$ using Proposition 2.1 and the Lipschitz condition on σ.

Since the convergence in law from $\mathcal{S}_n(J)$ to $\mathcal{N}(0,\sigma^2(J))$ implies $\lim_{n\to\infty} D_2(\mathcal{L}(\mathcal{S}_n(J)),\mathcal{N}(0,\sigma^2(J_1))) = 0$ (see Denker and Rösler (1985)), the theorem follows.

3. Convergence of the linear rank statistic.

Recall that the sequence $\{X_{ni}\}$ is φ-mixing if $\sup_{n\geq 1} \sup_{1\leq j\leq n-m} \{\sup\{|P(B|A)-P(B)|;\ A\in\sigma(X_{ni}, 1\leq i\leq j),\ B\in\sigma(X_{ni}, i\geq j+m)\}\} = \varphi(m)\downarrow 0$ for positive integers j and m.

Here $\sigma(X_{n1},\ldots,X_{nj})$ and $\sigma(X_{n,j+m},\ldots,X_{nn})$ are the σ–fields generated by $(X_{n1},\ldots,X_{nj})$ and $(X_{n,j+m},\ldots,X_{nn})$ respectively.

Also recall that $\{X_{ni}\}$ satisfies the strong mixing condition if

$$\sup_{n\geq 1}\ \sup_{1\leq j\leq n-m} \{\sup\{|P(A\cap B)-P(A)P(B)|;\ A\in\sigma(X_{ni},\ 1\leq i\leq j,\ B\in\sigma(X_{ni},\ i\geq j+m)\}\} = \alpha(m)\downarrow 0\ .$$

Since $\alpha(m)\leq\varphi(m)$, it follows that if $\{X_{ni}\}$ is φ–mixing, then it is also strong mixing.

We will study the asymptotic behavior of $\mathscr{S}_n(J)$ when the r.v.'s $\{X_{ni}\}$ are φ–mixing with rates

$$\sum_{m=1}^{+\infty} m(\varphi(m))^{(2+3\delta)/(4+2\delta)} < +\infty \quad \text{for some } 0\leq\delta<2 \tag{3.1}$$

or strong mixing with rates

$$\sum_{m=1}^{+\infty} m^2\alpha\,(m)^{\delta/(2+\delta)} < +\infty \quad \text{for some } \delta>0\ . \tag{3.2}$$

Let $F_{n,i,j}$ be the d.f. of (X_{ni},X_{nj}), $1\leq i<j\leq n,\ n\geq 1$. For any sequence of d.f.'s $\{G^*_\ell,\ \ell\geq 1\}$ on $[0,1]^2$ with uniform marginals, we denote

$$\sigma^2_J(\{G^*_\ell\}) = \lim_{n\to\infty}\{\int_0^1 f^2(u)du + 2\sum_{\ell=1}^{n}\int_0^1\int_0^1 f(u)f(v)dG^*_\ell(u,v)\} \tag{3.3}$$

if the limit exists, where

$$f(u) = \int_0^1 (I_{[u\leq v]} - v)dJ(v) + J(u) - \int_0^1 J(v)dv \tag{3.4}$$

and $J\in\mathscr{H}_\delta$ for some $\delta>0$.

THEOREM 3.1. <u>Suppose the sequence</u> $\{X_{ni}\}$ <u>is</u> φ–<u>mixing with rate</u> (3.1) <u>or strong mixing with rate</u> (3.2), <u>the function</u> g <u>which defines the regression constants in</u> (1.1) <u>belongs to</u> $C^*_{1,b}$ <u>the space of functions which admit a derivative of bounded variation, and suppose that for any</u> $\ell>1$, <u>there exists a continuous d.f.</u> G^*_ℓ <u>on</u> $[0,1]^2$ <u>with uniform marginals such that</u>

(3.5) $$\lim_{n\to\infty} \max_{1\le i<j\le n} |F_{n,i,j}(F_n^{-1}(u),F_n^{-1}(v)) - G^*_{j-i}(u,v)| = 0$$

for all $(u,v) \in [0,1]^2$.

Then, for every $J\in\mathcal{G}_\delta$ with $2>\delta\geq 0$ if we have (3.1), and $\delta>0$ if we have (3.2)

(3.6) $$\lim_{n\to\infty} D_2(\mathcal{L}(\mathcal{S}_n(J)),\mathcal{N}(0,\tilde{\sigma}_J^2(\{G_\ell^*\})) = 0$$

where

(3.7) $$\tilde{\sigma}_J^2(\{G_\ell^*\}) = \sigma_J^2(\{G_\ell^*\})(\int_0^1\int_0^1 (u\wedge v)g'(u)g'(v)dudv - 2\int_0^1 ug'(u)du + g^2(1))$$

and $\tilde{\sigma}_J^2(\{G_\ell^*\}) < \infty$.

REMARK 3.1. Let the sequence of distribution functions $\{F_{n,i,j}\}$ satisfy the following conditions:

(i) There exists a sequence of d.f.'s F_ℓ^* on $\mathbb{R}^2$ such that

$$\lim_{n\to\infty} \max_{1\le i<j\le n} |F_{n,i,j}(x_1,x_2)-F^*_{j-i}(x_1,x_2)| = 0 \text{ for all } (x_1,x_2)\in\mathbb{R}^2,$$

(ii) $$F_{ni} = F_n \text{ for all } 1\le i\le n,\ n\ge 1,$$

then the condition (3.5) is satisfied whenever the sequence $\{X_{ni}\}$ is strong mixing.

PROOF. To prove Theorem 3.1, we first need a few lemmas.

For any n $(n\geq 1)$, any i $(1\le i\le n)$ and any $J\in C_{2,b}$, let

(3.8) $$A_{ni}(J) = \int_0^1 (I_{[F_n(X_{ni})\le t]} - G_{ni}(t))J'(t)dG_{ni}(t) + J(F_n(X_{ni})) - \int_0^1 J(t)dG_{ni}(t).$$

It is obvious that $E(A_{ni}) = 0$.

Now consider for any $J\in C_{2,b}$, the process $L_n(J)(s)$ defined on C_1 the space of continuous functions on $[0,1]$, by

(3.9) $$L_n(J)(s) = n^{-\frac{1}{2}}(\sum_{i=1}^{[ns]} A_{ni} + (ns-[ns])A_{n,[ns]+1})$$

where $[ns]$ denotes the integer part of the real number ns.

LEMMA 3.1. Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 3.1 and J belongs to $C_{2,b}$, then the process $L_n(J)(s)$ converges weakly in uniform topology to a

Gaussian process $L_0(J)(s)$ with trajectories a.s. in C_1 with mean 0 and variance $s\sigma_J^2(\{G_\ell^*\})$ where $\sigma_J^2(\{G_\ell^*\})$ is defined in (3.3), and $\sigma_J^2(\{G_\ell^*\}) < \infty$.

PROOF. The process $L_n(J)$ defines a probability measure P_n on C_1. From Theorem 8.1 of Billingsley (1968), we have to prove that (i) the finite dimensional distribution of P_n converges in law to a normal distribution and (ii) P_n is tight.

First we prove (i) which is equivalent to proving that $\sum_{\ell=1}^{p} \lambda_\ell L_n(J)(s_\ell)$ converges in law to a normal distribution for any $p\in\mathbb{N}^*$, any $\ell\in[0,1]$ and any $\lambda_\ell\in\mathbb{R}$ $(1\le\ell\le p)$. Without loss of generality, we can take $p=2$ and suppose that $s_1<s_2$.

We have

$$(3.10)\quad \sum_{\ell=1}^{2} \lambda_\ell L_n(J)(s_\ell) = n^{-\frac{1}{2}}[\sum_{i=1}^{[ns_1]} (\lambda_1+\lambda_2)A_{ni}(J) + \sum_{i=[ns_1]+1}^{[ns_2]} \lambda_2 A_{ni}(J)$$

$$+ \lambda_1(ns_1-[ns_1])A_{n,[ns_1]+1}(J) + \lambda_2(ns_2-[ns_2])A_{n,[ns_2]+1}(J)] .$$

We define the sequence of r.v.'s $\{B_{ni}(J)\}$ by

$$(3.11)\quad B_{ni}(J) = \begin{cases} (\lambda_2+\lambda_2)A_{ni}(J) & \text{if } i\le[ns_1] \\ \lambda_2 A_{ni}(J) & \text{if } [ns_1]<i\le[ns_2] \\ 0 & \text{if } i>[ns_2] \end{cases} .$$

As J and J' are bounded, we deduce

$$(3.12)\quad \sum_{\ell=1}^{2} \lambda_\ell L_n(J)(s_\ell) = n^{-\frac{1}{2}} \sum_{i=1}^{n} B_{ni}(J) + 0(n^{-\frac{1}{2}}) .$$

From Corollary 1 of Withers (1975) we have to verify that

$$(3.13)\quad E(\sum_{i=1}^{n} B_{ni}(J))^2/n \to \{(\lambda_1+\lambda_2)^2 s_1 + \lambda_2^2(s_2-s_1)\}\sigma_J^2(\{G_\ell^*\}) \text{ as } n\to\infty .$$

We have

(3.14) $E(\sum_{i=1}^{n} B_{ni}(J))^2/n =$

$$= n^{-1}[(\lambda_1+\lambda_2)^2 \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} E(A_{ni}(J)A_{nj}(J)) + (\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} E(A_{ni}(J)A_{nj}(J))$$

$$+ \lambda_2^2 \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} E(A_{ni}(J)A_{nj}(J))] .$$

Suppose the sequence $\{X_{ni}\}$ is φ–mixing, then from the boundedness of J and J' and from the well known inequality on the moment of φ–mixing r.v.'s (see Doukhan and Portal (1987), Proposition 2.2), we obtain

(3.15) $$n^{-1}|(\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} E(A_{ni}(J)A_{nj}(J))|$$

$$\le M/n \, (\sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} \varphi^{1/p}(j-i))$$

where M is some constant > 0 and $p = (4+2\delta)/(2+3\delta)$. From (3.1), the last expression goes to zero as $n\to\infty$.

If the sequence $\{X_{ni}\}$ is strong mixing, the left hand of (3.15) is majorized by

$$(M'/n)(\sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} \alpha^{\delta/(2+\delta)}(j-i))$$

where M' is some constant > 0 and from (3.2), this converges to 0 as $n\to\infty$.

It remains to prove that

(3.16) $$n^{-1} \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} E(A_{ni}(J)A_{nj}(J)) \to s_1\sigma_J^2(\{G_\ell^*\}) \text{ as } n\to\infty$$

and

$$n^{-1} \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} E(A_{ni}(J)A_{nj}(J)) \longrightarrow (s_2-s_1)\sigma_J^2(\{G_\ell^*\}) \text{ as } n\to\infty .$$

We first prove the convergence of (3.16) when J and J' are replaced by indicator functions. Suppose

$$J(t) = I_{[a\le t\le b]} \text{ and } J' = I_{[a'\le t\le b']} \,.$$

Then we can write

$A_{ni}(J) =$

$$= G_{ni}(b') - G_{ni}(\{a'\vee F_n(X_{ni})\}\wedge b'\}) - \int_a^b t\, dG_{ni}(t) + I_{[a\le F_n(X_{ni})\le b]} - (G_{ni}(b) - G_{ni}(a))$$

$= D_{ni}(X^*_{ni})$ where $X^*_{ni} = F_n(X_{ni})$.

Let $G_{n,i,j}$ be the d.f. of $(F_n(X_{ni}),F_n(X_{nj}))$. Then, we have

$$E(A_{ni}(J)A_{nj}(J)) = \int_0^1\int_0^1 D_{ni}(u)D_{nj}(v)dG_{n,i,j}(u,v) \,.$$

From condition (3.5) we easily deduce that

$$\text{(3.17)}\qquad \lim_{n\to\infty}\max_{1\le i<j\le n} |E(A_{ni}(J)A_{nj}(J)) - \int_0^1\int_0^1 D(u)D(v)dG^*_{j-i}(u,v)| = 0$$

where

$$D(u) = I_{[a\le u\le b]} - (b-a) + b' - \{(a'\vee u)\wedge b'\} - \tfrac{1}{2}(b'-a')^2 \,.$$

We obtain the same result if J and J' are replaced by step functions.

As J and J' are continuous and bounded, we can uniformly approach them by step functions and we deduce that

$$\text{(3.18)}\qquad \lim_{n\to\infty}\max_{1\le i<j\le n} |E(A_{ni}(J)A_{nj}(J)) - \int_0^1\int_0^1 f(u)f(v)dG^*_{j-i}(u,v)| = 0$$

where $f(u)$ is defined in (3.4).

Now denote $\rho(0) = \int_0^1 f^2(u)du$ and $\rho(i) = 2\int_0^1\int_0^1 f(u)f(v)dG^*_i(u,v)$, $i\ge 1$. Then,

$$|n^{-1}\sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} E(A_{ni}(J)A_{nj}(J)) - [ns_1]n^{-1}\sum_{i=0}^{+\infty}\rho(i)|$$

$$\leq |n^{-1}[ns_1]([ns_1])^{-1} \sum_{i=0}^{[ns_1]-1} \sum_{j=1}^{[ns_1]-i} E(A_{nj}(J)A_{n,j+i}(J))$$

$$- [ns_1](n[ns_1])^{-1} \sum_{i=0}^{[ns_1]} ([ns_1]-i)\rho(i)| + [ns_1]n^{-1} \sum_{i=[ns_1]+1}^{\infty} |\rho(i)|$$

$$+ [ns_1]n^{-1} \sum_{i=0}^{[ns_1]} \sum_{k=i}^{\infty} |\rho(k)|$$

$$= |A_n| + B_n + C_n .$$

From (3.18) we deduce that $|A_n| \to 0$ as $n \to \infty$ and from the well known inequalities on the moment of mixing r.v.'s (see Proposition 2.2 and 2.8 of Doukhan and Portal (1987)) and (3.1) or (3.2) we deduce that $B_n \to 0$ and $C_n \to 0$ as $n \to \infty$.

It is also immediate that

$$|[ns_1]n^{-1} \sum_{i=0}^{+\infty} \rho(i) - s_1 \sum_{i=0}^{+\infty} \rho(i)| \to 0 \text{ as } n \to \infty .$$

We conclude that $n^{-1} \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} E(A_{ni}(J)A_{nj}(J))$ converges to $s_1 (\sum_{i=0}^{+\infty} \rho(i))$ as $n \to \infty$ where $\sum_{i=0}^{+\infty} \rho(i)$ is equal to $\sigma_J^2(\{G_\ell^*\})$. Similarly

$$n^{-1} \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} E(A_{ni}(J)A_{nj}(J)) \longrightarrow (s_2-s_1)\sigma_J^2(\{G_\ell^*\}) \text{ as } n \to \infty$$

(3.16) is proved.

From (3.14) – (3.16), we deduce (3.13) and we conclude that $E(\sum_{\ell=1}^{2} \lambda_\ell L_n(J)(s_\ell))^2$ converges to $\{(\lambda_1+\lambda_2)^2 s_1+\lambda_2^2(s_2-s_1)\}\sigma_J^2(\{G_\ell^*\})$ which implies that $\sum_{\ell=1}^{2} \lambda_\ell L_n(J)(s_\ell)$ converges in law to the normal distribution with mean 0 and variance $\{(\lambda_1+\lambda_2)^2 s_1+\lambda_2^2(s_2-s_1)\}\sigma^2(\{G_\ell^*\})$ and (i) is proved.

We now prove (ii).

From Theorem 8.2 of Billingsley (1968) we have to verify that $\forall\ \epsilon>0$, $\exists\ \eta>0$,

$(0<\eta<1)$ and an integer N_0 such that $\forall\ n \geq N_0$

(3.19) $$P[\sup_{|s-s'|<\eta} |L_n(J)(s) - L_n(J)(s')| \geq \epsilon] \leq \epsilon .$$

If ns and ns' are integers, by using Theorem 11 for q=4 of Doukhan and Portal (1987) for the strong mixing case and Lemma 5.1 in Harel (1988) for q=2 for the φ-mixing case, we obtain for $s \geq s'$

(3.20) $$E(L_n(J)(s) - L_n(J)(s'))^4 \leq ((s-s')^2 + n^{-1}(s-s'))MC(\beta)$$

where

(3.21) $$C(\beta) = \sum_{m=1}^{+\infty} m^{-1}\varphi^{1/4}(m)$$

if the sequence $\{X_{ni}\}$ is φ-mixing, and

(3.22) $$C(\beta) = \sum_{m=1}^{+\infty} m^2\alpha^{\delta/(2+\delta)}(m)$$

if the sequence $\{X_{ni}\}$ is strong mixing and M is some positive constant.

If $s>s'$ and ns and ns' are integers, we have $s-s' \geq n^{-1}$ and

$$E(L_n(J)(s) - L_n(J(s'))^4 \leq 2M(s-s')^2C(\beta) .$$

From Lemma 2 of Balacheff and Dupont (1980) we obtain that $\forall\ \epsilon>0$, $\exists\ \eta>0$, and an integer N_0 sufficiently large such that $\forall\ n \geq N_0$,

(3.23) $$P[\sup_{|\frac{[ns]}{n} - \frac{[ns']}{n}|<2\eta} |L_n(J)(\tfrac{[ns]}{n}) - L_n(J)(\tfrac{[ns']}{n})| > \epsilon/2] \leq 2MC(\beta)K\eta^2\epsilon^{-4}$$

where K is some positive constant.

From the definition of $L_n(J)(s)$ in (3.9), we obtain

(3.24) $$\sup_{|s-s'|<\eta} |L_n(J)(s) - L_n(J)(s')| \leq 2 \max_{|\frac{[ns]}{n} - \frac{[ns']}{n}|<2\eta} |L_n(J)(\tfrac{[ns]}{n}) - L_n(J)(\tfrac{[ns]}{n})| .$$

By using (3.23) and (3.24), we deduce

(3.25) $$P[\sup_{|s-s'|<\eta} |L_n(J)(s) - L_n(J)(s')| \geq \epsilon] \leq MC(\beta)K'\eta^2\epsilon^{-4}$$

where K' is some positive constant and (3.19) is proved. The fact that $\sigma_J^2(\{G_\ell^*\}) < \infty$ is a simple consequence of $J \in C_{2,b}$ and (3.1) or (3.2).

Now we consider for any $J \in C_{2,b}$ the r.v. $V_n(J)$ defined by

(3.26) $$V_n(J) = n^{-\frac{1}{2}} \sum_{i=1}^{n} c_{ni} A_{ni} .$$

LEMMA 3.2. <u>Suppose that</u> $\{X_{ni}\}$ <u>satisfies the conditions of Theorem</u> 3.1. J <u>belongs to</u> $C_{2,b}$ <u>and</u> g <u>admits a derivative</u> g'. <u>Then</u> $V_n(J)$ <u>converges in law to the normal distribution with mean</u> 0 <u>and variance</u> $\tilde{\sigma}_J^2(\{G_\ell^*\})$ <u>where</u> $\tilde{\sigma}_J^2(\{G_\ell^*\})$ <u>is defined in</u> (3.7) <u>and</u> $\tilde{\sigma}_J^2(\{G_\ell^*\}) < \infty$.

PROOF. For any n define a measure λ_n on [0,1] by setting

$$\lambda_n(\{\tfrac{i}{n}\}) = c_{ni} - c_{n,i+1} , \quad 1 \le i \le n-1 \quad \text{and} \quad \lambda_n(\{1\}) = c_{nn} .$$

By definition, we have

$$V_n(J) = \int_0^1 L_n(J)(u)\lambda_n(du) .$$

We now prove that

(3.27) $$\int_0^1 L_n(J)(u)\lambda_n(du) \text{ converges in law to } -\int_0^1 L_0(J)(u)g'(u)du + L_0(J)(1)g(1) \text{ as } n\to\infty .$$

Let $h_n : C_1 \to \mathbb{R}$, $n \ge 1$ be defined as $h_n(f) = \int_0^1 f(u)\lambda_n(du)$ and $h_0 : C_1 \to \mathbb{R}$ be defined as $h_0(f) = -\int_0^1 fg'(u)du + f(1)g(1)$. Let $\{f_n, n \ge 1\}$ be a sequence of functions in C_1 and suppose that $f_n \to f_0$ in uniform topology as $n\to\infty$ where $f_0 \in C_1$. We show that $h_n(f_n) \to h_0(f_0)$ as $n\to\infty$.

$$\left|\int_0^1 f_n(u)\lambda_n(du) - (-\int_0^1 f_0(u)g'(u) + f_0(1)g(1)\right|$$

$$\le \left|\int_0^1 (f_n(u)-f_0(u))\lambda_n(du)\right| + \left|\int_0^1 f_0\lambda_n(du) + \int_0^1 f_0(u)g'(u) + f_0(1)g(1)\right|$$

$$\le \sup_{u\in[0,1]} |f_n(u)-f_o(u)| \, |\int_0^1 \lambda_n(du)| + (\sum_{i=1}^{n-1} f_o(i/n)|g(i/n)-g(i+1/n)) + f_o(1)g(1)$$

$$+ \int_0^1 f_o(u)g'(u)du - f_o(1)g(1)| \to 0 \text{ as } n\to\infty$$

from the hypothesis $f_n \to f_o$ in uniform topology, g' is an integrable function and $\sup_{n\in\mathbb{N}^*} \Lambda_n < \infty$.

Consequently $h_n(f_n) \to h_o(f_o)$ as $n\to\infty$ and by Theorem 5.5 of Billingsley (1968), (3.27) follows.

It remains to show that $E(-\int_0^1 L_o(J)(u)g'(u)du + L_o(J)(1)g(1))^2 = \tilde{\sigma}(\{G_\ell^*\}) < \infty$.

We have

$$E(-\int_0^1 L_o(J)(u)g'(u)du + L_o(J)(1)g(1))^2 =$$

$$= \int_0^1 \int_0^1 E[(L_o)(J)(u)(L_o)(J)(v)]g'(u)g'(v)dudv - 2\int_0^1 E[L_o(J)(1)L_o(J)(u)]g'(u)du$$

$$+ E[L_o(J)(1)L_o(J)(1))]g^2(1) .$$

As $E[L_o(J)(u)L_o(J)(v)] = (u \wedge v)\sigma_J^2(\{G_\ell^*\}) < \infty$, the property follows and Lemma 3.2 is proved.

PROOF OF THE THEOREM. We first prove that the theorem is true for $J\in C_{2,b}$. We have the following decomposition

$$\mathscr{S}_n(J) = V_n(J) + A_n(J) + B_n(J) + C_n(J) ,$$

where

$$A_n(J) = n^{\frac{1}{2}} \int_0^1 J'(I_n(t))(\hat{I}_n(t)-I_n(t))d(\hat{G}_n-G_n)(t)$$

$$B_n(J) = -(n+1)^{-1}n^{\frac{1}{2}} \int_0^1 J'(I_n(t))\hat{I}_n(t)d\hat{G}_n(t)$$

$$C_n(J) = 2^{-1}n^{\frac{1}{2}} \int_0^1 J''(\theta_n(I_n(t)))(n(n+1)^{-1}(\hat{I}_n(t)-I_n(t)))^2 d\hat{G}_n(t)$$

where $\theta_n(I_n(t)) \in [I_n(t)\wedge\hat{I}_n(t), I_n(t)\vee\hat{I}_n(t)]$.

Suppose $J \in C_{2,b}$. Then the weak convergence of $V_n(J)$ is established in Lemma 3.2. The random variables $A(J)$, $B(J)$, $X(J)$ converge to zero in probability and in L_2 , since

(3.28) $$E(A_n^2(J)) \le Kn^{-1} \sup_{t\in[0,1]} |J'(t)|^2 \sup_{n\in\mathbb{N}}{}^{*} \Lambda_n^2 C(\beta)$$

(3.29) $$|B_n(J)| \le Kn^{-\frac{1}{2}} \sup_{t\in[0,1]} |J'(t)| \sup_{n\in\mathbb{N}}{}^{*} \Lambda_n$$

(3.30) $$E(C_n^2(J)) \le Kn^{-1} \sup_{t\in[0,1]} |J'(t)|^2 \sup_{n\in\mathbb{N}}{}^{*} \Lambda_n^2 C(\beta)$$

where $C(\beta) = \sum_{m=1}^{+\infty} m(\varphi(m))^{(2+3\delta)/(4+2\delta)}$ if we have (3.1), and $C(\beta) = m^2 \sum_{m=1}^{+\infty} (\alpha(m))^{\delta/(2+\delta)}$ if we have (3.2), and K is some positive constant.

We only prove the inequality (3.28) for the φ-mixing case, because the method is similar to the proof of the three inequalities in Denker and Rösler (1985, p. 66). For any $(i,j,\ell,q) \in \mathbb{N}^4$, we put

$$\beta(i,j,\ell,q) = \left(\int_0^1 J'(I_n(t))c_{ni}(I_{[X_{ni}\le t]} - F_{ni}(t))d(I_{[X_{nj}\le t]} - F_{nj}(t))\right)\left(\int_0^1 J'(I_n(u))c_{n\ell}\right.$$
$$\left. I_{[X_{n\ell}\le u]} - F_{n\ell}(u))d(I_{X_{nq}\le u]} - F_{nq}(u))\right) .$$

Suppose $i\le j\le \ell \le q$ and let $p = \frac{2+3\delta}{4+2\delta}$, then from the condition of φ-mixing, we have the following three inequalities:

$$\beta(i,j,\ell,q) \le \sup_{t\in[0,1]} |J'(t)|^2 \Lambda_n^2 \varphi(j-i)$$
$$\beta(i,j,\ell,q) \le (2 \sup_{t\in[0,1]} |J'(t)|^2 \Lambda_n^2)(\varphi(\ell-j) + 4\varphi^{1/p}(j-i)\varphi^{1/p}(q-\ell))$$
$$\beta(i,j,\ell,q) \le \sup_{t\in[0,1]} |J'(t)|^2 \Lambda_n^2 \varphi(q-\ell)$$

If i,j,ℓ,q are differently ordered, we obtain similar inequalities.

From this, we deduce

$$E(A_n^2(J)) \le 4!\, n^{-3} \sup_{t\in[0,1]} |J'(t)|^2)\Lambda_n^2 \sum_{i,j,\ell,q} \beta(i,j,\ell,q)$$

Put $j'=j-i$, $\ell'=\ell-j$ and $q'=q-\ell$. We have

$$E(A_n^2(J)) \le 4!\, n^{-3}\Big(\sup_{t\in[0,1]} |J'(t)|^2\Big)\Lambda_n^2 \sum_{i=1}^{n} \Big(\sum_{\substack{0\le \ell'\le j' \\ 0\le q'\le j'}} 2\varphi(j') + \sum_{\substack{0\le j'\le q' \\ 0\le \ell'\le q'}} 2\varphi(q') + \sum_{\substack{0\le j'\le \ell' \\ 0\le q'\le \ell'}} 2\varphi(\ell')$$

$$+ 4\varphi^{1/p}(j')\varphi^{1/p}(q')\Big)\Big)$$

and after some computations, we obtain

$$E(A_n^2(J)) \le 288\, n^{-3} \Big(\sup_{t\in[0,1]} |J'(t)|^2\Big)\Lambda_n^2 \Big(\sum_{m\ge 1} \varphi^{1/p}(m)\Big)\Big(\sum_{m\ge 1} m\varphi^{1/p}/m)\Big)(n+n^2)$$

$$\le K n^{-1} \Big(\sup_{t\in[0,1]} |J'(t)|^2\Big) \sup_{n\in\mathbb{N}}{}_{*}\, \Lambda_n^2\, C(\beta)$$

where K is some positive constant and $C(\beta) = \sum_{m\ge 1} m(\varphi(m))^{(2+3\delta)/(4+2\delta)}$. (3.28) is proved for the φ–mixing case. Hence the theorem is true for $J\in C_{2,b}$.

Following Proposition 2.3 it remains to prove that the operator $\sigma : \mathcal{H}_\delta \to \mathbb{R}$ defined by $\sigma(J) = \tilde{\sigma}_J(\{G_\ell^*\})$ satisfies the Lipschitz condition for the $\|\ \|_\delta$ norm and the condition (2.5) is satisfied. The first property follows easily from the definition of $\tilde{\sigma}_J(\{G_\ell^*\})$ in (3.7) and the definition of $\sigma_J(\{G_\ell^*\})$ in (3.3) and (3.4).

We now prove (2.5) if we have (3.1). For $p = (4+2\delta)/(2+3\delta)$ and $q = (1-p^{-1})^{-1}$ we have

$$nE(\hat{G}_n(t)-G_n(t))^2 =$$

$$= nE\Big[n^{-1} \sum_{i=1}^{n} c_{ni}(I_{[F_n(X_{ni})\le t]} - G_{ni}(t))\Big]^2 = n^{-1}E\Big[\sum_{i=1}^{n} c_{ni}(I_{[F_n(X_{ni})\le t]} - G_{ni}(t))\Big]^2$$

$$= n^{-1} \sum_{1\le i,j\le n} E[c_{ni}c_{nj}(I_{[F_n(X_{ni})\le t]} - G_{ni}(t))(I_{[F_n(X_{nj})\le t]} - G_{nj}(t))]$$

$$\le n^{-1}\Lambda_n^2 \sum_{1\le i,j\le n} |E[(I_{[F_n(X_{ni})\le t]} - G_{ni}(t))(I_{[F_n(X_{nj})\le t]} - G_{nj}(t))]$$

$$\le 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1}\sum_{j=1}^{n-i} E|(I_{[F_n(X_{ni})\le t]} - G_{ni}(t))(I_{[F_n(X_{n,j+i})\le t]} - G_{n,j+i}(t))|$$

$$\le 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \varphi^{1/p}(i) \sum_{j=1}^{n-i} (G_{n,j}(t)(1-G_{n,j}(t)))^{1/p}(G_{n,j+i}(t)(1-G_{n,j+i}(t)))^{1/q}$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=0}^{n-1}\varphi^{1/p}(i)\Big(\sum_{j=1}^{n-i}(G_{n,j}(t)(1-G_{n,j}(t)))^{1/p}\Big(\sum_{j=1}^{n-i}G_{n,j+i}(t)(1-G_{n,j+i}(t))\Big)^{1/q}$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=1}^{n-1}\varphi^{1/p}(i)\Big(\sum_{j=1}^{n}(G_{n,j}(t)(1-G_{n,j}(t)))^{1/p}\Big(\sum_{j=1}^{n}G_{n,j}(t)(1-G_{n,j}(t))\Big)^{1/q}$$

$$= n^{-1}\Lambda_n^2\sum_{i=1}^{n-1}\varphi^{1/p}(i)\Big(\sum_{j=1}^{n}(G_{n,j}(t)(1-G_{n,j}(t))\Big)$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=1}^{n-1}\varphi^{1/p}(i)n\Big(\frac{1}{n}\sum_{j=1}^{n}G_{nj}(t)\Big)\Big(1-n^{-1}\sum_{j=1}^{n}G_{nj}(t)\Big)$$

$$= 2\,\Lambda_n^2\sum_{i=1}^{n-1}\varphi^{1/p}(i)t(1-t)\ \le 2\Big(\sum_{i=1}^{+\infty}\varphi^{1/p}(i)\Big)\Lambda_n^2 t(1-t)$$

which implies (2.5) if we have (3.1).

Finally, we prove (2.5) if we have (3.2). We have

$$nE(\hat{G}(t)-G_n(t))^2 =$$

$$= n^{-1}\sum_{1\le i,j\le n}E[c_{ni}c_{nj}(I_{[F_n(X_{ni})\le t]}-G_{ni}(t))(I_{[F_n(X_{nj})\le t]}-G_{nj}(t))]$$

$$\le n^{-1}\Lambda_n^2\sum_{1\le i,j\le n}|E[(I_{[F_n(X_{ni})\le t]}-G_{ni}(t))(I_{[F_n(X_{nj})\le t]}-G_{nj}(t))]$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=0}^{n-1}\sum_{j=1}^{n-i}E|(I_{[F_n(X_{ni})\le t]}-G_{ni}(t))(I_{[F_n(X_{n,j+i})\le t]}-G_{n,j+i}(t))|$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)}\sum_{j=1}^{n-i}(G_{nj}(t)(1-G_{nj}(t)))^{1/(2+\delta)}(G_{n,j+i}(t)(1-G_{n,j+i}(t)))^{1/(2+\delta)}$$

$$\le 2n^{-1}\Lambda_n^2\sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)}\Big(\sum_{j=1}^{n-i}(G_{nj}(t)(1-G_{nj}(t)))^{1/(2+\delta)}\times$$

$$\times\Big(\sum_{j=1}^{n-i}G_{n,j+i}(t)(1-G_{n,j+i}(t))\Big)^{1/(2+\delta)}$$

$$\leq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)}(\sum_{j=1}^{n}(G_{nj}(1-G_{nj}(t)))^{1/(2+\delta)}(\sum_{j=1}^{n} G_{n,j}(t)(1-G_{n,j}(t)))^{1/(2+\delta)}$$

$$= 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)}(\sum_{j=1}^{n} G_{nj}(t)(1-G_{nj}(t)))^{2/(2+\delta)}$$

$$\leq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)} n(n^{-1}\sum_{j=1}^{n} G_{nj}(t)(1-n^{-1}\sum_{j=1}^{n} G_{nj}(t)))^{2/(2+\delta)}$$

$$= 2\Lambda_n^2 \sum_{i=0}^{n-1}(\alpha(i))^{\delta/(2+\delta)}(t(1-t))^{2/(2+\delta)}$$

$$\leq 2\Lambda_n^2 \sum_{i=0}^{+\infty}(\alpha(i))^{\delta/(2+\delta)}(t(1-t))^{1-2\eta} \quad \text{for} \quad \eta=\delta(4+2\delta)^{-1} .$$

Thus (2.5) is proved if we have (3.2).

COROLLARY 3.1. If among conditions in Theorem 3.1 the function g is replaced by a function for which there exists a decomposition $g = g_c + g_d$ where $g_c \in C^*_{1,b}$ and g_d is a step function with p jumps, say at $a_1,..,a_p$, such that $a_i \in (0,1)$ $(1\leq i\leq p)$, then the conclusion of Theorem 3.1 remains true but $\tilde{\sigma}_J(\{G^*_\ell\})$ defined in (3.7) is replaced by $\hat{\sigma}^2_J(\{G^*_\ell\})$ where

(3.31) $\hat{\sigma}^2_J(\{G^*_\ell\}) =$

$$= \sigma^2_J(\{G^*_\ell\}) (\int_0^1\int_0^1 (u\wedge v)g_c'(u)g_c'(v)dudv - 2\sum_{i=1}^{p} (g_d(a_i-) - g_d(a_i+))\int_0^1 (u\wedge a_i)g_c'(u)du$$

$$- 2\int_0^1 ug_c'(u)du + \sum_{1<i,j\leq p} (a_i\wedge a_j)(g_d(a_i-) - g_d(a_i+))(g_d(a_j-) - g_d(a_j+))$$

$$+ 2\sum_{i=1}^{p} a_i(g_d(a_i-) - g_d(a_i+)) + g^2(1)$$ where $\sigma^2_J(\{G^*_\ell\})$ is defined in (3.3).

4. Convergence of the two sample linear rank statistic. Let $\{Y_{n_1 i}\}$, $1\leq i\leq n_1$ and $\{Z_{n_2 j}\}$, $1\leq j\leq n_2$ be two independent sequences of weakly dependent random variables with

continuous d.f.'s $F^{(1)}_{n_1i}(x)$ and $F^{(2)}_{n_2j}(x)$ respectively, $x\in\mathbb{R}$. Given $n = n_1+n_2$ we set $X_{ni} = Y_{n_1i}$ if $i\le n_1$ and $X_{ni} = Y_{n_2,i-n_1}$ if $i>n_1$. Denote by $\hat{F}^{(1)}_{n_1}(x) = n_1^{-1}\sum_{i=1}^{n_1} c_{n_1i} I_{[X_{ni}\le x]}$ the empirical process based on the first sequence of r.v.'s $\{Y_{n_1i}\}$ and weighted by the regression constants c_{n_1i}. We put $F^{(1)}_{n_1} = E(\hat{F}^{(1)}_{n_1})$.

Then $\mathscr{S}^*_n(J)$ defined by

(4.1) $$\mathscr{S}^*_n(J) = n^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} J(\tfrac{n}{n+1}\hat{F}_n(x))d\hat{F}^{(1)}_{n_1}(x) - \int_{-\infty}^{+\infty} J(F_n(x))dF^{(1)}_{n_1}(x)\Big)$$

is the two–sample linear rank statistic. We suppose that the regression constants c_{n_1i} $(1\le i\le n_1)$ are defined by a function h on $[0,1]$ as

$$c_{n_1i} = h(i/n_1)\,,\quad 1\le i\le n_1,\ n_1\ge 1\,.$$

We assume that $n_1n^{-1} \to \lambda_0 \in (0,1)$.

We have $F_n = n^{-1}\big(\sum_{i=1}^{n_1} F^{(1)}_{n_1i} + \sum_{j=1}^{n_2} F^{(2)}_{n_2j}\big)$.

Let $F^{(1)}_{n_1,i,\ell}$ be the d.f. of $(Y_{n_1i},Y_{n_1\ell})$ and $F^{(2)}_{n_2,j,k}$ be the d.f. of (Z_{n_2j},Z_{n_2k}).

THEOREM 4.1. <u>Suppose the sequences</u> $\{Y_{n_1i}\}$ <u>and</u> $\{Z_{n_2j}\}$ <u>are φ–mixing with rate</u> (3.1) <u>or strong mixing with rate</u> (3.2), <u>the function</u> h <u>satisfies</u> $h = h_c+h_d$ <u>with</u> $h_c\in C^*_{1,b}$ <u>and</u> h_d <u>is a step function and if for each</u> $p>1$, <u>there exist two continuous d.f.'s</u> $\hat{G}^{(1)}_p$ <u>and</u> $\hat{G}^{(2)}_p$ <u>on</u> $\mathbb{R}^2$ <u>with marginals</u> $F^{(1)}$ <u>and</u> $F^{(2)}$ <u>such that</u>

(4.2) $$\lim_{n\to\infty}\max_{1\le i<j\le n} |F^{(\ell)}_{n_\ell i,j}(F_n^{-1}(t_1),F_n^{-1}(t_2)) - \hat{G}^{(\ell)}_{j-i}(H^{-1}(t_1),H^{-1}(t_2))| = 0$$

<u>for all</u> $(t_1,t_2) \in [0,1]^2$, $\ell=1,2$ <u>where</u>

(4.3) $$H = \lambda_0F^{(1)} + (1-\lambda_0)F^{(2)}\,.$$

<u>Then, for every</u> $J\in\mathscr{G}_\delta$ <u>with</u> $2>\delta\ge 0$ <u>if we have</u> (3.1) <u>and</u> $\delta>0$ <u>if we have</u> (3.2).

(4.4) $$\lim D_2(\mathscr{L}(\mathscr{S}^*_n(J)),\ \mathscr{N}(0,\tilde{\sigma}_J(\{G^{(\ell)}_p\})) = 0$$

<u>where</u>

$$(4.5)\qquad \tilde{\sigma}_J^2(\{\hat{G}_p^{(\ell)}\}) = \bar{\sigma}_J^2(\{\hat{G}_p^{(\ell)}\})L(h)$$

where

$$(4.6)\ \bar{\sigma}_J^2(\{G_p^{(\ell)}\}) =$$

$$= \lambda_0^{-1}\{\int_0^1 f_1^2(u)d(F^{(1)}\circ H^{-1})(u) + 2\sum_{p\geq 2}\int_0^1\int_0^1 f_1(u)f_1(v)d(\hat{G}_p^{(1)}(H^{-1}(u),H^{-1}(v))\}$$

$$+ (1-\lambda_0)\{\int_0^1 f_2^2(u)d(F^{(2)}\circ H^{-1})(u) + 2\sum_{p\geq 2}\int_0^1\int_0^1 f_2(u)f_2(v)d(\hat{G}_p^{(2)}(H^{-1}(u),H^{-1}(v))\}\ .$$

Here

$$f_1(u) = f_1^*(u) - \int_0^1 f_1^*(v)d(F^{(1)}\circ H^{-1}(v))$$

$$f_2(u) = f_2^*(u) - \int_0^1 f_2^*(v)d(F^{(2)}\circ H^{-1}(v))$$

with

$$f_1^*(u) = J(u) + \lambda_0\int_u^1 J'(v)d(F^{(1)}\circ H^{-1}(v))$$

$$f_2^*(u) = \int_u^1 J'(v)d(F^{(2)}\circ H^{-1}(v))$$

and

$$(4.7)\ L(h) = \int_0^1\int_0^1 h_c'(u)h_c'(v)dudv - 2\sum_{i=1}^p(h_d(a_i-) - h_d(a_i+))\int_0^1 h_c(u)du - 2\int_0^1 h_c'(u)du$$

$$+ \sum_{1\leq i,j\leq p}(h_d(a_i-) - h_c(a_i+))(h_d(a_j-) - h_d(a_j+)) + 2\sum_{i=1}^p h_d(a_i-) - h_d(a_i+) + h_c^2(1)$$

where a_i, $1\leq i\leq p$ are the discontinuous points of h_d.

REMARK. Let the sequences $\{F_{n_1,i,j}^{(1)}\}$ and $\{F_{n_2,i,j}^{(2)}\}$ satisfy the following conditions: (i) there exist two sequences of d.f.'s $G_p^{(\ell)}$ on $\mathbb{R}^2$, $\ell=1,2$ such that

$$\lim_{n\to\infty}\max_{1\leq i<j\leq n}|F_{n_\ell,i,j}^{(\ell)}(x_1,x_2) - G_{j-i}^{(\ell)}(x_1,x_2)| = 0$$

for all $(x_1,x_2)\in\mathbb{R}^2$, $\ell=1,2$,

(ii) $F^{(1)}_{n_1 i} = F^{(1)}_{n_1}$ for all $1 \le i \le n_1$, and

(iii) $F^{(2)}_{n_2 j} = F^{(2)}_{n_2}$ for all $1 \le j \le n_2$.

Then the condition (4.2) is satisfied when the sequences $\{Y_{n_1 i}\}$ and $\{Z_{n_2 j}\}$ are strong mixing.

PROOF OF THEOREM 4.2. For any $n_1 \ge 1$, for any i $(1 \le i \le n_1)$ and for any $J \in C_{2,b}$, let

$$(4.8)\ B_{n_1 i}(J) = n_1 n^{-1} \int_0^1 (I_{[F_n(Y_{n_1 i}) \le t]} - G^{(1)}_{n_1 i}(t)) J'(t) dG^{(1)}_{n_1 i}(t) + J(F_n(Y_{n_1 i})) - \int_0^1 J(t) dG^{(1)}_{n_1 i}(t)$$

where $G^{(1)}_{n_1 i} = F^{(1)}_{n_1 i} \circ F_n^{-1}$, and for any $n_2 \ge 1$, any j $(1 \le j \le n_2)$, any $J \in C_{2,b}$ and any $u \in [0,1]$, let

$$(4.9)\ C_{n_2 j}(J)(u) = n_2^{-1} \sum_{\ell=1}^{[n_2 u]} \int_0^1 (I_{[F_n(Z_{n_2 j}) \le t]} - G^{(2)}_{n_2 j}(t)) J'(t) dG^{(2)}_{n_2 \ell}(t) + n_2^{-1}(n_2 u - [n_2 u]) \int_0^1 (I_{[F_n(Z_{n_2 j}) \le t]} - G^{(2)}_{n_2 j}(t)) dG^{(2)}_{n_2, [n_2 u]+1}(t)$$

where $G^{(2)}_{n_2 j} = F^{(2)}_{n_2 j} \circ F_n^{-1}$.

Now consider for any $J \in C_{2,b}$ the processes $W_{n_1}(J)(s)$ and $W^*_{n_2}(J)(s,u)$ defined respectively on C_1 and C^*_1 (= the space of continuous functions on $[0,1]^2$) by

$$(4.10) \qquad W_{n_1}(J)(s) = n_1^{\frac{1}{2}} \Big(\sum_{i=1}^{[n_1 s]} B_{n_1 i}(J) + (n_1 s - [n_1 s]) B_{n_1, [n_1 s]+1}(J) \Big)$$

$$(4.11) \qquad W^*_{n_2}(J)(s,u) = n_2^{\frac{1}{2}} \Big(\sum_{j=1}^{[n_2 s]} C_{n_2 j}(J)(u) + (n_2 s - [n_2 s]) C_{n_2, [n_2 s]+1}(J)(u) \Big) .$$

By similar techniques as in Lemma 3.1 one can prove that the process $W_{n_1}(J)(s)$ converges weakly in uniform topology to a Gaussian process $W_0(J)(s)$ with trajectories a.s. in C_1 with mean 0 and variance

$$\text{(4.12)}\quad s[\int_0^1\int_0^1 f_1^2(u)d(F^{(1)}\circ H^{-1})(u) + 2\sum_{p\geq 1}\int_0^1\int_0^1 f_1(u)f_1(v)d(G_p^{(1)}(H^{-1}(u),H^{-1}(v)))]$$

and $W^*_{n_2}(J)(s,u)$ converges weakly in uniform topology to a Gaussian process $W^*_0(J)(s,u)$ with trajectories a.s. in C_1^* with mean 0 and variance

$$\text{(4.13)}\quad su[\int_0^1 f_2^2(u)d(F^{(2)}\circ H^{-1})(u) + 2\sum_{p\geq 1}\int_0^1\int_0^1 f_2(u)f_2(v)d(G_p^{(2)}(H^{-1}(u),H^{-1}(v)))]\,.$$

From this and following Lemma 3.2, it is easy to prove that $V^*_n(J)$ converges in law to the normal distribution with mean 0 and variance $\breve{\sigma}_J(\{\hat{G}_p^{(\ell)}\})$ where $\breve{\sigma}_J(\{\hat{G}_p^{(\ell)}\})$ is defined in (4.5) and $V^*_n(J)$ is a random variable defined by

$$\text{(4.14)}\qquad V^*_n(J) = n^{\frac{1}{2}}(\int_{-\infty}^{+\infty} J(\hat{F}_n(x))d(\hat{F}_n^{(1)}(x) - F_n^{(1)}(x)) + \int_{-\infty}^{+\infty} J'(\hat{F}_n(x))(\hat{F}_n(x) - F_n(x))dF_n^{(1)}(x))\,.$$

Since $\mathscr{S}^*_n(J) = V^*_n(J) + U_n(J)$ where $E(U_n(J))^2 = 0(n^{-\frac{1}{2}})$ we prove Theorem 4.1 following the line of argument in the proof of Theorem 3.1.

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Chapitre 3

Invariance faible de la U-statistique et d'une statistique de rang en condition d'absolue régularité.

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Limiting Behavior of U-Statistics, V-Statistics, and One Sample Rank Order Statistics for Nonstationary Absolutely Regular Processes

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The weak convergences of U- and V-statistics were established by Yoshihara (1976, *Z. Warsch. Verw. Gebiete* **35** 237–252) for stationary absolutely regular processes. Later Yoshihara (1978, *Z. Warsch. Verw. Gebiete* **43** 101–127) also proved the weak convergence of one sample rank order statistics under similar conditions. In this paper, we extend some of Yoshihara's results to the nonstationary cases. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let X_{ni}, $1 \leqslant i \leqslant n$, $n \geqslant 1$ be real-valued r.v.'s (random variables) with continuous d.f.'s (distribution functions) $F_{ni}(x)$, $x \in \mathbb{R}$. The d.f. of the $\mathbb{R}^n$-valued r.v. $(X_{n1}, ..., X_{nn})$ is denoted by $\mathbf{F}_n$.

In the first part of the paper, we will study the asymptotic behavior of the U-statistic

$$U(\mathbf{F}_n) = \binom{n}{k}^{-1} \sum_{(i)}^{(n)} g(X_{ni_1}, ..., X_{ni_k}), \qquad n \geqslant k \geqslant 1, \tag{1.1}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and g: $\mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function which is symmetric in its k $(\geqslant 1)$ arguments.

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From the study of $U(\mathbf{F}_n)$, we will deduce the asymptotic behavior of the V-static

$$V(\mathbf{F}_n)=n^{-k}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} g(X_{ni_1},...,X_{ni_k}) \tag{1.2}$$

called the Von Mises' statistic.

We assume that the underlying r.v.'s are absolutely regular with rates

$$\beta(m)=O(m^{-(2+\delta)/\delta}) \qquad \text{for some} \quad \delta>0. \tag{1.3}$$

Recall that the sequence is absolutely regular if

$$\sup_{m\leqslant n}\ \max_{1\leqslant j\leqslant n-m}\ E\{\sup_{A\in\sigma(X_{ni},i\geqslant j+m)} |P(A\mid\sigma(X_{ni},1\leqslant i\leqslant j))-P(A)|\}=\beta(m)\downarrow 0.$$

Here $\sigma(X_{ni},\ 1\leqslant i\leqslant j)$ and $\sigma(X_{ni},\ i\geqslant j+m)$ are the σ-fields generated by $(X_{n1},...,X_{nj})$ and $(X_{n,j+m},X_{n,j+m+1},...,X_{nn})$, respectively. Also recall that $\{X_{ni}\}$ satisfies the strong mixing condition if $\sup_{m\leqslant n}\sup_{1\leqslant j\leqslant n-m}\{|P(A\cap B)-P(A)\,P(B)|;\ A\in\sigma(X_{ni},\ 1\leqslant i\leqslant j),\ B\in\sigma(X_{ni},\ i\geqslant j+m)\}=\alpha(m)\downarrow 0$. Since $\alpha(m)\leqslant\beta(m)$, it follows that if $\{X_{ni}\}$ is absolutely regular, then it is also strong mixing.

In the second part of the paper, we will study the asymptotic behavior of the one sample rank order statistic $\mathscr{S}_{n,m}$

$$\mathscr{S}_{n,m}=\sum_{i=1}^{m} c_{ni}s(X_{ni})\,J\left(\frac{R_{n,m,i}}{m+1}\right), \qquad n\geqslant m\geqslant 1, \tag{1.4}$$

where J is a score function, $s(x)=\operatorname{sgn}(x)$, and the c_{ni} are regression constants defined by a continuous function $h(x)$ on $[0, 1]$ as

$$c_{ni}=h\left(\frac{i}{n}\right), \qquad 1\leqslant i\leqslant n,\ n\geqslant 1 \tag{1.5}$$

and $R_{n,m,i}=\sum_{j=1}^{m} I_{[|X_{nj}|\leqslant|X_{ni}|]}$, $1\leqslant i\leqslant m\leqslant n$, where $I_{[\ ]}$ denotes the indicator function. We assume also that the r.v.'s are absolutely regular but with rates

$$\beta(m)=O(m^{-4}). \tag{1.6}$$

The convergence of the U-statistic as well as the V-statistic was established by Yoshihara [8] and afterwards, for the one sample rank order statistic also, by Yoshihara [9] but only for stationary r.v.'s. Later, Denker and Keller [3] proved some limit theorems of the above-mentioned statistics for the processes which are uniformly mixing in both directions of time. In this paper we extend the results of Yoshihara [8, 9] from the stationary cases to the nonstationary cases.

Our results are the adaptions of some of the ideas of Yoshihara (loc. cit.) and a new central limit theorem (Lemma 2.3) for nonstationary unbounded strongly mixing random variables which is an extension of a central limit theorem due to Withers [6] for nonstationary uniformly bounded strongly mixing random variables.

To prove the weak convergence of the U-statistic $U(\mathbf{F}_n)$ in Theorem 3.1, we first decompose it as $\theta(\mathbf{F}_n)+\sum_{p=1}^{k}\binom{k}{p}U_n^{(p)}$, where $\theta(\mathbf{F}_n)$ is a centering constant and the $U_n^{(p)}$'s are some random variables (defined in (2.2)). Using Lemma 2.3, we prove that for $p=1$, $n^{1/2}U_n^{(p)}$ has a limiting normal distribution, and for $p\geqslant 2$, $n^{1/2}U_n^{(p)}\to 0$ in probability, as $n\to\infty$ (Lemma 2.2). The limiting behavior of the Von Mises V-statistic $V(\mathbf{F}_n)$ is the same as that of the U-statistic $U(\mathbf{F}_n)$, since $n^{1/2}\,|U(\mathbf{F}_n)-V(\mathbf{F}_n)|\to 0$ in probability (Corollary 3.1). To prove the weak convergence of rank statistics in Theorem 4.1, we have to verify that the finite-dimensional distributions of the process $L_n(t)$ defined in (4.7) satisfy the conditions of our Lemma 2.3, and the probability measure defined by $L_n(t)$ is tight.

For reasons of brevity and to avoid repetitious arguments, we have either omitted or given brief outlines of the proofs that are similar to those of Yoshihara.

2. Preliminaries

2.1. *Definitions*

For every p $(1\leqslant p\leqslant k)$ and $n\geqslant k$, let $1\leqslant i_1\neq\cdots\neq i_p\leqslant n$ be arbitrary integers. Put

$$g_{p,n}^{(i_1,\ldots,i_p)}(x_1,\ldots,x_p)=\sum_{(i_{p+1},\ldots,i_k)\in I_{p,n}(i_1,\ldots,i_p)}\lambda(x_1,\ldots,x_p), \tag{2.1}$$

where

$$\lambda(x_1,\ldots,x_p)=\int_{\mathbb{R}^{k-p}} g(x_1,\ldots,x_k)\,dF_{ni_{p+1}}(x_{p+1})\cdots dF_{ni_k}(x_k),$$

and $I_{p,n}(i_1,\ldots,i_p)=\{(i_{p+1},\ldots,i_k);\ 1\leqslant i_{p+1}\neq\cdots\neq i_k\leqslant n,\ i_l\notin\{i_1,\ldots,i_p\},\ p+1\leqslant l\leqslant k\}$ and $g_{0,n}=\sum_{(i_1,\ldots,i_k)\in I_{0,n}}\int_{\mathbb{R}^k} g(x_1,\ldots,x_k)\,dF_{ni_1}\cdots dF_{ni_k}$, where $I_{0,n}=\{(i_1,\ldots,i_k);\ 1\leqslant i_1\neq\cdots\neq i_k\leqslant n\}$.

For every p $(1\leqslant p\leqslant k)$, set

$$U_n^{(p)}=n^{-[k]}\sum_{1\leqslant i_1\neq\cdots\neq i_p\leqslant n}\int_{\mathbb{R}^p} g_{p,n}^{(i_1,\ldots,i_p)}(x_1,\ldots,x_p)\times\prod_{j=1}^{p} d(I_{[X_{ni_j}\leqslant x_j]}-F_{ni_j}(x_j)) \tag{2.2}$$

where $n^{-[k]}=n(n-1)\cdots(n-k+1)$.

Also for every p, $(0 \leqslant p \leqslant k)$, we write

$$g^*_{p,n}(x_1, ..., x_p) = \sum_{(i_{p+1}, ..., i_k) \in \{1, ..., n\}^{k-p}} \times \int_{\mathbb{R}^{k-p}} g(x_1, ..., x_k)\, dF_{ni_{p+1}} \cdots dF_{ni_k} \tag{2.3}$$

and

$$V_n^{(p)} = n^{-k} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} g^*_{p,n}(x_1, ..., x_p) \times \prod_{j=1}^{p} d(I_{[X_{ni_j} \leqslant x_j]} - F_{ni_j}(x_j)). \tag{2.4}$$

Let $p \geqslant 1$ and $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$ be arbitrary integers. For any j $(1 \leqslant j \leqslant p-1)$, $P_{j,n}^{(i_1, ..., i_p)}$ is the probability measure defined by

$$P_{j,n}^{(i_1, ..., i_p)}(A^{(j)} \times B^{(p-j)}) = P[(X_{ni_1}, ..., X_{ni_j}) \in A^{(j)}] \times P[(X_{ni_{j+1}}, ..., X_{ni_p}) \in B^{(p-j)}]$$

and $P_{0,n}^{(i_1, ..., i_p)}$ is the probability measure defined by

$$P_{0,n}^{(i_1, ..., i_p)}(A^{(p)}) = P[(X_{ni_1}, ..., X_{ni_p}) \in A^{(p)}]$$

for any $A^{(j)} \in \sigma(X_{ni_1}, ..., X_{ni_j})$ $(1 \leqslant j \leqslant p)$ and any $B^{(p-j)} \in \sigma(X_{ni_{j+1}}, ..., X_{ni_p})$ $(1 \leqslant j \leqslant p-1)$.

For any function h: $\mathbb{R}^k \to \mathbb{R}$, we denote by $h \otimes h$ the function $h \otimes h$: $\mathbb{R}^{2k} \to \mathbb{R}$ defined by $h \otimes h(x_1, ..., x_{2k}) = h(x_1, ..., x_k)\, h(x_{k+1}, ..., x_{2k})$.

2.2. *Basic Lemmas*

LEMMA 2.1. *For every $p \geqslant 1$ and $(i_1, ..., i_p)$ such that $i_1 < i_2 < \cdots < i_p$ and any j $(0 \leqslant j \leqslant p-1)$, let $h(x_1, ..., x_p)$ be a Borel function such that*

$$\int_{\mathbb{R}^p} |h(x_1, ..., x_p)|^{1+\delta}\, dP_{j,n}^{(i_1, ..., i_p)} \leqslant M$$

for some $\delta > 0$, then

$$\left| \int_{\mathbb{R}^p} h(x_1, ..., x_p)\, dP_{0,n}^{(i_1, ..., i_p)} - \int_{\mathbb{R}^p} h(x_1, ..., x_p)\, dP_{j,n}^{(i_1, ..., i_p)} \right| \leqslant 4M^{1/(2+\delta)} \beta^{\delta/(1+\delta)}(i_{j+1} - i_j). \tag{2.5}$$

As a special case, if $h(x_1, ..., x_p)$ is bounded, say, $|h(x_1, ..., x_p| \leqslant M$, then we can replace the right side of (2.5) *by $2M\beta(i_{j+1} - i_j)$.*

Proof. Follows from Lemma 1 of Yoshihara [8].

LEMMA 2.2. *If there is a positive number δ' such that for $r = 2 + \delta'$,*

$$\sup_n \max_{(i_1,...,i_k) \in \{1,...,n\}^k} \int_{\mathbb{R}^k} |g(x_1, ..., x_k)|^r \, dF_{ni_1} \cdots dF_{ni_k} \leqslant M_0 < \infty \quad (2.6)$$

$$\sup_n \max_{(i_1,...,i_k) \in \{1,...,n\}^k} E(|g(X_{ni_1}, ..., X_{ni_k})|^r) \leqslant M_0 < \infty \quad (2.7)$$

and for some δ $(0 < \delta < \delta')$, $\beta(n) = O(n^{-(2+\delta)/\delta})$, then we have

$$E(U_n^{(p)})^2 = O(n^{-1-\gamma}), \qquad 2 \leqslant p \leqslant k, \quad (2.8)$$

$$E(V_n^{(p)})^2 = O(n^{-1-\gamma}), \qquad 1 \leqslant p \leqslant k, \quad (2.9)$$

where $\gamma = 2(\delta - \delta')/\delta'(2 + \delta) > 0$.

Proof. It suffices to prove (2.8) because the proof of (2.9) is similar. Also it suffices to prove (2.8) for $p = 2$; the proofs for $p = 3, ..., k$ are analogous and are therefore omitted. We first note that

$$U_n^{(2)} = n^{-[k]} \sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \int_{\mathbb{R}^2} g_{2,n}^{(i_1,i_2)}(x_1, x_2) \, d(I_{[X_{ni_1} \leqslant x_1]} - F_{ni_1}(x_1)) \, d(I_{[X_{ni_2} \leqslant x_2]} - F_{ni_2}(x_2)), \quad (2.10)$$

so we have

$$E(U_n^{(2)})^2 = (n^{-[k]})^2 \sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \sum_{1 \leqslant j_1 \neq j_2 \leqslant n} J((i_1, i_2), (j_1, j_2)), \quad (2.11)$$

where

$$\begin{aligned} J((i_1, i_2), (j_1, j_2)) = E \Big\{ & \int_{\mathbb{R}^2} g_{2,n}^{(i_1,i_2)}(x_1, x_2) \, d(I_{[X_{ni_1} \leqslant x_1]} - F_{ni_1}(x_1)) \\ & \times d(I_{[X_{ni_2} \leqslant x_2]} - F_{ni_2}(x_2)) \Big\} \\ & \times \Big\{ \int_{\mathbb{R}^2} g_{2,n}^{(j_1,j_2)}(y_1, y_2) \, d(I_{[X_{nj_1} \leqslant y_1]} - F_{nj_1}(y_1)) \\ & \times d(I_{[X_{nj_2} \leqslant y_2]} - F_{nj_2}(y_2)) \Big\} \end{aligned} \quad (2.12)$$

and

$$g_{2,n}^{(i_1,i_2)}(x_1, x_2) = \sum_{(i_3,\ldots,i_k)\in I_{2,n}(i_1,i_2)} \int_{\mathbb{R}^{k-2}} g(x_1, {}_2, \ldots, x_k)\, dF_{ni_3}(x_3)\cdots dF_{ni_k}(x_k).$$

We deduce

$$\begin{aligned} J((i_1, i_2), (j_1, j_2)) = & \sum_{(i_3,\ldots,i_k)\in I_{2,n}(i_1,i_2)} \sum_{(j_3,\ldots,j_k)\in I_{2,n}(j_1,j_2)} \\ & \times E\Bigg(\Bigg\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^{k-2}} g(x_1, x_2, \ldots, x_k) \\ & \times dF_{ni_3}(x_3)\cdots dF_{ni_k}(x_k)(d(I_{[X_{ni_1}\leqslant x_2]} - F_{ni_1}(x_1)) \\ & \times d(I_{[X_{ni_2}\leqslant x_2]} - F_{ni_2}(x_2))\Bigg\} \\ & \times \Bigg\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^{k-2}} g(y_1, y_2, \ldots, y_k)\, dF_{nj_3}(y_3)\cdots dF_{nj_k}(y_k) \\ & \times (d(I_{[X_{nj_1}\leqslant y_1]} - F_{nj_1}(y_1)) \\ & \times d(I_{[X_{nj_2}\leqslant y_2]} - F_{nj_2}(y_2))\Bigg\}\Bigg), \end{aligned} \tag{2.13}$$

since for any $(i_1, \ldots, i_k)\in I_{0,n}$,

$$\begin{aligned} \int_{\mathbb{R}} \Bigg(& \int_{\mathbb{R}^{k-2}} g(x_1, x_2, \ldots, x_k)\, dF_{ni_3}(x_3)\cdots dF_{ni_k}(x_k) \\ & - \int_{\mathbb{R}^{k-1}} g(x_1, x_2, \ldots, x_k)\, dF_{ni_2}(x_2)\cdots dF_{ni_k}(x_k) \\ & - \int_{\mathbb{R}^{k-1}} g(x_1, x_2, \ldots, x_k)\, dF_{ni_1}(x_1)\, dF_{ni_3}(x_3)\cdots dF_{ni_k}(x_k) \\ & + \int_{\mathbb{R}^k} g(x_1, x_2, \ldots, x_k)\, dF_{ni_1}(x_1)\cdots dF_{ni_k}(x_k)\Bigg)\, dF_{ni_j}(x_j) = 0, \qquad j = 1, 2. \end{aligned}$$

So from Lemma 2.1 we have the inequalities

(i) If $1 \leqslant i_1 < i_2 \leqslant j_1 < j_2 \leqslant n$ and $j_2 - j_1 \geqslant i_2 - i_1$, then

$$\begin{aligned} & J((i_1, i_2), (j_1, j_2)) \\ & \quad \leqslant \sum_{(i_3,\ldots,i_k)\in I_{2,n}(i_1,i_2)} \sum_{(j_3,\ldots,j_k)\in I_{2,n}(j_1,j_2)} M_0^{1/(1+\delta')}\beta^{\delta'/(2+\delta')}(j_2 - j_1) \\ & \quad = \left(\frac{n^{[k]}}{n(n-1)}\right)^2 M_0^{1/(1+\delta')}\beta^{\delta'/(2+\delta')}(j_2 - j_1) \end{aligned} \tag{2.14}$$

and similarly if $1 \leqslant i_1 < i_2 \leqslant j_1 < j_2 \leqslant n$ and $i_2 - i_1 \geqslant j_2 - j_1$, then

$$J((i_1, i_2), (j_1, j_2)) \leqslant \left(\frac{n^{[k]}}{n(n-1)}\right)^2 M_0^{1/(1+\delta')} \beta^{\delta'/(2+\delta')}(i_2 - i_1). \quad (2.15)$$

Thus, using (2.14), (2.15), and (1.3), we obtain

$$\left| \sum_{1 \leqslant i_1 < i_2 \leqslant j_1 < j_2 \leqslant n} J((i_1, i_2), (j_1, j_2)) \right| = O(n^{2k-1-\gamma}). \quad (2.16)$$

Similarly,

$$\left| \sum_{1 \leqslant i_1 < j_1 \leqslant i_2 < j_2 \leqslant n} J((i_1, i_2), (j_1, j_2)) \right| = O(n^{2k-1-\gamma}) \quad (2.17)$$

$$\left| \sum_{1 \leqslant i_1 < j_1 < j_2 < i_2 \leqslant n} J((i_1, i_2), (j_1, j_2)) \right| = O(n^{2k-1-\gamma}) \quad (2.18)$$

$$\left| \sum_{1 \leqslant i_1, j_1 \leqslant n} \sum_{i_2 = 1}^{n} J((i_1, i_2), (j_1, i_2)) \right| = O(n^{2k-2}). \quad (2.19)$$

From (2.16)–(2.19) and (2.11) we have (2.8) for $p = 2$.

LEMMA 2.3. *Let $\{X^*_{ni}\}$ be r.v.'s (with mean 0) which are strong mixing with rates satisfying*

$$\sum_{m=1}^{\infty} (\alpha(m))^{\delta''/(2+\delta'')} < \infty \qquad \textit{for some} \quad \delta'' > 0. \quad (2.20)$$

Suppose that for any $K > 0$ there exists a sequence $\{Y^K_{ni}\}$ of r.v.'s satisfying (2.20) such that

$$\sup_{n \in \mathbb{N}^*} \max_{1 \leqslant i \leqslant n} |Y^K_{ni}| \leqslant B_K < \infty, \forall K > 0, \quad (2.21)$$

where B_K is some constant > 0;

$$\sup_{n \in \mathbb{N}^*} \max_{1 \leqslant i \leqslant n} E|X^*_{ni} - Y^K_{ni}|^{2+\delta''} \to 0 \qquad \textit{as} \quad K \to \infty \quad (2.22)$$

$$E\left(\sum_{i=1}^{n} X^*_{ni}\right)^2 \Big/ n \to C < \infty \textit{ as } n \to \infty, \textit{ where } C \textit{ is some constant} > 0; \quad (2.23)$$

$$E \sum_{i=1}^{n} (Y^K_{ni} - E(Y^K_{ni}))^2/n \to C_K < \infty \qquad \forall K > 0 \textit{ as } n \to \infty, \quad (2.24)$$

where C_K is some constant > 0;

$$C_K \to C \qquad \textit{as} \qquad K \to \infty; \quad (2.25)$$

then $n^{-1/2}\sum_{i=1}^{n} X_{ni}^{*}$ *converges in law to the normal distribution with mean* 0 *and variance* C^2.

Proof. From Corollary 1 of Withers [6], Lemma 2.3 holds if $\{X_{ni}^{*}\}$ is uniformly bounded. Thus, $\forall K>0$, $n^{-1/2}\sum_{i=1}^{n}(Y_{ni}^{K}-E(Y_{ni}^{K}))$ converges in law to the normal distribution with mean 0 and variance C_K^2.

Now denoting $Z_{ni}^{K}=X_{ni}^{*}-Y_{ni}^{K}$ and using Ibragimov [5], we represent the sum $S_n=n^{-1/2}C^{-1}(\sum_{i=1}^{n}X_{ni}^{*})$ as $S_n=S'_n+S''_n$, where $S'_n=n^{-1/2}C_K^{-1}\sum_{i=1}^{n}(Y_{ni}^{K}-E(Y_{ni}^{K}))\,C_K C^{-1}$ and $S''_n=n^{-1/2}C^{-1}\sum_{i=1}^{n}(Z_{ni}^{K}-E(Z_{ni}^{K}))$. Then, using the well-known inequality on moments of strong mixing sequences of r.v.'s [4, Proposition 8] and omitting the details of computations, we obtain

$$E\,|S''_n|^2\leqslant 2n^{-1}C^{-2}\sum_{i=0}^{n-1}\alpha(i))^{\delta''/(2+\delta'')}(n-i)2^{2+2\delta''}$$
$$\times\sup_{n\in\mathbb{N}^*}\max_{1\leqslant i\leqslant n}(E\,|Z_{ni}^{K}|^{2+\delta''})^{2/(2+\delta'')}.$$

Let $D_K=\sup_{n\in\mathbb{N}^*}\max_{1\leqslant i\leqslant n}(E\,|Z_{ni}^{K}|^{2+\delta''})^{2/(2+\delta'')}$. Then, from (2.24), it follows that

$$E\,|S''_n|^2\leqslant 2^{3+\delta''}C^{-2}\left(\sum_{i=0}^{\infty}(\alpha(i))^{\delta''/(2+\delta'')}\right)D_K\to 0\qquad\text{as}\quad K\to\infty.$$

For any $\varepsilon>0$, we can always choose K so large that $E\,|S''_n|^2\leqslant\varepsilon$ and $|1-C_K^2/C^2|\leqslant\varepsilon$. If now $f_n(t)=E(e^{itS_n})$ is the characteristic function of S_n, then

$$|f_n(t)-e^{-t^2/2}|\leqslant|e^{-(t^2/2)(C_K^2/C^2)}-e^{-t^2/2}|$$
$$+|e^{-(t^2/2)(C_K^2/C^2)}-Ee^{itS'_n}|+|t|\;E\,|S''_n|$$
$$\leqslant|t|\sqrt{\varepsilon}+\varepsilon t^2+|e^{-(t^2/2)(C_K^2/C^2)}-Ee^{itS'_n}|$$

from the convergence in law of S'_n to the normal law, the last expression converges to zero as $n\to\infty$.

3. Convergence of the U-Statistic and the V-Statistic

Let $\mathbf{F}_{n,i,j}$ be the d.f. of (X_{ni},X_{nj}), $1\leqslant i<j\leqslant n$, $n\geqslant 1$.

Theorem 3.1. *If there are two positive numbers* δ, δ' $(0<\delta<\delta')$, *such that* (1.3), (2.6), *and* (2.7) *hold, and if, further, for any* $i, j\in\mathbb{N}^*$, *with* $i<j$,

there exists a continuous d.f. $\mathbf{F}_{ij}$ *on* $\mathbb{R}^2$ *with continuous marginals* F_i^* *and* F_j^* *such that*

$$\lim_{n\to\infty} \max_{1\leqslant i<j\leqslant n} |\mathbf{F}_{n,i,j}(x_1, x_2) - \mathbf{F}_{ij}(x_1, x_2)| = 0 \qquad \textit{for all} \quad (x_1, x_2) \in \mathbb{R}^2 \tag{3.1}$$

and

$$\mathbf{F}_{ij} = \mathbf{F}_{1,j-i+1} \qquad \textit{for all} \quad i<j. \tag{3.2}$$

Furthermore, if

$$\int_{\mathbb{R}^k} |g(x_1, ..., x_k)|^r \prod_{j=1}^{k} dF(x_j) \leqslant M_0 < \infty, \tag{3.3}$$

where M_0 *is some constant* >0 *and* $F = F_i^*$ *for all* $i \in \mathbb{N}^*$ *and g is right continuous and has left-hand limits (rcll) or left continuous and has right-hand limits (lcrl), then* $n^{1/2}(U(F_n) - \theta(F_n))$ *converges in lax to the normal distribution with mean* 0 *and variance* $k^2\sigma^2$ *if* $\sigma^2 > 0$, *where* $\theta(F_n) = n^{-[k]} g_{0,n}$ *and*

$$\begin{aligned}\sigma^2 = &\left[\int_{\mathbb{R}^k} g^2(x_1, ..., x_k) \prod_{l=1}^{k} dF(x_l) - \theta^2(F)\right] \\ &+ 2\sum_{i=2}^{\infty} \left[\int_{\mathbb{R}^{2k}} g(x_1, ..., x_k)\, g(x_{k+1}, ..., x_{2k}) \right. \\ &\left. \times dF_{1i}(x_1, x_{k+1}) \prod_{l=2}^{k} dF(x_l) \prod_{p=k+2}^{2k} dF(x_p) - \theta^2(F)\right] < \infty,\end{aligned} \tag{3.4}$$

where $\theta(F) = \int_{\mathbb{R}^k} g(x_1, ..., x_k) \prod_{l=1}^{k} dF(x_l)$.

Note. The condition (3.2) implies that the limiting distribution $F_{ij}(x, y)$ of some random variable, say (X_i, X_j) is such that the marginal d.f. F_i^* of X_i are identical to F_1^* $(=F)$ for all $i \in \mathbb{N}^*$.

Proof. We rewrite (1.1) as

$$\begin{aligned}U(F_n) &= n^{-[k]} \sum_{(i_1, ..., i_k) \in I_{0,n}} \int_{\mathbb{R}^k} g(x_1, ..., x_k) \prod_{j=1}^{k} d(I_{[X_{ni_j} \leqslant x_j]}) \\ &= \theta(F_n) + \sum_{p=1}^{k} \binom{k}{p} U_n^{(p)}.\end{aligned}$$

From (2.8) of Lemma 2.2, it is sufficient to prove that $n^{1/2} U_n^{(1)}$ converges in law to the normal distribution with mean 0 and variance σ^2.

From (2.2), we note that $U_n^{(1)} = n^{-1} \sum_{i=1}^n X_{ni}^*$, where

$$X_{ni}^* = n^{-[k-1]} \left(g_{1,n}^{(i)}(X_{ni}) - \int_{\mathbb{R}} g_{1,n}^{(i)}(x_1)\, dF_{ni}(x_1) \right).$$

To prove the assertion, we have to show that $\{X_{ni}^*\}$ satisfies the conditions of Lemma 2.3. Note that (2.20) comes from (1.3) for $\delta < \delta'' < \delta'$. Now define Y_{ni}^K for any $K > 0$ by

$$\begin{aligned} Y_{ni}^K = n^{-[k-1]} & \sum_{(i_2, \ldots, i_k) \in I_{1,n}(i)} \\ & \times \left(\int_{\mathbb{R}^{k-1}} g^K(X_{ni}, x_2, \ldots, x_k) \prod_{j=2}^k dF_{ni_j}(x_j) \right. \\ & \left. - \int_{\mathbb{R}^k} g^K(x_1, \ldots, x_k) \prod_{j=2}^k dF_{ni_j}(x_j)\, dF_{ni}(x_1) \right), \end{aligned} \tag{3.5}$$

where g^K is defined by

$$g^K = \begin{cases} g & \text{if } |g| \leqslant K \\ 0 & \text{if } |g| > K. \end{cases} \tag{3.6}$$

Then $\sup_{n \in \mathbb{N}^*} \max_{1 \leqslant i \leqslant n} |Y_{ni}^K| \leqslant 2K$ which proves (2.21).

We now prove (2.22). Set $g^{-K} = g - g^K$. Then

$$\begin{aligned} & (E|X_{ni}^* - Y_{ni}^K|^{2+\delta''})^{1/(2+\delta'')} \\ & \leqslant n^{-[k-1]} \sum_{(i_2, \ldots, i_k) \in I_{1,n}(i)} \left(\int_{\mathbb{R}} 2^{1+\delta''} \left(\int_{\mathbb{R}^{k-1}} |\bar{g}^K(x_1, \ldots, x_k)|^{2+\delta''} \right. \right. \\ & \quad \left. + \int_{\mathbb{R}} |\bar{g}^K(y_1, x_2, \ldots, x_k)|^{2+\delta''} dF_{ni}(y_1) \right) \\ & \quad \left. \times \prod_{j=2}^k dF_{ni_j}(x_j))\, dF_{ni}(x_1) \right)^{1/(2+\delta'')}. \end{aligned}$$

From (2.6) we deduce

$$(E|X_{ni}^* - Y_{ni}^K|^{2+\delta''})^{1/(2+\delta'')} \leqslant 2^{2+\delta''} \frac{M_0}{(K^{2+\delta''})^{\varepsilon}} \to 0 \qquad \text{as} \quad K \to \infty,$$

where $(2+\delta'')(1+\varepsilon) = 2+\delta'$ and (2.22) is proved.

To prove (2.23) and (2.24) we first prove the following lemma.

LEMMA 3.1. *Let h: $\mathbb{R}^k \to \mathbb{R}$ be a function such that h is rcll (or lcrl) and suppose that*

$$\sup_n \max_{(i_1, ..., i_k) \in \{1, ..., n\}^k} \int_{\mathbb{R}^k} |h(x_1, ..., x_k)|^r \prod_{j=1}^{k} dF_{ni_j}(x_j) < M < \infty \tag{3.7}$$

$$\int_{\mathbb{R}^k} |h(x_1, ..., x_k)|^r \prod_{j=1}^{k} dF(x_j) < M < \infty, \tag{3.8}$$

where M is some constant > 0, then

$$\lim_{n \to \infty} \max_{1 \leqslant i_1 < i_{k+1} \leqslant n} \Bigg| (n^{-[k-1]})^2 \Bigg(\sum_{(i_2, ..., i_k) \in I_{1,n}(i_1)} \sum_{(i_{k+2}, ..., i_{2k}) \in I_{1,n}(i_{k+1})}$$

$$\times \int_{\mathbb{R}^{2k}} (h \otimes h)(x_1, ..., x_{2k})$$

$$\times \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF_{ni_j}(x_j)\, d\mathbf{F}_{n, i_1, i_{k+1}}(x_1, x_{k+1}) \Bigg)$$

$$- \int_{\mathbb{R}^{2k}} (h \otimes h)(x_1, ..., x_{2k})$$

$$\times \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF(x_j)\, d\mathbf{F}_{i_1 i_{k+1}}(x_1, x_{k+1}) \Bigg| = 0. \tag{3.9}$$

Proof. From (3.1) and (3.2) it follows that (3.9) is true if h is a step function. For any $K > 0$, we define the function h^K by

$$h^K = \begin{cases} h & \text{if } |h| \leqslant K \\ 0 & \text{if } |h| > K. \end{cases}$$

As h^K is rcll (or lcrl) and bounded, it is well known that h^K can be uniformly approached by a step function. Let $\varepsilon > 0$. Choose K sufficiently large so that

$$\max \left\{ \frac{M^2}{K^{r-1}}, \frac{M}{K^{r-2}} \right\} < \frac{\varepsilon}{6}. \tag{3.10}$$

Then, there exists a step function g_ε^K such that

$$\sup_{(x_1, ..., x_k)} |h^K(x_1, ..., x_k) - g_\varepsilon^K(x_1, ..., x_k)| < \frac{\varepsilon}{6K} \tag{3.11}$$

and there exists $N_0 \in \mathbb{N}$ such that $\forall n \geqslant N_0$,

$$\max_{1 \leqslant i_1 < i_{k+1} \leqslant n} \Bigg| (n^{-[k-1]})^2 \Bigg(\sum_{(i_2, ..., i_k) \in I_{1,n}(i_1)} \sum_{(i_{k+2}, ..., i_{2k}) \in I_{1,n}(i_{k+1})}$$

$$\times \int_{\mathbb{R}^{2k}} (g_\varepsilon^K \otimes g_\varepsilon^K)(x_1, ..., x_{2k})$$

$$\times \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF_{ni_j}(x_j) \, d\mathbf{F}_{n, i_1, i_{k+1}}(x_1, x_{k+1}) \Bigg)$$

$$- \int_{\mathbb{R}^{2k}} (g_\varepsilon^K \otimes g_\varepsilon^K)(x_1, ..., x_{2k})$$

$$\times \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF_{i_1 i_{k+1}}(x_1, x_{k+1}) \Bigg| < \varepsilon/3. \tag{3.12}$$

From (3.7), (3.8), (3.10)–(3.12) and the decomposition $h \otimes h = h \otimes h - h \otimes h^K + h \otimes h^K - h^K \otimes h^K + h^K \otimes h^K - h^K \otimes g_\varepsilon^K + h^K \otimes g_\varepsilon^K - g_\varepsilon^K \otimes g_\varepsilon^K + g_\varepsilon^K \otimes g_\varepsilon^K$, we deduce that $\forall n \geqslant N_0$,

$$\max_{1 \leqslant i_1 < i_{k+1} \leqslant n} \Bigg| (n^{-[k-1]})^2 \Bigg(\sum_{(i_2, ..., i_k) \in I_{1,n}(i_1)} \sum_{(i_{k+2}, ..., i_{2k}) \in I_{1,n}(i_{k+1})}$$

$$\times \int (h \otimes h)(x_1, ..., x_{2k}) \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF_{ni_j}(x_j)$$

$$\times dF_{n, i_1, i_{k+1}}(x_1, x_{k+1}) \Bigg) - \int_{\mathbb{R}^{2k}} (h \otimes h)(x_1, ..., x_{2k})$$

$$\times \prod_{\substack{j \neq 1 \\ j \neq k+1}} dF(x_j) \, dF_{i_1 i_{k+1}}(x_1, x_{k+1}) \Bigg|$$

$$< \frac{M^2}{K^{r-1}} + \frac{M}{K^{r-2}} + K\frac{\varepsilon}{6K} + K\frac{\varepsilon}{6K} + \varepsilon/3 < \varepsilon.$$

As ε is chosen arbitrarily, (3.9) is proved.

We now prove (2.23). Denote

$$\rho(1) = \int_{\mathbb{R}^k} g^2(x_1, ..., x_k) \prod_{l=1}^{k} dF(x_l) - \theta^2(F)$$

$$\rho(i) = 2 \Bigg[\int_{\mathbb{R}^{2k}} g(x_1, ..., x_k) \, g(x_{k+1}, ..., x_{2k}) \, d\mathbf{F}_{1i}(x_1, x_{k+1})$$

$$\times \prod_{l=2}^{k} dF(x_l) \prod_{p=k+2}^{2k} dF(x_p) - \theta^2(F) \Bigg], \qquad \forall i \geqslant 2,$$

$$\psi(i, i) = \int_{\mathbb{R}} (g_{1,n}^{(i)}(x_1))^2 \, dF_{ni}(x_1) - \left(\int_{\mathbb{R}} (g_{1,n}^{(i)}(x_1) \, dF_{ni}(x_1) \right)^2, \qquad \forall i \geqslant 1,$$

$$\psi(i, j) = 2 \int_{\mathbb{R}^2} (g_{1,n}^{(j)}(x_1) \, g_{1,n}^{(i+j)}(y_1) \, dF_{n,j,i+j}(x_1, y_1) - \left(\int_{\mathbb{R}} (g_{1,n}^{(j)}(x_1 \, dF_{nj}(x_1)) \times \left(\int_{\mathbb{R}} g_{1,n}^{(i+j)}(y_1) \, dF_{n,i+j}(y_1) \right), \forall (i, j) \in \mathbb{N}^* \right)^2, \qquad i < j.$$

Then,

$$\begin{aligned} &\left| E\left(\left(\sum_{i=1}^{n} X_{ni}^* \right)^2 \Big/ n \right) - \sigma^2 \right| \\ &= \left| \frac{1}{n} (n^{-[k-1]})^2 \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \psi(i, j) - \sum_{i=1}^{\infty} \rho(i) \right| \\ &\leqslant \left| \frac{1}{n} (n^{-[k-1]})^2 \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \psi(i, j) - \frac{1}{n} \sum_{i=1}^{n} (n-i) \, \rho(i) \right| \\ &\quad + \sum_{i=n+1}^{\infty} |\rho(i)| + \sum_{i=1}^{n} \sum_{k=i}^{\infty} |\rho^k| \\ &= |A_n| + B_n + C_n. \end{aligned}$$

From property (3.9) of Lemma 3.1, it follows that $|A_n| \to 0$ as $n \to \infty$.

From (3.3) and the well-known inequality on moments of strong mixing sequences of r.v.'s, we deduce $|\rho(i)| \leqslant (\alpha(i))^{\delta'/(2+\delta')} (M_0)^{2/(2+\delta')}$ which implies $B_n \to 0$ and $C_n \to 0$ as $n \to \infty$ by using (1.3). (2.23) is proved.

The proof of (2.24) follows analogously, and is therefore omitted.

COROLLARY 3.1. *Under the conditions of Theorem* 3.1,

$$n^{1/2} \, |U(\mathbf{F}_n) - V(\mathbf{F}_n)| \to 0 \qquad \textit{in probability as} \quad n \to \infty.$$

Hence $n^{1/2}(V(\mathbf{F}_n) - \theta(\mathbf{F}_n))$ *converges in law to the normal distribution with mean* 0 *and variance* $k^2\sigma^2$, *where* σ^2 *is defined in* (3.4).

Proof. From (2.2) and (2.4), we have $n^{1/2}(U_n^{(1)} - V_n^{(1)}) = O_P(n^{-1/2})$, $E(V_n^1) = n^{-[k]} g_{0,n}^*$ and, from (2.1) and (2.3),

$$n^{1/2}(\theta(\mathbf{F}_n) - n^{-[k]} g_{0,n}^*) = O(n^{-1/2}),$$

which implies that $n^{1/2}(V_n^{(1)} - \theta(\mathbf{F}_n))$ converges in law to the normal distribution with mean 0 and variance σ^2.

On the other hand, since

$$U(\mathbf{F}_n) = \theta(\mathbf{F}_n) + \sum_{p=1}^{k} \binom{k}{p} U_n^{(p)} \quad \text{and} \quad V(\mathbf{F}_n) = n^{-[k]} g_{0,n}^* + \sum_{p=1}^{k} \binom{k}{p} V_n^{(p)},$$

we conclude (by using properties (2.8) and (2.9) of Lemma 2.2) that $n^{1/2} |U(\mathbf{F}_n) - V(\mathbf{F}_n)| \to 0$ in probability as $n \to \infty$.

4. Convergence of the Rank Statistic $\mathcal{S}_{n,m}$

From now on we assume that $F_{ni} = F_n$ for any i $(1 \leqslant i \leqslant n)$. For any real number x, define $H_n(|x|)$ and $Y_{ni}(x)$ as

$$H_n(|x|) = F_n(|x|) - F_n(-|x|), \quad Y_{ni}(x) = I_{[|X_{ni}| \leqslant |x|]} - H_n(|x|).$$

Let

$$X_{n,m,i}^* = \frac{1}{m+1} \{(m-1) H_n(|X_{ni}|) + 1\},$$

$$Z_{n,m,i} = \frac{R_{n,m,i}}{m+1} - X_{n,m,i}^* = \frac{1}{m+1} \sum_{\substack{1 \leqslant j \leqslant m \\ j \neq i}} Y_{nj}(X_{ni}).$$

Let F be a d.f. on $\mathbb{R}$ and define the d.f. H on $\mathbb{R}^+$ by $H(|x|) = F(|x|) - F(-|x|)$.

For a score function $J(u)$ which is square integrable put

$$\mu_n = \mu_J(F_n) = \int_{\mathbb{R}} s(x) J(H_n(|x|) \, dF_n(x). \tag{4.1}$$

For h defined in (1.4) put

$$\mu_{n,m} = \mu_n \sum_{i=1}^{m} c_{ni} = \mu_n \sum_{i=1}^{m} h\left(\frac{i}{n}\right), \qquad m \leqslant n. \tag{4.2}$$

For any sequence of d.f.'s $\{\mathbf{F}_{1i}; i \geqslant 2\}$ on $\mathbb{R}^2$ with marginals F, we denote

$$\sigma^2(\{\mathbf{F}_{1i}\}) = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} g^2(x) \, dF(x) + 2 \sum_{l=2}^{n} \int_{\mathbb{R}^2} g(x) \, g(y) \, d\mathbf{F}_{1l}(x, y) \right\} \tag{4.3}$$

if the limit exists, where

$$g(x)=\int_{\mathbb{R}} s(y)\{I_{[|x|\leqslant|y|]}-H(|y|)\}\, J'(H(|y|)\, dF(y)$$
$$+\, s(x)\, J(H(|x|))-\int_{\mathbb{R}} s(y)\, J(H(|y|)\, dF(y).$$

For every $n\geqslant 1$, let

$$V_n(t)=\begin{cases}0 & \text{for}\quad t=0\\ (\mathscr{S}_{n,k}-\mu_{n,k})/\sigma n^{1/2} & \text{for}\quad t=k/n,\ k=1,...,n\\ \text{linearly interpolated} & \text{for}\quad t\in[k-1/n,\ k/n],\ k=1,...,n,\end{cases} \tag{4.4}$$

where σ is the positive constant defined in (4.3).

The process $V_n(t)=\{V_n(t),\ 0\leqslant t\leqslant 1\}$ belongs to the space C_1 of all continuous functions on [0, 1] with which we associate the usual uniform metric.

Now we give the following theorem which is a generalization from the stationary case to the nonstationary case of Theorem 3.1 in Yoshihara [9].

THEOREM 4.1. *Suppose the sequence* $\{X_{ni}\}$ *is absolutely regular with rate* (1.6) *and the sequence* $\{F_n\}$ *satisfies the conditions* (3.1) *and* (3.2) *of Theorem* 3.1. *Let J be a score function having a bounded second derivative. If* $\sigma^2(\{\mathbf{F}_{1i}\})$ *defined by* (4.3), (*below*) *is strictly positive, then* V_n *defined in* (4.3) *converges weakly in the uniform topology on* C_1 *to the process* $V_0=\{V_0(t),\ 0\leqslant t\leqslant 1\}$, *where*

$$V_0(t)=\int_0^t h(u)\, dW(u),\qquad 0\leqslant t\leqslant 1 \tag{4.5}$$

and $W=\{W(t),\ 0\leqslant t\leqslant 1\}$ *is a standard Brownian motion process, and* $\sigma^2(\{\mathbf{F}_{1i}\}<\infty$.

Proof. We need at first some lemmas. For any n ($n\geqslant 1$) and for any i ($1\leqslant i\leqslant n$) let

$$A_{ni}=\int_{\mathbb{R}} s(x)\{I_{[|X_{ni}|\leqslant|x|\}}-H_n(|x|)\}\, J'(H_n(|x|)\, dF_n(x)$$
$$+\, \{s(X_{ni})\, J(H_n(|X_{ni}|))-E(s(X_{ni})\, J(H_n(|X_{ni}|)))\}. \tag{4.6}$$

It is obvious that $E(A_{ni})=0$.

Now we consider the process $L_n(t)$ defined on C_1 by

$$L_n(t) = n^{-1/2}\left(\sum_{i=1}^{[nt]} A_{ni} - (nt - [nt])\, A_{n,[nt]+1}\right), \tag{4.7}$$

where $[nt]$ denotes the integer part of the real number nt.

LEMMA 4.1. *Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 4.1. Then, the process $L_n(t)$ converges weakly in uniform topology to a Gaussian process $L_0(t)$ with trajectories a.s. in C_1 with mean 0 and variance $t\sigma^2(\{\mathbf{F}_{1i}\})$, where $\sigma^2(\{F_{1i}\})$ is defined in* (4.3).

Proof. The process L_n defines a probability measure P_n on C_1. From Theorem 8.1 of Billingsley [2], we have to prove that (i) the finite-dimensional distributions of P_n converge in law to normal distributions and (ii) P_n is tight.

First we prove (i) which is equivalent to proving that $\sum_{l=1}^{p} \lambda_l L_n(t_l)$ converges in law to a normal distribution for any $p \in \mathbb{N}^*$, any $t_l \in [0, 1]$ and any $\lambda_l \in \mathbb{R}$ $(1 \leqslant l \leqslant p)$. Without loss of generality we can take $p = 2$ and suppose that $t_1 < t_2$.

As J and J' are bounded, the sequence of r.v.'s $\{A_{ni}\}$ are uniformly bounded and we have

$$\begin{aligned}\sum_{l=1}^{2} \lambda_l L_n(t_l) = n^{-1/2}\Bigg[&\sum_{i=1}^{[nt_1]} (\lambda_1 + \lambda_2) A_{ni} \\ &+ \sum_{i=[nt_1]+1}^{[nt_2]} \lambda_2 A_{ni} + (nt_1 - [nt_1]) A_{n,[nt_1]+1} \\ &+ (nt_2 - [nt_2]) A_{n,[nt_2]+1}\Bigg].\end{aligned} \tag{4.8}$$

We define the sequence of r.v.'s $\{B_{ni}\}$ by

$$B_{ni} = \begin{cases} (\lambda_1 + \lambda_2) A_{ni} & \text{if } \; i \leqslant [nt_1] \\ \lambda_2 A_{ni} & \text{if } \; [nt_1] < i \leqslant [nt_2] \\ 0 & \text{if } \; i > [nt_2] \end{cases} \tag{4.9}$$

and then we have

$$\sum_{l=1}^{2} \lambda_l L_n(t_l) = n^{-1/2} \sum_{i=1}^{n} B_{ni} + O_p(n^{-1/2}). \tag{4.10}$$

From Lemma (2.3) we have only to verify (2.20) and (2.23). (2.20) is immediate from (1.6) for $\delta'' > \frac{2}{3}$.

Now we prove (2.23). We have

$$E\left(\sum_{i=1}^{n} B_{ni}\right)^2 \Big/ n = \frac{1}{n}\Bigg[(\lambda_1+\lambda_2)^2\left(\sum_{i=1}^{[nt_1]}\sum_{j=1}^{[nt_1]} E(A_{ni}A_{nj})\right) + (\lambda_1+\lambda_2)\lambda_2\left(\sum_{i=1}^{[nt_1]}\sum_{j=[nt_1]+1}^{[nt_2]} E(A_{ni}A_{nj})\right) + \lambda_2^2 \sum_{i=[nt_1]+1}^{[nt_2]}\sum_{j=[nt_1]+1}^{[nt_2]} E(A_{ni}A_{nj}). \tag{4.11}$$

As the $\{A_{ni}\}$ are uniformly bounded, for any $\delta''>0$, we can find a constant $M>0$ such that

$$\frac{1}{n}\left|(\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[nt_1]}\sum_{j=[nt_1]+1}^{[nt_2]} E(A_{ni}A_{nj})\right| \leq \left(\sum_{i=1}^{[nt_1]}\sum_{j=[nt_1]+1}^{[nt_2]} \alpha^{\delta''/(2+\delta'')}(j-i)\right)M/n \tag{4.12}$$

From condition (1.6), the last expression converges to 0 as $n\to\infty$ for δ'' sufficiently large.

It remains to prove that $(1/n)\sum_{i=1}^{[nt_1]}\sum_{j=1}^{[nt_1]} E(A_{ni}A_{nj})$ converges to some finite constant.

We first prove the convergence by replacing J' and J in the expression of A_{ni} in (4.6) by indicator functions such as $J(t)=I_{[a\leq t\leq b]}$ and $J'(t)=I_{[a'\leq t\leq b']}$ and we then can write

$$\begin{aligned} A_{ni} = {} & \int_{(a'\vee H_n(|X_{ni}|))\wedge b'}^{b'} d(F_n\circ H_n^{-1}(t)) - \int_{a'}^{b'} t\, d(F_n\circ H_n^{-1}(t)) \\ & - \int_{(a'\vee H_n(|X_{ni}|))\wedge b'}^{b'} d(F_n\circ H_n^{*}(t)) \\ & + \int_{a'}^{b'} t\, d(F_n\circ H_n^{*}(t)) + s(X_{ni})I_{[a\leq H_n(|X_{ni}|)\leq b]} \\ & - \int_a^b d(F_n\circ H_n^{-1}(t)) + \int_a^b d(F_n\circ H_n^{*}(t)), \end{aligned}$$

where $H_n^*(t) = -H_n^{-1}(t)$.

Denote

$$D_{ni}(X_{ni}) = s(X_{ni}) I_{[a \leqslant H_n(|X_{ni}|) \leqslant b]} - \int_a^b d(F_n \circ H_n^{-1}(t)) + \int_a^b d(F_n \circ H_n^*(t))$$

and $C_{ni}(|X_{ni}|) = A_{ni} - D_{ni}(X_{ni})$.

Let $\bar{F}_{n,i,j}$ be the d.f. of $(|X_{ni}|, |X_{nj}|)$ (resp. $\mathbf{F}^1_{n,i,j}$ of $(|X_{ni}|, X_{nj})$ and $\mathbf{F}^2_{n,i,j}$ of $(X_{ni}, |X_{nj}|)$) and $\bar{\mathbf{F}}_{ij}$ (resp. $\mathbf{F}^1_{ij}$ and $\mathbf{F}^2_{ij}$) be the d.f. on $\mathbb{R}^+ \times \mathbb{R}^+$ (resp. $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R}^+$) defined by $\mathbf{F}_{ij}(x, y) = \mathbf{F}_{ij}(x, y) - \mathbf{F}_{ij}(-x, y) - \mathbf{F}_{ij}(x, -y) + \mathbf{F}_{ij}(-x, -y)$ for any $x \geqslant 0$, $y \geqslant 0$ (resp. $\mathbf{F}^1_{ij}(x, y) = \mathbf{F}_{ij}(x, y) - \mathbf{F}_{ij}(-x, y)$ for any $x \geqslant 0$, $y \in \mathbb{R}$, and $\mathbf{F}^2_{ij}(x, y) = \mathbf{F}_{ij}(x, y) - \mathbf{F}_{ij}(x, -y)$ for any $x \in \mathbb{R}$, $y \geqslant 0$) and $(i, j) \in \mathbb{N}^* \times \mathbb{N}^*$, where $i < j$.

We then have

$$E(A_{ni} A_{nj}) = \int_0^{+\infty} \int_0^{+\infty} C_{ni}(x)\, C_{nj}(y)\, d\bar{F}_{n,i,j}(x, y) + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_{ni}(x)\, D_{nj}(y)\, dF_{n,i,j}(x, y) + \int_0^{+\infty} \int_{-\infty}^{+\infty} C_{ni}(x)\, D_{nj}(y)\, dF^1_{n,i,j}(x, y) + \int_{-\infty}^{+\infty} \int_0^{+\infty} D_{ni}(x)\, C_{nj}(y)\, dF^2_{n,i,j}(x, y).$$

By using properties (3.1) and (3.2) and the following inequality

$$|F_n \circ H_n^{-1}(t) - F \circ H^{-1}(t)| \leqslant |F_n \circ H_n^{-1}(t) - F \circ H_n^{-1}(t)| + |F \circ H_n^{-1}(t) - F \circ H^{-1}(t)|$$

which implies

$$\sup_t |F_n \circ H_n^{-1}(t) - F \circ H^{-1}(t)| \xrightarrow[n \to \infty]{} 0$$

we deduce that

$$\lim_{n \to \infty} \max_{1 \leqslant i < j \leqslant n} \left| E(A_{ni} A_{nj}) - \int_0^{+\infty} \int_0^{+\infty} C(x)\, C(y)\, d\bar{\mathbf{F}}_{ij}(x, y) - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D(x)\, D(y)\, d\mathbf{F}_{ij}(x, y) - \int_0^{+\infty} \int_{-\infty}^{+\infty} C(x)\, D(y)\, d\mathbf{F}^1_{ij}(x, y) - \int_{-\infty}^{+\infty} \int_0^{+\infty} D(x)\, C(y)\, d\mathbf{F}^2_{ij}(x, y) \right| = 0,$$

where

$$C(x)=\int_{(a'\vee H(x))\wedge b'}^{b'} d(F\circ H^{-1}(t))-\int_{a'}^{b'} t\, d(F\circ H^{-1}(t))$$
$$-\int_{(a'\vee H(x))\wedge b'}^{b'} d(F\circ H^{*}(t))+\int_{a'}^{b'} t\, d(F\circ H^{*}(t))$$
$$D(x)=s(x)I_{[a\leqslant H(|x|)\leqslant b]}-\int_{a}^{b} d(F\circ H^{-1}(t))+\int_{a}^{b} d(F\circ H^{*}(t)),$$

where $H^*(t)=-H^{-1}(t)$.

We obtain a similar property if J and J' are replaced by step functions. As J and J' are continuous, we can uniformly approach them by a step function and we deduce that

$$\lim_{n\to\infty}\max_{1\leqslant i<j\leqslant n}\Big|E(A_{ni}A_{nj})-\Big(\int_0^{+\infty}\int_0^{+\infty} C^*(x)\,C^*(y)\,d\bar{\mathbf{F}}_{ij}(x,y)$$
$$+\int_0^{+\infty}\int_{-\infty}^{+\infty} C^*(x)\,D^*(y)\,d\mathbf{F}_{ij}^1(x,y)$$
$$+\int_{-\infty}^{+\infty}\int_0^{+\infty} D^*(x)\,C^*(y)\,d\mathbf{F}_{ij}^2(x,y)$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D^*(x)\,D^*(y)\,d\mathbf{F}_{ij}(x,y)\Big)\Big|=0, \tag{4.13}$$

where

$$C^*(x)=\int_{-\infty}^{+\infty} s(y)\{I_{[x\leqslant|y|]}-H(|y|)\}\,J'(H(|y|))\,dF(y),$$

and

$$D^*(x)=\Big\{s(x)\,J(H(|x|))-\int_0^{+\infty} J(H(x))\,dF(x)$$
$$+\int_{-\infty}^{0} J(H(-x))\,dF(x)\Big\}.$$

If we denote

$$\rho(1)=\int_0^{+\infty}(C^*(x))^2\,dH(x)+\int_{-\infty}^{+\infty}(D^*(x))^2\,dF(x)$$
$$+2\int_{-\infty}^{+\infty} C(|x|)\,D(x)\,dF(x)$$

$$\rho(i)=2\left(\int_0^{+\infty}\int_0^{+\infty} C^*(x)\,C^*(y)\,d\bar{\mathbf{F}}_{1i}(x,y)\right.$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D^*(x)\,D^*(y)\,d\mathbf{F}_{1i}(x,y)$$
$$+\int_0^{+\infty}\int_{-\infty}^{+\infty} C^*(x)\,D^*(y)\,d\mathbf{F}^1_{1i}(x,y)$$
$$\left.+\int_{-\infty}^{+\infty}\int_0^{+\infty} D^*(x)\,C^*(y)\,d\mathbf{F}^2_{1i}(x,y)\right),$$

we can write

$$\left|\frac{1}{n}\sum_{i=1}^{[nt_1]}\sum_{j=1}^{[nt_1]} E(A_{ni}A_{nj})-\frac{[nt_1]}{n}\sum_{i=1}^{+\infty}\rho(i)\right|$$
$$\leqslant\left|\frac{[nt_1]}{n}\frac{1}{[nt_1]}\sum_{i=0}^{[nt_1]-1}\sum_{j=1}^{[nt_1]-i}\varphi(i)\,E(A_{nj}A_{n,j+i})\right.$$
$$\left.-\frac{[nt_1]}{n}\frac{1}{[nt_1]}\sum_{i=1}^{[nt_1]}([nt_1]-i)\,\rho(i)\right|$$
$$+\frac{[nt_1]}{n}\sum_{i=[nt_1]+1}^{\infty}|\rho(i)|+\frac{[nt_1]}{n}\sum_{i=1}^{[nt_1]}\sum_{k=i}^{\infty}|\rho^k|$$
$$=|A_n|+B_n+C_n,$$

where

$$\varphi(i)=\begin{cases}1 & \text{if } i=0\\ 2 & \text{if } i\neq 0.\end{cases}$$

From (4.13) we deduce that $|A_n|\to 0$ and from the inequality on moments of strong mixing sequence of r.v.'s we deduce $|\rho(i)|\leqslant (\alpha(i))^{\delta''/(2+\delta'')}M$ for some constant $M>0$ which implies that $B_n\to 0$ and $C_n\to 0$ (as $n\to\infty$) by (1.6) for $\delta''>\frac{3}{2}$. It is also immediate that

$$\left|\frac{[nt_1]}{n}\sum_{i=1}^{+\infty}\rho(i)-t_1\left(\sum_{i=1}^{+\infty}\rho(i)\right)\right|\to 0 \qquad \text{as} \quad n\to\infty. \tag{4.14}$$

We conclude that $(1/n)\sum_{i=1}^{[nt_1]}\sum_{j=1}^{[nt_1]} E(A_{ni}A_{nj})$ converges to $t_1(\sum_{i=1}^{+\infty}\rho(i))$ as $n\to\infty$, where $\sum_{i=1}^{+\infty}\rho(i)$ is equal to $\sigma^2(\{\mathbf{F}_{1i}\})$ defined in (4.3).

From (4.8)–(4.12) and (4.14), we deduce that $E(\sum_{l=1}^2\lambda_l L_n(t_l))^2$ converges to $\{(\lambda_1+\lambda_2)^2 t_1\sigma^2(\{\mathbf{F}_{1i}\})$ $+\lambda_2^2(t_2-t_1)\}$ which implies that $\sum_{l=1}^2\lambda_l L_n(t_l)$ converges in law to the normal distribution with mean 0 and variance $\{(\lambda_1+\lambda_2)^2 t_1\sigma^2(\{\mathbf{F}_{1i}\})$ $+\lambda_2^2(t_2-t_1)\}$ and (i) is proved.

To prove (ii), we have to verify (cf. Billingsley [2]) that $\forall\varepsilon>0$, $\exists\eta>0$ $(0<\eta<1)$, and an integer N_0 such that $\forall n\geqslant N_0$,

$$P[\sup_{|s-t|<\eta} |L_n(t)-L_n(s)|\geqslant\varepsilon]\leqslant\varepsilon' \tag{4.15}$$

which follows by using Theorem 10 of Doukhan and Portal [4], Lemma 2 of Balacheff and Dupont [1], and routine analysis.

We now consider the process $\hat{L}_n(t)$ defined in C_1 by

$$\hat{L}_n(t)=n^{-1/2}\left(\sum_{i=1}^{[nt]} c_{ni}A_{ni}-c_{n,[nt]+1}(nt-[nt])\right.$$
$$\left.\times A_{n,[nt]+1})(\sigma(\{\mathbf{F}_{1i}\})\right)^{-1}. \tag{4.16}$$

LEMMA 4.2. *Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 4.1. Then, the process $\hat{L}_n(t)$ converges weakly in uniform topology to a Gaussian process $\hat{L}_0(t)=\int_0^t h(u)\,dW(u)$ with trajectories a.s. in C_1, where W is a standard Brownian motion process.*

Proof. From Theorem 1 of Yoshihara and Negishi [7] we have to verify that $\forall\varepsilon>0$

$$\lim_{\eta\to 0}\limsup_{n\to\infty} P\left[\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}\eta_i A_{ni}(\sigma(\{\mathbf{F}_{1i}\})^{-1}\right|\geqslant\varepsilon M\sqrt{n}\right]=0, \tag{4.20}$$

where η_i is any set of real numbers, $\eta=\sup_i\eta_i$ and $M=\sup_n\max_{1\leqslant i\leqslant n}E(A_{ni}^2(\sigma^2(\{\mathbf{F}_{1i}\}))^{-1})$. By using Theorem 10 of Doukhan and Portal [4] for $q=2$, we have

$$E\left(\sum_{i=1}^{k}\eta_i A_{ni}\right)^4\leqslant C_4(\alpha)(Mk\eta+M^2k^2\eta^2)$$

which implies from Theorem 12.2 of Billingsley for $\eta\geqslant n^{-2}$ that

$$P\left[\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}\eta_i A_{ni}\right|\geqslant\varepsilon M\sqrt{n}\right]\leqslant K\varepsilon^{-4}(M^{-3}+M^{-2})\eta^{3/2}. \tag{4.21}$$

Since the right side of (4.21) converges to zero as $\eta\to 0$, (4.20) follows.

Now we proceed as in Yoshihara [9]. As the score function J possesses a bounded derivative, we have by the Taylor expansion

$$J\left(\frac{R_{n,m,i}}{m+1}\right)=J(X^*_{n,m,i})+Z_{n,m,i}J'(X^*_{n,m,i})+\frac{1}{2}Z^2_{n,m,i}J''(\mu_{n,m,i}), \tag{4.22}$$

where

$$\mu_{n,m,i} \in \left[\frac{R_{n,m,i}}{m+1} \wedge X^*_{n,m,i}, \frac{R_{n,m,i}}{m+1} \vee X^*_{n,m,i}\right].$$

We note that

$$|X^*_{n,m,i} - H_n(|X_{ni}|)| \leqslant \frac{1}{m+1}.$$

So from (1.4), (4.1), (4.6), and (4.22) it follows (omitting the routine computations) that

$$\left|\mathcal{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni} A_{ni}\right| \leqslant |V_m| + M_1 \sum_{i=1}^{m} Z^2_{n,m,i} + M_2,$$

where M_1 and M_2 are some constants > 0, $M_0 = \sup_{t \in [0,1]} |J''(t)|$ and

$$V_m = \frac{1}{m+1} \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} c_{ni} \Big[s(X_{ni}) \{ I_{[|X_{ni}| \leqslant |X_{nj}|]} - H_n(|X_{ni}|)\} J'(H_n(|X_{ni}|)) - \int_{\mathbb{R}} s(x) \{ I_{[|X_{ni}| \leqslant |x|]} - H_n(|x|)\} J'(H_n(|x|))\, dF_n(x)\Big].$$

LEMMA 4.3. *Under the conditions of Theorem* 4.1, *we have*

$$E(|V_m|^2) = O(1), \qquad E(|V_m|^4) = O(1). \tag{4.23}$$

Proof. Follows using the same technique as in Lemma 2.2.

LEMMA 4.4. *Under the conditions of Theorem* 4.1, *we have*

$$E\left(\left|\sum_{i=1}^{m} Z^2_{n,m,i}\right|^2\right) = O(m^{-1}) \tag{4.24}$$

$$E\left(\sum_{i=1}^{m} Z^2_{n,m,i}\right) = O(m^{-1/2}). \tag{4.25}$$

Proof. Put $g_n(x, y) = I_{[|y| \leqslant |x|]} - H_n(|x|)$. Let i_{rs} ($\leqslant m$) ($r = 1, ..., 3$; $s = 1, 2$) be mutually different positive integers. Reorder $\{i_{rs}\}$ as $1 \leqslant k_1 < k_2 < \cdots < k_6 \leqslant m$ and put $E[\prod_{j=1}^{2} g_n(X_{ni_{j1}}, X_{ni_{j2}})\, g_n(X_{ni_{j3}}, X_{ni_{j2}})] = E[L(X_{nk_1}, ..., X_{nk_6})] = M(k_1, ..., k_6)$.

Let $d^{(c)}$ be the cth largest difference among $k_{j+1} - k_j$ ($j = 1, ..., 5$). Since $\int_{\mathbb{R}} g_n(x, y)\, dF_n(x) = 0$, we deduce from Lemma 2.1 that

$$M(k_1, \ldots, k_6) \leqslant K \sum_{\alpha=1}^{3} \beta(k_{j_\alpha+1} - k_{j_\alpha}),$$

where K is some constant >0 if for some j_α $(1 \leqslant j_\alpha \leqslant 5)$, $k_{j_\alpha+1} - k_{j_\alpha} = d^{(\alpha)}$ $(1 \leqslant \alpha \leqslant 3)$. Consequently,

$$\sum_{1 \leqslant k_1 < \cdots < k_6 \leqslant m} M(k_1, \ldots, k_6) \leqslant K' m^3 \sum_{j=1}^{m} (j+1)^2 \beta(j),$$

where K' is some constant >0. From (1.6) we deduce $\sum_{j=1}^{\infty} (j+1)^2 \beta(j) < +\infty$.

Using similar arguments we estimate the sums in the other cases and obtain (4.24). (4.25) follows immediately from (4.24).

LEMMA 4.5. *Under the conditions of Theorem* 4.1, *we have*

$$\forall \varepsilon > 0, \qquad P[\sup_{t \in [0,1]} |V_n(t) - \hat{L}_n(t)| > \varepsilon] \to 0 \qquad \textit{as} \quad n \to \infty. \quad (4.26)$$

Proof. To prove (4.26), it is sufficient to prove

$$P\left[\max_{1 \leqslant m \leqslant n} \left| \mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni} A_{ni} \right| \geqslant 3\varepsilon n^{1/2} \right] \to 0 \qquad \text{as} \quad n \to \infty,$$

for every $\varepsilon > 0$, which follows by using (4.23) and (4.25). (Since the method is the same as in the proof of the Lemma 3.5 of Yoshihara [9], the details are omitted.)

Theorem 4.1 now follows from Lemmas 4.2 and 4.5.

Define the score function $J_n(u)$ by

$$J_n(u) = \begin{cases} J(u) & \text{if } \ 0 \leqslant u \leqslant n/n+1 \\ J(n/n+1) & \text{otherwise.} \end{cases}$$

COROLLARY 4.1. *If the conditions of the Theorem* 4.1 *are satisfied except that the score function J is replaced by J_n in the expression of $\mathscr{S}_{n,m}$ defined in* (1.4) *and if J is twice differentiable and satisfies the condition*

$$|J''(u)| \leqslant M n^{1/2 - \delta} \qquad \textit{if} \quad 0 \leqslant u \leqslant \frac{n}{n+1} \ \textit{for some}\ \delta (0 < \delta < \tfrac{1}{2}), \quad (4.27)$$

where M is some constant >0, then the conclusion of Theorem 4.1 *remains true.*

Proof. Same as that of Theorem 3.2 of Yoshihara [9].

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Statistique/*Statistics*

Comportement limite de la U-statistique, de la V-statistique et d'une statistique de rang à un échantillon pour des processus absolument réguliers non stationnaires

Michel HAREL et Madan PURI

Résumé — La convergence faible des U statistiques et V statistiques fut établie par Yoshihara [2] pour des processus absolument réguliers et stationnaires. Par la suite Yoshihara [3] a montré la convergence faible d'une statistique de rang à un échantillon sous des conditions similaires. Dans cette Note, nous étendons certains résultats de Yoshihara aux cas non stationnaires.

Limiting behavior of U-statistics, V-statistics and one sample rank order statistics for non stationary absolutely regular processes

Abstract — *The weak convergences of* U *and* V *statistics were established by Yoshihara* [2] *for stationary absolutely regular processes. Later Yoshihara* [3] *also proved the weak convergence of one sample rank order statistics under similar conditions. In this paper, we extend some of Yoshihara's results to the non stationary cases.*

1. INTRODUCTION ET NOTATIONS. — Soient X_{ni}, $1 \leqq i \leqq n$, $n \geqq 1$ des variables aléatoires réelles avec des fonctions de répartition continues $F_{ni}(x)$, $x \in \mathbb{R}$. La fonction de répartition de la variable aléatoire $(X_{n1}, \ldots, X_{nn})$ à valeurs dans $\mathbb{R}^n$ est notée $\mathbf{F}_n$.

Dans la première partie de cette Note, nous étudierons le comportement asymptotique de la U-statistique

$$(1) \qquad U(\mathbf{F}_n) = \binom{n}{k}^{-1} \sum_{(i)}^{(n)} g(X_{ni_1}, \ldots, X_{ni_k}), \qquad n \geqq k \geqq 1$$

où la sommation $\sum_{(i)}^{(n)}$ recouvre toutes les inégalités $1 \leqq i_1 < \ldots < i_k \leqq n$ et $g : \mathbb{R}^k \to \mathbb{R}$ est une fonction Borel mesurable qui est symétrique dans ses k ($k \geqq 1$) arguments.

De l'étude de $U(\mathbf{F}_n)$, nous déduirons le comportement asymptotique de la V-statistique

$$(2) \qquad V(\mathbf{F}_n) = n^{-k} \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} g(X_{ni_1}, \ldots, X_{ni_k})$$

appelée statistique de Von Mises.

Les propriétés de convergence sont étudiées quand la suite $\{X_{ni}\}$ est absolument régulière avec le taux

$$(3) \qquad \beta(m) = O(m^{-(2+\delta)/\delta}), \qquad \delta > 0$$

(pour une définition de l'absolue régularité *voir* [2]) et quand la fonction g vérifie les conditions d'intégrabilité suivantes

$$(4) \qquad \sup_n \sup_{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k} \int_{\mathbb{R}^k} |g(x_1, \ldots, x_k)|^{2+\delta'} dF_{ni_1} \ldots dF_{ni_k} < \infty$$

$$(5) \qquad \sup_n \sup_{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k} E(|g(X_{ni_1}, \ldots, X_{ni_k})|^{2+\delta'}) < \infty$$

avec $0 < \delta < \delta'$.

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Dans la seconde partie de cette Note, nous étudierons le comportement asymptotique de la statistique de rang à un échantillon $\mathfrak{S}_{n,m}$

(6) $$\mathfrak{S}_{n,m}=\sum_{i=1}^{m} c_{ni}\, s(X_{ni})\, J\left(\frac{R_{n,m,i}}{m+1}\right), \qquad n \geqq m \geqq 1$$

où J est une fonction de score dérivable, $s(x)=\operatorname{sgn}(x)$ et les c_{ni} sont des constantes de régression définies par une fonction continue $h(x)$ sur [0,1] comme

(7) $$c_{ni}=h\left(\frac{i}{n}\right), \qquad 1 \leqq i \leqq n, \quad n \geqq 1$$

et $R_{n,m,i}=\sum_{j=1}^{m} I_{[|X_{nj}| \leqq |X_{ni}|]}$, $1 \leqq i \leqq m \leqq n$ où $I_{[\quad]}$ désigne la fonction indicatrice.

Nous supposerons aussi que la suite $\{X_{ni}\}$ est absolument régulière mais avec le taux

(8) $$\beta(m)=O(m^{-4}).$$

La convergence des U-statistiques et V-statistiques furent établies par Yoshihara [2] et après, pour la statistique de rang, aussi par Yoshihara [3] mais seulement dans le cas stationnaire.

Par la suite, Denker et Keller [1] montrèrent certains théorèmes limites des statistiques mentionnées ci-dessus pour des processus qui sont uniformément mélangeants dans les deux directions du temps. Dans cette Note, nous étendons les résultats de Yoshihara du cas stationnaire au cas non stationnaire.

2. Convergence de la U-statistique et de la V-statistique. — On définit $g_{0,n}$ par

(9) $$g_{0,n}=\sum_{(i_1,\ldots,i_k)\in I_{0,n}} \int_{\mathbb{R}^k} g(x_1,\ldots,x_k)\, dF_{ni_1}\ldots dF_{ni_k}$$

où

$$I_{0,n}=\{(i_1,\ldots,i_k);\ 1 \leqq i_1 \neq \ldots \neq i_k \leqq n\}$$

et on note $F_{n,i,j}$ la fonction de répartition de (X_{ni}, X_{nj}), $1 \leqq i<j \leqq n$, $n \geqq 1$.

Théorème 1. — *On suppose que les conditions suivantes sont satisfaites :*

(*a*) *Il existe deux constantes* δ, δ' $(0<\delta<\delta')$ *telles que* (3), (4) *et* (5) *soient vérifiées.*

(*b*) *Pour tout* $m \in \mathbb{N}^*$, *il existe une fonction de répartition continue* G_m *sur* $\mathbb{R}^2$ *avec des marges continues* F_m^* *telles que :*

(10) $$\lim_{n\to\infty} \max_{1 \leqq i<j \leqq n} |F_{n,i,j}(x_1, x_2)-G_{j-i}(x_1, x_2)|=0$$

pour tout $(x_1, x_2)\in\mathbb{R}^2$,

(11) $$\int_{\mathbb{R}^k} |g(x_1,\ldots,x_k)|^{2+\delta'} \prod_{j=1}^{k} dF(x_j)<\infty$$

où $F=F_1^*$.

(*c*) *g est continue à droite et a des limites à gauche ou est continue à gauche et a des limites à droite.*

Alors $n^{1/2}(U(F_n)-\theta(F_n))$ *converge en loi vers la distribution normale de moyenne nulle et variance* $k^2\sigma^2$ *si* $\sigma^2>0$ *où* $\theta(F_n)=((n-k)!/n!)\,g_{0,n}$ *et*

$$(13)\quad \sigma^2=\left[\int_{\mathbb{R}^k} g^2(x_1,\ldots,x_k)\prod_{l=1}^k dF(x_l)-\theta^2(F)\right]$$
$$+2\sum_{i=1}^{\infty}\left[\int_{\mathbb{R}^{2k}} g(x_1,\ldots,x_k)\,g(x_{k+1},\ldots,x_{2k})\,dG_i(x_1,x_{k+1})\prod_{l=2}^k dF(x_l)\prod_{p=k+2}^{k} dF(x_p)-\theta^2(F)\right]<\infty$$

où $\theta(F)=\int_{\mathbb{R}^k} g(x_1,\ldots,x_k)\prod_{l=1}^k dF(x_l)$.

Note. — La condition (10) implique que pour tout $m\geqq 1$, $F_m^*=F_1^*(=F)$.

COROLLAIRE 1. — *Sous les conditions du théorème* 1

$$n^{1/2}\,|U(F_n)-V(F_n)|\to 0 \textit{ en probabilité quand } n\to\infty.$$

Par conséquent $n^{1/2}(V_n(F_n)-\theta(F_n))$ *converge en loi vers la distribution normale de moyenne nulle et variance* $k^2\sigma^2$ *où* σ^2 *est défini en* (13).

3. CONVERGENCE DE LA STATISTIQUE DE RANG $\mathfrak{S}_{n,m}$. — Maintenant nous supposons que $F_{ni}=F_n$ pour tout $i\,(1\leqq i\leqq n)$.

Pour tout nombre réel x, on définit $H_n(|x|)$ par

$$H_n(|x|)=F_n(|x|)-F_n(-|x|).$$

Soit F une fonction de répartition sur $\mathbb{R}$ et on définit la fonction de répartition H sur $\mathbb{R}^+$ par $H(|x|)=F(|x|)-F(-|x|)$.

Pour une fonction de score $J(u)$ dérivable et de carré intégrable, on note

$$(14)\qquad \mu_n=\mu_J(F_n)=\int_{\mathbb{R}} s(x)\,J(H_n(|x|)\,dF_n(x).$$

Pour h définie en (7), on pose

$$(15)\qquad \mu_{n,m}=\mu_n\sum_{i=1}^m c_{n,i}=\mu_n\sum_{i=1}^m h\left(\frac{i}{n}\right),\qquad m\leqq n,$$

et

$$(16)\quad \varphi(x)=\int_{\mathbb{R}} s(y)\{I_{[|x|\leqq|y|]}-H(|y|)\}\,J'(H(|y|)\,dF(y)+s(x)\,J(H(|x|))-\int_{\mathbb{R}} s(y)\,J(H(|y|))\,dF(y).$$

Pour toute suite de fonction de répartition $\{G_m,\ m\geqq 1\}$ sur $\mathbb{R}^2$ de marges F, on note :

$$(17)\qquad \tilde\sigma^2(\{G_m\})=\lim_{n\to\infty}\left[\int_{\mathbb{R}}\varphi^2(x)\,dF(x)+2\sum_{l=1}^n\left[\int_{\mathbb{R}^2}\varphi(x)\,\varphi(y)\,dG_l(x,y)\right]\right.$$

si la limite existe.

Pour tout $n\geqq 1$, soit

$$(18)\qquad V_n(t)=(\tilde\sigma\,n)^{-1/2}\,[A_{n,[nt]}+(nt-[nt])\,A_{n,[nt]+1}]$$

où $\tilde{\sigma}$ est la constante positive définie en (17), $A_{n,i} = \mathfrak{S}_{n,i} - \mu_{n,i}$ et $[nt]$ désigne la partie entière de nt.

Le processus $V_n(t) = \{ V_n(t),\ 0 \leqq t \leqq 1 \}$ appartient à l'espace C_1 des fonctions continues sur $[0,1]$, auquel nous associons la métrique uniforme.

THÉORÈME 2. — *Supposons que la suite* $\{ X_{ni} \}$ *soit absolument régulière avec le taux* (8) *et que la suite* F_n *satisfasse la condition* (10) *du théorème* 1. *Soit* J *une fonction de score ayant une dérivée seconde bornée. Si* $\tilde{\sigma}^2(\{ G_m \})$ *est strictement positive, alors* V_n *converge faiblement pour la métrique uniforme sur* C_1 *vers le processus* $V_0 = \{ V_0(t),\ 0 \leqq t \leqq 1 \}$, *où* $V_0(t) = \int_0^t h(u)\, dW(u)$, $0 \leqq t \leqq 1$ *et* $W = \{ W(t),\ 0 \leqq t \leqq 1 \}$ *est le processus Brownien standard, et on a* $\tilde{\sigma}^2(\{ G_m \}) < \infty$.

On définit la fonction de score $J_n(u)$ par

$$J_n(u) = \begin{cases} J(u) & \text{si } 0 \leqq u \leqq \dfrac{n}{n+1} \\ J\left(\dfrac{n}{n+1}\right) & \text{autrement.} \end{cases}$$

COROLLAIRE 2. — *On suppose satisfaites les hypothèses du théorème* 2, *si ce n'est que la fonction de score* J *est remplacée par* J_n *dans l'expression de* $\mathfrak{S}_{n,m}$ *et dans le calcul de* $\tilde{\sigma}^2$, *que* J *est seulement supposée deux fois différentiable et qu'il existe* $\alpha\,(0 < \alpha < 1/2)$ *tel que* $\sup_n n^{\alpha - 1/2} \sup_{0 \leqq u \leqq n/n+1} | J''(u) | < \infty$. *Alors la conclusion du théorème* 2 *reste vraie.*

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Weak Convergences of the U–statistic and Weak Invariance of the One–sample Rank Order Statistic for Markov Processes and ARMA Models

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Harel and Puri (1988) studied the asymptotic behavior of the U–statistic and the one–sample rank order statistic for nonstationary absolutely regular processes. In this note, we present some applications of these results for Markov processes as well as ARMA processes.

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1. INTRODUCTION

Let X_{ni} , $1\le i\le n$, $n\ge 1$ be real–valued r.v.'s (random variables) with continuous d.f.'s (distribution functions) $F_{ni}(x)$, $x\in\mathbb{R}$, and let $\mathbf{F}_n$ denote the d.f. of the $\mathbb{R}^n$–valued r.v. $(X_{n1},\dots,X_{nn})$.

Consider the U–statistic

$$U(\mathbf{F}_n) = \binom{n}{k}^{-1} \sum_{(i)}^{(n)} g(X_{ni_1},\dots,X_{ni_k}) , \quad n\ge k\ge 1 \tag{1.1}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1\le i_1<\dots<i_k\le n$ and $g : \mathbb{R}^k\to\mathbb{R}$ is a Borel measurable function which is symmetric in its $k(\ge 1)$ arguments.

Consider also the one–sample rank order statistic $\mathscr{S}_{n,m}$

$$\mathscr{S}_{n,m} = \sum_{i=1}^{m} c_{ni}\, s(X_{ni}) J\left(\frac{R_{n,m,i}}{m+1}\right) , \quad n\ge m\ge 1 \tag{1.2}$$

where J is a score function, $s(x) = \operatorname{sgn}(x)$ and the c_{ni} are regression constants generated by a continuous function $h(x)$ on $[0,1]$:

$$c_{ni} = h(\tfrac{i}{n}) , \ 1\le i\le n,\ n\ge 1 , \ R_{n,m,i} = \sum_{j=1}^{m} I_{[|X_{nj}|\le |X_{ni}|]} , \quad 1\le i\le m\le n \tag{1.3}$$

where $I_{[\,]}$ denotes the indicator function.

In Harel and Puri (1988), we studied the asymptotic behavior of the statistics $U(\mathbf{F}_n)$ and $\mathscr{S}_{n,m}$ when the underlying r.v.s are nonstationary, absolutely regular with rates

$$\beta(m) = 0(m^{-(2+\delta)/\delta}) \quad \text{for some} \quad \delta>0 \tag{1.4}$$

in the case of the U–statistic (1.1), and

$$\beta(m) = 0(m^{-4}) \tag{1.5}$$

in the case of the one–sample rank statistic (1.2). In this paper, we provide applications of some of the results of Harel and Puri (loc. cit.) for some Markov processes as well as ARMA processes.

For the ease of convenience and easy reference we state the two main results from

Harel and Puri (1988).

THEOREM 1.1. *(Convergence of the U–statistic). Let* $F_{n,i,j}$ *be the df of* (X_{ni},X_{nj}) , $1\le i<j\le n$. *Suppose that in addition to the assumption* (1.4), *the assumptions* (2.6) *and* (2.7) *of Harel and Puri* (1988) *are satisfied. Furthermore assume that for any* $i,j\in\mathbb{N}^*$, *with* $i<j$, *there exists a continuous d.f.* F_{ij} *on* $\mathbb{R}^2$ *with continuous marginals* F_i^* *and* F_j^* *such that*

$$\lim_{n\to\infty}\max_{1\le i<j\le n} |F_{n,i,j}(x_1,x_2)-F_{ij}(x_1,x_2)| = 0 \text{ for all } (x_1,x_2)\in\mathbb{R}^2 \text{ and} \tag{1.6}$$

$$F_{ij} = F_{1,j-i+1} \text{ for all } i<j \tag{1.7}$$

and

$$\int_{\mathbb{R}^k} |g(x_1,\dots,x_k)|^r \prod_{j=1}^k dF(x_j) \le M_0 < \infty \tag{1.8}$$

where M_0 *is some constant* >0 *and* $F=F_i^*$ *for all* $i\in\mathbb{N}^*$ *and* g *is right continuous and has left–hand limits* (r.c.ℓ.ℓ.) *or left continuous and has right–hand limits* (ℓ.c.r.ℓ.). *Then* $n^{\frac{1}{2}}(U(F_n)-\theta(F_n))$ *converges in law to the normal distribution with mean* 0 *and variance* $k^2\sigma^2$ *if* $\sigma^2>0$ *where* $\theta(F_n) = n^{-[k]} \sum_{(i_1,\dots,i_k)\in I_{0,n}} \int_{\mathbb{R}^k} g(x_1,\dots,x_k)dF_{ni_1}\dots dF_{ni_k}$ *where* $I_{0,n} = \{i_1,\dots,i_k);1\le i_1\ne\dots\ne i_k\le n\}$,

$$\sigma^2 = [\int_{\mathbb{R}^k} g^2(x_1,\dots,x_k) \prod_{\ell=1}^k dF(x_\ell)-\theta^2(F)] + 2\sum_{i=2}^{\infty} [\int_{\mathbb{R}^{2k}} g(x_1,\dots,x_k)g(x_{k+1},\dots,x_{2k}) \tag{1.9}$$
$$dF_{1i}(x_1,x_{k+1}) \prod_{\ell=2}^k dF(x_\ell) \prod_{p=k+2}^{2k} dF(x_p) - \theta^2(F)] < \infty$$

where $\theta(F) = \int_{\mathbb{R}^k} g(x_1,\dots,x_k) \prod_{\ell=1}^k dF(x_\ell)$.

We now assume that $F_{ni} = F_n$ for any i $(1\le i\le n)$. For any real number x , define $H_n(|x|)$ as $H_n(|x|) = F_n(|x|) - F_n(-|x|)$.

Let F be a d.f. on $\mathbb{R}$ and define the d.f. H on $\mathbb{R}^+$ by $H(|x|) = F(|x|)-F(-|x|)$.

For a score function J(u) which is square integrable put

$$\mu_n = \mu_J(F_n) = \int_{\mathbb{R}} s(x)J(H_n(|x|)dF_n(x) . \tag{1.10}$$

For h defined in (1.3) put

$$\mu_{n,m} = \mu_n \sum_{i=1}^{m} c_{ni} = \mu_n \sum_{i=1}^{m} h(\tfrac{i}{n}) \ , \ m \le n \ . \tag{1.11}$$

For any sequence of d.f.s $\{F_{1i};\ i\ge 2\}$ on $\mathbb{R}^2$ with marginals F we denote

$$\sigma^2(\{F_{1i}\}) = \lim_{n\to\infty} \{\int_{\mathbb{R}} g^2(x)dF(x) + 2\sum_{\ell=2}^{n} \int_{\mathbb{R}^2} g(x)g(y)dF_{1\ell}(x,y)\} \tag{1.12}$$

if the limit exists, where

$$g(x) = \int_{\mathbb{R}} s(y)\{I_{[|x|\le|y|]} - H(|y|)\}J'(H(|y|)dF(y) + s(x)J(H(|x|)) - \int_{\mathbb{R}} s(y)J(H(|y|)dF(y).$$

For every $n\ge 1$, let

$$V_n(t) = \begin{cases} 0 & \text{for } t=0 \\ (\mathscr{S}_{n,k}-\mu_{n,k})/\sigma n^{\frac{1}{2}} & \text{for } t=k/n,\ k=1,\dots,n \\ \text{linearly interpolated} & \text{for } t\in[k-1/n,k/n],\ k=1,\dots,n \end{cases} \tag{1.13}$$

where σ is the positive constant defined in (1.12).

The process $V_n(t) = \{V_n(t), 0\le t\le 1\}$ belongs to the space C_1 of all continuous functions on $[0,1]$ with which we associate the usual uniform metric.

Then, we have

THEOREM 1.2. (*Convergence of the rank–order statistic*). *Suppose the sequence* $\{X_{ni}\}$ *is absolutely regular with rate* (1.5) *and the sequence* $\{F_n\}$ *satisfies the conditions* (1.6) *and* (1.7) *of Theorem* 1.1. *Let* J *be a score function having a bounded second derivative. If* $\sigma^2(\{F_{1i}\})$ *defined by* (1.12) *is strictly positive, then* V_n *defined in* (1.13) *converges weakly in the uniform topology on* C_1 *to the process* $V_0 = \{V_0(t), 0\le t\le 1\}$ *where*

$$V_0(t) = \int_0^t h(u)dW(u) \ , \ 0\le t\le 1 \tag{1.14}$$

and $W = \{W(t), 0\le t\le 1\}$ *is a standard Brownian motion process, and* $\sigma^2(\{F_{1i}\} < \infty$.

2. APPLICATIONS TO MARKOV PROCESSES AND ARMA PROCESSES.

2.1 Markov processes.

Consider a sequence $\{X_{ni}; i\in\mathbb{Z}\}$ of $\mathbb{R}$–valued processes such that for all $n\in\mathbb{N}^*$,

$\{X_{ni}\}$ is a Markov process with stationary transition probabilities $P_n(x;A)$ where $A \in \mathcal{B}$, $\mathcal{B}$ is the Borel σ–field of $\mathbb{R}$, and $x \in \mathbb{R}$.

Recall that the Markov process is *geometrically ergodic* if it is ergodic and if there exists $0<\rho_n<1$ such that

$$\|P_n^m(x;\cdot) - \mu_n(\cdot)\| = 0(\rho_n^m) \quad \text{for all a.s. } x \in \mathbb{R}$$

where $\|\cdot\|$ denotes the norm of total variation. (ρ_n is called the rate) and P_n^m is the m–step transition probability.

THEOREM 2.1. *Let* $\{X_{ni};i \in \mathbb{Z}\}$ *be a Markov process such that for every* $n \in \mathbb{N}^*$, $\{X_{ni}\}$ *is either* (a) *aperiodic, Harris recurrent and geometrically ergodic with rates* $0<\rho_n<\rho_o$, $\rho_o \in (0,1)$ *or* (b) *aperiodic and Doeblin recurrent.*

Suppose there exists a probability measure μ_o *on* $\mathbb{R}$ *and a transition probability* P_o *such that*

$$\mu_n((-\infty,x]) \longrightarrow \mu_o((-\infty,x]) \quad as \quad n\to\infty \tag{2.1}$$

for all $x \in \mathbb{R}$.

$$P_n(x;(-\infty,y]) \longrightarrow P_o(x;(-\infty,y]) \quad as \quad n\to\infty \tag{2.2}$$

for all $(x,y) \in \mathbb{R}^2$.

Then for a function g *satisfying the conditions of Theorem* 1.1 *and for a* U*–statistic defined in* (1.1), *the conclusion of Theorem* 1.1 *holds.*

Also for a score function J *having a second bounded derivative and the process* V_n *defined in* (1.13), *the conclusion of Theorem* 1.2 *holds.*

Proof. (i) Suppose (a) holds. From Theorems 1.1 and 1.2, we have to show that the conditions (1.6) and (1.7) are verified and the sequence $\{X_{ni}\}$ is absolutely regular with the rate (1.4) or (1.5).

From Nummelin and Tuominen (1982), a Markov process which is aperiodic, Harris recurrent and geometrically ergodic with rates ρ_n, satisfies

$$\int_{\mathbb{R}} \|P_n^m(x;\cdot) - \mu_n(\cdot)\| \mu_n(dx) = 0(\rho_n^m) \, .$$

From Proposition 1 of Davydov (1973) we have

$$\beta(m) = \sup_{n\in\mathbb{N}^*} \int_{\mathbb{R}} \|P_n^m(x;\cdot) - \mu_n(\cdot)\| \mu_n(dx)$$

and we deduce that the sequence $\{X_{ni}\}$ is absolutely regular with a geometrical rate ρ_0 which implies (1.4) or (1.5).

We now show that (1.6) is satisfied. For any $n\geq 1$, it can easily be seen that the d.f. $F_{n,i,j}$ has the same marginals F_n, say. For any $m\geq 1$, let P_0^m be the m–step transition probability of P_0.

Let G_m be the d.f. associated with the probability measure Q_m where

$$Q_m(A_1\times A_2) = \int_{A_1}\int_{A_2} P_0^m(x;dy)\mu_0(dx) \quad \forall\,(A_1,A_2)\in\mathscr{B}^2 .$$

For any $(i,j)\in(\mathbb{N}^*)^2$ with $i<j$, we consider the d.f. F_{ij} defined by $F_{ij} = G_{j-i}$. From the definition of F_{ij}, it is clear that the marginal d.f.s of F_{ij} are identical. We denote them by F.

Let $(x,y)\in\mathbb{R}^2$ be fixed, and note the following inequality

$$|F_{n,i,j}(x,y) - F_{ij}(x,y)| \leq F_{n,i,j}(x,y) - F_n(x)F_n(y)| + |F_n(x)-F(x)|F_n(y)$$
$$+ F(x)|F_n(y)-F(y)| + |F_{ij}(x,y)-F(x)F(y)| . \tag{2.3}$$

As the sequence $\{X_{ni}\}$ is absolutely regular with the geometrical rate ρ_0, it is also strong mixing with the same rate and from the definition of strong mixing we deduce that $\forall\,\epsilon>0\ \exists\, m_0\in\mathbb{N}^*$ such that $\forall\,(i,j)\in(\mathbb{N}^*)^2$ with $i<j$ and $j-i\geq m_0$.

$$|F_{n,i,j}(x,y) - F_n(x)F_n(y)| < A\rho_0^m \leq \epsilon/4 \tag{2.4}$$

where A is some positive constant, and from (2.1) and (2.2) we also obtain

$$|F_{ij}(x,y) - F(x)F(y)| < \epsilon/4 . \tag{2.5}$$

Now from conditions (2.1) and (2.2), we also deduce that $\forall\,\epsilon>0\ \exists\, n_0$ such that $\forall\, n\geq n_0$ and $\forall\,(i,j)\in(\mathbb{N}^*)^2$ with $i<j$ and $j-i<m_0$

$$|F_{n,i,j}(x,y) - F_{ij}(x,y)| < \epsilon \tag{2.6}$$

$$|F_n(x) - F(x)| < \epsilon/4 \tag{2.7}$$

$$|F_n(y) - F(y)| < \epsilon/4 \tag{2.8}$$

From (2.3)–(2.5) and (2.7), (2.8) we deduce that $\forall\ n \geq n_o$, $\forall\ (i,j) \in (\mathbb{N}^*)^2$ with $i<j$ and $j-i \geq m_o$

$$|F_{n,i,j}(x,y) - F_{ij}(x,y)| < \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon. \tag{2.9}$$

(2.6) and (2.9) yield (1.6). As the d.f. F_{ij} has the same marginals F, we obtain (1.7).

(ii) Suppose (b) holds. From Davydov (1973), a Markov process which is Doeblin recurrent and aperiodic is geometrically φ–mixing. This implies that $\{X_{ni}\}$ is also absolutely regular with a geometrical rate.

EXAMPLE 2.1. Consider the process $\{X_{ni};i\in\mathbb{Z}\}$ where $X_{n,i+1} = a_1^{(n)}X_{ni} + a_2^{(n)}X_{ni}\epsilon_{i+1} + a_3^{(n)}\epsilon_{i+1} + a_4^{(n)}\epsilon_{i+1}^2 + a_5^{(n)}$ where the a's are real numbers and $\{\epsilon_i;i\in\mathbb{Z}\}$ is a white noise with strictly positive density. Then Mokkadem (1985) has shown that if $(a_1^{(n)})^2 + (a_2^{(n)})^2E(\epsilon_1^2) < 1$ and $E(\epsilon_1^4) < \infty$, then the process $\{X_{ni};i\in\mathbb{Z}\}$ is geometrically ergodic.

If we have

$$\forall\ j\in\{1,\dots,5\}\ \exists\ a_j\in\mathbb{R}$$

such that

$$\lim_{n\to\infty} a_j^{(n)} = a_j$$

and

$$a_1^2 + a_2^2E(\epsilon_1^2) < 1$$

then the conditions (2.1) and (2.2) are satisfied and we can apply Theorem 2.1.

EXAMPLE 2.2. Consider the process $\{X_{ni};i\in\mathbb{Z}\}$ where $X_{n,i+1} = f_n(X_{ni}) + \epsilon_{i+1}$ where the ϵ's are independent and identically distributed random variables with strictly positive density and $f_n : \mathbb{R}\to\mathbb{R}$ is bounded and continuous. (This model was studied by Collomb and Doukhan (1983)). It is easy to check that this model is Doeblin recurrent and aperiodic and we deduce that $\{X_{ni};i\in\mathbb{Z}\}$ is geometrically φ–mixing and if f_n converges simply to a bounded and continuous function f_o, we can apply Theorem 2.1.

2.2 ARMA processes.

Consider a sequence of ARMA processes $\{X_{ni};i\in\mathbb{Z}\}$, $n\in\mathbb{N}^*$

$$X_{n,i+1} = \tilde{a}_1^{(n)} X_{ni} + \tilde{a}_2^{(n)} \epsilon_{i+1} \tag{2.10}$$

where $\{\epsilon_i;i\in\mathbb{Z}\}$ is a sequence of independent random variables such that $E(\epsilon_i) = 0$.

THEOREM 2.2. *Let* $\{X_{ni};i\in\mathbb{Z}\}$ *be a sequence of ARMA processes given by* (2.10). *Suppose* $\{X_{ni}\}$ *satisfies the following conditions:*

$\{\epsilon_i;i\in\mathbb{Z}\}$ *is a sequence of independent and identically distributed random variables with strictly positive density.* (2.11)

$$\exists\ (\tilde{a}_1,\tilde{a}_2) \in (-1,1)\times\mathbb{R} \tag{2.12}$$

such that

$$\lim_{n\to\infty} \tilde{a}_1^{(n)} = \tilde{a}_1 \quad \textit{and} \quad \lim_{n\to\infty} \tilde{a}_2^{(n)} = \tilde{a}_2 .$$

Then for a function g *satisfying the conditions of Theorem* 1.1 *and for a* U*–statistic defined in* (1.1) *the conclusions of Theorem* 1.1 *hold.*

Also for a score function J *having a second bounded derivative and for the process* V_n *defined in* (1.13) *the conclusions of Theorem* 1.2 *hold.*

Proof. Particular case of Theorem 2.1.

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LIMITING BEHAVIOR OF ONE SAMPLE RANK–ORDER STATISTICS WITH UNBOUNDED SCORES FOR NONSTATIONARY ABSOLUTELY REGULAR PROCESSES

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Abstract. Harel and Puri (1989a) established the weak convergence of one sample rank order statistics for nonstationary absolutely regular processes with bounded scores by a generalization of the results of Yoshihara (1978) in the stationary case. This paper extends the results to the case when the score functions and regression constants are not necessarily bounded.

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1. Introduction. Let X_{ni} , $1\le i\le n$,, $n\ge 1$ be real–valued r.v.'s (random variables) with continuous d.f's (distribution functions) $F_{ni}(x)$, $x\in\mathbb{R}$.

We will study the asymptotic behavior of the one sample rank order statistic $\mathscr{S}_{n,m}$ where

$$(1.1) \qquad \mathscr{S}_{n,m} = \sum_{i=1}^{m} c_{ni} s(X_{ni}) J\left(\frac{R_{n,m,i}}{m+1}\right), \quad n\ge m\ge 1 ,$$

J is a score function, $s(x) = \operatorname{sgn}(x)$ and the c_{ni} are regression constants defined by a continuous function $h(x)$ on $[0,1]$ as

$$(1.2) \qquad c_{ni} = h\left(\frac{i}{n}\right), \quad 1\le i\le n,\ n\ge 1$$

and $R_{n,m,i} = \sum_{j=1}^{m} I_{[|X_{nj}|\le|X_{ni}|]}$, $1\le i\le m\le n$ where $I_{[\,]}$ denotes the indicator function.

We assume that the underlying r.v.'s are absolutely regular with rates

$$(1.3) \qquad \beta(m) = 0(m^{-24(2-\delta)/\delta}), \quad \delta>0 .$$

Recall that the sequence $\{X_{ni}\}$ is absolutely regular if

$$\sup_n \max_{1\le j\le n-m} [E\{\sup_{A\in\sigma(X_{ni},i\ge j+m)} |P(A|\sigma(X_{ni},1\le i\le j)) - P(A)|\}] = \beta(m)\downarrow 0 .$$

Here $\sigma(X_{ni},1\le i\le j)$ and $\sigma(X_{ni},i\ge j+m)$ are the σ–fields generated by $(X_{n1},\dots,X_{nj})$ and $(X_{n,j+m},X_{n,j+m+1},\dots,X_{nn})$. Also recall that $\{X_{ni}\}$ satisfies the strong mixing condition if

$$\sup_n \max_{1\le j\le n-m} [\sup\{|P(A\cap B) - P(A)P(B)|;\ A\in\sigma(X_{ni},1\le i\le j,\ B\in\sigma(X_{ni},1\ge j+m\}] = \alpha(m)\downarrow 0 .$$

Since $\alpha(m) \le \beta(m)$, it follows that if $\{X_{ni}\}$ is absolutely regular, then it is also strong mixing.

The convergence of the one sample rank order statistic was established when both the score functions and the regression constants are bounded and $F_{ni} = F_n$ for all i $(1\le i\le n)$ in Harel and Puri (1989a) by a generalization from the stationary case to the nonstationary case of the results of Yoshihara (1978). This paper extends the results of Harel and Puri (loc. cit.) to the case when the score functions as well as the regression constants are not necessarily bounded and also for hypothesis of nonstationarity more general than in Harel and Puri (1989a). Later Harel and Puri (1989b) gave some applications of the results of Harel and Puri (1989a) for Markov processes as well as ARMA processes. The conclusions of Theorem 3.1, and Corollaries 3.1 and 3..2 in this paper hold also for the models given in section 2 in Harel and Puri (1989b). For applications of unsigned serial linear rank statistics introduced by Hallin, Ingenbleek and Puri (1985) in time series and Markov processes, the reader is referred to Harel and Puri (1989c).

2. Preliminaries. For any real x, define $F_n(x)$, $H_n(|x|)$ and $H_{ni}(|x|)$ as follows:

$$F_n(x) = n^{-1}\sum_{i=1}^{n} F_{ni}(x),\ H_{ni}(|x|) = F_{ni}(|x|) - F_{ni}(-|x|) \text{ and } H_n(|x|) = n^{-1}\sum_{i=1}^{n} H_{ni}(|x|).$$

Also for any $t \in [0,1]$, define $Y_{ni}(t)$ as $Y_{ni}(t) = I_{[H_n(|X_{ni}|) \le t]} - H_{ni} \circ H_n^{-1}(t)$,

and put

$$X_{ni}^{*} = H_n(|X_{ni}|),\quad H_{ni}^{*} = H_{ni} \circ H_n^{-1},\quad 1 \le i \le n .$$

Definition. We say that the sequence $\{H_{ni}^{*}\}$ of d.f. is μ–bounded if there exists a finite and positive measure μ such that for every $n \ge 1$ and $1 \le i \le n$, the probability measure μ_{ni} associated with H_{ni}^{*} satisfies $\mu_{ni}(B) \le \mu(B)$ for all intervals B in $[0,1]$.

2.1. Basic lemmas.

LEMMA 2.1. Let $c>0$ and $\alpha\ (0<\alpha<1)$ be fixed. Assume that $\{\hat{X}_{ni}\}$ are independent real random variables with the same d.f.'s F_{ni} as X_{ni} and let

$$Z_{ni}(t) = I_{[H_n(|\hat{X}_{ni}|) \le t]} - H_{ni}^{*}(t) .$$

If the sequence $\{H_{ni}^{*}\}$ is μ–bounded and μ is a uniform measure on $[0,1]$ then, there exists positive constants M_1 and M_2 such that for any $I \subset \{1,\ldots,n\}$ such that card $I = m$, we have

$$\text{(2.1)}\qquad P[\sup_{t \le cm^{-\alpha}} |m^{-1}\sum_{i \in I} Z_{ni}(t)| \ge m^{-(1+\alpha)/2}u] \le M_1 e^{-M_2 u}$$

and

(2.2) $$P[\sup_{t\geq 1-cm^{-\alpha}} |m^{-1}\sum_{i\in I} Z_{ni}(t)| \geq m^{-(1+\alpha)/2}u] \leq M_1 e^{-M_2 u} .$$

PROOF. It is sufficient to prove (2.1), the proof for (2.2) is similar and is therefore omitted.

Without loss of generality we take $I=\{1,...,m\}$. Since $\{Z_{ni}(t)/(1-H^*_{ni}(t)),0\leq t<1\}$ is a martingale, it is easy to see that $\sum_{i=1}^{m} Z_{ni}(t)/(1-H^*_{ni}(t))$ is also a martingale.

For sufficiently large m_o , we have $\forall\ m\geq m_o$

(2.3)
$$\begin{aligned}
&P[\sup_{t\leq cm^{-\alpha}} |m^{-1}\sum_{i=1}^{m} Z_{ni}(t)| \geq m^{-\frac{1}{2}(1+\alpha)}u] \\
&\leq P[\sup_{t\leq cm^{-\alpha}} |\sum_{i=1}^{m} Z_{ni}(t)/(1-H^*_{ni}(t))| \geq m^{\frac{1}{2}(1-\alpha)}u] \\
&\leq e^{-\frac{1}{2}u} E(\exp(\tfrac{1}{2}m^{-\frac{1}{2}(1-\alpha)}|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})/(1-H^*_{ni}(cm^{-\alpha}))|)) \\
&\leq e^{-\frac{1}{2}u} E(\exp(m^{-\frac{1}{2}(1-\alpha)}|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})|)) .
\end{aligned}$$

Thus

(2.4)
$$\begin{aligned}
&E(\exp(m^{-\frac{1}{2}(1-\alpha)}|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})|)) \\
&= 1 + \int_0^{\infty} e^x P[m^{-\frac{1}{2}(1-\alpha)}|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})|\geq x]dx \\
&= 1 + \int_0^{\infty} e^x P[|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})|\geq xm^{\frac{1}{2}(1-\alpha)}]dx
\end{aligned}$$

and by using a Bernstein–type inequality (see Bennett (1962))

(2.5) $$E(\exp(m^{-\frac{1}{2}(1-\alpha)}|\sum_{i=1}^{m} Z_{ni}(cm^{-\alpha})|)) \leq 1 + 2\int_0^{\infty} e^x \exp(-g_m(x))dx$$

where $g_m(x) = x^2 m^{1-\alpha}/2[\sum_{i=1}^{m} mH^*_{ni}(cm^{-\alpha})(1-H^*_{ni}(cm^{-\alpha}) + \frac{1}{3}xm^{\frac{1}{2}(1-\alpha)}M]$ for any constant M which satisfies $\max_{1\leq i\leq m} [\max\{H^*_{ni}(cm^{-\alpha}),1-H^*_{ni}(cm^{-\alpha})\}] \leq M$.

As $\{H^*_{ni}\}$ is μ–bounded, we obtain

(2.6) $$g_m(x) \geq x^2 m^{1-\alpha}(2\ Acm^{1-\alpha} + \tfrac{1}{3}xm^{\frac{1}{2}(1-\alpha)}A\ \max\{cm^{-\alpha},1-cm^{-\alpha}\})$$

$g_m(x) \geq 2x$ for $x \geq x_o$ and $m \geq m_1$ where x_o and m_1 are chosen sufficiently large and $A = \mu([0,1])$.

From (2.3) – (2.6), we deduce (2.1) for $m \geq m_1$ and we can choose M_1 and M_2 such that (2.1) also holds for $m < m_1$. (2.1) is thus proved.

LEMMA 2.2. Let $\alpha\ (0<\alpha<1)$ be fixed. Choose a number $\rho\ (0<\rho<\alpha)$. Assume that the sequence $\{X_{ni}\}$ is absolutely regular with rates $\beta(m)$ and the sequence $\{H^*_{ni}\}$ is μ–bounded where μ is a uniform measure on $[0,1]$.

If t is a number such that $t \leq m^{-\alpha}$ or $t \geq 1-m^{-\alpha}$, $1 \leq m \leq n$, then

$$(2.7)\quad P(m^{-1} | \sum_{i=1}^{m} Y_{ni}(t) | \geq u[\exp_m\{-(1-\rho)(1+\alpha)/2\}] \leq M_1 m^{\rho}\{e^{-M_2 u} + m^{1-\rho}\beta([m\rho]+1)\}$$

where $\exp_m(x) = m^x$, $[m\rho]$ denotes the integral part of $m\rho$, M_1 and M_2 are some constants > 0.

PROOF. Put $k = k_m = [m\rho] + 1$ and write

$$(2.8)\qquad S_{m,n}(t) = \sum_{j=1}^{k} U_{n,j}(t)$$

where

$$(2.9)\qquad U_{n,j}(t) = \sum_{p=0}^{\ell_j} Y_{n,j+pk}(t)$$

and ℓ_j is the largest integer such that $j+\ell_j k \leq m$. Then

$$(2.10)\qquad P[m^{-1}|S_{m,n}(t)| \geq u[\exp_m\{-(1-\rho)(1+\alpha)/2\}]$$
$$\leq P(m^{-1}\sum_{j=1}^{k} |U_{n,j}(t)| \geq u[\exp_m\{-(1-\rho)(1+\alpha)/2\}])$$
$$\leq \sum_{j=1}^{k} P(m^{-(1-\rho)}|U_{n,j}(t)| \geq u[\exp_m\{-(1-\rho)(1+\alpha)/2\}]) .$$

For any j, $1 \leq j \leq k$, define

$$A_j = \{(y_1,\dots,y_{\ell_j}) : | \sum_{p=1}^{\ell_j} y_p | \geq u[\exp_m\{-(1-\rho)(1+\alpha)/2\}]\}$$

and put

$$g(y_1,\dots,y_{\ell_j}) = \begin{cases} 1 & \text{if } (y_1,\dots,y_{\ell_j})\in A_j \\ 0 & \text{otherwise.} \end{cases}$$

By using Lemma 2.1 in Harel and Puri (1988),

$$P[|U_{n,j}(t)|\ge u\exp_m((1-\rho)(1-\alpha)/2] = Eg(Y_{n,j}(t),\dots,Y_{n,j+\ell_j k}(t))$$

$$\le P(|\sum_{p=o}^{\ell_j} Z_{n,j+pk}(t)|\ge u[\exp_m\{(1-\rho)(1+\alpha)/2\}]) + 2\ell_j\beta(k)$$

$$\le M_1 e^{-M_2 u} + 2m^{1-\rho}\beta([m\rho]+1) \tag{2.11}$$

by using Lemma 2.1.

From (2.10) and (2.11), we deduce (2.7) and the lemma is proved.

For any n,m $(m\le n)$ and any i $(1\le i\le m)$ let

$$X^*_{n,m,i} = \frac{1}{m+1}\Big(\sum_{\substack{j\le m\\ j\ne i}} H^*_{nj}(X^*_{ni}) + 1\Big) \tag{2.12}$$

and

$$Z_{n,m,i} = \frac{R_{n\,,\,m,i}}{m+1} - X^*_{n,m,i} = \frac{1}{m+1}\sum_{\substack{j\le m\\ j\ne i}} Y_{nj}(X^*_{ni})\,. \tag{2.13}$$

LEMMA 2.3. <u>Under the assumptions of Lemma 2.2 there exists an</u> $n_o = n_o(u)$ <u>such that</u>

$$P[|Z_{n,m,i}|\ge 5u[\exp_m\{-(1-\rho)(1+\alpha)/2\}],\ X^*_{ni}\le m^{-\alpha} \text{ or } X^*_{ni}\ge 1-m^{-\alpha}] \tag{2.14}$$
$$\le M_1 m^{-\alpha+\rho}\{e^{-M_1 u} + m^{1-\rho}\beta([m\rho]+1)\}\,.$$

PROOF. The proof is similar to the proof of Lemma 2.3 of Yoshihara (1978) and follows by using Lemma 2.2.

LEMMA 2.4. <u>Let</u> $\{\tilde{X}_{ni}\}$ <u>be r.v.'s (with means 0) which are strong mixing with rates satisfying</u>

$$\sum_{m\ge 1}(\alpha(m))^{\delta/(2+\delta)} < \infty \text{ for some } \delta>0\,. \tag{2.15}$$

<u>Suppose that for any</u> $K>0$, <u>there exists a sequence</u> $\{\tilde{X}^K_{ni}\}$ <u>of r.v.'s satisfying</u>

(2.15) and such that

(2.16) $$\sup_{n\in\mathbb{N}}{}_* \max_{1\le i\le n} |\tilde{X}^K_{ni}| \le B_K < \infty \ \forall K>0$$

where B_K is some constant > 0

(2.17) $$\sup_{n\in\mathbb{N}}{}_* \max_{1\le i\le n} E|\tilde{X}_{ni} - \tilde{X}^K_{ni}|^{2+\delta} \to 0 \text{ as } K\to\infty$$

(2.18) $$E(\sum_{i=1}^{n} \tilde{X}^K_{ni} - E(\tilde{X}^K_{ni}))^2/n \to C^2_K \ \forall K>0 \text{ as } n\to\infty$$

where C_K is some constant > 0.

(2.19) $$E(\sum_{i=1}^{n} \tilde{X}_{ni})^2/n \to C^2 < \infty \text{ as } n\to\infty$$

where C is some constant > 0.

(2.20) $$C_K \to C \text{ as } K\to\infty .$$

Then $n^{-\frac{1}{2}} \sum_{i=1}^{n} \tilde{X}_{ni}$ converges in law to the normal distribution with mean 0 and variance C^2.

PROOF. See Harel and Puri (1989a).

3. Weak convergence of the one sample rank order statistic $\mathscr{S}_{n,m}$.

Let F be a d.f. on $\mathbb{R}^+$ and define the d.f. H on $\mathbb{R}^+$ by $H(|x|) = F(|x|) - F(-|x|)$.

For a score function J which is square integrable, put

(3.1) $$\mu^*_{n,i} = \mu_J(F_{ni}) = \int_{\mathbb{R}} s(x)J(H_n(|x|)dF_{ni}(x) , \ i\le n .$$

For h defined in (1.2) put

(3.2) $$\mu_{n,m} = \sum_{i=1}^{m} c_{n,i}\mu^*_{n,i} = \sum_{i=1}^{m} h(\tfrac{i}{n})\mu^*_{n,i} , \ m\le n .$$

For any sequence of d.f.'s $\{F_\ell\}_{\ell\ge 1}$ on $\mathbb{R}^2$ with marginals F and assuming that J admits a derivative J' which is also integrable, we denote

(3.3) $$\sigma^2(\{F_\ell\}) = \lim_{n\to\infty} \{\int_{\mathbb{R}} g^2(x)dF(x) + 2 \sum_{\ell=2}^{n} \int_{\mathbb{R}^2} g(x)g(y)dF_\ell(x,y)\}$$

if the limit exists, where

(3.4) $$g(x) = \int_{\mathbb{R}} s(y)\{I_{[|x|\le|y|]} - H(|y|)\}J'(H(|y|)dF(y) + s(x)J(H(|x|)) - \int_{\mathbb{R}} s(y)J(H(|y|)dF(y)\ .$$

For every $n \ge 1$, let

(3.5) $$V_n(s) = (\sigma n)^{-\frac{1}{2}}[A_{n,[ns]} + (ns-[ns])A_{n,[ns]+1}]$$

where σ is the positive constant defined in (3.3) and $A_{n,i} = \mathscr{S}_{n,i} - \mu_{n,i}$.

The process $V_n(s) = \{V_n(s),\ 0 \le s \le 1\}$ belongs to C_1, the space of all continuous functions on $[0,1]$ on which we associate the usual uniform metric.

Let $\mathscr{L}_\delta$, $(0<\delta<1)$ consist of the set of functions L on (0,1) such that

(i) L is twice differentiable on (0,1)

(ii) L is nondecreasing on (0,1)

(iii) As $u \downarrow 0$ or $u \uparrow 1$ $(d^{(i)}/d_u^{(i)})(L(u)) = 0([u(1-u)]^{-\frac{1}{2}+i+\delta})$, $i=0,1,2$.

Let $\mathscr{L}_\delta^1$ consist of the set of functions L on (0,1] such that

(i') L is twice differentiable on (0,1],

(i'i') L in nondecreasing on (0,1],

(i'i'i') $(d^{(i)}/d_u^{(i)})(L(u)) \le Au^{-\frac{1}{2}-i+\delta}$, $i=0,1,2$ where A is some constant > 0.

Let $\mathscr{L}_{\delta'}^2$ consist of the set of functions L on [0,1] such that

(i") L is twice differentiable on [0,1)

(i"i") L is nondecreasing on [0,1)

(i'''i''') $(d^{(i)}/(d_u^{(i)}(L(u)) \le A(1-u)^{-\frac{1}{2}-i+\delta}$, $i=0,1,2$ where A is some constant > 0.

Let $F_{n,i,j}$ be the d.f. of (X_{ni}, X_{nj}). Then, we have the following theorem which is a generalization from the stationary case to the nonstationary case of the Theorem 4.3 in Yoshihara (1978).

THEOREM 3.1. Suppose the sequence $\{X_{ni}\}$ is absolutely regular with rate (1.3) where $3/10 < \delta < 1/2$ and the sequence $\{H_{ni}^*\}$ is μ–bounded where μ is a diffuse measure. Furthermore assume that for any $\ell \in \mathbb{N}^*$ with $1<\ell$, there exists a continuous d.f.

F_ℓ on $\mathbb{R}^2$ with continuous marginals F such that

$$(3.6)\quad \lim_{n\to\infty} \max_{1\le i<j\le n} |F_{n,i,j}(\epsilon H_n^{-1}(t_1),\epsilon' H_n^{-1}(t_2)) - F_{j-i}(\epsilon H^{-1}(t_1),\epsilon' H^{-1}(t_2))| = 0$$

for all $(t_1,t_2) \in [0,1]^2$ and $(\epsilon,\epsilon') \in \{0,1\}^2$

and

$$(3.7)\qquad \max_{1\le i\le n} \sup_{x\in\mathbb{R}^+} |H_{ni}(x) - H(x)| = 0(n^{-\gamma}) \text{ where } \gamma>1 .$$

Let J be a score function which belongs to $\mathscr{L}_\delta$ and let h belong to $\mathscr{L}^1_{\delta'}$ $(0 < \frac{1}{2}-\delta' < \delta^2/18)$. If $\sigma^2(\{F_\ell\})$ defined by (3.3) is strictly positive, then V_n defined in (3.5) converges weakly in uniform topology on C_1 to the process $V_0 = \{V_0(s), 1\le s\le 1\}$ where

$$(3.8)\qquad V_0(s) = \int_0^s h(u)dW(u) \text{ , } 0\le s\le 1$$

and $W = \{W(s), 0\le s\le 1\}$ is a standard Brownian motion process, and $\sigma^2(\{F_\ell\}) < \infty$.

REMARK 3.1. Yoshihara (1978) in Theorem 4.3 assumes that $J\in\mathscr{L}^2_\delta$ and h satisfies the Lipschitz condition (which implies the regression constants are bounded). Here we only assume that $J\in\mathscr{L}_\delta$ and $h\in\mathscr{L}^1_{\delta'}$. Thus we have more general score functions and our regression constants are unbounded.

REMARK 3.2. Let the sequence $\{X_{ni}\}$ be absolutely regular, and let the sequence of distribution functions $\{F_{n,i,j}\}$ satisfy the following conditions:

(i) There exists a sequence $\{F_\ell\}$ of distribution functions defined on $\mathbb{R}^2$ such that

$$(1)\qquad \lim_{n\to\infty} \max_{1\le i<j\le n} |F_{n,i,j}(x_1,x_2) - F_{j-i}(x_1,x_2)| = 0 \text{ for all } (x_1,x_2)\in\mathbb{R}^2$$

and

$$(b)\qquad F_{ni} = F_n \text{ for all } 1\le i\le n, n\ge 1 .$$

Then the condition (3.6) is satisfied, and moreover, the condition (3.7) is not necessary for the proof of Theorem 3.1.

PROOF. In what follows we assume that the measure μ is a uniform measure on [0,1] and we show that this condition is sufficient.

If μ is only diffuse, let ν be the uniform probability on $[0,1]$ and let $\mu^* = \mu+\nu$ and $K_\mu = 1+\mu([0,1])$. If H^* is the d.f. of the measure μ^* , define the map $\psi : [0,1]\to[0,1]$ by

$$\psi(t) = H^*(t)/K_\mu .$$

Let $\mu'^* = \psi(\mu^*)$ and $H'_{ni} = \psi(\mu_{ni})$, then it is obvious that μ'^* is uniform on $[0,1]$ and the sequence $\{H'_{ni}\}$ is μ'^*–bounded.

We then have

$$(3.9)\ Y_{ni}(t) = I_{[H_n(|X_{ni}|)\le t]} - H^*_{ni}(t) = I_{[\psi\circ H_n(|X_{ni}|)\le\psi(t)]} - H'_{ni}(\psi(t)) = Y'_{ni}(\psi((t)) .$$

As ψ is a nondecreasing bijection on $[0,1]$, we can work with $Y'_{ni}(\psi(t))$ instead of $Y_{ni}(t)$.

We need some lemmas. For any $n\ (n\ge1)$ and for any $i\ (1\le i\le n)$ let

$$(3.10)\ B^*_{ni}(X_{ni}) = B_{ni} = \int_{\mathbb{R}} s(x)\{I_{[|X_{ni}|\le|x|]} - H_{ni}(|x|)J'(H_n(|x|)dF_{ni}(x)$$

$$+ \{s(X_{ni})J(H_n(|X_{ni}|) - E(s(X_{ni})J(H_n(|X_{ni}|))\}$$

$$B_{ni} = \int_{[0,1]} Y_{ni}(t)J'(t)d\hat{H}_{ni}(t) + s(X_{ni})J(X^*_{ni}) - \int_{[0,1]} J(t)d\hat{H}_{ni}(t)$$

where $\hat{H}_{ni}(t) = F_{ni}(H^{-1}_{ni}(t)) + F_{ni}(-H^{-1}_{ni}(t)) - 1$. It is obvious that $E(B_{ni}) = 0$.

Now consider the process $L_n(t)$ defined on C_1 by

$$(3.11)\qquad L_n(s) = n^{-\frac{1}{2}}\Big(\sum_{i=1}^{[ns]} B_{ni} + (ns-[ns])B_{n,[ns]+1}\Big) .$$

LEMMA 3.1. <u>Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 3.1. Then the process $L_n(s)$ converges weakly in uniform topology to a gaussian process $L_0(s)$ with trajectories a.s. in C_1 with mean 0 and variance $s\sigma^2(\{F_\ell\})$ where $\sigma^2(\{F_\ell\})$ is defined in</u> (3.3).

PROOF. The process L_n defines a probability measure P_n on C_1 . From Theorem 8.1 of Billingsley (1968), we have to prove that (i) the finite dimensional distribution of P_n converges in law to a normal distribution and (ii) P_n is tight.

First we prove (i) which is equivalent to proving that $\sum_{\ell=1}^{p} \lambda_\ell L_n(s_\ell)$ converges in law to a normal distribution for any $p \in \mathbb{N}^*$, any $s_\ell \in [0,1]$ and any $\lambda_\ell \in \mathbb{R}$ $(1 \le \ell \le p)$. Without loss of generality we can take $p=2$ and suppose that $s_1 < s_2$. We then have

$$\sum_{\ell=1}^{2} \lambda_\ell L_n(s_\ell) = n^{-\frac{1}{2}}\Big[\sum_{i=1}^{[ns_1]} (\lambda_1+\lambda_2)B_{ni} + \sum_{i=[ns_1]+1}^{[ns_2]} \lambda_2 B_{ni} \tag{3.12}$$
$$+ \lambda_1(ns_1-[ns_1])B_{n,[ns_1]+1} + \lambda_2(ns_2-[ns_2])B_{n,[ns_2]+1}\Big] .$$

We define the sequence of r.v.'s $\{C_{ni}\}$ by

$$C_{ni} = \begin{cases} (\lambda_1+\lambda_2)B_{ni} & \text{if } i \le [ns_1] \\ \lambda_2 B_{ni} & \text{if } [ns_1] < i \le [ns_2] \\ 0 & \text{if } i > [ns_2] \end{cases} . \tag{3.13}$$

As J belongs to $\mathscr{L}_\delta$ and μ is a uniform measure, it is not difficult to prove that

$$\sup_{n\in\mathbb{N}} \max_{1\le i\le n} E(B_{ni})^{2+\gamma} \le M < \infty \quad \text{for} \quad \gamma \le 3 \tag{3.14}$$

and by the Markov inequality, we deduce

$$\sum_{\ell=1}^{2} \lambda_\ell L_n(s_\ell) = n^{-\frac{1}{2}} \sum_{i=1}^{n} C_{ni} + O_P(n^{-\frac{1}{2}}) \tag{3.15}$$

From Lemma 2.4 we have to verify (2.15)–(2.20).

(2.15) comes from (1.3).

We have

$$E\Big(\sum_{i=1}^{n} C_{ni}\Big)^2/n = \frac{1}{n}\Big[(\lambda_1+\lambda_2)^2\Big(\sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} E(B_{ni}B_{nj})\Big) \tag{3.16}$$
$$+ (\lambda_1+\lambda_2)\lambda_2\Big(\sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} E(B_{ni}B_{nj}) + \lambda_2^2 \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} E(B_{ni}B_{nj})\Big] .$$

From (3.14), we obtain

$$(3.17)\ \frac{1}{n}\Big|(\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} E(B_{ni}B_{nj})\Big|$$

$$\leq (|\lambda_1|+|\lambda_2|)|\lambda_2| M^{2/(2+\gamma)}\, n^{-1} \Big(\sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} \alpha^{\gamma/(2+\gamma)}(j-i)\Big)\,.$$

From (1.3), the last expression converges to 0 as $n\to\infty$ for $\gamma=1$.

It remains to prove that $\frac{1}{n}\sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} E(B_{ni}B_{nj})$ and $\frac{1}{n}\sum_{i=[ns_1]+1}^{[ns_2]}\sum_{j=[ns_1]+1}^{[ns_2]} E(B_{ni}B_{nj})$ converge respectively to some finite constants. Without loss of generality, we only prove this if J' is a nondecreasing function.

For every $K>0$, define the sequence $\{B^K_{ni}\}$ of r.v.'s from (3.10) with J and J' replaced by J^K and J'^K where

$$J^K = \begin{cases} J & \text{if } |J|\leq K \\ 0 & \text{if } |J|>K \end{cases}$$

and

$$J'^K = \begin{cases} J' & \text{if } t(1-t)|J'|\leq K \\ 0 & \text{if } t(1-t)|J'|>K \end{cases}$$

From (3.14), we easily deduce that (2.16) and (2.17) are satisfied.

Now we prove that $\frac{1}{n}\sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} E(B^K_{ni}B^K_{nj})$ converges to some finite constant.

We first prove the convergence by replacing J and J' in the expression of B_{ni} in (3.10) by indicator functions such that $J(t) = I_{[a\leq t\leq b]}$ and $J'(t) = I_{[a'\leq t\leq b']}$ and we can then write

$$B_{ni} = \hat{H}_{ni}(b') - \hat{H}_{ni}(\{(a\vee X^*_{ni})\wedge b'\}) - \int_{a'}^{b'} t\,d(\hat{H}_{ni}(t)) + s(X_{ni})I_{[a\leq X^*_{ni}\leq b]}$$
$$- (\hat{H}_{ni}(b) - \hat{H}_{ni}(a))\,.$$

Thus

$$E(B_{ni}B_{nj}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} B^*_{ni}(x)B^*_{nj}(y)\,dF_{n,i,j}(x,y)\,.$$

From the condition (3.6) we deduce that

$$\lim_{n\to\infty} \max_{1\leq i<j\leq n} |E(B_{ni}B_{nj}) - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(x)C(y)dF_{j-i}(x,y)| = 0$$

where

$$C(x) = H^*(b') - H^*(\{a'\vee H(|x|)\wedge b'\}) - \int_{a'}^{b'} t dH^*(t) + s(x)(I_{[a\leq H(|x|)\leq b]} - (H^*(b)-H^*(a))$$

and

$$H^*(t) = F(H^{-1}(t)) + F(-H^{-1}(t)) - 1$$

We obtain a similar property if J and J' are replaced by step functions.

As J and J' are continuous, we can uniformly approach J^K and J'^K by step functions and deduce that

$$(3.18)\qquad \lim_{n\to\infty} \max_{1\leq i<j\leq n} |E(B^K_{ni}B^K_{nj}) - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^K(x)g^K(y)dF_{j-i}(x,y)| = 0$$

where $g^K(x)$ is defined as in (3.4) by replacing J and J' respectively by J^K and J'^K

Let us now denote

$$\rho(1) = \int_{-\infty}^{+\infty} g^K(x)dF(x) \quad\text{and}\quad \rho(i) = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^K(x)g^K(y)dF_i(x,y)\ .$$

Then, we have

$$|\frac{1}{n}\sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} E(B^K_{ni}B^K_{nj}) - \frac{[ns_1]}{n}\sum_{i=1}^{+\infty}\rho(i)|$$

$$\leq |\frac{[ns_1]}{n}\frac{1}{[ns_1]}\sum_{i=0}^{[ns_1]-1}\sum_{j=1}^{[ns_1]-i} E(B^K_{ni}B^K_{n,j+i}) - \frac{[ns_1]}{n}\frac{1}{[ns_1]}\sum_{i=1}^{[ns_1]}([ns_1]-i)\rho(i)|$$

$$+\frac{[ns_1]}{n}\sum_{i=[ns_1]+1}^{\infty}|\rho(i)| + \frac{[ns_1]}{n}\sum_{i=1}^{[ns_1]}\sum_{k=i}^{\infty}|\rho(k)| = |A_n| + B_n + C_n\ .$$

From (3.18), we deduce that $|A_n|\to 0$ and from the inequality of strong mixing sequences of r.v.'s and (3.14) we deduce that $|\rho(i)| \leq \alpha(i)^{\gamma/(2+\gamma)}M^{2/(2+\gamma)}$ which implies that $B_n\to 0$ and $C_n\to 0$ (as $n\to\infty$) for $\gamma=1$.

It is also immediate that

$$\left|\frac{[ns_1]}{n}\sum_{i=1}^{+\infty}\rho(i) - s_1\left(\sum_{i=1}^{+\infty}\rho(i)\right)\right| \to 0 \text{ as } n\to\infty .$$

We conclude that $\frac{1}{n}\sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} E(B^K_{ni}B^K_{nj})$ converges to $s_1\left(\sum_{i=1}^{\infty}\rho(i)\right)$ as $n\to\infty$ where $\sum_{i=1}^{\infty}\rho(i)$ equals $\sigma^2_K(\{F_\ell\})$ defined in (3.3) with J and J' replaced by J^K and J'^K respectively.

Now from the decomposition

$$\sum_{i=1}^{[ns_2]}\sum_{j=1}^{[ns_2]} = \sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} + \sum_{i=1}^{[ns_1]}\sum_{j=[ns_1]+1}^{[ns_2]} + \sum_{i=[ns_1]+1}^{[ns_2]}\sum_{j=1}^{[ns_1]} + \sum_{i=[ns_1]+1}^{[ns_2]}\sum_{j=[ns_1]+1}^{[ns_2]}$$

and using (3.17) and the preceding result, we conclude that

$$\frac{1}{n}\sum_{i=[ns_1]+1}^{[ns_2]}\sum_{j=[ns_1]+1}^{[ns_2]} E(B^K_{ni}B^K_{nj}) \longrightarrow (s_2-s_1)\sigma^2_K(\{F_\ell\}) \text{ as } n\to\infty$$

and (2.18) is proved.

We now easily deduce from this and (3.14) that (2.19) and (2.20) are satisfied.

Finally from (3.12), (3.13), (3.15)–(3.17) we conclude that $E\left(\sum_{\ell=1}^{2}\lambda_\ell L_n(s_\ell)\right)^2$ converges to $\{(\lambda_1+\lambda_2)^2 s_1 + \lambda_2^2(s_2-s_1)\}\sigma^2(\{F_\ell\})$ which implies that $\sum_{\ell=1}^{2}\lambda_\ell L_n(s_\ell)$ converges in law to the normal distribution with mean 0 and variance $\{(\lambda_1+\lambda_2)^2 s_1 + \lambda_1^2(s_2-s_1)\}\sigma^2(\{F_\ell\})$ and (i) is proved.

We now prove (ii).

From Theorem 8.2 of Billingsley (1968) we have to verify that $\forall\ \epsilon>0$, $\exists\ \eta>0$, $(0<\eta<1)$ and an integer N_0 such that $\forall\ n\geq N_0$

(3.19) $$P[\sup_{|s-s'|<\eta} |L_n(s)-L_n(s')|\geq\epsilon] \leq \epsilon .$$

If ns and ns' are integers, we obtain the following inequality by using Theorem 2.10 for $q=4$ of Doukhan and Portal (1987) for $s>s'$

$$E(L_n(s)-L_n(s'))^4 \leq ((s-s')^2 + \frac{s-s'}{n})M^{4/(2+\gamma)} \sum_{m=0}^{+\infty} m^2\alpha^{(\gamma-2)/(2+\gamma)}(m) .$$

For $\gamma=3$, (1.3) entails

$$\sum_{m=0}^{+\infty} m^2\alpha^{(\gamma-2)/(2+\gamma)}(m) < M_0 < +\infty .$$

If $s>s'$ and ns and ns' are integers, we have $s-s' \geq n^{-1}$ and

(3.20) $$E(L_n(s)-L_n(s'))^4 \leq 2M^{4/(2+\gamma)}(s-s')^2M_0 .$$

From Lemma 2 of Balacheff and Dupont (1980) we obtain that $\forall\ \epsilon>0$, $\exists\ \eta>0$ and there exists an integer N_0 sufficiently large such $\forall\ n\geq N_0$

$$P[\sup_{|\frac{[ns]}{n}-\frac{[ns']}{n}|<2\eta} |L_n(\frac{[ns]}{n}) - L_n(\frac{[ns']}{n})| > \epsilon/2] \leq 2M^{4/(2+\gamma)}M_0\eta^2K16\epsilon^{-4}$$

where K is some constant > 0 .

From the definition of $L_n(s)$ in (3.11) we obtain

(3.21) $$\sup_{|s-s'|<\eta} |L_n(s)-L_n(s')| \leq 2 \max_{|\frac{[ns]}{n}-\frac{[ns']}{n}|<2\eta} |L_n(\frac{[ns]}{n} - \frac{[ns']}{n})| .$$

By using (3.20) and (3.21), we deduce

(3.22) $$P[\sup_{|s-s'|<\eta} |L_n(s)-L_n(s')|\geq\epsilon] \leq 64M^{4/(2+\gamma)}M_0K\epsilon^{-4}\eta^2$$

and (3.19) is proved. Lemma 3.1. follows.

DEFINITION. A function $r : [0,1]\to\mathbb{R}^+$ is called a weight function if it satisfies the following conditions:

(j) r belongs to C_1

(jj) $r(0)=0$ and $r(1)=0$.

LEMMA 3.2. _Suppose that_ $\{X_{ni}\}$ _satisfies the conditions of Theorem_ 3.1. _Then for any weight function_ r _satisfying_

(3.23) $$r(s) \geq As^{\frac{1}{2}-\delta''} \quad \text{where } A>0 ,\ 0<\delta''<\delta'$$

$L_n(s)\cdot\frac{1}{r(s)}$ _converges weakly in uniform topology a.s. in_ C_1 _to the gaussian process_ $L_0(s)\cdot\frac{1}{r(s)}$. _(By convention_ $\frac{1}{r(s)} = 0$ _if_ $r(s) = 0$).

PROOF. At first we prove that $\forall\ \eta>0$, $\exists\ \theta>0$ and $\exists\ N_0$ such that $\forall\ n\geq N_0$

(3.24) $$P[\sup_{s\notin[\theta,1-\theta]} |L_n(s)\cdot\frac{1}{r(s)}|\geq\eta] \leq \eta .$$

By symmetry and without loss of generality, it is sufficient to prove (3.23) for $r(s) = s^{\frac{1}{2}-\delta''}$.

If ns and ns' are integers, we obtain the following inequality by using Theorem 2.10 for $q=4$ of Doukhan and Portal (1987) for $s'>s$

$$E(\frac{L_n(s')}{r(s')}-\frac{L_n(s)}{r(s)})^4 \leq 8E(\frac{L_n(s')-L_n(s)}{r(s')})^4 + 8E(L_n(s)(\frac{1}{r(s')}-\frac{1}{r(s)}))^4$$
$$\leq 32[s-s']^{4\delta''}M^{4/(2+\gamma)} \sum_{m=0}^{+\infty} m^2\alpha^{(\gamma-2)/(2+\gamma)}(m) .$$

As $4\delta''>1$, we deduce from Theorem 12.2 of Billingsley that $\forall\ \eta>0$, $\forall\ \theta>0$, $\exists\ N_0$ such that $\forall\ n\geq N_0$

(3.25) $$P[\sup_{\substack{s\leq 2\theta \\ ns\ \text{integer}}} |L_n(s)\cdot\frac{1}{r(s)}| \geq \eta/2] \leq K16\eta^{-4}(2\theta)^{4\delta''}$$

where K is some constant > 0 .

From the definition of $L_n(s)$ in (3.11) and r is nondecreasing, we obtain also

(3.26) $$P[\sup_{s\leq\theta} |L_n(s)\cdot\frac{1}{r(s)}| \geq \eta] \leq 16K\eta^{-4}(2\theta)^{4\delta''} ,$$

From (3.25) we easily deduce (3.23).

We now prove Lemma 3.2.

The process $L_n\cdot\frac{1}{r}$ defines a probability measure Q_n on C_1 and we have to show that (a) the finite dimensional of Q_n converges in law to a normal distribution and (b) Q_n is tight.

(a) is deduced easily from (i) from the proof of Lemma 3.1. We have only to prove (b). For any $\theta>0$, set $C_\theta = \{t;t\leq\theta\}$. Then,

$$\sup_{\substack{|s-s'|<\eta \\ (s,s')\in C_\theta}} |L_n(s)\cdot\frac{1}{r(s)} - L_n(s')\,\frac{1}{r(s')}| \leq$$

$$\leq \sup_{s\in C_{\theta-\eta}} \sup_{|s-s'|<\eta} |L_n(s)\frac{1}{r(s)} - L_n(s')\frac{1}{r(s')}| + \sup_{s\notin C_{\theta-\eta}} \sup_{|s-s'|<\eta} |(L_n(s) - L_n(s')| \frac{1}{r(s)}$$

$$+ L_n(s')|\frac{1}{r(s)} - \frac{1}{r(s')}|$$

$$\leq 2 \sup_{s\in C_\theta} |L_n(s)\frac{1}{r(s)}| + \frac{1}{m} \sup_{|s-s'|<\eta} |L_n(s)-L_n(s')|$$

$$+ \frac{1}{m^2} \sup_{|s-s'|<\eta} |r(s)-r(s')| \sup_{s\in[0,1]} |L_n(s)| \quad \text{where } m = \min_{s\notin C_{\theta-2\eta}} r(t)$$

and from (3.22), (3.25) and (3.26) we easily deduce that $\forall \epsilon>0$, $\exists \eta>0$, $(0<\eta<1)$ and an integer N_0 such that $\forall n\geq N_0$

$$P[\sup_{|s-s'|<\eta} |L_n(s)\cdot\frac{1}{r(s)} - L_n(s')\frac{1}{r(s')}|\geq\epsilon] \leq \epsilon . \tag{3.28}$$

From (3.27) and Theorem (8.2) of Billingsley (1968) we deduce that Q_n is tight and 3.2 is proved.

We now consider the process $\hat{L}_n$ defined on C_1 by

$$\hat{L}_n(s) = n^{-\frac{1}{2}}(\sum_{i=1}^{[ns]} c_{ni}B_{ni} + c_{n,[ns]+1}(ns-[ns])B_{n,[ns]+1})(\sigma(\{F_\ell\}))^{-1} . \tag{3.29}$$

LEMMA 3.3. <u>Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 3.1. Then the process $\hat{L}_n(s)$ converges weakly in uniform topology to a gaussian process $\hat{L}_0(s) = \int_0^s h(u)dW(u)$ with trajectories a.s. in C_1 where W is a standard Brownian motion process.</u>

PROOF. For any n, define a measure λ_n on $[0,1]$ by setting

$$\lambda_n(\{\tfrac{i}{n}\}) = (c_{ni} - c_{n,i+1})(\sigma(\{F_\ell\}))^{-1} , \quad 1\leq i\leq n-1$$

$$\lambda_n(\{1\}) = (c_{nn})(\sigma(\{F_\ell\})^{-1} .$$

By definition, we have

$$\hat{L}_n(s) = \int_0^s L_n(u)\lambda_n(du) + L_n(s)c_{n,[ns]+1} = \int_0^1 L_n(u)\lambda_n(du) + L_n(s)h(\tfrac{[ns]+1}{n}) . \tag{3.30}$$

We now prove that

$$\hat{L}_n(s) \to (-\int_0^s L_0(u)h'(u)du + L_0(s)h(s))(\sigma(\{F_\ell\}))^{-1} \tag{3.31}$$

in uniform topology.

Let $g_n : C_1 \to D_1$ be defined by

$$g_n(f)(s) = \int_0^s f(u)r(u)\lambda_n(du) - f(s)r(s)h(\frac{[ns]+1}{n}) \quad \forall\, n \geq 1$$

where D_1 is the set of functions $[0,1] \to [0,1]$ which are right–continuous and have left–hand limits (see Billingsley (1968)) for any $f \in C_1$ where r is a weight function satisfying (3.23) and $g_o : C_1 \to C_1$ is defined by

$$g_o(f)(s) = \int_0^s f(u)r(u)h'(u)du - f(s)r(s)h(s)$$

for any $f \in C_1$.

Let $\{f_n; n \geq 1\}$ be a sequence of functions in C_1 and suppose that

(3.32) $\quad \frac{f_n}{r} \to \frac{f_o}{r}$ in uniform topology where $f_o \in C_1$ and $\frac{f_o}{r} \in C_1$.

We show that $g_n(f_n \cdot \frac{1}{r}) \to g_o(f_o \cdot \frac{1}{r})$ as $n \to \infty$ in uniform topology and we deduce (3.31) by Theorem 5.5 of Billingsley (1968). We have

$$|g_n(f_n \tfrac{1}{r})(s) - g_o(f_o \tfrac{1}{r})(s)| =$$

$$= |\int_0^s f_n(u) \frac{1}{r(u)} r(u)\lambda_n(du) - f_n(s)\cdot\frac{1}{r(s)} r(s)h(\frac{[ns]+1}{n}) - \int_0^s f_o(u) \frac{1}{r(u)} r(u)h'(u)du$$

$$+ f_o(s)\frac{1}{r(s)} r(s)h(s)|$$

$$\leq |\int_0^s (f_n(u) \frac{1}{r(u)} - f_o(u) \frac{1}{r(u)}) r(u)\lambda_n(du)| + |\int_0^s |f_o(u) \frac{1}{r(u)} r(u)(\lambda_n - h'(u))(du)|$$

$$+ |f_n(s)\frac{1}{r(s)} r(s)h(\frac{[ns]+1}{n}) - f_o(s) \frac{1}{r(s)} r(s)h(\frac{[ns]+1}{n})| + |f_o(s)\frac{1}{r(s)} (h(\frac{[ns]+1}{n}) - h(s))|r(s)$$

$$\leq \sup_{v\in[0,1]} |(f_n \tfrac{1}{r})(v) - (f_o \tfrac{1}{r})(v)| \int_0^s r(u)\lambda_n(du) + \sup_{v\in[0,1]} |f_o \tfrac{1}{r})(v)| \, |\int_0^s r(u)(\lambda_n - h'(u))du|$$

$$+ \sup_{v\in[0,1]} |h(v)r(v)| \sup_{w\in[0,1]} |(f_n \cdot \tfrac{1}{r})(w) - (f_o \cdot \tfrac{1}{r})(w)|$$

$$+ \sup_{v\in[0,1]} |(f_o \tfrac{1}{r})(v)| \, |h(\frac{[ns]+1}{n}) - h(s)|r(s) .$$

From (3.23), (3.32) and the properties of the function h , we deduce that the last expression converges uniformly to zero as $n \to \infty$, and hence we deduce (3.31) by Theorem 5.5 of

Billingsley (1968).

We can write

$$(-\int_0^s L_o(u)h'(u)du + L_o(s)h(s))(\sigma(\{F_\ell\})^{-1} = \int_0^s h(u)(\sigma(\{F_\ell\}))^{-1}dL_o(u) = \int_0^s h(u)dW(u) .$$

From (3,31) we conclude that

$$\hat{L}_n(s) \to \int_0^s h(u)dW(u) \quad \text{as} \quad n\to\infty$$

and Lemma 3.3 is proved.

LEMMA 3.4. Under the conditions of Theorem 3.1, we have

(3.33) $$\forall\ \epsilon>0\ ,\ P[\sup_{s\in[0,1]} |V_n(s)-\hat{L}_n(s)|>\epsilon] \to 0 \ \ \underline{\text{as}}\ \ n\to\infty .$$

PROOF. To prove (3.33), it is sufficient to prove

(3.34) $$P[\max_{1\le m\le n} |\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}| \ge \epsilon n^{\frac{1}{2}}] \to 0 \quad \text{as} \quad n\to\infty .$$

We now proceed as in Yoshihara (1978). Let $\alpha = 1 - \delta/3$ and define d as the smallest integer j for which $\alpha^j \le 1/5$ holds. For any n, let

$$E_1 = E_{n,1} = \{u : u\le n^{-\alpha} \text{ or } u\ge 1-n^{-\alpha}\}$$

$$E_j = E_{n,j} = \{u : n^{-\alpha^{j-1}} < u \le n^{-\alpha^j} \text{ or } 1-n^{-\alpha^j} < u \le 1-n^{-\alpha^{j-1}}\}\ ,\ j=2,\dots,d$$

and

$$E_{d+1} = E_{n,d+1} = \{u : n^{-\alpha^d} \le u \le 1-n^{-\alpha^d}\} .$$

For every $n\ge 1$, we define functions $K_j(u) = K_{n,j}(u)$ $(j=1,\dots,d+1)$ on $[0,1]$ by

$$K_1(u) = K_{n1}(u) = \begin{cases} 0 & \text{if } n^{-\alpha} \le u \le 1-n^{-\alpha} \\ J(u) & \text{if } 1/n+1 \le u \le n^{-\alpha} \text{ or } 1-n^{-\alpha} \le u \le n/n+1 \\ J(1/n+1) & \text{if } u \le 1/n+1 \\ J(n/n+1) & \text{if } u \ge n/n+1 \end{cases}$$

$K_j(u) = K_{n,j}(u) = J(u)I_{E_j}(u)$, $j=2,\dots,d+1$ where I_{E_j} is the indicator function of the set E_j . Then for almost all $u\in[1/n+1, n/n+1]$,

$$J(u) = \sum_{j=1}^{d+1} K_j(u)\ ,\quad J'(u) = \sum_{j=1}^{d+1} K_j'(u) \quad \text{and} \quad J''(u) = \sum_{j=1}^{d+1} K_j''(u)$$

and there exist two positive constants M_1 and M_2 such that

$$(3.35)\quad M_1 I_{E_j}[\exp_n\{(\tfrac{1}{2}+i-\delta)\alpha^j\}] \le d^{(i)}/d_u^{(i)} K_j(u) \le M_2 I_{E_j}[\exp_n\{(\tfrac{1}{2}+i-\delta)\alpha^{j-1}\}]$$

$i=0,1,2;\ j=1,\dots,d+1$. Set $h_n^{(m)} = \sup_{1\le i\le m} |h(i/n+1)|$.

From (3.14), it is obvious that

$$(3.36)\qquad E(\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni})^2 \le (h_n^{(m)})^2 M m^2$$

where M is some constant > 0 . So putting $n_o = [\exp_n \alpha^d]$, we have for any $\epsilon>0$

$$(3.37)\qquad P[\max_{1\le m\le n_o} |\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}| \ge \epsilon n^{\frac{1}{2}}]$$

$$\le \sum_{m=1}^{n_o} P[|\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}| \ge \epsilon n^{\frac{1}{2}}]$$

$$\le M(h_n^{(n_o)})^2 n_o^3 (\epsilon n^{\frac{1}{2}})^{-2} \le AM\, n^{\frac{1}{2}-\delta'} n_o^3 (\epsilon n^{\frac{1}{2}})^{-2} = 0(n^{-\gamma'})$$

for some $\gamma'>0$.

Thus, to prove (3.34), it is enough to show that for any $\epsilon>0$, there exists an integer $N=N(\epsilon)$ such that for all $n\ge N$,

$$P[\max_{n_o\le m\le n} |\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}| \ge \epsilon n^{\frac{1}{2}}] \le \epsilon\ .$$

From (1.1) and (3.10) we have that

$$(3.38)\quad |\mathscr{S}_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}| =$$

$$= |\sum_{i=1}^{m} c_{ni}s(X_{ni})(J(X^*_{n,m,i}) + Z_{n,m,i}J'(X^*_{n,m,i}) + Z^2_{n,m,i}k_{n,m,i}(X^*_{n,m,i})$$

$$- \mu_{n,m} - \sum_{i=1}^{m} c_{ni}B_{ni}|$$

(where $k_{n,m,i}$ is defined like $k_{m,i}$ in (3.8) of Yoshihara (1978))

$$\le h_n^{(m)} \sum_{i=1}^{m} |J(X^*_{n,m,i}) - J(X^*_{ni})| + |\sum_{i=1}^{m} c_{ni}[s(X_{ni})Z_{n,m,i}J'(X^*_{n,m,i})$$

$$-\int_{[0,1]} Y_{ni}(t)J'(t)dH^*_{ni}(t)]| + \sum_{j=1}^{d+1}\sum_{i=1}^{m} |c_{ni}|Z^2_{n,m,i}k^{(j)}_{n,m,i}(X^*_{n,m,i})$$

$$\le h_n^{(m)} \sum_{i=1}^{m} |J(X^*_{n,m,i}) - J(X^*_{n,i})| + \sum_{j=1}^{d+1} [|V^{(j)}_{n,m}| + \frac{2}{m+1}|\sum_{i=1}^{m} |c_{ni}| \int_{[0,1]} Y_{ni}(t)K_j'(t)dH^*_{ni}(t)]$$

$$+ h_n^{(m)} \sum_{j=1}^{d+1}\sum_{i=1}^{m} Z^2_{n,m,i}|K_j''(X^*_{n,m,i})|$$

where

$$(3.39)\quad V^{(j)}_{n,m} = \frac{1}{m+1} \sum_{1\le i\le m}\ \sum_{1\le \ell\le m} c_{ni}[s(X_{ni})Y_{n\ell}(X^*_{ni})K_j'(X^*_{ni}) - \int_{[0,1]} Y_{ni}(t)K_j'(t)dH^*_{ni}(t)]$$

and $k^{(j)}_{n,m,i}(u) = k_{n,m,i}(u)I_{E_j}(u)$.

We remark that if $X^*_{ni} = H_n(|X_{ni}|) \in E_j$, j=1,...,d , then $X^*_{ni} > 1/2$ or $X^*_{ni} < 1/2$, and so

$$X^*_{n,m,i} = \frac{1}{m+1}(\sum_{\substack{j\le m\\ j\ne i}} H^*_{nj}(X^*_{ni})+1)$$

$$= \frac{1}{m+1}(\sum_{\substack{j\le m\\ j\ne i}} H^*_{nj}(X^*_{ni})+1) - \frac{m-1}{m+1}H(|X_{ni}|) + \frac{m-1}{m+1}H(|X_{ni}|)$$

$$-\frac{m-1}{m+1}H_n(|X_{ni}|) + \frac{m-1}{m+1}H_n(|X_{ni}|)$$

and from (3.8), we deduce

$$(3.40)\qquad X^*_{n,m,i} \begin{cases} \le X^*_{ni} + An^{-\gamma} \text{ if } X^*_{ni} > \frac{1}{2}, \text{ where } A \text{ is some constant} > 0\ . \\ \ge X^*_{ni} - An^{-\gamma} \text{ if } X^*_{ni} < \frac{1}{2} \end{cases}$$

Now also from (3.8),

$$(3.41)\qquad \sum_{i=1}^{m} |J(X^*_{n,m,i}) - J(X^*_{ni})| = \sum_{i=1}^{m} (X^*_{n,m,i}-X^*_{ni})J'(\theta_{n,m,i})$$

$$\le B \sum_{j=1}^{d+1} \sum_{i=1}^{m} (m+1)^{-1} K_j'(\theta_{n,m,i})$$

where B is some constant > 0 and $\theta_{n,m,i} \in [X^*_{n,m,i} \wedge X^*_{ni}, X^*_{n,m,i} \vee X^*_{ni}]$.

From (3.40) and the fact that μ is a uniform measure we deduce for each $j=1,\dots,d$ and $i=1,\dots,m$

$$E(I_{E_j}(\theta_{n,m,i}) \le P[\theta_{n,m,i} \in E_j] \le P[(X^*_{ni} \le n^{\alpha^j} + An^{-\gamma}) \cup (X^*_{ni} \ge 1-n^{-\alpha^j} - An^{-\gamma}] \le Cn^{\alpha^j}$$

where C is some constant > 0 . And proceeding as in Yoshihara (1978), p. 113, 114 we obtain for each $j=1,\dots,d$

$$(3.42)\qquad P[\max_{n_0 \le m \le n} h_n^{(m)} \sum_{i=1}^{m} (m+1)^{-1} K_j'(\theta_{n,m,i}) \ge \epsilon n^{\frac{1}{2}}] = 0(n^{-\gamma}) .$$

On the other hand for $j=d+1$, we have that for any $\epsilon>0$

$$(3.43)\qquad h_n^{(m)} \sum_{i=1}^{m} (m+1)^{-1} K'_{d+1}(\theta_{n,m,i}) \le C[\exp_n \{\tfrac{1}{5}(\tfrac{3}{2} - \delta)\}] n^{\frac{1}{2}-\delta'} \le \epsilon n^{\frac{1}{2}}$$

for n sufficiently large where C is some constant > 0 .

Hence from (3.42) and (3.43), we obtain

$$(3.44)\qquad P[\max_{n_0 \le m \le n} h_n^{(m)} \sum_{i=1}^{m} |J(X^*_{n,m,i}) - J(X^*_{ni})| > \epsilon n^{\frac{1}{2}}] \to 0 \quad \text{as } n \to \infty .$$

As J belongs to $\mathscr{L}_\delta$, $h \in \mathscr{L}^1_{\delta'}$, and using the property (1.3), we obtain

$$(3.45)\qquad E\Big(\sum_{i=1}^{m} |c_{ni}| \int_{[0,1]} Y_{ni}(t) K_j'(t) dH^*_{ni}(t)\Big)^2 \le (h_m^{(n)})^2 M \le n^{1-2\delta'} M$$

where M is some constant > 0 , which implies

$$(3.46)\quad P[\max_{n_o \le m \le n} 2(m+1)^{-1} |\sum_{i=1}^{m} |c_{ni}| \int_{[0,1]} Y_{ni}(t) K_j'(t) dH_{ni}^*(t)| > \epsilon n^{\frac{1}{2}}]$$

$$\le \sum_{m=n_o}^{n} P[(m+1)^{-1} |\sum_{i=1}^{m} |c_{ni}| \int_{[0,1]} Y_{ni}(t) K_j'(t) dH_{ni}^*(t)| > \epsilon n^{\frac{1}{2}}]$$

$$\le M(\epsilon^2 n)^{-1} n^{1-2\delta} \sum_{m=n_o}^{n} m^{-1} = 0(n^{-2\delta} \log n)\ , \quad j=1,\dots,d+1\ .$$

From Lemma 2.1 in Harel and Puri (1989a) and using the same techniques as in the proof of Lemma 2.2 in Harel and Puri (1989a), we obtain

$$E|V_{n,m}^{(j)}|^4 = 0(n^{2-4\delta})$$

and so

$$(3.47)\qquad P[\max_{n_o \le n \le m} |V_{n,m}^{(j)}| \ge \epsilon n^{\frac{1}{2}}] \le M(\epsilon^2 n)^{-2} \sum_{m=1}^{n} E|V_{n,m}^{(j)}|^4 = 0(n^{1-4\delta})\ .$$

Finally let $n_j = \exp_n\{\alpha^{d-j}\}$, $j=1,\dots,d+1$. Let $\{T_{n,m,i}^1 : n_o \le m \le n_{d-j+1}\}$, $(j=1,\dots,d)$ and $\{T_{n,m,i}^2 : n_{d-j+1} \le m \le n\}$, $(j=1,\dots,d)$ be collections of random variables, defined respectively, by

$$(3.48)\qquad T_{n,m,i}^{1,j} = \begin{cases} 1 & \text{if } |Z_{n,m,i}| \le (\log m)^2 [\exp_n\{-(1-\rho)(1+\alpha)\alpha^{j-1}/2\}] \\ 0 & \text{otherwise.} \end{cases}$$

$$(3.49)\qquad T_{n,m,i}^{2,j} = \begin{cases} 1 & \text{if } |Z_{n,m,i}| \le (\log m)^2 [\exp_m\{-(1-\rho)(1+\alpha^j)/2\}] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $j\ (1 \le j \le d)$ and for any $m\ (n_o \le m \le n_{d-j+1})$

$$(3.50)\quad h_n^{(m)} \sum_{i=1}^{m} |k_{n,m,i}^{(j)}(X_{n,m,i}^*)| Z_{n,m,i}^2 \le h_n^{(m)} \sum_{i=1}^{m} I_{E_j}(X_{n,m,i}^*)\{(\log n)^4 [\exp_n\{(\tfrac{5}{2} - \delta)\alpha^{j-1}$$
$$- (1-\rho)(1+\alpha)\alpha^{j-1}\}] + (1 - T_{n,m,i}^{1,j})[\exp_n\{\tfrac{5}{2} - \delta\}\alpha^{j-1}]\}\ .$$

Accordingly for each $j\ (1 \le j \le d)$,

$$(3.51)\ P[\max_{n_o\le m\le n_{d-j+1}} h_n^{(m)} \sum_{i=1}^{m} |c_{ni}|\,|k_{n,m,i}^{(j)}(X_{n,m,i}^*)|\,Z_{n,m,i}^2 \ge 2\epsilon n^{\frac12}]$$

$$\le P[\max_{n_o\le m\le n_{d-j+1}} h_n^{(m)} \sum_{i=1}^{n_{d-j+1}} I_{E_j}(X_{n,m,i}^*)(\log n)^4(\exp_n\{(\tfrac12-\tfrac{\delta}{2})\alpha^{j-1}+\tfrac{\delta\alpha^j}{6})>\epsilon n^{\frac12}]$$

$$+ P[\max_{n_o\le m\le n} h_n^{(m)} \sum_{i=1}^{n_{d-j+1}} I_{E_j}(X_{n,m,i}^*)(1-T_{n,m,i}^{1,j})(\exp_n\{(\tfrac52-\delta)\alpha^{j-1}\})\ge\epsilon n^{\frac12}] = I_1 + I_2 .$$

Since for any i and j $(1\le j\le d)$ from (3.40)

$$P[X_{n,m,i}^*\in E_j] \le Cn^{-\alpha^j}$$

so for any j $(1\le j\le d)$, we have

(3.52)

$$I_1 \le \max_{n_o\le m\le n_{d-j+1}} h_n^{(m)} n^{-\frac12} \sum_{i=1}^{n_{d-j+1}} P(X_{n,m,i}^*\in E_j)(\log n)^4(\exp_n\{\tfrac12-\tfrac{\delta}{2})\alpha^{j-1}+\tfrac{\delta\alpha^j}{6}\})$$

$$\le M \max_{n_o\le m\le n_{d-j+1}} h_n^{(m)} (\log n)^4[\exp_n\{-\tfrac12+\alpha^{j-1}-\alpha^j+(\tfrac12-\tfrac{\delta}{2})\alpha^{j-1}+\tfrac{\delta\alpha^j}{6}]$$

$$= 0(n^{(\frac12-\delta')-\delta^2/18})$$

where M is some constant > 0.

On the other hand from Lemma 2.2 and (1.3)

$$(3.53)\ I_2 \le \sum_{m=n_o}^{n_{d-j+1}} P[h_n^{(m)} \sum_{i=1}^{m} I_{E_j}(X_{n,m,i}^*)(1-T_{n,m,i}^{1,j})[\exp_n\{\tfrac52-\delta\}\alpha^{j-1}\}]\ge\epsilon n^{\frac12}]$$

$$\le M_1 n^{-\frac12} \sum_{m=n_o}^{n_{d-j+1}} h_n^{(m)}[\exp_n\{\tfrac52-\delta\}\alpha^{j-1}] \sum_{i=1}^{m} E(I_{E_j}(X_{n,m,i}^*)(1-T_{n,m,i}^{1,j})$$

$$\le M_2 n^{-\frac12} n^{(\frac12-\delta')} \sum_{m=n_o}^{n_{d-j+1}} [\exp_n\{(\tfrac52-\delta)\alpha^{j-1}\}]m^{1-\alpha+\rho}\{e^{-M_3(\log m)^2}+m^{1-\rho}\beta(m^\rho)\}$$

$$\le M_2 n^{(\frac12-\delta')} [\exp_n\{-\tfrac12+(\tfrac52-\delta)\alpha^{j-1}\}n_o^{-5-\alpha+4\delta} = 0(n^{-\gamma'})$$

for any j ($1 \le j \le d$) and for some $\gamma' > 0$ (where M_1, M_2 and M_3 are some constant > 0).

From (3.52) and (3.53) it follows that

$$(3.54) \qquad P[\max_{n_0 \le m \le n_{d-j+1}} \sum_{i=1}^{m} |c_{ni}| \, |k_{n,m,i}^{(j)}(X_{n,m,i}^{*})| Z_{n,m,i}^{2} > 2\epsilon n^{\frac{1}{2}}] = 0(n^{-\gamma''})$$

for any j ($1 \le j \le d$) and for some $\gamma'' > 0$.

Similarly, for j ($1 \le j \le d$)

$$(3.55) \quad P[\sum_{n_{d-j+1} \le m \le n} \sum_{i=1}^{m} |c_{ni}| \, |k_{n,m,i}^{(j)}(X_{n,m,i}^{*})| Z_{n,m,i}^{2} > 2\epsilon n^{\frac{1}{2}}]$$

$$\le P[\max_{n_{d-j+1} \le m \le n} h_n^{(m)} \sum_{i=1}^{n} I_{E_j}(X_{n,m,i}^{*})(\log n)^4 (\exp_n\{(\tfrac{5}{2} - \delta)\alpha^{j-1} - (1-\rho)(1+\alpha^j)\}) > \epsilon n^{\frac{1}{2}}]$$

$$+ P[\max_{n_{d-j+1} \le m \le n} h_n^{(m)} \sum_{i=1}^{n} I_{E_j}(X_{n,m,i}^{*})(1 - T_{n,m,i}^{1,j})(\exp_n\{(\tfrac{5}{2} - \delta)\alpha^{j-1}\}) > \epsilon n^{\frac{1}{2}}]$$

$= 0(n^{-\gamma_1})$ for some $\gamma_1 > 0$.

For j=d+1 and for all m ($n_0 \le m \le n$) it follows from Lemma 4.4 in Harel and Puri (1988) that

$$E(\sum_{i=1}^{m} |c_{ni}| \, |k_{n,m,i}^{(d+1)}(X_{n,m,i}^{*})| Z_{n,m,i}^{2})^2 \le (h_n^{(m)})^2 (\exp_n\{2(\tfrac{5}{2} - \delta)\alpha^{d+1}\}) E(\sum_{i=1}^{m} Z_{n,m,i}^{2})^2$$
$$\le M n^{1-2\delta'} (\exp_n(5-2\delta)\alpha^{d}) m^{-1}$$

Since from the definition of d , $(5-2\delta)\alpha^d - 1 < 0$ so

$$(3.56) \qquad P[\max_{n_0 \le m \le n} \sum_{i=1}^{m} |c_{ni}| \, |k_{n,m,i}^{(d+1)}(X_{n,m,i}^{*})| Z_{n,m,i}^{2} \ge 2\epsilon n^{\frac{1}{2}}]$$

$$\le M n^{-1} n^{1-2\delta'} (\exp_n\{(5-2\delta)\alpha^d\}) \sum_{m=n_0}^{m} m^{-1} \text{ (where } M \text{ is some constant} > 0)$$

$$= 0(n^{-\gamma_2}) \text{ for some } \gamma_2 > 0 .$$

From (3.54)–(3.56) we deduce that for any j ($1 \le j \le d+1$) ,

$$\text{(3.57)} \qquad P[\max_{n_0 \le m \le n} \sum_{i=1}^{m} |c_{ni}|\,|k^{(j)}_{n,m,i}(X^*_{n,m,i})|\,Z^2_{n,m,i} \ge 2\epsilon n^{\frac{1}{2}}] = 0(n^{-\gamma_3})$$

for some $\gamma_3 > 0$.

Finally from (3.36), (3.38), (3.44), (3.46), (3.47) and (3.57), we obtain (3.34) and Lemma 3.4 is proved.

The proof of Theorem 3.1 now follows as an easy consequence of Lemma 3.3 and Lemma 3.4.

REMARK 3.3. Theorem 3.1 works when the function $h(x)$ defined in (1.2) is unbounded only when $x \to 0$ and not when $x \to 1$. If $h(x)$ is unbounded when $x \to 1$ but not when $x \to 0$, then we have the result as stated in Corollary 3.1 below.

For every $n \ge 1$, let

$$\text{(3.58)} \qquad \tilde{V}_n(s) = (\sigma n)^{-\frac{1}{2}}[\tilde{A}_{n,[ns]^*} + ([ns]^* - ns)\tilde{A}_{n,[ns]^*-1}]$$

where σ is the positive constant defined in (3.3) and where

$$[ns]^* = \begin{cases} ns & \text{if } ns \text{ is an integer} \\ [ns]+1 & \text{otherwise} \end{cases}, \qquad \tilde{A}_{n,i} = \tilde{\mathscr{S}}_{n,i} - \tilde{\mu}_{n,i},$$

$$\tilde{\mathscr{S}}_{n,i} = \sum_{j=i}^{m} c_{nj} s(X_{nj}) J(\frac{R_{n,i,j}}{n-i+1}) \text{ and } \tilde{\mu}_{n,i} = \sum_{j=i}^{m} c_{nj}\mu^*_{n,j}.$$

COROLLARY 3.1. <u>If among the conditions in Theorem 3.1, the function</u> h <u>is replaced by a function</u> $\tilde{h}$ <u>which belongs to</u> $\mathscr{L}^2_{\delta'}$ $(0 < \frac{1}{2} - \delta' < \delta^2/18)$, <u>then</u> $\tilde{V}_n$ <u>defined in (3.58) converges weakly in uniform topology on</u> C_1 <u>to the process</u> $\tilde{V}_0 = \{\tilde{V}_0(s), 0 \le s \le 1\}$ <u>where</u>

$$\tilde{V}_0(s) = \int_s^1 \tilde{h}(u)dW(u) \qquad 0 \le s \le 1$$

<u>and</u> $\sigma^2(\{F_\ell\}) < \infty$.

PROOF. Consequence of Theorem 3.1 by symmetrization.

REMARK 3.4. Finally, if $h(x)$ is unbounded both when $x \to 0$ as well as when $x \to 1$, then we have the result as stated in Corollary 3.2.

For every $n \geq 1$ let

$$\hat{V}_n(s) = \begin{cases} V_n(s) & \text{if } s \leq \frac{1}{2} \\ \tilde{V}_n(s) & \text{if } s > \frac{1}{2} . \end{cases} \tag{3.59}$$

The process $\hat{V}_n(s) = \{V_n(s), 0 \leq s \leq 1\}$ belongs to C_1^* the space of all functions f in $[0,1]$ for which $f/[0,\frac{1}{2}]$ is continuous and $f/(\frac{1}{2},1]$ admits a continuous prolongation on $[\frac{1}{2},1]$.

COROLLARY 3.2. *If among the conditions in Theorem* 3.1, *the function* h *is replaced by a function* $\hat{h}$ *which belongs to* $\mathscr{L}_{\delta'}$ $(0 < \frac{1}{2} - \delta' < \delta^2/18)$, *then* $\hat{V}_n$ *defined in* (3.59) *converges weakly in uniform topology on* C_1^* *to the process* $\hat{V}_0 = \{\hat{V}_0(s), 0 \leq s \leq 1\}$ *where*

$$\hat{V}_0(s) = \begin{cases} V_0(s) & \text{if } s \leq \frac{1}{2} \\ \tilde{V}_0(s) & \text{if } s > \frac{1}{2} \end{cases}$$

and $\sigma^2(\{F_\ell\}) < \infty$.

PROOF. Easy consequence from Theorem 3.1 and Corollary 3.1.

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WEAK INVARIANCE OF GENERALIZED U–STATISTICS FOR NONSTATIONARY ABSOLUTELY REGULAR PROCESSES

Short Title: NONSTATIONARY ABSOLUTELY REGULAR PROCESSES.

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Abstract. Yoshihara [7] established the weak invariance theorem of the generalized U–statistic for absolutely regular random variables but only for the stationary case. In this paper we extend the results from the stationary case to the nonstationary case.

weak convergence * generalized U–statistic * absolute regularity * strong mixing * Brownian process * Skorohod topology.

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1. Introduction. Let $c\ (\geq 2)$ be an integer and for each $j\ (1 \leq j \leq c)$ consider an integer $m_j \geq 1$; let g be a Borel measurable function $g : (\mathbb{R}^k)^{m_o} \longrightarrow \mathbb{R}$ where $m_o = m_1 + \ldots + m_c$ and assume that g is symmetric in $m_j\ (\geq 1)$ arguments of the j–th set for $j=1,\ldots c$, i.e. g is symmetric in each of the c sets of arguments. We consider c independent triangular arrays of r.v.'s (random variables) with values in $\mathbb{R}^k$ and let the random variables corresponding to the j–th triangular array be denoted by $\underset{\sim}{X}_{n,i,j}$ $(1 \leq i \leq n, n \geq 1)$. Let $\underset{\sim}{X}_{n,i,j}$ have the continuous d.f. (distribution function) $F_{n,i,j}$, $1 \leq i \leq n$, $1 \leq j \leq c$, $n \geq 1$.

Denote $\mathbf{n} = (n_1,\ldots,n_c)$ and assume that for any j, $n_j \geq m_j$, let $\mathscr{S}_{\mathbf{n}} = \prod_{j=1}^{c} \mathscr{S}_{n_j,j}$ where $\mathscr{S}_{n_j,j}$ is the set of all the strictly increasing sequences of m_j elements in $\{1,\ldots,n_j\}$ $(1 \leq i_{j1} < i_{j2} < \ldots < i_{jm_j} \leq n_j)$.

The generalized U–statistic of degree $\mathbf{m}\ (=(m_1,\ldots,m_o))$ is defined as

$$U(\mathbf{n}) = \left[\prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \right] \sum_{\mathscr{S}_{\mathbf{n}}} g(\underset{\sim}{X}_{n_j,i_{j\ell},j},\ 1 \leq \ell \leq m_j,\ 1 \leq j \leq c)\ . \tag{1.1}$$

Generalizing the definition given by Yoshihara [7] in the stationary case, we say that a triangular array of r.v.'s $\eta_{n,i}$ $(1 \leq n, 1 \leq i \leq n)$ is absolutely regular with rates β if

$$\beta(m) = \sup_{m \leq n} \sup_{1 \leq \ell \leq n-m} E\{\sup_{A \in \mathscr{M}^{n}_{n,\ell+m}} |P(A\,|\,\mathscr{M}^{\ell}_{n,1}) - P(A)|\} \downarrow 0 \quad \text{as} \quad m \to \infty\,,$$

where $\mathscr{M}^{b}_{n,a}$ is the σ–field generated by $(\eta_{n,a},\ldots,\eta_{n,b})$.

Also recall that $\{\eta_{n,i}\}$ is strong mixing if

$$\alpha(m) = \sup_{m \leq n} \sup_{1 \leq \ell \leq n-m} \{\sup P\,|P(A \cap B) - P(A)P(B)|;\ A \in \mathscr{M}^{\ell}_{n,1}, B \in \mathscr{M}^{n}_{n,\ell+m}\}\} \downarrow 0 \quad \text{as} \quad m \to \infty$$

since $\alpha(m) \leq \beta(m)$, it follows that if $\{\eta_{n,i}\}$ is absolutely regular, then it is also strong mixing.

We now suppose that for each j, the triangular array of r.v.'s $\xi_{n,i,j}$ is absolutely regular with rate β_j (consequently, it is strong mixing with rate $\alpha_j \leq \beta_j$).

Our aim is to study the asymptotic behavior of $U(\mathbf{n})$ when the $n_j \to \infty$ such that for any j, $n_j/(n_1+\dots+n_c) \longrightarrow \lambda_j$ $(0<\lambda_j<1)$, $1\le j\le c$.

The convergence of the generalized U–statistic was established by Yoshihara [7] for absolutely regular r.v.'s but only for the stationary case. In this paper, we generalize the results of Yoshihara [7] from the stationary case to the nonstationary case. Our methods are the adaptations of some of the ideas of Yoshihara (loc. cit.) and a new central limit theorem (Lemma 2.3) for nonstationary unbounded strongly–mixing random variables which is an extension of a central limit theorem due to Withers [6] for nonstationary uniformly bounded strongly–mixing random variables.

2. Preliminaries.

2.1 Notations. For each $\mathbf{d}=(d_1,\dots,d_c)$ with $0\le d_j\le m_j$, $1\le j\le c$, we set $J_n(\mathbf{d}) = \{\mathbf{i};\ \mathbf{i}=(\mathbf{i}_1,\dots,\mathbf{i}_c),\ \mathbf{i}_j=(i_{j1},\dots,i_{jd_j}),\ 1\le i_{j1}\ne\dots\ne i_{jd_j}\le n_j$ if $d_j\ne 0$ and $i_{jd_j}=0$ if $d_j=0$, $1\le j\le c\}$. Let $\mathbf{i}\in J_n(\mathbf{d})$ exist, and let

$$g^{(\mathbf{i})}_{\mathbf{d},\mathbf{n}}(\underset{\sim}{x}_{i,j}, i=1,\dots,d_j, j=1,\dots,c) \tag{2.1}$$
$$= \sum_{\mathbf{i}^*\in I^{(\mathbf{i})}_{\mathbf{d},\mathbf{n}}} \int_{\mathbb{R}^*_c} g(\underset{\sim}{x}_{i,j}, i=1,\dots,m_j, j=1,\dots,c) \prod_{j=1}^{c}\prod_{\nu=d_j+1}^{m_j} dF_{n_j,i_{j\nu},j}(\underset{\sim}{x}_{\nu,j})$$

where

$$I^{(\mathbf{i})}_{\mathbf{d},\mathbf{n}} = \{\mathbf{i}^*=(\mathbf{i}^*_1,\dots,\mathbf{i}^*_c), \mathbf{i}^*_j=(i_{j,d_j+1},\dots,i_{jm_j}),\ 1\le i_{j,d_j+1}\ne\dots\ne i_{jm_j}\le n_j, \tag{2.2}$$

$i_{j\ell}\notin\{i_{j1},\dots,i_{jd_j}\}$, if $d_j+1\le\ell\le m_j, 1\le j\le c\}$, $\mathbb{R}^*_c=(\mathbb{R}^k)^{\sum_{j=1}^{c}(m_j-d_j)}$ and we set $g^{\mathbf{i}}_{0,\mathbf{n}}=g_{\mathbf{d},\mathbf{n}}$ for $\mathbf{d}=(0,\dots,0)$. By convention if $\mathbf{d}=(d_1,\dots,d_c)$ where $d_j=1$ and $d_\ell=0$, $\ell\ne j$, then $\mathbf{d}$ will be denoted by $\mathbf{d}^{(j)}$.

For every $\mathbf{d}$, we put

$$U^{(\mathbf{d})}_{\mathbf{n}} = \prod_{j=1}^{c} n_j^{-[m_j]} \sum_{\mathbf{i}\in J_n(\mathbf{d})} \int_{\tilde{\mathbb{R}}_c} g^{(\mathbf{i})}_{\mathbf{d},\mathbf{n}}(\underset{\sim}{x}_{\ell,j}, \ell=1,\dots,d_j, j=1,\dots,c) \prod_{j=1}^{c}\prod_{\ell=1}^{d_j} d\big(I_{[\underset{\sim}{X}_{n_j,i_{j\ell},j}\le \underset{\sim}{x}_{\ell,j}]} - F_{n_j,i_{j\ell},j}(\underset{\sim}{x}_{\ell,j})\big) \tag{2.3}$$

where $I_{[\ ]}$ denotes the indicator function on $\mathbb{R}^k$, $\underset{\sim}{a}\le\underset{\sim}{b}$ means $a_\ell\le b_\ell$, $\ell=1,\dots,k$, $\tilde{\mathbb{R}}_c=(\mathbb{R}^k)^{\sum_{j=1}^{c} d_j}$ and $m^{-[p]}=m(m-1)\dots(m-p+1)$.

Let $a=(a_1,\dots,a_c)$ with $a_j\ge1$, $1\le j\le c$ and denote $L_n(a)=\{i;i=(i_1,\dots i_c)$, $i_j=(i_{j1},\dots,i_{ja_j})$, $1\le i_{j1}<\dots<i_{ja_j}\le n_j$, $j=1,\dots,c\}$.

Let $i\in L_n(a)$ and let for any j $(1\le j\le c)$ and any ℓ_j $(1\le\ell_j\le a_j)$, $\overset{i}{P}_{j,\ell_j,n}$ be the probability measure induced by the random vectors

$$\overset{i}{Y}_{n,j,\ell_j}=(\underset{\sim}{X}_{n_1,i_{11},1},\dots,\underset{\sim}{X}_{n_1,i_{1a_1},1},\underset{\sim}{X}_{n_2,i_{21},2},\dots,\underset{\sim}{X}_{n_{j-1},i_{j-1a_{j-1}},j-1},\underset{\sim}{X}_{n_j,i_{j1},j},\dots,\underset{\sim}{X}_{n_j,i_{j\ell_j},j})$$

and

$$\overset{*i}{Y}_{n,j,\ell_j}=(\underset{\sim}{X}_{n_j,i_{j,\ell_j+1},j},\dots,\underset{\sim}{X}_{n_j,i_{ja_j},j},\underset{\sim}{X}_{n_{j+1},i_{j+1,1},j+1},\dots,\underset{\sim}{X}_{n_{j+1},i_{ca_c},c})$$

respectively and defined by

$$\overset{i}{P}_{j,\ell_j,n}(A_{\ell_j}^{(j)}\times A_{\ell_j}^{*(j)})=P[\overset{i}{Y}_{n,j,\ell_j}\in A_{\ell_j}^{(j)}]\times P[\overset{*i}{Y}_{n,j,\ell_j}\in A_{\ell_j}^{*(j)}]$$

and

$$\overset{i}{P}_{o,n}(A^{(c)})=P[(\underset{\sim}{X}_{n_1,i_{11},1},\dots,\underset{\sim}{X}_{n_c,i_{ca_c},c})\in A^{(c)}]$$

where

$$A_{\ell_j}^{(j)}\in\sigma(\overset{i}{Y}_{n,j,\ell_j}),\quad A_{\ell_j}^{*(j)}\in\sigma(\overset{*i}{Y}_{n,j,\ell_j}),\quad A^{(c)}\in\sigma(\underset{\sim}{X}_{n_1,i_{11},1},\dots,\underset{\sim}{X}_{n_c,i_{ca_c},c})$$

and where $\sigma(\underset{\sim}{X})$ is the σ-field generated by $\underset{\sim}{X}$.

For any function $h:(\mathbb{R}^k)^{m_o}\to\mathbb{R}$, we denote $h\otimes h$ the function $h\otimes h:(\mathbb{R}^k)^{2m_o}\to\mathbb{R}$ defined by $h\otimes h(x_1,\dots,x_{2km_o})=h(x_1,\dots,x_{km_o})h(x_{km_o+1},\dots,x_{2km_o})$.

For any $p\ge1$, let $f:[0,1]^p\to\mathbb{R}$. For $\rho=(\rho(1),\dots,\rho(p))\in\{0,1\}^p$, define

$$f_\rho(t)=\lim_{\substack{s_i\uparrow t_i\ \ \rho(i)=1\\ s_i\downarrow t_i\ \ \rho(i)=0}} f(s),\ s\in[0,1]^p,\ t\in[0,1]^p$$ if it exists where $s=(s_1,\dots,s_p)$, $t=(t_1,\dots,t_p)$,

and denote by D_p (D_p^* resp.) the space of all maps $f : [0,1]^p \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^p$, f_ρ exists and $f_\rho = f$ for $\rho=(0,\dots,0)$ ($\rho=(1,\dots,1)$ resp.). We associate on D_p the Skorohod topology (for a definition of Skorohod topology see Neuhaus [4] and Balacheff and Dupont [1]).

2.2 Basic lemmas. We need the following Lemmas:

LEMMA 2.1. For every $a=(a_1,\dots,a_c)$ with $a_j=b_j m_j$, $b_j \ge 1$, $1 \le j \le c$ let $i \in L_n(a)$ exist, and for every j $(1 \le j \le c)$ and any ℓ_j $(1 \le \ell_j \le a_j)$, let $h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1},\underset{\sim}{x}_{b_1m_1+1},\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c})$ be a Borel function such that

$$\int_{\mathbb{R}_{a,c}} |h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c})|^{1+\delta}\, dP^i_{j,\ell_j,n} \le M, \tag{2.4}$$

$$\int_{\mathbb{R}_{a,c}} |h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c})|^{1+\delta} dP^i_{o,n} \le M \text{ for some } \delta>0, \tag{2.5}$$

where $\mathbb{R}_{a,c} = (\mathbb{R}^k)^{\sum_{i=1}^{c} b_i m_i}$. Then

$$\left|\int_{\mathbb{R}_{a,c}} h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c}) dP^i_{j,\ell_j,n} - \int_{\mathbb{R}_{a,c}} h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c}) dP^i_{o,n}\right|$$
$$\le 4M^{1/(2+\delta)} \beta_j^{\delta/(1+\delta)}(i_{j,\ell_j+1} - i_{j,\ell_j}). \tag{2.6}$$

As the special case, if h is bounded, say, $|h(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{b_1m_1+\dots+b_cm_c})| \le M$, then we can replace the right side of (2.6) by $2M\beta_j(i_{j,\ell_j+1} - i_{j,\ell_j})$.

PROOF. Follows from Lemma 1 of Yoshihara [7].

Let $\mathscr{S}_j^*$ be the set of all the nondecreasing sequences of m_j elements in $\{1,\dots,n_j\}$ $(1 \le i_{j,1} \le i_{j,2} \le \dots \le i_{j,m_j} \le n_j)$ and let $\mathscr{S}_n^* = \prod_{j=1}^{c} \mathscr{S}_j^*$ exist.

For any n and any family $s = (i_{j,\ell}, 1 \le \ell \le m_j, 1 \le j \le c)$ which belongs to $\mathscr{S}_n^*$, we denote

$$a_{n,s}(h,r) = \int |h(\underset{\sim}{x}_{\ell,j}, 1 \le \ell \le m_j, 1 \le j \le c)|^r \prod_{j=1}^{c} \prod_{\ell=1}^{m_j} dF_{n_j,i_{j,\ell}^j}(\underset{\sim}{x}_{\ell,j}) \tag{2.7}$$

and

$$b_{n,s}(h,r) = E(|h(\underset{\sim}{X}_{n_j,i_{j\ell},j})|)^r \tag{2.8}$$

for any $r>0$ and any Borel function $h : (\mathbb{R}^k)^{m_o} \longrightarrow \mathbb{R}$.

LEMMA 2.2. <u>If there exists a positive number</u> δ' <u>such that</u> (i) <u>for</u> $r = 4+\delta'$, <u>the family</u> $(a_{n,s}(h,r),\ s\in\mathscr{S}_n^*,\ n\in(\mathbb{N}^*)^c)$ <u>and</u> $(b_{n,s}(h,r),\ s\in\mathscr{S}_n^*,\ n\in\mathbb{N}^*)^c)$ <u>are bounded by a finite constant</u> M , (ii) <u>for some</u> δ , $\delta>0$, $\delta' > 12\delta^{-1} - 2$, <u>and</u> (iii) <u>the rate of absolute regularity satisfies</u> $\max_{1\le j\le c} \beta_j(m) = 0(m^{-6-\delta})$,<u>then</u>

$$E(U_n^{(d)})^4 = 0(n^{-6}) \text{ for any } d \text{ such that } \sum_{j=1}^{c} d_j=2 \text{ and } \max_{1\le j\le c} d_j=1 \tag{2.9}$$

$$E(U_n^{(d)})^4 = 0(n^{-4}) \text{ for any } d \text{ such that } \sum_{j=1}^{c} d_j=2 \text{ and } \max_{1\le j\le c} d_j=2 . \tag{2.10}$$

$$E(U_n^{(d)})^2 = 0(n^{-2-c(d)}) \text{ for any } d \text{ such that } \sum_{j=1}^{c} d_j>2 \tag{2.11}$$

<u>where</u> $c(d) = \text{card}\,\{j\in\{1,\dots,c\},d_j\neq 0\}$.

PROOF. It is sufficient to prove (2.9) for $d=(1,1,0,\dots,0)$, the proof for $d\neq(1,1,0,..,0)$ is analogous and is therefore omitted.

Suppose $d=(1,1,0,\dots,0)$. Then, we first note that

$$U_n^{(d)} = \prod_{j=1}^{c} n_j^{-[m_j]} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_1} \int_{\mathbb{R}^{2k}} g_{d,n}^{(i)}(\underset{\sim}{x}_{1,1},\underset{\sim}{x}_{1,2})d(I_{[\underset{\sim}{X}_{n_1,i_1,1}\le \underset{\sim}{x}_{1,1}]} - F_{n_1,i_1,1}(\underset{\sim}{x}_{1,1}))d(I_{[\underset{\sim}{X}_{n_2,i_2,2}\le\underset{\sim}{x}_{1,2}]} - F_{n_2,i_2,2}(\underset{\sim}{x}_{1,2})) \tag{2.12}$$

where $i=(i_1,i_2,0,\dots,0)$, so that

$$E(U_n^{(d)})^4 = (\prod_{j=1}^{c} n_j^{-[m_j]})^4 \sum{}^{*} J(i_{j\ell},j=1,2,\ \ell=1,\dots,4) \tag{2.13}$$

where Σ^* is the sum over all $1\le i_{j\ell}\le n_j$, $j=1,2$, $\ell=1,\dots,4$.

$$J(i_{j\ell},j=1,2,\ \ell=1,\dots,4) = E\{\prod_{\ell=1}^{4} \int_{\mathbb{R}^{2k}} g_{d,n}^{(\ell)}(\underset{\sim}{x}_{\ell,1},\underset{\sim}{x}_{\ell,2})d(I_{[\underset{\sim}{X}_{n_1,i_{1\ell},1}\le\underset{\sim}{x}_{\ell,1}]} - F_{n_1,i_{1\ell},1}(\underset{\sim}{x}_{\ell,1}))\, d(I_{[\underset{\sim}{X}_{n_2,i_{2\ell},2}\le\underset{\sim}{x}_{\ell,2}]} - F_{n_2,i_{2\ell},2}(\underset{\sim}{x}_{\ell,2}))\} , \tag{2.14}$$

$\ell=(\ell_1,\dots,\ell_c)$, $\ell_m=i_{m\ell}$, $m=1,2$, $\ell_m=0$, $m\neq 1,2$, $\ell=1,\dots,4$ and

$$g_{d,n}^{(\ell)} = \sum_{\ell^* \in I_{d,n}^{(\ell)}} \int_{(\mathbb{R}^k)^{(\sum_{j=1}^{c} m_j)-2}} g(\underset{\sim}{x}_{i,j}, i=1,\dots,m_j, j=1,\dots,c) \prod_{j=1}^{c} \prod_{\nu=\epsilon(j)}^{m_j} dF_{n_j,i_{j\nu},j}(\underset{\sim}{x}_{\nu,j}) ,$$

where $\epsilon(j)=1$ if $j\neq 1,2$ and $\epsilon(j)=2$ if $j=1,2$.

Suppose $n_1 \leq n_2$. If $i_{j\ell}$ ($j=1,2$, $\ell=1,\dots,4$) are mutually different, reorder $\{i_{j\ell}\}$ as $1\leq k_1<k_2<\dots<k_8\leq n$ and put

$$J(i_{j\ell}, j=1,2, \ell=1,\dots,4) = H(k_1,\dots,k_8) . \tag{2.15}$$

Let $m^{(p)}$ be the p–th largest difference among $k_{j+1}-k_j$ ($j=1,\dots,7$) if for some j_α ($1\leq j_\alpha\leq 7, 1\leq\alpha\leq 2$) , $k_{j_\alpha+1} - k_\alpha = m^{(\alpha)}$. Then, from Lemma 2.1, (2.14) and proceeding as in the proof of Lemma 2.2, we obtain

$$H(k_1,\dots,k_p) \leq M^{1/(1+\delta')}(\prod_{j=1}^{c} n_j^{[m_j]})^4(n_1n_2)^4 \sum_{\alpha=1}^{2} \max_{1\leq j\leq c} \beta_j^{(2+\delta')/(4+\delta')}(k_{j_\alpha+1}-k_\alpha)$$

and then

$$\sum_{1\leq k_1<\dots<k_8\leq n_2} H(k_1,\dots,k_8) \leq 2M^{1/(1+\delta')}(\prod_{j=1}^{c} n_j^{[m_j]})^4(n_1n_2)^{-4}n_2^2 \sum_{\ell=1}^{n_2}(\ell+1)^5$$

$$\max_{1\leq j\leq c} \beta_j^{(2+\delta')/(4+\delta')}(\ell) = 0(n^{4\sum_{j=1}^{c} m_j-6}) . \tag{2.16}$$

We can use a similar method to estimate the sums in the other cases and so from (2.13) and (2.16), we have (2.9) for $\mathbf{d}=(1,1,0,\dots,0)$.

Now suppose $\mathbf{d}=(2,0,\dots,0)$ and then we have

$$E(U_n^{(d)})^4 = (\prod_{j=1}^{c} n_j^{-[m_j]})^4 \sum{}^{*} J(i_{j\ell}, j=1,\dots 4, \ell=1,2)$$

where Σ^* is the sum over all $1\leq i_{j1}\leq i_{j2}\leq n_1$, $j=1,\dots,4$

$$J(i_{j\ell}, j=1,\dots,r\ \ell=1,2) = E\{\prod_{j=1}^{4} \int_{\mathbb{R}^{2k}} g_{d,n}^{(j)}(\underset{\sim}{x}_{1,j},\underset{\sim}{x}_{2,j}) d(I_{[\underset{\sim}{X}_{n_1,i_{j1},1}\leq\underset{\sim}{x}_{1,j}]} - F_{n_1,i_{j1},1}(\underset{\sim}{x}_{1,j}))$$

$$d(I_{[\underset{\sim}{X}_{n_1,i_{j2},1}\leq\underset{\sim}{x}_{2,j}]} - F_{n_1,i_{j2},1}(\underset{\sim}{x}_{2,j}))\}$$

$j=(j_1,\dots,j_c)$, $j_1=(i_{j1},i_{j2})$, $1\le i_{j1}\le i_{j2}\le n_1$, $j_\ell=0$, $\ell\ne 1$.

If $i_{j\ell}$ ($j=1,\dots,4$, $\ell=1,2$) are mutually different, reorder $\{i_{j\ell}\}$ as $1\le k_1<k_2<\dots<k_8\le n_1$ and put

$$J(i_{j\ell},\ j=1,\dots,4,\ \ell=1,2) = H(k_1,\dots,k_8)\ .$$

Then, proceeding as in the preceding case, we obtain

$$\sum_{1\le k_1<\dots<k_8\le n_1} H(k_1,\dots,k_8) \le 4M^{1/(1+\delta')}\Big(\prod_{j=1}^{c} n_j^{[m_j]}\Big)^4 (n_1(n_1-1)]^{-4}\, n_1^4 \sum_{\ell=1}^{n_1}(\ell+1)^3$$

$$\max_{1\le j\le c} \beta_j^{(2+\delta')/(4+\delta')}(\ell) = 0\big(n^{4\sum_{j=1}^{c} m_j-4}\big) \qquad (2.17)$$

and similar results for the other cases. Thus (2.10) is proved for $d=(2,0,\dots,0)$.

The proof of (2.11) is similar to that of (2.9) and (2.10) and is therefore omitted.

LEMMA 2.3. <u>Let</u> $\{X^*_{ni}\}$ $1\le i\le n$, $n\ge 1$, <u>be</u> <u>r.v.'s</u> (<u>with</u> <u>means</u> 0) <u>which</u> <u>are</u> <u>strong</u> <u>mixing</u> <u>with</u> <u>rate</u> $\alpha(m)$ <u>satisfying</u>

$$\sum_{m=1}^{\infty} (\alpha(m))^{\delta''/(2+\delta'')} < \infty \ \text{for some}\ \delta''\ . \qquad (2.18)$$

<u>Suppose</u> <u>that</u> <u>for</u> <u>any</u> $K>0$, <u>there</u> <u>exists</u> <u>a</u> <u>sequence</u> $\{Y^K_{ni}\}$ <u>of</u> <u>real</u> <u>r.v.'s</u> <u>satisfying</u> (2.18) <u>such</u> <u>that</u>

$$\sup_{n\in\mathbb{N}^*} \max_{1\le i\le n} |Y^K_{ni}| \le B_K < \infty \quad \forall K>0 \qquad (2.19)$$

<u>where</u> B_K <u>is</u> <u>some</u> <u>constant</u> >0 .

$$\sup_{n\in\mathbb{N}^*} \max_{1\le i\le n} E|X^*_{ni}-Y^K_{ni}|^{2+\delta''} \to 0\ ,\quad K\to\infty \qquad (2.20)$$

$$E\Big(\sum_{i=1}^{n} X^*_{ni}\Big)^2/n \to C^2 < \infty\ ,\quad n\to\infty \qquad (2.21)$$

<u>where</u> C <u>is</u> <u>some</u> <u>constant</u> >0 .

$$E\Big(\sum_{i=1}^{n} Y^K_{ni}-E(Y^K_{ni})\Big)^2/n \to C_K^2 < \infty\ ,\quad n\to\infty \quad \forall K>0 \qquad (2.22)$$

<u>where</u> C_K <u>is</u> <u>some</u> <u>constant</u> >0 .

$$C_K \to C\,, \quad K \to \infty \tag{2.23}$$

<u>then, $n^{-\frac{1}{2}} \sum_{i=1}^{n} X^*_{ni}$ converges in law to the normal distribution with mean 0 and variance C^2.</u>

REMARK. Actually we are considering the sequence $\{X^*_{n_N i}\}$, $1 \le i \le n_N$ where $n_N \to \infty$ as $N \to \infty$, but for convenience we are writing $\{X^*_{ni}\}$.

PROOF. From Corollary 1 of Withers [6], since $\{Y^K_{ni}\}$ is uniformly bounded, we deduce that $\forall\, K>0$, $n^{-\frac{1}{2}} \sum_{i=1}^{n} (Y^K_{ni} - E(Y^K_{ni}))$ converges in law to the normal distribution with mean 0 and variance C_K^2.

Now denote $Z^K_{ni} = X^*_{ni} - Y^K_{ni}$ and following Ibragimov [5], we represent the sum $S_n = n^{-\frac{1}{2}} C^{-1} (\sum_{i=1}^{n} X^*_{ni})$ as $S_n = S'_n + S''_n$ where $S'_n = n^{-\frac{1}{2}} C_K^{-1} (\sum_{i=1}^{n} Y^K_{ni} - E(Y^K_{ni})) C_K C^{-1}$ and $S''_n = n^{-\frac{1}{2}} C^{-1} (\sum_{i=1}^{n} Z^K_{ni} - E(Z^K_{ni}))$. Evaluating $E|S''_n|^2$, we have

$$E|S''_n|^2 \le 2n^{-1} C^{-2} \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} |E(Z^K_{nj} - E(Z^K_{nj}))(Z^K_{n,j+i} - E(Z^K_{n,j+i}))|$$

$$\le 2n^{-1} C^{-2} \sum_{i=0}^{n-1} (\alpha(i))^{\delta''/(2+\delta'')} \sum_{j=1}^{n-i} 2^{2+2\delta''} E|Z^K_{nj}|^{2+\delta''} E|Z^K_{n,j+i}|^{2+\delta''})^{1/(2+\delta'')}$$

(from the well known inequality on moments of strong mixing sequences of r.v.'s (Doukhan and Portal, Proposition 2.8, [3]))

$$\le 2n^{-1} C^{-2} \sum_{i=0}^{n-1} (\alpha(i))^{\delta''/(2+\delta'')} (n-i)\, 2^{2+2\delta''} \sup_{n\in\mathbb{N}^*} \max_{1\le i\le n} E|Z^K_{ni}|^{2+\delta''}.$$

Denoting $D_K = \sup_{n\in\mathbb{N}^*} \max_{1\le i\le n} E|Z^K_{ni}|^{2+\delta''}$, we deduce from (2.18) and (2.20) that $E|S''_n|^2 \le 2^{3+\delta''} C^{-2} (\sum_{i=0}^{\infty} (\alpha(i))^{\delta''/(2+\delta'')}) D_K \underset{K\to\infty}{\to} 0$. For any $\epsilon>0$, we can always choose K so large that

$$E|S''_n|^2 \le \epsilon \quad \text{and} \quad |1 - C_K^2/C^2| \le \epsilon\,.$$

If now $f_n(t) = E(e^{itS_n})$ is the characteristic function of the quantity S_n, then

$$|f_n(t)-e^{-t^2/2}| \le$$
$$\le |e^{-(t^2/2)(C_K^2/C^2)}-e^{-t^2/2}| + |e^{-(t^2/2)(C_K^2/C^2)}-Ee^{itS_n'}|+|t|E|S_n''|\le$$
$$\le |t|\sqrt{\epsilon}+\epsilon t^2 + |e^{-(t^2/2)(C_K^2/C^2)} - Ee^{itS_n'}|$$

from the convergence in law of S_n' to the normal law, the last expression converges to zero as $n\to\infty$, and Lemma 2.3 is proved.

Let $G_{n,i,\ell,j}$ be the d.f. of $(\underset{\sim}{X}_{n,i,j},\underset{\sim}{X}_{n,\ell,j})$, $1\le i<\ell\le n_j$, $1\le j\le c$. In what follows $\lim_n$ means $n_j\to\infty\ \forall\ j=1,\dots,c$, and $\frac{n_j}{\sum_{i=1}^{c} n_i} \longrightarrow \lambda_j$ as $n_j\to\infty$, $j=1,\dots,c$.

LEMMA 2.4. <u>Let</u> $h:(\mathbb{R}^k)^{m_o}\to\mathbb{R}$ <u>be a function which belongs to</u> D_{km_o} <u>or</u> $D^*_{km_o}$ <u>and suppose that for any</u> $\ell\in\mathbb{N}^*$ <u>with</u> $1<\ell$ <u>and any</u> $j\in\{1,\dots,c\}$ <u>there exists a d.f.</u> $G_{\ell,j}$ <u>on</u> $\mathbb{R}^{2k}$ <u>with marginals</u> F_j. <u>Furthermore assume that</u>

$$\lim_n \max_{1\le j\le c} \max_{1\le i<\ell\le n} |G_{n_j,i,\ell,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) - G_{\ell-i,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_2)| = 0 \tag{2.24}$$

<u>for all</u> $(\underset{\sim}{x}_1,\underset{\sim}{x}_{2k})\in\mathbb{R}^{2k}$, <u>the family</u> $(a_{n,s}(h,r),\ s\in\mathscr{S}_n^*,\ n\in(\mathbb{N}^*)^c)$ <u>is bounded by a finite constant</u> M_o, <u>and</u>

$$c(h,r) = \int_{(\mathbb{R}^k)^{m_o}} |h(\underset{\sim}{x}_{1,1},\dots,\underset{\sim}{x}_{1,2},\dots,\underset{\sim}{x}_{c,m_c})|^r \prod_{j=1}^{c}\prod_{\ell=1}^{m_j} dF_j(\underset{\sim}{x}_{\ell,j}) < M_o < \infty \text{ for } r>2. \tag{2.25}$$

<u>Then for any</u> $j\in\{1,\dots,c\}$

$$\lim_n \max_{(i,\ell)\in J_n(d^{(j)})} |(n_j\prod_{j=1}^{c} n_j^{-[m_j]})^2 (\sum_{i^*\in I^{(i)}_{d^{(j)},n}} \sum_{\ell^*\in I^{(\ell)}_{d^{(j)},n}} \int_{\mathbb{R}^{2km_o}} (h\otimes h)(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{2m_o}) \tag{2.26}$$
$$dG_{n_j,i,\ell,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_{m_o+1})\, dF^*_{n,i^*}(\underset{\sim}{x}_2,\dots,\underset{\sim}{x}_{m_o})\, dF^*_{n,\ell^*}(\underset{\sim}{x}_{m_o+2},\dots,\underset{\sim}{x}_{2m_o}))$$
$$-\int_{\mathbb{R}^{2km_o}} (h\otimes h)(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_{2m_o}) dG_{\ell_j-i_j,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_{m_o+1}) dF^*(\underset{\sim}{x}_2,\dots,\underset{\sim}{x}_{m_o}) dF^*(\underset{\sim}{x}_{m_o+2},\dots,\underset{\sim}{x}_{2m_o}))| = 0$$

<u>where</u> $F^*_{n,i^*} = (\prod_{r\neq j}\prod_{p=1}^{m_r} F_{n_p,i_{rp},r}) \prod_{p=2}^{m_j} F_{n_j,i_{jp},j}$ <u>if</u> $i^* = (i_1^*,\dots,i_c^*)$, $i_r^* = (i_{r1},\dots,i_{rm_r})$ $r\neq j$, $i_j^* = (i_{j2},\dots,i_{jm_j})$, $F^* = (\prod_{r\neq j} F_r^{m_r})\times F_j^{m_j-1}$.

PROOF. From (2.24) we easily deduce that (2.26) is true if h is a step function.

Without loss of generality, we only prove (2.26) if h is a nondecreasing function. For any $K>0$, we denote by h^K the function defined by

$$h^K = \begin{cases} h \text{ if } |h| \le K \\ 0 \text{ if } |h| > K \end{cases}.$$

As h belongs to D_{km_o} or $D^*_{km_o}$ and is bounded, it is well known that h^K can be uniformly approached by a step function. Let $\epsilon>0$ exist.

Choose K sufficiently large so that

$$\max\left\{\frac{M_o^2}{K^{r-1}}, \frac{M_o}{K^{r-2}}\right\} < \frac{\epsilon}{6}. \tag{2.27}$$

There exists a step function g_ϵ^K such that

$$\sup_{(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{m_o})} |h^K(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{m_o}) - g_\epsilon^K(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{m_o})| < \epsilon/6K$$

and there exists $N_o \in \mathbb{N}$ such that $\forall\, n \ge N_o$,

$$\max_{(i,\ell)\in J_n(d^{(j)})} |(n_j \prod_{j=1}^{c} n_j^{-[m_j]})^2 (\sum_{i^* \in I^{(i)}_{d^{(j)},n}} \sum_{\ell^* \in I^{(\ell)}_{d^{(j)},n}} \int_{\mathbb{R}^{2km_o}} (g_\epsilon^K \otimes g_\epsilon^K)(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{2m_o})$$

$$dG_{n_j,i,\ell,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_{m_o+1})dF^*_{n,i^*}(\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_{m_o})dF^*_{n,\ell^*}(\underset{\sim}{x}_{m_o+2},\ldots,\underset{\sim}{x}_{2m_o}))$$

$$-\int (g_\epsilon^K \otimes g_\epsilon^K)(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{2m_o})dG_{\ell_j-i_j,j}(\underset{\sim}{x}_1,\underset{\sim}{x}_{m_o+1})dF^*(\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_{m_o})dF^*(\underset{\sim}{x}_{m_o+2},\ldots,\underset{\sim}{x}_{2m_o})| < \epsilon/3\,.$$

(2.28)

From (2.27)–(2.28) and the decomposition

$$h\otimes h = h\otimes h - h\otimes h^K + h\otimes h^K - h^K\otimes h^K + h^K\otimes h^K - h^K\otimes g_\epsilon^K + h^K\otimes g_\epsilon^K - g_\epsilon^K\otimes g_\epsilon^K + g_\epsilon^K\otimes g_\epsilon^K,$$

we deduce that $\forall\, n \ge N_o$ that the left side of (2.28) in which we replace $g_\epsilon^K \otimes g_\epsilon^K$ by $h\otimes h$ is less than

$$\frac{M_o^2}{K^{r-1}} + \frac{M_o}{K^{r-2}} + K\frac{\epsilon}{6K} + K\frac{\epsilon}{6K} + \epsilon/3 < \epsilon\,.$$

As ϵ is chosen arbitrarily, (2.26) is proved.

3. Convergence of the U–statistic.

3.1 Definitions. For any double sequence of distribution functions $G_{\ell,j}$ on $\mathbb{R}^{2k}$ with marginals F_j, $1<\ell$, $j\in\{1,\dots,c\}$, let $F=(F_1,\dots,F_c)$ and $G_\ell=(G_{\ell_1,1},\dots,G_{\ell_c,c})$, where $\ell=(\ell_1,\dots,\ell_c)$, we define

$$\theta(F)=\int_{(\mathbb{R}^k)^{m_0}} g(\underset{\sim}{x}_{11},\dots,\underset{\sim}{x}_{m_1 1},\underset{\sim}{x}_{1,2},\dots,\underset{\sim}{x}_{m_c c})\prod_{j=1}^{c}\prod_{i=1}^{m_j} dF_j(\underset{\sim}{x}_{i,j}) \tag{3.1}$$

and

$$\sigma_j^2(\{G_\ell\})=\lim_{n_j\to\infty}\{[\int_{\mathbb{R}^k}(e_j(\underset{\sim}{x}))^2 dF_j(\underset{\sim}{x})-\theta^2(F)]+2\sum_{\ell_j=2}^{n_j}[\int_{\mathbb{R}^{2k}}(e_j(\underset{\sim}{x})e_j(\underset{\sim}{z}))dG_{\ell_j,j}(\underset{\sim}{x},\underset{\sim}{z})-\theta^2(F)]\} \tag{3.2}$$

if it exists, $j=1,\dots,c$ where $e_j=g^F_{\delta_{j1}\dots\delta_{jc}}$, $\delta_{ab}=1$ or 0 according $a=b$ or not and

$$g^F_{d_1,\dots,d_c}(\underset{\sim}{x}_{i,j},i=1,\dots,d_j,j=1,\dots,c)=\int_{\mathbb{R}^*_c} g(\underset{\sim}{x}_{i,j},i=1,\dots,m_j,j=1,\dots,c)\prod_{j=1}^{c}\prod_{i=d_j+1}^{m_j} dF_j(\underset{\sim}{x}_{i,j}).$$

For convenience we shall write σ_j for $\sigma_j(\{G_\ell\})$, if no confusion is possible. For every $t=(t_1,\dots,t_c)\in[0,1]^c$ and $n=(n_1,\dots,n_c)$, denote $[nt]=([n_1t_1],\dots,[n_ct_c])$ where $[s]$ is the largest integer $\le s$.

Let $Z(n)=\{Z(t;n),t\in[0,1]^c\}$ be the process defined by

$$Z(t;n)=\begin{cases}\psi([nt];n)[U([nt])-\theta(F_n)] & \text{for all } [nt]\ge m\\ 0 & \text{otherwise.}\end{cases} \tag{3.3}$$

where for $p=(p_1,\dots,p_c)$ $(p_j>0,j=1,\dots,c)$ and $n=n_1+\dots+n_c$

$$\psi(p;n)=n^{-\frac{1}{2}}\Big(\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}}\Big)\Big(\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}}p_j^{-1}\Big)^{-1}$$

σ_j is the constant defined in (3.2), $\theta(F_n)=\Big(\prod_{j=1}^{c} n_j^{-[m_j]}\Big)g_{0,n}$ and $a\le b$ means $a_j\le b_j$ for all $j\in\{1,\dots,c\}$.

Denote by $\{W_j;j=1,\dots,c\}=\{W_j(t_j);0\le t_j\le 1,j=1,\dots,c\}$ c independent copies of a standard Brownian motion on $[0,1]$.

3.2 Convergence of the U–statistic.

THEOREM 3.1. Assume that there are two positive numbers δ,δ', where $\delta' > \max(12\delta^{-1}-2,\ 2(5+\delta)^{-1})$ such that the families $(a_{n,s}(g,r),\ s\in\mathscr{S}_n^*,\ n\in(\mathbb{N}^*)^c)$ and $(b_{n,s}(g,r),\ s\in\mathscr{S}_n^*,\ n\in(\mathbb{N}^*)^c)$ are bounded, $c(g,r)$ is finite where $r=4+\delta'$ and the rates of absolute regularity satisfy

$$\max_{1\le j\le c} \beta_j(m) = 0(m^{-6-\delta}) . \tag{3.4}$$

Further assume the condition (2.24) is satisfied and g belongs to D_{km_0} or $D^*_{km_0}$.

Then, the limits in (3.2) exist.

Furthermore if $\max_{1\le j\le c} \sigma_j^2(\{G_\ell\}) > 0$, then $Z(n)$ converges in law with respect to the Skorohod topology on D_c to a Gaussian process $W = \{W(t);t\in[0,1]^c\}$ where

$$W(t) = \begin{cases} (\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}})[\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}}W_j(t_j)], & t>0 \\ 0 \text{ with probability one if } t_j=0 \text{ for } j\ (1\le j\le c) . \end{cases} \tag{3.5}$$

PROOF. We rewrite (1.1) as

$$U(n) = \prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \sum_{i^*\in I^{(i)}_{d^{(0)},n}} \int_{\mathbb{R}^{2km_0}} g(x_{i,j},i=1,\dots,m_j,j=1,\dots,c) \prod_{j=1}^{c} \prod_{\nu=1}^{m_j} d(I_{[X_{n_j,i_{j\nu},j}\le x_{\nu,j}]})$$

$$= \prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \sum_{i^*\in I^{(i)}_{d^{(0)},n}} \int_{\mathbb{R}^{2km_0}} g(x_{i,j},i=1,\dots,m_j,j=1,\dots,c) \prod_{j=1}^{c} \prod_{\nu=1}^{m_j} d(I_{[X_{n_j,i_{j\nu},j}\le x_{\nu j}]}$$

$$- F_{n_j,i_{j\nu},j}(x_{\nu,j}) + F_{n_j,i_{j\nu},j}(x_{\nu,j}))$$

$$= \theta(F_n) + \sum_{d\in I_0(m)} \prod_{j=1}^{c} \binom{m_j}{d_j} U_n^{(d)}$$

where $I_0(m) = \{d=(d_1,\dots,d_c);\ 0\le d_j\le m_j,\ j=1,\dots,c,\ d\ne(0,\dots,0)\}$, $d^0=(0,\dots,0)$ and $i\in J_n(d^{(0)})$.

For any $j\in\{1,\dots,c\}$ we define the process $V_{n,j}(t_j)$ by

$$V_{n,j}(t_j) = n_j^{-\frac{1}{2}}(\sum_{i=1}^{[n_jt_j]} A_{n,i,j}) \tag{3.6}$$

where $A_{n,i,j} = n_j \left(\prod_{j=1}^{c} n_j^{-[m_j]} \right) \int_{\mathbb{R}^k} g_{d^{(j)},n}^{(i)}(\underset{\sim}{x}) d(I_{[\underset{\sim}{X}_{n_j,i,j} \leq \underset{\sim}{x}]} - F_{n_j,i,j}(\underset{\sim}{x}))$, $i = (i_1,\dots,i_c)$, $i_1 = i$, $i_j = 0$, $j \neq 1$.

We remark that $V_{n,j}(1) = n_j^{\frac{1}{2}} U_n^{(d^{(j)})}$.

First we prove the following lemma.

LEMMA 3.1. Under the conditions of Theorem 3.1 for any $j \in \{1,\dots,c\}$, $V_{n,j}(t_j)$ converges in law to the Skorohod topology on D_1 to a Gaussian process $V_{0,j}(t_j)$ with trajectories a.s. continuous with mean 0 and variance $t_j \sigma_j^2(\{G_\ell\})$ where $\sigma_j^2(\{G_\ell\})$ is defined in (3.2).

PROOF. It suffices to prove for one value of j and by convenience, we take $j=1$. The process $V_{n,1}$ defines a probability measure P_n on D_1 . By Theorem 15.1 of Billingsley [2], we have to prove that (i) the finite dimensional distributions of P_n converge in law to normal distributions and (ii) P_n is tight.

To prove (i), we have to show that $\sum_{\ell=1}^{p} \lambda_\ell V_{n,1}(t_1^{(\ell)})$ converges in law to a normal distribution for any $p \in \mathbb{N}^*$, any $t_1^{(\ell)} \in [0,1]$ and any $\lambda_\ell \in \mathbb{R}$ $(1 \leq \ell \leq p)$. Without loss of generality we can take $p=2$ and suppose $t_1^{(1)} < t_1^{(2)}$.

Then, we have

$$\sum_{\ell=1}^{2} \lambda_\ell V_{n,1}(t_1^{(\ell)}) = n_1^{-\frac{1}{2}} \left[\sum_{i=1}^{[n_1 t_1^{(1)}]} (\lambda_1+\lambda_2) A_{n,i,1} + \sum_{i=[n_1 t_1^{(1)}]+1}^{[n_1 t_1^{(2)}]} \lambda_2 A_{n,i,1} \right] . \tag{3.7}$$

We define the sequence of r.v.'s $\{B_{n,i,1}\}$ by

$$B_{n,i,1} = \begin{cases} (\lambda_1+\lambda_2) A_{n,i,1} & \text{if } i \leq [n_1 t_1^{(1)}] \\ \lambda_2 A_{n,i,1} & \text{if } [n_1 t_1^{(1)}] < i \leq [n_1 t_1^{(2)}] \\ 0 & \text{if } i > [n_1 t_1^{(2)}] \end{cases} \tag{3.8}$$

and we have

$$\sum_{\ell=1}^{2} \lambda_\ell V_{n,1}(t_1^{(\ell)}) = n_1^{-\frac{1}{2}} \sum_{i=1}^{n_1} B_{n,i,1} \, . \tag{3.9}$$

From Lemma 2.3 we have to verify (2.18)–(2.23). (2.18) is immediate from (3.4) for $\delta'' > 2(5+\delta)^{-1} \vee (12\delta^{-1}-2)$. Now we choose such a δ'' for which $\delta' > \delta''$.

We define $A^K_{n,i,1}$ for any $K>0$ by

$$A^K_{n,i,1} = n_1(\prod_{j=1}^{c} n_j^{-[m_j]}) \int_{\mathbb{R}^k} (g^K)^{(i)}_{d^{(1)},n}(\underset{\sim}{x}) d(I_{[\underset{\sim}{X}_{n_j,i,1} \leq \underset{\sim}{x}]} - F_{n_1,i,1}(\underset{\sim}{x}))$$

where g^K is defined by

$$g^K = \begin{cases} g & \text{if } |g| \leq K \\ 0 & \text{if } |g| > K \end{cases} \tag{3.10}$$

$(g^K)^{(i)}_{d^{(1)},n}$ is defined as in (2.1) in which we replace g by g^K and $B_{n,i,1}$ by

$$B^K_{n,i,1} = \begin{cases} (\lambda_1+\lambda_2)A^K_{n,i,1} & \text{if } i \leq [n_1 t_1^{(1)}] \\ \lambda_2 A^K_{n,i,1} & \text{if } [n_1 t_1^{(1)}] < i \leq [n_1 t_1^{(2)}] \\ 0 & \text{if } i > [n_1 t_1^{(2)}] \quad . \end{cases} \tag{3.11}$$

We have $\sup_n \max_{1 \leq i \leq n_1} |B^K_{n,i,1}| \leq 2(|\lambda_1|+|\lambda_2|)K$ which proves (2.19).

We now prove (2.20). If $\bar{g}^K = g - g^K$, then we have

$$(E|B_{n,i,1} - B^K_{n,i,1}|^{2+\delta''})^{1/(2+\delta'')} \leq$$

$$\leq (|\lambda_1|+|\lambda_2|)n_1(\prod_{j=1}^{c} n_j^{-[m_j]}) \sum_{i^* \in I^{(i)}_{d^{(1)},n}} (\int_{\mathbb{R}^k} | \int_{\mathbb{R}^{km_o-k}} (\bar{g}^K(\underset{\sim}{x}_{\ell,j}, \ell=1,\ldots,m_j, j=1,\ldots,c)$$

$$- \int_{\mathbb{R}^{km_o}} (\bar{g}^K(\underset{\sim}{z}_{\ell,j}, \ell=1,\ldots,m_j, j=1,\ldots,c) \prod_{j=1}^{c} \prod_{\ell=1}^{m_j} dF_{n_j,i_{j\ell},j}(\underset{\sim}{z}_{\ell,j})) \prod_{j=2}^{c} \prod_{\ell=1}^{m_j}$$

$$dF_{n_j,i_{j\ell},j}(\underset{\sim}{x}_{\ell,j}) \prod_{\ell \neq 1} dF_{n_1,i_{1\ell},1}(\underset{\sim}{x}_{\ell,1})|^{2+\delta''} dF_{n_1,i,1}(\underset{\sim}{x}_{i,1}))^{1/(2+\delta'')}$$

$$\leq (|\lambda_1|+|\lambda_2|)n_1(\prod_{j=1}^{c} n_j^{-[m_j]}) \sum_{i^* \in I^{(i)}_{d^{(1)},n}} (\int_{\mathbb{R}^{km_o}} 2^{2+\delta''} |\bar{g}^K(\underset{\sim}{x}_{\ell,j}, \ell=1,\ldots,m_j, j=1,\ldots,c)|^{2+\delta''}$$

$$\prod_{j=1}^{c} \prod_{\ell=1}^{m_j} dF_{n_j,i_{j\ell},j}(\underset{\sim}{x}_{\ell,j}))^{1/(2+\delta'')} \, .$$

If the family defined in (2.7) is bounded by the constant M_0 ($<\infty$), we deduce

$$(E|B_{n,i,1} - B^K_{n,i,1}|^{2+\delta''})^{1/(2+\delta'')} \le 2^{2+\delta''} \frac{M_0}{(K^{2+\delta''})^{\epsilon}} \longrightarrow 0 \text{ as } K\to\infty .$$

where ϵ is given by $(2+\delta'')(1+\epsilon) = 2+\delta'$. (2.20) is proved.

We now prove (2.21). Note that

$$E(\sum_{i=1}^{n_1} B_{n,i,1})^2/n_1 =$$

$$= \frac{1}{n_1}[(\lambda_1+\lambda_2)^2(\sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=1}^{[n_1t_1^{(1)}]} E(A_{n,i,1}A_{n,j,1}) + (\lambda_1+\lambda_2)\lambda_2(\sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} E(A_{n,i,1}A_{n,j,1}) + \lambda_2^2 \sum_{i=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} E(A_{n,i,1}A_{n,j,1}). \quad (3.12)$$

As the family defined in (2.7) is bounded by M_0, we deduce from Proposition 2.8 of Doukhan and Portal [3] that

$$\frac{1}{n_1}|(\lambda_1+\lambda_2)\lambda_2(\sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} E(A_{n,i,1}A_{n,j,1}))|$$

$$\le (|\lambda_1|+|\lambda_2|)|\lambda_2|\, M_0^{2/(2+\delta')}n_1^{-1}(\sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} \alpha_1^{\delta'/(2+\delta')}(j-i)). \quad (3.13)$$

From condition (3.4) the last expression converges to 0 as $n_1\to\infty$.

It remains to prove that $\frac{1}{n_1}\sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=1}^{[n_1t_1^{(1)}]} E(A_{n,i,1}A_{n,j,1})$ and $\frac{1}{n_1}\sum_{i=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} E(A_{n,i,1}A_{n,j,1})$ converge respectively to some finite constants as $n_1\to\infty$. Now denote

$$\rho(1) = \int_{\mathbb{R}^k} (e_1(\underset{\sim}{x}))^2 dF_1(\underset{\sim}{x}) - \theta^2(F) ,$$

and

$$\rho(i) = 2\left[\int_{\mathbb{R}^{2k}} (e_j(\underset{\sim}{x})e_j(\underset{\sim}{z}))dG_{i,1}(\underset{\sim}{x},\underset{\sim}{z}) - \sigma^2(F)\right], \quad i \geq 2 .$$

Then

$$\left|\frac{1}{n_1}\left(\sum_{i=1}^{[n_1 t_1^{(1)}]} \sum_{j=1}^{[n_1 t_1^{(1)}]} E(A_{n,i,1}A_{n,j,1}) - \frac{[n_1 t_1^{(1)}]}{n_1} \sum_{i=1}^{+\infty} \rho(i)\right|$$

$$\leq \left|\frac{[n_1 t_1^{(1)}]}{n_1} \frac{1}{[n_1 t_1^{(1)}]} \sum_{i=0}^{[n_1 t_1^{(1)}]-1} \sum_{j=1}^{[n_1 t_1^{(1)}]-i} \varphi(i)E(A_{n,j,1}A_{n,j+i,1})\right.$$

$$\left. - \frac{[n_1 t_1^{(1)}]}{n_1} \frac{1}{[n_1 t_1^{(1)}]} \sum_{i=1}^{[n_1 t_1^{(1)}]} ([n_1 t_1^{(1)}]-i)\rho(i)\right| + \frac{[n_1 t_1^{(1)}]}{n_1} \sum_{i=[n_1 t_1^{(1)}]+1}^{\infty} |\rho(i)|$$

$$+ \frac{[n_1 t_1^{(1)}]}{n_1} \sum_{i=1}^{[n_1 t_1^{(1)}]} \sum_{k=i}^{\infty} |\rho(i)| = |A_{n,1}| + B_{n_1} + C_{n_1} \tag{3.14}$$

where $\varphi(i) = \begin{cases} 1 & \text{if } i=0 \\ 2 & \text{if } i \neq 0 \end{cases}$.

From Lemma 2.4 we deduce that $|A_{n,1}| \to 0$ as $n_1 \to \infty$ and from the well known inequality on moments for strong mixing sequences of r.v.'s (Doukhan and Portal, Proposition 2.8, [3]) we deduce

$$|\rho(i)| \leq (\alpha_1(i))^{\delta/(2+\delta)} M_0^{2/(2+\delta)} \tag{3.15}$$

which entails $B_{n_1} \to 0$ and $C_{n_1} \to 0$ as $n_1 \to \infty$ by (3.4).

It is also immediate that

$$\left|\frac{[n_1 t_1^{(1)}]}{n_1} \sum_{i=1}^{+\infty} \rho(i) - t_1^{(1)} \sum_{i=1}^{+\infty} \rho(i)\right| \to 0 , \quad \text{as } n_1 \to \infty .$$

We conclude that

$$\frac{1}{n_1} \sum_{i=1}^{[n_1 t_1^{(1)}]} \sum_{j=1}^{[n_1 t_1^{(1)}]} E(A_{n,i,1}A_{n,j,1}) \longrightarrow t_1^{(1)}\left(\sum_{i=1}^{+\infty} \rho(i)\right) \text{ as } n_1 \to \infty \tag{3.16}$$

where $\sum_{i=1}^{+\infty} \rho(i) = \sigma_1^2(\{G_\ell\})$ is defined in (3.2).

Now from the decomposition

$$\sum_{i=1}^{[n_1t_1^{(2)}]}\sum_{j=1}^{[n_1t_1^{(2)}]} = \sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=1}^{[n_1t_1^{(1)}]} + \sum_{i=1}^{[n_1t_1^{(1)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} + \sum_{i=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}\sum_{j=1}^{[n_1t_1^{(1)}]}$$
$$+ \sum_{i=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}$$

and using (3.13) and (3.16), we conclude that

$$\frac{1}{n_1}\sum_{i=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]}\sum_{j=[n_1t_1^{(1)}]+1}^{[n_1t_1^{(2)}]} E(A_{\mathbf{n},i,1}A_{\mathbf{n},j,1}) \longrightarrow (t_1^{(2)}-t_1^{(1)})\sigma_1^2(\{G_\ell\}) \text{ as } n_1\to\infty \quad (3.17)$$

From (3.8) and (3.12)–(3.16) we deduce that

$$E(\sum_{i=1}^{n_1}(B_{\mathbf{n},i,1}-E(B_{\mathbf{n},i,1})))^2/n_1 \to (\lambda_1+\lambda_2)^2t_1^{(1)}\sigma_1^2(\{G_\ell\}) + \lambda_2^2(t_1^{(2)}-t_1^{(1)})\sigma_1^2(\{G_\ell\})$$

where $\sigma_1^2(\{G_\ell\})$ is defined in (3.2), and (2.21) is proved.

To prove (2.22), we replace g by g^K defined in (3.10) and proceeding as above we deduce that

$$E(\sum_{i=1}^{n_1}B^K_{\mathbf{n},i,1}-E(B^K_{\mathbf{n},i,1}))^2/n_1 \to \{(\lambda_1+\lambda_2)^2t_1^{(1)} + \lambda_2^2(t_1^{(2)}-t_1^{(1)})\}\sigma^2_{1,K}(\{\mathbf{G}_\ell\}) \quad \forall\, K>0$$

as $n_1\to\infty$,

where $B^K_{\mathbf{n},i,1}$ is defined in (3.11), and

$$\sigma^2_{1,K}(\{\mathbf{G}_\ell\}) = \lim_{n_1\to\infty}\{[\int_{\mathbb{R}^k}(e_1^K(\underset{\sim}{x}))^2dF_1(\underset{\sim}{x})-\theta_K^2(F)] + 2\sum_{\ell_1=1}^{n_1}[\int(e_1^K(\underset{\sim}{x})e_1^K(\underset{\sim}{z}))dG_{\ell_1,1}(\underset{\sim}{x},\underset{\sim}{z})-\theta_K^2(F)]$$

$$e_1^K(\underset{\sim}{x}_{1,1}) = \int_{\mathbb{R}^{km_o-k}} g^K(\underset{\sim}{x}_{i,j},i=1,\dots,m_j,j=1,\dots,c)\prod_{j=2}^{c}\prod_{i=1}^{m_j}dF_j(\underset{\sim}{x}_{i,j})\prod_{i=2}^{m_1}dF_1(\underset{\sim}{x}_{i,1})$$

$$\theta_K^2(F) = \int_{\mathbb{R}^{km_o}} g^K(\underset{\sim}{x}_{i,j},i=1,\dots,m_j,j=1,\dots,c)\prod_{j=1}^{c}\prod_{i=1}^{m_j}dF_j(\underset{\sim}{x}_{i,j})\,.$$

From the definition of $\sigma^2_{1,K}(\{\mathbf{G}_\ell\})$ and $\sigma_1^2(\{\mathbf{G}_\ell\})$, the properties (3.4) and (2.25), it easily follows that $\sigma^2_{1,K}(\{\mathbf{G}_\ell\})$ converges to $\sigma_1^2(\{\mathbf{G}_\ell\})$ as $K\to\infty$, that is (2.23) and the fact that $\sigma_1^2(\{G_\ell\}) < \infty$ is also an easy consequence of (3.4) and (2.25). Conditions (2.18)–(2.23) of Lemma 2.2 are verified and we conclude that $\sum_{\ell=1}^{2} \lambda_\ell V_{n,1}(t_1^{(\ell)})$ converges in law to the normal distribution with mean 0 and variance $\{(\lambda_1+\lambda_2)^2 t_1^{(1)} + \lambda_2^2(t_1^{(2)} - t_1^{(1)})\}\sigma_1^2(\{G_\ell\})$. (i) is proved.

We now prove (ii).

From Theorem 8.2 of Billingsley [2] we have to verify that $\forall\, \epsilon>0$, $\exists\, \eta>0$ $(0<\eta<1)$ and an integer N_0 such that $\forall\, n_1 \geq N_0$,

$$P[\sup_{|s_1-t_1|<\eta} |V_{n,1}(t_1) - V_{n,1}(s_1)| \geq \epsilon] < \epsilon . \tag{3.18}$$

If n_1t_1 and n_1s_1 are integers and $t_1 \geq s_1$, we obtain the following inequality by using Theorem 10 for q=4 of Doukhan and Portal [3]

$$E(V_{n,1}(t_1) - V_{n,1}(s_1))^4 \leq ((t_1-s_1)^2 + \frac{t_1-s_1}{n_1})M \sum_{m=0}^{+\infty} m^2 \alpha_1^{\delta''/(2+\delta'')}(m)$$

where M is some constant >0 .

From (3.4), we have $\sum_{m=0}^{\infty} m^2\alpha_1^{\delta''/(2+\delta'')}(m) < M_1 < +\infty$ where M_1 is some constant > 0 . If $t_1>s_1$, and n_1t_1 and n_1s_1 are integers, we have $t_1-s_1 \geq 1/n_1$. We deduce

$$E(V_{n,1}(t_1) - V_{n,1}(s_1))^4 \leq 2MM_1(t_1-s_1)^2 . \tag{3.19}$$

From (3.18) and Lemma 2 of Balacheff and Dupont [1], we obtain for n_1 sufficiently large

$$P[\sup_{|s_1-t_1|<\eta} |V_{n,1}(t_1) - V_{n,1}(s_1)| \geq \epsilon]$$

$$\leq P[\sup_{\left|\frac{[n_1s_1]}{n_1} - \frac{[n_1t_1]}{n_1}\right|<2\eta} |V_{n,1}(\frac{[n_1t_1]}{n_1}) - V_{n,1}(\frac{[n_1s_1]}{n_1})| > \frac{\epsilon}{2}] \leq \frac{16}{\epsilon^4} K\eta$$

where K is some constant >0 , which entails (3.18) for η sufficiently small and N_0 sufficiently large. This proves (ii) and also Lemma 3.1.

Now, we prove Theorem 3.1.

For any $j\in\{1,\dots,c\}$, let $W_{n,j}(t_j)$ be defined by

$$W_{n,j}(t_j) = \begin{cases} ([n_jt_j]/m_j\sigma_jn_j^{\frac{1}{2}})m_jU^{d(j)}_{[nt^{(j)}]} & m_j\le[n_jt_j]\le n_j \\ 0 & t_j\le(m_j-1)n_j^{-1} \end{cases}$$

where $t^{(j)} = (1,\dots,1,t_j,1,\dots,1)$ and put

$$w_j(t,n) = \psi([nt];n)\{m_j\sigma_jn_j^{\frac{1}{2}}/[n_jt_j]\} .$$

Then, since

$$\lim_n w_j(t,n) = (\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1})^{-1}(m_j\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1}) ,$$

we deduce (from Lemma 3.1) that $\sum_{j=1}^{c} w_j(t,n)W_{n,j}(t_j)$ converges in law to $W(t)$ defined in (3.5).

As

$$U(n) - \theta(F_n) - \sum_{j=1}^{c} m_jU_n^{d^{(j)}} = \sum_{d\in I_1(m)} \prod_{j=1}^{c} \binom{m_j}{d_j}U_n^{(d)}$$

where $I_1(m) = \{d=(d_1,\dots,d_c);\ \sum_{j=1}^{c} d_j>1\}$, we have only to prove that

$$\lim_n \max_{m\le p\le n} \psi(p;n)|\sum_{d\in I_1(m)} \prod_{j=1}^{c} \binom{m_j}{d_j}U_n^{(d)}| = 0_P . \tag{3.20}$$

Let $C(1)$ and $C(2)$ be defined by

$$C(1) = \{d=(d_1,\dots,d_c);\ \sum_{\ell=1}^{c} d_\ell=2\},\ C(2) = \{d=(d_1,\dots,d_c),\ \sum_{\ell=1}^{c} d_\ell>2\} .$$

Then, we have the following decomposition

$$\sum_{d\in I_1(m)} \prod_{j=1}^{c} \binom{m_j}{d_j}U_n^{(d)} = \sum_{j=1}^{2} \sum_{d\in C(j)} \prod_{\ell=1}^{c} \binom{m_\ell}{d_\ell}U_n^{(d)} . \tag{3.21}$$

We also have

$$\psi(p;n)(\psi(n;n))^{-1} = 0(1) \quad \text{for} \quad p\le n \tag{3.22}$$

and

$$\psi(n;n) = n^{\frac{1}{2}}\Big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{\frac{1}{2}}\Big)\Big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}}\Big)^{-1}(1+0(1))\,. \tag{3.23}$$

Consequently, using (3.22) and (3.23) we deduce that $\forall\ \epsilon>0$ and for n sufficiently large, we can find a constant K such that

$$P\Big[\max_{m\le p\le n} \psi(p;n)\Big|\sum_{d\in I_1(m)} \prod_{j=1}^{c} \binom{m_j}{d_j} U_p^{(d)}\Big| > \epsilon\Big]$$

$$\le P\Big[n^{\frac{1}{2}}\Big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{\frac{1}{2}}\Big)\Big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}}\Big)^{-1} \max_{m\le p\le n}\Big|\sum_{d\in I_1(m)} \prod_{j=1}^{c}\binom{m_j}{d_j} U_p^{(d)}\Big| > K\epsilon\Big]\,. \tag{3.24}$$

For convenience, let us write $K_0 = \big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{\frac{1}{2}}\big)\big(\sum_{j=1}^{c} m_j\sigma_j\lambda_j^{-\frac{1}{2}}\big)^{-1}$. Then, from (3.21), we can write

$$P\Big[n^{\frac{1}{2}}K_0 \max_{m\le p\le n}\Big|\sum_{d\in I_1(m)} \prod_{j=1}^{c}\binom{m_j}{d_j} U_p^{(d)}\Big| > K\epsilon\Big] \le \sum_{j=1}^{2} P\Big[n^{\frac{1}{2}}K_0 \max_{m\le p\le n}\Big|\sum_{d\in C(j)} \prod_{\ell=1}^{c}\binom{m_\ell}{d_\ell} U_p^{(d)}\Big| > K\epsilon 2^{-1}\Big] \tag{3.25}$$

From (2.9) and (2.10) of Lemma 2.2, we deduce that

$$P\Big[n^{\frac{1}{2}} K_0 \max_{m\le p\le n}\Big|\sum_{d\in C(1)} \prod_{\ell=1}^{c}\binom{m_\ell}{d_\ell} U_p^{(d)}\Big| > 2^{-1}K\epsilon\Big]$$

$$\le P\Big[n^{\frac{1}{2}} K_0 \max_{m\le p\le n}\Big|\sum_{d\in D(1)} \prod_{\ell=1}^{c}\binom{m_\ell}{d_\ell} U_p^{(d)}\Big| > 4^{-1}K\epsilon\Big]$$

$$+ P\Big[n^{\frac{1}{2}} K_0 \max_{m\le p\le n}\Big|\sum_{d\in C(1)-D(1)} \prod_{\ell=1}^{c}\binom{m_\ell}{d_\ell} U_p^{(d)}\Big| > 4^{-1}K\epsilon\Big]$$

$$\le \Big(4 \sum_{\substack{i,j=1\\ i\ne j}}^{c} \binom{m_i}{1}\binom{m_j}{1}\Big)^4 (K_0\epsilon)^{-4}\, n^2\times n^2\times 0(n^{-6})$$

$$+ \Big(4\sum_{i=1}^{c}\binom{m_i}{2}\Big)^4 (K_0\epsilon)^{-4}\, n^2 n\times 0(n^{-4}) = 0(n^{-1}) \tag{3.26}$$

where $D(1) = \{d\in C(1);\ \max_{1\le j\le c} d_j = 1\}$.

From (2.11) of Lemma 2.2 we obtain proceeding as above that

$$P[n^{\frac{1}{2}} K_0 \max_{m\le p\le n} | \sum_{d\in C(2)} \binom{m_\ell}{d_\ell} U_p^{(d)} | > 2^{-1} K\epsilon] = 0(n^{-1}) . \qquad (3.27)$$

From (3.24)–(3.27) we deduce (3.20). Theorem 3.1. follows.

Let $Z^*(n) = \{Z(t;n), t\in[0,1]^c\}$ be the process defined by

$$Z^*(t,n) = [\text{Var}\ (U(n))]^{-\frac{1}{2}}[U([n/t]) - \theta(F_n)]$$

where $[n/t] = ([n_1/t_1]),\ldots,[n_c/t_c])$.

Further let

$$W^* = \{W^*(t), t\in[0,1]^c\} \quad (W^*(t)=w'\hat{W}(t), t\in[0,1]^c) \qquad (3.28)$$

be the process defined by $w = (w_1,\ldots,w_c)'$, $w_j = m_j\sigma_j\lambda_j^{-\frac{1}{2}}(\sum_{j=1}^{c} m_j^2\sigma_j^2\lambda_j^{-1})^{-\frac{1}{2}}$ $j\in\{1,\ldots,c\}$ and $\hat{W}(t) = (W_1(t_1),\ldots,W_c(t_c))$ $t\in[0,1]^c$.

COROLLARY 3.1. <u>Under the conditions of Theorem 3.1,</u> $Z^*(n)$ <u>converges in law with respect to the Skorohod topology on</u> D_c <u>to the Gaussian process</u> W^*.

PROOF. From Lemmas 2.2, 3.1 and the decomposition

$$U(n) = \theta(F_n) + \sum_{d\in I_0(m)} \prod_{j=1}^{c} \binom{m_j}{d_j} U_n^{(d)}$$

we easily deduce that

$$\text{Var}\ (U(n)) = \left[\sum_{j=1}^{c} n_j^{-1} m_j^2 \sigma_j^2\right](1 + o_n(1)) + 0(n^{-2}) \qquad (3.29)$$

by showing $E(U_n^{(d)})^2 = 0(n^{-2})$ for any d such that $\sum_{j=1}^{c} d_j = 2$.

Let us now define

$$W_{n,j}^*(t_j) = \begin{cases} (m_j\sigma_j)^{-1} n_j^{\frac{1}{2}} U^{d(j)}_{[n/t^{(j)}]} & 0<t_j\le 1 \\ 0 & t_j=0 \end{cases}$$

and $w_j^*(t,n) = [\text{Var}\ (U(n))]^{-\frac{1}{2}} m_j\sigma_j n_j^{-\frac{1}{2}}$. From Lemma 3.1 and (3.30) we deduce $\sum_{j=1}^{c} w_j^*(t,n) W_{n,j}^*(t_j)$ converges in law to $W^*(t)$ defined in (3.28). We finish the proof of the Corollary by showing that

$$\lim_{n} \max_{m \le p \le n} [\mathrm{Var}\,(U(n))]^{-\frac{1}{2}} \Big| \sum_{d \in I_1(m)} \prod_{j=1}^{c} \binom{m_j}{d_j} U_n^{(d)} \Big| = 0_P,$$

which follows proceeding as in the proof of Theorem 3.1.

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Statistique/*Statistics*

Convergence faible de la U-statistique généralisée pour des processus non stationnaires absolument réguliers

Michel HAREL et Madan PURI

Résumé — Yoshihara [5] a établi la convergence faible des U-statistiques pour des variables aléatoires absolument régulières. Dans cette Note, nous étendons ces résultats du cas stationnaire au cas non stationnaire.

Weak convergence of generalized U-statistics for non stationary absolutely regular processes

Abstract — *Yoshihara* [5] *establihed the weak convergence of the generalized* U-*statistics for absolutely regular random variables. In this Note, we extend these results from the stationary case to the non stationary case.*

1. INTRODUCTION. — Soient un entier $c \geqq 2$ et, pour tout $j\,(1 \leqq j \leqq c)$, un entier $m_j \geqq 1$; soit g une application de $\prod_{j=1}^{c} (\mathbb{R}^k)^{m_j}$ dans $\mathbb{R}$, symétrique pour tout j [c'est-à-dire que $g(x_1, \ldots, x_c)$ est invariant pour toute permutation sur les m_j indices de x_j].

On considère c tableaux triangulaires indépendants de variables aléatoires à valeurs dans $\mathbb{R}^k$, notées $\xi_{j,n,i}\,(1 \leqq j \leqq c,\ 1 \leqq n,\ 1 \leqq i \leqq n)$, à fonctions de répartition continues $F_{j,n,i}$.

Pour tout $\mathbf{n}\ (=(n_1, \ldots, n_c))$, tel que, pour tout j, $n_j \geqq m_j$, soit $\mathscr{S}_{\mathbf{n}} = \prod_{j=1}^{c} \mathscr{S}_{j,n_j}$, où $\mathscr{S}_{j,n_j}$ est l'ensemble de toutes les suites strictement croissantes à m_j éléments dans $\{1, \ldots, n_j\}$ $(1 \leqq i_{j,1} < i_{j,2} < \ldots < i_{j,m_j} \leqq n_j)$. La U-statistique généralisée de degré $(m_1, \ldots, m_j)$ est définie par

$$\text{(1)} \qquad \mathrm{U}(\mathbf{n}) = \left[\prod_{j=1}^{c} \binom{n_j}{m_j}^{-1}\right] \sum_{\mathscr{S}_n} g(\xi_{j,n_j,i_{j,l}},\ 1 \leqq l \leqq m_j,\ 1 \leqq j \leqq c).$$

Généralisant la définition donnée par Yoshihara dans le cas stationnaire [5], nous dirons qu'un tableau triangulaire de variables aléatoires $\eta_{n,i}\,(1 \leqq n,\ 1 \leqq i \leqq n)$ est absolument régulier de taux β si

$$\beta(m) = \sup_{m \leqq n} \ \sup_{1 \leqq l \leqq n-m} \ \mathrm{E}\{ \sup_{\mathrm{A} \in \mathscr{M}_{n,l+m}^{\infty}} |\mathrm{P}(\mathrm{A} \mid \mathscr{M}_{n,1}^{l}) - \mathrm{P}(\mathrm{A})| \} \downarrow 0,$$

où $\mathscr{M}_{n,a}^{b}$ est la σ-algèbre engendrée par $(\eta_{n,a}, \ldots, \eta_{n,b})$. On suppose que, pour tout j, le tableau triangulaire des variables aléatoires $\xi_{j,n,i}\,(1 \leqq n,\ 1 \leqq i \leqq n)$ est absolument régulier de taux noté β_j.

On étudie le comportement asymptotique de $\mathrm{U}(\mathbf{n})$ quand les n_j tendent vers l'infini de telle sorte que, pour tout j, $n_j/n_1 + \ldots + n_c$ tende vers λ_j $(0 < \lambda_j < 1)$; les λ_j seront fixés dans toute la suite, et on note $\lim_{\mathbf{n}}$ ce type de convergence.

Note présentée par Paul-André MEYER.

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Pour tout $p \geqq 1$, pour toute application f de $[0, 1]^p$ dans $\mathbb{R}$, et pour tout $\rho = (\rho(1), \ldots, \rho(p)) \in [0, 1]^p$, on définit

$$f_\rho(t) = \lim_{\substack{s_i \uparrow t_i\ \rho(i)=1 \\ s_i \downarrow t_i\ \rho(i)=0}} f(s), \qquad (s, t) \in [0, 1]^p$$

si cette limite existe avec $s = (s_1, \ldots, s_p)$, $t = (t_1, \ldots, t_p)$.

On note D_p (resp. D_p^*) l'espace de toutes les applications $f : [0, 1]^p \to \mathbb{R}$ telles que, pour tout $\rho \in \{0, 1\}^p$, f_ρ existe et $f_\rho = f$ pour $\rho = (0, \ldots, 0)$ [resp. $\rho = (1, \ldots, 1)$].

On associe à D_p la topologie de Skorokhod multidimensionnelle, voir Neuhaus [4] et Balacheff et Dupont [1].

Soit $\mathscr{S}_j^*$ l'ensemble de toutes les suites croissantes à m_j éléments dans $\{1, \ldots, n_j\}$ $(1 \leqq i_{j,1} \leqq i_{j,2} \leqq \ldots \leqq i_{j,m_j} \leqq n_j)$ et soit $\mathscr{S}_{\mathbf{n}}^* = \prod_{j=1}^{c} \mathscr{S}_j^*$.

Pour tout $\mathbf{n}$ et toute famille $s = (i_{j,l},\ 1 \leqq l \leqq m_j,\ 1 \leqq j \leqq c)$ appartenant à $\mathscr{S}_n^*$, notons

$$a_{\mathbf{n},s} = \int |g(x_{j,l},\ 1 \leqq l \leqq m_j,\ 1 \leqq j \leqq c)|^{4+\delta} \prod_{k=1}^{c} \prod_{l=1}^{m_j} dF_{j,n_j,i_{j,l}}(x_{j,l})$$

et

$$b_{\mathbf{n},s} = E(|g(\xi_{\mathbf{n},j,i_{j,l}})|)^{4+\delta}.$$

Nous établissons la convergence de la U-statistique lorsque les conditions suivantes d'intégrabilité sont vérifiées.

(3) Il existe $\delta(>0)$ tel que les familles $(a_{\mathbf{n},s},\ s \in \mathscr{S}_{\mathbf{n}}^*,\ \mathbf{n} \in (\mathbb{N}^*)^c)$ et $(b_{\mathbf{n},s},\ s \in \mathscr{S}_{\mathbf{n}}^*,\ \mathbf{n} \in (\mathbb{N}^*)^c)$ soient bornées.

La convergence de la U-statistique généralisée fut établie par Yoshihara [5] pour des variables aléatoires absolument régulières mais seulement dans le cas stationnaire. Dans cette Note, nous généralisons les résultats de Yoshihara du cas stationnaire au cas non stationnaire.

2. Convergence faible de la U-statistique généralisée. — Pour toute suite de fonctions de répartition sur $\mathbb{R}^{2k}$, $G_{j,l}$ de marges F_j, $l > 1$, $1 \leqq j \leqq c$, soient $\mathbf{F} = (F_1, \ldots, F_c)$ et $\mathbf{G_l} = (G_{1,l_1}, \ldots, G_{c,l_c})$, $\mathbf{l} = (l_1, \ldots, l_c)$; on définit :

$$\theta(\mathbf{F}) = \int_{(\mathbb{R}^k)^{m_0}} g(x_{j,l},\ 1 \leqq l \leqq m_j,\ 1 \leqq j \leqq c) \prod_{j=1}^{c} \prod_{l=1}^{m_j} dF_j(x_{j,l})$$

où $m_0 = m_1 + \ldots + m_c$ et

$$(4) \quad \sigma_j^2(\{\mathbf{G_l}\}) = \lim_{n_i \to \infty} \left\{ \left[\int_{\mathbb{R}^k} (e_j(x))^2 \, dF_j(x) - \theta^2(\mathbf{F}) \right] + 2 \sum_{l_j=1}^{n_j} \left[\int_{\mathbb{R}^{2k}} (e_j(x)\, e_j(z))\, dG_{j,l_j}(x, z) - \theta^2(\mathbf{F}) \right] \right\}$$

si la limite existe, avec, pour tout j_0 $(1 \leqq j_0 \leqq c)$,

$$e_{j_0}(x_{j_0,1}) = \int_{\mathbb{R}_c^*} g(x_{j,l},\ 1 \leqq j \leqq c,\ 1 \leqq l \leqq m_j) \prod_{j=1}^{c} \prod_{l=\delta_{j,j_0}+1}^{m_j} dF_j(x_{j,l})$$

où $\mathbb{R}_c^* = (\mathbb{R}^k)^{\sum_{j=1}^{c} (m_j - \delta_{j,j_0})}$ et $\delta_{j,j_0} = 1$ ou 0 selon que $j = j_0$ ou $j \neq j_0$.

Comme dans Yoshihara [5], on définit maintenant le processus suivant :

Pour tout $\mathbf{t}=(t_1, \ldots, t_c)\in[0, 1]^c$ et $\mathbf{n}=(n_1, \ldots, n_c)$ on note $[\mathbf{n}\,\mathbf{t}]=([n_1 t_1), \ldots, [n_c, t_c])$ où $[s]$ désigne la partie entière de s.

Soit $X(\mathbf{n})=\{X(\mathbf{t}; \mathbf{n}), \mathbf{t}\in[0, 1]^c\}$ le processus défini par

$$X(\mathbf{t}; \mathbf{n})=\begin{cases}\psi([\mathbf{n}\,\mathbf{t}]; n)(U([\mathbf{n}\,\mathbf{t}])-\theta(\mathbf{F_n})) \text{ pour tout } [\mathbf{n}\,\mathbf{t}]\geqq \mathbf{m}\\ 0 \text{ autrement,}\end{cases}$$

où pour tout $\mathbf{p}=(p_1, \ldots, p_c)$ $(p_j>0,\ 1\leqq j\leqq c)$, $n=n_1+\ldots+n_c$

$$\psi(\mathbf{p}; n)=n^{-1/2}\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{1/2}\right)\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{1/2}p_j^{-1}\right)^{-1},$$

σ_j est la constante définie en (4),

$$\theta(\mathbf{F_n})=\left(\prod_{j=1}^{c}n_j^{-[m_j]}\right)g_{\mathbf{n},0},$$

$$g_{\mathbf{n},0}=\sum_{(n)}^{*}\int_{\mathbb{R}^{km_0}}g(x_{j,l}, 1\leqq l\leqq m_j, 1\leqq j\leqq c)\prod_{j=1}^{c}\prod_{l=1}^{m_j}dF_{n,j,i_{j,l}}(x_{j,l}),$$

où la sommation $\sum_{(n)}^{*}$ recouvre toutes les suites

$$(1\leqq i_{j,1}\neq i_{j,2}\neq\ldots\neq i_{j,m_j}\leqq n_j)\,(1\leqq j\leqq c),$$

$a\leqq b$ signifie $a_j\leqq b_j$ pour tout $j=1, \ldots, c$ et $m^{-[p]}=m(m-1)\ldots(m-p+1)$.

On note également $\{W_j; j=1, \ldots, c\}=\{W_j(t_j); 0\leqq t_j\leqq 1, 1\leqq j\leqq c\}$ c copies indépendantes d'un mouvement brownien sur [0, 1].

Soit $G_{j,n,i,l}$ la fonction de répartition de la variable aléatoire bidimensionnelle $(\xi_{j,n,i}, \xi_{j,n,l})$.

THÉORÈME. — *On suppose qu'il existe une famille de fonctions de répartition sur* $\mathbb{R}^2$, $G_{j,l}$, *de marges* F_j $(1\leqq j\leqq c, l>1)$ *telle que, pour tout* $(x_1, x_2)\in\mathbb{R}^{2k}$,

$$\lim_{\mathbf{n}}\max_{1\leqq j\leqq c}\max_{1\leqq i<l\leqq n_j}|G_{j,n,i,l}(x_1, x_2)-G_{j,l-i}(x_1, x_2)|=0.$$

On suppose que g appartient à D_{k,m_0} ou D^*_{k,m_0}, et qu'il existe deux constantes δ et δ', avec $\delta'>6\delta^{-1}-3$, telles que (3) *soit vérifiée et que les taux d'absolue régularité vérifient*

$$\max_{1\leqq j\leqq c}\beta_j(m)=O(m^{-6-\delta'}). \tag{5}$$

Alors la limite en (4) *existe.*

Si de plus $\max_{1\leqq j\leqq c}\sigma_j^2(\{G_l\})>0$, *alors* $X(\mathbf{n})$ *converge en loi pour la topologie de Skorokhod sur* D_c *vers le processus gaussien* $W=\{W(\mathbf{t}), \mathbf{t}\in[0, 1]^c\}$ *où*

$$W(\mathbf{t})=\begin{cases}\left(\sum_{j=1}^{c}m_j\sigma_j\lambda_j^{1/2}\right)\left(\sum_{j=1}^{c}m_j\sigma_j\lambda_j^{-1/2}\right)\left[\sum_{j=1}^{c}m_j\sigma_j\lambda^{-1/2}{}_j W_j(t_j)\right] & \mathbf{t}\geqq 0\\ 0 \text{ avec la probabilité 1 si } t_j=0 \text{ pour au moins un } j,\ 1\leqq j\leqq c\end{cases}$$

et σ_j défini en (4).

Preuve. — On généralise le théorème 3.1 dans Harel et Puri [3] ou le théorème 1 dans Harel et Puri [2].

Remarque. — La condition (5) est plus faible que la condition (6.11) de Yoshihara [5] dans le cas stationnaire.

Soit maintenant :

$X^*(\mathbf{n}) = \{X^*(\mathbf{t}; \mathbf{n}), \mathbf{t} \in [0, 1]^c\}$ le processus défini par

$$X^*(\mathbf{t}; \mathbf{n}) = r^{-1}(\mathbf{n}) [U(\mathbf{n}/\mathbf{t}] - \theta(F_{\mathbf{n}})],$$

où $r^{-1}(\mathbf{n}) = \mathrm{Var}(U(\mathbf{n}))$ et $[\mathbf{n}/\mathbf{t}] = ([n_1/t_1], \ldots, [n_c/t_c])$.

De plus soit

$$W^* = \{W^*(\mathbf{t}), \mathbf{t} \in [0, 1]^c\} \quad (W^*(\mathbf{t}) = \mathbf{w}' \hat{W}(\mathbf{t}), \mathbf{t} \in [0, 1]^c),$$

le processus défini par

$$\mathbf{w} = (w_1, \ldots, w_c)',$$

$$w_j = m_j \sigma_j \lambda_j^{-1/2} \left(\sum_{j=1}^{c} \sigma_i^2 \lambda_j^{-1} \right)^{-1/2} \quad 1 \leqq j \leqq c,$$

$$\hat{W}(\mathbf{t}) = (W_1(\mathbf{t}_1), \ldots, W_c(t_c)) \, \mathbf{t} \in [0, 1]^c.$$

Corollaire. — *Sous les conditions du théorème,* $X^*(\mathbf{n})$ *converge en loi pour la topologie de Skorokhod sur* D_c *vers le processus gaussien* W^*.

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