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Réarrangement relatif : propriétés et applications aux équations aux dérivées partielles

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Jean-Michel RAKOTOSON

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ABSTRACT.

The purpose of this thesis is to introduce some properties of the relative rearrangement and their applications to variational inequalities and to partial differential equations of elliptic and parabolic type. The relative rearrangement comes from the computation of the directional derivative of the monotone rearrangement. This rearrangement is a generalization of the classical rearrangement (see Chap. I and II).

Thanks to an integral formula of Federer type (see Chap.IV to VI) and the properties of this rearrangement, we develop a new method to get a priori estimates for elliptic and parabolic problems. These estimates lead to additional regularity for the solutions of P.D.E. (see Chap. IV and VI). We prove also a regularity theorem for a family of symmetrized functions.

PLAN

1ère PARTIE : *Réarrangement relatif des fonctions. Définition. Propriétés.*

Chapitre I. Isoperimetric Inequalities in Parabolic Equations (*Ann. della Scuola Norm. Sup. di Pisa, class. di Sc.,* série IV, Vol. XIII, n° 1 (1986), p. 51-73 (Avec J. Mossino).

Chapitre II. Some properties of the relative rearrangement. A paraître dans *Journal of Mathematical Analysis and its Applications.* (21 pages). Note aux C.R.A.S., Paris, t. 302, série I, n°15, (1986).

Chapitre III. Un modèle non local en physique des plasmas : résulution par une méthode de degré topologique. *Acta Applicandae Mathematicae,* 4 (1985), p. 1-14.

2e PARTIE : *Applications dans les équations aux dérivées partielles.*

Equations quasilinéaires elliptiques.

Chapitre IV. Réarrangement relatif dans les équations elliptiques quasilinéaires avec un second membre distribution : application à un théorème d'existence et de régularité. A paraître dans *Journal of Differential Equations,* (39 pages), Note aux C.R.A.S., Paris, t. 302, série I, n° 16 (1986).

Chapitre V. Existence of bounded solutions of some degenerate quasilinear elliptic equations. A paraître dans *Communications in Partial Differential Equations* (53 pages).

Inéquations quasilinéaires.

Chapitre VI. Relative rearrangement in quasilinear variational inequalities. (avec R. Temam). A paraître dans *Indiana Mathematical Journal* (98 pages). Note aux C.R.A.S. à paraître.

Problèmes quasilinéaires type parabolique.

Chapitre VII. Time behaviour of solution of parabolic problems. ***Applicable Analysis***, vol. 21 (1986), p. 13-29.

Remarques diverses (voir Remarque 1).

INTRODUCTION

L'objet de cette thèse est de présenter quelques propriétés du réarrangement relatif et leurs applications dans les inéquations variationnelles et les équations aux dérivées partielles de types elliptique et parabolique.

Tout d'abord rappelons que le réarrangement relatif est né à partir du calcul de la dérivée directionnelle de l'application $u \longrightarrow u_*$ (u_* est le réarrangement décroissante de u). A son origine, il a été introduit de façon formelle par R.Temam ([28, Chap. VI] pour résoudre un problème de type nouveau apparaissant en physique des plasmas, à savoir les équations de Grad-Mercier appelées Queer Differential Equations par H.Grad. Le calcul rigoureux a été donné pour la première fois par J.Mossino et R.Temam ([6, Chap. I] dans la direction des fonctions bornées. Ils ont appelé réarrangement relatif la quantité définie pour $v \in L^\infty(\Omega)$: $v_{*u} = \lim\limits_{\lambda \searrow 0} \dfrac{(u + \lambda v)_* - u_*}{\lambda}$, Ω étant un borné mesurable de $\mathbb{R}^N$.

Dans l'article (Chap. I) en collaboration avec J. Mossino, nous avons étendu ce calcul dans la direction des fonctions v de $L^p(\Omega)$ ($1 \leq p \leq + \infty$) et nous avons établi des propriétés supplémentaires de cette dérivée. Entre autres, nous avons montré que le réarrangement monotone est aussi un réarrangement relatif. Toutefois, il ne présente pas exactement les mêmes propriétés. Par exemple, ce réarrangement n'est pas monotone (on montre que si F est borélienne alors $F(u)_{*u} = F(u_*)$ dès que $F(u) \in L^1(\Omega)$). Il ne conserve pas les normes mais on a :

$$\|v_{*u}\|_{L^p(\Omega^*)} \leq \|v\|_{L^p(\Omega)} \qquad (0.1)$$

($\Omega^* =]0$, mesure de $\Omega[$ et u est une fonction mesurable sur Ω).

L'un des avantages réside dans le fait que la fonction u joue un rôle de paramètre qui n'intervient pas dans le second membre de l'inégalité (0.1).

Nous avons considéré des fonctions u(t,x) dépendant d'un paramètre t . Nous avons montré que si $u_*(t,.)$ est le réarrangement décroissante de

$u(t,.)$ alors $\frac{\partial u_*}{\partial t}$ a un sens dès que $\frac{\partial u}{\partial t}$ en a un. Plus précisément, on a

$$\frac{\partial u_*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_{*u} \tag{0.2}$$

Il s'ensuit immédiatement le théorème de régularité suivant :

Si $u \in W^{1,r}(L^p(\Omega)) = X \quad (1 \leqslant r \leqslant +\infty,\ 1 \leqslant p \leqslant +\infty)$ alors

$u_* \in W^{1,r}(L^p(\Omega^*)) = \tilde{X}$

et

$$\|u_*\|_{\tilde{X}} \leqslant \|u\|_X \tag{0.3}$$

Une des conséquences de ce théorème est : si $u \in X$ alors pour presque tout t, $\frac{\partial u}{\partial t}$ $(t,.)$ est constant (presque partout) où $u(t,.) = c^{te}$. Une telle proposition est à rapprocher de celle donnée par Stampacchia [19, Chap. IV] pour des fonctions ne dépendant que des variables d'espace.

Dans le Chapitre II, nous avons développé des propriétés supplémentaires de ce réarrangement. Le but est de montrer que ce réarrangement généralise effectivement le réarrangement monotone. Entre autres, nous avons montré que l'on obtient une généralisation de l'inégalité de Hardy--Littlewood.

Dans le cadre des applications des réarrangements monotones, G. Talenti [8, Chap. I] a montré que l'on pouvait utiliser le réarrangement monotone dans les équations elliptiques linéaires moyennant un choix de fonction test. L'idée d'utiliser le réarrangement monotone est due semble-t-il à H. Weinberger. G. Talenti a comparé la solution de l'équation elliptique (coercive)

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(a_{ij}(x) \cdot \dfrac{\partial u}{\partial x_i}\right) = f \text{ dans } \Omega \quad (\text{ouvert borné de } \mathbb{R}^N) \\ \qquad u \in H^1_0(\Omega) \end{cases} \tag{0.4}$$

à celle de l'équation dite symétrisée

$$- \Delta u = \underset{\sim}{f} \ , \ u \in H^1_0(\tilde{\Omega})$$

où $\tilde{\Omega}$ est la boule de même mesure que Ω, $\underset{\sim}{f}$ le réarrangement sphérique de f.

C. Bandle ([2], Chap. I) a obtenu le même type de résultat dans le cadre des équations paraboliques linéaires mais pour des solutions fortes. Dans le Chapitre I, nous avons étendu ce dernier résultats pour des solutions faibles.

Au Chapitre IV, nous avons montré que pour une classe de problèmes quasilinéaires dégénérés (ou non) de type parabolique, régis par un opérateur A du type Leray-Lions, $Au = - \partial/\partial x_i \; a_i(t,x,u,\frac{\partial u}{\partial t}, \nabla u)$ satisfaisant à : $\forall z = (z_i) \in \mathbb{R}^N$, $a_i(t,x,u,u',z)z_i \geqslant 0$, on peut donner une estimation de la norme de la solution dans les espaces L(p,q). En particulier, nous avons montré que la norme de la solution décroît en temps.

Plus précisément, l'application $t \longrightarrow \int_0^s u_*(t,\sigma)d\sigma$ est décroissante $\forall s \in \bar{\Omega}^*$. Ceci implique entre autres le comportement à $t = 0$ dans les espaces L(p,q).

L'article de L.Boccardo, F.Murat et J.P.Puel ([3], chap. IV) a montré que l'obtention (éventuelle) d'une estimation L^∞ pouvait conduire à des théorèmes d'existence dans les problèmes quasilinéaires.

Ladyzenskaja et al ([10], Chap. IV) ayant étendu une technique de De Giorgi ont montré que l'obtention d'une estimation L^∞ pouvait conduire à montrer des propriétés de régularité supplémentaires dans les E.D.P.

Au Chapitre IV, nous avons considéré une équation quasilinéaire pouvant dégénérer à l'infini avec un second membre peu régulier :

$$\begin{cases} - \partial/\partial x_i (a_i(x,u,\nabla u)) + F(x,u,\nabla u) = T \text{ dans } \Omega \text{ (ouvert borné de } \mathbb{R}^N) \\ \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \qquad 1 < p < +\infty \end{cases} \tag{0.5}$$

avec $a_i(x,u,\xi)\xi_i \geqslant \nu(|u|)|\xi|^p$ + (autre terme) ici, $\nu > 0$. F a une croissance au plus d'ordre p par rapport au gradient et illimité par rapport u,

$T \in W^{-1,r}(\Omega)$, $r > N/p-1$. En considérant le réarrangement relatif par rapport à $|u|$ de certaines quantités liées à a_i et à T, nous avons obtenu des estimations a priori dans $L^{\infty}(\Omega)$ (estimation optimale). Ce qui nous a donné une troncature convenable du problème (0.5) et permis de montrer que : il existe une solution de (0.5) et que de plus, cette solution est α-höldérienne à l'intérieur de Ω.

Au Chapitre V, nous avons étendu cette technique à des équations quasilinéaires dégénérées du même type que (0.5) (où $\nu(0) = 0$) lorsque $T \in W^{-1,r}(\Omega) \cap L^1(\Omega)$ et F a une croissance d'ordre $< p$. Nous y avons montré l'existence d'une solution et de plus, grâce au réarrangement, nous avons donné une expression explicite de la norme de la solution aussi bien dans $L^{\infty}(\Omega)$ que dans $W_0^{1,p}(\Omega)$.

Au Chapitre VI, en collaboration avec R.Temam, en raffinant cette technique du réarrangement relatif, nous avons montré un théorème général d'estimation L^{∞}. Nous l'avons appliqué à des inéquations variationnelles et quasivariationnelles avec divers types de convexes. Nous avons prouvé dans le cas particulier d'une contrainte unilatérale, que l'on peut s'en servir pour des théorèmes d'existence et de régularité.

CHAPITRE I

Isoperimetric Inequalities in Parabolic Equations

(*Ann. della Scuola Norm. Sup. di Pisa, class. di Sc.,* série IV,

Vol. XIII, n° 1 (1986), p. 51-73 (Avec J. Mossino).

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Isoperimetric Inequalities in Parabolic Equations.

J. MOSSINO - J. M. RAKOTOSON

0. – Introduction.

Consider the parabolic equation

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathfrak{A}_t(u) + cu = f & \text{in } Q = (0, T) \times \Omega \,, \\ u = 0 & \text{on } \Sigma = (0, T) \times \partial\Omega \,, \\ u(0, \cdot) = u_0 & \text{for } \ t = 0 \,, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ $(N \geqq 1)$,

$$\mathfrak{A}_t(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \,,$$

a_{ij} satisfy the uniform ellipticity condition (with constant one)

$$\sum_{i,j=1}^{N} a_{ij}(t, x) \xi_i \xi_j \geqq |\xi|^2 \,, \qquad \forall \xi \in \mathbb{R}^N;$$

c, u_0 and f are non-negative functions; their regularity will be precised later on.

Consider also the equation

$$(\tilde{1}) \qquad \begin{cases} \dfrac{\partial U}{\partial t} - \Delta U = \underset{\sim}{f} & \text{in } \tilde{Q} = (0, T) \times \tilde{\Omega} \,, \\ U = 0 & \text{on } \tilde{\Sigma} = (0, T) \times \partial\tilde{\Omega} \,, \\ U(0, \cdot) = \underset{\sim}{u}_0 \,, \end{cases}$$

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where $\tilde{\Omega}$ is the ball of $\mathbf{R}^N$, centered at the origin, which has the same measure as Ω, and $\underset{\sim}{u}_0$ (resp. $\underset{\sim}{f}(t,\cdot)$) is the rearrangement of u_0 (resp. $f(t,\cdot)$) in $\tilde{\Omega}$, which decreases along the radii. This rearrangement is defined as follows.

If v is a real measurable (1) function defined in Ω, the decreasing rearrangement of v is defined in $\bar{\Omega}^* = [0, |\Omega|]$, by

$$v_*(s) = \operatorname{Inf}\{\theta \in \mathbf{R},\ |v > \theta| \leqq s\} \tag{2}$$

where $|v > \theta| = \operatorname{meas}\{x \in \Omega,\ v(x) > \theta\}$ (for any measurable set E, we denote $|E|$ its measure). The spherical rearrangement of v in $\bar{\tilde{\Omega}}$, which decreases along the radii is

$$\underset{\sim}{v}(x) = v_*(\alpha_N |x|^N)\ ,\quad \text{for } x \in \bar{\tilde{\Omega}}, \tag{3}$$

where α_N is the measure of the unit ball of $\mathbf{R}^N$. If v is defined in $(0, T)\times\Omega$, and is measurable with respect to the space variable x of Ω, we consider its rearrangement with respect to x:

$$v_*(t, s) = (v(t,\cdot))_*(s) = \operatorname{Inf}\{\theta \in \mathbf{R},\ |v(t,\cdot) > \theta| \leqq s\}\ , \tag{4}$$

$$\underset{\sim}{v}(t, x) = v_*(t, \alpha_N |x|^N)\ . \tag{5}$$

C. Bandle [2] proved that every *strong* solution u of problems (1) satisfies

$$\forall t \in [0, T],\ \forall s \in \bar{\Omega}^*\ ,\qquad \int_0^s u_*(t, \sigma)\, d\sigma \leqq \int_0^s U_*(t, \sigma)\, d\sigma\ , \tag{6}$$

which leads to

$$\forall t \in [0, T],\ \forall r \in [1, \infty]\ ,\qquad \|u(t,\cdot)\|_{L^r(\Omega)} \leqq \|U(t,\cdot)\|_{L^r(\tilde{\Omega})}\ . \tag{7}$$

J. L. Vasquez [9] obtained the same result, if u is a weak solution of a degenerate parabolic equation, the equation of porous media:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\varphi(u) & \text{in } Q = (0, \infty)\times\mathbf{R}^N\ , \\ u(0,\cdot) = u_0 & \text{for } t = 0\ , \end{cases} \tag{8}$$

where $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ is increasing and continuous, $\varphi(0) = 0$. He used the

(1) In the whole paper, we consider the Lebesgue measure.

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semigroups theory, and the isoperimetric inequalities for *elliptic* equations (see [7] for example).

In this paper, we give a direct proof of (6), (7), valid for every *weak* solution of problems (1) (see Section 2).

Our method relies on the calculation of the directional derivative of the mapping $u \to u_*$, that is $v_{*u} = \lim_{\lambda \downarrow 0} ((u + \lambda v)_* - u_*)/\lambda$. This calculation was made first by J. Mossino and R. Temam [6], with a direction v in $L^\infty(\Omega)$. In the first section, we extend their result to functions v in $L^p(\Omega)$ $(1 \leq p \leq +\infty)$. Moreover we prove that, if u belongs to $H^1(0, T; L^p(\Omega))$, then u_* belongs to $H^1(0, T; L^p(\Omega^*))$ and

$$\frac{\partial u_*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_{*u}.$$

Besides, $(\partial u/\partial t)(t, \cdot)$ is shown to be constant on every set where $u(t, \cdot)$ is constant. The last formula is a crucial point in Section 2.

1. – Directional derivative of the rearrangement mapping.

In this Section 1, we assume that Ω is a measurable subset of $\mathbf{R}^N$ ($|\Omega| < \infty$, $N \geq 1$). For the sake of completeness, we first recall some properties of rearrangements (see the proofs in [7] for example), and a result of [6].

1.1. *Properties of rearrangements.*

Let u be a measurable function: $\Omega \to \mathbf{R}$ and u^* be its increasing rearrangement, defined by (2) and

$$u^* = -(-u)_* . \tag{1.1}$$

An essential property of rearrangement is that u and u^* are equi-measurable:

$$\forall \theta \in \mathbf{R}, \quad |u < \theta| (= \text{meas}\,\{x \in \Omega, u(x) < \theta\}) = |u^* < \theta| ,$$

which implies

$$\int_\Omega F(u)\, dx = \int_{\Omega^*} F(u^*)\, ds , \tag{1.2}$$

for every Borel measurable $F: \mathbf{R} \to \mathbf{R}^+$. Here are some other properties of the increasing rearrangement mapping.

(*a*) If u_1, u_2 are two measurable functions such that $u_1 \leqq u_2$ almost everywhere, then $u_1^* \leqq u_2^*$ everywhere.

(*b*) For all constants C, $(u + C)^* = u^* + C$.

(*c*) More generally, if φ is an increasing function from **R** into **R**, then $(\varphi(u))^* = \varphi(u^*)$ almost everywhere.

(*d*) The mapping $u \to u^*$ applies $L^p(\Omega)$ into $L^p(\Omega^*)$ $(1 \leqq p \leqq \infty)$. It is contracting and norm-preserving.

(*e*) If u is in $L^p(\Omega)$, v in $L^{p'}(\Omega)(1/p + 1/p' = 1)$, then

$$\int_\Omega uv\, dx \leqq \int_{\Omega^*} u^* v^*\, ds = \left(\int_{\tilde{\Omega}} \tilde{u}\tilde{v}\, dx \right). \tag{1.3}$$

This inequality is due to Hardy and Littlewood.
We shall use a slight extention of (*d*):

LEMMA 1.1. *Let* $u\colon \Omega \to \mathbf{R}$ *be measurable,* v *in* $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$. *Then* $(u + v)^* - u^*$ *belongs to* $L^p(\Omega^*)$ *and*

$$\|(u + v)^* - u^*\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)} .$$

This lemma was proved in [7]. For convenience, we reproduce the proof here.

(i) If $p = \infty$, we have

$$u - \|v\|_{L^\infty(\Omega)} \leqq u + v \leqq u + \|v\|_{L^\infty(\Omega)} , \qquad \text{a.e.} .$$

By properties (*a*) and (*b*) above,

$$u^* - \|v\|_{L^\infty(\Omega)} \leqq (u + v)^* \leqq u^* + \|v\|_{L^\infty(\Omega)},$$

that is

$$\|(u + v)^* - u^*\|_{L^\infty(\Omega^*)} \leqq \|v\|_{L^\infty(\Omega)} .$$

(ii) If $p < \infty$, we use the truncation

$$f_n(t) = \begin{cases} -n & \text{if } t \leqq -n , \\ t & \text{if } -n \leqq t \leqq n , \\ n & \text{if } t \geqq n . \end{cases}$$

Then $f_n(u)$ and $f_n(u+v)$ are in $L^\infty(\Omega)$. By (*c*) and (*d*), $(f_n(u))^* = f_n(u^*)$, $(f_n(u+v))^* = f_n((u+v)^*)$, these functions are in $L^\infty(\Omega^*)$, and

$$\|f_n((u+v)^*) - f_n(u^*)\|_{L^p(\Omega^*)} = \|(f_n(u+v))^* - (f_n(u))^*\|_{L^p(\Omega^*)} \leqq \|f_n(u+v) - f_n(u)\|_{L^p(\Omega)} \leqq \|v\|_{L^p(\Omega)}$$

(as f_n is contracting). Then, using Fatou lemma,

$$\begin{aligned}\|v\|^p_{L^p(\Omega)} &\geqq \varliminf_{n\to\infty} \int_{\Omega^*} |f_n((u+v)^*) - f_n(u^*)|^p\, ds \\ &\geqq \int_{\Omega^*} \lim |f_n((u+v)^*) - f_n(u^*)|^p\, ds \\ &= \int_{\Omega^*} |(u+v)^* - u^*|^p\, ds\,.\end{aligned}$$

1.2. *Directional derivative of the rearrangement mapping. Relative rearrangement.*

First, we shall recall a result due to J. Mossino and R. Temam [6].

Consider a couple of functions (u, v), $u\colon \Omega \to \mathbf{R}$ is measurable, v is in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$, and a parameter $\lambda > 0$. By Lemma 1.1, $(u + \lambda v)^* - u^*$ belongs to $L^p(\Omega^*)$, and we can define

$$w_\lambda(s) = \int_0^s \frac{(u+\lambda v)^* - u^*}{\lambda}\, d\sigma\,. \tag{1.4}$$

Thus, $dw_\lambda/ds = ((u+\lambda v)^* - u^*)/\lambda$. By Lemma 1.1.,

$$\left\|\frac{dw_\lambda}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)}\,. \tag{1.5}$$

We are going to show that dw_λ/ds tends (in the sense of distributions) to dw/ds, where

$$w(s) = \begin{cases} \displaystyle\int_{u<u^*(s)} v\, dx & \text{if } |u = u^*(s)| = 0\,, \\ \displaystyle\int_{u<u^*(s)} v\, dx + \int_0^{s-|u<u^*(s)|} (v|_{P(s)})^*\, d\sigma\,, & \text{otherwise}\,, \end{cases} \tag{1.6}$$

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$v|_{P(s)}$ is the restriction of v to $P(s) = \{u = u^*(s)\}$. The following was proved in [6].

THEOREM 1.1. *If u is a measurable function from Ω into* **R**, *v is in $L^\infty(\Omega)$, then w is lipschitz,*

$$\left\|\frac{dw}{ds}\right\|_{L^\infty(\Omega^*)} \leqq \|v\|_{L^\infty(\Omega)},$$

and, when λ decreases to zero,

(i) $w_\lambda \to w$ in $\mathcal{C}^0([0, |\Omega|])$ that is uniformly;

(ii) $\dfrac{dw_\lambda}{ds} = \dfrac{(u+\lambda v)^* - u^*}{\lambda} \to \dfrac{dw}{ds}$ in $L^\infty(\Omega^*)$ weak $*$. □

We shall extend Theorem 1.1 to functions v in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$.

THEOREM 1.1 bis. *Let u, v be two measurable functions from Ω into* **R**, *v in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$. Then w belongs to $W^{1,p}(\Omega^*)$,*

$$\left\|\frac{dw}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)}, \tag{1.7}$$

and, when λ decreases to zero

(i) $w_\lambda \to w$ *in* $\mathcal{C}^0([0, |\Omega|])$;

(ii) $\dfrac{dw_\lambda}{ds} = \dfrac{(u+\lambda v)^* - u^*}{\lambda} \to \dfrac{dw}{ds}$ *in the sense of distributions:*

$$\forall \varphi \in \mathcal{D}(\Omega^*)\,, \quad \int_{\Omega^*} \frac{dw_\lambda}{ds}\varphi\, ds \to \int_{\Omega^*} \frac{dw}{ds}\varphi\, ds\,.$$

(In particular, $dw_\lambda/ds \to dw/ds$ in $L^p(\Omega^*)$-weak if $1 < p < \infty$, in $L^\infty(\Omega^*)$-weak $*$ if $p = \infty$). □

PROOF. Consider v_n in $L^\infty(\Omega)$; $w_{\lambda,n}$, w_n are associated to (u, v_n) as in (1.4), (1.6). We have

$$|w_\lambda(s) - w(s)| \leqq |w_\lambda(s) - w_{\lambda,n}(s)| + |w_{\lambda,n}(s) - w_n(s)| + |w_n(s) - w(s)|\,.$$

By Lemma 1.1,

$$|w_{\lambda,n}(s) - w_\lambda(s)| = \left|\int_0^s \frac{(u+\lambda v_n)^* - (u+\lambda v)^*}{\lambda}\, d\sigma\right| \leqq \|v_n - v\|_{L^1(\Omega)},$$

and, clearly,

$$|w_n(s) - w(s)| \leqq \|v_n - v\|_{L^1(\Omega)} .$$

Then

$$\operatorname*{Sup}_s |w_\lambda(s) - w(s)| \leqq \operatorname*{Sup}_s |w_{\lambda,n}(s) - w_n(s)| + 2\|v_n - v\|_{L^1(\Omega)} .$$

By Theorem 1.1. (i), $w_{\lambda,n}$ tends to w_n in $\mathcal{C}^0([0, |\Omega|])$. When λ decreases to zero,

$$\overline{\lim_{\lambda \downarrow 0}} \operatorname*{Sup}_s |w_\lambda(s) - w(s)| \leqq 2\|v_n - v\|_{L^1(\Omega)} .$$

We deduce (i). Evidently (ii) follows, as, with φ in $\mathfrak{D}(\Omega^*)$,

$$\int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi\, ds = -\int_{\Omega^*} w_\lambda \frac{d\varphi}{ds} ds \to -\int_{\Omega^*} w \frac{d\varphi}{ds} ds \qquad \text{(by (i))}$$

$$= \int_{\Omega^*} \frac{dw}{ds} \varphi\, ds .$$

Now, we shall prove that dw/ds is in $L^p(\Omega^*)$, and satisfies (1.7). Taking again φ in $\mathfrak{D}(\Omega^*)$, we have by (1.5)

$$\left| \int_{\Omega^*} w_\lambda \frac{d\varphi}{ds} ds \right| = \left| \int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi\, ds \right| \leqq \|v\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega^*)}$$

$(1/p + 1/p' = 1)$. From (i), it follows

$$\left| \int_{\Omega^*} w \frac{d\varphi}{ds} ds \right| \leqq \|v\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega^*)} .$$

If $p > 1$, $L^p(\Omega^*)$ is the dual of $L^{p'}(\Omega^*)$, and we get immediately (1.7). In any case $(p \geqq 1)$, we can use the following argument. Let v_n be a sequence in $L^\infty(\Omega)$. As previously, one can prove that

$$\left\| \frac{dw_m}{ds} - \frac{dw_n}{ds} \right\|_{L^p(\Omega^*)} \leqq \|v_m - v_n\|_{L^p(\Omega)} , \tag{1.8}$$

for any $p > 1$, and, consequently, (passing to the limit) for $p \geqq 1$. Now consider v_1, v_2 in $L^p(\Omega)$ $(p \geqq 1)$, v_{in} $(i = 1, 2)$ in $L^\infty(\Omega)$, $v_{in} \to v_i$ in $L^p(\Omega)$; w_i, w_{in} are associated to (u, v_i) and (u, v_{in}) respectively as in (1.6). By (1.8),

dw_{in}/ds is a Cauchy sequence in $L^p(\Omega^*)$. As $|w_{in}(s) - w_i(s)| \leqq \|v_{in} - v_i\|_{L^1(\Omega)}$, w_{in} tends to w_i in $\mathfrak{C}^0([0, |\Omega|])$, $dw_{in}/ds \to dw_i/ds$ in $L^p(\Omega^*)$, and, by passing to the limit in

$$\left\|\frac{dw_{1n}}{ds} - \frac{dw_{2n}}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v_{1n} - v_{2n}\|_{L^p(\Omega)},$$

we get

$$\left\|\frac{dw_1}{ds} - \frac{dw_2}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v_1 - v_2\|_{L^p(\Omega)}. \tag{1.9}$$

With $v_1 = v$, $v_2 = 0$, we get evidently (1.7). □

Relative rearrangement.

DEFINITION. According to J. Mossino and R. Temam [6] the function dw/ds is called *the rearrangement of v with respect to u*, and is denoted by v_u^*.

The usual rearrangement of a function is also the rearrangement of this function with respect to a constant ($u_c^* = u^*$) or with respect to itself ($u_u^* = u^*$). More generally, if a Borel function F: $\mathbf{R} \to \mathbf{R}$, and a measurable function u: $\Omega \to \mathbf{R}$, are such that $F(u)$ is in $L^p(\Omega)$, then $F(u^*)$ is in $L^p(\Omega^*)$ (by (1.2)) and $(F(u))_u^* = F(u^*)$. In fact $(F(u))_u^* = dw/ds$, with

$$w(s) = \begin{cases} \displaystyle\int_{u<u^*(s)} F(u)\,dx & \text{if } |u = u^*(s)| = 0, \\ \displaystyle\int_{u<\alpha} F(u)\,dx + \int_0^{s-s_\alpha} (F(u)|_{P_\alpha})^*\,d\sigma & \text{otherwise}, \end{cases}$$

with $\alpha = u^*(s)$, $P_\alpha = \{u = \alpha\}$, $|P_\alpha| \neq 0$, $s_\alpha = |u < \alpha|$,

$$= \begin{cases} \displaystyle\int_0^s F(u^*)\,d\sigma & \text{if } |u = u^*(s)| = 0, \\ \displaystyle\int_0^{s_\alpha} F(u^*)\,ds + F(\alpha)(s - s_\alpha) = \int_0^s F(u^*)\,d\sigma & \text{otherwise} \end{cases}$$

(by (1.2))

$$= \int_0^s F(u^*)\,d\sigma.$$

However, generally, v_u^* is not an increasing function, the property of equimeasurability and properties (*c*), (*e*) above, for the usual rearrangement, do not seem to have their analogue for the rearrangement of a function with respect to another one. But we have, if v is in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$, $u\colon \Omega \to \mathbf{R}$ is measurable

(*a'*) $v_1 \leqq v_2$ a.e. implies $(v_1)_u^* \leqq (v_2)_u^*$ a.e.

In fact, with φ in $\mathfrak{D}(\Omega^*)$, $\varphi \geqq 0$,

$$\int_{\Omega^*}[(v_2)_u^* - (v_1)_u^*]\varphi\, ds = \lim_{\lambda \downarrow 0}\int_{\Omega^*}\varphi\,\frac{(u+\lambda v_2)^* - (u+\lambda v_1)^*}{\lambda}\, ds \geqq 0$$

by property (*a*).

(*b'*) For all constants C, $(v+C)_u^* = v_u^* + C$.

In fact, with φ in $\mathfrak{D}(\Omega^*)$,

$$\int_{\Omega^*}(v+C)_u^*\varphi\, ds = \lim_{\lambda \downarrow 0}\int_{\Omega^*}\frac{(u+\lambda(v+C))^* - u^*}{\lambda}\,\varphi\, ds$$

$$= \lim_{\lambda \downarrow 0}\int_{\Omega^*}\frac{(u+\lambda v)^* - u^*}{\lambda}\,\varphi\, ds + \int_{\Omega^*}C\varphi\, ds$$

(by property (*b*))

$$= \int_{\Omega^*}(v_u^* + C)\varphi\, ds\,.$$

(*d'*) If $u\colon \Omega \to \mathbf{R}$ is measurable, the mapping $v \to v_u^*$ is a contraction from $L^p(\Omega)$ into $L^p(\Omega^*)$ $(1 \leqq p \leqq \infty)$ as we have seen in (1.9).

(*f'*) Besides, the mapping $v \to v_u^*$ $(L^1(\Omega) \to L^1(\Omega^*))$ preserves the integral:

$$\int_{\Omega^*}v_u^*\, ds = \int_{\Omega^*}\frac{dw}{ds}\, ds = w(|\Omega|) - w(0) = w(|\Omega|) = \int_{\Omega}v\, dx\,. \qquad \square$$

One can also define another rearrangement v_{*u} which is relative to the directional derivative of the mapping $u \to u_*$ (the decreasing rearrangement of u):

$$v_{*u} = \frac{dw}{ds} = \lim_{\lambda \downarrow 0}\frac{dw_\lambda}{ds}$$

(the limit is taken in the sense of distributions),

$$\frac{dw_\lambda}{ds} = \frac{(u + \lambda v)_* - u_*}{\lambda} = \frac{-(-u - \lambda v)^* + (-u)^*}{\lambda}$$

(by (1.1)). Thus

$$v_{*u} = -(-v)^*_{-u} . \tag{1.10}$$

1.3. *Symmetrization of a family of functions.*

In this Section, $u: [0, T] \times \Omega \to \mathbb{R}$ will be a function defined everywhere in $[0, T]$, and almost everywhere in $\Omega \subset \mathbb{R}^N$. For all t in $[0, T]$, we denote by $u(t): \Omega \to \mathbb{R}$, the function $u(t)(x) = u(t, x)$. (For a fixed t, if no confusion is possible, we shall sometimes write u instead of $u(t)$.). We assume that $u(t)$ is measurable for every t in $[0, T]$. Then, we can define the function $u^*: [0, T] \times \bar{\Omega}^* \to \mathbb{R}$, the increasing rearrangement of u with respect to the x variable in Ω, that is:

$$\forall t \in [0, T],\ \forall s \in \bar{\Omega}^*, \qquad u^*(t, s) = (u(t))^*(s) (= u^*(t)(s)) . \tag{1.11}$$

We consider now another real function v defined almost everywhere in $Q = (0, T) \times \Omega$, such that, for almost every t in $(0, T)$, $v(t)$ is in $L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$. Then, we can define as in Section 1.2, $(v(t))^*_{u(t)}$, which is in $L^p(\Omega^*)$

$$\|(v(t))^*_{u(t)}\|_{L^p(\Omega^*)} \leqq \|v(t)\|_{L^p(\Omega)} . \tag{1.12}$$

We denote by v^*_u the function defined almost everywhere in $Q^* = (0, T) \times \Omega^*$ by

$$\text{a.e. } t \in (0, T),\ \text{a.e. } s \in \Omega^*, \qquad v^*_u(t, s) = (v(t))^*_{u(t)}(s) . \tag{1.13}$$

The aim of this Section 1.3 is to study the regularity of u^* with respect to t (assuming a certain regularity of u with respect to t), and to compute $\partial u^*/\partial t$. We have

THEOREM 1.2. *If u belongs to $H^1(0, T; L^p(\Omega))$ $(1 \leqq p \leqq \infty)$, then u^* belongs to $H^1(0, T; L^p(\Omega^*))$, and*

$$\|u^*\|_{H^1(0,T;L^p(\Omega^*))} \leqq \|u\|_{H^1(0,T;L^p(\Omega))} . \tag{1.14}$$

Moreover

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w}{\partial s} \quad \textit{(in the sense of distributions)} \tag{1.15}$$

where

$$(1.16)\quad w(t, s) = \begin{cases} \displaystyle\int\limits_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx & \text{if } |u(t) = u(t)^*(s)| = 0\,, \\ \displaystyle\int\limits_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx + \int\limits_0^{s-|u(t)<u(t)^*(s)|} \left(\frac{\partial u}{\partial t}\Big|_{u(t)=u(t)^*(s)}\right) d\sigma & \textit{otherwise}\,. \end{cases}$$

PROOF. As $\|u(t)^*\|_{L^p(\Omega^*)} = \|u(t)\|_{L^p(\Omega)}$ (by (1.2)), we have

$$\|u^*\|_{L^2(0,T;\,L^p(\Omega^*))} = \|u\|_{L^2(0,T;\,L^p(\Omega))}\,.$$

Besides, by (1.12), (1.13)

$$\left\|\left(\frac{\partial u}{\partial t}\right)^*_u(t)\right\|_{L^p(\Omega^*)} \leqq \left\|\frac{\partial u}{\partial t}(t)\right\|_{L^p(\Omega)},$$

$$\left\|\left(\frac{\partial u}{\partial t}\right)^*_u\right\|_{L^2(0,T;L^p(\Omega^*))} \leqq \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;\,L^p(\Omega))}.$$

Thus, we have only to prove (1.15), (1.16). Our proof uses the following lemma (see its proof in the Appendix).

LEMMA 1.2. *Let u be in $H^1(0, T; L^p(\Omega))$ $(1 \leqq p \leqq \infty)$,*

$$r_h = \frac{u(t+h) - u(t)}{h} - \frac{\partial u}{\partial t}\,.$$

Consider a fixed number $\varepsilon > 0$. When h tends to zero, r_h tends to zero in $L^\alpha(Q_\varepsilon)$, with $\alpha = \mathrm{Min}\,(p, 2)$, $Q_\varepsilon = (\varepsilon, T-\varepsilon)\times\Omega$.

Let φ be in $\mathfrak{D}(Q^*)$, and let $\varepsilon > 0$ be such that the support of φ is included into $Q^*_\varepsilon = (\varepsilon, T-\varepsilon)\times\Omega^*$. Consider $0 < h < \varepsilon$. We have

$$\int\limits_{Q^*}\frac{u(t+h)^* - u(t)^*}{h}\,\varphi(t)\, ds\, dt = \int\limits_{Q^*}\frac{(u + h(\partial u/\partial t))^* - u^*}{h}\,\varphi\, ds\, dt$$

$$+ \int\limits_{Q^*}\frac{(u + h(\partial u/\partial t + r_h))^* - (u + h(\partial u/\partial t))^*}{h}\,\varphi\, ds\, dt\,.$$

The first integral in the right hand side is $\int_0^T A_h(t)\, dt$, where

$$\text{a.e. } t\,, \quad A_h(t) = \int_{\Omega^*} \frac{(u(t) + h(\partial u/\partial t)(t))^* - u(t)^*}{h}\, \varphi(t)\, ds \xrightarrow[(h\to 0)]{} \int_{\Omega^*} \left(\frac{\partial u}{\partial t}\right)^*_u (t)\, \varphi(t)\, ds$$

(by Theorem 1.1 bis), and

$$|A_h(t)| \leq \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \|\varphi(t)\|_{L^{p'}(\Omega^*)},$$

with $1/p + 1/p' = 1$ (by property (d') above). Using Lebesgue theorem

$$\int_0^T A_h(t)\, dt \xrightarrow[(h\to 0)]{} \int_{Q^*} \left(\frac{\partial u}{\partial t}\right)^*_u \varphi\, ds\, dt\,.$$

The other integral is majorized by

$$\int_\varepsilon^{T-\varepsilon} \|r_h(t)\|_{L^\alpha(\Omega)} \|\varphi(t)\|_{L^{\alpha'}(\Omega^*)}\, dt$$

with $\alpha = \mathrm{Min}\,(p, 2)$, $(1/\alpha + 1/\alpha' = 1)$

$$\leq \|r_h\|_{L^\alpha(Q_\varepsilon)} \|\varphi\|_{L^{\alpha'}(Q^*_\varepsilon)},$$

which tends to zero with h, by Lemma 1.2. Thus

$$\int_{Q^*} \frac{u(t+h)^* - u(t)^*}{h}\, \varphi(t)\, ds\, dt \xrightarrow[(h\to 0)]{} \int_{Q^*} \left(\frac{\partial u}{\partial t}\right)^*_u \varphi\, ds\, dt\,.$$

But, classically,

$$\int_{Q^*} \frac{u(t+h)^* - u(t)^*}{h}\, \varphi(t)\, ds\, dt \to -\int_{Q^*} u^* \frac{\partial \varphi}{\partial t}\, ds\, dt\,.$$

We conclude that, in the sense of distributions,

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^*_u\,. \quad \square$$

A direct consequence of Theorem 1.2 is the following

PROPOSITION. *Assume u belongs to $H^1(0, T; L^1(\Omega))$. Then, for almost every t in $(0, T)$, $\partial u/\partial t(t, \cdot)$ is constant (almost everywhere) on any set where $u(t, \cdot)$ is constant (almost everywhere).*

PROOF. If, in the proof of Theorem 1.2, we consider $h < 0$, we get

$$A_h(t) = \int_{\Omega^*} \frac{(u(t) + h(\partial u/\partial t)(t))^* - u(t)^*}{h} \varphi(t)\, ds = -\int_{\Omega^*} \frac{(u(t) + (-h)(-\partial u/\partial t))^* - u(t)^*}{-h} \varphi(t)\, ds\,,$$

which tends to $-\int_{\Omega^*}(-\partial u/\partial t)^*_u \varphi(t)\, ds$. Thus, one has, in the sense of distributions,

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^*_u = -\left(-\frac{\partial u}{\partial t}\right)^*_u \quad \left(= \left(\frac{\partial u}{\partial t}\right)_{*-u} \text{ by } (1.10)\right),$$

and

$$\left(\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w}{\partial s} \quad (w \text{ defined in } (1.16)), \qquad -\left(-\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w'}{ds}\,,$$

with

$$w'(t, s) = \begin{cases} \displaystyle\int_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx & \text{if } |u(t) = u(t)^*(s)| = 0\,, \\ \displaystyle\int_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx + \int_0^{s-|u(t)<u(t)^*(s)|} -\left(-\frac{\partial u}{\partial t}\bigg|_{u(t)=u(t)^*(s)}\right)^* d\sigma\,, & \text{otherwise}. \end{cases}$$

The last integral is also

$$\int_0^{s-|u(t)<u(t)^*(s)|} \left(\frac{\partial u}{\partial t}\bigg|_{u(t)=u(t)^*(s)}\right)_* d\sigma \quad (\text{by } (1.1))\,.$$

Now, fix t in $(0, T)$, such that $\partial w/\partial s = \partial w'/\partial s$ (in $L^1(\Omega^*)$) (this is true for almost every t in $(0, T)$), and consider a flat region of $u(t)$: $P_\theta(t) = \{u(t) = \theta\}$, $|P_\theta(t)| \neq 0$. Set $s_\theta = |u(t) < \theta|$, $s'_\theta = |u(t) \leqq \theta|$. As $w(t, 0) = 0 = w'(t, 0)$, one has

$$w(t, s) = \int_0^s \frac{\partial w}{\partial s}(t, \sigma)\, d\sigma = \int_0^s \frac{\partial w'}{\partial s}(t, \sigma)\, d\sigma = w'(t, s)\,,$$

for all s in $\bar{\Omega}^*$. Moreover, for all s in $[s_0, s'_\theta]$, one has, by definition of w and w',

$$\frac{\partial w}{\partial s}(t, s) = \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)^*(s - s_0)$$
$$= \frac{\partial w'}{\partial s}(t, s) = \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)_*(s - s_\theta) .$$

In particular,

$$\left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)_*(0) = \operatorname*{Sup\,ess}_{P_\theta(t)} \frac{\partial u}{\partial t}$$
$$= \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)^*(0) = \operatorname*{Inf\,ess}_{P_\theta(t)} \frac{\partial u}{\partial t} ,$$

that is $(\partial u/\partial t)(t,\cdot)$ is constant almost everywhere on $P_\theta(t)$. □

We shall give now the application to parabolic equations.

2. – Isoperimetric inequalities for linear parabolic equations.

Let us consider first the parabolic equation

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathfrak{A}_t(u) + cu = f & \text{in } Q = (0, T)\times\Omega ,\\ u = 0 & \text{on } \Sigma = (0, T)\times\partial\Omega ,\\ u(0,\cdot) = u_0 & \text{in } \Omega , \end{cases}$$

where Ω is a bounded regular open set in $\mathbf{R}^N$,

$$\mathfrak{A}_t(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(t, x)\frac{\partial u}{\partial x_i}\right).$$

We denote by $A(= A(t, x))$ the matrix $(a_{ij}(t, x))$, as well as the bilinear form on $\mathbf{R}^N$ associated with A, and we assume that A satisfies the unifomr (with respect to (t, x)) ellipticity condition:

$$A(\xi, \xi) = \sum_{i,j=1}^{N} a_{ij}(t, x)\xi_i\xi_j \geq |\xi|^2 , \qquad \forall \xi \in \mathbf{R}^N.$$

Furthermore, we assume that the data satisfy:

c, f, u_0 are non-negative functions; c, a_{ij} are in $L^\infty(Q)$;

$\partial a_{ij}/\partial t$ are continuous in $\bar{Q}$, f is in $L^2(Q)$, and u_0 is in $H_0^1(\Omega)$.

Then, the solution u is in $L^\infty(0, T; H_0^1(\Omega))$, $\partial u/\partial t$ is in $L^2(Q)$ (see [5], pp. 113-114, and [4] if $a_{ij} \neq a_{ji}$).

Let us introduce the problem

$$(\widetilde{2.1}) \quad \begin{cases} \dfrac{\partial U}{\partial t} - AU = \underset{\sim}{f} & \text{in } \tilde{Q} = (0, T) \times \tilde{\Omega}, \\ U = 0 & \text{on } \tilde{\Sigma} = (0, T) \times \partial\tilde{\Omega}, \\ U(0, \cdot) = \underset{\sim}{u}_0 & \text{in } \tilde{\Omega}, \end{cases}$$

$\tilde{\Omega}$, $\underset{\sim}{f}$, $\underset{\sim}{u}_0$ are as in the Introduction.

We are going to compare the solution u of (2.1) with the solution U of $(\widetilde{2.1})$. More precisely, we have

THEOREM 2.1. *With the assumptions above,*

$$(2.2) \qquad \forall t \in [0, T],\ \forall s \in \bar{\Omega}^*, \qquad \int_0^s u_*(t, \sigma)\, d\sigma \leq \int_0^s U_*(t, \sigma)\, d\sigma \leq \int_0^s g(t, \sigma)\, d\sigma$$

where $g(t, s) = \int_0^t f_*(\tau, s)\, d\tau + (u_0)_*(s)$. *We deduce*

$$(2.3) \qquad \forall t \in [0, T],\ \forall r \in [1, \infty], \qquad \|u(t, \cdot)\|_{L^r(\Omega)} \leq \|U(t, \cdot)\|_{L^r(\tilde{\Omega})} \leq \|g(t, \cdot)\|_{L^r(\Omega^*)} (\leq +\infty),$$

PROOF. For a fixed $t \in [0, T]$, we denote for convenience $u = u(t)$, $f = f(t)$ .... We argue as for the elliptic problem (see [8], [7]). By the maximum principle, we have $u \geq 0$. For any $\theta > 0$, we get from (2.1),

$$(2.4) \qquad \int_\Omega A(\nabla u, \nabla (u - \theta)_+)\, dx = \int_\Omega \left(f - cu - \frac{\partial u}{\partial t}\right)(u - \theta)_+\, dx.$$

Thus, as in [8], [7], a simple derivation gives:

$$(2.5) \qquad -\frac{d}{d\theta} \int_{u > \theta} A(\nabla u, \nabla u)\, dx = \int_{u > \theta} \left(f - cu - \frac{\partial u}{\partial t}\right) dx.$$

The uniform ellipticity condition and the Cauchy-Schwartz inequality lead to

$$\left[-\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\right]^2 \leqq \mu'(\theta)\frac{d}{d\theta}\int_{u>\theta}A(\nabla u,\nabla u)\,dx$$

where $\mu(\theta) = |u > \theta|$, and, by (2.5),

$$\left[-\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\right]^2 \leqq -\mu'(\theta)\int_{u>\theta}\left(f - cu - \frac{\partial u}{\partial t}\right)dx\,. \tag{2.6}$$

Using a result of Fleming-Rishel, and the isoperimetric inequality for the perimeter in the sense of De Giorgi, we find

$$N\alpha_N^{1/N}\mu(\theta)^{1-(1/N)} \leqq -\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\,. \tag{2.7}$$

Hence, combining (2.6), (2.7),

$$N^2\alpha_N^{2/N}\mu(\theta)^{2-(2/N)} \leqq -\mu'(\theta)\int_{u>\theta}\left(f - cu - \frac{\partial u}{\partial t}\right)dx\,. \tag{2.8}$$

By the inequality (1.3) of Hardy-Littlewood,

$$\int_{u>\theta}(f - cu)\,dx \leqq \int_{u>\theta} f\,dx \leqq \int_0^{\mu(\theta)} f_*\,ds = F(t,\mu(\theta)) \tag{2.9}$$

if we set

$$F(t,s) = \int_0^s f_*(t,\sigma)\,d\sigma\,. \tag{2.10}$$

For almost every θ, $|u = \theta| = 0$, and $u_*(\mu(\theta)) = \theta$ because u_* is continuous in $]0, |\Omega|]$ (as u is in $H_0^1(\Omega)$, u non-negative, then $\tilde{u}$ is in $H_0^1(\tilde{\Omega})$, see [7], for example). By Theorem 1.2

$$\int_{u>\theta}\frac{\partial u}{\partial t}\,dx = \int_{u>u_*(\mu(\theta))}\frac{\partial u}{\partial t}\,dx = w(t,\mu(\theta))\,,$$

with

$$w(t, s) = \int_{u(t)>u(t)_*(s)} \frac{\partial u}{\partial t}\, dx \qquad \text{if } |u(t) = u(t)_*(s)| = 0\,,$$

$$\frac{\partial w}{\partial s} = \left(\frac{\partial u}{\partial t}\right)_{*u} = \frac{\partial u_*}{\partial t}\,.$$

Thus

$$\int_{u>\theta} \frac{\partial u}{\partial t}\, dx = \int_0^{\mu(\theta)} \frac{\partial u_*}{\partial t}\, ds = \frac{\partial k}{\partial t}\,(t, \mu(\theta))$$

if we set

$$k(t, s) = \int_0^s u_*(t, \sigma)\, d\sigma \tag{2.11}$$

$\Big($as $(\partial k/\partial t)(t, s) = \int_0^s (\partial u_*/\partial t)(t, \sigma)\, d\sigma\Big)$.

Thus

$$\int_{u>\theta} \frac{\partial u}{\partial t}\, dx = \frac{\partial k}{\partial t}\,(t, \mu(\theta))\,, \qquad \text{a.e. } \theta > 0\,. \tag{2.12}$$

From (2.8), (2.9), (2.12), we get

$$1 \leqq -N^{-2} \alpha_N^{-2/N} \mu(\theta)^{(2/N)-2} \left[F(t, \mu(\theta)) - \frac{\partial k}{\partial t}\,(t, \mu(\theta))\right] \mu'(\theta)\,. \tag{2.13}$$

As $F(t,\cdot) - \partial k/\partial t(t,\cdot)$ is continuous in $\bar{\Omega}^*$, then, the function $H(t,\cdot)$ defined in Ω^* by

$$H(t, s) = s^{(2/N)-2}\left[F(t, s) - \frac{\partial k}{\partial t}\,(t, s)\right]$$

is continuous in $]0, |\Omega|]$. By integrating (2.13), we get, for any $0 \leqq \theta \leqq \theta'$,

$$\theta' - \theta \leqq -N^{-2} \alpha_N^{-2/N} \int_{\mu(\theta)}^{\mu(\theta')} s^{(2/N)-2} \left[F(t, s) - \frac{\partial k}{\partial t}\,(t, s)\right] ds\,. \tag{2.14}$$

Thus, as in [7], one has for almost every s in Ω^*,

$$0 \leqq -\frac{\partial^2 k}{\partial s^2} = -\frac{d}{ds}\,(u(t))_* \leqq N^{-2} \alpha_N^{-2/N} s^{(2/N)-2} \left[F(t, s) - \frac{\partial k}{\partial t}\,(t, s)\right]. \tag{2.15}$$

Hence, k satisfies

$$
(2.16)\qquad \begin{cases} \dfrac{\partial k}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \dfrac{\partial^2 k}{\partial s^2} \leqq F & \text{a.e. in } Q^* = (0, T) \times \Omega^* , \\ k(t, 0) = 0 , \quad \dfrac{\partial k}{\partial s}(t, |\Omega|) = 0 , & \forall t \in [0, T] , \\ k(0, s) = \displaystyle\int_0^s (u_0)_* \, d\sigma = k_0(s) , & \forall s \in \bar{\Omega}^* . \end{cases}
$$

Let $K(t, s) = \int_0^s U_*(t, \sigma)\, d\sigma$, where U is the solution of $(\widetilde{2.1})$. We are going to show that the equality is achieved in (2.16) for K instead of k.

By the maximum principle, $U(t, \cdot)$ decreases along the radii in $\tilde{\Omega}$, and (2.1) can be written

$$
\frac{\partial U_*}{\partial t} - \frac{\partial}{\partial s}\left(N^2 \alpha_N^{2/N} s^{2-(2/N)} \frac{\partial U_*}{\partial s}\right) = f_* \quad \text{in } \Omega^* .
$$

By integrating between 0 and s, using the fact that $s^{2-(2/N)}(\partial U_*/\partial s) = O(s)$ when s tends to zero (see the remark below) we obtain

$$
\frac{\partial K}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \frac{\partial^2 k}{\partial s^2} = F \quad \text{in } Q^* .
$$

REMARK 2.1. Using Cauchy-Schwartz inequality in the first line of (2.16), we get

$$
0 \leqq - s^{2-(2/N)} \frac{\partial u_*}{\partial s} \leqq N^{-2} \alpha_N^{-(2/N)} s^{1/2} \left[\|f(t)\|_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)} \right] . \quad \square
$$

Now, setting $\chi = k - K$,

$$
(2.17)\qquad \begin{cases} \dfrac{\partial \chi}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \dfrac{\partial^2 \chi}{\partial s^2} \leqq 0 & \text{a.e. in } Q^* , \\ \chi(t, 0) = 0 , \quad \dfrac{\partial \chi}{\partial s}(t, |\Omega|) = 0 & \forall t \in [0, T] , \\ \chi(0, s) = 0 , \quad \forall s \in \bar{\Omega}^* . \end{cases}
$$

The first inequality in (2.2) will result from a maximum principle for χ:

Lemma 2.1. *Let* $\chi(t, s) = (k - K)(t, s) = \int_0^s (u_* - U_*)(t, \sigma)\, d\sigma$. *One has* $\chi \leqq 0$ *everywhere in* $\overline{Q^*}$.

Proof of Lemma 2.1. Multiplying the inequality in (2.17) by $s^{(2/N)-2}\chi_+$ we get

$$s^{(2/N)-2}\frac{\partial \chi}{\partial t}\chi_+ \leqq N^2 \alpha_N^{2/N}\frac{\partial^2 \chi}{\partial s^2}\chi_+ \qquad \text{a.e. in } Q^* . \tag{2.18}$$

For fixed t, we shall denote $u, \chi \ldots$, for simplicity, instead of $u(t), \chi(t) \ldots$. We shall also denote by [ ] a function of t, independent of s. First we prove that $(\partial^2\chi/\partial s^2)\chi_+$ is in $L^1(\Omega^*)$. In fact, by Remark 2.1,

$$\begin{aligned} &\left|\frac{\partial u_*}{ds}\right| \leqq [\ ]\, s^{(2/N)-(3/2)} ,\\ &\left|\frac{\partial^2 \chi}{\partial s^2}\right| \leqq \left|\frac{\partial u_*}{\partial s}\right| + \left|\frac{\partial U_*}{\partial s}\right| \leqq [\ ]\, s^{(2/N)-(3/2)} . \end{aligned} \tag{2.19}$$

On the other hand

$$\begin{aligned} &|k| \leqq \int_0^s |u_*|\, d\sigma \leqq s^{1/2}\|u\|_{L^2(\Omega)} = [\ ]s^{1/2}\\ &|\chi_+| \leqq |\chi| \leqq |k| + |K|\\ &|\chi_+| \leqq [\ ]s^{1/2} . \end{aligned} \tag{2.20}$$

Thus,

$$\left|\frac{\partial^2 \chi}{\partial s^2}\chi_+\right| \leqq [\ ]s^{(2/N)-1} ,$$

which belongs to $L^1(\Omega^*)$. By integrating by parts, we are going to prove that $\int_{\Omega^*}(\partial^2\chi/\partial s^2)\chi_+\, ds$ is non-positive. For $a > 0$, as χ belongs to $W^{2,\infty}(a, |\Omega|)$ by (2.19), the following integration by parts is justified

$$\int_a^{|\Omega|}\frac{\partial^2 \chi}{\partial s^2}\chi_+\, ds = -\int_a^{|\Omega|}\left(\frac{\partial \chi_+}{\partial s}\right)^2 ds - \frac{\partial \chi}{\partial s}(a)\,\chi_+(a) \tag{2.21}$$

(we used the fact that $(\partial\chi/\partial s)(|\Omega|) = 0$ by (2.17)). When a tends to zero, the two integrals tend respectively to $\int_{\Omega^*}(\partial^2\chi/\partial s^2)\chi_+\, ds$ and $\int_{\Omega^*}(\partial\chi_+/\partial s)^2\, ds$ (χ_+, as χ, belongs to $H^1(\Omega^*)$). Now we prove that $(\partial\chi/\partial s)(a)\chi_+(a)$ tends to zero with a. One has

$$\left|\frac{\partial \chi}{\partial s}(a)\right| = \left|\int_{|\Omega|}^{a}\left(\frac{\partial u_*}{\partial s} - \frac{\partial U_*}{\partial s}\right) ds\right| \leqq [\ ]|a^{(2/N)-(1/2)} - |\Omega|^{(2/N)-(1/2)}| \qquad \text{(by (2.19))}.$$

By (2.20),

$$\left|\frac{\partial \chi}{\partial s}(a)\,\chi_+(a)\right| \leqq [\ ]|a^{2/N} - |\Omega|^{(2/N)-(1/2)} a^{1/2}|$$

which tends to zero with a. From (2.21), we get

$$\int_{\Omega^*} \frac{\partial^2 \chi}{\partial s^2}\,\chi_+\,ds = -\int_{\Omega^*}\left(\frac{\partial \chi_+}{\partial s}\right)^2 ds \leqq 0\,.$$

From (2.18),

$$0 \geqq 2\int_0^t \int_{\Omega^*} s^{(2/N)-2}\frac{\partial \chi}{\partial t}\,\chi_+\,ds\,d\tau = \int_0^t \int_{\Omega^*} s^{(2/N)-2}\frac{\partial}{\partial t}(\chi_+^2)\,ds\,d\tau = \int_{\Omega^*} s^{(2/N)-2}\chi_+^2\,ds\,.$$

It follows $\chi_+ \equiv 0$ in $\bar{Q}^*$. □

Now we shall prove the second inequality in (2.2). Let us consider the equation satisfied by K in Q^*:

$$F - \frac{\partial K}{\partial t} = -N^2 \alpha_N^{2/N} s^{2-(2/N)} \frac{\partial U_*}{\partial s} \geqq 0\,.$$

Thus

$$\int_0^s f_*(t,\sigma)\,d\sigma \geqq \frac{\partial}{\partial t}\int_0^s U_*(t,\sigma)\,d\sigma\,.$$

By integration, we find

$$\int_0^s U_*(t,\sigma)\,d\sigma - \int_0^s u_{0*}\,d\sigma \leqq \int_0^s d\sigma \int_0^t f_*(\tau,\sigma)\,d\tau\,. \qquad \square$$

Now, (2.3) is a simple consequence of a lemma in [2] (p. 174), for all r in $[1, \infty[$, and then for $r = \infty$.

REMARK 2.2. If $f_*(t)$ is absolutely continuous in $[0, |\Omega|]$, for almost every t in $(0, T)$, then we can obtain an isoperimetric energy inequality:

we get from (2.1)

$$\int_{\Omega}\left(\frac{\partial u}{\partial t}u + A(\nabla u, \nabla u) + cu^2\right)dx = \int_{\Omega} fu\, dx$$

$$\leq \int_{\Omega^*} f_* u_*\, ds \quad \text{(by Hardy-Littlewood inequality)}$$

$$= -\int_{\Omega^*} \frac{\partial f_*}{\partial s} k\, ds + f_*(|\Omega|)\, k(|\Omega|)$$

$$\leq -\int_{\Omega^*} \frac{\partial f_*}{\partial s} K\, ds + f_*(|\Omega|)\, K(|\Omega|) \qquad \text{(by Theorem 2.1)}$$

$$= \int_{\tilde{\Omega}^*} f_*\, U_*\, ds$$

$$= \int_{\tilde{\Omega}}\left(\frac{\partial U}{\partial t} U + |\nabla U|^2\right) dx\,.$$

Using the uniform ellipticity condition, we have

$$\int_{\Omega}\left(\frac{\partial u}{\partial t} u + |\nabla u|^2\right) dx \leq \int_{\tilde{\Omega}}\left(\frac{\partial U}{\partial t} U + |\nabla U|^2\right) dx\,,$$

and, by integration

$$\frac{1}{2}\int_{\Omega} u(T)^2\, dx + \int_{Q} |\nabla u|^2\, dx\, dt \leq \frac{1}{2}\int_{\tilde{\Omega}} U(T)^2\, dx + \int_{\tilde{Q}} |\nabla U|^2\, dx\, dt\,.$$

Appendix.

In the proof of Lemma 1.2, we shall use the following lemma, whose proof is easy (see [1] for example).

LEMMA A. *Let v in $W^{1,\alpha}(0, T)$ $(1 \leq \alpha \leq \infty)$. If $0 < |h| < \varepsilon$, we have*

$$\int_{\varepsilon}^{T-\varepsilon} \left|\frac{v(t+h) - v(t)}{h}\right|^{\alpha} dt \leq \left\|\frac{dv}{dt}\right\|^{\alpha}_{L^{\alpha}(\varepsilon-|h|,\, T-\varepsilon+|h|)}\,.$$

PROOF OF LEMMA 1.2. If u belongs to $H^1(0, T; L^p(\Omega))$, then u and $\partial u/\partial t$ belong to $L^2(0, T; L^p(\Omega)) \subset L^\alpha(Q)$;

$$\text{a.e. } x\,, \quad u(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x) \in L^\alpha(0, T)\,,$$

that is

$$\text{a.e. } x\,, \quad u(x) \in W^{1,\alpha}(0, T)\,.$$

We can apply Lemma A, with $v = u(x)$. For $0 < |h| < \varepsilon$, we have, with $q_h(t, x) = (u(t + h) - u(t))/h$,

$$\text{a.e. } x\,, \quad \int_\varepsilon^{T-\varepsilon} |q_h(t, x)|^\alpha\, dt \leq \left\|\frac{\partial u}{\partial t}\right\|^\alpha_{L^\alpha(\varepsilon-|h|, T-\varepsilon+|h|)}.$$

By integrating over Ω, we get

$$\overline{\lim_{h\to 0}}\, \|q_h\|_{L^\alpha(Q_\varepsilon)} \leq \left\|\frac{\partial u}{\partial t}\right\|_{L^\alpha(Q_\varepsilon)}. \tag{A.1}$$

1) If $p > 1$ (then $\alpha > 1$), it is easy to prove that $q_h \to \partial u/\partial t$ in $L^\alpha(Q_\varepsilon)$, weakly. Then, classically, by (A.1), $q_h \to \partial u/\partial t$ in $L^\alpha(Q_\varepsilon)$ for the strong topology.

2) If $p = 1$, then $\alpha = 1$. There exists a sequence u_n in $H^1(0, T; L^\infty(\Omega))$ such that $u_n \to u$ in $H^1(0, T; L^1(\Omega))$. Let

$$r_{hn} = \frac{u_n(t + h) - u_n(t)}{h} - \frac{\partial u_n}{\partial t}\,.$$

We have, with the $L^1(Q_\varepsilon)$ norms,

$$\|r_h\| \leq \|r_{hn}\| + \|r_{hn} - r_h\|\,.$$

We have just seen (case $p > 1$) that $r_{hn} \to 0$ in $L^2(Q_\varepsilon)$ (and consequently in $L^1(Q_\varepsilon)$). Besides, by Lemma A,

$$\left\|\frac{(u_n - u)(t + h) - (u_n - u)(t)}{h}\right\|_{L^1(Q_\varepsilon)} \leq \left\|\frac{\partial(u_n - u)}{\partial t}\right\|_{L^1(Q)}.$$

Thus

$$\|r_{hn} - r_h\|_{L^1(Q_\varepsilon)} \leq 2\left\|\frac{\partial(u_n - u)}{\partial t}\right\|_{L^1(Q)}$$

which tends to zero with n. It follows that $r_h \to 0$ in $L^1(Q_\varepsilon)$, and Lemma 1.2 is proved.

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CHAPITRE II

Some properties of the relative rearrangement.

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ANALYSE MATHÉMATIQUE. — *Quelques propriétés du réarrangement relatif.* Note de **Jean-Michel Rakotoson**, présentée par Jacques-Louis Lions.

On développe quelques propriétés du réarrangement relatif introduit en [3] et étudié en [1], [2]. Quelques-unes de ces propriétés généralisent des inégalités bien connues telles que l'inégalité de Hardy-Littlewood.

MATHEMATICAL ANALYSIS. — Some properties of the relative rearrangement.

We develop some new properties of the relative rearrangement which was introduced in [3] *and further studied in* [1], [2]. *Some of these properties generalize well-known results as the Hardy-Littlewood inequality.*

The aim of this Note is to give new developments concerning the relative rearrangement ([2], [3]). *As shown in* [2], *this notion can be seen as an extension of the usual rearrangement. Let us recall the definition as it appeared in* [3].

DEFINITION 1. — *Let Ω be a bounded measurable* [1] *subset of $\mathbb{R}^N$, u a measurable function from Ω into $\mathbb{R}$ and $v \in L^1(\Omega)$. We will denote by u_* the decreasing rearrangement of u, that is*

$$u_*(s) = \operatorname{Inf}\{\theta \in \mathbb{R} \mid |u > \theta| \leqq s\} \quad \textit{if } s \in]0, |\Omega|] \quad (^2), \qquad u_*(0) = \operatorname*{Sup\,ess}_{\Omega} u.$$

We define a function w on $\Omega^* = [0, |\Omega|]$ by the formula (0.1).

We have shown in [2] *the following theorem:*

THEOREM 1. — Let u, v be two measurable functions defined in Ω, $v \in L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$. Then

(i) $$w \in W^{1,p}(\Omega^*), \qquad \text{where} \quad \Omega^* =]0, |\Omega|[.$$

(ii) $$\left\| \frac{dw}{ds} \right\|_{L^p(\Omega^*)} \leqq \| v \|_{L^p(\Omega)}.$$

The function dw/ds is called the relative rearrangement of v with respect to u and is denoted v_{*u}.

1. CONTINUITY OF THE RELATIVE REARRANGEMENT IN $L^p(\Omega)$ AND ORLICZ-SPACES. — *The following theorem insures the continuity of the mapping $v \to v_{*u}$ in Orlicz spaces in particular in $L^p(\Omega)$.*

THEOREM 2. — Let $\varphi(t)$, $t \geqq 0$ be a convex non negative and non decreasing function, $(v_1, v_2) \in L^1(\Omega) \times L^1(\Omega)$ and u a measurable function. Then

$$\int_{\Omega^*} \varphi(|v_{1*u} - v_{2*u}|)\, d\sigma \leqq \int_{\Omega} \varphi(|v_1 - v_2|)\, dx.$$

2. AN INEQUALITY OF HARDY-LITTLEWOOD TYPE. — *We show that the following inequality generalizes the Hardy-Littlewood inequality.*

THEOREM 3. — Let u be a real measurable function on Ω, v_1 in $L^p(\Omega)$ and $v_2 \in L^q(\Omega^*)$, $(1/p) + (1/q) = 1$. Then $\int_{\Omega^*} v_{1*u} v_2\, d\sigma \leqq \int_{\Omega^*} v_{1*} v_{2*}\, d\sigma.$

In particular, the following one is crucial in the proof of Theorem 3.

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LEMMA 1. — Let u be a real measurable function defined in Ω and $v \in L^1(\Omega)$, then, for all measurable set E in Ω^*, we have $\int_E v_{*_u}(\sigma)\,d\sigma \geqq \int_0^{|E|} v^*(\sigma)\,d\sigma = \int_{|\Omega|-|E|}^{|\Omega|} v_*(\sigma)\,d\sigma$, where v^* denotes the increasing rearrangement of v.

All of the details of the proofs will be given in [4].

([1]) *We use only Lebesgue measure.*

([2]) *For any measurable subset* E *of* $\mathbb{R}^N$, *we denote by* $|E|$ *its measure.*

0. RAPPEL DE DÉFINITIONS ET DE PROPRIÉTÉS. — Dans cette Note, nous n'utilisons que la mesure de Lebesgue sur $\mathbb{R}^N$. Pour un ensemble E mesurable dans $\mathbb{R}^N$, on désigne par $|E|$ sa mesure. Considérons Ω un ensemble borné mesurable de $\mathbb{R}^N$ et u une fonction mesurable de Ω dans $\mathbb{R}$. On désigne par u_* le réarrangement décroissant de u défini sur $\bar{\Omega}^* = [0, |\Omega|]$ par :

$$u_*(s) = \operatorname{Inf}\{\theta \in \mathbb{R} \mid |u > \theta| \leqq s\} \quad \text{si } s \in]0, |\Omega|], \qquad u_*(0) = \operatorname*{Sup\,ess}_{\Omega} u.$$

Le réarrangement croissant de u, noté u^*, est alors défini par $u^*(s) = u_*(|\Omega| - s)$ pour $s \in \bar{\Omega}^*$. Pour $v \in L^1(\Omega)$, u mesurable, on définit une fonction w sur $\bar{\Omega}^*$ en posant :

$$(0.1) \qquad w(s) = \begin{cases} \displaystyle\int_{u > u_*(s)} v(x)\,dx & \text{si } |u = u_*(s)| = 0, \\ \displaystyle\int_{u > u_*(s)} v(x)\,dx + \int_0^{s - |u > u_*(s)|} (v_{|P(s)})_*(\sigma)\,d\sigma & \text{sinon.} \end{cases}$$

où $(v_{|P(s)})_*$ est le réarrangement décroissant de v restreint à $P(s) = \{u = u_*(s)\}$.

Le théorème suivant a été démontré dans [2].

THÉORÈME 0. — *Soit u une fonction mesurable sur Ω, $v \in L^p(\Omega)$, $1 \leqq p \leqq +\infty$. Alors*

(i) $\qquad w \in W^{1,p}(\Omega^*)$ *avec* $\Omega^* =]0, |\Omega|[$,

(ii) $$\left\| \frac{dw}{ds} \right\|_{L^p(\Omega^*)} \leqq \| v \|_{L^p(\Omega)}.$$

La définition qui suit a été introduite dans [3].

DÉFINITION 0. — *Réarrangement relatif.* — La fonction dw/ds est appelée le réarrangement de v par rapport à u et l'on note $v_{*_u} = dw/ds$.

Les propriétés suivantes sont démontrées dans [1], [2] et montrent en particulier que le réarrangement usuel d'une fonction v est aussi un réarrangement relatif (de v par rapport à v ou par rapport aux constantes).

PROPRIÉTÉS 0. — Si u est une fonction mesurable sur Ω, $v \in L^1(\Omega)$, alors :

(i_1) Pour toute constante c, $v_{*_c} = v_*$, $c_{*_u} = c$.

(i_2) $v_{*_v} = v_*$.

(i_3) Si l'on considère le réarrangement relatif v_u^* associé au réarrangement croissant u^*, alors nous avons $v_u^* = -(-v)_{*(-u)}$.

(i_4) Si $v \geqq 0$ p.p. dans Ω, alors $v_{*_u} \geqq 0$ p. p. dans Ω^*.

Le lemme suivant concerne l'opérateur moyenne $M_{u,v}$ qui à une fonction g mesurable sur Ω^* associe une fonction mesurable $M_{u,v}(g)$ définie sur Ω et où (u, v) sont deux

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fonctions mesurables sur Ω. Nous renvoyons à [3] pour une définition précise de l'opérateur $M_{u,v}$. Le lemme 0 suivant se démontre comme dans [1], [3].

LEMME 0. — *Soit u, v deux fonctions mesurables de* Ω *dans* $\mathbb{R}$, $v \in L^p(\Omega)\,(1 < p \leqq +\infty)$ *et* $g \in L^q(\Omega^*)$, $(1/p)+(1/q)=1$. *Alors* $M_{u,v}(g) \in L^q(\Omega)$ *et*

$$\int_{\Omega^*} g\, v_u^*\, d\sigma = \int_\Omega M_{u,v}(g)\, v\, dx. \tag{0.2}$$

Si v est seulement dans $L^1(\Omega)$ *et que* $g \in C^0(\bar{\Omega}^*)$ *alors la relation* (0.2) *reste valide.* □

Dans la suite, Ω désignera un ensemble borné mesurable de $\mathbb{R}^N$.

1. CONTINUITÉ DU RÉARRANGEMENT RELATIF DANS LES ESPACES D'ORLICZ. — Nous énonçons deux théorèmes qui assurent la continuité de l'application $v \to v_{*u}$ dans les espaces $L^p(\Omega)$ et plus généralement dans les espaces d'Orlicz.

THÉORÈME 1. — *Soient* φ *une fonction réelle convexe définie sur* $\mathbb{R}$, $(v_1, v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *et u une fonction mesurable définie sur* Ω. *Alors*

$$\int_{\Omega^*} \varphi(v_{1*u} - v_{2*u})\, d\sigma \leqq \int_\Omega \varphi(v_1 - v_2)\, dx. \tag{1.1}$$

Si de plus, φ *satisfait à la condition de croissance :* $\exists\, \alpha > 0$, $\exists\, \beta \in \mathbb{R}$ *tel que* $(\forall\, t \in \mathbb{R})$, $|\varphi(t)| \leqq \alpha |t|^p + \beta$ $(1 \leqq p < +\infty)$, *alors la formule* (1.1) *est valable pour* $(v_1, v_2) \in L^p(\Omega) \times L^p(\Omega)$.

THÉORÈME 2. — *Soient* $\varphi(t)$, $t \geqq 0$, *une fonction convexe positive et croissante,* $(v_1, v_2) \in L^1(\Omega) \times L^1(\Omega)$ *et u une fonction mesurable sur* Ω. *Alors*

$$\int_{\Omega^*} \varphi(|v_{1*u} - v_{2*u}|)\, d\sigma \leqq \int_\Omega \varphi(|v_1 - v_2|)\, dx. \tag{1.2}$$

2. UNE GÉNÉRALISATION DE L'INÉGALITÉ DE HARDY-LITTLEWOOD. — Nous généralisons ici l'inégalité de Hardy-Littlewood classique relative à deux fonctions $v \in L^p(\Omega)$, $h \in L^q(\Omega)$, $(1/p)+(1/q)=1$:

$$\int_\Omega hv\, dx \leqq \int_{\Omega^*} h^* v^*\, d\sigma. \tag{2.1}$$

Nous avons le :

THÉORÈME 3. — *Soit u une fonction mesurable sur* Ω, v_1 *dans* $L^p(\Omega)$ *et* $v_2 \in L^q(\Omega^*)$, $(1/p)+(1/q)=1$. *Alors*

$$\int_{\Omega^*} v_{1u}^* v_2\, d\sigma \leqq \int_{\Omega^*} v_1^* v_2^*\, d\sigma. \tag{2.2}$$

Le passage du théorème 3 à (2.1) se fait de la manière suivante : en vertu de la continuité de l'application $(h, v) \in L^q(\Omega) \times L^p(\Omega) \to \left[\int_\Omega hv\, dx, \int_{\Omega^*} h^* v^*\, d\sigma\right]$, il nous suffit de considérer $h \in C^0(\bar{\Omega})$ et $v \in L^1(\Omega)$. On applique alors le théorème 3 avec $u = h$, $v_1 = v$ et $v_2 = h^*$. On obtient : $\int_{\Omega^*} v_h^* . h^*\, d\sigma \leqq \int_{\Omega^*} v^* h^*\, d\sigma$. Par application du lemme 0 sur l'opérateur moyenne $M_{h,v}$ on déduit que

$$\int_\Omega M_{h,v}(h^*)\, v\, dx = \int_{\Omega^*} v_h^* h^*\, d\sigma.$$

Puisque h est continue, alors $M_{h,\,v}(h^*)(x)=h(x)$, pour $x\in\Omega$. Ainsi

$$\int_\Omega h(x)\,v(x)\,dx=\int_\Omega v_h^*\,h^*\,d\sigma\leqq\int_\Omega v^*\,h^*\,d\sigma.$$

Les lemmes suivants servent à démontrer le théorème 3. Auparavant, nous introduisons la :

DÉFINITION 1. — Soit u une fonction réelle mesurable sur Ω; on dira que u n'a pas de palier si $(\forall t\in\mathbb{R})$, $|u=t|=\text{mes}\{x\in\Omega,\ u(x)=t\}=0$.

LEMME 1 (densité). — *Soit $v\in L^p(\Omega)$ $(1\leqq p\leqq+\infty)$; alors il existe une suite v_n de $L^p(\Omega)$ n'ayant pas de palier et qui converge vers v dans $L^p(\Omega)$.*

LEMME 2. — *Soit $v\in L^p(\Omega)$, $1\leqq p\leqq+\infty$, telle que v soit sans palier. Alors :*

(i) *pour tout $a\in\Omega^*$, il existe un ensemble mesurable $E(a)$ de Ω tel que $|E(a)|=a$ et w définie par* (0. 1) *vérifie $w(a)=\int_{E(a)}v(x)\,dx$. De plus,*

(ii) *si $a<b$, alors $E(a)\subset E(b)$.*

LEMME 3. — *Soit u une fonction mesurable sur Ω et v dans $L^1(\Omega)$. Alors pour tout sous-ensemble E mesurable de Ω^*, nous avons*

$$\left(\int_{|\Omega|-|E|}^{|\Omega|} v_*(\sigma)\,d\sigma\right)=\int_0^{|E|} v^*(\sigma)\,d\sigma\leqq\int_E v_{*u}(\sigma)\,d\sigma.\quad\square$$

Lorsque u est assez régulière, on peut donner une expression ponctuelle de v_{*u} comme le montre la remarque suivante :

Remarque 1. — On suppose que Ω est un ouvert borné de $\mathbb{R}^N$; soit $u\in C^\infty(\Omega)$ tel que $1/|\nabla u|\in L^1(\Omega)$. Alors pour tout $v\in L^1(\Omega)$:

$$v_{*u}(s)=\frac{\displaystyle\int_{u=u_*(s)} v(x)\,d\Gamma(x)/|\nabla u(x)|}{\displaystyle\int_{u=u_*(s)} d\Gamma(x)/|\nabla u(x)|}\quad\text{p. p. dans }\Omega^*,$$

où $d\Gamma$ désigne la mesure de Lebesgue $(N-1)$-dimensionnelle.

Les détails des preuves seront données dans [4]. $\square$

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SOME PROPERTIES OF THE RELATIVE REARRANGEMENT

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Introduction

The notion of relative rearrangement was introduced first by J. Mossino and R. Temam [7] . It has been developped in a recent paper [6] , and several applications in partial differential equations can be found in [6] , [7] , [8] , [9] . In a further work [10] , we will use it to get some optimal bounds on non linear elliptic p.d.e's, when the second member is a distribution.

Some properties of the relative rearrangement have been given in [5] , [6] , [8] . In this paper, we prove some additional properties, which generalize well known results for the usual rearrangement.

Section 1 : Definitions and preliminary results

In this paper, we use only Lebesgue measure.

Let Ω be a bounded measurable set of $\mathbb{R}^N$. For any measurable subset E of Ω , we denote by $|E|$ its measure.

Let u be a real measurable function defined in Ω. We will say that u has a flat region of value t if meas $\{x \in \Omega,\ u(x) = t\} = |u=t|$ is strictly positive. There may exist a countable family of flat regions $P_i = \{u = t_i\}$. We denote $P = \bigcup_{i \in D} P_i$ the union of all flat regions of u.

Definition 1

The decreasing rearrangement of u is defined on $\overline{\Omega}^* = [0, |\Omega|]$ by :

$$u_*(s) = \mathrm{Inf}\ \{\ \theta \in \mathbb{R},\ |u > \theta| \leq s\ \}$$

we will also consider the increasing rearrangement of u : $u^*(s) = u_*(|\Omega| - s)$. Now, we recall the notion of relative rearrangement, as it appeared in [7]. Let $v \in L^1(\Omega)$, we define a function w in $\overline{\Omega}^*$ by :

$$w(s) = \begin{cases} \displaystyle\int_{u > u_*(s)} v(x)\, dx & \text{if } |u = u_*(s)| = 0 \\ \displaystyle\int_{u > u_*(s)} v(x)\, dx + \int_0^{s-|u>u_*(s)|} (v_{|P(s)})_*(\sigma)\, d\sigma & \text{otherwise} \end{cases}$$

(Here the last integrand is the decreasing rearrangement of the restriction of v to the set $P(s) = \{u = u_*(s)\}$ supposed to be of positive measure). The following theorem was proved in [6].

Theorem 1

Let u *be a measurable function defined in* Ω *,* v *in* $L^p(\Omega)$ $(1 \leq p \leq +\infty)$ *then :*

(i) $w \in W^{1,p}(\Omega^*)$

(ii) $\left\| \frac{dw}{ds} \right\|_{L^p(\Omega^*)} \leq \|v\|_{L^p(\Omega)}$

where $\Omega^* =]0, |\Omega|[$.

Definition 2 : Relative rearrangement

The function $\frac{dw}{ds}$ is called the rearrangement of v with respect to u and is denoted v_{*u}.

We will need the following properties proved in [6] (see also [5]) .

Proposition 1

If u is a measurable function defined in Ω , v in $L^1(\Omega)$ then :

(i-1) for all constant c , $v_{*c} = v_*$, $c_{*u} = c$

(i-2) $\int_{\Omega^*} v_{*u}(\sigma)\, d\sigma = \int_\Omega v(x)\, dx$

(i-3) If we consider the relative rearrangement v_u^* (see [5] , [6]) associated to the increasing rearrangement, we have :

$$v_u^* = -(-v)_* - u$$

The following definitions concern the mean value operators introduced in [7] .

Definition 3

Let g be a measurable real function, almost everywhere defined in Ω^* . To the functions u and g , we can associate another function $M_u(g) : \Omega \to \overline{\mathbb{R}}$ defined by :

$$\text{for a.e. } x \qquad M_u(g)(x) = \begin{cases} g(\underline{\beta}(u)(x)) & \text{if } x \in \Omega \setminus P \\ \dfrac{1}{|P_i|} \displaystyle\int_{s'_i}^{s''_i} g(\sigma)\, d\sigma & \text{if } x \in P_i \end{cases}$$

where $s'_i = |u < t_i|$, $s''_i = |u \leq t_i|$, $\underline{\beta}(u)(x) = |u < u(x)|$

Definition 4

Now, let us consider two measurable real functions defined in Ω . We denote by v_i the restriction of v to a flat region P_i of u . To the function g defined above, we can associate the function $M_{u,v}(g) : \Omega \to \overline{\mathbb{R}}$ by

$$M_{u,v}(g)(x) = \begin{cases} M_u(g)(x) & \text{if } x \in \Omega \setminus P \\ M_{v_i}(h_i)(x) & \text{if } x \in P_i \end{cases}$$

where $h_i(s) = g(s'_i + s)$ if $s \in [0, |P_i|]$

M_{v_i} is defined as M_u (with Ω replaced by P_i).
The proof of the following lemma is given in [5] , [7] .

Lemma 1

Let u , v *be two measurable functions from* Ω *into* $\mathbb{R}$,
$v \in L^p(\Omega)$ $(1 < p \leq +\infty)$ *and* $g \in L^q(\Omega^*), \frac{1}{p} + \frac{1}{q} = 1$ *then*

$$M_{u,v}(g) \in L^q(\Omega) \text{ and } \int_{\Omega^*} g\, v_u^*\, d\sigma = \int_\Omega M_{u,v}(g)\, v\, dx \qquad (1\text{-}1)$$

Remark 1

If $v \in L^1(\Omega)$, $g \in \mathcal{C}^0(\overline{\Omega^*})$ the relation (1 - 1) holds.
In fact, if we consider first $g \in \mathcal{D}(\Omega^*)$, we can argue as in [7] to get relations (1 - 1) . As $M_{u,v}$ belongs to $\mathcal{L}(L^\infty(\Omega^*), L^\infty(\Omega))$ (see [5]) and the mapping $g \in L^\infty(\Omega^*) \to \int_{\Omega^*} g\, v_u^*\, d\sigma$ is continuous,we can conclude by density of $\mathcal{D}(\Omega^*)$ in $\mathcal{C}^0(\overline{\Omega^*})$.

Remark 2

One can give an explicit expression of v_{*u} when u is a regular function. (R - 1) assume that Ω is a bounded open set of $\mathbb{R}^N$ and u is an element of $\mathcal{C}^{\infty}(\Omega)$ such that $\frac{1}{|\nabla u|} \in L^1(\Omega)$

then for any $v \in L^1(\Omega)$

$$v_{*u}(s) = \frac{\displaystyle\int_{u=u_*(s)} \frac{v(x)d\Gamma(x)}{|\nabla u(x)|}}{\displaystyle\int_{u=u_*(s)} \frac{d\Gamma(x)}{|\nabla u(x)|}} \quad \text{a.e. in } \Omega^* \qquad (1\text{-}2)$$

where $d\Gamma$ denote the $(N - 1)$ dimensional Lebesgue measure.

Proof of remark 2

We recall (see [1]) that a real t is called a regular value of u if

$u^{-1}(t)$ is a compact $(N - 1)$ dimensional manifold on which $\nabla u(x) \neq 0$.

A real t is said to be a critical value of u if it is not a regular value.

The set of critical values is denoted by $\mathcal{C}$. According to Sard's Theorem (see [1]) , if $u \in \mathcal{C}^{\infty}(\Omega)$ then the Lebesgue measure of $\mathcal{C}$ is zero.

We denote by $\mu(t) = |u > t|$. We observe that u has no flat region because $\frac{1}{|\nabla u|}$ is in $L^1(\Omega)$ and on flat regions $\nabla u(x) = 0$ a.e. We then have

$$\text{for all } s \in \overline{\Omega}^* : \mu(u_*(s)) = s \qquad (1\text{-}3)$$

The function μ is absolutely continuous. In fact, let us take $(a,b) \in \mathbb{R}^2$ $(a < b)$. The function u is lipchitz and $\frac{1}{|\nabla u|}$ is integrable, we can use Federer 's theorem [2] to get that :

$$\mu(a) - \mu(b) = \int_{a \leq u \leq b} dx = \int_a^b dp \int_{u=p} \frac{d\Gamma}{|\nabla u(x)|} \quad \text{and} \quad \mu'(p) = \int_{u=p} \frac{d\Gamma}{|\nabla u(x)|}$$

These last relations prove that μ is absolutely continuous. As absolutely continuous functions map null sets into null sets, we deduce meas $\mu(\mathcal{C}) = |\mu(\mathcal{C})| = 0$.

Using relation (1-3) , we get that : $\{ s \in \bar{\Omega}^*, u_*(s) \in \mathcal{C}\}$ is included into $\mu(\mathcal{C})$. Thus, for almost every s in Ω^*, $u_*(s)$ is a regular value of u . In the following, we consider only such points s . The following computation is then true for almost every s of Ω^* .

Let $h > 0$, as u has no flat region, we get :

$$w(s+h) - w(s) = \int_{u_*(s+h) \leq u \leq u_*(s)} v(x)\, dx$$

One can check that for small h , $\nabla u(x) \neq 0$ for all x in the compact

$K_h = \{u_*(s+h) \leq u \leq u_*(s)\}$, $\frac{1}{|\nabla u|} \in L^\infty(K_h)$ and $\frac{v}{|\nabla u|}$ is in $L^1(K_h)$.

We use Federer's theorem [2] to get :

$$\frac{w(s+h) - w(s)}{h} = -\frac{1}{h} \int_{u_*(s)}^{u_*(s+h)} dp \int_{u=p} \frac{v(x)\, d\Gamma}{|\nabla u(x)|}$$

Let us write $u_*(s+h) = u_*(s) + R(s,h)$ where $R(s,h) = h . \frac{du_*}{ds} + o(h)$ ($\frac{du_*}{ds}$ exist a.e as u_* is decreasing and $R(s,h) \neq 0$ since u has no flat region). Then, we get :

$$\frac{w(s+h) - w(s)}{h} = - \frac{R(s,h)}{h} \cdot \frac{1}{R(s,h)} \int_{u_*(s)}^{u_*(s)+R(s,h)} dp \int_{u=p} \frac{v(x)\, d\Gamma}{|\nabla u(x)|}$$

And when h tends to zero :

$$v_{*u}(s) = \frac{dw}{ds} = - \frac{du_*}{ds} \int_{u=u_*(s)} \frac{v(x)\, d\Gamma}{|\nabla u(x)|} \quad \text{a.e in } \Omega^*$$

This last relation is true for all $v \in L^1(\Omega)$. In particular if $v = 1$ ($v_{*u} = 1$ see proposition 1) and thus :

$$\frac{du_*}{ds} = - \frac{1}{\displaystyle\int_{u=u_*(s)} \frac{d\Gamma}{|\nabla u(x)|}} \quad \text{a.e in } \Omega^*$$

These last formulas lead to (1-2) .

Section 2 : General properties of the relative rearrangement

The following is a generalization of the property of contraction for v_{*u} .

Theorem 2

Let ρ *be a convex function defined in* $\mathbb{R}$, $(v_1, v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ u *a measurable function defined in* Ω *then :*

$$\int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma \leq \int_\Omega \rho(v_1 - v_2)\, dx \qquad (1\text{-}4)$$

This last formula is also valid for $(v_1,v_2) \in L^p(\Omega) \times L^p(\Omega)$ $(1 \leq p < +\infty)$ *if* ρ *satisfies :*

$$\exists\ \alpha \geq 0\ ,\quad \exists\ \beta \in \mathbb{R}\ ,\quad \forall\ t \in \mathbb{R}\ ,\ |\rho(t)| \leq \alpha|t|^p + \beta$$

Remark 3

According to Kranoselskii [4] , the last condition for ρ is necessary and sufficient to insure that the mapping $v \to \rho(v)$ is continuous from $L^p(\Omega)$ (resp. $L^p(\Omega^*)$) into $L^1(\Omega)$ (resp. $L^1(\Omega^*)$) .

Remark 4

Under the assumptions of Remark 2 , if moreover ρ satisfies $|\rho(t)| \leq \alpha|t|^{\cdot} + \beta$, we have the ponctual inequality :

$$\text{a.e in } \Omega^* \quad \rho(v_{1*u}(s) - v_{2*u}(s)) \leq [\rho(v_1 - v_2)]_{*u}(s) \tag{1-5}$$

for any $(v_1,v_2) \in L^1(\Omega) \times L^1(\Omega)$

In fact, by relation (1-2) :

$$v_{1*u}(s) - v_{2*u}(s) = \frac{\displaystyle\int_{u=u_*(s)} (v_1 - v_2)(x) \frac{d\Gamma(x)}{|\nabla u(x)|}}{\displaystyle\int_{u=u_*(s)} \frac{d\Gamma(x)}{|\nabla u(x)|}}$$

Setting $d\nu = \dfrac{d\Gamma(x)}{|\nabla u(x)|} \cdot \dfrac{1}{\displaystyle\int_{u=u_*(s)} \frac{d\Gamma}{|\nabla u(x)|}}$, one can use Jensen inequality to get :

$$\rho(v_{1*u}(s) - v_{2*u}(s)) = \rho\left(\int_{u=u_*(s)} (v_1 - v_2)(x)\, d\nu(x)\right) \leq \int_{u=u_*(s)} \rho(v_1 - v_2)\, d\nu$$

and

$$\int_{u=u_*(s)} \rho(v_1 - v_2)\, d\nu = [\rho(v_1 - v_2)]_{*u}(s) \quad \text{(by relation (1-2))}$$

This ponctual relation leads to (1-4) since by integration :

$$\int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma \leq \int_{\Omega^*} [\rho(v_1 - v_2)]_{*u}(\sigma)\, d\sigma = \int_\Omega \rho(v_1 - v_2)\, dx$$

(by proposition 1 , (i-2))

Remark 5

The ponctual relation (1-5) is not valid for any u . To see this, let us take, u = constante = 1 , and $v_1 = v \in L^\infty(\Omega)$, $v_2 = 0$. Then, if it is true, we will get :

$$\rho(v_{*1}) \leq [\rho(v)]_{*1} .$$

By proposition 1 (i-1) , $v_{*1} = v_*$, $[\rho(v)]_{*1} = [\rho(v)]_*$ thus $\rho(v_*) \leq [\rho(v)]_*$. By equimesurability , we deduce : $\rho(v_*) = [\rho(v)]_*$ for any $v \in L^\infty(\Omega)$ and any convex function ρ : it is not difficult to see that this is impossible (for example, take ρ decreasing).

The proof of theorem 2 needs the following lemma whose proof can be easily deduced from G. Chiti's result [3] .

Lemma 2

Let ρ be a convex function defined in $\mathbb{R}$, u and v two measurable functions defined in Ω , $v \in L^\infty(\Omega)$ then we have

$$\int_{\Omega^*} \rho((u + v)_* - u_*)\, d\sigma \leq \int_\Omega \rho(v)\, dx$$

Proof of theorem 2

Since ρ is a convex function, the mapping: $v \in L^{\infty}(\Omega) \to \int_{\Omega} \rho(v)\, dx$ is L.S.C. for the weak star topology. Hence, if (v_1, v_2) are two elements of $L^{\infty}(\Omega)$, we know (see [5]) that for all $\lambda > 0$ and u measurable defined in Ω

$$\left\| \frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \right\|_{\infty} \leq \| v_1 - v_2 \|_{\infty}$$

and

$$\frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \underset{\lambda \to 0}{\rightharpoonup} v_{1*u} - v_{2*u} \quad \text{in } L^{\infty}(\Omega^*)$$

weak star

we deduce from lemma 2 and the remark above :

$$\int_{\Omega} \rho(v_1 - v_2)\, dx \geq \underline{\lim}_{\lambda} \int_{\Omega^*} \rho\left[\frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda}\right] d\sigma \geq \int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma$$

Assume that ρ satisfies the growth condition in theorem 2. Since the mapping $v \in L^p(\Omega) \to v_{*u} \in L^p(\Omega^*)$ is continuous (see [6]), we deduce by remark 3 that the mapping $v \in L^p(\Omega) \to \left(\int_{\Omega} \rho(v)\, dx \ , \int_{\Omega^*} \rho(v_{*u})\, d\sigma\right)$ is. continuous. We can conclude using the density of $L^{\infty}(\Omega)$ into $L^p(\Omega)$

Collary 1

Let $\rho(t)$, $t \geq 0$ *be convex , non-negative, non-decreasing,* (v_1, v_2) *in* $L^1(\Omega) \times L^1(\Omega)$ *and* u *a measurable function. Then,*

$$\int_{\Omega^*} \rho(|v_{1*u} - v_{2*u}|)\, d\sigma \leq \int_{\Omega} \rho(|v_1 - v_2|)\, dx.$$

Proof :

We argue as in [3]. We consider the real lipchitz function T_n defined by :

$$T_n(z) = \begin{cases} n & \text{if } z \geq n \\ z & \text{if } |z| \leq n \\ -n & \text{if } z \leq -n \end{cases}$$

Then the functions $v_{in} = T_n(v_i)$ $i=1,2$ are in $L^\infty(\Omega)$ and satisfie $|v_{1n}-v_{2n}| \leq |v_1-v_2|$ a.e.. Since the function ρ is non-decreasing, we deduce that

$$\int_\Omega \rho(|v_{1n}-v_{2n}|)\, dx \leq \int_\Omega \rho(|v_1-v_2|)\, dx$$

and that the function $\rho(|t|)$ is convex, we apply theorem 2 to get that :

$$\int_{\Omega^*} \rho(|v_{1n*u}-v_{2n*u}|)\, d\sigma \leq \int_\Omega \rho(|v_1-v_2|)\, dx$$

Since the sequence v_{in} tends to v_i in $L^1(\Omega)$ $i=1,2$ and the mapping $v \in L^1(\Omega) \to v_{*u} \in L^1(\Omega^*)$ is continuous, we can substract a sequence(**) denoted also v_{in*u} which converges almost everywhere in Ω^*. We apply Fatou's lemma to get that

$$\int_{\Omega^*} \rho(|v_{1*u}-v_{2*u}|)\, d\sigma \leq \underline{\lim}_{n} \int_{\Omega^*} \rho(|v_{1n*u}-v_{2n*u}|)\, d\sigma \leq \int_\Omega \rho(|v_1-v_2|)\, dx$$

□

These last results illustrate the convergence of the relative rearrangement in Orlicz spaces if the original functions belong to $L^1(\Omega)$ and converge in Orlicz spaces.

■

(**) that is from the sequence (v_{1n*u}, v_{2n*u})

2.2 : A generalization of the Hardy-Littlewood inequality

Before proving the result of generalization, we will need some lemmas:

Lemma 3

Let $v \in L^p(\Omega)$ $(1 \le p \le +\infty)$ *then there exists a sequence* $v_n \in L^p(\Omega)$ *such that* v_n *has no flat region and* v_n *tends to* v *in* $L^p(\Omega)$

Proof :

Let $P = \bigcup_{i \in D} P_i$ where $P_i = \{v = \theta_i\}$, $|P_i| \neq 0$ and $\theta_i \neq 0$.

We denote by χ_A the characteristic function of a measurable set A .
We put $\lambda_n(x) = \frac{1}{n} \cdot \frac{1}{1 + |x|}$ for any $x \in \Omega$. We observe that :

$$|\{x \in \Omega , \lambda_n(x) = \alpha\}| = |\{x \in \Omega , e^{-\lambda_n(x)} = \beta\}| = 0 \quad \forall (\alpha,\beta) \in \mathbb{R}^2$$

We define :

$$v_n(x) = e^{-\lambda_n(x)\,\chi_P(x)}\,(v(x) + \lambda_n(x)\ \chi_{\{v=0\}}(x))$$

One can check that :

$$|v_n(x) - v(x)| \le \frac{1}{n}\ (|v(x)| + 1)$$

So, v_n tends to v in $L^p(\Omega)$ $(1 \leq p \leq +\infty)$.

Let us prove that for all $t \in \mathbb{R}$, $|v_n = t| = 0$. We remark that :

$$\{v_n = t\} = \{x \in \Omega \setminus P,\ v(x) \neq 0,\ v_n(x) = t\} \cup \left(\bigcup_{i \in D} \{x \in P_i,\ v_n(x) = t\} \right)$$

$$\cup\ \{x \in \Omega\ ,\ v(x) = 0\ ,\ v_n(x) = t\}$$

We deduce then :

$$|v_n = t| = |\{x \in \Omega \setminus P\ ,\ v(x) \neq 0\ ,\ v(x) = t\}| + \sum_{i \in D} |\{x \in P_i,\ e^{-\lambda_n(x)} = \frac{t}{\theta_i}\}|$$

$$+\ |\{x \in \Omega\ ,\ v(x) = 0\ ,\ \lambda_n(x) = t\}|$$

By the remark above, each term of the summation vanishes.

<u>*Lemma* 4</u>

Let $v \in L^p(\Omega)$, $1 \leq p \leq +\infty$ *such that* v *has no flat region then :*

(i) for all $a \in \overline{\Omega}^*$, *there exists a measurable set* $E(a)$ *such that*

$$w(a) = \int_{E(a)} v(x)\,dx \qquad \textit{and} \quad |E(a)| = a$$

In addition,

(ii) if $a < b$ *then* $E(a) \subset E(b)$

<u>Proof</u> :

(i) Let $a \in \overline{\Omega}^*$

if $|u = u_*(a)| = 0$, we set $E(a) = \{u > u_*(a)\}$

then $|u > u_*(a)| = |u \geq u_*(a)| = a$

if $|u = u_*(a)| \neq 0$, we denote v_a the restriction of v to the set $\{u = u_*(a)\}$ then by equimesurability :

$$\int_0^{a-|u>u_*(a)|} (v_a)_* (\sigma)\, d\sigma = \int_{v_a > (v_a)_* (a-|u>u_*(a)|)} v(x)\, dx$$

The set $\{v_a > (v_a)_* (a - |u > u_*(a)|)\}$ and $\{u > u_*(a)\}$ are disjoint. Hence, we have :

$$|E(a)| = |u > u_*(a)| + |v_a > (v_a)_*(a - |u > u_*(a)|)| = a$$

if we set $E(a) = \{u > u_*(a)\} \cup \{v_a > (v_a)_* (a - |u > u_*(a)|)\}$

(ii) Let $a < b$

If $u_*(a) = u_*(b)$ then $v_a = v_b = k$ we deduce

$$k_*(a - |u > u_*(a)|) \geq k_*(b - |u > u_*(b)|)$$

so $E(a) \subset E(b)$

If $u_*(a) > u_*(b)$: $E(a) \subset \{u \geq u_*(a)\} \subset \{u > u_*(b)\} \subset E(b)$

Remark 6

If $]a,b[\cap]c,d[= \emptyset$, if we set

$E(a,b) = E(b) \setminus E(a)$

$E(c,d) = E(d) \setminus E(c)$

then

$$E(a,b) \cap E(c,d) = \emptyset$$

The following lemma is crucial to prove the result of generalization :

Lemma 5

Let u *be a real measurable function defined in* Ω *and* $v \in L^1(\Omega)$ *then for all measurable set* E *in* Ω^* *, we have :*

$$\int_E v_{*u}(\sigma)\, d\sigma \geq \int_0^{|E|} v^*(\sigma)\, d\sigma \quad \left(= \int_{|\Omega|-|E|}^{|\Omega|} v_*(\sigma)\, d\sigma \right)$$

where v^* *denote the increasing rearrangement of* v .

Proof lemma 5

As the mappings $v \in L^1(\Omega) \to v_1^*$ or $v_{*u} \in L^1(\Omega^*)$ are continuous, thanks to lemma 3 , we can restrict to the case when v has no flat region.

Let $\mathcal{O}$ be an open set of Ω^* then $\mathcal{O}$ is the union (at most countable) of his disjoint connected components : $\mathcal{O} = \bigcup_{i\in D}]a_i, b_i[$

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma = \sum_{i\in D} \int_{a_i}^{b_i} \frac{dw}{ds}\, d\sigma$$

According to lemma 4

$$\sum_{i\in D} \int_{a_i}^{b_i} \frac{dw}{ds}\, d\sigma = \sum_{i\in D} \int_{E(a_i,b_i)} v(x)\, dx ,$$

Since $E(a_i, b_i) \cap E(a_j, b_j) = \emptyset$ for $i \neq j$ (see Remark 5), we deduce via Hardy-Littlewood inequality :

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma = \int_{\bigcup_{i\in D} E(a_i,b_i)} v(x)\, dx \geq \int_0^{|\bigcup_{i\in D} E(a_i,b_i)|} v^*(\sigma)\, d\sigma ,$$

$$|\bigcup_{i\in D} E(a_i, b_i)| = \sum_{i\in D} (b_i - a_i) = |\mathcal{O}| ,$$

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma \geq \int_0^{|\mathcal{O}|} v^*(\sigma)\, d\sigma \tag{1-6}$$

If E is a measurable set of Ω^*, then there exist a sequence $\mathcal{O}_p$ of open set such that : $E \subset \mathcal{O}_{p+1} \subset \mathcal{O}_p$ and $|\mathcal{O}_p| \underset{p\to+\infty}{\to} |E|$

Then by (1-6)

$$\int_{\mathcal{O}_p} \frac{dw}{ds}\, d\sigma \geq \int_0^{|\mathcal{O}_p|} v^*(\sigma)\, d\sigma$$

When we pass to the limit :

$$\int_E v_{*u}(\sigma)\, d\sigma = \int_E \frac{dw}{ds}\, d\sigma \geq \int_0^{|E|} v^*(\sigma)\, d\sigma$$

The following theorem is the generalization of the Hardy-Littlewood inequality :

Theorem 3

Let u *be a real measurable function on* Ω, v_1 *in* $L^p(\Omega)$ *and* $v_2 \in L^q(\Omega^*)$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_{\Omega^*} v_{1*u}\, v_2\, d\sigma \leq \int_{\Omega^*} v_1^*\, v_2^*\, d\sigma \left(= \int_{\Omega^*} v_{1*}\, v_{2*}\, d\sigma\right) \tag{1-7}$$

Remark 7

If we change v_1 into $-v_1$, v_2 into $-v_2$ and u into $-u$, using proposition 1(i-3) , we get :

$$\int_{\Omega^*} v_{1u}^* \, v_2 \, d\sigma \leq \int_{\Omega^*} v_1^* \, v_2^* \, d\sigma \qquad (1\text{-}8)$$

The proof of this theorem needs the following lemma whose proof is in [5] .

Lemma 6

Let $f \in L^\infty(\Omega^*)$, $a \leq f \leq b$, $g \in L^1(\Omega^*)$ *then*

$$\int_{\Omega^*} fg \, d\sigma = b \int_{\Omega^*} g \, d\sigma - \int_a^b dt \int_{f<t} g \, d\sigma$$

Proof of theorem 3

We begin with the case $v_1 \in L^p(\Omega)$ and $v_2 \in L^\infty(\Omega^*)$ then $v_{1*u} \in L^1(\Omega^*)$ and by lemma 6 :

$$\int_{\Omega^*} v_{1*u} \, v_2 \, d\sigma = b \int_{\Omega^*} v_{1*u} \, d\sigma - \int_a^b dt \int_{v_2<t} v_{1*u} \, d\sigma$$

By proposition 1 (i-2) and equimesurability :

$$\int_{\Omega^*} v_{1*u} \, d\sigma = \int_{\Omega^*} v_1^* \, d\sigma$$

By lemma 4

$$\int_{v_2<t} v_{1*u} \, d\sigma \geq \int_0^{|v_2^*<t|} v_1^*(\sigma) \, d\sigma = \int_{v_2^*<t} v_1^*(\sigma) \, d\sigma$$

Hence

$$\int_{\Omega^*} v_{1*u}\, v_2\, d\sigma \leq b \int_{\Omega^*} v_1^*\, d\sigma - \int_a^b dt \int_{v_2^*<t} v_1^*(\sigma)\, d\sigma = \int_{\Omega^*} v_1^*\, v_2^*\, d\sigma$$

The result for $q < +\infty$ easily follows by density.

Remark 8

If $v_2 \geq 0$ is only measurable, $v_1 \geq 0$ and $v_1 \in L^1(\Omega)$ the relation (1-7) (or (1-8)) remains valid. In fact, there exists an increasing sequence $v_{2n} \in L^\infty(\Omega^*)$ such that :

$$\lim_n v_{2n}(\sigma) = v_2(\sigma) \text{ and } 0 \leq v_{2n}(\sigma) \leq v_2(\sigma) \text{ a.e.}$$

then

$$\int_{\Omega^*} v_{2n}(\sigma)\, v_{1*u}(\sigma)\, d\sigma \leq \int_{\Omega^*} v_{2n}^*\, v_1^*\, d\sigma \leq \int_{\Omega^*} v_2^*(\sigma)\, v_1^*(\sigma)\, d\sigma$$

As $v_1 \geq 0$ implies $v_{1*u} \geq 0$ (see [6]) then by Fatou's lemma :

$$\int_{\Omega^*} v_{1*u}\, v_2\, d\sigma \leq \int_{\Omega^*} v_1^*\, v_2^*\, d\sigma$$

Remark 9

As a corollary, we recover the well-known Hardy-Littlewood theorem : For all $v \in L^p(\Omega)$, $h \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$

we have :

$$\int_\Omega hv\, dx \leq \int_{\Omega^*} h^* v^*\, d\sigma$$

Proof :

We begin with the case $h \in \mathscr{C}^0(\bar{\Omega})$ and $v \in L^1(\Omega)$. Using remark 1 and theorem 3 :

$$\int_\Omega M_{h,v}(h^*)\, v\, dx \leq \int_{\Omega^*} h^* v^* d\sigma \tag{1-9}$$

By definition 4 :

$$M_{h,v}(h^*)(x) = \begin{cases} h^*(\underline{\beta}(h)(x)) & \text{if } x \in \Omega \smallsetminus P \\ \\ M_{v_i}(g)(x) & \text{if } x \in P_i = \{h = t_i\} \end{cases}$$

$$g(s) = h^*(s'_i + s) \quad \text{for } s \in [0, s''_i - s'_i]$$

$$s'_i = |h < t_i| \ , \quad s''_i = |h \leq t_i|$$

Then, we have : $g(s) = h^*(s'_i + s) = h^*(s'_i) = h^*(\underline{\beta}(h)(x))$

In any case $M_{h,v}(h^*)(x) = h^*(\underline{\beta}(h)(x)) = h(x)$ (since h is continuous).

By (1-9), we get the Hardy-Littlewood inequality.

By density, the inequality remains valid for $h \in L^q(\Omega)$ and $v \in L^p(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ and then for $q = 1$

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CHAPITRE III

Un modèle non local en physique des plasmas : résolution par une méthode de degré topologique.

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Un modèle non local en physique des plasmas : résolution par une méthode de degré topologique

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INTRODUCTION.

Différents modèles de la physique des plasmas ont été introduits dans la littérature (cf. par exemple [2],[6],[8],[9],[10],[14]).

On peut les résumer sous la forme générale suivante : trouver une fonction u définie sur un ouvert borné régulier Ω de $\mathbb{R}^N$ ($N \geq 1$) telle que :

$$(0) \quad \begin{cases} (0.1) & -\Delta u + g(x,u(x),\underline{\delta}(u)(x)) = 0 \quad \text{dans } \{u<0\} \\ (0.2) & \Delta u = 0 \quad \text{dans } \{u\geq 0\} \\ (0.3) & u = \text{constante (inconnue) sur } \partial\Omega \\ (0.4) & \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I > 0 \end{cases}$$

avec $\underline{\delta}(u)(x) = \text{mes}\,\{y \in \Omega \ / \ u(x) < u(y) < 0\}$. I est une constante positive donnée. L'ensemble $\{u<0\}$ représente la région occupée par le plasma, tandis que l'ensemble $\{u\geq 0\}$ représente une région vide. Ces régions sont connues dès que l'on résoud le problème (0). Pour plus de détails sur le problème physique, on peut consulter [5],[7] et aussi l'appendice de [13].

La fonction g est définie sur $\overline{\Omega} \times \mathbb{R} \times [o,|\Omega|]^{(*)}$; sa forme n'est jusqu'à présent pas précise : c'est l'une des difficultés physiques. Suivant la forme de g , les modèles traités antérieurement peuvent être classés en deux catégories :

(*) $|\Omega|$ = mesure de Ω

M1./ $g(x,u(x),\underline{\delta}(u)(x)) = \lambda g(x,u(x))$

M2./ $g(x,u(x),\underline{\delta}(u)(x)) \equiv \lambda g(\underline{\delta}(u)(x))$

où λ est un paramètre fixé.

R. Temam était le premier à prouver l'existence d'une solution pour le modèle M1 [14], par une méthode variationnelle. Tandis que le second modèle M2 a été résolu par H. Gourgeon et J. Mossino [4], par un procédé de symétrisation et les techniques de l'analyse multivoque déjà utilisées dans [8].

Ici nous résolvons un modèle relativement général qui est une forme affaiblie du problème (0). C'est le problème (1) suivant : trouver $u \in H^2(\Omega)$ vérifiant :

$$(1) \quad \begin{cases} (1.1) & \Delta u \in g(x,u(x),\delta(u)(x)) \quad \text{dans } \{u<0\} \\ (1.2) & \Delta u = 0 \quad \text{dans } \{u\geq 0\} \\ (1.3) & u = \text{constante (inconnue) sur } \partial\Omega \\ (1.4) & \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I > 0 \end{cases}$$

où $\delta(u)(x) = [\underline{\delta}(u)(x), \overline{\delta}(u)(x)]$, $\overline{\delta}(u)(x) = \text{mes}\,\{y \in \Omega \,/\, u(x) \leqq u(y) \leqq 0\}$.

Dans la Section 1, nous précisons les hypothèses sur g et nous introduisons un problème régularisé $(1)_\varepsilon$ (ε paramètre positif) associé au problème (1). Dans la Section 2, nous montrons l'existence de solutions pour ce problème régularisé par des arguments topologiques. Cette méthode nécessite quelques estimations a priori, ce qui se fait dans cette même section. Enfin, dans la Section 3, nous passons à la limite dans le problème $(1)_\varepsilon$ pour montrer l'existence de solutions du problème (1).

Ce travail est une partie d'une Thèse de 3e Cycle soutenue à l'Université de Paris XI-Orsay [12] .

Je tiens à remercier J. Mossino de sa généreuse collaboration.

1. HYPOTHESES - UN PROBLEME GENERALISE.

Considérons le problème (1) ci-dessus, où g est une fonction définie sur $\overline{\Omega} \times \mathbb{R} \times [o,|\Omega|]$ vérifiant :

(H1) g peut se décomposer : $g(x,u,\delta) = g_1(x,u,\delta) + g_2(x,u,\delta)$, avec

$g_i \in C^o(\overline{\Omega} \times \mathbb{R} \times [o,|\Omega|])$ $i = 1,2$

$g_1 \leqq 0$ et $g_2 \geqq 0$.

Il existe deux constantes $a \geqq 0$ et $b > 0$ t.q

$|g_1(x,u,\delta)| \leqq au_- + b \quad \forall\ (x,u,\delta)\ \overline{\Omega} \times \mathbb{R} \times [o,|\Omega|]$

(H2) $g(x,u,o) = o \quad \forall\, u \geqq 0 , \forall\, x \in \overline{\Omega}$

(H3) Il existe deux constantes $\mu_o > 0$, $\mu_o' \leqq 0$ t.q

$\mu_o u_- + \mu_o' \leqq g(x,u,\delta) \ \forall\ (x,u,\delta) \in \overline{\Omega} \times \mathbb{R} \times [o,|\Omega|]$

(H4) $\lim\limits_{|u| \to +\infty} \dfrac{|g(x,u,\delta)|}{|u|^p} = 0$ uniformément en x et en δ avec

$p = \frac{N}{N-2}$ si $N \geqq 3$, p (quelconque) > 1 si $N \leqq 2$.

□

Remarque 0 . Ces hypothèses sont vérifiées pour le modèle M1 où $g(x,u,\delta) \equiv \lambda u_-$ $(\lambda > 0)$, tandis que le modèle M2 ne satisfait pas l'hypothèse (H3). Il peut cependant être résolu par la présente méthode complétée d'un nouveau passage à la limite (cf. Remarque 3 à la fin de l'article).

Pour $\varepsilon > 0$, on associe au problème (1) le problème $(1)_\varepsilon$ régularisé suivant :

Trouver $u_\varepsilon \in H^2(\Omega)$ solution de

$$(1)_\varepsilon \begin{cases} (1\text{-}1)_\varepsilon & -\Delta u_\varepsilon + g(x,u_\varepsilon(x),\delta_\varepsilon(u_\varepsilon)(x)) = 0 \text{ dans } \Omega \\ (1\text{-}2)_\varepsilon & u_\varepsilon = \text{constante (inconnue) sur } \partial\Omega \\ (1\text{-}3)_\varepsilon & \displaystyle\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\, d\sigma = I > 0 \ . \end{cases}$$

où δ_ε est un opérateur de $L^2(\Omega)$ dans $L^\infty(\Omega)$ défini par : pour $v \in L^2(\Omega)$,

$$\delta_\varepsilon(v)(x) = \left[\int_\Omega \underline{h}_\varepsilon(-v(y))dy - \int_\Omega \bar{h}_\varepsilon(v(x) - v(y))dy\right]_+ \quad , \text{ avec}$$

$$\underline{h}_\varepsilon(t) = \begin{cases} 1 \text{ si } t \geq \varepsilon \\ \frac{t}{\varepsilon} \text{ si } 0 \leq t \leq \varepsilon \\ 0 \text{ si } t \leq 0 \end{cases} \qquad \bar{h}_\varepsilon(t) = \begin{cases} 1 \text{ si } t \geq 0 \\ 1 + \frac{t}{\varepsilon} \text{ si } -\varepsilon \leq t \leq 0 \\ 0 \text{ si } t \leq -\varepsilon \end{cases}$$

Nous notons que $\delta_\varepsilon(c) = 0$ lorsque c est une constante $\delta_\varepsilon(v)(x) = 0$ en tout point x où $v(x) \geq 0$ et que

(1.1) $$|\delta_\varepsilon(v)|_{L^\infty(\Omega)} \leq |\Omega|$$

2. EXISTENCE DE SOLUTIONS POUR LE PROBLEME $(1)_\varepsilon$

2.1. Préliminaires.

On désigne par $W = H^1_o(\Omega) \oplus \mathbb{R}$ le sous-espace fermé de $H^1(\Omega)$ défini par

$$H^1_o(\Omega) \oplus \mathbb{R} = \{v \in H^1(\Omega),\ v = \text{constante sur } \partial\Omega\}$$

On le munit de la norme de $H^1(\Omega)$.

Soit L l'opérateur complètement continu (i.e. continu et compact) de $L^2(\Omega)$ dans W défini par :

$$Lf = u \Leftrightarrow \begin{cases} -\Delta u + u = f \text{ dans } \Omega \\ u = \text{constante sur } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = 0 \end{cases}$$

Soit ϕ_o l'élément de $H^1_o(\Omega) \oplus \mathbb{R}$ satisfaisant à

$$\begin{cases} -\Delta\phi_o + \phi_o = o \\ \phi_o \in W \\ \int_{\partial\Omega} \frac{\partial \phi_o}{\partial n}\, d\sigma = I \end{cases}$$

Alors si nous définissons $S_\varepsilon(1,v) = L(v-g(.,v(.),\delta_\varepsilon(v)(.)) + \phi_o$, on remarque que toute solution du problème $(1)_\varepsilon$ est un point fixe de $S_\varepsilon(1,.)$: $u_\varepsilon = S_\varepsilon(1,u_\varepsilon)$.

Par ailleurs, l'application $S_\varepsilon(1,.)$ est complètement continue de W dans lui-même. Pour cela, il nous suffit de démontrer le

<u>Lemme 2.1.</u> <u>L'application</u> $v \in W \to g(.,v(.),\delta_\varepsilon(v)(.)) \in L^2(\Omega)$ <u>est continue et bornée (i.e. transforme les sous-ensembles bornés de</u> W <u>en des sous-ensembles bornés de</u> $L^2(\Omega)$).

Dans la suite, les c_i désignent des constantes ne dépendant que des données I, N, Ω, g .

<u>Démonstration</u>. Par (H1) et (H4), il existe une constante $c_1>0$ t.q

(2.1) $$|g(x,u,\delta)| \leq c_1(1 + |u|^p) \ \forall\ (x,u,\delta) \in \overline{\Omega} \times \mathbb{R} \times [o,|\Omega|]$$

Ainsi, pour $v \in W$, $|g(x,v(x),\delta_\varepsilon(v)(x))| \leq c_1(1 + |v(x)|^p)$ p.p. Sachant que $W \subsetneq L^{2p}(\Omega)$, on déduit que $g(.,v(.),\delta_\varepsilon(v)(.)) \in L^2(\Omega)$ et que l'application en question est bornée.

La continuité de l'application $v \in W \to g(x,v(.),\delta_\varepsilon(v)(.)) \in L^2(\Omega)$ revient à dire que si v_m tend vers v dans W-fort , alors la différence $g(.,v_m(.),\delta_\varepsilon(v_m)(.)) - g(.,v(.),\delta_\varepsilon(v)(.))$ tend vers 0 dans $L^2(\Omega)$-fort lorsque m tend vers $+\infty$.

Posons $w_m(x) = |g(x,v_m(x),\delta_\varepsilon(v_m)(x)) - g(x,v(x),\delta_\varepsilon(v)(x))|^2$. Par la relation (2.1), il existe une constante $c_2 > 0$ t.q

$$0 \leq w_m(x) \leq c_2(1 + |v(x)|^{2p} + |v_m(x)|^{2p}) \quad \text{p.p.}$$

Puisque $v_m \to v$ dans $L^{2p}(\Omega)$, il existe $h \in L^{2p}(\Omega)$ et une sous-suite $(v_{m'})$ de (v_m) t.q :

$$|v_{m'}(x)| \leq h(x) \quad \text{p.p. dans } \Omega \quad , (\forall\, m')$$

et

$$\lim_{m'} v_{m'}(x) = v(x) \quad \text{p.p.}$$

Par l'hypothèse (H1), on déduit $\lim_{m'} w_{m'}(x) = 0$ p.p. dans Ω et $0 \leq w_m(x) \leq c_2\left(1 + |v(x)|^{2p} + [h(x)]^{2p}\right)$. Par le théorème de la convergence dominée : $\lim_{m'} \int_\Omega w_{m'}(x)dx = 0$. Nécessairement, toute la suite w_m tend vers 0 dans $L^1(\Omega)$.

□

Maintenant, nous allons déterminer une boule B_k de rayon k dans W sur laquelle nous pouvons définir le degré topologique $d(I-S_\varepsilon(1,.),B_k,o)$.

2.2. Estimations a priori.

Définissons pour $t \in [o,1]$, l'application g_t par : pour $(x,u,\delta) \in \bar{\Omega} \times \mathbb{R} \times [o,|\Omega|]$

$$g_t(x,u,\delta) = tg(x,u,\delta) + (1-t)\ \mu_1 u_-$$

où μ_1 est la première valeur propre du problème de Dirichlet homogène.

Soit E_t l'ensemble défini par :

$$E_t = \{v \in W \ / \ j_t(v) = e_t(v) = 0\}$$

où

$$e_t(v) = -I + \int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))dx$$

$$j_t(v) = |\nabla v|^2_{L^2} - Iv_{|\partial\Omega} - \int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))v_-(x)dx .$$

On pose $\underline{\mu} = \min(\mu_o,\mu_1) > 0$.

Proposition 2.1. Il existe une constante k ne dépendant que de I, N, Ω, g telle que

$$\forall\, v \in \bigcup_{t\in[o,1]} E_t \qquad |v|_{H^1} < k$$

La démonstration de cette proposition utilise toute une série de lemmes.

Lemme 2.2. $\forall\, t \in [o,1]$, $E_t \neq \emptyset$

Démonstration. Considérons la fonction d'une variable réelle $A(u) = -I + \int_\Omega g_t(x,u,o)dx$. Par (H2), $A(u) = -I$ pour $u \geq 0$. Par ailleurs, en utilisant (H3), on trouve que

$$\int_\Omega g_t(x,u,o)dx \geq \underline{\mu}\ |\Omega|u_- + \mu_o' t|\Omega| \ : \ \lim_{u\to-\infty} \int_\Omega g_t(x,u,o)dx = +\infty \ .$$

Par continuité de A , il existe $u_t < 0$ t.q $A(u_t) = 0$; ainsi

$\int_\Omega g_t(x,u_t,\delta_\varepsilon(u_t))dx = I$: $u_t \in E_t$.

Lemme 2.3. Il existe une constante $c_3 > 0$ telle que

$$\forall\, v \in \bigcup_{t\in[o,1]} E_t \qquad (2.2) \qquad |v_-|_{L^1} \leq c_3$$

$$(2.3) \qquad |v_-|_{L^{2p}} \leq c_3(1 + |\nabla v|_{L^2}) \ .$$

Démonstration. Par (H3), pour $v \in E_t$:

$$t\,\mu_o' \ |\Omega| + \underline{\mu}\ |v_-|_{L^1} \leq \int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))dx = I$$

d'où (2.2).

Pour obtenir (2.3), on utilise l'inégalité de Sobolev-Poincaré [3] : $\forall u \in H^1(\Omega)$ $|u|_{L^{2p}} \leq c(|u|_{L^1} + |\nabla u|_{L^2})$ en prenant $u = v_-$ et en utilisant la relation (2.2) on obtient (2.3).

□

Lemme 2.4. Pour tout $\eta>0$, il existe une constante c_η^1 ne dépendant que de η, I, N, Ω, g, telle que

$$\forall\, v \in H^1(\Omega) \qquad |g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^2}^{\frac{1}{p}} \leq \eta|v_-|_{L^{2p}} + c_\eta^1$$

Dans la suite, les c_η^i désignent des constantes ne dépendant que de η, I, N, Ω, g .

<u>Démonstration</u>. Par (H1) et (H4), pour $\eta>0$, il existe c_η^2 tel que l'on ait $\forall\ (x,u,\delta) \in \bar{\Omega} \times\ ]-\infty,o] \times [o,|\Omega|]$:

(2.4) $$|g(x,u,\delta)|^2 \leq \eta\, u_-^{2p} + c_\eta^2$$

Il existe c_η^3 ne dépendant que de η tel que

(2.5) $$\mu_1^2\, u_-^2 \leqq \eta\, u_-^{2p} + c_\eta^3$$

Des relations (2.4) et (2.5), on déduit :

(2.6) $$|g_t(x,u,\delta)|^2 \leqq \eta\, u_-^{2p} + c_\eta^4$$

Ce qui entraîne par (H2) , pour $v \in H^1(\Omega)$:

$$\int_\Omega |g_t(x,v(x),\delta_\varepsilon(v)(x))|^2 dx \leq \eta \int_\Omega |v_-(x)|^{2p} dx + c_\eta^5$$

d'où le lemme.

<u>Lemme 2.5</u>. <u>Pour tout</u> $\eta>0$, <u>il existe une constante</u> c_η^6 <u>telle que</u> $\forall\ v \in \bigcup_{t\in[o,1]} E_t$

$$|g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^{q'}} \leqq \eta\, |v_-|_{L^{2p}} + c_\eta^6$$

<u>avec</u>

$$\frac{1}{2p} + \frac{1}{q'} = 1.$$

<u>Démonstration</u>. Par le théorème d'interpolation de Riesz ($\theta = \frac{1}{p}$)

$$|g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^{q'}} \leqq |g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^1}^{1-\theta} \cdot |g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^2}^{\theta}$$

Sachant que $g_2 \geqq 0$ (par (H1)), nous avons

$$|g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^1} \leqq t\int_\Omega g_2(x,v(x),\delta_\varepsilon(v)(x))dx + \mu_1 |v_-|_{L^1} +$$

$$+ \int_\Omega |g_1(x,v(x),\delta_\varepsilon(v)(x))|dx .$$

Par (H1) et sachant que $e_t(v) = 0$,

$$\int_\Omega |g_1(x,v(x),\delta_\varepsilon(v)(x))|dx \leq a|v_-|_{L^1} + b|\Omega|$$

et

$$t\int_\Omega g_2(x,v(x),\delta_\varepsilon(v)(x))\,dx \leq I + a\,|v_-|_{L^1} + b|\Omega|$$

En utilisant le lemme (2.3) :

$$|g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^1} \leq c_4$$

et donc

$$|g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^{q'}} \leq c_4^{1-\theta}\; g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^2}^{\frac{1}{p}}\,.$$

On conclut avec le Lemme (2.4).

□

<u>Lemme 2.6</u>. <u>Pour tout</u> $\eta>0$, <u>il existe</u> c_η^7 <u>tel que</u> $\forall\, v \in \bigcup_{t\in[o,1]} E_t$

$$\int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))v_-(x)dx \leq \eta\,|\nabla v|_{L^2}^2 + c_\eta^7$$

<u>Démonstration</u>. Par l'inégalité de Hölder :

$$\int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))v_-(x)dx \leq |g_t(.,v(.),\delta_\varepsilon(v)(.))|_{L^{q'}}|v_-|_{L^{2p}}$$

Ainsi par le Lemme (2.5) :

$$\int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))v_-(x)dx \leq \eta\,|v_-|_{L^{2p}}^2 + c_\eta^8$$

et par le Lemme (2.3) :

$$\int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x))v_-(x)dx \leq \eta\,|\nabla v|_{L^2}^2 + c_\eta^7$$

□

<u>Lemme 2.7</u>. <u>Pour tout</u> $\eta>0$, <u>il existe</u> c_η^9 <u>tel que</u> $\forall\, v \in \bigcup_{t\in[o,1]} E_t$

$$Iv_{|\partial\Omega} \leq \eta\,|\nabla v|_{L^2}^2 + c_\eta^9$$

Démonstration. Soit $v \in E_t$, il suffit de démontrer le lemme pour $\gamma = v_{|\partial\Omega} > 0$. Reprenons la relation (2.6) du lemme (2.4), il vient $\forall (x,u,\delta) \in \bar{\Omega} \times]-\infty,o] \times [o,|\Omega|]$, $\forall\, t \in [o,1]$

$$|g_t(x,u,\delta)| \leq \eta\, u_-^p + c_\eta^{10}$$

Alors presque partout dans $\{v<0\}$:

$$\gamma^p\, g_t(x,v(x),\delta_\varepsilon(v)(x)) \leq \eta\, \gamma^p\, v_-^p(x) + \gamma^p\, c_\eta^{10}$$

On trouve après intégration que :

$$\gamma^p \int_{v<0} g_t(x,v(x),\delta_\varepsilon(v)(x))dx \leq \eta \int_{v<0} |\gamma + v_-|^{2p} dx + c_\eta^{11}$$

Sachant que $e_t(v) = 0$:

$$\gamma^p I \leq \eta \int_{v<0} |v - \gamma^{2p}| + c_\eta^{11} \leq \eta \int_\Omega |v - \gamma|^{2p} dx + c_\eta^{12}$$

donc, par un théorème d'injection de Sobolev :

$$\gamma^p I \leq \eta\, |\nabla v|_{L^2}^{2p} + c_\eta^{13}$$

d'où le Lemme. □

Lemme 2.8. Il existe une constante $c_5>0$ telle que

$\forall\, v \in \bigcup_{t\in[o,1]} E_t$

(2.7) $$|\nabla v|_{L^2} \leq c_5$$

(2.8) $$|v_{|\partial\Omega}| \leq c_5 \quad .$$

Démonstration. Soit $v \in E_t$ alors $j_t(v) = 0$ entraîne que $|\nabla v|_{L^2}^2 = I v_{|\partial\Omega} + \int_\Omega g_t(x,v(x),\delta_\varepsilon(v)(x)v_-(x)dx$. En utilisant les Lemmes (2.6) et (2.7), on a la relation (2.7) avec $\eta = \frac{1}{4}$. Par ailleurs, si $v_{|\partial\Omega} \geq 0$, alors la relation (2.8) est une conséquence immédiate du Lemme (2.7) et de la relation (2.7). Si $v_{|\partial\Omega} < 0$, alors on écrit que :

$$|v_{|\partial\Omega}| = |v_{-|\partial\Omega}| \leq c(|v_-|_{L^2} + |\nabla v_-|_{L^2}) \leq c(|v_-|_{L^2} + |\nabla v|_{L^2})$$

Par le Lemme (2.3) : $|v_-|_{L^2} \leq c'|v_-|_{L^{2p}} \leq c''(1 + |\nabla v|_{L^2})$, on déduit le lemme en appliquant la relation (2.7).

□

Démonstration de la Proposition 2.1. Avec la relation (2.7) du Lemme (2.8), il nous suffit de prouver qu'il existe $c_6 > 0$ t.q $\forall\, v \in \bigcup_{t\in[o,1]} E_t$:

$|v|_{L^2} \leq c_6$. Nous déduisons du lemme de Poincaré que :

$$|v|_{L^2} \leq c(|v_{|\partial\Omega}| + |\nabla v|_{L^2}) .$$

En utilisant le Lemme (2.8), on a le résultat.

□

2.3. <u>Calcul d'un degré topologique. Existence de solutions pour le problème</u> $(1)_\varepsilon$.

<u>Proposition 2.2</u>. <u>Pour tout</u> $t \in [o,1]$, <u>le degré topologique</u> $d(I - S_\varepsilon(t,.),B_k,o)$ <u>est bien défini et vaut</u> -1 (k <u>étant la même constante que celle de la Proposition</u> (2.1) <u>et</u> B_k <u>la boule ouverte de</u> W <u>centrée en</u> 0 <u>de rayon</u> k).

Avant de démontrer cette proposition, rappelons deux lemmes dus à H. Berestycki-H. Brézis [2] .

<u>Lemme 2.9</u>. Si on note λ_1, λ_2 $(\lambda_1 < \lambda_2)$ les deux premières valeurs caractéristiques de L, et μ_1 la première valeur propre du problème de Dirichlet homogène, alors

i) $\lambda_1 = 1$ est simple et est associé à des fonctions propres qui sont des constantes.

ii) $\lambda_2 > 1 + \mu_1$.

<u>Lemme 2.10</u>. Pour $\tau \in [o,1]$, on considère l'application complètement continue :

ψ_τ : W $\rightarrow$ W définie par

$$\psi_\tau(u) = (1 + \mu_1)Lu - \tau\mu_1 \; L(u_+) + \phi_o$$

et soit v_1 l'unique fonction propre du problème de Dirichlet satisfaisant à $\int_\Omega \frac{\partial v_1}{\partial n} d\sigma = I$

i.e.
$$\begin{cases} -\Delta v_1 = \mu_1 v_1 \text{ dans } \Omega \\ v_1 = 0 \text{ sur } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial v_1}{\partial n} d\sigma = I \end{cases}$$

alors, pour tout $\tau \in [o,1]$, v_1 est l'unique solution de l'équation $u - \psi_\tau(u) = 0$.

Démonstration de la Proposition 2.2. Grâce au Lemme (2.1), l'application qui à $v \in W$ associe $S_\varepsilon(t,v) = L(v - g_t(.,v(.),\delta_\varepsilon(v)(.)) + \phi_o$ est complètement continue de W dans lui-même. Par ailleurs, $\{u = S_\varepsilon(t,u)\} \subset E_t \subset B_k$.

En effet

$$u = S_\varepsilon(t,u) \Leftrightarrow \begin{cases} (2.1)' \quad -\Delta u + g_t(x,u(x),\delta_\varepsilon(u)(x)) = 0 \text{ dans } \Omega \\ (2.2)' \quad u = \text{constante sur } \partial\Omega \\ (2.3)' \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma = I \end{cases}$$

En intégrant la relation (2.1)', on trouve $e_t(u) = 0$. En multipliant par u et en intégrant sur Ω , la relation (2.1)' donne $j_t(u) = 0$, ainsi $u \in E_t$. Par la Proposition (2.1), $u \in B_k$: $u - S_\varepsilon(t,u) \neq 0$ sur ∂B_k .

Par invariance homotopique, $d(I - S_\varepsilon(t,.),B_k,o) = d(I - S_\varepsilon(o,.),B_k,o)$ mais $S_\varepsilon(o,v) = L(v - \mu_1 \; v_-) + \phi_o = \psi_1(v)$ (ψ_1 étant l'opérateur introduit au Lemme (2.10)). Ce Lemme (2.10) assure alors que $d(I - S_\varepsilon(o,.),B_k,o) = i(I - \psi_1,v_1,o)$. De nouveau par invariance homotopique, $i(I - \psi_1, v_1, o) = i(I - \psi_o, v_1, o)$. Par le théorème du calcul d'index par linéarisation [11], $i(I - \psi_o, v_1, o) = (-1)^\beta$ où β est la somme des multiplicités des valeurs caractéristiques réelles $\mu \in [o,1]$ de $\psi'_o = (1+\mu_1)L$. Le Lemme (2.9) montre que $\beta = 1$. Ainsi nous avons obtenu : $d(I - S_\varepsilon(t,.), B_k, o) = (-1)^\beta = -1$.

□

<u>Proposition 2.3. Il existe</u> u_ε <u>solution du problème</u> $(1)_\varepsilon$. <u>De plus</u> u_ε <u>demeure dans un borné de</u> $W^{2,s}(\Omega)$ <u>pour tout</u> $s \in [1,+\infty[$ <u>lorsque</u> ε <u>tend vers</u> 0.

<u>Démonstration</u>. Puisque $d(I - S_\varepsilon(1,.), B_k, o) = -1 \neq 0$, il existe $u_\varepsilon \in B_k$ tel que $u_\varepsilon = S_\varepsilon(1,u_\varepsilon)$: u_ε solution du problème $(1)_\varepsilon$. Puisque u_ε reste dans un borné de $H^1(\Omega) \hookrightarrow L^{2p}(\Omega)$, la relation (2.1) implique que $g(.,u_\varepsilon(.),\delta_\varepsilon(u_\varepsilon)(.))$ est dans un borné de $L^2(\Omega)$. D'après S. Agmon-Douglis-L. Niremberg [1], u_ε est dans un borné de $H^2(\Omega)$. Par l'injection continue de Sobolev : $W^{2,2}(\Omega) \hookrightarrow L^{q_1}(\Omega)$, q_1 étant tel que : si $4 \geq N$: $1 \leq q_1 < +\infty$ alors u_ε est dans un borné de $L^{q_1}(\Omega)$. Par la relation (2.1), $g(.,u_\varepsilon(.),\delta_\varepsilon(u_\varepsilon)(.))$ est dans un borné de $L^s(\Omega)$. D'après [1], u_ε est dans un borné de $W^{2,s}(\Omega)$ $\forall$ $s \in [1,+\infty[$. Si $N>4$, on prend q_1 t.q $\frac{1}{q_1} = \frac{1}{2} - \frac{2}{N}$. On reprend le même raisonnement pour montrer que u_ε est dans un borné de $W^{2,\frac{q_1}{p}}(\Omega) \hookrightarrow L^{q_2}(\Omega)$ avec $\frac{1}{q_2} = \frac{p}{q_1} - \frac{2}{N}$ si $2q_1 < pN$ et $1 \leq q_2 < +\infty$ sinon. Les q_j étant liés de façon générale par $\frac{1}{q_j} = \frac{p}{q_{j-1}} - \frac{2}{N}$: $\exists$ k t.q $2q_k \geq pN$.

3. <u>PASSAGE A LA LIMITE</u>.

On désigne par u_ε une solution de $(1)_\varepsilon$ vérifiant, lorsque ε tend vers 0 :

a) $u_\varepsilon \to u$ dans $W^{2,s}$-faible $(s>N)$

b) $u_\varepsilon \to u$ dans $C^1(\bar{\Omega})$-fort

c) $\int_\Omega h_{-\varepsilon}(-u_\varepsilon)(y)dy \to \alpha$ dans $\mathbb{R}$

d) $g(.,u\ (.),\delta_\varepsilon(u_\varepsilon)(.)) \to G$ dans L^∞-faible * .

Le passage à la limite dans $(1)_\varepsilon$ se fait sans peine et donne :

$$(1)' \quad \begin{cases} (1\text{-}1)' & -\Delta u + G(x) = 0 \text{ dans } \Omega \\ (1\text{-}2)' & u = \text{constante sur } \partial\Omega \\ (1\text{-}3)' & \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I \end{cases}$$

Nous allons préciser la fonction G , grâce au

Lemme 2.11.

i) $\alpha \in [|u<0|, |u\leq 0|]$ (*)

ii) G vérifie les relations suivantes :

ii-1) $G(x) = 0$ dans $\{u\geq 0\}$ p.p

ii-2) $G(x) \in g(x,u(x),\delta(u)(x))$ p.p dans $\{u<0\}$

avec $\delta(u)(x) = [\underline{\delta}(u)(x), \overline{\delta}(u)(x)]$

Démonstration.

i) Pour presque tout x dans $\{u<0\}$ (resp. dans $\{u>0\}$) $\lim\limits_{\varepsilon\to 0} \underline{h}_\varepsilon(-u_\varepsilon)(x) = 1$ (resp. 0). Le théorème de la convergence dominée entraîne $\lim\limits_{\varepsilon\to 0} \displaystyle\int_{u<0} \underline{h}_\varepsilon(-u_\varepsilon)(x)dx = |u<0|$ (resp. $\lim\limits_{\varepsilon\to 0} \displaystyle\int_{u>0} \underline{h}_\varepsilon(-u_\varepsilon)(y)dy = 0$) sachant que

$$\int_{u<0} \underline{h}_\varepsilon(-u_\varepsilon)(y)dy \leq \int_\Omega \underline{h}_\varepsilon(-u_\varepsilon)(y)dy \leq |u\leq 0| + \int_{u>0} \underline{h}_\varepsilon(-u_\varepsilon)(y)dy$$

le résultat en découle.

ii) Posons $\beta_\varepsilon(v)(x) = \displaystyle\int_\Omega \overline{h}_\varepsilon(v(x) - v(y))dy$

$$\overline{\beta}(v)(x) = \int_\Omega \overline{h}(v(x) - v(y))dy = |v\leq v(x)|$$

($\overline{h} \equiv$ fonction de Heaveside prenant la valeur 1 en 0).

Soit $\varepsilon_0>0$ fixé mais destiné à tendre vers 0, alors pour tout $0<\varepsilon\leq\varepsilon_0$:

(2.9) $\quad \overline{h} \leq \overline{h}_\varepsilon \leq \overline{h}_{\varepsilon_0}$, ceci entraîne que pour tout x dans Ω

(*) Lorsque $E \subset \mathbb{R}^N$ mesurable, on note $|E|$ sa mesure

(2.10) $\qquad \overline{\beta}(u_\varepsilon)(x) \leqq \beta_\varepsilon(u_\varepsilon)(x) \leqq \beta_{\varepsilon_0}(u_\varepsilon)(x)$

En outre, la convergence uniforme de u_ε vers u implique qu'il existe $0<\varepsilon_1 \leqq \varepsilon_0$ t.q $\forall\ \varepsilon \leqq \varepsilon_1$ on ait : $|u_\varepsilon(x) - u(x)| \leqq \varepsilon_0 (\forall\ x \in \Omega)$. De ce fait, nous obtenons que, $\forall\ x \in \Omega$

(2.11) $\qquad |u \leqq u(x) - 2\varepsilon_0| \leqq \overline{\beta}(u_\varepsilon)(x)$

(2.12) $\qquad |u_\varepsilon \leqq u_\varepsilon(x) + \varepsilon_0| \leqq |u \leqq u(x) + 3\varepsilon_0|$

(2.13) $\qquad u(x) - \varepsilon_0 \leqq u_\varepsilon(x) \leqq u(x) + \varepsilon_0$

$(\forall\ t)\ \overline{h}_{\varepsilon_0}(t) \leqq \overline{h}(t+\varepsilon_0)$ ainsi $\beta_{\varepsilon_0}(u_\varepsilon)(x) \leqq |u_\varepsilon \leqq u_\varepsilon(x) + \varepsilon_0| \leqq |u \leqq u(x) + 3\varepsilon_0|$.
Cette dernière relation, (2.10) et (2.11) impliquent :

$$|u \leqq u(x) - 2\varepsilon_0| \leqq \beta_\varepsilon(u_\varepsilon)(x) \leqq |u \leqq u(x) + 3\varepsilon_0|$$

ainsi $\forall\ x \in \Omega$

(2.14) $$\left[\int_\Omega \underline{h}_\varepsilon(-u_\varepsilon)dy - |u \leqq u(x) + 3\varepsilon_0|\right]_+ \leqq \delta_\varepsilon(u_\varepsilon)(x) \leqq$$
$$\leqq \left[\int_\Omega \underline{h}_\varepsilon(-u_\varepsilon)dy - |u \leqq u(x) - 2\varepsilon_0|\right]_+$$

Posons :

$$\underline{\delta}(x,\varepsilon_0) = [|u \leqq 0| - |u \leqq u(x) + 3\varepsilon_0| - \varepsilon_0]_+$$
$$\overline{\delta}(x,\varepsilon_0) = [|u \leqq 0| - |u \leqq u(x) - 2\varepsilon_0| + \varepsilon_0]_+$$

Quitte à remplacer ε_1 par $\varepsilon_2 \leqq \varepsilon_1$, nous avons, grâce à (2.14) et à la relation i), que

(2.15) $\qquad \underline{\delta}(x,\varepsilon_0) \leqq \delta_\varepsilon(u_\varepsilon)(x) \leqq \overline{\delta}(x,\varepsilon_0) \quad (\forall\ x \in \Omega)\ (\forall\ \varepsilon \leqq \varepsilon_2)$

Soit

$$U(x,\varepsilon_0) = [u(x) - \varepsilon_0, u(x) + \varepsilon_0]$$

et

$$D(x,\varepsilon_0) = [\underline{\delta}(x,\varepsilon_0), \overline{\delta}(x,\varepsilon_0)] \ .$$

Par les relations (2.13) et (2.15), nous avons pour tout x

(2.16) $\qquad \underline{\chi}_{\varepsilon_0}(x) \leqq g(x,u_\varepsilon(x),\delta_\varepsilon(u\ )(x)) \leqq \overline{\chi}_{\varepsilon_0}(x)$

où

$$\underline{\chi}_{\varepsilon_o}(x) = \underset{(u,\delta) \in U(x,\varepsilon_o) \times D(x,\varepsilon_o)}{\text{Min}} g(x,u,\delta)$$

$$\overline{\chi}_{\varepsilon_o}(x) = \underset{(u,\delta) \in U(x,\varepsilon_o) \times D(x,\varepsilon_o)}{\text{Max}} g(x,u,\delta) \quad (^*)$$

Mais l'ensemble $\{\psi \in L^\infty(\Omega)$ tel que $\psi \in [\underline{\chi}_{\varepsilon_o}\ \overline{\chi}_{\varepsilon_o}]$ p.p$\}$ est un convexe fermé de $L^\infty(\Omega)$ donc faiblement-$*$ fermé, on déduit de (2.16) que

(2.17) $\qquad \underline{\chi}_{\varepsilon_o}(x) \leqq G(x) \leqq \overline{\chi}_{\varepsilon_o}(x) \quad$ p.p

Par ailleurs, il existe $(u(x,\varepsilon_o),d(x,\varepsilon_o)) \in U(x,\varepsilon_o) \times D(x,\varepsilon_o)$ tel que

$$\underline{\chi}_{\varepsilon_o}(x) = g(x,u(x,\varepsilon_o),d(x,\varepsilon_o)).$$

Or $u(x,\varepsilon_o) \underset{\varepsilon_o \downarrow 0}{\rightarrow} u(x)$ et quitte à extraire une sous-suite, on peut supposer que : $d(x,\varepsilon_o) \underset{\varepsilon_o \downarrow 0}{\rightarrow} \delta_m(x) \in [\underline{\delta}(u)(x),\overline{\delta}(u)(x)]$. Par continuité (H1) ,

$\underline{\chi}_{\varepsilon_o}(x) \underset{\varepsilon_o \downarrow 0}{\rightarrow} g(x,u(x),\delta_m(x))$. De la même façon, nous obtenons que

$$\overline{\chi}_{\varepsilon_o}(x) \underset{\varepsilon_o \downarrow 0}{\rightarrow} g(x,u(x),\delta_M(x)) : \delta_M(x) \in [\underline{\delta}(u)(x),\overline{\delta}(u)(x)]$$

La relation (2.17) et ces deux dernières relations impliquent :

$$G(x) \in g(x,u(x),\delta(u)(x)) \text{ p.p}$$

Il est évident que dans $\{u>0\}$, $\overline{\delta}(u)(x) = 0 = \underline{\delta}(u)(x)$ donc par (H2), $G(x) = 0$. Dans $\{u=0\}$, on a p.p $G(x) = 0$: en effet, si mes $\{u=0\} \neq 0$ d'après Stampacchia, $\Delta u = 0$ dans $\{u=0\}$, l'équation (1.1)' donne $G(x) = 0$

Nous pouvons donc annoncer le résultat principal :

<u>Théorème 1</u>. <u>Il existe</u> $u \in C^{1,\alpha}(\overline{\Omega}) \cap W^{2,s}(\Omega)$ <u>avec</u> $0 \leqq \alpha < 1$ <u>et</u> $1 \leqq s < +\infty$ <u>solution du problème</u> (1) <u>i.e.</u> :

$$\begin{cases} \Delta u \in g(x,u(x),\delta(u)(x)) \ \underline{\text{dans}}\ \{u<0\} & (1.1) \\ \Delta u = 0 \ \underline{\text{dans}}\ \{u \geqq 0\} & (1.2) \\ u = \underline{\text{constante sur}}\ \partial\Omega & (1.3) \\ \int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = I\ . & (1.4) \end{cases}$$

□

(*) $\overline{\chi}_{\varepsilon_o}$ et $\underline{\chi}_{\varepsilon_o}$ sont mesurables (cf. Remarque ci-après).

<u>Remarque 1</u>. Si les ensembles $\{u=t\}$ sont de mesure nulle pour tout $t \leq 0$ alors u (solution de (1)) est solution du problème (0). Cette hypothèse sera certainement vérifiée pour $t<0$, si par exemple $0 \notin g(\overline{\Omega} \times]-\infty, 0[\times [0,|\Omega|])$ (puisque d'après Stampacchia, $\Delta u = 0$ p.p. dans $\{u=t\}$) .

<u>Remarque 2</u>. Les fonctions $\underline{\chi}_{\varepsilon_o}$ et $\overline{\chi}_{\varepsilon_o}$ sont mesurables. Posons

$$U_1(x,\varepsilon_o) = U(x,\varepsilon_o) \cap \mathbb{Q}$$

$$D_1(x,\varepsilon_o) = D(x,\varepsilon_o) \cap \mathbb{Q}$$

On a

$$U_1(x,\varepsilon_o) \subset I = [\underset{\overline{\Omega}}{\mathrm{Min}}\, u - \varepsilon_o, \underset{\overline{\Omega}}{\mathrm{Max}}\, u + \varepsilon_o] \cap \mathbb{Q}$$

$$D_1(x,\varepsilon_o) \subset J = [o, |\Omega|] \cap \mathbb{Q}$$

Soit

$$a = \underset{\overline{\Omega} \times \overline{I} \times \overline{J}}{\mathrm{Max}} g(x,u,\delta) + 1$$

Puisque $I \times J$ est dénombrable, soit φ une bijection de $\mathbb{N}$ dans $I \times J$: $n \in \mathbb{N}$, $\varphi(n) \in I \times J$. Pour chaque $n \in \mathbb{N}$, associons le sous-ensemble mesurable de Ω suivant :

(2.18) $$M_n = \{x \in \Omega \;/\; \varphi(n) \in U(x,\varepsilon_o) \times D(x,\varepsilon_o)\} =$$
$$= \{x \in \Omega \;/\; \varphi(n) \in U_1(x,\varepsilon_o) \times D_1(x,\varepsilon_o)\}$$

Soit χ_n la fonction mesurable définie sur Ω par :

$$\chi_n(x) = \begin{cases} g(x, \varphi(n)) & \text{si } x \in M_n \\ a & \text{si } x \notin M_n \;. \end{cases}$$

On sait que $\chi(x) = \underset{n \in \mathbb{N}}{\mathrm{Inf}}\, \chi_n(x)$ est mesurable. Montrons que $\chi = \underline{\chi}_{\varepsilon_o}$. Par continuité de g, $\underline{\chi}_{\varepsilon_o}(x) = \mathrm{Inf}\,\{g(x,u,\delta) : (u,\delta) \in U_1(x,\varepsilon_o) \times D_1(x,\varepsilon_o)\}$.
Par définition de φ, on peut écrire

$$\underline{\chi}_{\varepsilon_o}(x) = \mathrm{Inf}\,\{g(x,\varphi(n))\;;\; n \in \mathbb{N},\, \varphi(n) \in U_1(x,\varepsilon_o) \times D_1(x,\varepsilon_o)\}$$
$$= \mathrm{Inf}\,\{g(x,\varphi(n))\;;\; n \in \mathbb{N} \;/\; x \in M_n\}$$
$$= \chi(x).$$

On ferait une démonstration analogue pour établir la mesurabilité de $\overline{\chi}_{\varepsilon_o}$

Remarque 3. On peut résoudre des problèmes du type M2 où l'équation (1.1) du problème (1) est remplacée par $\Delta u \in g(\delta(u)(x))$ où g est une fonction continue sur $[o,|\Omega|]$ ne s'annulant qu'en 0 . Pour ce faire, on introduit : $g_\mu(x,u,\delta) = g(\delta) + \mu u_-$ $(\mu>0)$. On résoud alors l'équation :

$$
(3)\begin{cases}
(3.1) & \Delta u \in g_\mu(x,u(x),\delta(u)(x)) \quad \text{dans } \{u<0\} \\
(3.2) & \Delta u = 0 \\
(3.3) & u = \text{constante (inconnue) sur } \partial\Omega \\
(3.4) & \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I > 0 .
\end{cases}
$$

Par le Théorème 1, il existe une solution u_μ de ce problème. On peut montrer sans difficulté que u_μ demeure dans un borné de $H^1(\Omega)$ (donc dans $W^{2,s}(\Omega)$ $\forall$ s fini) lorsque μ tend vers 0 . En définissant $u \in W^{2,s}(\Omega)$ t.q $u_\mu \underset{\mu\to 0}{\to} u$ dans $W^{2,s}(\Omega)$ faible (s>1). On passe à la limite, pour aboutir à la solution du modèle (M2).

Remarque 4. Les mêmes résultats peuvent être obtenus en remplaçant l'opérateur $-\Delta$ de (1.1) par un opérateur elliptique du second ordre auto-adjoint du type

$$Bu = -\sum_{i,j}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) \quad \text{avec} \quad a_{ij} = a_{ji} \in C^2(\overline{\Omega})$$

et

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\,\xi_j \geq \nu|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N ,\ (\forall\, x \in \overline{\Omega}) ,\ (\nu>0) ,$$

ou encore par l'opérateur utilisé dans les plasmas [13] donné en coordonnées cylindriques par

$$\mathscr{L}u = -\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial u}{\partial r}\right) - \frac{1}{r}\cdot\frac{\partial^2 u}{\partial r^2}$$

De même, comme dans [8], on peut résoudre un problème analogue en remplaçant les opérateurs $\underline{\delta}$ et $\overline{\delta}$ par les opérateurs $\underline{\beta}$, $\overline{\beta}$: $\underline{\beta}(u)(x) = |u<u(x)|$ et $\overline{\beta}(u)(x) = |u\leq u(x)|$.

□

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CHAPITRE IV

Réarrangement relatif dans les équations elliptiques quasilinéaires avec un second membre distribution : application à un théorème d'existence et de régularité.

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ÉQUATIONS AUX DÉRIVÉES PARTIELLES. — *Résultats de régularité et d'existence pour certaines équations elliptiques quasi linéaires.* Note de **Jean-Michel Rakotoson,** présentée par Jacques-Louis Lions.

En utilisant les techniques du réarrangement relatif ([3], [4]), on démontre des résultats de régularité et d'existence pour des équations elliptiques quasi linéaires avec des hypothèses minimales sur les données.

PARTIAL DIFFERENTIAL EQUATIONS. — Regularity and existence results for some elliptic quasi-linear equations.

Using the techniques of the relative rearrangement (see [3], [4]), *we prove a regularity and an existence results for some elliptic quasi-linear equations with minimal assumptions on the data.*

In this Note, we are interested in the following problem:

($\mathscr{P}$) Find $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\quad A u + F(u, \nabla u) = T$ in Ω,

where Ω *is a bounded open set of* $\mathbb{R}^N$, A *is an operator of Leray-Lions type* [5]. *All the assumptions on* A, F *and* T *are given by* (H1) *to* (H4).

One of the main result that we obtain is:

THEOREM 1. — Assume (H1) to (H4), then there exists at least one solution u of the problem ($\mathscr{P}$).

Moreover, if $r > p'$, $f_0 \in L^{r/p'}(\Omega)$ *and* $a_0 \in L^r(\Omega)$, *then the solution u satisfies the Hölder condition inside* Ω, *for some exponent* $\alpha > 0$.

It is crucial to get an a priori *estimate of the* L^∞*-norm of the solution u of* ($\mathscr{P}$). *So, we use the techniques of the relative rearrangement* ([3], [4], [6], [7]) *and the isoperimetric inequality of De Giorgi-Fleming Rischel to get such estimate* (see *Lemma* 1).

The proof of the Hölder continuity is essentially based on the fact that the solution u belongs to a class of function $\mathscr{B}_p(\Omega, M, \gamma, \delta, 1/(p-1)r)$ *introduced by Ladyzen'skaja* [2].

If the domain Ω *is smooth enough, then the solution* $u \in C^{0,\alpha}(\bar{\Omega})$.

The details of the proofs will be given in [12].

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0. INTRODUCTION. — On se place sur un ouvert borné Ω de $\mathbb{R}^N (N \geqq 1)$ et l'on considère le problème ($\mathscr{P}$), trouver $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $1 < p < +\infty$ t. q.

$$A u + F(u, \nabla u) = T \quad \text{dans } \Omega,$$

où A est un opérateur du type Leray-Lions [5] pouvant s'écrire

$$A u = -\sum_{i=1}^{N} (\partial/\partial x_i)\, a_i(x, u(x), \nabla u(x)) \qquad \text{pour} \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega);$$

F est une application non linéaire de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$ à croissance d'ordre p au plus par rapport au gradient et T appartient à

$$W^{-1,r}(\Omega) \qquad \text{avec} \quad r > N/(p-1), \qquad r \geqq p/(p-1) = p'.$$

Les fonctions a_i sont de Caratheodory de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$, c'est-à-dire qu'elles satisfont aux deux conditions suivantes

$$\forall \eta \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \qquad x \to a_i(x, \eta, \xi) \text{ est mesurable,}$$

$$\text{p. p. en } x \in \Omega, \qquad (\eta, \xi) \to a_i(x, \eta, \xi) \text{ est continue.}$$

En outre, on suppose que les a_i vérifient les hypothèses suivantes :

(H1) Il existe deux fonctions réelles continues $\nu_1 > 0$ et $\nu_2 \geqq 0$ définies sur $\mathbb{R}_+$ et une fonction positive k de $L^{r/p'}(\Omega)$ et une constante $c_0 \geqq 0$ telles que :

presque partout en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\xi_i \geqq \nu_1(|\eta|)|\xi|^p - \nu_2(|\eta|)k(x)^{1/p'}|\xi|.$$

avec $[\nu_2(|\eta|)]^{p'/p} \leqq \nu(|\eta|) + c_0$ et $\nu(|\eta|) = \int_0^{|\eta|} \nu_1(t)^{p'/p}\,dt . \nu(+\infty) = +\infty$.

(H2) Il existe une fonction croissante a de $\mathbb{R}_+$ dans $\mathbb{R}_+$ et un élément a_0 positif de $L^{p'}(\Omega)$ tels que :

presque partout en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|a_i(x, \eta, \xi)| \leqq a(|\eta|)\,[|\xi|^{p-1} + a_0(x)].$$

(H3) Presque partout en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$

$$\sum_{i=1}^{N} (a_i(x, \eta, \xi) - a_i(x, \eta, \xi'))(\xi_i - \xi'_i) > 0.$$

(H4) La fonction F est de Carathéodory et vérifie :

presque partout en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

(i) $\eta\, F(x, \eta, \xi) \geqq 0$;

(ii) $|F(x, \eta, \xi)| \leqq f(|\eta|)\,[f_0(x) + |\xi|^p]$,

où f_0 est une fonction positive de $L^1(\Omega)$ et f une fonction croissante de $\mathbb{R}_+$ dans $\mathbb{R}_+$.

L'un des principaux résultats de notre étude est le :

THÉORÈME 1. — *Sous les hypothèses* (H1) *à* (H4), *le problème* ($\mathscr{P}$) *admet au moins une solution. De plus, si* $r > p'$, $f_0 \in L^{r/p'}(\Omega)$ *et* a_0 *dans* $L^r(\Omega)$, *alors la solution u est* α-*höldérienne à l'intérieur de* Ω *pour un certain* $\alpha > 0$.

Notre résultat améliore ceux donnés antérieurement (*cf.* [1], [9], [10], [11]...) par plusieurs des propriétés suivantes :

L'opérateur A [*voir* (H1)] peut dégénérer avec la solution, i. e. on peut avoir $\lim_{\eta \to +\infty} \nu_1(\eta) = 0$, alors que les travaux précédents supposent $\nu_1(\eta) = \nu_0 = \text{Cte}$. On peut tenir compte des « termes non coercifs » dans le terme contenant ν_2.

Notons aussi que les travaux les plus récents ([9], [11]) supposent l'existence d'une sous- et sur-solutions, ce qui n'est pas nécessaire ici.

Par ailleurs, dans le théorème 2, nous donnons une démonstration directe de la régularité $L^\infty(\Omega)$ des équations du type ($\mathscr{P}$). Les majorations que nous apportons sont explicites. Nos conditions sur T, i. e. $r > N/(p-1)$, coïncident dans le cas linéaire avec la condition nécessaire et suffisante due à Stampacchia [13], pour avoir des solutions bornées. Signalons enfin un résultat récent [8] dans le cas $p = 2$ sur des problèmes quasi linéaires, qui donne aussi la même condition avec une méthode différente de la nôtre mais toujours dans le cas où $\nu_1(\eta)$ est constante.

Notre démonstration s'appuie sur des estimations *a priori* des solutions de ($\mathscr{P}$) obtenues à l'aide du réarrangement relatif qui fait l'objet de la Note [6]. De plus amples détails sur le réarrangement sont donnés dans [3], [4], [6] et [7].

1. UN THÉORÈME DE RÉGULARITÉ $L^\infty(\Omega)$. — On se donne la famille de fonctions lipschitziennes, notées $S_{\theta, h}$, associée à deux nombres positifs $\theta > 0$, $h > 0$ et définies, pour $\tau \in \mathbb{R}$, par $S_{\theta, h}(\tau) = \text{sign}\ \tau$ si $|\tau| \geqq \theta + h$, 0 si $|\tau| \leqq \theta$, affine ailleurs.

On se donne $u \in W_0^{1,p}(\Omega)$ (non nulle) et l'on note $\bar{V} = S_{\theta, h}(u)$. On suppose que u satisfait à : $\forall \theta \in]0$, sup ess $|u|[$, $\forall h \in]0$, sup ess $|u| - \theta[$; on a

$$(1.1) \qquad \sum_{i=1}^{N} \int_{\Omega} a_i(x, u(x), \nabla u(x)) . \frac{\partial \bar{V}}{\partial x_i} dx + \int_{\Omega} F(x, u(x), \nabla u(x)) \bar{V} \, dx = \langle T, \bar{V} \rangle,$$

où $\langle \quad \rangle$ désigne le crochet de dualité entre $W^{-1,p'}(\Omega)$ et $W_0^{1,p}(\Omega)$. On note

$$\gamma = \left[\int_{\Omega^*} \sigma^{((1/N)-1)((p-1)r/(p-1)r-1)} d\sigma \right]^{1-(1/(p-1)r)} \qquad \left(\text{on a } \gamma < +\infty, \text{ car } r > \frac{N}{p-1} \right),$$

$$\lambda_0 = \gamma (c_0 \| k \|_{L^{r/p'}(\Omega)}^{1/p} + \| T \|_{W^{-1,r}(\Omega)}^{p'/p}),$$

$$\lambda_1 = \exp(\lambda_2 \gamma \| k \|_{L^{r/p'}(\Omega)}^{1/p}), \qquad \lambda_2 = 2^*/N \, \alpha_N^{1/N} \quad \text{avec } 2^* = 2^{((p'/p)-1)_+},$$

α_N = mesure de la boule unité de $\mathbb{R}^N$.

Nous avons le théorème suivant qui est indépendant du théorème 1 (et est utilisé dans la preuve de celui-ci) :

THÉORÈME 2. — *On suppose* (H1), (H2) *et* (H4) *et l'on considère* $u \in W_0^{1,p}(\Omega)$ *satisfaisant* (1.1). *Alors :*

(i) $u \in L^\infty(\Omega)$ *et la norme de* u *dans* $L^\infty(\Omega)$ *est telle que*

$$\nu(\| u \|_\infty) \leqq \lambda_0 \lambda_1 \lambda_2 \quad \Leftrightarrow \quad \| u \|_\infty \leqq \nu^{-1}(\lambda_0 \lambda_1 \lambda_2) = M.$$

On note

$$\alpha = \min_{0 \leqq \theta \leqq M} \nu_1(\theta)^{p'/p}, \qquad \beta = \max_{0 \leqq \theta \leqq M} \nu_2(\theta)^{p'/p}.$$

Alors nous avons aussi l'estimation du gradient dans $L^p(\Omega)$

$$\text{(ii)} \qquad \| \nabla u \|_{L^p(\Omega)} \leqq \frac{2^*}{\alpha} (\beta \| k \|_{L^1(\Omega)}^{1/p} + |\Omega|^{(r-p')/pr} \| T \|_{W^{-1,r}(\Omega)}^{p'/p}) = M_0. \quad \square$$

Pour u solution de (1.1), on note $v = |u|$ et l'on considère $(g_i)_{i=1, \ldots, N}$, dans $L^r(\Omega)$ t. q.

$$T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i} \quad [\text{au sens de } \mathcal{D}'(\Omega)] \qquad \text{et} \qquad \| T \|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \| g_i \|_{L^r(\Omega)}.$$

On pose $g = \left(\sum_{i=1}^{N} g_i^2 \right)^{p'/2}$ et on note g_{*v} son réarrangement relatif par rapport à v et k_{*v} celui de k. Par la propriété 0 de [6], g_{*v} et k_{*v} sont des fonctions positives. Définissons maintenant deux fonctions K et b par :

$$\text{pour } (t, s) \in \bar{\Omega}^* \times \bar{\Omega}^*,$$

$$K(t, s) = \exp\left(\lambda_2 \int_t^s \sigma^{(1/N)-1} k_{*v}(\sigma)^{1/p} d\sigma \right), \qquad b(s) = s^{(1/N)-1} (c_0 k_{*v}(s)^{1/p} + g_{*v}(s)^{1/p}.$$

Le théorème 0 de [6] assure que K et b ont un sens. Plus précisément, nous avons

$$K \in L^\infty(\Omega^* \times \Omega^*), \quad b \in L^1(\Omega^*) \qquad \text{et} \qquad \| b \|_{L^1(\Omega^*)} \leqq \lambda_0, \quad \| K \|_{L^\infty(\Omega^* \times \Omega^*)} \leqq \lambda_1.$$

On démontre alors le lemme suivant qui implique la partie (i) du théorème 2.

LEMME 1. — *Sous les mêmes hypothèses que le théorème* 2, *on a*

$$\forall s \in \bar{\Omega}^*, \quad v_*(s) \leqq \nu^{-1}\left(\lambda_2 \int_s^{|\Omega|} K(s, \sigma) b(\sigma) d\sigma \right). \quad \square$$

Le lemme fondamental qui conduit au résultat de régularité höldérienne est :

LEMME 3. — *Pour $\delta>0$, il existe une constante γ ne dépendant que de δ et des données a_i, F et T, telles que toute solution u de ($\mathscr{P}$) appartient à $\mathscr{B}_p(\Omega, M, \gamma, \delta, 1/(p-1)r)$ où M est le nombre donné au théorème 2.*

(1) On n'utilise que la mesure de Lebesgue dans $\mathbb{R}^N$, et pour tout sous-ensemble mesurable E, on note $|E|$ sa mesure. On note $\Omega^*=]0, |\Omega|[$.

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Réarrangement relatif dans les équations elliptiques quasi-linéaires avec un second membre distribution: Application à un théorème d'existence et de régularité

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In this paper we are concerned with quasilinear elliptic equations, that is $(\mathscr{P})$ $Au + F(u, \nabla u) = T$ in $\Omega \subset \mathbb{R}^N$, $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; where A is an operator of Leray-Lions type which is defined on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ $(1 < p < +\infty)$, F is a non-linear map having a suitable growth, and T is a distribution of $W^{-1,r}(\Omega)$, $r > N/(p-1)$ and $r \geqq p/(p-1)$. Using the techniques of the relative rearrangement (*Ann. Scuola Norm. Sup. Pisa Cl. Sci* (4), in press), we give a precise a priori estimate of the solution u of $(\mathscr{P})$ in L^∞-norm. These estimates allow us to prove an existence theorem for $(\mathscr{P})$ and to get the Hölder continuity of the solution u.

0. INTRODUCTION

Soit Ω un ouvert borné de $\mathbb{R}^N$ $(N \geqq 1)$, de frontière $\Gamma = \partial\Omega$ et soit $p \in]1, +\infty[$. On veut étudier le problème

$$\begin{cases} Au + F(u, \nabla u) = T, \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega); \end{cases} \tag{0.1}$$

où A est un opérateur de type Leray-Lions [13] de $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ dans $W^{-1,p'}(\Omega)$ avec $1/p + 1/p' = 1$.

L'application non linéaire F de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$ est à croissance d'ordre p au plus par rapport à ∇u et vérifie une "condition d'un seul côté" que nous préciserons ultérieurement.

Le second membre T est une distribution de $W^{-1,r}(\Omega)$ avec $r > N/(p-1)$ et $r \geqq p'$. Plus précisément, on suppose que l'opérateur A s'écrit: $Au = -\sum_{i=1}^N (\partial/\partial x_i)\, a_i(x, u(x), \nabla u(x))$ pour $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, où les a_i sont

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des fonctions de Cara théodory de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$, c'est-à-dire qu'elles satisfont aux deux conditions suivantes:

$$\forall(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N, \qquad x \to a_i(x, \eta, \xi) \text{ est mesurable;} \tag{0.2}$$

$$\text{p.p. en } x \in \Omega, \qquad (\eta, \xi) \to a_i(x, \eta, \xi) \text{ est continue.} \tag{0.3}$$

Il existe deux fonctions continues $\nu_1 > 0$ et $\nu_2 \geqq 0$ définies sur $\mathbb{R}_+$, une fonction positive $k \in L^{r/p'}(\Omega)$ et une constante $c_0 \geqq 0$, tels que p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\, \xi_i \geqq \nu_1(|\eta|)\, |\xi|^p - \nu_2(|\eta|)\, k(x)^{1/p'} |\xi|^1 \tag{0.4}$$

avec

$$[\nu_2(|\eta|)]^{p'/p} \leqq \nu(|\eta|) + c_0$$

et où

$$\nu(|\eta|) = \int_0^{|\eta|} [\nu_1(t)]^{p'/p}\, dt.$$

On suppose que $\nu(+\infty) = +\infty$.

Remarque. Le terme $\nu_2(|\eta|)\, k(x)^{1/p'}\, |\xi|$ sert à tenir compte des termes "non coercifs" de l'opérateur. Prenons un cas simple:

$$Au = -\Delta u - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(b_i(x)\, u), \qquad b_i \in L^2(\Omega);$$

alors pour presque tout $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\, \xi_i = |\xi|^2 + \eta \left(\sum_{i=1}^{N} b_i(x)\, \xi_i \right).$$

Posons $b(x) = \sum_{i=1}^{N} b_i(x)^2$, on obtient via l'inégalité de Cauchy–Schwartz:

$$\eta \left(\sum_{i=1}^{N} b_i(x)\, \xi_i \right) \geqq -|\eta|\, b(x)^{1/2} |\xi|.$$

Ainsi

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\, \xi_i \geqq |\xi|^2 - |\eta|\, b(x)^{1/2} |\xi|.$$

[1] On notera que le cas le plus intéressant est quand $\lim_{\eta \to +\infty} \nu_1 = 0$.

On suppose que les a_i vérifient l'hypothèse de croissance suivante:

Il existe une fonction croissante a: $\mathbb{R}_+ \to \mathbb{R}_+$ et un élément $a_0 \geqq 0$ de $L^{p'}(\Omega)$ tels que p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$|a_i(x, \eta, \xi)| \leqq a(|\eta|)[|\xi|^{p-1} + a_0(x)]. \tag{0.5}$$

p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$, (0.6)

$$\sum_{i=1}^{N} [a_i(x, \eta, \xi) - a_i(x, \eta, \xi')][\xi_i - \xi'_i] > 0.$$

La fonction F est une fonction de Caratheodory de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$ vérifiant la "condition d'un seul côté (one-sided condition) suivante: p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$\eta F(x, \eta, \xi) \geqq 0. \tag{0.7}$$

De plus, F admet la croissance suivante: il existe une fonction croissante f de $\mathbb{R}_+$ dans $\mathbb{R}_+$ et une fonction positive f_0 de $L^1(\Omega)$ telles que p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$|F(x, \eta, \xi)| \leqq f(|\eta|)(f_0(x) + |\xi|^p).$$

Comme l'ont fait remarquer Boccardo–Murat–Puel [3], le fait de supposer f croissante n'est pas une restriction car on peut toujours la remplacer par $\tilde{f}(s) = \text{Sup}_{0 \leq t \leq s} f(t)$:

$$T \in W^{-1,r}(\Omega) \qquad \text{avec} \quad r \geqq p' \text{ et } r > N/(p-1). \tag{0.8}$$

La condition $r > N/(p-1)$ semble être une condition critique pour avoir une solution bornée. En effet, Boccardo, Murat, et Puel [5] ont montré que si $p = 2$, $T \in H^{-1}(\Omega)$, $N \leqq 2$, et $v_1(|\eta|) = v_0 > 0$ (constante), alors les solutions de (0.1) existent et sont non bornées. L'un des principaux théorèmes que nous allons démontrer est le suivant:

THÉORÈME. *Sous les hypothèses* (0.2)–(0.8), *le problème* (0.1) *admet au moins une solution. De plus si* $r > p'$, $f_0 \in L^{r/p'}(\Omega)$, *et* $a_0 \in L^r(\Omega)$, *alors la solution* u *est* α*-höldérienne à l'intérieur de* Ω *pour un certain* $\alpha > 0$.

La régularité α-höldérienne a été démontrée dans le cas linéaire par De Giorgi [8] (voir aussi [12]). Pour les cas non linéaires voir [7, 11].

Notre résultat améliore ceux données antérieurement (cf. [3, 4, 6, 9]) par plusieurs des propriétés suivantes:

• L'opérateur A (voir (0.4)) peut dégénérer avec la solution, i.e., on peut avoir $\lim_{\eta \to +\infty} v_1(\eta) = 0$ alors que les travaux précédents supposent $v_1(\eta) = v_0 =$ constante. On peut tenir compte des "termes non coercifs" dans le terme contenant v_2.

• Notons aussi que les travaux les plus récents ([4, 9]) supposent l'existence d'une sous- et sur-solutions, ce qui n'est pas nécessaire ici.

• Par ailleurs, dans le Théorème 2, nous donnons une démonstration directe de la régularité $L^\infty(\Omega)$ des équations du type $(\mathscr{P})$. Les majorations que nous apportons sont explicites. Nos conditions sur $T \in W^{-1,r}(\Omega)$, $r \geqq p/(p-1)$, $r > N/(p-1)$ coïncident dans le cas linéaire avec la condition nécessaire et suffisante due à Stampacchia [21], pour avoir des solutions bornées. Signalons enfin un résultat récent [2] dans le cas $p = 2$ sur des problèmes quasi-linéaires, qui donne aussi la même condition avec une méthode différente de la nôtre mais toujours dans le cas où $v_1(\eta)$ est constante.

Le plan de ce travail sera le suivant: La première section sera consacrée à une estimation a priori de la norme de la solution (éventuelle) de (0.1) dans $L^\infty(\Omega)$ et de son gradient dans $L^p(\Omega)$. Nous donnerons alors une précision sur la dépendance des majorants en fonction des données. Ces précisions nous permettront de définir de façon convenable un problème équivalent de (0.1).

Dans la seconde section, nous commençons par une troncature des coefficients a_i analogue à celle donnée par Hess [9]. On introduit alors un problème (0.1′) équivalent au problème (0.1). On perturbe ensuite la fonction F en une fonction F_ε et l'on regarde un problème approximé $(0.1'_\varepsilon)$ de (0.1′). On montre alors que l'opérateur A_ε ainsi introduit dans $(0.1'_\varepsilon)$ est un opérateur de type calcul de variation. La solution de $(0.1'_\varepsilon)$ est alors donnée par un théorème dans [14]. On montre, grâce aux estimations de la Section 1 que la solution u_ε de $(0.1'_\varepsilon)$ reste dans un borné de $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ lorsque ε varie. On démontre que cette suite de solutions u_ε converge dans $W_0^{1,p}(\Omega)$ vers un élément u solution de (0.1′).

Enfin, nous terminons par une étude de régularité en montrant que toute solution de (0.1) est α-höldérienne à l'intérieur de Ω pour un certain exposant $\alpha > 0$. Dans la suite, on désigne par $\|\cdot\|_V$ la norme dans un Banach V. En particulier, on notera quelquefois $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$.

Un travail récent de Boccardo et Giachetti [2] prouve la régularité de $L^\infty(\Omega)$ des solutions u dans $W_0^{1,p}(\Omega)$ des équations du type (0.1) lorsque l'opérateur est un laplacien (cas $p = 2$).

La méthode de la première section utilise largement la notion de réarrangement relatif dont nous rappelons brièvement les lignes essentielles pour notre travail. De plus amples détails sont donnés dans [15, 16, 20].

Rappels. Soit Ω un ensemble borné mesurable[2] de $\mathbb{R}^N$. Pour tout sous-ensemble E mesurable de Ω, on désigne par $|E|$ sa mesure. Soit u: $\Omega \to \mathbb{R}$ une fonction mesurable, on appelle réarrangement décroissant de u la fonction définie sur $[0, |\Omega|]$ par

$$u_*(s) = \operatorname{Inf}\{\theta \in \mathbb{R} \backslash |u > \theta| \leqq s\},$$
$$u_*(0) = \operatorname*{Sup\,ess}_{\Omega} u.$$

Soit maintenant $v \in L^1(\Omega)$ et u une fonction mesurable, on leur associe la fonction w définie sur $[0, |\Omega|]$ par

$$w(s) = \int_{u > u_*(s)} v(x)\, dx \qquad \text{si} \quad |u = u_*(s)| = 0,$$

$$w(s) = \int_{u > u_*(s)} v(x)\, dx + \int_0^{s - |u > u_*(s)|} (v_{|P(s)})_* (\sigma)\, d\sigma \qquad \text{sinon,}$$

où $(v_{|P(s)})_*$ est le réarrangement décroissant de la restriction de v à $P(s) = \{u = u_*(s)\}$. Le théorème suivant a été démontré dans [16].

THÉORÈME 0. *Soit* $v \in L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, *u une fonction mesurable de* Ω *dans* $\mathbb{R}$, *alors*

(i) $w \in W^{1,p}(\Omega^*)$ avec $\Omega^* =]0, |\Omega|[$,

(ii) $\|dw/ds\|_{L^p(\Omega^*)} \leqslant \|v\|_{L^p(\Omega)}$.

La définition suivante a été introduite dans [17].

DÉFINITION 0. On appelle réarrangement relatif de v par rapport à u, la fonction dw/ds et l'on note

$$v_{*u} = dw/ds.$$

Ce type de réarrangement jouit de la propriété suivante (cf. [16]).

PROPRIÉTÉ 1. Si $v \geqq 0$ p.p. dans Ω, alors $v_{*u} \geqq 0$ p.p. dans Ω^*.

Remarque 2. Considérons Ω un ouvert borné de $\mathbb{R}^N$, $v \in W_0^{1,1}(\Omega)$, $v \geqq 0$, et $f \in L^1(\Omega)$. On pose $\mu(t) = |v > t|$, alors pour presque tout $t \in (0, \operatorname{Sup\,ess} v)$, nous avons

$$\frac{d}{dt} \int_{v > t} f(x)\, dx = \mu'(t) \cdot f_{*v}(\mu(t)). \qquad \textbf{(R.1)}$$

[2] On n'utilise que la mesure de Lebesgue.

Et si $\bar{f}$ est une fonction positive, localement intégrable dans Ω^* alors: $\forall(s, s') \in \bar{\Omega}^* \times \bar{\Omega}^*$, $s \leqq s'$, on a

$$\int_{v_*(s')}^{v_*(s)} \bar{f}(\mu(\theta))(-\mu'(\theta))\, d\theta \leqq \int_s^{s'} \bar{f}(\sigma)\, d\sigma. \tag{R.2}$$

La démonstration de cette remarque sera faite en appendice, à la fin de cet article.

Voici une propriété classique des distributions dont nous nous servirons constamment dans la suite:

Caractérisation d'un élément T de $W^{-1,r}(\Omega)$ (cf. [10]).

Soit Ω un ouvert borné de $\mathbb{R}^N$. Un élément T appartient à $W^{-1,r}(\Omega)$ si et seulement si, il existe N fonctions $g_1, \ldots, g_N$ de $L^r(\Omega)$ telles que

$$T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i} \qquad \text{(au sens de } \mathscr{D}'(\Omega)) \tag{R.3}$$

et

$$\|T\|_{W^{-1,r}(\Omega)} = \operatorname{Inf} \sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)},$$

L'infinimum étant pris sur tous les N-uplets satisfaisant (R.3).

Le calcul classique de minimisation de fonctionelles montre que l'infinimum est atteint. Dans la suite, on considèrera $(g_i)_{i=1,\ldots,N}$ tel que

$$T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i} \qquad \text{et} \qquad \|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)}.$$

1. Réarrangement relatif dans les équations elliptiques quasilinéaires: Estimations a priori

Cette section sera vouée en grande partie à une estimation a priori de la solution (0.1). Pour ce faire, nous allons considérer une fonction réelle lipchitzienne $S_{\theta,h}$ associée à deux nombres $\theta > 0$, $h > 0$ et définie de la façon suivante:

$$S_{\theta,h}(\tau) = \begin{cases} 1 & \text{si } \tau \geqq \theta + h, \\ (\tau - \theta)\, 1/h & \text{si } \theta \leqq \tau \leqq \theta + h, \\ 0 & \text{si } |\tau| \leqq \theta, \\ (\tau + \theta)\, 1/h & \text{si } -\theta - h \leqq \tau \leqq -\theta, \\ -1 & \text{si } \tau \leqq -\theta - h. \end{cases}$$

On se donne $u \in W_0^{1,p}(\Omega)$, on note $\bar{v} = S_{\theta,h}(u)$[3]. D'après Stampacchia [21], $\bar{v} \in W^{1,p}(\Omega)$ et vérifie presque partout dans Ω:

$$\frac{\partial \bar{v}}{\partial x_i} = \begin{cases} \dfrac{1}{h} \cdot \dfrac{\partial u}{\partial x_i} & \text{si} \quad \theta < |u| \leqq \theta + h, \\ 0 & \text{sinon.} \end{cases}$$

De plus, $\bar{v} \in L^\infty(\Omega)$. On suppose que u satisfait à

$$\forall \theta \in]0, \operatorname{Sup\,ess} |u|[, \qquad \forall h \in]0, \operatorname{Sup\,ess} |u| - \theta[,$$

on a[4]

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u(x), \nabla u(x)) \cdot \frac{\partial \bar{v}}{\partial x_i} \, dx + \int_\Omega F(x, u(x), \nabla u(x) \cdot \bar{v} \, dx = \langle T, \bar{v} \rangle \tag{1.1}$$

$\langle \ \rangle$ désignant un crochet de dualité entre $W^{-1,p'}(\Omega)$ et $W_0^{1,p}(\Omega)$.

Remarque 3. (1) Toute solution éventuelle du problème (0.1) vérifie l'équation (1.1).

(2) Le produit de dualité entre T et $\bar{v}$ a un sens puisque $r \geqq p'$ donc $W^{-1,r}(\Omega) \subset W^{-1,p'}(\Omega)$.

L'un des principaux résultats de notre étude concerne la régularité $L^\infty(\Omega)$ des solutions u de l'équation (1.1).

THÉORÈME 1. *On suppose* (0.2)–(0.5), (0.7), *et* (0.8) *et l'on considère* $u \in W_0^{1,p}(\Omega)$ *satisfaisant à l'équation* (1.1); *alors*

(i) $u \in L^\infty(\Omega)$ *et la norme u dans* $L^\infty(\Omega)$ *est telle que*

$$\nu(\|u\|_\infty) \leqq \lambda_0 \lambda_1 \lambda_2 \Leftrightarrow \|u\|_\infty \leqq \nu^{-1}(\lambda_0 \lambda_1 \lambda_2) = M,^5$$

où

$$\lambda_0 = \gamma(c_0 \|k\|^{1/p}_{L^{r/p'}(\Omega)} + \|T\|^{p/p'}_{W^{-1,r}(\Omega)}),$$
$$\lambda_1 = \exp(\lambda_2 \gamma \|k\|^{1/p}_{L^{r/p'}(\Omega)}),$$
$$\gamma = \left[\int_{\Omega^*} \sigma^{((1/N)-1)((p-1)r/(p-1)r-1)} \, d\sigma \right]^{1-1/(p-1)r}$$

[3] Dans les démonstrations qui vont suivre, on suppose que u est non nulle. Les résultats sont triviaux dans le cas contraire.

[4] L'hypothèse (0.7) assure que $F(x, u(x), u(x))\, \bar{v} \geqq 0$ p.p.

[5] Cette estimation est optimale (voir [24]).

avec $\Omega^* =]0, |\Omega|[$ *(notons que* $\gamma < +\infty$ *car* $r > N/(p-1)$,

$$\lambda_2 = 2^*/N\alpha_N^{1/N} \qquad \text{avec} \quad 2^* = 2^{(p'/p-1)_+},$$

$\alpha_N =$ *mesure de la boule unité de* $\mathbb{R}^N$, ν^{-1} *étant l'inverse de la fonction* ν *de* $\mathbb{R}_+$ *dans* $\mathbb{R}_+$.

On note $\alpha = \mathrm{Min}_{0 \leqq \theta \leqq M} \nu_1(\theta)^{p'/p} > 0$, $\beta = \mathrm{Max}_{0 \leqq \theta \leqq M} \nu_2(\theta)^{p'/p}$. *M étant le majorant de* $\|u\|_\infty$ *donné ci-dessus, alors nous avons l'estimation du gradient dans* $L^p(\Omega)$:

(ii) $\|\nabla u\|_{L^p(\Omega)} \leqq 2^*/\alpha(\beta\|k\|_{L^1(\Omega)}^{1/p} + |\Omega|^{(r-p')/pr} \cdot \|T\|_{W^{-1,r}(\Omega)}^{p'/p}) = M_0$.

On considère $u \in W_0^{1,p}(\Omega)$ satisfaisant à l'équation (1.1) et soit $(g_i)_{i=1,\dots,N}$ des éléments de $L^r(\Omega)$ tels que $T = -\sum_{i=1}^N \partial g_i/\partial x_i$ et $\|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^N \|g_i\|_{L^r(\Omega)}$. On note $v = |u|$ et $g = (\sum_{i=1}^N g_i^2)^{p'/2}$. On définit les fonctions suivantes:

$$K(t,s) = \exp\left(\lambda_2 \int_t^s \sigma^{(1/N)-1} k_{*v}(\sigma)^{1/p}\, d\sigma\right) \qquad \text{pour} \quad (t,s) \in \overline{\Omega^*} \times \overline{\Omega^*},$$

$$b(s) = s^{(1/N)-1}(c_0\, k_{*v}(s)^{1/p} + g_{*v}(s)^{1/p},$$

$$\lambda_2 = 2^*/N\alpha_N^{1/N} \qquad \text{avec} \quad 2^* = 2^{((p'/p)-1)_+}.$$

Nous avons alors le lemme suivant qui nous servira de base pour ces estimations:

LEMMA 1. *On suppose* (0.2)–(0.5), (0.7), *et* (0.8) *ont lieu. Alors* $\forall s \in \overline{\Omega^*}$,

$$v_*(s) \leqq \nu^{-1}\left(\lambda_2 \int_s^{|\Omega|} K(s,\sigma)\, b(\sigma)\, d\sigma\right). \tag{1.2}$$

Remarque 4. Les termes K et b ont un sens. Plus précisément, $K \in L^\infty(\Omega^* \times \Omega^*)$ et $b \in L^1(\Omega^*)$. En effet, par la propriété 1 (voir Rappels), on a $b \geqq 0$ et quand on applique l'inégalité de Hölder:

$$0 \leqq \int_{\Omega^*} b(\sigma)\, d\sigma \leqq \gamma \|c_0 k_{*v}^{1/p} + g_{*v}^{1/p}\|_{L^{(p/p')r}(\Omega^*) = L^{(p-1)r}(\Omega^*)}$$

où

$$\gamma = \left[\int_{\Omega^*} \sigma^{((1/N)-)((p-1)r/(p-1)r-1)} d\sigma\right]^{1-(1/(p-1)r)}$$

En utilisant le Théorème 0 (voir Rappels):

$$\|c_0 k_{*v}^{1/p} + g_{*v}^{1/p}\|_{L^{(p-1)r}(\Omega^*)} \leqslant c_0 \|k\|_{L^{r/p'}(\Omega)}^{1/p} + \|g\|_{L^{r/p'}(\Omega)}^{1/p},$$

or,

$$\|g\|_{L^{r/p'}(\Omega)} \leqq \left(\sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)} \right)^{p'} = \|T\|^{p'}_{W^{-1,r}(\Omega)}.$$

Ainsi

$$\|b\|_{L^1(\Omega^*)} \leqq \gamma(c_0\|k\|^{1/p}_{L^{r/p'}(\Omega)} + \|T\|^{p'/p}_{W^{-1,r}(\Omega)}) = \lambda_0. \tag{1.3}$$

Pour des raisons analogues, on a:

$$\|K\|_{L^\infty(\Omega^* \times \Omega^*)} \leqq \exp(\lambda_2 \gamma \|k\|^{1/p}_{L^{r/p'}(\Omega)}) = \lambda_1. \tag{1.4}$$

On notera que les λ_i ne dépendent que de c_0, p, r, N, Ω, k, et T. L'hypothèse (0.6) est inutile dans cette section.

Démonstration du Lemme 1. Pour des raisons de commodité, nous poserons:

$$a_i(x, u(x), \nabla u(x)) = a_i, \qquad F(x, u(x), \nabla u(x)) = F,$$
$$g = \left(\sum_{i=1}^{N} g_i^2 \right)^{p'/2} \qquad \text{et} \qquad v = |u|.$$

Puisque u vérifie (1.1), nous avons donc pour $\theta \in]0, \sup\text{ess}\, v[$, $h \in]0, \sup\text{ess}\, v - \theta[$:

$$\frac{1}{h}\int_{\theta < v \leqq \theta + h} \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i}\, dx + \int_\Omega F S_{\theta,h}(u)\, dx = \frac{1}{h}\int_{\theta < v \leqq \theta + h} \sum_{i=1}^{N} g_i \frac{\partial u}{\partial x_i}\, dx. \tag{1.5}$$

Puisque $S_{\theta,h}(0) = 0$, nous avons $\int_\Omega F S_{\theta,h}(u)\, dx = \int_{u \neq 0} u \cdot F \cdot S_{\theta,h}(u)/u\, dx$. La fonction $S_{\theta,h}$ étant croissante et s'annulant en 0, on a donc $S_{\theta,h}(u)/u \geqq 0$. En vertu de l'hypothèse (0.7), $uF \geqq 0$, on obtient ainsi $\int_\Omega F S_{\theta,h}(u)\, dx \geqq 0$. Compte tenu de l'hypothèse (0.4) sur a_i, la relation (1.5) entraîne alors

$$\frac{1}{h}\int_{\theta < v \leqq \theta + h} \nu_1(v)|\nabla u|^p\, dx \leqq \frac{1}{h}\int_{\theta < v \leqq \theta + h} \nu_2(v)\, k(x)^{1/p'}|\nabla u|\, dx + \frac{1}{h}\int_{\theta < v \leqq \theta + h} \sum_{i=1} g_i \frac{\partial u}{\partial x_i}\, dx. \tag{1.6}$$

Puisque les fonctions v_i sont bornées sur $[\theta, \theta+h]$ (voir (0.4)), nous obtenons

$$\operatorname*{Min}_{\theta \leqq \eta \leqq \theta+h} v_1(\eta) \cdot \int_{\theta < v \leqq \theta+h} |\nabla u|^p \, dx \leqq \int_{\theta < v \leqq \theta+h} v_1(v)|\nabla u|^p \, dx \tag{1.7}$$

et

$$\int_{\theta < v \leqq \theta+h} v_2(v)\, k(x)^{1/p'} |\nabla u| \, dx \leqq \operatorname*{Max}_{\theta \leqq \eta \leqq \theta+h} v_2(\eta) \cdot \int_{\theta < v \leqq \theta+h} k(x)^{1/p'} |\nabla u| \, dx. \tag{1.8}$$

En utilisant l'inégalité de Hölder, il vient

$$\int_{\theta < v \leqq \theta+h} k(x)^{1/p'} |\nabla u| \, dx \leqq \left(\int_{\theta < v \leqq \theta+h} k(x)\, dx\right)^{1/p'} \left(\int_{\theta < v \leqq \theta+h} |\nabla u|^p \, dx\right)^{1/p} \tag{1.9}$$

et

$$\int_{\theta < v \leqq \theta+h} \sum_{i=1}^{N} g_i \frac{\partial u}{\partial x_i} \, dx \leqq \left(\int_{\theta < v \leqq \theta+h} g(x)\, dx\right)^{1/p'} \left(\int_{\theta < v \leqq \theta+h} |\nabla u|^p \, dx\right)^{1/p} \tag{1.10}$$

Quand h tend vers zéro, les relations (1.6)–(1.10) entraînent que pour presque tout $\theta \in \,]0, \sup\text{ess}\, v[$,

$$v_1(\theta)\left(-\frac{d}{d\theta}\int_{v>\theta} |\nabla u|^p \, dx\right) \leqq \left[v_2(\theta)\left(-\frac{d}{d\theta}\int_{v>\theta} k(x)\, dx\right)^{1/p'} + \left(-\frac{d}{d\theta}\int_{v>\theta} g(x)\, dx\right)^{1/p'}\right] \times \left(-\frac{d}{d\theta}\int_{v>\theta} |\nabla u|^p \, dx\right)^{1/p}.$$

Soit en simplifiant et en tenant compte de la relation (R.1) de la remarque 2:

$$v_1(\theta)\left(-\frac{d}{d\theta}\int_{v>\theta} |\nabla u|^p \, dx\right)^{1/p'} \leqq [v_2(\theta)\, k_{*v}(\mu(\theta))^{1/p'} + g_{*v}(\mu(\theta))^{1/p'}](-\mu'(\theta))^{1/p'}. \tag{1.11}$$

Elevons le tout à la puissance $p'/p = 1/(p-1)$. Deux cas sont à distinguer: Si $p \geqq 2$, alors $p'/p \leqq 1$, en utilisant le fait que $(a+b)^{p'/p} \leqq a^{p'/p} + b^{p'/p}$ pour $a \geqq 0$, $b \geqq 0$. Nous obtenons que

$$v_1(\theta)^{p'/p}\left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p\,dx\right]^{1/p}$$
$$\leqq [v_2(\theta)^{p'/p}k_{*v}(\mu(\theta))^{1/p} + g_{*v}(\mu(\theta))^{1/p}](-\mu'(\theta))^{1/p}.$$

On a utilisé en passant le fait que k_{*v} et g_{*v} sont des fonctions positives (voir propriété 1). Si $1 < p < 2$, alors on utilise l'inégalité $(a+b)^{p/p'} \leqq 2^{(p/p')-1}(a^{p/p'} + b^{p/p'})$ provenant de la convexité de la fonction $t \to t^{p/p'}$.

On observe alors que dans tous les cas, on obtient, pour presque tout $\theta \in]0, \text{supp ess } v[$,

$$v_1(\theta)^{p/p'}\left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p\,dx\right]^{1/p}$$
$$\leqq 2^*[v_2(\theta)^{p/p'}k_* v(\mu(\theta))^{1/p} + g_{*v}(\mu(\theta))^{1/p'}](-\mu'(\theta))^{1/p'} \quad (1.12)$$

où $2^* = 2^{((p'/p)-1)_+}$. En utilisant l'inégalité de Hölder au niveau des primitives avant de passer àla limite ($h \to 0$), on trouve:

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|\,dx \leqq [-\mu'(\theta)]^{1/p'}\left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p\,dx\right]^{1/p}. \quad (1.13)$$

Par la formule de Flemming et Rishel (cf. [15]) nous avons:

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|\,dx = P_\Omega \qquad (v>\theta),$$

où P_Ω $(v > \theta)$ est le périmètre suivant Ω de $\{v > \theta\}$. Par l'inégalité isopérimétrique de De Giorgi (cf. [15]), on déduit:

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|\,dx = P_\Omega\,(v>\theta) \geqq N\alpha_N^{1/N}(\mu(\theta))^{1-(1/N)}, \quad (1.14)$$

où α_N = mesure de la boule unité de $\mathbb{R}^N$.

Sachant que $\theta = v_*(\mu(\theta))$, la relation (1.12) jointe à (1.13) et (1.14) donne l'inéquation. Pour presque tout $\theta \in]0, \text{Sup ess } v[$,

$$1 \leqq \lambda_2 \frac{(\mu(\theta))^{(1/N)-1}}{v_1(v_*(\mu(\theta)))^{p/p'}}[v_2(\mu(\theta))^{p/p'}k_{*v}(\mu(\theta))^{1/p} + g_{*v}(\mu(\theta))^{1/p}](-\mu'(\theta)),$$
$$(1.15)$$

où $\lambda_2 = 2^*/N\alpha_N^{1/N}$.

Puisque les fonctions $v_1 > 0$ et v_2 sont continues (voir (0.4)) et que les fonctions $k_{*v}^{1/p}$ et $g_{*v}^{1/p}$ sont intégrables (voir Théorème 0), on déduit que l'application:

$$\sigma \in \Omega^* \to \frac{\sigma^{(1/N)-1}}{v_1(v_*(\sigma))^{p'/p}} [v_2(v_*(\sigma))^{p'/p} k_{*v}(\sigma)^{1/p} + g_{*v}(\sigma)^{1/p}]$$

est localement intégrable.

Par application de larelation (R.2) de la remarque 2 on obtient pour $0 < s < s + \varepsilon < |\Omega|$:

$$\frac{v_*(s) - v_*(s+\varepsilon)}{\varepsilon} \leqq \frac{\lambda_2}{\varepsilon} \int_\varepsilon^{s+\varepsilon} \frac{\sigma^{(1/N)-1}}{v_1(v_*(\sigma))^{p'/p}} [v_2(v_*(\sigma))^{p'/p} k_{*v}(\sigma)^{1/p} + g_{*v}(\sigma)^{1/p}]\, d\sigma.$$

Quand ε tend vers 0, on obtient pour presque tout s dans Ω^*,

$$-v_1(v_*(s))^{p'/p} \frac{dv_*}{ds} \leqq \lambda_2 s^{(1/N)-1} [v_2(v_*(s))^{p'/p} k_{*v}(s)^{1/p} + g_{*v}(s)^{1/p}]. \tag{1.16}$$

Par définition de v (voir (0.4)), nous avons $v(v_*(s)) = \int_0^{v_*(s)} [v_1(t)]^{p'/p}\, dt$. Comme $v_* \in W_{\text{loc}}^{1,p}(\Omega^*)$, il en est de même de $v(v_*(s))$ et l'on a presque partout en s:

$$\frac{d}{ds} v(v_*(s)) = v_1(v_*(s))^{p'/p} \frac{dv_*}{ds}. \tag{1.17}$$

Par hypothèse sur v_2 (voir (0.4)),

$$v_2(v_*(s))^{p'/p} \leqq v(v_*(s)) + c_0. \tag{1.18}$$

En posant $V(s) = v(v_*(s))$, les relations (1.16)–(1.18) montrent que V satisfait à une inégalité de type Gronwall:

$$-\frac{dV}{ds} \leqq \lambda_2 s^{(1/N)-1} k_{*v}(s)^{1/p}\, V(s) + \lambda_2 s^{(1/N)-1}(c_0 k_{*v}(s)^{1/p} + g_{*v}(s)^{1/p}). \tag{1.19}$$

En définissant

$$K(t, s) = \exp\left(\lambda_2 \int_t^s \sigma^{(1/N)-1} k_{*v}(\sigma)^{1/p}\, d\sigma\right) \qquad \text{pour} \quad (t, s) \in \Omega^* \times \Omega^*$$

et

$$b(s) = s^{(1/N)-1}(c_0 k_{*v}(s)^{1/p} + g_{*v}(s)^{1/p}),$$

la relation (1.19) est alors équivalente à

$$-\frac{d}{ds}[K(0,s)\,V(s)] \leqq \lambda_2 K(0,s)\,b(s). \tag{1.20}$$

Ainsi par intégration, nous obtenons que

$$V(s) \leqq \lambda_2 \int_s^{|\Omega|} K(s,\sigma)\,b(\sigma)\,d\sigma. \tag{1.21}$$

(On remarquera que $K(0,\cdot)\cdot V(\cdot) \in W^{1,p}_{\mathrm{loc}}(\Omega^*)$ et que $V(|\Omega|)=0$ car $v_*(|\Omega|) = \inf \operatorname{ess} |u| = 0$.)

Comme la fonction v est continue et est strictement croissante de $\mathbb{R}_+$ dans $\mathbb{R}_+$, elle est donc inversible. La relation (1.21) implique pour tout $s \in \bar{\Omega}^*$,

$$v_*(s) \leqq v^{-1}\left(\lambda_2 \int_s^{|\Omega|} K(s,\sigma)\,b(\sigma)\,d\sigma\right). \quad \blacksquare$$

Démonstration du Théorème 1. Puisque $K \in L^\infty(\Omega^* \times \Omega^*)$ et $b \in L^1(\Omega^*)$ (voir Remarque 4), on déduit

$$\int_s^{|\Omega|} K(s,\sigma)\,b(\sigma)\,d\sigma \leqq \|K\|_{L^\infty(\Omega^*\times\Omega^*)} \cdot \|b\|_{L^1(\Omega^*)}.$$

De (1.3) et (1.4), on a

$$\int_s^{|\Omega|} K(s,\sigma)\,b(\sigma)\,d\sigma \leqq \lambda_0 \lambda_1.$$

Ainsi, $\forall s \in \bar{\Omega}^*$: $v_*(s) \leqq v^{-1}(\lambda_0\lambda_1\lambda_2) = M$, i.e., $\|u\|_\infty \leqq M$. Pour montrer la relation (ii), posons

$$\alpha = \operatorname*{Min}_{0 \leqq \theta \leqq M} v_1(\theta)^{p'/p}, \qquad \beta = \operatorname*{Max}_{0 \leqq \theta \leqq M} v_2(\theta)^{p'/p},$$

où M est le nombre établi ci-dessus; on remarquera que $\alpha > 0$ car $v_1 > 0$ et est continue. En reprenant la relation (1.12), on obtient

$$\alpha\left[-\frac{d}{d\theta}\int_{v>\theta} |\nabla u|^p\,dx\right]^{1/p} \\ \leqq 2^*[\beta k_{*v}(\mu(\theta))^{1/p} + g_{*v}(\mu(\theta))^{1/p}](-\mu'(\theta))^{1/p}. \tag{1.22}$$

On élève la relation (1.22) à la puissance p, il vient

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p\,dx$$
$$\leqq\left(\frac{2^*}{\alpha}\right)^p[\beta k_{*v}(\mu(\theta))^{1/p}+g_{*v}(\mu(\theta))^{1/p}]^p(-\mu'(\theta)). \tag{1.23}$$

Psuique la fonction $\theta\to\int_{v>\theta}|\nabla u|^p\,dx$ est absolument continue, une intégration de la relation (1.23) donne, via la remarque 2 (voir (R.2))

$$\left(\int_\Omega|\nabla u|^p\,dx\right)^{1/p}\leqq\frac{2^*}{\alpha}\,\|\beta k_{*v}^{1/p}+g_{*v}^{1/p}\|_{L^p(\Omega^*)}. \tag{1.24}$$

Par le Théorème 0,

$$\|\beta k_{*v}^{1/p}+g_{*v}^{1/p}\|_{L^p(\Omega^*)}\leqq\beta\|k\|_{L^1(\Omega)}^{1/p}+\|g\|_{L^1(\Omega)}^{1/p}.$$

Par l'inégalité de Hölder,

$$\|g\|_{L^1(\Omega)}\leqq|\Omega|^{1-p'/r}\|g\|_{L^{r/p'}(\Omega)}.$$

Comme

$$\|g\|_{L^{r/p'}(\Omega)}\leqq\left(\sum_{i=1}^N\|g_i\|_{L^r(\Omega)}\right)^{p'}=\|T\|_{W^{-1,r}(\Omega)}^{p'}.$$

Ainsi, de (1.24), on aboutit

$$\|\nabla u\|_{L^p(\Omega)}\leqq\frac{2^*}{\alpha}\,(\beta\|k\|_{L^1}^{1/p}+|\Omega|^{(r-p')/pr}\|T\|_{W^{-1,r}(\Omega)}^{p'/p}).\quad\blacksquare$$

2. Application à un théorème d'existence et de régularité d'une équation elliptique quasilinéaire

Le but de cette section sera de montrer le

THÉORÈME 2. (i.1) *On suppose que* (0.2)–(0.8) *ont lieu, alors il existe une solution* $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *du problème* (0.1).

(i.2) *Une estimation de la norme de la solution* u *dans* $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *est donnée par le Théorème* 1.

(i.3) *De plus si* $r > p'$, $f_0 \in L^{r/p'}(\Omega)$ (*voir* (0.7)) *et* $a_0 \in L^r(\Omega)$ (*voir* (0.4)) *alors toute solution u est une fonction* α*-höldérienne à l'intérieur de* Ω *pour un certain exposant* $\alpha > 0$.

L'estimation $L^\infty(\Omega)$ de la première section jouera un rôle prépondérant au cours de cette section. La démonstration se fera en plusieurs étapes.

2.1. *Troncature du Problème* (0.1)

Comme dans Hess [8], on introduit une troncature des coefficients a_i définie par p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$a_i'(x, \eta, \xi) = \begin{cases} a_i(x, \eta, \xi) & \text{si} \quad |\eta| \leqq M, \\ a_i(x, M, \xi) & \text{si} \quad \eta > M, \\ a_i(x, -M, \xi) & \text{si} \quad \eta < -M \end{cases}$$

(M étant le nombre donné dans le Théorème 1).

On définit aussi une troncature de v_1, v_2 par

$$\forall \eta \in \mathbb{R}, \qquad v_i'(|\eta|) = \begin{cases} v_i(|\eta|) & \text{si} \quad |\eta| \leqq M, \\ v_i(M) & \text{si} \quad |\eta| \geqq M, \end{cases} \tag{2.1}$$

On vérifie sans peine que si

$$v'(|\eta|) = \int_0^{|\eta|} [v_1'(t)]^{p'/p} dt, \quad \text{alors} \quad [v_2'(|\eta|)]^{p'/p} \leqq v'(|\eta|) + c_0 \tag{2.2}$$

et que p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$,

$$\sum_{i=1}^N [a_i'(x, \eta, \xi) - a_i'(x, \eta, \xi')][\xi_i - \xi_i'] > 0; \tag{2.3}$$

p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$,

$$\sum_{i=1}^N a_i'(x, \eta, \xi)\, \xi_i \geqq v_1'(|\eta|)|\xi|^p - v_2'(|\eta|)\, k(x)^{1/p'}|\xi|. \tag{2.4}$$

On considère maintenant le problème

$$\begin{gathered} -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i'(x, u(x), \nabla u(x)) + F(u, \nabla u) = T \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{gathered} \tag{0.1'}$$

LEMME 1. *Toute solution de* (0.1′) *est solution de* (0.1) *et réciproquement.*

Preuve. Si u est solution de (0.1′) alors u satisfait à l'équation (1.1). On applique le Théorème 1 en remplaçant v par v' alors

$$v'(\|u\|_\infty) \leqq \lambda_0 \lambda_1 \lambda_2 = v(M). \tag{2.5}$$

Par définition de v', on remarque que $v(M) = v'(M)$ et comme v' est inversible, on déduit de la relation (2.5) que $\|u\|_\infty \leqq M$. Ainsi, $\forall v' \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega a_i'(x, u(x), \nabla u(x)) \frac{\partial v'}{\partial x_i}\, dx \\ = \int_{|u| \leqq M} a_i(x, u(x), \nabla u(x)) \cdot \frac{\partial v'}{\partial x_i}\, dx \\ = \int_\Omega a_i(x, u(x), \nabla u(x)) \cdot \frac{\partial v'}{\partial x_i}\, dx.\end{aligned}$$

La réciproque se démontre de façon analogue.

2.2. *Un problème approché de* (0.1′)

Soit $\varepsilon > 0$, on définit F_ε par p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$:

$$F_\varepsilon(x, \eta, \xi) = \frac{F(x, \eta, \xi)}{1 + \varepsilon |F(x, \eta, \xi)|}.$$

On remarque alors que

$$|F_\varepsilon(x, \eta, \xi)| \leqq 1/\varepsilon \qquad \text{et} \qquad |F_\varepsilon| \leqq |F|. \tag{2.6}$$

Considérons l'opérateur

$$A_\varepsilon u = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i'(x, u(x), \nabla u(x)) + F_\varepsilon(x, u(x), \nabla u(x))$$

pour $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. On cherche à résoudre le problème

$$\begin{aligned} A_\varepsilon u_\varepsilon = T, \\ u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{0.1'_\varepsilon}$$

Nous avons le

LEMME 2. (i) *Pour tout* $\varepsilon > 0$, *il existe une solution* u_ε *de* $(0.1'_\varepsilon)$.

(ii) *De plus,* u_ε *reste dans un borné de* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *lorsque* ε *varie.*

Démonstration. La partie (i) est un des cas concrets de théorème d'existence concernant les opérateurs de calcul de variations (cf. [14, pp. 182–183, Théorème 2.8]). Pour vérifier que les hypothèses de ce théorème sont satisfaites, il suffit de voir que les fonctions a'_i et F_ε sont de Caracthéodory et satisfont (2.3) et que p.p. en $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$: $|a'_i(x, \eta, \xi)| \leqq a(M)(|\xi|^{p-1} + a_0(x))$ (voir (0.6)), et que

$$\sum_{i=1}^{N} a'_i(x, \eta, \xi)\, \xi_i \geqq \alpha_1 |\xi|^p - \beta_1 |\xi|\, k(x)^{1/p'}, \tag{2.7}$$

où $\alpha_1 = \mathrm{Min}_{0 \leqq \eta \leqq M}\, v'_1(\eta)$ et $;_1 = \mathrm{Max}_{0 \leqq \eta \leqq M}\, v'_2(\eta)$. On notera aussi (2.6).

La partie (ii) découle immédiatement du Théorème 1: $\|u_\varepsilon\|_\infty \leqq M$ et $\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqq M_0$. On supposera dans la suite que u_ε tend vers u dans $W_0^{1,p}(\Omega)$-faible, dans L^∞-faible $*$ et simplement quand ε tend vers 0.

2.3. *Passage à la limite*

Nous avons la convergence forte suivante:

LEMME 3. *u_ε tend vers u dans $W_0^{1,p}(\Omega)$, quand ε tend vers* 0.

Puisque $\|u_\varepsilon\|_\infty \leqq M$, nous pouvons utiliser dans la suite a_i à la place de a'_i. Pour alléger l'écriture, on note $F_\varepsilon(x, u(x), \nabla u(x)) = F_\varepsilon u$, $\tilde{A}(u, \nabla u)$ le vecteur de $\mathbb{R}^N$ de composante $a_i(x, u(x), \nabla u(x))$ ainsi,

$$\sum_{i=1}^{N} a_i(x, u(x), \nabla u(x)) \cdot \frac{\partial u}{\partial x_i} = \tilde{A}(u, \nabla u) \cdot \nabla u.$$

Démonstration. L'idée de la démonstration repose sur la propriété (S_+) du théorème de Browder [6, p. 27]. Pour prouver le Lemme 3, il suffit alors de montrer que

$$\limsup_{\varepsilon \to 0} \int_\Omega [\tilde{A}(u_\varepsilon, \nabla u_\varepsilon) - \tilde{A}(u, \nabla u)] \cdot \nabla(u_\varepsilon - u)\, dx \leqq 0.$$

La démonstration qui suit s'inspire de [19]. On définit la fonction réelle $h_\lambda(t) = te^{\lambda t^2}$ où $\lambda > 0$ est choisi de sorte que

$$\alpha_1 h'_\lambda(t) - f(M)\, |h_\lambda(t)| \geqq (\alpha_1/2)\ (\forall t \in \mathbb{R}). \tag{2.8}$$

Ainsi, λ ne dépend que de α_1 et de $c = f(M)$. On remarque alors que $\lim_{\varepsilon \to 0} h_\lambda(u - u_\varepsilon) = 0$ simplement et donc dans $L^q(\Omega)$ pour tout q, $1 \leqq q < +\infty$. Considérons ε et δ deux nombres strictement positifs et destinés à tendre vers 0, u_ε et u_δ satisfont à

$$A_\varepsilon u_\varepsilon = T, \qquad u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{0.1_ε}$$

$$A_\delta u_\delta = T, \qquad u_\delta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{0.1_δ}$$

En faisant le différence entre les équations (0.1_ε) et (0.1_δ), et en multipliant par la fonction test $\alpha_1 h_\lambda(u_\delta - u_\varepsilon)$, il vient

$$\alpha_1 \int_\Omega (\tilde{A}(u_\delta, \nabla u_\delta) - \tilde{A}(u_\varepsilon, \nabla u_\varepsilon)) \cdot \nabla(u_\delta - u_\varepsilon)\, h'_\lambda(u_\delta - u_\varepsilon)$$

$$+ \alpha_1 \int_\Omega (F_\delta u_\delta - F_\varepsilon u_\varepsilon)(h_\lambda(u_\delta - u_\varepsilon)\, dx = 0. \tag{2.9}$$

Par l'hypothèse (0.7) sur la croissance de F, on déduit (en posant $c = f(M)$):

$$\left|\int_\Omega (F_\delta u_\delta - F_\varepsilon u_\varepsilon)\, h_\lambda(u_\delta - u_\varepsilon)\, dx\right| \leqq c \int_\Omega |\nabla u_\delta|^p\, |h_\lambda(u_\delta - u_\varepsilon)|$$

$$+ c \int_\Omega |\nabla u_\varepsilon|^p\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx + 2c \int_\Omega f_0(x)\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx. \tag{2.10}$$

Par l'hypothèse (0.4), sachant que $\beta_1 = \mathrm{Max}_{0 \leqq \eta \leqq M}\, \nu_2(\nu)$ on obtient pour u_ε (resp. pour u_δ) que, presque partout dans Ω:

$$\tilde{A}(u_\varepsilon, \nabla u_\varepsilon)\, \nabla u_\varepsilon + \beta_1 k(x)^{1/p'}\, |\nabla u_\varepsilon| \geqq \alpha_1\, |\nabla u_\varepsilon|^p. \tag{2.11}$$

De par cette relation (2.11), on majore alors les gradients dans le second membre de (2.10) pour obtenir

$$\alpha_1 \left|\int_\Omega (F_\delta u_\delta - F_\varepsilon u_\varepsilon) h_\lambda(u_\delta - u_\varepsilon)\, dx\right|$$

$$\leqq c \int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\delta\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx$$

$$+ c \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx$$

$$+ c\beta_1 \int_\Omega k(x)^{1/p'}\, |\nabla u_\delta|\, (h_\lambda(u_\delta - u_\varepsilon)|\, dx$$

$$+ c\beta_1 \int_\Omega k(x)^{1/p'}\, |\nabla u_\varepsilon|\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx$$

$$+ 2c\alpha_1 \int_\Omega f_0(x)|\, h_\lambda(u_\delta - u_\varepsilon)|\, dx. \tag{2.12}$$

Remarque. Dans la suite, on va faire tendre ε et δ vers 0. Par le théorème de la convergence dominée on note alors que

$$\int_\Omega f_0(x)\, |h_\lambda(u_\delta - u_\varepsilon)|\, dx \xrightarrow[(\varepsilon,\delta) \to (0,0)]{} 0.$$

De même, puisque $|\nabla u_\delta|$ (resp. $|\nabla u_\varepsilon|$) est dans un borné de $L^p(\Omega)$, on déduit alors

$$\int_\Omega k(x)^{1/p'} |\nabla u_\delta| \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx \xrightarrow[(\varepsilon,\delta)\to(0,0)]{} 0 \qquad \text{(resp. pour } u_\varepsilon\text{)}.$$

Pour alléger la démonstration, nous pouvons supposer dans la suite que $f_0(x) = k(x) = 0$.

Compte tenu de cette remarque, la relation (2.9) avec la relation (2.12) entraînent:

$$\begin{aligned}\alpha_1\Big[\int_\Omega (\tilde{A}(u_\delta, \nabla u_\delta) - \tilde{A}(u_\varepsilon, \nabla u_\varepsilon)) \cdot \nabla(u_\delta - u_\varepsilon)\, h'_\lambda(u_\delta - u_\varepsilon)\, dx\Big] \\ \leqq c\Big[\int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\delta \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx \\ + \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx\Big]. \qquad (2.13)\end{aligned}$$

Ecrivons alors

$$\begin{aligned}\tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\delta &= \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla(u_\delta - u_\varepsilon) + \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\varepsilon \\ \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon &= -\tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u_\delta - u_\varepsilon) + \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\delta.\end{aligned} \qquad (2.14)$$

Ainsi le second membre de la relation (2.13) s'écrit

$$\begin{aligned}c\Big[\int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\delta \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx + \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon | h_\lambda(u_\delta - u_\varepsilon)) \, dx\Big] \\ = c\int_\Omega [\tilde{A}(u_\delta, \nabla u_\delta) - \tilde{A}(u_\varepsilon, \nabla u_\varepsilon)] \cdot \nabla(u_\delta - u_\varepsilon) \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx \\ + c\int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\varepsilon \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx \\ + c\int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\delta \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx. \qquad (2.15)\end{aligned}$$

Compte tenu du choix de λ (voir la relation (2.8)), la relation (2.13) jointe à (2.15) entraînent que

$$\begin{aligned}\frac{\alpha_1}{2}\int_\Omega (\tilde{A}(u_\delta, \nabla u_\delta) - \tilde{A}(u_\varepsilon, \nabla u_\varepsilon)) \cdot \nabla(u_\delta - u_\varepsilon) \, dx \\ \leqq c\int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\varepsilon \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx \\ + c\int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\delta \, |h_\lambda(u_\delta - u_\varepsilon)| \, dx. \qquad (2.16)\end{aligned}$$

Nous avons $\tilde{A}(u_\delta, \nabla u_\delta)$ qui est dans un borné de $(L^{p'}(\Omega))^N$. Quitte à extraire une sous-suite, on peut supposer alors que

$$\tilde{A}(u_\delta, \nabla u_\delta) \xrightarrow[\delta \to 0]{} \xi \qquad \text{dans} \quad (L^{p'}(\Omega))^N\text{-faible.}$$

Par le théorème de la convergence dominée et la définition de la convergence faible, le passage à la limite supérieure quand δ tend vers 0 (pour ε fixé), conduit à

$$\begin{aligned}&\frac{\alpha_1}{2} \overline{\lim} \int_\Omega \tilde{A}(u_\delta, \nabla u_\delta) \cdot \nabla u_\delta \, dx - \frac{\alpha_1}{\alpha} \int_\Omega \xi \cdot \nabla u_\varepsilon \, dx \\ &\quad - \frac{\alpha_1}{2} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u dx + \frac{\alpha_1}{\alpha} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \, dx \\ &\leqq c \int_\Omega \xi \cdot \nabla u_\varepsilon \, |h_\lambda(u - u_\varepsilon)| \, dx + c \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u \, |h_\lambda(u - u_\varepsilon)| \, dx.\end{aligned} \tag{2.17}$$

Faisons tendre ε vers 0 dans (2.17), pour obtenir

$$\begin{aligned}&\frac{\alpha_1}{\alpha} \limsup_{\delta \to 0} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\delta) \cdot \nabla u_\delta \, dx + \frac{\alpha_1}{\alpha} \limsup_{\varepsilon \to 0} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \\ &\leqq \frac{\alpha_1}{\alpha} \int_\Omega \xi \cdot \nabla u dx + \frac{\alpha_1}{\alpha} \int_\Omega \xi \cdot \nabla u \, dx.\end{aligned} \tag{2.18}$$

(On remarquera que les termes du second membre de (2.17) tendent vers 0 quand ε tend vers 0.)

Ainsi,

$$\limsup_{\varepsilon \to 0} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \leqq \int_\Omega \xi \cdot \nabla u \cdot dx. \tag{2.19}$$

Sachant que $\lim_{\varepsilon \to 0} \tilde{A}(u_\varepsilon, \nabla u) = \tilde{A}(u, \nabla u)$ dans $(L^{p'}(\Omega))^N$-fort (par le théorème de la convergence dominée), on déduit

$$\begin{aligned}&\limsup_{\varepsilon \to 0} \int_\Omega [\tilde{A}(u_\varepsilon, \nabla u_\varepsilon) - \tilde{A}(u, \nabla u)] \cdot \nabla(u_\varepsilon - u) \, dx \\ &\quad \leqq \limsup_{\varepsilon \to 0} \int_\Omega \tilde{A}(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \\ &\qquad - \int_\Omega \xi \cdot \nabla u dx - \int_\Omega \tilde{A}(u, \nabla u) \cdot \nabla u dx + \int_\Omega \tilde{A}(u, \nabla u) \cdot \nabla u dx.\end{aligned} \tag{2.20}$$

Par la relation (2.19), le second membre de cette inégalité est négatif, ce qui conduit à

$$\limsup_{\varepsilon \to 0} \int_\Omega [\tilde{A}(u_\varepsilon, \nabla u_\varepsilon) - \tilde{A}(u, \nabla u)] \cdot \nabla(u_\varepsilon - u) \, dx \leqq 0. \tag{2.21}$$

Puisque v_ε tend vers 0 faiblement dans $W_0^{1,p}(\Omega)$, le Lemme 3 résulte alors de (2.21).

Démonstration de la partie (i.1) *et* (i.2) *du Théorème* 2. Grâce au Lemme 3, on peut passer à la limite (quand $\varepsilon \downarrow 0$) dans $(0, 1'_\varepsilon)$ d'où (i.1). La partie (i.2) résulte immédiatement du Théorème 1.

2.4. *Continuité höldérienne des solutions du Problème* (0.1)

Dans cette sous-section, nous démontrons la partie (i.3) du Théorème 2. Auparavant, remarquons que si $r = p'$ et $r > N/(p-1)$ alors $p > N$, dans ce cas le théorème reste valable puisque $W_0^{1,p}(\Omega) \subset C^{0,\alpha}(\bar{\Omega})$ pour $\alpha = 1 - N/p$.

Dans la suite, nous supposerons que $p \leqq N$ et $N \geqq 2$. L'outil essentiel pour la démonstration est l'ensemble $\mathscr{B}_p(\Omega, M', \gamma', \delta, 1/q)$ introduit par Ladyzen'skaja et Ural'ceva ([12, Chap. II, p. 76]). Dans le rappel qui suit, nous utiliserons leurs notations.

DÉFINITION (Ensemble $\mathscr{B}_p(\Omega, M', \gamma', \delta, 1/q)$). Etant donnés M', γ', δ, q quatre nombres strictement positifs avec $q > N$, on définit l'ensemble $X = \mathscr{B}_p(\Omega, M', \gamma', \delta, 1/q)$ par $u \in X$ si et seulement si:

(a) $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ et $\|u\|_\infty \leqq M'$.

(b) u vérifie: pour toute boule K_ρ de rayon ρ contenue dans Ω et pour tout $\sigma \in]0, 1[$, on a

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla u|^p \, dx \leqq \gamma' \left[\frac{1}{\sigma^p \rho^{p(1-N/q)}} \operatorname*{Sup\,ess}_{A_{k,\rho}} [u(x) - k]^p + 1 \right] (\text{mes } A_{k,\rho})^{1-(p/q)}, \tag{2.22}$$

lorsque $k \geqq \operatorname{sup\,ess}_{K_\rho} u - \delta$, où $A_{k,\rho} = \{x \in K_\rho \text{ tel que } u(x) > k\}$ et $K_{\rho-\sigma\rho}$ est une boule concentrique à la boule K_ρ.

(c) La fonction $-u$ vérifie la même hypothèse (b) que la fonction u.

Une condition suffisante pour obtenir la relation (2.22) est

LEMME 4. *Soit $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ vérifiant: Il existe $\gamma'' > 0$ tel que $\psi \in \mathscr{D}(\Omega)$ avec Support $\psi \subset K_\rho$, $0 \leqq \psi \leqq 1$, on ait*

$$\int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx \leqq \gamma'' \left[\int_{A_{k,\rho}} (u(x) - k)^p |\nabla \psi|^p \, dx + (\text{mes } A_{k,\rho})^{1-(p/q)} \right], \tag{2.23}$$

pour $k \geqq \operatorname{sup\,ess}_{K_\rho} u - \delta$; alors u satisfait à la relation (2.22).

Démonstration. Comme à la remarque faite dans [12, p. 76–77], il suffit de bien choisir ψ. Il est loisible de construire une fonction $\psi \in \mathscr{D}(\Omega)$ telle que

— $\psi = 1$ sur $K_{\rho - \sigma\rho}$,
— $0 \leqq \psi \leqq 1$ dans Ω, support $\psi \subset K_\rho$,
— $|\nabla\psi| \leqq c/\sigma\rho$ où la constante c ne dépend que de N.

Pour un exemple de construction de ψ, on peut consulter, par exemple [23, p. 195]. ∎

On montre alors (voir [12, Théorème 6.1, p. 83]) que si $u \in X$ alors u est α-höldérienne à l'interieur de Ω pour un certain exposant $\alpha > 0$ (α ne dépendant que de M', γ', δ, q).

Le lemme suivant nous conduit alors au résultat voulu:

LEMME 5. *$\forall \delta > 0$, il existe une constante $\gamma' > 0$ ne dépendant que de δ et des données sur a_i, F, et T, telle que toute solution u de* (0.1) *appartient à $\mathscr{B}_p(\Omega, M, \gamma', \delta, 1/q) = X$, où $q = (p-1)r$ et M le nombre donné au Théorème* 1.

La partie (a) de la définition de X est satisfaite par la fonction u. Pour montrer la partie (b) de la définition, il suffit de vérifier le

LEMME 6. *Soit $\delta > 0$ alors il existe une constante $c_1 > 0$ ne dépendant que de δ et des données sur a_i, F, et T et telle que $\forall \psi \in \mathscr{D}(\Omega)$ avec support $\psi \subset K_\rho$, $0 \leqq \psi \leqq 1$, on ait*

$$\int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx \leqq c_1 \left[\int_{A_{k,\rho}} (u(x) - k)^p |\nabla\psi|^p \, dx + (\text{mes } A_{k,\rho})^{1 - (p'/r)} \right].$$

Remarque 5. Dans la suite, c_i et $c_{i\varepsilon}$ désigneront des constantes ne dépendant que de δ, des données sur a_i, F, T, et éventuellement d'un paramètre ε que l'on précisera ultérieurement. Souvent, on fera usage de l'inégalité de Young sous la forme suivante: $\forall \varepsilon > 0$, il existe $c_\varepsilon > 0$ tel que si $a \geqq 0$, $b \geqq 0$ alors

$$ab \leqq \varepsilon a^p + c_\varepsilon b^{p'}.$$

On omettra les signes de sommation et l'on écrira: $a_i(u, \nabla u) = a_i$, $F(u, \nabla u) = F$, $\psi(x) = \psi$,....

Démonstration du Lemme 6. Soit $\delta > 0$ (fixé), $\psi \in \mathscr{D}(\Omega)$, $0 \leqq \psi \leqq 1$, et support $\psi \subset K_\rho$ et l'on considère une fonction réelle $l_\lambda(t) = t_+ e^{\lambda(t_+)^2}$, où λ est choisi positif et de telle sorte que

$$\alpha_1 l'_\lambda(t) - f(M)\, l_\lambda(t) \geqq (\alpha_1/2)\, \chi_{\{t>0\}} \qquad (\forall t \neq 0), \tag{2.25}$$

avec $\alpha_1 = \text{Min}_{0 \leqq s \leqq M} \nu_1(s)$ et $\chi_{\{t>0\}}$ la fonction caractéristique de $\mathbb{R}_+ \setminus \{0\}$. Le réel λ ne dépend que de α_1 et de $f(M)$.

Pour $k \geqq \sup \mathrm{ess}_{K_\rho} u - \delta$, on définit

$$u_k(x) = l_\lambda(u-k)(x)$$

et

$$\psi_k(x) = \psi^p(x)\, u_k(x) \qquad \text{pour} \quad x \in \Omega.$$

Il existe deux constantes c_2, c_3 strictement positives t.q. $\forall k \geqq -M-\delta$,

$$0 \leqq u_k(x) \leqq c_2 \cdot (u-k)_+$$

et

$$\left|\frac{\partial u_k}{\partial x_i}\right| = \left|\frac{\partial u}{\partial x_i} l'_\lambda(u-k)\right| \leqq c_3 \left|\frac{\partial u}{\partial x_i}\right| \qquad \text{p.p. dans } \Omega. \tag{2.26}$$

En effet, puisque $k \geqq -M-\delta$, on obtient $(u-k)_+ \leqq 2M+\delta$ d'où la relation (2.26). Puisque $\psi_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, de l'équation (0.1), nous obtenons

$$\langle Au, \psi_k \rangle + (F, \psi_k) = \langle T, \psi_k \rangle. \tag{2.27}$$

$\langle\ \rangle$ désignant un produit de dualité entre un élément de $W^{-1,p'}(\Omega)$ et d'un élément de $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Introduisons trois lemmes concernant chacun des termes de l'égalité de (2.27).

LEMME 7. *$\forall \varepsilon > 0$, il existe $c_{3,\varepsilon} > 0$ tel que*

$$|\langle T, \psi_k \rangle| \leqq \varepsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx$$
$$+ c_{3,\varepsilon} \left(\int_{A_{k,\rho}} (u(x)-k)^p |\nabla \psi|^p \, dx + (\mathrm{mes}\, A_{k,\rho})^{1-(p'/r)} \right).$$

Démonstration du Lemme 7. Ecrivons $T = -\sum_{i=1}^N \partial g_i / \partial x_i$ avec

$$\|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^N \|g_i\|_{L^r(\Omega)} \qquad \text{et posons} \qquad g_0(x) = \left(\sum_{i=1}^N g_i^2(x)\right)^{1/2}.$$

Alors

$$\langle T, \psi_k \rangle = Y_1 + Y_2, \tag{2.28}$$

où

$$Y_1 = \int_{A_{k,\rho}} g_i \frac{\partial u_k}{\partial x_i} \psi^p \, dx, \qquad Y_2 = p \int_{A_{k,\rho}} g_i \frac{\partial \psi}{\partial x_i} \cdot \psi^{p-1} u_k \, dx.$$

Majorons chacun des termes Y_i, puisque $0 \leqq \psi \leqq 1$ et que $|\partial u_k/\partial x_i| \leqq c_2|\partial u/\partial x_i|$ (voir (2.26)), on déduit:

$$\left| g_i \cdot \frac{\partial u_k}{\partial x_i} \psi^p \right| \leqq c_2 \, g_0(x) \cdot |\nabla u| \cdot \psi.$$

Par l'inégalité de Young,

$$c_2 \, g_0(x)|\nabla u| \; \psi \leqq \varepsilon |\nabla u|^p \psi^p + c_{0\varepsilon} \, g_0(x)^{p'}.$$

Ainsi

$$|Y_1| \leqq \varepsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx + c_{0\varepsilon} \int_{A_{k,\rho}} g_0(x)^{p'} \, dx. \tag{2.29}$$

Par l'inégalité de Hölder,

$$\int_{A_{k,\rho}} g_0(x)^{p'} \, dx \leqq \left(\int_{A_{k,\rho}} g_0(x)^r \, dx \right)^{p'/r} (\text{mes } A_{k,\rho})^{1-(p'/r)}$$

$$\leqq \|T\|^{p'}_{W^{-1,r}(\Omega)} (\text{mes } A_{k,\rho})^{1-(p'/r)}.$$

Il existe donc $c_{1\varepsilon} > 0$, t.q.

$$|Y_1| \leqq \varepsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx + c_{1\varepsilon} (\text{mes } A_{k,\rho})^{1-(p'/r)}. \tag{2.30}$$

Pour majorer Y_2, on constate que

$$\left| p g_i \frac{\partial \psi}{\partial x_i} \psi^{p-1} u_k \right| \leqq c_4 \, g_0(x) |\nabla \psi| \cdot (u-k)_+ \qquad \text{(voir (2.26))}.$$

Par l'inégalité de Young

$$c_4 \, g_0(x)|\nabla \psi|(u-k)_+ \leqq g_0(x)^{p'} + c_5 |\nabla \psi|^p (u-k)_+^p .$$

Ainsi

$$|Y_2| \leqq c_5 \int_{A_{k,\rho}} |\nabla \psi|^p (u-k)^p \, dx + \int_{A_{k,\rho}} g_0(x)^{p'} \, dx. \tag{2.31}$$

On conclut comme ci-dessus avec l'inégalité de Hölder,

$$|Y_2| \leqq c_5 \int_{A_{k,\rho}} |\nabla \psi|^p (u-k)^p \, dx + c_6 (\text{mes } A_{k,\rho})^{1-(p'/r)}. \tag{2.32}$$

Le Lemme 7 découle des relations (2.28), (2.30), et (2.32). ∎

LEMME 8. *Il existe une constante* $c_9 > 0$, *t.q.*

$$|(F, \psi_k)| \leqq f(M) \int_{A_{k,\rho}} |\nabla u|^p \psi^p l_\lambda(u-k) \, dx + c_9 (\text{mes } A_{k,\rho})^{1-(p'/r)}.$$

Démonstration du Lemme 8. Par l'hypothèse (0.7) sur F, nous avons

$$|(F, \psi_k)| \leqq f(M) \int_{A_{k,\rho}} |\nabla u|^p \psi^p l_\lambda(u-k)\, dx$$

$$+ f(M) \int_{A_{k,\rho}} f_0(x)\, \psi^p u_k(x)\, dx. \tag{2.33}$$

Puisque $(u-k)_+ \leqslant 2M+\delta$ alors $u_k \leqq (2M+\delta)\, e^{\lambda(2M+\delta)^2}$, il existe alors $c_8 > 0$ telle que

$$f(M) \int_{A_{k,\rho}} f_0(x)\, \psi^p u_k\, dx \leqq c_8 \int_{A_{k,\rho}} f_0(x)\, dx. \tag{2.34}$$

Par hypothèse, $f_0 \in L^{r/p'}(\Omega)$, l'inégalité de Hölder conduit à

$$\int_{A_{k,\rho}} f_0(x)\, dx \leqq \|f_0\|_{L^{r/p'}(\Omega)} \cdot (\text{mes}\, A_{k,\rho})^{1-(p'/r)}. \tag{2.35}$$

Les relations (2.33), (2.34), et (2.35) entraînent le Lemme 8.

LEMME 9. *$\forall \varepsilon > 0$, il existe $c_{8\varepsilon} > 0$ tel que*

$$\langle Au, \psi_k \rangle \geqq \alpha_1 \int_{A_{k,\rho}} \psi^p |\nabla u|^p l'_\lambda(u-k)\, dx - \varepsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p\, dx$$

$$- c_{8\varepsilon} \left(\int_{A_{k,\rho}} (u-k)^p |\nabla \psi|^p\, dx + (\text{mes}\, A_{k,\rho})^{1-(p'/r)} \right),$$

où $\alpha_1 = \text{Min}_{0 \leqq \eta \leqq M}\, \nu_1(\eta) > 0$.

Démonstration du Lemme 9. On a

$$\langle Au, \psi_k \rangle = X_1 + X_2 \tag{2.36}$$

où

$$X_1 = \int_{A_{k,\rho}} a_i \psi^p \frac{\partial u_k}{\partial x_i}\, dx, \qquad X_2 = p \int_{A_{k,\rho}} a_i \frac{\partial \psi}{\partial x_i} \psi^{p-1} u_k\, dx.$$

On va minorer chacun des termes X_i $(i=1,2)$. On note $\beta_1 = \text{Max}_{0 \leqq \eta \leqq M}\, \nu_2(\eta)$ alors en écrivant $\partial u_k/\partial x_i = (\partial u/\partial x_i)\, l'_\lambda(u-k)$, l'hypothèse (0.4) implique

$$X_1 \geqq \alpha_1 \int_{A_{k,\rho}} \psi^p |\nabla u|^p l'_\lambda(u-k)\, dx - \beta_1 \int_{A_{k,\rho}} k(x)^{1/p'} |\nabla u|\, \psi^p l'_\lambda(u-k)\, dx. \tag{2.37}$$

Par (2.26), il existe $c_{10} > 0$ tel que

$$\beta_1 k(x)^{1/p'} |\nabla u|\, \psi \leqq c_{10} k(x)^{1/p'} |\nabla u|\, \psi \qquad \text{p.p.}$$

Par l'inégalité de Young, on trouve $c_{4\varepsilon} > 0$ t.q.

$$c_{10} k(x)^{1/p'} |\nabla u|\, \psi \leqq (\varepsilon/2) |\nabla u|^p \psi^p + c_{4\varepsilon} k(x) \qquad \text{p.p.}$$

Puisque $k \in L^{r/p'}(\Omega)$ (voir (0.4)), l'inégalité de Hölder conduit à

$$\int_{A_{k,\rho}} k(x)\,dx \leqq \|k\|_{L^{r/p'}(\Omega)}(\text{mes } A_{k,\rho})^{1-(p'/r)}.$$

Ainsi

$$-\beta_1 \int_{A_{k,\rho}} k(x)^{1/p'}|\nabla u|\,\psi^p l'_\lambda(u-k)\,dx$$
$$\geqq -\frac{\varepsilon}{2}\int_{A_{k,\rho}} |\nabla u|^p \psi^p\,dx - c_{5\varepsilon}(\text{mes } A_{k,\rho})^{1-(p'/r)}. \tag{2.38}$$

Pour minorer le terme X_2, utilisons l'hypothèse (0.5) et la relation (2.26) pour obtenir

$$\left|pa_i \frac{\partial\psi}{\partial x_i}\psi^{p-1}u_k\right| \leqq Z_1 + Z_2 \qquad \text{p.p.,} \tag{2.39}$$

où

$$Z_1 = c_{11}|\nabla u|^{p-1}\psi^{p-1}|\nabla\psi|(u-k)_+,$$
$$Z_2 = c_{12}\psi^{p-1}|\nabla\psi|(u-k)_+ a_0 \qquad \text{p.p.}$$

Par l'inégalité de Young, le terme Z_1 se majore par

$$Z_1 \leqq \frac{\varepsilon}{2}|\nabla u|^p\psi^p + c_{6\varepsilon}|\nabla\psi|^p(u-k)_+^p$$

et le terme Z_2 par

$$Z_2 \leqq c_{13}a_0^{p'} + c_{14}|\nabla\psi|^p(u-k)_+^p \qquad \text{p.p.}$$

Par hypothèse $a_0 \in L^r(\Omega)$, l'inégalité de Hölder entraîne

$$\int_{A_{k,\rho}} a_0^{p'}\,dx \leqq \|a_0\|_{L^r(\Omega)}^{p'}(\text{mes } A_{k,\rho})^{1-(p'/r)}.$$

Finalement, après intégration de la relation (2.39) et compte tenu des majorations de Z_1 et de Z_2:

$$X_2 \geqq -\frac{\varepsilon}{2}\int_{A_{k,\rho}} |\nabla u|^p\psi^p\,dx - c_{7\varepsilon}$$
$$\times\left(\int_{A_{k,\rho}} |\nabla\psi|^p(u-k)^p\,dx + (\text{mes } A_{k,\rho})^{1-(p'/r)}\right) \tag{2.40}$$

La relation (2.40) jointe aux relations (2.36), (2.37), et (2.38) impliquent le Lemme 9.

Fin de la démonstration du Lemme 6. De (2.27), on obtient

$$\langle Au, \psi_k\rangle \leqq |(F, \psi_k)| + |\langle T, \psi_k\rangle|. \tag{2.41}$$

Compte tenu des Lemmes 7, 8, 9, il existe $c_{9\varepsilon} > 0$ tel que

$$\int_{A_{k,\rho}} [\alpha_1 l'_\lambda(u-k) - f(M)\, l_\lambda(u-k)]\, \psi^p |\nabla u|\, dx$$
$$\leqq 2\varepsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p\, dx + c_{9\varepsilon}\left(\int_{A_{k,\rho}} |\nabla\psi|^p (u-k)^p\, dx + (\text{mes } A_{k,\rho})^{1-(p'/r)}\right).$$

Par le choix de λ (voir (2,25)):

$$\int_{A_{k,\rho}} [\alpha_1 l'_\lambda(u-k) - f(M)\, l_\lambda(u-k)]\, \psi^p |\nabla u|^p\, dx \geqq \frac{\alpha_1}{2} \int_{A_{k,\rho}} |\nabla u|^p \psi^p\, dx.$$

Ainsi, $\forall \varepsilon > 0$,

$$\left(\frac{\alpha_1}{2} - 2\varepsilon\right) \int_{A_{k,\rho}} |\nabla u|^p \psi^p\, dx$$
$$\leqq c_{9\varepsilon}\left(\int_{A_{k,\rho}} |\nabla\psi|^p (u-k)^p\, dx + (\text{mes } A_{k,\rho})^{1-(p'/r)}\right).$$

En choisissant $\varepsilon = \alpha_1/8$, cette dernière relation entraîne le Lemme 6. ∎

Pour terminer la démonstration du Lemme 5, il suffit de prouver que $-u$ satisfait la même propriété que u à savoir le

LEMME 6 BIS. *Soit $\delta > 0$ alors il existe une constante $c_{11} > 0$ telle que $\forall \psi \in \mathscr{D}(\Omega)$, avec Support $\psi \subset K_\rho$, $0 \leqq \psi \leqq 1$, on ait*

$$\int_{A'_{k,\rho}} |\nabla u|^p \psi^p\, dx \leqq c_{11}\left[\int_{A'_{k,\rho}} (-u-k)^p |\nabla\psi|^p\, dx + (\text{mes } A'_{k,\rho})^{1-p'/r}\right],$$

où $A'_{k,\rho} = \{x \in K_\rho,\ -u(x) > k\}$, $k \geqq \text{Sup ess}_{K_\rho}(-u) - \delta$.

Démonstration. On remplace ψ_k par $\bar\psi_k$ définie par $\bar\psi_k(x) = -\psi^p(x)\, l_\lambda(-u-k)$. Des considérations analogues à celles données ci-dessus conduisent au Lemme 6 bis. ∎

APPENDICE

Démonstration de la remarque 2. Puisque $v \in W_0^{1,1}(\Omega)$, $v \geqq 0$, alors $v_* \in W^{1,1}(a, |\Omega|)$ pour tout $a > 0$ (cf. [15]). Ainsi on déduit que

(i) $t = v_*(\mu(t))$ $\forall t \in [0, \text{sup ess } v]$;

(ii) v_* est absolument continue sur tout intervalle $[a, |\Omega|]$, avec $a > 0$.

Les relations (i) et (ii) assurent que pour tout sous ensemble négligeable E de Ω^*, on a

IV.32

(iii) $v_*(E)$ est de mesure nulle (on rapelle qu'une fonction absolument continue envoie un ensemble négligeable en un ensemble négligeable).

(iv) L'ensemble $\{t \geqq 0 \text{ t.q. } \mu(t) \in E\}$ est inclus dans $v_*(E)$ et donc lui-même est de mesure nulle.

Par ailleurs, pour presque tout t, nous avons $0 = |v = t| = |v = v_*(\mu(t))|$. Ainsi, en utilisant la fonction w définie au début des rappels (associeé à f et v), nous obtenons que pour presque tout t:

$$\int_{v>t} f(x)\, dx = w(\mu(t)). \tag{R.3}$$

Les fonctions $\int_{v>t} f(x)\, dx$, w, et μ sont presque partout différentiables. L'assertion (iv) et la relation (R.3) impliquent alors que pour presque tout t:

$$\frac{d}{dt}\int_{v>t} f(x)\, dx = \mu'(t)\frac{dw}{ds}(\mu(t)) = \mu'(t)\, f_{*v}(\mu(t)).$$

Quant à la relation (R.2), si $\bar f$ est continue alors c'est une conséquence immédiatement de l'inégalité classique (cf. [15 ou 22]):

$$\int_a^b \bar f(\mu(\theta))(-\mu'(\theta))\, d\theta \leqq \int_{\mu(a)}^{\mu(b)} \bar f(\sigma)\, d\sigma.$$

Si $\bar f$ est seulement localement intégrable, alors pour $0 < s \leqq s' < |\Omega|$, il existe une suite de fonctions continues $\bar f_n \geqq 0$ telle que $\bar f_n(\sigma) \underset{\infty}{\rightrightarrows} \bar f(\sigma)$ p.p. en $\sigma \in [s, s']$ et $\bar f_n$ tend vers $\bar f$ dans $L^1(s, s')$. L'assertion (iv) assure alors que pour presque tout θ dans $[v_*(s'), v_*(s)]$, $\lim_n \bar f_n(\mu(\theta)) = \bar f(\mu(\theta))$.

Par le Lemme de Fatou et la remarque ci-dessus pour $\bar f_n$ continue, on déduit que

$$\int_{v_*(s')}^{v_*(s)} \bar f(\mu(\theta))(-\mu'(\theta)) \leqq \liminf_n \int_{v_*(s')}^{v_*(s)} \bar f_n(\mu(\theta))(-\mu'(\theta))\, d\theta$$

et

$$\liminf_n \int_{v_*(s')}^{v_*(s)} \bar f_n(\mu(\theta))(-\mu'(\theta))\, d\theta \leqslant \lim_n \int_s^{s'} \bar f_n(\sigma)\, d\sigma = \int_s^{s'} \bar f(\sigma)\, d\sigma. \tag{R.4}$$

Si $s = 0$ (ou ($s' = |\Omega|$), la relation reste valable par passage à la limite dans R.4), i.e., $s \to 0$ (resp. $s' \to |\Omega|$).

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CHAPITRE V

Existence of bounded solutions of some degenerate quasilinear elliptic equations.

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Existence of Bounded Solutions of Some Degenerate Quasilinear Elliptic Equations

Jean Michel Rakotoson*

0. Introduction

We consider the equation:

$$\Delta(|u|^{m-1}u) + f(x,u) = 0 \qquad (E)$$

on a bounded open set Ω of $\mathbb{R}^N$ with Dirichlet boundary condition. This equation is in a closed relation with the stationary case of the porous media equation ([1], [2], [18], [19], [20]). Ju. A. Dubinskii [4], had studied such equation in a direct way, when the growth of f is at most polynomial with respect to u. The problem of uniqueness of solutions has been treated by J. Spruck [21].

The purpose of this article is to give a direct method to study a generalization of such an equation (E). That is, we want to solve the following problem:

$$(\mathcal{P}) \begin{cases} \text{Find } u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \quad 1 < p < +\infty \\ Au + F(u,\nabla u) = G \text{ in } \Omega \end{cases}$$

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Here, A is an operator of Leray-Lions type [7] which maps $W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$ into $W^{-1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and which degenerates with respect to u. The nonlinearity F will depend on x, u but also on ∇u and G will be some element of $W^{-1,p'}(\Omega) \cap L^1(\Omega)$. A typical sample of equation that we can solve is:

$$-\Delta(|u|^{m-1}u) + a(x)\, u\, |\nabla u|^{2-\epsilon} + b(x)\, u = G \qquad (0.1)$$

As in [14], one of the main novelties of this article is to derive the L^∞-estimate of the solution u of $(\mathcal{P})$. So in Section I, we use the techniques of the relative rearrangement (see [9], [10], [11], [12], [13]) to get a priori estimates in $L^\infty(\Omega)$. This maximum principle will be extended in a forthcoming work to elliptic inequalities ([17]) and to some elliptic systems ([16]).

A second difficulty is to find, despite the degeneracy, a solution in $W_o^{1,p}(\Omega)$. To overcome such difficulty, we introduce in Section II, a family of modified problems $(\mathcal{P}_n)_{n\in\mathbb{N}}$. As for the problem $(\mathcal{P})$, we prove (by the same techniques) that the solution v_n of the perturbed problems $(\mathcal{P}_n)$ are in a bounded set of $L^\infty(\Omega)$ which is independent of n.

This estimate provides us a suitable truncation of the problem $(\mathcal{P}_n)$ denoted $(\mathcal{P}'_n)$, the problem $(\mathcal{P}'_n)$ has the same solutions as the problem $(\mathcal{P}_n)$ and satisfies the standard conditions of Leray-Lions' theorem (see [8]).

The last section is devoted to the passage to the limit in $(\mathcal{P}'_n)$ (using a compactness argument) in order to prove the existence of solution for $(\mathcal{P})$.

For the L^∞-estimate, we use essentially the notion of relative rearrangement. This notion introduced in [11] is widely studied and developed in [10], [12], [13]. Further applications are given in [14], [15], [16], [17].

For convenience, we begin by recalling the definitions and some properties.

1. Relative rearrangement: Definition Properties

Let Ω be an open bounded set of $\mathbb{R}^N$ $(N\geq 1)$ and u a real measurable[(1)] function from Ω into $\mathbb{R}$.

The decreasing rearrangement of u is defined on $\overline{\Omega^*} = [0,|\Omega|]$[(2)] by:

$$u_*(s) = \operatorname{Inf} \{\delta \in \mathbb{R},\ |u>\delta| \leq s\} \quad \text{if} \quad s \in \,]0,|\Omega|]$$

$$u_*(0) = \operatorname*{ess\,sup}_{\Omega} u$$

Let $v \in L^1(\Omega)$, we define a function w in Ω^* by:

$$w(s) = \begin{cases} \displaystyle\int_{u>u_*(s)} v(x)\,dx & \text{if } |u=u_*(s)| = 0 \\ \displaystyle\int_{u>u_*(s)} v(x)\,dx + \int_0^{s-|u>u_*(s)|} (v|_{P(s)})_*(t)\,dt & \text{otherwise} \end{cases}$$

Here the last integrand is the decreasing rearrangement of the restriction of v to the set $P(s) = \{u = u_*(s)\}$ supposed to be of positive measure.

The following theorem is proved in [10]:

THEOREM 1:

Let u *be a measurable function defined on* Ω , v *in* $L^p(\Omega)$ $(1 \leq p \leq +\infty)$. *Then :*

(i) $w \in W^{1,p}(\Omega^*)$ *where* $\Omega^* =]o, |\Omega|[$

(ii) $\|\frac{dw}{ds}\|_{L^p(\Omega^*)} \leq \|v\|_{L^p(\Omega)}$

<u>Definition</u> 1 : <u>Relative</u> <u>rearrangement</u> (see [11])

The function $\frac{dw}{ds}$ is called the rearrangement of v with respect to u and is denoted v_{*u} . The following properties are proved in [14].

PROPOSITION 1

Let v *be an element of* $W_o^{1,1}(\Omega)$, $v \geq o$ *and* $f \in L^1(\Omega)$.

Then:

(i_1) $\frac{d}{dt}\int_{v>t} f(x)dx = \mu'(t)\, f_{*v}\,(\mu(t))$ *for* a.e t *in*

$]o\ ,\ \operatorname{ess\,sup}_{\Omega} v\ [$, *where* $\mu(t) = |v > t|$

(i_2) *Let* $\bar{f}$ *be a positive locally integrable function on* Ω^* .

Then, for all $(s,s') \in \bar{\Omega}^* \times \bar{\Omega}^*$, $s \leq s'$

$$\int_{v_*(s')}^{v_*(s)} \bar{f}(\mu(\theta))\ (-\mu'(\theta))\ d\theta \leq \int_{s'}^{s} \bar{f}(\sigma)\ d\sigma \qquad \square$$

We also need the following proposition (see [10]) .

PROPOSITION 2:

Let u *be a measurable function defined on* Ω *and* $v \in L^1(\Omega)$. *If* $v \geq o$, a.e. *in* Ω , *then* $v_{*u} \geq o$ a.e. *in* Ω^*.

Let E *be a measurable subset of* $\mathbb{R}^N$; $P_{\mathbb{R}^N}(E)$ *denotes the perimeter of* E *(in the sense of De Giorgi* [5]) *in* $\mathbb{R}^N$ *Then, either*

(1.1) $P_{\mathbb{R}^N}(E) \geq N\, \alpha_N^{1/N}\, |E|^{1-\frac{1}{N}}$

or

(1.2) $P_{\mathbb{R}^N}(E) \geq N\, \alpha_N^{1/N}\, |\mathbb{R}_N \setminus E|^{1-\frac{1}{N}}$

where α_N *is the measure of the unit ball of* $\mathbb{R}^N$

THEOREM 3 : The Fleming-Rischel Formula ([6])

For any f *in* $W^{1,1}(\Omega)$, *we have*

$$\int_\Omega |\nabla f|\, dx = \int_{-\infty}^{+\infty} P_\Omega\ (f>t)\ dt \qquad (1.3)$$

where $P_\Omega(f>t)$ *is the perimeter of* $\{f>t\}$ *in* Ω .

From Theorem 2 and Theorem 3, we get the

COROLLARY 1

Let u *be in* $W_o^{1,p}(\Omega)$ *and* $v = |u|$ *Then, for* a.e. θ *in*]o , $\operatorname{ess\,sup}_\Omega v$ [.

$$\frac{d}{d\theta} \int_{v>\theta} |\nabla u| dx \geq N \alpha_N^{1/N} |v>\theta|^{1-\frac{1}{N}} \tag{1.4}$$

PROOF

Let us observe first that $\int_{v>\theta} |\nabla u| dx = \int_{v>\theta} |\nabla v| dx$ (1.5)

In Theorem 3, we choose $f = (v-\theta)_+ + \theta$, then

$$\int_{v>\theta} |\nabla v| dx = \int_\Omega |\nabla f| dx = \int_{-\infty}^{+\infty} P_\Omega(f>t)\, dt \tag{1.6}$$

One can check that:

$$P_\Omega(f>t) = \begin{cases} P_\Omega(\Omega) = o & \text{if } t \leq \theta \\ P_\Omega(v>t) & \text{if } t > \theta \end{cases}$$

Hence, from (1.5) (1.6) and (1.7), we find :

$$\frac{d}{d\theta}\int_{v>\theta}|\nabla u|\,dx = P_\Omega(v>\theta) \qquad (1.8)$$

Let $\bar{v}$ be a function defined in $\mathbb{R}^n$ by:

$$\bar{v} = \begin{cases} v & \text{in } \Omega \\ o & \text{otherwise} \end{cases}$$

Then, we get early $P_\Omega(v>\theta) = P_{\mathbb{R}^N}(\bar{v}>\theta)$ and $|v>\theta| = |\bar{v}>\theta|$ for any $\theta > o$. By Theorem 2, we have

$$P_{\mathbb{R}^N}(v>\theta) \geq N\,\alpha_N^{1/N}|v>\theta|^{1-1/N} \qquad (1.9)$$

and then

$$\frac{d}{d\theta}\int_{v>\theta}|\nabla u|\,dx = P_\Omega(v>\theta) = P_{\mathbb{R}^N}(\bar{v}>\theta) \geq N\,\alpha_N^{1/N}|v>\theta|^{1-1/N}$$

The corollary 1 is used in [9] for $u \geq o$. □

2. HYPOTHESES The Main Results

Let Ω be an open bounded set of $\mathbb{R}^N$, we consider the problem $(\mathcal{P})$ with:

(H.1) $G \in W^{1,r}(\Omega) \cap L^1(\Omega)$, $r > \frac{N}{p-1}$, $r \geq p'$

(H.2) The map F is a Caratheodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ i.e.:

For all (η,ξ) fixed in $\mathbb{R} \times \mathbb{R}^N$, $x \to F(x,\eta,\xi)$ is measurable for a. a x fixed in Ω, $(\eta,\xi) \to F(x,\eta,\xi)$ is continuous.

Furthermore, we assume that F satisfies the two following conditions:

(i) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\eta \, F(x, \eta, \xi) \geq o$$

(ii) There exists an increasing function f from $\mathbb{R}_+$ into $\mathbb{R}_+$ vanishing and continuous at o, and a positive function f_o of $L^1(\Omega)$, such that for a. e $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|F(x,\eta,\xi)| \leq f(|\eta|) \, [|\xi|^{p-\epsilon} + f_o(x)], \; o < \epsilon \leq p$$

(H.3) Each $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function satisfying the following degeneracy condition :

There exists a continuous function ν from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that : $\nu(o) = o$ and $\nu(\eta) > o$ if $\eta > o$ and for a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$:

$$\sum_{i=1}^{N} a_i(x,\eta,\xi) \, \xi_i \geq \nu(|\eta|) \, |\xi|^p$$

$$\int_o^{+\infty} \nu(t)^{-1} dt = \int_o^{+\infty} \nu(t)^{p'-1} dt = +\infty$$

(H.4) Furthermore, we assume that there exists a Caratheodory function $\bar{a}_i$ $(i=1,\dots,N)$ such that:

For a. e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$

(i) $a_i(x,\eta,\xi) = \nu(|\eta|)\ \bar{a}_i\ (x,\eta,\xi)$

(ii) $\sum_{i=1}^{N} [\bar{a}_i(x,\eta,\xi) - \bar{a}_i(x,\eta,\xi')]\ [\xi_i - \xi'_i] > o$

(iii) The functions $\bar{a}_i$ are positively homogeneous of degree (p-1) with respect to the variable ξ i.e.,

$$(\forall t \geq o)\quad \bar{a}_i(x,\eta,t\xi) = t^{p-1}\ \bar{a}_i(x,\eta,\xi)$$

(iv) There exists an increasing function h from $\mathbb{R}_+$ into $\mathbb{R}_+$ and a positive element h of $L^{p'}(\Omega)$ such that:

$$|\bar{a}_i(x,\eta,\xi)| \leq h(|\eta|)\ [|\xi|^{p-1} + h(x)]$$

Let us consider a few examples of equations for which their conditions are satisfied:

1st Example:

We consider the equation (0.1), ie.,

$$-\Delta(|u|^{m-1}u) + a(x)\ u\ |\nabla u|^{2-\epsilon} + b(x)\ u = G$$

We assume that $G \in L^{N+\epsilon_1}(\Omega)$ for some $\epsilon_1 > o$, a and b are two positive functions with $a \in L^{\infty}(\Omega)$, $b \in L^1(\Omega)$ and $1 < m < 2$.

Let us remark that $|u|^{m-1}u = m \int_o^u |t|^{m-1}dt$. So, we write:

$Au = -\Delta(|u|^{m-1}u) = -m\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[|u|^{m-1}\frac{\partial u}{\partial x_i}\right]$. If we set $a_i(x,\eta,\xi) = m\,|\eta|^{m-1}\xi_i$ for $(x,\eta,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$, then we can take $\nu(|\eta|) = m\,|\eta|^{m-1}$. All the assumptions on ν are satisfied if and only if $1 < m < 2$. Hence, $\bar{a}_i(x,\eta,\xi) = \xi_i$ and $F(x,\eta,\xi) = a(x)\,\eta\,|\xi|^{2-\epsilon} + b(x)\,\eta$ satisfy all the required assumptions.

2nd EXAMPLE:

Let a_o be a function of $L^\infty(\Omega)$, $a_o \geq o$. Consider the equation

$$- \operatorname{div}\left[\frac{1-e^{-|u|^{1/2}}}{1+a_o(x)|u|^{p-1}}\;|\nabla u|^{p-2}\,\nabla u\right] + (e^u-1)|\nabla u|^{p-\epsilon} = G$$

where $G \in L^{\frac{N+p}{p-1}}(\Omega)$. We notice that , for (x,η,ξ) in $\Omega\times\mathbb{R}\times\mathbb{R}^N$: $F(x,\eta,\xi) = (e^\eta-1)\,|\xi|^{p-\epsilon}$; F satisfy the assumptions (H.2).

One can check that

$$a_i\,(x,\eta,\xi) = \frac{1-e^{-|\eta|^{1/2}}}{1+a_o(x)|\eta|^{p-1}}\;|\xi|^{p-2}\,\xi_i$$

and

$$\sum_{i=1}^{N} a_i\,(x,\eta,\xi)\,\xi_i \geq \frac{1-e^{-|\eta|^{1/2}}}{1+\|a_o\|_\infty\,|\eta|^{p-1}}\;|\xi|^p$$

Hence, we can take $\nu(|\eta|) = \dfrac{1-e^{-|\eta|^{1/2}}}{1+\|a_o\|_\infty |\eta|^{p-1}}$.

The assumptions on ν are fulfilled since, $\nu(|\eta|) \sim |\eta|^{1/2}$ as $\eta \to o$ and $\nu(\eta) \sim \dfrac{1}{1+\|a_o\|_\infty |\eta|^{p-1}}$ as $\eta \to +\infty$. The function $\bar{a}_i\ (x,\eta,\xi) = \dfrac{1+\|a_o\|_\infty |\eta|^{p-1}}{1+a_o(x)|\eta|^{p-1}} |\xi|^{p-2}\xi_i$ satisfies the assumptions (H.4).

Other possible examples include $a_i(x,\eta,\xi) = |\eta|^{1/2}e^{|\eta|}\ |\xi|^{p-2}\xi_i$ then $\nu(|\eta|) = |\eta|^{1/2}$ and $\bar{a}_i\ (x,\eta,\xi) = e^{|\eta|}\ |\xi|^{p-2}\xi_i$. □

Similar problems have been treated in [14] in the case that $\nu > o$ but the growth of F is at most of the order of $|\nabla u|^p$ and G is only in $W^{-1,r}(\Omega)$, $r > N|p-1$ $r \geq p'$ while here, we assume that $\nu(o) = o$ and G is also a function of $L^1(\Omega)$.

The main results that we obtain are the following:

THEOREM 4:

We set $k(\eta) = \int_o^\eta \nu(t)^{p'-1}\,dt$ *for any* $\eta \geq 0$. *Under the previous assumptions* (H1) *to* (H4) , *any solution* v *of* $(\mathcal{P})$ (H.1) *to* (H.4) *satisfies the following estimate:*

$$\|v\|_{L^\infty(\Omega)} \leq k^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}}\|G\|_{W^{-1,r}(\Omega)}^{p'/p}\right] = M$$

where $\mathcal{R}^{-1}$ *is the inverse function of* $\mathcal{R}$ *mapping* $\mathbb{R}_+$ *onto* $\mathbb{R}_+$, α_N *is the measure of the unit ball, and*

$$\delta = \left[\int_0^{|\Omega|} t^{\left(\frac{1}{N} - 1\right)\left[\frac{(p-1)r}{(p-1)r-1}\right]} dt\right]^{1-\frac{1}{(p-1)r}}$$

is finite since $r > \frac{N}{p-1}$. □

THEOREM 5:

Under the previous assumptions (H1) *to* (H4), *there exists at least one solution* u *of problem* $(\mathcal{P})$ *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

These theorems are proved in §3 to 5 .

3. A PRIORI ESTIMATE IN $L^\infty(\Omega)$.

As in [14], we introduce a more general lemma which is independent of the previous problem $(\mathcal{P})$ and which will prove Theorem 4.

For this purpose, we consider two real (fixed) numbers $\theta > 0$, $h > 0$ and we associate to them the family of real Lipschitz functions $S_{\theta,h}$ by setting :

$$S_{\theta,h}(\eta) = \begin{cases} 1 & \text{if } \eta \geq \theta + h \\ \frac{1}{h}(\eta - \theta) & \text{if } \theta \leq \eta \leq \theta + h \\ 0 & \text{if } |\eta| \leq \theta \\ \frac{1}{h}(\eta + \theta) & \text{if } -\theta - h \leq \eta \leq -\theta \end{cases} \qquad (3.0)$$

$$\begin{cases} -1 & \text{if } \eta \leq -\theta - h \end{cases}$$

LEMMA 1

For a. e. x *in* Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$ $\forall \theta > 0$, $\forall h > 0$

$$S_{\theta,h}(\eta)\ F(x,\eta,\xi) \geq 0 \tag{3.1}$$

Proof

If $\eta = 0$, $S_{\theta,h}(0) = 0$, the relation (3.1) is then true.

If $\eta \neq 0$, $\frac{S_{\theta,h}}{\eta} > 0$, since $S_{\theta,h}$ is an increasing function vanishing at 0 and

$$S_{\theta,h}(\eta)\ F(x,\eta,\xi) = \frac{S_{\theta,h}(\eta)}{\eta}\ \eta\ F(x,\eta,\xi) \geq 0 \quad \text{(By H.2)}$$

□

Now, we consider a function $u \in W_0^{1,p}(\Omega)$ and we set

$$\bar{u} = S_{\theta,h}(u)\ ,\ v = |u|\ .$$

Moreover, we assume that $\forall \theta \in]0, \operatorname{ess\,sup}_\Omega v[$, $\forall h \in]0, \operatorname{ess\,sup}_\Omega v - \theta[$, $\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) \frac{\partial \bar{u}}{\partial x_i}\, dx +$

$$\int_\Omega \bar{u}\ F(x,u,\nabla u)\ dx = \langle G,\bar{u}\rangle \tag{3.2}$$

$\langle\ \rangle$ denoting the scalar product between an element of

$W^{1,p}(\Omega)$. Concerning relation (3.2) we make the following observations:

Remark 1 :

i) Each term of the equation (3.2) makes sense since

$$\bar{u} \in W_o^{1,p}(\Omega) \quad \text{and} \quad \frac{\partial \bar{u}}{\partial x_i} = \begin{cases} 1/h \ \dfrac{\partial u}{\partial x_i} & \text{if } \theta < |u| \leq \theta + h \\ o & \text{otherwise} \end{cases}$$

On the other hand, by (3.1) $\bar{u}\, F(x,u,\nabla u) \geq o$.

ii) Any solution of ($\mathcal{P}$) satisfies (3.2).

iii) We note also that $W^{-1,r}(\Omega) \subset W^{-1,p'}(\Omega)$.

□

In the sequel, we consider $g_i \in L^r(\Omega)$ such that $G = - \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}$ (in the sense of $\mathcal{D}'(\Omega)$) and $\|G\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)}$. We set $\bar{g} = \left[\sum_{i=1}^{N} g_i^2\right]^{p'/2}$, $\bar{g}_{*v}$ is the relative rearrangement of $\bar{g}$ with respect to $v = |u|$.

LEMMA 2

Assume (H.1) *to* (H.3) *and let* u *be a solution of* (3.2). *Then,* $v_* = |u|_*$ *satisfies the following relations:*

i) $$\frac{dv_*}{ds}\, \nu(v_*(s))^{1/p} \leq \frac{s^{\frac{1}{N}-1}\, [\bar{g}_{*v}(s)]^{\frac{1}{p}}}{N\alpha_N^{\ 1/N}} \tag{3.4}$$

for a.e. s *in* $\Omega^* =]0, |\Omega|[$,

ii) *For all* s *in* $\bar{\Omega}^*$:

$$v_*(s) \le k^{-1}\left[\frac{1}{N\alpha_N}\int_s^{|\Omega|} \sigma^{\frac{1}{N}-1}\,[\bar{g}_{*v}(\sigma)]^{\frac{1}{p}}\,d\sigma\right] \tag{3.5}$$

□

Remark 2:

The integral $\int_0^{|\Omega|} \sigma^{\frac{1}{N}-1}\,[\bar{g}_{*v}(\sigma)]^{\frac{1}{p}}\,d\sigma$ makes sense. In fact, by Proposition 2 , $\bar{g}_{*v} \ge 0$ since $\bar{g} \ge 0$ and by the Hölder inequality, we get:

$$\int_0^{|\Omega|} \sigma^{\frac{1}{N}-1}\,[\bar{g}_{*v}(\sigma)]^{\frac{1}{p}}\,d\sigma \le \left[\int_0^{|\Omega|} \sigma^{(\frac{1}{N}-1)\left[\frac{(p-1)r}{(p-1)r-1}\right]}\,d\sigma\right]^{1-\frac{1}{(p-1)r}} \|\bar{g}_{*v}\|^{\frac{1}{p}}_{L^{r/p'}(\Omega^*)} \tag{3.6}$$

By Theorem 1 and the choice of $(g_i)_{i=1,,N}$, we have:

$$\|\bar{g}_{*v}\|_{L^{r/p'}(\Omega)} \le \|\bar{g}\|_{L^{r/p'}(\Omega)} \le \left[\sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)}\right]^{p'} = \|G\|^{p'}_{W^{-1,r}(\Omega)} \tag{3.7}$$

By setting

$$\delta = \left[\int_0^{|\Omega|} \sigma^{\left(\frac{1}{N}-1\right)\left[\frac{(p-1)r}{(p-1)r-1}\right]} d\sigma \right]^{1-\frac{1}{(p-1)r}}$$

(δ is finite since $r > \frac{N}{p-1}$) . From (3.6) and (3.7), we deduce:

$$\int_0^{|\Omega|} \sigma^{\left(\frac{1}{N}-1\right)} [\bar{g}_{*v}(\sigma)]^{\frac{1}{p}} \, d\sigma \le \delta \, \|G\|^{\frac{p'}{p}}_{W^{-1,r}(\Omega)} \qquad (3.8)$$

Proof of LEMMA 2

For convenience, we will omit the sign $\sum_{i=1}^{N}$. From the equation (3.2), we obtain:

$$\frac{1}{h} \int_{\theta < v \le \theta + h} a_i(x,u,\nabla u) \frac{\partial u}{\partial x_i} dx + \int_\Omega \bar{u} \, F(x,u,\nabla u) dx =$$

$$\frac{1}{h} \int_{\theta < v \le \theta + h} g_i \frac{\partial u}{\partial x_i} \, dx \qquad (3.9)$$

From (H.3), we get

$$\operatorname{Min}_{\theta \le \eta \le \theta + h} \nu(\eta) \int_{\theta < v \le \theta + h} |\nabla u|^p dx \le \int_{\theta < v \le \theta + h} \nu(v) \, |\nabla u|^p dx \le \int_{\theta < v \le \theta + h} a_i(x,u,\nabla u) \cdot \frac{\partial u}{\partial x_i} dx \qquad (3.10)$$

By Lemma 1, we deduce:

$$\int_{\Omega} \bar{u}\, F(x,u,\nabla u)\, dx \geq o \tag{3.11}$$

By the Hölder inequality, we have:

$$\frac{1}{h} \int g_i \frac{\partial u}{\partial x_i}\, dx \leq \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} \bar{g}(x) dx\right]^{1/p'} \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} |\nabla u|^p dx\right]^{1/p} \tag{3.12}$$

From (3.9) to (3.12)

$$\underset{\theta \leq \eta \leq \theta + h}{\text{Min}}\ \nu(\eta) \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} |\nabla u|^p dx\right] \leq \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} \bar{g}(x) dx\right]^{1/p'}$$

$$\left[\frac{1}{h} \int_{< v \leq \theta + h} |\nabla u|^p dx\right]^{1/p} \tag{3.13}$$

After simplification, we find:

$$\underset{\theta \leq \eta \leq \theta + h}{\text{Min}}\ \nu(\eta) \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} |\nabla u|^p dx\right]^{1/p'} \leq \left[\frac{1}{h} \int_{\theta < v \leq \theta + h} \bar{g}(x) dx\right]^{1/p'} \tag{3.14}$$

When h tends to zero, we find that for a.e. θ on

$]o\ ,\ \text{ess}\sup_{\Omega} v\ [$

$$\nu(\theta)\left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p dx\right]^{1/p'} \le \left[-\frac{d}{d\theta}\int_{v>\theta}\bar{g}(x)dx\right]^{1/p'} \qquad (3.15)$$

By the Hölder inequality:

$$\frac{1}{h}\int_{\theta<v\le\theta+h}|\nabla u|dx \le \left[\frac{1}{h}\int_{\theta<v\le\theta+h}|\nabla u|^p dx\right]^{1/p}\left[\frac{1}{h}\int_{\theta<v\le\theta+h}dx\right]^{1/p'} \qquad (3.16)$$

When, h tends to zero in (3.16), we arrive to

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|dx \le \left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p dx\right]^{1/p}(-\mu'(\theta))^{1/p'} \qquad (3.17)$$

Here, $\mu'(\theta)$ is the derivative (in the usual sense) of the non-increasing function $\mu(\theta) = |v > \theta|$.

By Corollary 1 of Theorem 2 and Theorem 3, we obtain

$$-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|dx \ge N\alpha_N^{1/N}[\mu(\theta)]^{1-\frac{1}{N}} \qquad (3.18)$$

We combine (3.17) and (3.18) to obtain:

$$\left[-\frac{d}{d\theta}\int_{v>\theta}|\nabla u|^p dx\right]^{1/p}(-\mu'(\theta))^{1/p} \ge N\alpha_N^{1/N}[\mu(\theta)]^{1-\frac{1}{N}} \qquad (3.19)$$

The relations (3.15) and (3.19) lead to:

$$N\alpha_N^{1/N}\ \mu(\theta)^{1-\frac{1}{N}}\ \nu_1(\theta)^{p'/p} \leq (-\mu'(\theta))^{1/p'}\left[\frac{d}{d\theta}\int_{v>\theta}\bar{g}(x)dx\right]^{1/p} \tag{3.20}$$

We use Proposition 1, (i_1) to get that, for almost all $\theta \in]o , \operatorname{ess\,sup}_\Omega v[$,

$$-\frac{d}{d\theta}\int_{v>\theta}\bar{g}(x)dx = -\mu'(\theta)\ \bar{g}_{*v}(\mu(\theta)) \tag{3.21}$$

Since $v_* \in \mathscr{C}^o(\Omega)$, we have for all $\theta \in]o , \operatorname{ess\,sup}_\Omega v[$

$$\theta = v_*\ (\mu(\theta)) \tag{3.22}$$

From (3.20) to (3.22), we have

$$1 \leq \frac{1}{N\alpha_N^{1/N}}\ [\mu(\theta)]^{\frac{1}{N}-1}\ [\nu(v_*(\mu(\theta))]^{-p/p'}\ \{\bar{g}_{*v}(\mu(\theta))\}^{1/p}(-\mu'(\theta)) \tag{3.23}$$

If $o \leq \sigma < |v > o|$, then $v_*(\sigma) > o : \nu(v_*(\sigma)) > o$

The mapping $\sigma \in]o , |v>o| [\rightarrow \dfrac{\sigma^{\frac{1}{N}-1} [\bar{g}_{*v}(\sigma)]^{1/p}}{\nu(v_*(\sigma))^{p'/p}}$ is then positive, locally integrable on $(o, |v > o|)$. We apply Proposition 1 to the relation (3.23), we deduce that for $o < s < s + \eta < |v > o| = \mu(o)$

$$\frac{v_*(s) - v_*(s+\eta)}{\eta} \leq \frac{1}{N\alpha_N^{1/N}} \frac{1}{\eta} \int_s^{s+\eta} \frac{\sigma^{\frac{1}{N}-1} [\bar{g}_{*v}(\sigma)]^{\frac{1}{p}}}{\nu(v_*(\sigma))^{p'/p}} d\sigma$$

When η tends to zero, we get for a. e. s on $(o,\mu(o))$

$$- \frac{dv_*}{ds} \nu(v_*(s))^{p'/p} \leq \frac{s^{\frac{1}{N}-1} [\bar{g}_{*v}(s)]^{\frac{1}{p}}}{N\alpha_N^{1/N}} \qquad (3.24)$$

if $\mu(o) = |v > o| \leq s \leq |\Omega|$, then $v_*(s) = o$, thus $\nu(v_*(s)) = o$.

Since $\frac{dv_*}{ds}$ exists almost everywhere in Ω^* and $\bar{g}_{*v} \geq o$, we deduce that the relation (3.24) is also valid for a.a. $s \in (\mu(o) , |\Omega|)$. This proves Lemma 2 , (i) .

To prove (ii), we consider $k(t) = \int_o^t \nu(\eta)^{p'/p} d\eta$ for any $t \geq o$ and we set $K(s) = k(v_*(s))$.

Since $v_* \in W^{1,p}_{loc}(\Omega^*)$, we have $K \in W^{1,p}_{loc}(\Omega^*)$ and

$$- \frac{dK}{ds} = - \frac{dv_*}{ds} \nu(v_*(s))^{p'/p} \quad \text{for a.e .s in } \Omega^* \tag{3.25}$$

From (3.24) and (3.25), we have for a.a.s $\in$]o , $|\Omega|$[= Ω^*

$$- \frac{dK}{ds} \leq \frac{s^{1/N-1}}{N\alpha_N^{1/N}} [\bar{g}_{*v}(s)]^{1/p} \tag{3.26}$$

Since $v_*(|\Omega|) = \operatorname{ess\,inf}_{\Omega} v = o$, a simple integration of (3.26) leads to:

$$K(s) \leq \frac{1}{N\alpha_N^{1/N}} \int_s^{|\Omega|} \sigma^{\frac{1}{N}-1} [\bar{g}_{*v}(\sigma)]^{1/p} d\sigma$$

for all $s \in \bar{\Omega}^*$. (3.27)

The function $\mathcal{k}$ is continuous and strictly increasing from $\mathbb{R}_+$ into $\mathbb{R}_+$, so it is invertible and its inverse $\mathcal{k}^{-1}$ is also strictly increasing. Hence, the relation (3.27) is equivalent to:

$$v_*(s) \leq \mathcal{k}^{-1} \left[\frac{1}{N\alpha_N^{1/N}} \int_s^{|\Omega|} \sigma^{\frac{1}{N}-1} [\bar{g}_{*v}(\sigma)]^{1/p} d\sigma\right]$$

for all $s \in \bar{\Omega}_*$. (3.28)

□

A consequence of lemma 2 is the following statement slightly stronger than Theorem 4:

If $u \in W_0^{1,p}(\Omega)$ is a solution of equation (3.2), then $u \in L^\infty(\Omega)$ and u satisfies the estimate:

$$\|u\|_{L^\infty(\Omega)} \leq \mathcal{k}^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)}\right] \tag{3.29}$$

where δ is the same constant as in Theorem 4:

PROOF:

From Remark 2, the relation (3.8) leads to

$$0 \leq \int_0^{|\Omega|} \sigma^{\frac{1}{N}-1} [\bar{g}_{*v}(\sigma)]^{1/p} d\sigma \leq \delta \|G\|^{p'/p}_{W^{-1,r}(\Omega)}$$

Since $\mathcal{k}^{-1}$ is increasing, we deduce from (3.28)

$$v_*(s) \leq \mathcal{k}^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)}\right]$$

for all $s \in \bar{\Omega}^*$. (3.30)

In particular

$v_*(0) = \operatorname{ess\,sup}_\Omega |u|$ is finite and satisfies (3.30). The proof of Theorem 4 is complete. □

APPENDIX TO SECTION 3:

For the convenience of the reader, we give here a brief summary of F.E. Browder's result that we used in Section 3.

DEFINITION A:

Let T be an operator which maps a separable reflexive Banach space V into its dual V'. The operator T is supposed to be continuous, coercive and maps bounded sets into bounded sets.

The Browder condition (S_+) is defined as follows:

(S_+) if $\{u_n\}$ is a weakly convergent sequence in V with limit u and if $\overline{\lim} < Tu_n - Tu, u_n - u > \leq o$, then u_n converges strongly to u.

THEOREM A:

Let $V = W_o^{1,p}(\Omega)$ $1 < p < +\infty$, $V' = W^{-1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

T is the Leray-Lions operator $Tv = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i^o(x, v(x), \nabla v(x))$ which maps V into V'. The functions a_i^o satisfies the standard conditions:

(i) a_i^o are caratheodory functions (see (H.2));

(ii) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, $\forall \xi' \in \mathbb{R}^N$

$\xi \neq \xi'$

$$\sum_{i=1}^{N} a_i^o(x,\eta,\xi)\, \xi_i \geq c_1 |\xi|^p - c_2 \qquad c_1 > o\ ,\ c_2 \geq o$$

$$\sum_{i=1}^{N} [a_i^o(x,\eta,\xi) - a_i^o(x,\eta,\xi')][\xi_i - \xi_i'] > o$$

(iii) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|a_i^o(x,\eta,\xi)| \leq c[|\xi|^{p-1} + |\eta|^{p-1} + a_o(x)]$$

with $c > o$ (constant) , $a_o \in L^{p'}(\Omega)$ then T satisfies the condition (S_+) of Definition A.

Remark:

F.E. Browder shows that the property is satisfied by a more general operator than in Theorem A.

4. A SEQUENCE OF MODIFIED PROBELMS $(\mathcal{P}_n)$

Before introducing the modified problem $(\mathcal{P}_n)$, we derive some preliminary results.

4.1 PRELIMINARY RESULTS:

LEMMA 3

There exists a function $g \in C^1(\mathbb{R})$, *such that*

$$g'(\eta) = \frac{dg}{d\eta} = \nu(|g(\eta)|) \quad \textit{in} \quad \mathbb{R}$$

$$g(o) = o \ , \ \lim_{\eta \to +\infty} g(\eta) = +\infty \quad \textit{and} \quad g \quad \textit{is odd} \qquad (4.1)$$

Proof

Let us consider $b(\eta) = \int_0^\eta \frac{dt}{\nu(t)}$ for $\eta \geq 0$ (By (H.3), this function makes sense). The function b is strictly increasing, continuous on $\mathbb{R}^+$ and $\lim_{\eta \to +\infty} b(\eta) = +\infty$ since $1/\nu \notin L^1(\mathbb{R}_+)$; thus b is invertible from $\mathbb{R}_+$ onto $\mathbb{R}_+$.

Moreover, this function is continuously differentiable every where in $\{\eta > 0\} = \mathbb{R}_+^*$. Hence, if we define the function:

$$g(\eta) = \begin{cases} b^{-1}(\eta) & \text{if } \eta \geq 0 \\ -b^{-1}(-\eta) & \text{otherwise} \end{cases}$$

We verify easily that g satisfies (4.1) . □

LEMMA 4

Let $S_{\theta,h}$ *be the function defined by the formula* (3.0). *Then, for* a.e. x *in* Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$ $\forall \theta > 0$, $\forall h > 0$

$$S_{\theta,h}(\eta)\, F(x, g(\eta), g'(\eta) \cdot \xi) \geq 0 \tag{4.2}$$

$$\eta\, F(x, g(\eta), g'(\eta) \cdot \xi) \geq 0 \tag{4.3}$$

PROOF

If $\eta = 0$, $S_{\theta,h}(0) = 0$, the relation (4.2) and (4.3) are Then true if $\eta \neq 0$, then $\frac{g(\eta)}{\eta} > 0$ (since g is a strictly increasing function satisfying $g(0) = 0$) . For the same reason, $\frac{S_{\theta,h}(\eta)}{\eta} \geq 0$ ($S_{\theta,h}$ is non-decreasing).

Let us write

$$S_{\theta,h}(\eta)\ F(x,g(\eta),\ g'(\eta)\cdot\xi) =$$

$$\frac{S_{\theta,h}(\eta)}{\eta}\cdot\frac{\eta}{g(\eta)}\ g(\eta)\ F(x,g(\eta),g'(\eta)\xi) \tag{4.4}$$

By the assumption (H.2) (i) on F :

$$g(\eta)\ F(x,g(\eta),\ g'(\eta)\xi) \geq o \tag{4.5}$$

Hence, the first inequality follows easily from (4.4) and (4.5) ,

To get (4.3) for $\eta \neq o$, we write:

$$\eta\ F(x,g(\eta),\ g'(\eta)\cdot\xi) =$$
$$\frac{\eta}{g(\eta)}\ g(\eta)\ F(x,g(\eta),\ g'(\eta)\cdot\xi)$$

We conclude by the same argument as before. □

LEMMA 5

We denote by $V(\eta) = \int_o^{\eta} \nu(g(t))^{p'}dt$ *for any* $\eta \geq o$.

Then,

(i) $\lim_{\eta\to+\infty} V(\eta) = +\infty$

(ii) V *is a one to one invertible mapping from* $\mathbb{R}_+$ *into* $\mathbb{R}_+$

Proof:

Let us set $u = g(t)$; then using Lemma 3 we have:

$$V(\eta) = \int_0^{g(\eta)} \nu(u)^{p'-1} du = \mathcal{K}(g(\eta))$$

By Lemma 3, $\lim_{\eta \to +\infty} g(\eta) = +\infty$ and by (H.3), we deduce,

$$\lim_{\eta \to +\infty} V(\eta) = \int_0^{+\infty} \nu(t)^{p'-1} dt = +\infty$$

To prove the second part (ii) , we remark that V is a strictly increasing, continous map from $\mathbb{R}_+$ into $\mathbb{R}_+$. The assertion (i) above insures that V maps $\mathbb{R}_+$ onto $\mathbb{R}_+$ and then it is invertible.

4.2 : A SEQUENCE OF MODIFIED PROBLEMS:

For each $n \in \mathbb{N}^*$, we define the functions F_n, $\bar{F}_n$, a_{in},:
For a e. x in Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, :

$$F_n(x,\eta,\xi) = \frac{F(x,\eta,\xi)}{1+\frac{1}{n}|F(x,\eta,\xi)|} \qquad (4.6)$$

$$\bar{F}_n(x,\eta,\xi) = F_n(x,g(\eta),g'(\eta)\ \xi) \qquad (4.7)$$

$$a_{in}(x,\eta,\xi) = \frac{1}{n}|\xi|^{p-2}\xi_i + a_i(x,g(\eta),g'(\eta)\cdot\xi) \qquad (4.8)$$

Remark 3:

The function $\bar{F}_n$ and a_{in} enjoy the following properties, for a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi , \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$.

(P_1) $$S_{\theta , h}(\eta)\ \bar{F}_n(x,\eta,\xi) \geq 0 \quad \text{for any } \theta > 0 ,\ h > 0$$
$$\eta\ \bar{F}_n(x,\eta,\xi) \geq 0$$

These inequalities follow from Lemma 4

(P_2) There exists an increasing function h_o from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that :

$$|a_{in}(x,\eta,\xi)| \leq h_o(|\eta|)\ [|\xi|^{p-1} + \bar{h}(x)]$$

In fact, by the assumptions (H.4) (see (i) and (iv)), we deduce that:

$$|a_i(x,g(\eta),g'(\eta)\xi)| \leq h(g(|\eta|))\ \nu(g(|\eta|))$$

$$[|g'(\eta)|^{p-1}|\xi|^{p-1} + \bar{h}(x)] \leq h(g(|\eta|))\ \nu(g|\eta|))$$

$$[|g'(\eta)|^{p-1}+1][|\xi|^{p-1}+\bar{h}(x)]$$

so, if we define on $\mathbb{R}^+$, the non-decreasing function :

$$h_o(\eta) = \sup_{|t|\leq\eta} [h(|g(t)|)\, \nu(|g(t)|)(|g'(t)|^{p-1}+1)] + 1$$

We get early:

$$|a_{in}(x,\eta,\xi)| \leq h_o(|\eta|)\ [|\xi|^{p-1} + \bar{h}(x)]$$

$$(P_3) \qquad \sum_{i=1}^{N} a_{in}(x,\eta,\xi)\ \xi_i \geq \left[\frac{1}{n} + \nu_1(|\eta|)\right] |\xi|^p$$

where $\nu_1\ (|\eta|) = \nu(|g(\eta)|)^p = [g'(\eta)]^p$ (see lemma 3).

To get (P_3), it suffices to prove that:

$$\sum_{i=1}^{N} a_i(x,g(\eta),\ g'(\eta)\ \xi)\ \xi_i \geq \nu_1(|\eta|)|\xi|^p \tag{4.9}$$

If $\eta = o$, $\nu_1(o) = a_i(x,o,o) = o$, the relation (4.9) is true if $\eta \neq o$, then $g'(\eta) > o$. Let us write:

$$\sum_{i=1}^{N} a_i(x,g(\eta),g'(\eta)\ \xi)\xi_i = \frac{\sum_{i=1}^{N} a_i(x,g(\eta),g'(\eta)\xi)\ g'(\eta)\ \cdot\xi_i}{g'(\eta)} \tag{4.10}$$

By the assumption (H.3), we have

$$\sum_{i=1}^{N} a_i(x,g(\eta),g'(\eta)\xi)\ g'(\eta)\xi_i \geq \nu(|g(\eta)|)|g'(\eta)|^p\ |\xi|^p \tag{4.11}$$

From Lemma 3, (4.10) and (4.11), we obtain (4.9) .

$$(P_4) \qquad \sum_{i=1}^{N} [a_{in}(x,\eta,\xi) - a_{in}(x,\eta,\xi')][\xi_i-\xi'_i] > o$$

This is a consequence of the two following relations:

$$(*) \quad \sum_{i=1}^{N} (|\xi|^{p-2}\xi_i - |\xi'|^{p-2}\xi'_i)(\xi_i-\xi'_i) > o \qquad (4.12)$$

$$(*) \quad \sum_{i=1}^{N} [a_i(x,g(\eta),g'(\eta)\xi) - a_i(x,g(\eta),g'(\eta)\xi')][\xi_i-\xi'_i] \geq o \qquad (4.13)$$

The proof of this last inequality is the same as (P_3) using the assumption (H.4) (i) and (ii). □

Now we consider the following problem:

$$(\mathcal{P}_n) \qquad \begin{cases} \text{Find } v_n \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ such that} \\ -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_{in}(x,v_n,\nabla v_n) + \bar{F}_n(x,v_n,\nabla v_n) = G \text{ in } \Omega \end{cases}$$

4.3 : A PRIORI ESTIMATE IN $L^\infty(\Omega)$ OF THE SOLUTION v_n of $(\mathcal{P}_n)$

The proof of the following Lemma is the same as in Lemma 2.

LEMMA 6

Let v_n *be a solution of* $(\mathcal{P}_n)$. *Then, for all* $s \in \bar{\Omega}^*$:

$$|v_n|_*(s) \leq V^{-1}\left[\frac{1}{N\alpha_N^{1/N}} \int_s^{|\Omega|} \sigma^{\frac{1}{N}-1} [\bar{g}_{*|v_n|}(\sigma)]^{\frac{1}{p}} d\sigma\right] \qquad (4.14)$$

V^{-1} is the inverse of the function $V(\eta) = \int_0^{\eta} \nu(g(t))^{p'} dt$ (see Lemma 5)

$$\bar{g} = \left[\sum_{i=1}^{N} g_i^2\right]^{p'/2} \quad \text{such that} \quad G = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i},$$

$\|G\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \|g_i\|_{L^{r'}}$, $\bar{g}_{*|v_n|}$ is the relative rearrangement of $\bar{g}$ with respect to $|v_n|$.

THEOREM 6

Any solution v_n *of* $(\mathcal{P}_n)$ *satisfies the following estimate:*

$$\|v_n\|_\infty \leq V^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)}\right] = M_o \qquad (4.15)$$

where α_N , δ *are the same constants as in Theorem 4*

□.

Theorem 6 is a direct consequence of Lemma 6 and the relation

(3.8) of Remark 2.

4.4 : A TRUNCATION OF a_{in} AND AN EQUIVALENT PROBLEM

We introduce the following truncation of g :

$$\hat{g}(\eta) = \begin{cases} g(\eta) & \text{if } |\eta| \leq M_o \\ g(M_o) & \text{if } \eta \geq M_o \\ g(-M_o) & \text{if } \eta \leq -M_o \end{cases} \tag{4.16}$$

This function $\hat{g}$ defines a truncation $\check{g}$ of g' by setting:

$$\check{g}(\eta) = \nu(|\hat{g}(\eta)|) = \begin{cases} g'(\eta) & \text{if } |\eta| \leq M_o \\ g'(M_o) & \text{if } \eta \geq M_o \\ g'(-M_o) & \text{if } \eta \leq -M_o \end{cases} \tag{4.17}$$

M_o is the number given in Theorem 6 .

Let us define a'_{in} , the truncation of a_{in} by setting:

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$a'_{in}(a,\eta,\xi) = \frac{1}{n} |\xi|^{p-2} \xi_i + a_i(x,\hat{g}(\eta),\check{g}(\eta)\xi) \tag{4.18}$$

The function a'_{in} enjoys the following properties:

$[Q_1]$ The function a'_{in} (x,η,ξ) satisfies the Caratheodory condition, this a consequence of the fact that $\hat{g}$, $\check{g}$ are continuous and a_i is a Caratheodory function.

The following properties are direct consequences of (P_2),

(P_3), and (P_4) :

$[Q_2]$ There exists a constant $c > o$ (depending only on M_o, h, , ν) such that :

For a.e. x in Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|a'_{in}(x,\eta,\xi)| \leq c[|\xi|^{p-1} + \bar{h}(x)]$$

$[Q_3]$ We denote by $\nu'_1(\eta) = \nu(|\hat{g}(\eta)|).^p$, then:

For a.e. x in Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\sum_{i=1}^{N} a'_{in}(x,\eta,\xi)\ \xi_i \geq [\frac{1}{n} + \nu'_1(|\eta|)]\ |\xi|^p$$

$[Q_4]$ For a.e. x in Ω , $\forall \eta \in \mathbb{R}$, $\forall (\xi,\xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$

$$\sum_{i=1}^{N} [a'_{in}(x,\eta,\xi) - a'_{in}(x,\eta,\xi')][\xi_i - \xi'_i] > o$$

Let us define the following problem $(\mathcal{P}'_n)$.

$$(\mathcal{P}'_n) \qquad \begin{cases} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \mathcal{A}'_{in}(x,v,\nabla v) + \bar{F}_n(x,v,\nabla v) = G \text{ in } \Omega \\ v \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

We then have:

LEMMA 7

Any solution of $(\mathcal{P}'_n)$ *is a solution of* $(\mathcal{P}_n)$ *and conversely.*

Proof:

Let v be a solution of $(\mathcal{P}'_n)$. As in Theorem ٤ one can check that v satisfies the following estimate.

$$\|v\|_\infty \le (V')^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}}\|G\|^{p'/p}_{W^{-1,r}(\Omega)}\right] \tag{4.19}$$

Here, $V'(\eta) = \int_o^\eta \nu_1'(t)^{p'/p}\,dt$, for any $\eta \ge o$.

By the definition of ν'_1, we observe that $V'(M_o) = V(M_o) =$

$$\frac{\delta}{N\alpha_N^{1/N}}\|G\|^{p'/p}_{W^{-1,r}(\Omega)} .$$

From (4.19), we deduce $\|v\|_\infty \le M_o$.

For all $\varphi \in W_o^{1,p}(\Omega)$

$$\int_\Omega a'_{in}(x,v,\nabla v)\,\frac{\partial\varphi}{\partial x_i}\,dx = \int_{|v|\le M_o} a'_{in}(x,v,\nabla v)\,\frac{\partial\varphi}{\partial x_i}\,dx =$$

$$\int_\Omega a_{in}(x,v,\nabla v)\cdot\frac{\partial\varphi}{\partial x_i}\,dx$$

The proof of the converse is the same as before. □

4.5 : EXISTENCE OF SOLUTIONS OF PROBLEM $(\mathcal{P}'_n)$

LEMMA 8:

(i) There exists a solution v_n of $W_o^{1,p}(\Omega)$ of the problem $(\mathcal{P}'_n)$;

(ii) Furthermore, $v_n \in L^\infty(\Omega)$ and

$$\|v_n\|_{L^\infty(\Omega)} \leq M_o \tag{4.20}$$

and

$$\frac{1}{n}\|\nabla v_n\|_{L^p(\Omega)} + \|\nabla g(v_n)\|^p_{L^p} \leq M_o \|G\|_{L^1(\Omega)} \tag{4.21}$$

M_o is the number given in Theorem 6 .

Proof:

The first part of Lemma 8 is a direct consequence of Leray-Lions' theorem (see [8], p. 182-183 theorem 2.8). The properties (Q_1) to (Q_4) and the relation $|\bar{F}_n| \leq n$ insure that the hypotheses of this theorem are fulfilled.

To show that v_n is in $L^\infty(\Omega)$, we first remark that $\forall\theta \in\,]o\,,\, \operatorname{ess\,sup}_\Omega v_n[\ ,\ \forall h \in\,]o\,,\, \operatorname{ess\,sup} v_n - \theta[$, the function $\bar{v}_n = S_{\theta,h}(v_n)$ belongs to $W_o^{1,p}(\Omega)$ (The definition of $S_{\theta,h}$ is given in formula 3.0) and then:

$$\int_\Omega a'_{in}\,(x,v_n,\nabla v_n)\ \frac{\partial \bar{v}_n}{\partial x_i} + \int_\Omega \bar{v}_n \bar{F}_n(x,v_n,\nabla v_n) = \langle G,\bar{v}_n\rangle \qquad (3)$$

So, v_n satisfies the same relation as (3.2) with a_i replaced By a'_{in} and F by $\bar{F}_n$. The proof is thus the same as in Theorem 6. The estimate follows from Theorem 6.

To get the relation (4.21), we multiply by v_n the equation of $(\mathcal{P}'_n)$, we find:

$$\frac{1}{n}\int_\Omega |\nabla v_n|^p\,dx + \int_\Omega a_i(x,g(v_n),g'(v_n)\,\nabla v_n)\,\frac{\partial v_n}{\partial x_i}\,dx +$$

$$\int_\Omega v_n\ \bar{F}_n(x,v_n,\nabla v_n)\ dx = \int_\Omega G v_n\ dx \qquad (4.22)$$

(Observe that we have used the fact that $\tilde{g}(v_n) = g(v_n)$, $\check{g}(v_n) = g'(v_n)$ a.e. in Ω, since $\|v_n\|_\infty \leq M_o$).

By the property (P_1) on $\bar{F}_n$: $\int_\Omega v_n\ \bar{F}_n(x.v_n,\nabla v_n)\ dx \geq o$ (4.23)

By the property (P_3), we have:

$$\int_\Omega a_i(x.g(v_n),g'(v_n)\ \nabla v_n)\ \frac{\partial v_n}{\partial x_i}\ dx \geq \int_\Omega \nu(|g(v_n)|)^p\ |\nabla v_n|^p dx \qquad (4.24)$$

By Lemma 3, $g'(v_n) = \nu(|g(v_n)|)$. Hence, from (4.24) we get:

$$\int_\Omega a_i(x,g(v_n),\ g(v_n)\ \nabla v_n)\ \frac{\partial v_n}{\partial x_i}\ dx \geq \int_\Omega |\nabla g(v_n)|^p dx \qquad (4.25)$$

Since $G \in L^1(\Omega)$, and $\|v_n\|_{L^\infty(\Omega)} \leq M_o$

$$\left|\int_\Omega G\, v_n\, dx\right| \leq \|v_n\|_{L^\infty(\Omega)} \|G\|_{L^1(\Omega)} \leq M_o \|G\|_{L^1(\Omega)} \tag{4.26}$$

The relations (4.22) to (4.26) imply:

$$\frac{1}{n} \|\nabla v_n\|^p_{L^p(\Omega)} + \|\nabla g(v_n)\|^p_{L^p(\Omega)} \leq M_o \|G\|_{L^1(\Omega)}$$

5 : EXISTENCE OF SOLUTIONS OF PROBLEM ($\mathcal{P}$)

Let us set $\hat{u}_n = g(v_n)$, then by Lemma 8 and Lemma 3 , we have $\hat{u}_n \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Moreover:

$$\|\hat{u}_n\|_\infty \leq k^{-1}\left[\frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)}\right] = M \tag{5.1}$$

(see Theorem 4 for the definition of k).

In fact, by definition $V(M_o) = \dfrac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)}$.

By Lemma 5, $V(M_o) = \int_0^{g(M_o)} \nu(t)^{p'-1} dt = k(g(M_o))$ and thus,

$$g(M_o) = k^{-1}(V(M_o)) = M \tag{5.2}$$

On the other hand, $\|\hat{u}_n\|_\infty = \|g(v_n)\|_\infty \leq \underset{|t| \leq M_o}{\text{Max}} |g(t)|$ (5.3)

Since g is odd and increasing:

$$\underset{|t| \leq M_o}{\text{Max}} |g(t)| = \underset{|t| \leq M_o}{\text{Max}}\, g(|t|) = g(M_o) \qquad (5.4)$$

From (5.2) to (5.4), we get: $\|\hat{u}_n\|_\infty \leq M$. □

By Lemma 8, $\|\nabla \hat{u}_n\|^p_{L^p(\Omega)} \leq M_o \|G\|_{L^1(\Omega)}$ (5.5)

So, we can assume that $\hat{u}_n$ converges to an element $u \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega)$ in:

$$W^{1,p}_o(\Omega) - \text{weak}$$
$$L^\infty(\Omega) - \text{weak-star}$$

and a.e. in Ω.

Furthermore, from the equation of $(\mathcal{P}_n)$, we see that $\hat{u}_n$ satisfies the equation:

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,\hat{u}_n,\nabla \hat{u}_n) + F_n(x,\hat{u}_n,\nabla \hat{u}_n) =$$

$$G + \frac{1}{n} \nabla(|\nabla v_n|^{p-2} \nabla v_n) \qquad (5.6)$$

5.1 : A STRONG CONVERGENCE RESULT IN $W^{1,p}_o(\Omega)$

LEMMA 9

For any $t \in \mathbb{R}$, *we set*

$$k(t) = \int_0^t \nu(|\eta|)^{p'-1}\, d\eta = \int_0^t \nu(|\eta|)^{\frac{1}{p-1}}\, d\eta$$

Then as $n \to \infty$, $\check{u}_n = k(\hat{u}_n)$ *converges to* $\check{u} = k(u)$ *in the norm of* $W_0^{1,p}(\Omega)$.

Proof:

Let us remark first that from the relation (5.1)

$\|\check{u}_n\|_{L^\infty(\Omega)} \leq \operatorname{Max}_{|t| \leq M} |k(t)| = k(M)$ (since k is odd and increasing)

and $k(M) = \frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)} = M_1$.

Moreover, $k(\hat{u}_n) \in W_0^{1,p}(\Omega)$ (since $\hat{u}_n \in W_0^{1,p}(\Omega)$) and a.e. in Ω, $\frac{\partial k(\hat{u}_n)}{\partial x_i} = \nu(|\hat{u}_n|)^{\frac{1}{p-1}} \cdot \frac{\partial \hat{u}_n}{\partial x_i}$). Hence, $\check{u}_n$ is in a bounded set of $W_0^{1,p}(\Omega)$ independently of n . Since $\lim_n k(\hat{u}_n) = k(u)$ a.e. in Ω , then $\check{u}_n = k(\hat{u}_n)$ tends to $\check{u} = k(u)$ weakly in $W_0^{1,p}(\Omega)$ and in $L^\infty(\Omega)$-weak * .

Let us construct an operator T which maps $W_0^{1,p}(\Omega)$ into $W_0^{-1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by the following way:

We define $\bar{a}_i'$ a truncation of $\bar{a}_i$ by setting:

$$\bar{a}'_i(x,\eta,\xi) = \begin{cases} \bar{a}_i(x,\eta,\xi) & \text{if } |\eta| \leq M \\ \bar{a}_i(x,M,\xi) & \text{if } \eta \geq M \\ \bar{a}_i(x,-M,\xi) & \text{if } \eta \leq -M \end{cases}$$

We check easily that the $\bar{a}'_i$ are Caratheodory functions and enjoy the same properties as $\bar{a}_i$, that is:

(R_1) For a.e. x in Ω, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\sum_{i=1}^{N} \bar{a}'_i(x,\eta,\xi)\,\xi_i \geq |\xi|^p \tag{5.7}$$

In fact, let us take $\eta \neq o$

By (H.4), (i) : $\nu(|\eta|) \sum_{i=1}^{N} \bar{a}_i(x,\eta,\xi)\xi_i = \sum_{i=1}^{N} a_i(x,\eta,\xi)\xi_i$ and by (H.3), $\sum_{i=1}^{N} a_i(x,\eta,\xi)\xi_i \geq \nu(|\eta|)\,|\xi|^p$.

Since $\nu(|\eta|) > o$ for $\eta \neq o$, we deduce from the two last relations that:

$$\sum_{i=1}^{N} \bar{a}_i(x,\eta,\xi)\xi_i \geq |\xi|^p \tag{5.8}$$

Each $\bar{a}_i$ is a Caratheodory function, thus the relation (5.8) is also valid for $\eta = o$. A simple argument shows that (5.8) implies (5.7).

The two following properties of $\bar{a}'_i$ are direct consequences of the assumption (H.4) (ii) and (iv) on $\bar{a}_i$:

(R_2) For a.e. x in Ω , $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, $\forall \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$

$$\sum_{i=1}^{N} [\bar{a}_i'(x,\eta,\xi) - \bar{a}_i'(x,\eta,\xi')][\xi_i - \xi_i'] > o$$

(R_3) There exists a constant $c_o > o$ such that:

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|\bar{a}_i'(x,\eta,\xi)| \leq c_o[|\xi|^{p-1} + \bar{h}(x)]$$

Let k^{-1} be the inverse of k from $\mathbb{R}$ onto $\mathbb{R}$ for any $v \in W_o^{1,p}(\Omega)$, we set:

$Tv = - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{a}_i'(x,k^{-1}(v), \nabla v)$. The property (R_3) insures that $Tv \in W^{-1,p'}(\Omega)$. The operator T is coercive since:

$$\langle Tv,v\rangle = \int_\Omega \bar{a}_i'(x,k^{-1}(v), \nabla v) \frac{\partial v}{\partial x_i} dx \geq \int_\Omega |\nabla v|^p dx$$

(by R_1) .

The operator T satisfies all the assumptions of the property (S_+) of F.E. Browder's compactness theorem (see appendix or [2], page 27). Hence, to prove that $\check{u}_n = k(\hat{u}_n)$ tends to $\check{u} = k(u)$, it suffices to show that :

$$\lim_{n\to+\infty} \langle T\check{u}_n , \check{u}_n - \check{u}\rangle = o \qquad (5.9)$$

Let us compute $\langle T\check{u}_n , \check{u}_n - \check{u}\rangle$:

$$\langle T\check{u}_n , \check{u}_n - \check{u}\rangle = \int_\Omega \bar{a}_i'(x,\hat{u}_n,k'(\hat{u}_n) \cdot \nabla \hat{u}_n) \frac{\partial}{\partial x_i}(\check{u}_n - \check{u})\, dx$$

$$= \int_\Omega \bar{a}_i(x,\hat{u}_n,k'(\hat{u}_n) \nabla \hat{u}_n) \cdot \frac{\partial}{\partial x_i}(\check{u}_n - \check{u})dx \qquad (5.10)$$

We have:

$$\bar{a}_i(x,\hat{u}_n,k'(\hat{u}_n) \cdot \nabla \hat{u}_n) = k'(\hat{u}_n)^{p-1}\, \bar{a}_i(x,\hat{u}_n,\nabla \hat{u}_n) \quad \text{(by (H.4)(iii))}$$

$$= \nu(|\hat{u}_n|)\, \bar{a}_i(x,\hat{u}_n,\nabla \hat{u}_n)$$

$$= a_i(x,\hat{u}_n,\nabla \hat{u}_n) \quad \text{By (H.4) (i)}$$

Hence,

$$\langle T\check{u}_n , \check{u}_n - \check{u}\rangle = \int_\Omega a_i(x,\hat{u}_n,\nabla \hat{u}_n) \frac{\partial}{\partial x_i}(\check{u}_n - \check{u})dx \qquad (5.11)$$

Since the function $\hat{u}_n$ satisfies the equation:

$$- \frac{\partial}{\partial x_i} a_i(x,\hat{u}_n,\nabla \hat{u}_n) = G + \frac{1}{n} \nabla \cdot (|\nabla v_n|^{p-2} \nabla v_n) - F_n(x,\hat{u}_n,\nabla \hat{u}_n) \qquad (5.12)$$

We obtain from (5.11) and (5.12):

$$\langle T\check{u}_n, \check{u}_n - \check{u}\rangle = \langle G, \check{u}_n - \check{u}\rangle - \frac{1}{n}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla(\check{u}_n - \check{u})dx$$
$$- \int_\Omega F_n(x, \hat{u}_n, \nabla\hat{u}_n) \cdot (\check{u}_n - \check{u})dx \qquad (5.13)$$

We then study the convergence of each term of (5.13):

By the definition of the weak convergence

$$\lim_n \langle G, \check{u}_n - \check{u}\rangle = o \qquad (5.14)$$

By the Hölder inequality:

$$\left|\frac{1}{n}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla(\check{u}_n - \check{u})dx\right|$$
$$\leq \frac{1}{n}\|\nabla(\check{u}_n - \check{u})\|_{L^p(\Omega)} \cdot \|\nabla v_n\|^{p/p'}_{L^p(\Omega)} \qquad (5.15)$$

By Lemma 8, we get:

$$\frac{1}{n}\|\nabla v_n\|^{p/p'}_{L^p(\Omega)} \leq \left[M_o \|G\|_{L^1(\Omega)}\right]^{\frac{1}{p'}} \cdot \left(\frac{1}{n}\right)^{\frac{1}{p}} \underset{n\to+\infty}{\to} 0 \qquad (5.16)$$

Since $\|\nabla(u_n - u)\|_{L^p(\Omega)}$ is in a bounded of $\mathbb{R}$ independently of n, we deduce from (5.15) and (5.16) :

$$\lim_{n\to+\infty} \frac{1}{n} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla(\check{u}_n - \check{u})dx = 0 \qquad (5.17)$$

Since $\|\hat{u}_n\|_{L^\infty(\Omega)} \leq M$, then by (H.2) (ii)

$$|F_n(x,\hat{u}_n,\nabla\hat{u}_n)| \leq |F(x,\hat{u}_n,\nabla\hat{u}_n)| \leq f(M)[|\nabla\hat{u}_n|^{p-\epsilon} + f_o(x)]$$

Hence, if we set $c_1 = f(M)$

$$|\int_\Omega F_n(x,\hat{u}_n,\nabla\hat{u}_n) \cdot (\check{u}_n - \check{u})dx| \leq$$

$$c_1 \int_\Omega |\nabla\hat{u}_n|^{p-\epsilon} |\check{u}_n - \check{u}|dx + c_1 \int_\Omega f_o|\check{u}_n - \check{u}|dx \qquad (5.18)$$

By the Lebesque dominated convergence theorem:

$$\lim_n \int_\Omega f_o(x)|\check{u}_n - \check{u}|dx = o \qquad (5.19)$$

and

$$\lim_n \int_\Omega |\check{u}_n - \check{u}|^q \, dx = o \quad \text{for all} \quad q \in [1, +\infty[\; .$$

Since $|\nabla\hat{u}_n|$ is in a bounded set of $L^p(\Omega)$, we deduce that there exists a constant $c_2 > o$ (independent of n) such that:

$$\int_\Omega |\nabla\hat{u}_n|^{p-\epsilon} \, |\check{u}_n - \check{u}| \, dx \leq c_2 \, \|\check{u}_n - \check{u}\|_{L^q(\Omega)}$$

for some $q \in [1, +\infty[$. We, then have:

$$\lim_{n} \int_{\Omega} |\nabla u_n|^{p-\epsilon} |\check{u}_n - \check{u}| dx = 0 \qquad (5.20)$$

From (5.18) to (5.20), we get:

$$\lim_{n} \int_{\Omega} F_n(x,\hat{u}_n,\nabla\hat{u}_n) (\check{u}_n - \check{u}) dx = 0 \qquad (5.21)$$

The relation (5.13) together with (5.14), (5.17), and (5.21) gives the expected result.

In the sequel, we will assume that

$$\nabla k(\hat{u}_n) \text{ tends to } \nabla k(u) \text{ a.e. in } \Omega$$

and

$$\nabla k(\hat{u}_n) \text{ tends to } \nabla k(u) \text{ in } (L^p(\Omega))^N\text{-strong}$$

LEMMA 10

We have the two following convergences in $L^1(\Omega)$, strongly:

$$\lim_{n\to+\infty} a_i(x,\hat{u}_n.\nabla\hat{u}_n) = a_i(x,u,\nabla u) \qquad (5.22)$$

$$\lim_{n\to+\infty} F_n(x,\hat{u}_n,\nabla\hat{u}_n) = F(x,u,\nabla u) \qquad (5.23)$$

Proof

For convenience, we write:

$$a_i(x,u,\nabla u) = a_i(u,\nabla u) \ , \ F(u,\nabla u) = F(x,u,\nabla u) \ , \ldots \text{ etc. }.$$

Let us consider:

$$\int_\Omega |a_i(\hat{u}_n,\nabla\hat{u}_n) - a_i(u,\nabla u)|dx = \int_{u\neq o} |a_i(\hat{u}_n,\nabla\hat{u}_n) - a_i(u,\nabla u)|dx$$

$$+ \int_{u=o} |a_i(\hat{u}_n,\nabla\hat{u}_n) - a_i(u,\nabla u)|dx \tag{5.24}$$

On the set $\{u=o\}$, $\nu(o) = o$ and then, $a_i(u,\nabla u) = \nu(o) \cdot \bar{a}_i(u,\nabla u) = o$, thus

$$\int_{u=o} |a_i(\hat{u}_n,\nabla\hat{u}_n) - a_i(u,\nabla u)|dx = \int_{u=o} |a_i(\hat{u}_n,\nabla\hat{u}_n)|dx \tag{5.25}$$

By (H.4) (i) and (iv), we have:

$$\int_{u=o} |a_i(\hat{u}_n,\nabla\hat{u}_n)|dx \leq \int_{u=o} \nu(|\hat{u}_n|)\ h(|\hat{u}_n|)\ |\nabla\hat{u}_n|^{p-1}dx +$$

$$\int_{u=o} \bar{h}(x)\ \nu(|\hat{u}_n|)\ h(|\hat{u}_n|)dx$$

Since $h(|\hat{u}_n|)\ |\nabla\hat{u}_n|^{p-1}$ is in a bounded set of $L^{p'}(\Omega)$ and $\lim_{n\to+\infty} \nu(|\hat{u}_n|) = \nu(o) = o$ on $\{u=o\}$. We deduce easily from Vitali's theorem:

$$\lim_{n\to+\infty} \int_{u=o} \nu(|\hat{u}_n|)\ h(|\hat{u}_n|)\ |\nabla\hat{u}_n|^{p-1}\ dx =$$

$$\lim_{n\to+\infty} \int_{u=o} \bar{h}(x)\ \nu(|\hat{u}_n|)\ h(|\hat{u}_n|)\ dx = o$$

Hence,

$$\lim_{n\to+\infty} \int_{u=o} |a_i(\hat{u}_n,\nabla\hat{u}_n)|dx = o \tag{5.26}$$

As for the term $\int_{u\neq o} |a_i(\hat{u}_n,\nabla\hat{u}_n) - a_i(u,\nabla u)|dx$, we will show that ∇u_n tends to ∇u almost everywhere in $\{u \neq o\}$ and the convergence of this term will follow easily from Vitali's theorem.

By the continuity of ν , $\lim_{n\to+\infty} \nu(|\hat{u}_n(x)|) = \nu(|u(x)|) > o$ for a.e. x in $\{u\neq o\}$. Then, for a sufficiently large n , we have for a.e. x (fixed) in $\{u\neq o\}$

$$\nabla u_n(x) = \frac{\nabla k(u_n(x))}{k'(u_n,x))} = \left[\frac{\nabla k(u_n(x))}{\nu(|u_n(x)|)^{p'-1}}\right]$$

For a.e. x in $\{u \neq o\}$: $\lim_{n\to+\infty} \nabla\hat{u}_n(x) = \frac{\nabla k(u(x))}{k'(u(x))} = \nabla u(x)$.

Since a_i satisfies the Caratheodory conditions, we deduce $\lim_{n\to+\infty} a_i(\hat{u}_n,\nabla\hat{u}_n)(x) = a_i(u,\nabla u)(x)$ a.e. x in $\{u \neq o\}$. By using Vitali's theorem, we get:

$$\lim_{n\to+\infty} \int_\Omega |F_n(\hat{u}_n,\nabla\hat{u}_n) - F(u,\nabla u)|dx = o \qquad (5.28)$$

(We remark in particular that $F(u,\nabla u) = o$ on $\{u = o\}$ since $f(o) = o$, (H.2) (ii)).

THE PASSAGE TO THE LIMIT:

For all $\varphi \in \mathcal{D}(\Omega)$, we get from $(\mathcal{P}_n)$

$$\int_\Omega a_i(\hat{u}_n,\nabla\hat{u}_n) \frac{\partial\varphi}{\partial x_i} dx + \int_\Omega F_n(\hat{u}_n,\nabla\hat{u}_n)\, \varphi\, dx =$$

$$\langle G,\varphi\rangle - \frac{1}{n} \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\, \nabla\varphi\, dx \qquad (5.29)$$

From Lemma 10, we get:

$$\lim_{n\to+\infty} \int_\Omega a_i(\hat{u}_n,\nabla\hat{u}_n) \frac{\partial\varphi}{\partial x_i}dx = \int_\Omega a_i(u,\nabla u) \frac{\partial\varphi}{\partial x_i} dx \qquad (5.30)$$

$$\lim_n \int_\Omega F_n(\hat{u}_n,\nabla\hat{u}_n) \cdot \varphi\, dx = \int_\Omega F(u,\nabla u) \cdot \varphi\, dx \qquad (5.31)$$

By the Hölder inequality and the relation (5.5)

$$\frac{1}{n} \left|\int_\Omega |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla\varphi\, dx \right| \le \left[\frac{1}{n}\right]^{1/p}\left[M_o \|G\|_{L^1(\Omega)}\right]^{1/p'} \|\nabla\varphi\|_{L^p(\Omega)} \to o$$

as $n \to +\infty$.

Hence, for all $\varphi \in \mathcal{D}(\Omega)$

$$\int_\Omega a_i(u,\nabla u) \cdot \frac{\partial \varphi}{\partial x_i}\, dx + \int_\Omega F(u,\nabla u)\, \varphi\, dx = \langle G,\varphi \rangle$$

□

Remark 4:

The solution u that we found here satisfies the estimate:

$$\|\nabla u\|_{L^p(\Omega)} \leq \left[M_o \|G\|_{L^1(\Omega)} \right]^{1/p} \tag{5.34}$$

where

$$M_o = V^{-1}\left(\frac{\delta}{N\alpha_N^{1/N}} \|G\|^{p'/p}_{W^{-1,r}(\Omega)} \right)$$

and V^{-1} is the inverse of the function V (see Lemma 5) .

□

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FOOTNOTES

(1) We use only Lebesgue measure.

(2) For any measurable set E, we denote by $|E|$ its measure.

(3) In the sequel we sometimes omit the sign Σ .

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CHAPITRE VI

Relative rearrangement in quasilinear variational inequalities

(avec R. Temam)

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ANALYSE MATHÉMATIQUE. — *Une nouvelle méthode d'estimation L^∞. Application aux inéquations variationnelles.* Note de **Jean-Michel Rakotoson** et **Roger Temam**, présentée par Jacques-Louis Lions.

Nous donnons ici un théorème d'existence et de régularité pour un problème d'inéquation variationnelle quasilinéaire avec une contrainte unilatérale. Ces résultats utilisent de manière essentielle une estimation *a priori* L^∞ de la solution qui est relativement explicite et qui est obtenue sous des hypothèses assez larges. Cette estimation fait appel à une formule intégrale qui s'apparente au théorème intégral de Federer [16] et à la formule de Fleming-Rishel [17] (*cf.* [21]).

MATHEMATICAL ANALYSIS. — A new method for L^∞ estimates. Application to variational inequalities.

We derive here an existence and regularity result for a quasilinear variational inequality with a unilateral constraint. These results rely in an essential manner on an a priori *L^∞ estimate of the solution which is somehow explicit and derived under fairly general hypotheses. This estimate is proved with the utilization of an integral formula related to the co-area formula of Federer* [16] *or to Fleming-Rishel formula* [17] (see [21]).

Let us consider the following problem $(\mathscr{P})$

Find $u\in\kappa(\varphi)\cap L^\infty(\Omega)$ *satisfying:*

$$\langle A u, v-u\rangle + (F(u,\nabla u), v-u) \geqq \langle T, v-u\rangle \quad \textit{for all } v\in\kappa(\varphi)\cap L^\infty(\Omega).$$

Here, Ω *is a bounded open set of* $\mathbb{R}^N$, A *is an operator of Leray-Lions type and* $\kappa(\varphi)=\{v\in W_0^{1,p}(\Omega),\ v\geqq\varphi \text{ a. e. in } \Omega,\ \varphi\in L^\infty(\Omega)\}$. *All the assumptions on* A, F *and* T *are given by* (H_0) *to* (H_5). *One of the main results that we obtain is*

THEOREM 1. — Assume (H_0) to (H_5), then there exists at least one solution u of $(\mathscr{P})$. Moreover, if $r>p'=p|p-1$, $\varphi\in W^{1,(p-1)r}_{loc}(\Omega)$, $f_0\in L^{r/r'}(\Omega)$, $a_0\in L^r(\Omega)$, then the solution u satisfies the Hölder condition inside of Ω for some exponent $\alpha>0$.

As in [12], *the solution u satisfies and* L^∞ *estimate* (see *Theorem* 2) *obtained by means of the technic of relative rearrangement* (see [9], [10], [11], [13], [14]).

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1. RÉARRANGEMENT RELATIF. — Notre étude utilise un principe du maximum (*voir* th. 2) prouvé à l'aide du réarrangement relatif. Pour plus de détail sur la notion de réarrangement relatif nous renvoyons le lecteur aux références [9], [10], [11], [13], [14]. Néanmoins nous rappelons ici une formule intégrale qui joue un rôle fondamental dans le travail (*cf.* [21]).

LEMME 1. — *Soit* Ω *un ouvert borné de* $\mathbb{R}^N$, $v\in W_0^{1,p}(\Omega)\oplus\mathbb{R}$ (*fonction à trace constante*) *avec* $1\leqq p\leqq+\infty$ *et* $f\in L^1(\Omega)$. *On note* v_* *le réarrangement décroissant de* v *et* f_{*v} *le réarrangement relatif de* f *par rapport à* v. *Alors*

(i) $v_*\in W^{1,p}_{loc}(\Omega^*)$ *avec* $\Omega^*=]0, \operatorname{mes}\Omega[$ ([1]).

(ii) *Presque partout pour* $t\in]\inf\operatorname{ess}_\Omega v, \sup\operatorname{ess}_\Omega v[$

on a

$$\frac{d}{dt}\int_{v>t} f(x)\,dx = \mu'(t)\, f_{*v}(\mu(t))$$

où les dérivés sont prises au sens usuel et $\mu(t) = \text{mes}\{x \in \Omega,\ v(x) > t\}$.

2. Soit Ω un ouvert borné de $\mathbb{R}^N$ et on considère A un opérateur de type Leray-Lions $A\,u = -\partial/\partial x_i\, a_i(x, u(x), \nabla u(x))$ (2) pour $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. On suppose que les fonctions a_i sont de Carathéodory de $\Omega \times \mathbb{R} \times \mathbb{R}^N$ dans $\mathbb{R}$ i. e. $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, $x \to a_i(x, \eta, \xi)$ *est mesurable, p. p. en* $x \in \Omega$, $(\eta, \xi) \to a_i(x, \eta, \xi)$ *est continue*, et que les hypothèses suivantes de monotonie, bornitude et croissance sont vraies pour presque tout $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, et $\forall \xi' \in \mathbb{R}^N (\xi \neq \xi')$.

(H_0) *Il existe une fonction croissante* $a : \mathbb{R}_+ \to \mathbb{R}_+$ *et un élément* $a_0 \in L^{p'}_+(\Omega)$:

$$|a_i(x, \eta, \xi)| \leqq a(\eta)[|\xi|^{p-1} + a_0(x)].$$

(H_1) *Il existe deux fonctions continues* $\nu_1 > 0$, $\nu_2 \geqq 0$ *définies sur* $\mathbb{R}_+$ *et une fonction* k *de* $L^{r/p'}(\Omega)$, $k \geqq 0$ *telle que*

$$a_i(x, \eta, \xi)\xi_i \geqq \nu_1(|\eta|)|\xi|^p - \nu_2(|\eta|)k(x)^{1/p'}|\xi|$$

avec $\nu_2(t)^{p'}/p \leqq \nu(t) + c_0$

$$\forall t \geqq 0,\quad c_0 > 0. \qquad \nu(t) = \int_0^t \nu_1(\tau)^{p'}/p\, d\tau \quad et \quad \nu(+\infty) = +\infty.$$

(H_2) $[a_i(x, \eta, \xi) - a_i(x, \eta, \xi')][\xi_i - \xi'_i] > 0$

(H_3) *On considère* F *une fonction de Carathéodory de* $\Omega \times \mathbb{R} \times \mathbb{R}^N$ *dans* $\mathbb{R}$ *telle que*

(i) $\eta F(x, \eta, \xi) \geqq 0$.

(ii) *Il existe une fonction croissante* f *de* $\mathbb{R}_+$ *dans* $\mathbb{R}_+$ *et un élément* $f_0 \in L^1_+(\Omega)$:

$$|F(x, \eta, \xi)| \leqq f(|\eta|)[|\xi|^p + f_0(x)].$$

(H_4) *Soit* T *un élément de* $W^{-1,r}(\Omega)$, $T = -\partial g_i/\partial x_i$ *avec* $g_i \in L^r(\Omega)$,

$$r > N/p - 1, \qquad r \geqq p', \qquad \|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1,\ldots,N} \|g_i\|_{L^r(\Omega)}.$$

(145)

(H_5) *On suppose que* $\varphi \in L^{\infty}(\Omega)$,

$$\kappa(\varphi) = \{v \in W_0^{1,p}(\Omega), v \geqq \varphi\ p.\ p.\} \neq \varnothing.$$

On considère le problème $(\mathscr{P})$ suivant :

Trouver u *dans* $\kappa(\varphi) \cap L^{\infty}(\Omega)$ *tel que pour tout* $v \in \kappa(\varphi) \cap L^{\infty}(\Omega)$ *on ait* : $\langle A\,u, v - u\rangle + (F(u, \nabla u), v - u) \geqq \langle T, v - u\rangle$.

THÉORÈME 1. — *On suppose* (H_0) *à* (H_5). *Alors il existe une solution* u *du problème* $(\mathscr{P})$. *De plus, si l'on suppose que* $r > p'$, $\varphi \in W^{1,(p-1)r}_{loc}(\Omega)$, $f_0 \in L^{r/p'}(\Omega)$ *et* a_0 *dans* $L^r(\Omega)$, *alors toute solution* u *est* α*-höldérienne à l'intérieur de* Ω *pour un exposant* $\alpha > 0$.

Ce résultat peut s'étendre à d'autres convexes et d'autres inéquations. Pour une telle extension et pour la preuve du théorème 1, nous avons établi deux résultats connexes. Le premier est un principe du maximum :

THÉORÈME 2. — *On suppose* (H_0), (H_1) *et* (H_4). *Soit* $u \in W^{1,p}(\Omega)$ *tel qu'il existe* $a \in \mathbb{R}_+$ *tel que* $\bar{u} = [|u| - a]_+ \in W_0^{1,p}(\Omega)$. *On suppose aussi que u satisfait la propriété suivante*

(A_∞) *Si* $\text{mes}\{x \in \Omega, \bar{u}(x) > 0\} \neq 0$ alors pour tout $\theta \in]0, \text{ess}\sup_\Omega \bar{u}[$, $h \in]0, \text{ess}\sup_\Omega \bar{u} - \theta[$, *on a*

$$\int_{\theta < \bar{u} \leqq \theta + h} a_i(x, u, \nabla u) . \frac{\partial u}{\partial x_i} dx \leqq \int_{\theta < \bar{u} \leqq \theta + h} g_i \frac{\partial u}{\partial x_i} dx$$

Alors, u est dans $L^{\infty}(\Omega)$ et de plus u satisfait à $\|u\|_{\infty} \leqq a + \nu_a^{-1}(\lambda_a \lambda_1 \lambda_2) = M(a)$. La fonction ν_a est définie par $\nu_a(t) = \nu(a+t) - \nu(a)$ et les constantes λ_a, λ_1, λ_2 sont données explicitement en fonction des données.

Soit $\theta > 0$, $h > 0$, on définit deux fonctions réelles lipchitziennes notées H_θ, $S_{\theta,h}$ par

$H_\theta(\tau) = 0$ *si* $\tau \leqq 0$, $= \theta$ *si* $\tau \geqq \theta$, *affine ailleurs*,

$S_{\theta,h}(\tau) = 0$ *si* $|\tau| \leqq \theta$, $\operatorname{sign} \tau$ *si* $|\tau| \geqq \theta + h$ *et affine ailleurs.*

Nous avons alors le théorème suivant :

THÉORÈME 3. — *Soit $u \in W^{1,p}(\Omega)$. Alors la fonction $\hat{u} = H_\theta(u-\varphi) S_{\theta_1,h}(u)$ est dans $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ pour $\varphi \in L^{\infty}(\Omega)$, $\theta_1 = \theta + \|\varphi\|_{\infty}$ et pour tout $\theta > 0$.*

Remarque 1. — On peut aussi traiter par les méthodes ci-dessus des inéquations du genre

$$\langle A u, v-u \rangle + (F(u, \nabla u), v-u) + \Phi(v) - \Phi(u) \geqq \langle T, v-u \rangle$$

avec $\operatorname{Dom} \Phi = \kappa \cap L^{\infty}(\Omega)$, ou aussi des inéquations quasi variationnelles [19].

De nombreux auteurs ont traité un théorème d'existence analogue au théorème 1, mais à notre connaissance les hypothèses sont en général plus fortes : ou bien $p=2$ (*cf.* par exemple [1], [5]), ou l'obstacle régulier (*cf.* [2]), ou le second membre est plus régulier que le nôtre (*cf.* par exemple [3]), ou la croissance de F est plus faible qu'ici (*cf.* [7]). Dans le théorème de régularité on peut supposer a_i, F uniquement borélienne et on peut aussi se passer de la monotonie de A et l'hypothèse (H_3) (i) sur F. De même les hypothèses utilisées pour la régularité höldérienne semblent en général plus fortes qu'ici (*cf.* par exemple [6]).

L'utilisation du réarrangement des fonctions pour l'obtention d'estimations dans les équations aux dérivées partielles est due semble-t-il à H. Weinberger [20] (*cf.* [14], [18] pour d'autres références bibliographiques). L'utilisation du réarrangement relatif a été faite pour la première fois dans [12] dans le cas des équations. Pour d'autres aspects du réarrangement, *cf.* [15].

Les détails de ce travail seront publiés ultérieurement.

(¹) On n'utilise que la mesure de Lebesgue.

(²) On omet le signe de sommation $\sum_{i=1}^{N}$.

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Relative Rearrangement in Quasilinear Elliptic Variational Inequalities

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Dedicated to the memory of Harold Grad

Introduction

Our aim in this article is to study some elliptic variational inequalities and derive results of existence and regularity of solutions of these problems. The main novelty of the present approach is the derivation of L^∞ estimates on the solutions which allow us to consider non-fully coercive elliptic operators and lead to further regularity results on the solution. The derivation of the L^∞ estimates is based on a seemingly new differentiation formula which resembles the Fleming-Rishel formula and the Federer co-area formula. This formula concerns the differentiation with respect to t of integrals of the form $\int_{u>t} v(x)\,dx$, where $t \in \mathbb{R}$, $x \in \Omega$ a bounded open set of $\mathbb{R}^N$ and u,v are two real functions defined on Ω. When u is smooth and t is a regular value of u (i.e. $\nabla u(x) \neq 0$, $\forall x \in u^{-1}(t)$), then it is elementary to check that:

$$\frac{d}{dt}\int_{u>t} v(x)\,dx = -\int_{u^{-1}(t)} \frac{v(x)}{|\nabla u(x)|}\, d\mathcal{H}_{N-1}, \tag{0.1}$$

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where $d\mathcal{H}_{N-1}$ is the N-1 dimensional measure on the smooth manifold $u^{-1}(t)$. A generalization of (0.1) can be derived under very mild assumptions on u and v by using the concept of relative rearrangement of functions. This concept has emerged in [19] in the study of variational problems related to the queer differential equations of plasma physics due to H. Grad and C. Mercier. These equations contain a function u defined on Ω and its rearrangement u_* in the sense of Polya and Szego. When studying variational problems related to these equations, J. Mossino and one of the authors were led in [19] to consider the directional derivative of the mapping:

$$u \to u_*$$

i.e to study the limit as $\lambda \to 0$ of the ratio

$$\frac{(u+\lambda v)_* - u_*}{\lambda}$$

This limit was called the relative rearrangement of v with respect to u and denoted v_{*u} . As it then appears in [22], (0.1) can be generalized in the form

$$\frac{d}{dt}\int_{u>t} v(x)dx = v_{*u}(\mu(t))\,\frac{d\mu(t)}{dt},\ \text{a.e. } t \qquad (0.2)$$

where μ is the distribution function of u i.e. $\mu(t)$ is the Lebesgue measure of the set $\{x \in \Omega\ ,\ u(x) > t\}$. Formula (0.2) is valid for $u \in W_0^{1,1}(\Omega)$, $u \geq 0$, and $v \in L^P(\Omega)$.

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The concept of relative rearrangement and formula (0.2) were applied in [22] to the study of the existence and regularity of solutions of nonlinear elliptic equations of the Leray-Lions type of the second order. Our aim here is to generalize this study to variational inequalities and free boundary value problems associated to such operators.

Let Ω denote a bounded domain of $\mathbb{R}^N$ and let K be a convex set of $W_o^{1,p}(\Omega)$ (or $W^{1,p}(\Omega)$, $1 < p < \infty$) . The problem we consider is the following.

$$(\mathcal{P}) \qquad \left| \begin{array}{l} \text{To find } u \in K \cap L^{\infty}(\Omega) \text{ such that} \\ \text{for all } v \in K \cap L^{\infty}(\Omega) \\ \langle Au, v-u\rangle + (F(u,\nabla u), v-u) \geq \langle T, v-u\rangle \end{array} \right.$$

Here A is an operator of Leray-Lions type which maps $W_o^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ into $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$, and $\langle .,.\rangle$ denotes the scalar product between $W_o^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$; T belongs to $W^{-1,p'}(\Omega)$ and F is a nonlinear map such that $F(u,\nabla u) \in L^1(\Omega)$ for $u \in K \cap L^{\infty}(\Omega)$.

A typical problem entering into the framework of this article is the following

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\sigma(|u|)\, |\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right] - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (b_i(x)\, |u|^{\alpha}), \quad 1 < p < \infty ,$$

$$F(u,\nabla u) = u|\nabla u|^p \quad , \quad T \in W^{-1,N+\epsilon}(\Omega) \ ,$$

$$K = \{v \in W_o^{1,p}(\Omega) \ , \ v \geq \varphi \quad \text{a.e. in} \quad \Omega\} \ ,$$

with $\sigma(u) > o$, $\forall u \geq o$. Similar problems have been considered, in particular when $p = 2$ by [2], [3], [4], [6], [13] but the growth of F is less than quadratic with respect to $|\nabla u|$ and T is often in $L^2(\Omega)$ (except in [2],[4],[14],[15]). In all these references the operator A is uniformly elliptic, but this assumption is not necessary in this approach.

Here we consider a general operator of the form $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u;\nabla u)$ whose constant of "coercivity" can depend on u, and the growth of F is at most of order p with respect to the gradient. We can also consider different convex K (see Section 6), but we emphasize on the case where

$$K = K(\varphi) = \{v \in W_o^{1,p}(\Omega) \ , \ v \geq \varphi \quad \text{a.e. in} \quad \Omega\} \ .$$

The obstacle φ is discontinuous (only in $L^\infty(\Omega)$). As mentioned above, using the concept of relative rearrangement and (0.2) we are able to derive an L^∞ estimate on the solutions of

The paper is organized as follows: Section 1 contains a review of definitions and properties on the relative rearrangement. Section 2 contains the general hypotheses and the main results for the convex $K = K(\mathcal{P})$. In Section 3 we state and prove a general theorem providing an L^∞-estimate for the solutions of $\mathcal{P}$; this theorem is applicable as well when $K = K(\mathcal{P})$ and for different convex K . In Section 3 we then establish the existence of solution of $\mathcal{P}$ when $K = K(\mathcal{P})$. Thanks to the L^∞-estimate of Section 3, we define a suitable truncation of the operator A and we introduce a truncated function F_n of F . This allows us to define a family of modified problems $\mathcal{P}_n$, $n \in \mathbb{N}^*$. The existence of solutions u_n for $\mathcal{P}_n$ is given by standard results on variational inequalities. Then passing to the limit $n \to \infty$, we obtain the existence of solution for $\mathcal{P}$. In Section 5, we consider various other convexes, but restrict ourselves to the necessary modifications for the derivation of the L^∞-estimate. Finally, in Section 6, we return to $K = K(\mathcal{P})$ and show in this case the Hölder continuity of the solutions.

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Plan

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1 *Review of Properties of the Relative Rearrangement*

Let Ω be an open bounded set of $\mathbb{R}^N (N \geq 1)$ and let u be a measurable function from Ω into $\mathbb{R}$. We denote by $|\Omega|$ the Lebesgue measure of Ω and $|E|$ denotes more generally the Lebesgue measure of a set E. We also write $|u > t|$ for the Lebesgue measure of $\{x \in \Omega , u(x) > t\}$; the definition of $|u \geq t|$, $|u = t|$, is similar.

We recall that the *distribution function* of u is the real function

$$t \in \mathbb{R} \to \mu(t) = |u > t|$$

and the *decreasing rearrangement* of u denoted u_* is the generalized inverse function of μ, defined on $\bar{\Omega}^*$, where $\Omega^* = (0, |\Omega|)$: for every $s \in [0,|\Omega|]$.

$$u_*(s) = \text{Inf}\ \{\theta \in \mathbb{R} , |u > \theta| \leq s\} ,$$

We recall that the functions u and u_* are equimeasurable, i.e.

$$|u > t| = |u_* > t| ,$$

$$|u \geq t| = |u_* \geq t| , \quad \forall t \in \mathbb{R}$$

which implies that for any Borel function f

$$\int_\Omega f(u(x))dx = \int_0^{|\Omega|} f(u_*(s))ds$$

In particular $(f(r) = |r|^p, 1 \leq p < \infty)$, we have

$$\|u\|_{L^p(\Omega)} = \|u_*\|_{L^p(\Omega^*)} , \forall p, 1 \leq p \leq \infty \qquad (1.1)$$

and the mapping

$$u \in L^p(\Omega) \to u_* \in L^p(\Omega^*) \tag{1.2}$$

is well-defined from $L^p(\Omega)$ into $L^p(\Omega^*)$. One can show that this mapping is a Lipschitz mapping, i.e., for every $u,v \in L^p(\Omega)$

$$\|u_* - v_*\|_{L^p(\Omega^*)} \leq \|u - v\|_{L^p(\Omega)} \tag{1.3}$$

The concept of relative rearrangement of a function $v \in L^p(\Omega)$ with respect to u was introduced in J. Mossino-R. Temam [19] in relation with the study of the directional derivative of the mapping

$$u \in L^p(\Omega) \to u_* \in L^p(\Omega^*)$$

motivated by the study of the queer differential equations of plasma physics (see [28]).

Let, as before, u be a measurable function from Ω into $\mathbb{R}$ and let $v \in L^p(\Omega)$. The relative rearrangement of v with respect to u , denoted v_{*u} is the function

$$v_{*u} = \frac{dw}{ds} \tag{1.4}$$

where w (depending on u and v) is the real function on $\bar{\Omega}^*$

defined by

$$w(s) = \begin{cases} \displaystyle\int_{u>u_*(s)} v(x)dx & \text{if } |u = u_*(s)| = o \\[2ex] \displaystyle\int_{u>u_*(s)} v(x)dx + \int_o^{s-|u>u_*(s)|} \left[v|_{P(s)}\right]_*(\sigma)d\sigma & \text{if not} \end{cases} \qquad (1.5)$$

Here $P(s)$ is the plateau of u corresponding to the value $u_*(s)$ of u :

$$P(s) = \{x \in \Omega \ , \ u(x) = u_*(s)\}$$

and the integral in (1.5) involves the decreasing rearrangement of the restriction of v to $P(s)$, $v|_{P(s)}$. This rearrangement is just defined as above for u , with Ω replaced by $P(s)$, u replaced by $v|_{P(s)}$; note that

$$|P(s)| = |u \geq u_*(s)| - |u > u_*(s)| \ .$$

This definition of v_{*u} was introduced in [19] in the case where $v \in L^\infty(\Omega)$ and then extended in [18] to the case $1 \leq p < \infty$.

Let us recall some of the properties of the relative rearrangement (see [18] [19] [22] [25] for more details):

If $v \in L^p(\Omega)$, then $v_{*u} \in L^p(\Omega^*)$ and

$$\|v_{*u}\|_{L^p(\Omega^*)} \leq \|v\|_{L^p(\Omega)} \tag{1.6}$$

$$v_{*v} = v_* \text{ , and } v_{*c} = v_*,\ c_{*u} = c,\ \forall c \in \mathbb{R} \tag{1.7}$$

Monotonicity: $v_1 \leq v_2 \rightarrow (v_1)_{*u} \leq (v_2)_{*u}$

$$\text{and in particular } v \geq o \rightarrow (v_*)_u \geq o \ . \tag{1.8}$$

If $v_1, v_2 \in L^p(\Omega)$,

$$\|(v_1)_{*u} - (v_2)_{*u}\|_{L^p(\Omega^*)} \leq \|v_1 - v_2\|_{L^p(\Omega)} \tag{1.9}$$

As far as the initial question of the directional derivative of the mapping (1.2) is concerned, we have:

If $v \in L^p(\Omega)$ (and u is measurable) then as $\lambda \rightarrow o$, $\lambda > o$:

$$\frac{(u+\lambda v)_* - u_*}{\lambda} \rightarrow v_{*u} \quad \text{in } L^p(\Omega^*) \text{ weakly}$$

(weak-star if $p = \infty$)

The property of the relative rearrangement which will play an essential role here (as in [22]) is the formula (1.11) below (see also (1.15)) which resembles the Fleming-Rishel formula [9] and the co-area formula of Federer. It is easy to derive this formula from (1.5) and for the convenience of the reader, we rapidly recall its proof given in [22].

THEOREM 1.1

Let Ω *be a bounded open set of* $\mathbb{R}^N$ *and let us assume that* $u \in W_o^{1,1}(\Omega)$, $u \geq o$ *and* $v \in L^1(\Omega)$. *Then for almost every* $t \in \mathbb{R}$:

$$\frac{d}{dt}\int_{u>t} v(x)dx = v_{*u}(\mu(t))\ \mu'(t)\ , \qquad (1.11)$$

where μ *is the distribution function of* u , $\mu(t) = |u>t|$.

Proof

We recall that the rearrangement u_* of u belongs to a Sobolev space with weight on $\overline{\Omega}^*$, namely u_* is absolutely continuous on $(o,|\Omega|]$ and

$$\| N\ \alpha_N^{1/N} \frac{du_*}{ds} \|_{L^p(\Omega^*)} \leq \|\text{grad } u\|_{L^p(\Omega)} \qquad (1.12)$$

where α_N is the measure of the unit ball of $\mathbb{R}^N$ (see, for instance, [31]) . We recall also that

$$t = u_*(\mu(t)) \tag{1.13}$$

for every t which is not a plateau value of t, i.e. $|u = t| = o$ (and there is at most a countable number of such t's).

Consider then the function w in (1.5). Because of (1.4) (1.6), it belongs to $W^{1,1}(\Omega^*)$, it is absolutely continuous and thus differentiable (in the usual sense) except on a set $E \subset \Omega^*$ of null measure. Due to (1.13) $\mu^{-1}(E) \subset u_*(E)$ and $u_*(E)$ has measure o since u_* is absolutely continuous. Now if E' is the union of $\mu^{-1}(E)$ and the countable plateau values of u, then $t \to w(\mu(t))$ is differentiable on $\mathbb{R}\backslash E'$, w is differentiable at $\mu(t)$, $t \notin E'$, and the chain differentiation rule applies and provides (1.11) for such t's $(t \notin E')$.

Remark 1.1

i) The relation (1.11) is not valid in the distribution sense. For example, if t_o is a plateau value of u then the distribution derivative of

$$\int_{u>t} v(x)dx$$

will contain in general a Dirac measure δ_{t_o} which does not appear in (1.11).

ii) It is easy to compute the left hand side of (1.11) when u is smooth, say $u \in \mathcal{C}^{\infty}(\Omega)$ and t is a regular value of u, i.e., $\nabla u(x) \neq 0$ on the set $u^{-1}(t)$. By Sard's theorem, this happens for almost every t and it is clear that in this case $u^{-1}(t)$ is a smooth manifold. An easy computation shows that:

$$\frac{d}{dt}\int_{u>t} v(x)dx = -\int_{u=t} \frac{v(x)}{|\nabla u(x)|}\, d\,\mathcal{H}_{N-1} \qquad (1.14)$$

where $d\,\mathcal{H}_{N-1}$ is the $(N-1)$ dimensional Hausdorff measure.

□

Another formula which is useful for our purpose is the following.

If $\bar{f} \geq 0$ is a locally integrable function on Ω^*, then for every s, s', $0 \leq s \leq s' \leq |\Omega|$,

$$\int_{u_*(s')}^{u_*(s)} \bar{f}(\mu(\theta))(-\mu'(\theta))d\theta \leq \int_s^{s'} \bar{f}(\sigma)\, d\sigma . \qquad (1.15)$$

This inequality follows immediately from a classical argument if $\bar{f} \in \mathcal{C}(\bar{\Omega}^*)$ and it is proved by approximation for a more general $\bar{f}$ (see [22]).

□

We now complete this section by recalling two classical results used in this article, the isoperimetric inequality of De Giorgi [8] and the (usual) Fleming-Rishel formula [9].

THEOREM 1.2 (De Giorgi [8])

Let E *be a measurable subset of* $\mathbb{R}^N$, *and let* $P_{\mathbb{R}^N}(E)$ *denote the perimeter of* E *in* $\mathbb{R}^N$. *Then, either*

$$P_{\mathbb{R}^N}(E) \geq N\alpha_N^{1/N} |E|^{1-1/N} \qquad (1.16)$$

or

$$P_{\mathbb{R}^N}(E) \geq N\alpha_N^{1/N} |R^N \setminus E|^{1-1/N} \qquad (1.17)$$

THEOREM 1.3 (Fleming-Rishel [9])

For any f *in* $W^{1,1}(\Omega)$, *we have*

$$\int_\Omega |\nabla f| dx = \int_{-\infty}^{\infty} P_\Omega(f>t)\, dt \qquad (1.18)$$

where $P_\Omega(f > t)$ *is the perimeter of* $\{f > t\}$ *in* Ω *and* Ω *is an open set of* $\mathbb{R}^N$.

2. HYPOTHESES. THE MAIN RESULTS (UNILATERAL CONVEX)

We first describe the precise assumptions concerning the problem $(\mathcal{P})$ mentioned in the Introduction.

We are given an open bounded set Ω of $\mathbb{R}^N$, and we make on T,F,A, the following assumptions (H.1) to (H.4):

(H.1) $$T \in W^{-1,r}(\Omega) \ , \ r > \frac{N}{p-1} \ , \ r \geq \frac{p}{p-1}$$

(H.2) The map F is a Caratheodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ i.e.

For all (η,ξ) fixed in $\mathbb{R} \times \mathbb{R}^N$, $x \to F(x,\eta,\xi)$ is measurable

For a.e. x fixed in Ω, $(\eta,\xi) \to F(x,\eta,\xi)$ is continuous

Furthermore, we assume that F satisfies the two following conditions:

i) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\eta F(x,\eta,\xi) \geq 0$$

ii) There exists an increasing function f from $\mathbb{R}_+$ into $\mathbb{R}_+$ and a positive function f_o of $L^1(\Omega)$, such that :

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|F(x,\eta,\xi)| \leq f(|\eta|) \ (f_o(x) + |\xi|^p)$$

The operator A can be written as follows:

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u(x), \nabla u(x)) \quad \text{for } u \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where the functions a_i satisfy:

(H.3) Each $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function satisfying the following growth property:
There exists an increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ and an element $a_o \geq o$ in $L^{p'}(\Omega)$ such that:

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|a_i(x,\eta,\xi)| \leq a(|\eta|) \, [|\xi|^{p-1} + a_o(x)]$$

(H.4) (Coerciveness) There exists two continuous functions $\nu_1 > o$ and $\nu_2 \geq o$ defined on $\mathbb{R}_+$ and a positive function k of $L^{r/p'}(\Omega)$ such that:

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$\sum_{i=1}^{N} a_i(x,\eta,\xi) \, \xi_\lambda \geq \nu_1(|\eta|) \, |\xi|^p - \nu_2(|\eta|) \, k(x)^{1/p'} \, |\xi|$$

with $\nu_2(t)^{p'/p} \leq \nu(t) + c_o$, $\forall t \geq o$. Here, c_o is a positive constant and $\nu(t) = \int_o^t (\nu_1(\tau))^{p'/p} \, d\tau$, satisfies:

$$\lim_{t \to \infty} \nu(t) = \int_o^\infty (\nu_1(\tau))^{p'/p} \, d\tau = +\infty \quad (1)$$

(H.5) (Monotonicity)

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi,\xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$

$$\sum_{i=1}^{N} [a_i(x,\eta,\xi) - a_i(x,\eta,\xi')]\,[\xi_i - \xi'_i] > o . \qquad \square$$

Before stating our main results, we present a few typical examples of operators satisfying the classical assumptions above.

1st EXAMPLE : THE LINEAR CASE

We set

$$Lu = - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_i}\right] - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (b_i(x)u)$$

$$a_{ij} \in L^{\infty}(\Omega) \ , \quad \sum_{ij=1}^{N} a_{ij}(x)\, \xi_i\, \xi_j \geq \alpha|\xi|^2 \ (\alpha > o) \quad \forall \xi \in \mathbb{R}^N ,$$

$$\text{a.e. } x \in \Omega ,$$

$b_i \in L^r(\Omega)$, $r > N$, $F = F(x,u,\nabla u) = c(x)u$, $c \in L^1(\Omega)$, $c(x) \geq o$ a.e. and $T \in W^{-1,r}(\Omega)$, $r > N$.

We note that $a_i(x,\eta,\xi) = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_j + \eta b_i$

$$\sum_{i=1}^{N} a_i(x,\eta,\xi)\, \xi_i \geq \alpha|\xi|^2 + \eta\left[\sum_{i=1}^{N} b_i \xi_i\right] \qquad (2.1)$$

By Cauchy-Schwarz inequality:

$$\left|\sum_{i=1}^{N} b_i \xi_i\right| \le \left[\sum_{i=1}^{N} b_i^2(x)\right]^{1/2} |\xi|$$

Let us set $b(x) = \sum_{i=1}^{N} b_i(x)^2$, then from (2.1), we deduce

$$\sum_{i=1}^{N} a_i(x,\eta,\xi)\ \xi \ge \alpha|\xi|^2 - |\eta|\, b(x)^{1/2}\ |\xi|$$

Here, $\nu_1(|\eta|) = \alpha$, $\nu(|\eta|) = \int_0^{|\eta|} \nu_2(t)dt = \alpha|\eta|$ and

$\nu_2(|\eta|) = \alpha|\eta|$, $k(x) = \frac{1}{\alpha^2} b(x)$.

One can check that there exists $c_o \geq o$ such that $\nu_2(|\eta|) \le \nu(|\eta|) + c_o$ and we have:

$$\sum_{i=1}^{N} [a_i(x,\eta,\xi) - a_i\ (x,\eta,\xi')]\ [\xi_i - \xi_i'] \ge \alpha|\xi - \xi'|^2 > o$$

for any $(\xi,\xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \ne \xi'$

2nd EXAMPLE

$$Au = -\operatorname{div}\left(a(x,u)\ |\nabla u|^{p-2}\nabla u\right) \text{ , where}$$

$a(x,\eta) = \dfrac{1}{1 + a_o(x)|\eta|^{p-1}}$, $a_o \in L^\infty(\Omega)$, $a_o \ge o$. One can check that:

$$a_i(x,\eta,\xi) = \frac{1}{1 + a_o(x)|\eta|^{p-1}} \quad |\xi|^{p-2}\xi_i$$

and

$$\sum_{i=1}^{N} a_i(x,\eta,\xi)\ \xi_i \geq \frac{1}{1 + \|a_o\|_\infty |\eta|^{p-1}} \quad |\xi|^p ,$$

$$\nu_1(|\eta|) = \frac{1}{1 + \|a_o\|_\infty |\eta|^{p-1}}$$

$$\sum_{i=1}^{N} (a_i(x,\eta,\xi) - a_i(x,\eta,\xi'))\ (\xi_i - \xi_i') = \frac{1}{1 + a_o(x)\ |\eta|^{p-1}}$$

$$\sum_{i=1}^{N} \left[|\xi|^{p-2}\ \xi_i - |\xi'|^{p-2}\ \xi_i' \right] (\xi_i - \xi_i') > 0$$

We notice that in this example, we have $\lim\limits_{\eta\to+\infty} \nu_1(\eta) = 0$ and $\nu_1(|\eta|) \underset{\eta\to+\infty}{\sim} \dfrac{1}{\|a_o\|_\infty\ |\eta|^{p-1}}$ if $a_o \neq 0$ and $\nu_1(\eta) = 1$ otherwise. This insures that the function $\nu(|\eta|) = \int_0^{|\eta|} \nu_1(t)^{p'/p}\, dt$ satisfies $\lim\limits_{\eta\to+\infty} \nu(\eta) = +\infty$.

For the function F , there are many possible choices. For instance, $F(x,\eta,\xi) = (e^\eta - 1)\ |\xi|^p + a(x)\ \mathrm{sign}\ (\eta)\ |e^{-\eta} - 1|$ $a \in L^1_+\ (\Omega)$ or $F(x,\eta,\xi) = \eta\ |\xi|^p$. Finally, $T \in W^{-1,r}(\Omega)$ $r >$ N/p-1 $r \geq p/-1$, for example $T \in L^{\frac{N+p}{p-1}}(\Omega)$.

3rd EXAMPLE

$$Au = - \operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (b_i(x) \; |u|^{\alpha})$$

Here $a(x,\eta) = a_o(x) \; e^{|\eta|}$, $a_o(x) \geq \beta > o$, $a_o \in L^{\infty}(\Omega)$, $b_i \in L^{\infty}(\Omega)$ and $\alpha > o$.

We can check that $a_i(x,\eta,\xi) = a(x,\eta) \; |\xi|^{p-2} \; \xi_i + b_i(x) \; |\eta|^{\alpha}$

$$\sum_{i=1}^{N} a_i(x,\eta,\xi) \; \xi_i \geq \beta \; e^{|\eta|} \; |\xi|^p - |\eta|^{\alpha} \; k(x)^{1/p'} \; |\xi|$$ where

$$k(x) = \Big[\sum_{i=1}^{N} b_i(x)^2 \Big]^{p'/2} \; , \; \nu_1(|\eta|) = \beta \; e^{|\eta|} \; , \; \nu_2(|\eta|) = |\eta|^{\alpha}$$

$$\sum_{i=1}^{N} (a_i(x,\eta,\xi) - a_i(x,\eta,\xi)) \; (\xi_i - \xi'_i) =$$

$$a(x,\eta) \Big[\sum_{i=1}^{N} \Big(|\xi|^{p-2} \; \xi_i - |\xi'|^{p-2} \; \xi'_i \Big) \Big(\xi_i - \xi'_i \Big) \Big] > 0 \; \square$$

For the set of constraint K , we consider in Sections 2 to 3, the convex set:

(H.6) $$K(\varphi) = \Big\{ v \in W_o^{1,p}(\Omega) \; , \; v \geq \varphi \text{ a.e. in } \Omega \Big\}$$

We assume that $\varphi \in L^{\infty}(\Omega)$ and $K(\varphi) \neq 0$. Other convex sets will be considered in Section 4 .

In the sequel, we will use the following quantities:
For any real $a \geq o$, we define:

$$\nu_a(t) = \int_a^{a+t} \nu_1(\tau)^{p'/p} d\tau \quad , \quad t \geq o \; , \tag{2.2}$$

$$c_a = c_o + \int_o^a \nu_1(\tau)^{p'/p} d\tau \tag{2.3}$$

$t(N,p) = \left[\frac{1}{N} - 1\right]\left[\frac{(p-1)r}{(p-1)r - 1}\right] + 1$ ($t(N,p) > o$ since $r > \frac{N}{p-1}$)

$$\gamma = \left[\frac{|\Omega|^{t(N,p)}}{t(N,p)}\right]^{1 - \frac{1}{(p-1)r}} \tag{2.4}$$

$$\lambda_a = \left[c_a \, \|k\|^{1/p}_{L^{r/p'}(\Omega)} + \|T\|^{p'/p}_{W^{-1,r}(\Omega)}\right] \tag{2.5}$$

$$\lambda_1 = \exp\left[\lambda_2 \, \gamma \, \|k\|^{1/p}_{L^{r/p'}(\Omega)}\right] \tag{2.6}$$

$$\lambda_2 = \frac{2^*}{N\alpha_N^{1/N}} \quad \text{with} \quad 2^* = 2^{(p'/p-1)_+} \tag{2.7}$$

α_N is the measure of the unit ball of $\mathbb{R}^N$.

The first main results that we obtain are the following:

THEOREM 2.1:

Assume (H.1) *to* (H.6) *and let* u *be a solution of* ($\mathcal{P}$) *in* $W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$. *Then,* u *satisfies the following estimate*

$$\|u\|_\infty \leq \|\varphi\|_\infty + \nu_{a'}^{-1}\ (\lambda_{a'}\ \lambda_1\ \lambda_2) = M\ ,$$

where $a' = \|\varphi\|_\infty$, $\nu_{a'}^{-1}$ *is the inverse function of* $\nu_{a'}$, *mapping* $\mathbb{R}_+$ *into* $\mathbb{R}_+$. *The constant* $\lambda_{a'}, \lambda_1, \lambda_2$ *are given by* (2.5) *to* (2.7).

THEOREM 2.2

Assume (H.1) *to* (H.6). *Then there exists at least one solution* u *of the problem* ($\mathcal{P}$) *in* $W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Moreover, if $\varphi \in W_{loc}^{1,(p-1)r}(\Omega)$, ($a_o \in L^r(\Omega)$, $f_o \in L^{r/p'}(\Omega)$) *then the solution* u *is Hölder continuous inside* Ω .

3. A GENERAL THEOREM FOR L^∞- ESTIMATE

3.1 *A Preliminary Result*

It is convenient for our purpose to present a more general theorem from which we will derive Theorem 2.1.

In the sequel, we will write $T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}$ with $g_i \in L^r(\Omega)$, $r > N/p-1$, $r \geq p/p-1$ and

$$\|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)} \qquad (3.1)$$

There exist functions g_i satisfying these conditions.

THEOREM 3.1

Assume (3.1), (H.3) and (H.4), let u *be a given element in* $W^{1,p}(\Omega)$, *and let* $a'' \in \mathbb{R}_+$ *such that* $\bar{u} = [|u| - a'']_+$ *belongs to* $W_o^{1,p}(\Omega)$. *We suppose that* u *satisfies the following property:*

(A_∞) If $|\bar{u} > o| \neq o$ then for any $\theta \in]o, \operatorname{ess\,sup}_\Omega \bar{u}[$ and $h \in]o, \operatorname{ess\,sup}_\Omega \bar{u} - \theta[$, one has:

$$\int_{\theta < \bar{u} \leq \theta + h} a_i(x,u,\nabla u) \frac{\partial u}{\partial x_i} dx \leq \int_{\theta < \bar{u} \leq \theta + h} g_i \frac{\partial u}{\partial x_i} dx \quad (^2) \qquad (3.2)$$

Then $u \in L^\infty(\Omega)$. Moreover, u satisfies the following estimate

$$\|u\|_\infty \leq a'' + \nu_{a''}^{-1}(\lambda_{a''}\ \lambda_1\ \lambda_2) = M(a'') \qquad (3.4)$$

where the λ_i's are given by (2.5) to (2.7).

Remark 3.1

i) If $|\bar{u} > o| = o$, then $|u| \leq a''$ a.e. in Ω and the conclusion of the theorem is true.

ii) We will prove later that any solution of $(\mathcal{P})$ satisfies the property (A_∞) and Theorem 2.1 follows in this manner from Theorem 3.1. □

Before proving Theorem 3.1, we introduce some notations. For $(g_i) \in L^r(\Omega)$, such that $T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}$, we note $g = \left[\sum_{i=1}^{N} g_i^2\right]^{p'/2}$, and $g_{*\bar{u}}$ is its relative rearrangement with respect to $\bar{u} = [|u| - a'']_+$. Similarly $k_{*\bar{u}}$ is the relative rearrangement of k with respect to $\bar{u}$.

For any $(t,s) \in \bar{\Omega}^* \times \bar{\Omega}^*$, we define:

$$K(t,s) = \exp\left[\lambda_2 \int_t^s \sigma^{\frac{1}{N}-1} \, k_{*\bar{u}}(\sigma)^{1/p}\right] d\sigma \tag{3.5}$$

$$b(s) = s^{\frac{1}{N}-1}\left[c_{a''} k_{*\bar{u}}(s)^{1/p} + g_{*\bar{u}}(s)^{1/p}\right] \tag{3.6}$$

and we recall that $\lambda_2 = \frac{2^{(p'/p-1)_+}}{N\alpha_N{}^{1/N}}$, $c_{a''} = c_o + \int_o^{a''} \nu_1(r))^{p'/p} d\tau$

We note that $K \in L^\infty(\Omega^* \times \Omega^*)$ and $b \in L^1(\Omega)$. In fact, by (1.8), $k_{*\bar{u}} \geq o$ and $g_{*\bar{u}} \geq o$ (since $k \geq o$, and $g \geq o$) . By Holder inequality

$$o \leq \int_{\Omega^*} b(\sigma)\, d\sigma \leq \gamma \| c_{a''} k_{*\bar{u}}^{1/p} + g_{*\bar{u}}^{1/p} \|_{L^{(p-1)r}(\Omega^*)} \qquad (3.7)$$

where,

$$\gamma = \left[\int_{\Omega^*} \sigma^{\left[\frac{1}{N}-1\right]\left[\frac{(p-1)r}{(p-1)r-1}\right]} d\sigma\right]^{1-\frac{1}{(p-1)r}}$$

$$= \left[\frac{|\Omega|^{t(N,p)}}{t(N,p)}\right]^{1-\frac{1}{(p-1)r}}$$

With (1.6), we deduce

$$\| c_{a''} k_{*\bar{u}}^{1/p} + g_{*\bar{u}}^{1/p} \|_{L^{(p-1)r}(\Omega^*)} \leq c_{a''} \|k\|_{L^{r/p'}(\Omega)}^{1/p} + \|g\|_{L^{r/p'}}^{1/p} \qquad (3.8)$$

Since

$$\|g\|_{L^{r/p'}(\Omega)} \leq \left[\sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)}\right]^{p'} = \|T\|_{W^{-1,r}(\Omega)}^{p'} \qquad (3.9)$$

we obtain from (3.6) to (3.9)

$$\|b\|_{L^1(\Omega^*)} \leq \gamma \left[c_{a''} \|k\|^{1/p}_{L^{r/p'}(\Omega)} + \|T\|^{p'/p}_{W^{-1,r}(\Omega)} \right] = \lambda_{a''} \quad (3.10)$$

By the same argument:

$$\|K\|_{L^\infty(\Omega^* \times \Omega^*)} \leq \exp\left[\lambda_2 \gamma \|k\|^{1/p}_{L^{r/p'}(\Omega)} \right] = \lambda_1 \quad (3.11)$$

The following lemma is crucial for the proof of Theorem 3.1.

Lemma 3.1

Assume (3.1), (H.3) and (H.4) and let u *be as in Theorem 3.1. We set* $\Psi(s) = \nu_{a''}(\bar{u}_*(s))$ $s \in \overline{\Omega^*}$, *where* $\bar{u}_*$ *is the decreasing rearrangement of* $\bar{u}$. *Then,* Ψ *satisfies the differential inequality of Gronwall type*

$$-\frac{d\Psi}{ds} \leq \lambda_2 \, s^{\frac{1}{N} - 1} \, k_{*\bar{u}}(s)^{1/p} \, \Psi(s) + \lambda_2 b(s), \text{ a.e. in } \Omega^* \quad (3.12)$$

and for all $s \in \overline{\Omega^*}$:

$$\bar{u}_*(s) \leq \nu^{-1}_{a''} \left[\lambda_2 \int_s^{|\Omega|} K(s,\sigma) \, b(\sigma) \, d\sigma \right] \quad (3.13)$$

Proof of Lemma 3.1.

Let u be a solution of the inequality (3.2). By (H.4)

$$\int_{\theta<u\leq\theta+h} a_i(x,u,\nabla u)\frac{\partial u}{\partial x_i}\,dx \geq \int_{\theta<u\leq\theta+h} \nu_1(|u|)\,|\nabla u|^p dx - \int_{\theta<u\leq\theta+h} \nu_2(|u|)k(x)^{1/p'}\,|\nabla u|dx \qquad (3.14)$$

By setting $\theta_1 = \theta + a''$, we see that

$$\operatorname*{Min}_{\theta_1\leq\eta\leq\theta_1+h} \nu_1(\eta) \int_{\theta<u\leq\theta+h} |\nabla u|^p dx \leq \int_{\theta<u\leq\theta+h} \nu_1(|u|)\,|\nabla u|^p dx \qquad (3.15)$$

and

$$\int_{\theta<u\leq\theta+h} \nu_2(|u|)\,k(x)^{1/p'}\,|\nabla u|dx \leq \operatorname*{Max}_{\theta_1\leq\eta\leq\theta_1+h} \nu_2(\eta) \int_{\theta<u\leq\theta+h} k(x)^{1/p'}\,|\nabla u|dx \qquad (3.16)$$

By the Holder inequality, we obtain:

$$\int_{\theta<u\leq\theta+h} k(x)^{1/p'}\,|\nabla u|\,dx \leq \left[\int_{\theta<u\leq\theta+h} k(x)\,dx\right]^{1/p'} \left[\int_{\theta<u\leq\theta+h} |\nabla u|^p dx\right]^{1/p'} \qquad (3.17)$$

$$\sum_{i=1}^{N} \int_{\theta<\bar{u}\le\theta+h} g_i \frac{\partial u}{\partial x_i}\, dx \le \left[\int_{\theta<\bar{u}\le\theta+h} g(x)\, dx\right]^{1/p'} \left[\int_{\theta<\bar{u}\le\theta+h} |\nabla u|^p dx\right]^{1/p} \qquad (3.18)$$

From (3.2), (3.14) to (3.18), we deduce

$$\underset{\theta_1\le\eta\le\theta_1+h}{\mathrm{Min}}\, \nu_1(\eta) \left[\int_{\theta<\bar{u}\le\theta+h} |\nabla u|^p dx\right] \le \left[\underset{\theta_1\le\eta\le\theta_1+h}{\mathrm{Max}}\, \nu_2(\eta) \left[\int_{\theta<\bar{u}\le\theta+h} k(x)dx\right]^{1/p'} + \right.$$

$$\left. \left[\int_{\theta<\bar{u}\le\theta+h} g(x)dx\right]^{1/p'}\right] \left[\int_{\theta<\bar{u}\le\theta+h} |\nabla u|^p dx\right]^{1/p} \qquad (3.19)$$

After simplification, we obtain

$$\underset{\theta_1\le\eta\le\theta_1+h}{\mathrm{Min}}\, \nu_1(\eta) \left[\int_{\theta<\bar{u}\le\theta+h} |\nabla u|^p dx\right]^{1/p'} \le \left[\left[\int_{\theta<\bar{u}\le\theta+h} k(x)dx\right]^{1/p'} \underset{\theta_1\le\eta\le\theta_1+h}{\max}\, \nu_2(\eta) \right.$$

$$\left. + \left[\int_{\theta<\bar{u}\le\theta+h} g(x)dx\right]^{1/p'}\right] \qquad (3.20)$$

Let us observe that

$$|\nabla u| = |\nabla \bar{u}| \quad \text{on} \quad \{\theta < \bar{u} \le \theta + h\} \qquad (3.21)$$

We multiply (3.20) by $\left[\frac{1}{h}\right]^{1/p'}$, and when h tends to zero, we obtain (using (3.21)) that for a.e. θ :

$$\nu_1(\theta + a'') \left[- \frac{d}{d\theta} \int_{\bar{u}>\theta} |\nabla u|^p dx \right]^{1/p'}$$
$$\leq \left[\nu_2(\theta + a'') \left[- \frac{d}{d\theta} \int_{\bar{u}>\theta} k(x)dx \right]^{1/p} + \left[- \frac{d}{d\theta} \int_{\bar{u}>\theta} g(x)dx \right]^{1/p'} \right] \tag{3.22}$$

By the assumption on $\bar{u}$, $\bar{u} \in W^{1,p}_o(\Omega)$ and $\bar{u} \geq o$, we can apply Theorem 1.1.:

$$- \frac{d}{d\theta} \int_{\bar{u}>\theta} k(x) \; dx = - \mu'(\theta) \; k_{*\bar{u}}(\mu(\theta)) \tag{3.23}$$

$$- \frac{d}{d\theta} \int_{\bar{u}>\theta} g(x)dx = - \mu'(\theta) \; g_{*\bar{u}}(\mu(\theta)) \tag{3.24}$$

where $\mu(\theta) = |\bar{u} > \theta|$. Then we infer from (3.22) to (3.24),

$$\nu_1(\theta + a'') \left[- \frac{d}{d\theta} \int_{\bar{u}>\theta} |\nabla \bar{u}|^p dx \right]^{1/p'} \leq$$
$$\left[\nu_2(\theta + a'') \; k_{*\bar{u}}(\mu(\theta))^{1/p'} + g_{*\bar{u}}(\mu(\theta))^{1/p'} \right] (-\mu'(\theta))^{1/p'}$$

We raise relation (3.25) at power $\frac{p'}{p} = \frac{1}{p-1}$ and observe that:

- if $p \geq 2$, then $\frac{p'}{p} \leq 1$ and $(a + b)^{p'/p} \leq a^{p'/p} + b^{p'/p}$, $\forall a , b \geq o$
- if $1 < p < 2$, the convexity of the function $t \to t^{p'/p}$ implies $(a + b)^{p'/p} \leq 2^{p'/p-1} (a^{p'/p} + b^{p'/p})$

- in both cases , $(a+b)^{p'/p} \leq 2^* \cdot (a^{p'/p} + b^{p'/p})$, $\forall a$, $b \geq o$, $2^* = 2^{(p'/p-1)_+}$.

Thus

$$\nu_1(\theta+a'')^{p'/p} \left[-\frac{d}{d\theta}\int_{\bar{u}>\theta}|\nabla\bar{u}|^p dx\right]^{1/p} \leq$$
$$2^*\left[\nu_2(\theta+a'')^{p'/p}\, k_{*\bar{u}}(\mu(\theta))^{1/p} + g_{*\bar{u}}(\mu(\theta))^{1/p}\right](-\mu'(\theta))^{1/p} \quad (3.26)$$

By the Hölder inequality, we have:

$$\frac{1}{h}\int_{\theta<\bar{u}\leq\theta+h}|\nabla\bar{u}|dx \leq \left[\frac{1}{h}\int_{\theta<\bar{u}\leq\theta+h}dx\right]^{1/p'}\left[\frac{1}{h}\int_{\theta<\bar{u}\leq\theta+h}|\nabla\bar{u}|^p dx\right]^{1/p} \quad (3.27)$$

and when $h > o$ tends to zero, we find:

$$-\frac{d}{d\theta}\int_{\bar{u}>\theta}|\nabla\bar{u}|dx \leq (-\mu'(\theta))^{1/p'}\left[-\frac{d}{d\theta}\int_{\bar{u}>\theta}|\nabla\bar{u}|^p dx\right]^{1/p} \quad (3.28)$$

Due to the Fleming-Rishel formula and the De Giorgi isoperimetric inequality (see Theorems 1.1 and 1.2), we can write:

$$-\frac{d}{d\theta}\int_{\bar{u}>\theta}|\nabla\bar{u}|dx \geq N\,\alpha_N^{1/N}\,[\mu(\theta)]^{1-1/N} \quad (3.29)$$

As $\bar{u}_* \in C^o(\Omega^*)$. we have $\theta = \bar{u}_*(\mu(\theta))$ and from (3.26), (3.28) and (3.29), we deduce that for a.e. $\theta \in]\,o\,,\ \operatorname{ess\,sup}_\Omega \bar{u}\,[$

$$1 \le \lambda_2 \frac{\mu(\theta)^{\frac{1}{N}-1}}{\nu_1(\bar{u}_*(\mu(\theta))+a'')} \left[\nu_2(\bar{u}_*(\mu(\theta))+a'')^{p'/p} k_{*\bar{u}}(\mu(\theta))^{1/p} + g_{*\bar{u}}(\mu(\theta))^{1/p}\right] (-\mu'(\theta))$$

Since the functions $\nu_1 > o$ and ν_2 are continuous (see H.4) and the functions $k_{*\bar{u}}^{1/p}$ and $g_{*\bar{u}}^{1/p}$ are integrable (see (1.6)), we deduce that the mapping:

$$\sigma \in \Omega^* \to \frac{\sigma^{\frac{1}{N}-1}}{\nu_1(\bar{u}_*(\sigma)+a'')} \left[\nu_2(\bar{u}_*(\sigma)+a'')^{p'/p} k_{*\bar{u}}(\sigma)^{1/p} + g_{*\bar{u}}(\sigma)^{1/p}\right]$$

is locally integrable.

We apply Inequation (1.15) to obtain that for $o < s < s + \epsilon < |\Omega|$:

$$\frac{\bar{u}_*(s) - \bar{u}_*(s+\epsilon)}{\epsilon} \le \frac{\lambda_2}{\epsilon} \int_s^{s+\epsilon} \frac{\sigma^{\frac{1}{N}-1}}{\nu_1(\bar{u}_*(\sigma)+a'')^{p'/p}} \cdot$$

$$\left[\nu_2(\bar{u}_*(\sigma)+a'')^{p'/p} k_{*\bar{u}}^{1/p}(\sigma) + g_{*\bar{u}}(\sigma)^{1/p}\right] d\sigma$$

When ϵ tends to zero, we obtain for a.e. s in Ω^* :

$$- \nu_1(\bar{u}_*(s)+a'')^{p'/p} \frac{d\bar{u}_*}{ds} \leq$$

$$\lambda_2 \, s^{\frac{1}{N}-1}\left[\nu_2(\bar{u}_*(s)+a'')^{p'/p} \, k_{*\bar{u}}^{1/p}(s) + g_{*\bar{u}}^{1/p}(s)\right] \qquad (3.30)$$

By (H.4), the function ν_2 satisfies:

$$\nu_2(\bar{u}_*(s)+a'')^{p'/p} \leq \int_0^{\bar{u}_*(s)+a''} \nu_1(\tau)^{p'/p} d\tau + c_0 = c_{a''} + \nu_{a''}(\bar{u}_*(s))$$

(3.31)

with

$$c_{a''} = c_0 + \int_0^{a''} \nu_1(t)^{p'/p} dt \ ,$$

$$\nu_{a''}(t) = \int_{a''}^{a''+t} \nu_1(r)^{p'/p} dr = \int_0^t \nu_1(r+a'')^{p'/p} dr$$

If we set $\psi(s) = \nu_{a''}(\bar{u}_*(s))$, we have $\psi \in W^{1,p}_{loc}(\Omega^*)$ since $\bar{u}_* \in W^{1,p}_{loc}(\Omega^*)$ and

$$- \frac{d}{ds}\psi = - \nu_1(\bar{u}_*(s) + a'')^{p'/p} \frac{d\bar{u}_*}{ds} \text{ a.e. in } \Omega^* \qquad (3.32)$$

From (3.30), (3.31) and (3.32), we find:

$$\frac{d}{ds}\psi \leq \lambda_2 \, s^{\frac{1}{N}-1} k_{*\bar{u}}(s)^{1/p} \psi +$$

$$\lambda_2 \, s^{\frac{1}{N}-1}\left[c_{a''} \, k_{*\bar{u}}(s)^{1/p} + g_{*\bar{u}}^{1/p}(s)\right] \qquad (3.33)$$

Set

$$K(t,s) = \exp\left[\lambda_2 \int_t^s \sigma^{\frac{1}{N}-1} k_{*\bar{u}}(s)^{1/p} d\sigma\right] \quad \text{for } (t,s) \in \bar{\Omega}^* \times \bar{\Omega}^*$$

and

$$b(s) = s^{\frac{1}{N}-1}\left[c_{a''}\, k_{*\bar{u}}(s)^{1/p} + g_{*\bar{u}}(s)^{1/p}\right]$$

We then obtain relation (3.12) of Lemma 3.1 and the relation (3.33) is equivalent to:

$$-\frac{d}{ds}\,[K(o,s)\,\Psi(s)] \leq \lambda_2\, K(o,s)\, b(s) \tag{3.34}$$

By an integration of relation (3.34), we get that for all $s \in \Omega^*$

$$\nu_{a''}(\bar{u}_*(s)) = \Psi(s) \leq \lambda_2 \int_s^{|\Omega|} K(s,\sigma)\, b(\sigma) d\sigma \tag{3.35}$$

(We note that $K(o,\cdot)\,\Psi \in W^{1,p}_{loc}(\Omega^*)$ and $\Psi(|\Omega|) = o$ since $o = \text{ess inf } \bar{u} = \bar{u}_*(|\Omega|))$.

The function $\nu_{a''}$ is strictly increasing from $\mathbb{R}_+$ onto $\mathbb{R}_+$ and we deduce from (3.35):

$$\bar{u}_*(s) \leq \nu_{a''}^{-1}\left[\lambda_2 \int_s^{|\Omega|} K(s,\sigma)\, b(\sigma) d\sigma\right] \quad \text{for all } s \in \bar{\Omega}^*$$

(3.36)

Proof of Theorem 3.1

From Remark 1.1, relations (3.10) and (3.11), we deduce that:

$$\int_s^{|\Omega|} K(s,\sigma)\ b(\sigma)\ d\sigma \leq \|K\|_{L^\infty(\Omega^* \times \Omega^*)} \|b\|_{L^1(\Omega^*)} \leq \lambda_{a''}\ \lambda_1 \qquad (3.37)$$

and then (3.36) and (3.37) imply

$$\bar{u}_*(s) \leq \nu_{a''}^{-1}\ (\lambda_{a''}\ \lambda_1\ \lambda_2) \quad (\nu_{a''}^{-1}\ \text{is also increasing})$$

Since $\bar{u} = [|u| - a'']_+ \geq |u| - a''$, we deduce that:

$$\bar{u}_* \geq [|u| - a'']_* = |u|_* - a'' \quad \text{for all} \quad s \in \bar{\Omega}^* \qquad (3.38)$$

Then, with (3.37) and (3.38), we see that for all $s \in \bar{\Omega}^*$

$$|u|_*(s) \leq a'' + \nu_{a''}^{-1}\ (\lambda_{a''}\ \lambda_1\ \lambda_2)$$

In particular, $\|u\|_\infty = |u|_*(o) \leq a'' + \nu_{a''}^{-1}\ (\lambda_{a''}\ \lambda_1\ \lambda_2) = M(a'')$.

□

3.2 Proof of the L^∞-estimate

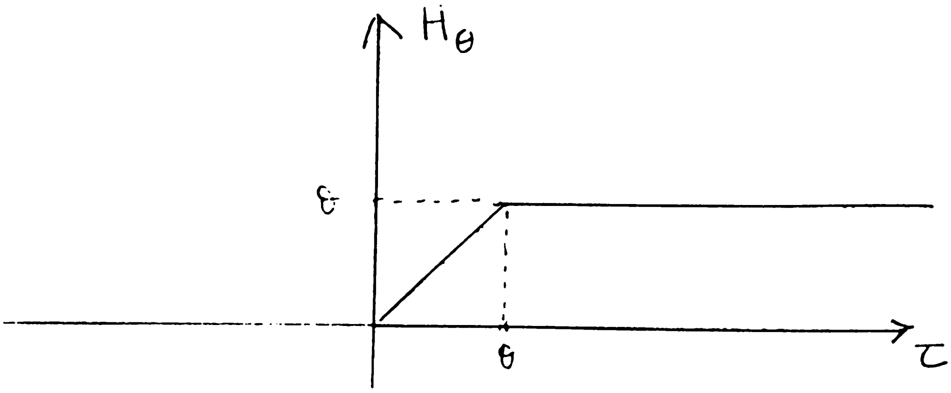
The next step is now to show that any solution u of $(\mathcal{P})$ satisfies the property (A_∞) . This will prove Theorem 2.1.

Lemma 3.2

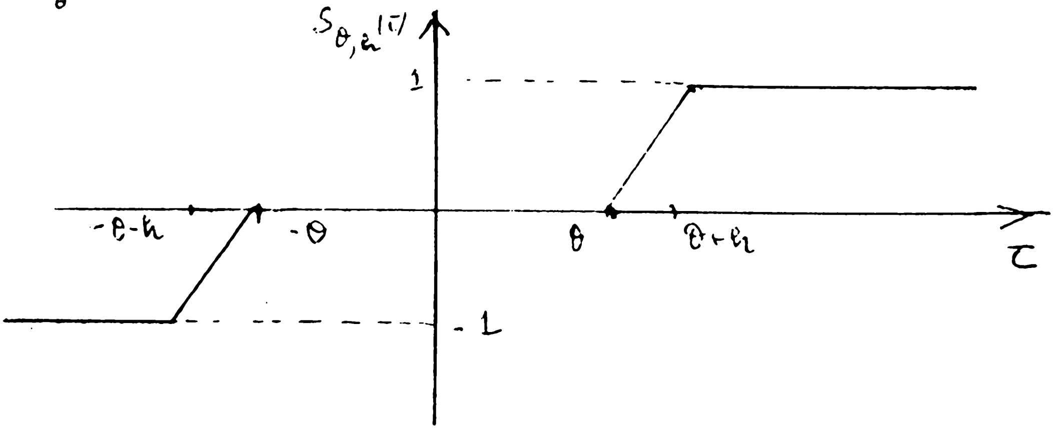
Any solution of $(\mathcal{P})$ *satisfies inequality* (3.2).

The proof of this lemma necessitates some preliminary lemmas that we now prove.

Let $\theta > o$ and $h > o$ be two fixed real numbers; we associate to them two Lipschitz real functions H_θ , $S_{\theta,h}$:



$H_\theta(\tau) = \theta$ if $\tau \geq \theta$, $= \tau$ if $o \leq \tau \leq \theta$, $= o$ otherwise



$S_{\theta,h}(\tau)$ = sign τ if $|\tau| \geq \theta + h$, $= o$ if $|\tau| \leq \theta$, and is linear continuous for $\theta \leq |\tau| \leq \theta + h$.

We consider also a sequence φ_n of $\mathcal{C}_o^\infty(\Omega)$ functions such that φ_n tends to φ a.e. in Ω as $n \to \infty$, and

$$\|\varphi_n\|_\infty \leq \|\varphi\|_\infty \ , \ \forall \ n \tag{3.39}$$

Lemma 3.3

Let $\theta > o$, $h > o$ *be two real numbers, and set* $\theta_1 = \theta + \|\varphi\|_\infty$. *For any* $u \in W^{1,p}_o(\Omega)$, *we define* $\hat{u}_n = H_\theta(u - \varphi_n) \ S_{\theta_1,h}(u)$. *Then,*

$$\hat{u}_n \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega) \tag{3.40}$$

and

$$\frac{\partial \hat{u}_n}{\partial x_i} = H_\theta(u - \varphi_n) \ S'_{\theta_1,h}(u) \ \frac{\partial u}{\partial x_i} \quad \text{a.e.}$$

$$\left\|\frac{\partial \hat{u}_n}{\partial x_i}\right\|_{L^p(\Omega)} \leq \frac{\theta}{h} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)} \tag{3.41}$$

Proof

We note that $S_{\theta_1,h}(u) \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega)$ and $H_\theta(u - \varphi_n) \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega)$, thus $\hat{u}_n \in V^{1,p}_o(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial \hat{u}_n}{\partial x_i} = S_{\theta_1,h}(u) \ H'_\theta(u-\varphi_n) \ \frac{\partial}{\partial x_i}(u-\varphi_n) + H_\theta(u - \varphi_n) \ S'_{\theta_i,h}(u)\frac{\partial u}{\partial x_i} \ \text{a.e.} \tag{3.42}$$

Let us show that $S_{\theta_1,h}(u) \ H'_\theta(u-\varphi_n) \ \frac{\partial}{\partial x_i} (u-\varphi_n) = o$ a.e. in Ω .

In fact, by the definition of H_θ , one can check that

$$H'_\theta(u-\varphi_n)\ \frac{\partial}{\partial x_i}(u-\varphi_n) = \begin{cases} o \text{ on } \{u-\varphi_n \le o\} \cup \{u-\varphi_n \ge \theta\} \\ \frac{\partial}{\partial x_i}(u-\varphi_n) \text{ elsewhere} \end{cases}$$

If $o \le u - \varphi_n \le \theta$ then $\varphi_n \le u \le \theta + \varphi_n$. Since $\|\varphi_n\|_\infty \le \|\varphi\|_\infty$, we deduce that

$$-\theta_1 = -\theta - \|\varphi\|_\infty \le -\|\varphi_n\|_\infty \le \varphi_n \le u \le \theta + \|\varphi_n\|_\infty \le \theta_1$$

Thus $|u| \le \theta_1$, and then from the definition of $S_{\theta_1,h}$: $S_{\theta_1,h}(u) = o$. This proves that $S_{\theta_1,h}(u)\ H'_\theta(u-\varphi_n)\ \frac{\partial u}{\partial x_i} = o$ a.e. on $\{o \le u - \varphi_n \le \theta\}$.

Since $|S'_{\theta_1,h}| \le \frac{1}{h}$ and $|H_\theta| \le \theta$, we deduce from the relation $\frac{\partial \hat{u}_n}{\partial x_i} = S'_{\theta_1,h}(u)\ H_\theta(u-\varphi_n)\ \frac{\partial u}{\partial x_i}$ that $\left|\frac{\partial \hat{u}_n}{\partial x_i}\right| \le \frac{\theta}{h} \left|\frac{\partial u}{\partial x_i}\right|$ a.e. □

Corollary of Lemma 3.3

For any $u \in W^{1,p}_o(\Omega)$, *we set* $\hat{u} = H_\theta(u-\varphi)\ S_{\theta_1,h}(u)$. *Then,* $\hat{u} \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega)$ *and*

$$\frac{\partial \hat{u}}{\partial x_i} = H_\theta(u-\varphi)\ S'_{\theta_1,h}(u)\ \frac{\partial u}{\partial x_i} = \begin{cases} \frac{1}{h} H_\theta(u-\varphi) \frac{\partial u}{\partial x_i} & \text{if } \theta_1 < |u| \le \theta_1 + h \\ o & \textit{otherwise} \end{cases} \tag{3.43}$$

Proof

As $n \to \infty$, $\hat{u}_n$ tends to $\hat{u}$ a.e. in Ω (because $\varphi_n \to \varphi$ a.e. in Ω) and $\hat{u}_n$ remains in a bounded set of $W^{1,p}_o(\Omega)$ (by (3.41)). We deduce that the sequence $\hat{u}_n$ tends to $\hat{u}$ weakly in $W^{1,p}_o(\Omega)$: $\hat{u} \in W^{1,p}_o(\Omega)$. Moreover, $\frac{\partial \hat{u}_n}{\partial x_i} = H_\theta(u-\varphi_n)\ S'_{\theta_1,h}(u)\ \frac{\partial u}{\partial x_i}$ tends to $H_\theta(u-\varphi)\ S'_{\theta_1,h}(u)\ \frac{\partial u}{\partial x_i}$ a.e. in Ω . This implies that $\frac{\partial \hat{u}}{\partial x_i} = H_\theta(u-\varphi)\ S'_{\theta_1,h}(u)\ \frac{\partial u}{\partial x_i}$. □

Lemma 3.4

For any $u \in K(\varphi)$, *we set*, $\check{u} = u - \hat{u}$, $\hat{u} = H_\theta(u-\varphi)\ S_{\theta_1,h}(u)$. *Then*, $\check{u} \in K(\varphi)$

Proof

We infer from the Corollary of Lemma 3.3 that $\check{u} \in W^{1,p}_o(\Omega)$ and we ought to prove that $\check{u} \geq \varphi$.

If $o \leq u - \varphi \leq \theta$, then $u \leq \|\varphi\|_\infty + \theta = \theta_1$ and $u \geq \varphi \geq - \|\varphi\|_\infty \geq - \theta_1$. Hence $|u| \leq \theta_1$, $S_{\theta_1,h}(u) = o$ and then $\check{u} = u \geq \varphi$.

If $u - \varphi \geq \theta$, we note that $\theta \geq H_\theta(u-\varphi)\ S_{\theta_1,h}(u)$, then $u \geq \varphi + \theta \geq \varphi + H_\theta(u - \varphi)\ S_{\theta_1,h}(u) = \varphi + \hat{u}$; thus $\check{u} \geq \varphi$.

Lemma 3.5

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall\ \xi \in \mathbb{R}^N$, $\forall \tau \in \mathbb{R}$

$$H_\theta(\tau)\ S_{\theta_1,h}(\eta)\ F(x,\eta,\xi) \geq o \tag{3.44}$$

Proof

$S_{\theta_1,h}(o) = o$. The relation (3.44) is thus true for $\eta = o$. If $\eta \neq o$, we have $\frac{1}{\eta}\ S_{\theta_1,h}(\eta) \geq o$.

$$S_{\theta_1,h}(\eta)\ F(x,\eta,\xi) = \left[\frac{1}{\eta}\ S_{\theta_1,h}(\eta)\right] \eta\ F(x,\eta,\xi) \tag{3.45}$$

$$\eta\ F(x,\eta,\xi) \geq o \quad \text{(By assumption (H.2))} \tag{3.46}$$

$$H_\theta(\tau) \geq o \quad \text{for any} \quad \tau \tag{3.47}$$

From (3.45) to (3.47) , we obtain (3.48) . □

Proof of Lemma 3.2

Let u be a solution of $(\mathcal{P})$, then by Lemma 3.4, the function $u - H_\theta(u - \varphi)\ S_{\theta_1,h}(u) \in K(\varphi) \cap L^\infty(\Omega)$ is a suitable test function v and we deduce that:

$$\langle Au\ ,\ \hat{u}\rangle + (F(u,\nabla u),\ \hat{u}) \leq \langle T\ ,\ \hat{u}\rangle \tag{3.48}$$

$\hat{u} = H_\theta(u - \varphi)\ S_{\theta_1,h}(u)$. With lemma 3.5, we see that

$$(F(u,\nabla u),\ \hat{u}) = \int_\Omega \hat{u}\ F(x,u,\nabla u)\ dx \geq 0 \tag{3.49}$$

By the Corollary of Lemma 3.3 , we obtain then

$$<Au,\ \hat{u}> = \int_\Omega a_i(x,u,\nabla u)\ \frac{\partial \hat{u}}{\partial x_i}\ dx$$

$$= \frac{1}{h} \int_{\theta_1<|u|\leq\theta_1+h} a_i(x,u,\nabla u)\ H_\theta(u - \varphi)\ \frac{\partial u}{\partial x_i}\ dx \tag{3.50}$$

and

$$<T,\hat{u}> = \int_\Omega g_i\ \frac{\partial \hat{u}}{\partial x_i}\ dx = \frac{1}{h} \int_{\theta_1<|u|\leq\theta_1+h} g_i H_\theta(u-\varphi)\ \frac{\partial u}{\partial x_i}\ dx \tag{3.51}$$

We observe that if $\theta + \|\varphi\|_\infty = \theta_1 < |u|$ then $\theta < |u| - |\varphi| \leq |u - \varphi| = u - \varphi$ and thus $H_\theta(u - \varphi) = \theta$ and $|u| - \|\varphi\|_\infty = [|u| - \|\varphi\|_\infty]_+$. If we set $\bar{u} = [|u| - \|\varphi\|_\infty]_+$, then $\bar{u} \in W^{1,p}_o(\Omega)$ and from (3.50) and (3.51), we find that:

$$<T,\ \hat{u}> = \frac{\theta}{h} \int_{\theta<\bar{u}\leq\theta+h} g_i\ \frac{\partial u}{\partial x_i}\ dx \tag{3.52}$$

$$<Au,\ \hat{u}> = \frac{\theta}{h} \int_{\theta<\bar{u}\leq\theta+h} a_i(x,u,\nabla u)\ \frac{\partial u}{\partial x_i}\ dx \tag{3.53}$$

From (3.48), (3.49), (3.52), and (3.53), we get easily

$$\int_{\theta<\bar{u}\leq\theta+h} a_i(x,u,\nabla u)\,\frac{\partial u}{\partial x_i}\,dx \leq \int_{\theta<\bar{u}\leq\theta+h} g_i\,\frac{\partial u}{\partial x_i}\,dx$$

So u satisfies the property (A_∞) of **Theorem** 3.1 , with $a'' = \|\varphi\|_\infty = a'$. By application of this theorem, we have the estimate:

$$\|u\|_\infty \leq a' + \nu_{a'}^{-1}\,(\lambda_{a'},\ \lambda_1\ \lambda_2) \qquad \square$$

4. A SEQUENCE OF MODIFIED PROBLEMS $(\mathcal{P}_n)$

4.1 A Problem Equivalent to $(\mathcal{P})$

In order to avoid the difficulties related to the possible growth of $a_i(x,\eta,\xi)$ as $|\eta| \to \infty$, we introduce some truncation a_i' of a_i defined as follows:

For a.e. x in Ω, $\forall\eta \in \mathbb{R}$, $\forall\xi \in \mathbb{R}^N$

$$a_i'(x,\eta,\xi) = \begin{cases} a_i(x,\eta,\xi) & \text{if } |\eta| \leq M \\ a_i(x,(\text{sign } \eta)\,M,\xi) & \text{otherwise} \end{cases} \qquad (4.1)$$

Here M is the number given by Theorem 2.1. The functions a'_i enjoy the following properties:

(P_1) Each a'_i is a Caratheodory function and there exists a constant $c = c(M)$ such that:

$$\text{For a.e. } x \in \Omega \;,\; \forall \eta \in \mathbb{R} \;,\; \forall \xi \in \mathbb{R}^N$$

$$|a'_i(x,\eta,\xi)| \leq c\,[|\xi|^{p-1} + a_o(x)]$$

This is a direct consequence of (H.3) . Then, if we define ν'_i (i=1,2) by:

$$\nu'_i(|\eta|) = \begin{cases} \nu_i(|\eta|) & \text{if } |\eta| \leq M \\ \nu_i(M) & \text{otherwise} \end{cases}$$

and

$$\nu'(|\eta|) = \int_o^{|\eta|} (\nu'_1(t))^{p'/p}\,dt \;,$$

we deduce easily from (H.4) that:

(P_2) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$:

$$\sum_{i=1}^{N} a'_i(x,\eta,\xi)\;\xi_i \geq \nu'_1(|\eta|)\;|\xi|^p - \nu'_2(|\eta|)\;k(x)^{1/p'}|\xi|$$

and

$$[\nu'_2(|\eta|)]^{p/p} \leq \nu'(|\eta|) + c_o$$

Moreover, setting $\alpha = \underset{o\leq\eta\leq M}{\text{Min}}\,\nu_1(\eta)$, $\beta = \underset{o\leq\eta\leq M}{\text{Max}}\,\nu_2(\eta)$ we see that:

$$\sum_{i=1}^{N} a'_i(x,\eta,\xi)\xi_i \geq \alpha|\xi|^p - \beta\;k(x)^{1/p'}|\xi|$$

It is easy to check that:

(P_3) For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall (\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \xi'$,

$$\sum_{i=1}^{N} [a'_i(x,\eta,\xi) - a'_i(x,\eta,\xi')] [\xi_i - \xi'_i] > 0$$

We define an operator A' which maps $W_o^{1,p}(\Omega)$ into $W^{-1,p}(\Omega)$ by setting:

$$A'u = - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a'_i(x,u(x), \nabla u(x)) \quad \text{for} \quad u \in W_o^{1,p}(\Omega)$$

Clearly the properties (P1) to (P3) amount to saying that A' satisfies the same hypotheses as $\mathcal{P}$ ((H.3) to (H.5)).

We then introduce the following problem $(\mathcal{P}')$

$$(\mathcal{P}') \quad \left| \begin{array}{l} \text{To find } u \in K \cap L^\infty(\Omega) \text{ such that} \\ \langle A'u, v-u\rangle + (F(u,\nabla u), v-u) \geq \langle T, v-u\rangle \\ \qquad \text{for all } v \in K \cap L^\infty(\Omega) \end{array} \right.$$

PROPOSITION 4.1

Any solution of $(\mathcal{P}')$ *is a solution of* $(\mathcal{P})$ *and conversely.*

PROOF

Let us define $\nu'_{a'} = \int_o^t \nu'_1(t+a')^{p'/p}\, dt$ ($a' = \|\varphi\|_\infty$, $t \geq o$); $\nu'_{a'}$ is invertible from $\mathbb{R}_+$ onto $\mathbb{R}_+$. On the other hand,

$$\nu'_{a'}(M-a') = \nu_{a'}(M-a') = \lambda_{a'}\, \lambda_1\, \lambda_2 \qquad (4.2)$$

In fact, $\nu'_{a'}(M-a') = \int_o^{M-a'} \nu'_1(t+a')^{p'/p}\, dt = \int_{a'}^{M} \nu'_1(t)^{p'/p}\, dt = \nu_{a'}(M-a')$.

Let u be a solution of $(\mathcal{P}')$. As in Theorem 2.1, one can check that:

$$\|u\|_\infty \leq a' + (\nu'_{a'})^{-1}\, (\lambda_{a'}\, \lambda_1\, \lambda_2) \qquad (4.3)$$

(We note $c_{a'} = c_o + \int_o^{\|\varphi\|} \nu'_1(t)^{p'/p}\, dt$ since $\|\varphi\|_\infty \leq M$).

From (4.2), we see that (4.3) is equivalent to:

$$\|u\|_\infty - a' \leq (\nu'_{a'})^{-1}\, (\lambda_{a'}\, \lambda_1\, \lambda_2) = M-a' \text{ , i.e., } \|u\|_\infty \leq M$$

Thus, $a'_i(x,u(x),\nabla u(x)) = a_i(x,u(x),\nabla u(x))$ for a.e. $x \in \Omega$ and we deduce that for all $v \in K \cap L^\infty(\Omega)$:

$$\langle A'u,\ v-u\rangle = \langle Au,\ v-u\rangle$$

The proof of the converse property is similar.

4.2 A SEQUENCE OF MODIFIED PROBLEMS APPROACHING $\mathcal{P}'$

The difficulties related to the possible growth of the functions a_i (as $|\eta| \to \infty$) have been overcome by replacing a_i by the truncated functions a_i' . For the growth of the functions $F(x,\eta,\xi)$ as $|\eta| \to \infty$, we introduce now another form of truncation which is more appropriate, but which will necessitate this time an approximation procedure and a passage to the limit.

For any $n \in \mathbb{N}^*$, for a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, we set

$$F_n(x,\eta,\xi) = \frac{F(x,\eta,\xi)}{1+\frac{1}{n}|F(x,\eta,\xi)|} \tag{4.4}$$

We can deduce easily from (H.2) that F_n satisfies the following properties:

(Q_1) Each F_n is a Caratheodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ and for a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

(i) $\eta F_n(x,\eta,\xi) \geq 0$

(ii) $|F_n(x,\eta,\xi)| \leq |F(x,\eta,\xi)|$

(Q_2) $|F_n(x,\eta,\xi)| \leq n$

Let us introduce then the following approximate problem

$$(\mathcal{P}_n) \quad \left| \begin{array}{l} \text{Find } u_n \in K(\varphi) \text{ such that} \\ \langle A'u_n, v-u_n\rangle + (F_n(u_n,\nabla u_n), v-u_n) \geq \langle T, v-u_n\rangle \\ \qquad \text{for all } v \in K \end{array} \right.$$

Then we have the:

Lemma 4.1

(i) *There exists at least one solution* u_n *of* $(\mathcal{P}_n)$ *in* $K(\varphi)$ *Moreover,*

(ii) *Any solution* u_n *of* $(\mathcal{P}_n)$ *is in* $L^\infty(\Omega)$ *and satisfies the following estimates:*

$$\|u_n\|_\infty \leq M$$

where M *is exactly the number given in Theorem 2.1.*

PROOF

The first part of Lemma 4.1 is a direct consequence of well-known results concerning the variational inequalities (see for instance [17], Theorem 8.2, page 247). The properties $(\mathcal{P}_1)$ to $(\mathcal{P}_3)$ and (Q_1), (Q_2) insure that the operator A_n, $A_n u = A'u + F_n(\cdot, u, \nabla u)$ is pseudo-monotone from the closed-convex set $K(\varphi)$ into $W^{-1,p'}(\Omega)$ and is coercive in the sense that:

$$\left| \begin{array}{l} \text{There exists} \quad v_o \in K(\varphi) \\ \dfrac{\langle A_n v, v - v_o \rangle}{\|v\|_{W_o^{1,p}(\Omega)}} \to \infty \quad \text{as} \quad \|v\|_{W_o^{1,p}(\Omega)} \to +\infty \end{array} \right.$$

We note that this last property of A_n is true for any $v_o \in K$.

To prove the part (ii) of the lemma, it suffices to show that u_n satisfies the property (A_∞) of Theorem 3.1 with a_i replaced by a_i' . For this purpose, we choose $a'' = \|\varphi\|_\infty = a'$ and $\bar{u}_n = [|u_n| - \|\varphi\|_\infty]_+$.

The proof is then similar to the proof of Lemma 3.2, i.e., we consider $\dot{u}_n = H_\theta(u_n - \varphi)\ S_{\theta_1,h}(u_n)$ with $\theta_1 = \theta + \|\varphi\|_\infty$ and then from Lemma 3.4, the function $v_n = u_n - \hat{u}_n$ belongs to $K(\varphi)$. So if we take $v = v_n$ in $(\mathcal{P}_n)$ we get easily

$$\langle A' u_n, \hat{u}_n \rangle + \int_\Omega \hat{u}_n\ F_n(x, u_n, \nabla u_n)\ dx \leq \langle T, \hat{u}_n \rangle \qquad (4.6)$$

As in Lemma 3.5, using the property $(Q_1)(i)$ of F_n we get easily:

$$\int_\Omega \hat{u}_n\ F_n(x, u_n, \nabla u_n)\ dx \geq 0 \qquad (4.7)$$

As in Lemma 3.2, we check that:

$$\langle A' u_n, \hat{u}_n \rangle = \frac{\theta}{h} \int_{\theta < \bar{u}_n \leq \theta + h} a_i'(x, u_n, \nabla u_n) \frac{\partial u_n}{\partial x_i} dx \tag{4.8}$$

$$\langle T, \hat{u}_n \rangle = \frac{\theta}{h} \int_{\theta < \bar{u}_n \leq \theta + h} g_i \frac{\partial u_n}{\partial x_i} dx \tag{4.9}$$

From (4.6) to (4.9), we find that:

$$\int_{\theta < \bar{u}_n \leq \theta + h} a_i'(x, u_n, \nabla u_n) \frac{\partial u_n}{\partial x_i} dx \leq \int_{\theta < \bar{u}_n \leq \theta + h} g_i \frac{\partial u_n}{\partial x_i} dx$$

for all $\theta \in \left] o , \operatorname{ess\,sup}_{\Omega} \bar{u}_n \right[$ and $h \in \left] o, \operatorname{ess\,sup}_{\Omega} \bar{u}_n - \theta \right[$.
We then infer from Theorem 3.1 that

$$\|u_n\|_\infty \leq a' + \nu_{a'}^{-1} (\lambda_{a'} \ \lambda_1 \ \lambda_2) = M \qquad \square$$

REMARK 4.1

Lemma 4.1 insures that $K \cap L^\infty(\Omega) \neq \emptyset$. $\square$

We now study the behaviour of u_n as $n \to \infty$, and we first show that the sequence u_n remains in a bounded set of $W_o^{1,p}(\Omega)$. We have:

Lemma 4.2

There exists a constant c_1 *independent of* n, *such that:*

$$\|\nabla u_n\|_{L^p(\Omega)} \leq c_1 \tag{4.10}$$

In the sequel, the c_i like c_1 will denote various constants which are independent of n .

Before proving lemma 4.2, we state two more lemmas.

LEMMA 4.3

Let $\mu_1 > 0$, $\mu_2 > 0$ *be given real numbers. Then there exists a function* $\sigma \in \mathcal{C}^1(\mathbb{R})$ *such that*

$$\begin{cases} \mu_1 \sigma'(t) - \mu_2 \, |\sigma(t)| = 1 \quad \text{in} \quad \mathbb{R} \\ \sigma(0) = 0 \ , \quad \sigma \text{ is odd} \end{cases} \tag{4.11}$$

PROOF

We consider the differential equation

$$\begin{cases} \mu_1 \bar{\sigma}' - \mu_2 \bar{\sigma} = 1 \quad \text{in} \quad \mathbb{R}_+ \\ \bar{\sigma}(0) = 0 \end{cases}$$

One can check that $\bar{\sigma}$ is given by $\bar{\sigma}(t) = \frac{1}{\mu_2} \left[e^{\frac{\mu_2 t}{\mu_1}} - 1 \right]$. We then define σ by the following formula:

$$\sigma(t) = \begin{cases} \bar{\sigma}(t) & \text{if } t \geq o \\ -\bar{\sigma}(-t) & \text{if } t < o \end{cases}$$

Then, we have

$$\begin{cases} \mu_1 \sigma' - \mu_2 |\sigma| = 1 & \text{in } \mathbb{R} \\ \sigma(o) = o \end{cases}$$ □

LEMMA 4.4

We consider the function σ *of Lemma 4.3 and let* $\Gamma > o$ *be defined by*

$$\Gamma = \sup_{|t| \leq 2M} \frac{|\sigma(t)|}{|t|} \tag{4.12}$$

where M *is the number given in Theorem 2.1. Then:*

$$v_{n,m} = u_m - \frac{1}{\Gamma} \sigma(u_n - u_m) \in K(\varphi) \quad \textit{for every} \quad n,m \in \mathbb{N}^* .$$

Here, u_n *(resp.* u_m*) is a solution of* $(\mathcal{P}_n)$ *(resp.* $\mathcal{P}_m$*) .*

PROOF

Since $u_n - u_m \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\sigma \in \mathscr{C}^1(\mathbb{R})$ satisfies $\sigma(o) = o$, we deduce that $\sigma(u_n - u_m) \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$. On the other hand, we can check that $\sigma' \geq \frac{1}{\mu_1} > o$. So $u_m \geq \varphi$ a.e. implies

$$\frac{1}{\Gamma}\,\sigma(u_n - u_m) \le \frac{1}{\Gamma}\,\sigma(u_n - \varphi) \text{ a.e} \tag{4.13}$$

Thus,

$$v_{n,m} - \varphi \ge u_n - \varphi - \frac{1}{\Gamma}\,\sigma(u_n - \varphi) \tag{4.14}$$

By Lemma 4.1, we get $|u_n - \varphi| = u_n - \varphi \le 2M$ and by the definition of Γ :

$$\sigma(u_n - \varphi) = |\sigma(u_n - \varphi)| \le (u_n - \varphi)\cdot\Gamma \tag{4.15}$$

Finally, from (4.14) and (4.15), we find

$$v_{n,m} - \varphi \ge u_n - \varphi - \frac{1}{\Gamma}\,\sigma(u_n - \varphi) \ge o \qquad \square$$

PROOF OF LEMMA 4.2

We set $\alpha = \underset{o\le\eta\le M}{\mathrm{Min}}\, \nu_1(\eta)$, $\beta = \underset{o\le\eta\le M}{\mathrm{Max}}\, \nu_2(\eta)$

In Lemma 4.3, we choose $\mu_1 = \alpha > o$, $\mu_2 = f(M) + 1$ (see (H.2) for the definition of f) and φ is the function associated with μ_1 , μ_2 . Let us take $v = v_{n,1} = u_n - \frac{1}{\Gamma}\,\sigma(u_n - u_1)$ in $(\mathcal{P}_n)$; we deduce:

$$\langle A'u_n,\, \sigma(u_n - u_1)\rangle + \int_\Omega F_n(x,u_n,\nabla u_n)\,\sigma(u_n - u_1)\,dx \le \langle T,\sigma(u_n - u_1)\rangle \tag{4.16}$$

We observe that $A'u_n = Au_n$ since $\|u_n\|_\infty \leq M$, and therefore

$$<A'u_n,\sigma(u_n - u_1)> = \int_\Omega a_i(x,u_n,\nabla u_n)\ \sigma'(u_n - u_1)\frac{\partial u_n}{\partial x_i}\,dx$$
$$- \int_\Omega a_i(x,u_n,\nabla u_n)\ \sigma'(u_n - u_1)\frac{\partial u_1}{\partial x_i}\,dx$$

$$<T,\ \sigma(u_n - u_1)> = \int_\Omega g_i\ \sigma'(u_n - u_1)\frac{\partial}{\partial x_i}\ (u_n - u_1)\ dx \quad (4.18)$$

Since $\sigma' > o$, by (H.4) , we have a e. in Ω :

$$a_i(x,u_n,\nabla u_n)\frac{\partial u_n}{\partial x_i}\ \sigma'(u_n - u_1) \geq \alpha\ \sigma'(u_n - u_1)\ |\nabla u_n|^p$$
$$- \beta\ k(x)^{1/p'}\,|\nabla u_n|\ \sigma'(u_n - u_1) \quad (4.19)$$

From (4.16) to (4.19), we deduce (using (Q_1)(ii)) :

$$\int_\Omega \alpha\sigma'(u_n - u_1)\ |\nabla u_n|^p\ dx \leq \int_\Omega |\sigma(u_n - u_1)\ F(x,u_n,\nabla u_n)|\ dx$$
$$+ \int g_i\ \sigma'(u_n - u_1)\frac{\partial}{\partial x_i}\ (u_n - u_1)\ dx$$
$$+ \int_\Omega a_i(x,u_n,\nabla u_n)\ \sigma'(u_n - u_1)\frac{\partial u_n}{\partial x_i}\,dx$$
$$+ \beta\int_\Omega k(x)^{1/p'}\,|\nabla u_n|\ \sigma'(u_n - u_1)dx \quad (4.20)$$

We infer from (H.2) that:

$$\int_\Omega |\sigma(u_n - u_1)|\ |F(x,u_n,\nabla u_n)|\ dx \le \mu_2 \int_\Omega |\sigma(u_n - u_1)|\ |\nabla u_n|^p\ dx$$
$$+ \mu_2 \int_\Omega f_o(x)\ |\sigma(u_n - u_1)|\ dx \qquad (4.21)$$

Since $|u_n - u_1| \le 2M$, we can find two numbers Γ_1 , $\Gamma_2 > o$ independent of n such that

$$|\sigma'(u_n - u_1)| \le \Gamma_1\ ,\ |\sigma(u_n - u_m)| \le \Gamma_2\ ,\ \text{a.e. in } \Omega \qquad (4.22)$$

From (4.21) and (4.22), we then find $c_3 > o$ such that:

$$\int_\Omega |\sigma(u_n - u_1)|\ |F(x,u_n,\nabla u_n)|\ dx \le$$
$$\int_\Omega \mu_2 |\sigma(u_n - u_1|\ (u_n - u_1)|\nabla u_n|^p\ dx + c_3 \qquad (4.23)$$

With the Hölder inequality, we deduce (using (4.22)), that there exists $c_4 > o$ such that:

$$\left|\int_\Omega g_i\ \sigma'(u_n - u_1)\frac{\partial}{\partial x_i}\ (u_n - u_1)\ dx\ \right| \le c_4\ \|\nabla(u_n - u_1)\|_{L^p(\Omega)} \qquad (4.24)$$

$$\left|\beta\int_\Omega k(x)^{1/p'}\,|\nabla u_n|\ \sigma'(u_n - u_1)\ dx\ \right| \le c_4\ \|\nabla u_n\|_{L^p(\Omega)} \qquad (4.25)$$

Due to (H.3) and (4.22)

$$\left|\int_\Omega a_i(x,u_n,\nabla u_n)\ \sigma'(u_n - u_1)\frac{\partial u_1}{\partial x_i}\ dx\ \right| \le c_5 \int_\Omega |\nabla u_n|^{p-1}|\nabla u_1|\ dx$$
$$+ c_5 \int_\Omega a_o(x)\ |\nabla u_1|\ dx \tag{4.26}$$

and with Hölder inequality:

$$\int_\Omega |\nabla u_n|^{p-1}|\nabla u_1|\ dx \le \|\nabla u_n\|^{p/p'}_{L^p(\Omega)} \|\nabla u_1\|_{L^p(\Omega)} \tag{4.27}$$

Hence from (4.26) and (4.27), we deduce:

$$\left|\int_\Omega a_i(x,u_n,\nabla u_n)\ \sigma'(u_n - u_1)\frac{\partial u_1}{\partial x_i}\ dx\ \right| \le c_6\ \|\nabla u_n\|^{p/p'}_{L^p(\Omega)} + c_7 \tag{4.28}$$

Now, we combine (4.20) to (4.28), to obtain that

$$\int_\Omega [\mu_1\ \sigma'(u_n - u_1) - \mu_2|\ \sigma(u_n - u_1)|]\ |\nabla u_n|^p\ dx$$
$$\le c_8\ \|\nabla u_n\|^{p/p'}_{L^p(\Omega)} + c_9\ \|\nabla u_n\|_{L^p(\Omega)} + c_{10} \tag{4.29}$$

By the choice of σ (see Lemma 4.3)

$$\int_\Omega [\mu_1\ \sigma'(u_n - u_1) - \mu_2|\ \sigma(u_n - u_1)|]\ |\nabla u_n|^p\ dx = \|\nabla u_n\|^p_{L^p(\Omega)} \tag{4.30}$$

So, (4.29) together with (4.30) give:

$$\|\nabla u_n\|^p_{L^p(\Omega)} \le c_8 \|\nabla u_n\|^{p/p'}_{L^p(\Omega)} + c_9 \|\nabla u_n\|_{L^p(\Omega)} + c_{10} \qquad (4.31)$$

A simple application of Young's inequality leads to the desired result. □

4.3 A STRONG CONVERGENCE RESULT IN $W^{1,p}_o(\Omega)$

We now want to pass to the limit $n \to \infty$. By Lemma 4.1 and 4.2 there exists $u \in W^{1,p}_o(\Omega) \cap L^\infty(\Omega)$ and a subsequence (still denoted n) such that, as $n \to \infty$, u_n converges to some limit u in the following sense:

$$\text{weakly in } W^{1,p}_o(\Omega), L^\infty(\Omega)\text{-weak}^* \text{ and a.e. in } \Omega \qquad (4.32)$$

Our aim is to pass to the limit $n \to \infty$ in $\mathcal{P}_n$.

For convenience, we denote by $\tilde{A}(u,\nabla u)$ the vector of $\mathbb{R}^N$ whose components are $a_i(x,u(x),\nabla u(x))$ and we write

$$\sum_{i=1}^{N} a_i(x,u(x), \nabla u(x))\frac{\partial u_n}{\partial x_i} = \tilde{A}(u,\nabla u) \cdot \nabla u \ , \ F(x,u,\nabla u) = F(u,\nabla u)$$

LEMMA 4.5

The sequence u_n *tends to* u *strongly in* $W^{1,p}_o(\Omega)$ *as* n *tends to infinity*

PROOF

The idea of the proof is essentially to use the property (S_+) of F.E. Browder ([5], page 27, see also [22], [26]).

In order to prove the lemma, it suffices then to prove that:

$$\limsup_{n\to+\infty} \int_\Omega [\tilde{A}(u_n,\nabla u_n) - \tilde{A}(u,\nabla u)] \cdot \nabla(u_n - u)\, dx \leq o \tag{4.33}$$

As in Lemma 4.2, we choose $\mu_1 = \alpha$, $\mu_2 = f(M) + 1$ and let σ be the associated function (according to Lemma 4.3). We denote by $\Gamma > o$, the number satisfying $|\sigma(t)| \leq |t| \cdot \Gamma$ for any $|t| \leq 2M$. From Lemma 4.4, we obtain $\forall n,m \in \mathbb{N}^*$.

$$v_{n,m} = u_n - \frac{1}{\Gamma}\sigma(u_n - u_m) \in K(\varphi)$$
$$v_{m,n} = u_m - \frac{1}{\Gamma}\sigma(u_m - u_n) \in K(\varphi) \tag{4.34}$$

We replace v by $v_{n,m}$ (resp. $v_{m,n}$) in $(\mathcal{P}_n)$ (resp. in $\mathcal{P}_m$) . We find:

$$\langle Au_n, \sigma(u_n - u_m)\rangle + (F_n(u_n,\nabla u_n), \sigma(u_n-u_m)) \leq \langle T, \sigma(u_n-u_m)\rangle \tag{4.35}$$

$$\langle Au_m, \sigma(u_m-u_n)\rangle + (F_m(u_m,\nabla u_m), \sigma(u_m-u_n)) \leq \langle T,\sigma(u_m-u_n)\rangle \tag{4.36}$$

Since σ is odd, we have $\sigma(u_m-u_n) = -\sigma(u_n-u_m)$. Thus adding (4.35) and (4.36), we find:

$$<Au_n - Au_m , \sigma(u_n-u_m)> + <F_n(u_n,\nabla u_n) - F_m(u_m,\nabla u_m), \sigma(u_n-u_m)> \leq o \tag{4.37}$$

Hence, we find:

$$\mu_1 \int_\Omega [\tilde{A}(u_n,\nabla u_n) - \tilde{A}(u_m,\nabla u_m)] \nabla(u_n-u_m) \sigma'(u_n-u_m) \leq$$
$$\mu_1 \int_\Omega [|F_n(u_n,\nabla u_n)| + |F_m(u_m,\nabla u_m)|] |\sigma(u_n-u_m)| dx \tag{4.38}$$

By assumption (H.2) on the growth of F , we see that

$$\int_\Omega (|F_n(u_n,\nabla u_n)| + |F_m(u_m,\nabla u_m)|) |\sigma(u_n-u_m)| dx \leq$$
$$\mu_2 \int_\Omega (|\nabla u_n|^p + |\nabla u_m|^p |\sigma(u_n-u_m)| dx +$$
$$2\mu_2 \int_\Omega f_o(x) |\sigma(u_n-u_m)| dx \tag{4.39}$$

Due to (H.4) , using the fact that $\|u_n\|_\infty \leq M$,(resp. $\|u_m\|_\infty \leq M$), we obtain:

$$\tilde{A}(u_n,\nabla u_n) \cdot \nabla u_n + \beta k(x)^{1/p'} |\nabla u_n| \geq \mu_1 |\nabla u_n|^p \tag{4.40}$$

$$\tilde{A}(u_m,\nabla u_m) \cdot \nabla u_m + \beta k(x)^{1/p'} |\nabla u_m| \geq \mu_1 |\nabla u_m|^p \tag{4.41}$$

We infer from (4.39) to (4.41) that:

$$\begin{aligned}
\mu_1 \int_\Omega [|F_n(u_n,\nabla u_n)| + |F_m(u_m,\nabla u_m)|]\ |\sigma(u_n-u_m)|\ dx \\
\le \mu_2 \int \tilde{A}(u_n,\nabla u_n)\ \nabla u_n\ |\sigma(u_n-u_m)|\ dx \\
+ \mu_2 \int_\Omega \tilde{A}(u_m,\nabla u_m)\nabla u_m\ |\sigma(u_n - u_m)|dx \\
+ \mu_2 \int_\Omega k(x)^{1/p'}\ |\nabla u_m|\ |\sigma(u_n-u_m)|dx \\
+ \mu_2\ \beta \int_\Omega k(x)^{1/p'}\ |\nabla u_n|\ |\sigma(u_n - u_m)|\ dx \\
+ 2\mu_1\ \mu_2 \int_\Omega f_o(x)\ |\sigma(u_n - u_m)|\ dx
\end{aligned} \tag{4.42}$$

Hereafter we will let $n \to \infty$ and then $m \to \infty$. Since $u_n \to u$ a.e. in Ω as $n \to \infty$ (and also $u_m \to u$ a.e. in Ω as $m \to \infty$) , we infer from the Lebesgue dominated convergence theorem that:

$$\lim_{m\to\infty} \int_\Omega f_o(x)\ |\sigma(u_n-u_m)|\ dx = \int_\Omega f_o(x)|\sigma(u - u_m)|dx$$

$$\lim_{m\to\infty} \int_\Omega f_o(x)|\sigma(u - u_m)|dx = o$$

Similarly, since $|\nabla u_n|$ and $|\nabla u_m|$ remain in a bounded set of $L^p(\Omega)$ we can write:

$$\limsup_{m \to \infty}\ \limsup_{n \to \infty} \int_\Omega k(x)^{1/p'} (|\nabla u_n| + |\nabla u_m|)\ |\sigma(u_n - u_m)|\ dx = o$$

For the sake of simplicity we will denote symbolically by θ_{nm} any quantity satisfying

$$\limsup_{n \to \infty} \theta_{nm} = \theta_m,\ \limsup_{m \to \infty} \theta_m = o\ .$$

With this remark, the relation (4.42), combined with the relation (4.38) implies:

$$\mu_1 \int_\Omega [\tilde{A}(u_n,\nabla u_n) - \tilde{A}(u_m,\nabla u_m)]\ \nabla(u_n - u_m)\ \sigma'(u_n - u_m)\ dx \le$$

$$\mu_2\Big[\Big|\int_\Omega \tilde{A}(u_n,\nabla u_n)\ \nabla u_n|\ \sigma(u_n - u_m)|\ dx +$$

$$\int_\Omega \tilde{A}(u_m,\nabla u_m)\ \nabla u_m|\sigma(u_n - u_m)|\ dx\Big] + \theta_{nm} \quad (4.43)$$

Let us write:

$$\tilde{A}(u_n,\nabla u_n)\cdot\nabla u_n = \tilde{A}(u_n,\nabla u_m)\cdot\nabla(u_n - u_m) + \tilde{A}(u_n,\nabla u_n)\cdot\nabla u_m \quad (4.44)$$

and

$$\tilde{A}(u_m,\nabla u_m)\cdot\nabla u_m = -\tilde{A}(u_m,\nabla u_m)\cdot\nabla(u_n-u_m) + \tilde{A}(u_m,\nabla u_m)\cdot\nabla u_n \quad (4.45)$$

From (4.43) to (4.45), using the fact that $\mu_1\ \sigma'(u_n - u_m) - \mu_2\ |\sigma(u_n - u_m)| = 1$ (see Lemma 4.3), we obtain

$$\int_\Omega [\tilde{A}(u_n,\nabla u_n) - \tilde{A}(u_m,\nabla u_m)]\cdot\nabla(u_n-u_m)\ dx$$
$$\le \mu_2 \int_\Omega \tilde{A}(u_n,\nabla u_n)\ \nabla u_m|\sigma(u_n - u_m)|\ dx$$
$$+ \mu_2 \int_\Omega \tilde{A}(u_m,\nabla u_m)\ \nabla u_m\ |\sigma(u_n - u_m)|\ dx + \theta_{nm} \quad (4.46)$$

Since $\tilde{A}(u_m, \nabla u_m)$ remains in a bounded set of $(L^{p'}(\Omega))^N$, we can extract a subsequence still denoted n such that:

$$\tilde{A}(u_n, \nabla u_n) \to U \text{ weakly in } (L^{p'}(\Omega))^N, \text{ as } n \to +\infty$$

For a fixed m, we take the lim sup as $n \to \infty$ of (4.46) and we obtain via the Lebesgue dominated convergence theorem ([3]):

$$\begin{aligned} &\limsup_{n \to +\infty} \int_\Omega \tilde{A}(u_n, \nabla u_n)\, \nabla u_n \, dx - \int_\Omega U\, \nabla u_m \, dx \\ &- \int_\Omega \tilde{A}(u_m, \nabla u_m)\, \nabla u \, dx + \int_\Omega \tilde{A}(u_m, \nabla u_m) \cdot \nabla u_m \, dx \\ \leq\ & \mu_2 \int_\Omega U\, \nabla u_m\, |\sigma(u - u_m)|\, dx + \mu_2 \int_\Omega \tilde{A}(u_m, \nabla u_m)\, \nabla u\, |\sigma(u - u_m)|\, dx \\ &+ \theta_m \end{aligned} \tag{4.47}$$

Then we take the lim sup as $m \to +\infty$ of (4.56), and we find:

$$\limsup_{n \to +\infty} \int_\Omega \tilde{A}(u_n, \nabla u_n) \cdot \nabla u_n \, dx \leq \int_\Omega U\, \nabla u \, dx \tag{4.48}$$

It follows from the Lebesgue dominated convergence theorem, that:

$$\lim_{n \to +\infty} \tilde{A}(u_n, \nabla u) = \tilde{A}(u, \nabla u) \text{ strongly in } \left[L^{p'}(\Omega)\right]^N,$$

and we conclude easily that:

$$\begin{aligned} &\limsup_{n \to +\infty} \int_\Omega [\tilde{A}(u_n, \nabla u_n) - \tilde{A}(u, \nabla u)] \cdot \nabla(u_n - u)\, dx \\ &\leq \limsup_{n \to +\infty} \int_\Omega \tilde{A}(u_n, \nabla u_n)\, \nabla u_n \, dx - \int_\Omega U\, \nabla u \, dx \\ &\quad - \int_\Omega \tilde{A}(u, \nabla u)\, \nabla u \, dx + \int_\Omega \tilde{A}(u, \nabla u)\, \nabla u \, dx \end{aligned} \tag{4.49}$$

From (4.48), the right-hand side of this inequality is negative, and this leads to the desired result, i.e., (4.33).

□

4.4 THE EXISTENCE OF SOLUTIONS FOR THE PROBLEM $(\mathcal{P})$:

We conclude the passage to the limit $n \to \infty$ in $\mathcal{P}_n$ and the proof of existence of solution of $\mathcal{P}$. The convergences (4.32) are now supplemented by the strong-convergence result given by Lemma 4.5, namely $u_n \to u$ strongly in $W_o^{1,p}(\Omega)$ as $n \to \infty$.

Let $v \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$, and write

$$\langle Au_n, v-u_n \rangle = \int_\Omega [a_i(u_n, \nabla u_n) - a_i(u, \nabla u)] \frac{\partial}{\partial x_i} (v-u_n)\, dx$$
$$+ \int_\Omega a_i(u, \nabla u) \frac{\partial}{\partial x_i} (v-u_n)\, dx$$

By the Vitali Theorem, we deduce (using Lemma 4.5) that $a_i(u_n, \nabla u_n) - a_i(u, \nabla u)$ tends to zero in $L^{p'}(\Omega)$-strongly and

$$\frac{\partial}{\partial x_i} (v-u_n) \to \frac{\partial}{\partial x_i} (v-u) \quad \text{in } L^p(\Omega) - \text{strongly}$$

Hence,

$$\lim_{n \to +\infty} \langle Au_n, v-u_n \rangle = \int_\Omega a_i(u, \nabla u) \cdot \frac{\partial}{\partial x_i} (v-u)\, dx = \langle Au, v-u \rangle$$

We deduce also from Vitali's Theorem that:

$$F_n(u_n,\nabla u) \underset{n\to+\infty}{\to} F(u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega)$$

and by Lebesgue theorem:

$$F(u,\nabla u)\, u_n \underset{n\to+\infty}{\to} F(u,\nabla u)u \quad \text{strongly in} \quad L^1(\Omega)$$

If we write:

$$\int_\Omega (v-u_n)\, F_n(u_n,\nabla u_n)\, dx = \int_\Omega (v-u_n)\, [F_n(u_n,\nabla u_n) - F(u,\nabla u)]\, dx + \int_\Omega (v-u_n)\, F(u,\nabla u)\, dx$$

we obtain that:

$$\lim_{n\to+\infty} \int_\Omega (v-u_n)\, F_n(u_n,\nabla u_n) = \int_\Omega (v-u)\, F(u,\nabla u)\, dx$$

We then conclude easily from $(\mathcal{P}_m)$ that:

$$\langle Au,v-u\rangle + (F(u,\nabla u),v-u) \geq \langle T,v-u\rangle \ , \ \forall v \in K \cap L^\infty(\Omega)$$

This proves the first part of Theorem 2.2. The statement in the second part of Theorem 2.2 is the same as Theorem 6.1 which will be proved in Section 6. Meanwhile in Section 5, we derive several extensions of the results above to other convex sets K. In each case, we restrict ourselves to the proof of the L^∞-estimates.

5. OTHER CONVEX SETS

The L^∞-estimate can be obtained for other convex sets K. The principle of the method is exactly the same as before, that is:

- Prove the property (A_∞) of Theorem 3.1 by choosing a suitable test function v. In general, this test function is of the form $v = u - \Lambda(u)\, S_{\theta,h}(u)$ the function Λ and the parameter θ depending on the convex.

We illustrate briefly this method with a few examples.

5.1 CASE OF BILATERAL CONSTRAINTS

Case 1: $K(\varphi,\psi) = \{v \in W_o^{1,p}(\Omega)\ ,\ \varphi \leq v \leq \psi \text{ a.e.in } \Omega\}$

Lemma 5.1

Consider the problem $(\mathcal{P})$ *with* $K = K(\varphi,\psi)$, *and assume that* $\varphi \in L^\infty(\Omega)$, ψ *is a measurable function and* $K(\varphi,\psi) \neq \emptyset$ *Then, any solution of* $(\mathcal{P})$ *satisfies*

$$\|u\|_\infty \leq a' + \nu_{a'}^{-1}(\lambda_{a'}, \lambda_1\, \lambda_2)$$

where $a' = \|\varphi\|_\infty$, *and the* λ_i *are the same constants as before.*

PROOF

Let us prove first that the function, $v = u - H_\theta(u-\varphi)$. $S_{\theta_1,h}(u)$, with $\theta_1 = \|\varphi\|_\infty + \theta$ belongs to $K(\varphi,\psi)$. In Lemma 3.4, we have shown that $v \geq \varphi$. To prove that $v \leq \psi$, let us check that $v \le u$ that is $H_\theta(u-\varphi)\, S_{\theta_1,h}(u) \ge o$. In fact, $u \ge \varphi$ implies that $u \ge -\|\varphi\|_\infty - \theta = -\theta_1$: $S_{\theta_1,h}(u) \ge o$ (by the definition of $S_{\theta_1,h}(.)$) and $H_\theta(u-\varphi) \ge o$ (since $H_\theta \ge o$). As $u \in K(\varphi,\psi)$, $v \le u \le \psi$. So the function $v = u - H_\theta(u-\varphi) \cdot S_{\theta_1,h}(u)$ is a suitable test function. The proof of Inequality (3.2) of Theorem 3.1 is then exactly the same as in Lemma 3.2.

□

LEMMA 5.2

Consider the problem $(\mathcal{P})$ *, with* $K = K(\varphi,\psi)$ *and assume that* φ *is measurable,* ψ *is in* $L^\infty(\Omega)$ *. Then the solution* u *of* $(\mathcal{P})$ *satisfies:*

$$\|u\|_\infty \le b' + \nu_{b'}^{-1}\,(\lambda_{b'}\ \lambda_1\ \lambda_2)$$

where $b' = \|\psi\|_\infty$ *and* $\lambda_{b'}\ \lambda_1\ \lambda_2$ *are the same as before.*

PROOF

We define $H_{-\theta}$ by $H_{-\theta} = -H_\theta$. We set $\theta_1 = \|\psi\|_\infty + \theta$, $\overset{\circ}{v} = H_{-\theta}(\psi-u)\, S_{\theta_1,h}(-u)$. As in the Corollary of Lemma 3.3, we have $\overset{\circ}{v} \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial \overset{\circ}{v}}{\partial x_i} = H_{-\theta}(\psi - u)\ \ S'_{\theta_1,h}(-u)\frac{\partial u}{\partial x_i}$$

$$= \begin{cases} -\frac{1}{h} H_\theta(\varphi-u) \frac{\partial u}{\partial x_i} & \text{if} \quad \theta_1 < |u| \leq \theta_1 + h \\ o & \text{otherwise} \end{cases}$$

Let us prove that $v = u - \overset{\circ}{v} \in K(\varphi,\varphi)$, and first that $v \leq \varphi$. In fact, if $\varphi - u \geq \theta$ then $\varphi \geq u + \theta$ and $u + \theta \geq u - H_{-\theta}(\varphi - u) \; S_{\theta_1,h}(-u)$ since

$$|H_{-\theta}(\varphi - u) \; S_{\theta_1,h}(-u)| \leq \theta \; ,$$

and if, $o \leq \varphi - u \leq \theta$ then $H_{-\theta}(\varphi - u) = u - \varphi$,

$$v - \varphi = (u - \varphi) \; [1 - S_{\theta_1,h}(-u)] \leq o \; .$$

To prove that $\varphi \leq v$, we show that $v \geq u$, i.e., $H_{-\theta}(\varphi - u) \cdot S_{\theta_1,h}(-u) \leq o$. This quantity vanishes if $\varphi - u \leq o$; if $\varphi - u \geq o$, then $-u \geq -\|\varphi\|_\infty \geq -\theta_1$, $S_{\theta_1,h}(-u) \geq o$ and since $H_{-\theta} \leq o$, we deduce that $H_{-\theta}(\varphi - u) \; S_{\theta_1,h}(-u) \leq o$.

□

We choose v as the test function and then we get:

$$- \int_{\theta_1<|u|\leq\theta_1+h} a_i(x,u,\nabla u) \; H_{-\theta}(\varphi - u) \; \frac{\partial u}{\partial x_i} \; dx$$

$$\leq - \int_{\theta_1<|u|\leq\theta_1+h} g_i \; \frac{\partial u}{\partial x_i} \; H_{-\theta}(\varphi - u) \; dx \qquad (5.1)$$

If $\theta_1 < |u|$ then $\theta < |u| - \|\psi\|_\infty \leq |u - \psi| = \psi - u$: $H_{-\theta}(\psi - u) = -\theta$. As in Lemma 3.2, setting $\bar{u} = [|u| - \|\psi\|_\infty]_+$ we see that the inequality (5.1) is equivalent to:

$$\int_{\theta<\bar{u}\leq\theta+h} a_i(x,u,\nabla u) \frac{\partial u}{\partial x_i} dx \leq \int_{\theta<\bar{u}\leq\theta+h} g_i \frac{\partial u}{\partial x_i} dx$$

Thus u satisfies the property (A_∞) .

5.2 BOUNDARY INEQUALITIES

Case 2: $K(\partial\Omega,\varphi,\psi) = \left\{v \in W^{1,p}(\Omega) \ , \ \varphi \leq v|_{\partial\Omega} \leq \psi \ \text{ a.e. on } \partial\Omega\right\}$

We assume that $\partial\Omega$ is smooth and let φ,ψ be two elements of $L^\infty(\partial\Omega)$ We consider the problem

$$(\mathcal{P}'') \quad \left| \begin{array}{l} \text{Find } u \in K(\partial\Omega,\varphi,\psi) \cap L^\infty(\Omega) \text{ such that} \\ \langle Au,v-u\rangle + (F(u,\nabla u),v-u) \geq \langle T,v-u\rangle \\ \text{for all } v \in K(\partial\Omega,\varphi,\psi) \cap L^\infty(\Omega) \end{array} \right.$$

LEMMA 5.3

Assume that $K(\partial\Omega, \varphi, \psi) \neq \emptyset$. *Then, any solution* u *of* $(\mathcal{P}'')$ *satisfies*

$$\|u\|_\infty \leq m + \nu_m^{-1}(\lambda_m \lambda_1 \lambda_2) \ ,$$

$$m = \text{Max}\ [\|\varphi\|_{L^\infty(\partial\Omega)} \ , \ \|\psi\|_{L^\infty(\partial\Omega)}]$$

PROOF

The solution u of $(\mathcal{P}'')$ satisfies the property (A_∞) of Theorem 3.1. In fact, if we set $\bar{u} = [|u| - m]_+$ and we choose as a test function $v = u - H_\theta(|u| - m)\ S_{\theta_1, h}(u)$, we check easily that:

$$v|_{\partial\Omega} = u|_{\partial\Omega} \Rightarrow v \in K(\partial\Omega, \varphi, \psi)$$

and

$$\int_{\theta < \bar{u} \le \theta + h} a_i(x,u,\nabla u) \cdot \frac{\partial u}{\partial x_i}\, dx \le \int_{\theta < \bar{u} \le \theta + h} g_i \frac{\partial u}{\partial x_i}\, dx \ .$$

□

Case 3:

$$K_q(\varphi) = \left\{ v \in W_o^{1,p}(\Omega) \ , \ \|\nabla v\|_{L^q(\Omega)} \le 1, \ v \ge \varphi \text{ a.e. in } \Omega \right\}$$

or

$$K_{o,q} = \left\{ v \in W_o^{1,p}(\Omega) \ , \ \|\nabla v\|_{L^q(\Omega)} \le 1 \right\} \ , \ q \le N$$

LEMMA 5.4

(i) *Let* $K = K_q(\varphi)$. *Then any solution* u *of* $(\mathcal{P})$ *satisfies the estimate:*

$$\|u\|_\infty \le \|\varphi\|_\infty + \nu_{a'}^{-1} (\lambda_{a'}\ \lambda_1\ \lambda_2) , \ \bar{a}' = \|\varphi\|_\infty$$

(ii) *Let* $K = K_{o,q}$, *then any solution* u *of* ($\mathcal{P}$) *satisfies:*

$$\|u\|_\infty \leq \nu_o^{-1} (\lambda_o \lambda_1 \lambda_2)$$

□

We prove below the part (i) of Lemma 5.4; the proof of part (ii) is the same.

LEMMA 5.5

Let u *be an element of* $W_o^{1,p}(\Omega)$ *and* $\hat{u} = H_\theta(u - \varphi)$. $S_{\theta_1,h}(u)$. *We consider* $\rho > o$ *such that:* $o < \frac{\theta\rho}{h} < 1$ $(\theta > 0, h > o)$. *Then,*

(i) $|\text{Grad}\,(u - \rho\hat{u})| \leq |\text{Grad}\,u|$ a.e. in Ω

(ii) *If* $u \in K_q(\varphi)$, $\rho < 1$, *then* $u - \rho\hat{u} \in K_q(\varphi)$

PROOF OF LEMMA 5.5

From the Corollary of Lemma 3.3, we have $u - \rho\hat{u} \in W_o^{1,p}(\Omega)$ **and** $\nabla\hat{u} = H_\theta\,(u - \varphi)\ S'_{\theta_1,h}(u)\,\nabla u$. **For convenience, we set** $a = H_\theta(u - \varphi)\ S'_{\theta_1,h}(u)$, **and observe that** $o \leq a \leq \frac{\theta}{h}$. **Clearly** $\text{grad}(u - \rho\hat{u}) = \text{grad}\,u$ **if** $|u| \leq \theta_1$ **or** $|u| > \theta_1 + h$ **while on the set** $\theta_1 < |u| \leq \theta_1 + h$:

$$|\text{Grad } (u - \rho\hat{u})|^2 = |\nabla u|^2 - 2a\rho\, |\nabla u|^2 + \rho^2 a^2\, |\nabla u|^2$$
$$= |\nabla u|^2 (1 - a\rho)^2 \leq |\nabla u|^2$$

To show that $u - \rho\hat{u} \in K_q(\varphi)$ if $\rho < 1$ and $u \in K_q(\varphi)$ it suffices to prove that $u - \rho\hat{u} \geq \varphi$ a.e. in Ω, since from (i)

$$\|\nabla(u - \rho\hat{u})\|_{L^q(\Omega)} \leq \|\nabla u\|_{L^q(\Omega)} \leq 1 .$$

The proof is as in Lemma 3.4, i.e.,:

-If $u - \varphi \geq \theta$, then $u \geq \varphi + \theta \geq \varphi + \rho\, H_\theta(u - \varphi)\ S_{\theta_1,h}(u)$ since $0 < \rho < 1$ and $|H_\theta(u - \varphi)\ S_{\theta_1,h}(u)| \leq \theta$: $u \geq \varphi + \rho\hat{u}$.

-If $o \leq u - \varphi \leq \theta$ then $u \leq \|\varphi\|_\infty + \theta = \theta_1$ and $u \geq \varphi \geq -\|\varphi\|_\infty - \theta = -\theta_1$: $|u| \leq \theta_1$ and $S_{\theta_1,h}(u) = o$ $u - \rho\hat{u} = u \geq \varphi$.

PROOF OF LEMMA 5.4

We have to check that u satisfies the property (A_∞) of Theorem 3.1. We choose $v = u - \rho\hat{u}$ in $(\mathcal{P})$ (that is suitable by Lemma 5.5) . Thus:

$$\langle Au,\hat{u}\rangle + (F(u,\nabla u),\ \hat{u}) \leq \langle T,\ \hat{u}\rangle$$

which is exactly the same as relation (3.48) in the proof of Lemma 3.2. So, we can conclude that u satisfies the property (A_∞) . □

REMARK 5.1

(i) For all these convexes, one can prove an existence theorem similar to Theorem 2.2 for which these L^∞-estimates play a fundamental role.

(ii) If the growth of F, with respect to the gradient, is of order less than p one can give a simpler proof of Lemma 4.2 (the estimate of $|\nabla u_n|_{L^p}$) and of Lemma 4.5 (the strong convergence in $W_o^{1,p}(\Omega)$) . In Lemma 4.2, for instance, one can choose $v = u_1$ as a test function in $(\mathcal{P}_n)$ and in Lemma 4.5, one can choose $v = u$ as a test function and it suffices to prove that:

$$\limsup_{n \to +\infty} \langle Au_n, u_n - u\rangle \leq o .$$

5.3 QUASI-VARIATIONAL INEQUALITIES

Case 4: A remark on a Quasi-variational elliptic inequality :

The method that we have presented can be also applied to some quasi-variational problems. Consider, for instance, the following problem (see A. Bensoussan ,J.L. Lions [29], L. Cafarelli-A. Friedman [7]):

$$(\mathcal{P}_Q) \quad \left| \begin{array}{l} \text{Find } u \in W_o^{1,p}(\Omega) \text{ such that } u \leq Mu \text{ a.e. in } \Omega \text{ and} \\ \langle Au, v-u\rangle + (F(u,\nabla u), v-u) \geq \langle T, v-u\rangle \\ \forall v \in W_o^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ such that } v \leq Mu , \end{array} \right.$$

where M is a suitable operator from $W^{1,p}(\Omega)$ into $\mathbb{R}$. For instance, the following operator appears in [29] and [7]:

$$Mu(x) = 1 + \operatorname*{Inf}_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi)$$

LEMMA 5.6

Assume that the solution u *of* $(\mathcal{P}_Q)$ *satisfies one of the following conditions: There exists* $u_o \in L^\infty(\Omega)$ *such that:*

(i) $\qquad u \geq u_o$ *a.e.*

(ii) $\qquad u \leq u_o$ or $Mu \leq u_o$ *a.e.*

Then, $u \in L^\infty(\Omega)$ *and:*

$$\|u\|_\infty \leq b'' + \nu_{b''}^{-1}\ (\lambda_{b''}\ \lambda_1\ \lambda_2) \qquad b'' = \|u_o\|_\infty$$

□

REMARK 5.2

In the linear case and when $T = f \in \mathcal{C}^{\alpha}(\bar{\Omega})$, L. Cafarelli and A. Friedman proved that $u \geq -1$ (see [7]) .

□

PROOF OF LEMMA 5.6

It suffices to show that the solution u satisfies the property (A_∞) of Theorem 3.1. For this purpose, we choose $\bar{u} = [|u| - \|u_o\|_\infty]_+$ and as test function v in $(\mathcal{P})$:

- in the case of (i) , $v = u - H_\theta(u - u_o)\ S_{\theta_1,h}(u)$ with

  $\theta_1 = b'' + \theta$

- in the case of (ii) , $v = u - H_{-\theta}(u_o - u)\ S_{\theta_1,h}(-u)$ with

  $\theta_1 = b'' + \theta$

The proof is then exactly as in Lemma 3.2.

□

6. HöLDER CONTINUITY OF THE SOLUTIONS OF THE VARIATIONAL QUASI-LINEAR INEQUALITY

6.1 Assumptions - The Main Result

The purpose of this section is prove the Hölder continuity of the solutions to problems:

$$(\mathcal{P}_o)\left|\begin{array}{l} u \in K \cap L^\infty(\Omega) \\ \langle Au, v-u\rangle + (F(u,\nabla u),\ v-u) \geq \langle T, v-u\rangle \\ \forall v \in K \cap L^\infty(\Omega) \end{array}\right.$$

Here, as in Sections 2 to 4, $K = \left\{v \in W_o^{1,p}(\Omega)\ ,\ v \geq \varphi \text{ a.e. in } \Omega\right\}$, K is supposed to be non-empty.

The question of the Hölder continuity of the solutions of $\mathcal{P}_o$ has been investigated in particular in [11], [12], [15],

[10]. The particularity of our study is that the right hand side is some element of $W^{-1,p'}(\Omega)$ $(1/p + 1/p' = 1)$, the operator A possesses minimal regularity properties and the growth of F is at most of order p with respect to the gradient.

More precisely, the only assumptions that we will need are the following:

(H.1)' T is in $W_o^{-1,r}(\Omega)$, $q = (p-1)r > \text{Max}\,(p,N)$

(H.2)' The function F is borelian and has the same growth as in (H.2), that is :

For a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$

$$|F(x,\eta,\xi)| \le f(|\eta|)\ [|\xi|^p + f_o(x)]$$

Here, f_o is a positive function in $L^{r/p'}(\Omega)$, f is a non-decreasing map from $\mathbb{R}_+$ into itself.

As for the operator $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u)$, we suppose that:

(H.3)' Each a_i is a Borel function such that:

For a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$

$$|a_i(x,\eta,\xi)| \le a(|\eta|)\ [|\xi|^{p-1} + a_o(x)]$$

Here, a_o is a positive function of $L^r(\Omega)$ and a is a non-decreasing map from $\mathbb{R}_+$ into itself

$$\sum_{i=1}^{N} a_i(x,\eta,\xi)\xi_i \ge \nu_1(|\eta|)\ |\xi|^p - \nu_2(|\eta|)\ k(x)^{1/p'}\ |\xi|$$

The functions $\nu_1 > 0$, $\nu_2 \geq 0$ are continuous and k is a positive function of $L^{r/p'}(\Omega)$. We then have the following result restating the second part of Theorem 2.2:

THEOREM 6.1

Assume (H.1)' *to* (H.3)' *and let* u *be a solution of problem* $(\mathcal{P}_0)$.

If the obstacle Ψ *is in* $W^{1,q}_{loc}(\Omega)$, $q = (p-1)r$, *then* u *is Hölder continuous in* Ω .

REMARK 6.1

If $r = \frac{p}{p-1}$ and $r > \frac{N}{p-1}$ then $p > N$ the theorem remains valid since $W^{1,p}_0(\Omega) \subset C^{0,\alpha}(\bar{\Omega})$.

In this case, $\Psi \in L^\infty(\Omega')$ for any Ω' such that $\overline{\Omega'} \subset \Omega$.

□

In the sequel, we will assume that $p \leq N$, $N \geq 2$ and $\|u\|_{L^\infty(\Omega)} \leq M$. We choose an open set Ω_0 such that $\overline{\Omega_0} \subset \Omega$, $\|\Psi\|_{L^\infty(\Omega_0)} \leq M_0$ (of course, we can choose M_0 satisfying $M_0 \leq M$).

As in [22], the main tool for the proof of Theorem 6.1 is the class of functions $\mathcal{B}_\rho(\Omega_0, M', \gamma', \delta, \frac{1}{q})$ introduced by O.A. Ladyzens'kaja and N.N. Uraltseva ([30], Chapter II, page 81). For the convenience of the reader, we recall the definition of the class and will use the same notations as in [30].

6.2 DEFINITION OF THE CLASS $\mathcal{B}_p(\Omega_o, M', \gamma', \delta, 1/q)$

Let M', γ', δ, q be four fixed positive numbers with $q > N$. A function u belongs to $\mathcal{B}_p(\Omega_o, M', \gamma', \delta, 1/q) = X$ if and only if

a) $u \in W_o^{1,p}(\Omega_o) \cap L^\infty(\Omega_o)$, $\|u\|_\infty \leq M'$

b) u satisfies: For any arbitrary sphere K_ρ of radius ρ contained in Ω_o , for any arbitrary $\sigma \in \,]o,1[$, we have

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla u|^p \, dx \leq \gamma' \left[\frac{1}{\sigma^p \rho^{p(1-N/q)}} \operatorname*{ess\,sup}_{A_{k,\rho}} (u(x) - k)^p + 1\right] . (\text{meas. } A_{k,\rho})^{1 - p'/p}$$

$$\text{for } k > \operatorname*{ess\,sup}_{K_\rho} u(x) - \delta \qquad (6.1)$$

Here, $A_{k,\rho} = \{x \in K_\rho \, , \, u(x) > k\}$, $K_{\rho-\sigma\rho}$ and K_ρ are concentric spheres.

c) the function $-u$ satisfies the same inequality b) as u .

In the applications, the property b) is replaced by the following sufficient condition (see [22], for the details) .

LEMMA 6.1

Let u *be an element of* $W_o^{1,p}(\Omega_o) \cap L^\infty(\Omega_o)$ *satisfying the following condition:*

There exists a constant $\gamma'' > o$ *such that, for any* $\varphi \in \mathcal{D}(\Omega_o)$ *with support* φ *included in* K_ρ , $o < \varphi < 1$ *on* Ω_o , *one has :*

$$\int_{A_{k,\rho}} |\nabla u|^p \, \psi^p \, dx \le \gamma'' \left[\int_{A_{k,\rho}} (u(x)-k)^p \, |\nabla \psi|^p \, dx + (\text{meas } A_{k,\rho})^{1-\frac{p}{q}} \right] \tag{6.2}$$

for $k > \operatorname{ess\,sup}_{K_\rho} u - \delta$. *Then* u *satisfies* (6.1) . □

It is proved in [30] (Theorem 6.1, page 90) that if $u \in X = \mathcal{B}_p\,(\Omega_o, M', \gamma', \delta, 1/q)$, then u satisfies the Hölder condition inside of Ω_o . So, Theorem 6.1 is an immediate consequence of the following lemma.

LEMMA 6.2

Under the same assumptions as in Theorem 6.1, for any $\delta > o$, *there exists a constant* $\gamma' > o$ *depending only on* δ *and the data* a_i, F *and* T , *such that any solution* u *of problem* $(\mathcal{P}_o)$ *belongs to* $\mathcal{B}_p(\Omega_o, \mu, \gamma', \delta, 1/q)$ *where* $q = (p-1)r$ *and* M *is a given number satisfying* $\|u\|_\infty \le M$ *(for instance the number given by Theorem 2.2)* .

□

Since the solution u of $(\mathcal{P}_o)$ belongs to $K \cap L^\infty(\Omega)$ and $\|u\|_\infty \le M$, the first part a) of the definition is fulfilled. To prove the part b) of the definition, it suffices to check the following:

LEMMA 6.3:

For an arbitrary $\delta > 0$, *there exists a constant* c'_1 *depending only on* $\delta > 0$ *and the data* a_i , F *and* T , *such that:*

For any $\varphi \in \mathcal{D}(\Omega_o)$, *with support* $\varphi \subset K_\rho$, $0 \leq \varphi \leq 1$, *one has:*

$$\int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx \leq c'_1 \left[\int_{A_{k,\rho}} (u(x)-k)^p \, |\nabla\varphi|^p \, dx + (\text{meas. } A_{k,\rho})^{1-p'/r}\right] \tag{6.3}$$

for $k > \operatorname{ess\,sup}_{K_\rho} u - \delta$.

REMARK 6.2

In the sequel, c'_i and $c'_{i\epsilon}$ will denote constants depending on δ , on the data a_i, F, T and on a parameter ϵ (in the case of $c'_{i\epsilon}$) . □

We will make use of the Young inequality in the following form:

If $a \geq 0$, $b \geq 0$, then for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that : $ab \leq \epsilon a^p + c_\epsilon b^{p'}$, $1/p + 1/p' = 1$.
We will omit the sign Σ , and will write $a_i(u,\nabla u) = a_i$, $F(u,\nabla u) = F$, $\varphi(x) = \varphi$ ... The function u is a solution of $(\mathcal{P}_o)$. □

We set $\mu_1 = \underset{o \leq \eta \leq M}{\text{Min}} \nu_1(\eta)$, $\mu_2 = f(M) + 1$ and we consider the function σ associated to μ_1 , μ_2 as in Lemma 4.3 .

Let $\delta > o$ be a fixed number $\varphi \in \mathcal{D}(\Omega_o)$ such that $o \leq \varphi \leq 1$ and support $\varphi \subset K_\rho \subset \Omega_o$. We denote by $\Gamma > o$, the number satisfying:

$$|\sigma'(t)| \leq \Gamma \quad \text{for} \quad |t| \leq 2M + \delta \tag{6.4}$$

LEMMA 6.4

Let $k \geq \underset{K_\rho}{\text{ess sup}}\, u - \delta$. *Then*

i) $o \leq \sigma((\varphi - k)_+) \leq \sigma((u-k)_+)$

ii) $\sigma((u-k)_+) \leq \Gamma\ (u - k)_+ \leq \Gamma\ (2M + \delta) = M_1$

iii) $H_\varphi(u - k) \overset{\text{def}}{=} \sigma(u - k)_+ - \sigma(\varphi - k)_+ \leq \Gamma(u - \varphi)$

PROOF

We have $\varphi \leq u$, and thus $(\varphi - k)_+ \leq (u - k)_+$. Since σ is increasing and $\sigma(o) = o$, we obtain

$$o \leq \sigma(\varphi - k)_+ \leq \sigma(u - k)_+$$

To prove ii) , we note that: $k \geq - M - \delta$, and deduce that $(u - k)_+ \leq (u + M + \delta)_+ \leq 2M + \delta$. By the definition of Γ , we find $\sigma(u - k)_+ \leq \Gamma\ (u - k)_+$.

From (6.4) and i) , we conclude

$$0 \leq \sigma(u - k)_+ - \sigma(\varphi - k)_+ \leq \Gamma|(u - k)_+ - (\varphi - k)_+| \leq \Gamma(u - \varphi)$$

(We have used the fact that $(\varphi - k)_+ \leq (u - k)_+ \leq 2M + \delta)$.

□

LEMMA 6.5

Let u *be a solution of* $(\mathcal{P}_o)$. *Then*

$$\check{u} = u - \frac{\psi^p}{\Gamma} H_\varphi(u - k) \in K \cap L^\infty(\Omega_o)$$

$$H_\varphi(u-k) \overset{\text{def}}{=} \sigma(u - k)_+ - \sigma(\varphi - k)_+$$

□

PROOF

Since $\psi \in \mathcal{D}(\Omega_o)$, one has $\check{u} \in W_o^{1,p}(\Omega)$. We have to check $\check{u} \geq \varphi$ and $\check{u} \in L^\infty(\Omega)$.

By the preceeding lemma, since $o \leq \psi \leq 1$, we obtain

$$o \leq \frac{\psi^p}{\Gamma} (\sigma(u-k)_+ - \sigma(\varphi-k)_+) \leq u - \varphi : \quad \check{u} \geq \varphi$$

As $\varphi \in L^\infty(\Omega_o)$ and $\psi \in \mathcal{D}(\Omega_o)$, one has $\psi^p H_\varphi(u - k) \in L^\infty(\Omega_o)$ so $\check{u} \in L^\infty(\Omega_o)$.

□

From Lemma 6.5, we can choose as a test function $v = u - \frac{\psi^p}{\Gamma}$ $(H_\varphi(u - k))$, and we deduce that:

$$\langle Au, \varphi^p H_\varphi(u - k)\rangle + \int_\Omega F(u,\nabla u)\ \varphi^p H_\varphi(u - k)\ dx$$
$$\leq \langle T, \varphi^p H_\varphi(u - k)\rangle \qquad (6.5)$$

Let us estimate each term of relation (6.5) .

LEMMA 6.6

There exists a constant $c_2' > o$ *such that*

$$\left|\int_\Omega F(u,\nabla u)\ \varphi^p H_\varphi(u - k)\right| \leq \mu_2 \int_{A_{k,\rho}} |\nabla u|^p\ \varphi^p (u - k)_+\ dx$$

$$+\ c'_2(\text{meas. } A_{k,\rho})^{1 - p'/r},\quad \mu_2 = f(M) + 1\ .$$

PROOF

From (H.2)' , since $\|u\|_\infty \leq M$ and f is non-decreasing , we obtain for a.e. x in Ω :

$$|F(u,\nabla u)| \leq \mu_2\ [|\nabla u|^p + f_o(x)] \qquad (6.6)$$

From Lemma 6.4,

$$o \leq H_\varphi(u - k) \leq \sigma(u - k)_+ \qquad (6.7)$$

Hence,

$$\left|\int_{\Omega} F(u,\nabla u)\ \psi^p H_\psi(u - k)\ dx\ \right| \leq \mu_2 \int_{A_{k,\rho}} |\nabla u|^p\ \psi^p\ \sigma(u - k)_+\ dx$$

$$+ \mu_2 \int_{A_{k,\rho}} f_o(x)\ \sigma(u - k)_+\ \psi^p\ dx \qquad (6.8)$$

From ii) of Lemma 6.4, since $o \leq \psi \leq 1$

$$\mu_2 \int_{A_{k,\rho}} f_o(x)\ \sigma(u - k)_+\ \psi^p\ dx \leq \mu_2\ M_1 \int_{A_{k,\rho}} f_o(x)\ dx \qquad (6.9)$$

By the Hölder inequality:

$$\int_{A_{k,\rho}} f_o(x)\ dx \leq \| f_o \|_{L^{r/p'}(\Omega)} (\text{meas. } A_{k,\rho})^{1-\ p'/r} \qquad (6.10)$$

With (6.8) to (6.10), the proof of Lemma 6.6 is completed.

LEMMA 6.7

For any $\epsilon > o$ *, there exists* $c'_\epsilon > o$ *such that:*

$$|\langle T,\ \psi^p H_\psi(u - k)\rangle| \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p\ \psi^p\ dx$$

$$+ c'_\epsilon \left[\int_{A_{k,\rho}} (u(x) - k)^p\ |\nabla\psi|^p\ dx + (\text{meas. } A_{k,\rho})^{1-p'/r}\right]$$

Proof

We set $T = -\sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}$, $\|T\|_{W^{-1,r}(\Omega)} = \sum_{i=1}^{N} \|g_i\|_{L^r(\Omega)}$ and $g_o = \left[\sum_{i=1}^{N} g_i^2\right]^{1/2}$. Then,

$$<T, \varphi^p H_\varphi(u-k)> = H_1 + H_2 \tag{6.11}$$

$$H_1 = p \int_{\Omega_o} g_i \frac{\partial \varphi}{\partial x_i} \varphi^{p-1} H_\varphi(u-k)\, dx,$$

$$H_2 = \int_{\Omega_o} g_i \frac{\partial}{\partial x_i} H_\varphi(u-k)\, \varphi^p \, dx \tag{6.12}$$

Since $o \leq H_\varphi(u-k) \leq \sigma(u-k)_+ \leq r(u-k)_+$ (see Lemma 6.4) and $o \leq \varphi \leq 1$, we have for a.e. x in Ω :

$$|p\, g_i \frac{\partial \varphi}{\partial x_i} \varphi^{p-1} H_\varphi(u-k)| \leq c_4' \, g_o \, |\nabla\varphi|(u-k)_+ \tag{6.13}$$

Hence, we have:

$$|H_1| \leq c_4' \int_{\Omega_o} g_o \, |\nabla\varphi|(u-k)_+ \, dx = c_4' \int_{A_{k,\rho}} g_o |\nabla\varphi|(u-k)_+ \, dx \tag{6.14}$$

By the Young inequality:

$$c_4' \, g_o|\nabla\varphi|(u-k)_+ \leq g_o(x)^{p'} + c_5' \, |\nabla\varphi|^p(u-k)_+^p \tag{6.15}$$

Therefore, we deduce from (6.14) and (6.15) :

$$|H_1| \le \int_{A_{k,\rho}} g_o(x)^{p'}\, dx + c_5 \int_{A_{k,\rho}} |\nabla \varphi|^p (u-k)_+^p\, dx \quad (6.16)$$

By the Hölder inequality:

$$\int_{A_{k,\rho}} g_o(x)^{p'}\, dx \le \|g_o\|_{L^{r/p'}(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r}$$
$$\le \|T\|^{p'}_{W^{-1,r}(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r}$$

From the last two relations, we have:

$$|H_1| \le c_6' (\text{meas. } A_{k,\varphi})^{1-p'/r} + c_5' \int_{A_{k,\rho}} |\nabla \varphi|^p (u-k)_+^p\, dx \quad (6.17)$$

As for the term H_2 , we can write $H_2 = H_2^1 + H_2^2$ where

$$H_2^1 = \int_{\{u>k\}\cap\Omega_o} g_i \frac{\partial u}{\partial x_i} \sigma'(u-k)\, \varphi^p\, dx\ ;$$

$$H_2^2 = - \int_{\{\varphi>k\}\cap\Omega_o} g_i \frac{\partial \varphi}{\partial x_i} \sigma'(\varphi - k)\, \varphi^p\, dx \quad (6.18)$$

Since $(u-k)_+ \le 2M + \delta$, we have $|\sigma'(u-k)_+| \le \Gamma$ and

$$|\sigma'(u-k)|\ \left|\frac{\partial u}{\partial x_i}\right|\ |g_i|\ \varphi^p \le \Gamma\ |\nabla u|\ g_o(x)\ \varphi \text{ a.e.} \quad (6.19)$$

From the Young inequality:

$$\Gamma |\nabla u| \; g_o(x) \; \varphi \leq \epsilon \; |\nabla u|^p \; \varphi^p + c'_{2\epsilon} \; g_o(x)^{p'} \qquad (6.20)$$

So, we find

$$|H_2'| \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p \; \varphi^p \; dx + c'_{2\epsilon} \int_{A_{k,\rho}} g_o(x)^{p'} \; dx \qquad (6.21)$$

By the Hölder inequality, we get as before:

$$\int_{A_{k,\rho}} g_o(x)^{p'} \; dx \leq \|T\|^{p'}_{W^{-1,r}(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r} \qquad (6.22)$$

Since $(\varphi - k)_+ \leq 2M + \delta$ (this stems from Lemma 6.4, and $(\varphi' - k)_+ \leq (u - k)_+)$ we have

$|\sigma'(\varphi - k) \leq \Gamma$ and

$$|H_2^2| \leq \Gamma \int_{\{\varphi>k\}\cap\Omega_o} |\nabla\varphi| \; g_o(x) \; \varphi^p \; dx \qquad (6.23)$$

On the other hand, $u \geq \varphi$ so this last quantity can be majorized by :

$$\int_{\{u>k\}\cap\Omega_o} |\nabla\varphi| \; g_o(x) \; \varphi^p \; dx = \int_{A_{k,\rho}} |\nabla\varphi| \; g_o(x) \; \varphi^p \; dx$$

By the Young inequality, we deduce:

$$\int_{A_{k,\rho}} |\nabla\varphi|\ g_o(x)\ \varphi^p\ dx \le \frac{1}{p} \int_{A_{k,\rho}} |\nabla\varphi|^p\ dx + \frac{1}{p'} \int_{A_{k,\rho}} g_o(x)^{p'}\ dx \tag{6.24}$$

By the Hölder inequality:

$$\frac{1}{p} \int_{A_{k,\rho}} |\nabla\varphi|^p\ dx \le \frac{1}{p} \|\nabla\varphi\|^p_{L^q(\Omega_o)} (\text{meas. } A_{k,\rho})^{1-p/q}, \ (q=(p-1)r)$$

$$\frac{1}{p'} \int_{A_{k,\rho}} g_o(x)^{p'}\ dx \le \frac{1}{p'} \|T\|^{p'}_{W^{-1,r}(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r}$$

The last two relations combined with (6.23) and (6.24) lead to:

$$|H_2^2| \le c_7' \ (\text{meas. } A_{k,\rho})^{1-p'/r} \tag{6.25}$$

From the estimate (6.22) and (6.25), we obtain

$$|\langle T, \varphi^p H_\varphi(u-k)\rangle| \le \epsilon \int_{A_{k,\rho}} |\nabla u|^p\ \varphi^p\ dx + c_{4\epsilon}' \ (\text{meas. } A_{k,\rho})^{1-p'/r} \tag{6.26}$$

□

LEMMA 6.8

For any $\epsilon > o$, *there exists* $c'_{5\epsilon} > o$ *such that :*

$$\langle Au, \varphi^p H_\varphi(u - k)\rangle \geq \mu_1 \int_{A_{k,\rho}} \varphi^p |\nabla u|^p \sigma'(u - k)_+ \, dx$$

$$- \epsilon \int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx - c'_{5\epsilon} \left[\int_{A_{k,\rho}} (u(x) - k)^p \, |\nabla\varphi|^p \, dx\right.$$

$$\left. + (\text{meas. } A_{k,\rho})^{1-p'/r}\right]$$

Proof

Let us write $\langle Au, \varphi^p H_\varphi(u - k)\rangle = A_1 + A_2 + A_3 + A_4$

$$A_1 = \int_{A_{k,\rho}} a_i \frac{\partial u}{\partial x_i} \varphi^p \sigma'(u - k)_+ \, dx ,$$

$$A_2 = p \int_{A_{k,\rho}} a_i \frac{\partial \varphi}{\partial x_i} \varphi^{p-1} \sigma(u - k)_+ \, dx$$

$$A_3 = p \int_{\Omega_o} a_i \frac{\partial \varphi}{\partial x_i} \varphi^{p-1} \sigma(\varphi - k)_+ \, dx ,$$

$$A_4 = -\int_{\Omega_o} a_i \frac{\partial \varphi}{\partial x_i} \varphi^p \sigma'(\varphi - k)_+ \, dx$$

For A_1 , we have by (H.3)' (ii) :

$$a_i \frac{\partial u}{\partial x_i} \geq \nu_1(|u|)\ |\nabla u|^p - k(x)^{1/p'}\ \nu_2(|u|)\ |\nabla u| \text{ a.e.} \tag{6.27}$$

So, if we set $\beta = \underset{0 \leq \eta \leq M^2}{\text{Max}}\ \nu_2(\eta)$, then

$$A_1 \geq \mu_1 \int_{A_{k,\rho}} |\nabla u|^p\ \varphi^p\ \sigma'(u-k)_+\ dx$$

$$- \beta \int_{A_{k,\rho}} k(x)^{1/p'}\ |\nabla u|\ \varphi^p\ \sigma'(u-k)_+\ dx \tag{6.28}$$

From the Young inequality, one has:

$$\beta \int_{A_{k,\rho}} k(x)^{1/p'}\ |\nabla u|\ \varphi^p\ \sigma'(u-k)_+\ dx \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p\ \varphi^p\ dx$$

$$+ c'_{5\epsilon} \int_{A_{k,\rho}} k(x)\ dx \tag{6.29}$$

From the Hölder inequality:

$$\int_{A_{k,\rho}} k(x)\ dx \leq \|k\|_{L^{r/p'}(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r} \tag{6.30}$$

From (6.28) to (6.30), we find

$$A_1 \geq \mu_1 \int_{A_{k,\rho}} |\nabla u|^p \psi^p \sigma'(u-k)_+ \, dx - \epsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx - c'_{6\epsilon}(\text{meas. } A_{k,\rho})^{1-p'/r} \tag{6.31}$$

For A_2, we use the assumption on the growth of a_i (see (H.3)' (i)) and the fact that $\|u\|_\infty \leq M$, $\sigma(u-k)_+ \leq \Gamma(u-k)_+$ to obtain

$$|A_2| \leq c'_8 \int_{A_{k,\rho}} |\nabla u|^{p-1} |\nabla\psi| \psi^{p-1}(u-k)_+ \, dx$$

$$+ c'_8 \int_{A_{k,\rho}} a_o(x) |\nabla\psi| \psi^{p-1}(u-k)_+ \, dx \tag{6.32}$$

From the Young inequality, we have:

$$c'_8 \int_{A_{k,\rho}} |\nabla u|^{p-1} |\nabla\psi| \psi^{p-1}(u-k)_+ \, dx \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p \psi^p \, dx$$

$$+ c'_{7\epsilon} \int_{A_{k,\rho}} |\nabla\psi|^p (u-k)_+^p \, dx \tag{6.33}$$

$$c'_8 \int_{A_{k,\rho}} a_o(x) |\nabla\psi| \psi^{p-1}(u-k)_+ \, dx \leq c'_9 \int_{A_{k,\rho}} |\nabla\psi|^p (u-k)_+^p \, dx$$

$$+ c'_{10} \|a_o\|^{p'}_{L^r(\Omega)} (\text{meas. } A_{k,\rho})^{1-p'/r} \tag{6.34}$$

From (6.32) to (6.34), we find:

$$A_2 \geq - \epsilon \int_{A_{k,\rho}} |\nabla u|^p \, \varphi^p \, dx - c'_{11}(\text{meas. } A_{k,\rho})^{1-p'/r}$$
$$- c'_{8\epsilon} \int_{A_{k,\rho}} |\nabla \varphi|^p (u - k)_+^p \, dx \tag{6.35}$$

For A_3, we notice from Lemma 6.4 that $o \leq \sigma(\varphi - k)_+ \leq \Gamma$ $(u - k)_+$, so

$$|A_3| \leq c'_{12} \int_{A_{k,\rho}} |\nabla u|^{p-1} \, |\nabla \varphi|^{p-1} (u - k)_+ \, dx$$
$$+ c'_{12} \int a_o(x) \, |\nabla \varphi| \, \varphi^{p-1} (u - k)_+ \, dx \tag{6.36}$$

This relation is the same as (6.32), so we conclude as in A_2, i.e.:

$$A_3 \geq - \epsilon \int_{A_{k,\rho}} |\nabla u|^p \, \varphi^p \, dx - c'_{13}(\text{meas. } A_{k,\rho})^{1-p'/r}$$
$$- c'_{9\epsilon} \int_{A_{k,\rho}} |\nabla \varphi|^p (u - k)_+^p \, dx \tag{6.37}$$

For A_4, we use the assumption on the growth of a_i (see (H.3)'(i)) to obtain:

$$|A_4| \leq c'_{14} \int_{\{\varphi > k\} \cap \Omega_o} |\nabla u|^{p-1} \, \varphi^p \, |\nabla \varphi| \, dx + c'_{14} \int_{\{\varphi > k\} \cap \Omega_o} a_o(x) \, \varphi^p \, |\nabla \varphi| \, dx$$

(6.38)

(since $\sigma'(\varphi - k)_+ \leq \Gamma$) .

Since $u \geq \varphi$, one can majorize (6.38) by:

$$|A_4| \leq c'_{14} \int_{\{u>k\}\cap\Omega_o} |\nabla u|^{p-1} \varphi^p |\nabla\varphi| \, dx + c'_{14} \int_{\{u>k\}\cap\Omega_o} a_o(x) \varphi^p |\nabla\varphi| \, dx$$

(6.39)

From the Young Inequality, we have:

$$c'_{14} \int_{\{u>k\}\cap\Omega_o} |\nabla u|^{p-1} \varphi^p |\nabla\varphi| \, dx \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx + c_\epsilon \int_{A_{k,\rho}} |\nabla\varphi|^p \, dx$$

(6.40)

$$c'_{14} \int_{\{u>k\}\cap\Omega_o} a_o(x) \varphi^p |\nabla\varphi| \, dx \leq c'_{15} \int_{A_{k,\rho}} |\nabla\varphi|^p \, dx + c'_{15} \int_{A_{k,\rho}} a_o^{p'}(x) \, dx$$

(6.41)

From the Hölder inequality:

$$c_\epsilon \int_{A_{k,\rho}} |\nabla\varphi|^p \, dx + c'_{15} \int_{A_{k,\rho}} |\nabla\varphi|^p \, dx + c'_{15} \int_{A_{k,\rho}} a_o^{p'}(x) \, dx$$

$$\leq c'_{9\epsilon} \, (\text{meas. } A_{k,\rho})^{1-p'/r} \qquad (6.42)$$

From (6.39) to (6.42), we have:

$$|A_4| \leq \epsilon \int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx + c'_{9\epsilon} \; (\text{meas. } A_{k,\rho})^{1-p'/r}$$

$$A_4 \geq -\epsilon \int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx - c'_{9\epsilon} \; (\text{meas. } A_{k,\rho})^{1-p'/r} \quad (6.43)$$

The result of Lemma 6.8 can be easily obtained from the decomposition $\langle Au, \varphi^p H_\varphi(u-k)\rangle = A_1 + A_2 + A_3 + A_4$ and the relations (6.28) , (6.35), (6.37) and (6.43).

□

PROOF OF LEMMA 6.3

From (6.5), we have:

$$\begin{aligned} \langle Au, \varphi^p H_\varphi(u-k)\rangle &\leq |\langle T, \varphi^p H_\varphi(u-k)\rangle| \\ &\quad + \left|\int_\Omega F(u,\nabla u) \; \varphi^p H_\varphi(u-k) \, dx\right| \end{aligned} \quad (6.44)$$

From Lemma 6.5, 6.6, 6.7, and relation (6.44), we find

$$\int_{A_{k,\rho}} [\mu_1 \, \sigma'(u-k)_+ - \mu_2 \sigma(u-k)_+] \; \varphi^p \; |\nabla u|^p \, dx$$

$$\leq 2\epsilon \int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx + c'_{10,\epsilon} \left[\int_{A_{k,\rho}} (u(x)-k)^p \; |\nabla \varphi|^p \, dx\right.$$

$$\left. + \; (\text{meas. } A_{k,\rho})^{1-p'/r}\right]$$

By the choice of σ, $\mu_1 \sigma'(u-k)_+ - \mu_2 \sigma(u-k)_+ = 1$ and if we set $\epsilon = \frac{1}{4}$, we find:

$$\int_{A_{k,\rho}} |\nabla u|^p \varphi^p \, dx \leq c'_{15} \left[\int_{A_{k,\rho}} (u(x) - k)^p \, |\nabla \varphi|^p \, dx + (\text{meas. } A_{k,\rho})^{1-p'/r} \right] \qquad \square$$

To complete the proof of Lemma 6.2, we have to check that u satisfies the same inequality as u. The following lemma leads to the desired result:

LEMMA 6.3:

For any $\delta > o$, *there exists a constant* $c'_{16} > o$ *such that for all* $\varphi \in \mathcal{D}(\Omega_o)$ *with support* $\varphi \subset K_\rho$, $o \leq \varphi \leq 1$, *one has:*

$$\int_{A'_{k,\rho}} |\nabla u|^p \varphi^p \, dx \leq c'_{16} \left[\int_{A'_{k,\rho}} (-u - k)^p \, |\nabla \varphi|^p \, dx + (\text{meas. } A'_{k,\rho})^{1-p'/r} \right]$$

where:

$$A'_{k,\rho} = \{x \in K_\rho \ , \ - u(x) > k\} \ , \ k \geq \operatorname*{ess\,sup}_{K_\rho} (-u) - \delta$$

Proof

The proof is the same as for Lemma 6.3, we only replace Lemma 6.5 by:

LEMMA 6.5

Let u *be a solution of* $(\mathcal{P}_o)$ *. Then:*

$$\check{u} = u + \frac{\varphi^p}{\Gamma}[\sigma(-u - k)_+ - \sigma(-\varphi - k)_+] \in K \cap L^\infty(\Omega)$$

Proof

Since $|\sigma(-u - k)_+ - \sigma(-\varphi - k)_+| \leq \Gamma(u - \varphi)$, we have

$$o \leq -\frac{\varphi^p}{\Gamma}[\sigma(-u - k)_+ - \sigma(-\varphi - k)_+] \leq u - \varphi \; : \; \check{u} \geq \varphi \; .$$

As in Lemma 6.5, we have $\check{u} \in K \cap L^\infty(\Omega)$.

□

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FOOTNOTES

(1) The function ν_1 may be only continuous on $]o , \infty[$; it suffices that $\nu_1(t)^{p'/p}$ be integrable at $t = o$, and $\operatorname{Inf}_{o \leq t \leq \tau} \nu_1(t) > o$, $\forall \tau > o$.

(2) In the sequel, we sometimes omit the sign Σ and summation is understood when repeated indices appear.

(3) Here we use also the following remarks: if $a_n \to a$ weakly in $L^{p'}(\Omega)$ as $n \to \infty$ and $b_m \to b$ weakly in $L^p(\Omega)$ as $m \to \infty$, then

$$\lim_{n\to\infty} \int_\Omega a_n \, b_m \, |\sigma(u_n - u_m)| \, dx = \int_\Omega a \, b_m | \, \sigma(u - u_m)| \, dx$$

$$\lim_{m\to\infty} \int_\Omega a \, b_m \, |\sigma(u - u_m)| \, dx = o$$

For the first convergence we notice that by Lebesgue's theorem, as $n \to \infty$:

$$b_m \, |\sigma(u_n - u_m)| \to b_m |\sigma(u - u_m)| \quad \text{strongly in } L^p(\Omega)$$

and for the second convergence we observe that

$$a \, |\sigma(u - u_m)| \to o \quad \text{strongly in } L^{p'}(\Omega) \ , \ m \to \infty \ .$$

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CHAPITRE VII

Time behaviour of solution of parabolic problems.

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Time Behaviour of Solutions of Parabolic Problems

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Abstract.

For a large class of parabolic problems, we give a priori L^p-estimates $(1 \leq p \leq +\infty)$ of the solution $u(t)$. We show that these L^p-norms may decrease exponentially when time goes to infinity.

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INTRODUCTION.

Let u be the solution of the heat equation :

$$\begin{cases} u' - \Delta u + cu = 0 & \text{in } \mathbb{R}_+ \times \Omega , \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega , \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is an open, bounded smooth domain of $\mathbb{R}^N$. It is well-known (see [1]) that the mapping $t \to \|u(t)\|_{L^p(\Omega)}$ decreases for all $p \in [1,+\infty]$. A recent paper [9] shows that for the degenerate porous medium equation :

$$\begin{cases} u' = \Delta\phi(u) & \text{in } \mathbb{R}_+ \times \Omega , \\ \frac{\partial}{\partial n}\phi(u) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega , \\ u(0,x) = u_0(x) & \text{in } \Omega , \end{cases}$$

the same result holds. The method uses the semi-groups theory.

In this work, we generalize such results using totally different methods. More precisely, we consider a regular function u defined on an open bounded set of $\mathbb{R}^N$, satisfying

$$\int_\Omega \frac{\partial u}{\partial t} h_\theta(u)dx + \sum_{i,j=1}^{N} \int_\Omega a_{ij}(t,x,u,\frac{\partial u}{\partial t},\nabla u) \frac{\partial u}{\partial x_i} \frac{\partial h_\theta(u)}{\partial x_j} dx +$$

$$+ \int_\Omega \Gamma(t,x,u,\frac{\partial u}{\partial t},\nabla u)h_\theta(u)dx = \int_\Omega fh_\theta(u)dx \qquad (*)$$

for a.e. $t \in (0,T)$, for any $\theta>0$, with $h_\theta(u) = (|u| - \theta)_+ \text{sign}\, u$, Γ having a suitable growth property. The coefficients a_{ij} are measurable and such that $\sum_{i,j=1}^{N} a_{ij}(t,x,u,u',p)\xi_i\xi_j \geq 0$, $\forall\, \xi = (\xi_i) \in \mathbb{R}^N$.

The solutions of the previous equations as well as the solutions of many parabolic equations or inequations satisfy (*).

Here, we give an explicit upper bound of $\|u(t)\|_{L^p(\Omega)}$ for all $t \in [0,T]$ and $p \in [1,+\infty]$. If $f \equiv 0$, for a large class of non linearities Γ, these norms decay exponentially i.e. $\|u(t)\|_{L^p(\Omega)} \leq e^{-\nu(t)} \|u_0\|_{L^p(\Omega)}$ where the rate ν depends only on Γ.

We use essentially the rearrangement of functions (see [7] for example). For convenience, we begin by some preliminary definitions and theorem. In the first section, we give the proof of the main result. In the second section, we give some applications.

0. DEFINITIONS AND PRELIMINARY RESULTS.

Let Ω be a bounded measurable (1) set of $\mathbb{R}^N$ $(N\geq 1)$, v a real measurable function defined on Ω and let Ω^* denote $(0,|\Omega|)$. The decreasing rearrangement of v is defined on $\overline{\Omega^*} = [0,|\Omega|]$ by

(1) We use only Lebesgue measure.

$v_*(s) = \text{Inf}\,\{\theta \in \mathbb{R},\ |v>\theta| \leq s\}$ where $|v>\theta| = \text{meas}\,\{x \in \Omega, v(x)>\theta\}$ (for any measurable set E, we denote by $|E|$ its measure). If v is defined on $(0,T)\times\Omega$ and is measurable with respect to the space variable x of Ω, we consider its rearrangement with respect to x : $v_*(t,s) = (v(t))_*(s)$ for all $s \in \overline{\Omega^*}$. The following result is immediate from [8] (in [8], u belongs to $H^1(0,T,L^p(\Omega))$ but the proof is the same).

<u>Theorem 0</u> <u>If</u> $u \in W^{1,r}(0,T,L^p(\Omega))$ ([2]) ($1\leq r\leq+\infty$, $1\leq p\leq+\infty$), <u>then</u> $u_* \in W^{1,r}(0,T,L^p(\Omega^*))$,

$$\|u_*\|_{W^{1,r}(0,T,L^p(\Omega^*))} \leq \|u\|_{W^{1,r}(0,T,L^p(\Omega))}$$

<u>and</u> $\frac{\partial u_*}{\partial t} = \frac{\partial w}{\partial s}$ <u>(in the distribution sense), where</u>

$$w(t,s) = \begin{cases} \displaystyle\int_{u(t)>u(t)_*(s)} \frac{\partial u}{\partial t}\,dx & \text{if } |u(t) = u(t)_*(s)| = 0 \\[2ex] \displaystyle\int_{u(t)>u(t)_*(s)} \frac{\partial u}{\partial t}dx + \\[2ex] \displaystyle\qquad + \int_0^{s-|u(t)>u(t)_*(s)|} \left(\frac{\partial u}{\partial t}\Big|_{u(t)=u(t)_*(s)}\right)_* d\sigma & \text{otherwise.} \end{cases}$$

(Here the last integrand is the decreasing-rearrangement of the restriction of $\frac{\partial u}{\partial t}$ to the set $\{x \in \Omega,\ u(t,x) = u(t)_*(s)\}$ supposed to be of positive measure).

1. <u>HYPOTHESIS. MAIN RESULT.</u>

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$, $0<T<+\infty$, and let f , u_o , Γ , a_{ij} $(1\leq i,j\leq N)$ be such that :

([2]) u and $\frac{\partial u}{\partial t} \in L^r(0,T,L^p(\Omega))$.

(H1) $u_o \in L^1(\Omega)$, $f \in L^2(Q)$ where $Q =]0,T[\times\Omega$

(H2) Γ is an operator which maps $D = [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ into $\mathbb{R}$ and satisfies the following condition : there exists some functions $\lambda_i (i=1,2,3)$ such that $\lambda_1 \in L^1(0,T)$, λ_2 is measurable on $(0,T)$ and $\lambda_2 \geq 0$ a.e., $\lambda_3 \in L^1(Q)$ and $\lambda_3 \leq 0$ a.e., and for a.e. $(t,x) \in Q$ and for all $(u,u',p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$

$$(\text{sign } u)\,\Gamma(t,x,u,u',p) \geq \lambda_1(t)|u| + \lambda_2(t)\, u'(\text{sign } u) + \lambda_3(t,x).$$

(H3) a_{ij} are measurable functions on D such that for a.e. $(t,x) \in Q$ and for all $(u,u',p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$

$$\sum_{i,j=1}^{N} a_{ij}(t,x,u,u',p)\xi_i\xi_j \geq 0 \quad \forall\, \xi = (\xi_i)\ \mathbb{R}^N .$$

Let u be a real function defined on Q satisfying the following assumptions :

(H4)-a. i) $u \in L^1(0,T,L^q(\Omega))$, $u' = \frac{\partial u}{\partial t} \in L^1(0,T,L^{q'}(\Omega))$ for some $q \geq 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

ii) $u(t) \in H^1(\Omega)$ for almost every t,

iii) $\Gamma(t,.,u(t), u'(t), \nabla u(t)) \in L^{q'}(\Omega)$ for a.e. t,

iv) $v(t)_* = |u(t)|_* \in C^o(\Omega^*)$ for a.e. t.

(H4)-b. For each $\theta > 0$, we consider the real lipschitz function $h_\theta(\tau) = (|\tau|-\theta)_+ \text{ sign } \tau$. We assume that for almost every t, the function $u = u(t)$ satisfies

$$\int_\Omega u' h_\theta(u)dx + \sum_{i,j=1}^{N} \int_\Omega a_{ij}(t,x,u,u',\nabla u)\frac{\partial u}{\partial x_i}\frac{\partial h_\theta}{\partial x_j}(u)dx +$$

$$+ \int_\Omega \Gamma(t,x,u,u',\nabla u)h_\theta(u)dx = \int_\Omega f h_\theta(u)dx$$

Remark 1. The property (H4)-a.i) infers that $u \in W^{1,1}(0,T,L^1(\Omega))$ and thus $u \in C^o(0,T,L^1(\Omega))$. As the rearrangement is contracting from $L^1(\Omega)$ into $L^1(\Omega^*)$ (see [7]), we deduce that the mapping $t \to \int_0^s |u(t)|_*(\sigma)d\sigma$ is continuous on $[0,T]$, for all $s \in [0,|\Omega|]$.
We notice that assumption (H4)-a.iv) is satisfied for example if $|u(t)| \in H_o^1(\Omega)$ (see [7]) or $u(t) \in C^o(\Omega)$.

In the following, we shall set

$$\nu(t) = \int_0^t \frac{\lambda_1(\tau)}{1+\lambda_2(\tau)}\,d\tau \qquad t \in [0,T] \tag{1.1}$$

$$q(t,s) = \frac{|f|_*(t,s) + (-\lambda_3)_*(t,s)}{1+\lambda_2(t)} \geq 0 \quad \text{(by (H2))} \tag{1.2}$$

$$g(t,s) = |u_0|_*(s) + \int_0^t e^{\nu(\tau)} q(\tau,s)\,d\tau . \tag{1.3}$$

Remark 2. The function ν is in $C^0([0,T])$ and g is in $C^0([0,T], L^1(\Omega^*))$. In fact, by assumption (H2), $\left|\frac{\lambda_1}{1+\lambda_2}\right| \leq |\lambda_1|$. On the other hand, by (H1) and (H2), f and λ_3 belong to $L^1(Q)$; so $|f|_*$, $(-\lambda_3)_*$ belong to $L^1(Q^*)$ where $Q^* =]0,T[\times\Omega^*$ (see appendix for their measurability) and $q \in L^1(Q^*)$. Let $(t_1,t_2) \in [0,T]^2, t_1 \leq t_2$ then $\int_{\Omega^*}(g(t_2,s) - g(t_1,s))\,ds = \int_{t_1}^{t_2} e^{\nu(\tau)}d\tau \int_{\Omega^*} q(\tau,s)\,d\tau$ =

$\int_{t_1}^{t_2} \frac{e^{\nu(\tau)}}{1+\lambda_2(\tau)} [||f(\tau)||_{L^1(\Omega)} + ||\lambda_3(\tau)||_{L^1(\Omega)}]d\tau$ (by equimesurability).

This last relation shows that $g \in C^0([0,T], L^1(\Omega^*))$.

Our main result is the following

Theorem 1. Under the previous assumptions , one has $\forall\, t \in [0,T]$, $\forall\, s \in \overline{\Omega^*}$

$$\int_0^s |u(t)|_*(\sigma)\,d\sigma \leq e^{-\nu(t)} \int_0^s g(t,\sigma)\,d\sigma \ ; \tag{1.4}$$

we deduce

$$||u(t)||_{L^p(\Omega)} \leq e^{-\nu(t)} ||g(t)||_{L^p(\Omega^*)} \text{ for all } p \in [1,+\infty] \ . \tag{1.5}$$

A direct consequence of Theorem 1 is

Corollary 1. Assume that $\lambda_3 = f = 0$ and $\lambda_1 \geq 0$ a.e., then the

<u>mapping</u> $t \to \int_0^s |u(t)|_*(\sigma)d\sigma$ <u>decreases for all</u> $s \in \overline{\Omega^*}$; <u>more precisely</u> : $\int_0^s |u(t)|_*(\sigma)d\sigma \le e^{-\nu(t)} \int_0^s |u_0|_* (\sigma)d\sigma$ <u>which leads to</u> : $\|u(t)\|_{L^p(\Omega)} \le e^{-\nu(t)} \|u_0\|_{L^p(\Omega)}$ for all $p \in [1,+\infty]$.

<u>Proof of Theorem 1</u>. For a fixed t , we denote $u = u(t)$, $v = |u(t)|$, $a_{ij} = a_{ij}(t,x,u,u',\nabla u)$, $\Gamma = \Gamma(t,x,u,u',\nabla u)$.

By (H4)-a.ii), as $u \in H^1(\Omega)$, according to Stampacchia [5], $h_\theta(u) \in H^1(\Omega)$, and $\frac{\partial h_\theta(u)}{\partial x_j} = 0$ if $|u| = \theta$, $\frac{\partial h_\theta(u)}{\partial x_j} = \frac{\partial u}{\partial x_j} h'_\theta(u)$ if $|u| \ne \theta$, with $h'_\theta(\sigma) = \begin{cases} 1 & \text{if } |\sigma| > \theta \\ 0 & \text{if } |\sigma| < \theta \end{cases}$.

By assumption (H4)-b, we deduce then

$$A(\theta) = \int_{v>\theta} \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} dx =$$

$$= \int_{v>\theta} (f - \frac{\partial u}{\partial t} - \Gamma)(\text{sign } u)(v-\theta)dx \quad . \tag{1.7}$$

By (H3), we have $\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \ge 0$ a.e., the function $A(\theta)$ is thus a decreasing function. As in [7], a simple derivation of the relation (1.7) leads to :

$$\int_{v>\theta} (f - \frac{\partial u}{\partial t} - \Gamma)(\text{sign } u)dx = - \frac{dA}{d\theta} \ge 0 \quad \text{a.e. } \theta \quad . \tag{1.8}$$

Set $\mu(\theta) = |v>\theta|$, $k(t,s) = \int_0^s v_*(t,\sigma)d\sigma$, $F(t,s) = \int_0^s |f(t)|_*(\sigma)d\sigma$.

By Hardy-Littlewood inequality (see [7]) :

$$\int_{v>\theta} f \text{ sign } u \, dx \le \int_0^{|v>\theta|} |f(t)|_*(\sigma)d\sigma = F(t,\mu(\theta)). \tag{1.9}$$

By assumption (H4)-a.iv), $v_* \in C^0(\Omega^*)$, then as in [7] , we obtain that for a.e. $\theta>0$, $\theta = v_*(\mu(\theta))$ and $|v_* = \theta| = 0$. On the other hand, by (H4)-a.i), $v \in W^{1,1}(0,T,L^1(\Omega))$. So, using Theorem 0 as in [8] , we get for a.e. $\theta>0$:

$$\int_{v>\theta} \frac{\partial u}{\partial t} \operatorname{sign} u \, dx = \int_{v>\theta} \frac{\partial v}{\partial t} \, dx = \int_{v>v_*(\mu(\theta))} \frac{\partial v}{\partial t} \, dx$$

$$= \frac{\partial k}{\partial t} (t,\mu(\theta)) \ . \tag{1.10}$$

Using the assumption (H2) on Γ and the relation (1.10), we get easily : $-\int_{v>\theta} \Gamma \operatorname{sign} u \, dx \leq -\lambda_1(t)k(t,\mu(\theta)) - \lambda_2(t) \frac{\partial k}{\partial t}(t,\mu(\theta)) - \int_{v>\theta} \lambda_3(t,x)dx$. Again, by Hardy-Littlewood inequality : $-\int_{v>\theta} \lambda_3(t,x)dx \leq \int_0^{|v>\theta|} (-\lambda_3)_*(t,\sigma)d\sigma$. Setting $\Lambda(t,s) = \int_0^s (-\lambda_3)_*(t,\sigma)d\sigma$, we then have :

$$-\int_{>\theta} \Gamma \operatorname{sign} u \, dx \leq -\lambda_1(t)k(t,\mu(\theta)) - \lambda_2(t) \frac{\partial k}{\partial t}(t,\mu(\theta)) +$$

$$+ \Lambda(t,\mu(\theta)) \tag{1.11}$$

Combining the relations (1.9), (1.10), (1.11), we get by (1.8) :

$$\frac{F(t,\mu(\theta)) + \Lambda(t,\mu(\theta))}{1+\lambda_2(t)} - \frac{\lambda_1(t)}{1+\lambda_2(t)} k(t,\mu(\theta)) - \frac{\partial k}{\partial t}(t,\mu(\theta)) \geq 0 \ . \tag{1.12}$$

Set $\lambda(t) = \dfrac{\lambda_1(t)}{1+\lambda_2(t)}$.

$$G(t,s) = \frac{F(t,s) + \Lambda(t,s)}{1+\lambda_2(t)} - \lambda(t)k(t,s) - \frac{\partial k}{\partial t}(t,s) =$$

$$= \int_0^s q(t,\sigma)d\sigma - \lambda(t)k(t,s) - \frac{\partial k}{\partial t}(t,s) \ .$$

Lemma 1. $G(t,.) \in C^o(\overline{\Omega^*})$.

Proof : By (H4)-a, $v \in W^{1,1}(0,T,L^1(\Omega))$. Hence, by Theorem 0, $v_* \in W^{1,1}(0,T,L^1(\Omega^*))$. For a fixed t, $k(t,s) = \int_o^s v_*(t,\sigma)d\sigma$ and $\frac{\partial k}{\partial t}(t,s) = \int_o^s \frac{\partial v_*}{\partial t} d\sigma$ belong to $C^o(\overline{\Omega^*})$. By remark 2, $q(t,.) \in L^1(\Omega^*)$ and then $\int_o^s q(t,\sigma)d\sigma$ belongs to $C^o(\overline{\Omega^*})$. □

By relation (1.12), $G(t,\mu(\theta)) \geq 0$ for a.e. $\theta>0$ and $G(t,0) = 0$ then we have :

Lemma 2. For almost every t , $G(t,s) \geq 0$ for all $s \in \overline{\Omega^*}$.

Proof of Lemma 2. $\mu(\theta) = |v>\theta|$ is a right continuous function. Hence,

$$G(t,\mu(\theta)) \geq 0 \quad \text{for all} \quad \theta \in [0,+\infty[\tag{1.13}$$

As in [7] , one can check easily that for $\theta>0$,

$$G(t,\bar{\mu}(\theta)) \geq 0 \tag{1.14}$$

where $\bar{\mu}(\theta) = |v\geq\theta|$.

Let P be the union of flat regions of v_* that is $P = \bigcup_{i\in D} P_i$ and $P_i = \{v_* = \theta_i\}$, $\theta_i \in \mathbb{R}_+$, $|P_i| \neq 0$, P_i is an interval of the form $(s_i = |v_*>\theta_i|,\ s_i' = |v_*\geq\theta_i|)$.

Let $s \in [0,|\Omega|]$.

If $s \notin P$, by equimesurability, we have $\mu(v_*(s)) = s$ and by (1.13), we get $G(t,s) \geq 0$.

If $s \in P_i = [s_i,s_i']$, by (1.13), we always have $G(t,s_i) \geq 0$. By Proposition 1 in [8], $\frac{\partial v_*}{\partial t} = \gamma_i$ = constant a.e. on P_i . So, if we write :

$$G(t,s) = G(t,s_i) + \int_{s_i}^{s} q(t,\sigma)d\sigma - \lambda(t)\int_{s_i}^{s} v_*(t,\sigma)d\sigma - \int_{s_i}^{s} \frac{\partial v_*}{\partial t}\, d\sigma$$

we obtain

$$\frac{\partial G}{\partial s} = \frac{|f|_* (t,s) + (-\lambda_3)_*(t,s)}{1 + \lambda_2(t)} - \lambda(t)\theta_i - \gamma_i .$$

As $|f(t)|_*(.) + (-\lambda_3)_*(t,.)$ is decreasing, $G(t,.)$ is then concave on $[s_i,s_i']$ and $G(t,s) \geq \mathrm{Min}(G(t,s_i),G(t,s_i'))$ for $s \in [s_i,s_i']$. We have to show that $G(t,s_i') \geq 0$.

If $\theta_i \neq 0$, $G(t,s_i') \geq 0$ by (1.14).

If $\theta_i = 0$, two cases can occur :

1) Measure $\{(t,s) \in Q^* = (0,T)\times\Omega^*,\ v_*(t,s) = 0\} = 0$, then the set of t in $(0,T)$ such that $|v(t)_* = 0| \neq 0$ is negligible.

2) Measure $\{(t,s) \in Q^*,\ v_*(t,s) = 0\} \neq 0$. As v belongs to $W^{1,1}(0,T,L^1(\Omega))$, using Theorem 0 and the same argument as Stampacchia [5], we obtain $\frac{\partial v_*}{\partial t} = 0$ a.e. on $\{(t,s) \in Q^*, v_*(t,s) = 0\}$. We deduce that for a.e. $t \in (0,T)$, $\frac{\partial v_*}{\partial t}(t) = 0$ a.e. on $\{v(t)_* = 0\}$. We then have $\theta_i = \gamma_i = 0$ and $G(t,s) = G(t,s_i) + \int_{s_i}^{s} q(t,\sigma)d\sigma \geq 0$ (since $q\geq 0$). □

This Lemma allows us to get the Gromwall inequality :

$$\frac{\partial k}{\partial t}(t,s) + \lambda(t)k(t,s) \leq \int_0^s q(t,\sigma)d\sigma \text{ for a.e. } t \qquad (1.15)$$

By Remark 2, $e^{\nu(t)}q(t,s)$ belongs to $L^1(Q^*)$. So, if we set

$$L(t,s) = \int_0^t \int_0^s e^{\nu(\tau)} q(\tau,\sigma)d\sigma d\tau ,$$

the relation (1.15) implies :

$$\frac{\partial}{\partial t}[e^{\nu(t)}k(t,s) - L(t,s)] \leq 0 \quad \text{for a.e. } t \ ;$$

then, for every t ,

$$e^{\nu(t)}k(t,s) - L(t,s) \leq k(o,s) = \int_o^s |u_o|_*(\sigma)d\sigma \quad . \tag{1.16}$$

As

$$\int_o^s g(t,\sigma)d\sigma = L(t,s) + \int_o^s |u_o|_*(\sigma)d\sigma,$$

the relation (1.4) of Theorem 1 follows from (1.16). The relation (1.5) is a simple consequence of a lemma in [1] (p. 174).

Remark 3. If u is only in $L^1(\tau,T,L^q(\Omega))$ and $\frac{\partial u}{\partial t} \in L^1(\tau,T,L^{q'}(\Omega))$ for some $\tau \in]0,T[$, $q\geq 2$, $\frac{1}{q}+\frac{1}{q'}=1$, then the result remains valid by replacing the origin of time 0 by τ that is :

$$\nu(t) = \int_\tau^t \frac{\lambda_1(\tau')}{1+\lambda_2(\tau')}\,d\tau' \ , \ t \in [\tau,T] \ ,$$

$$g(t,s) = |u(\tau)|_*(s) + \int_\tau^t e^{\nu(\tau')}\,\frac{|f|_*(\tau',s) + (-\lambda_3)_*(\tau',s)}{1+\lambda_2(\tau')}\,d\tau'$$

and then for all $t \in [\tau,T]$, $s \in \overline{\Omega^*}$:

$$\int_o^s |u(t)|_*(\sigma)d\sigma \leq e^{-\nu(t)} \int_o^s g(t,\sigma)d\sigma \ .$$

2. SOME APPLICATIONS.

1st example : the porous medium equation.

Let us consider the system

$$(2.1)\quad \begin{cases} u' = \Delta\phi(u) & \text{in } \mathbb{R}_+\times\Omega \\ \frac{\partial}{\partial n}\phi(u) = 0 & \text{on } \mathbb{R}_+\times\partial\Omega \\ u(o,x) = u_o(x) & \text{in } \Omega \end{cases}$$

where $\phi(u) = |u|^{m-1}u$ $(m>1)$, $u_o \in C^{2,\mu}(\overline{\Omega})$, $\mu>0$, $u_o\geq 0$, Ω is a smooth bounded set of $\mathbb{R}^N$.

Lemma 3. Under these assumptions on u_0, Problem (2.1) admits a positive unique classical solution.

The proof was given by Ladyzenskaya et al. [6].

Theorem 2. The positive solution of (2.1) satisfies :

$$s \in \overline{\Omega^*}\ ,\ \forall\, t \geq 0 :\ \int_0^s u(t)_*(\sigma)d\sigma \leq \int_0^s u_{0*}(\sigma)d\sigma\ ,$$

and we get

$$\|u(t)\|_{L^p(\Omega)} \leq \|u(o)\|_{L^p(\Omega)} \quad \forall\, p \in [1,+\infty]\ .$$

Proof of Theorem 2. As u is a classical solution, all the assumtions on the regularity of u are satisfied. As $u \geq 0$, $\phi(u) = u^m$, we deduce easily that :

$$-\int_\Omega \Delta u^m h_\theta(u)dx = \int_\Omega \sum_{i=1}^N m\, u^{m-1} \frac{\partial u}{\partial x_i} \frac{\partial h_\theta(u)}{\partial x_i} dx\ .$$

So the assumptions (H3)(H4) are then satisfied with $a_{ij} = m\,\delta_{ij}|u|^{m-1}$, $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$. Here, $\Gamma = f = 0$, we apply then Corollary 1.

Remark 4. Let us consider the porous medium equation with Dirichlet boundary condition :

$$\begin{cases} u' = \Delta\phi(u) & \text{in } (0,T)\times\Omega = Q \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases} \tag{2.2}$$

With $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, Ω and ϕ are as in Lemma 3. According to P. Benilan [2], the unique solution u satisfies

i) $u \geq 0$ a.e.

ii) $u \in L^\infty(Q) \cap C^o([0,T],L^1(\Omega))$

iii) $\frac{\partial u}{\partial t} \in L^1(\tau,T,L^1(\Omega))$ for any $\tau > 0$.

Assume, in addition that $u(t) \in H_o^1(\Omega)$ for a.e. $t \in]0,T[$. Then

$u(t) \in C^o(\Omega^*)$ for a.e. t (see [7]). As $u \in L^1(\tau,T,L^\infty(\Omega))$ and $\frac{\partial u}{\partial t} \in L^1(\tau,T,L^1(\Omega))$ for any $\tau>0$, one can apply Remark 3 to get

$$\int_o^s u(t)_*(\sigma)d\sigma \le \int_o^s u(\tau)_*(\sigma)d\sigma \quad \text{for any } t \in [\tau,T] \text{ and } s \in [0,|\Omega|] .$$

Since $u \in C^o([0,T],L^1(\Omega))$, by Remark 1, we deduce

$$\int_o^s u(t)_*(\sigma)d\sigma \le \lim_{\tau\to o} \int_o^s u(\tau)_*(\sigma)d\sigma = \int_o^s u(o)_*(\sigma)d\sigma ,$$

thus Theorem 2 holds.

2nd example. Parabolic inequality.

As previously, Ω is a bounded smooth domain of $\mathbb{R}^N$, $0<T<+\infty$. We consider the functions $a_{ij}(t,x)$, $a_o(t,x)$ with

(E.1) $a_{ij}, a_o \in L^\infty(Q)$, $Q = (0,T)\times\Omega$

(E.2) $\sum_{i,j=1}^N a_{ij}(t,x)\xi_i\xi_j \ge |\xi|^2$ a.e. in Q, $\forall\ \xi \in \mathbb{R}^N$

(E.3) $\frac{\partial a_{ij}}{\partial t}, \frac{\partial a_o}{\partial t} \in L^\infty(Q)$

(E.4) $f, \frac{\partial f}{\partial t} \in L^2(Q)$.

We denote by

$$A_1(t)v = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} (a_{ij}(t,x). \frac{\partial v}{\partial x_j})$$

$$A(t)v = A_1(t)v + a_o v$$

(E.5) Let $u_o \in H_o^1(\Omega)$, $u_o \ge o$, $A(T)u_o \in L^2(\Omega)$.

Lemma 4. Under the assumptions (E.1),...,(E.5), there exists one unique solution u of

$$\begin{cases} \frac{\partial u}{\partial t} + Au - f \geq 0 \quad , \quad u \geq 0 & \text{in } Q \\ (\frac{\partial u}{\partial t} + Au - f)u = 0 & \text{in } Q \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases} \tag{2.3}$$

Moreover, u satisfies $u \in L^{\infty}(0,T,H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^{\infty}(0,T,L^2(\Omega))$.

The proof of Lemma 4 is contained in Bensoussan-Lions [3].

Theorem 3. Let u be the solution of Problem 3 and let $\lambda_1(t) = \operatorname{ess\,inf}_{\Omega} a_0(t,x)$, $\nu(t) = \int_0^t \lambda_1(\tau)d\tau$, then for all $s \in \overline{\Omega^*}$, $t \in [0,T]$:

$$\int_0^s u(t)_*(\sigma)d\sigma \leq e^{-\nu(t)} \int_0^s [u_{0_*}(\sigma) + \int_0^t e^{\nu(\tau)}|f(\tau)|_*(\sigma)d\tau]d\sigma$$

which leads to

$$||u(t)||_{L^p(\Omega)} \leq e^{-\nu(t)} ||g(t)||_{L^p(\Omega^*)} \quad \forall\, p \in [1,+\infty] \tag{2.4}$$

where

$$g(t)(s) = u_{0_*}(s) + \int_0^t e^{\nu(\tau)}|f(\tau)|_*(s)d\tau \quad .$$

Proof. In order to apply Theorem 1, we check all the assumptions. The regularity on u is satisfied because of Lemma 4. By Remark 1, $u(t)_* \in C^0(\Omega^*)$. The operator Γ is reduced to $\Gamma = a_0(t,x)v$. So, $(\text{sign } v)\Gamma \geq \lambda_1(t)|v|$. One can chek that the condition (H.4)-b) is satisfied. We apply Theorem 1 and get the result.

3rd example. The semi-linear heat equation.

Let g be a function of class $C^1(\mathbb{R},\mathbb{R})$ satisfying

i) There exists $\varepsilon>0$, such that $xg(x) \geq (2+\varepsilon)G(x)$, $\forall\, x \in \mathbb{R}$, where $G(x) = \int_0^x g(s)ds$.

ii) There exist three constants A, B, and p such that

$$|g(x)| \leq A|x| + B|x|^p \quad (\forall\, x \in \mathbb{R}),\quad 1 < p < \frac{N+2}{N-2} \text{ if } N\geq 3,\quad 1 < p < +\infty \text{ if } N=1,2.$$

iii) g is non increasing.

The following proposition is proved in [4].

Proposition 1. Assume that g satisfies i), ii), iii), and m is a nonnegative constant, then every solution $u \in L^\infty_{loc}([0,+\infty[, H^1_o(\Omega))$ of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + mu = g(u) & \text{in } \mathbb{R}_+\times\Omega \\ u = 0 & \text{on } \mathbb{R}_+\times\partial\Omega \\ u(o,x) = u_o(x) & \text{in } \Omega \text{ for } t = 0 \end{cases} \tag{2.5}$$

is such that

a) $u(t) \in C^2(\overline{\Omega})$, $\frac{\partial u}{\partial t}(t) \in C^o(\overline{\Omega})$ for any $t>0$ and

$$\sup_{t\geq\tau>o} \|u(t)\|_{C^2(\overline{\Omega})} < +\infty \text{ for any } \tau>0 .$$

b) $$\sup_{t\geq o} \|u(t)\|_{L^2(\Omega)} < +\infty .$$

□

Remark 5. The hypothesis on u and g infer that u is weakly continuous with values in $H^1_o(\Omega)$.

□

An improvement of the statement b) is the following

Proposition 2. Every solution $u \in L^\infty_{loc}([0,+\infty[,H^1_o(\Omega))$ of Problem (2.5) satisfies :

c) $$\|u(t)\|_{L^p(\Omega)} \leq e^{-mt}\|u_o\|_{L^p(\Omega)} \quad \forall\, p \in [1,+\infty],\ \forall\, t \geq 0.$$

Proof. Let T be a positive number. By Proposition 1-a), we have u and Δu in $L^\infty([\tau,T]\times\Omega)$ for any $\tau>0$. By ii), $g(u) \in L([\tau,T]\times\Omega)$

and then $\frac{\partial u}{\partial t} = \Delta u + g(u) - mu \in L^{\infty}([\tau,T]\times\Omega)$. By Remark 1, $|u(t)|_* \in C^o(\Omega^*)$. If we set $\Gamma = mu - g(u)$, we check easily that (sign u)$\Gamma \geq m|u|$ (using iii) and $g(o) = o$). The assumption (H4)-b can be verified by a simple integration by parts. We then apply Remark 3 and Corollary 1 to get :

$$\int_o^s |u(t)|_*(\sigma)d\sigma \leq e^{-m(t-\tau)} \int_o^s |u(\tau)|_*(\sigma)d\sigma \quad \forall\, t \geq \tau \quad . \qquad (2.6)$$

By remark 5, we deduce that $u \in C^o([0,T],L^1(\Omega))$ and by Remark 1, relation (2.6) infers (τ tends to zero) :

$$\int_o^s |u(t)|_*(\sigma)d\sigma \leq e^{-mt} \int_o^s |u(o)|_*(\sigma)d\sigma;$$

We apply Corollary 1.

□

APPENDIX

Lemma. Let Ω be a measurable subset of $\mathbb{R}^N$ (N≥1), whose measure is finite, T>0 and let f be a measurable function defined on $Q =]0,T[\times\Omega$ then the decreasing rearrangement of f , f_* , is also measurable on $Q^* =]0,T[\times\Omega^*$. Moreover, if $f \in L^p(0,T,L^q(\Omega)) = V$ $(1 \leq p \leq +\infty,\ 1 \leq q \leq +\infty)$. Then $f_* \in L^p(0,T,L^q(\Omega^*)) = W$ and $\|f\|_V = \|f_*\|_W$.

Proof. Firstly, if $f \in L^1(Q)$, then there exists a sequence f_n in $C^o([0,T],L^1(\Omega))$ such that f_n tends to f in $L^1(Q)$. If we set

$$k(t,s) = \int_o^s f_*(t,\sigma)d\sigma \ , \ k_n(t,s) = \int_o^s f_{n*}(t,\sigma)d\sigma$$

for $s \in \overline{\Omega^*}$, then $k_n \in C^o(\overline{Q^*})$ and k is measurable. In fact, let us take $(t_1,s_1) \in \overline{Q^*}$, for any $(t,s) \in \overline{Q^*}$:

$$|k_n(t_1,s_1) - k_n(t,s)| \leq |k_n(t_1,s_1) - k_n(t_1,s)| + |k_n(t_1,s) - k_n(t,s)|.$$

Since the rearrangement is contracting from $L^1(\Omega)$ into $L^1(\Omega^*)$

$$|k_n(t_1,s) - k_n(t,s)| \leq \int_\Omega |f_{n_*}(t_1,\sigma) - f_{n_*}(t,\sigma)|d\sigma \leq$$

$$\leq \|f_n(t_1) - f_n(t)\|_{L^1(\Omega)} .$$

Hence

$$|k_n(t_1,s_1) - k_n(t,s)| \leq \left|\int_{s_1}^{s} (f_n(t_1))_*(\sigma)d\sigma\right| + \|f_n(t_1) - f_n(t)\|_{L^1(\Omega)} . \quad \text{(A.1)}$$

Since $f_n \in C^0([0,T],L^1(\Omega))$ then $f_n(t_1)_* \in L^1(\Omega^*)$ (by equimesurability), so the relation (A.1) implies that $k_n \in C^0(\overline{Q^*})$. On the other hand, by the same argument as before, for every s ,

$$|k_n(t,s) - k(t,s)| \leq \|f(t) - f_n(t)\|_{L^1(\Omega)} \underset{n\to+\infty}{\to} 0$$

a.e. t (since $f_n \underset{n}{\to} f$ in $L^1(Q)$). Thus, $\lim_{n\to+\infty} k_n(t,S) = k(t,s)$: k is then measurable. As

$$\lim_{\substack{h\to 0\\ h>0}} \frac{k(t,s+h) - k(t,s)}{h} = f_*(t,s) :$$

f_* is also measurable. Secondly, if f is only measurable on Q , then we consider the real function

$$T_n(\tau) = \begin{cases} n & \text{if } \tau \geq n \\ \tau & \text{if } |\tau| \leq n \\ -n & \text{if } \tau \leq -n \end{cases}$$

$T_n(f) \in L^\infty(Q)$ and $T_n(f_*) = (T_n(f))_*$ is measurable according to the remark avove. Since, $\lim_{n\to+\infty} T_n(f_*) = f_*$. We deduce that

f_* is measurable. If $f \in V$, f_* is measurable and the relation $\|f\|_V = \|f_*\|_W$ is a simple consequence of the equimesurability. □

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Remark 1.

Let u be a measurable function on Ω and $v \in L^1(\Omega)$. Then

$$\frac{dw_\lambda}{ds} = \frac{(u + \lambda v)_* - u_*}{\lambda}$$

tends to v_{*u} weakly in $L^1(\Omega^*)$ as λ tends to zero. That is for the topology $\sigma(L^1, L^\infty)$.

Proof. We will use the Dunford-Pettis' theorem [a]. Let $\epsilon > 0$ and $v_n \in L^\infty(\Omega)$ such that v_n tends to v in $L^1(v)$ as n tends to infinity. By Lemma 1.1 in [b], we have for any $\lambda > 0$

$$\int_{\Omega^*} \left| \frac{(u + \lambda v_n)_* - (u + \lambda v)_*}{\lambda} \right| d\sigma \leqslant \|v_n - v\|_{L^1(\Omega)} \qquad \text{(R.1)}$$

and a.e. in Ω^*

$$\left| \frac{(u + \lambda v_n)_* - u_*}{\lambda} \right| (s) \leqslant \|v_n\|_\infty \qquad \text{(R.2)}$$

Let us choose $n_0 \in \mathbb{N}$, $\|v_{n_0} - v\|_{L^1(\Omega)} \leqslant \epsilon/2$ and $\delta = \frac{\epsilon}{2} \cdot \frac{1}{1 + \|v_{n_0}\|_\infty}$. If A is a measurable subset of Ω^* such $|A| \leqslant \delta$, then we deduce from (R.2) and the choice of δ :

$$\int_A \left| \frac{(u + \lambda v_{n_0})_* - u_*}{\lambda} \right| d\sigma \leqslant |A| . \|v_{n_0}\|_\infty \leqslant \epsilon/2 .$$

Since,

$$\int_A \left| \frac{(u + \lambda v)_* - u_*}{\lambda} \right| d\sigma \leqslant \int_A \left| \frac{(u + \lambda v)_* - (u + \lambda v_{n_0})_*}{\lambda} \right| d\sigma +$$

$$+ \int_A \left| \frac{(u + \lambda v_n)_* - u_*}{\lambda} \right| d\sigma$$

we then have

$$\int_A \left| \frac{(u + \lambda v)_* - u_*}{\lambda} \right| d\sigma \leqslant \|v - v_{n_0}\|_{L^1(\Omega)} + \epsilon/2$$

$$\leqslant \epsilon/2 + \epsilon/2 = \epsilon .$$

We conclude with Theorem 1.1(bis, in [b].

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