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Etude d'un problème de contrôle optimal convexe mal posé

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RESUME DE THESE

Soit Ω un ouvert borné de l'espace euclidien R^N, $N \geq 3$, dont le frontière Γ est la réunion $\Gamma_0 \cup \Gamma_1$ de deux variétés de classe $\mathcal{C}^\infty$, compactes connexes et disjointes. Ω est supposé être localement d'un seul côté de Γ. Soit $\alpha, \alpha', \beta, \beta'$ quatre réels strictement positifs. Notre propos est d'étudier le problème de contrôle optimal correspondant aux données suivantes :

* L'ensemble des contrôles admissibles $\mathcal{U}_{ad}$ est

$$\mathcal{U}_{ad} = \{(v_0, v_1) \in (L^2(\Gamma_0))^2,\ v_0(x) \ (\text{resp. } v_1(x)) \in [-\alpha, \beta]$$
$$(\text{resp. } [-\alpha', \beta'])\ \text{p.p. } x \in \Gamma_0\}.$$

* L'état z du sytème vérifie les conditions :

$$(c_0) \qquad z \in L^2(\Omega) \ ; \Delta z = 0 \ ; \ z = v_0 \text{ sur } \Gamma_0 \ ; \ \frac{\partial z}{\partial \nu} = v_1 \text{ sur } \Gamma_0 .$$

* La fonction coût, que l'on cherche à minimiser, est donnée par

$$j(z, v_0, v_1) = \frac{1}{2} |z - z_d|^2_{L^2(\Omega)} + \frac{1}{2} |v_0|^2_{L^2(\Gamma_0)} + \frac{1}{2} |v_1|^2_{L^2(\Gamma_0)}$$

(z_d étant un élément fixe de $L^2(\Omega)$.

On voit par des méthodes classique que ce problème de contrôle admet une solution unique. Toutefois ce problème est mal posé, dans la mesure où l'état adjoint ne peut être déterminé par les méthodes usuelles ce qui rend difficile l'écriture des conditions d'optimalité. Nous sommes amenés ici à étudier directement ce problème de contrôle optimal comme un problème d'optimisation convexe dans l'espace

$$W = \{z \in L^2(\Omega), \Delta z \in L^2(\Omega),\ (z, \frac{\partial z}{\partial \nu}) \in (L^2(\Gamma_0))^2\}$$

dont la formulation est évidemment la suivante :

$$(\mathcal{P}) \qquad \text{Inf } \{\frac{1}{2} |z - z_d|^2_{L^2(\Omega)} + \frac{1}{2} |z|^2_{L^2(\Gamma_0)} + \frac{1}{2} |\frac{\partial z}{\partial \nu}|^2_{L^2(\Gamma_0)} \}$$

pour $z \in W$, $\Delta z = 0$, $z(x)$ (resp. $\frac{\partial z}{\partial \nu}(x)$) $\in [-\alpha,\beta]$ (resp. $[-\alpha',\beta']$) pour presque tout $x \in \Gamma_0$.

La détermination de l'état adjoint de la solution de ce problème est équivalente à la recherche de solutions du problème dual de $(\mathcal{P})$, soit $(\mathcal{P}^*)$:

$$(\mathcal{P}^*) \qquad \operatorname{Sup} \{ -\frac{1}{2} |\Delta q|^2_{L^2(\Omega)} + \int_\Omega \Delta q . z_d \, dx - \int_{\Gamma_0} h_{\alpha,\beta}(-\frac{\partial q}{\partial \nu}) d\Gamma - \int_{\Gamma_0} h_{\alpha',\beta'}(q) d\Gamma \}$$

pour $q \in W$, $q = \frac{\partial q}{\partial \nu} = 0$ sur Γ_1, et $h_{\alpha,\beta}$ définie par :

$$h_{\alpha,\beta}(s) \begin{cases} = \frac{s^2}{2}, & s \in [-\alpha,\beta] \\ = \beta(s - \frac{\beta}{2}), & s \geqslant \beta \\ = -\alpha(s + \frac{\alpha}{2}), & s \leqslant -\alpha . \end{cases}$$

L'étude des problèmes $(\mathcal{P})$ et $(\mathcal{P}^*)$: existence de solutions, système des conditions d'optimalité, va donc constituer la majeure partie de notre travail.

Dans la Section I, nous utilisons les résultats classiques d'optimisation convexe, dans le cas coercif sci, pour établir l'existence et l'unicité de la solution z_0 de $(\mathcal{P})$, ainsi que l'égalités des extremas de $(\mathcal{P})$ et $(\mathcal{P}^*)$. Etant donné la définition de $h_{\alpha,\beta}$, nous ne savons pas si $(\mathcal{P}^*)$ admet des solutions dans W, par manque de coercivité. L'étude des suites maximisantes de $(\mathcal{P}^*)$ nous montre que ces dernières vérifient les conditions suivantes :

(c_1) Δq_n bornée dans $L^2(\Omega)$,

(c_2) $(q_n, \frac{\partial q_n}{\partial \nu})$ bornée dans $M_1(\Gamma_0) \times M_1(\Gamma_0)$,

où $M_1(\Gamma_0)$ désigne l'ensemble des mesures bornées sur Γ_0. Nous souhaitons donc étendre l'ensemble des éléments admissibles pour $(\mathcal{P}^*)$ à un espace dans lequel les suites maximisantes de $(\mathcal{P}^*)$ seront bornées.

Avant de définir précisément l'extension souhaitée du problème $(\mathcal{P}^*)$, nous donnons dans la Section II quelques résultats d'approximation et de régularisation de fonctions et de mesures sur une variété riemanienne compacte, puis nous définissons les fonctionnelles $\mu \to \int h(\mu)$, où μ est une mesure bornée, et h une fonction convexe vérifiant certaines conditions. La Section III est consacrée à l'étude des représentations intégrales des solutions d'un problème de Neumann avec données au bord qui sont des mesures. En particulier, nous précisons l'appartenance de ces fonctions ainsi que de leurs traces sur Γ, à certains espaces L^p. Nous sommes en mesure, dans la Section IV, de définir l'espace $BT(\Omega)$ ("bounded trace") qui constituera le domaine généralisé d'étude pour le problème $(\mathcal{P}^*)$; de manière informelle, $BT(\Omega)$ sera défini comme l'ensemble des fonctions définies sur Ω, à laplacien dans $L^2(\Omega)$, et dont la dérivée normale appartient à $M_1(\Gamma_0) \times H^{1/2}(\Gamma_1)$ (*). Les résultats de la section III nous permettent de caractériser les inclusions de $BT(\Omega)$ dans certains espaces de Lebesgue sur Ω et Γ, ainsi que de donner des résultats d'approximation de fonctions de $BT(\Omega)$ par des éléments de $\mathcal{C}^\infty(\bar{\Omega})$.

Grâce à la notion de fonctionnelle convexe d'une mesure bornée, étudiée à la Section II, nous pouvons introduire, à la Section V, le problème généralisé $(\mathcal{Q}^*)$:

$$(\mathcal{Q}^*) \qquad \operatorname{Sup} \left\{- \frac{1}{2} |\Delta u|^2_{L^2(\Omega)} + \int_\Omega \Delta u . z_d - \int_{\Gamma_0} h_{\alpha,\beta}\left(- \frac{\partial u}{\partial v}\right) - \int_{\Gamma_0} h_{\alpha',\beta}(u) d\Gamma\right\}$$

pour $u \in BT(\Omega)$, $u = \frac{\partial u}{\partial v} = 0$ sur Γ_1. Ici, "$\int_{\Gamma_0} h_{\alpha,\beta}(- \frac{\partial u}{\partial v})$" représente l'intégrale sur Γ_0 de la mesure bornée $h_{\alpha,\beta}(- \frac{\partial u}{\partial v})$. Le principal résultat de cette section, obtenu par une méthode directe d'approximation est l'égalité suivante : $\operatorname{Inf} (\mathcal{P}) = \operatorname{Sup} (\mathcal{P}^*) = \operatorname{Sup} (\mathcal{Q}^*)$.

La Section VI achève l'étude des problèmes $(\mathcal{P})$ et $(\mathcal{Q}^*)$, et rassemble donc les résultats les plus importants de notre travail. Nous y démontrons l'existence d'un état adjoint, c'est-à-dire une solution dans $BT(\Omega)$ du problème $(\mathcal{Q}^*)$; ensuite, nous décrivons le système d'optimalité, qui donne les conditions nécessaires et suffisantes pour qu'un élément z de W soit l'état optimal. Ces résultats sont obtenus par la méthode directe du calcul des variations, dont l'utilisation est rendue possible grâce à une "formule de Green" généralisée liant un élément z admissible pour $(\mathcal{P})$ et un éléments U de $BT(\Omega)$.

Dans la Section VII, nous proposons une famille de problèmes perturbés $(\mathcal{P}_\epsilon)$, $\epsilon>0$, pour laquelle nous prouvons les propriétés suivantes:

(1) Il y a existence et unicité de l'état optimal z_ϵ, ainsi que de l'état adjoint p_ϵ,

(2) z_ϵ converge vers z_0 pour la topologie forte de W,

(3) la famille $(p_\epsilon)_{\epsilon>0}$ admet des points d'accumulation pour la topologie faible de $BT(\Omega)$ qui sont des états adjoints de z_0.

Finalement, la Section VIII est consacrée à l'étude d'un problème similaire à $(\mathcal{P})$, correspondant au cas d'une fonction coût sur la frontière.

$$(\mathfrak{R}) \qquad \text{Inf } \{\frac{1}{2} |z - y_d|^2_{L^2(\Gamma_1)} + \frac{1}{2} |z|^2_{L^2(\Gamma_0)} + \frac{1}{2} |\frac{\partial z}{\partial \nu}|^2_{L^2(\Gamma_0)}\}$$

pour $z \in W$, $\Delta z = 0$, $z \in L^2(\Gamma_1)$, $z(x)$ (resp. $\frac{\partial z}{\partial \nu}(x)$) $\in [-\alpha,\beta]$ (resp. $[-\alpha',\beta']$) pour presque tout $x \in \Gamma_0$; y_d est une fonction donnée dans $L^2(\Gamma_1)$.

Nous reconduisons quasiment pas à pas la méthode utilisée pour l'étude de $(\mathcal{P})$, ce qui nous permet d'obtenir, dans l'espace adapté $BT_1(\Omega)$, l'existence de solutions généralisées du problème dual $(\mathfrak{R}^*)$, ainsi que le système d'optimalité de ce nouveau problème de contrôle optimal.

Note de bas de page

(*) La condition "$\frac{\partial u}{\partial \nu}_{|\Gamma_1} \in H^{1/2}(\Gamma_1)$", techniquement nécessaire, n'impose pas de restrictions, étant donné le critère d'admissibilité "$\mu = \frac{\partial u}{\partial \nu} = 0$ sur Γ_1".

SOMMAIRE

INTRODUCTION

When one is interested in problems of optimal control, the existence, and even uniqueness of an optimal state are note quite satisfactory ; one also requires some informations about the existence of an adjoint state, so as to establish the optimality system of the problem. When such informations are missing, the problem is said non-well posed [L], and one has to find a suitable way of overcoming this difficulty. In this work, our purpose is to study a non-well-posed problem of convex optimal control, by developping a method of generalized duality for a pair of conjugate problems in convex optimization. Our processes are based on some previous results for the study of plasticity (R. Temam [T], R. Kohn-R. Temam [KT]). Let us now present the mathematical setting of the problem we want to study. Let Ω be an open bounded set of the euclidean space $\mathbb{R}^N$, $N \geq 3$, whose boundary is the union $\Gamma_0 \cup \Gamma_1$ of two disjoint connected $(N-1)$ dimensional $\mathcal{C}^\infty$ submanifolds of $\mathbb{R}^N$ and let $(\alpha, \alpha', \beta, \beta')$ be four strictly positive real numbers.

We set : $\mathcal{U}_{\alpha,\beta} = \{q \in L^2(\Gamma_0)\,,\ q(x) \in [-\alpha, \beta] \text{ for a . e } x \in \Gamma_0\}$

The problem we want to study is :

to find $(z_0, u_0, u_1) \in L^2(\Omega) \times \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'}$ minimizing the cost function

$$J(z,u,v) = 1/2\ |z - z_d|^2_{L^2(\Omega)} + 1/2\ |u|^2_{L^2(\Gamma_0)} + 1/2\ |v|^2_{L^2(\Gamma_0)}$$

(P)

the infimum being taken for $z \in L^2(\Omega)$, $\Delta z = 0$ in Ω, $z = u$ on Γ_0 ,

$$\partial z/\partial r = v \text{ on } \Gamma_0 \ , \ (u,v) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'} \ .$$

(z_d a given function in $L^2(\Omega)$).

If we set $W = \{z \in L^2(\Omega),\ \Delta z \in L^2(\Omega),\ (z,(\partial z/\partial r)) \in L^2(\Gamma_0) \times L^2(\Gamma_0)\}$, then (P) is obviously equivalent to the problem of convex minimization :

$$(P) \qquad \mathrm{Inf}\{1/2|z-z_d|^2_{L^2(\Gamma_0)} + 1/2\ |z|^2_{L^2(\Gamma_0)} + 1/2\ |(\partial z/\partial r)|^2_{L^2(\Gamma_0)}\}$$

for $z \in W,\ \Delta z = 0\ ,\ (z,(\partial z/\partial r)) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'}$.

The determination of an adjoint state and of the optimality system for (P) is equivalent to solving the conjugate problem of *(P)* , [E-T], that is to say :

$$(P^*) \qquad \mathrm{Sup}\ \{-1/2|\Delta q|^2_{L^2(\Omega)} - \int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r)d\Gamma - \int_{\Gamma_0} h_{\alpha',\beta'}(q)d\Gamma\}$$

$$\text{for } q \in W,\ q = \partial q/\partial r = 0 \text{ on } \Gamma_1 .$$

$h_{\alpha,\beta}$ is the function defined on $\mathbb{R}^1$ by : $h_{\alpha,\beta}(s)\ \begin{cases} = (s^2/2), & \text{if } s \in [-\alpha,\beta] \\ = \beta(s-\beta/2), & \text{if } s \geqslant \beta \\ = -\alpha(s+\alpha/2), & \text{if } s \leqslant -\alpha \end{cases}$

Problem *(P)* is easily seen to have a unique solution z_0 in W [E.T]. Unfortunately, by lack of coercivity (see the definition of $h_{\alpha,\beta}$), we do not know wether Problem *(P*)* has a solution in W.

Nevertheless, one can remark that a minimizing sequence q_n of (P^*) is bounded in the following sense :

$$\Delta q_n \text{ is bounded in } L^2(\Omega)$$
$$(q_n,(\partial q_n/\partial r)) \text{ is bounded in } M_1(\Gamma_0)\times M_1(\Gamma_0)$$

where $M_1(\Gamma_0)$ is the set of bounded measures on Γ_0. As was the case for the problem considered by R. Temam in [T], that remark is the starting point of our method for studying problem *(P^*)* : as a matter of fact, $M_1(\Gamma_0)$ has the good property of vague compactness for bounded sets, which makes it a nice substitute for the study of optimization problems that are coercive in the L^1-norm.

That is why we are led to introducing the space $BT(\Omega)$, which seems to be well adapted to the study of *(P^*)*. The main results for $BT(\Omega)$, extending some classical results for solutions of elliptic PDE [M1] are the following.

<u>Theorem A</u> :

a). Let $u\in BT(\Omega)$; u satisfies the properties (i) to (iv)

(i) $u\in L^s(\Omega)$, for $1\leqslant s<N/N-2$; (ii) $\Delta u\in L^2(\Omega)$

(iii) $u_{|\Gamma}\in L^t(\Gamma_0)\times H^{3/2}(\Gamma_1)$, for $1\leqslant t<N-1/N-2$

(iv) $\partial u/\partial r \in M_1(\Gamma_0)\times H^{1/2}(\Gamma_1)$.

b) Let τ_1 be the (weak) topology defined on $BT(\Omega)$ by :

$u_n\to u$ for τ_1 if f : $\Delta u_n\to\Delta u$ in $L^2(\Omega)$ weakly

$\partial u_n/\partial r \to \partial u/\partial r$ in $M_1(\Gamma_0)$ vaguely and in $H^{1/2}(\Gamma_1)$ weakly.

Then every bounded set of $BT(\Omega)$ is relatively compact for τ_1.

By means of the notion of convex function of a measure, introduced by F. Demengel and R. Temam [DT1], we can define the generalized problem (Q^*), which is an extension of *(P^*)* to $BT(\Omega)$. The study of (Q^*), thanks to the generalized duality method, enables us to prove the following results, which solve Problem (P) :

<u>Theorem B</u> :

a) There exist adjoint states, in $BT(\Omega)$, for the optimal state z_0 ; they are the solutions of (Q^*).

b) Let z be an admissible state ; z is optimal iff there exists an adjoint state $u \in BT(\Omega)$ satisfying the optimality conditions

(i) $\Delta u = -(z - z_d)$ in Ω

(ii) $h_{\alpha',\beta'}(u) = u.(\partial z/\partial r) - 1/2\, |\partial z/\partial r|^2$ a.e. on Γ.

(iii) $h_{\alpha,\beta}(-\partial u/\partial r) = -z\, (\partial u/\partial r) - 1/2 |z|^2$ in $M_1(\Gamma_0)$.

We then turn to a study of a penalized form *(P_ε)* of *(P)*, having the property that its conjugate *(P_ε^*)* is coercive on W. The results we prove for *(P_ε)* and *(P_ε^*)* are summed up in the

Theorem C :

a) There exists a unique optimal state (resp. adjoint state) z_ε (resp. p_ε), solution in W of the optimization problem (P_ε) (resp. (P_ε^*) .

b) $z_\varepsilon \rightarrow z_0$ in W as $\varepsilon \rightarrow 0$.

c) $\{p_\varepsilon\}_{\varepsilon>0}$ has cluster points for the τ_1 - topology of $BT(\Omega)$, that are adjoint states of z_0 .

Finally, we use our method to study another problem of optimal control (R) , similar to (P) , in which the cost function involves the boundary value $z|_{\Gamma_1}$ of z . As for (P) , we prove the existence of a generalized adjoint state for the optimal state, and establish the set of optimality conditions.

The article is organized as follows. Section I presents the first elementary results on *(P)* and *(P*)*. Section II and III contains the technical results needed for the introduction of $BT(\Omega)$ and the generalized problem (Q^*). Section IV is devoted to the study of $BT(\Omega)$, while, in Section V, we define and study the generalized problem (Q^*). In section VI, we develop the generalized duality method, and prove our main results for problem (P). Section VII presents the penalized problems (P_ε) and (P_ε^*); and, in Section VIII, we study problem (R).

A NON WELL-POSED PROBLEM IN CONVEX OPTIMAL CONTROL.

0. Hypothesis, notations.

0.1. Geometrical assumptions.

Let Ω be a bounded open set of the euclidean linear space R^N, $N \geq 3$. Its boundary Γ is supposed to be the union $\Gamma_0 \cup \Gamma$ of two $(N-1)$ dimensional disjoint differentiable manifolds, each of them being compact, connected and indefinitely differentiable.

As usual, we shall write dx (resp. $d\Gamma$) for the Lebesgue measure on Ω (resp. Γ), induced by the canonical riemannian structure of R^N.

0.2. Notations ; the function spaces.

We shall always consider real-valued functions, and generally use the classical notations of the theory of partial differential equations $(\Delta, \partial/\partial x_i, \partial/\partial r, \ldots)$.

Let B be a Banach space ; the norm of an element x of B will be denoted by $\|x\|_B$; if $\mathcal{C}, \mathcal{C}^*$ are two conjugate topological vector spaces, $\langle x, x^* \rangle_{\mathcal{C} \times \mathcal{C}^*}$ will stand for the value at x of the linear form x^*.

Let X be an indefinitely differentiable submanifold of R^N, and μ a measure on X. We shall write $L^p(X;d\mu)(1 \leq p \leq +\infty)$, $\mathcal{C}^k(X)(0 \leq k \leq +\infty)$, $\mathcal{D}(X)$, $H^s(X)(-\infty < s < +\infty)$, $M_1(X)$ for the spaces respectively defined as :

* (classes of) measurable functions whose p^{th} power is summable on X with respect to μ (for $p = \infty$, essentially bounded measurable functions).

- k - times continuously differentiable functions on X
- indefinitely differentiable compactly supported functions on X.
- distributions on X (dual of the former).
- Sobolev space of order s built on $L^2(X;dx)$ (dx : Lebesgue measure on X).
- Bounded measures on X,

each of these being endowed with its usual topology.

When μ is the Lebesgue measure on X, we shall write $L^p(X)$ instead of $L^p(X,d\mu)$.

The Letter C will be allowed to represent as many different positive constants as necessary.

I. Exposition of the problem.

1.1. Problem (P) and its variational formulation.

Let us define the space

$$W = \{z \in L^2(\Omega),\ \Delta z \in L^2(\Omega)\ \ (z, \partial z/\partial r) \in L^2(\Gamma_0) \times L^2(\Gamma_0)\},$$

which is a Hilbert space when endowed with the norm

$$\|z\|_W = \{\|z\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Gamma_0)} + \|\partial z/\partial r\|^2_{L^2(\Gamma_0)}\}^{1/2}.$$

Remark I.1.1. According to [L2], p. 148, $z|_{\Gamma_1}$ and $\partial z/\partial r|_{\Gamma_1}$ are well-defined in $H^{-1/2}(\Gamma_1)$ and $H^{-3/2}(\Gamma_1)$, the trace applications being linear continuous.

Let α,β be two strictly positive real numbers ; we set :

$$\mathcal{U}_{\alpha,\beta}=\{p\in L^2(\Gamma_0),\ p^+=\mathrm{Sup}(p,o)\leqslant\beta,\ \ p^-=-\mathrm{Inf}(p,o)\leqslant\alpha \ \text{ a.e. on } \Gamma_0\}$$

$\mathcal{U}_{\alpha,\beta}$ is a closed convex set of $L^2(\Gamma_0)$.

Given four such numbers $\alpha,\alpha',\beta,\beta'$, we want to study the following problem of optimal control.

to find $(z_0, u_0, v_0)\in L^2(\Omega)\times\mathcal{U}_{\alpha,\beta}\times\mathcal{U}_{a',\beta'}$ minimizing the cost function :

(P) $$J(z,u,v)=1/2\ \|z-z_d\|^2_{L^2(\Omega)}+1/2\|u\|^2_{L^2(\Gamma_0)}+1/2\|v\|^2_{L^2(\Gamma_0)}$$

the infimum being taken for $z\in L^2(\Omega)$, $\Delta z=0$ in Ω, $z=u$ on Γ_0, $\partial z/\partial r=v$ on

Γ_0, $(u,v)\in\mathcal{U}_{\alpha,\beta}\times\mathcal{U}_{\alpha',\beta'}$.

(z_d is a given function in $L^2(\Omega)$).

In order to establish the variational formulation of (P), we define the functional :

$$j(z)\begin{cases} =1/2(\|z-z_d\|^2_{L^2(\Omega)}+\|z\|^2_{L^2(\Gamma_0)}+\|\partial z/\partial r\|^2_{L^2(\Gamma_0)}) \ \text{ if } z\in W \\ \quad\Delta z=0 \ \text{ in } \Omega \\ \quad(z,\partial z/\partial r)\in\mathcal{U}_{\alpha,\beta}\times\mathcal{U}_{\alpha'\beta'} \\ =+\infty \ \text{ otherwise}\end{cases}\qquad(1.1.1)$$

The functional j is strictly convex and coercive, that is :

$$\exists C>0, \quad \forall z \in W, \quad j(z) \geqslant C\|z\|_W^2 \qquad (I.1.2)$$

Moreover, j is lower semicontinuoous (ℓ sc) on W, for $\mathcal{U}_{\alpha,\beta}$ is a closed convex set. Problem (P) is obviously equivalent to the convex minimization problem *(P)*:

$$(P) \qquad \inf_{z \in W}\{j(z)\} \qquad (I.1.3)$$

We can now state our first result for (P) :

<u>Proposition I.1.1</u>. There exist a unique optimal state z_0 of problem (P) in W.

Proof : This a classical result ([E.T.], ch. II, Prop. 1.2) for the minimization of a strictly convex ℓsc coercive functional on a Hilbert space.

The determination of an adjoint state for the optimal state z_0, and the establishement of the optimality system for (P) come down to solving the dual problem of *(P)* , and writing the optimality conditions that the minimizer of *(P)* and a solution of its dual problem must satisfy.

1.2. The dual problem.

We shall now give the expression of the dual problem of (I.2.3), using the general duality frame of [E.T].

Let Y be the space $(L^2(\Omega))^2 \times (L^2(\Gamma_0))^2$; Y will be identified with its conjugate Y^*.

Let Λ be the linear operator from W to Y, defined by :

$$\Lambda z = (z, \Delta z, z_{|\Gamma_0}, \partial z/\partial r_{|\Gamma_0}) \qquad (1.2.1)$$

For z in W, and $p = (p_i)_{1 \leq i \leq 4}$ in Y, we define the following functions :

$F(z) = 0$, $G(p) = \sum_{i=1}^{4} G_i(p_i)$ with :

$$G_1(p_1) = 1/2 \|p_1 - z_d\|^2_{L^2(\Omega)} \qquad (1.2.2)$$

$$G_2(p_2) \begin{cases} = 0, \text{ if } p_2 = 0 \\ = +\infty \text{ otherwise} \end{cases} \qquad (1.2.3)$$

$$G_3(p_3) \begin{cases} = 1/2 \|p_3\|^2_{L^2(\Gamma_0)}, \text{ if } p_3 \in \mathcal{U}_{\alpha,\beta} \\ = +\infty \text{ otherwise} \end{cases} \qquad (1.2.4)$$

$$G_4(p_4) \begin{cases} = 1/2 \|p_4\|^2_{L^2(\Gamma_0)}, \text{ if } p_4 \in \mathcal{U}_{\alpha',\beta'} \\ = +\infty \text{ otherwise} \end{cases} \qquad (1.2.5)$$

(P) may be written under the following form :

$$(P) \qquad \inf_{z\in W} \{F(z)+G(\Lambda z)\} \qquad (I.2.6)$$

and its dual problem is ([E.T], p. 61) :

$$(P^*) \qquad \sup_{q\in Y^*} \{-G^*(-q)-F^*(\Lambda^* q)\} \qquad (I.2.7)$$

G^* (resp. F^*) being the conjugate function of G (resp. F), and Λ^*, the transposed operator of Λ.

Remark I.2.1. The identification of Y and its conjugate, thanks to its hilbertian structure, permits to write *(P*)* under the following form :

$$(P^*) \qquad \sup_{p\in Y} \{-G^*(-p)-F^*(\Lambda^* p)\} \qquad (I.2.8)$$

We now compute F^* and G^*.

Let $h_{\alpha,\beta}$ be the function defined by :

$$h_{\alpha,\beta}(s) \begin{cases} = s^2/2, & \text{if } s\in[-\alpha,\beta] \\ = \beta(s-\beta/2), & \text{if } s\geqslant\beta \\ = -\alpha(s+\alpha/2), & \text{if } s\leqslant-\alpha \end{cases} \qquad (I.2.9)$$

We can easily compute G^* ([E.T], p. 62) :

$$G^*(q) = \sum_{i=1}^{4} G_i^*(q_i),$$ with the following expressions for the G_i^*'s ;

$$G_1^*(q_1) = 1/2\|q_1\|^2_{L^2(\Omega)} + \int_\Omega q_1 \cdot z_d\, dx \qquad (I.2.10)$$

$$G_2^*(q_2) = 0 \qquad (I.2.11)$$

$$G_3^*(q_3) = \int_{\Gamma_0} h_{\alpha,\beta}(q_3)\, d\Gamma \qquad (I.2.12)$$

$$G_4^*(q_4) = \int_{\Gamma_0} h_{\alpha',\beta'}(q_4)\, d\Gamma \qquad (I.2.13)$$

For F^*, we have $(p \in Y)$:

$$F^*(\Lambda^* p) = \sup_{z \in W} \{\int_\Omega p_1 z\, dx + \int_\Omega p_2 . \Delta z\, dx + \int_{\Gamma_0} p_3 . z\, d\Gamma + \int_{\Gamma_1} p_4 (\partial z/\partial r). d\Gamma\} \qquad (I.2.14)$$

hence

$$F^*(\Lambda^* p) \geqslant \sup_{z \in \mathcal{D}(\Omega)} [\int_\Omega p_1 . z\, dx + \int_\Omega p_2 . \Delta z\, dx + \int_{\Gamma_0} p_3 . z\, d\Gamma + \int_{\Gamma_0} p_4 (\partial z/\partial r) d\Gamma] ,$$

$\geqslant$ (given that $z = \partial z/\partial r = 0$ on Γ_0),

$$\geqslant \begin{cases} 0 \text{ if } \Delta p_2 + p_1 = 0 \text{ in } \Omega \\ +\infty \text{ otherwise} \end{cases} \qquad (I.2.15)$$

We now assume that $\Delta p_2 + p_1 = 0$ in Ω.

p_2 is such that : $$p_2 \in L^2(\Omega), \Delta p_2 \in L^2(\Omega) \quad (I.2.16)$$

Thanks to [L1], p. 148, the following Green's formula is valid, for v in $H^2(\Omega)$:

$$\int_\Omega (\Delta p_2 . v - \Delta v . p_2) dx = \langle (\partial p_2/\partial r), v \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}$$

$$- \langle p_2, (\partial v/\partial r) \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \quad (I.2.17)$$

Thus, we can derive the following inequalities :

$$F^*(\Lambda^* p) \geqslant \sup_{v \in H^2(\Omega)} \left[\int_\Omega p_1 . v dx + \int_\Omega p_2 . \Delta v dx + \int_{\Gamma_0} p_3 . v d\Gamma + \int_{\Gamma_0} p_4 . \partial v/\partial r \, d\Gamma \right]$$

$\geqslant$ (Given (I.2.17))

$$\geqslant \sup_{v \in H^2(\Omega)} [\langle (\partial v/\partial r), p_4 + p_2 \rangle_{H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)}$$

$$+ \langle v, p_3 - (\partial p_2/\partial r) \rangle_{H^{3/2}(\Gamma_0) \times H^{-3/2}(\Gamma_0)}$$

$$+ \langle (\partial v/\partial r), p_2 \rangle_{H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)} + \langle v, (\partial p_2/\partial r) \rangle_{H^{3/2}(\Gamma_1) \times H^{-3/2}(\Gamma_1)}]$$

v, $\partial v/\partial r$ being arbitrary in $H^2(\Omega)$, we finally get :

$$F^*(\Lambda^* p) \geqslant \begin{cases} 0 \text{ if} & \Delta p_2 + p_1 = 0 \text{ in } \Omega \\ & p_2 = \partial p_2/\partial r = 0 \text{ in } \Gamma_1 \\ & p_4 + p_2 = p_3 - \partial p_2/\partial r = 0 \text{ on } \Gamma_o . \\ +\infty \text{ otherwise} & \end{cases} \tag{I.2.18}$$

p_2 satisfies the condition (I.2.16), and also :

$$\begin{aligned} & p_2 = \partial p_2/\partial r = 0 \text{ on } \Gamma_1 \\ & (p_2, \partial p_2/\partial r) \in [L^2(\Gamma_o)]^2 \end{aligned} \tag{I.2.19}$$

Let us admit (for the moment) the

Lemma I.2.1. Let (z,p) belong to $W \times W$, such that $(p, \partial p/\partial r)$ is in $H^{3/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)$.

The following Green's formula is valid :

$$\int_\Omega (\Delta p . z - \Delta z . p) dx = \int_{\Gamma_o} (z . \partial p/\partial r - \partial z/\partial r . p) d\Gamma$$

$$+ \langle \partial p/\partial r, z \rangle_{H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)} - \langle p, \partial z/\partial r \rangle_{H^{3/2}(\Gamma_1) \times H^{-3/2}(\Gamma_1)} . \tag{I.2.20}$$

The expression of F^* obviously becomes :

$$F^*(\Lambda^* p) \begin{cases} = 0 \text{ if} & \Delta p_2 + p_1 = 0 \text{ in } \Omega \\ & p_2 = \partial p_2/\partial r = 0 \text{ on } \Gamma_1 \\ & p_4 + p_2 = p_3 - \partial p_2/\partial r = 0 \text{ on } \Gamma_0 . \\ = +\infty \text{ otherwise} & \end{cases} \qquad (1.2.21)$$

and the dual problem of (1.1.3) is then :

$$\operatorname*{Sup}_{p \in Y} \{-1/2\|p_2\|^2_{L^2(\Omega)} + \int_\Omega p_1 . z_d dx - \int_{\Gamma_0} h_{\alpha,\beta}(-p_3) d\Gamma - \int_{\Gamma_0} h_{\alpha',\beta'}(-p_4) d\Gamma\} \qquad (1.2.22)$$

$\Delta p_2 + p_1 = 0$ in Ω

$p_2 = \partial p_2/\partial r = 0$ on Γ_1

$p_4 + p_2 = p_3 - \partial p_2/\partial r = 0$ on Γ_0.

We can easily eliminate p_1, p_3, p_4, and reformulate (1.2.22) as follows :

$$(P^*) \quad \operatorname*{Sup}_{q \in W} \{-1/2\|\Delta q\|^2_{L^2(\Omega)} - \int_\Omega \Delta q . z_d dx - \int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r) d\Gamma - \int_{\Gamma_0} h_{\alpha',\beta'}(q) d\Gamma\}$$

$$q = \partial q/\partial r = 0 \text{ on } \Gamma_1 \qquad (1.2.23)$$

We do not know wether there exists an adjoint state in W for the optimal state z_o : as a matter of fact, *(P*)* is not coercive on W (see the definition of $h_{\alpha,\beta}$), and we cannot state the existence of a solution of *(P*)* in W. One of our major aims is to establish the existence of generalized solutions of problem (I.2.23), and to relate them to the optimal state z_0. Right now, we prove Lemma I.2.1, then we give the first results about the relationships between *(P)* and *(P*)*.

<u>Proof of Lemma I.2.1</u> : Let (z,p) be as in the assumptions of Lemma I.2.1. We now prove the existence of two sequences $(z_n)_{n\in\mathbb{N}}$ and $(p^n)_{n\in\mathbb{N}}$ of $\mathcal{C}^\infty(\overline{\Omega}))^N$ approximating z and p respectively as follows :

$$z_n \to z \text{ in } W \tag{I.2.24}$$
$$p^n \to p \text{ in } W$$

$$(p^n, (\partial p^n/\partial r)) \to (p,(\partial p/\partial r)) \text{ in } H^{3/2}(\Gamma_1)\times H^{1/2}(\Gamma_1). \tag{I.2.25}$$

Let Φ be an element of $\mathcal{D}(\overline{\Omega})$, identically equal to 0 (resp. 1) in a neighbourhood of Γ_0 (resp. Γ_1), such that $0 \leqslant \Phi \leqslant 1$

We set

$$p = \Phi p + (1-\Phi)p = p_a + p_b$$
$$z = \Phi z + (1-\Phi)z = z_a + z_b$$

We now show that z_a, z_b, p_a, p_b are in W. For instance, let us prove this result for z_a.

First of all, z_a obviously belongs to $L^2(\Omega)$.

We also have, in $\mathcal{D}'(\Omega)$, the following equality, [Sch] ch. II :

$$\Delta z_a = \Delta z.\Phi + \Delta\Phi.z + 2\sum_{i=1}^{N} \partial\Phi/\partial x_i \;.\; \partial z/\partial x_i \tag{I.2.26}$$

It is now sufficient to prove that $\partial\Phi/\partial x_i \;.\; \partial z/\partial x_i$ belongs to $L^2(\Omega)$, for $1 \leqslant i \leqslant N$.

We know that $\partial\Phi/\partial x_i$ belongs to $\mathcal{D}(\Omega)$.

Moreover, we have ([L.M], ch. 2, § 2.3) that z is locally in $H^2(\Omega)$, that is to say :

$$\forall \Theta \in \mathcal{D}(\Omega),\ (\Theta.z) \in H^2(\Omega).$$

Writing $\partial\Phi/\partial x_i \;.\; \partial z/\partial x_i = \partial/\partial x_i(\partial\Phi/\partial x_i.z) - \partial^2\Phi/\partial x_i^{\,2}.z$, we can see immediately that $\partial\Phi/\partial x_i.\partial z/\partial x_i$ belongs to $L^2(\Omega)$. Therefore, z_a belongs to W, and so do z_b, p_a, p_b. In order to approximate p_a, we use the conditions $[p_a \in L^2(\Omega),\ \Delta p_a \in L^2(\Omega),\ p_a = \partial p_a/\partial r = 0$ on Γ_1, $(p_a, \partial p_a/\partial r) \in H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)]$ to find a sequence $(p_a^{(n)})_{n \in \mathbb{N}}$ of $(\mathcal{D}(\overline{\Omega}))^{\mathbb{N}}$ converging to p_a in $H^2(\Omega)$ [L.M], ch. II, §5).

For z_a, we use [L1], p. 148, and the condition $(\mathrm{Supp.}\ z_a) \cap \Gamma_0 = \emptyset$ to find a sequence $z_a^{(n)}$ in $\mathcal{D}(\overline{\Omega})$ such that :

$$\lim_{n\to\infty} \|z_a^{(n)}-z_a\|_{L^2(\Omega)}=\lim_{n\to\infty} \|\Delta z_a^{(n)}-\Delta z_a\|_{L^2(\Omega)}=0$$
$$z_a^{(n)}= \partial z_a^{(n)}/\partial r=0 \text{ on } \Gamma_o . \quad (I.2.27)$$

Thanks to Remark I.1.1, we have :

$$(z_a^{(n)},\partial z_a^{(n)}/\partial r)\to(z_a,\partial z_a/\partial r) \text{ in } H^{-1/2}(\Gamma_1)\times H^{-3/2}(\Gamma_1) \quad (I.2.28)$$

As for p_b, it satisfies the conditions ($p_b \in L^2(\Omega)$, $\Delta p_b \in L^2(\Omega)$, $p_b=\partial p_b/\partial r=0$ on Γ_1, $(p_b,\partial p_b/\partial r)\in(L^2(\Gamma_o))^2$). To approximate such a function, we solve the Neumann problem :

$$\Delta u=g_n \text{ in } L^2(\Omega)$$
$$u+\partial u/\partial r=h_n \text{ on } \Gamma_o \quad (I.2.29)$$
$$u+\partial u/\partial r=0 \text{ on } \Gamma_1$$

$(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ being such that :

$$g_n \in \mathcal{D}(\Omega), \text{ and } \lim_{n\to\infty} \|g_n-\Delta p_b\|_{L^2(\Omega)}=0 \quad (I.2.30)$$

$$h_n\in \mathcal{D}(\Gamma_o), \text{ and } \lim_{n\to\infty} \|h_n-(p_b+\partial p_b/\partial r)\|_{L^2(\Gamma_o)}=0 \quad (I.2.31)$$

According to [L.M], ch.II, §7, Problem (I.2.29) admits a unique solution $p_b^{(n)}$, converging to p_b in $H^{3/2}(\Omega)$, such that $\lim_{n\to\infty}\|\Delta p_b^{(n)}-\Delta p_b\|_{L^2(\Omega)}=0$.

We approximate z_b exactly as we did for p_b, and find the corresponding sequence $z_b^{(n)}$.

If we now set $z_n = z_a^{(n)} + z_b^{(n)}$, $p^n = p_a^{(n)} + p_b^{(n)}$, we have obtained two sequences fulfilling the conditions (1.2.24) and (1.2.25); the Green's formula (1.2.20) for z and p now results from a straightforward passage to the limit in the Green's formula for z_n and p^n, and Lemma 1.2.1 is proved.

1.3. Comparison of the extremas of *(P)* and *(P*)*.

We now prove the following result :

Theorem 1.3.1. *(P)* and *(P*)* have the same extrema

$$-\infty < \text{Sup } (1.2.23) = \text{Inf } (1.1.3). \qquad (1.3.1)$$

Proof : For p in Y, $p = (p_i)_{1 \leq i \leq 4}$, we consider the penalized problem :

$$(P_p) \qquad \inf_{z \in W} \{1/2\|p_1 - z + z_d\|^2_{L^2(\Omega)} + 1/2\|z - p_3\|^2_{L^2(\Gamma_o)} + 1/2\|\partial z/\partial r - p_4\|^2_{L^2(\Gamma_o)}\} \qquad (1.3.2)$$

$$\Delta z = p_2 \text{ in } \Omega$$

$$(z - p_3, \partial z/\partial r - p_4) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'}$$

We set $\Phi(p) = \text{Inf } (P_p)$.

We already know that $\Phi(0) < +\infty$; using the method of [E.T], we now show that Φ is ℓsc at zero, which will prove Theorem I.3.1.

Thanks to [E.T], p.34, we have :

$$\forall p \in Y,\ \exists! z_p \in W,\ \Phi(p) = \mathrm{Inf}\ (\mathcal{P}_p) = F(z_p + G(\Lambda z_p - p) \qquad (I.3.3)$$

Let us suppose : $$\varliminf_{p \to 0} \Phi(p) < \Phi(0) \qquad (I.3.4)$$

We then have :

$$\exists (p_n)_{n \in \mathbf{N}} \in Y^{\mathbf{N}},\ \lim_{n \to \infty} (p_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \Phi(p_n) = \lambda < \Phi(0) \qquad (I.3.5)$$

We consider the sequence $(z_n)_{n \in \mathbf{N}}$ such that $\Phi(p_n) = F(z_n) + G(\Lambda z_n - p_n)$. Given that p_n tends to zero, z_n is bounded in W. We can therefore choose a subsequence, still denoted by z_n, converging to an element y of W for the weak topology of W. As p_n tends to zero in Y,

y satisfies the conditions
$$\begin{cases} y \in W \\ \Delta y = 0 \text{ in } \Omega \qquad (I.3.6) \\ (y, (\partial y/\partial r)) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'} \\ (\text{for } \mathcal{U}_{\alpha,\beta}, \text{ being convex, is weakly closed}). \end{cases}$$

Therefore we have :

$$j(y) < +\infty \qquad (I.3.7)$$

$$1/2(\|y - z_d\|^2_{L^2(\Omega)} + \|y\|^2_{L^2(\Gamma_0)} + \|\partial y/\partial r\|^2_{L^2(\Gamma_0)})$$
$$\leq \varliminf_{n \to \infty} (F(z_n) + G(\Lambda z_n - p_n)) < \Phi(0) \qquad (I.3.8)$$

but (1.3.7) and (1.3.8) are contradictory, because of the definition of Φ. We have thus proved that Φ is ℓsc at zero, and Theorem 1.3.1 is valid.

II. Approximation and regularization on a riemannian manifold.

1. Some classical results.

We recall the following results, for which we refer to [B.G.M], [MK.S], [D], [Min]. Let M be a compact connected ℓ-dimensional riemannian manifold, indefinitely differentiable. The canonical measure on M is denoted by $d\lambda$. If M is a $(N-1)$-dimensional submanifold of R^N, then $d\lambda$ is the Lebesgue measure on M.

One can define ([B.G.M], p.126 to 144) the Laplacean Δ_M ; Δ_M is a second order differential operator, self-adjoint with respect to the hilbertian structure of $L^2(M;d\lambda)$. Its spectrum is a sequence

$$0 = \omega_0 < \omega_1 < \ldots < \omega_n \underset{n\to\infty}{\to} +\infty$$

of positive real numbers, associated with a family $\{X_n\}_{n\in\mathbb{N}}$ of eigenfunctions, orthonormal in $L^2(M;d\lambda)$ and dense in that space.

Following [B.G.M], [Min], one can construct a fundamental solution $e(x,y,t)$ of the heat equation $(\partial/\partial t + \Delta_M = 0)$ on M.

$e(x,y,t)$ satisfies the following conditions [MK.S], [Min]:

$$e(x,y,t) \in \mathcal{C}^\infty (M \times M \times]0, +\infty[) \qquad (II.1.1)$$

$$e(x,y,t) = \sum_{n=0}^{\infty} e^{-\omega_n t} X_n(x) X_n(y) \qquad (II.1.2)$$

$$e(x,y,t) > 0 \;;\; e(x,y,t) = e(y,x,t), \quad (x,y,t) \in M^2 \times]0, +\infty[\qquad (II.1.3)$$

$$\int_M e(x,y,t)d\lambda(x) = \int_M e(x,y,t)d\lambda(y) = 1 \qquad \text{(II.1.4)}$$

for $f \in \mathcal{C}(M) : \lim_{t \to 0} (\sup_{x \in M} |\int_M e(x,y,t)\, f(y)d\lambda(y) - f(x)| = 0$ (II.1.5)

2. Approximation and regularization of a function defined on M.

Let f be an element of $L^p(M;d\lambda)$, $1 \leqslant p \leqslant +\infty$; we set

$$\forall x \in M\ ,\quad f_t(x) = \int_M e(x,y,t)\, f(y)dy \qquad \text{(II.2.1)}$$

Thanks to (II.1.1), f_t belong to $\mathcal{C}^\infty(M)$.

We have the following results, whose proofs are given for the commodity of the reader :

Proposition II.2.1. if $1 \leqslant p < +\infty$, then $\lim_{t \to 0} \|f - f_t\|_{L^p(M;d\lambda)} = 0$

Proposition II.2.2. if $p = +\infty$, then $(f_t)_{t>0}$ is such that :

$$\|f_t\|_{L^\infty(M;d\lambda)} \leqslant \|f\|_{L^\infty(M;d\lambda)} \qquad \text{(II.2.i)}$$

$f_t \to f$ for the weak $_*$ topology of $L^\infty(M;d\lambda)$ (II.2.ii)

Proof of II.2.1 :Let p be a real number, $1 \leqslant p < +\infty$.

We first recall that $\mathcal{C}(M)$ is dense in $L^p(M;d\lambda)$ for the norm topology.

If we suppose $1<p<+\infty$, we have the following inequalities :

$$|f_t(x)|=|\int_M e(x,y,t)\, f(y)d\lambda(y)|$$

$\leqslant$ (by Hölder's inequality)

$$\leqslant |\int_M |e(x,y,t)|d\lambda(y)|^{1-1/p} |\int_M |e(x,y,t)\, f(y)|^p d\lambda(y)|^{1/p}$$

$\leqslant$ (Thanks to (II.1.3) and (II.1.4))

$$\leqslant \{\int_M e(x,y,t)|f(y)|^p d\lambda(y)\}^{1/p}$$

hence : $$\int_M |f_t(x)|^p\, d\lambda(x) \leqslant \int_M \{\int_M e(x,y,t)\, |f(y)|^p d\lambda(y)\} d\lambda(x)\,.$$

Using Fubini's Theorem, and (II.1.4), we get :

$$\|f_t\|_{L^p(M;d\lambda)} \leqslant \|f\|_{L^p(M\,;\,d\lambda)} \tag{II.2.2}$$

Let us now take $\varepsilon>0$; we first choose f' in $\mathcal{C}(M)$, such that $\|f-f'\|_{L^p(M;d\lambda)}<\varepsilon/3$. The function f_t' is defined as in (II.2.1).

We may write :

$$\|f-f_t\|_{L^p(M;d\lambda)} \leqslant \|f_t-f'_t\|_{L^p(M;d\lambda)}+\|f_t'-f'\|_{L^p(M;d\lambda)}+\|f'-f\|_{L^p(M;d\lambda)}$$

$$\leqslant \quad J_1 \quad + \quad J_2 \quad + \quad J_3$$

Using (II.2.2), we have $0 \leqslant J_1 \leqslant J_3 \leqslant \varepsilon/3$. Thanks to (II.1.5), we can choose t_0 such that $J_2 < \varepsilon/3$ for $t < t_0$ (remember that M is compact) and this ends the proof when $1 < p < +\infty$.

The inequality (II.2.2) being obvious if $p = 1$, Proposition II.2.1 is proved.

Proof of II.2.2 : We first notice that (II.2.i) is a trivial consequence of (II.1.3) and (II.1.4).
To prove (II.2.ii), we have to show the following result :
for $f \in L^{\infty}(M;d\lambda)$, and $g \in L^1(M;d\lambda)$,

$$\lim_{t \to 0} \int_M g(f_t - f)\,d\lambda = 0 \tag{II.2.4}$$

We have :

$$\begin{aligned}
\int_M g(x)[f_t(x) - f(x)]d\lambda(x) &= \int_M [\int_M e(x,y,t)f(y)d\lambda(y) - f(x)]g(x)d\lambda(x) \\
&= \int_M g(x)\{\int_M e(x,y,t)f(y)d\lambda(y)\}d\lambda(x) - \int_M g(x)f(x)d\lambda(x) \\
&= \text{(using Fubini's Theorem)} \\
&= \int_M f(y)[\int_M e(x,y,t)g(x)d\lambda(x) - g(y)]d\lambda(y) \\
&= \text{(given (II.1.3))} \\
&= \int_M f(x)[g_t(x) - g(x)]d\lambda(x)\ .
\end{aligned}$$

Thanks to Proposition II.2.1, we have

$$\lim_{t \to 0} \|g_t - g\|_{L^1(M;d\lambda)} = 0 \qquad (II.2.5)$$

and Proposition II.2.2 is proved.

Remark II.2.1 : for $1 \leq p \leq +\infty$ we have :

$$\lim_{t \to 0} f_t(x) = f(x) \quad d\lambda \text{ - almost everywhere on } M.$$

3. Approximation and regularization of a bounded measure on M .

Let $M_1(M)$ be the space of bounded measures on $M[B]$, and μ an element of $M_1(M)$. We define a regularization μ_t of μ :

$$\forall x \in M \ , \ \mu_t(x) = \int_M e(x,y,t)\, d\mu(y) \qquad (II.3.1)$$

We know, as before, that μ_t is indefinitely differentiable on M.

We now prove the

Proposition II.3. Let μ be in $M_1(M)$; the family $(\mu_t)_{t>0}$ defined by (II.3.1) converges to μ for the vague topology of $M_1(M)$. Moreover, we have the following results :

$$\forall t \in]0,+\infty[, \quad \int_M \mu_t(x) d\lambda(x) = \int_M \mu \qquad (II.3.i)$$

if μ is a positive measure, then so is μ_t for every t in $]0,+\infty[$

(II.3.ii)

Proof of II.3.1 :

(II.3.i) and (II.3.ii) are obvious, thanks to (II.1.3) and (II.1.4).

Let f be in $\mathcal{C}(M)$; we have

$$|\int_M f(x)\mu_t(x)d\lambda(x) - \int_M f(x)d\mu(x)| = |\int_M f(x)\{\int_M e(x,y,t)d\mu(y)\}d\lambda(x) - \int_M f(x)d\mu(x)|$$

$$= \text{(using Fubini's Theorem)}$$

$$= |\int_M \{\int_M e(x,y,t)f(x)d\lambda(x) - f(y)\}d\mu(y)|$$

$$\leq \|\mu\|_{M_1(M)} \sup_{y\in M} |\int_M e(x,y,t)f(x)d\lambda(x) - f(y)| .$$

and Proposition II.3.1 is now a straighforward consequence of (II.1.5).

4. Convex function of a bounded measure on M.

Let h be a convex function of one real variable ; its conjugate h^* is defined by

$$\forall s\in R,\ \ h^*(s) = \sup_{t\in R}\ [st - h(t)] \qquad \text{(II.4.1)}$$

Let $K = \{s\in R,\ h^*(s) < +\infty\}$ be the domain of h^* .

We shall make, from now on, the following assumptions on h :

$$\exists C>0\ ,\quad |h(s)|\leqslant C(1+|s|) \tag{II.4.2}$$

$$\exists C>0,\ \ \forall\ s\in K,\ \ |h^*(s)|\leqslant C \tag{II.4.3}$$

Last, we define the "principal part" h_∞ of h as follows :

$$\forall s\in R,\ \ h_\infty(s)=\operatorname*{Sup}_{t\in K}\,(s.t) = \lim_{t\to+\infty}\ h(st)/t \tag{II.4.4}$$

Let μ be an element of $M_1(M)$. μ admits a decomposition with respect to the measure $d\lambda$ of the following form :

$$\mu=gd\lambda+\mu^{s} \tag{II.4.5}$$

where g is locally summable on M, and μ^s is singular.

We denote by Θ_s the function defined by $\mu^s=\Theta_s|\mu^s|$ ([B]) .
We can now define the measure $h(\mu)$, [D.T (1), (2)], [T] :

$$h(\mu)=(hog)d\lambda+(h_\infty o\Theta_s)|\mu^s| \tag{II.46}$$

We recall the following results [D.T (1), (2)], [T] :

(II.4.i) $h(\mu)$ is a bounded measure, absolutely continuous with respect to μ ; moreover, the application $\mu\to h(\mu)$ is continuous for the strong topology of $M_1(M)$.

(II.4.ii) $h(\mu)$ has the fundamental property :

$$\forall f \in \mathcal{C}(M),\quad \langle h(\mu),f\rangle = \underset{\substack{g \in \mathcal{C}^{\infty}(M) \\ h^*(g)\in L^1(M;d\lambda)}}{\mathrm{Sup}} \left\{\int g.f.d\mu - \int_M h^*(g).f d\lambda\right\}$$

(II.4.iii) If $\mu_n \to \mu$ vaguely, and $h(\mu_n) \to v$ vaguely, then $h(\mu) \leqslant v$.

If furthermore h is positive and $\lim\limits_{n\to\infty} \int_M h(\mu_n) d\lambda \leqslant \int_M h(\mu)$,

then $h(\mu) = v$.

<u>Remark II.4.1</u>. In (II.4.7), the supremum may be taken indifferently in $\mathcal{C}^{\infty}(M)$ or $\mathcal{C}(M)$.

We now assume that h is positive, and prove the following result :

<u>Theorem II.4.1</u>. Let h be a convex function satisfying the conditions (II.4.2), (II.4.3), such that h is positive. For any measure μ in $M_1(M)$, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $(\mathcal{C}^{\infty}(M))^{\mathbb{N}}$, such that $(\mu_n, h(\mu_n))$ tends to $(\mu, h(\mu))$ for the vague topology of $M_1(M)\times M_1(M)$.

Proof : We show that the family $(\mu_t)_{t>0}$ defined by (II.3.1) satisfies the required conditions.

Let g be in $\mathcal{C}^{\infty}(M)$, such that $h^*(g)$ belongs to $L^1(M;d\lambda)$.

We have :

$$\int_M g(x)\mu_t(x)d\lambda(x) - \int_M h^*(g)d\lambda = \int_M \left[\int_M e(x,y,t)d\mu(y) \right] g(x)d\lambda(x)$$

$$- \int_M h^*(g)d\lambda \qquad \text{(II.4.8)}$$

$$= \text{(using Fubini's Theorem)}$$

$$= \int_M \left[\int_M e(x,y,t)g(y)d\lambda(y) \right] d\mu(x) - \int_M h^*(g)d\lambda$$

with (II.1.4) :

$$h^*\left(\int_M e(x,y,t)g(y)d\lambda(y)\right) = \operatorname*{Sup}_{\gamma\in R} \int_M [\gamma g(y) - h(\gamma)] e(x,y,t)d\lambda(y)$$

$$\leqslant \text{(Thanks to (II.1.3))}$$

$$\leqslant \int_M h^*(g)(y)e(x,y,t)d\lambda(y)$$

and therefore

$$\int_M h^*\left(\int_M e(x,y,t)g(y)d\lambda(y)\right)d\lambda(x) \leqslant \int_M d\lambda(x) \left(\int_M h^*(g)(y)e(x,y,t)d\lambda(y)\right)$$

$$\leqslant \text{(with (II.1.4))}$$

$$\leqslant \int_M h^*(g)d\lambda$$

hence (II.4.8) implies :

$$\int_M h^*(x)\mu_t(x)d\lambda(x) - \int_M h^*(g)d\lambda \leqslant \int_M \left(\int_M e(x,y,t)g(y)d\lambda(y)\right)d\mu(x) -$$

$$- \int_M h^*\left(\int_M e(x,y,t)g(y)d\lambda(y)\right)d\lambda$$

and we have :

$$\int_M g\mu_t d\lambda - \int_M h^*(g)d\lambda \leq \underset{\substack{g' \in \mathcal{C}^\infty(M) \\ h^*(g') \in L^1(M,d\lambda)}}{\text{Sup}} \left\{ \int_M g' d\mu - \int_M h^*(g')d\lambda \right\}$$

$$\leq \int_M h(\mu) .$$

As we have $\lim\limits_{t \to \infty} \int_M h(\mu_t) \leq \int_M h(\mu)$, we can extract a sequence $(\mu_{(n)})_{n \in \mathbb{N}}$

satisfying the required conditions, thanks to (II.4.iii) .
In fact, a contradiction argument shows that the whole family $(\mu_t)_{t>0}$ tends to μ in the sense of Theorem II.4.1.

Corollary II.4.2. The families $(\mu_t)_{t>0}$ and $(h(\mu_t))_{t>0}$ tends respectively towards μ and $h(\mu)$ in the sense of the tight convergence of measures.

Proof : The tight and vague convergences are equivalent, on a compact space, for positive measures.

III. Integral representations for the solutions of a Neuman problem with irregular boundary data.

1. The classical case.

We are interested in the study of the following Neumann problem :

Let f,g be two functions respectively defined on Ω and Γ ;

$$\text{to find } u \text{ solution of: } \begin{cases} \Delta u = f \text{ in } \Omega \\ u + \partial u/\partial r = g \text{ on } \Gamma \end{cases} \qquad \text{(III.1.1)}$$

We refer to $[M_1]$, ch. 1 to 3, for the following definitions and results.
Let $F(x,y)$ be the Green function of Problem (II.1.1) ; F is defined in $\overline{\Omega}\times\overline{\Omega}/_{\{(x,x),x\in\overline{\Omega}\}}$, indefinitely differentiable for $x\neq y$ and symmetrical in (x,y). We set $H(x,y)=(1/(N-2)\sigma_N)\,(x-y)^{2-N}$, σ_N being the volume of the unit sphere in R^N.

F is such that :

$$\forall\,(x,y)\in\overline{\Omega}\times\overline{\Omega},\ x\neq y \qquad \begin{cases} F(x,y)-H(x,y)=0\,(\,|x-y|^{3-N}) \\ \partial/\partial x_i[F-\ H](x,y)=0(|x-y|^{2-N}) \\ \partial^2/\partial x_i\partial x_R[F-H](x,y)=0(|x-y|^{\,1-N}) \end{cases} \qquad \text{(III.1.2)}$$

(with the notations of Landau).

When f and g are summable respectively on Ω and Γ, the following result is valid :

<u>Proposition III.1.1</u>. Problem (III.1.1) admits a unique solution u given by :

$$\forall x\in\Omega,\quad u(x)=-\int_\Omega F(x,y)f(y)dy+\int_\Gamma F(x,y)g(y)dy \qquad \text{(III.1.3)}$$

If (f,g) is in $\mathcal{C}^\infty(\overline{\Omega})\times\mathcal{C}^\infty(\Gamma)$, we have [$M_2$] :

<u>Proposition III.1.2</u>. The function u defined by (III.1.3) belongs to $\mathcal{C}^\infty(\overline{\Omega})$.

Last, we recall two theorems of Sobolev's ([M_2], Th. 12.VII, [S]p.25-30)

<u>Proposition III.1.3</u>. The operator $\lambda_1 : L^r(\Omega)\to\mathcal{C}(\overline{\Omega})$

$$f\to\int_\Omega F(.,y)f(y)dy$$

is compact for $r>N/2$.

<u>Proposition III.1.4</u>. The operator λ_2: $L^q(\Gamma)\to\mathcal{C}(\overline{\Omega})$

$$g\to\int_\Gamma F(.,y)g(y)dy$$

is compact, for $q>(N-1)$.

2. Neumann problem with data in $M_1(\Gamma)$.

We now wish to study the following problem :

Let (f,μ) belong to $L^2(\Omega)\times M_1(\Gamma)$

$$\text{to find } u \text{ solution of: } \quad \begin{cases} \Delta u = f \text{ in } \Omega \\ u + \partial u/\partial r = \mu \text{ in } M_1(\Gamma) \end{cases} \qquad \text{(III.2.1)}$$

Let us define the function v by :

$$\forall x\in\Omega, \quad v(x) = -\int_\Omega F(x,y)f(y)dy + \int_\Gamma F(x,y)d\mu(y) \qquad \text{(III.2.2)}$$

v is obviously well defined in Ω;we shall prove that v is actually the unique solution of the problem (III.2.1).

We shall study separately the domain potential :

$$v_1 = -\int_\Omega F(.,y)\ f(y)dy \qquad \text{(III.2.3)}$$

and the single layer potential

$$v_2 = \int_\Gamma F(.,y)d\mu(y) \qquad \text{(III.2.4)}$$

3. The domain potential.

Using [M_1], Th. 12.VIII and Th. 13.III, we know that v_1 belongs to $H^2(\Omega)$. Furthermore, v_1 is the solution of the homogeneous Neumann problem,

$$\Delta u = f \text{ in } \Omega$$
$$u + \partial u/\partial r = 0 \text{ on } \Gamma. \qquad \text{(III.3.1)}$$

The condition on Γ is true almost everywhere on Γ (actually, cf [L.M], ch II in $H^{1/2}(\Gamma)$).

4. The single layer potential.

We first point out the fact that v_2 is indefinitely differentiable on Ω, and satisfies the condition :

$$\Delta v_2 = 0 \text{ in } \Omega \qquad \text{(III.4.1)}$$

We now prove the following propositions :

Proposition III.4.1. The operator $\lambda_1' : M_1(\Gamma) \to L^s(\Omega)$
$$\mu \to \int_\Omega F(.,y)\,d\mu(y)$$
is compact for $1 \leq s < N/N-2$.

Proposition III.4.2. The operator $\lambda_2' : M_1(\Gamma) \to L^t(\Gamma)$
$$\mu \to \int_\Gamma F(.,y)\,d\mu(y)$$
[that is, the value on Γ of the function, defined on Ω, $\int_\Gamma F(.,y)\,d\mu(y)$]
is compact for $1 \leq t < N-1\ /N-2$.

Proposition III.4.3. The operator $\lambda_3' : M_1(\Gamma) \to M_1(\Gamma)$
$$\mu \to \partial/\partial r \left[\int_\Gamma F(.,y)\,d\mu(y) \right]$$

[that is, the normal derivative, in the distributional sense, of $\int_\Gamma F(.,y)d\mu(y)$]
is continuous, and the following equality is valid :

$$(\lambda_2'+\lambda_3')\mu = \mu \quad \text{in } M_1(\Gamma) \tag{III.4.2}$$

The equalities (III.4.1) and (III.4.2) show that v_2 is the solution of the Neumann problem :

$$\begin{aligned} &\Delta u=0 \quad \text{in } \Omega \\ &u+\partial u/\partial r=\mu \quad \text{in } M_1(\Gamma) \end{aligned} \tag{III.4.3}$$

Proof of Proposition III.4.1 :

Let $(s;r)$ be a pair of conjugate exponents $(1/r+1/s=1)$, such that $1 \leqslant s < N/N-2$.

For $g \in L^r(\Omega)$ ($r > N/2$ in this case), we have :

$$\int_\Omega v_2(x)g(x)dx = \int_\Omega g(x)[\int_\Gamma F(x,y)d\mu(y)]dx \tag{III.4.4}$$

Proposition III.1.3 allows us to apply Fubini's theorem, and we get :

$$\begin{aligned} |\int_\Omega v(x)g(x)dx| &= |\int_\Gamma [\int_\Omega g(x)F(x,y)dx]d\mu(y)| \\ &\leqslant C\|\mu\|_{M_1(\Gamma)}\|g\|_{L^r()} \end{aligned}$$

A standard argument of functional analysis gives the desired conclusion, taking into account the fact that the adjoint of a compact operator is itself compact, and Proposition III.4.1 is proved.

<u>Remark III.4.1.</u> We can prove similarly that the partial derivatives $\partial v_2/\partial x_i (1 \leq i \leq N)$ belong to $L^{\delta}(\Omega)$, for $1 \leq \delta < N/N-1$.

Proof of Proposition III.4.2 and III.4.3 :

We first remark that the boundary values $v_{2|\Gamma}$ and $\partial v_2/\partial r_{|\Gamma}$ are defined in the distributional sense on Γ. As a matter of fact, v_2 belongs to $L^s(\Omega)$, for $1 \leq s < N/N-2$, and satisfies : $\Delta v_2 = 0$ in Ω. If we choose K such that $L^s(\Omega)$ is imbedded in $H^{-K}(\Omega)$ we can see that $(v_2, \partial v_2/\partial r)$ belongs to $H^{-K-1/2}(\Gamma) \times H^{-K-3/2}(\Gamma)$ ([L.M], ch. II, Th. 6.5).

Let us now choose a sequence μ_n of smooth functions tightly converging to μ in $M_1(\Gamma)$, and set :

$$f_n = \int_\Gamma F(.,y)\, \mu_n(y) dy \qquad \text{(III.4.5)}$$

We know that f_n converges pointwise to v_2 in Ω, and that the family $(f_n)_{n \in \mathbb{N}}$ is relatively compact in $L^s(\Omega)$. Hence there exists a subsequence, still denoted by f_n, converging to v_2 in $L^s(\Omega)$ for the norm topology ; according to [L.M], ch. II, the boundary values $f_n, \partial f_n/\partial r$ converge respectively to v_2, $\partial v_2/\partial r$ in $\mathcal{D}'(\Gamma)$. The sequence μ_n being bounded in $L^1(\Gamma)$, that proves (thanks to Sobolev's theorems, [M1] Th. 12.VII) that v_2, $\partial v_2/\partial r$ belong respectively to $L^t(\Omega)$, for $1 \leq t < N-1/N-2$, and $M_1(\Gamma)$, and that the operators λ'_2, λ'_3 are continuous.

Moreover, the sequence f_n solves the Neumann problem :

$$\Delta f_n = 0 \text{ in}$$
$$f_n + \partial f_n / \partial r = \mu_n \text{ on } \Gamma \qquad \text{(III.4.6)}$$

which shows that v_2 satisfies :

$$\Delta v_2 = 0 \text{ in } \Omega$$
$$(\lambda'_2 + \lambda'_3) v = \mu \text{ on } \Gamma \qquad \text{(III.4.7)}$$

The compactness of λ'_2 is a straightforward consequence of the fact that λ'_2 is the adjoint operator of λ_2 (defined in Prop. III.1.4), which is itself a compact operator.

IV. A function space adapted to ($\mathcal{P}^*$).

1. Definition of BT(Ω).

Let $(q_n)_{n\in\mathbb{N}}$ be a maximizing sequence of Problem (I.2.23) ; its satisfies the following conditions :

Δq_n is bounded in $L^2(\Omega)$ (IV.1.1)

$q_n=\partial q_n/\partial r=0$ on Γ_1 (IV.1.2)

$(q_n, \partial q_n/\partial r)$ is bounded in $L^1(\Gamma_0)$ (IV.1.3)

Hence, it is natural to widen the class of admissible elements for Problem (I.2.23) so as to include functions having their Laplacean in $L^2(\Omega)$, and whose boundary value together with their normal derivative are bounded measures on Γ_0. As for the boundary value and normal derivative on Γ_1, we shall require that they should be sufficiently regular to allow a Green's formula to hold with any admissible state for Problem ($\mathcal{P}$).

We consider the operator $\mathcal{A}$, whose domain is $L^2(\Omega)\times H^{1/2}(\Gamma_1)\times M_1(\Gamma_0)$, defined by :

for $(f,g,\mu)\in L^2(\Omega)\times H^{1/2}(\Gamma_1)\times M_1(\Gamma_0)$,

$$v=\mathcal{A}(f,g,\mu)=-\int_{\Omega}F(.,y)f(y)dy+\int_{\Gamma_1}F(.,y)g(y)dy+\int_{\Gamma_0}F(.,y)d(y) \qquad \text{(IV.1.4)}$$

According to Section III, v is the solution of the boundary value problem :

$$\Delta u = f \text{ in } \Omega$$
$$u + \partial u/\partial r = g \text{ on } \Gamma_1 \qquad \text{(IV.1.5)}$$
$$u + \partial u/\partial r = \mu \text{ on } \Gamma_0$$

the equality "$u+\partial u/\partial r=\mu$" being understood in the sense of the distributions on Γ_0.

$\mathcal{A}$ is a one-to-one linear operator, and we define the space $BT(\Omega)$ (BT stands for "bounded trace") as the range of $\mathcal{A}$. $BT(\Omega)$ is a Banach space, when endowed with the graph norm :

$$\|u\|_{BT(\Omega)} = \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma_1)} + \|\mu\|_{M_1(\Gamma_1)}, \text{ for } u = \mathcal{A}(f,g,\mu).$$

We shall now prove the following propositions :

<u>Proposition IV.1.1</u>. The space $\mathcal{C}^\infty(\overline{\Omega})$ is imbedded in $BT(\Omega)$.

<u>Proposition IV.1.2</u> Let s be a real number, $1 \leqslant s < N/N-2$.

The embedding of $BT(\Omega)$ into $L^s(\Omega)$ is compact

<u>Proposition IV.1.3</u>. The operator $\gamma_0{}^0$ (resp. $\gamma_0{}^1$), which assigns to a function in $\mathcal{C}^\infty(\overline{\Omega})$ its boundary value on Γ_0 (resp. Γ_1) can be extended to a compact (resp. continuous) operator from $BT(\Omega)$ to $L^\tau(\Gamma_0)$, for $1 \leqslant \tau < N-1 / N-2$ (resp. $H^{3/2}(\Gamma_1)$).

<u>Proposition IV.1.4</u>. The operator $\gamma_1^{\,o}$ (resp. $\gamma_1^{\,1}$), which assigns to a function in $\mathcal{C}^\infty(\overline{\Omega})$ its normal derivative on Γ_0 (resp. Γ_1) can be extended to a continuous operator from $BT(\Omega)$ to $M_1(\Gamma_0)$ (resp. $H^{1/2}(\Gamma_1)$).

Proof of Propositions IV.1.1 to IV.1.4:

For (f,g,μ) in $L^2(\Omega)\times H^{1/2}(\Gamma_1)\times M_1(\Gamma_0)$, let u be equal to $\mathcal{A}(f,g,\mu)$.

We set : $u=u_1+u_2$.

Where
$$u_1=-\int_\Omega F(.,y)f(y)dy+\int_{\Gamma_1} F(.,y)g(y)dy \qquad \text{(IV.1.6)}$$

$$u_2=\int_{\Gamma_0} F(.,y)d(y) \qquad \text{(IV.1.7)}$$

We first remark that u_1 solves the Neumann problem

$$\begin{aligned} &\Delta v=f \text{ in } \Omega\\ &v+\partial v/\partial r=0 \text{ on } \Gamma_0\\ &v+\partial v/\partial r=g \text{ on } \Gamma_1, \end{aligned} \qquad \text{(IV.1.8)}$$

and therefore ([L.M], ch.II) belongs to $H^2(\Omega)$, which is obviously compactly imbedded in $L^s(\Omega)$ for $1\leqslant s<N/N-2$. Similarly, the trace (resp. normal derivative) of u_1 on Γ belongs to $H^{3/2}(\Gamma)$ (resp. $H^{1/2}(\Gamma)$), and obviously $H^{3/2}(\Gamma)$ is compactly imbedded in $L^\tau(\Gamma)$, for $1\leqslant\tau<N-1/N-2$, while

$H^{1/2}(\Gamma)$ is embedded in $M_1(\Gamma)$.
Furthermore, one can see that, given an open set A such that $\bar{A}$ does not intersect Γ_0, u_2 belongs to $\mathcal{C}^\infty(\bar{A})$. The proof of Propositions IV.1.1 to IV.1.4 is now over, thanks to these remarks, and using Propositions III.1.2, III.4.1, III.4.2, III.4.3.

2. Approximation of a function in BT(Ω) by smooth functions.

We know that $\mathcal{C}^\infty(\bar{\Omega})$ is not dense in $BT(\Omega)$ for the norm topology, for $\mathcal{C}^\infty(\Gamma)$ is not dense in $M_1(\Gamma)$.
Nevertheless, the following result is valid :

Theorem IV.2.1. Given u in $BT(\Omega)$ and two numbers $(s,\tau), 1\leqslant s<N/N-2$, $1\leqslant\tau<(N-1)/N-2$, there exists a sequence $(\Phi_n)\ n\in\mathbb{N}$ in $(\mathcal{C}^\infty(\bar{\Omega}))^{\mathbb{N}}$ such that :

$$\Phi_n \to u \text{ in } L^s(\Omega) \qquad \text{(IV.2..i)}$$

$$\Delta\Phi_n \to \Delta u \text{ in } L^2(\Omega) \qquad \text{(IV.2.ii)}$$

$$(\gamma_0{}^0\Phi_n, \gamma_0{}^1\Phi_n) \to (\gamma_0{}^0 u, \gamma_0{}^1 u) \text{ in } L^\tau(\Gamma_0)\times H^{3/2}(\Gamma_1) \qquad \text{(IV.2.iii)}$$

$$\gamma_1{}^0\Phi_n \to \gamma_1{}^0 u \text{ vaguely in } M_1(\Gamma_0). \qquad \text{(IV.2.iv)}$$

$$\gamma_1{}^1\Phi_n \to \gamma_1{}^1 u \text{ in } H^{1/2}(\Gamma_1). \qquad \text{(IV.2.v)}$$

Proof : We set $u=\mathcal{A}(f,g,\mu)$, $(f,g,\mu)\in L^2(\Omega)\times H^{1/2}(\Gamma_1)\times M_1(\Gamma_0)$.
We consider a sequence f_n (resp. g_n) of functions in $\mathcal{C}^\infty(\overline{\Omega})$ (resp. $\mathcal{C}^\infty(\Gamma_1)$). which tends to f (resp. g) in $L^2(\Omega)$ (resp. $H^{1/2}(\Gamma_1)$).
We now choose $(t_n)_{n\in\mathbb{N}}$, a sequence of positive real number tending to zero, and we set $\mu^{(n)}=\mu_{t_n}$, where μ_{t_n} is defined by (II.3.1).
The approximation sequence Φ_n is then :

$$\Phi_n=-\int_\Omega F(.,y)f_n(y)dy+\int_{\Gamma_1}F(.,y)g_n(y)dy+\int_{\Gamma_0}F(.,y)\mu^{(n)}(y)dy \quad .(IV.2.1)$$

Thanks to Proposition III.1.2, we know that Φ_n belongs to $\mathcal{C}^\infty(\overline{\Omega})$.
Let us now split Φ_n into

$$\Phi_n^{(1)}=-\int_\Omega F(.,y)f_n(y)dy+\int_{\Gamma_1}F(.,y)g_n(y)dy \qquad (IV.2.2)$$

and

$$\Phi_n^{(2)}=\int_{\Gamma_0}F(.,y)\mu^{(n)}(y)dy \qquad (IV.2.3)$$

We already know that $\Phi_n^{(1)}$ converges towards u_1 (defined by (IV.1.6)) in $H^2(\Omega)$. Thanks to Propositions IV.2.2 and IV.2.3, we can extract from $\Phi_n^{(2)}$ a subsequence, still denoted by $\Phi_n^{(2)}$, satisfying the following conditions :

$$\Phi_n^{(2)}\to u_2 \text{ in } L^s(\Omega) \qquad (IV.2.4)$$

$$\gamma_0{}^0\Phi_n^{(2)}\to\gamma_0{}^0u_2 \text{ in } L^\tau(\Gamma_0) \qquad (IV.2.5)$$

$$\gamma_1{}^{o}\Phi_n{}^{(1)} \to \gamma_1{}^{o}u_2 \text{ in } M_1(\Gamma_0) \text{ vaguely} \tag{IV.2.6}$$

(u_2 is defined by (IV.1.7)).

Only two statements are now left to prove, namely :

$$\begin{aligned} &\gamma_0{}^{1}\Phi_n{}^{(2)} \to \gamma_0{}^{1}u_2 \text{ in } H^{3/2}(\Gamma_1) \\ &\gamma_1{}^{1}\Phi_n{}^{(2)} \to \gamma_1{}^{1}u_2 \text{ in } H^{1/2}(\Gamma_1) \end{aligned} \tag{IV.2.7}$$

Let us admit (for the moment) the following result :

Lemma IV.2.2 : Let A be an open set in Ω, such that $\bar{A} \cap \Gamma_0 = \emptyset$. The family $(\Phi_n{}^{(2)})_{n \in \mathbb{N}}$ is equicontinuous in $\bar{A}$, and so is any of the families of its successive derivatives.

We can now extract another subsequence, still denoted by $\Phi_n{}^{(2)}$, from $\Phi_n{}^{(2)}$ which tends to u_2 for the topology of $\mathcal{C}^2(\bar{A})$, and a fortiori in $H^2(A)$, for Ω is bounded.

The sequence $\Phi_n = \Phi_n{}^{(1)} + \Phi_n{}^{(2)}$ satisfies the required conditions, and Theorem IV.2.1 is proved.

Proof of Lemma IV.2.2

Let (x,x') be in $\bar{A} \times \bar{A}$;

we have :

$$|\Phi_n^{(2)}(x) - \Phi_n^{(2)}(x')| = \left| \int_{\Gamma_0} (F(x,y) - F(x',y))\, \mu^{(n)}(y)dy \right|$$

$$\leqslant \int_{\Gamma_0} |F(x,y) - F(x',y)|\, |\mu^{(n)}(y)|\, dy$$

$\leqslant$ (with Theorem II.4.1)

$$\leqslant \|\mu\|_{M_1(\Gamma_0)} \sup_{y \in \Gamma_0} |F(x,y) - F(x',y)|$$

Hence, the equicontinuity of $\Phi_n^{(2)}$ is obvious, for $F(x,y)$ is indefinitely differentiable in $\bar{A} \times \Gamma_0$, and so Lemma IV.2.2 is proved.

Finally, we prove the following result, which will be of great use in the further study of Problem (I.2.23) :

Corollary IV.2.3. Let h be a convex function of one real variable satisfying the assumptions of Theorem II.4.1. The sequence $(\Phi_n)_{n \in \mathbb{N}}$ defined by (IV.2.1) is such that :

$$h(\gamma_1 {}^\circ \Phi_n) \rightarrow h(\gamma_1 {}^\circ u) \text{ vaguely in } M_1(\Gamma_0) \qquad \text{(IV.2.vi)}$$

Proof : Thanks to Proposition III.4.3, we know that $\gamma_1 \circ \Phi_n = \mu^{(n)} - \gamma_0 \circ \Phi_n$. Using Theorems IV.2.1 and II.4.1, we know that $h(\mu^{(n)})$ tends to $h(\mu)$ vaguely in $M_1(\Gamma_0)$, while $\gamma_0 \circ \Phi_n$ tends to $\gamma_0 \circ u$ in $L^1(\Gamma_0)$, and this proves Corollary IV.2.3.

V. Extension of *(P*)* to the space BT(Ω) ; the generalized problem (Q*).

1. Definition of (Q*)

We recall the formulation of Problem (I.2.23)

$$(P^*) \qquad \sup_{q \in W} \{-j^*(q)\} \tag{V.1.1}$$

where j^* is the convex functional defined on W by

$$j^*(q) \begin{cases} = 1/2\|\Delta q\|^2_{L^2(\Omega)} + \int_\Omega \Delta q \, . \, z_d dx + \int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r) d\Gamma \\ \qquad + \int_{\Gamma_0} h_{\alpha',\beta'}(q) d\Gamma \\ \text{if } q = \partial q/\partial r = 0 \ \text{ on } \Gamma_1 \\ = +\infty \ \text{ otherwise.} \end{cases} \tag{V.1.2}$$

Thanks to results of II and IV, we can see that j^* can be naturally extended to a functional J^* whose domain is BT(Ω). As a matter of fact, every element u of BT(Ω) has a Laplacean Δu in $L^2(\Omega)$, a trace $\gamma_0{}^o u$ in $L^1(\Gamma_0)$ and a normal derivative $\gamma_1{}^o u$ in $M_1(\Gamma_0)$. We may then define $\int_{\Gamma_0} h_{\alpha,\beta}(-\gamma_1{}^o u)$ as the integral on Γ_0 of the bounded measure $h_{\alpha,\beta}(-\gamma_1{}^o u)$. Moreover, owing to the compactness of Γ_0, this integral is equal to the duality product $< 1, h_{\alpha,\beta}(-\gamma_1{}^o u)>_{\mathcal{C}(\Gamma_0)\times M_1(\Gamma_0)}$.

We now define the maximization problem (Q*) :

$$(Q^*) \qquad \operatorname*{Sup}_{u \in BT(\Omega)} \{-J^*(u)\} \qquad (V.1.3)$$

in which the functional J^* is defined as follows :

$$J^*(u) \begin{cases} = 1/2\|\Delta u\|^2_{L^2(\Omega)} + \int_\Omega \Delta u . z_d . dx + \int_{\Gamma_0} h_{\alpha,\beta}(-\gamma_1{}^o u) \\ \qquad + \int_{\Gamma_0} h_{\alpha',\beta'}(\gamma_0{}^o u) d\Gamma \\ \text{if } u = \partial u/\partial r = 0 \text{ on } \Gamma_1 \\ = +\infty \text{ otherwise.} \end{cases} \qquad (V.1.4)$$

The space W is obviously imbedded in $BT(\Omega)$, and we have, for q in W :

$$j^*(q) = J^*(q) \qquad (V.1.5)$$

As a consequence, the following inequality is satisfied :

$$\operatorname*{Sup}_{u \in BT(\Omega)} (-J^*(u)) \geqslant \operatorname*{Sup}_{q \in W} (-j^*(q)) . \qquad (V.1.6)$$

Actually, we shall now prove a stronger result :

<u>Theorem V.1.1</u>. The extremas of problems (V.1.3) and (I.2.23) are equal.

Proof : We first give an approximation result :

<u>Lemma V.1.2</u>. Let u be in $BT(\Omega)$: there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of $(\mathcal{C}^\infty(\overline{\Omega}))^{\mathbb{N}}$ which tends to u in the sense of Theorem IV.2.1 and Corollary

IV.2.3, such that $\gamma_0{}^1 u_n = \gamma_1{}^1 u_n = 0$ for every integer n.
Let us suppose that Lemma V.1.2 is proved.

For any u in $BT(\Omega)$, such that $J^*(u) < +\infty$, one can then find a sequence u_n of smooth functions, admissible for Problem (I.2.23), which satisfies :

$$\lim_{n\to\infty} j^*(u_n) = J^*(u) \qquad (V.1.7)$$

That is obviously sufficient to prove Theorem V.1.1.
Proof of Lemma V.1.22 :
We consider ([L.M], ch.II, ∈8) a family $(\Gamma_1{}^\xi)_{0\leqslant\xi\leqslant 1}$ of indefinitely differentiable surfaces parallel to $\Gamma_1(\Gamma_1{}^0 = \Gamma_1)$.
Let A_ξ be the open set limited by Γ_1 and $\Gamma_1{}^\xi$, and A be the union $\cup A_\xi$, $0 \leqslant \xi \leqslant 1$. A is supposed to be such that $\bar{A} \cap \Gamma_0 = \emptyset$.
The following result is then valid ([A], p.216) :

<u>Proposition V.1.3.</u> There exists a family of truncating functions $\{\psi_\xi\}_{0\leqslant\psi_\xi\leqslant 1}$, elements of $C^\infty(\bar{\Omega})$, satisfying the conditions :

$$0 \leqslant \psi_\xi \leqslant 1 \ ; \ \psi_\xi = 0 \text{ in } \tilde{A}_{\xi/2} \ ; \ \psi_\xi = 1 \text{ in } \Omega/_{A_\xi} \qquad (V.1.8)$$

and such that

$$\forall v \in H^2(\Omega),\ v = \partial v/\partial r = 0 \text{ on } \Gamma_1,\ \lim_{\xi\to 0} \|(\psi_\xi - 1)v\|_{H^2(\Omega)} = 0 \qquad (V.1.9)$$

In order to prove Lemma V.1.2, we shall truncate in the neighbourhood of Γ_1 the functions Φ_n constructed in Theorem IV.2.1.

We set : $$\Phi_{n,\xi} = \psi_\xi \Phi_n \tag{V.1.10}$$

It is now sufficient to prove

<u>Proposition V.1.4</u> : Let s be a real number, $1 \leqslant s < N/N-2$, the following assertion is true : $\forall \varepsilon > 0,\ \exists \xi_0 \in]0,1[,\ \exists n_0 \in \mathbb{N}$,

$$\forall n > n_0,\quad \|\Phi_{n,\xi} - u\|_{L^s(\Omega)} < \varepsilon \tag{V.1.i}$$

$$\|\Delta \Phi_{n,\xi} - \Delta u\|_{L^2(\Omega)} < \varepsilon \tag{V.1.ii}$$

Proof of (V.1.i) :

We have

$$\begin{aligned}
\|\Phi_{n,\xi} - u\|_{L^s(\Omega)} &= \{\int_\Omega |\Phi_n(x)\Psi_\xi(x) - u(x)|^s\, dx\}^{1/s} \\
&\leqslant \{\int_\Omega |\Phi_n(x)\Psi_\xi(x) - \Phi_n(x)|^s dx\}^{1/s} + \{\int_\Omega |\Phi_n(x) - u(x)|^s dx\}^{1/s} \\
&\leqslant \quad I_1 \quad + \quad I_2
\end{aligned}$$

Thanks to the definition of Φ_n, we first choose n_1 such that $I_2<\varepsilon/2$ for $n>n_1$.

For I_1, we remark that Φ_n is bounded in $L^{s_1}(\Omega)$, $1\leqslant s<s_1<N/N-2$, and therefore :

$$(I_1)^s = \int_\Omega |1-\Psi_\xi(x)|^s |\Phi_n(x)|^s dx$$

$$\leqslant \text{(by Holder's inequality)}$$

$$\leqslant \left(\int_\Omega |\Phi_n(x))^{s_1} dx\right|^{s/s_1} \left(\int_\Omega |1-\psi_\xi(x)|^{s_1/s_1-s}\right)^{s_1-s/s_1}.$$

$$\leqslant C.\,(\text{mes } A_\xi)^{s_1-s/s_1}.$$

C being independent of n. We now choose ξ_1 such that $C(\text{mes } A_\xi)<\varepsilon/2$ for $\xi\leqslant\xi_1$, and (V.1.i) is proved.

Proof of (V.1.ii) :

$$\|\Delta(\Phi_n\Psi_\xi)-\Delta u\|_{L^2(\Omega)} = \|\Delta(\psi_\xi\Phi_n) - \Delta(\psi_\xi u)\|_{L^2(\Omega)} + \|\Delta(\psi_\xi u)-\Delta u\|_{L^2(\Omega)}$$

$$\leqslant \text{(Thanks to Proposition V.1.3)}$$

$$\leqslant \|\psi_\xi\Phi_n-\psi_\xi u\|_{H^2(A)} + \|\Delta\Phi_n-\Delta u\|_{L^2(\Omega)} + \|\psi_\xi u-u\|_{H^2(A)}$$

$$\leqslant \qquad J'_1 \quad + \quad J'_2 \quad + \quad J'_3$$

Proposition V.1.3 enables us to choose ξ_2 such that $J'_3<\varepsilon/3$ for $\xi\leqslant\xi_2$. Owing to the definition of Φ_n, we can choose n_2 such that $J'_2<\varepsilon/3$ for $n>n_2$. Finally, ξ_2 being fixed, we can find n_3 such that $J'_1<\varepsilon/3$ for

$n > n_3$, thanks to Lemma IV.2.2.

We have finally obtained :

$$\forall \varepsilon > 0, \quad \exists \xi_0 = \text{Inf}\ (\xi_1, \xi_2),\ \exists n_0 = \text{Sup}(n_1, n_2, n_3)\ ,$$
$$n > n_0 \Rightarrow (\text{V.1.i}) \text{ and } (\text{V.1.ii}) \qquad (\text{V.1.11})$$

and Lemma V.1.2 is proved.

<u>Remark V.1.1</u>. We have used, without mentioning it, the fact that $BT(\Omega)$ is a local space (see Lemma I.2.1), and that u belongs to $H^2(A)$.

VI. Existence of and adjoint state for Problem (P) ; System of optimality conditions.

In this section, we give the main results of our work : the existence in $BT(\Omega)$, of an adjoint state for the optimal state of (P), and the optimality system for (P). We introduce a generalized duality between *(P)* and (Q^*), which enables to use the direct method of calculus of variations and give very simple proofs for our results. Most of the latter are based on the generalized Green's formula hereafter.

1. Green's formula.

In this section, we prove the following result :

Proposition VI.1.1 Let (u,z) be an element of $BT(\Omega)\times W$, z being an admissible state for problem (P). Then z satisfies the conditions :

$$z\in{}^2(\Omega)\ ;\ \Delta z\in L^\infty(\Omega)\ ;\ (\gamma_0{}^\circ z,\gamma_1{}^\circ z)\in \mathcal{C}(\Gamma_0)\times L^\infty(\Gamma_0);(\gamma_1{}^\circ z,\gamma_1{}^1 z)$$
$$\in H^{-1/2}(\Gamma_1)\times H^{-3/2}(\Gamma_1)$$

and the Green's formula (V.1.i) is valid :

$$\int_\Omega(\Delta z\ .\ u-\Delta u\ .\ z)dx=\int_\Gamma((\partial z/\partial r).u-(\partial u/\partial r).z)d\Gamma \qquad \text{(V.1.i)}$$

Remark V.1.1 : The precise formulation of (V.1.i) is :

$$\int_{\Omega}(\Delta z\,.\,u-\Delta u\,.\,z)dx=\int_{\Gamma_0}\gamma_1{}^0 z\,\gamma_0{}^0 u.d\Gamma+<\gamma_1{}^1 z\,,\gamma_0{}^1 u>_{H^{-3/2}(\Gamma_1)\times H^{3/2}(\Gamma_1)}-$$
$$-<\gamma_0{}^0 z,\gamma_1{}^0 u>_{\mathcal{C}(\Gamma_0)\times M_1(\Gamma_0)}-<\gamma_0{}^1 z,\gamma_1{}^1 u>_{H^{-1/2}(\Gamma_1)\times H^{-3/2}(\Gamma_1)}$$

Proof of Proposition III.1.1 :

We first point out the fact that all the expressions in (VI.1.i) are meaningful, and we proceed to prove an approximation result for z, namely :

<u>Proposition VI.1.2</u>. Let z belong to W, z being admissible for Problem (I.1.3), and let ρ be a real number, $2\leq\rho<+\infty$. There exists a sequence Z_n of functions of $\mathcal{C}^{\infty}(\overline{\Omega})$, such that :

$\Delta Z_n\rightarrow\Delta z$ in $L^{\rho}(\Omega)$; (VI.1.a)

$Z_n\rightarrow z$ in $L^2(\Omega)$; (VI.1.b)

$\gamma_1{}^1 Z_n\rightarrow\gamma_1{}^1 z$ in $H^{-1/2}(\Gamma_1)$ (VI.1.c)

$\gamma_1{}^0 Z_n\rightarrow\gamma_1{}^0 z$ in $L^r(\Gamma_0)$, for $1\leq r<+\infty$, and in $L^{\infty}(\Gamma_0)$ $*$-weakly : (VI.1.d)

$(\gamma_0{}^0 Z_n,\gamma_0{}^1 Z_n)\rightarrow(\gamma_0{}^0 z,\gamma_0{}^1 z)$ in $\mathcal{C}(\Gamma_0)\times H^{-1/2}(\Gamma_1)$ (VI.1.e)

From this result, the Green's formula (VI.1.i) will be proved by a straight forward passage to the limit, using Theorem IV.2.1.

We start with a Lemma :

Lemma VI.1.3. We set $\chi_\rho(\Omega)=\{v\in L^2(\Omega),\ \ \Delta v\in L^\rho(\Omega)\}$, for $2\leq\rho<+\infty$; $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $\chi_\rho(\Omega)$ for the norm topology defined by :

$$\|v\|_{\chi_\rho(\Omega)}=\|v\|_{L^2(\Omega)}+\|\Delta v\|_{L^\rho(\Omega)} \tag{VI.1.1}$$

Proof : Let b be a continuous linear form on $\chi_\rho(\Omega)$, whose restriction to $\mathcal{C}^\infty(\overline{\Omega})$ is the null form.

Thanks to the Hahn-Banach Theorem, there exists an element (f,g), of $L^2(\Omega)\times L^{\rho'}(\Omega)$, $1/\rho+1/\rho'=1$, such that

$$\langle v,b\rangle_{\chi_\rho(\Omega)\times[\chi_\rho(\Omega)]^*}=\int_\Omega f.v dx+\int_\Omega g.\Delta v dx \tag{VI.1.2}$$

b being null on $\mathcal{C}^\infty(\overline{\Omega})$, we have, for v in $\mathcal{D}(\Omega)$:

$$\int_\Omega [f.v+g.\Delta v]dx=0 \tag{VI.1.3}$$

and hence :

$$f+\Delta g=0 \quad \text{in } \mathcal{D}'(\Omega) \tag{VI.1.4}$$

g is then a function in $L^{\rho'}(\Omega)$, whose Laplacean is in $L^2(\Omega)$; we can define ([L.M], ch.II), in the distributional sense on Γ, its boundary values $\gamma_0 g$ and $\gamma_1 g$.

For v in $\mathcal{C}^{\infty}(\overline{\Omega})$, we shall have the Green's formula :

$$\int_{\Omega}(\Delta v.g \ , \ \Delta g.v)dx = <\gamma_1 v \ , \ \gamma_0 g>_{\mathcal{D}(\Gamma)\times \mathcal{D}'(\Gamma)} - <\gamma_0 v \ , \ \gamma_1 g>_{\mathcal{D}(\Gamma)\times \mathcal{D}'(\Gamma)} \qquad \text{(VI.1.5)}$$

but the assumptions on b now imply that $\gamma_0 g = \gamma_1 g = 0$.

From this, we obtain that g is in $H^2(\Omega)$, and we may write the Green's formula for $v \in \chi_\rho(\Omega)$ ([L], p. 148) :

$$\int_{\Omega}(g.\Delta v - v.\Delta g) = <\gamma_0 g, \gamma_1 v>_{H^{3/2}(\Gamma)\times H^{-3/2}(\Gamma)} - <\gamma_1 g \ , \gamma_0 v>_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = 0 \qquad \text{(VI.1.6)}$$

We may therefore conclude that b is the null form on $X_\rho(\Omega)$; and $\mathcal{C}^{\infty}(\overline{\Omega})$ is dense in $X_\rho(\Omega)$.

Let us now prove Proposition VI.1.2.

Let z belong to W, such that $j(z) < +\infty$. We have necessarily :

$$\Delta z = 0 \ \text{ in } \Omega \ ; \ (\gamma_0{}^0 z, \gamma_0{}^1 z) \in (L^{\infty}(\Gamma_0))^2 \qquad \text{(V.1.7)}$$

Owing to the hypoellipticity of the Laplacean, we know that z is indefinitely differentiable in Ω.

We now consider the same function Φ as in the proof of Lemma I.1.1 and we set as before $z = \Phi z + (1-\Phi)z$.

$$= z_1 + z_2 .$$

By using the fact that z and all its partial derivatives are bounded in any relatively compact open set of Ω, we can show exactly as we did for Lemma I.2.1 that z_1 and z_2 are such that :

$$(z_1,z_2)\in(L^2(\Omega))^2 \ ; \ (\Delta z_1,\Delta z_2)\in(L^r(\Omega))^2 \quad (1\leqslant r\leqslant+\infty) \quad \text{(VI.1.8)}$$

In order to approximate z_2, we use Lemma VI.1.3 in an open set ω whose closure does not intersect Γ_0, and obtain a sequence $Z_n^{(2)}$.
For z_1, we have the additional conditions :

$$(\gamma_0{}^0 z_1,\gamma_1{}^0 z_1)=(\gamma_0{}^0 z,\gamma_1{}^0 z)\in L^\infty(\Gamma_0))^2 \ ; \gamma_1{}^0 z_1=\gamma_1{}^1 z_1=0 \ \text{ on } \ \Gamma_1 \quad \text{(VI.1.9)}$$

Therefore, z_1 is the only solution of the Neumann problem :

$$\Delta z_1=\Delta(\Phi z) \ \text{ in } \ \Omega$$
$$z_1+\partial z_1/\partial r=z+\partial z/\partial r \ \text{ on } \ \Gamma_0$$
$$z_1+\partial z_1/\partial r=0 \ \text{ on } \ \Gamma_1$$

and this implies, using III, that z_1 is defined by :

$$z_1 = - \int_\Omega F(.y)[\Delta(\Phi z)](y)dy + \int_{\Gamma_0} F(.y)[z + \partial z/\partial r](y)dy \quad \text{(VI.1.11)}$$

Thanks to Proposition III.1.3, and $[M_1]$, Th.12.VIII, z_1 is continuous on $\overline{\Omega}$; if we use the approximation of $(\gamma_0{}^o z + \gamma_1{}^o z)$ defined by (II.2.1), and any sequence of smooth functions converging to $\Delta(\Phi z)$ in $L^\rho(\Omega)$, we can construct an approximating sequence $Z_n^{(2)}$ which is easily shown to satisfy the conditions :

$$Z_n^{(1)} \to z_1 \text{ in } \mathcal{C}(\overline{\Omega}) \;;\; \Delta Z_n^{(1)} \to \Delta z_1 \text{ in } L^\rho(\Omega)$$
$$(\gamma_1{}^o Z_n^{(1)} ; \gamma_1{}^1 Z_n^{(1)}) \to (\gamma_1{}^o z_1, \gamma_1{}^1 z_1) \text{ in } L^r(\Gamma_0), \text{ for } 1 \leq r < +\infty \;,$$
$$\text{and in } L^\infty(\Gamma_0) \;_* \text{ - weakly}$$

If we now set $Z_n = Z_n^{(1)} + Z_n^{(2)}$, it is clear that Z_n approximates z in the sense of Proposition V.1.2 ; we just have to take into account the fact that $\gamma_0{}^o Z_n^{(2)} = \gamma_1{}^o Z_n^{(2)} = 0$ for every integer n.

Now, we can conclude the proof of Proposition VI.1.1 ; for this, we consider the sequence u_n of Theorem IV.2.1, with $\tau = 1$, $1 \leq s < N/N-2$, and the sequence Z_n for $\rho = s/s-1$.

We then have the classical Green's formula :

$$\int_{\Omega}(\Delta u_n.Z_n - \Delta Z_n.u_n)dx = \int_{\Gamma}((\partial u_n/\partial r).Z_n - u_n.(\partial Z_n/\partial r))d\Gamma \qquad \text{(VI.1.12)}$$

and one can easily check that the conditions satisfied by u_n and Z_n ensures that equality (VI.1.12) is still valid after passing to the limit.

Remark VI.1.1 If we only make the assumption that Δz is bounded in Ω, we can nevertheless prove analogous results. This is done by using the results of L^{ϱ} regularity for the solutions of elliptic boundary value problems ([G.G]).

2. Existence in BT(Ω) of an adjoint state for z_0.

First, we prove two auxiliary results

Lemma VI.2.1. Let (u,z) belong to $BT(\Omega)\times W$, u (resp. z) being admissible for Problem (V.1.3) (resp. I.1.3)). The following inequality is satisfied

$$-J^*(u) \leq j(z) - 1/2\|\Delta u-(z-z_d)\|^2_{L^2(\Omega)} \qquad \text{(VI.2.1)}$$

Proof : we recall that $\gamma_0{}^0 z$ is continuous on Γ_0, and that $\gamma_0{}^0(x)$ (resp. $\gamma_1{}^0(x)$) belongs to $[-\alpha,\beta]$ (resp. $[-\alpha',\beta']$) for almost every x in Γ_0.
From the definition of the conjugate of a convex function, and thanks to (II.4.7) we have :

$$h_{\alpha',\beta'}(\gamma_0{}^o u)(x) \geq (\gamma_0{}^o u.\gamma_1{}^o z)(x) - 1/2(\gamma_1{}^o z(x))^2 \text{ a.e. on } \Gamma_0 \qquad \text{(VI.2.2)}$$

and

$$<h_{\alpha,\beta}(-\gamma_1{}^o u),1>_{M_1(\Gamma_0)\times \mathcal{C}(\Gamma_0)} \geq (-\gamma_1{}^o u,\gamma_0{}^o z)_{M_1(\Gamma_0)\times \mathcal{C}(\Gamma_0)} - 1/2\|\gamma_0{}^o z\|^2_{L^2(\Gamma_0)} \qquad \text{(VI.2.3)}$$

Therefore, we obtain :

$$-J^*(u) \leq -1/2\|\Delta u\|^2_{L^2(\Omega)} - \int_\Omega \Delta u.z_d + 1/2\|z\|^2_{L^2(\Gamma_0)} + 1/2\|\partial z/\partial r\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0}((\partial u/\partial r).z - (\partial z/\partial r).u)d\Gamma \qquad \text{(VI.2.4)}$$

$\leq$ (thanks to Proposition VI.1.1)

$$\leq -1/2\|\Delta u\|^2_{L^2(\Omega)} - \int_\Omega \Delta u.z_d dx + \int_\Omega(\Delta u.z - \Delta z.u)dx + 1/2\|\partial z/\partial r\|^2_{L^2(\Gamma_0)} + 1/2\|z\|^2_{L^2(\Gamma_0)}$$

$\leq$(given that $\Delta z = 0$ in Ω)

$$\leq 1/2\|z - z_d\|^2_{L^2(\Omega)} + 1/2\|z\|^2_{L^2(\Gamma_0)} + 1/2\|\partial z/\partial r\|^2_{L^2(\Gamma_0)} - 1/2\|\Delta u - (z - z_d)\|^2_{L^2(\Omega)}$$

and the last inequality is precisely the result we wished to prove.

We now define on $BT(\Omega)$ the weak topology τ_1 as follows :

$$u_n \to u \text{ for } \tau_1 \text{ if } \quad \begin{array}{l} \Delta u_n \to \Delta u \text{ weakly in } L^2(\Omega) \\ \gamma_0{}^0 u_n \to \gamma_0{}^0 u \text{ in } L_1(\Gamma_0) \\ \gamma_1{}^0 u_n \to \gamma_1{}^0 u \text{ vaguely in } M_1(\Gamma_0) \\ (\gamma_0{}^1 u_n, \gamma_1{}^1 u_n) \to (\gamma_0{}^1 u, \gamma_1{}^1 u) \text{ in } L^2(\Gamma_1). \end{array} \qquad \text{(VI.2.5)}$$

Thanks to the results of III and IV, we know that every bounded set of $BT(\Omega)$ is relatively compact for the τ_1 topology.

For the τ_1 topology on $BT(\Omega)$, we have :

Lemma VI.2..2. The functional J^* is lower semi continuous on $BT(\Omega)$ for τ_1.

Proof : this result is obvious, thanks to (II.4.i), (II.4.iii) and to the lower semi continuity of the norm for the weak topology of $L^2(\Omega)$.

We can now prove the existence result for an adjoint state of z_0 :

Theorem VI.2.3. There exists in $BT(\Omega)$ an adjoint state for the optimal state z_0 of (P) ; it is a solution of Problem (V.1.3). Moreover, any maximizing sequence of (1.2.23) has a cluster point for the τ_1 topology of $BT(\Omega)$ which is an adjoint state of z_0. Last, any solution of (V.1.3) can be obtained as a cluster point for such a sequence.

Proof : Let $(w_n)_{n\in\mathbb{N}}$ be a maximizing sequence of Problem (V.1.3). Thanks to Lemma VI.2.1, w_n is bounded in $BT(\Omega)$. Thus, it has a cluster point w for

the τ_1 topology, given Proposition III.4.1 and the definition of BT(Ω) ; moreover w is admissible for Problem (V.1.3), for the convex set ($\gamma_0{}^o u = \gamma_1{}^1 u = 0$ on Γ_1) is closed for the τ_1 topology.
Owing to Lemma VI.2.2, we then have :

$$(-J^*(w)) \geq \lim_{n \to \infty} (-J^*(w_n)) = \text{Sup(V.1.3)} \qquad \text{(VI.2.6)}$$

and (VI.2.6) obviously implies that w is a solution of Problem (V.1.3). The second part of Theorem VI.2.3 is a consequence of Theorem V.1.1, while the last assertion is obvious, using the approximation result of Lemma V.1.2.

3. The optimality system.

We can now establish the set of optimality conditions for Problem (P).

<u>Theorem VI.3.1</u>. Let z be an admissible state for (P) ; then z is the optimal state if and only if there exists w in BT(Ω) admissible for problem (V.1.3), and such that :

$$\Delta w = z - z_d \text{ in } \Omega \qquad \text{(VI.3.i)}$$

$$h_{\alpha',\beta'}(\gamma_0{}^o w) = \gamma_0{}^o w . \gamma_1{}^o z - 1/2(\gamma_1{}^o z)^2 \text{ almost everywhere on } \Gamma_0 \qquad \text{(VI.3.ii)}$$

$$h_{\alpha,\beta}(-\gamma_1{}^o w) = -\gamma_1{}^o w . \gamma_0{}^o z - 1/2(\gamma_0{}^o z)^2 \text{ in } M_1(\Gamma_0) \qquad \text{(VI.3.iii)}$$

Moreover, if equalities (VI.3.i) to (VI.3.iii) are satisfied, z and w are a pair of solutions of the dual optimization problems *(P)* and (Q^*).

Proof : If we look at the proof of Lemma VI.2.1, we remark that (VI.3.i), (VI.3.ii), (VI.3.iii) are nothing but the necessary and sufficient conditions for $j(z)$ and $(-J^*(w))$ to be equal.

VII. Another problem of optimal control.

In this section, we apply the method of generalized duality to a problem of convex optimal control, in which the cost function involves the boundary value of the considered state.

1. Variational formulation.

We set

$W_1 = \{y \in W,\ y \in L^2(\Gamma_1)\}$; W_1 is a Hibert space when endowed with the norm :

$$\|y\|_{W_1} = \{\|y\|^2_{L^2(\Gamma_1)} + \|y\|_W^2\}^{1/2}.$$

Moreover, the trace operator $y \to \partial y/\partial r_{|\Gamma_1}$ is linear continuous from W_1 into $H^{-1}(\Gamma_1)$, and W_1 is imbedded in $H^{1/2}(\Omega)$ ([L.M], ch. II). The problem we wish to study is :

to find $(y_0, u'_0, v'_0) \in W_1 \times \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'}$, minimizing the cost function

$$J_1(y,u,v,) = 1/2\{\|y - y_d\|^2_{L^2(\Gamma_1)} + \|u\|^2_{L^2(\Gamma_0)} + \|v\|^2_{L^2(\Gamma_0)}$$

(R)

the infimum being taken for $y \in W_1$, $\Delta y = 0$ in Ω, $y = u$ on Γ_0,

$\partial y/\partial r = v$ on Γ_0,

$(u,v) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'}$.

(y_d is a given function in $L^2(\Gamma_1)$).

We define the functional

$$
j_1(y) \begin{cases} = 1/2\{\|y-y_d\|^2_{L^2(\Gamma_1)} + \|y\|^2_{L^2(\Gamma_o)} + \|\partial y/\partial r\|^2_{L^2(\Gamma_o)}\} \text{ if } y \in W_1, \\ \Delta y = 0 \text{ in } \Omega,\ (y, \partial y/\partial r) \in \mathcal{U}_{\alpha,\beta} \times \mathcal{U}_{\alpha',\beta'} \\ = +\infty \text{ otherwise}. \end{cases} \quad \text{(VII.1.1)}
$$

The variational formulation of Problem (R) is :

$$
(R) \qquad \operatorname*{Inf}_{y \in W_1} \{j_1(y)\} \quad \text{(VII.1.2)}
$$

Thanks to the results of [L.M], ch. II, we know that j_1 is coercive on W_1 :

$$
\exists C,\ \forall y \in W_1,\ j_1(y) \geqslant C\|y\|^2_{W_1} \quad \text{(VII.1.3)}
$$

and therefore the following result is valid (see Section I):

Proposition VII.1.1. There exists a unique optimal state y_o of (R).

We now turn to the problem of existence of an adjoint state for y_o; this leads us to studying the dual problem *(R*)* of *(R)*, which we now formulate.

We set $Y_1 = L^2(\Omega) \times (L^2(\Gamma_o))^2 \times L^2(\Gamma_1)$; Y_1 will be identified with its

conjugate Y^*_1. Λ_1 is the operator from W_1 to Y_1 defined by
$\Lambda_1(g)=(\Delta y, y_{|\Gamma_0}, \partial y/\partial r_{|\Gamma_0}, y_{|\Gamma_1})$.

Let F',G' be the functionals respectively defined on W_1 and Y_1 by :

$$F'(y)=0\ ,\ G'(p)=\sum_{i=1}^{4} G'_i(p_i)\ ,$$

$$G_1'(p_1)\quad \begin{cases} =0, \text{ if } p_1=0 \\ =+\infty \text{ otherwise} \end{cases} \tag{VII.1.4}$$

$$G_2'(p_2)\quad \begin{cases} =1/2\|p_2\|^2_{L^2(\Gamma_0)}, \text{ if } p_2\in \mathcal{U}_{\alpha,\beta} \\ =+\infty \text{ otherwise} \end{cases} \tag{VII.1.5}$$

$$G_3'(p_3)\quad \begin{cases} =1/2\|p_3\|^2_{L^2(\Gamma_0)}, \text{ if } p_3\in \mathcal{U}_{\alpha',\beta'} \\ =+\infty \text{ otherwise} \end{cases} \tag{VII.1.6}$$

$$G_4'(p_4)\quad =1/2\|p_4-y_d\|^2_{L^2(\Gamma_1)} \tag{VII.1.7}$$

the dual problem *(R*)* of Problem (VII.1.2) is then :

$$(R^*)\qquad \sup_{q\in Y_1}\ \{-G'^*(-q)-F'^*(\Lambda^*_1 q)\} \tag{VII.1.8}$$

We compute the conjugate functions F'^* and G'^* as in Section I ; for F'^*, we shall need the following approximation result :

Lemma VII.1.2. Let (y,p) belong to $W_1 \times W_1$, such that $\partial p/\partial r$ belongs to $L^2(\Gamma_1)$.

(i) there exist two sequences y_n and p'_n of smooth functions, such that :

$$\begin{aligned} & p'_n \to p \text{ in } H^{3/2}(\Omega) \\ & \Delta p'_n \to \Delta p \text{ in } L^2(\Omega) \end{aligned} \tag{VII.1.9}$$

$$y_n \to y \text{ in } W_1 \tag{VII.1.10}$$

(ii) The following Green's formula is valid :

$$\begin{aligned} & \int_\Omega (\Delta p . y - \Delta y . p) dx = \\ & \int_{\Gamma_0} ((\partial p/\partial r) y - (\partial y/\partial r) . p) d\Gamma + \int_{\Gamma_1} \partial p/\partial r \, . \, y d\Gamma \\ & \qquad - \langle p, (\partial y/\partial r) \rangle_{H^1(\Gamma_1) \times H^{-1}(\Gamma_1)} \end{aligned} \tag{VII.1.11}$$

Proof : We first remark that the existence of p'_n has been shown in the proof of Lemma I.2.1. So as to approximate y, we write $y = y_1 + y_2$, y_1 and y_2 being such that :

$$\begin{aligned} & (y_1, \Delta y_1) \in (L^2(\Omega))^2 \\ & (y_1, (\partial y_1/\partial r)) \in (L^2(\Gamma_0))^2 \\ & (\operatorname{Supp} y_1) \cap \Gamma_1 = \emptyset \end{aligned} \tag{VII.1.12}$$

$$\begin{aligned}&(y_2,\Delta y_2)\in(L^2(\Omega))^2\\&y_2\in L^2(\Gamma_1)\\&\text{Supp }(y_2)\cap\Gamma_0=\emptyset\end{aligned}\tag{VII.1.13}$$

(cf. the proof of Lemma I.2.1).

The function y_1 can approximated in the same way as p ; for y_2 , we first choose an indefinitely differentiable open set ω_1 of Ω , whose closure does not intersect Γ_0 , such that supp(y_1) is contained in ω_1. We then solve a Dirichlet problem in ω_1 , namely :

$$\begin{aligned}&\Delta y_{2n}=r_n \text{ in } \omega_1\\&y_{2n}=t_n \text{ on } \Gamma_1\\&y_{2n}=0 \text{ on } \Gamma_0\end{aligned}\tag{VII.1.14}$$

r_n (resp. t_n) being a sequence of smooth functions approximating Δy_2 (resp. y_2) in $L^2(\Omega)$ (resp. $L^2(\Gamma_1)$). We know that y_{2n} tends to y_2 in $H^{1/2}(\omega_1)$, while $\partial y_{2n}/\partial r$ tends to $\partial y_2/\partial r$ in $H^{-1}(\Gamma_1)$; moreover, Δy_{2n} obviously tends to Δy_2. It we now set $y_n = y_{1n} + y_{2n}$, we obtain the sequence we were looking for, and Lemma VII.1.2 is proved.

The expression of $F'^*(\Lambda^*{}_1 q)$ is then easily shown to be :

$$F'^*(\Lambda^*{}_1 q) \begin{cases} = 0 \text{ if } & \Delta q_1 = 0 \text{ in } \Omega \\ & q_1 + q_3 = q_2 - \partial q_1/\partial r = 0 \text{ on } \Gamma_0 \\ & q_4 - \partial q_1/\partial r = q_1 = 0 \text{ on } \Gamma_1 \\ = +\infty \text{ otherwise.} & \end{cases} \tag{VII.1.15}$$

Computing the $G_i'^*$ is easy, and we find for *(R*)* :

$$(R^*) \qquad \sup_{\substack{q \in W_1 \\ \Delta q = 0 \text{ in } \Omega \\ q = 0 \text{ on } \Gamma_1 \\ \partial q/\partial r \in L^2(\Gamma_1)}} \Big\{ - \int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r)\,d\Gamma - \int_{\Gamma_0} h_{\alpha',\beta'}(q)\,d\Gamma - 1/2\|\partial q/\partial r\|^2_{L^2(\Gamma_1)} + \int_{\Gamma_1} (\partial q/\partial r).y_d\,d\Gamma \Big\} \tag{VII.1.16}$$

The difficulty we meet here is obviously of the same kind as in Section I : the functional j_1^* defining *(R*)* is not coercive on W_1 and hence, we do not know wether there exists an adjoint state in W_1 for the optimal state y_0.

Nevertheless, we have the following comparison result :

Theorem VII.1.3. The extremas of Problems (VII.1.2) and (VII.1.16) are equal.

Proof : We consider the perturbations (R_p) of (R), for p given in y_1 :

$$(R_p) \qquad \inf_{z \in W_1} \{F^*(z) + G^*(\Lambda_1 z - p)\}$$

Given $p^{(n)}$, a sequence in Y_1 converging to zero, let $y^{(n)}$ be the solution of $(R_{p^{(n)}})$. As in the proof of Theorem I.3.1, we show that we can extract from $y^{(n)}$ a subsequence weakly converging in W_1 to the solution y_0 of Problem (VII.1.2), the latter being therefore normal ([E.T]). Hence, the equality $\mathrm{Inf}\,(R) = \mathrm{Sup}\,(R^*)$ is true.

2. A function space adapted to Problem (VII.1.16).

We now introduce a new function space, modelled on $BT(\Omega)$, on which we shall define a natural extension of (R^*).

Let us first remark that a maximizing sequence $(q_n)_{n \in N}$ of (R^*) satisfies the conditions :

$$\Delta q_n = 0 \text{ in } \Omega \,;\; (q_n, (\partial q_n/\partial r)) \text{ is bounded in } L^1(\Gamma_0)$$

$$\partial q_n/\partial r \text{ is bounded in } L^2(\Gamma_1) \,;\; q_n = 0 \text{ on } \Gamma_1 . \qquad \text{(VII.2.1)}$$

Let $\mathcal{A}_1$ be the operator defined on $L^2(\Omega) \times L^2(\Gamma_1) \times M_1(\Gamma_0)$ by :

$$\mathcal{A}_1(f,g,\mu) = - \int_\Omega F(.,y)f(y)dy + \int_{\Gamma_1} F(.,y)g(y)dy + \int_{\Gamma_0} F(.,y)d\mu(y) \qquad \text{(VII.2.2)}$$

where F is the Green function of the Neumann problem (III.1.1). We define the space $BT_1(\Omega)$ as the range of $\mathcal{A}_1$, endowed with the graph norm. $BT_1(\Omega)$ is a Banach space, and its elements are all the functions defined on Ω whose Laplacean is in $L^2(\Omega)$, and whose normal derivative on Γ_0 (resp. Γ_1) is a bounded measure (resp. in $L^2(\Gamma_1)$). The properties of $BT_1(\Omega)$ are very similar to those of $BT(\Omega)$; actually, all the results of Section IV are valid for $BT_1(\Omega)$, after replacing $H^{1/2}(\Gamma_1)$ and $H^{3/2}(\Gamma_1)$ respectively by $L^2(\Gamma_1)$ and $H^1(\Gamma_1)$.

The extended functional J_1^* is then defined on $BT_1(\Omega)$ as follows :

$$J^*_1(q) \begin{cases} = \int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r) + \int_{\Gamma_0} h_{\alpha',\beta'}(q)d\Gamma + 1/2 \|\partial q/\partial r\|^2_{L^2(\Gamma_1)} \\ \qquad - \int_{\Gamma_1} \partial q/\partial r . y_d \, d\Gamma \\ \\ \text{if } \Delta q = 0 \text{ in } \Omega, \ q = 0 \text{ on } \Gamma_1 \qquad\qquad \text{(VII.1.3)} \\ \\ = +\infty \text{ otherwise.} \end{cases}$$

As before, $\int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r)$ stands for the integral on Γ_0 of the bounded measure $h_{\alpha,\beta}(-\partial q/\partial r)$.

The generalised problem $(\mathcal{S}^*)$ is :

$$(\mathcal{S}^*) \qquad \underset{q \in BT_1(\Omega)}{\text{Sup}} \quad \{-J_1^*(q)\} \qquad \text{(III.2.4)}$$

The following inequality is obvious :

$$\text{Sup (III.1.16)} \leqslant \text{Sup(VII.2.4)} \qquad \text{(II.2.5)}$$

In order to carry out the generalized duality method, as we did in Section VI, we shall need the following results :

<u>Proposition III.2.1</u>. Let (y,q) belong to $W_1 \times BT_1(\Omega)$, y (resp. q) being admissible for Problem (VII.1.2) (resp. (VII.2.4)). We have the Green's formula :

$$\begin{aligned} 0 = \int_\Omega (\Delta q.y - \Delta y.q)dx = &< (\partial q/\partial r\, ,\, y >_{M_1(\Gamma_0) \times \mathcal{C}(\Gamma_0)} \\ &- \int_{\Gamma_0} \partial y/\partial r\, .q\, d\Gamma + \int_{\Gamma_1} (\partial q/\partial r).y\, d\Gamma \\ &- < q,(\partial y/\partial r) >_{H^1(\Gamma_1) \times H^{-1}(\Gamma_1)} \end{aligned} \qquad \text{(VII.2.6)}$$

Proof : As for Lemma VI.1.1, all we need to prove is an approximation result for y and q. For q, we use the modified of version of Theorem IV.2.1, with $H^{3/2}(\Gamma_1)$ (resp. $H^{1/2}(\Gamma_1)$) replaced by $H^1(\Gamma_1)$ (resp. $L^2(\Gamma_1)$); for y, we prove the existence of a sequence y' satisfying the conditions :

$$y_n' \to y_n \text{ in } L^2(\Omega)\,;\ \Delta y_n' \to \Delta y_n \text{ in } L^\beta(\Omega) \ (\beta \text{ a given number, } 2 \leqslant \beta < +\infty)\,;$$
$$y_n' \to y \text{ in } \mathcal{C}(\Gamma_0) \text{ and } L^2(\Gamma_1)\,;\ \partial y_n'/\partial r \to \partial y/\partial r \text{ in } L^r(\Gamma_0),$$
$$\text{for } 1 \leqslant r < +\infty, \text{ and } L^\infty(\Gamma_0)_* \text{weakly}\,;$$
$$\partial y_n'/\partial r \to \partial y/\partial r \text{ in } H^{-1}(\Gamma_1). \qquad \text{(VII.2.7)}$$

To prove this, we set $y = y_1 + y_2$ as in Lemma (VII.1.2) with y_1, y_2 such that :

$$\begin{aligned} &\mathrm{Supp}(y_1) \cap \Gamma_1 = \emptyset \\ &(y_1, \Delta y_1) \in L^2(\Omega) \times L^\beta(\Omega) \\ &(y_1, \partial y_1/\partial r) \in L^\infty(\Gamma_0) \times L^\infty(\Gamma_0) \end{aligned} \qquad \text{(VII.2.8)}$$

$$\begin{aligned} &\mathrm{Supp}(y_2) \cap \Gamma_0 = 0 \\ &(y_2, \Delta y_2) \in L^2(\Omega) \times L^\beta(\Omega) \\ &y_2 \in L^2(\Gamma_1) \end{aligned} \qquad \text{(VII.2.9)}$$

We approximate y_1 as in the proof of Proposition VI.1.1 ; For y_2 we solve a Dirichlet problem similar to (VII.1.14), with a sequence r_n converging to Δy_2 in $L^\beta(\Omega)$.

The Green's formula (VII.2.6) is then proved, and both expressions appearing in it are null, thanks to : $\Delta y = \Delta q = 0$ in Ω.

3. Existence in $BT_1(\Omega)$ of an adjoint state for y_0.

We first prove two auxiliary results :

Lemma VII.3.1. Let (y,q) be a couple of admissible elements for Problems (VII.1.2) and (VII.2.4) ; we have the following inequality :

$$-J_1^*(q) \leqslant j_1(y) \qquad \text{(VII.3.1)}$$

Proof : by the definition of $h_{\alpha,\beta}, h^*_{\alpha,\beta}$, and thanks to (II.4.ii), we have

$$\int_{\Gamma_0} h_{\alpha,\beta}(-\partial q/\partial r) \geqslant - \int_{\Gamma_0} (\partial q/\partial r) y \, d\Gamma - 1/2 \int_{\Gamma_0} (y)^2 \, d\Gamma \qquad \text{(VII.3.2)}$$

$$\int_{\Gamma_0} h_{\alpha',\beta'}(q) \, d\Gamma \geqslant \int_{\Gamma_0} q.(\partial y/\partial r).d\Gamma - 1/2 \int_{\Gamma_0} (\partial y/\partial r)^2 d\Gamma \qquad \text{(VII.3.3)}$$

and therefore we derive the following inequalities :

$$\begin{aligned}
-J_1^*(q) &\leqslant -1/2 \| -\partial q/\partial r \|^2_{L^2(\Gamma_0)} + \int_{\Gamma_1} \partial q/\partial r . y_d d\Gamma \\
&\quad + \int_{\Gamma_0} ((\partial q/\partial r).y - (\partial y/\partial r).q) \, d\Gamma + 1/2 \|y\|^2_{L^2(\Gamma_0)} \\
&\quad + 1/2 \|\partial y/\partial r\|^2_{L^2(\Gamma_0)} . \\
&\leqslant 1/2 \|y - y_d\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0} ((\partial q/\partial r).y - (\partial y/\partial r).q) \, d\Gamma + \int_{\Gamma_1} \partial q/\partial r . y \, d\Gamma \\
&\quad + 1/2 \|y\|^2_{L^2(\Gamma_0)} + 1/2 \| \partial y/\partial r \|^2_{L^2(\Gamma_0)} \\
&\leqslant (\text{for } q = 0 \text{ on } \Gamma_1) \\
&\leqslant j_1(y) + \int_{\Gamma} ((\partial q/\partial r).y - (\partial y/\partial r).q) \, d\Gamma
\end{aligned}$$

$\leqslant$ (by Proposition VII.2.1)

$\leqslant j_1(y)$.

Lemma VII.3.2. Let τ_1' be the weak topology defined on $BT_1(\Omega)$ by :

$$q_n \to q \quad \text{for } \tau_1' \text{ if} \quad \begin{cases} \Delta q_n \to \Delta q \quad \text{in } L^2(\Omega) \text{ weakly} \\ q_n \to q \quad \text{in } L^1(\Gamma_0) \text{ and } L^2(\Gamma_1). \\ \partial q_n/\partial r \to \partial q/\partial r \quad \text{weakly in } L^2(\Gamma_1) \\ \partial q_n/\partial r \to \partial q/\partial r \quad \text{vaguely in } M_1(\Gamma_0). \end{cases} \qquad \text{(VII.3.4)}$$

Then any bounded set of $BT_1(\Omega)$ is compact for τ_1' ; moreover, the functional J_1^* is lower semi continuous on $BT_1(\Omega)$ endowed with the τ_1' topology.

The proof of Lemma VII.3.2 is obvious.

We now state the main result for Problem (VII.2.4), namely :

Theorem VII.3.3. (i) Inf (VII.1.2) = Sup (VII.1.16) = Sup(VII.2.4)

(ii) Problem (VII.2.4) has a solution in $BT_1(\Omega)$, which is an adjoint state of the optimal state y_0.

Proof : Theorem VII.3.3 is proved exactly as was Theorem VI.2.3, using the direct method of Calculus of Variations. The use of that method is made possible by Lemmas VII.3.1 and VII.3.2.

4. System of optimality conditions.

We now present the system of optimality conditions for Problem (R).

<u>Theorem VII.4.1.</u> Let y be an admissible state for (R) ; then y is the optimal state y_0 if and only if there exists q in $BT_1(\Omega)$, admissible for Problem *(S*)*, such that (y,q) satisfy the following equalities :

$$h_{\alpha',\beta'}(q) = q.(\partial y/\partial r) - 1/2(\partial y/\partial r)^2 \text{ on } \Gamma_0 \qquad \text{(VII.4.i)}$$

$$h_{\alpha,\beta}(-\partial q/\partial r) = -(\partial q/\partial r).y - 1/2(y)^2 \text{ in } M_1(\Gamma_0) \qquad \text{(VII.4.ii)}$$

$$\partial q/\partial r = y - y_d \text{ on } \Gamma_1 \qquad \text{(VII.4.iii)}$$

Moreover, Equalities (VII.4.i) to (VII.4.iii) are equivalent to the condition that y (resp. q) should be a solution of *(R)* (resp. *(S*)*).

Proof : obvious, thanks to lemma VII.3.1.

VIII. Penalization of Problem (P).

We now return to the study of Problem (P), for which we propose a family of penalized problems $(P_\varepsilon)_{\varepsilon>0}$. We prove the existence and unicity in W of the optimal state z_ε of (P_ε) and of its adjoint state p_ε, and give the system of optimality conditions. We then study the asymptotic behaviour of the families $(z_\varepsilon)_{\varepsilon>0}$ and $(p_\varepsilon)_{\varepsilon>0}$ as ε tends to zero.

1. Definition and variational formulation of (P_ε).

Let α,β,ε be three strictly positive real numbers, we set

$$ {}_\alpha^\beta h_\varepsilon(s) \begin{cases} = s^2/2, \text{ if } s\in[-\alpha,\beta] \\ = \beta(s-\beta/2) + \varepsilon/2\,(s-\beta)^2, \text{ if } s\geqslant\beta \\ = -\alpha(s+\alpha/2) + \varepsilon/2\,(s+\alpha)^2, \text{ if } s\leqslant-\alpha \end{cases} \tag{VIII.1.1} $$

${}_\alpha^\beta h_\varepsilon$ is a C^1 convex function that converges to $h_{\alpha,\beta}$ (def. (I.2.9)) as ε tends to zero.

Its conjugate function is given by :

$$ {}_\alpha^\beta h_\varepsilon^*(s) \begin{cases} = s^2/2, \text{ if } s\in[-\alpha,\beta] \\ = 1/2\varepsilon\,(s-\beta)^2 + \beta(s-\beta/2), \text{ if } s\geqslant\beta \\ = 1/2\varepsilon\,(s+\alpha)^2 - \alpha(s+\alpha/2), \text{ if } s\leqslant-\alpha \end{cases} \tag{VII.1.2} $$

${}_{\alpha}{}^{\beta}h_{\varepsilon}^{*}$ is a C^1 convex function converging pointwise to the conjugate $h^{*}_{\alpha,\beta}$ of $h_{\alpha,\beta}$, defined by :

$$h^{*}_{\alpha,\beta}(s) \begin{cases} = s^2/2, \text{ if } s \in [-\alpha,\beta] \\ = +\infty \text{ otherwise} \end{cases} \qquad \text{(VIII.1.3)}$$

The penalized form of Problem (P) is :

to find $(z_{\varepsilon}, u_{0\varepsilon}, v_{0\varepsilon}) \in W \times L^2(\Gamma_0) \times L^2(\Gamma_0)$ minimizing the cost function

$$(P_{\varepsilon}) \qquad J_{\varepsilon}(z,u,v) = 1/2\|z-z_d\|^2_{L^2(\Omega)} + \int_{\Gamma_0} {}_{\alpha}{}^{\beta}h_{\varepsilon}^{*}(u)d\Gamma + \int_{\Gamma_0} {}_{\alpha}{}^{\beta}h_{\varepsilon}^{*}(v)d\Gamma + 1/2\varepsilon \| \Delta z \|^2_{L^2(\Omega)} .$$

the infimum being taken for $z \in W$, $z = u$ on Γ_0, $\partial z/\partial r = v$ on Γ_0 (z_d as in Section I).

<u>Remark VIII.1.1</u> .(P_{ε}) is a problem without constraints.

Let us now give the variational formulation of (P_{ε})..

In order to do so, we define on W the functional :

$$j_{\varepsilon}(z) = 1/2\|z-z_d\|^2_{L^2(\Omega)} + 1/2\varepsilon\|\Delta z\|^2_{L^2(\Omega)} + \int_{\Gamma_0} {}_{\alpha}{}^{\beta}h_{\varepsilon}^{*}(\partial z/\partial r)d\Gamma + \int_{\Gamma_0} {}_{\alpha}{}^{\beta}h_{\varepsilon}^{*}(z)d\Gamma \qquad \text{(VIII.1.5)}$$

and (P_{ε}) is equivalent to the minimization problem :

(P_ε) $\qquad \inf_{z \in W} \{j_\varepsilon(z)\}$ (VIII.1.6)

We have the following result [E.T]:

Proposition VIII.1.. There exists a unique optimal state z_ε for (P_ε).

The dual problem (P^*_ε) of (P_ε) is determined as in Section I, and its expression is

$$(P_\varepsilon^*) \quad \sup_{\substack{p \in W \\ p = \partial p/\partial r = 0 \text{ on } \Gamma_1}} \{-\varepsilon/2\|p\|^2_{L^2(\Omega)} - 1/2\|\Delta p\|^2_{L^2(\Omega)} - \int_\Omega \Delta p . z_d dx - \int_{\Gamma_0} {}_\alpha^\beta h_\varepsilon(-\partial p/\partial r) d\Gamma - \int_{\Gamma_0} {}_{\alpha'}^{\beta'} h_\varepsilon(p) d\Gamma\} \qquad \text{(VIII.1.7)}$$

Thanks to the definition of ${}_\alpha^\beta h_\varepsilon$, (P_ε^*) is a coercive problem on W ; hence, we can state :

Proposition VIII.1.1. There exists a unique adjoint state P_ε for the optimal state z_ε .

Moreover, the set of optimality conditions for z_ε and p_ε is easily obtained ([E.T], Th. 4.2 and Rem. 4.2), and is summed up in the following proposition :

Proposition VIII.1..2. An element z of w is an optimal state for (P_ε) if and only if there exists p in w, such that (z,p) satisfy the equalities :

$$\Delta p = z - z_d \text{ in } \Omega ; \qquad \text{(VIII.1.i)}$$

$$p = -1/\varepsilon\, \Delta z \text{ in } \Omega ; \qquad \text{(VIII.1.ii)}$$

$${}_\alpha^\beta h_\varepsilon\,(-\partial p/\partial r) = -\partial p/\partial r.z - {}_\alpha^\beta h_\varepsilon{}^*(z) \text{ on } \Gamma_0 \qquad \text{(VIII.1.iii)}$$

$${}_{\alpha'}^{\beta'} h_\varepsilon(p) = p.\partial z/\partial r - {}_{\alpha'}^{\beta'} h_\varepsilon{}^*(\partial z/\partial r) \text{ on } \Gamma_0 \qquad \text{(VIII.1.iv)}$$

Moreover, Equalities (VII.1.i) to (VIII.1.iv) are equivalent to the condition that z (resp. p) should be equal to the solution z_ε (resp. p_ε) of *(P_ε)* (resp. *$(P_\varepsilon{}^*)$*).

2. Asymptotic behaviour of the optimal state z_ε .

We now prove the :

Theorem VIII.2.1. The family $\{z_\varepsilon\}_{\varepsilon > 0}$ tends to the optimal state z_0 of (P) for the strong topology of W as ε tends to zero.

Proof :

We first notice that the family $\{j_\varepsilon(z_\varepsilon)\}_{\varepsilon > 0}$ is decreasing and satisfies :

$$j(z_0) = j_\varepsilon(z_0) \geqslant j_\varepsilon(z_\varepsilon) \qquad \text{(VIII.2.1)}$$

hence, there exists a number ζ with $\zeta = \lim_{\varepsilon \to 0} (j_\varepsilon(z_\varepsilon))$.

Thanks to (VIII.2.1), z_ε is bounded in W, and also satisfies :

$$\lim_{\varepsilon \to 0} \|\Delta z_\varepsilon\|^2_{L^2(\Omega)} = 0 \qquad \text{(VI.2.2)}$$

We now extract a subsequence still denoted by z_ε which converges weakly in W towards an element z'_0 of W. Let us now show that z'_0 is admissible for problem (I.1.3).

First of all, we obviously have :

$$\Delta z_0' = 0 \text{ in } \Omega \qquad \text{(VIII.2.3)}$$

Let us choose a fixed η ; we have :

$$\int_{\Gamma_0} {}^\beta_\alpha h_\eta^*(z'_0) \leqslant \underline{\lim}_{\varepsilon \to 0} \int_{\Gamma_0} {}^\beta_\alpha h_\eta^*(z_\varepsilon) d\Gamma \qquad \text{(VIII.2.4)}$$

thanks to the weak convergence of z_ε towards z_0'. But ${}_\alpha^\beta h_\varepsilon^*$ is a decreasing function of ε, and therefore we obtain from (VIII.2.4) :

$$\int_{\Gamma_0} {}^\beta_\alpha h_\eta^*(z_0') \leqslant \underline{\lim}_{\varepsilon \to 0} \int_{\Gamma_0} {}^\beta_\alpha h_\varepsilon^*(z_\varepsilon) d\Gamma \qquad \text{(VIII.2.5)}$$

The sequence $(\int_{\Gamma_0} {}^{\beta}{}_{\alpha} h_{\eta}^{*}(z'_0)d\Gamma)$ is then bounded, and this necessarily implies that z_0' belongs to $\mathcal{U}_{\alpha,\beta}$; we prove in the same way that $\partial z'_0/\partial r$ belongs to $\mathcal{U}_{\alpha',\beta'}$, and z_0' is therefore admissible for problem (I.1.3).

By lower semi continuity we have :

$$j(z'_0) \leq \varliminf_{\varepsilon \to 0} (1/2 \,\| z_\varepsilon - z_d \|^2_{L^2(\Omega)} + 1/2 \,\| \partial z_\varepsilon/\partial r \|^2_{L^2(\Gamma_0)}$$

$$+ 1/2 \,\| z_\varepsilon \|^2_{L^2(\Gamma_0)})$$

$$\leq \varliminf_{\varepsilon \to 0} (j_\varepsilon(z_\varepsilon))$$

and, using (VIII.2.1) :

$$j(z_0') \leq \varliminf_{\varepsilon \to 0} j_\varepsilon(z_\varepsilon) \leq \lim_{\varepsilon \to 0} j_\varepsilon(z_\varepsilon) \leq j(z_0) \qquad \text{(VIII.2.6)}$$

so that we finally obtain :

$$j(z'_0) = \lim_{\varepsilon \to 0} j_\varepsilon(z_\varepsilon) = j(z_0) \qquad \text{(VIII.2.7)}$$

Thus, z'_0 and z_0 are equal ; let us now prove the strong convergence of z_ε to z_0. We set :

$$H(z_0,z_\varepsilon) = j(z_0) - 1/2 \,\|z_\varepsilon - z_d\|^2_{L^2(\Omega)} - 1/2 \,\|z_\varepsilon\|^2_{L^2(\Gamma_0)}$$

$$-1/2 \,\|\partial z_\varepsilon/\partial r\|^2_{L^2(\Gamma_0)} \qquad \text{(VIII.2.8)}$$

$H(z_0,z_\varepsilon)$ is positive, and tends to zero with ε. Owing to the weak convergence of z_ε to z_0, we have :

$$\lim_{\varepsilon\to 0} (\|z_0\|^2_{L^2(\Omega)} - \|z_\varepsilon\|^2_{L^2(\Omega)}) = \lim_{\varepsilon\to 0} (\|z_0\|^2_{L^2(\Gamma_0)} - \|z_\varepsilon\|^2_{L^2(\Gamma_0)})$$

$$= \lim_{\varepsilon\to 0} (\|\partial z_0/\partial r\|^2_{L^2(\Gamma_0)} - \|\partial z_\varepsilon/\partial r\|^2_{L^2(\Gamma_0)}) = 0 \qquad \text{(VIII.2.9)}$$

and (VIII.2.9), together with the weak convergence of z_ε, is obviously enough to prove the strong convergence of z_ε towards z_0, for W is a Hilbert space.

Finally, we conclude by an easy contradiction argument that the whole familly $(z_\varepsilon)_{\varepsilon>0}$ converges to z_0, and Theorem VIII.2.1. is proved.

3. Asymptotic behaviour of the adjoint state p_ε.

Theorem VIII.3.1. The family $\{p_\varepsilon\}_{\varepsilon\geqslant 0}$ has a cluster point in $BT(\Omega)$ which is an adjoint state of the optimal state z_0 of (P).

Proof : Thanks to the equality Inf (VIII.1.6) = Sup(VIII.1.7), we know that $\{p_\varepsilon\}_{\varepsilon>0}$ is bounded in $BT(\Omega)$, and therefore compact for the τ_1 topology.

Let w be a cluster point of this family for τ_1 ; the lower semi continuity of J^* ensures that w is a solution of Problem (V.1.3). .

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