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Sur quelques problèmes non linéaires en physique des plasmas. Sur des problèmes de diffusion non linéaires en hydrologie et en dynamique des populations. Sur quelques méthodes de parallélisation automatique de programmes

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DANIELLE HILHORST-GOLDMAN

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. SUR DES PROBLÈMES DE DIFFUSION NON LINÉAIRES EN HYDROLOGIE ET EN DYNAMIQUE DES POPULATIONS.

2ème Thèse : SUR QUELQUES MÉTHODES DE PARALLÉLISATION AUTOMATIQUE DE PROGRAMMES.

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SUBJECT

- On some nonlinear problems arising in plasma physics.
- On some nonlinear diffusion problems in hydrology and in population dynamics.

ABSTRACT

In the first part of this thesis we consider certain nonlinear problems arising in plasma physics. We first study a singular two-point nonlinear boundary value problem on an interval $(0,R)$; we prove that it has a unique solution and study its limiting behavior as $R \to \infty$ and as a small parameter $\varepsilon \downarrow 0$. We also study the large time behavior of a related evolution problem. We then extend our study to more general boundary value problems in higher dimension and show that as $\varepsilon \downarrow 0$ their solution converges to the solution of a free boundary problem.

The second part of the thesis concerns the study of certain nonlinear diffusion problems. We first show the existence and uniqueness of the solution of boundary value problems related to a doubly nonlinear diffusion equation in hydrology and study its asymptotic behavior as $t \to \infty$. We then consider a system of nonlinear degenerate parabolic equations which models the time evolution of the densities of two interacting biological populations. We suppose that their supports are initially disjoint. Our results concern the time evolution and the large time behavior of those populations and of their supports, and the regularity of the boundaries of the supports.

KEY WORDS

Nonlinear elliptic equations, Singular perturbations, Nonlinear diffusion equations, Free boundary problems.

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INTRODUCTION

INTRODUCTION À LA PREMIÈRE PARTIE : SUR QUELQUES PROBLÈMES NON LINÉAIRES EN PHYSIQUE DES PLASMAS.

Le problème que nous étudions provient de la physique des plasmas. On considère une assemblée d'ions et d'électrons. Les ions, lourds et lents, sont considérés comme fixes dans l'échelle de temps considérée et l'on se propose de déterminer la densité des électrons, leur charge totale étant supposée connue. Si l'on prend comme inconnue le potentiel électrique, on obtient le problème

$$\overline{\text{BVP}} \quad \begin{cases} -\Delta u + e^{u/\varepsilon} = f \quad \text{dans} \quad \Omega \\ \displaystyle\int_\Omega e^{u(x)/\varepsilon}\, dx = C \qquad\qquad C > 0 \\ u_{|\partial\Omega} = \text{constante (inconnue)} \quad , \end{cases}$$

où Ω est un domaine de $\mathbb{R}^n$, borné ou non, et ε une constante proportionnelle à la température. La fonction f correspondant à la densité ionique est supposée connue ; la quantité $e^{u/\varepsilon}$ correspond à la densité des électrons et la condition intégrale exprime le fait que leur charge totale est connue. On suppose de plus que le domaine Ω est entouré d'un conducteur électrique, ce qui implique la condition $u_{|\partial\Omega}$ = constante.

La situation expérimentale étudiée par Bastien et Marode est celle d'une décharge filiforme entre deux électrodes ; le système physique présente une symétrie de révolution par rapport à l'axe x_3 et l'on suppose que les fonctions considérées ne dépendent que de $r = \sqrt{x_1^2 + x_2^2}$. Dans le Chapitre I, nous explicitons les lois physiques qui déterminent le problème et nous résumons les résultats des Chapitres II-IV en termes physiques.

Dans le Chapitre II, nous considérons le problème aux limites qu'on obtient en posant $y(x) = \int_0^{\sqrt{x}} e^{u(r)/\varepsilon}\, r\, dr$ et $g(x) = \int_0^{\sqrt{x}} f(r)\, r\, dr$, soit

$$P(\varepsilon,R) \begin{cases} \varepsilon\, x\, y'' + (g(x) - y)y' = 0 \;, & x \in (0,R) \\ y(0) = 0 \qquad y(R) = C \;, & \end{cases}$$

et nous supposons que la fonction g est suffisamment régulière, strictement croissante et strictement concave et telle que $C \in (0,g(\infty))$. Nous montrons que $P(\varepsilon,R)$ a une solution unique y qui converge vers une limite $\bar{y}$ quand $R \to \infty$. Si $\varepsilon \leqq g(\infty) - C$, $\bar{y}$ coïncide avec la solution de $P(\varepsilon,\infty)$; si $\varepsilon > g(\infty) - C$, $P(\varepsilon,\infty)$ n'a pas de solution. Nous prouvons ensuite que, quand $\varepsilon \downarrow 0$, y converge vers la fonction $\min(g(x),C)$. Pour les démonstrations nous employons essentiellement des arguments liés au principe du maximum et à la construction de sur- et sous-solutions.

Nous nous intéressons ensuite à la stabilité de la solution du Problème $P(\varepsilon,\infty)$, ce qui nous amène, dans le Chapitre III, à l'étude du problème d'évolution non linéaire

$$P_1 \begin{cases} v_t = \varepsilon x v_{xx} + (g(x)-v)v_x & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ v(0,t) = 0 & t \in [0,\infty) \\ v(x,0) = \psi(x) & x \in \mathbb{R}^+ \end{cases}$$

où ψ est une fonction croissante telle que $\psi(0) = 0$ et $\psi(\infty) = C$. Nous démontrons que le Problème P_1 admet une solution classique unique v et nous étudions le comportement asymptotique de v quand $t \to \infty$. En particulier, on déduit que, si $\varepsilon < g(\infty) - C$, la solution y de $P(\varepsilon,\infty)$ est algébriquement stable. Nous considérons également le cas limite $\varepsilon \downarrow 0$; quand $\varepsilon \downarrow 0$, v converge vers la solution généralisée $\bar{v}$ du problème hyperbolique correspondant et quand $t \to \infty$, $\bar{v}$ converge algébriquement vers sa limite.

Si l'on ne suppose plus que la fonction g est monotone, le problème réduit correspondant à $P(\varepsilon,R)$ a en général une infinité de solutions ; il s'agit de déterminer laquelle est la limite de y quand $\varepsilon \downarrow 0$. Dans le but de rendre les démonstrations moins techniques nous considérons dans le Chapitre IV le problème $\varepsilon y'' + (g-y)y' = 0$, $y(0) = 0$ et $y(1) = 1$ où l'on suppose $g \in L^2(0,1)$. Pour étudier ce problème nous utilisons la théorie des opérateurs maximaux monotones et aboutissons finalement à une caractérisation concrète de la limite de y quand $\varepsilon \downarrow 0$.

Dans le Chapitre V, nous étudions le comportement limite, quand $\varepsilon \downarrow 0$, de la solution u_ε du problème

$$\text{BVP}\quad \begin{cases} -\Delta u + h\left(\frac{u}{\varepsilon}\right) = f & \text{dans } \Omega \\ \int_\Omega h\left(\frac{u(x)}{\varepsilon}\right)dx = C & h(-\infty) < C/|\Omega| < h(+\infty) \\ u_{|\partial\Omega} = \text{constante (inconnue)}, & \end{cases}$$

où nous supposons Ω borné et la fonction h continue, strictement croissante et telle que $D(h) = \mathbb{R}$ et $h(0) = 0$. Ce problème correspond à $\overline{\text{BVP}}$ dans le cas où $h(s) = e^s - 1$. On utilise une méthode variationnelle et la théorie de la dualité pour montrer l'existence et l'unicité de la solution u_ε de ce problème. Quand $\varepsilon \downarrow 0$, u_ε converge vers la solution d'un problème à frontière libre, que l'on peut mettre sous la forme d'une relation d'inclusion, si la fonction h est bornée, ou d'une inéquation variationnelle si $h(+\infty) = +\infty$ ou $h(-\infty) = -\infty$. Les résultats obtenus concordent avec ceux que Brauner et Nicolaenko obtiennent pour un problème provenant de la théorie cinétique des enzymes.

INTRODUCTION À LA DEUXIÈME PARTIE : SUR DES PROBLÈMES DE DIFFUSION NON LINÉAIRES EN HYDROLOGIE ET EN DYNAMIQUE DES POPULATIONS.

Cette partie porte sur des problèmes de diffusion non linéaires dégénérés.

Dans le Chapitre VI, nous étudions un modèle introduit par de Josselin de Jong pour décrire l'infiltration d'eau salée dans les nappes aquifères au voisinage des zones côtières. Plus précisément nous nous intéressons à l'évolution dans le temps de l'interface entre eau salée et eau douce. Mathématiquement, cela nous amène à étudier l'équation

$$E \qquad u_t = (D(u)\varphi(u_x))_x$$

où D et φ sont des fonctions suffisamment régulières telles que $D > 0$ sur $(0,1)$ et $D(0) = D(1) = 0$, $D'' \leqq 0$ sur $(0,1)$, $\varphi(0) = 0$, $\varphi' > 0$ sur $(-1,1)$, $\varphi'(-1) = \varphi'(1) = 0$. L'équation E dégénère à la fois aux points où $u = 0$ ou $u = 1$ et à ceux où $u_x = 1$ ou $u_x = -1$. Nous montrons l'existence et l'unicité de la solution de problèmes de Neumann, Cauchy et Cauchy-Dirichlet liés à cette équation et nous étudions le comportement asymptotique de la solution quand $t \to \infty$. Pour les démonstrations nous adaptons en particulier la méthode de monotonie et nous utilisons des arguments liés à un principe de comparaison et au fait que la solution vérifie une propriété de contraction dans L^1 .

Nous étudions ensuite un système d'équations de diffusion non linéaires intervenant en dynamique des populations. Il a pour origine un modèle de Gurtin et Pipkin décrivant la dispersion de deux populations biologiques en interaction qui se dispersent sous l'influence de la pression de population. Ce modèle est basé sur l'hypothèse que les vitesses de dispersion individuelles sont proportionnelles à $-(u+v)_x$ où u et v sont les densités correspondantes. Si l'on suppose de plus que l'habitat Ω est isolé, on obtient le problème d'évolution suivant

$$I \quad \left\{ \begin{array}{ll} u_t = (u(u+v)_x)_x & \text{dans } \Omega \times \mathbb{R}^+ \\ v_t = k(v(u+v)_x)_x & \text{dans } \Omega \times \mathbb{R}^+ \\ u(u+v)_x = 0 \text{ , } v(u+v)_x = 0 & \text{sur } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & \text{dans } \Omega \end{array} \right.$$

où $k \geqq 0$, $\Omega = (-L,L)$ avec $L > 0$ et où u_0 et v_0 sont des fonctions suffisamment régulières telles que $u_0, v_0 \geqq 0$.

Dans les Chapitres VII et VIII, nous considérons le cas où $k = 0$; l'équation en v du problème I implique alors $v = v(x)$ et l'on a donc une population mobile u en présence de la population sédentaire v . Le problème I se réduit alors au problème

$$\text{II}\quad\begin{cases} u_t = (u(u+v)_x)_x & \text{dans } \Omega\times\mathbb{R}^+ \\ u(u+v)_x = 0 & \text{sur } \partial\Omega\times\mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{dans } \Omega \end{cases}$$

Dans le Chapitre VII, nous étudions un problème plus général en dimension N quelconque

$$\text{III}\quad\begin{cases} u_t = \Delta\varphi(u) + \mathrm{div}(u \,\mathrm{grad}\, v) & \text{dans } \Omega\times\mathbb{R}^+ \\ \frac{\partial}{\partial\nu}\varphi(u) + u\frac{\partial v}{\partial\nu} = 0 & \text{sur } \partial\Omega\times\mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{dans } \Omega \end{cases}$$

où Ω est un domaine borné de $\mathbb{R}^n$, ν le vecteur unitaire normal à $\partial\Omega$, φ est une fonction régulière telle que $\varphi(0) = \varphi'(0) = 0$, $\varphi'(s) > 0$ pour tout $s > 0$, la fonction initiale $u_0 \in L^\infty(\Omega)$ est telle que $u_0 \geqq 0$ et v est une fonction donnée suffisamment régulière. Ce chapitre est composé de deux parties. Dans la deuxième partie nous montrons que le Problème III admet une solution généralisée unique $u(t,u_0)$. Dans la première partie, nous prouvons que $u(t,u_0)$ se stabilise quand $t \to \infty$. L'une des difficultés mathématiques provient du fait que l'on est en présence d'un continuum de solutions stationnaires ; pour les démonstrations on s'inspire d'une méthode de Osher et Ralston. L'idée est de prouver tout d'abord que u satisfait une propriété de contraction dans L^1 , de montrer que dans certains cas cette contraction est stricte et d'utiliser cette propriété et la structure du continuum de solutions stationnaires pour construire une fonctionnelle de Lyapounov.

Dans le Chapitre VIII on revient au Problème II et l'on s'intéresse à la question suivante : si à l'instant initial, le support de u est d'un côté du support de v , une partie de la population u peut-elle atteindre l'autre côté de la population sédentaire v pour t suffisamment grand, ou bien la population v constitue-t-elle une barrière infranchissable pour les membres de la colonie u ? La réponse dépend des proportions relatives des deux populations. Nous montrons que si la population mobile u est suffisamment importante par rapport à la population sédentaire v, alors u occupe tout l'habitat à partir d'un certain moment ; si par contre $\|u_0\|_{L^\infty(\Omega)} \leqq \|v\|_{L^\infty(\Omega)}$ alors la population v représente effectivement une barrière pour la population u. La plupart des démonstrations de ce chapitre s'appuient sur des résultats obtenus au Chapitre VII.

On s'intéresse ensuite au Problème I dans le cas où $k > 0$ et l'on suppose que les fonctions initiales u_0 et v_0 sont séparées, c'est-à-dire qu'il existe $a \in \Omega$ tel que

$$u_0(x) \equiv 0 \quad \text{si} \quad x > a \quad \text{et} \quad v_0(x) \equiv 0 \quad \text{si} \quad x < a \ .$$

On se propose de montrer que le Problème I admet une seule solution généralisée (u,v) telle que $u(t)$ et $v(t)$ sont séparées à tout instant t. L'idée est la suivante. On pose $U = \int_\Omega u_0$, $V = \int_\Omega v_0$ et l'on suppose que (u,v) est une solution du Problème I telle que $u(t)$ et $v(t)$ sont séparées à chaque instant t. Alors la fonction $z : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ définie par

$$(1) \qquad z(x,t) = -U + \int_{-L}^{x} (u(s,t) + v(s,t))ds$$

satisfait formellement le problème

$$\text{IV} \quad \begin{cases} (c(z))_t = (|z_x|^{m-1} z_x)_x & \text{dans } \Omega \times \mathbb{R}^+ \\ z(-L,t) = -U \quad z(L,t) = V & \text{pour } t \in \mathbb{R}^+ \\ z(x,0) = z_0(x) & \text{dans } \Omega \end{cases}$$

où

$$z_0(x) = -U + \int_{-L}^{x} (u_0 + v_0) \ ,$$

$m = 2$ et $c : \mathbb{R} \to \mathbb{R}$ est défini par

$$c(s) = \begin{cases} c^- s & \text{si } s \leqq 0 \\ c^+ s & \text{si } s \geqq 0 \end{cases}$$

avec $c^- = 2$ et $c^+ = 2/k$.

Dans le Chapitre IX, nous étudions le Problème IV avec $m > 1$, $c^- > 0$, $c^+ > 0$ quelconques et nous supposons que la fonction initiale z_0 est suffisamment régulière, non décroissante et qu'elle satisfait $z_0(-L) = -U$ et $z_0(L) = V$. Nous montrons que le Problème IV admet une solution généralisée unique $z(t,z_0)$ qui converge dans $C^1(\overline{\Omega})$ vers l'unique solution stationnaire quand $t \to \infty$. Le but principal de ce chapitre est de donner une description détaillée de l'ensemble

$$\mathbb{N}(z) := \{(x,t) \in \overline{\Omega} \times [0,\infty) \quad : z(x,t) = 0\} .$$

On obtient en particulier le résultat suivant : il existe des fonctions continues $\zeta^{\pm} : [0,\infty) \to \Omega$ telles que

$$\mathbb{N}(z) = \{(x,t) \in \overline{\Omega} \times [0,\infty) \quad : \quad \zeta^-(t) \leqq x \leqq \zeta^+(t)\} .$$

De plus il existe $T^* \geqq 0$ tel que

ζ^- est croissante, ζ^+ est décroissante sur $[0,T^*]$

et

$\zeta(t) := \zeta^-(t) = \zeta^+(t)$ pour $t \geqq T^*$, $\zeta \in C^1((T^*,\infty))$

et $z_x(\zeta(t),t) > 0$ pour $t > T^*$.

Un élément essentiel de la démonstration est constitué par un changement de coordonnées introduit par Gurtin, MacCamy et Socolovsky.

Dans le Chapitre X on montre que le Problème IV est équivalent au problème de rechercher une solution (u,v) du Problème I telle que $u(t)$ et $v(t)$ soient séparées à tout instant t. Donc, si les fonctions initiales u_0 et v_0 sont séparées, le Problème I admet une seule solution (u,v) telle que $u(t)$ et $v(t)$ sont séparées pour tout t et l'on peut immédiatement déduire des résultats du Chapitre IX des informations sur les supports de u et v : l'ensemble $\mathbb{N}(z)$ constitue le front de séparation entre les deux populations ; à partir de l'instant T^*, les supports de u et v ne sont donc plus séparés que par une courbe de classe C^1 (la courbe $x = \zeta(t)$) qui représente une ligne de discontinuité pour les deux fonctions u et v.

CHAPITRE I

RIGOROUS RESULTS ON A TIME-DEPENDENT INHOMOGENEOUS COULOMB GAS PROBLEM

par

D. Hilhorst, H.J. Hilhorst et E. Marode.

I.1.

RIGOROUS RESULTS ON A TIME-DEPENDENT INHOMOGENEOUS COULOMB GAS PROBLEM

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We report results obtained by rigorous analysis of a nonlinear differential equation for the electron density n_e in a specific type of electrical discharge. The problem is essentially two-dimensional. We discuss in particular (i) the escape of electrons to infinity above a critical temperature; and (ii) the boundary layer exhibited by n_e near zero temperature.

In a filamentary discharge studied by Marode et al. [1,2] electrons and ions are produced with number densities n_e and n_i, respectively. The charged particles move in a background of neutrals. The discharge area is cylindrical and has its radial dimension much smaller than its longitudinal dimension. Since to a good approximation the physical situation is cylindrically symmetric, it suffices to consider a two-dimensional cross section perpendicular to the cylinder axis, in which all quantities involved are functions only of the distance r to the axis. As the ions are heavy and slow, $n_i(r, t) \equiv n_i(r)$ may be regarded as fixed on the time scale of interest. For the density $n_e(r, t)$ Marode et al. [3] used the following three equations:

(i) Coulomb's law:

$$r^{-1}\, \partial[rE(r,t)]/\partial r = 4\pi e\,[n_i(r) - n_e(r,t)]\,, \tag{1}$$

where E is the electric field and $-e$ the electron charge;

(ii) a constitutive equation for the current density $j(r)$, consisting of a drift term and a diffusion term,

$$j(r,t) = e\mu n_e(r,t)E(r,t) + eD\, \partial n_e(r,t)/\partial r\,, \tag{2}$$

where μ is the electron mobility and D the diffusion constant; and

(iii) the continuity equation

$$e\, \partial n_e(r,t)/\partial t = r^{-1}\, \partial[rj(r,t)]/\partial r\,. \tag{3}$$

Both E and j are radially directed.

From eqs. (1)–(3) a nonlinear partial differential equation for a single function can be derived. To this end we set [4]

$$u(x,t) = \int_0^{\sqrt{x}} \rho n_e(\rho,t)\,\mathrm{d}\rho\,, \tag{4a}$$

$$g(x) = \int_0^{\sqrt{x}} \rho n_i(\rho)\,\mathrm{d}\rho\,. \tag{4b}$$

Upon employing for the diffusion constant the Einstein relation $D = k_B T\mu/e$ (where k_B is Boltzmann's constant and T the electron temperature), putting $\epsilon = k_B T/(2\pi e^2)$, and absorbing a factor $8\pi\mu e$ in the time scale we deduce that u satisfies

$$u_t = \epsilon x u_{xx} + (g - u)u_x\,, \tag{5}$$

$$u(0,t) = 0\,. \tag{6}$$

By its definition $g(0) = 0$. Typically, as r increases, I.2.

$n_i(r)$ rapidly falls off to zero, and hence $g(x)$ attains a limit value $g(\infty)$. The nonlinear term in eq. (5) represents the interaction between the electrons. Without it, this equation would reduce to a linear one studied by McCauley [5] and describing the brownian motion of a pair of opposite two-dimensional charges in each other's field. As it stands, eq. (5) is rather reminiscent of the nonlinear equations occurring in the Thomas–Fermi theory of the atom (see, e.g., ref. [6]).

In the experimental situation that we are describing the total charge in the discharge area is positive and conserved in time. This is expressed by

$$u(\infty, t) = N_e, \quad \text{for } 0 \leqslant t < \infty, \tag{7}$$

with $0 \leqslant N_e < g(\infty)$. One of the authors has investigated [4,7,8], by rigorous mathematical methods, the solution of eqs. (5) and (6) for a given initial distribution $u(x, 0) = u_0(x)$ and subject to condition (7) on the total charge. Here we present the main results in physical language.

1. We take g concave and in $C^2([0,\infty))$. Then at given ϵ (i.e. at given temperature), there exists [4] a unique stationary solution $u_{st}(x)$ if the total number of electrons N_e is such that $N_e \leqslant g(\infty) - \epsilon$. In particular, when $\epsilon \geqslant g(\infty)$, thermal motion prevents any electrons to be bound to the fixed ionic background. The existence of such a critical temperature is characteristic of two-dimensional Coulomb systems [9]. The main mathematical tools in treating the stationary problem are maximum principle arguments and the construction of upper and lower solutions.

2. The solution u_{st}, when it exists, has the following properties [4].

(i) It belongs to $C^2([0,\infty))$. It is strictly increasing, concave, and bounded from above by the function $\min(g(x), N_e)$. As $x \to \infty$, $u_{st}(x)$ approaches its limiting value N_e at least fast enough so that

$$n_e(r) \leqslant n_e(r_1)(r^2/x_1)^{-[g(x_1)-N_e]/\epsilon}, \quad r \to \infty, \tag{8}$$

where $r_1^2 \equiv x_1 > 0$ is arbitrary. Such power law decay is again typical of Coulomb systems in two dimensions.

(ii) As $\epsilon \downarrow 0$, $u_{st}(x)$ converges to $\min(g(x), N_e)$ uniformly on $[0,\infty)$, and we have for the zero temperature limit of the electron density

$$\lim_{\epsilon\downarrow 0} n_e(r) = n_i(r), \quad r < r_0, \\ = 0, \quad r > r_0, \tag{9}$$

where the critical radius r_0 is defined by the relation $g(r_0) = N_e$. At small ϵ there is a transition layer of width $\sim\epsilon^{1/2}$, located at r_0, analogous to a Debye shielding length [3]. A uniformly valid approximate stationary solution for $\epsilon \ll 1$ is given in ref. [4]. It is obtained by the method of matched asymptotic expansions.

3. We consider now the time evolution problem of eqs. (5) and (6). Suppose that the initial condition u_0 is sufficiently smooth, nondecreasing, with bounded derivative, and with $u_0(0) = 0$ and $u_0(\infty) = N_e$. Mathematically one has to find a way to deal with the degeneracy of the parabolic equation (5) in the origin. In ref. [7] this is done via a sequence of regularized problems. The following is shown.

(i) The time evolution problem has a unique solution $u(x, t)$ such that u and u_x are bounded. In fact it satisfies $0 \leqslant u(x,t) \leqslant N_e$, it is nondecreasing in x for all t, and for each $t \geqslant 0$ we have $u(\infty, t) = N_e$.

(ii) In order to discuss the behavior of $u(x, t)$ as $t \to \infty$ we consider the function $\bar{u}_{st}$ which satisfies the steady-state equation and has boundary values $\bar{u}_{st}(0) = 0$ and

$$\bar{u}_{st}(\infty) = N_e, \quad \text{if } N_e \leqslant g(\infty) - \epsilon, \tag{10a}$$

$$= g(\infty) - \epsilon, \quad \text{if } 0 < g(\infty) - \epsilon < N_e, \tag{10b}$$

$$= 0, \quad \text{otherwise}. \tag{10c}$$

We know from section 1 that $\bar{u}_{st}$ exists and is unique. In particular, in the case of eq. (10c), $\bar{u}_{st}(x) \equiv 0$. Our result is that the solution $u(x, t)$ of the evolution problem converges to $\bar{u}_{st}(x)$ as $t \to \infty$, uniformly on all compact subsets of $[0,\infty)$; in the case of eq. (10a) the convergence is actually uniform on $[0,\infty)$. The proofs are based upon the use of upper and lower solutions of the stationary problem and on a comparison theorem. Thus we have proved that all the electrons stay attached to the ions for $t \leqslant \infty$ at temperatures such that $\epsilon \leqslant g(\infty) - N_e$ [case (10a)]. If the temperature rises above this critical value, then some of the electrons diffuse away to infinity [case (10b)], and if it rises above a second critical value, viz. $\epsilon = g(\infty)$, then all electrons escape to infinity [case (10c)].

(iii) For the case of eq. (10a) (with the inequality strictly satisfied) we have derived results about the rate of convergence of u to $\bar{u}_{st}$. Let the initial state have the property that $N_e - u_0(x) \leqslant N_e(x_1/x)^\nu$ for some $x_1, \nu > 0$ satisfying $\epsilon \leqslant (\nu + 1)^{-1}[g(x_1) - N_e]$.

Then $u(x,t)$ converges to $\bar{u}_{\mathrm{st}}(x)$ at least as fast as $t^{-1/(2p)}$ with $p = [1/\nu] + 1$, for all finite x. Furthermore, if $\nu > 1$ and $\epsilon < \frac{1}{2}[g(\infty) - N_{\mathrm{e}}]$, then u converges to $\bar{u}_{\mathrm{st}}$ at least as fast as $t^{-1/2}$.

4. Negative regions in the background charge density. We have considered an interesting modification of the above problem obtained by also allowing negative ions to be present in the fixed background [8]. This leads to a function g which can assume minima and maxima. We studied the stationary state on a bounded domain $[0,R]$ with boundary condition $u_{\mathrm{st}}(R) = N_{\mathrm{e}}$. For non-monotone g it is nontrivial to find the zero temperature ($\epsilon \to 0$) limit of $u_{\mathrm{st}}(x)$ [and thus of $n_{\mathrm{e}}(r)$], since the solution of the reduced differential equation (i.e. the one obtained by setting $\epsilon = 0$) is no longer unique. To solve this problem we observe that for $\epsilon > 0$ the solution $u_{\mathrm{st}}(x;\epsilon)$ minimizes the free-energy functional

$$F_\epsilon[u] = \epsilon \int_0^R u_x \ln u_x \,\mathrm{d}x + \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} \,\mathrm{d}x\,, \qquad (11)$$

which is readily recognized as the sum of an entropy and an electrostatic energy term.

In ref. [8] two alternative methods were used to study the minimization of F_ϵ: one based on the theory of maximal monotone operators and one on duality theory. Both yield

$$\lim_{\epsilon \downarrow 0} u_{\mathrm{st}}(x;\epsilon) = \inf_{0 \leqslant u \leqslant N_{\mathrm{e}},\, u' \geqslant 0} \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} \,\mathrm{d}x\,, \qquad (12)$$

i.e. the limit solution of the differential equation is the physically expected minimum-energy configuration. The function $u_{\mathrm{st}}(x;0)$ is continuous [10] and can be characterized as follows: there exist intervals $[a_1, b_1], [a_2, b_2], ..., [a_s, b_s], s \geqslant 0$, where $u_{\mathrm{st}}(x;0)$ takes constant values $c_1, c_2, ..., c_s$, respectively, and where, therefore, $n_{\mathrm{e}}(r) = 0$. Outside those intervals $u_{\mathrm{st}}(x;0) = g(x)$. The constants a_i, b_i and c_i, $i = 1, 2, ..., s$, can be shown, finally, to be uniquely determined by the set of implicit inequalities

$$\left.\begin{aligned} &\int_x^{b_i} \frac{c_i - g(\xi)}{\xi}\,\mathrm{d}\xi \geqslant 0\,, \quad \text{if } c_i \neq N_{\mathrm{e}} \\ &\int_{a_i}^{x} \frac{c_i - g(\xi)}{\xi}\,\mathrm{d}\xi \leqslant 0\,, \quad \text{if } c_i \neq 0 \end{aligned}\right\} \qquad (13)$$

for all $x \in [a_i, b_i]$, $i = 1, 2, ..., s$.

To verify this characterization of $u_{\mathrm{st}}(x;0)$, one checks [8] that this function satisfies a variational inequality related to the minimization problem (12). In particular, if $0 < c_i < N_{\mathrm{e}}$, we have the equal area construction $\int_{a_i}^{b_i}[c_i - g(\xi)]\,\xi^{-1}\,\mathrm{d}\xi = 0$. The interpretation is that the points $x = a_i$ and $x = b_i$ are at equal potential and separated by a potential barrier. Eqs. (13) may serve as the basis for a numerical algorithm to compute a_i, b_i and c_i.

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CHAPITRE II

A SINGULAR BOUNDARY VALUE PROBLEM ARISING IN A PRE-BREAKDOWN GAS DISCHARGE

par

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A SINGULAR BOUNDARY VALUE PROBLEM ARISING IN A PRE-BREAKDOWN GAS DISCHARGE*

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Abstract. We consider the nonlinear two-point boundary value problem $\varepsilon xy''+(g(x)-y)y'=0$, $y(0)=0$, $y(R)=k$, where g is a given function. We prove that the problem has a unique solution and we study the limiting behavior of this solution as $R\to\infty$ and as $\varepsilon\downarrow 0$.

Furthermore, we show how a so-called pre-breakdown discharge in an ionized gas between two electrodes can be described by an equation of this form, and we interpret the results physically.

1. Introduction. In this paper we study the two-point boundary value problem

$$\varepsilon xy''+(g(x)-y)y'=0, \qquad x\in(0,R), \tag{1.1}$$

in which R is a positive number, which may be infinite, and g a given function, which satisfies the hypotheses

$$H_g\colon g\in C^2(\mathbb{R}_+), \qquad g(0)=0, \qquad g'(x)>0 \quad \text{and} \quad g''(x)<0 \quad \text{for all } x\geqq 0.$$

We are interested in solutions of (1.1) which satisfy the boundary conditions

$$y(0)=0, \tag{1.2}$$

$$y(R)=k \tag{1.3}$$

in which $k\in(0,g(\infty))$ and $R>x_0$, x_0 being the (unique) root of the equation $g(x)=k$.

In § 2 we shall sketch how problem (1.1)–(1.3) arises in the study of electrical discharges in an ionized gas. It will appear that y' and g' are measures for, respectively, the electron and ion densities, and that the parameter ε is proportional to the temperature of the gas.

In § 3 we begin the mathematical analysis of problem (1.1)–(1.3). We derive some a priori estimates and then prove the existence of a solution. Subsequently, in § 4 we prove that the solution is unique.

The main objective of this paper is the study of the dependence of the solution on the parameters ε and R. In § 4 we prove that the solution is a monotone function of ε and R. From the physical point of view the interesting regions of the parameters are small ε and large R. In § 5 we analyze the limiting behavior of the solution when R tends to infinity and ε is kept fixed. It turns out that the solution converges uniformly in x to a function $\bar{y}$ which satisfies (1.1)–(1.2) and the limiting form of (1.3), i.e., $\bar{y}(\infty)=k$, if and only if $\varepsilon\leqq g(\infty)-k$. If on the other hand, this inequality is violated, then the solution converges uniformly on compact sub-sets to a function $\bar{y}$ which satisfies (1.1)–(1.2) and $\bar{y}(\infty)=\max\{g(\infty)-\varepsilon,0\}$. In particular this implies that $\bar{y}$ is identically zero if $\varepsilon\geqq g(\infty)$.

In § 6 we analyze the limiting behavior of the solution when ε tends to zero and R is kept fixed. It turns out that the solution y converges uniformly for $x\in[0,R]$ to the function $\tilde{y}(x)=\min\{g(x),k\}$, but that its derivative y' converges uniformly to $\tilde{y}'$ only on compact subsets of $[0,R]$ which do not contain the transition point x_0.

In § 7 we discuss in greater detail the behavior of y' near the point x_0 as $\varepsilon\downarrow 0$. By the standard method of matched asymptotic expansions we formally obtain in § 8 an approximation y_a. In § 9 we prove that for each $n>1$

$$y-y_a=O(\varepsilon^{n+1/2}), \qquad y'-y_a'=O(\varepsilon^{n-1/2}) \quad \text{as } \varepsilon\downarrow 0,$$

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uniformly on $[0, R]$, where n counts the number of terms included in the approximation. In this part of our treatment of the singular perturbation problem we derived much inspiration from reading bits and pieces of van Harten's thesis [9].

Since the limits $\varepsilon \downarrow 0$ and $R \to \infty$ (for $\varepsilon \leqq g(\infty)-k$) are interchangeable, the two separate limits give a complete picture of the limiting behavior with respect to both parameters.

Finally, in § 10, we consider problem (1.1)–(1.3) under the much weaker condition on g:

$$\tilde{H}_g\colon g \in C^1([0, R]), \qquad g(0) = 0, \qquad g(R) \geqq k,$$

$$g \text{ has only finitely many local extrema on } [0, R].$$

Again, the existence and uniqueness of a solution $y(x; \varepsilon)$ is established and it is shown that $y' > 0$. In addition

$$y(x; \varepsilon) \to u(x) \quad \text{as } \varepsilon \downarrow 0,$$

uniformly on $[0, R]$, where the function u, which is continuous, consists of pieces where $u(x) = g(x)$ and pieces where $u(x)$ is a constant. The arguments we employ here are borrowed from the theory of dynamical systems and are somewhat unusual in this context.

Problems like the one treated in this paper have also been considered by Hallam and Loper [8], Howes and Parter [11] (also see Howes [10]), Clément and Emmerth [4] and Clément and Peletier [5]. Both of the first two papers deal with one particular equation and the second two papers deal with concave solutions y_ε of a general class of equations. In all of these $\lim_{\varepsilon \downarrow 0} y_\varepsilon$ is determined. In this paper we do the same by the method of upper and lower solutions, which was also used by Howes and Parter, and in addition we give precise estimates of the behavior of y_ε and y'_ε as $\varepsilon \downarrow 0$.

2. Physical background.

2.1. An electrical discharge. Marode et al. [14] consider an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(r)$ and $n_e(r)$ respectively, where $r = (x_1, x_2, x_3)$. The ions are heavy and slow, and the density $n_i(r)$ may therefore be regarded as fixed. The electrons are highly mobile and assume a spatial distribution in thermal equilibrium with the ions. The problem is then to find $n_e(r)$ for given $n_i(r)$.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. Gallimberti [7] and Marode [13]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $\rho := (x_1^2 + x_2^2)^{1/2}$ only. Using Coulomb's law and a constitutive equation for the electric current, which contains both a diffusion and a conduction term, Marode et al. [14] derived that the electron density $n_e(\rho)$ should satisfy the equation

$$-\frac{\varepsilon}{2}\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho}{n_e(\rho)}\frac{d}{d\rho}n_e(\rho)\right) = n_i(\rho) - n_e(\rho), \tag{2.1}$$

where ε is a combination of physical constants which is proportional to the temperature. In addition n_e has to satisfy the boundary condition

$$\frac{dn_e}{d\rho}(0) = 0 \tag{2.2}$$

and the condition

$$\int_0^\infty \{n_i(\rho)-n_e(\rho)\}\rho d\rho = N > 0, \tag{2.3}$$

where N is a measure for the excess of ions.

In the experiment the ions are concentrated near the center of the discharge. Hence we shall take for n_i a function which decreases monotonically to zero as ρ tends to infinity. In this paper we study the solution n_e of (2.1)–(2.3) and in particular its behavior as $\varepsilon \downarrow 0$.

In order to cast (2.1) in a more convenient form, we make the change of variable

$$x = \rho^2 \tag{2.4}$$

and we define the new dependent variable

$$y(x) = \int_0^{x^{1/2}} n_e(s)s\,ds. \tag{2.5}$$

Thus, $y(x)$ represents the number of electrons contained in a cylinder of unit height and radius $x^{1/2}$. Analogously, we define

$$g(x) = \int_0^{x^{1/2}} n_i(s)s\,ds. \tag{2.6}$$

If we now multiply (2.1) by ρ, integrate from $\rho = 0$ to $\rho = x^{1/2}$ and use (2.4)–(2.6) we obtain (1.1). The boundary condition (1.2) is implied by (2.5) and the boundary condition (1.3), with $R = \infty$, follows from (2.3):

$$y(\infty) = k := g(\infty) - N,$$

where clearly $k \in (0, g(\infty))$.

2.2. The two-dimensional Coulomb gas. Equation (1.1) describes the equilibrium distribution of electrons interacting, via the Coulomb potential, with themselves and with a fixed positive background in a two-dimensional geometry. Theoretically one can generalize Coulomb's law to a space of arbitrary dimension d and then the corresponding equation would become

$$\varepsilon x^{2((d-1)/d)}y'' + (g(x) - y)y' = 0 \tag{2.7}$$

in which ε is again a positive constant which is proportional to the temperature.

The behavior of an assembly of charges depends on the competition between the electrostatic forces, which tend to bind positive and negative charges together, and the thermal motion which drives them apart. By physical arguments one can show that for $d > 2$ the thermal motion wins: at no nonzero temperature are the electrons bound to the ions. For $d < 2$, the electrostatic forces win, and whatever the temperature the charges are bound together (see Chui and Weeks [3]).

For the model problem consisting of (2.7) supplemented with the boundary conditions (1.2) and (1.3), with $R = \infty$, we find these matters reflected in the fact that for arbitrary positive ε, no solution exists when $d > 2$ whereas, on the contrary, a unique solution exists when $d < 2$. One can prove this along the lines indicated in § 5.

The marginal case $d = 2$ is of greatest interest. Presumably there is a critical value of the temperature at which a transition occurs from bound to unbound charges and recently there has been much interest in the precise nature of this transition (see Kosterlitz and Thouless [12]).

II.4.

In our study of the two-dimensional case we find indeed, in § 5, a critical value of ε (and hence of the temperature)

$$\varepsilon_1 = g(\infty) - k = N$$

at which the nature of the solution n_e changes, corresponding to the loss (towards infinity) of part of the negative charge. Beyond a still higher value of ε:

$$\varepsilon_2 = g(\infty)$$

there appears to be no solution, indicating that the negative charge is no longer bound to the positive background.

2.3. Low temperatures. We also have studied the equations in the low temperature regime, i.e. for $\varepsilon \downarrow 0$. Physically one then expects all the electrons to gather in the region of lowest energy, that is in the center of the ion distribution. Indeed we have found that for $\varepsilon \downarrow 0$ the solution of (2.1) exhibits transition behavior

$$\lim_{\varepsilon \downarrow 0} n_e(\rho) = \begin{cases} n_i(\rho), & \rho < \rho_0, \\ 0, & \rho > \rho_0, \end{cases}$$

where ρ_0 is determined by the boundary condition (2.3). There appears to be a transition layer of width of order $\varepsilon^{1/2}$ which, according to Marode et al. [14], has the form of a Debye shielding length.

3. A priori estimates and the existence of a solution. In this section we consider the problem (1.1)–(1.3) for fixed values of the parameters ε and R. By a solution we shall mean a function $y \in C^2([0, R])$ which satisfies (1.1)–(1.3). We first derive some a priori estimates for a solution and its first two derivatives. Subsequently we prove that a solution actually exists by constructing an upper and lower solution and by verifying the appropriate Nagumo condition.

THEOREM 3.1. *Let y be a solution; then for all* $x \in (0, R)$

(i) $0 < y(x) < \min\{g(x), k\}$;

(ii) $0 < y'(x) < g'(0)$;

(iii) $-(g'(0))^2/\varepsilon < y''(x) < 0$.

Proof. Let us first prove that $y'(x) > 0$ for all $x \in (0, R)$. Suppose that $y'(x_1) = 0$ for some $x_1 > 0$; then the standard uniqueness theorem for ordinary differential equations implies that $y(x) = y(x_1)$ for all x. Since this is not compatible with the two boundary conditions we conclude that y' is sign-definite. Invoking the boundary conditions once more, we see that the sign has to be positive.

The positivity of y' implies that $0 < y(x) < k$ for $x \in (0, R)$. Next we shall prove that $y(x) < g(x)$. We begin by observing that this inequality holds for $x \geqq x_0$. Suppose there is an interval $[x_1, x_2] \subset [0, x_0]$ such that $y - g$ is strictly positive in the interior of $[x_1, x_2]$ and $y(x_1) - g(x_1) = y(x_2) - g(x_2) = 0$. Then $y'(x_2) \leqq g'(x_2) < g'(x_1) \leqq y'(x_1)$. On the other hand (1.1) implies that $y''(x) > 0$ for $x \in (x_1, x_2)$ and hence $y'(x_2) = y'(x_1) + \int_{x_1}^{x_2} y''(\xi)\, d\xi > y'(x_1)$. So our assumption must be false since it leads to a contradiction. Thus, $y(x) \leqq g(x)$. Now, let us suppose that $y(x_1) = g(x_1)$ for some $x_1 > 0$, then necessarily $y'(x_1) = g'(x_1)$. However, because $y''(x_1) = 0$ (by (1.1)) and $g''(x_1) < 0$, this would imply that $y(x) > g(x)$ in a right-hand neighborhood of x_1, which is impossible. Hence the inequality is strict for $x \in (0, R]$, and this completes the proof of (i).

From (i), $y'(x) > 0$ and (1.1) we deduce that $y''(x) < 0$ for $x \in (0, R)$. Hence $y'(x) < y'(0) \leqq g'(0)$ for $x \in (0, R)$ which completes the proof of (ii).

Finally, we note that H_R implies that $g(x) \leqq g'(0)x$ and hence that $y''(x) = (\varepsilon x)^{-1}(y(x)-g(x))y'(x) > -(\varepsilon x)^{-1}g(x)g'(0) \geqq -\varepsilon^{-1}(g'(0))^2$. This proves property (iii). □

THEOREM 3.2. *There exists a function* $y \in C^2([0, R])$ *which satisfies* (1.1)–(1.3).

Proof. We define two functions α and β by $\alpha(x) := 0$ and $\beta(x) := g(x)$ for $x \in [0, R]$. Moreover, we define a function f by $f(x, y, y') := (\varepsilon x)^{-1}(y-g(x))y'$. Then $\alpha''(x) = 0 \geqq 0 = f(x, \alpha(x), \alpha'(x))$ and $\beta''(x) = g''(x) < 0 = f(x, \beta(x), \beta'(x))$ for $x \in (0, R)$. Hence α and β are, respectively, a lower and an upper solution of (1.1). The existence of a solution now follows from [1, Thm. 1. 5.1] if we can show that f satisfies a Nagumo condition with respect to the pair α, β. This amounts to finding a positive continuous function h on $[0, \infty)$ such that $|f(x, y, y')| \leqq h(|y'|)$ for all $x \in [0, R]$, $\alpha(x) \leqq y \leqq \beta(x)$ and $y' \in \mathbb{R}$ and, furthermore, such that

$$\int_{R^{-1}\beta(R)}^{\infty} \frac{s}{h(s)}\,ds > \beta(R),$$

cf. [1, Def. 1.4.1]. The function h defined by $h(s) := \varepsilon^{-1}g'(0)(s+1)$ satisfies all these conditions. □

4. A comparison theorem. In order to emphasize that we are going to study the dependence of a solution on the parameters ε and R, we introduce the notation $P(\varepsilon, R)$ for the problem (1.1)–(1.3). The main result of this section is a comparison theorem which is proved by standard maximum principle arguments. As corollaries we obtain that the solution is unique and that it depends in a monotone fashion on both ε and R.

THEOREM 4.1. *Let* y_i *be a solution of* $P(\varepsilon_i, R_i)$ *for* $i = 1, 2$ *and suppose that* $R_2 \geqq R_1 > x_0$ *and* $\varepsilon_2 \geqq \varepsilon_1$. *Then* $y_1(x) \geqq y_2(x)$ *for* $0 < x < R_1$. *Moreover, if one of the inequalities for the parameters is strict, then so is the inequality for the solutions.*

Proof. Let the function m be defined by $m(x) := y_1(x) - y_2(x)$. Suppose that m achieves a nonpositive minimum on $(0, R_1)$, i.e. suppose that for some $x_1 \in (0, R_1)$, $m(x_1) \leqq 0$, $m'(x_1) = 0$ and $m''(x_1) \geqq 0$. By subtracting the equation for y_2 from the one for y_1 we obtain

$$\varepsilon_1 x_1 m''(x_1) - (\varepsilon_2 - \varepsilon_1)x_1 y_2''(x_1) - y_1'(x_1)m(x_1) = 0.$$

However, all the terms on the left-hand side of this equality are nonnegative and if either $\varepsilon_2 > \varepsilon_1$ or $m(x_1) < 0$ at least one of them is positive. If $\varepsilon_1 = \varepsilon_2$ and $m(x_1) = 0$ then the uniqueness theorem for ordinary differential equations implies that $m(x) = 0$ for all $x \in [0, R_1]$, which cannot be true if $R_2 > R_1$. So we see that m cannot achieve a negative minimum and that m cannot become zero on $(0, R_1)$ if one of the inequalities for the parameters is strict. Since $m(0) = 0$ and $m(R_1) \geqq 0$ this proves the theorem. □

COROLLARY 4.2. *The problem* $P(\varepsilon, R)$ *has one and only one solution.*

Proof. We know that at least one solution exists (Theorem 3.2). Let both y_1 and y_2 satisfy $P(\varepsilon, R)$, then Theorem 4.1 implies that $y_1(x) \geqq y_2(x)$ but likewise that $y_2(x) \geqq y_1(x)$. Hence, $y_1(x) = y_2(x)$ for $x \in [0, R]$. □

COROLLARY 4.3. *Let* $y = y(x; \varepsilon, R)$ *be the solution of* $P(\varepsilon, R)$. *Then y is a monotone decreasing function of* ε *for each* $R > x_0$ *and each* $x \in (0, R)$ *and y is a monotone decreasing function of R for each* $\varepsilon > 0$ *and each* $x \in (0, R)$.

5. The limiting behavior as $R \to \infty$. In this section we study the limiting behavior as $R \to \infty$ of the solution $y = y(x; \varepsilon, R)$ of the problem $P(\varepsilon, R)$. Since y is a bounded and monotone function of R, the definition $\bar{y}(x; \varepsilon) := \lim_{R\to\infty} y(x; \varepsilon, R)$ makes sense for all $x, \varepsilon > 0$. This definition implies at once that $\bar{y}(0, \varepsilon) = 0$ and that $\bar{y}$ is a nondecreasing function of x and a nonincreasing function of ε

From the estimates in Theorem 3.1 we obtain, via the Arzela–Ascoli theorem, that both $y(\cdot\,;\varepsilon, R)$ and $y'(\cdot\,;\varepsilon, R)$ converge uniformly on compact subsets. Invoking (1.1) we see that the same must be true for $y''(\cdot\,;\varepsilon, R)$. It follows that $\bar{y}(\cdot\,;\varepsilon)$ belongs to $C^2(\mathbb{R}_+)$ and satisfies (1.1). Now it remains to determine $\bar{y}(\infty, \varepsilon)$. We will estimate $\bar{y}(\infty, \varepsilon)$ from below by constructing a more subtle lower solution for y. But first we prove a result which can be used to estimate $\bar{y}(\infty, \varepsilon)$ from above.

LEMMA 5.1. *Let* $z \in C^2(\mathbb{R}_+)$ *satisfy* (1.1) *and* $z(0) = 0$. *Suppose that* $z(\infty) := \lim_{x\to\infty} z(x)$ *exists and satisfies* $0 < z(\infty) < \infty$. *Then* $z(\infty) \leqq g(\infty) - \varepsilon$.

Proof. Both z and z' are positive on $(0, \infty)$ (cf. the proof of Theorem 3.1). For the purpose of contradiction, let us suppose that $z(\infty) > g(\infty) - \varepsilon$. Let x_1 be such that $\beta := \varepsilon^{-1}(z(x_1) - g(\infty)) > -1$. Then $z(x) - g(x) \geqq z(x_1) - g(\infty) = \varepsilon\beta$ for all $x \geqq x_1$. Integrating (1.1) twice from x_1 to x we obtain

$$z(x) = z(x_1) + z'(x_1)\int_{x_1}^{x} \exp\left(\int_{x_1}^{\xi} \frac{z(\eta) - g(\eta)}{\varepsilon\eta}\, d\eta\right) d\xi.$$

Thus, for $x \geqq x_1$,

$$z(x) \geqq z'(x_1)\int_{x_1}^{x} \exp\left(\beta \ln \frac{\xi}{x_1}\right) d\xi = \frac{x_1 z'(x_1)}{\beta + 1}\left(\left(\frac{x}{x_1}\right)^{\beta+1} - 1\right).$$

Since $\beta + 1 > 0$ this would imply that $z(x) \to \infty$ as $x \to \infty$. Hence the assumption that $z(\infty) > g(\infty) - \varepsilon$ must be false. □

We define a function $s = s(x; \lambda, x_1, \nu)$ by

$$s(x; \lambda, x_1, \nu) := \lambda\left(1 - \left(\frac{x}{x_1}\right)^{-\nu}\right) \tag{5.1}$$

and we investigate which conditions for the parameters λ, x_1 and ν guarantee that $s'' \geqq f(x, s, s')$ for $x \geqq x_1$ (recall that $f(x, y, y') = (\varepsilon x)^{-1}(y - g(x))y'$). A simple computation shows that this inequality holds indeed for all $x \geqq x_1$ if and only if $g(x_1) - \lambda - \varepsilon\nu - \varepsilon \geqq 0$, or equivalently, $\nu \leqq \varepsilon^{-1}(g(x_1) - \lambda) - 1$. The latter inequality can be satisfied for some *positive* value of ν if and only if $\lambda < g(x_1) - \varepsilon$. In its turn this inequality can be satisfied for sufficiently large x_1 and some *positive* value of λ if and only if $g(\infty) - \varepsilon > 0$.

We now have all the ingredients at hand to prove the following theorem.

THEOREM 5.2.

(i) *If* $\varepsilon \leqq g(\infty) - k$ *then* $\bar{y}(\infty, \varepsilon) = k$ *and* $\lim_{R\to\infty} \sup_{0\leqq x\leqq R} |y(x; \varepsilon, R) - \bar{y}(x; \varepsilon)| = 0$;

(ii) *if* $g(\infty) - k < \varepsilon < g(\infty)$ *then* $\bar{y}(\infty; \varepsilon) = g(\infty) - \varepsilon$;

(iii) *if* $\varepsilon \geqq g(\infty)$ *then* $\bar{y}(x; \varepsilon) = 0$ *for all* $x \geqq 0$.

Proof. (i) For any $\lambda < k$ we can choose x_1 such that $\lambda < g(x_1) - \varepsilon$ and subsequently ν such that $0 < \nu \leqq \varepsilon^{-1}(g(x_1) - \lambda) - 1$. For these values of the parameters, s is a lower solution on the interval $[x_1, R]$. The function t defined by $t(x) := k$ is an upper solution and f satisfies a Nagumo condition with respect to the pair s, t and the interval $[x_1, R]$. It follows that the inequality

$$s(x; \lambda, x_1, \nu) \leqq y(x; \varepsilon, R) \leqq k,$$

which holds for $x = x_1$ and for $x = R$, actually is satisfied for all $x \in [x_1, R]$. By taking first the limit $R \to \infty$ and then the limit $x \to \infty$ we obtain

$$\lambda \leqq \bar{y}(\infty; \varepsilon) \leqq k.$$

Since this inequality holds for $\lambda < k$, necessarily $\bar{y}(\infty, \varepsilon) = k$. This result and the monotonicity of y with respect to x together imply that the convergence of y to $\bar{y}$ is in fact uniform in x (we refer to [6, Lemma 2.4] for the proof of this statement).

(ii) If $g(\infty) - k < \varepsilon < g(\infty)$, we can make s into a lower solution by a suitable choice of x_1 and ν if and only if $\lambda < g(\infty) - \varepsilon$. The argument we used in the proof of (i) now shows that $\bar{y}(\infty; \varepsilon) \geqq g(\infty) - \varepsilon$. On the other hand, Lemma 5.1 implies that $\bar{y}(\infty; \varepsilon) \leqq g(\infty) - \varepsilon$. So $\bar{y}(\infty; \varepsilon) = g(\infty) - \varepsilon$.

(iii) From Lemma 5.1 we deduce that no solution of (1.1) with a positive limit at infinity can exist if $\varepsilon \geqq g(\infty)$. Hence $\bar{y}(\infty; \varepsilon) = 0$ and consequently $\bar{y}(x; \varepsilon) = 0$ for all $x \geqq 0$. □

The results of this section are at the same time results concerning the existence and nonexistence of a solution of the problem $P(\varepsilon, \infty)$ defined by (1.1), (1.2) and $\lim_{x\to\infty} y(x) = k$. By exactly the same arguments which we used before one can derive the bounds of Theorem 3.1 and one can show that there exists at most one solution of $P(\varepsilon, \infty)$. For convenience we formulate this result in the following theorem.

THEOREM 5.3. *There exists a function* $y \in C^2(\mathbb{R}_+)$ *which satisfies* (1.1), (1.2) *and the condition* $\lim_{x\to\infty} y(x) = k$ *if and only if* $\varepsilon \leqq g(\infty) - k$. *If it exists, it is unique and it satisfies the inequalities given in Theorem* 3.1.

6. The limiting behavior as $\varepsilon \downarrow 0$. Throughout this section $R > x_0$ will be fixed and we will suppress the dependence on R in the notation, because it is inessential. The solution y of (1.1)–(1.3) is a bounded and monotone function of ε and we define $\bar{y}(x) := \lim_{\varepsilon\downarrow 0} y(x; \varepsilon)$. From Theorem 3.1(i) and (ii) and the Arzela–Ascoli theorem we deduce that $\bar{y}$ is continuous and that in fact

$$\lim_{\varepsilon\downarrow 0} \sup_{0\leqq x\leqq R} |\bar{y}(x) - y(x; \varepsilon)| = 0.$$

THEOREM 6.1. $\bar{y}(x) = \min\{g(x), k\}$.

Proof. From Theorem 3.1(i) we know that $\bar{y}(x) \leqq \min\{g(x), k\}$. Take any $x < x_0$, then $y(x) < k$. We claim that this implies that $\lim_{\varepsilon\downarrow 0} \inf y'(x; \varepsilon) > 0$. Indeed, suppose that the sequence $\{\varepsilon_i\}$ is such that $\varepsilon_i \downarrow 0$ and $y'(x; \varepsilon_i) \downarrow 0$ as $i \to \infty$, then by taking the limit $i \to \infty$ in the relation

$$k = y(R; \varepsilon_i) = y(x; \varepsilon_i) + \int_x^R y'(\xi; \varepsilon_i)\, d\xi \leqq y(x; \varepsilon_i) + (R - x) y'(x; \varepsilon_i),$$

we arrive at the conclusion that $\bar{y}(x) \geqq k$, which is impossible.

Integrating (1.1) from 0 to x we obtain

$$\varepsilon(y'(x; \varepsilon) - y'(0; \varepsilon)) = \int_0^x \frac{y(\xi; \varepsilon) - g(\xi)}{\xi} y'(\xi, \varepsilon)\, d\xi. \tag{6.1}$$

Suppose that $x < x_0$ and $\max_{0\leqq\xi\leqq x} |\bar{y}(\xi) - g(\xi)| > 0$; then, since $g'(0) > y'(\xi; \varepsilon) \geqq y'(x; \varepsilon)$ for $0 < \xi \leqq x$ and $\lim_{\varepsilon\downarrow 0} \inf y'(x; \varepsilon) > 0$, the right-hand side of (6.1) is bounded away from zero as $\varepsilon \downarrow 0$. However, this is impossible since the left-hand side tends to zero as $\varepsilon \downarrow 0$. So $\bar{y}(x) = g(x)$ for all $x < x_0$, and by continuity $\bar{y}(x_0) = k$. The function $\bar{y}$, being the limit of monotone functions, is monotone nondecreasing. Hence $\bar{y}(x) \geqq k$ for $x > x_0$ and consequently $\bar{y}(x) = k$ for $x > x_0$. □

By taking $\varepsilon = 0$ in (1.1) we obtain the reduced equation

$$(g(x) - y)y' = 0. \tag{6.2}$$

The limiting function $\tilde{y}$ satisfies the boundary conditions (1.2) and (1.3) and (6.2) except at the point $x = x_0$, where $\tilde{y}'$ is not defined. Motivated in part by the physical application (cf. § 2) we shall now investigate the limiting behavior of $y'(x; \varepsilon)$ as $\varepsilon \downarrow 0$. It will then become even more apparent that $x = x_0$ is an exceptional point. The following lemma is needed in the proof of Theorem 6.3, but it is of some interest in itself.

LEMMA 6.2. *Let $\delta > 0$ be arbitrary. For any $\varepsilon_0 > 0$ there exists an $M > 0$ such that $0 < g(x) - y(x; \varepsilon) < M\varepsilon x$ for all $x \in [0, x_0 - \delta]$ and all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let $\delta > 0$ and $\varepsilon_0 > 0$ arbitrary. We define

$$m(\varepsilon) := \min_{x_0 - \delta \leqq x \leqq x_0 - \frac{1}{2}\delta} \{g(x) - y(x; \varepsilon)\}.$$

Then there exist positive constants C_i, $i = 1, 2, 3$, such that for $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} m(\varepsilon) &\leqq C_1 \int_{x_0 - \delta}^{x_0 - \delta/2} (g(\xi) - y(\xi; \varepsilon))\, d\xi \\ &\leqq C_2 \int_{x_0 - \delta}^{x_0 - \delta/2} \frac{g(\xi) - y(\xi; \varepsilon)}{\xi} y'(\xi; \varepsilon)\, d\xi \leqq C_3 \varepsilon \end{aligned}$$

(see the proof of Theorem 6.1 and in particular formula (6.1)). Let the function $v = v(x; \varepsilon)$ be defined by $v(x; \varepsilon) := g(x) - y(x; \varepsilon) - M\varepsilon x$, where the constant $M > 0$ is still at our disposal. Then v satisfies the equation

$$\varepsilon x v'' - y'(x; \varepsilon) v = \varepsilon x (g''(x) + M y'(x; \varepsilon))$$

and consequently $\varepsilon x v'' - \mu v > 0$ if $M > \gamma \mu^{-1}$, $\varepsilon \in (0, \varepsilon_0)$ and $x \in (0, x_0 - \frac{1}{2}\delta]$, where the positive numbers γ and μ are defined by

$$\gamma := -\inf_{0 < x \leqq x_0 - \frac{1}{2}\delta} g''(x)$$

and

$$\mu := \inf_{0 < \varepsilon < \varepsilon_0} y'\left(x_0 - \frac{\delta}{2}; \varepsilon\right).$$

So if $M > \gamma \mu^{-1}$ and $\varepsilon \in (0, \varepsilon_0)$, then v cannot assume a nonnegative maximum on $(0, x_0 - \frac{1}{2}\delta)$. Let $x(\varepsilon)$ be such that $g(x) - y(x; \varepsilon)$ achieves its minimum on the set $[x_0 - \delta, x_0 - \frac{1}{2}\delta]$ in the point $x = x(\varepsilon)$. Then $v(x(\varepsilon); \varepsilon) = m(\varepsilon) - M\varepsilon x(\varepsilon) < 0$ if $M > (x_0 - \delta)^{-1} C_3$. Since $v(0; \varepsilon) = 0$, this implies that for $M > \max\{\gamma\mu^{-1}, (x_0 - \delta)^{-1} C_3\}$, $v(x; \varepsilon) < 0$ for $x \in (0, x(\varepsilon))$ and a fortiori for $x \in (0, x_0 - \delta)$. □

THEOREM 6.3. *Let $\delta > 0$ be arbitrary. Then*

(i) $\lim_{\varepsilon \downarrow 0} \sup_{0 \leqq x \leqq x_0 - \delta} |g'(x) - y'(x; \varepsilon)| = 0$;

(ii) $\lim_{\varepsilon \downarrow 0} \sup_{x_0 + \delta \leqq x \leqq R} |y'(x; \varepsilon)| = 0$.

Proof. (i) From (1.1), Theorem 3.1(ii) and Lemma 6.2 we deduce that $-g'(0)M < y''(x; \varepsilon) < 0$ for $x \in [0, x_0 - \delta]$ and $\varepsilon \in (0, \varepsilon_0)$. By the Arzela–Ascoli theorem this implies that the limit set of $\{y'(\cdot\,; \varepsilon) \mid \varepsilon > 0\}$ as $\varepsilon \downarrow 0$ is nonempty in $C([0, x_0 - \delta])$. The result now follows from the fact that y tends to g on $[0, x_0 - \delta]$ as $\varepsilon \downarrow 0$.

(ii) Integrating (1.1) from $x_0 + \frac{1}{2}\delta$ to x we obtain

$$\varepsilon(y'(x; \varepsilon) - y'(x_0 + \tfrac{1}{2}\delta; \varepsilon)) = \int_{x_0 + \frac{1}{2}\delta}^{x} \frac{y(\xi; \varepsilon) - g(\xi)}{\xi} y'(\xi; \varepsilon)\, d\xi.$$

For $x \in [x_0 + \delta, R]$ the right-hand side is smaller than $\frac{1}{2}\delta R^{-1}(k - g(x_0 + (\delta/2)))y'(x; \varepsilon)$. Consequently $0 < y'(x; \varepsilon) < 2g'(0)\varepsilon R \delta^{-1}(g(x_0 + (\delta/2)) - k)^{-1}$. □

In the next section we shall concentrate on a formal approximation for y and y' in the neighborhood of $x = x_0$.

In § 5 it was shown that the problem $P(\varepsilon, \infty)$ has a unique solution for ε sufficiently small. The analysis of this section can be repeated, mutatis mutandis, to derive the analogous results concerning the limiting behavior of this solution as $\varepsilon \downarrow 0$. In particular this implies that the limits $\varepsilon \downarrow 0$ and $R \to \infty$ are interchangeable.

7. The transition layer. In Theorem 6.3 we have shown that y' converges nonuniformly on the interval $[0, R]$ as $\varepsilon \downarrow 0$. This feature is typical for a singular perturbation problem. In this section we use the standard method of the stretching of a variable to obtain more information about the behavior of y' near the transition point $x = x_0$.

By the stretching of the variable x near x_0 we mean the introduction of a local coordinate ξ according to $x = x_0 + \varepsilon^{\alpha}\xi$. At the same time we introduce a local dependent variable η according to

$$y(x) = g(x_0) + \varepsilon^{\beta}\eta(\xi).$$

If we make these substitutions in the equation, and subsequently only retain the terms of lowest order in ε, it depends on the values of α and β what the resulting equation will be. One easily verifies that the choice $\alpha = \beta = \frac{1}{2}$ leads to a significant equation, namely to

$$x_0\eta_1'' + (\xi g'(x_0) - \eta_1)\eta_1' = 0, \tag{7.1}$$

where we have introduced the subscript 1 to indicate that we consider in fact a first approximation. To this equation we add the condition that its solution should match the limits of y to the left and to the right of x_0, respectively, up to the appropriate order in $\sqrt{\varepsilon}$. This amounts to the conditions

$$\begin{aligned} &\eta_1(\xi) = g'(x_0)\xi + o(1) \quad \text{as } \xi \to -\infty, \\ &\eta_1(\xi) = o(1), \quad \text{as } \xi \to +\infty. \end{aligned} \tag{7.2}$$

A straightforward application of the maximum principle (see Theorem 4.1) shows that the problem (7.1)–(7.2), which we shall denote by Π_1, admits at most one solution.

The problem Π_1 is nonautonomous. However, if we set $\eta_1' = z_1$, divide the equation by z_1 and then differentiate it, we formally obtain an autonomous problem, which we denote by $\tilde{\Pi}_1$:

$$x_0\left(\frac{z_1'}{z_1}\right)' + g'(x_0) - z_1 = 0, \tag{7.3}$$

$$\begin{aligned} &z_1(\xi) = g'(x_0) + o(1) \quad \text{as } \xi \to -\infty, \\ &z_1(\xi) = o(1) \quad \text{as } \xi \to +\infty. \end{aligned} \tag{7.4}$$

One should note that, at least formally up to first order in $\sqrt{\varepsilon}$, z_1 describes the shape of y' in the neighborhood of x_0. In the remainder of this section we shall discuss the existence of a family of solutions of problem $\tilde{\Pi}_1$, and we shall show how this family can be used to obtain the solution of problem Π_1.

One way to handle problem $\tilde{\Pi}_1$ is to write (7.3) as a two-dimensional first order system and analyze the trajectories in the phase plane. It turns out that the singular point $(z_1, z_1') = (g'(x_0), 0)$ is a saddle point and that one branch of the unstable manifold lies in the half-plane $z_1' < 0$ and enters the (singular) singular point $(0, 0)$. Hence $\tilde{\Pi}_1$ has a one-parameter family of strictly decreasing solutions, where the parameter describes simply the translation of one particular solution.

However, it so happens that $\tilde{\Pi}_1$ can be solved explicitly for ξ in terms of z_1. To this end we put

$$z_1 = g'(x_0)e^v \quad \text{and} \quad \xi' = \sqrt{\frac{2g'(x_0)}{x_0}}\,\xi.$$

Then $v = v(\xi')$ has to satisfy

$$2v'' + 1 - e^v = 0,$$

$$v(-\infty) = 0, \qquad v(+\infty) = -\infty,$$

and we obtain, after multiplication by v' and one integration,

$$(v')^2 + v - e^v = -1$$

and finally

$$\xi' = \int_v^C \frac{dw}{\sqrt{e^w - w - 1}}, \tag{7.5}$$

where the parameter C corresponds to the free translation parameter. From this expression we easily obtain the asymptotic behavior of the solutions:

$$z_1(\xi) \sim g'(x_0) + \exp\left(\frac{\sqrt{g'(x_0)}}{x_0}(\xi - C)\right), \qquad \xi \to -\infty,$$

$$z_1(\xi) \sim g'(x_0)\exp\left(-\frac{g'(x_0)}{2x_0}(\xi - C)^2\right), \qquad \xi \to +\infty.$$

As candidates for a solution of Π_1 we take the functions

$$\psi(\xi, C) = \int_\infty^\xi \tilde{z}_1(\tau + C)\, d\tau = \int_\infty^{\xi + C} \tilde{z}_1(\tau)\, d\tau,$$

where $\tilde{z}_1$ is the particular solution of $\tilde{\Pi}_1$ which satisfies $\tilde{z}_1(0) = \frac{1}{2}g'(x_0)$ (or, in other words, which corresponds with $C = \frac{1}{2}g'(x_0)$ in (7.5)). Using (7.3) we obtain after some manipulation

$$(x_0\psi'' + (\xi g'(x_0) - \psi)\psi')' = \frac{\psi''}{\psi'}(x_0\psi'' + (\xi g'(x_0) - \psi)\psi'),$$

where primes denote differentiation with respect to ξ and where we have suppressed the dependence on C in the notation. Hence

$$x_0\psi'' + (\xi g'(x_0) - \psi)\psi' = K_1\psi'.$$

Furthermore, we deduce from $\tilde{\Pi}_1$ that

$$\psi(\xi; C) = g'(x_0)\xi + K_2 + o(1), \qquad \xi \to -\infty.$$

Since ψ''/ψ' tends to zero as $\xi \to -\infty$ it follows that $K_2 = -K_1$.

Of course the constants K_1 and K_2 depend on C and it remains to show that we can choose C in such a way that they both become zero. We observe that

$$K_1(C) = x_0\frac{\psi''(0; C)}{\psi'(0; C)} - \psi(0; C)$$

$$= x_0\frac{\tilde{z}_1'(C)}{\tilde{z}_1(C)} - \int_\infty^C \tilde{z}_1(\tau)\, d\tau.$$

From the known asymptotic behavior of $\tilde{z}_1$ we deduce that K_1 tends to $\pm\infty$ as C tends to $\mp\infty$. Moreover

$$\frac{dK_1}{dC}(C) = x_0\left(\frac{\tilde{z}_1'}{\tilde{z}_1}\right)'(C) - \tilde{z}_1(C) = -g'(x_0) < 0.$$

Thus, K_1 is a strictly decreasing function with range $(-\infty, \infty)$ and we conclude that there exists a unique value of C, C_1 say, such that $K_1(C) = 0$. Consequently $\eta_1 := \psi(\cdot\,; C_1)$ is the solution of problem Π_1. Furthermore, the properties of $\tilde{z}_1$ imply that (i) η_1 is negative, strictly increasing and concave, (ii) $\eta_1(\xi) \to 0$ faster than exponentially as $\xi \to +\infty$, (iii) the function $\eta_1(\xi) - g'(x_0)\xi$, as well as all its derivatives, converge exponentially to zero as $\xi \to -\infty$.

The idea of singular perturbation theory is that $\tilde{z}_1(\cdot + C_1)$ describes the transition of y' near $x = x_0$ for small values of ε, and that one can approximate y' uniformly on $[0, R]$ by using the building-stones $\tilde{z}_1(\cdot + C_1)$ and $\tilde{y}'$. In the following sections we shall elaborate this idea and we shall prove its correctness. It turns out that this will require the construction of at least five terms in a uniform asymptotic expansion. Since for us, as for many mathematicians, five is almost equal to infinity we shall first discuss the construction of a complete asymptotic expansion.

8. Matched asymptotic expansions. Throughout this and the next section we shall assume that $g \in C^\infty([0, R])$.

On the interval $[0, x_0 - \delta]$ we look for an asymptotic expansion of the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x). \tag{8.1}$$

We find that $y_0(x) = g(x)$ and that y_n is defined recursively by

$$y_n(x) = (y_0'(x))^{-1}\left\{x y_{n-1}''(x) - \sum_{k=1}^{n-1} y_k(x) y_{n-k}'(x)\right\}, \qquad n \geqq 1. \tag{8.2}$$

In order to calculate the matching conditions for the transition layer expansion, we expand each y_n in a Taylor series

$$y_n(x) = \sum_{k=0}^{\infty} (\sqrt{\varepsilon})^k \frac{y_n^{(k)}(x_0)}{k!}\xi^k,$$

where, as before, $\xi = (x - x_0)/\sqrt{\varepsilon}$. If we substitute this in the expansion for y and rearrange the resulting expression by collecting terms with like powers of $\sqrt{\varepsilon}$, we obtain

$$y(x) = \sum_{m=0}^{\infty} (\sqrt{\varepsilon})^m u_m(\xi) \tag{8.3}$$

where, by definition,

$$u_m(\xi) = \sum_{n=0}^{[m/2]} \frac{y_n^{(m-2n)}(x_0)}{(m-2n)!}\xi^{m-2n}. \tag{8.4}$$

On the interval $[x_0 + \delta, R]$ one can also introduce a series expansion in powers of ε, but it will quickly turn out that all the terms, except the one of zero'th order which is k, are zero.

Next we introduce the transition layer expansion

$$y(x) = \sum_{n=0}^{\infty} (\sqrt{\varepsilon})^n \eta_n(\xi), \tag{8.5}$$

where $\eta_0(\xi)=g(x_0)$ and η_1 is the solution of the problem Π_1 discussed in § 7. Substitution in the equation yields an equation for each η_n. Together with the matching condition which is obtained by formal identification of (8.5), as $\xi\to-\infty$, with (8.3), this yields for $n\geqq 2$ a *linear* problem Π_n defined recursively by

$$\begin{aligned} &x_0\eta''_n+(g'(x_0)\xi-\eta_1)\eta'_n-\eta'_1\eta_n=q_n,\\ &\eta_n(\xi)=u_n(\xi)+o(1)\quad \text{as } \xi\to-\infty,\\ &\eta_n=o(1)\quad \text{as } \xi\to+\infty, \end{aligned} \tag{8.6}$$

where

$$q_n(\xi):=-\frac{g^{(n)}(x_0)}{n!}\xi^n\eta'_1-\xi\eta''_{n-1}-\sum_{k=2}^{n-1}\eta'_{n+1-k}\left(\frac{g^{(k)}(x_0)}{k!}\xi^k-\eta_k\right). \tag{8.7}$$

As before the maximum principle implies that problem Π_n can have at most one solution. In order to discuss the existence of a solution we first rewrite the equation by making use of (7.1) for η_1:

$$x_0\left(\frac{\eta'_n}{\eta'_1}\right)'-\eta_n=\frac{q_n}{\eta'_1}.$$

Introducing $z_1:=\eta'_1$, $\zeta_n:=(z_1)^{-1}\eta'_n$ and $h_n:=((z_1)^{-1}q_n)'$, we obtain by differentiation

$$x_0\zeta''_n-z_1\zeta_n=h_n. \tag{8.8}$$

At this point it is important to observe that we know a particular solution of the homogeneous equation $x_0\phi''-z_1\phi=0$, namely

$$\phi(\xi):=\frac{z'_1(\xi)}{z_1(\xi)} \tag{8.9}$$

(one can verify this by differentiation of (7.3)). Hence we can construct solutions of (8.8) through the method of variation of constants, and we find

$$\zeta_n(\xi;C)=\frac{\phi(\xi)}{x_0}\int_0^{\xi}\phi^{-2}(\tau)\int_{-\infty}^{\tau}\phi(\sigma)h_n(\sigma)\,d\sigma\,d\tau+C\phi(\xi) \tag{8.10}$$

(note that we do not consider the general solution of the homogeneous equation since only ϕ has the right asymptotic behavior as $\xi\to-\infty$). For any C, the function defined in (8.10) is of polynomial growth as $\xi\to+\infty$ and behaves like $g'(x_0)u'_n$ as $\xi\to-\infty$. The last statement can be verified by working out the consistency relations between q_n and u_n which follow from the identity

$$x_0u''_n-g'(x_0)u_n=-\frac{g^{(n)}(x_0)}{n!}\xi^ng'(x_0)-\xi u''_{n-1}-\sum_{k=2}^{n-1}u'_{n+1-k}\left(\frac{g^{(k)}(x_0)}{k!}\xi^k-u_k\right)$$

and by making use of the known asymptotic behavior of ϕ.

Finally, we define

$$\eta_n(\xi;C)=\int_{\infty}^{\xi}z_1(\tau)\zeta_n(\tau;C)\,d\tau=\eta_n(\xi;0)+C\eta'_1(\xi). \tag{8.11}$$

Then $\eta_n(\xi;C)=u_n(\xi)+B_n+g'(x_0)C+o(1)$, $\xi\to-\infty$, where B_n is some number, which does not depend on C. It follows that there exists a unique constant, say C_n, for which the matching condition is satisfied and consequently $\eta_n(\xi;C_n)$ is the unique solution of the problem Π_n. [illegible]

To conclude this section we construct a uniform approximation of formal order $2n+1$ in $\sqrt{\varepsilon}$. We introduce two C^{∞}-functions H and J defined on $\mathbb{R}$ (so-called cut-off functions) with the following properties

$$H(x)=\begin{cases}0 & \text{if } |x-x_0|\geqq \delta_1,\\ 1 & \text{if } |x-x_0|\leqq \dfrac{\delta_1}{2},\end{cases}$$

$$J(x)=\begin{cases}0 & \text{if } |x|\leqq \delta_2,\\ 1 & \text{if } |x|\geqq 2\delta_2,\end{cases}$$

where δ_1 and δ_2 are suitable constants which do not depend on ε. Then the formal approximation $y_a(x)$ is defined by

$$(8.12)\qquad y_a(x)=\begin{cases}J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\displaystyle\sum_{m=1}^{n}\varepsilon^m y_m(x)+H(x)\sum_{m=1}^{2n+1}(\sqrt{\varepsilon})^m\left(\eta_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\right.\\ \qquad\qquad\left.-J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)u_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\right) & \text{for } x\leqq x_0,\\ J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)k(1-H(x))+H(x)\displaystyle\sum_{m=1}^{2n+1}(\sqrt{\varepsilon})^m\eta_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right) & \text{for } x\geqq x_0.\end{cases}$$

Apart from the cut-off functions this formula is the usual one, expressing a uniform approximation as the sum of approximations in the different regions minus the matching terms, which are contained in two approximations and hence should be subtracted in order to avoid double counting. The cut-off functions are used to achieve two ends: the approximation should satisfy the boundary conditions and it should be smooth at $x=x_0$. Moreover, the cut-off functions are harmless in the sense that they are multipled by factors which are small (if ε is small) in regions where the cut-off functions are different from one. In the next section we shall prove that y_a and y_a' are indeed uniform approximations of y and y' up to the order $\varepsilon^{n+(1/2)}$ and $\varepsilon^{n-(1/2)}$, respectively.

9. A proof of the validity of the formal construction. We begin by deriving an estimate for the difference

$$(9.1)\qquad z(x):=y(x)-y_a(x).$$

It follows from the equation for y and from the construction of y_a that z satisfies

$$(9.2)\qquad \begin{aligned}&\varepsilon\times z''+(g-y)z'-y'z+zz'=r,\\ &z(0)=0,\qquad z(R)=0,\end{aligned}$$

where the remainder term r, defined by

$$(9.3)\qquad r(x):=-(\varepsilon x y_a''+(g-y_a)y_a'),$$

can be shown, after an elaborate computation, to satisfy

$$(9.4)\qquad r(x)=O(x\varepsilon^n)\quad \text{as } \varepsilon\downarrow 0 \text{ and/or } x\downarrow 0.$$

If we multiply the equation for z by z and integrate from 0 to R we obtain after some integrations by parts and an application of the Cauchy–Schwarz inequality

$$\varepsilon\int_0^R x(z'(x))^2\,dx+\frac{1}{2}\int_0^R (g'(x)+y'(x))z^2(x)\,dx\leqq\|z\|\|r\|,$$

where $\|\cdot\|$ denotes the L_2-norm. Since $g'(x)+y'(x)\geqq g'(R)$ this implies, first of all, that

$$\|z\|\leqq\frac{2}{g'(R)}\|r\|$$

and hence that

$$\varepsilon\int_0^R x(z'(x))^2\,dx+\frac{g'(R)}{2}\|z\|^2\leqq\frac{2}{g'(R)}\|r\|^2.$$

Now, fix $\delta\in(0,x_0)$. The estimate above is easily translated into an estimate for the $H^1(\delta,R)$-norm of z, where H^1 denotes the usual Sobolev space of L_2-functions which have a generalized derivative belonging to L_2. Thus, by the continuous imbedding of H^1 into the space of continuous functions we obtain

$$|z(x)|\leqq C(\varepsilon^{-1}\|r\|^2)^{1/2}\leqq C\varepsilon^{n-1/2},\qquad \delta\leqq x\leqq R,$$

where C depends on δ. Having established this estimate on the interval $[\delta,R]$, we can extend it to the interval $[0,R]$ by means of the maximum principle in exactly the same way as we proved Lemma 6.2.

Next, it is advantageous to take explicitly into account the dependence on the parameter n, which counts the number of terms included in the approximation. So putting $z=z_n$ we write the estimate obtained so far as

$$|z_n(x)|\leqq Cx\varepsilon^{n-1/2},\qquad 0\leqq x\leqq R,\quad n\in\mathbb{N}.$$

Then, observing that

$$|z_{n+1}(x)-z_n(x)|\leqq Cx\varepsilon^{n+1},$$

we deduce the sharper estimate

$$|z_n(x)|\leqq|z_n(x)-z_{n+1}(x)|+|z_{n+1}(x)|\leqq Cx\varepsilon^{n+1/2}$$

(This is the familiar "throwing away" of terms which are needed in the proof, but do not contribute to the result.) We state this as a theorem.

THEOREM 9.1. *There exist constants* $\varepsilon_0>0$ *and* $C>0$ *such that*

$$|y(x)-y_a(x)|\leqq Cx\varepsilon^{n+1/2}$$

for $0<\varepsilon<\varepsilon_0$ *and* $0\leqq x\leqq R$.

Our next objective is to show that the derivative of y_a is a good approximation for the derivative of y (recall that y_a is more or less constructed through the integration of its derivative, and that in our application the derivative is the function which has a direct physical meaning). Our proof will be based on the following interpolation inequality.

LEMMA 9.2. *There exist constants* $\mu_0>0$ *and* $D>0$ *such that for any* $\phi\in C^2([0,R])$ *and each* $\mu\in(0,\mu_0)$

$$\sup|\phi'(x)|\leqq D\{\mu\sup|\phi''(x)|+\mu^{-1}\sup|\phi(x)|\},$$

where the suprema are taken over the interval $[0,R]$.

Proof. See Besjes [2]. The proof is based on a result to be found in Miranda [15, 33, III, p. 149]. □

THEOREM 9.3. *There exist constants* $\varepsilon_0>0$ *and* $C>0$ *such that*

$$|y'(x)-y_a'(x)|\leqq C\varepsilon^{n-1/2}$$

for $0<\varepsilon<\varepsilon_0$ *and* $0\leqq x\leqq R$.

Proof. From the equation for z (see (9.2)) we deduce that

$$|z''(x)| \leqq \varepsilon^{-1}\left\{\left|\frac{r(x)}{x}\right| + C_1|z'(x)| + C_2\left|\frac{z(x)}{x}\right|\right\},$$

where

$$C_1 := \sup_{0 \leqq x \leqq R} \frac{g(x)-y(x)}{x}, \qquad C_2 := \sup_{0 \leqq x \leqq R} |y_a'(x)|.$$

Next we apply Lemma 9.2 with $\mu = \varepsilon(2C_1D)^{-1}$ to obtain

$$\sup|z''(x)| \leqq 2\varepsilon^{-1}\left\{\sup\left|\frac{r(x)}{x}\right| + 2(C_1D)^2\varepsilon^{-1}\sup|z(x)| + C_2\sup\left|\frac{z(x)}{x}\right|\right\}.$$

By Theorem 9.1 and the estimate (9.4) this implies that

$$\sup|z''(x)| = O(\varepsilon^{n-3/2}).$$

Then a second application of Lemma 9.2, this time with $\mu = \varepsilon$, leads to the desired result. □

10. Some remarks about the case where g is neither everywhere increasing nor everywhere concave. In this section we shall discuss some extensions of our results to equations in which the conditions on the function g are considerably relaxed. In fact we shall merely assume that g satisfies the following hypotheses

$$\tilde{H}_g\colon g \in C^1([0, R]), \qquad g(0) = 0, \quad g(R) \geqq k,$$

$$g \text{ has only finitely many local extrema on } [0, R].$$

Thus, in particular the sign conditions on g' and g'' are dropped.

First of all we observe that the existence of a solution of (1.1)–(1.3) can be proved as in Theorem 3.2 by using zero as a lower solution and G as an upper solution, where G is any increasing, concave and smooth function such that $G(0) = 0$ and $G(x) \geqq g(x)$ on $[0, R]$.

As before we find that if $y = y(x; \varepsilon)$ is a solution then $y' > 0$ and hence sign y'' = sign $(y - g)$; subsequently, reasoning along the lines indicated in the proofs of Theorem 3.1 one can show that for any $\varepsilon > 0$,

$$0 < y'(x; \varepsilon) \leqq \sup_{0 \leqq \xi \leqq R} g'(\xi). \tag{10.1}$$

This in turn enables one to prove by means of the maximum principle that (1.1)–(1.3) can have at most one solution, and that the mapping $\varepsilon \mapsto y(\cdot\,; \varepsilon)$ is continuous from $\mathbb{R}_+$ into $C = C([0, R])$.

By (10.1) the set $\{y(\cdot\,; \varepsilon) | \varepsilon > 0\}$ is a precompact subset of C. Let X denote its limit set, as $\varepsilon \downarrow 0$, in C. Taking into account the continuity with respect to ε, we conclude that X is a nonempty, compact and connected subset of C (see Sell [16, p. 20]).

Any element u of X is a nondecreasing function with $u(0) = 0$ and $u(R) = k$. Our first objective is to give further characteristics of the elements of X.

LEMMA 10.1. *Let $u \in X$. Then there exist a nonempty, open set A and a closed set B such that*

(i) *$u(x) = g(x)$ if $x \in A$,*

(ii) *u is constant on each connected component of B,*

(iii) *$A \cap B = \varnothing$, $A \cup B = [0, R]$.*

Proof. Since $u \in X$, there exists a sequence $\{\varepsilon_n\}$ such that as $n \to \infty$, $\varepsilon_n \downarrow 0$ and $y(\cdot\,;\varepsilon_n) \to u$ strongly in C. By (10.1) $\{y(\cdot\,;\varepsilon_n)\}$ is bounded in $H^1 = H^1(0, R)$ and hence it is possible to pick a subsequence, again denoted by $\{\varepsilon_n\}$, such that as $n \to \infty$, $y(\cdot\,;\varepsilon_n) \to u$ weakly in H^1.

Next, we multiply (1.1) by an arbitrary function $\phi \in H^1$, integrate from 0 to R, integrate the first term by parts and let n tend to infinity. This yields the identity

$$\int_0^R (g(x) - u(x))u'(x)\phi(x)\,dx = 0,$$

whence

$$(g(x) - u(x))u'(x) = 0 \quad \text{a.e. on } [0, R]. \tag{10.2}$$

Define the sets A and B by

$$A = \{x \in [0, R] \mid u = g \text{ in a neighborhood of } x\}, \qquad B = [0, R]\backslash A,$$

then clearly $u'(x) = 0$ a.e. on B. In view of the continuity of g and u the sets A and B have all the properties listed in the lemma. □

LEMMA 10.2. *Let $u \in X$ and let I be a connected component of B such that $I \subset (0, R)$. Then*

$$\int_I \frac{u(x) - g(x)}{x}\,dx = 0. \tag{10.3}$$

Before proving this lemma, we prove an auxiliary result.

LEMMA 10.3. *Suppose that, as $n \to \infty$, $\varepsilon_n \downarrow 0$ and $y(x;\varepsilon_n) \to g(x)$ uniformly on $[a, b] \subset [0, R]$. Then*

$$\varepsilon_n \ln y'(x;\varepsilon_n) \to 0 \quad \text{as } n \to \infty$$

uniformly on $[a, b]$.

Proof. Choose a subinterval $[c, d]$ of $[a, b]$ and a positive constant $\delta > 0$ such that $g'(x) \geqq \delta$ on $[c, d]$. Define for each $n \geqq 1$, a point $\xi_n \in [c, d]$ such that

$$y'(\xi_n;\varepsilon_n) = \max\{y'(x;\varepsilon_n) \mid c \leqq x \leqq d\}.$$

Then it follows that there exists an $N_1 \geqq 1$ such that

$$y'(\xi_n;\varepsilon_n) \geqq \tfrac{1}{2}\delta \quad \text{for } n \geqq N_1.$$

If we divide (1.1) by xy' and integrate from ξ_n to x we obtain

$$\varepsilon_n \ln y'(x;\varepsilon_n) = \varepsilon_n \ln y'(\xi_n;\varepsilon_n) + \int_{\xi_n}^{x} \frac{y(\tau;\varepsilon_n) - g(\tau)}{\tau}\,d\tau.$$

Since the right-hand side tends to zero as $n \to \infty$, the same must be true for the left-hand side and the result follows. □

Proof of Lemma 10.2. Let $I = (e, f)$, where, by assumption, $0 < e < f < R$. Manipulating as above we obtain

$$\varepsilon_n \ln y'(e;\varepsilon_n) - \varepsilon_n \ln y'(f;\varepsilon_n) = \int_e^f \frac{y(\tau, \varepsilon_n) - g(\tau)}{\tau}\,d\tau.$$

Applying Lemma 10.3 to a left-hand neighborhood of e and to a right-hand neighborhood of f, we deduce that the left-hand side of this identity tends to zero as $n \to \infty$. So taking the limit $n \to \infty$ leads to the desired result. □

We now collect the information we have obtained about an arbitrary element u of X: u is a continuous, nondecreasing function with $u(0)=0$ and $u(R)=k$, which is composed out of pieces where $u(x)=g(x)$ and pieces where $u(x)$ is constant. Moreover, if I is a maximal interval on which u is constant, and I does not contain 0 or R, then (10.3) has to be satisfied. For convenience of formulation we shall call the set of functions having all these characteristics Y.

Our next objective is to show that Y is finite. First we shall illustrate our approach by discussing one example in full detail.

Consider a function g satisfying $\tilde{H}_g$ and such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0<b<c<R$ and $0<g(c)<g(b)<k$. Let g_1^{-1} denote the inverse of g on $[0, b]$ and g_2^{-1} the inverse of g on $[c, R]$.

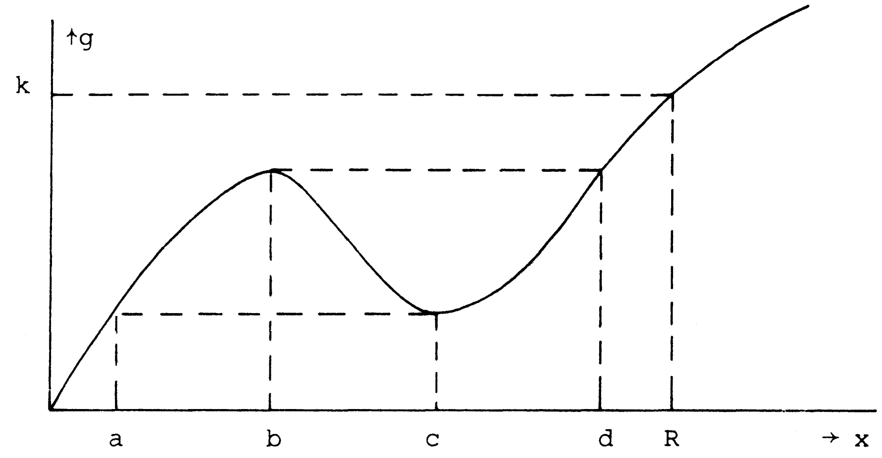

FIG. 1

Define two points a and b by

$$a=g_1^{-1}(g(c)), \qquad d=g_2^{-1}(g(b)).$$

Then $g([a,b])=g([c,d])$. (See Fig. 1.)

On $[a,b]$ we define a mapping F by

$$F(x)=\int_x^{g_2^{-1}(g(x))} \frac{g(x)-g(\tau)}{\tau}\, d\tau.$$

Then on (a,b)

$$F'(x)=g'(x)\int_x^{g_2^{-1}(g(x))} \frac{d\tau}{\tau}>0$$

and $F(a)<0$, $F(b)>0$. Consequently F has a unique zero on $[a,b]$.

Let w be an arbitrary element of Y. Then w has to coincide with g on $[0,a]$ and $[d, g_2^{-1}(k)]$ and it has to be equal to k on $[g_2^{-1}(k), R]$. Since w is nondecreasing the inverse function of w must "jump" from a point on $[a,b]$ to a point on $[c,d]$. In view of (10.3) this jump can only take place at the *unique* zero of F. Thus Y consists of one and only one element.

Returning to a general function g satisfying $\tilde{H}_g$ we define E to be the set of local maxima and minima of g and D to be the closure of the set $\{x \mid g$ is increasing in a neighborhood of $x\}$. Let D_c be one of the finitely many connected components of D. The set $g^{-1}(E)\cap D_c$ is finite. Take two successive points α_0 and β_0 in this set. To $[\alpha_0, \beta_0]$ there correspond finitely many disjunct intervals $[\alpha_i, \beta_i]\subset D$ such that $\alpha_i > \alpha_0$ and $g([\alpha_0, \beta_0]) = g([\alpha_i, \beta_i])$. Define g_i^{-1} on $[g(\alpha_0), g(\beta_0)]$ as the inverse of g with range in $[\alpha_i, \beta_i]$. On $[\alpha_0, \beta_0]$ we define mappings F_i by

$$F_i(x) = \int_x^{g_i^{-1}(g(x))} \frac{g(x)-g(\tau)}{\tau}\, d\tau.$$

Since F_i is monotone, it has at most one zero.

As already noted above the condition (10.3) implies that a point where the inverse function of an element of Y makes a jump should be a zero of some F_i for some connected component D_c of D and some pair of points α_0, β_0. Hence the set of possible "jump" points is finite and likewise the set Y is finite.

Thus X, being a subset of Y, must be discrete. Because it is also connected it can only consist of a single element. Consequently $y(\cdot\,; \varepsilon)$ converges in C to this function as $\varepsilon \downarrow 0$. We summarize the results in the following theorem.

THEOREM 10.4. *There exists a function $u \in Y$ such that*

$$\lim_{\varepsilon \downarrow 0} y(x;\varepsilon) = u(x), \quad \textit{uniformly on } [0, R].$$

In some cases the conditions determine the limit uniquely. For instance, this happens in the example we discussed at length and, more generally if the local extrema are ordered in such a way that with each connected component of D there corresponds precisely one possible "jump" point. In other cases our analysis is not constructive in the sense that, although we have shown that convergence occurs as $\varepsilon \downarrow 0$, we are not able to describe the limit completely. (See Fig. 2.) We intend to investigate whether this ambiguity can be resolved by using variational principles. See note added in proof.

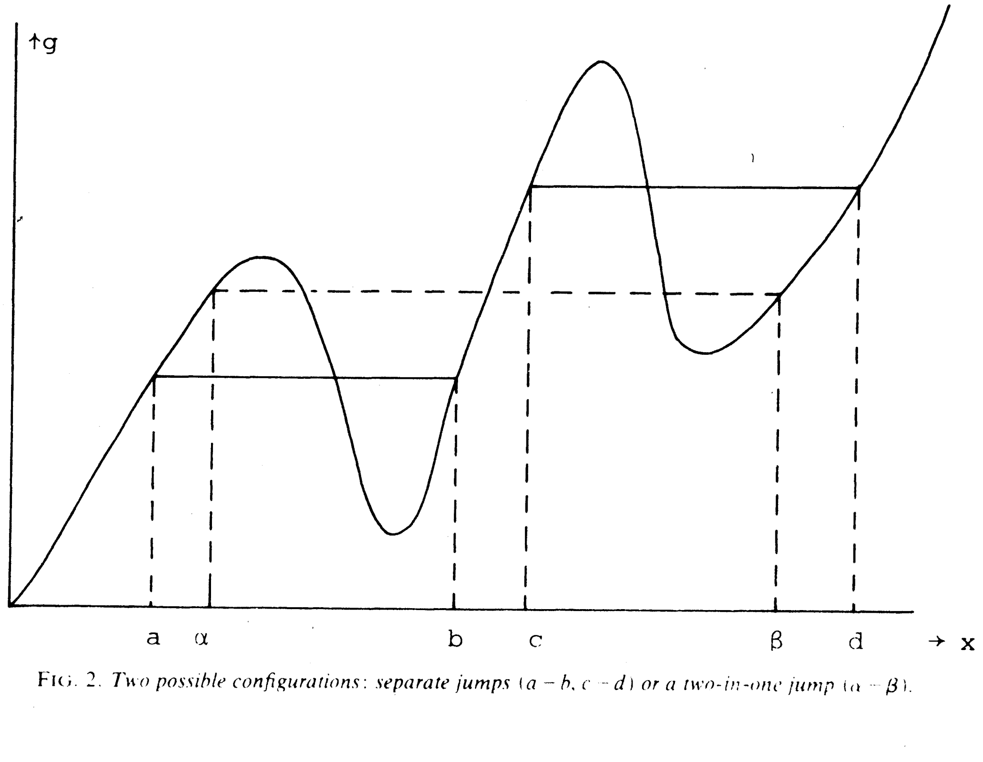

FIG. 2. *Two possible configurations: separate jumps $(a-b, c-d)$ or a two-in-one jump $(\alpha-\beta)$.*

In conclusion we remark that the hypothesis $g(R) \geqq k$ was made in order to obtain the uniform convergence on $[0, R]$. If $g(R) < k$ the solution will exhibit boundary layer behavior near the right endpoint. However, outside a small neighborhood of this endpoint, the solution will behave in exactly the same way as we have shown for the case $g(R) > k$.

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Note added in proof. It has been possible indeed to resolve the ambiguity connected with the limit $\varepsilon \to 0$ by means of a variational formulation of the problem (O. Diekmann and D. Hilhorst, *How many jumps? Variational characterization of the limit solution of a singular perturbation problem*, Proceedings of the Fourth Scheveningen Conference on Differential Equations, 1979, Springer, to appear).

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CHAPITRE III

A NONLINEAR EVOLUTION PROBLEM ARISING IN THE PHYSICS OF IONIZED GASES

par

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A NONLINEAR EVOLUTION PROBLEM ARISING IN THE PHYSICS OF IONIZED GASES*

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Abstract. We consider a Coulomb gas in a special experimental situation: the pre-breakdown gas discharge between two electrodes. The equation for the negative charge density can be formulated as a nonlinear parabolic equation degenerate at the origin. We prove the existence and uniqueness of the solution as well as the asymptotic stability of its unique steady state. Also some results are given about the rate of convergence.

1. Introduction. In this paper we study the nonlinear evolution problem

$$\mathrm{P}\qquad \begin{array}{ll} u_t = \varepsilon x u_{xx} + (g(x)-u)u_x & \text{on } D=(0,\infty)\times(0,T), \\ u(0,t)=0 & \text{for } t\in[0,T], \\ u(x,0)=\psi(x) & \text{for } x\in(0,\infty), \end{array}$$

where ε is a positive constant, g is a given function which satisfies the hypothesis $H_g: g\in C^2([0,\infty))$; $g(0)=0$; $g'(x)>0$ and $g''(x)<0$ for all $x\geqq 0$ and the initial function ψ satisfies the hypothesis H_ψ:

(i) ψ is continuous, with piecewise continuous derivative on $[0,\infty)$;
(ii) $\psi(0)=0$ and $\psi(\infty)=K\in(0,g(\infty))$;
(iii) there exists a constant $M_\psi\geqq g'(0)$ such that $0\leqq\psi'(x)\leqq M_\psi$ at all points x where ψ' is defined.

In § 2 we briefly describe how the problem arises in physics and give the derivation of the equations.

In § 3 we present maximum principles for certain linear and nonlinear problems related to P; the uniqueness of the solution of P follows directly from those principles.

In § 4 we prove that P has a classical solution which satisfies furthermore the condition

$$(*)\qquad u(\infty,t)=K \quad \text{for } t\in[0,T],\quad T<\infty.$$

The methods used here are inspired by those of van Duyn [7], [8] and Gilding and Peletier [13]. We also consider the limit case $\varepsilon\downarrow 0$ and prove that u tends to the generalized solution of the corresponding hyperbolic problem.

We then investigate the behavior of u as $t\to\infty$ and prove that it converges towards the unique solution Φ of the problem P_0 defined as follows

$$\mathrm{P}_0\qquad \begin{array}{l} \varepsilon x\Phi''+(g(x)-\Phi)\Phi'=0, \\ \Phi(0)=0, \qquad \Phi(\infty)=\lambda_0=:\min(\max(g(\infty)-\varepsilon,0),K). \end{array}$$

Qualitative properties of Φ have been extensively studied by Diekmann, Hilhorst and Peletier [6]. Here we analyze its stability. In § 5, following a method of Aronson and Weinberger [2] based on the knowledge about lower and upper solutions for the steady state problem P_0, we prove that Φ is asymptotically stable.

In § 6 we investigate the rate of convergence of u towards its steady state. The function Φ turns out to be exponentially stable when the function g grows fast enough to infinity as $x\to\infty$; the proof, based on constructing upper and lower solutions for the function $u-\Phi$, follows the same lines as that of Fife and Peletier [10]. We also

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consider the case when g increases less fast and show that provided $\varepsilon < g(\infty) - K$ and Φ converges algebraically fast to K as $x \to \infty$ the function $u - \Phi$ decays algebraically fast; this is done by obtaining first that property for a weighted integral of $u - \Phi$ according to a method of Il'in and Oleinik [14] and van Duyn and Peletier [9]. Finally we consider the corresponding hyperbolic problem and obtain a similar result of algebraic convergence.

2. Physical derivation of the equations. The physical context of the present problem has been described in some detail by Diekmann, Hilhorst and Peletier [6]. Here we shall summarize it again and explain how one can obtain the time evolution problem P.

One considers an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(\mathbf{r})$ and $n_e(\mathbf{r}, t)$ respectively, where $\mathbf{r} = (x_1, x_2, x_3)$. The ions are heavy and slow and the density $n_i(\mathbf{r})$ may therefore be regarded as fixed. The electrons are highly mobile. The problem is then to find $n_e(\mathbf{r}, t)$ for given $n_i(\mathbf{r})$ and in particular to find out whether given an initial electron distribution, the electrons stabilize and if so to evaluate the time needed for such a stabilization.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. Marode [17] and Marode, Bastien and Bakker [18]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $r = (x_1^2 + x_2^2)^{1/2}$ only. We thus have effectively a two-dimensional Coulomb gas with circular symmetry. The starting equations are

(i) Coulomb's law for the electric field E,

$$\frac{1}{r}\frac{\partial}{\partial r} rE = -C_d(n_e - n_i), \tag{2.1}$$

where C_d is a fixed constant;

(ii) a constitutive equation for the electric current j,

$$j = n_e \mu E + kT\frac{\partial n_e}{\partial r}, \tag{2.2}$$

in which the first term represents Ohm's law and the second term is due to thermal diffusion, μ being the mobility, k Boltzmann's constant and T the temperature; and

(iii) the continuity equation for the electron density,

$$\frac{\partial n_e}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rj. \tag{2.3}$$

If we set

$$u(x, t) = \int_0^{\sqrt{x}} n_e(r, t) r\, dr$$

and

$$g(x) = \int_0^{\sqrt{x}} n_i(r) r\, dr,$$

we obtain, after redefining the constants, the equation

$$u_t = \varepsilon x u_{xx} + (g(x) - u)u_x, \tag{2.4}$$

where $\varepsilon = 2kT/(\mu C_d)$, and the boundary condition

$$u(0, t) = 0. \tag{2.5}$$

Furthermore one makes the hypothesis that the total charge is positive and fixed, that is

$$\int_0^\infty (n_i(r) - n_e(r, t))r\,dr = N > 0,$$

from which we deduce the boundary condition at infinity;

$$u(\infty, t) = K := g(\infty) - N. \tag{2.6}$$

Clearly $K \in (0, g(\infty))$.

Equations (2.4) and (2.5) together with the initial condition

$$u(x, 0) = \psi(x) \tag{2.7}$$

constitute the mathematical formulation of the problem which we propose to study in this paper. Furthermore the condition (2.6) will turn out to be satisfied at all finite times t and also, for low enough values of the small parameter ε, at the time $t = \infty$. This latter property expresses the fact that all the electrons stay attached to the ions at low enough temperature; we shall also see that if the temperature rises above a critical value then some of the electrons escape to infinity, and if it rises even further above a second critical value then all the electrons escape to infinity.

3. Maximum principles for some degenerate parabolic operators–uniqueness theorem. In this section we prove maximum principles for some linear and nonlinear operators which have a degeneracy at the origin; these principles hold for functions $u \in C^{2,1}(D) \cap C(\bar{D})$, where $C^{2,1}(D)$ is the set of continuous functions on D with two continuous x-derivatives and one continuous t-derivative. It will follow easily from those maximum principles that P can have at most one solution $u \in C^{2,1}(D) \cap C(\bar{D})$ such that u_x is bounded in $\bar{D}$.

We begin by defining a linear operator L as follows

$$Lu = \varepsilon x u_{xx} + b(x, t)u_x + c(x, t)u - u_t, \tag{3.1}$$

where the functions b and c are continuous on D and such that the quantities $b/(1+x)$ and c are bounded on $\bar{D}$. First we consider the bounded domain $D_R := (0, R) \times (0, T)$, where R is a positive constant. In the same way as for a uniformly parabolic operator one can prove the following maximum principle which holds in fact for a much wider class of degenerate parabolic operators (see, for example, Ippolito [15] or Cosner [4])

THEOREM 3.1. *Suppose* $c \leqq 0$. *Let* $u \in C^{2,1}(D_R) \cap C(\bar{D}_R)$ *satisfy* $Lu \geqq 0$ *on* $(0, R) \times (0, T]$. *Then if u has a positive maximum in* $\bar{D}_R$, *that maximum is attained on* $((0, R) \times \{0\}) \cup (\{0, R\} \times [0, T])$.

Next, following a method due to Aronson and Weinberger [2], we derive a comparison theorem for a class of nonlinear evolution problems.

THEOREM 3.2. *Let u and* $v \in C^{2,1}(D_R) \cap C(\bar{D}_R)$ *and suppose that either* u_x *or* v_x *is bounded on* $\bar{D}_R$. *Let u and v satisfy*

$$Lv - vv_x \geqq Lu - uu_x \quad \text{on } (0, R) \times (0, T],$$

and let

$$0 \leqq v \leqq u \leqq K \quad \text{on } (0, R) \times \{0\} \text{ and } \{0, R\} \times [0, T].$$

Then $v \leqq u$ *in* $(0, R)\times(0, T]$.

Proof. Let

$$w=(v-u)\,e^{-\alpha t},$$

where

$$\alpha=\max_{(x,t)\in\bar{D}}\,(c(x,t)-u_x(x,t))$$

(in the case where u_x is bounded). Then w satisfies

$$\varepsilon x w_{xx}+(b(x,t)-v)w_x+(c(x,t)-u_x-\alpha)w-w_t\geqq 0$$

and

$$w\leqq 0 \quad \text{on } (0,R)\times\{0\} \text{ and } \{0,R\}\times[0,T].$$

Thus we deduce from Theorem 3.1 that

$$w\leqq 0 \quad \text{in } (0,R)\times(0,T],$$

which completes the proof of Theorem 3.2. □

Now let us consider the unbounded domain D. To begin with we Phragmèn–Lindelöf principle which is a special case of a theorem due to Cosner [4].

THEOREM 3.3. *Suppose that* $b/(1+x)$ *and* c *are continuous and bounded in* $\bar{D}$. *Let* $u\in C^{2,1}(D)\cap C(\bar{D})$ *satisfy* $Lu\geqq 0$ *on* $(0,\infty)\times(0,T]$ *and the growth condition*

$$\liminf_{\mathcal{R}\to\infty} e^{-B\mathcal{R}}\big[\max_{0\leqq t\leqq T} u(\mathcal{R},t)\big]\leqq 0 \tag{3.2}$$

for some positive constant B. *If* $u\leqq 0$ *for* $t=0$ *and on* $\{0\}\times[0,T]$ *then* $u\leqq 0$ *in* $(0,\infty)\times(0,T]$.

Making use of Theorem 3.3 one can prove a comparison theorem on the unbounded domain D.

THEOREM 3.4. *Let* u *and* $v\in C^{2,1}(D)\cap C(\bar{D})$ *be such that either* u_x *and* v *or* u *and* v_x *are bounded on* $\bar{D}$ *and that*

$$|u(x,t)|,\ |v(x,t)|\leqq Ce^{B_1x}$$

for some positive constants C *and* B_1 *and uniformly in* $t\in[0,T]$. *Suppose that*

$$Lv-vv_x\geqq Lu-uu_x \quad \text{on } (0,\infty)\times(0,T]$$

and that

$$0\leqq v\leqq u\leqq K \quad \text{on } (0,\infty)\times\{0\} \text{ and } \{0\}\times[0,T].$$

Then $v\leqq u$ *in* $(0,\infty)\times(0,T]$.

Finally let us come to the question of uniqueness of the solution of problem P.

DEFINITION. We shall say that u is a classical solution of problem P if it is such that (i) $u\in C^{2,1}(D)\cap C(\bar{D})$, (ii) u and u_x are bounded in $\bar{D}$, (iii) u satisfies the equation in D, (iv) u satisfies the initial and boundary conditions.

THEOREM 3.5. *Problem* P *can have at most one solution.*

Proof. Apply Theorem 3.4 twice to deduce that if u and v are two such solutions then their difference $w=u-v$ satisfies $w\geqq 0$ and $w\leqq 0$ and thus $w\equiv 0$. □

4. Existence and regularity of the solution. In order to be able to prove the existence of a solution of the nonlinear degenerate parabolic problem P, we consider

III.5.

certain related nonlinear uniformly parabolic problems on bounded domains and observe that they have a unique solution; we then deduce that P has a generalized solution, in a certain sense. It finally turns out that this solution is in fact a classical solution of P and thus the unique solution of P and that it also satisfies condition $(*)$. Finally we consider its limiting behavior as $\varepsilon \downarrow 0$.

4.1. Existence. Let us first introduce some notation. Let $D_n := (0, n)\times(0, T)$. We denote by $C_{2+\alpha}([0, n])$ the space of functions v which are twice differentiable and such that v'' is Hölder continuous on $[0, n]$ with exponent α. We also use the spaces $\bar{C}_\alpha(D_n)$, $\overline{C_{2+\alpha}}(D_n)$ and $C_{2+\alpha}(D_n)$, defined in Friedman [11, pp. 62, 63].

Consider the problem

$$\mathrm{P}_n \qquad \begin{array}{ll} u_t = \varepsilon(x+1/n)u_{xx} + (g(x)-u)u_x & \text{in } D_n, \\ u(0,t)=0, \qquad u(n,t)=K, & t\in[0,T], \\ u(x,0)=\psi_n(x), & x\in(0,n), \end{array}$$

with $n \geqq g^{-1}(K)$ and where ψ_n is such that

(i) $\psi_n \in C^\infty([0, \infty])$;
(ii) ψ_n satisfies H_ψ;
(iii) $\psi_n''(0)=0$ and $\psi_n(x)=K$ for $x\in[n-1,\infty)$.

In what follows we shall denote by H_n properties (i) – (iii). The following theorem holds:

THEOREM 4.1. *There exists a unique solution $u_n \in \overline{C_{2+\alpha}}(D_n)$ of P_n for any $\alpha \in (0, 1)$; furthermore u_n satisfies the inequalities*

$$0 \leqq u_n(x,t) \leqq \min(M_{\psi_n}x, K), \tag{4.1}$$

$$0 \leqq u_{nx}(x,t) \leqq M_{\psi_n}, \tag{4.2}$$

for all $(x, t)\in \bar{D}_n$.

Proof. The existence and uniqueness of $u_n \in \overline{C_{2+\alpha}}(D_n)$ is a consequence of Theorem 5.2 of Ladyženskaja [16, pp. 564–565]. The inequalities in (4.1) can be deduced by means of a comparison theorem analogous to Theorem 3.2. From the linear theory (Friedman [11, p. 72]) we deduce that the function $w := u_{nx} \in C_{2+\alpha}(D_n)$; thus $w \in C^{2,1}(D_n)\cap C(\bar{D}_n)$. Furthermore w satisfies

$$\begin{array}{l} w_t = \varepsilon(x+1/n)w_{xx} + (g(x)-u_n+\varepsilon)w_x + (g'(x)-w)w, \\ 0\leqq w(0,t)\leqq M_{\psi_n}, \qquad 0\leqq w(n,t)\leqq M_{\psi_n}, \\ w(x,0)=\psi_n'(x). \end{array} \tag{4.3}$$

The bounds on the function $w(n, t)$ follow from the fact that the function $\max(0, M_{\psi_n}(x-n)+K)$ is a lower solution of the boundary value problem

$$\varepsilon\left(x+\frac{1}{n}\right)\phi'' + (g(x)-\phi)\phi' = 0, \qquad \phi(0)=0, \quad \phi(n)=K$$

and consequently a lower bound for u_n. Clearly the set

$$\{w \in C([0, n]) \text{ such that } 0 \leqq w(x) \leqq M_{\psi_n}\}$$

is invariant with respect to the problem (4.3), and thus the inequalities (4.2) are satisfied.

Next we deduce, from Theorem 4.1, the existence of solution of P. We begin by approximating the initial function ψ by a sequence of smooth functions $\{\psi_n\}$.

III.6.

LEMMA 4.2. *Let the function* ψ *satisfy* H_ψ. *Then there exists a sequence* $\{\psi_n\}$ *which satisfies the properties* H_n *given at the beginning of this section with* $M_{\psi_n}=M_\psi$ *for all* n, *such that* $\psi_n \to \psi$ *as* $n\to\infty$, *uniformly on* $[0,\infty)$.

Proof. Let $n_0 \geqq g^{-1}(K)$ be such that for all $n \geqq n_0$ the point x_{1n} defined by $M_\psi(x_{1n}-1/n)=\psi(x_{1n})$ is such that $1/n<x_{1n}\leqq n-2$ and the point x_{2n} defined by $x_{2n}=n-2+(K-\psi(n-2))/M_\psi$ satisfies $n-2<x_{2n}<n-1$. Also define

$$\psi_n^*(x)=\begin{cases}0, & -\infty<x\leqq\dfrac{1}{n},\\[2mm] M_\psi\left(x-\dfrac{1}{n}\right), & \dfrac{1}{n}<x\leqq x_{1n},\\ \psi(x), & x_{1n}<x\leqq n-2,\\ M_\psi(x-n+2)+\psi(n-2), & n-2<x\leqq x_{2n},\\ K, & x_{2n}<x<+\infty.\end{cases}$$

Note that, for all x,

$$|\psi_n^*(x)-\psi(x)|\leqq \max\left(\frac{M_\psi}{n}, K-\psi(n-2)\right).$$

Next introduce the function

$$\rho(x)=\begin{cases}0 & \text{if } |x|\geqq 1,\\ C\exp\left(\dfrac{1}{|x|^2-1}\right) & \text{if } |x|<1,\end{cases}$$

where the constant C is such that $\int_{\mathbb{R}}\rho\,dx=1$, and let

$$\rho_\delta(x)=\frac{\rho(x/\delta)}{\delta}.$$

Finally define

$$\psi_n(x)=\int_{\mathbb{R}}\rho_{\delta_n}(x-y)\psi_n^*(y)\,dy,\qquad x\in[0,n],$$

with $\delta_n=\min(1/n, x_{1n}-1/n, n-2-x_{1n}, x_{2n}-n+2, n-1-x_{2n})/10$. We now show that ψ_n has the desired properties. Firstly $\psi_n\in C^\infty([0,n])$. The uniform convergence of $\{\psi_n\}$ to ψ follows from the continuity of ψ_n^*, uniformly in n and in x and the uniform convergence of ψ_n^* to ψ as $n\to\infty$. Finally properties (ii) and (iii) of H_n can be deduced for ψ_n from the fact that ψ also satisfies them.

Next we prove the following theorem.

THEOREM 4.3. P *has a unique classical solution. Furthermore this solution also satisfies condition* $(*)$:

$$(*)\qquad \lim_{x\to\infty} u(x,t)=K\quad \textit{for each } t\in(0,T].$$

Proof. We rewrite the parabolic equation of problem P_n as

$$(4.4)\qquad u_t=\varepsilon(x+1/n)u_{xx}+c(x,t)u_x,$$

where

$$c(x, t) = g(x) - u_n(x, t).$$

From Theorem 4.1 we know that for all (x', t), $(x'', t) \in \bar{D}_n$ and for all $n \geqq n_0$

$$|u_n(x', t) - u_n(x'', t)| \leqq M_\psi |x' - x''|. \tag{4.5}$$

Now fix $I \geqq n_0$; (4.4) and (4.5) enable us to apply a theorem of Gilding [12] about the Hölder continuity of solutions of parabolic equations, and we obtain

$$|u_n(x, t') - u_n(x, t'')| \leqq C |t' - t''|^{1/2}$$

for all $n \geqq I$ and for all (x, t'), $(x, t'') \in \bar{D}_I$, with $|t' - t''| \leqq 1$. Here the constant C depends on I but not on n. The set $\{u_n(x, t)\}_{n=I}^{\infty}$ is bounded and equicontinuous in D_I, and thus there exists a continuous function $u_I(x, t)$ and a convergent subsequence $\{u_{n_k}(x, t)\}$ with $n_k \geqq I$ such that $u_{n_k}(x, t) \to u_I(x, t)$ as $n_k \to \infty$, uniformly on $\bar{D}_I$. Then, by a diagonal process, it follows that there exists a function $u(x, t)$ defined on $\bar{D}$ and a convergent subsequence, denoted by $\{u_j(x, t)\}$ such that $u_j(x, t) \to u(x, t)$ as $j \to \infty$, pointwise on $\bar{D}$. Since this convergence is uniform on any bounded subset of $\bar{D}$, the limit function u is continuous on $\bar{D}$.

It remains to show that u is a solution of P; to that purpose we shall proceed in two steps: firstly we show that u is a generalized solution of P in a certain sense and then we conclude that it is in fact a classical solution. We shall say that u is a generalized solution of P if it has the following properties:

(i) u is continuous and uniformly bounded in $\bar{D}$;
(ii) $u(0, t) = 0$ for all $t \in [0, T]$;
(iii) u has a bounded generalized derivative with respect to x in D;
(iv) u satisfies the identity

$$\iint_D [u\phi_t - \varepsilon(xu_x - u)\phi_x - (g - u/2)u\phi_x - ug'\phi]\, dx\, dt + \int_0^\infty \psi(x)\phi(x, 0)\, dx = 0 \tag{4.6}$$

for all $\phi \in C^1(\bar{D})$ which vanish for $x = 0$, large x and $t = T$.

Let us check that u satisfies those properties.

(i) We already know that u is continuous on $\bar{D}$ and furthermore, since $u(x, t) = \lim_{j\to\infty} u_j(x, t)$, we have that $0 \leqq u \leqq K$.

(ii) This property follows from a similar boundary condition in P_n.

(iii) Let ϕ be an admissible test function and let $L \geqq n_0$ be such that supp $\phi \subset D_L$. Since $|u_{jx}|$ is uniformly bounded with respect to $j \geqq L$ for all $(x, t) \in D_L$, it follows that there exists a subsequence $\{(u_{j_k})_x\}$ and a bounded function $p \in L^2(D_L)$ such that

$$(u_{j_k})_x \rightharpoonup p \quad \text{in } L^2(D_L) \text{ as } j_k \to \infty.$$

Now let $\zeta \in C_0^1(\bar{D}_L)$. Then

$$((u_{j_k})_x, \zeta) \to (p, \zeta) \quad \text{as } j_k \to \infty, \tag{4.7}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(D_L)$. But since $u_{j_k} \to u$ as $j_k \to \infty$, uniformly on $\bar{D}_L$, we have

$$(u_{j_k}, \zeta_x) \to (u, \zeta_x) \quad \text{as } j_k \to \infty. \tag{4.8}$$

Hence, combining (4.7) and (4.8), we find that p is the generalized derivative of u.

(iv) Since u_{j_k} is a classical solution of P_n it follows that

$$\iint\limits_{D_L} \left[u_{j_k}\phi_t - \varepsilon\left(\left(x+\frac{1}{j_k}\right)(u_{j_k})_x - u_{j_k}\right)\phi_x - \left(g-\frac{u_{j_k}}{2}\right)u_{j_k}\phi_x - u_{j_k}g'\phi\right] dx\,dt \tag{4.9}$$
$$+\int_0^L \psi_{j_k}(x)\phi(x,0)\,dx = 0.$$

The sequences $\{u_{j_k}\}$ and $\{u_{j_k}^2\}$ converge to u and u^2, respectively, strongly in $L^2(D_L)$ as $j_k \to \infty$. Furthermore since $(u_{j_k})_x$ is uniformly bounded we have

$$\iint\limits_{D_L} \frac{1}{j_k}(u_{j_k})_x\phi_x\,dx\,dt \to 0 \quad \text{as } j_k \to \infty.$$

Thus letting $j_k \to \infty$ we obtain (4.6). Because ϕ has been chosen arbitrarily, we may conclude that u is indeed a generalized solution of P.

It remains to show that u is a classical solution of P. One can do it by using a classical bootstrap argument (see, for example, Gilding and Peletier [13]) to show that for whatever $\eta, L > 0$ there exists $\alpha(\eta, L) \in (0, 1)$ such that

$$u \in \overline{C_{2+\alpha}}((\eta, L)\times(\eta, T)), \tag{4.10}$$

where α and $\|u\|_{C_{2+\alpha}}$ may be estimated independently of T. In particular,

$$u \in C^{2,1}(D)\cap C(\bar{D}).$$

Since furthermore u and u_x are uniformly bounded u is a classical solution of problem P and by Theorem 3.5 it is the unique solution of P.

Finally let us analyze the behavior of u for large x; since we have $0 \leqq u \leqq K$ and $u_x \geqq 0$, $u(\infty, t) = \lim_{x\to\infty} u(x, t)$ is well defined for all $t \in [0, T]$ and such that $0 \leqq u(\infty, t) \leqq K$. Next we show that $u(\infty, t) \equiv K$ by constructing a time dependent lower solution for P. Consider the problem

$$\begin{aligned} &u_t = \varepsilon x u_{xx} + (K-u)u_x, \\ &u(x_0, t) = 0, \qquad x_0 \geqq g^{-1}(K), \\ &u(x, 0) = \psi(x). \end{aligned} \tag{4.11}$$

Since $u_x \geqq 0$ we have that

$$\begin{aligned} \varepsilon x u_{xx} + (g(x)-u)u_x - u_t &= \varepsilon x u_{xx} + (K-u)u_x - u_t + (g(x)-K)u_x \\ &\geqq \varepsilon x u_{xx} + (K-u)u_x - u_t \quad \text{for all } x \geqq g^{-1}(K). \end{aligned}$$

Thus a lower solution $\hat{u}$ of (4.11) with $\hat{u}_x \geqq 0$ is also a lower solution of P on $[x_0, \infty)\times[0, T]$. We search such functions $\hat{u}_k$ which satisfy furthermore

$$\hat{u}_k(\infty, t) = K - k \quad \text{for all } t \in [0, T] \text{ and with } k \in (0, K).$$

Writing

$$\hat{v} = K - \hat{u},$$

reduces this to finding an upper solution $\hat{v}_k$ of

$$\begin{aligned} &v_t = \varepsilon x v_{xx} + v v_x, \\ &v(x_0, t) = K, \qquad v(\infty, t) = 0. \end{aligned}$$

Next we look for such a function $\hat{v}_k$, also requiring that

$$\hat{v}_k(x, t) = \hat{f}_k\left(\frac{x}{t+1}\right).$$

Setting

$$\eta = \frac{x}{t+1},$$

one can easily derive that $\hat{f}_k$ should be an upper solution for the boundary value problem

$$\pi \qquad \begin{aligned} &\varepsilon\eta f'' + (f+\eta)f' = 0, \\ &f(x_0) = K, \qquad f(\infty) = 0. \end{aligned}$$

Let $x_0 > \max(\varepsilon, g^{-1}(K))$, and take

$$\hat{f}_k(\eta) = k + (K-k)\left(\frac{\eta}{x_0}\right)^{1-x_0/\varepsilon}.$$

One can check that indeed $\hat{f}_k$ is an upper solution for problem π and consequently that $\hat{u}_k(x, t) = K - \hat{f}_k(x/(t+1))$ is a lower solution for problem P on the sector $\{t \geqq 0, x \geqq x_0(t+1)\}$ provided that x_0 is large enough. Since k can be chosen arbitrarily in $(0, K)$ it follows that $u(\infty, t) = K$ for all $t < \infty$. □

4.2. The limiting behavior as $\varepsilon \downarrow 0$. In this section we study the limiting behavior of the solution u^{ε} of P as $\varepsilon \downarrow 0$. To begin with, we consider the following hyperbolic problem:

$$\text{H} \qquad \begin{aligned} &u_t = (g(x) - u)u_x \quad \text{in } D, \\ &u(x, 0) = \psi(x) \qquad \text{for all } x \in (0, \infty), \end{aligned}$$

and make some heuristic considerations about the solution $\bar{u}$ of problem H; they are due to Wilders [23]. One possible configuration of g and ψ is drawn in Fig. 1; the corresponding characteristics are represented in Fig. 2. Their equations are

$$\frac{dx}{dt} = -(g(x) - \psi(x(0))).$$

Along those characteristics $\bar{u}$ is constant, i.e., $\bar{u} = \psi(x(0))$. Also, since $\psi(0) = 0$ it follows that the line $x = 0$ is the characteristic passing through the point $(0, 0)$ and

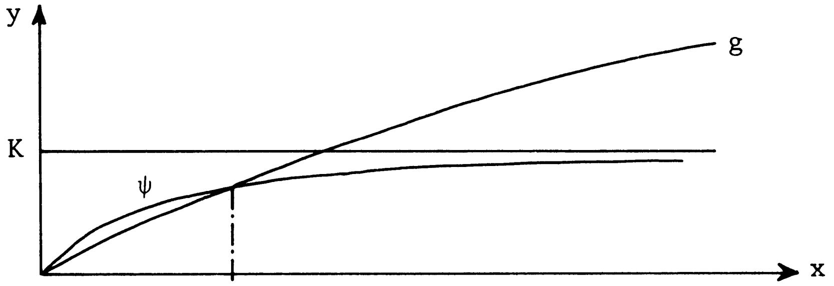

FIG. 1

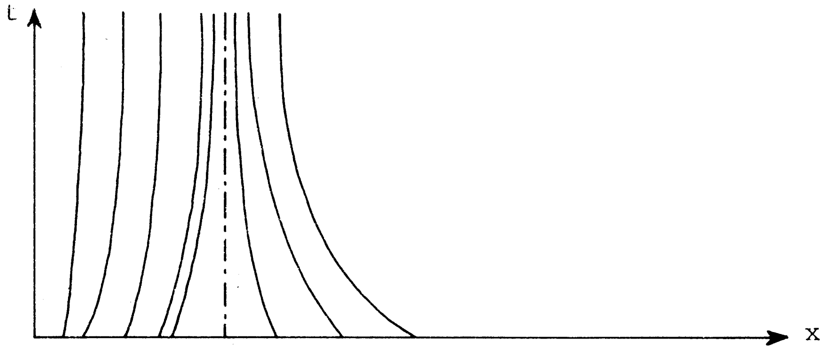

FIG. 2

consequently that $\bar{u}$ automatically satisfies a boundary condition of the form $\bar{u}(0, t) = 0$. Next we deduce from the fact that ψ is nondecreasing that two characteristics do not intersect. Suppose that there exist two characteristics issuing from the points $x = a$ and $x = b (a < b)$ on the initial line, intersecting each other at the point $(x, t) = (x^*, t^*)$. Then if they would intersect transversally we would have $-(g(x^*) - \psi(a)) > -(g(x^*) - \psi(b))$ and hence $\psi(a) > \psi(b)$, which is impossible. Now if the characteristics would be tangent to each other at the point (x^*, t^*) we would have $-(g(x^*) - \psi(a)) = -(g(x^*) - \psi(b))$ and consequently $\psi(a) = \psi(b)$; both characteristics would then be described by the same differential equation $dx/dt = -(g(x) - \psi(a))$, which, by the standard uniqueness theorem for ordinary differential equations, implies $a = b$. Finally we conclude that since the initial condition ψ is continuous and nondecreasing, no shock wave can occur and $\bar{u}(\cdot, t)$ is continuous at all times.

In [19] Oleinik proved existence and uniqueness of the generalized solution of Cauchy problems and boundary value problems related to problem H but since the boundary line $x = 0$ is a characteristic for H (which is reflected in the relation $g(0) - \bar{u}(0, 0) = 0$), problem H does not satisfy all the assumptions made in [19]. This leads us to give here a proof of the existence of a solution of problem H, by showing that the solution u of problem P tends to a limit as $\varepsilon \downarrow 0$; the uniqueness is a consequence of [19]. Following [19, Lemmas 18 and 19], we say that $\bar{u}$ is a generalized solution of H if it satisfies

(i) $\bar{u}$ is bounded and measurable in $\bar{D}$;

(ii) $\dfrac{\bar{u}(x_1, t) - \bar{u}(x_2, t)}{x_1 - x_2} \leqq M_\psi$ for all points $(x_1, t), (x_2, t) \in \bar{D}$;

(iii) $\bar{u}$ satisfies the identity

$$\iint_D \left[\bar{u}\phi_t - \left(g - \frac{\bar{u}}{2}\right)\bar{u}\phi_x - \bar{u}g'\phi\right] dx\, dt + \int_0^\infty \psi(x)\phi(x, 0)\, dx = 0 \tag{4.12}$$

for all $\phi \in C^1(\bar{D})$ which vanish for large x and $t = T$.

Next we shall prove the following theorem.

THEOREM 4.4. *The solution $u(x, t)$ of* P *tends uniformly on all compact subdomains of D to a limit $\bar{u}$ as $\varepsilon \downarrow 0$, where $\bar{u}$ is the unique generalized solution of* H. *The function $\bar{u}$ is furthermore continuous, nondecreasing in x at all times $t \in [0, T]$ and satisfies the boundary conditions $\bar{u}(0, t) = 0$ and $\bar{u}(\infty, t) = K$.*

Before proving Theorem 4.4, let us introduce a class of upper and lower solutions for problem P which depend neither on ε nor on time. They will turn out to be very useful both to prove that $\bar{u}(\infty, t) = K$ in Theorem 4.4 and to study the asymptotic behavior of u as $t \to \infty$ in the next sections. Next we define

$$s^+(x) := \min (M_\psi x, K)$$

and

$$s^-(x, \lambda, x_1, \nu) := \max \left(0, \lambda\left(1 - \left(\frac{x}{x_1}\right)^{-\nu}\right)\right),$$

where the constants $\lambda \in [0, K]$, $\nu > 0$ and $x_1 > 0$ are chosen in the following manner:

(a) If $\varepsilon < g(\infty)$, we choose $x_1 > 0$ so that $g(x_1) > \varepsilon$, then $\lambda > 0$ so that $\lambda < g(x_1) - \varepsilon$ and finally $\nu > 0$ so that

$$\nu \leqq \varepsilon^{-1}(g(x_1) - \lambda) - 1. \tag{4.13}$$

(b) If $\varepsilon \geqq g(\infty)$, we set $\lambda = 0$, which amounts to setting $s^- \equiv 0$.

It is easily seen that s^- satisfies the inequality

$$\hat{\varepsilon} x (s^-)'' + (g - s^-)(s^-)' \geqq 0 \quad \text{for all } x \in [0, \infty) \backslash \{x_1\},\ \hat{\varepsilon} \in (0, \varepsilon).$$

Thus if $\varepsilon < g(\infty)$, given any $\hat{\lambda} < \lambda_0 = \min (g(\infty) - \varepsilon, K)$, one can find $\hat{x}_1$ and $\hat{\nu}$ satisfying (4.13) and such that $s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}) \leqq \psi$. Applying the comparison Theorem 3.4 we deduce that $s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}) \leqq u$ (and thus that $\lambda_0 \leqq u(\infty, t)$ for all $t \leqq \infty$). Similarly one can check that $u \leqq s^+$.

Proof of Theorem 4.4. The uniqueness of the solution of problem H can be proven along the same lines as in the proof of [19, Thm. 1, Lemma 21]. Next we show its existence. Fix $I \geqq 1$. Since u and u_x are bounded uniformly in ε we deduce from Gilding [12] that u is equicontinuous on $\bar{D}_I$; thus, there exists a subsequence $\{u_{\varepsilon_n}\}_{n=I}^{\infty}$ of u and a function $\bar{u}_I \in C(\bar{D}_I)$, such that $u_{\varepsilon_n} \to \bar{u}_I$ as $\varepsilon_n \downarrow 0$ uniformly in $\bar{D}_I$ and such that for all $\lambda < K$, one can find x_1 and ν satisfying (4.13) and $s^-(\cdot, \lambda, x_1, \nu) \leqq \bar{u}_I(\cdot, t) \leqq s^+(\cdot)$. Then by a diagonal process, it follows that there exists a bounded continuous function $\bar{u}$ and a converging subsequence denoted by $\{u_{\varepsilon_k}\}$ such that $u_{\varepsilon_k} \to \bar{u}$ as $\varepsilon_k \downarrow 0$, pointwise on D and uniformly on all compact subsets of D. Since $0 \leqq (u_{\varepsilon_k})_x \leqq M_\psi$, $\bar{u}$ is nondecreasing in the x-direction and satisfies (ii); $u_{\varepsilon_k}(0) = 0$ implies the same property for $\bar{u}$. The boundary condition $\bar{u}(\infty, t) = K$ follows from the inequalities $s^-(\cdot, \lambda, x_1, \nu) \leqq \bar{u}(\cdot, t) \leqq s^+(\cdot)$ for all $\lambda < K$.

It remains to show that $\bar{u}$ is a generalized solution of H. Let $\phi \in C^1(\bar{D})$ vanish for large x and $t = T$, and let $L \geqq 1$ be such that ϕ vanishes in the neighborhood of $x = L$ and for $x > L$. Because the functions u_{ε_k} are classical solutions of P, we have

$$\iint_{D_L} \left[u_{\varepsilon_k}\phi_t - \varepsilon_k (x u_{\varepsilon_k x} - u_{\varepsilon_k})\phi_x - \left(g - \frac{u_{\varepsilon_k}}{2}\right) u_{\varepsilon_k}\phi_x - u_{\varepsilon_k} g'\phi \right] dx\, dt$$
$$+ \int_0^L \psi(x)\phi(x, 0)\, dx = 0.$$

Now letting $\varepsilon_k \downarrow 0$ we deduce that $\bar{u}$ satisfies (4.12); because ϕ has been chosen arbitrarily we conclude that $\bar{u}$ is indeed the generalized solution of H and that $\{u_\varepsilon\}$ converges to $\bar{u}$ as $\varepsilon \downarrow 0$. □

5. Asymptotic stability of the steady state. Adapting a method due to Aronson and Weinberger [2] we investigate the stability of the solution Φ of problem P_0. To that purpose we consider the solution u of the corresponding evolution problem P; since its dependence on ψ plays a central role in what follows, we denote this solution by $u(x, t, \psi)$. We show that for all the functions ψ satisfying the hypothesis H_ψ given in the introduction we have that

$$u(x, t, \psi) \to \Phi(x) \quad \text{as } t \to \infty.$$

To begin with we prove two auxiliary lemmas.

LEMMA 5.1. (i) *Let* $\varepsilon < g(\infty)$ *and* $\hat{\lambda}, \hat{x}_1, \hat{\nu}$ *satisfy* (4.13). *The function* $u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}))$ *is nondecreasing in time and such that*

$$\lim_{t\to\infty} u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu})) = \phi_{\hat{\lambda}}(x), \tag{5.1}$$

where $\phi_{\hat{\lambda}}$ *is the unique solution of*

$$\begin{gathered} \varepsilon x\phi'' + (g(x) - \phi)\phi' = 0, \\ \phi(0) = 0, \qquad \phi(\infty) = \hat{\lambda}. \end{gathered} \tag{5.2}$$

(ii) *The function* $u(x, t, s^+)$ *is nonincreasing in time. Furthermore*

$$\lim_{t\to\infty} u(x, t, s^+) = \Phi. \tag{5.3}$$

Proof. First note that it follows from the proofs in § 4 that problem P with initial value $s^-(x, \hat{\lambda}, \hat{x}_1, \hat{\nu})$ has a unique classical solution $u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}))$ with $u(\infty, t) = \hat{\lambda}$ for all $t \leqq \infty$. Applying repeatedly Theorem 3.4, one can show that $u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}))$ is nondecreasing in time and that $u(x, t, s^+)$ is nonincreasing in time; it also follows from Theorem 3.4 that

$$u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu})) \leqq \phi_{\hat{\lambda}}(x),$$

and that

$$u(x, t, s^+) \geqq \Phi(x).$$

Now for each x, $u(x, t, s^-(\cdot, \hat{\lambda}, \hat{x}_1, \hat{\nu}))$ is nondecreasing in t and bounded from above. Therefore it has a limit $\tau-(x)$ as $t \to \infty$ and one can use standard arguments (see for example Aronson and Weinberger [2]) to show that $\tau^- \in C_{2+\alpha}((0, \infty)) \cap C([0, \infty))$ and satisfies the differential equation in (5.2) and the boundary conditions $\tau^-(0) = 0$ and $\tau^-(\infty) = \hat{\lambda}$. Finally since $\phi_{\hat{\lambda}}$ is the unique solution of problem (5.2) we have that $\tau^- = \phi_{\hat{\lambda}}$. Similarly one can show that $u(x, t, s^+)$ converges to a function $\tau^+ \in C_{2+\alpha}((0, \infty)) \cap C([0, \infty))$ which satisfies the steady state equation, the boundary condition $\tau^+(0) = 0$ and the condition $\Phi(\infty) \leqq \tau^+(\infty) \leqq K$. The fact that $\tau^+(\infty) = \Phi(\infty)$ follows from [6, Lemma 5.1]. Consequently $\tau^+ = \Phi$.

LEMMA 5.2. $\phi_{\hat{\lambda}}$ *is an increasing and continuous function of* $\hat{\lambda}$. *More precisely if* $\hat{\lambda}_1 \geqq \hat{\lambda}_2$ *we have*

$$0 \leqq \phi_{\hat{\lambda}_1} - \phi_{\hat{\lambda}_2} \leqq \hat{\lambda}_1 - \hat{\lambda}_2.$$

Proof. Let $m = \phi_{\hat{\lambda}_1} - \phi_{\hat{\lambda}_2}$. It satisfies the differential equation

$$\varepsilon x m'' + (g - \phi_{\hat{\lambda}_1})m' - \phi'_{\hat{\lambda}_2} m = 0$$

and the boundary conditions $m(0)=0$ and $m(\infty)=\hat{\lambda}_1-\hat{\lambda}_2 \geqq 0$. Suppose that m attains a negative minimum at a certain point $\xi \in (0,\infty)$; then $m(\xi)<0$, $m'(\xi)=0$ and $m''(\xi)\geqq 0$ which is in contradiction with $\varepsilon\xi m''(\xi)=\phi'_{\hat{\lambda}_2}(\xi)m(\xi)$. Thus $m \geqq 0$. In the same way one can show that m cannot attain a positive maximum, which implies $m \leqq \hat{\lambda}_1-\hat{\lambda}_2$.

Finally we are in a position to prove the following theorem.

THEOREM 5.3. *Let* $\Phi(x)$ *be the solution of problem* P_0. *Suppose* ψ *satisfies the hypothesis* H_ψ, *then for each* $x \geqq 0$

$$\lim_{t\to\infty} u(x,t,\psi)=\Phi(x).$$

If $\varepsilon \leqq g(\infty)-K$ *the convergence is uniform on* $[0,\infty)$; *if* $\varepsilon > g(\infty)-K$ *it is uniform on all compact intervals of* $[0,\infty)$.

Proof. Since the functions u and u_x are bounded uniformly in t, we apply the Arzela-Ascoli theorem and a diagonal process to deduce that there exists a function $\tau \in C([0,\infty))$ and a sequence $\{u(t_n)\}$ with $u(t_n)=u(\cdot,t_n,\psi)$ such that $u(t_n)\to\tau$ as $t_n\to\infty$, uniformly on all compact subsets of $[0,\infty)$. Let $\varepsilon<g(\infty)$; then for each $\hat{\lambda}<\lambda_0=\min(g(\infty)-\varepsilon,K)$ one can find $\hat{\nu}$ and $\hat{x}_1$ satisfying (4.13) and such that $s^-(\cdot,\hat{\lambda},\hat{x}_1,\hat{\nu})\leqq\psi$. Applying Theorem 3.4 we obtain

$$u(x,t,s^-(\cdot,\hat{\lambda},\hat{x}_1,\hat{\nu}))\leqq u(x,t,\psi)\leqq u(x,t,s^+). \tag{5.4}$$

Letting $t\to\infty$ in (5.4) and applying Lemma 5.1 we obtain

$$\phi_{\hat{\lambda}}\leqq\tau\leqq\Phi \quad \text{for all } \hat{\lambda}<\lambda_0.$$

Next we deduce from Lemma 5.2 that

$$\Phi-\tau<\lambda_0-\hat{\lambda} \quad \text{for all } \hat{\lambda}<\lambda_0$$

and thus, that $\tau=\Phi$. If $\varepsilon\geqq g(\infty)$, then the inequalities

$$0\leqq u(x,t,\psi)\leqq u(x,t,s^+)$$

imply

$$0\leqq\tau\leqq\Phi=0.$$

Thus also in this case we have that $\tau=\Phi$. Finally we conclude that as $t\to\infty$, $u(\cdot,t,\psi)$ converges to Φ, uniformly on all compact intervals of $[0,\infty)$. This convergence result can be made slightly stronger in the case that $\varepsilon\leqq g(\infty)-K$: since then $\Phi(\infty)=K$ and since u is nondecreasing in x one can apply Diekmann [5, Lemma 2.4] to deduce that the convergence is uniform on $[0,\infty)$. □

6. Rate of convergence of the solution towards the steady state. In this section we analyze the rate of convergence of the solution u of P towards its steady state Φ. The results which we are able to derive depend strongly on the behavior of g as $x\to\infty$. If g tends to infinity fast enough, we can prove exponential convergence with a certain weighted norm. In the more general case, when $\varepsilon<g(\infty)-K$ we find that the solution converges algebraically fast towards its steady state on all finite x-intervals. No results are available in the case $\varepsilon\geqq g(\infty)-K$, which coincides with the physical situation when some (or all the) electrons escape to infinity.

We write

$$u(x,t,\psi)=\Phi(x)+v(x,t).$$

III.14.

Then v satisfies the problem

$$
\begin{aligned}
&v_t = \varepsilon x v_{xx} + (g-\Phi)v_x - \Phi' v - v v_x,\\
(6.1)\qquad &v(0,t)=0,\\
&v(x,0)=\psi(x)-\Phi(x).
\end{aligned}
$$

Now let us make the change of function

$$v(x,t)=\exp\left(-\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde v(x,t).$$

Problem (6.1) becomes

$$
\begin{aligned}
&\tilde v_t = \varepsilon x \tilde v_{xx} - q(x)\tilde v + h(x,\tilde v,\tilde v_x),\\
(6.2)\qquad &\tilde v(0,t)=0,\\
&\tilde v(x,0)=\exp\left(\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)(\psi(x)-\Phi(x)),
\end{aligned}
$$

where

$$q(x)=\frac{(g(x)-\Phi(x))^2}{4\varepsilon x}+\frac{g'(x)+\Phi'(x)}{2}-\frac{g(x)-\Phi(x)}{2x}$$

and

$$h(x,\tilde v,\tilde v_x)=-\exp\left(-\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde v\left(\tilde v_x-\frac{g(x)-\Phi(x)}{2\varepsilon x}\tilde v\right).$$

In particular, there exists $M>0$ such that

$$|h(x,\tilde v,\tilde v_x)|\leqq M(\|\tilde v\|^2+\|\tilde v_x\|^2),\qquad 0<x<\infty,$$

where the notation $\|\cdot\|$ indicates the sup-norm.

In what follows we shall distinguish two cases: (i) the case when $\liminf_{x\to\infty} q(x)=\delta>0$: this is so if $g(x)\geqq C_0\sqrt{x}$ for all $x\geqq x_2$ for some positive constants C_0 and x_2; (ii) the case when $\liminf_{x\to\infty} q(x)=0$.

6.1. Case when g tends to infinity at least as fast as $\sqrt{x}$ for $x\to\infty$. The theorem we give next is very similar in its form and in its proof to a theorem of Fife and Peletier [10].

THEOREM 6.1. *Suppose that there exist constants x_2, $C_0\geqq 0$ such that*

$$(6.3)\qquad g(x)\geqq C_0\sqrt{x}\quad \textit{for all } x\geqq x_2.$$

Then there exist positive constants δ, μ, C such that if

$$\left\|\exp\left(\int_0^{\cdot} \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)(\psi-\Phi)\right\|\leqq\delta$$

then

$$\left\|\exp\left(\int_0^{\cdot} \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)(u(\cdot,t,\psi)-\Phi)\right\|\leqq Ce^{-\mu t},\qquad t\geqq 0,$$

where the notation $\|\cdot\|$ indicates the sup-norm.

III.15.

Proof. To begin with we note that with the hypothesis of Theorem 6.1 we have that $v(\infty, t) = 0$ (since $\varepsilon < g(\infty) - K$) or equivalently

$$\lim_{x\to\infty} \exp\left(-\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde{v}(x, t) = 0.$$

Next let us consider the boundary value problem

$$\text{(6.4)}\qquad \varepsilon x w'' - (q(x)+\lambda)w = -\theta(\Phi'(\mathcal{R})+\lambda)\min(\tilde{\Phi}(x), (x/\mathcal{R})^{-\nu_0}\tilde{\Phi}(\mathcal{R})), w(0) = 0,$$

where

$$\tilde{\Phi}(x) = \exp\left(\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\Phi(x).$$

The right-hand side of the differential equation in (6.4) has been chosen in a special manner so that one can exhibit upper and lower solutions for a problem closely related to (6.4); more precisely we shall prove in the appendix that this problem has at least one solution $w \in C^2([0, \infty))$ with w, w' and w'' bounded such that

$$0 < w(x) \leqq \min\left(\tilde{\Phi}(x), \left(\frac{x}{\mathcal{R}}\right)^{-\nu_0}\tilde{\Phi}(\mathcal{R})\right)$$

for all constants $\nu_0 > 1$ provided that the constants $\theta \in (0, 1)$, $\mathcal{R} > 0$ and $\lambda < 0$ satisfy certain conditions. We adjust θ such that $\|w\| + \|w'\| \leqq 1$.

We are now in a position to prove Theorem 6.1. Let

$$z(x, t) = \beta(w(x)+\gamma)\,e^{-\mu t},$$

in which β, γ and μ are positive constants still to be determined, and let

$$\mathcal{M}z = \varepsilon x z_{xx} - q(x)z + h(x, z, z_x) - z_t.$$

(i) The function q is positive for x near zero and, because of condition (6.3), also for large x; thus there exists $\bar{q}_0 > 0$ and ζ_1, $\zeta_2 \in (0, \infty)$ such that $\bar{q}_0 = \min\{q(x): x \in [0, \zeta_1] \cup [\zeta_2, \infty)\}$ is positive; therefore

$$\mathcal{M}z \leqq \beta\,e^{-\mu t}((\lambda+\mu)w + \gamma(-\bar{q}_0+\mu) + M\beta(1+\gamma)^2).$$

Choose

$$0 < \mu < \min(-\lambda, \bar{q}_0);$$

assume that γ is known (we shall specify it later), and choose

$$\beta = \frac{\gamma(\bar{q}_0-\mu)}{M(1+\gamma)^2}.$$

Then $\mathcal{M}z \leqq 0$ for all $x \in [0, \zeta_1] \cup [\zeta_2, \infty)$ and $t \geqq 0$,

(ii) Let $\zeta_1 \leqq x \leqq \zeta_2$; since $w(x) > 0$ on $(0, \infty)$, and since w is continuous we have

$$m = \min\{w(x): \zeta_1 \leqq x \leqq \zeta_2\} > 0.$$

Therefore

$$\mathcal{M}z \leqq \beta\,e^{-\mu t}((\lambda+\mu)m + \gamma(-\bar{q}+\mu) + M\beta(1+\gamma)^2),$$

where $\bar{q}$ is an arbitrary constant such that

$$\bar{q} < \min\{q(x): x \in [0, \infty)\}.$$

Hence

$$\mathcal{M}z \leqq \beta e^{-\mu t}((\lambda+\mu)m+\gamma(-\bar{q}+\bar{q}_0)).$$

Therefore if we choose

$$\gamma=-\frac{\lambda+\mu}{-\bar{q}+\bar{q}_0}m$$

we have

$$\mathcal{M}z \leqq 0 \quad \text{for } \zeta_1 \leqq x \leqq \zeta_2 \text{ and } t \geqq 0.$$

Thus for the above choice of β, γ and μ the function z is an upper solution of the equation $\mathcal{M}\tilde{v}=0$. Let

$$\sup_{[0,\infty)} \tilde{v}(x,0) \leqq \delta,$$

where $\delta=\beta\gamma$. Then

$$\tilde{v}(x,0) \leqq z(x,0) \quad \text{for all } x \in [0,\infty),$$

and hence by Theorem 3.4

$$\tilde{v}(x,t) \leqq z(x,t) \quad \text{for all } x \in [0,\infty), \quad t \geqq 0.$$

In a similar manner one can show that if

$$\inf_{[0,\infty)} \tilde{v}(x,0) \geqq -\delta$$

then

$$\tilde{v}(x,t) \geqq -z(x,t) \quad \text{for all } x \in [0,\infty), \quad t \geqq 0.$$

Hence if $\|\tilde{v}(\cdot,0)\| \leqq \delta$ then $\|\tilde{v}(\cdot,t)\| \leqq Ce^{-\mu t}$ where we define

$$C=\beta(1+\gamma)=(1+1/\gamma)\delta. \qquad \square$$

6.2. Algebraic decay rate in the case that $\varepsilon < g(\infty)-K$. Provided that $\varepsilon < g(\infty)-K$ and that the initial function ψ converges algebraically fast to K as $x \to \infty$, we prove that the solution u of P converges algebraically fast to the steady state solution Φ for all finite values of x. To that purpose we show that a certain weighted space integral of the function $|u-\Phi|^p$, for some integer $p \geqq 1$, decays algebraically in time; a similar proof, with exponent $p=1$, has been given, for example, by van Duyn and Peletier [9].

THEOREM 6.2. *Provided that* $\varepsilon < g(\infty)-K$ *and that* $\psi \geqq s^-(\cdot, K, \bar{x}_1, \bar{\nu})$ *for some* $\bar{x}_1$, $\bar{\nu}$ *satisfying* (4.13) *with* $\lambda=K$, *we have that*

$$\int_0^\infty (g'(x)+(p-1)\Phi'(x))|u(x,t,\psi)-\Phi(x)|^p\,dx \leqq \left[\int_0^\infty ((s^+-\Phi)^p+(\Phi-s^-)^p)\,dx\right]\Big/t \tag{6.5}$$

for all $t>0$ *and* $p=[1/\bar{\nu}]+1$.

Proof. Since $|v(x,t)|^p \leqq (s^+(x)-s^-(x,K,\bar{x}_1,\bar{\nu}))^p$ it follows that $\int_0^\infty (v(x,t))^p\,dx$ is defined for all $t \geqq 0$. If $p \geqq 2$ let us multiply the differential equation in (6.1) by v^{p-1}

and integrate with respect to x; we obtain

$$\frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = [\varepsilon x v_x v^{p-1}]_0^\infty - \left[\varepsilon \frac{v^p}{p}\right]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2\,dx$$

$$+\left[g\frac{v^p}{p}\right]_0^\infty - \int_0^\infty (g' + \Phi'(p-1))\frac{v^p}{p}\,dx - \left[\Phi\frac{v^p}{p} + \frac{v^{p+1}}{p+1}\right]_0^\infty$$

Since v tends to zero at least as fast as $x^{-\bar{\nu}}$ as $x \to \infty$, the equation above can be written in the simpler form

$$\frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = \left[\varepsilon x v_x v^{p-1}\right]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2\,dx$$

(6.6)

$$-\int_0^\infty (g' + \Phi'(p-1))\frac{v^p}{p}\,dx.$$

Now let us define the functions v^+ and v^- as the solutions of (6.1) with initial values $v^+(x, 0) = s^+(x) - \Phi(x)$ and $v^-(x, 0) = s^-(x, K, \bar{x}_1, \bar{\nu}) - \Phi(x)$, respectively. By Theorem 3.4 we know that $v^+ \geqq 0$ and $v^- \leqq 0$. Furthermore, it follows from Lemma 5.1 that v^+ is nonincreasing in time and v^- nondecreasing. Of course both v^+ and v^- satisfy (6.6) and in order to simplify this expression we use the following lemma which we shall prove later.

LEMMA 6.3. *Let* $\varepsilon < g(\infty) - K$. *Then* $\lim_{x\to\infty} x\Phi'(x) = 0$. *If furthermore* $\psi \geqq s^-(\cdot, K, \bar{x}_1, \bar{\nu})$ *for some* $\bar{x}_1$, $\bar{\nu}$ *satisfying* (4.13) *with* $\lambda = K$ (*we suppose furthermore that* $\bar{\nu} > 1$ *if* $\varepsilon < (g(\infty) - K)/2$ *and* $\psi \in C_{1,\alpha}([x_3, \infty))$ *for some* $\alpha, x_3 > 0$, *then* $\lim_{x\to\infty} x u_x(x, t) = 0$ *for all* $t \in (0, \infty)$.

From Lemma 6.3 and formula (6.6) we deduce that v^+ satisfies

$$\frac{d}{dt}\int_0^\infty \frac{(v^+)^p}{p}\,dx = -\varepsilon(p-1)\int_0^\infty x(v^+)^{p-2}(v_x^+)^2\,dx - \int_0^\infty (g' + \Phi'(p-1))\frac{(v^+)^p}{p}\,dx.$$

If $p = 1$, similar calculations yield

$$\frac{d}{dt}\int_0^\infty v^+ dx = -\int_0^\infty g' v^+\,dx.$$

Since $0 < g'(x) < g'(0)$ and $0 < \Phi'(x) < \Phi'(0)$, we have for all $p \geqq 1$

$$\int_0^\infty (v^+(x, t))^p\,dx \geqq \frac{1}{g'(0) + (p-1)\Phi'(0)}\int_0^\infty (g'(x) + (p-1)\Phi'(x))(v^+(x, t))^p\,dx,$$

and thus

$$\int_0^\infty (g'(x) + (p-1)\Phi'(x))(v^+(x, t))^p\,dx$$

$$\leqq (g'(0) + (p-1)\Phi'(0))\int_0^\infty (v^+(x, 0))^p\,dx$$

$$-(g'(0) + (p-1)\Phi'(0))\int_0^t d\tau\int_0^\infty (g'(x) + (p-1)\Phi'(x))(v^+(x, \tau))^p\,dx.$$

In what follows we apply the following lemma that we shall prove later.

LEMMA 6.4. *Let $y \in C([0,\infty))$ with $y' \in L^1((0,\infty))$ and $y' \leqq 0$ such that*

$$0 \leqq y(t) \leqq N - M\int_0^t y(\tau)\,d\tau \tag{6.7}$$

for some constants $N \geqq 0, M > 0$. Then

$$y(t) \leqq \frac{N}{Mt}. \tag{6.8}$$

Since the function $\int_0^\infty (g'(x)+(p-1)\Phi'(x))(v^+(x,t))^p\,dx$ is continuous and nonincreasing (because v^+ is nonincreasing), we deduce from Lemma 6.4 that

$$\int_0^\infty (g'(x)+(p-1)\Phi'(x))(v^+(x,t))^p\,dx \leqq \left(\int_0^\infty (v^+(x,0))^p\,dx\right)\Big/ t.$$

Similarly one can show that

$$\int_0^\infty (g'(x)+(p-1)\Phi'(x))(-v^-(x,t))^p\,dx \leqq \left(\int_0^\infty (-v^-(x,0))^p\,dx\right)\Big/ t.$$

Formula (6.5) is then deduced from the fact that

$$|v(x,t)|^p \leqq \max\left((v^+(x,t))^p, (-v^-(x,t))^p\right) \leqq (v^+(x,t))^p + (-v^-(x,t))^p. \qquad \square$$

Proof of Lemma 6.3. We first show that $\lim_{x\to\infty} x\Phi'(x) = 0$. Since

$$\varepsilon x\Phi'(x) = \varepsilon\Phi(x) - \int_0^x (g(\zeta)-\Phi(\zeta))\Phi'(\zeta)\,d\zeta \leqq \varepsilon K,$$

we have

$$0 \leqq x\Phi'(x) \leqq K.$$

Furthermore

$$(x\Phi')' = x\Phi'' + \Phi' = -\frac{g-\Phi-\varepsilon}{\varepsilon}\Phi' \leqq 0 \quad \text{for } x \text{ large enough.}$$

Since the function $x\Phi'$ is bounded and decreasing for large x, we deduce that there exists $E \in [0, K]$ such that

$$\lim_{x\to\infty} x\Phi'(x) = E,$$

which implies

$$\Phi(x) \sim E \ln x + C \quad \text{as } x \to \infty.$$

Since

$$\lim_{x\to\infty} \Phi(x) = K,$$

we deduce that $E = 0$.

Next we show that $\lim_{x\to\infty} xu_x = 0$ by making use of Bernstein's argument, in a similar way as in Aronson [1] and Peletier and Serrin [21].

Let

$$R_n = \left(\frac{n}{2}, \frac{3n}{2}\right) \times (0, T], \qquad n > 3x_3$$

and let

$$\phi(r)=\frac{Nr(4-r)}{3},$$

where $N=\sup_{\bar{R}_n}u-\inf_{\bar{R}_n}u$. The function ϕ increases from 0 to N as r increases from 0 to 1. Note that $\phi'(r)=2N(2-r)/3>0$ and $\phi''(r)=-2N/3<0$ and define a new function w such that

$$u=\inf_{\bar{R}_n}u+\phi(w).$$

Then w satisfies the differential equation

$$w_t=\varepsilon xw_{xx}+\varepsilon x\frac{\phi''(w)}{\phi'(w)}(w_x)^2+(g-\phi(w)-\inf_{\bar{R}_n}u)w_x.$$

Set $p=w_x$ and differentiate the last equation with respect to x; we get

$$p_t=\varepsilon xp_{xx}+\varepsilon p_x+\varepsilon\frac{\phi''}{\phi'}p^2+2\varepsilon x\frac{\phi''}{\phi'}pp_x+\varepsilon x\left(\frac{\phi''}{\phi'}\right)'p^3$$
$$+(g-\phi-\inf_{\bar{R}_n}u)p_x+(g'-\phi'p)p,$$

and thus

$$\frac{1}{2}(p^2)_t-\varepsilon xpp_{xx}=\varepsilon x\left(\frac{\phi''}{\phi'}\right)'p^4+\varepsilon\left(\frac{\phi''}{\phi'}-\phi'\right)p^3 \tag{6.9}$$
$$+2\varepsilon x\frac{\phi''}{\phi'}p^2p_x+(g-\phi-\inf_{\bar{R}_n}u+\varepsilon)pp_x+g'p^2.$$

Let $R_n^*=(3n/4,5n/4)\times(0,T]$, and let $\zeta=1-4(x-n)^2/n^2$. Set $z=\zeta^2p^2$.

(i) If z attains its maximum value at the lower boundary of R_n we have

$$\sup_{\bar{R}_n^*}z\leqq z(\tilde{x},0)\quad\text{where }\tilde{x}\in\left[\frac{n}{2},\frac{3n}{2}\right].$$

Hence,

$$\sup_{\bar{R}_n^*}\zeta|w_x|\leqq\zeta(\tilde{x})|w_x(\tilde{x},0)|.$$

Since $\zeta\geqq 3/4$ in $(3n/4,5n/4)$ and since $u_x=\phi'(w)w_x$ we find

$$\sup_{\bar{R}_n^*}|u_x|\leqq\frac{4}{3}\frac{\sup\phi'}{\inf\phi'}|\psi'(\tilde{x})|\leqq\frac{8M_\psi}{3}.$$

(ii) If z attains its maximum value at an interior point $(\tilde{x},\tilde{t})$ of R_n we have at that point

$$z_x=2\zeta\zeta'p^2+2\zeta^2pp_x=0, \tag{6.10}$$
$$\varepsilon xz_{xx}-z_t\leqq 0.$$

The last inequality can be cast in the more explicit form

$$\zeta^2(\tfrac{1}{2}(p^2)_t-\varepsilon xpp_{xx})\geqq\varepsilon x(\zeta'^2p^2+\zeta\zeta''p^2+4\zeta\zeta'pp_x+\zeta^2p_x^2).$$

Using (6.9), (6.10) and the inequality

$$|4\zeta\zeta' pp_x| \leqq \zeta^2 p_x^2 + 4\zeta'^2 p^2$$

we obtain

$$-\zeta^2\varepsilon\left(\frac{\phi''}{\phi'}\right)' p^4 \leqq \left(-2\varepsilon\zeta\zeta'\frac{\phi''}{\phi'} + \varepsilon\frac{\zeta^2}{x}\frac{\phi''}{\phi'} - \frac{\zeta^2}{x}\phi'\right)p^3$$

$$+\left(\zeta^2\frac{g'}{x} + 3\varepsilon\zeta'^2 - \varepsilon\zeta\zeta'' - \frac{g-\phi-\inf_{\bar{R}_n} u+\varepsilon}{x}\zeta\zeta'\right)p^2.$$

Since $(\phi''/\phi')' \leqq -\frac{1}{4}$, this implies

$$2\zeta^2 p^4 \leqq \mathscr{C}_1 p^2 + \zeta\mathscr{C}_2|p|^3,$$

where the $\mathscr{C}_i$'s are positive and depend only on N and n. Since

$$\zeta\mathscr{C}_2|p|^3 \leqq \zeta^2 p^4 + \frac{\mathscr{C}_2^2}{4}p^2$$

it follows that

$$z(x,t) \leqq \max_{\bar{R}_n}(z(x,t)) \leqq \mathscr{C}_1 + \frac{\mathscr{C}_2^2}{4} \equiv \mathscr{C}_3.$$

Therefore

$$\max_{\bar{R}_n^*}|w_x| \leqq \frac{4\mathscr{C}_3^{1/2}}{3}.$$

Finally $u_x = \phi'(w)w_x$ and $\phi' \leqq 4N/3$ imply that

$$\max_{\bar{R}_n^*}|u_x| \leqq 16N\mathscr{C}_3^{1/2}/9.$$

Note that $N \leqq \sup_{\bar{R}_n}(K - s^-(x, K, \bar{x}_1, \bar{\nu}))$ (which behaves as x , where $\bar{\nu} > 0$) is furthermore such that $\bar{\nu} > 1$ if $\varepsilon < (g(\infty) - K)/2$.
Thus

$$\max_{\bar{R}_n^*}|u_x| \leqq 16\mathscr{C}_3^{1/2}\sup_{\bar{R}_n}\frac{(K - s^-(x, K, \bar{x}_1, \bar{\nu}))}{9}. \tag{6.11}$$

If $\varepsilon < (g(\infty) - K)/2$ $\mathscr{C}_3$ is bounded uniformly in n, and we deduce that xu_x tends to zero as $x \to \infty$. If on the other hand $(g(\infty) - K)/2 \leqq \varepsilon < g(\infty) - K$, then we only have that $\bar{\nu} > 0$ in (6.11) and $\sup_{\bar{R}_n}(K - s^-(x, K, \bar{x}_1, \bar{\nu}))$ tends to zero as $x \to \infty$. However $\mathscr{C}_3^{1/2}$ tends to zero as $1/x$ when $x \to \infty$, which also yields the result. □

Proof of Lemma 6.4. Integrating by parts we get

$$\int_0^t y(\tau)\,d\tau = ty(t) - \int_0^t \tau y'(\tau)\,d\tau \geqq ty(t).$$

Also we deduce from (6.7) that

$$\int_0^t y(\tau)\,d\tau \leqq \frac{N}{M},$$

and thus (6.8) follows. □

Next we deduce from Theorem 6.2 that there is also pointwise convergence. More precisely we prove the following theorem.

THEOREM 6.5. *Provided that* $\varepsilon < g(\infty) - K$ *and that* $\psi \geqq s^-(\cdot, K, \bar{x}_1, \bar{\nu})$ *for some* $\bar{x}_1$, $\bar{\nu}$ *satisfying* (4.13) *with* $\lambda = K$, *we have that*

$$\|(g'(\cdot) + (p-1)\Phi'(\cdot))^{1/p}(u(\cdot, t, \psi) - \Phi)\| \leqq \frac{C_\varepsilon}{t^{1/2p}} \quad \textit{for all } t > 0, \tag{6.12}$$

and $p = [1/\bar{\nu}] + 1$, *where*

$$C_\varepsilon = \Big[2\Big(\Big(K^{p-1}p^2 + K^p\frac{p-1}{\varepsilon}\Big)(g'(0))^2 + K^p \sup_{x\in[0,\infty)} |g''(x)|\Big) \cdot \int_0^\infty ((s^+ - \Phi)^p + (\Phi - s^-)^p)\, dx\Big]^{1/2p}. \tag{6.13}$$

In particular, if $\varepsilon < (g(\infty) - K)/2$ and $\bar{\nu} > 1$, then $p = 1$ and formulas (6.12) and (6.13) simplify as follows

$$\|g'(\cdot)(u(\cdot, t, \psi) - \Phi)\| \leqq \frac{C}{\sqrt{t}} \quad \text{for all } t > 0, \tag{6.14}$$

where

$$C = \Big[2\Big((g'(0))^2 + K \sup_{x\in[0,\infty)} |g''(x)|\Big)\int_0^\infty (s^+(x) - s^-(x, K, \bar{x}_1, \bar{\nu}))\, dx\Big]^{1/2}.$$

Proof. To prove Theorem 6.5 we need the following auxiliary lemma.

LEMMA 6.6. *Let* ϕ *be defined for* $0 \leqq x < \infty$ *and satisfy the conditions*

(i) $\phi(x) \geqq 0$ *and* $\phi(0) = 0$;
(ii) ϕ *is Lipschitz continuous with constant* l;
(iii) $\int_0^\infty \phi(x)\, dx \leqq \mathcal{N}$.

Then

$$\sup_{0\leqq x<\infty} |\phi(x)| \leqq \sqrt{2\mathcal{N}l}.$$

We omit here the demonstration of this lemma since the main ideas of the proof are given in the proof of Peletier [20, Lemma 3].

Now let us apply Lemma 6.6 to the function $(g' + (p-1)\Phi')|u - \Phi|^p$; it is nonnegative, equal to zero at the origin and its derivative is continuous by parts and bounded by

$$\Big\{\Big(K^{p-1}p^2 + K^p\frac{p-1}{\varepsilon}\Big)(g'(0))^2 + K^p \sup_{x\in[0,\infty)} |g''(x)|\Big\}$$

at all points where it is defined. Finally the bound on its integral is given in Theorem 6.2. Inequality (6.12) follows. □

6.3. Asymptotic behavior of the solution $\bar{u}$ of the hyperbolic problem H as $t \to \infty$.

THEOREM 6.7. *Let* ψ *satisfy* H_ψ *and be such that* $\psi \geqq s^-(\cdot, K, \bar{x}_1, \bar{\nu})$ *for some* $\bar{x}_1 > 0$, $\bar{\nu} > 1$ *satisfying* (4.13) *with* $\lambda = K$ *and define* $\bar{\Phi}(x) = \min(g(x), K)$. *Then*

$$\|g'(\cdot)(\bar{u}(\cdot, t, \psi) - \bar{\Phi})\| \leqq \frac{C}{\sqrt{t}} \quad \textit{for all } t > 0,$$

where C is the constant defined in Theorem 6.5.

Proof. Let $\varepsilon \in (0, (g(\infty)-K)/2)\downarrow 0$ in inequality (6.14), note that the constant C does not depend in ε, and use the fact that Φ converges to $\tilde{\Phi}$ uniformly on $[0, \infty)$ as $\varepsilon \downarrow 0$ (see [6]). □

Appendix. In what follows we shall prove the following theorem:

THEOREM A1. *Suppose that there exist constants x_2, $C_0 > 0$ such that the condition* (6.3) *is satisfied. There exist $\theta \in (0, 1)$, $\mathscr{R} > 0$ and $\lambda < 0$ such that the Cauchy-Dirichlet problem* (6.4) *has at least one solution $w \in C^2([0, \infty))$ with w, w', w'' bounded and*

$$0 < w(x) \leqq \min(\tilde{\Phi}(x), (x/\mathscr{R})^{-\nu_0}\tilde{\Phi}(\mathscr{R})) \quad \textit{for all } x \in (0, \infty).$$

Proof. Let $n \geqq 1$, and consider the boundary value problem

$$\text{(A1)}\quad \varepsilon\left(x+\frac{1}{n}\right)w''-(q_n(x)+\lambda)w = -\theta(\Phi'(\mathscr{R})+\lambda)\min(\tilde{\Phi}_n(x), (x/\mathscr{R})^{-\nu_0}\tilde{\Phi}_n(\mathscr{R})),$$

$$\text{(A2)}\quad w(0)=0,$$

where

$$\tilde{\Phi}_n(x) = \exp\left(\int_0^x \frac{g(\zeta)-\Phi(\zeta)}{2\varepsilon(\zeta+1/n)}\,d\zeta\right)\Phi(x),$$

and

$$q_n(x) = \frac{(g(x)-\Phi(x))^2}{4\varepsilon(x+1/n)} + \frac{g'(x)+\Phi'(x)}{2} - \frac{g(x)-\Phi(x)}{2(x+1/n)},$$

$\nu_0 > 1$ is arbitrary and where the constants $\theta \in (0, 1)$, $\mathscr{R} > 0$ and $\lambda \in (-\Phi'(\mathscr{R}), 0)$ satisfy some additional conditions which will be given later. Obviously zero is a lower solution for the differential equation in (A1). We shall now construct an upper solution. Firstly we deduce from the asymptotic behavior of g that there exists $\mathscr{R}_1 \geqq 1$ and $q_0 > 0$ such that $q_n(x) \geqq 2q_0$ for $x \geqq \mathscr{R}_1$. Also if $\lambda > \max(-q_0, -\Phi'(\mathscr{R}))$ and $\theta < (q_0+\lambda)/(\Phi'(\mathscr{R})+\lambda)$, then the function $(x/\mathscr{R})^{-\nu_0}\tilde{\Phi}_n(\mathscr{R})$ is an upper solution of the differential equation (A1) for $x \geqq \mathscr{R} := \max(\mathscr{R}_1, 2\varepsilon\nu_0(\nu_0+1)/q_0)$. Next we note that $\tilde{\Phi}_n$ is an upper solution of (A1) on $[0, \mathscr{R}]$ and thus that $\min(\tilde{\Phi}_n(x), (x/\mathscr{R})^{-\nu_0}\tilde{\Phi}_n(\mathscr{R}))$ is an upper solution of (A1) on $[0, \infty)$. Finally we conclude that there exists at least one solution $w_n \in C^2([0, \infty))$ of (A1), (A2) [3, Thm. 1.7.1], such that

$$0 \leqq w_n(x) \leqq \min\left(\tilde{\Phi}_n(x), \left(\frac{x}{\mathscr{R}}\right)^{-\nu_0}\tilde{\Phi}_n(\mathscr{R})\right),$$

which, since $\tilde{\Phi}_n \leqq \tilde{\Phi}$, implies that

$$\text{(A3)}\qquad 0 \leqq w_n(x) \leqq \min\left(\tilde{\Phi}(x), \left(\frac{x}{\mathscr{R}}\right)^{-\nu_0}\tilde{\Phi}(\mathscr{R})\right).$$

Furthermore, the inequalities (A3) and

$$\text{(A4)}\qquad |q_n(x)| \leqq \frac{(g-\Phi)^2}{4\varepsilon x} + \frac{g'+\Phi'}{2}$$

yield, together with (A1),

$$|w_n''(x)| \leqq C \quad \text{for all } x \in [0, \infty),$$

where $C>0$ is independent of n. Now let us integrate (A1); we get

$$w_n'(x)=w_n'(0)$$

$$+\int_0^x \frac{(q_n(\zeta)+\lambda)w_n(\zeta)-\theta(\Phi'(\mathcal{R})+\lambda)\min(\tilde{\Phi}_n(\zeta),(\zeta/\mathcal{R})^{-\nu_0}\tilde{\Phi}_n(\mathcal{R}))\,d\zeta}{\varepsilon(\zeta+1/n)}, \tag{A5}$$

and again using (A3) and (A4) we obtain

$$|w_n'(x)|\leqq C \quad \text{for all } x\in[0,\infty].$$

Using the Arzela–Ascoli theorem and a diagonal process, we deduce that there exist a function $w\in C^1([0,\infty))$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k}\to w$ as $n_k\to\infty$ uniformly in $C^1([0,\infty))$ on all compact subsets of $[0,\infty)$. Also setting $n=n_k$ in (A5) and letting $n_k\to\infty$, we deduce that w satisfies the differential equation

$$\varepsilon x w''-(q(x)+\lambda)w=-\theta(\Phi'(\mathcal{R})+\lambda)\min(\tilde{\Phi}(x),(x/\mathcal{R})^{-\nu_0}\tilde{\Phi}(\mathcal{R})) \tag{A6}$$

and the boundary condition

$$w(0)=0.$$

It follows from (A6) that $w\in C^2((0,\infty))$, and since

$$\lim_{x\to\infty} w''(x)=[(\Phi'(0)+\lambda)w'(0)-\theta(\Phi'(\mathcal{R})+\lambda)\tilde{\Phi}'(0)]/\varepsilon,$$

we deduce that in fact $w\in C^2([0,\infty))$. Finally the strict inequality $w>0$ is proven by means of a maximum principle argument. ☐

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CHAPITRE IV

HOW MANY JUMPS ?

VARIATIONAL CHARACTERIZATION OF THE LIMIT SOLUTION OF A SINGULAR PERTURBATION PROBLEM.

par

O. Diekmann et D. Hilhorst.

1. INTRODUCTION

Consider the two-point boundary value problem

$$\varepsilon y'' + (g-y)y' = 0,$$

BVP

$$y(0) = 0, \quad y(1) = 1,$$

where $g \in L_2 = L_2(0,1)$ is a given function and $y \in H^2$ is unknown. As we shall show, there exists for each $\varepsilon > 0$ a unique solution y_ε, which is increasing. We are interested in the limiting behaviour of y_ε as $\varepsilon \downarrow 0$.

Motivated by a physical application we previously studied a similar problem in a joint paper with L.A. PELETIER [2]. Using the maximum principle as our main tool we were able to establish the existence of a unique limit solution y_0 under certain, physically reasonable, assumptions on the function g. In some cases we could characterize y_0 completely, in others, however, some ambiguity remained.

Here, inspired by the work of GRASMAN & MATKOWSKY [4], we shall resolve this ambiguity by using a variational formulation of the problem. The method we use is based on the theory of maximal monotone operators. It has been suggested to us by Ph. Clément.

During our investigation of BVP we experienced that it could serve as a fairly simple, yet nontrivial, illustration of concepts and methods from abstract functional analysis. In order to demonstrate this aspect of the problem we shall spell out our arguments in some more detail than is strictly necessary.

The organization of the paper is as follows. In Section 2 we prove, by means of Schauder's fixed point theorem, that BVP has a solution y_ε for each $\varepsilon > 0$. Moreover, we show that BVP is equivalent to an abstract equation AE, involving a maximal monotone operator A, and to a variational problem VP, involving a convex, lower semi-continuous functional W.

In Section 3 we exploit these formulations in the investigation of the limiting behaviour of y_ε as $\varepsilon \downarrow 0$. It turns out that y_ε converges in L_2 to a limit y_0. Moreover, y_0 is abstractly characterized as the projection (in L_2) of g on $\overline{\mathcal{D}(A)}$. We conclude this section with some results about uniform convergence under restrictive assumptions.

In Section 4 we give concrete form to the characterization of y_0. In particular we present sufficient conditions for a function to be y_0 and we show, by means of examples, how these criteria can be used in concrete cases. The first part of the title originated from Example 4.

In Section 5 we make various remarks about generalizations and limitations of our approach.

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2. THREE EQUIVALENT FORMULATIONS

In order to demonstrate the existence of a solution of BVP, let us first look at the auxiliary problem

$$u'' + (g-w)u' = 0,$$

$$u(0) = 0, \quad u(1) = 1,$$

where $w \in L_2$ is a given function. The solution of this linear problem is given explicitly by

$$u(x) = C(w) \int_0^x \exp(\int_0^\zeta (w(\xi) - g(\xi))d\xi)d\zeta$$

with

$$C(w) = (\int_0^1 \exp(\int_0^\zeta (w(\xi) - g(\xi))d\xi)d\zeta)^{-1}.$$

From this expression it can be concluded that $u' > 0$ and $0 \le u \le 1$. So if we write $u = Tw$, then T is a compact map of the closed convex set

$\{w \in L_2 \mid 0 \le w \le 1\}$ into itself and hence, by Schauder's theorem, T must have a fixed point. Clearly this fixed point corresponds to a solution of BVP. Thus we have proved

PROPOSITION 2.1. *For each* $\varepsilon > 0$ *there exists a solution* $y_\varepsilon \in H^2$ *of* BVP. *Moreover, any solution* $y \in H^2$ *satisfies* (i) $y' > 0$ *and* (ii) $0 \le y \le 1$.

The a priori knowledge that y' is positive allows us to divide the equation by y'. In this manner we are able to reformulate the boundary value problem as an equivalent abstract equation

AE $\qquad (I + \varepsilon A)y = g$

where the (unbounded, nonlinear) operator $A: \mathcal{D}(A) \to L_2$ is defined by

(2.1) $$Au = -\frac{u''}{u'} = -(\ell n\ u')'$$

with

(2.2) $$\mathcal{D}(A) = \{u \in L_2 \mid u \in H^2,\ u' > 0,\ u(0) = 0,\ u(1) = 1\}.$$

PROPOSITION 2.2. *The operator* A *is monotone. Hence the solution of* AE *(and* BVP*) is unique.*

PROOF. Let $u_i \in \mathcal{D}(A)$ for $i = 1,2$ then

$$(Au_1 - Au_2, u_1 - u_2) = -\int((\ell n\ u_1')' - (\ell n\ u_2')')(u_1 - u_2)$$
$$= \int(\ell n\ u_1' - \ell n\ u_2')(u_1' - u_2') \ge 0$$

(because $z \mapsto \ell n\ z$ is monotone on $(0,\infty)$; note that here and in the following we write $\int\phi$ to denote $\int_0^1 \phi(x)dx$.) Next, suppose $\varepsilon Ay_i = g - y_i$, $i = 1,2$, then $0 \le \varepsilon(Ay_1 - Ay_2, y_1 - y_2) = (g - y_1 - g + y_2, y_1 - y_2) = -\|y_1 - y_2\|^2$ and hence $y_1 = y_2$. □

We recall that a monotone operator A defined on a Hilbert space H is is called *maximal monotone* if it admits no proper monotone extension (i.e., it is maximal in the sense of inclusion of graphs). It is well known that A is maximal monotone if and only if $\mathcal{R}(I + \varepsilon A) = H$ for each $\varepsilon > 0$ (see BRÉZIS [1]). In our case, with $H = L_2$ and A defined in (2.1), this is just a reformulation of the existence result Proposition 2.1. Consequently we know

PROPOSITION 2.3. A *is maximal monotone.*

In search for yet another formulation let us write the equation in the form

$$-\varepsilon(\ell n\ y')' + y - g = 0$$

Hence, for any $\phi \in H_0^1$,

$$\varepsilon \int \phi'(\ell n\ y' + 1) + \int \phi(y - g) = 0.$$

Motivated by this calculation we define a functional $W: L_2 \to \overline{\mathbb{R}}$ by

$$W(u) = \varepsilon\Psi(u) + \frac{1}{2}\|u - g\|^2 \tag{2.3}$$

where

$$\Psi(u) = \begin{cases} \int u'\ \ell n\ u' & \text{if } u \in \mathcal{D}(\Psi), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\mathcal{D}(\Psi) = \{u \in L_2 \mid u \text{ is AC}, u' \geq 0, u'\ \ell n\ u' \in L_1, u(0) = 0, \\ u(1) = 1\} \tag{2.5}$$

(here AC means absolutely continuous.) Also we define a variational problem

VP $\quad \operatorname{Inf}_{L_2} W.$

We note that the mappings $z \mapsto z\ \ell n\ z$ and $z \mapsto z^2$ are (strictly) convex (on

$[0,\infty)$ and $(-\infty,\infty)$ respectively) and that W inherits this property because $\mathcal{D}(\Psi)$ is convex as well. Hence VP has at most one solution. For further use we observe that the convexity of $z \mapsto z\,\ell n\, z$ implies, for $z \geq 0$ and $\zeta > 0$, the inequality

$$z\,\ell n\, z - \zeta\,\ell n\,\zeta \geq (1 + \ell n\,\zeta)(z - \zeta).$$

PROPOSITION 2.4. y_ε *solves* VP.

PROOF. Firstly we note that $y_\varepsilon \in \mathcal{D}(\Psi)$. So for any $u \in \mathcal{D}(\Psi)$

$$\begin{aligned} W(u) - W(y_\varepsilon) &= \varepsilon \int (u'\,\ell n\, u' - y_\varepsilon'\,\ell n\, y_\varepsilon') + \frac{1}{2}\|u - g\|^2 - \frac{1}{2}\|y_\varepsilon - g\|^2 \\ &\geq \varepsilon \int (1 + \ell n\, y_\varepsilon')(u' - y_\varepsilon') + \int (y_\varepsilon - g)(u - y_\varepsilon) \\ &= \int (-\varepsilon \frac{y_\varepsilon''}{y_\varepsilon'} + y_\varepsilon - g)(u - y_\varepsilon) = 0. \qquad \square \end{aligned}$$

We recall that the *subgradient* $\partial\Psi$ of the convex functional Ψ is defined by

$$\partial\Psi(u) = \{\zeta \in L_2 \mid \Psi(v) - \Psi(u) \geq (\zeta, v - u),\ \forall v \in \mathcal{D}(\Psi)\}.$$

A calculation like the one above shows that, for $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(\Psi)$,

$$\Psi(v) - \Psi(u) \geq (Au, v - u).$$

Hence $A \subset \partial\Psi$, but, since $\partial\Psi$ is monotone and A is *maximal* monotone, we must have $A = \partial\Psi$. Likewise it follows that $\partial W = \varepsilon A + I - g$. These observations should clarify the relation between VP and AE.

One can show that Ψ (and hence W as well) is lower semicontinuous and subsequently one can use this knowledge to give a direct variational proof of the existence of a solution of VP.

We summarize the main results of this section in the following theorem.

THEOREM 2.5. *The problems* BVP, AE *and* VP *are equivalent. In fact, for each*

$\varepsilon > 0$, *there exists* $y_\varepsilon \in \mathcal{D}(A)$ *which solves each problem and no problem admits any other solution.*

3. LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

The fact that y_ε solves AE can be expressed as

$$y_\varepsilon = (I + \varepsilon A)^{-1} g.$$

Subsequently, the observation that A is maximal monotone provides a key to describing the limiting behaviour. For, it is known from the general theory of such operators (see BRÉZIS [1, Section II.4, in particular Th. 2.2]) that

$$\lim_{\varepsilon \downarrow 0} (I + \varepsilon A)^{-1} g = \mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g,$$

where the expression at the right-hand side denotes the projection (in the sense of the underlying Hilbert space, hence L_2 in this case) of g on the closed convex set $\overline{\mathcal{D}(A)}$, or, in other words,

$$\mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g = y_0$$

where y_0 denotes the unique solution of the variational problem

$$\mathrm{Min}_{\overline{\mathcal{D}(A)}}\, W_0$$

with

$$W_0(u) = \| u - g \|^2.$$

Below we shall give a proof of this result for this special case, using techniques as in Brézis' book, but exploiting the fact that A is the subdifferential of the functional Ψ.

<u>THEOREM 3.1.</u>

$$\lim_{\varepsilon \downarrow 0} \| y_\varepsilon - y_0 \| = 0.$$

PROOF. First of all we note that $\|y_\varepsilon\| \le 1$. We shall split the proof into three steps.

Step 1. Take any $z \in \mathcal{D}(A)$ then from

$$\Psi(y_\varepsilon) - \Psi(z) \ge (Az,\ y_\varepsilon - z)$$

it follows that

$$\lim_{\varepsilon \downarrow 0} \inf \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) \ge 0.$$

Step 2. By definition,

$$0 \ge W(y_\varepsilon) - W(z) = \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) + \frac{1}{2}\|g - y_\varepsilon\|^2 - \frac{1}{2}\|g - z\|^2.$$

Hence

$$\lim_{\varepsilon \downarrow 0} \sup \|g - y_\varepsilon\|^2 \le \|g - z\|^2, \qquad \forall z \in \mathcal{D}(A).$$

But then, in fact, the same must hold for all $z \in \overline{\mathcal{D}(A)}$.

Step 3. Since $\|y_\varepsilon\| \le 1$, $\{y_\varepsilon\}$ is weakly precompact in L_2. Take any $\{\varepsilon_n\}$ and $\tilde{y}$ such that $y_{\varepsilon_n} \rightharpoonup \tilde{y}$ in L_2, then

$$(*) \qquad \|g-\tilde{y}\|^2 \le \lim_{n \to \infty} \inf \|g-y_{\varepsilon_n}\|^2 \le \lim_{n \to \infty} \sup \|g-y_{\varepsilon_n}\|^2 \le \|g-z\|^2,$$

$$\forall z \in \overline{\mathcal{D}(A)}.$$

Consequently $\tilde{y} = y_0$, which shows that the limit does not depend on the subsequence under consideration. Hence $y_\varepsilon \rightharpoonup y_0$. Finally, by taking $z = y_0$ in (*) it follows that in fact $y_\varepsilon \to y_0$. □

We note that

$$\overline{\mathcal{D}(A)} = \{u \in L_2 \mid u \text{ is nondecreasing, } 0 \le u \le 1\}.$$

So in general y_0 need not be continuous (nor does it need to satisfy the boundary conditions). However it is possible, as our next result shows, to establish uniform convergence to a continuous limit at the price of some conditions on g.

THEOREM 3.2. *Suppose* $g \in C^1$, $g(0) < 0$ *and* $g(1) > 1$. *Then* $y_0 \in C$ *and*

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le 1} |y_\varepsilon(x) - y_0(x)| = 0.$$

PROOF. The idea is to derive a uniform bound for y'_ε. We know already that $y'_\varepsilon > 0$ and we are going to show that $y'_\varepsilon \le \sup g'$. To this end we first observe that $g(0) - y_\varepsilon(0) < 0$ and $g(1) - y_\varepsilon(1) > 0$, which, combined with the differential equation, shows that $y''_\varepsilon(0) > 0$ and $y''_\varepsilon(1) < 0$. Hence y'_ε assumes its maximum in an interior point, say $\bar{x}$. Next, differentiation of the differential equation followed by substitution of $y''_\varepsilon(\bar{x}) = 0$, $y'''_\varepsilon(\bar{x}) \le 0$, leads to the conclusion that $y'_\varepsilon(\bar{x}) \le g'(\bar{x})$. The uniform bound for y'_ε implies, by virtue of the Arzela-Ascoli theorem, that the limit set of $\{y_\varepsilon\}$ in the space of continuous functions is nonempty. Combination of this result with Theorem 3.1 leads to the desired conclusion. □

In Section 4 we shall show that y_0 can be calculated in many concrete examples. Quite often it will turn out that y_0 is continuous (or piecewise continuous). This motivates our next result.

THEOREM 3.3. *Suppose* y_0 *is continuous. Then* y_ε *converges to* y_0 *uniformly on compact subsets of* (0,1).

PROOF. Let $I \subset (0,1)$ be a compact set. Put $\beta(\varepsilon) = \max\{y_\varepsilon(x) - y_0(x) \mid x \in I\}$ and let $\bar{x}(\varepsilon) \in I$ be such that $y_\varepsilon(\bar{x}(\varepsilon)) - y_0(\bar{x}(\varepsilon)) = \beta(\varepsilon)$. Suppose $\limsup_{\varepsilon \downarrow 0} \beta(\varepsilon) = \beta > 0$ and let $\{\varepsilon_n\}$ be such that $\beta(\varepsilon_n) \to \beta$ as $n \to \infty$. Choose $\delta \in (0,\delta_1)$, where δ_1 denotes the distance of 1 to I, such that $|y_0(x) - y_0(\xi)| \le \frac{1}{4}\beta$ if $|x - \xi| \le \delta$. Also, choose n_0 such that $\beta(\varepsilon_n) \ge \frac{3}{4}\beta$ for $n \ge n_0$. Then for $x \in [\bar{x}(\varepsilon_n), \bar{x}(\varepsilon_n) + \delta]$ and $n \ge n_0$ the following inequality holds:

$$y_{\varepsilon_n}(x) - y_0(x) \geq y_{\varepsilon_n}(\bar{x}(\varepsilon_n)) - y_0(\bar{x}(\varepsilon_n)) + y_0(\bar{x}(\varepsilon_n)) - y_0(x)$$
$$\geq \frac{3}{4}\beta - \frac{1}{4}\beta = \frac{1}{2}\beta.$$

However, this leads to

$$\|y_{\varepsilon_n} - y_0\|^2 \geq \frac{1}{4}\delta\beta^2$$

which is in contradiction with Theorem 3.1. Hence our assumption $\beta > 0$ must be false and we arrive at the conclusion that $\limsup_{\varepsilon\downarrow 0} \max\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \leq 0$. Essentially the same argument yields that $\liminf_{\varepsilon\downarrow 0} \min\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \geq 0$. Taking both statements together yields the result. □

It should be clear that appropriate analogous results can be proved if y_0 is piece-wise continuous. In Theorem 3.3 the sense of convergence is sharpened "a posteriori", that is, once the continuity of y_0 is established by other means. Note that our proof exploits the uniform one-sided bound $y'_\varepsilon > 0$.

4. CALCULATION OF y_0

We recall that y_0 is the unique solution of the variational problem $\min_{\overline{\mathcal{D}(A)}} W_0$, where $W_0(u) = \|u-g\|^2$. It is well known (for instance, see EKELAND & TÉMAM [3, II, 2.1]) that one can equivalently characterize y_0 as the unique solution of the *variational inequality:*

(4.1) find $y \in \overline{\mathcal{D}(A)}$ such that $(y-g, v-y) \geq 0$, $\forall v \in \overline{\mathcal{D}(A)}$.

Already from the reduced differential equation $(g-y)y' = 0$, it can be guessed that y_0 is possibly composed out of pieces where it equals g and pieces where it equals a constant. Of course, if $y_0 = g$ in some open interval, g has to be nondecreasing in that interval. The characterization of y_0 by (4.1) can be used to find conditions on the "allowed" constants.

<u>THEOREM 4.1</u>. *Suppose* $y \in \overline{\mathcal{D}(A)}$ *has the following property: there exists a partition* $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ *of* $[0,1]$ *and a subset* L *of* $\{0,1,\dots,n-1\}$ *such that:*

(i) *if* $i \notin L$ *then* $y(x) = g(x)$ *for* $x \in [x_i,x_{i+1}]$,

(ii) *if* $i \in L$ *then* $y(x) = C_i$ *for* $x \in [x_i,x_{i+1}]$ *and*

$$\int_x^{x_{i+1}} (C_i - g(\xi))d\xi \ge 0, \qquad \forall x \in [x_i,x_{i+1}], \quad \text{if } C_i \in [0,1),$$

$$\int_{x_i}^{x} (C_i - g(\xi))d\xi \le 0, \qquad \forall x \in [x_i,x_{i+1}], \quad \text{if } C_i \in (0,1],$$

(*so in particular, if* $C_i \in (0,1)$, $\int_{x_i}^{x_{i+1}}(C_i - g(\xi))d\xi = 0$).

Then $y = y_0$.

<u>PROOF</u>. According to (4.1) it is sufficient to check that

$$I(v) = \int (y - g)(v - y) \ge 0, \qquad \forall v \in \overline{\mathcal{D}(A)}.$$

In fact it is sufficient to check this for all $v \in \overline{\mathcal{D}(A)} \cap H^1$ (since this set is dense in $\overline{\mathcal{D}(A)}$ and I is continuous). We note that $I(v) = \Sigma_{i\in L}\, I_i(v)$, where

$$I_i(v) = \int_{x_i}^{x_{i+1}} (C_i - g(\xi))(v(\xi) - C_i)d\xi.$$

If $C_i = 0$ then

$$I_i(v) = -v(x_i) \int_{x_i}^{x_{i+1}} g(\xi)d\xi - \int_{x_i}^{x_{i+1}} v'(\xi) \int_{\xi}^{x_{i+1}} g(x)dxd\xi \ge 0.$$

If $C_i \in (0,1)$ then

$$I_i(v) = \int_{x_i}^{x_{i+1}} v'(\xi) \int_{\xi}^{x_{i+1}} (C_i - g(x))dxd\xi \ge 0.$$

If $C_i = 1$ then

$$I_i(v) = (v(x_{i+1}) - 1)\int_{x_i}^{x_{i+1}} (C_i - g(\xi))d\xi - \int_{x_i}^{x_{i+1}} v'(\xi)\int_{x_i}^{\xi} (C_i - g(x))dxd\xi \geq 0 .$$

Hence indeed $I(v) \geq 0$, $\forall v \in \overline{\mathcal{D}(A)} \cap H^1$. □

The sufficient conditions of the theorem can be used as a kind of algorithm to compute y_0 in concrete cases. We shall illustrate this idea by means of a number of examples (some of which are almost literally taken from [2]).

EXAMPLE 1. Suppose g is nondecreasing, then

$$y_0(x) = \begin{cases} 0 & \text{if } g(x) \leq 0, \\ g(x) & \text{if } 0 \leq g(x) \leq 1, \\ 1 & \text{if } g(x) \geq 1. \end{cases}$$

EXAMPLE 2. Suppose g is nonincreasing, then $y_0(x) = C$ with

$$C = \begin{cases} 0 & \text{if } \int g \leq 0, \\ \int g & \text{if } 0 \leq \int g \leq 1, \\ 1 & \text{if } \int g \geq 1. \end{cases}$$

EXAMPLE 3. Suppose that $g \in C^1$ is such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0 < b < c < 1$ and $0 < g(c) < g(b) < 1$. Let g_1^{-1} denote the inverse of g on $[0,b]$ and g_2^{-1} the inverse of g on $[c,1]$. Define two points a and d by

$$a = g_1^{-1}(g(c)), \qquad d = g_2^{-1}(g(b)).$$

Then $g([a,b]) = g([c,d])$. (See Figure 1).

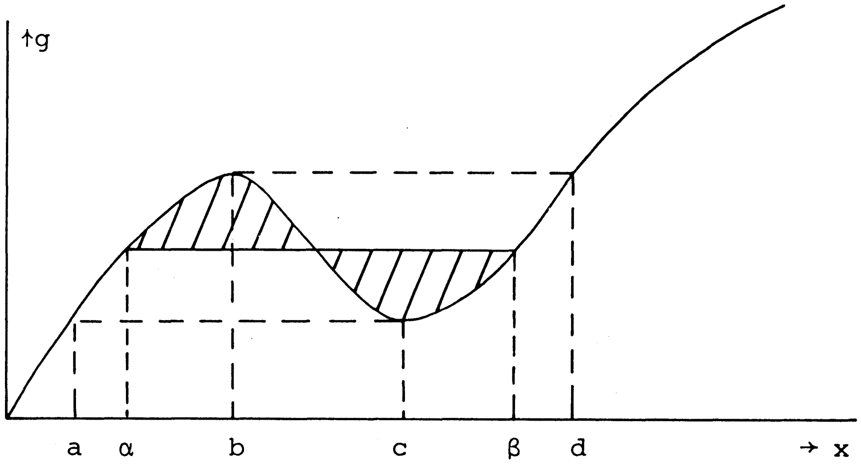

Figure 1

On [a,b] we define a mapping G by

$$G(x) = \int_{x}^{g_2^{-1}(g(x))} (g(x) - g(\xi))d\xi.$$

Then G(a) < 0, G(b) > 0 and on (a,b)

$$G'(x) = g'(x) \int_{x}^{g_2^{-1}(g(x))} d\xi > 0.$$

Consequently G has a unique zero on [a,b], say for x = α. The function y_0 has the tendency to follow g as much as possible. However, it also has to be nondecreasing. So the inverse function of y_0 must "jump" from a point on [a,b] to a point on [c,d]. In view of Theorem 4.1 this jump can only take place between α and $\beta = g_2^{-1}(\alpha)$. We leave it to the reader to verify (by checking all requirements of Theorem 4.1) that

$$y_0(x) = \begin{cases} 0 & \text{if } x \leq \alpha \text{ and } g(x) \leq 0, \\ g(x) & \text{if } x \leq \alpha \text{ and } g(x) \geq 0, \\ g(\alpha) & \text{if } \alpha \leq x \leq \beta, \\ g(x) & \text{if } x \geq \beta \text{ and } g(x) \leq 1, \\ 1 & \text{if } x \geq \beta \text{ and } g(x) \geq 1. \end{cases}$$

It should be clear that the differentiability of g is not strictly necessary for our arguments to apply. In fact the monotonicity of G follows from straightforward geometrical considerations and the condition $G(\alpha) = 0$ has a corresponding interpretation (see Figure 1).

EXAMPLE 4. If g has more maxima and minima the construction of candidates for y_0 can be based on essentially the same idea as outlined in Example 3. However, it becomes more complicated since the number of possibilities becomes larger (see [2] for some more details). For instance, if g has a graph as shown in Figure 2, looking at zeroes of functions like G above leaves us with two possible candidates: one with two "jumps" (a-b,c-d) and one with a "two-in-one jump" (α-β).

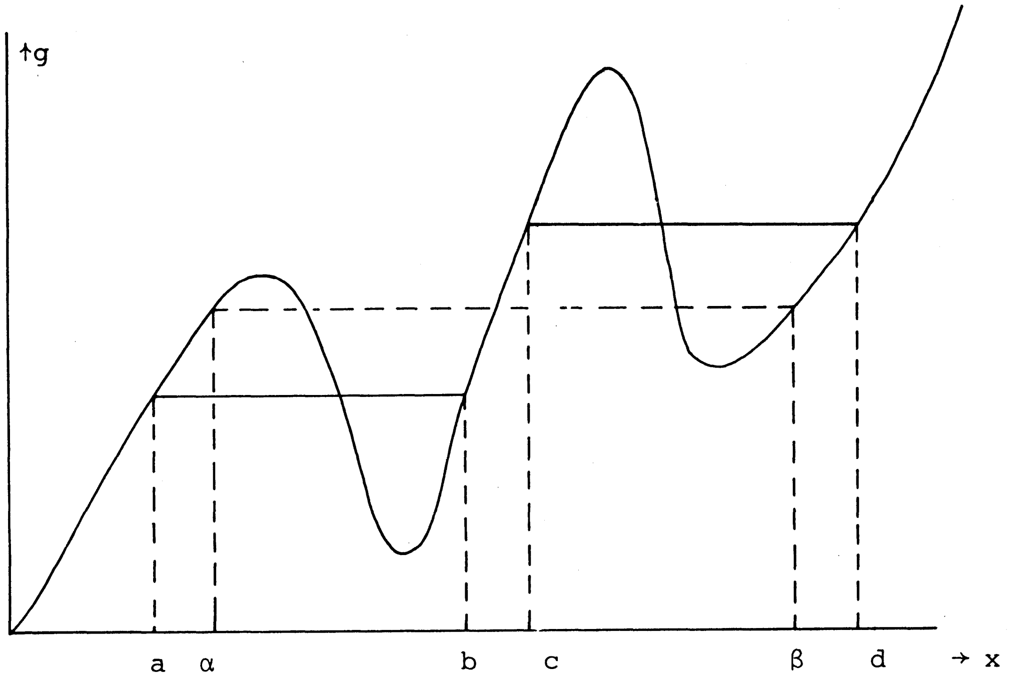

Figure 2

In [2] we were unable to decide in such a situation which was the actual limit. But now it can be read off from the picture that only the one with two "jumps" satisfies the requirements of Theorem 4.1, and hence this one must actually be y_0. (The other one corresponds to a saddle point of the functional W_0 restricted to $\overline{\mathcal{D}(A)}$.) It is in this sense that y_0 must have as many "jumps" as possible.

5. CONCLUDING REMARKS

(i) In all our examples y_0 satisfies the reduced equation $(g-y)y' = 0$. However this equation is by no means sufficient to characterize y_0 completely. Our analysis clearly shows that the reduced variational problem $\mathrm{Min}_{\overline{\mathcal{D}(A)}} W_0$ contains much more information than the reduced differential equation.

(ii) In [2] we were actually interested in a boundary value problem of the type

$$(5.1) \qquad \varepsilon x y'' + (g-y)y' = 0, \qquad 0 < x < 1,$$

$$(5.2) \qquad y(0) = 0, \quad y(1) = 1,$$

which arises from the assumption of radial symmetry in a two-dimensional geometry. This problem can be analysed in completely the same way as we did with BVP in this paper, by choosing as the underlying Hilbert space the weighted L_2-space corresponding to the measure $d\mu(x) = x^{-1}dx$. For instance, the operator $\tilde{A}$ defined by

$$(\tilde{A}u)(x) = -x\,\frac{u''(x)}{u'(x)}$$

with

$$\mathcal{D}(\tilde{A}) = \{u \in L^2(d\mu) \mid u' \in C(0,1],\ u' > 0,\ u(1) = 1,$$
$$i\,\frac{u''}{u'} \in L^2(d\mu)\}\ ,$$

where i denotes the function $i(x) = x$, is clearly monotone in this space. The surjectivity of $I + \varepsilon\tilde{A}$ can be proved with the aid of an auxiliary problem and Schauder's fixed point theorem. (Note that some care is needed in checking that the functions which occur belong to the right space and that the solution operator is compact. This turns out to be all right. We refer to Martini's thesis [5] where related problems are treated in full detail.) Hence $\tilde{A}$ is maximal monotone. Subsequently it follows that, for given $g \in L_2(d\mu)$, the solution y_ε tends, as $\varepsilon \downarrow 0$, to a limit y_0 in $L_2(d\mu)$ and that y_0 is the projection in $L_2(d\mu)$ of g onto the closed convex set

$$\overline{\mathcal{D}(\tilde{A})} = \{u \in L_2(d\mu) \mid u \text{ is nondecreasing, } \quad 0 \le u \le 1\}.$$

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CHAPITRE V

VARIATIONAL ANALYSIS OF A PERTURBED FREE BOUNDARY PROBLEM

par

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VARIATIONAL ANALYSIS OF A PERTURBED FREE BOUNDARY PROBLEM

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1. INTRODUCTION

In this paper we study the nonlinear boundary value problem

$$\text{BVP}\quad\begin{cases} -\Delta u + h(\frac{u}{\varepsilon}) = f & \text{in } \Omega \\ \int\limits_{\Omega} h(\frac{u(x)}{\varepsilon})dx = C & \\ u|_{\partial\Omega} \text{ is constant (but unknown)} & \end{cases}$$

where

(i) Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$

(ii) ε is a small positive parameter

(iii) $h : \mathbb{R} \to \mathbb{R}$ is a given continuous, strictly monotone increasing function with $h(0) = 0$

(iv) f is a given distribution in $H^{-1}(\Omega)$

(v) C is a given constant which satisfies the compatibility condition

$$h(-\infty)|\Omega| < C < h(+\infty)|\Omega|.$$

Here $|\Omega|$ denotes the measure of Ω.

V.2.

The motivation for studying BVP partly stems from the physics of ionized gases and in this respect we continue earlier work [18, 19, 24, 25]. We refer to [25] and Appendix 2 for a discussion of this connection.

Our basic tools are the calculus of variations, convex analysis and the maximum principle.

We prove that BVP admits for each $\varepsilon > 0$ a unique solution u_ε which converges as $\varepsilon \downarrow 0$ to a limit u_0. Moreover, we give a variational characterization of u_0 which leads to the conclusion that u_0 solves a free boundary problem.

Our findings fit in with those of BRAUNER & NICOLAENKO [8, 9] in their study of related Dirichlet problems (we certainly have been inspired by their paper). In this connection it is also worth mentioning the work of FRANK & VAN GROESEN [21] and FRANK & WENDT [22] which analyses in particular the coincidence set. In Appendix 1 we give the analysis of the homogeneous Dirichlet problem.

In a recent paper [10] BRAUNER & NICOLAENKO stress the following point. Suppose one wants to analyse some free boundary problem, then it may be possible to view this problem as the limit when $\varepsilon \downarrow 0$ of a problem like BVP (with ε occurring in the argument of a smooth function). This smooth regularization can be used to solve problems of existence, regularity and approximation and it forms an alternative version of the usual penalization method. (see also [7]).

In the physical problem of Appendix 2 the parameter ε naturally appears in the same way as in BVP. In other situations one may arrive at the equation

$$-\varepsilon\Delta v + h(v) = f.$$

Then our results bear on $\varepsilon v_\varepsilon$ and $h(v_\varepsilon)$. The model of a confined plasma introduced by TEMAM [29,30] is of this type (with $f = 0$) but with h decreasing. The limiting behaviour of its solutions v_ε as $\varepsilon \downarrow 0$ is studied by CAFFARELLI & FRIEDMAN [15] and BERGER & FRAENKEL [6]. It may be possible that an adapted version of our duality approach can be applied to this problem. One would then have to use Toland's non convex duality as given by DAMLAMIAN [16].

After these general remarks, let us describe the contents of the paper in some more detail. We shall interpret BVP as the subdifferential equation $\partial V_\varepsilon(u) = 0$, where V_ε is a proper, strictly convex, lower semicontinuous and coercive functional defined on the direct sum of $H_0^1(\Omega)$ and the constant functions on Ω. This is rather easy if h satisfies certain growth restrictions. For the general case we heavily lean upon some results of BREZIS [12]. These and some other preliminaries are collected in section 2. The functional V_ε is defined in section 3 and from its properties we deduce the existence and uniqueness of a solution u_ε for each $\varepsilon > 0$.

The functional V_ε depends monotonously on ε and therefore has a well-defined limit V_0. Moreover, V_ε is coercive uniformly in ε and consequently we deduce in section 4 that as $\varepsilon \downarrow 0$ u_ε converges to u_0, the minimizer of V_0. The subdifferential ∂V_0 is multivalued. We find that u_0 satisfies an operator inclusion relation if h is bounded and a variational inequality if h is unbounded. We emphasize that u_0 depends only on f,C and $h(\pm\infty)$.

Problem BVP has the form

$$Lu + N\left(\frac{u}{\varepsilon}\right) = f$$

where both L and N are maximal monotone operators. The variational approach suggests the introduction of a dual formulation (in section 5) which turns out to be of the form

$$(\varepsilon A + I)p = g$$

where A is a maximal monotone operator on $(L_2(\Omega))^n$ with a special structure, and where g is related to f by div g = f. This gives some further insight into the convergence. The limit p_0 equals the projection of g onto the closed convex set $\overline{\mathcal{D}(A)}$. Duality theory yields a characterization of $\overline{\mathcal{D}(A)}$ by inequalities which seems difficult to obtain directly. Duality theory has been applied to related problems by ARTHURS & ROBINSON [4] and ARTHURS [3]. For the basic theory we refer to EKELAND & TEMAM [20]

In section 6 we assume $f \in L_\infty(\Omega)$. We employ maximum principle arguments and make some estimates. We prove that u_ε and u_0 belong to $W^{2,p}(\Omega)$ for each $p \geq 1$ and that u_ε converges weakly to u_0 in $W^{2,p}(\mathcal{O})$ for each $\mathcal{O}$ with $\bar{\mathcal{O}} \subset \Omega$. Either one has convergence in $W^{2,p}(\Omega)$ itself, or a boundary layer develops as $\varepsilon \downarrow 0$. We present criteria in terms of the data f, $h(\pm\infty)$ and C from which it can be decided in many cases which of these two possibilities actually occurs. In section 7 we briefly discuss the one-dimensional case.

Our analysis reveals that BVP and the homogeneous Dirichlet problem have exactly the same variational structure. In order to emphasize this point we analyse the latter problem in Appendix 1. Finally, we discuss the physical background of BVP in Appendix 2.

2. PRELIMINARIES

In this section we collect some definitions and results from the literature which we will use later. We state these in the form we need, which is not always the most general.

Let B be a Banach space and B^* its dual. Let $F : B \to (-\infty,+\infty]$ be a proper (i.e. $F \not\equiv +\infty$), lower semicontinuous (ℓ.s.c.), convex functional. The polar (or conjugate) functional $F^* : B^* \to (-\infty,+\infty]$ is defined by

$$F^*(u^*) = \sup\{<u^*,u> - F(u) \mid u \in \mathcal{D}(F)\} \tag{2.1}$$

where

$$\mathcal{D}(F) = \{u \mid F(u) < +\infty\} \tag{2.2}$$

and where $<\cdot,\cdot>$ denotes the duality pairing between B^* and B. The subdifferential ∂F is a, possibly multivalued, mapping of X into X^* defined by

$$u^* \in \partial F(u) \text{ if and only if } F(v) - F(u) \geq <u^*,v-u>, \ \forall v \in B. \tag{2.3}$$

LEMMA 2.1.

$u^* \in \partial F(u)$ *if and only if* $F(u) + F^*(u^*) = <u^*,u>$.

LEMMA 2.2.

$u^* \in \partial F(u)$ *if and only if* $u \in \partial F^*(u^*)$.

A convenient reference for these items is EKELAND & TEMAM [20].

If B is a Hilbert space one can identify B and B^* and then ∂F becomes a mapping of B into itself. It is well-known that ∂F is maximal monotone.

LEMMA 2.3. *Let* H *be a Hilbert space and* A *a maximal monotone operator on* H. *Then, for each* $\varepsilon > 0$, $(I+\varepsilon A)^{-1}$ *is a contraction defined on all of* H *and* $\lim_{\varepsilon\downarrow 0} (I+\varepsilon A)^{-1} h =$ *projection of* h *on* $\overline{\mathcal{D}(A)}$.

For this standard result we refer to BREZIS [11].

Let, as before, Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We shall write H_0^1, L_2 etc. to denote $H_0^1(\Omega), L_2(\Omega)$ etc. Also, we write $\int u$ to denote $\int_\Omega u(x)dx$.

Let $j : \mathbb{R} \to [0,+\infty]$ be a convex, l.s.c. function such that $j(0) = 0$. The convex, l.s.c. functional $J : H_0^1 \to [0,+\infty]$ is defined by

$$(2.4) \qquad J(u) = \begin{cases} \int j(u) & \text{if } j(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

The following two lemmas are special cases of results due to BREZIS [12].

LEMMA 2.4. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then*

$$J^*(w) = \begin{cases} \int j^*(w) & \text{if } w \in H^{-1} \cap L_1 \text{ and } j^*(w) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

LEMMA 2.5. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then* $w \in \partial J(u)$ *if and only if* $w \in H^{-1} \cap L_1$, $w.u \in L_1$ *and* $w(x) \in \partial j(u(x))$ *for almost all* $x \in \Omega$.

Finally, we quote a special case of a result of BREZIS & BROWDER [13,14].

LEMMA 2.6. *Assume* $w \in H^{-1} \cap L_1$ *and* $u \in H_0^1$ *are such that* $w(x)u(x) \geq g(x)$ *for almost all* $x \in \Omega$ *and some* $g \in L_1$. *Then* $w.u \in L_1$ *and*

$$<w,u> = \int w.u.$$

Here and in the following $<\cdot,\cdot>$ denotes the duality pairing of H^{-1} and H_0^1. We observe that Lemma 2.6 implies that the condition $w.u \in L_1$ in Lemma 2.5 is automatically satisfied.

3. VARIATIONAL FORMULATION

Let X be the direct sum of H_0^1 and the constant functions: $X = H_0^1 \oplus \mathbb{R}$. If u is some element of X, we write $u = \tilde{u} + u|_{\partial\Omega}$ for its decomposition. X is, provided with the topology inherited of H^1, a Hilbert space. Moreover, X is isomorphic to $H_0^1 \times \mathbb{R}$ and the H^1-norm is equivalent with the norm $\|\tilde{u}\|_{H_0^1} + | u|_{\partial\Omega} |$ on X. So we can realize the dual space X^* by

$$X^* = H^{-1} \times \mathbb{R}$$

V.6.

the pairing being given by

$$<(w,k),u>_X = <w,\tilde{u}> + ku|_{\partial\Omega}.$$

Consider the functional W defined on X by

$$(3.1) \qquad W(u) = \begin{cases} \int H(u) - C\, u|_{\partial\Omega} & \text{if } H(u) \in L_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where by definition

$$(3.2) \qquad H(y) = \int_0^y h(\eta)d\eta.$$

LEMMA 3.1.

$$W^*(w,k) = \begin{cases} \int H^*(w) & \textit{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \textit{ and } \int w = k + C, \\ +\infty & \textit{otherwise.} \end{cases}$$

PROOF. The idea is to take first the supremum with respect to the H_0^1-component and to use Lemma 2.4.

$$\sup\{<w,\tilde{u}> + k\, u|_{\partial\Omega} - \int H(\tilde{u} + u|_{\partial\Omega}) + C\, u|_{\partial\Omega} \;\Big|\; \tilde{u} \in H_0^1,\ u|_{\partial\Omega} \in \mathbb{R}\}$$

$$= \begin{cases} \sup\{\int H^*(w) - u|_{\partial\Omega} \int w + (k + C)\, u|_{\partial\Omega} \;\Big|\; u|_{\partial\Omega} \in \mathbb{R}\} \\ \qquad \text{if } w \in L_1 \cap H^{-1} \text{ and } H^*(w) \in L_1 \\ +\infty \qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} \int H^*(w) & \text{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \text{ and } \int w = k + C \\ +\infty & \text{otherwise.} \end{cases}$$

LEMMA 3.2.

$$\partial W(u) = \begin{cases} (h(u), \int h(u) - C) & \textit{if } h(u) \in H^{-1} \cap L_1 \\ \emptyset & \textit{otherwise.} \end{cases}$$

PROOF. (i) Let $(w,k) \in \partial W(u)$ then

$$W(\tilde{v}+ v|_{\partial\Omega}) - W(\tilde{u}+ u|_{\partial\Omega}) \geq <w,\tilde{v}- \tilde{u}> + k(v-u)|_{\partial\Omega}$$

for all $\tilde{v} \in H_0^1$ and all $v|_{\partial\Omega} \in \mathbb{R}$. By first taking $v|_{\partial\Omega} = u|_{\partial\Omega}$, we see that necessarily w belongs to the subdifferential of the functional $\tilde{u} \to W(\tilde{u}+u|_{\partial\Omega})$ defined on H_0^1. Hence, by Lemma 2.5, $w = h(u)$ and $w \in L_1$. Next, a combination of Lemma 2.1 and Lemma 3.1 shows that necessarily $k = \int w - C = \int h(u) - C$.
(ii) Conversely, let $h(u) \in H^{-1} \cap L_1$. Since h is the derivative of H we have

$$H(v) - H(u) \geq h(u)(v-u) = h(u)(\tilde{v}- \tilde{u}+ (v-u)|_{\partial\Omega}).$$

So if $H(v)$ and $H(u) \in L_1$, we can invoke Lemma 2.6 and conclude that $h(u)(\tilde{v}- \tilde{u}) \in L_1$ and that the integral equals the duality pairing. Integration of the inequality then yields, after adding a term $-C(v-u)|_{\partial\Omega}$,

$$W(v) - W(u) \geq <h(u),\tilde{v}- \tilde{u}> + (\int h(u) - C)(v-u)|_{\partial\Omega}. \qquad \square$$

We remark that, by Lemma 2.2, $\partial H^* = h^{-1}$. So, since h is strictly monotone,

$$(3.3) \qquad H^*(y) = \int_0^y h^{-1}(\eta)d\eta.$$

Let $g \in (L_2)^n$ be such that div $g = f$. The functional $G : (L_2)^n \to \mathbb{R}$ defined by

$$(3.4) \qquad G(p) = \int (\tfrac{1}{2}p^2 + g.p)$$

is Fréchet-differentiable with derivative p+g. The polar functional $G^* : (L_2)^n \to \mathbb{R}$ is given by

$$(3.5) \qquad G^*(p) = \tfrac{1}{2} \int (p-g)^2$$

and its derivative is p-g.

We define the bounded linear mapping $T : X \to (L_2)^n$ by

$$(3.6) \qquad T\ u = -\ \text{grad}\ u.$$

Its adjoint $T^*:(L_2)^n \to X^*$ is given by

$$(3.7) \qquad T^*p = (\text{div } p, 0).$$

Clearly the functional $u \mapsto G(-Tu)$ defined on X is differentiable with derivative $-T^*G'(-Tu) = (-\Delta u - f, 0)$.

Finally, let us put together the materials constructed above. Define $V_\varepsilon : X \to (-\infty, +\infty]$ by

$$(3.8) \qquad V_\varepsilon(u) = G(-Tu) + \varepsilon W(\tfrac{u}{\varepsilon}).$$

Then

$$(3.9) \qquad \partial V_\varepsilon(u) = \begin{cases} (-\Delta u - f + h(\frac{u}{\varepsilon}), \int h(\frac{u}{\varepsilon}) - C) & \text{if } h(\frac{u}{\varepsilon}) \in H^{-1} \cap L_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and, consequently, the problem BVP is equivalent with the variational problem

$$\text{VP} \qquad \operatorname*{Inf}_{u \in X} \; V_\varepsilon(u).$$

<u>THEOREM 3.3</u>. VP *has a unique solution* u_ε.

<u>PROOF</u>. G is convex, W is strictly convex and both functionals are l.s.c. (by Fatou's lemma). It remains to verify that V_ε is coercive on X. It is convenient to rewrite the functional V_ε as

$$V_\varepsilon(u) = \int (\tfrac{1}{2}(\text{gradu})^2 + (g-a) \cdot \text{gradu} + \varepsilon\, H(\tfrac{u}{\varepsilon}) - \tfrac{C}{|\Omega|}\, u)$$

where $|\Omega|$ denotes the measure of Ω and a is such that diva $= C|\Omega|^{-1}$ (for instance take $a = C(n|\Omega|)^{-1}\,(x_1, \ldots, x_n)$). Since $C|\Omega|^{-1} \in (h(-\infty), h(+\infty))$, there exist positive constants δ and M_1 such that

$$\varepsilon\, H(\tfrac{y}{\varepsilon}) - \tfrac{C}{|\Omega|}\, y \geq \delta|y| - M_1.$$

By the inequalities of Hölder and Poincaré there exists a positive constant $M_2 = M_2(\Omega)$ such that

$$\int |\tilde{u}| \leq \sqrt{|\Omega|}\, \|\tilde{u}\|_{L_2} \leq M_2 \|\text{grad } \tilde{u}\|_{L_2} = M_2 \|\text{grad } u\|_{L_2}.$$

Hence, using Hölder's inequality once more, we find

$$V_\varepsilon(u) \geq \tfrac{1}{2}\| \text{grad}\, u \|^2_{L_2} - \| g-a \|_{L_2} \| \text{grad}\ u \|_{L_2} + \delta |\Omega| \, | u|_{\partial\Omega} | - \delta \int |\tilde{u}| - M_1$$

$$\geq \tfrac{1}{4}\| \text{grad}\ u \|^2_{L_2} + \delta |\Omega| \, | u|_{\partial\Omega} | - M_3$$

for some constant M_3. It should be noted that the right hand side is independent of ε. □

4. LIMITING BEHAVIOUR OF u_ε AS $\varepsilon \downarrow 0$

In this section we show that u_ε converges as $\varepsilon \downarrow 0$. The limit u_0 is characterized as the unique solution of a variational problem. Equivalently one can characterize u_0 by an operator inclusion relation if h is bounded and by a variational inequality if h is unbounded. It turns out that u_0 depends only on $h(\pm\infty)$, f and C.

As $\varepsilon \downarrow 0$, the function $h(\frac{y}{\varepsilon})$ converges to the multivalued function

$$(4.1) \qquad h_0(y) = \begin{cases} h(+\infty), & y > 0 \\ [h(-\infty), h(+\infty)], & y = 0 \\ h(-\infty), & y < 0. \end{cases}$$

in the sense that each point on the graph of h_0 is the limit of points on the graph of $h(\frac{\cdot}{\varepsilon})$. We define

$$(4.2) \qquad H_0(y) = \begin{cases} h(+\infty)y, & y > 0 \\ 0\ , & y = 0 \\ h(-\infty)y, & y < 0 \end{cases}$$

<u>LEMMA 4.1</u>. $\varepsilon\, H(\frac{y}{\varepsilon})$ *converges monotonously increasing to* $H_0(y)$.

<u>PROOF</u>. $h(\frac{\eta}{\varepsilon})$ increases towards $h_0(\eta)$ for $\eta > 0$ and decreases towards $h_0(\eta)$ for $\eta < 0$. Since $\varepsilon\, H(\frac{y}{\varepsilon}) = \int_0^y h(\frac{\eta}{\varepsilon})d\eta$ we can use Lebesgue's monotone convergence theorem. □

We note that, by Dini's theorem, the convergence is uniform on compact subsets if h is bounded and, for instance, uniform on compact subsets of $(-\infty,0)$ if $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. Motivated by Lemma 4.1 we define

$$(4.3) \qquad W_0(u) = \begin{cases} \int H_0(u) - C\, u|_{\partial\Omega} & \text{if } H_0(u) \in L_1 \\ +\infty & \text{otherwise} \end{cases}$$

and we introduce the reduced variational problem

$$\text{RVP} \quad \operatorname*{Inf}_{u \in X} \; G(-Tu) + W_0(u).$$

Exactly as in the proof of Theorem 3.3 it follows that RVP has a solution. The functional $G(-Tu) + W_0(u)$ is convex, but not strictly convex. Still we have

LEMMA 4.2. *RVP has a unique solution* u_0.

PROOF. Since $G(\operatorname{grad} u)$ is strictly convex on H_0^1, two minimizers can only differ by a constant. For arbitrary $u \in X$ define

$$\Omega_+(u) = \{x \mid u(x) > 0\},\; \Omega_0(u) = \{x \mid u(x) = 0\},\; \Omega_-(u) = \{x \mid u(x) < 0\}.$$

Then

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(+\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C$$

and

$$\lim_{\delta \uparrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(-\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C.$$

So if $W_0(u+\ell)$ is constant for $|\ell| \le \eta$ then necessarily for those values of ℓ

$$h(+\infty)|\Omega_+(u+\ell)| + h(+\infty)[\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| =$$

$$h(+\infty)|\Omega_+(u+\ell)| + h(-\infty)|\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| = C.$$

Since $h(+\infty) > h(-\infty)$ this implies that

$$\{x \mid -\eta \le u(x) \le \eta\}$$

has measure zero. Then, however, u has to be sign-definite (this follows, for instance, from the connection between Sobolev and Beppo Levi spaces; see DENY & LIONS [17]) and we arrive at the conclusion that either $h(+\infty)|\Omega| = C$ or $h(-\infty)|\Omega| = C$. Finally, the compatibility condition excludes both of these possibilities. □

THEOREM 4.3.

$$\lim_{\varepsilon \downarrow 0} \| u_\varepsilon - u_0 \|_X = 0.$$

PROOF.

Step 1. We know that V_ε is coercive uniformly in ε (see the proof of Theorem 3.3). Hence $\|u_\varepsilon\|_X \le M$ for some constant M independent of ε and, consequently, the weak limit set of $\{u_\varepsilon\}$ is nonempty.

Step 2. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$ and suppose that $h(+\infty) = +\infty$. We claim that $\bar{u} \le 0$. Define $Q_0^\delta = \{x \mid \bar{u}(x) \ge \delta > 0\}$ and $Q_n^\delta = \{x \in Q_0^\delta \mid u_{\varepsilon_n}(x) \ge \frac{1}{2}\delta\}$. Then

$$\int |u_{\varepsilon_n} - \bar{u}|^2 \ge \int_{Q_0^\delta \backslash Q_n^\delta} |u_{\varepsilon_n} - \bar{u}|^2 \ge \frac{\delta^2}{4} |Q_0^\delta \backslash Q_n^\delta| .$$

Hence, since $u_{\varepsilon_n} \to \bar{u}$ strongly in L_2, necessarily $|Q_n^\delta| \to |Q_0^\delta|$. Furthermore,

$$\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right) \ge \varepsilon_n \int_{Q_n^\delta} H\left(\frac{\delta}{2\varepsilon_n}\right) = \varepsilon_n H\left(\frac{\delta}{2\varepsilon_n}\right) |Q_n^\delta|$$

Since $\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right)$ is bounded uniformly in n and since $\varepsilon_n H\left(\frac{\delta}{2\varepsilon_n}\right) \to +\infty$ as $n \to +\infty$, necessarily $|Q_n^\delta| \to 0$ as $n \to +\infty$. So we must have $|Q_0^\delta| = 0$. Since $\delta > 0$ was arbitrary we conclude that $\bar{u} \le 0$. Similarly, $h(-\infty) = -\infty$ implies $\bar{u} \ge 0$.

Step 3. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. We claim that $V_{\varepsilon_n}(u_{\varepsilon_n}) \to V_0(\bar{u})$. From $V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \langle \partial V_{\varepsilon_n}(\bar{u}), u_{\varepsilon_n} - \bar{u} \rangle_X$ we obtain, using step 2,

$$V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \int (\operatorname{grad} \bar{u} + g)(\operatorname{grad} u_{\varepsilon_n} - \operatorname{grad} \bar{u})$$

$$+ \int (h(\frac{\bar{u}}{\varepsilon_n})(u_{\varepsilon_n} - \bar{u})) - C(u_{\varepsilon_n} - \bar{u})|_{\partial\Omega}$$

Since the right-hand side converges to zero as $n \to +\infty$ we find

$$\liminf_{n \to +\infty} V_{\varepsilon_n}(u_{\varepsilon_n}) \ge \lim_{n \to \infty} V_{\varepsilon_n}(\bar{u}) = V_0(\bar{u}).$$

On the other hand, since u_{ε_n} minimizes V_{ε_n} and since $V_\varepsilon(v)$ is, for fixed v, monotone with respect to ε (Lemma 4.1), we have

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \le V_{\varepsilon_n}(\bar{u}) \le V_0(\bar{u}).$$

<u>Step 4</u>. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. Then

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \le V_{\varepsilon_n}(u_0) \le V_0(u_0)$$

and therefore $V_0(\bar{u}) \le V_0(u_0)$. Hence $\bar{u} = u_0$.

<u>Step 5</u>. We now know that u_0 is the only point in the weak limit set of $\{u_\varepsilon\}$ and thus $u_\varepsilon \rightharpoonup u_0$ as $\varepsilon \downarrow 0$. From

$$\varepsilon \int (H(\frac{u_\varepsilon}{\varepsilon}) - H(\frac{u_0}{\varepsilon})) \ge \int h(\frac{u_0}{\varepsilon})(u_\varepsilon - u_0)$$

and Step 2 we conclude that

$$\liminf_{\varepsilon \downarrow 0} \int \varepsilon H(\frac{u_\varepsilon}{\varepsilon}) \ge \int H_0(u_0).$$

It then follows from the weak l.s.c. of G and Step 3 that necessarily $\|\text{grad } u_\varepsilon\|_{L_2} \to \|\text{grad } u_0\|_{L_2}$ as $\varepsilon \downarrow 0$. Consequently u_ε converges in fact strongly in X to u_0. □

In order to get more information about u_0 we first determine W_0^* and ∂W_0. We write $u \ge 0$ for some $u \in X$ if and only if $u(x) \ge 0$ for almost all $x \in \Omega$. Let C denote the closed, convex, positive cone corresponding to this ordering. By duality C induces a cone C^* in X^*: we write $(w,k) \ge 0$ if and only if $\langle (w,k),u\rangle_X \ge 0$ for all $u \in C$. For any $u \in X$ we define $u_+ = \max(u,0)$ and $u_- = \max(-u,0)$. Then $u_+ \in X$, $u_- \in X$ and at least one of these belongs to H_0^1 (see, for instance, KINDERLEHRER & STAMPACCHIA [26, Ch. II, Proposition 5.3]).

In the following we slightly abuse notation if $h(+\infty) = +\infty$ and/or $h(-\infty) = -\infty$. However this should not lead to confusion.

<u>LEMMA 4.4</u>.

$$W_0^*(w,k) = \begin{cases} 0 \quad \textit{if both} & (h(+\infty)-w, h(+\infty)|\Omega| - C - k) \in C^* \\ & (w - h(-\infty), k - h(-\infty)|\Omega| + C) \in C^* \\ +\infty \quad \textit{otherwise}. & \end{cases}$$

<u>PROOF</u>.

$$W_0^*(w,k) = \sup\{\langle (w,k),u\rangle_X - \int h(+\infty)u_+ + \int h(-\infty)u_- + C\, u|_{\partial\Omega} \mid u \in X\}$$

$$= \sup\{\langle (w-h(+\infty), k - h(+\infty)|\Omega| + C),\ u_+\rangle_X$$

$$- \langle (w-h(-\infty), k - h(-\infty)|\Omega| + C), u_-\rangle_X \mid u \in X\}. \qquad □$$

LEMMA 4.5. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then*

$$\partial W_0(u) = \{(w,k) \mid w \in L_1,\ w(x) \in h_0(u(x)) \text{ for a.e. } x \in \Omega,\ k = \int w - C\}.$$

PROOF. (i) Suppose $(w,k) \in \partial W_0(u)$. As in the proof of Lemma 3.2 it follows that $w \in L_1$ and $w(x) \in h_0(u(x))$ a.e.. Let v_n be the solution of

$$\begin{cases} -\frac{1}{n}\Delta v_n + v_n = 0 \\ \quad v_n|_{\partial\Omega} = 1. \end{cases}$$

Then $v_n \geq 0$ and, as $n \to \infty$, v_n converges strongly in L_2 to zero. By Lemmas 2.1 and 4.4 we know that

$$<(h(+\infty) - w,\ h(+\infty)|\Omega| - C - k),\ v_n>_X \geq 0$$

and

$$<(h(-\infty) - w,\ h(-\infty)|\Omega| - C - k),\ v_n>_X \leq 0.$$

Taking into account that $w \in L_\infty$ (since $w \in h_0(u)$), we rewrite these inequalities as

$$\int (h(+\infty) - w)(v_n - 1) + h(+\infty)|\Omega| - C - k \geq 0$$

and

$$\int (h(-\infty) - w)(v_n - 1) + h(-\infty)|\Omega| - C - k \leq 0.$$

Upon passing to the limit $n \to +\infty$ we find that $\int w - C - k \geq 0$ and $\int w - C - k \leq 0$.

(ii) is exactly the same as the second part of the proof of Lemma 3.2. □

COROLLARY 4.6. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then* RVP *is equivalent with the reduced boundary value problem*

$$\text{RBVP} \quad \begin{cases} \Delta u + f \in h_0(u) \\ \int (\Delta u + f) = C \\ u|_{\partial\Omega} \text{ is constant (but unknown).} \end{cases}$$

Finally, let us consider a function h which is unbounded. We concentrate on the case $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. From the proof of Theorem 4.3 we know that $u_0 \leq 0$. Consequently RVP is equivalent to minimizing a differentiable functional on the cone $-$ C and, therefore, with the variational inequality:

V.14.

$$\text{VI}\quad \begin{cases} \text{Find } u \in -C \text{ such that for all } v \in -C \\ <(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v - u>_X \geq 0. \end{cases}$$

Unfortunately we cannot use Lemma 2.5 in this situation (see, however, [23]) but still we have

<u>LEMMA 4.7</u>. *Suppose* $h(-\infty) > -\infty$ *and* $h(+\infty) = +\infty$. *Then*

$$\partial W_0(u) = \begin{cases} \{(w,k) \mid (w-h(-\infty),\ k - h(-\infty)|\Omega| + C) \in C^* \text{ and } \\ \quad <(w-h(-\infty),\ k - h(-\infty)|\Omega| + C), u>_X = 0\} \text{ if } -u \in C \\ \emptyset \text{ otherwise.} \end{cases}$$

<u>PROOF</u>. This follows directly from Lemma 2.1, Lemma 4.4 and the fact that W_0 is linear on the negative cone. □

5. THE DUAL FORMULATION

So far we have used polar functionals repeatedly, but we have not yet given a systematic presentation of duality theory as applied to our problem. This will be done now. We follow closely EKELAND & TEMAM [20, Ch. III, section 4, in particular Remarque 4.2].

The dual formulation of VP, corresponding to the splitting $V_\varepsilon(u) = G(-Tu) + \varepsilon W(\frac{u}{\varepsilon})$, is given by

$$\text{VP}^* \quad \operatorname*{Inf}_{p \in (L_2)^n} \varepsilon\, W^*(T^*p) + G^*(p).$$

Since VP is stable (use [20, Proposition III.2.3]), VP^* has a (unique) solution p_ε. Furthermore, the infima are equal to each other and u_ε and p_ε are related by the so-called extremality relations

$$(5.1) \qquad T^* p_\varepsilon = \partial W\left(\frac{u_\varepsilon}{\varepsilon}\right)$$

$$(5.2) \qquad p_\varepsilon = \partial G(-Tu_\varepsilon).$$

By Lemma 3.2 and (3.4) these can be rewritten as

$$(5.3) \qquad \operatorname{div} p_\varepsilon = h\left(\frac{u_\varepsilon}{\varepsilon}\right) \text{ and } \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) = C$$

$$(5.4) \qquad p_\varepsilon = g + \operatorname{grad} u_\varepsilon.$$

Note that g is not uniquely determined by div g = f but that (5.3) and (5.4) define $p_\varepsilon - g$ and div p_ε unambiguously. One can view (5.3) and (5.4) as a canonical splitting of BVP into first order equations. Indeed, elimination of p_ε leads to BVP. On the other hand, we can also eliminate u_ε to find the subdifferential equation satisfied by p_ε:

$$\varepsilon\, T(\partial W)^{-1}(T^{*}p_\varepsilon) + p_\varepsilon = g \tag{5.5}$$

or, more explicitly,

$$\mathrm{BVP}^{*}\quad \begin{cases} -\varepsilon\,\mathrm{grad}(h^{-1}(\mathrm{div}\ p_\varepsilon)) + p_\varepsilon = g \\ \displaystyle\int \mathrm{div}\ p_\varepsilon = C \\ h^{-1}(\mathrm{div}\ p_\varepsilon) \in X \end{cases}$$

By Lemmas 2.2, 3.2 and [20, Proposition I.5.7] the operator A from $(L_2)^n$ into itself defined by

$$\begin{cases} Ap = -\mathrm{grad}(h^{-1}(\mathrm{div}\ p)) \\ \mathcal{D}(A) = \{p \in (L_2)^n \mid \mathrm{div}\ p \in L_1,\ \displaystyle\int \mathrm{div}\ p = C,\ \mathrm{div}\ p = h(u) \text{ for} \\ \qquad\qquad \text{some } u \in X\} \end{cases} \tag{5.6}$$

is the subdifferential of the convex l.s.c. functional $p \mapsto W^{*}(T^{*}p)$. Consequently, A is maximal monotone. (See Weyer [31] for related results). Rewriting (5.5) as

$$(\varepsilon A + I)p_\varepsilon = g \tag{5.7}$$

and invoking Lemma 2.3, we find that p_ε converges, as $\varepsilon \downarrow 0$, strongly in $(L_2)^n$ to the projection of g onto $\overline{\mathcal{D}(A)}$. It does not seem easy to characterize $\overline{\mathcal{D}(A)}$ directly from (5.6). Therefore we use duality theory once more, but now for the reduced problem.

The dual formulation of RVP is given by

$$\mathrm{RVP}^{*}\qquad \underset{p\in(L_2)^n}{\mathrm{Inf}}\quad W_0^{*}(T^{*}p) + G^{*}(p).$$

By (3.5) and Lemma 4.4 the solution of RVP^{*} is the projection of g onto the closed convex set

$$(5.8) \qquad Q = \{p \in (L_2)^n \mid (h(+\infty) - \text{div } p,\ h(+\infty)|\Omega| - C) \in C^*$$

$$\text{and } (\text{div } p - h(-\infty),\ C - h(-\infty)|\Omega|) \in C^*\}$$

Denoting the (unique) solution of RVP* by p_0, we have the extremality relations

$$(5.9) \qquad T^* p_0 \in \partial W_0(u_0)$$

$$(5.10) \qquad p_0 = \partial G(-T u_0).$$

The second one, $p_0 = g + \text{grad } u_0$, is identical to the extremality relation $p_\varepsilon = g + \text{grad } u_\varepsilon$. Hence the fact that u_ε converges strongly in X to u_0, implies that p_ε converges strongly in $(L_2)^n$ to p_0. So we find that p_ε converges to a limit which is at the same time the projection of g onto $\overline{\mathcal{D}(A)}$ and onto Q. Since g is an arbitrary element of $(L_2)^n$, necessarily $\overline{\mathcal{D}(A)} = Q$. Thus we have shown that (5.8) gives an explicit characterization of $\overline{\mathcal{D}(A)}$.

The extremality relation (5.9) is easy to work with only in the case that h is bounded (see Lemmas 4.5 and 4.7). It then follows that RBVP is equivalent to (5.9) - (5.10). Likewise one can, by elimination of u_0, derive a subdifferential equation for p_0 similar to BVP*.

If $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ we deduce from Lemma 4.7 that u_0 is the solution of the following variant of VI:

$$\begin{cases} \text{Find } u \in -C \text{ such that} \\ \text{(i)} \quad <(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v>_X \le 0,\ \forall v \in C, \\ \text{(ii)} \quad <(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ u>_X = 0. \end{cases}$$

6. THE REDUCED PROBLEM AS A FREE BOUNDARY PROBLEM

In this section we assume that $f \in L_\infty$. We shall deal with the regularity of u_0 (and u_ε), with the free boundary value problem satisfied by u_0 and with sharp convergence results versus the occurrence of boundary layers. We shall write $C^{1,\alpha}$ to denote the Hölder space $C^{1,\alpha}(\bar{\Omega})$ and $W^{2,p}$ to denote the usual Sobolev space. We recall that $W^{2,p}$ is imbedded into $C^{1,\alpha}$ if $p(1-\alpha) \ge n$.

THEOREM 6.1. *If* h *is bounded,* u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \ge 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

PROOF.

$$\|\Delta u_\varepsilon\|_{L_\infty} \leq \max\{-h(-\infty),h(+\infty)\} + \|f\|_{L_\infty}. \qquad \square$$

We can now interpret RBVP as a free boundary problem. The domain Ω consists of three subdomains:

$$\Omega_+ = \{x \in \Omega \mid u_0(x) > 0\} \text{ where } -\Delta u_0 + h(+\infty) = f \quad \text{a.e.}$$

$$\Omega_- = \{x \in \Omega \mid u_0(x) < 0\} \text{ where } -\Delta u_0 + h(-\infty) = f \quad \text{a.e.}$$

$$\Omega_0 = \{x \in \Omega \mid u_0(x) = 0\} \text{ which has to be a subset of}$$
$$\{x \in \Omega \mid h(-\infty) \leq f(x) \leq h\ (+\infty)\}.$$

These subdomains are unknown, possibly empty and such that

$$h(+\infty)|\Omega_+| + h(-\infty)|\Omega_-| + \int_{\Omega_0} f = C.$$

From the proof of Theorem 4.3 we know that $u_0 = 0$ if $h(\pm\infty) = \pm\infty$. So in that case we cannot have convergence in $W^{2,p}$ unless $\int f = C$.

Next, we concentrate on the most interesting case in which h is bounded from one and only one side. In the remaining part of this section we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. We emphasize that all theorems below have a counterpart in the case $h(-\infty) = -\infty$ and $h(+\infty) < +\infty$.

THEOREM 6.2. $u_\varepsilon \in W^{2,p}$ *for each* $p \geq 1$.

PROOF. We shall show that Δu_ε is bounded by finding an upper bound for u_ε. Let $\zeta \in H_0^1$ be the solution of $-\Delta\zeta + h(-\infty) = f$. Then, in fact, since $\Delta\zeta$ is bounded, we have $\zeta \in C^{1,\alpha}$. Define $\psi \in H_0^1$ by $\psi = u_\varepsilon - u_\varepsilon|_{\partial\Omega} - \zeta$. Then

$$\Delta\psi = \Delta u_\varepsilon - \Delta\zeta = h(\frac{u_\varepsilon}{\varepsilon}) - h(-\infty) \geq 0$$

and hence, by the weak maximum principle, $\psi \leq 0$. So u_ε is bounded from above by the bounded function $u_\varepsilon|_{\partial\Omega} + \zeta$. $\square$

THEOREM 6.3. *If* $C \leq \int f$, u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \geq 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

PROOF. We show that $h(\frac{u_\varepsilon}{\varepsilon})$ and hence Δu_ε is bounded. Choose $\delta > 0$ and define

$$\Omega_\varepsilon = \{x \in \Omega \,|\, h(\frac{u_\varepsilon(x)}{\varepsilon}) > \|f\|_{L_\infty} + \delta\}.$$

The points of $\partial\Omega_\varepsilon$ either belong to $\partial\Omega$ or are such that $h(\frac{u_\varepsilon(x)}{\varepsilon}) = \|f\|_{L_\infty} + \delta$. If $|\Omega_\varepsilon| \neq 0$ and $\partial\Omega_\varepsilon \cap \partial\Omega = \emptyset$, we find that simultaneously $\Delta u_\varepsilon > 0$ in Ω_ε and u_ε assumes, with respect to Ω_ε, its maximum in an interior point. Since this is impossible we conclude that either $|\Omega_\varepsilon| = 0$ or $\partial\Omega_\varepsilon \cap \partial\Omega \neq \emptyset$ and u_ε assumes its maximum at $\partial\Omega$ with $h(\frac{u_\varepsilon|\partial\Omega}{\varepsilon}) > \|f\|_{L_\infty} + \delta$.

Suppose $|\Omega_\varepsilon| \neq 0$. Let $\tilde{\Omega}_\varepsilon$ be a domain with boundary $\partial\Omega \cup \Gamma$ and strictly contained in Ω_ε. We define $\tilde{u}_\varepsilon$ to be the solution of $\Delta\tilde{u} = \delta$, $\tilde{u}(x) = u_\varepsilon(x)$, $x \in \partial\tilde{\Omega}_\varepsilon$. Then $\tilde{u}_\varepsilon$ attains its maximum on $\partial\Omega$ and it follows from the Hopf maximum principle [27, Thm 7, p. 65] that $\frac{\partial\tilde{u}_\varepsilon}{\partial n}|_{\partial\Omega} > 0$. Also we have that $\Delta(\tilde{u}_\varepsilon - u_\varepsilon) = \delta - h(\frac{u_\varepsilon}{\varepsilon}) + f \le 0$ and therefore $\tilde{u}_\varepsilon - u_\varepsilon \ge 0$ and, finally,

$$\frac{\partial u_\varepsilon}{\partial n}\Big|_{\partial\Omega} \ge \frac{\partial \tilde{u}_\varepsilon}{\partial n}\Big|_{\partial\Omega} > 0.$$

This leads to the contradiction

$$C - \int f = \int \Delta u_\varepsilon = \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} > 0. \qquad \square$$

The proof above shows that, if $h(\frac{u_\varepsilon}{\varepsilon})$ blows up somewhere, it does so at the boundary. If $u_0|_{\partial\Omega} < 0$ this can not happen, so we also have

THEOREM 6.4. *If* $u_0|_{\partial\Omega} < 0$ *then* u_ε *converges to* u_0 *weakly in* $W^{2,p}$, $p \ge 1$, *and strongly in* $C^{1,\alpha}$, $\alpha \in [0,1)$.

THEOREM 6.5. $u_0 \in W^{2,p}$ *for each* $p \ge 1$.

PROOF. If $u_0|_{\partial\Omega} < 0$ we can apply Theorem 6.4. If $u_0|_{\partial\Omega} = 0$, then u_0 is completely characterized by the restriction of RVP to H_0^1. The result then follows, for instance, from Appendix 1. $\square$

THEOREM 6.6. u_0 *is completely characterized by*

$$\begin{cases} -\Delta u_0 + h(-\infty) - f \le 0 & \text{a.e.} \\ u_0 \le 0 & \text{a.e.} \\ u_0(-\Delta u_0 + h(-\infty) - f) = 0 & \text{a.e.} \\ \int(\Delta u_0 + f) - C \le 0 & \\ u_0|_{\partial\Omega}(\int(\Delta u_0 + f) - C) = 0. & \end{cases}$$

PROOF. Because of Theorem 6.5 we can rewrite the variant of VI given at the end of section 5 in the form

$$\int(\Delta u_0 - h(-\infty) + f)v + (C - \int(\Delta u_0 + f))v|_{\partial\Omega} \ge 0, \quad \forall v \in C,$$

$$\int(\Delta u_0 - h(-\infty) + f)u_0 + (C - \int(\Delta u_0 + f))u_0|_{\partial\Omega} = 0,$$

and from this formulation the result easily follows. □

If $\int f \ge C$ then Theorem 6.3 implies that actually $\int(\Delta u_0 + f) = C$. We emphasize that $\int f < C$ does not preclude the possibility that $u_0|_{\partial\Omega} < 0$ and $\int(\Delta u_0 + f) = C$. However, if $\int(\Delta u_0 + f) < C$ we cannot have weak convergence in $W^{2,p}$. Next, we present some conditions on the data $h(-\infty)$, f and C under which this happens.

THEOREM 6.7. *Any of the three assumptions*

(i) $f(x) \le h(-\infty)$ a.e.

(ii) $f(x) \ge h(-\infty)$ a.e. *and* $\int f < C$

(iii) $\int_{\tilde\Omega} f < C$ *for all* $\tilde\Omega \subset \Omega$

implies that $\int(\Delta u_0 + f) < C$.

PROOF. (i) Let $v \in H_0^1$ be the solution of $\Delta v = h(-\infty) - f$. Then $v \le 0$ and $\int(\Delta v + f) = h(-\infty)|\Omega| < C$. By Theorem 6.6 $u_0 = v$.

(ii) Again by Theorem 6.6, $u_0 = 0$.

(iii)

$$\int(\Delta u_0 + f) = \int_{\bar\Omega} h(-\infty) + \int_{\Omega\setminus\bar\Omega} f = h(-\infty)|\bar\Omega| + \int_{\Omega\setminus\bar\Omega} f < C$$

where $\bar\Omega = \{x \mid u_0(x) < 0\}$. □

In the proof of Theorem 6.3 it was already shown that if u_ε displays a layer of rapid change somewhere, it certainly does so near to the boundary. Next we prove that it can do so *only* near to the boundary. The estimates below have been indicated to us by H. BREZIS.

THEOREM 6.8. *Assume* h *is* C^1. *Then* u_ε *converges to* u_0 *weakly in* $W^{2,p}(O)$ *for any open set* O *with* $\bar O \subset \Omega$ *and any* $p \ge 1$.

PROOF.

Step 1. Since $h(y) > h(-\infty)$ we have

$$\int \left|h\left(\frac{u_\varepsilon}{\varepsilon}\right)\right| \le \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) - 2h(-\infty)|\Omega| = C - 2h(-\infty)|\Omega|.$$

Step 2. Since u_ε is bounded uniformly in ε in H^1, it follows from the Sobolev imbedding theorem (see, for instance, ADAMS [1, p. 97]) that u_ε is bounded uniformly in ε in $L_r(\Omega)$, where $r = \frac{2n}{n-2}$ if $n > 2$ and $r \geq 1$ if $n \leq 2$.

Step 3. (Proof by recursion). We suppose that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in ε in $L_q(\mathcal{U}_1)$ for some $q \geq 1$ and $\mathcal{U}_1$ such that $\overline{\mathcal{U}_1} \subset \Omega$. Let ζ be a C^∞-function with compact support in $\mathcal{U}_1$. We multiply the differential equation by $|h(\frac{u_\varepsilon}{\varepsilon})|^{t-2}\, h(\frac{u_\varepsilon}{\varepsilon})\, |\zeta|^t$ and we integrate. Thus we obtain

$$\int - \Delta u_\varepsilon \, |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} h(\frac{u_\varepsilon}{\varepsilon})|\zeta|^t + \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \leq \int |f\zeta| \, |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^{t-1}.$$

Integrating the first term by parts and using the inequality $ab \leq \frac{1}{\alpha} a^\alpha + \frac{1}{\beta} b^\beta$ with $a,b > 0$, $\alpha,\beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, for the term at the right hand side we deduce

$$\frac{t-1}{\varepsilon}\int |\mathrm{grad}\; u_\varepsilon|^2 \; |\zeta|^t \; |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} \; h'(\frac{u_\varepsilon}{\varepsilon}) + \frac{1}{t}\int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$$

$$\leq \frac{1}{t}\int |f\zeta|^t - \int |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} \; h(\frac{u_\varepsilon}{\varepsilon}) \; \mathrm{grad}\; u_\varepsilon \, . \, \mathrm{grad}\; |\zeta|^t.$$

We observe that the first term at the left hand side is nonnegative (so we delete this term). Now let $\gamma(x) = |h(x)|^{t-2}\, h(x)$ and $\Gamma(x) = \int_0^x \gamma(\tau)d\tau$. Then $\Gamma(x) \leq x\gamma(x)$ for all x and hence

$$-\int \gamma(\frac{u_\varepsilon}{\varepsilon})\mathrm{grad}\; u_\varepsilon \, . \, |\mathrm{grad}\; \zeta|^t = \varepsilon\int \Gamma(\frac{u_\varepsilon}{\varepsilon})\Delta|\zeta|^t \leq \int u_\varepsilon \gamma(\frac{u_\varepsilon}{\varepsilon})\Delta|\zeta|^t$$

So finally

$$\int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \leq K_1 + K_2 \int_{\mathcal{U}_1} |u_\varepsilon| \; |h(\frac{u_\varepsilon}{\varepsilon})|^{t-1}. \tag{6.1}$$

We now distinguish different cases:

1st case $q = 1$. If $n > 2$, we choose $t = 1 + \frac{n+2}{2n}$ in (6.1) and apply Hölder's inequality with conjugate exponents $\frac{2n}{n-2}$ and $\frac{2n}{n+2}$; also using the results of Steps 1 and 2 we deduce that $\int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$ is bounded uniformly in ε. If $n \leq 2$, we choose $t = 1 + \frac{r-1}{r}$ for some $r > 1$ and apply Hölder's inequality with conjugate exponents r and $\frac{r}{r-1}$ to obtain a similar result. So we know in both cases that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$ for some $t > 1$ and any open set $\mathcal{U}_2$ with $\overline{\mathcal{U}_2} \subset \mathcal{U}_1$. Consequently u_ε is bounded uniformly in $W^{2,t}(\mathcal{U}_2)$ (cf. AGMON [2]).

2nd case $q > \frac{n}{2}$. It follows from the Sobolev imbedding theorem that u_ε is bounded uniformly in ε in $L_\infty(\mathcal{U}_1)$. Choosing $t = q+1$ in (6.1), we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in ε in $L^{q+1}(\mathcal{U}_2)$. The result of the theorem follows then from a bootstrap argument.

3rd case $q \leq \frac{n}{2}$. By the Sobolev imbedding theorem u_ε is bounded uniformly in $L_{q^*}(\mathcal{U}_1)$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$ (or $\frac{1}{q^*} = \frac{1}{q} - \alpha$ for any $\alpha \in (0,\frac{1}{q})$ if $q = \frac{2}{n}$). Let q^{**} be the conjugate exponent of q^* and choose $t = 1 + \frac{q}{q^{**}}$. Applying Hölder's inequality (with exponents q^* and q^{**}) to (6.1) we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$. Now a bootstrap argument either yields the result or leads to the 2nd case. □

7. THE ONE DIMENSIONAL CASE

Again we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. The results of section 5 imply that p_0 is the projection of g onto the set

$$\overline{\mathcal{D}(A)} = \{p \in L_2 \mid (p' - h(-\infty),\ C - h(-\infty)|\Omega|) \in \mathcal{C}^*\}.$$

A simple calculation shows that, with $\Omega = (-1,+1)$,

$$\overline{\mathcal{D}(A)} \cap H^1 = \{p \mid p' \geq h(-\infty) \text{ and } p(1) - p(-1) \leq C\}.$$

We found in section 6 that $p_0 \in \overline{\mathcal{D}(A)} \cap H^1$ if $f \in L_\infty$. So we can find p_0 by minimizing the L_2-distance to g subject to two constraints: an inequality for the derivative and a bound for the total variation. This is more or less a combinatorial problem which is rather easy to solve for some given smooth g, but whose general solution is cumbersome. We refer to [19, section 4] for a more detailed discussion of the symmetric case, noting that the result presented there covers the general case after some minor modifications. Finally, we remark that, once p_0 is found, u_0 can be calculated from the extremality relations.

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APPENDIX 1. THE HOMOGENEOUS DIRICHLET PROBLEM

In this appendix we present some results about the problem

$$-\Delta u + h(\tfrac{u}{\varepsilon}) \ni f,$$

where by assumption h is the subdifferential of a convex, l.s.c. function $H : \mathbb{R} \to [0,\infty)$, with $H(0) = 0$ and $H(y) < +\infty$ for all $y \in \mathbb{R}$. Here $f \in H^{-1}$ is given and $u \in H_0^1$ is sought. We use some of the notation defined in the preceding pages and omit all proofs since these are similar to (and in fact easier than) those already given. In contravention of prior definitions we now have:

$$T : H_0^1 \to (L_2)^n, \qquad Tu = -\text{grad } u$$

$$T^*: (L_2)^n \to H^{-1}, \qquad T^*p = \text{div } p$$

$$W : H_0^1 \to [0,\infty], \qquad W(u) = \begin{cases} \int H(u) & \text{if } H(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

The problem can be rewritten as

$$\partial V_\varepsilon(u) \ni 0$$

where

$$V_\varepsilon(u) = G(-Tu) + \varepsilon\, W(\tfrac{u}{\varepsilon}).$$

It admits a unique solution u_ε which converges as $\varepsilon \downarrow 0$ strongly in H_0^1 to u_0, the unique solution of

$$\operatorname*{Inf}_{u \in H_0^1} \; G(-Tu) + W_0(u).$$

If h is bounded u_0 satisfies

$$-\Delta u + h_0(u) \ni f$$

and if, for instance, $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ then u_0 solves the variational inequality: find $u \leq 0$ such that

$$<-\Delta u + h(-\infty) - f, \; v-u> \; \geq 0, \quad \forall v \leq 0.$$

The dual formulation is obtained by the transformations

$$p = g - Tu$$

$$u \in \varepsilon h^{-1}(T^* p)$$

$$f = T^* g$$

and reads

$$\varepsilon \; T(h^{-1}(T^* p)) + p \ni g$$

or, equivalently,

$$(\varepsilon A + I)p \ni g$$

where $A : (L_2)^n \rightarrow (L_2)^n$ is defined by

$$Ap = T(h^{-1}(T^* p))$$

$\mathcal{D}(A) = \{p \in (L_2)^n \mid T^* p \in L_1$ and there exists $u \in H_0^1$ such that $T^* p \in h(u)\}$.

As $\varepsilon \downarrow 0$, p_ε converges to the projection of g onto

$$\overline{\mathcal{D}(A)} = \{p \in (L_2)^n \mid h(-\infty) \leq T^* p \leq h(+\infty)\},$$

where the inequalities are defined by the positive cone in H_0^1 and the duality of H_0^1 and H^{-1}.

If $f \in L_\infty$, u_ε converges to u_0 weakly in $W^{2,p}$ for each $p \geq 1$ and strongly in $C^{1,\alpha}$ for each $\alpha \in [0,1)$. This follows most easily from the observation that, by the maximum principle, u_ε equals the solution of the "truncated" problem

$$-\Delta u + \tilde{h}\,(\frac{u}{\varepsilon}) \ni f$$

where

$$\tilde{h}(y) = \begin{cases} \|f\|_{L_\infty} & \text{if } h(y) \geq \|f\|_{L_\infty} \\ h(y) & \text{if } -\|f\|_{L_\infty} \leq h(y) \leq \|f\|_{L_\infty} \\ -\|f\|_{L_\infty} & \text{if } h(y) \leq -\|f\|_{L_\infty}. \end{cases}$$

For sharper estimates under additional assumptions we refer to [8], [9], [5] and [28].

APPENDIX 2. THE PHYSICAL BACKGROUND OF THE PROBLEM

Consider a bounded domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a charge distribution inside Ω with two components:

(i) a fixed ionic charge density en_i

(ii) a mobile electronic charge density $-en_e$ such that

$$\text{(A.1)} \qquad \int n_e = N_e.$$

Here e is the unit charge, n_i and n_e are number densities and N_e is a number. N_e and n_i are given, but n_e is unknown.

Let the region outside Ω be a conductor. Then we have the condition

(A.2) the potential Φ is constant outside Ω.

Physically this condition is realized by the formation of a surface charge density which, however, will be of no further concern.

The equation for the potential Φ in Ω can be deduced from two physical laws:

$$\text{(A.3)} \qquad \Delta\Phi = -4\pi e(n_i - n_e), \qquad \text{Poisson's equation,}$$

and

(A.4) $n_e = K\, e^{\frac{e\Phi}{k_B T}}$, Boltzmann's formula.

Here K is a normalization constant, T is the temperature of the system and k_B is Boltzmann's constant.

Substituting (A.4) into (A.3) and (A.1) we obtain the problem

$$\begin{cases} -\Delta\Phi + 4\pi e K\, e^{\frac{e\Phi}{k_B T}} = 4\pi e\, n_i \\ K \int e^{\frac{e\Phi}{k_B T}} = N_e \\ \Phi|_{\partial\Omega} \text{ is constant (but unknown)} \end{cases}$$

which, up to a renaming of the constants and variables, is the special case of BVP in which $h(y) = e^y - 1$.

Alternatively, one can argue that n_e should be such that the free energy F of the system be minimized under the constraint (A.1). The free energy is defined by

$$F = U - TS$$

where U is the electrostatic energy given by

$$U = \frac{1}{8\pi} \int (\text{grad } \Phi)^2,$$

T is the temperature and S the entropy given by

$$S = -k_B \int n_e\, \ell n\, n_e .$$

So if E_i denotes the electric field created by the ions and E_e the electric field created by the electrons, it comes to solve the minimization problem

$$\operatorname*{Inf}_{E_e} k_B T \int \text{div } E_e\, \ell n\, (\text{div } E_e) + \frac{1}{8\pi} \int (E_i - E_e)^2$$

subject to the constraint

$$\int \text{div } E_e = N_e .$$

Clearly this problem corresponds to VP^*.

V.28.

The main results of this paper concern the limiting behaviour of the potential Φ and the electrical field E_e due to the electrons, as the temperature T tends to zero. For instance, we find that at $\partial\Omega$ no boundary layer occurs if the total charge density $\int n_i$ of the ions exceeds N_e. In the limit $T \to 0$ there may be regions where electrons are absent. If such a region $\bar{\Omega}$ is strictly contained in Ω it necessarily must be such that $\int_{\bar{\Omega}} n_i = 0$. For such a region which extends up to $\partial\Omega$ there is a more complicated condition. If $n_i \geq 0$ and $\int n_i < N_e$, necessarily a boundary layer arises: the electrons are repelled against the conductor.

CHAPITRE VI

ON A DOUBLY NONLINEAR DIFFUSION EQUATION IN HYDROLOGY

par

C.J. Van Duyn et D. Hilhorst.

1. Introduction.

In this paper we consider the nonlinear partial differential equation

$$(1.1) \qquad u_t = (D(u)\phi(u_x))_x$$

where the functions D and ϕ satisfy the hypotheses

H_ϕ : $\phi \in C^1([-1,1]) \cap C^2((-1,1))$, $\phi(0) = 0$, $\phi'(-1) = \phi'(1) = 0$,
$\phi' > 0$ on $(-1,1)$.

and

H_D : $D \in C^1([0,1]) \cap C^2((0,1))$, $D > 0$ on $(0,1)$, $D(0) = D(1) = 0$,
$D'' \leq 0$ on $(0,1)$.

The main difficulty in studying equation (1.1) is that it has two kinds of degeneracies, namely one in points where $u = 0$ or 1 and one in points where $u_x = 1$ or -1. An equation of type (1.1) arises in the theory of hydrology, with $D(s) = s(1-s)$ and $\phi(s) = s/(1+s^2)$. We give its derivation in section 2.

We are interested in the following three problems related to equation (1.1) : the Neumann problem on (-1,1) with the natural boundary conditions $D(u)\phi(u_x)(\pm 1,t) = 0$ for $t > 0$, the Cauchy problem and a related Cauchy Dirichlet problem on $(0,\infty)$ with the boundary condition $u(0,t) = A$ for $t > 0$, $A \in (0,1)$. For each problem we assume that the initial function u_0 is Lipschitz continuous and such that $0 \le u_0 \le 1$ and $-1 \le u_0' \le 1$ a.e. in the corresponding domain. For the precise assumptions we refer to section 3.

In section 4, we show that solutions of the three problems satisfy a contraction property in L^1. It then follows immediately that each problem has at most one solution.

Considering related uniformly parabolic problems and using the monotony of the function ϕ , we prove that there exists a solution of each problem (for the Cauchy-Dirichlet problem under some extra assumptions on the data). This is done in section 5.

In section 6 we study the large time behaviour. In the case of the Neumann problem, we show with the help of a suitably chosen Lyapunov functional that the solution converges to a constant as $t \to \infty$. For the Cauchy problem, we give conditions on the initial function under which the solution converges to a similarity solution as $t \to \infty$ in the case that $D(u) = u(1-u)$. Finally, we show by means of a method based on the comparison principle that the solution of the Cauchy-Dirichlet problem converges to the unique stationary solution as $t \to \infty$.

Other doubly degenerate problems, with differential equations of the form

$$(1.2) \qquad u_t = (\psi((\Phi(u))_x))_x$$

have been considered by several authors : in the case of the Cauchy problem, Kalashnikov [15,16] gives a method for studying the existence of a solution and proves some properties related to the support of u. Bamberger [4] constructs a solution of the Dirichlet problem and remarks that it has at most one solution such that $u_t \in L^1$. For the study of the semi-group solution we refer to Benilan & Crandall [6] and Cortazar [7].

Atkinson & Bouillet [2,3] study similarity solutions for the Cauchy-Dirichlet problem and give a comparison principle for solutions satisfying $u_t \in L^1$. We remark that the methods used in this paper to study the large time behaviour of solutions could also be applied to solutions of the differential equation (1.2).

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2. The physical derivation of the problem.

Consider the two-dimensional flow of fresh and salt groundwater in a homogeneous coastal aquifer, which is vertically confined (with height H) and horizontally extended. The fresh and salt groudwater have a different specific weight, γ_f and γ_s respectively. In addition to external factors, the difference in specific weight induces a flow and thus a movement of both fluids.

It is common practice in hydrology to assume that the fluids are separated by a sharp *interface*, e.g. see Bear [5]. Adopting this assumption, it is then sufficient to know the evolution of this fresh-salt interface in order to determine the movement of the fluids.

In this section, a derivation of a differential equation is given which describes the fresh-salt interface as a function of position and time. The analysis is based on the work of de Josselin de Jong [14], further references are given there.

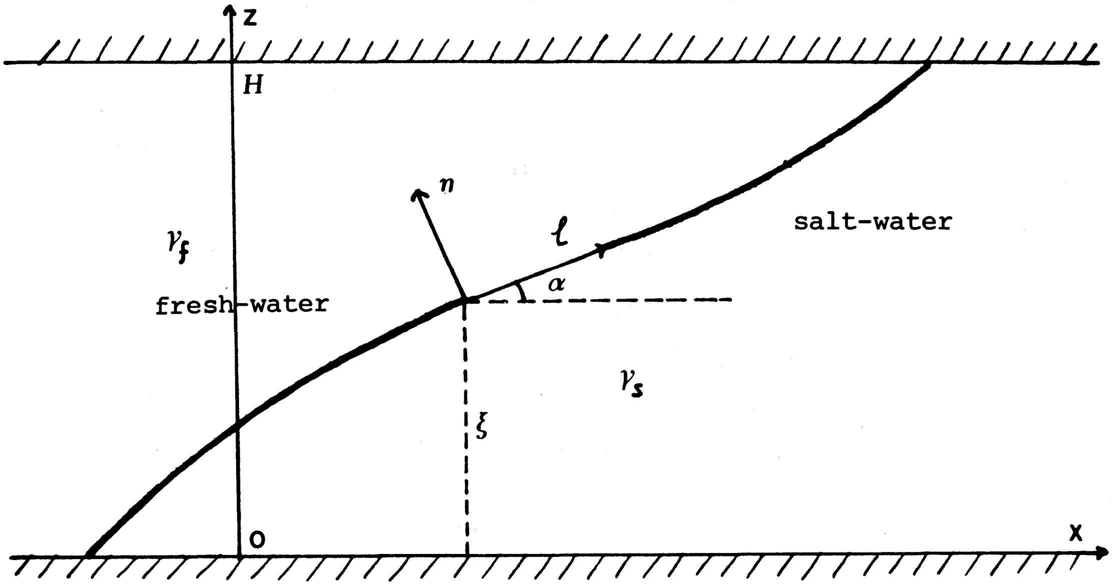

Fig.1. The distribution of fresh and salt water in an aquifer.

Let the flow take place in the xz-plane. The height of the interface is denoted by $\xi(x,t)$: when $0 \leq z < \xi(x,t)$ only salt water is present, when $\xi(x,t) < z \leq H$ only fresh water is present. Here t denotes the time.

Further, let α denote the angle of the tangent at the interface with the horizontal and let n ands ℓ denote the local orthogonal coordinates, normal and tangential to the interface (see fig.1). In both fluids, the specific discharge $\underline{q}$, the pressure p and specific weight γ are related through Darcy's law as :

$$\frac{\mu}{k} q_i + \text{grad } p_i + \gamma_i \underline{e}_z = 0 \text{ ,} \quad i = f,s, \tag{2.1}$$

where μ is the dynamic viscosity of the fluids (which is assumed here to be the same for both fluids), k is the intrinsic permeability of the porous medium and $\underline{e}_z$ is the unit vector in the positive z-direction.

If the fluids are incompressible, the following *continuity* condition is required at the interface :

$$q_{f_n} - q_{s_n} = 0 \quad \text{at } z = \xi. \tag{2.2}$$

At the interface, the fluids must also be in *equilibrium*. This means that the pressure on either side of the interface must be equal : $p_f - p_s = 0$ along the interface. This implies that

$$\frac{\partial p_f}{\partial \ell} - \frac{\partial p_s}{\partial \ell} = 0 \quad \text{at } z = \xi. \tag{2.3}$$

Equations (2.2) and (2.3), written out in x and z coordinates become

$$(q_{f_x} - q_{s_x}) \sin\alpha - (q_{f_z} - q_{s_z}) \cos\alpha = 0 \tag{2.4}$$

and

$$\left(\frac{\partial p_f}{\partial x} - \frac{\partial p_s}{\partial x} \right) \cos\alpha - \left(\frac{\partial p_f}{\partial z} - \frac{\partial p_s}{\partial z} \right) \sin\alpha = 0 \quad . \tag{2.5}$$

Substituting Darcy's law in (2.5) yields

$$(q_{f_x} - q_{s_x}) \frac{\mu}{k} \cos\alpha + (q_{f_z} - q_{s_z}) \frac{\mu}{k} \sin\alpha = \sin\alpha \, (\gamma_s - \gamma_f) \quad . \tag{2.6}$$

Then from (2.4) and (2.6), the unknown $(q_{f_x} - q_{s_x})$ can be solved:

(2.7) $$q_{f_x} - q_{s_x} = \Gamma \frac{\tan\alpha}{1+\tan^2\alpha} \quad \text{at } z = \xi,$$

where $\Gamma = \frac{k}{\mu}(\gamma_s - \gamma_f)$. Here (2.7) represents the x-component of the discontinuity which occurs in the tangential component of $q_f - q_s$ at the interface : this is the *shear flow* observed by de Josselin de Jong.

The total fresh water discharge through the aquifer in the positive x-direction is given by

(2.8) $$Q_{f_x}(x,t) = \int_{\xi(x,t)}^{H} q_{f_x}(x,z,t)dz \quad .$$

The corresponding expression for the saltwater is

$$Q_{s_x}(x,t) = \int_{0}^{\xi(x,t)} q_{s_x}(x,z,t)dz.$$

When the aquifer is confined in the sense that $q_{f_z} = q_{s_z} = 0$ when $z = 0$ or $z = H$ then the following continuity equations hold:

(2.9) $$\frac{\partial Q_{f_x}}{\partial x} = n\frac{\partial \xi}{\partial t}$$

and

$$\frac{\partial Q_{s_x}}{\partial x} = -n\frac{\partial \xi}{\partial t}$$

where n denotes the porosity of the medium. Consequently, the total discharge

(2.10) $$Q = Q_{f_x} + Q_{s_x} \quad ,$$

does not depend on x : it is considered here as a given constant.

Next a simplification is being made which is related to the *Dupuit-approximation* in hydrology : it is assumed here that the horizontal components of the specific discharges q_{f_x} and q_{s_x} are constant over the height of the aquifer. Thus

$$q_{f_x}(x,z,t) = q_{f_x}(x,\xi,t) \qquad \text{for } \xi \leq z \leq H \tag{2.11}$$

and

$$q_{s_x}(x,z,t) = q_{s_x}(x,\xi,t) \qquad \text{for } 0 \leq z \leq \xi \quad . \tag{2.12}$$

Strictly speaking, this simplification is only valid when the interface is rather flat : thus for large angles α we expect this model to break down.

The total discharge can now be written as

$$Q = Q_{f_x} + Q_{s_x} = q_{f_x}(x,\xi,t)(H-\xi) + q_{s_x}(x,\xi,t)\xi \quad . \tag{2.13}$$

From equations (2.7) and (2.13) the unknowns $q_{f_x}(x,\xi,t)$ and $q_{s_x}(x,\xi,t)$ can be solved : for q_{f_x} one finds,

$$q_{f_x}(x,\xi,t) = Q + \Gamma\xi \frac{\tan\alpha}{1+\tan^2\alpha} \quad . \tag{2.14}$$

Finally, expression (2.14) is substituted into equation (2.8) and the result into equation (2.9). This gives the partial differential equation

$$n\xi_t = \{(H-\xi)Q + \Gamma(H-\xi)\xi \frac{\xi_x}{1+\xi_x^2}\}_x \quad , \tag{2.15}$$

where the subscripts t and x denote differentiation with respect to these variables and $\tan\alpha = \xi_x$ is used.
Setting $n = H = \Gamma = 1$, $Q = -\lambda$ and $\xi(x,t) = u(x,t)$, (2.15) becomes

$$u_t = \left(u(1-u)\frac{u_x}{1+u_x^2}\right)_x + \lambda u_x \quad . \tag{2.16}$$

Observe that in the case of the Cauchy problem, the term λu_x can be eliminated. We set $\bar{x} = x + \lambda t$ and $u(x,t) = \bar{u}(\bar{x},t)$. Then since $u_t = +\lambda\bar{u}_x + \bar{u}_t$, we have that

$$\bar{u}_t = \left(\bar{u}(1-\bar{u})\ \frac{\bar{u}_x}{1+\bar{u}_x^2}\right)_x \quad .$$

In this paper, we study the following problems : the Cauchy problem

$$C\begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}\ , \end{cases}$$

the Neumann problem which is interesting in its own right and which is useful for understanding Problem C

$$N\begin{cases} u_t = \left(u(1-u)\ \dfrac{u_x}{1+u_x^2}\right)_x & \text{for } (x,t) \in (-1,1) \times \mathbb{R}^+ \\ \\ u(1-u)\ \dfrac{u_x}{1+u_x^2} = 0 & \text{for } (x,t) \in \{-1,1\} \times \mathbb{R}^+ \\ \\ u(x,0) = u_0(x) & x \in (-1,1)\ , \end{cases}$$

and the Cauchy Dirichlet problem

$$\text{CD}\quad\begin{cases} u_t = \left(u(1-u)\dfrac{u_x}{1+u_x^2}\right)_x + \lambda u_x & \text{for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,t) = A & t \in \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^+ \quad , \end{cases}$$

with $\lambda \geq 0$ and $A \in (0,1)$.

Instead of studying these three problems with this specific differential equation, we consider the more general case where the nonlinear term has been replaced by $(D(u)\phi(u_x))_x$, where D and ϕ are given real functions such that D defined on the interval $[0,1]$ and ϕ on $[-1,1]$ satisfy the hypotheses H_D and H_ϕ.

3. Definitions.

Let us first give a definition of a solution of the three problems and state for each of these problems the precise hypotheses on the initial function u_0.

The Neumann problem

$$N \begin{cases} u_t = (D(u)\phi(u_x))_x & (x,t) \in (-1,1) \times \mathbb{R}^+ \\ D(u)\ \phi(u_x) = 0 & (x,t) \in \{-1,1\} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in (-1,1) \quad , \end{cases}$$

where u_0 satisfies the hypothesis

H_{0N} : $u_0 \in W^{1,\infty}(-1,1)$, $0 \le u_0 \le 1$, $-1 \le u_0' \le 1$ a.e.

Definition 3.1. We say that u is a weak solution of Problem N if it satisfies for every $T > 0$

(i) $u \in L^\infty(0,T;W^{1,\infty}(-1,1)), u_t \in L^2(Q_{NT})$ with $Q_{NT} := (-1,1)\times (0,T)$;

(ii) $0 \le u \le 1$, $-1 \le u_x \le 1$ a.e. in Q_{NT} ;

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint\limits_{Q_{NT}} \{ u_t\ \psi + D(u)\phi(u_x)\ \psi_x\} = 0$ for all $\psi \in L^2(0,T;H^1(-1,1))$.

The Cauchy problem

$$C \begin{cases} u_t = (D(u)\phi(u_x))_x & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R} \quad , \end{cases}$$

where u_0 satisfies

H_{0C} : $u_0 \in W^{1,\infty}(\mathbb{R})$, $0 \le u_0 \le 1$, $-1 \le u_0' \le 1$ a.e. and $u_0 - H \in L^1\ (\mathbb{R})$
where H denotes the Heaviside function: $H(x) = 1$ when $x>0$ and $H(x) = 0$ when $x\le 0$.

Definition 3.2. We say that u is a weak solution of Problem C if it satisfies for every $T > 0$

(i) $u \in L^{\infty}(0,T; W^{1,\infty}(\mathbb{R}))$, $u_t \in L^2((-R,R) \times (0,T))$ for all $R > 0$;

(ii) $0 \leq u \leq 1$, $-1 \leq u_x \leq 1$ a.e. in Q_{CT} where $Q_{CT} := \mathbb{R} \times (0,T)$;

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint_{Q_{CT}} \{u_t \psi + D(u)\phi(u_x)\,\psi_x\} = 0$ for all $\psi \in L^2(0,T\,;\,H^1(\mathbb{R}))$

such that ψ vanishes for large $|x|$.

The Cauchy-Dirichlet problem

$$\text{CD}\begin{cases} u_t = (D(u)\phi(u_x))_x + \lambda u_x & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,t) = A & t \in \mathbb{R}^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^+ \,, \end{cases}$$

where the constants λ and A are such that $\lambda \geq 0$ and $0 < A < 1$ and where u_0 satisfies the hypothesis

H_{0D} : $u_0 \in W^{1,\infty}(0,\infty) \cap L^1(0,\infty)$, $0 \leq u_0 \leq 1$, $-1 \leq u_0' \leq 1$ a.e., $u_0(0) = A$.

Definition 3.3. We say that u is a weak solution of Problem CD if it satisfies for every $T > 0$

(i) $u - A \in L^{\infty}(0,T\,;\,V)$ where $V := \{v \in W^{1,\infty}(\mathbb{R}^+), v(0) = 0\}$ and $u_t \in L^2((0,R) \times (0,T))$ for all $R > 0$;

(ii) $0 \leq u \leq 1$, $-1 \leq u_x \leq 1$ a.e. in Q_{DT} where $Q_{DT} := \mathbb{R}^+ \times (0,T)$;

(iii) $u(.,0) = u_0(.)$;

(iv) $\iint_{Q_{DT}} \{u_t \psi + (D(u)\phi(u_x) + \lambda u)\,\psi_x\} = 0$ for all $\psi \in L^2(0,T; H_0^1(0,\infty))$

such that ψ vanishes for large x.

We remark that if u is a solution of any of the three problems then $u(t) \in W^{1,\infty}(\Omega)$ for all $t > 0$, where Ω denotes either $(-1,1)$ or $\mathbb{R}$ or $\mathbb{R}^+$. This is a consequence of a result given by Temam [19, Lemma 1.4 p.263].

4. Contraction property and uniqueness of the solution.

In this section we show how solutions of each problem satisfy a contraction property in L^1. The uniqueness of the weak solution follows immediately.

Lemma 4.1. Let u be a solution of any of the three problems. Then $D(u)\phi(u_x)(t) \in C(\overline{\Omega})$ for a.e. $t>0$ where Ω denotes either $(-1,1)$ or $\mathbb{R}$ or $\mathbb{R}^+$.

Proof. We prove Lemma 4.1 in the case of Problem N. By Definition 3.1 $u_t \in L^2(-1,1)$ and $u_t = (D(u)\phi(u_x))_x$ for a.e. $t > 0$. Thus

$$(D(u)\phi(u_x))_x \in L^2(-1,1) \qquad \text{for a.e. } t > 0$$

and consequently

$$D(u)\phi(u_x) \in C([-1,1]) \qquad \text{for a.e. } t > 0 . \qquad \square$$

Remark 4.2. Let t be such that $D(u)\phi(u_x)(t) \in C(\overline{\Omega})$. Then $u_x(t)$ is continuous as a function of x in every point x such that $u(x,t) \in (0,1)$.

Theorem 4.3. Let u and v be solutions of Problem N with initial conditions u_0 and v_0 respectively. Then

$$\|u(t) - v(t)\|_{L^1(-1,1)} \le \|u_0-v_0\|_{L^1(-1,1)} \qquad \text{for all } t > 0 .$$

Proof. Let W denote either u or v. By Definition 3.1 W satisfies for a.e. $t > 0$

$$(4.1) \quad \begin{cases} W_t \in L^2(-1,1) \quad \text{and} \quad W_t = (D(W)\phi(W_x))_x \\ \text{and} \\ D(W)\phi(W_x)(\pm 1,t) = 0 . \end{cases}$$

Multiplying by sgn(u-v) the difference of the equations for u and v yields

$$(4.2) \quad \int_{-1}^{1} (u-v)_t \,\text{sgn}(u-v) = \int_{-1}^{1} \{D(u)\phi(u_x) - D(v)\phi(v_x)\}_x \,\text{sgn}(u-v)$$

$$\text{for a.e. } t > 0$$

where sgn s = -1 if $s < 0$, 0 if $s = 0$ and 1 if $s > 0$.

Next we use the following lemma, given for instance by Crandall & Pierre [8].

Lemma 4.4. Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. If $w \in W^{1,1}(0,T;L^1(-1,1))$, then $G(w) \in W^{1,1}(0,T;L^1(-1,1))$ and $\frac{d}{dt} G(w) = G'(w) \frac{dw}{dt}$ a.e.

It follow from Lemma 4.4 that

$$(u-v)_t \operatorname{sgn}(u-v) = |u-v|_t \quad \text{a.e.}$$

so that (4.2) implies that

$$\frac{d}{dt} \|u-v\|_{L^1(-1,1)} = \int_{-1}^{1} \{D(u)\phi(u_x) - D(v)\phi(v_x)\}_x \operatorname{sgn}(u-v) \tag{4.3}$$

$$\text{for a.e. } t > 0.$$

We show below that the right hand side of (4.3) is nonpositive for a.e. $t > 0$. This corresponds to the accretivity in $L^1(-1,1)$ of the operator $Au = -(D(u)\phi(u'))'$ when defined on a suitable domain.

Let t be such that (4.1) holds ; since u(t) and v(t) are Lipschitz continuous the open interval $(-1,1) \setminus \{x | u(x,t) - v(x,t) = 0\}$ is the union of open intervals where either $u-v > 0$ or $u-v < 0$. Since the proofs for both kinds of intervals are similar, we only consider the intervals where $u-v > 0$. In order to simplify the notations, we omit the variable t in what follows.

(i) if $[a,b] \subset (-1,1)$ is such that $u-v > 0$ on (a,b) and $u = v$ in a and in b then

$$\begin{aligned} &\int_a^b (D(u)\phi(u_x) - D(v)\phi(v_x))_x \operatorname{sgn}(u-v) \\ &= \{D(u)(\phi(u_x) - \phi(v_x))\}(b) - \{D(u)(\phi(u_x) - \phi(v_x))\}(a) . \end{aligned} \tag{4.4}$$

Then if $u(b) = 0$ or 1 the first term on the right-hand side of (4.4) is equal to zero and if $0 < u(b) < 1$ it follows from Remark 4.2 that $u_x(b)$ and $v_x(b)$ are well defined; then $u_x(b) \leq v_x(b)$ and this term is nonpositive; similarly one can see that the second term on the right-hand-side of (4.4) is also nonpositive.

(ii) if $(-1,c] \subset (-1,1)$ is such that $u - v > 0$ in $(-1,c)$ and $u(c) = v(c)$ then, in view of the boundary condition,

$$\int_{-1}^{c} \left(D(u)\phi(u_x) - D(v)\phi(v_x)\right)_x \operatorname{sgn}(u - v) = \left\{D(u)(\phi(u_x) - \phi(v_x))\right\}(c)$$

which, similarly as in the case (i), is nonpositive.

Finally, one finds that

$$\frac{d}{dt}\|u - v\|_{L^1(-1,1)} \leq 0 \qquad \text{for a.e. } t > 0.$$

and thus

$$\|u(t) - v(t)\|_{L^1(-1,1)} \leq \|u_0 - v_0\|_{L^1(-1,1)} \qquad \text{for all } t > 0. \qquad \square$$

Corollary 4.5. The solution of Problem N is unique.

In what follows, we prove similar properties for the problems C and CD.

Lemma 4.6. Let u be a solution of Problem C.

Then $u(t) - H \in L^1(\mathbb{R})$ for all $t>0$.

Proof. We show below that $\int_{-\infty}^{0} u(t)<\infty$. The proof that $\int_{0}^{+\infty} (1-u(t)) < \infty$ is similar. It follows from Definition 3.2 (iv) that u satisfies

$$\int_{\mathbb{R}} u(t)\psi = \int_{\mathbb{R}} u_0\psi + \int_0^t \int_{\mathbb{R}} \Big(D(u)\,\phi(u_x)\Big)_x \psi$$

for all $\psi \in H^1(\mathbb{R})$ with compact support and all $t > 0$. Let $R > 0$ be arbitrary. The characteristic function $\chi_{[-R,0]}$ of the interval $[-R,0]$ can be constructed as the limit in $L^2(\mathbb{R})$ of H^1 functions with compact support. Thus

$$\int_{\mathbb{R}} u(t)\chi_{[-R,0]} = \int_{\mathbb{R}} u_0\chi_{[-R,0]} + \int_0^t \int_{\mathbb{R}} \Big(D(u)\phi(u_x)\Big)_x \chi_{[-R,0]}$$

which implies that

$$\int_{\mathbb{R}} u(t)\chi_{[-R,0]} \le \int_{-\infty}^{0} u_0 + \int_0^t D(u)\phi(u_x)\Big]_{x=-R}^{x=0} .$$

Finally applying Lebesgue's monotone convergence theorem one finds

$$\int_{-\infty}^{0} u(t) \le \int_{-\infty}^{0} u_0 + Ct. \qquad \square$$

Corollary 4.7. Let u be a solution of Problem C. Then, for all $t > 0$, $u(x,t) \to 0$ as $x \to -\infty$, $u(x,t) \to 1$ as $x \to +\infty$.

Proof. Corollary 4.7 follows from Lemma 4.6 together with the fact that u(t) is Lipschitz continuous for all $t > 0$. □

We are now in a position to show that the solution u of Problem C defines a contraction in $L^1(\mathbb{R})$.

Theorem 4.8. Let u and v be solutions of Problem C with initial functions u_0 and v_0 respectively. Then

$$\|u(t) - v(t)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})} \quad \text{for all } t > 0.$$

Corollary 4.9. The solution of Problem C is unique.

Proof of Theorem 4.8. Let $R > 1$ be arbitrary. Then, for a.e. $t > 0$

$$\frac{d}{dt}\|u - v\|_{L^1(-R,R)} = \int_{-R}^{R} (D(u)\phi(u_x) - D(v)\phi(v_x))_x \operatorname{sgn}(u - v) .$$

Using the proof of Theorem 4.3 we deduce that for all t

$$\int_{-\infty}^{+\infty} |u(t) - v(t)|\chi_{[-R,R]} \le \int_0^t \Big\{(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(R,t) +$$

$$(4.5)\qquad + (|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(-R,t)\Big\}dt + \|u_0 - v_0\|_{L^1(\mathbb{R})} .$$

Let us denote by f_R the integrand in the first term at the right-hand-side of (4.5). Since by Corollary 4.7, f_R tends to zero as $R \to \infty$ for a.e. $\tau \in (0,t)$ and since $\|f_R\|_{L^1(0,t)} \le C$, it follows from the dominated convergence theorem that $\int_0^t f_R$ tends to zero as $R \to \infty$, which completes the proof. □

Similar results hold for the Cauchy-Dirichlet Problem CD, namely that

(i) $u(t) \in L^1(\mathbb{R}^+)$ for all $t > 0$;

(ii) $u(x,t) \to 0$ as $x \to +\infty$ for all $t > 0$;

(iii) If u and v are solutions of Problem CD with initial functions u_0 and v_0, then

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^+)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^+)} \quad \text{for all } t > 0 \ .$$

Since the proof of properties (i) - (iii) is very similar to the one given above, we omit it here.

5. Existence of solution.

In this section, we adapt a proof of Kalashnikov [17] in order to show that there exists a solution of each of the three problems (in the case of Problem CD under an extra assumption relating D,ϕ,λ and A). We first consider the Cauchy and the Neumann problem; we then show how one can modify the proof in order to obtain the existence result for the Cauchy Dirichlet problem as well.

We consider the following problems, with $n \in \mathbb{N}$ large enough

$$P_n \begin{cases} u_t = \left(D_n(u)\phi_n(u_x)\right)_x & \text{in } Q_{nT} := (-n,n) \times (0,T) \\ u_x(-n,t) = 0 \quad u_x(n,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) & \text{for } x \in (-n,n) \quad , \end{cases}$$

where

$$D_n \in C^2(\mathbb{R}) \text{ is such that } D_n(s) = D(s) \text{ for } s \in \left[\frac{1}{n}\ ,\ 1-\frac{1}{n}\right]$$

$$\text{and } \frac{1}{2} \inf_{[\frac{1}{n},1-\frac{1}{n}]} D \le D_n \le \sup_{[0,1]} D \quad \text{on } \mathbb{R} \quad ,$$

and

$$\phi_n \in C^2(\mathbb{R}) \text{ is such that } \phi_n(s) = \phi(s) \text{ for } s \in [-1+\frac{1}{n},\ 1-\frac{1}{n}]$$

$$\text{and } \frac{1}{2} \inf_{[-1+\frac{1}{n},1-\frac{1}{n}]} \phi' \le \phi_n' \le \sup_{[-1,1]} \phi' \quad ,$$

and where

H_{0n} : $u_{0n} \in C^\infty(\mathbb{R})$, $\frac{1}{n} \le u_{0n} \le 1 - \frac{1}{n}$, $|u_{0n}'| \le 1 - \frac{1}{n}$, u_{0n}' $(x) = 0$ for $|x| \ge n$, u_{0n} converges uniformly to u_0 on all compact subsets of $\mathbb{R}$ as $n \to \infty$.

We show in the appendix that given u_0 satisfying the hypothesis H_{0C}, we can construct a sequence of functions $\{u_{0n}\}$ satisfying H_{0n}. First we prove a comparison principle.

Lemma 5.1. Let u_1 and $u_2 \in C^{2,1}(\bar{Q}_{nT})$ be two solutions of Problem P_n with corresponding initial functions $u_{01} \le u_{02}$. Then $u_1(t) \le u_2(t)$ for every $t \ge 0$.

Proof. The function $z := u_1 - u_2$ satisfies the problem

$$\begin{cases} z_t = \left(D_n(u_1)A_n(x,t)z_x\right)_x + \left(B_n(x,t)\phi_n(u_{2x})z\right)_x & \text{in } Q_{nT} \\ z_x(-n,t) = 0 \qquad z_x(n,t) = 0 & \text{for } t \in (0,T] \\ z(x,0) = u_{01}(x) - u_{02}(x) & \text{for } x \in (-n,n) \end{cases}$$

where

$$A_n(x,t) = \int_0^1 \phi_n'(\Theta u_{1x}(x,t) + (1 - \Theta)\, u_{2x}(x,t))d\Theta$$

and

$$B_n(x,t) = \int_0^1 D_n'(\Theta u_1(x,t) + (1 - \Theta)\, u_2(x,t))d\Theta \quad .$$

Since $z(0) \le 0$, it is a consequence from the standard maximum principle that $z(t) \le 0$ for all $t \in (0,T]$. □

Lemma 5.2. Problem P_n has a unique classical solution $u_n \in C^{2+\alpha}(\bar{Q}_{nT})$ for each $\alpha \in (0,1)$. Furthermore we have that $\frac{1}{n} \le u_n \le 1 - \frac{1}{n}$ and $-1 + \frac{1}{n} \le u_{nx} \le 1 - \frac{1}{n}$ in $\bar{Q}_{nT}$.

Proof. The existence and uniqueness of the solution of Problem P_n follows from [18, Theorem 7.4 p.491 and a remark at the end of Section 7 p.492]. Also we remark that both $\frac{1}{n}$ and $1 - \frac{1}{n}$ satisfy problem P_n which, by Lemma 5.1 implies that $\frac{1}{n} \le u_n \le 1 - \frac{1}{n}$. Next we show that $|u_{nx}| \le 1 - \frac{1}{n}$. We set $w = u_{nx}$. Using the linear theory (see for instance in [12, p.72]) we deduce that $w \in C^{2+\alpha}(Q_{nT})$. Thus $w \in C(\bar{Q}_{nT}) \cap C^{2,1}(Q_{nT})$. Differentiating the differential equation in P_n yields

$$w_t = D(u_n)\phi_n'(w)w_{xx} + \Big(D(u_n)\phi_n''(w)w_x + 2D'(u_n)w\phi_n'(w)$$

$$+ D'(u_n)\phi_n(w)\Big)w_x + D''(u_n)\phi_n(w)w^2 \qquad \text{in } Q_{nT}$$

$$w(-n,t) = 0 \quad w(n,t) = 0 \qquad \text{for } t \in (0,T]$$

$$w(x,0) = u_{0n}'(x) \qquad \text{for } x \in (-n,n) \ .$$

In order to simplify the notation, we rewrite the equation above as

$$w_t = a(x,t)w_{xx} + b(x,t)w_x - c(x,t)w$$

where a,b,c are continuous functions and $a > 0$, $c \geq 0$.
The function $w - 1 + \frac{1}{n}$ satisfies

$$a(x,t)(w-1+\tfrac{1}{n})_{xx} + b(x,t)(w-1+\tfrac{1}{n})_x - c(x,t)(w-1+\tfrac{1}{n}) - (w-1+\tfrac{1}{n})_t \geq 0$$

$$\text{in } Q_{nT}$$

$$w(-n,t) - 1 + \frac{1}{n} \leq 0 \qquad w(n,t) - 1 + \frac{1}{n} \leq 0 \qquad \text{for } t \in (0,T]$$

$$w(x,0) - 1 + \frac{1}{n} \leq 0 \qquad \text{for } x \in (-n,n) \ .$$

Thus by the maximum principle $w - 1 + \frac{1}{n} \leq 0$, that is $u_{nx} \leq 1 - \frac{1}{n}$.
The bound $u_{nx} \geq -1 + \frac{1}{n}$ follows in the same way. □

Lemma 5.3. $\int_0^T \int_{-R}^{R} u_{nt}^2 \leq C(R,T)$ for all $R \leq n-2$

where the constant C(R,T) does not depend on n.

Proof. Let $m,n \in \mathbb{N}$ such that $0 < m < n$. Set

$$\zeta_m(x) = \begin{cases} 1 & x \leq m-1 \\ m-x & m-1 \leq x \leq m \\ 0 & x \geq m \\ \zeta_m(-x) & x \leq 0 \ . \end{cases}$$

We multiply the differential equation for u_n by $\zeta_m{}^2\, u_{nt}$ to obtain the equality

$$\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 = \iint_{Q_{nT}} \Big(D(u_n)\phi(u_{nx}) \Big)_x\, u_{nt}\, \zeta_m^2$$

that is, after integration by parts

$$\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 = - \iint_{Q_{nT}} D(u_n)\phi(u_{nx})\, u_{nx\,t}\, \zeta_m^2 - 2 \iint_{Q_{nT}} D(u_n)\phi(u_{nx}) u_{nt}\, \zeta_m\, \zeta'_m \ .$$

Thus

$$\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 = - \iint_{Q_{nT}} D(u_n)\, \frac{\partial}{\partial t}\, F(u_{nx})\zeta_m^2 - 2 \iint_{Q_{nT}} D(u_n)\phi(u_{nx}) u_{nt}\, \zeta_m\, \zeta'_m$$

where the positive function F is defined by $F(s) = \int_0^s \phi(\tau)d\tau$.
Thus

$$\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 + \int_{-n}^{n} D(u_n(T))F(u_{nx}(T))\zeta_m^2$$

$$\leq \int_{-n}^{n} D(u_{0n})F(u'_{0n})\zeta_m^2 + \Bigg\{ 2 \sup_{[0,1]} D \sup_{[-1,1]} \phi \Bigg(\iint_{Q_{nT}} \zeta'^2_m \Bigg)^{\frac{1}{2}} + \sup_{[0,1]} D' \sup_{[-1,1]} F \Bigg(\iint_{Q_{nT}} \zeta_m^2 \Bigg)^{\frac{1}{2}} \Bigg\} \Bigg(\iint_{Q_{nT}} u_{nt}^2 \zeta_m^2 \Bigg)^{\frac{1}{2}}$$

which we rewrite as

$$\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 \leq C_1 m + C_2 \sqrt{mT} \Bigg(\iint_{Q_{nT}} u_{nt}^2\, \zeta_m^2 \Bigg)^{\frac{1}{2}} \ .$$

Finally, we find that

$$\int_0^T \int_{-m+1}^{m-1} u_{nt}^2 \leq C(m,T)$$

which concludes the proof of Lemma 5.3. □

Theorem 5.4. There exists a solution of Problem C.

Proof. Fix $R > 0$. Since $1/n \leq u_n \leq 1 - 1/n$ and since $|u_{nx}| \leq 1 - 1/n$, it follows from Gilding [13] that

$$|u_n(x,t') - u_n(x,t)| \leq C|t - t'|^{\frac{1}{2}}$$

for all $n > R$ and for all (x,t), $(x,t') \in \bar{Q}_{RT} := [-R,R] \times [0,T]$. Here the constant C depends on R but does not depend on n. The set $\{u_n\}_{n> R}$ is bounded and equicontinuous in Q_{RT}. Thus there exists a continuous function u_R and a subsequence $\{u_{nk}\}$, with $n_k > R$ such that u_{nk} converges uniformly to u_R in $\bar{Q}_{RT}$ as $n_k \to \infty$. Then by a diagonal process there exists a function $u \in C(\bar{Q}_{CT})$ and a converging subsequence $\{u_j\}$such that u_j converges to u as $j \to \infty$, pointwise on $\bar{Q}_{CT}$ and uniformly on all bounded subsets of $\bar{Q}_{CT}$. Also it follows from the estimates above that $u_{jx} \rightharpoonup u_x$ and $u_{jt} \rightharpoonup u_t$ weakly in $L^2(Q_{RT})$ for all $R > 0$ as $j \to \infty$. Thus u satisfies conditions (i),(ii), (iii) of Definition 3.2. In what follows we check that u also satisfies (iv). The function u_j satisfies for j sufficiently large:

$$(5.1) \qquad \iint\limits_{Q_{CT}} \left\{u_{jt}\psi + D(u_j)\phi(u_{jx})\ \psi_x\right\} = 0$$

for all $\psi \in V := \{v \in L^2(0,T\ ;\ H^1(\mathbb{R}))$ such that $v = 0$ for large $|x|\}$. Since $\| \phi(u_{jx})\|_{L^\infty(Q_{CT})} \leq \sup\limits_{[-1,1]} |\phi|$, there exists $\chi \in L^\infty(Q_{CT})$ and a subsequence of $\{u_j\}$, that we denote again by $\{u_j\}$such that

$$\phi(u_{jx}) \rightharpoonup \chi \text{ weakly in } L^2(Q_{RT}) \ ,$$

for any $R > 0$ as $j\to\infty$. Letting now $j\to\infty$ in (5.1) yields

$$(5.2) \qquad \iint\limits_{Q_{CT}} \{u_t\ \psi + D(u)\chi\ \psi_x\} = 0 \quad \text{for all } \psi \in V$$

ext we show that .

(5.3) $$\iint_{Q_{CT}} D(u)(\chi - \phi(u_x))\,\psi_x = 0 \text{ for all } \psi \in V \quad .$$

We first write an inequality which is based on the monotonicity of the function ϕ and which involves the functions u_j. This was also done by Kalashnikov [17], for example.

Let us extend ϕ on $\mathbb{R}$ such that $\phi(s) = \phi(1)$ for $s > 1$ and $\phi(s) = \phi(-1)$ for $s < -1$ and let v be such that $v - u \in V$. We have, with $m < n$, $Q_{mT} := (-m,m) \times (0,T)$ and $\delta \in (0,1]$

$$\iint_{Q_{CT}} D(u)(\phi(u_{jx}) - \phi(v_x))\left(\zeta_m^2(u_j - v)\right)_x = \iint_{Q_{mT}} \zeta_m^2 \; D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x)$$

$$+ 2 \iint_{Q_{mT}} \zeta_m \zeta_m' \; D(u)(\phi(u_{jx}) - \phi(v_x))(u_j - v)$$

$$\geq \iint_{Q_{mT}} \zeta_m^2 \; D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x)$$

$$- \delta \iint_{Q_{mT}} \zeta_m^2 \; D(u)(\phi(u_{jx}) - \phi(v_x))^2 - \frac{1}{\delta}\iint_{Q_{mT}} D(u)(u_j - v)^2$$

$$\geq (1 - \delta \sup_{[-1,1]} \phi') \iint_{Q_{mT}} \zeta_m^2 \; D(u)(\phi(u_{jx}) - \phi(v_x))(u_{jx} - v_x) - \frac{1}{\delta}\iint_{Q} D(u)(u_j - v)^2 .$$

Setting $\delta = 1/\sup_{[-1,1]} \phi'$ yields

(5.4) $$\iint_{Q_{mT}} D(u)\left(\phi(u_{jx}) - \phi(v_x)\right)\left(\zeta_m^2(u_j - v)\right)_x \geq -C \iint_{Q_{mT}} D(u)(u_j - v)^2 \quad .$$

Since u_n satisfies the differential equation in Problem P_n, we also have

$$(5.5)\qquad \iint_{Q_{mT}} D(u)\phi(u_{jx})(\zeta_m^2 u_j)_x$$

$$= \iint_{Q_{mT}} (D(u) - D(u_j))\phi(u_{jx})(\zeta_m^2 u_j)_x - \iint_{Q_{mT}} u_{jt}\, \zeta_m^2 u_j \quad .$$

Letting j → ∞ while keeping m fixed in (5.4) and (5.5) yields

$$- \iint_{Q_{mT}} u_t u\, \zeta_m^2 - \iint_{Q_{mT}} D(u)\, \chi\, (\zeta_m^2 v)_x$$

$$(5.6)\qquad - \iint_{Q_{mT}} D(u)\phi(v_x)((\zeta_m^2 u)_x - (\zeta_m^2 v)_x) \geq - C \iint_{Q_{mT}} D(u)(u - v)^2 \ .$$

Replacing ψ by $\zeta_m^2 u$ in (5.2) yields

$$(5.7)\qquad \iint_{Q_{mT}} (u_t(\zeta_m^2 u) + D(u)\, \chi\, (\zeta_m^2 u)_x) = 0 \quad .$$

Next we add (5.6) and (5.7) to obtain

$$\iint_{Q_{mT}} D(u)(\chi - \phi(v_x))((\zeta_m^2 u)_x - (\zeta_m^2 v)_x) \geq -C \iint_{Q_{mT}} D(u)(u - v)^2 \ .$$

We set $v = u - \mu\xi$ with $\mu > 0$, $\xi \in L^2(0,T\ ;\ H^1(\mathbb{R}))$ with $\xi(x) = 0$ for $|x| \geq m - 1$. Then

$$\iint_{Q_{CT}} D(u)(\chi-\phi(u_x - \mu\xi_x))\ \mu\xi_x \geq -\mu^2 C \iint_{Q_{DT}} D(u)\ \xi^2 \quad .$$

Dividing by μ and letting $\mu \downarrow 0$ yields

$$\iint_{Q_{CT}} D(u)(\chi - \phi(u_x))\, \xi_x \geq 0$$

which in turn implies (5.3). □

In section 4 we proved that $u(t)-H \in L^1(\mathbb{R})$. We show below some extra invariance properties of Problem C, namely that if $u_0' \in L^2(\mathbb{R}))$, $u_x(t) \in L^2(\mathbb{R})$ for all $t > 0$ and that $u_t \in L^2(Q_{CT})$. We suppose that the approximating functions u_{0n} are such that

$\limsup_{n\to\infty} \int_{-n}^{n} u_{0n}'^{\,2} \le \int_{\mathbb{R}} u_0'^{\,2}$ (it follows from the appendix that the construction of such functions is possible)

Lemma 5.5. Let $u_0' \in L^2(\mathbb{R})$. Then $\int_{\mathbb{R}} u_x^2(t) \le \int_{\mathbb{R}} u_0'^{\,2}$ for all $t > 0$.

Proof. We multiply the differential equation

$$u_{nxt} = \left(D(u_n)\phi(u_{nx})\right)_{xx}$$

by u_{nx} and integrate on $Q_{nt} := (-n,n) \times (0,t)$ to obtain

$$\iint_{Q_{nt}} u_{nxt}\, u_{nx} = \iint_{Q_{nt}} \left(D(u_n)\phi(u_{nx})\right)_{xx} u_{nx},$$

that is

$$\int_{-n}^{n} u_{nx}^2(t) - \int_{-n}^{n} u_{0n}'^{\,2} = -2 \iint_{Q_{nt}} \left(D(u_n)\phi(u_{nx})\right)_x u_{nxx}$$

(5.8)

$$= -2 \iint_{Q_{nt}} D'(u_n)\phi(u_{nx})u_{nx}\, u_{nxx} - 2 \iint_{Q_{nt}} D(u_n)\phi'(u_{nx})\, u_{nxx}^2 \quad .$$

Next we define the monotone function $\Phi(s) := \int_0^s \phi(\tau)\tau \, d\tau$. It follows from (5.8) that

$$\int_{-n}^{n} u_{nx}^2(t) - \int_{-n}^{n} u_{0n}'^2 \leq -2 \iint_{Q_{nt}} D'(u_n) \frac{\partial}{\partial x} \Phi(u_{nx})$$

$$\leq 2 \iint_{Q_{nt}} D''(u_n)\Phi(u_{nx})u_{nx} \leq 0.$$

Thus

$$\int_{-n}^{n} u_{nx}^2(t) \leq \int_{-n}^{n} u_{0n}'^2 .$$

Since the L^2-norm is w.l.s.c., this implies that

$$\int_{-R}^{R} u_x^2(t) \leq \int_{\mathbb{R}} u_0'^2 \text{ for all } R > 0$$

and finally that $\|u_x^2(t)\|_{L^2(\mathbb{R})} \leq \|u_0'\|_{L^2(\mathbb{R})}$. □

Lemma 5.6. Let $u_0' \in L^2(\mathbb{R})$. Then $u_t \in L^2(Q_{CT})$.

Proof. The proof is close to that of Lemma 5.3. Here one multiplies the equation for u_n by u_{nt}.

□

Remark 5.7. If one assumes that $u_0' \in L^2(\mathbb{R})$, it is not necessary to use the cut-off function ζ_m in the proof of Theorem 5.4.

Theorem 5.8. There exists a solution of Problem N.

Proof. The proof is very similar to that of Theorem 5.4. It uses the same auxiliary problem P_n, but now on the fixed domain Q_{NT}. □

We shall now study the existence of a solution of Problem CD. An essential problem here is to find a lower bound on $u_x(0,t)$, which is obtained by considering a suitable lower solution for the problem. Therefore, we first study the corresponding stationary problem

$$S_\lambda \begin{cases} \Big(D(y)\phi(y')\Big)' + \lambda y' = 0 & \text{in } \mathbb{R}^+ \\ y(0) = A \; ; \; y(\infty) = 0 \quad \text{if } \lambda > 0 & \end{cases}$$

Definition 5.9. A function y_λ is said to be a weak solution of Problem S_λ if it satisfies

(i) $y_\lambda \in H^1(0,R)$ for all $R > 0$

(ii) $0 \leq y_\lambda \leq 1$, $-1 \leq y_\lambda' \leq 1$ a.e. in $(0,\infty)$

(iii) $y_\lambda(0) = A$; $y_\lambda(\infty) = 0$ if $\lambda > 0$

(iv) $\int_{\mathbb{R}^+} (D(y_\lambda)\phi(y_\lambda') + \lambda y_\lambda)\psi' = 0$ for all $\psi \in H_0^1(\mathbb{R}^+)$ such that ψ vanishes for large x.

Remark 5.10. If y_λ is a weak solution of Problem S_λ, then y_λ satisfies the differential equation a.e.

Lemma 5.11. Let y_λ be a weak solution of Problem S_λ for $\lambda > 0$. On the set where y_λ is positive, it satisfies: (i) y_λ' is continuous, (ii) y_λ is strictly decreasing, (iii) y_λ is convex.

Proof. If y_λ is a weak solution of S_λ for $\lambda > 0$, then it satisfies

$$\begin{cases} D(y_\lambda)\phi(y_\lambda') + \lambda y_\lambda = 0 & \text{in } \mathbb{R}^+ \\ y_\lambda \;(0) = A \quad y_\lambda'(\infty) = 0 & . \end{cases}$$

Thus

$$\phi(y'_\lambda) = -\lambda\frac{y_\lambda}{D(y_\lambda)} \tag{5.9}$$

at points where $0 < y_\lambda < 1$, which implies that y'_λ is continuous and strictly negative in those points. Next, we show that y_λ is convex in a neighborhood of each point where it is positive. We define $d(x) = -\frac{x}{D(x)}$ on $(0,1)$. Since $d'(x) = -\frac{D(x) - xD'(x)}{D^2(x)}$, it follows from the concavity of D that d is nonincreasing on (0,1). Let $0 < x_1 < x_2$ be such that $y_\lambda(x_1), y_\lambda(x_2) \in (0,A)$. Then

$$y_\lambda(x_2) < y_\lambda(x_1)$$

and thus

$$\frac{-y_\lambda(x_2)}{D(y_\lambda(x_2))} \geq \frac{-y_\lambda(x_1)}{D(y_\lambda(x_1))}$$

which yields

$$\phi(y'_\lambda(x_2)) \geq \phi(y'_\lambda(x_1))$$

and finally

$$y'_\lambda(x_2) \geq y'_\lambda(x_1).$$

□

Lemma 5.12. Suppose $\lambda > 0$. Problem S_λ has a unique weak solution if and only if $\lambda \leq \lambda_{\max} := \frac{D(A)}{A}(-\phi(-1))$.

proof. It follows from (5.9) that S_λ has no solution if the condition $\lambda \le \frac{D(A)}{A}(-\phi(-1))$ is not satisfied. Next we supppose that this condition holds and construct a solution y_λ which will turn out to be the unique solution of Problem S_λ.
We deduce from Lemma 5.11 that if y_λ is a solution, there exists $L_\lambda \in (0,\infty]$ such that y_λ is positive and strictly decreasing on $(0,L_\lambda)$ and that $y_\lambda(L_\lambda) = 0$. In order to calculate L_λ and y_λ on the interval $(0,L_\lambda)$ we take as new unknown on that interval the inverse function $x := \sigma(y_\lambda)$. It comes to solve

$$\begin{cases} \lambda = -\dfrac{D(y_\lambda)}{y_\lambda}\phi\left(\dfrac{1}{\sigma'(y_\lambda)}\right) & \text{on } (0,A) \\ \sigma(A) = 0 \end{cases}$$

Thus

$$\sigma(y_\lambda) = -\int_{y_\lambda}^{A} \frac{ds}{\phi^{-1}(-\lambda s/D(s))} \tag{5.10}$$

and

$$L_\lambda = \sigma(0) = -\int_{0}^{A} \frac{ds}{\phi^{-1}(-\lambda s/D(s))} \tag{5.11}$$

so that L_λ ,which by (5.11) has a finite value, and the function y_λ on the interval $(0,L_\lambda)$ are uniquely determined. Note that $y_\lambda(x) = 0$ for all $x > L_\lambda$ since otherwise there would be a point $\tilde{x}$ such that $0 < y_\lambda(\tilde{x}) < 1$ and $y_\lambda'(\tilde{x}) > 0$ which contradicts (5.9). □

Corollary 5.13. Let $\tilde{\delta} \in [0,L_\lambda]$ be such that $y' > -1$ on $(\tilde{\delta},L_\lambda)$. Then $y_\lambda \in C^2((\tilde{\delta},L_\lambda))$.

Proof. Corollary 5.13 follows from the fact that the elliptic equation in Problem S_λ is non degenerate in $(\tilde{\delta},L_\lambda)$. □

We remark that if $\lambda < \lambda_{max}$, then $y_\lambda'(0) > -1$ and $y_\lambda \in C^2([0,L_\lambda))$.

Lemma 5.14. If $\lambda = 0$, the unique solution of Problem S_λ is $y_0 = A$.

Proof. Integrating the differential equation in S_λ yields

$$\begin{cases} D(y_0)\phi(y_0') = C & \text{in } \mathbb{R}^+ \\ y_0(0) = A & . \end{cases}$$

If $C = 0$, then $y_0 = A$. We claim that C must be equal to zero. Suppose not and let $C > 0$. Then $y_0 \in C^1([0,+\infty))$, $0 < y_0 < 1$ and $y_0' > 0$. Since y_0 is increasing and bounded from above, it tends to a constant as $x \to \infty$. Hence there exists a subsequence $\{x_n\}$ such that $y_0'(x_n) \to 0$ as $x_n \to \infty$. This is in contradiction with the differential equation. Similarly, one can show that C cannot be negative. □

Lemma 5.15. (i) y_λ is decreasing in λ .

(ii) Let $\lambda_n := \lambda_{max} - 1/n$; as $n \to \infty$, y_{λ_n} converges to $y_{\lambda_{max}}$ uniformly on compact subsets of $\overline{\mathbb{R}^+}$.

Proof. Property (i) follows from (5.10). As $n \to \infty$, $y_{\lambda n}$ decreases to a limit $\tilde{y}$, which satisfies Properties (i), (ii) and (iii) of Definition 5.9. In order to show that $\tilde{y} = y_{\lambda\cdot max}$, one also has to show that $\tilde{y}$ satisfies the integral relation (iv) in Definition 5.9. This is done in a similar way as in the proof of Theorem 5.4. □

We are now in a position to prove that there exists a solution of Problem CD, however with some extra assumptions on λ, A and u_0.

$$H_{CD}\begin{cases} 0 \le \lambda \le \lambda_{max} = \frac{D(A)}{A} \quad (-\phi(-1))\,;\; u_0 \le A \text{ on } \mathbb{R}^+\,; \\ \text{if } \lambda = 0,\ A \le \bar{s} := \sup\,\{s \in (0,1) \text{ such that } D'(s) = 0\}\,; \\ \text{if } \lambda > 0,\ u_0 \ge y_{\lambda_{max}} \quad \text{on } \mathbb{R}^+\,. \end{cases}$$

Next we consider the regularized problems, with $n \in \mathbb{N}$ large enough

$$CD_n\begin{cases} u_t = \left(D_n(u)\phi_n\,(u_x)\right)_x + \tilde{\lambda}_n u_x \text{ in } Q_T^n := (0,n) \times (0,T) \\ u(0,t) = A \quad u_x(n,t) = 0 \qquad \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) \qquad\qquad\quad \text{for } x \in (0,n) \end{cases}$$

where $\tilde{\lambda}_n := \min\,(\lambda,\lambda_n)$ and where u_0 satisfies

H_0^n : $u_{0n} \in C^\infty(\mathbb{R}^+)$, $u_{0n}(x) = A$ for x in a neighborhood of zero, $u_{0n} \le A$, $u'_{0n}(x) = 0$ for $x \ge n$, u_{0n} converges uniformly to u_0 on compact subsets of $\mathbb{R}^+$ as $n \to \infty$. If $\lambda > 0$, then $u_{0n} \ge \max\{\,A(1+y'_{\lambda n}(0)),\ y_{\lambda_n}\}$ and $|u'_{0n}| \le -y'_{\lambda_n}(0)$; if $\lambda = 0$, then $u_{0n} \ge 1/n$ and $|u'_{0n}| \le 1-1/n$.

We show in the appendix that, given an initial function u_0 which satisfies H_{0D} and H_{CD} , one can construct a sequence $\{u_{0n}\}$ which satisfies the above properties.

As in the case of Problem P_n, one can show that a comparison principle holds and that Problem CD_n has a unique solution u_n which is such that $A\,(1+y'_{\lambda_n}(0)) \le u_n \le A$ and $u_{nx} \le -y'_{\lambda_n}(0)$ if $\lambda > 0$ (resp. $1/n \le u_n \le A$ and $u_{nx} \le 1-1/n$ if $\lambda = 0$). In order to show that $u_{nx} \ge y'_{\lambda_n}(0)$ if $\lambda > 0$ (resp. $u_{nx} \ge -1 + \frac{1}{n}$ if $\lambda = 0$) a lower bound for $u_{nx}(0,.)$ is necessary.

Lemma 5.16. Suppose that $\lambda > 0$. Then $u_n \ge y_{\lambda_n}$ in $\overline{Q_T^n}$, which implies that $u_{nx}(0,t) \ge y'_{\lambda_n}(0)$ for all $t \in [0,T]$.

Proof. Assume that n is large enough so that $L_{\lambda_n} < n$. Since $u_n \geq 0$ in $\overline{Q^n_T}$, we have that $u_n \geq y_{\lambda_n}$ in $[L_{\lambda_n}, n] \times [0,T]$. Also, since y_{λ_n} satisfies

$$(D(y_{\lambda_n})\phi(y'_{\lambda_n}))' + \tilde{\lambda}_n y'_{\lambda_n} = (\tilde{\lambda}_n - \lambda_n)y'_{\lambda_n} \geq 0 \qquad \text{for } x \in (0,L_{\lambda_n})$$

$$y_{\lambda_n}(0) = u_n(0,t) = A \qquad y_{\lambda_n}(L_{\lambda_n}) = 0 \leq u_n(L_{\lambda_n},t) \quad \text{for } t \in (0,T]$$

$$y_{\lambda_n} \leq u_{0n} \qquad \text{for } x \in (0,L_{\lambda_n})$$

it follows from a comparison principle argument that $u_n \geq y_{\lambda_n}$ on $[0,L_{\lambda_n}] \times [0,T]$. □

Lemma 5.17. If $\lambda = 0$, $u_n \geq \underline{y} := \max\,(A-(1-1/n)x,\ 0)$ in $\overline{Q^n_T}$.

Proof. Again suppose that n is large enough so that $n > A/(1-1/n)$. Obviously $u_n \geq \underline{y}$ for $x \in [A/(1-1/n),n]$. It remains to show that

$$(D(\underline{y})\phi(\underline{y}'))' \geq 0 \text{ for } x < A/(1-1/n)$$

that is

$$D'(\underline{y})\phi(1/n-1)(1/n-1) \geq 0$$

which follows from the assumption that $A \leq \bar{s}$. □

Theorem 5.18. There exists a solution of Problem CD.

Proof. From Lemma 5.16 (resp. Lemma 5.17 if $\lambda = 0$), it follows that $u_{nx} \geq y'_{\lambda_n}(0)$ (resp $u_{nx} \geq 1/n-1$ if $\lambda = 0$). In particular, $|u_{nx}| < 1$ on Q^n_T, thus there exists a sequence $\{n_k\}$ and a function $u \in C(\bar{Q}_{DT})$ such that u_{n_k} tends to u as $n_k \to \infty$ uniformly on compact subsets of $\overline{Q_{DT}}$. It follows at once that u satisfies condition (i)-(iii) of the definition of a solution of Problem CD (Definition 3.3). The proof that u also satisfies the integral condition (iv) of Definition 3.3 is quite similar to that of Theorem 5.4. However it is convenient to consider the function $\bar{u} := u - y_\lambda$ as the new unknown function, in order to have a homogeneous boundary condition in the point $x = 0$. □

Note that the following results hold .

Theorem 5.19. Let u be the solution of any of the three problems. Then $u \in C(\bar{Q})$ where Q denotes either $(-1,1) \times \mathbb{R}^+$ or $\mathbb{R} \times \mathbb{R}^+$ or $\mathbb{R}^+ \times \mathbb{R}^+$.

Theorem 5.20. (Comparison principle).

(i) Let u_1 and u_2 be the solutions of any of the three problems with initial functions $u_{01} \le u_{02}$. Then $u_1(t) \le u_2(t)$ for all $t > 0$.

(ii) Let $\lambda \in [\lambda_2, \lambda_1] \subset [0, \lambda_{max}]$ and let u be the solution of Problem CD. Then if $y_{\lambda_1} \le u_0 \le y_{\lambda_2}$, $y_{\lambda_1} \le u(t) \le y_{\lambda_2}$ for all $t > 0$.

Finally we remark that the hypothesis $D'' \le 0$ is necessary to obtain the uniform bounds on u_{nx} in the proof of Lemma 5.2.

6. The large time behavior

6.1. The case of the Neumann problem. Convergence to a constant.

In this subsection, we show that the solution of Problem N converges to a constant as $t \to \infty$; we adapt a proof of Alikakos & Rostamian [1] and Dafermos [9] based on the use of a suitable Lyapunov functional . We first give a result in the case that u is bounded away from 0 and 1. We denote by $u(t,u_0)$ the solution of Problem N with initial function u_0.

Theorem 6.1. Let $\delta \le u_0 \le 1-\delta$ for some $\delta \in (0,1/2)$. Then there exist constants $K > 0$ and $\sigma = \sigma(\delta) > 0$ such that

$$\|u(t,u_0) - \frac{1}{2}\int_{-1}^{1} u_0\|_{L^\infty(-1,1)} \le Ke^{-\sigma t} \qquad t \ge 0.$$

Proof. We first consider the solution u_n of the problem N_n

$$N_n \begin{cases} u_t = \Big(D(u)\phi(u_x)\Big)_x & \text{in } Q_{NT} \\ u_x(-1,t) = 0 \quad u_x(1,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = u_{0n}(x) & \text{for } x \in (-1,1) \end{cases} ,$$

where u_{0n} satisfies

$u_{0n} \in C^\infty([-1,1])$, $\delta \le u_{0n} \le 1-\delta$, $|u'_{0n}| \le 1-1/n$,
$u'_{0n}(-1) = u'_{0n}(1) = 0$, u_{0n} converges to u_0 as $n \to \infty$
uniformly in $[-1,1]$.

Using the methods developped in section 5, one can show that the solution u_n of Problem N_n converges to the solution u of Problem N uniformly in $\overline{Q_{NT}}$.

Let $v_n = u_n - \frac{1}{2}\int_{-1}^{1} u_{0n}$. Then v_n satisfies the problem

$$v_t = \left(D(u_n)\phi(v_x)\right)_x \qquad \text{in } Q_{NT}$$

$$v_x(-1,t) = 0 \qquad v_x(1,t) = 0 \qquad \text{for } t \in (0,T]$$

$$v(x,0) = v_{0n}(x) := u_{0n}(x) - \frac{1}{2}\int_{-1}^{1} u_{0n} \qquad \text{for } x \in (-1,1) \quad ,$$

and is such that $\int_{-1}^{1} v_n(t) = 0$ for all $t \in [0,T]$. We multiply by v_n the equation for v_n and integrate by parts. This yields

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} v_n^2 = -\int_{-1}^{1} D(u_n)\phi(v_{nx})v_{nx} \quad .$$

Now since $\phi' \geq 0$ with $\phi'(0) > 0$, there exists $\mu > 0$ such that $|\phi(s)| \geq \mu|s|$ for $s \in [-1,1]$. Thus

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} v_n^2 \leq -\mu \inf_{[\delta,1-\delta]} D \int_{-1}^{1} v_{nx}^2 \leq -\frac{\mu}{4} \inf_{[\delta,1-\delta]} D \int_{-1}^{1} v_n^2$$

which implies that

$$\int_{-1}^{1} v_n^2 \leq \left(\int_{-1}^{1} v_{0n}^2\right) e^{-\frac{\mu}{2}\inf_{[\delta,1-\delta]} Dt} \quad .$$

Letting $n \to \infty$, we obtain

$$\text{(6.1)} \qquad \|v(t)\|_{L^2(-1,1)} \leq K_1 e^{-\sigma_1 t}$$

where $\sigma_1 = \frac{\mu}{4} \inf_{[\delta,1-\delta]} D$. Next, observe that since $v(t)$ is Lipschitz continuous with respect to the space variable, it satisfies the inequality

$$\frac{1}{2}\|v(t)\|^2_{L^\infty(-1,1)} \le \int_{-1}^{1}|v(t)| \le \sqrt{2}\,\|v(t)\|_{L^2(-1,1)}$$

which combined with (6.1) yields

$$\|v(t)\|_{L^\infty(-1,1)} \le 2^{3/4}\sqrt{K_1}e^{-\sigma_1 t/2} . \qquad \square$$

Theorem 6.2. Let $\delta \le u_0 \le 1$ for some $\delta > 0$. When $t \to \infty$, $u(t,u_0)$ converges to the constant $\frac{1}{2}\int_{-1}^{1} u_0$ uniformly on$[-1,1]$.

Proof. Since $\{u(t),\ t \ge 0\}$ is precompact in $C([-1,1])$, there exists a sequence $\{t_n\}$ and a function $q \in C([-1,1])$ such that

$$u(t_n) \to q \text{ as } t_n \to \infty \text{ uniformly on } [-1,1] \quad .$$

In particular $u(t_n)$ converges to q in $L^1(-1,1)$. Defining the ω-limit set of u_0 by

$$\omega(u_0) = \{w \in L^1(-1,1) \ : \text{ there exists a sequence } t_n \to \infty \text{ such that}$$
$$u(t_n) \to w \text{ in } L^1(-1,1) \text{ as } t_n \to \infty\}$$

we conclude that $\omega(u_0)$ is not empty. Define $V : L^1(-1,1) \to \overline{\mathbb{R}}$ by

$$V(v) = \begin{cases} -\operatorname*{ess\,inf}_{x\in(-1,1)} v(x) & \text{if } -\operatorname*{ess\,inf}_{x\in(-1,1)} v(x) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Since $u(t_0) \ge \operatorname*{ess\,inf}_{x\in(-1,1)} u(t_0)$, we have that, for $t \ge t_0$, $u(t) \ge \operatorname*{ess\,inf}_{x\ (-1,1)} u(t_0)$. This follows from the comparison principle and the fact that constants are solutions of Problem N. Thus

$$V(u(t)) \le V(u(t_0)) \quad \text{for all } t,t_0 \text{ such that } t \ge t_0$$

which shows that V is a Lyapunov functional for Problem N. Since this functional is lower semi-continuous in $L^1(-1,1)$ and since the orbits are Lyapunov stable (because u satisfies a contraction property in $L^1(-1,1)$) , it follows from DaFermos [9, Proposition 4.1] that V is constant on $\omega(u_0)$, say $V = -W$. Next, we show that for any $w_0 \in \omega(u_0)$

$$(6.2) \qquad w_0(x) = W \qquad \text{for all} \quad x \in (-1,1).$$

Since

$$\int_{-1}^{1} w_0 = \int_{-1}^{1} u_0 < 2$$

and since $w_0 \le 1$, it follows that $W < 1$. Now suppose that (6.2) is not true. Then for sufficiently small $\mu \in (0,1-W)$ the set

$$\Omega_\mu = \{x \in (-1,1) : w_0(x) \ge W + \mu\}$$

has a positive measure. Define

$$\underline{w}_0(x) = \begin{cases} w_0(x) & \text{if } w_0(x) < W + \mu \\ W + \mu & \text{if } w_0(x) \ge W + \mu \end{cases}$$

and let $\underline{w}$ and w be the solutions of Problem N with intial values $\underline{w}_0$ and w_0 respectively. Since $\delta \le \underline{w}_0 \le W + \mu < 1$, we have that $\delta \le \underline{w} \le W + \mu < 1$. Thus, by Theorem 6.1, $\underline{w}(t)$ converges to $\frac{1}{2}\int_{-1}^{1} \underline{w}_0$ as $t \to \infty$, uniformly on [-1,1].

Hence for given $\eta > 0$, there exists $T(\eta,\mu)$ such that

$$\underline{w}(t) \geq \frac{1}{2} \int_{-1}^{1} \underline{w}_0 - \eta \text{ for } t \geq T(\eta,\mu) .$$

Since $\underline{w}_0 \leq w_0$, the comparison principle implies that $\underline{w} \leq w$. Therefore for $t \geq T(\eta,\mu)$

$$w(t) \geq \frac{1}{2} \int_{-1}^{1} \underline{w}_0 - \eta = \frac{1}{2} \int_{(-1,1)\setminus\Omega_\mu} \underline{w}_0 + \frac{1}{2} \int_{\Omega_\mu} \underline{w}_0 - \eta.$$

Thus

$$w(t) \geq \frac{1}{2} (2 - |\Omega_\mu|)W + \frac{1}{2} |\Omega_\mu| (W + \mu) - \eta = W - \eta + \frac{|\Omega_\mu|}{2} \mu.$$

For fixed μ , we choose η sufficiently small so that

$w(t) \geq W + \nu$ for some $\nu > 0$.

Then

$V(w(t)) < -W$, for t sufficiently large,

which is a contradiction. □

Theorem 6.3. When $t \to \infty$, $u(t,u_0)$ converges to the constant $\frac{1}{2} \int_{-1}^{1} u_0$ uniformly on $[-1,1]$.

Proof. We now take as the Lyapunov functional

$$\bar{V}(v) = \begin{cases} \underset{x \in (-1,1)}{\text{ess sup}}\ v(x) & \text{if } \underset{x \in (-1,1)}{\text{ess sup}}\ v(x) < +\infty \\ +\infty & \text{otherwise} \end{cases} .$$

Then $\bar{V}$ is constant on $\omega(u_0)$, say $\bar{V} = \bar{W}$. The reasoning then follows as in the proof of Theorem 6.2. The auxiliary function is now defined as

$$\bar{w}_0(x) = \begin{cases} w_0(x) & \text{if } w_0(x) > \bar{W} - \bar{\mu} \\ \bar{W} - \bar{\mu} & \text{if } w_0(x) \le \bar{W} - \bar{\mu} \end{cases}$$

with $\bar{\mu} \in (0,\bar{W})$. Then $\bar{w}_0 > 0$ and by Theorem 6.2 the solution $\bar{w}$ of Problem N with initial function $\bar{w}_0$ converges to $\frac{1}{2} \int_{-1}^{1} \bar{w}_0$ as $t \to \infty$, which in turn implies the contradiction

$$\bar{V}(w(t)) < \bar{W}$$

for t sufficiently large. □

Corollary 6.4. There exists $t_0 > 0$, $K > 0$ and $\sigma = \sigma(t_0) > 0$ such that

$$\|u(t,u_0) - \frac{1}{2} \int_{-1}^{1} u_0\|_{L^\infty(-1,1)} \le Ke^{-\sigma t} \text{ for all } t \ge t_0.$$

Proof. Corollary 6.4 follows from the uniform convergence of u(t) to $\frac{1}{2} \int_{-1}^{1} u_0 \in (0,1)$ as $t \to \infty$. □

6.2. The Cauchy problem in the case that D(u) = u(1 - u). Convergence to similarity solutions.

In this section, we first construct a class of similarity solutions and then give a convergence theorem.

Following de Josselin de Jong [14] and van Duyn [10] , we look for a similarity solution of Problem C of the form

$$(6.3) \qquad u_s(x,t) = f(\eta) = \begin{cases} 0 & \text{if } \eta < \frac{1}{2} \\ \frac{1}{2} + \eta & \text{if } -\frac{1}{2} \le \eta \le \frac{1}{2} \\ 1 & \text{if } \eta > \frac{1}{2} \end{cases}$$

with $\eta = x/g(t)$ where the function g is still unknown and has to be determined. Substituting (6.3) in equation (1.1) with $D(s) = s(1-s)$, we formally deduce that g must satisfy the differential equation

$$(6.4) \qquad g'(t) = 2\phi\left(\frac{1}{g(t)}\right)$$

which we solve below together with the initial condition

$$g(0) = g_0 \ge 1 .$$

Note that $1/g_0$ corresponds to the slope of the initial value $u_s(x,0)$ for $x \in (-g_0/2, g_0/2)$.
We set

$$\Phi(\tau) = \int_1^\tau \frac{ds}{\phi(\frac{1}{s})} = \int_{1/\tau}^1 \frac{du}{\phi(u)u^2} \quad \text{for } \tau \ge 1.$$

Remark that since $\phi'(0) > 0$, $\Phi(+\infty) = +\infty$. Thus the function Φ , which is strictly increasing, maps $[1,\infty)$ on to $[0,\infty)$. Integrating the differential equation (6.4) yields

$$\Phi(g(t)) = 2t + \Phi(g_0)$$

and thus

$$g(t) = \Phi^{-1}(2t + \Phi(g_0)).$$

The function u_s is such that

$$\begin{aligned} u_s(x,t) &= 0 && \text{for } x \le S_f(t) \\ 0 < u_s(x,t) &< 1 && \text{for } S_f(t) < x < S_s(t) \\ u_s(x,t) &= 1 && \text{for } x \ge S_s(t) \end{aligned}$$

where

$$S_f(t) = -g(t)/2 \quad \text{and } S_s(t) = g(t)/2$$

and the velocity of the two fronts is given by

$$S_f'(t) = -\phi\left(\frac{1}{g(t)}\right) \text{ and } S_s'(t) = \phi\left(\frac{1}{g(t)}\right) .$$

It remains to show that u_s is a weak solution of Problem C. It is immediate that u_s satisfies properties (i) and (ii) of Definition 3.2. Since $\left(u_s(1-u_s)\phi(u_{sx})\right)_x \in L^2(Q_{CT})$ and since u_s satisfies equation (1.1) a.e., it easily follows that u_s satisfies the integral equation (iv) of Definition 3.2.

Next, we give a convergence theorem which extends a result of van Duyn [11] in the case that $\phi(s) = s$.

Theorem 6.5. Suppose that $D(u) = u(1-u)$. Let u_0 be such that $u_0(x) = 0$ for $x < x_1$ and $u_0(x) = 1$ for $x > x_2$ with $-\infty < x_1 < x_2 < +\infty$. Then there exists $C > 0$ such that

$$\| u(t,u_0) - f(./g(t)) \|_{L^\infty(\mathbb{R})} \le C/g(t) \qquad \text{for all } t \ge 0 .$$

Proof. We choose $g_0 = 1$; then $g(t) = \Phi^{-1}(2t)$. In view of the hypothesis on u_0, there exists $d > 0$ such that

$$f(x-d) \le u_0(x) \le f(x+d) \qquad \text{for } x \in \mathbb{R} .$$

Then by the comparison theorem

$$f((x-d)/g(t)) \le u(x,t) \le f((x+d)/g(t)) \quad \text{for all } (x,t) \in \bar{Q}_{CT},$$

which implies that

$$|u(x,t) - f(x/g(t))| \le |f((x+d)/g(t)) - f((x-d)/g(t))| \le 2d/g(t)$$

for all $x \in \mathbb{R}$ and $0 \le t \le T < \infty$. □

6.3 The Cauchy-Dirichlet problem : convergence to the stationary solution.

In what follows, we show that the solution of Problem CD stabilizes as $t \to \infty$. The idea of considering sets of the form $\mathbb{R}^+ x(t,t+\tau)$ was suggested to us by M.Bertsch.

Theorem 6.6.(i) If $\lambda > 0$ and if u_0 satisfies the hypothesis H_{CD} and is such that $u_0 \le y_{\bar{\lambda}}$ for some $\bar{\lambda} \in (0,\lambda]$, then the solution $u(t,u_0)$ of Problem CD converges to the stationary solution y_λ as $t \to \infty$, uniformly on $\overline{\mathbb{R}^+}$.

(ii) If $\lambda = 0$ and u_0 satisfies H_{CD}, $u(t,u_0)$ converges to A as $t \to \infty$, uniformly on compact subsets of $\overline{\mathbb{R}^+}$.

Proof. (i) It follows from the comparison theorem 5.20 that

$$y_{\lambda_{\max}} \le u(t,y_{\lambda_{\max}}) \le u(t,u_0) \le u(t,y_{\bar{\lambda}}) \le y_{\bar{\lambda}} \quad \text{for all } t \ge 0.$$

The proof will be completed if we show that $u(t,y_{\lambda_{\max}})$ and $u(t,y_{\bar{\lambda}})$ converge to the stationary solution y_λ as $t \to \infty$. Since both proofs are similar, we only show the convergence result for $u(t,y_{\lambda_{\max}})$.

Using again the comparison principle we deduce that

$$u(\tau, y_{\lambda_{\max}}) \leq u(t+\tau, y_{\lambda_{\max}}) \qquad \text{for all } t,\ \tau \geq 0$$

and thus that $u(.,y_{\lambda_{\max}})$ is nondecreasing. Since furthermore $u(.,y_{\lambda_{\max}}) \leq A$, there exists a function $q \in C^{0,1}(\overline{\mathbb{R}^+})$ such that

$$u(t, y_{\lambda_{\max}}) \to q \text{ as } t \to \infty$$

uniformly on compact subsets of $\overline{\mathbb{R}^+}$. It remains to show that $q = y_\lambda$. Obviously $q(x) = 0$ for large x and q satisfies properties (i),(ii),(iii) of Definition (5.9). Next, we show that q also satisfies the integral relation (iv). In order to have a homogeneous boundary condition in 0, it is convenient to make the change of functions

$$\bar{u} = u - y_\lambda \text{ and } \bar{q} = q - y_\lambda .$$

Then $\bar{u}$ satisfies the differential equation

$$\bar{u}_t = \left(D(\bar{u}+y_\lambda)\,\phi\,(\bar{u}_x + y_\lambda')\right)_x + \lambda(\bar{u}_x + y_\lambda') \tag{6.5}$$

for $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Set

$$w^{(t)}(x,s) = \bar{u}(x,s+t).$$

Then $w^{(t)}$ satisfies the differential equation (6.5) as well as the boundary condition $w^{(t)}(0,.) = 0$. Let $\tau > 0$ be given. Then

$$\int_{\mathbb{R}^+} w^{(t)}(\tau)\psi(\tau) - \int_{\mathbb{R}^+} w^{(t)}(0)\psi(0) = \tag{6.6}$$

$$= \int_0^\tau \int_{\mathbb{R}^+} \left\{ w^{(t)} \psi_t - (D(w^{(t)} + y_\lambda) \phi(w_x^{(t)} + y_\lambda') + \lambda(w^{(t)} + y_\lambda)) \psi_x \right\}$$

for all $\psi \in L^2(0,\tau;H_0^1(\mathbb{R}^+))$ such that $\psi_t \in L^2(\mathbb{R}^+ \times (0,\tau))$.
Note that

(i) $w^{(t)} \to \bar{q}$ as $t \to \infty$, uniformly in $\overline{\mathbb{R}^+}$

(ii) there exists a function $\bar{\chi} \in L^\infty((0,\infty) \times (0,\tau))$ and a sequence $\{t_n\}$ such that

$$\phi(w_x^{(t_n)} + y_\lambda') \rightharpoonup \bar{\chi} \text{ weakly in } L^2(0,\tau;\ L^2(0,\infty))$$

Letting $t \to \infty$ in (6.6) and setting $\psi = \psi(x)$ yields

$$(6.7) \qquad \int_0^\tau \int_{\mathbb{R}^+} (D(\bar{q} + y_\lambda)\ \bar{\chi} + \lambda(\bar{q} + y_\lambda))\psi' = 0 \quad \text{for all } \psi \in H_0^1(\mathbb{R}^+).$$

We show below that

$$(6.8) \qquad \int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\bar{\chi} - \phi(\bar{q}' + y_\lambda'))\psi' = 0 \text{ for all } \psi \in H_0^1(\mathbb{R}^+).$$

Let $v \in L^2(0,\tau;H_0^1(\mathbb{R}^+))$. Since ϕ is monotone, we have that

$$(6.9) \qquad \int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\phi(w_x^{(t_n)} + y_\lambda') - \phi(v_x + y_\lambda'))(w_x^{(t_n)} - v_x) \geq 0$$

and since $w^{(t_n)}$ satisfies (6.6), we also have, putting $\psi = w^{(t_n)}$

$$(6.10) \qquad \int_0^\tau \int_{\mathbb{R}^+} \{D(\bar{q} + y_\lambda)\ \phi(w_x^{(t_n)} + y_\lambda') + \lambda(w^{(t_n)} + y_\lambda)\} w_x^{(t_n)} =$$

$$= \int_0^\tau \int_{\mathbb{R}^+} \{D(\bar{q} + y_\lambda) - D(w^{(t_n)} + y_\lambda)\}\phi(w_x^{(t_n)} + y_\lambda')w_x^{(t_n)}$$

$$-\frac{1}{2}\int_{\mathbb{R}^+}(w^{(t_n)}(\tau))^2 + \frac{1}{2}\int_{\mathbb{R}^+}(w^{(t_n)}(0))^2 .$$

Letting t_n tend to infinity in (6.9) and (6.10) yields

$$(6.11) \qquad -\int_0^\tau \int_{\mathbb{R}^+} \{D(\bar{q} + y_\lambda)\ \bar{\chi}\ v_x + \lambda(\bar{q} + y_\lambda)\bar{q}'\}$$

$$-\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)\phi(v_x + y_\lambda')(\bar{q}' - v_x) \geq 0.$$

Replacing ψ by $\bar{q}$ in (6.7) gives

$$(6.12) \qquad \int_0^\tau \int_{\mathbb{R}^+} \{D(\bar{q} + y_\lambda)\ \bar{\chi}+\lambda(\bar{q} + y_\lambda)\}\bar{q}' = 0 .$$

Next we add (6.11) and (6.12) to obtain

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\bar{\chi} - \phi(v_x + y_\lambda'))(\bar{q}' - v_x) \geq 0 .$$

We set $v = \bar{q} - \mu\xi$ with $\mu > 0$ and $\xi \in H_0^1(\mathbb{R}^+)$.
Then

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\bar{\chi} - \phi(\bar{q}' + y_\lambda' - \mu\xi')\ \mu\xi' \geq 0$$

Dividing by μ and letting $\mu \downarrow 0$, we obtain

$$\int_0^\tau \int_{\mathbb{R}^+} D(\bar{q} + y_\lambda)(\bar{\chi} - \phi(\bar{q}' + y_\lambda')\xi' \geq 0 \text{ for all } \xi \in H_0^1(\mathbb{R}^+)$$

which in turn implies (6.8). Combining (6.7) and (6.8) we obtain

$$\int_0^\tau \int_{\mathbb{R}^+} (D(\bar{q} + y_\lambda)\phi(\bar{q}' + y_\lambda') + \lambda(\bar{q} + y_\lambda))\psi' = 0 \text{ for all } \psi \in H_0^1(\mathbb{R}^+)$$

from which we deduce that $q = \bar{q} + y_\lambda$ satisfies the integral relation (iv) of Definition 5.9. Thus $q = y_\lambda$ and $\bar{q} = 0$.

(ii) The proof of (ii) is quite similar to that of (i). However, since we do not suppose here that u has compact support, one has to use cut-off functions in several formulas. □

A P P E N D I X

We prove below two approximation lemmas

Lemma A1. Let u_0 satisfies H_{0C}. Then there exists $\{u_{0n}\}$ satisfying H_{0n}. If $u_0' \in L^2(\mathbb{R})$, $\{u_{0n}\}$ satisfies in addition

$$\limsup_{n\to\infty} \int_{-n}^{n} u_{0n}'^2 \leq \int_{\mathbb{R}} u_0'^2 .$$

Proof. Set

$$\tilde{u}_{0n} = (1 - 2/n)u_0 + 1/n$$

and

$$\hat{u}_{0n}(x) = \begin{cases} \max(\frac{1}{n}, \ (1-\frac{1}{n})(x+n/4) + \tilde{u}_{0n}(-n/4)) & \text{if } x \leq -n/4 \ , \\ \tilde{u}_{0n}(x) & \text{if } |x| \leq n/4 \ , \\ \min(1-\frac{1}{n}, (1-\frac{1}{n})(x-n/4) + \tilde{u}_{0n}(n/4)) & \text{if } x \geq n/4 \ . \end{cases}$$

Using the function

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \ , \\ C \exp(1/(|x|^2 - 1)) & \text{if } |x| < 1 \ , \end{cases}$$

where C is a constant such that $\int_{\mathbb{R}} \rho(x)dx = 1$, we define the sequence

$$u_{0n}(x) = n \int_{\mathbb{R}} \rho(n(x-y))\hat{u}_{0n}(y)dy .$$

Then one can check that if n is large enough, u_{0n} satisfies the hypothesis H_{0n}. Also if $u_0' \in L^2(\mathbb{R})$, then

$$\int_{-n}^{+n} u_{0n}'^2 \leq \int_{-n}^{n} \hat{u}_{0n}'^2 \leq \|u_0'\|^2_{L^2(\mathbb{R})} + C(n)$$

where $\lim_{n\to\infty} C(n) = 0$. □

Lemma A2. Let u_0 satisfy H_{0D} and H_{CD}. Then there exists $\{u_{0n}\}$ satisfying H_0^n.

Proof. (i) The case $\lambda = 0$. Set

$$\tilde{u}_{0n} = (1 - 1/(An))(u_0 - A) + A$$

and

$$\hat{u}_{0n}(x) = \begin{cases} A & \text{if } x \le 1/n \\ \max(-(1 - 1/n)(x - 1/n) + A,\ \tilde{u}_{0n}(x)) & \text{if } 1/n < x < n/4 \\ \max(-(1 - 1/n)(x - n/4) + \tilde{u}_{0n}(n/4),\ 1/n) & \text{if } x \ge n/4 \end{cases}$$

where we assume that n is large enough so that $\hat{u}_{0n} = \tilde{u}_{0n}$ in an interval of positive measure. Let

$$u_{0n}(x) = 2n \int_{\mathbb{R}} \rho\ (2n(x-y))\ \hat{u}_{0n}(y)dy \qquad \text{for } x \ge 0$$

Then one can check that u_{0n} satisfies the hypothesis H_0^n for n sufficiently large.

(ii) The case $\lambda > 0$. We first construct an approximation of $y_{\lambda_{max}}$. We set

$$\tilde{y}_n = \max(y_{\lambda_n}\ ,\ A(1 + y'_{\lambda_n}(0)))$$

and

$$\hat{y}_n(x) = \begin{cases} A & \text{if } x < 1/n \\ \tilde{y}_n(x - \frac{1}{n}) & \text{if } x \ge 1/n. \end{cases}$$

Let $\{\varepsilon_n\}$ be such that $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$ and let

$$y_n(x) = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \rho\left(\frac{x-z}{\varepsilon_n}\right) \hat{y}_n(z)dz \qquad \text{for } x \geq 0$$

Then one can check that for n large enough and for ε_n small enough y_n satisfies the hypothesis H_0^n in the case that $u_0 = y_{\lambda\max}$.

In the general case that $u_0 \geq y_{\lambda\max}$ on $\overline{\mathbb{R}^+}$, we set

$$\tilde{u}_{0n}(x) = \begin{cases} \max(y_{\lambda_n}(x),\ -y'_{\lambda_n}(0)(u_0(x) - A) + A) & \text{if } x \leq n/4 \\ \max(A(1 + y'_{\lambda_n}(0)),\ y'_{\lambda_n}(0)(x - n/4) + \tilde{u}_{0n}(n/4)) & \text{if } x > n/4 \end{cases}$$

and

$$\hat{u}_{0n}(x) = \begin{cases} A & \text{if } x < 1/n \\ \tilde{u}_{0n}(x - 1/n) & \text{if } x \geq 1/n \end{cases}$$

Again one can check that the function u_{0n}, defined by

$$u_{0n}(x) = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \rho\left(\frac{x-y}{\varepsilon_n}\right) \hat{u}_{0n}(y)dy \qquad \text{for } x \geq 0$$

which is such that $u_{0n} \geq y_{\lambda_n}$, satisfies the hypothesis H_0^n for n large enough and ε_n small enough.

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CHAPITRE VII

A DENSITY DEPENDENT DIFFUSION EQUATION IN POPULATION DYNAMICS : STABILIZATION TO EQUILIBRIUM

par

M. Bertsch et D. Hilhorst

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$. We consider the nonlinear evolution problem

$$(P) \begin{cases} u_t = \Delta\varphi(u) + \operatorname{div}(u \operatorname{grad} v) & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial}{\partial\nu}\varphi(u) + u\frac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Here ν denotes the outward normal at $x \in \partial\Omega$, the function φ is a smooth function such that $\varphi(0) = 0$, $\varphi'(s) > 0$ for $s > 0$ and $\varphi'(0) = 0$, the initial function $u_0 \in L^\infty(\Omega)$ is nonnegative and $v \in W^{1,\infty}(\Omega)$ is a given function (for the precise assumptions we refer to section 3).

In section 2, we show how Problem P arises in the theory of population dynamics in the case that $\varphi(s) = \frac{1}{2}s^2$ and interpret some of our results in terms of the geographical location of two biological populations.

This paper is divided into two main parts.
In part I we discuss the large time behaviour of the solution of Problem P. In part II we collect the basic results about Problem P: existence, uniqueness and regularity of the solution.

In part I we prove that the solution $u(t;u_0)$ of problem P stabilizes to equilibrium. Let E denote the set of equilibrium solutions; then there exists a function $q \in E$ such that

$$u(t;u_0) \to q \text{ in } C(\bar{\Omega}) \text{ as } t \to \infty$$

where q satisfies

$$\int_\Omega q dx = \int_\Omega u_0 dx.$$

In addition we give a characterization of E: we show that E coincides with the set

$$S = \{w \in C(\bar{\Omega}) : w \geq 0 \text{ in } \Omega, \text{ and for every } x \in \Omega \text{ either } w(x) = 0 \text{ or } \Phi(w)+v = \text{constant in a neighbourhood of } x\}. \qquad (1.1)$$

Here

$$\Phi\ (s) = \int_0^s \frac{\varphi'(\tau)}{\tau}\, d\tau, \qquad s \geq 0. \qquad (1.2)$$

The proof of these results is given in the sections 4 and 5. In section 4 we show that solutions of Problem P satisfy a contraction property in $L^1(\Omega)$. In section 5 we follow an idea of Osher and Ralston [18] and exploit this contraction property, combined with the structure of the set S, to construct a Lyapunov functional.
A remarkable detail of the proof is that we do not study the elliptic problem to prove that E = S. Also this fact follows from the contraction property and the structure of the set S.

In section 6 we extend the above results to the case when the natural boundary condition is replaced by a homogeneous Dirichlet condition.

In part II, we show that Problem P has a unique solution in some generalized sense. In section 7 we construct a solution $u(t;u_0)$ of Problem P as the limit of solutions of related uniformly parabolic problems. It turns out that the set $\{u(t;u_0); t \geq 1\}$ is precompact in $C(\bar{\Omega})$, thanks to a regularity result of DiBenedetto [7].

In order to show that the solution of Problem P is unique, we are led to use another sequence of regularized problems, following closely a method of Kalashnikov [12,13]. This is done in section 8.

Finally, in section 9, we give the corresponding results about the Dirichlet problem.

Studies concerning the existence and uniqueness of the solution of problems related to Problem P have also been done by Aronson, Crandall and Peletier [3], Diaz and Kersner [6], Gagneux [10], Madaune [17] and Touré [21].

There exists an extensive literature about the large time behaviour of solutions of degenerate parabolic equations. However there are not many articles where one constructs a Lyapunov functional in order to establish the stabilization to equilibrium. We have already mentioned the work of Osher and Ralston [18]. Such a method is also used by Aronson, Crandall and Peletier [3], Schatzmann [20] and Alikakos and Rostamian [1,2].

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2. BIOLOGICAL CONTEXT

Problem P arises in the theory of population dynamics. Consider a population in a finite habitat Ω which consists of two different groups, for instance age groups. Let $u(x,t)$ and $v(x,t)$ denote the density of these groups. In order to model their evolution with time, Gurtin and Pipkin [11] propose the following system of equations

$$\begin{cases} u_t = \operatorname{div}(u \operatorname{grad} (u+v)) & \text{in } \Omega \times \mathbb{R}^+ \\ v_t = k \operatorname{div}(v \operatorname{grad} (u+v)) & \text{in } \Omega \times \mathbb{R}^+, \end{cases}$$

where k is some positive constant. The flow of the populations is described by the dispersal velocities: grad(u+v) for the u-individuals and k grad (u+v) for the v-individuals. In particular, when the parameter k is small, the v-individuals disperse much slower than the u-individuals.

In this article, we study the problem in the limit $k = 0$. The second equation yields at once that v is constant in time; remains the equation in u which coincides with the differential equation in Problem P if we set $\varphi(s) = \frac{1}{2}s^2$. The boundary condition expresses the fact that no individuals can leave or enter the habitat.

An interesting consequence of our results is the following. It follows from (1.1) that, for any non-constant function $v(x)$, the set $E = S$ contains a non-trivial function $q(x)$ such that

$$q \equiv 0 \text{ in } \Omega_0 \subset \Omega \text{ , } q > 0 \text{ in } \Omega\backslash\Omega_0$$

for some nonempty subset Ω_0. If $u_0 \leq q$ in Ω, then $u(t;u_0) \leq q$ in Ω for all $t > 0$. In particular

$$u(t;u_0) \equiv 0 \text{ in } \Omega_0 \text{ for } t \geq 0.$$

From a biological point of view this phenomenon of *localization* is interesting: the v-individuals can stop the spread of the u-individuals.

A detailed analysis of this model in one space dimension was given in [4].

3. PRELIMINARIES

Let us first state the precise hypotheses on φ, u_0 and v and give a definition of a solution of Problem P.

H1. $\varphi \in C^3(\mathbb{R}^+) \cap C^1(\overline{\mathbb{R}^+})$, $\varphi(0) = \varphi'(0) = 0$, $\int_0^1 \tau^{-1}\varphi'(\tau)d\tau < \infty$, $\varphi'(s) > 0$ for $s > 0$, $\varphi''(s) \geq 0$ for $s \in (0,s_0)$ for some $s_0 > 0$.

H2a. $v \in W^{1,\infty}(\tilde{\Omega})$ for some smooth domain $\tilde{\Omega} \supset \overline{\Omega}$, and $\Delta v \geq -M$ in $\tilde{\Omega}$ in the sense of distributions for some $M > 0$.

H2b. If $N \geq 2$, $v \in W^{2,p}(\tilde{\Omega})$ for some $p > N$.

H2c. v has finitely many local strict minima.

H3. If $N = 1$ either $\varphi(s) = \frac{1}{2}s^2$ or $v'' \in L^1(\Omega)$.

H4. $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ a.e. in Ω.

We use the notations $Q_t = \Omega \times (0,t]$ for $t > 0$ and $Q = \Omega \times \mathbb{R}^+$.

DEFINITION 3.1. *We say that a function* $u : [0,\infty) \to L^1(\Omega)$ *is a generalized solution of Problem P if it satisfies:*

(i) $u \in C([0,t]; L^1(\Omega)) \cap L^\infty(Q_t)$ *for all* $t > 0$;

(ii) $\int_\Omega u(t)\psi(t) = \int u_0\psi(0) + \iint_{Q_t}\{\varphi(u)\Delta\psi + u\psi_t - u \text{ grad } v \text{ grad}\psi\}$

for all $t > 0$ *and all* $\psi \in C^{2,1}(\overline{Q})$ *such that* $\psi \geq 0$ *in* Q *and* $\frac{\partial\psi}{\partial\nu} = 0$ *on* $\partial\Omega \times \mathbb{R}^+$.

A generalized subsolution (resp. supersolution) of Problem P is defined by (i) *and* (ii) *with equality replaced by* $\leq$ *(resp.* $\geq$*).*

In the sequel we shall often omit the word generalized.

In part II, we shall prove the following results.
We suppose that the hypotheses H1-H2a-H4 are satisfied.

PROPOSITION 3.2. *There exists a unique solution of Problem P.*

PROPOSITION 3.3. *(Regularity). Let* u *be the solution of Problem P. Then* $u \in C(\overline{\Omega}\times(0,\infty))$ *and the set* $\{u(t);\ t \geq 1\}$ *is bounded and equicontinuous. Furthermore if* $u_0 \in C(\overline{\Omega})$, *then* $u \in C(\overline{\Omega}\times[0,\infty))$.

PROPOSITION 3.4. *(Comparison Principle). Let* $\underline{u}(t)$ *and* $\bar{u}(t)$ *be respectively a subsolution and a supersolution of Problem P with initial functions* $\underline{u}_0$ *and* $\bar{u}_0$ *such that* $\underline{u}_0 \leq \bar{u}_0$. *Then* $\underline{u}(t) \leq \bar{u}(t)$ *in* Ω *for* $t \geq 0$.

PART I

4. CONTRACTION IN $L^1(\Omega)$.

In this section we prove a contraction theorem which turns out to be our main tool when studying the asymptotic behaviour of $u(t)$ as $t \to \infty$.

THEOREM 4.1. *Let* $u_1(t)$ *and* $u_2(t)$ *be the solutions of Problem P with initial functions* u_{01} *and* u_{02} *respectively and suppose that the hypotheses* H1-H2a-H4 *are satisfied.*

(i) *Then*

$$||u_1(t)-u_2(t)||_{L^1(\Omega)} \leq ||u_{01}-u_{02}||_{L^1(\Omega)} \qquad \text{for any } t > 0.$$

(ii) *Let* v *satisfy in addition the hypotheses* H2b *and* H3. *If* u_{01} *and* $u_{02} \in C(\bar{\Omega})$ *and if there exists a connected subdomain* $U \subset \Omega$ *such that*

$$u_{01} > 0 \qquad u_{02} > 0 \text{ in } \bar{U} \tag{4.1}$$

and

$u_{01} - u_{02}$ *changes sign in* U,

then

$$||u_1(t)-u_2(t)||_{L^1(\Omega)} < ||u_{01}-u_{02}||_{L^1(\Omega)} \qquad \text{for any } t > 0.$$

REMARK 4.2. Condition (4.1) is necessary because the parabolic equation in Problem P is degenerate at points where $u = 0$.

Due to the degeneracy of the equation and the fact that v is not smooth, the proof of Theorem 4.1 is fairly technical. The idea of the proof is due to Osher and Ralston [18].

PROOF OF (i). In part II of this article we show that we can approximate $u_i (i = 1,2)$ by solutions of uniformly parabolic problems: let $u_{i\varepsilon} (\varepsilon > 0)$ be the classical solution of the problem

$$\begin{cases} u_t = \Delta\varphi_\varepsilon(u) + \operatorname{div}(u \operatorname{grad} v_\varepsilon) & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial}{\partial\nu}\varphi_\varepsilon(u) + u\dfrac{\partial v_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_{0i\varepsilon}(x) & \text{in } \Omega, \end{cases}$$

where φ_ε is a smooth function such that $\varphi'_\varepsilon(s) \geq c(\varepsilon) > 0$ for $s \geq 0$ and $\varphi_\varepsilon(s) \to \varphi(s)$ uniformly on compact subsets of $[0,\infty)$ and where v_ε and $u_{0i\varepsilon}$ are smooth functions such that $v_\varepsilon \to v$ in $H^1(\Omega)$ and $u_{0i\varepsilon} \to u_{0i}$ in $L^2(\Omega)$ as

$\varepsilon \downarrow 0$. In part II we show that $\{u_{i\varepsilon}\}$ is uniformly bounded and equicontinuous in compact subsets of $\bar{\Omega}\times(o,\infty)$. Using in addition the uniqueness of the solution $u_i (i = 1,2)$, we conclude that

$$u_{i\varepsilon}(t) \to u_i(t) \text{ in } C(\bar{\Omega}) \text{ as } \varepsilon \to 0 \text{ for } t > 0,\ i = 1,2. \tag{4.2}$$

We define

$$z_\varepsilon(x,t) = u_{1\varepsilon}(x,t) - u_{2\varepsilon}(x,t), \quad x \in \bar{\Omega},\ t \geq 0.$$

Then z_ε is the solution of the linear problem

$$(L_\varepsilon)\begin{cases} z_t = \Delta(a_\varepsilon z) + \operatorname{div}(z \text{ grad } v_\varepsilon) & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial}{\partial\nu}(a_\varepsilon z) + z\dfrac{\partial v_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ z(x,0) = z_{0\varepsilon}(x) \equiv u_{01\varepsilon}(x) - u_{02\varepsilon}(x) & \text{in } \Omega, \end{cases}$$

where

$$a_\varepsilon(x,t) = \int_0^1 \varphi'_\varepsilon(\Theta u_{1\varepsilon}(x,t) + (1-\Theta)u_{2\varepsilon}(x,t))d\Theta.$$

For smooth initial functions $z_{0\varepsilon}$ which satisfy the compatibility conditions at $\partial\Omega \times \{0\}$, the existence of a unique solution $z_\varepsilon \in C^{2,1}(\bar{Q})$ of Problem L_ε is proved in [16, p. 320 Th. 5.3]. Below we shall need an existence and uniqueness result if $z_{0\varepsilon}$ is merely continuous in $\bar{\Omega}$. To obtain this result we can proceed in the same way as we sketched above (and as we shall prove in section 7) for the more difficult nonlinear and degenerate Problem P: we approximate $z_{0\varepsilon}$ uniformly by smooth initial functions $z_{0\varepsilon n}$ $(n = 1,2....)$. Then the corresponding solutions $z_{\varepsilon n}$ are uniformly bounded and equicontinuous in $\bar{\Omega} \times [0,t]$ for $t > 0$ and $z_{\varepsilon n}$ converges uniformly to a generalized solution $z_\varepsilon \in C(\bar{\Omega}\times[0,t])$ of Problem L_ε as $n \to \infty$. By standard regularity results [16, 8], $z_\varepsilon \in C^{2,1}(\Omega \times (0,t])$. In addition these solutions satisfy the comparison principle; in particular they are uniquely determined by the initial function. The proof rests on the same testfunction argument which is used in section 8 for the nonlinear problem and which is extremely easy in this linear case.

For any initial function $z_0 \in C(\bar{\Omega})$, we denote the unique solution of Problem L_ε by $z_\varepsilon(t) = T_\varepsilon(t)z_0$. We set $a^+ = \max\{a,0\}$ and $a^- = \max\{-a,0\}$. Then for any $t > 0$

$$\|z_\varepsilon(t)\|_{L^1(\Omega)} - \|z_{0\varepsilon}\|_{L^1(\Omega)} = |\|T_\varepsilon(t)z^+_{0\varepsilon} - T_\varepsilon(t)z^-_{0\varepsilon}\|_{L^1(\Omega)} - \|z_{0\varepsilon}\|_{L^1(\Omega)} =$$

$$= \int_\Omega \{\max_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon} - \min_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon}\}\, dx - \|z_{0\varepsilon}\|_{L^1(\Omega)} =$$

$$= \int_\Omega \{\max_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon} + \min_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon}\}\, dx - \int_\Omega (z^+_{0\varepsilon} + z^-_{0\varepsilon})dx$$

$$-2\int_\Omega \min_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon}dx = \int_\Omega \{T_\varepsilon(t)z^+_{0\varepsilon} + T_\varepsilon(t)z^-_{0\varepsilon} - z^+_{0\varepsilon} - z^-_{0\varepsilon}\}\, dx$$

$$-2\int_\Omega \min_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon}dx = -2\int_\Omega \min_{+,-} T_\varepsilon(t)z^\pm_{0\varepsilon}\, dx,$$

since

$$\int_\Omega T_\varepsilon(t)z^\pm_{0\varepsilon}dx = \int_\Omega z^\pm_{0\varepsilon}\, dx.$$

It follows from the comparison principle that $T_\varepsilon(t)z^\pm_{0\varepsilon} \geq 0$. Thus for any $\varepsilon > 0$

$$\|u_{1\varepsilon}(t)-u_{2\varepsilon}(t)\|_{L^1(\Omega)} - \|u_{01\varepsilon}-u_{02\varepsilon}\|_{L^1(\Omega)} \leq 0. \tag{4.3}$$

Clearly Theorem 4.1 (i) follows from (4.2) and (4.3).

PROOF OF (ii). Let $u_{i\varepsilon}(i = 1,2)$ and z_ε be defined as above. Since $u_{0i} \in C(\bar{\Omega})$ we may assume that $u_{0i\varepsilon} = u_{0i}$. Let $\delta > 0$, and write

$$z_\varepsilon(t) = T^\delta_\varepsilon(t)z_\varepsilon(\delta) \quad \text{for } t \geq \delta.$$

Then, by the proof of (i),

$$\|z_\varepsilon(t)\|_{L^1(\Omega)} - \|z_\varepsilon(\delta)\|_{L^1(\Omega)} = -2\int_\Omega \min_{+,-} (T^\delta_\varepsilon(t)z^\pm_\varepsilon(\delta)),\ t \geq \delta,$$

and it is enough to prove that, for sufficiently small values of δ, there exists a $t_1 = t_1(\delta) > \delta$ such that

$$\int_\Omega \min_{+,-} (T^\delta_\varepsilon(t)z^\pm_\varepsilon(\delta)) \geq \eta(t,\delta) > 0 \quad \text{for } t \in (\delta,t_1)$$

for all small $\varepsilon > 0$.

Consider the Cauchy-Dirichlet problem

$$(\tilde{L}_\varepsilon^\delta)\begin{cases} z_t = \Delta(a_\varepsilon z) + \operatorname{div}(z \operatorname{grad} v_\varepsilon) & \text{in } \tilde{U} \times (\delta,\infty) \\ z = 0 & \text{on } \partial\tilde{U} \times (\delta,\infty) \\ z(\cdot,\delta) = z_\varepsilon^\pm(\delta) & \text{in } \tilde{U}, \end{cases}$$

where $\tilde{U} \subset U$ is such that $\operatorname{dist}(\tilde{U}, \delta U) > 0$ and $z_{0\varepsilon} = u_{01} - u_{02}$ changes sign in $\tilde{U}$. We denote the solution of Problem $\tilde{L}_\varepsilon^\delta$ by

$$\tilde{z}_\varepsilon^\pm(t) = \tilde{T}_\varepsilon^\delta(t) z_\varepsilon^\pm(\delta) \quad \text{in} \quad \tilde{U} \times (\delta,\infty).$$

Then, by the maximum principle,

$$\tilde{T}_\varepsilon^\delta(t) z_\varepsilon^\pm(\delta) \le T_\varepsilon^\delta(t) z_\varepsilon^\pm(\delta) \quad \text{in } \tilde{U} \times (\delta,\infty).$$

Thus, it is enough to prove that

$$\int_{\tilde{U}} \min_{+,-} (\tilde{T}_\varepsilon^\delta(t) z_\varepsilon^\pm(\delta)) \ge \eta\,(t,\delta) > 0 \quad \text{for } t \in (\delta, t_1) \tag{4.4}$$

for every $\varepsilon \in (0,\varepsilon_0)$ for some $\varepsilon_0 > 0$.

This will be done by means of the following lemma, which is an immediate consequence of Harnack's inequality [16, p.209-210].

LEMMA 4.3. *Let* $\varepsilon_0 > 0$ *and* $t_1 > \delta > 0$ *be constants, and let the following assumptions be satisfied for all* $\varepsilon \in (0,\varepsilon_0)$:

(a) $\quad ||\tilde{T}_\varepsilon^\delta(t)\ z_\varepsilon^\pm(\delta)||_{L^\infty(\tilde{U})} \ge \mu_0 > 0 \quad$ *for* $t \in [\delta, t_1]$,

(b) *When* $N = 1$, *then* $||a_\varepsilon||_{L^\infty(\delta,t_1;H^1(U))} \le C$ *and* $||v_\varepsilon||_{W^{1,\infty}(U)} \le C$,

(c) *When* $N \ge 2$, *then* $||a_\varepsilon||_{L^\infty(\delta,t_1;W^{1,\infty}(U))} \le C$ *and* $||v_\varepsilon||_{W^{2,p}(U)} \le C$,

for some constants $\mu_0 > 0$, $C > 0$ *and* $p > N$. *Then, for all* $\varepsilon \in (0,\varepsilon_0)$,

$$\tilde{T}_\varepsilon^\delta(t)\ z_\varepsilon^\pm(\delta) \ge \mu(x,t;\delta) > 0 \quad \text{in } \tilde{U} \times (\delta, t_1]$$

for some function μ *which does not depend on* ε.

Assuming that (a), (b) and (c) are satisfied for small $\delta > 0$, (4.4) follows. Thus, to complete the proof we need to verify these conditions.

In view of Proposition 3.3 and the assumption $u_{oi} \in C(\bar{\Omega})$, we have $u_i \in C(\bar{\Omega} \times [0,\infty))$. Hence there exists a $t_0 > 0$ such that $u_i > 0$ in $\bar{U} \times [0,t_0]$ and $z(t) = u_1(t) - u_2(t)$ changes sign in $\tilde{U}$ for $t \in [0,t_0]$.

Since $u_{i\varepsilon} \to u_i$ and $z_\varepsilon \to z$ in $C(\bar{\Omega} \times [0,t_0])$ there exist positive numbers μ_0, ν_0 and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$

$$u_{i\varepsilon} \geq \nu_0 \quad \text{in } \bar{U} \times [0,t_0] \quad , \quad i = 1,2, \tag{4.5}$$

and

$$||z_\varepsilon^{\pm}(t)||_{L^\infty(\tilde{U})} \geq 2\mu_0 \quad \text{for} \quad t \in [0,t_0]. \tag{4.6}$$

Let $\delta \in (0,t_0)$ be fixed.

When $N = 1$, Lemma 7.7 below, combined with (4.5), implies that $u_{i\varepsilon}$ is uniformly bounded in $L^\infty(\delta,t_0;H^1(U))$. Hence it follows from the definition of a_ε that condition (b) is satisfied for all $t_1 \in (\delta,t_0]$ (The hypothesis H_3 is necessary in the proof of Lemma 7.7).

When $N \geq 2$ we may assume that v_ε is uniformly bounded in $W^{2,p}(\Omega)$. It follows from (4.5) and [16, Th 3.1, p.437] that $u_{i\varepsilon}$ is uniformly bounded in $L^\infty(\delta,t_0,W^{1,\infty}(U))$. Thus condition (c) is satisfied for all $t_1 \in (\delta,t_0]$.

It remains to show that for some $t_1 \in (\delta,t_0]$ condition (a) is satisfied. In view of the conditions (b) and (c), which we proved to be satisfied for $t_1 \in (\delta,t_0]$ we deduce from [16, Th.7.1, p.181] that $\tilde{T}_\varepsilon^\delta(t)\ z_\varepsilon^{\pm}(\delta)$ is uniformly bounded in $\bar{\tilde{U}} \times [\delta,t_0]$. In addition, (4.5) and [16 , Th.1.1, p.419] imply that $u_{i\varepsilon}(\delta)$ is uniformly Hölder continuous in $\tilde{U}$. Finally it follows from [16, Th.10.1, p.204] that $T_\varepsilon^\delta(t)\ z_\varepsilon^{\pm}(\delta)$ is Hölder continuous in $\bar{\tilde{U}} \times [\delta,t_0]$, uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$. Hence, by (4.6), there exists a $t_1 \in (\delta,t_0]$ such that condition (a) of Lemma 4.3 is satisfied for all $\varepsilon \in (0,\varepsilon_0)$.

This completes the proof of Theorem 4.1.

5. STABILIZATION TO EQUILIBRIUM

In the present section we prove the main result of this paper, namely that u stabilizes to equilibrium as $t \to \infty$.

Let the set E be defined by

$$E = \{q \in C(\bar{\Omega}) \ : \ q \geq 0 \text{ and } \int_\Omega (\varphi(q)\Delta\eta - q \text{ grad } v \text{ grad } \eta) = 0$$
$$\text{for all } \eta \in C^2(\bar{\Omega}) \text{ with } \frac{\partial\eta}{\partial\nu} = 0 \text{ on } \partial\Omega\}.$$

It follows from the definition of a solution of Problem P (Definition 3.1) that

$$E = \{q \in C(\bar{\Omega}) : q \geq 0 \text{ and } u(t;q) = q \text{ for } t \geq 0\}. \tag{5.1}$$

Let S be defined by (1.1).

THEOREM 5.1. *If the hypotheses* H1-H2abc-H3-H4 *are satisfied. Then*
(i) E = S;
(ii) *There exists a function* $q \in E$ *such that*

$$u(t;u_0) \to q \textit{ in } C(\bar{\Omega}) \textit{ as } t \to \infty$$

where q *satisfies*

$$\int_\Omega q dx = \int_\Omega u_0 dx \tag{5.2}$$

REMARK 5.2. For some functions v and initial functions u_0, condition (5.2) characterizes q completely (see [4]).

The main tools in the proof of Theorem 5.1 are the contraction property which we proved in section 4, and the following Lemma about the structure of the set S.

LEMMA 5.3. *Let* $q \in C(\bar{\Omega})$ *be nonnegative. Then either* $q \in S$, *or there exists a function* $w \in S$ *such that* w-q *changes sign in a connected subdomain* $U \subset \Omega$ *such that* $w,q > 0$ *in* $\bar{U}$.

Thus S is a continuum in the space of nonnegative continuous functions on $\bar{\Omega}$.

PROOF OF LEMMA 5.3. Suppose that there is no $w \in S$ such that w-q changes sign in some connected subdomain $U \subset \Omega$ such that $w,q > 0$ in $\bar{U}$. We shall prove that $q \in S$.

If $q \equiv 0$ in Ω, then $q \in S$. So let $q(x_1) > 0$ for some $x_1 \in \Omega$. We set $C_1 = \Phi(q(x_1)) + v(x_1)$, where the function Φ is defined by (1.2). Let

$P_1 \subset \bar{\Omega}$ be the connected component of the set $\{x \in \bar{\Omega}: v(x) < C_1\}$ which contains x_1. We claim that

$$\Phi(q(x)) = C_1 - v(x) \text{ in } P_1. \tag{5.3}$$

Suppose that P_1 contains a point where $\Phi(q) < C_1 - v$. Then

$$\Phi(q(\tilde{x})) + v(\tilde{x}) = C_1 - \varepsilon_0 \text{ and } q(\tilde{x}) > 0$$

for some $\tilde{x} \in P_1$ and $\varepsilon_0 > 0$. Let $\tilde{P}_\varepsilon \subset P_1$ $(0 < \varepsilon < \varepsilon_0)$ be the connected component of the set $\{x \in \bar{\Omega}: v(x) < C_1 - \varepsilon\}$ which contains $\tilde{x}$. We fix $\varepsilon \in (0,\varepsilon_0)$ so small, that $\tilde{P}_\varepsilon$ contains x_1. Define w by

$$\Phi(w(x)) = \begin{cases} C_1 - \varepsilon - v(x) & \text{for } x \in \tilde{P}_\varepsilon \\ 0 & \text{for } x \in \bar{\Omega} \setminus \tilde{P}_\varepsilon. \end{cases}$$

Then $w \in S$. Let Γ be a curve in $\tilde{P}_\varepsilon$ which connects $\tilde{x}$ and x_1. Since $w > 0$ on Γ, and since, by construction, $w-q$ changes sign on Γ, there exists a connected closed subset $\Gamma_0 \subset \Gamma$ such that

$$w,q > 0 \text{ on } \Gamma_0 \text{ and } w-q \text{ changes sign on } \Gamma_0.$$

Hence there exists a neighbourhood $\bar{U}$ of Γ_0 in $\tilde{P}_\varepsilon$, where $w,q > 0$ and $w-q$ changes sign. Thus we have a contradiction and P_1 does not contain points where $\Phi(q) < C_1 - v$.

A similar, but easier proof yields that P_1 does not contain points where $\Phi(q) > C_1 - v$, and (5.3) follows.

If $\bar{P}_1 = \bar{\Omega}$ or if $q \equiv 0$ in $\bar{\Omega} \setminus P_1$, then $q \in S$. So suppose that $q(x_2) > 0$ in $\bar{\Omega} \setminus P_1$. Set $C_2 = \Phi(q(x_2)) + v(x_2)$ and let $P_2 \subset \bar{\Omega}$ be the connected component of the set $\{x \in \bar{\Omega}: C_2 - v(x) > 0\}$ which contains x_2. Then again we conclude that

$$\Phi(q(x)) = C_2 - v(x) \text{ in } P_2$$

and clearly $P_1 \cap P_2 = \emptyset$.

Continuing this process, we construct sets P_i, $i = 1,2,\ldots$ Since v has a local strict minimum in each connected P_i and since the number of local strict minima of v in Ω is finite, this process is finite. Thus $q \in S$.

PROOF OF THEOREM 5.1.(i). We first show that $S \subset E$. Let $w \in S$. Since v has a finite number of local strict minima, it follows from (1.1) that there exists a finite number of continuous functions $\Phi_i(x)$ $(i = 1,\ldots,i_0)$ with connected and mutually disjoint support such that

$$\Phi(w(x)) = \sum_{i=1}^{i_0} \Phi_i(x) \tag{5.4}$$

and

$$\Phi_i(x) = C_i - v(x) \text{ for } x \in \operatorname{supp} \Phi_i$$

for some constants C_i. Since $v \in W^{1,\infty}(\Omega)$ it follows from a standard result (see for instance [14, Th. A1, p. 50] that $\Phi(w(\cdot)) \in W^{1,\infty}(\Omega)$ and

$$\operatorname{grad} \Phi(w) = \begin{cases} -\operatorname{grad} v & \text{in } \{x\colon w(x) > 0\} \\ 0 & \text{elsewhere} \end{cases}$$

Next we show that $\varphi(w(\cdot)) \in W^{1,\infty}(\Omega)$ and that

$$\operatorname{grad} \varphi(w) = \begin{cases} -w \operatorname{grad} v & \text{in } \{x\colon w(x) > 0\} \\ 0 & \text{elsewhere.} \end{cases} \tag{5.5}$$

To do so we first prove that $\varphi(w(\cdot))$ is a Lipschitz continuous function. Then, by Rademacher's theorem [19], $\varphi(w(\cdot)) \in W^{1,\infty}(\Omega)$ as well. Let $||w||_{L^\infty(\Omega)} \le D$. Then, for $x_1 \neq x_2 \in \Omega$, we have that

$$\left|\varphi(w(x_1)) - \varphi(w(x_2))\right| = \left|\int_{w(x_1)}^{w(x_2)} \varphi'(s)ds\right| \leq D \left|\int_{w(x_1)}^{w(x_2)} \frac{\varphi'(s)}{s} ds\right|$$

$$= D \left|\Phi(w(x_1)) - \Phi(w(x_2))\right| \leq C \text{ dist } (x_1,x_2)$$

since $\Phi(w(\cdot))$ is Lipschitz. Thus $\varphi(w(\cdot))$ is Lipschitz. It follows from [14, Lemma A.4, p. 63] that grad $\varphi(w) = 0$ a.e. in $\{x: w(x) = 0\}$. Let $x \in \{x: w(x) > 0\}$ and let U be a neighbourhood of x where $w(x) \geq \delta > 0$. Then, by (5.4), $w \in W^{1,\infty}(U)$. Thus

$$\text{grad } \varphi(w) = \varphi'(w)\text{grad } w = w \text{ grad } \Phi(w) \text{ in } \{x: w(x) > 0\}$$

and (5.5) follows.
Let $\eta \in C^2(\bar{\Omega})$ with $\partial\eta/\partial\nu = 0$ on $\partial\Omega$. Then, by (5.5),

$$\int_\Omega (\varphi(w)\Delta\eta - w\text{gradv}\text{grad}\eta) = -\int_\Omega (\text{grad}\varphi(w) + w\text{gradv})\text{grad } \eta = 0.$$

Thus $w \in E$.

Next we show that $E \subset S$. Let $q \in E$ and suppose that $q \notin S$. Then, by Lemma 5.3, there exists a $w \in S$ such that $w-q$ changes sign in a connected subdomain $U \subset \Omega$ in which $w,q > 0$. Since $q \in E$ and $w \in S \subset E$, it follows from (5.1) and Th. 4.1(ii) that

$$||q-w||_{L^1(\Omega)} = ||u(t;q) - u(t;w)||_{L^1(\Omega)} < ||q-w||_{L^1(\Omega)}, \quad t > 0.$$

Thus we have obtained a contradiction and $q \in S$.

REMARK 5.4. When $N = 1$, Th. 5.1(i) follows at once by integrating the differential equation (see [4]).

PROOF OF THEOREM 5.1.(ii). We define the ω-limit set

$$\omega(u_0) = \{w \in L^1(\Omega): \text{there exists a sequence } t_n \to \infty \text{ such that } u(t_n) \to w \text{ in } L^1(\Omega) \text{ as } t \to \infty\}.$$

By Proposition 3.2, the set $\{u(t;u_0)\ ;\ t \geq 1\}$ is precompact in $C(\bar{\Omega})$ (and hence in $L^1(\Omega)$). Thus $\omega(u_0)$ is nonempty and $\omega(u_0) \subset C(\bar{\Omega})$.

Let $q \in \omega(u_0)$. We show first that q satisfies (5.2), then that $q \in E$, and finally that $\omega(u_0) = \{q\}$.

Setting $\psi(x,t) \equiv 1$ in Definition 3.1, we find that

$$\int_\Omega u(t;u_0) = \int_\Omega u_0 \qquad \text{for all } t > 0$$

and (5.2) follows.

In order to show that $q \in E$, we argue by contradiction: suppose that $q \notin E$. Then, by Theorem 5.1(i), $q \notin S$. Thus, by Lemma (5.3), there exists a function $w \in S$ such that q-w changes sign in a connected subdomain $U \subset \Omega$ in which $q,w > 0$. We use w to define the functional $V: L^1(\Omega) \to [0,\infty)$:

$$V(u) = \|u-w\|_{L^1(\Omega)}, \quad u \in L^1(\Omega).$$

Since $w \in E$, it follows from Theorem 4.1(i) that the solution u(t) of Problem P satisfies

$$V(u(t_1)) \leq V(u(t_2)) \text{ for all } t_1 \geq t_2 \geq 0.$$

Thus V is a Lyapunov functional for Problem P. Since $u \in C([0,\infty) : L^1(\Omega))$ and V is continuous, it follows from [5, Prop. 2.1 and 2.2] that $u(t;q) \in \omega(u_0)$ and that V is constant on $\omega(u_0)$. Hence

$$V(u(t;q)) = V(q) \text{ for all } t \geq 0. \tag{5.6}$$

On the other hand, since q and $w \in C(\bar{\Omega})$, it follows from the choice of w and Theorem 4.1(ii) that

$$V(u(t;q)) < V(q) \quad \text{for all } t > 0$$

which contradicts (5.6). Thus $q \in E$.

Finally we show that $\omega(u_0) \equiv \{q\}$.

Suppose that $\tilde{q} \in \omega(u_0)$ and that $u(t_n;u_0) \to q$ as $t_n \to \infty$ and $u(s_n;u_0) \to \tilde{q}$ as $s_n \to \infty$ where the sequences $\{t_n\}$ and $\{s_n\}$ are chosen such that $s_n < t_n$ for all $n \geq 1$. Then, using Theorem 4.1(i), we find that

$$||q-\tilde{q}||_{L^1(\Omega)} = \lim_{n\to\infty} ||u(t_n;u_0) - \tilde{q}||_{L^1(\Omega)} \leq$$

$$\leq \lim_{n\to\infty} ||u(s_n;u_0) - \tilde{q}||_{L^1(\Omega)} = 0.$$

Thus $\tilde{q} = q$, which completes the proof of Theorem 5.1.

6. THE DIRICHLET PROBLEM

In this section we show how the results about Problem P can be extended to the case of homogeneous Dirichlet boundary conditions. We consider the problem

$$(P_D) \begin{cases} u = \Delta\varphi(u) + \mathrm{div}(u\mathrm{grad}v) & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

We define a (generalized) solution $u(t;u_0)$ of Problem P_D in a similar way as for Problem P, taking testfunctions $\psi \in C^{2,1}(\bar{Q})$ such that $\psi = 0$ on $\partial\Omega \times \mathbb{R}^+$. The Propositions 3.1, 3.2 and 3.3 as well as Theorem 4.1 remain valid in the case of Problem P_D. In particular

$$u(t;u_0) \in C(\bar{\Omega}) \text{ and } u(t;u_0) = 0 \text{ on } \partial\Omega \text{ for } t > 0. \tag{6.1}$$

Let the set of steady-state solutions, E_D, be defined by

$$E_D = \{q \in C(\bar{\Omega}) : q \geq 0 \text{ and } \int_\Omega (\varphi(q)\Delta\eta - q\mathrm{grad}v\mathrm{grad}\eta) = 0$$

$$\text{for all } \eta \in C^2(\bar{\Omega}) \text{ such that } \eta = 0 \text{ on } \partial\Omega\}.$$

LEMMA 6.1. *Let* E *be defined as in section* 5. *Then*

$E_D \subset E$.

PROOF. Let $q \in E_D$. Then

$$u(t;q) = q \text{ for } t > 0. \qquad (6.2)$$

Let $\tilde{u}(t;q)$ denote the solution of Problem P with initial function q. Since, by (6.1) and (6.2),

$$\tilde{u}(t;q) \geq u(t;q) = q = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+,$$

$\tilde{u}(t;q)$ is a supersolution of Problem P_D. Hence, by (6.2),

$$\tilde{u}(t;q) \geq q \quad \text{in } \Omega \times \mathbb{R}^+. \qquad (6.3)$$

On the other hand we have that

$$\int_\Omega \tilde{u}(t;q)\,dx = \int_\Omega q\,dx, \quad t \geq 0.$$

Combined with (6.3) this yields $\tilde{u}(t;q) = q$ for $t \geq 0$. Thus $q \in E$. We define S_D by

$$S_D = \{q \in S, \text{ such that } q = 0 \text{ on } \partial\Omega\},$$

where S is defined by (1.1). In what follows we prove the following theorem.

THEOREM 6.2. *Let the hypotheses* H1-H2abc-H3-H4 *be satisfied. Then*

(i) $E_D = S_D$;

(ii) E_D *contains a maximal element* q_{max}, i.e. $q \geq q_{max}$ *in* Ω *for any* $q \in E_D$;

(iii) *There exists a function* $q \in E_D$ *such that* $u(t;u_0) \to q$ *in* $C(\bar{\Omega})$ *as* $t \to \infty$.

If in addition $u_0 \leq q_{max}$, *then* q *satisfies* $\int_\Omega q\,dx = \int_\Omega u_0\,dx$.

PROOF. (i) By Lemma 6.1 and Theorem 5.1(i), $E_D \subset S$. Hence $E_D \subset S_D$. The proof of the inclusion $S_D \subset E_D$ is identical to the proof of $S \subset E$, given in section 5. Thus $E_D = S_D$.

(ii) Let $C > ||v||_{L^\infty(\Omega)}$ be constant and let $w \in S$ be defined by

$$\Phi(w(x)) = C - v(x), \ x \in \bar{\Omega}.$$

Then $w > 0$ in $\bar{\Omega}$ and it follows from the definition of the set S_D that $q \leq w$ in Ω for any $q \in S_D$. Hence, by (i),

$$q \leq w \text{ in } \Omega \text{ for any } q \in E_D. \tag{6.4}$$

Since $w \in S = E$, w is a supersolution of Problem P_D. Hence the solution $u(t;w)$ of Problem P_D is nonincreasing in t and we may define

$$0 \leq p(x) = \lim_{t\to\infty} u(x,t;w), \ x \in \bar{\Omega}.$$

By (6.4) and the comparison principle

$$q \leq p \text{ in } \Omega \text{ for any } q \in E_D. \tag{6.5}$$

Below we prove that $p \in E_D$. Then the result follows at once from (6.5), with $q_{max} = p$.

Let $\eta(x) \geq 0$ be a smooth testfunction on $\bar{\Omega}$ such that $\eta = 0$ on $\partial\Omega$.

Then $u = u(\cdot;w)$ satisfies

$$\int_\Omega u(t)\eta = \int_\Omega w\eta + \iint_{Q_t} (\varphi(u)\Delta\eta - u\text{gradvgrad}\eta).$$

Thus

$$\frac{d}{dt}\int_\Omega u(t)\eta = \int_\Omega (\varphi(u(t))\Delta\eta - u(t)\text{gradvgrad}\eta). \tag{6.6}$$

Since $u(t;w)$ decreases to p as $t \to \infty$, the left-hand side of (6.6) is nonpositive and there exists a sequence $t_n \to \infty$ such that

$$\frac{d}{dt}\int_\Omega u(t_n)\eta \to 0 \text{ as } t_n \to \infty . \tag{6.7}$$

On the other hand, the right hand side of (6.6) converges to

$$\int_\Omega (\varphi(p)\Delta\eta - p\text{gradvgrad}\eta) \text{ as } t_n \to \infty .$$

Hence, by (6.6) and (6.7),

$$\int_{\Omega} \varphi(p)\Delta\eta - pgradvgrad\eta) = 0$$

and thus $p \in E_D$.

(iii). Given an initial function u_0, one can find a function $w \in S$ such that $u_0 \leq w$ in Ω. Using the above argument and the comparison principle, we find that

$$\limsup_{t\to\infty} u(x,t;u_0) \leq q_{max}(x),\ x \in \Omega.$$

Hence

$$q \in \omega(u_0) \text{ implies that } q \leq q_{max}. \tag{6.8}$$

In order to prove that $u(t;u_0)$ stabilizes to equilibrium, we use the same arguments as for the proof of Theorem 5.1 (ii), but now based on the fact that S_D is a continuum between zero and q_{max}, on (6.8), and on the contraction property of u. If furthermore $u_0 \leq q_{max}$, then the solution $\tilde{u}(t;u_0)$ of Problem P satisfies $\tilde{u}(t;u_0) \leq q_{max}$ for $t \geq 0$ and in particular $\tilde{u}(t;u_0) = 0$ on $\partial\Omega \times \mathbb{R}^+$. Thus $\tilde{u}(t;u_0)$ coincides with the solution $u(t;u_0)$ of Problem P_D (see Lemma 9.4 below). Then, if $q = \lim_{t\to\infty} u(t;u_0)$, we have, by Theorem 5.1 (ii)

$$\int_{\Omega} q dx = \int_{\Omega} u_0 dx.$$

PART II

7. EXISTENCE AND REGULARITY

In this section we prove the existence of a solution of Problem P which satisfies this problem in a somewhat stronger sense than that of Definition 3.1. We first recall some usual definitions and then give an alternative definition of a solution, involving the gradient of $\varphi(u)$. The existence proof itself is based on the study of uniformly parabolic problems which are related to Problem P.

We denote by $L^2(0,T;\ H^1(\Omega))$ the Hilbert space with inner product

$$(u,v)_{L^2(0,T;H^1(\Omega))} = \iint_{Q_T} uv + \iint_{Q_T} \mathrm{grad}u\,\mathrm{grad}v$$

and by $V_2(Q_T)$ the Banach space with norm

$$|u|^2_{V_2(Q_T)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega u^2(t) + \iint_{Q_T} (\mathrm{grad}u)^2.$$

DEFINITION 7.1. *We say that* $u\colon [0,\infty) \to L^1(\Omega)$ *is a weak solution of Problem P if it satisfies*

(i) $u \in C([0,t];\ L^1(\Omega)) \cap L^\infty(Q_t)$ *for all* $t \in (0,\infty)$;
(ii) $\varphi(u) \in V_2(Q_t)$ *for all* $t \in_t (0,\infty)$;
(iii) $\int_\Omega u(t)\psi(t) = \int_\Omega u_0\psi(0) + \int_0^t \int_\Omega \{u\psi_t - (\mathrm{grad}\varphi(u) - u\,\mathrm{grad}v)\mathrm{grad}\psi\}$
for all $\psi \in C^1(\bar{Q})$ *and all* $t \in (0,\infty)$.

LEMMA 7.2. *A weak solution of Problem P is a generalized solution as well.*

PROOF. Take $\psi \in C^{2,1}(\bar{Q})$ with $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega \times \mathbb{R}^+$ and integrate by parts.

In what follows, we show that Problem P has a weak solution. To that purpose, we consider the problems

$$\begin{cases} u_t = \Delta\varphi_\varepsilon(u) + \mathrm{div}(u\,\mathrm{grad}v_\varepsilon) & \text{in } Q_T = \Omega \times (0,T] \\ \frac{\partial}{\partial\nu}\varphi_\varepsilon(u) + u\frac{\partial v_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_{0\varepsilon}(x) & \text{in } \Omega \end{cases}$$

where

$$\varphi_\varepsilon \in C^\infty(\mathbb{R}^+),\ \varphi_\varepsilon(0) = 0,\ \varphi_\varepsilon'(s) \geq C(\varepsilon) > 0 \text{ for } s \in [0,K]$$

$(\varphi_\varepsilon^{-1}(s))' \leq (\varphi^{-1}(s))'$ for $s \in [0,\varphi(2K)]$ where K is the uniform L^∞-bound of u_ε that we find in the proof of Lemma 7.4 below and φ_ε and φ_ε' converge to φ and φ' on all compact subsets of $\mathbb{R}^+$ as $\varepsilon\downarrow 0$,

where

$v \in C^\infty(\bar\Omega)$, $||v_\varepsilon||_{C^1(\bar\Omega)} \leq C$ for some constant $C > 0$,

$$||v_\varepsilon||_{C(\bar\Omega)} \leq ||v||_{L^\infty(\Omega)} \text{ and } ||v_\varepsilon - v||_{H^1(\Omega)} \to 0 \text{ as } \varepsilon\downarrow 0$$

and where

$u_{0\varepsilon} \in C^\infty(\bar\Omega)$, $0 \leq u_{0\varepsilon} \leq ||u_0||_{L^\infty(\Omega)}$, $u_{0\varepsilon}$ satisfies the compatibility condition $\frac{\partial}{\partial\nu}\varphi_\varepsilon(u_{0\varepsilon}) + u_{0\varepsilon}\frac{\partial v_\varepsilon}{\partial\nu} = 0$ on $\partial\Omega$ and $||u_{0\varepsilon} - u_0||_{L^2(\Omega)} \to 0$ as $\varepsilon\downarrow 0$.

Since it is standard that one can construct the approximations φ_ε of the function φ having the properties indicated above, we do not do it here. On the other hand we construct explicitely in Appendix A approximations v_ε and $u_{0\varepsilon}$ of the functions v and u_0.

To begin with we give a comparison principle, which turns out to be basic in the study of Problems P_ε and P.

LEMMA 7.3. *Let* u_1 *and* $u_2 \in C^{2,1}(\bar Q_T)$ *be two solutions of Problem* P_ε *with initial functions* $u_{01} \leq u_{02}$. *Then* $u_1(t) \leq u_2(t)$.

PROOF. Let $z = u_1 - u_2$. Then z satisfies the linear problem L_ε which we discussed in section 4 and Lemma 7.1 follows from the comparison principle for that problem.

Before proving the existence of a solution of Problem P, we first give some a priori estimates.

LEMMA 7.4. *Let* $u_\varepsilon \in C^{2,1}(\bar Q_T)$ *be a solution of Problem* P_ε. *Then*

$$0 \leq u_\varepsilon \leq K \text{ in } \bar Q_T \tag{7.1}$$

where the constant K *does not depend on* T.

PROOF. We first note that zero is a solution of Problem P_ε. Consequently, since $u_{0\varepsilon} \geq 0$, we have that $u_\varepsilon \geq 0$. In order to find an upper bound for u_ε, we now search for a large enough stationary solution of Problem P. Let $a_0 = \|u_0\|_{L^\infty(\Omega)}$. We define

$$\Phi_0(s) = \int_{a_0}^{s} \frac{\varphi'(\tau)}{\tau}\, d\tau \text{ and } \Phi_{0,\varepsilon}(s) = \int_{a_0}^{s} \frac{\varphi'_\varepsilon(\tau)}{\tau}\, d\tau \text{ for } s \geq a_0 .$$

The function

$$\beta_\varepsilon = \Phi_{0,\varepsilon}^{-1} \left(\|v\|_{L^\infty(\Omega)} - v_\varepsilon \right)$$

is a solution of Problem P_ε and it is such that

$$u_{0\varepsilon} \leq \beta_\varepsilon \leq \Phi_{0,\varepsilon}^{-1} \left(\|v\|_{L^\infty(\Omega)} \right).$$

Also, since φ'_ε converges uniformly to φ' on compact subsets of $\mathbb{R}^+$ as $\varepsilon \downarrow 0$, we have that $\Phi_{0,\varepsilon}^{-1}$ converges uniformly to Φ_0^{-1} on compact subsets of $\mathbb{R}^+$ as $\varepsilon \downarrow 0$. Thus there exists $C > 0$ such that

$$u_{0\varepsilon} \leq \beta_\varepsilon \leq \Phi_0^{-1} \left(\|v\|_{L^\infty(\Omega)} \right) + C$$

which completes the proof of (7.1).

LEMMA 7.5. *Problem* P_ε *has a unique classical solution* $u_\varepsilon \in C^{2+\alpha}(\bar{Q}_T)$ *for each* $\alpha \in (0,1)$.

PROOF. See [16, Th. 7.4, p. 491].

In what follows we give some more a priori estimates for u_ε.

LEMMA 7.6. *Let* $0 \leq t-\tau < t < T$. *Then there exists* $C(\tau) > 0$ *such that*

$$\int_{t-\tau}^{t} \int_{\Omega} (\text{grad}\varphi_\varepsilon(u_\varepsilon))^2 \leq C(\tau).$$

In particular the constant $C(\tau)$ *does not depend on* T *or on* ε.

PROOF. We multiply the differential equation by $\varphi_\varepsilon(u_\varepsilon)$ and integrate by parts over $\Omega \times (t, t+\tau)$. This yields, setting $F_\varepsilon(s) = \int_0^s \varphi_\varepsilon(\tau)\, d\tau$

$$\int_\Omega F_\varepsilon(u_\varepsilon)(t+\tau) - \int_\Omega F_\varepsilon(u_\varepsilon)(t) + \int_t^{t+\tau} \int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon))^2$$

$$= - \int_t^{t+\tau} \int_\Omega \text{grad } v_\varepsilon \; u_\varepsilon \; \text{grad } \varphi_\varepsilon(u_\varepsilon).$$

Since $0 \leq u_\varepsilon \leq C$, we deduce the result by applying the Cauchy-Schwarz inequality.

Next we give an estimate which is useful for the proof of Theorem 4.1 (ii); we adapt a proof from Gagneux [10].

LEMMA 7.7. *We suppose that either* $\varphi(s) = \frac{s|s|}{2}$ *or* $\Delta v \in L^1(\Omega)$. *Then*

$$||\varphi_\varepsilon(u_\varepsilon)||_{L^\infty(T-\tau,T;H^1(\Omega))} \leq C(\tau), \; 0 < \tau \leq T. \tag{7.2}$$

The constant $C(\tau)$ *does not depend on* T.

PROOF. We first show that for $0 \leq t-\tau < t \leq T$, the following estimate holds

$$\int_{t-\tau}^{t} \int_\Omega \varphi'_\varepsilon(u_\varepsilon)(\text{grad } u_\varepsilon)^2 \leq C(\tau). \tag{7.3}$$

For that purpose we multiply the differential equation by u_ε and integrate by parts; we obtain

$$\frac{1}{2}\int_\Omega u_\varepsilon^2(t) - \frac{1}{2}\int_\Omega u_\varepsilon^2(t-\tau) + \int_{t-\tau}^{t} \int_\Omega \varphi'_\varepsilon(u_\varepsilon)(\text{grad } u_\varepsilon)^2$$

$$= \int_{t-\tau}^{t} \int_\Omega \text{grad } v_\varepsilon \; \text{grad}(\tfrac{1}{2}u_\varepsilon^2).$$

When (i) : $\varphi(s) = \frac{s^2}{2}$, we have

$$\int_{t-\tau}^{t} \int_{\Omega} \text{grad } v_{\varepsilon} \text{ grad}(\tfrac{1}{2}u_{\varepsilon}^{2}) \leq C\sqrt{\tau}\ ||\varphi(u_{\varepsilon})||_{L^{2}(t-\tau,t;H^{1}(\Omega))} \leq C\sqrt{\tau}$$

by Lemma 7.4; and when (ii) : $\Delta v \in L^{1}(\Omega)$, then

$$\int_{t-\tau}^{t} \int_{\Omega} \text{grad } v_{\varepsilon} \text{ grad}(\tfrac{1}{2}u_{\varepsilon}^{2}) = \int_{t-\tau}^{t} \int_{\partial\Omega} \frac{u_{\varepsilon}^{2}}{2} \frac{\partial v_{\varepsilon}}{\partial \nu} - \int_{t-\tau}^{t} \int_{\Omega} \Delta v_{\varepsilon} \frac{u_{\varepsilon}^{2}}{2} \leq C(\tau)$$

which completes the proof of (7.3).

In order to prove (7.2) we multiply the differential equation by $(s-t+\tau)(\varphi_{\varepsilon}(u_{\varepsilon}))_{t}$ and integrate by parts. We obtain

$$\int_{t-\tau}^{t} \int_{\Omega} (s-t+\tau)\ \varphi_{\varepsilon}'(u_{\varepsilon})(u_{\varepsilon t})^{2} + \frac{\tau}{2}\int_{\Omega} (\text{grad } \varphi_{\varepsilon}(u_{\varepsilon})(t))^{2}$$

$$= - \int_{t-\tau}^{t} \int_{\Omega} (s-t+\tau)u_{\varepsilon} \text{ grad } v_{\varepsilon}(\text{grad } \varphi_{\varepsilon}(u_{\varepsilon}))_{t}. \tag{7.4}$$

Integrating the right-hand side of (7.4) by parts and applying the Cauchy-Schwarz inequality we obtain

$$|\text{RHS}| = \Big| + \int_{\Omega} \int_{t-\tau}^{t} \text{grad } v_{\varepsilon}(u_{\varepsilon} + (s-t+\tau)\ u_{\varepsilon t}) \text{ grad } \varphi_{\varepsilon}(u_{\varepsilon})$$

$$- \tau \int_{\Omega} \text{grad } v_{\varepsilon}\ u_{\varepsilon}(t) \text{ grad } \varphi_{\varepsilon}(u_{\varepsilon})(t) \Big|$$

$$\leq \sqrt{\tau}\ ||\text{grad } v_{\varepsilon}||_{L^{2}(\Omega)}\ ||\varphi_{\varepsilon}(u_{\varepsilon})||_{L^{2}(t-\tau,t;H^{1}(\Omega))}$$

$$||u_{\varepsilon}||_{L^{\infty}(Q_{T})} + ||\text{grad } v_{\varepsilon}||_{L^{\infty}(\Omega)} \Big(\int_{t-\tau}^{t} \int_{\Omega} (s-t+\tau)\ \varphi_{\varepsilon}'(u_{\varepsilon})$$

$$(u_{\varepsilon t})^{2}\Big)^{\frac{1}{2}} \Big(\int_{t-\tau}^{t} \int_{\Omega} (s-t+\tau)\ \varphi_{\varepsilon}'(u_{\varepsilon})(\text{grad } u_{\varepsilon})^{2}\Big)^{\frac{1}{2}}$$

$$+ \tau \,||\text{grad } v_\varepsilon||_{L^2(\Omega)} \; ||u_\varepsilon||_{L^\infty(\Omega)} \; (\int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon)(t))^2)^{\frac{1}{2}}.$$

This inequality combined with (7.4) implies that

$$\int_{t-\tau}^{t} \int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon))^2 \leq C(\tau)$$

and the proof of Lemma 7.6 is complete.

We shall need a result of DiBenedetto [7, Th. 6.2] to deduce a strong estimate, namely the equicontinuity of u_ε.

LEMMA 7.8 (i). *For every* $\tau > 0$ *there exists a continuous nondecreasing function* $\omega_\tau(\cdot)$, $\omega_\tau(0) = 0$ *such that*

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \leq \omega_\tau(|x_1-x_2| + |t_1-t_2|^{\frac{1}{2}})$$

for all $(x_i,t_i) \in \bar{\Omega} \times [\tau,T]$, $i = 1,2$.

The function ω_τ *does not depend on* T *and* ε.

(ii) *If* $u_0 \in C(\bar{\Omega})$, *then* $\{u_\varepsilon\}$ *is equicontinuous on* $\bar{\Omega} \times [0,T]$.

We are now in a position to prove the existence theorem.

THEOREM 7.9. *We suppose that* H1 *and* H4 *are satisfied and that* $v \in W^{1,\infty}(\Omega)$. *Then there exists a weak solution of Problem P which satisfies*

$$0 \leq u \leq C \quad \text{on } Q_T$$

and is continuous in any set $\bar{\Omega} \times [\tau,T]$ *with* $\tau > 0$. *The constant* C *and the modulus of continuity do not depend on* T.

PROOF. From the estimates above we deduce that there exist a function $u \in L^\infty(Q_T) \cap C(\bar{\Omega} \times (0,T])$ and a subsequence of $\{u_\varepsilon\}$ which we denote again by $\{u_\varepsilon\}$ such that

(i) $u_\varepsilon \to u$ uniformly on all sets of the form $\bar{\Omega} \times [\tau,T]$ with $\tau > 0$ (by Lemma 7.8);

(ii) $u_\varepsilon \to u$ strongly in $L^2(Q_T)$ and a.e. (this is a consequence of (i) and the uniform bound of u_ε in $L^\infty(\Omega)$);

(iii) $\varphi_\varepsilon(u_\varepsilon) \rightharpoonup \varphi(u)$ weakly in $L^2(0,T;H^1(\Omega))$ (this follows from Lemma 7.6; one checks that the limit is $\varphi(u\ )$ by observing that by (ii) and Lebesgue's dominated convergence theorem $\varphi_\varepsilon(u_\varepsilon) \to \varphi(u)$ strongly in $L^2(Q_T)$);

(iv) u_ε grad $v_\varepsilon \to u$ grad v strongly in $L^1(Q_T)$.

It remains to check that u is a solution of Problem P. Is is easy to deduce from (i) - (iv) that u satisfies the intergral equation in Definition 3.1 since u_ε satisfies a similar equation. Also $u \in C((0,T];L^1(\Omega))$. In order to show that $||u(t)||_{L^1(\Omega)}$ is continuous at zero we use the contraction Theorem 4.1 (i). Let $\tilde{u}_\varepsilon$ be a solution of Problem P with initial function $u_{0\varepsilon}$ obtained as a limit of solutions of Problem P_ε. Then

$$||u(t) - u_0||_{L^1(\Omega)} \leq ||u(t) - \tilde{u}_\varepsilon(t)||_{L^1(\Omega)} + ||\tilde{u}_\varepsilon(t) - u_{0\varepsilon}||_{L^1(\Omega)}$$

$$+ ||u_{0\varepsilon} - u_0||_{L^1(\Omega)} .$$

Let $\eta > 0$ be arbitrary. Since $u_{0\varepsilon}$ converges to u_0 in $L^1(\Omega)$, one can fix ε such that $||u_{0\varepsilon} - u_0||_{L^1(\Omega)} \leq \eta/3$. Then by Theorem 4.1 $||u(t) - \tilde{u}_\varepsilon(t)||_{L^1(\Omega)} \leq \eta/3$. Finally we deduce from Lemma 7.8 (ii) that one can find t_o such that $||\tilde{u}_\varepsilon(t) - u_{0\varepsilon}||_{L^1(\Omega)} \leq \eta/3$ for all $t \leq t_0$.

REMARK 7.10. If the function φ is defined on $\mathbb{R}$ with $\varphi'(s) > 0$ for $s < 0$, the condition $u_0 \geq 0$ is not necessary to obtain the results of section 7.

8. UNIQUENESS OF THE SOLUTION

In order to show that the solution of Problem P is unique, we apply a method due to Kalashnikov [13] which consists of comparing an arbitrary solution of Problem P with a solution obtained as the limit of a sequence of classical solutions of the parabolic equation in Problem P. We do so below and for technical reasons which will appear later we impose the condition $\Delta v \geq -M$ in the sense of distributions.

We approximate Problem P in two steps, first by the problem

$$(P_n) \begin{cases} u_t = \Delta\varphi(u) + \mathrm{div}(u \,\mathrm{grad}\, v) & \text{in } Q_T \\ \frac{\partial}{\partial\nu}\varphi(u) + u\frac{\partial v}{\partial\nu} = \frac{A}{n}e^{-Mt} & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_{0n}(x) := u_0(x) + \frac{1}{n} & \text{in } \Omega \end{cases}$$

which in turn we approximate by the problem

$$(P_{nj}) \begin{cases} u_t = \Delta\varphi(u) + \mathrm{div}(u \,\mathrm{grad}\, v_j) & \text{in } Q_T \\ \frac{\partial}{\partial\nu}\varphi(u) + u\frac{\partial v_j}{\partial\nu} = \frac{A}{n}\ e^{-Mt} & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_{0j}(x) + \frac{1}{n} & \text{in } \Omega \end{cases}$$

where

$v_j \in C^\infty(\bar{\Omega})$, $||v_j||_{C^1(\bar{\Omega})} \leq C_1$ for some constant $C_1 > 0$,

$\Delta v_j \geq -M$ and $||v_j - v||_{H^1(\Omega)} \to 0$ as $j \to \infty$

where the constant A is such that $A \geq C_1$ and

$u_{0j} \in C^{2+\alpha}(\bar{\Omega})$, $0 \leq u_{0j} \leq C_2$ for some constant $C_2 > 0$,

u_{0j} satisfies the compatibility condition

$$\varphi'(u_{0j} + \frac{1}{n}) \frac{\partial u_{0j}}{\partial \nu} + \frac{\partial v_j}{\partial \nu} u_{0j} + \frac{1}{n} (\frac{\partial v_j}{\partial \nu} - A) = 0 \text{ on } \partial\Omega$$

and is such that $||u_{0j} - u_0||_{L^2(\Omega)} \to 0$ as $j \to \infty$,

We show in the appendix that one can construct such functions v_j and u_{0j}.

We first give uniform upper and lower bounds for the solution u_{nj} of Problem P_{nj}; the fact that u_{nj} turns out to be bounded away from zero ensures that Problem P_{nj} is uniformly parabolic.

LEMMA 8.1. *Let* $u_{nj} \in C^{2,1}(\bar{Q}_T)$ *be a solution of Problem* P_{nj}. *Then, for* n *large enough,*

$$\frac{1}{n} e^{-Mt} \leq u_{nj}(x,t) \leq C \text{ for all } (x,t) \in \bar{Q}_T.$$

where the constant C *does not depend on time.*

The main tool of the proof is the following comparison principle which is an immediate generalization of Lemma 7.3.

LEMMA 8.2. *Let* u_1 *and* $u_2 \in C^{2,1}(\bar{Q}_T)$ *and assume that* u_1 *and* u_2 *are positive on* $\overline{Q}_T$. *If*

$$\Delta\varphi(u_1) + \mathrm{div}(u_1 \text{ grad } v_j) - u_{1t} \geq \Delta\varphi(u_2) + \mathrm{div}(u_2 \text{ grad } v_j) - u_{2t} \textit{ in } Q_T$$

$$\frac{\partial}{\partial\nu} \varphi(u_1) + u_1 \frac{\partial v_j}{\partial \nu} \leq \frac{\partial}{\partial\nu} \varphi(u_2) + u_2 \frac{\partial v_j}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T]$$

$$u_1(x,0) \leq u_2(x,0) \qquad \textit{in } \Omega.$$

Then

$$u_1 \leq u_2 \qquad \textit{in } \bar{Q}_T.$$

PROOF OF LEMMA 8.1. We first observe that the function $s^-(x,t) := \frac{1}{n} e^{-Mt}$ is a lower solution of Problem P_{nj} since it satisfies

$$\begin{cases} \Delta\varphi(s^-) + \operatorname{div}(s^- \operatorname{grad} v_j) - s_t^- = \frac{1}{n} e^{-Mt} (\Delta v_j + M) \geq 0 \text{ in } Q_T \\ \frac{\partial}{\partial\nu} \varphi(s^-) + s^- \frac{\partial v_j}{\partial\nu} = \frac{e^{-Mt}}{n} \frac{\partial v_j}{\partial\nu} \leq \frac{A}{n} \; e^{-Mt} \qquad \text{on } \partial\Omega \times (0,T] \\ s^-(x,0) = \frac{1}{n} \leq u_{0j}(x) + \frac{1}{n} \qquad \text{in } \Omega, \end{cases}$$

Thus, by lemma 8.2, $u_n \geq \frac{1}{n} e^{-Mt}$. Next we seek an upper solution in the form

$$s^+(x,t) = \Phi^{-1}(C - v_j - e^{-Mt} h(x))$$

where h is a smooth function such that $1 \leq h \leq 2$ and the constant C is choosen large enough so that

$$C_2 + 1 < \Phi^{-1}(C - C_1 - 2)$$

and hence

$$s^+(x,0) \geq u_{0j}(x) + \frac{1}{n} \text{ for all } n \geq 1,\ j \geq 1.$$

Thus we must choose the function h such that

$$\Delta\varphi(s^+) + \operatorname{div}(s^+ \operatorname{grad} v_j) - s_t^+ \leq 0 \text{ in } \bar{Q}_T \tag{8.1}$$

and

$$\frac{\partial}{\partial\nu} \varphi(s^+) + s^+ \frac{\partial v_j}{\partial\nu} \geq \frac{A}{n} e^{-Mt} \text{ on } \partial\Omega \times (0,T]. \tag{8.2}$$

We rewrite (8.1) as

$$\operatorname{div} [s^+ \operatorname{grad}(\Phi(s^+) + v_j)] - s_t^+ \leq 0$$

and we substitute the expression for s^+ to obtain

$$-\operatorname{div}[\Phi^{-1}(C - v_j - e^{-Mt} h) \operatorname{grad} h] \leq Mh(x) \frac{\Phi^{-1}(C-v_j-e^{-Mt}h)}{\varphi'[\Phi^{-1}(C-v_j-e^{-Mt}h)]} .$$

This inequality is satisfied if

$$|\Delta h| + |\text{grad } h| + |\text{grad } h|^2 \leq C_3 \qquad \text{in } \bar{Q}_T \tag{8.3}$$

for a sufficiently small constant C_3. On the otherhand condition (8.2) is equivalent to

$$-e^{-Mt}\,\Phi^{-1}(C - v_j - e^{-Mt}\,h)\,\frac{\partial h}{\partial \nu} \geq \frac{A}{n}\;e^{-Mt} \qquad \text{on } \partial\Omega \times (0,T]$$

which holds if

$$-\frac{\partial h}{\partial \nu} \geq \frac{A}{(C_2+1)n}\,, \tag{8.4}$$

Let $\tilde{h}$ be defined by $\Delta\tilde{h} = -1$ in Ω, $\tilde{h} = 0$ on $\partial\Omega$, and set $h = 1 + \alpha\tilde{h}$ where α is a positive constant such that $\alpha\tilde{h} \leq 1$ and (8.3) is satisfied. Since $\frac{\partial h}{\partial \nu} < 0$ on $\partial\Omega$, (8.4) is satisfied for n large enough. Thus we have found a function h such that s^+ is a supersolution. It follows that $u_{nj} \leq s^+ \leq \Phi^{-1}(C)$ in Q_T.

By the method of section 7 one can obtain further a priori estimates for solutions of the Problems P_{nj} and P_n and use them to show that a subsequence $\{u_{nj}\}$ of solutions of Problems P_{nj} converge to a generalized solution of Problem P_n as $j \to \infty$ and then that a subsequence $\{u_{\tilde{n}}\}$ of solutions of Problems $P_{\tilde{n}}$ converge to a solution of Problem P. In addition, following DiBenedetto again, we find that the sequence $\{u_{nj}\}$ is equicontinuous. In particular one can show that there exists a solution u of Problem P and a subsequence of the solutions u_n of Problems P_n (which we denote again by $\{u_n\}$) which converges to u as $n \to \infty$.
Below we use this construction to prove the following result.

THEOREM 8.3. *We suppose that the hypotheses* H1-H2a-H4 *are satisfied. Let* u *be the solution of Problem* P *obtained above and let* $\underline{u}$ *(resp.* $\bar{u}$*) be a subsolution (resp. supersolution) of* P *with initial function* $\underline{u}_0$ *(resp.* $\bar{u}_0$*). Then for every* $t \in (0,T]$ *we have that*

$$\int_\Omega (\underline{u}(t) - u(t))^+ \leq \int_\Omega (\underline{u}_0 - u_0)^+ \tag{8.5}$$

and

$$\int_\Omega (u(t) - \bar{u}(t))^+ \le \int_\Omega (u_0 - \bar{u}_0)^+. \tag{8.6}$$

COROLLARY 8.4. *If the hypotheses* H1-H2a-H4 *are satisfied, Problem* P *has a unique solution.*

COROLLARY 8.5. *Let* $\underline{u}(t)$ *and* $\bar{u}(t)$ *be respectively a subsolution and a supersolution of Problem* P *with initial functions* $\underline{u}_0$ *and* $\bar{u}_0$ *such that* $\underline{u}_0 \le \bar{u}_0$. *Then* $\underline{u}(t) \le \bar{u}(t)$ *for every* $t \in (0,T]$.

PROOF OF THEOREM 8.3. The proof follows closely that of Diaz and Kersner [6]. Let ψ be a test function. Then

$$\int_\Omega (\underline{u} - u_n)(t)\,\psi\,(t) - \int_\Omega (\underline{u}_0 - u_{0n})\,\psi\,(0)$$

$$\le \int_0^t \int_\Omega \{(\underline{u} - u_n)(t)\,\psi_t + (\varphi(\underline{u}) - \varphi(u_n))\Delta\psi - (\underline{u} - u_n)\text{ grad } v \text{ grad } \psi\}$$

$$- \frac{A}{n} \int_0^t \int_{\partial\Omega} e^{-Mt}\,\psi$$

$$\le \int_0^t \int_\Omega (\underline{u} - u_n)\,\{\psi_t + A_n\Delta\psi - \text{grad } v \text{ grad } \psi\}$$

where

$$A_n(x,t) = \int_0^1 \varphi'(\Theta\underline{u}(x,t) + (1-\Theta)\,u_n(x,t))\,d\Theta.$$

Since $u_n \ge \frac{1}{n}e^{-Mt}$, there exists $\varepsilon(n) > 0$ such that $A_n \ge \varepsilon(n) > 0$. We now define a sequence of smooth functions $A_{nj} \ge \varepsilon(n)$ such that

$$A_{nj} \to A_n \text{ strongly in } L^2(Q_T) \text{ as } j \to \infty.$$

Let ψ_{nj} be the solution of the problem

$$(L_{nj}) \begin{cases} \psi_t + A_{nj}\Delta\psi - \text{grad } v_j \text{ grad } \psi = 0 & \text{in } \Omega \times (0,t) \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega \times [0,t) \\ \psi(x,t) = \chi(x) & \text{in } \Omega, \end{cases}$$

where χ is a smooth function such that $0 \leq \chi \leq 1$. As a consequence of the maximum principle we have that $0 \leq \psi_{nj} \leq 1$. We set $\psi = \psi_{nj}$. Then

$$\int_\Omega (\underline{u} - u_n)\chi \leq \int_\Omega (\underline{u}_0 - u_{0n})^+ + \int_0^t \int_\Omega (\underline{u} - u_n)\, \{(A_n - A_{nj})\Delta\psi_{nj} - (\mathrm{grad} v - \mathrm{grad} v_j)\ \mathrm{grad}\psi_{nj}\}. \tag{8.7}$$

In what follows we first keep n fixed. In order to show that the second term of the right-hand side of (8.7) vanishes as $j \to \infty$, it is sufficient to prove that there exists a constant $C(n,t)$ such that

$$\int_0^t \int_\Omega (\mathrm{grad}\ \psi_{nj})^2 \leq C(n,t) \text{ and } \int_0^t \int_\Omega (\Delta\psi_{nj})^2 \leq C(n,t). \tag{8.8}$$

These estimates follow from multiplying the differential equation in Problem L_{nj} by $\Delta\psi_{nj}$ and integrating it on $\Omega \times (0,T)$. For details we refer to Aronson, Crandall & Peletier [3] where a similar calculation is made. Inequality (8.7) together with (8.8) yields

$$\int_\Omega (\underline{u}(t) - u_n(t))\chi \leq \int_\Omega (\underline{u}_0 - u_{0n})^+$$

for all smooth χ such that $0 \leq \chi \leq 1$ and hence, since $u_{0n} \to u_0$ in $C(\bar{\Omega})$ and $u_n \to u$ in $C(\bar{Q}_T)$ we have

$$\int_\Omega (\underline{u}(t) - u(t))\chi \leq \int_\Omega (\underline{u}_0 - u_0)^+. \tag{8.9}$$

Next we consider a sequence of smooth functions χ_m such that χ_m converges in $L^2(\Omega)$ to a limit $\bar{\chi}$ defined by

$$\bar{\chi}(x) = \begin{cases} 1 & \text{in } \{x \mid \underline{u}(x,t) \geq u(x,t)\} \\ 0 & \text{elsewhere} \end{cases}$$

Taking $\chi = \chi_m$ in (8.9) and letting $m \to \infty$ yield (8.5). Finally one can show (8.6) in a similar way.

9. SOME REMARKS ABOUT THE DIRICHLET PROBLEM

In this section we shall discuss the existence, uniqueness and regularity of solutions of Problem P_D, which we introduced in section 6.

THEOREM 9.1. *(Existence + Regularity). Let* H1 *and* H4 *be satisfied, and let* $v \in W^{1,\infty}(\Omega)$. *Then Problem* P_D *possesses a solution u which is uniformly bounded in* Q *and which is continuous in any set* $\overline{\Omega} \times [\tau,T]$ *with* $\tau > 0$. *The modulus of continuity does not depend on* T.

The proof of Theorem 9.1 is quite similar to the proof of Theorem 7.9 and we omit it.

THEOREM 9.2. *(Uniqueness + Comparison Principle) Let* H1, H2a *and* H4 *be satisfied.*

(i) *Problem* P_D *possesses at most one solution.*

(ii) *Let* $\underline{u}(t)$ *and* $\bar{u}(t)$ *be respectively a subsolution and a supersolution of Problem* P_D *with respect to the initial functions* $\underline{u}_0$ *and* $\bar{u}_0$. *If* $\underline{u}_0 \leq \bar{u}_0$ *in* Ω, *then* $\underline{u}(t) \leq \bar{u}(t)$ *in* Ω *for* $t \geq 0$.

In order to prove Theorem 9.2 we proceed as in section 8. Let u be a solution of Problem P_D which is obtained as the limit function of a sequence $\{u_n\}$, where u_n is the solution of

$$(P_{D,n}) \begin{cases} u_t = \Delta\varphi(u) + \operatorname{div}(u \operatorname{grad} v) & \text{in } Q \\ u = (1/n)e^{-Mt} & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(\cdot,0) = u_0 + 1/n & \text{in } \Omega, \end{cases}$$

such that

$$(1/n)e^{-Mt} \leq u_n \leq C \qquad \text{in } Q. \tag{9.1}$$

Then it is sufficient to prove that for any $t > 0$

$$\int_\Omega \{\underline{u}(t) - u(t)\}^+\chi \leq \int_\Omega (\underline{u}_0 - u_0)^+ \tag{9.2}$$

and

$$\int_\Omega \{(u(t) - \bar{u}(t)\}^+\chi \le \int_\Omega (u_0 - \bar{u}_0)^+, \tag{9.3}$$

for any $\chi \in C^\infty(\bar{\Omega})$ with compact support in Ω such that $0 \le \chi \le 1$ in Ω.

The proof of (9.2) follows the same lines as the proof which we gave in section 8. However, the proof of (9.3) requires handling a boundary term which was absent in section 8. This aspect we discuss below.
Let ψ_{nj} be the solution of

$$(L_D)\begin{cases} \psi_t + A_{nj}\Delta\psi - \text{grad } v_j \ . \ \text{grad } \psi = 0 & \text{in } \Omega \times [0,t) \\ \psi = 0 & \text{on } \partial\Omega \times [0,t) \\ \psi(\cdot\,,t) = \chi & \text{in } \Omega, \end{cases}$$

where $A_{nj} \to A_n$ in $L^2(Q_t)$ as $j \to \infty$, with

$$A_n = \begin{cases} \{\varphi(u_n) - \varphi(\bar{u})\}/(u_n - \bar{u}) & \text{if } u_n \ne \bar{u} \\ \varphi'(u_n) & \text{if } u_n = \bar{u}, \end{cases}$$

and where v_j is as in section 8. Then (cf. (8.7))

$$\int_\Omega \{u_n(t) - \bar{u}(t)\}^+\chi \le \int_\Omega (u_{0n} - \bar{u}_0)^+ - \int_0^t \int_{\partial\Omega} \varphi(\frac{1}{n} e^{-Mt}) \frac{\partial\psi_{nj}}{\partial\nu}$$

$$+ \int_0^t \int_\Omega (u_n - \bar{u})\{(A_n - A_{nj})\Delta\psi_{nj} - (\text{grad } v - \text{grad } v_j)\text{grad } \psi_{nj}\}. \tag{9.4}$$

As in section 8, the third term at the right-hand side vanishes as $j \to \infty$ for fixed n. In order to handle the second term, we need the following lemma.

LEMMA 9.3. *Let* ψ_{nj} *be the solution of Problem* L_D. *If* $0 < \varepsilon(n) \le A_{nj} \le K_1$ *in* Q_t *and* $|\text{grad } v_j| \le K_2$ *in* Ω, *then*

$$\frac{\partial\psi_{nj}}{\partial\nu} \geq - C/\varepsilon(n) \text{ on } \partial\Omega \times [0,t],$$

where C *depends on* Ω, χ, K_1 *and* K_2.

The proof of Lemma 9.3 is lengthybut fairly standard. For completeness, we give it in Appendix B.

Since $\varphi'(0) = 0$ and $\varphi''(s) \geq 0$ for $0 < s \leq s_0$, we deduce from (9.1) that

$$A_n \geq \varphi(\frac{1}{n} e^{-Mt})/(\frac{1}{n} e^{-Mt}) \text{ in } Q_t$$

if n is big enough. Since we may assume that $A_{nj} \geq A_n$, it follows from Lemma 9.3 that

$$\varphi(\frac{1}{n} e^{-Mt}) \frac{\partial\psi_{nj}}{\partial\nu} \geq - \frac{C}{n} e^{-Mt} \text{ on } \partial\Omega \times [0,t).$$

Hence (9.3) follows if we let first $j \to \infty$ and then $n \to \infty$ in (9.4). Finally we give a result which we used in section 6.

LEMMA 9.4. *If the solution* $u(t;u_0)$ *of Problem* P *satisfies* $u(t;u_0) = 0$ *on* $\partial\Omega$ *for any* $t \geq 0$, *then* $u(t;u_0)$ *is a solution of Problem* P_D.

PROOF. Let $\psi \in C^{2,1}(\bar{Q})$ with $\psi = 0$ on $\partial\Omega \times \mathbb{R}^+$. Then $u(t;u_0)$ satisfies the integral equality (iii) of definition 7.1. Integrating by parts yields

$$\int_\Omega u(t)\psi(t) = \int_\Omega u_0\psi(0) - \int_0^t \int_{\partial\Omega} \varphi(u) \frac{\partial\psi}{\partial\nu} + \int_0^t \int_\Omega \{\varphi(u)\Delta\psi + u\psi_t$$
$$- u \text{ grad } v \text{ grad } \psi\}.$$

Since $\varphi(u) = 0$ on $\partial\Omega \times \mathbb{R}^+$, the second term at the right-hand side vanishes. Thus $u(t;u_0)$ is a solution of Problem P_D.

APPENDIX A

In this appendix we collect various approximation results which are used in this article.

A1. APPROXIMATION OF v

LEMMA A1. *Let* $v \in W^{1,\infty}(\Omega)$. *Then there exists a sequence* $\{v_\varepsilon\} \subset C^\infty(\bar{\Omega})$ *such that* $||v_\varepsilon||_{C^1(\bar{\Omega})} \leq C$, $||v_\varepsilon||_{C(\bar{\Omega})} \leq ||v||_{L^\infty(\Omega)}$ *and* $||v_\varepsilon - v||_{H^1(\Omega)} \to 0$ *as* $\varepsilon \downarrow 0$.

PROOF. Let $\tilde{\Omega} \supset \Omega$ with $\mathrm{dist}(\Omega, \partial\tilde{\Omega}) > 0$. Then one can extend v by a function $\tilde{v} \in W^{1,\infty}(\tilde{\Omega})$ such that $\tilde{v} = v$ in Ω and $||\tilde{v}||_{L^\infty(\tilde{\Omega})} \leq ||v||_{L^\infty(\Omega)}$. We define the function

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ C \exp\{\dfrac{1}{|x|^2-1}\} & \text{if } |x| < 1 \end{cases}$$

where C is a constant such that $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$. Let

$$v_\varepsilon(x) = \varepsilon^{-N} \int_{\tilde{\Omega}} \rho(\frac{x-y}{\varepsilon})\, \tilde{v}(y)\,dy \qquad \text{for } x \in \tilde{\Omega}.$$

In particular note that $||v_\varepsilon||_{C(\bar{\Omega})} \leq ||v||_{L^\infty(\Omega)}$. Let us suppose that $\varepsilon < \mathrm{dist}(\Omega, \partial\tilde{\Omega})$. Then it is a standard result (see for instance Kufner et.al. [15]) that

$$\text{grad } v_\varepsilon(x) = \varepsilon^{-N} \int_{\tilde{\Omega}} \rho(\frac{x-y}{\varepsilon}) \text{ grad } v(y)\,dy \text{ for } x \in \Omega, \tag{A.1}$$

Then $||\text{grad } v_\varepsilon||_{C(\bar{\Omega})} \leq ||\text{grad } v||_{L^\infty(\tilde{\Omega})}$ and $||v_\varepsilon - v||_{W^{1,p}(\Omega)} \to 0$ as $\varepsilon \downarrow 0$ for every $p \in [1,\infty)$.

LEMMA A2. *Let* $v \in W^{1,\infty}(\tilde{\Omega})$ *be such that* $\Delta v \geq -M$ *in the sense of distributions in* $\tilde{\Omega} \supset \Omega$ *with* $\mathrm{dist}(\Omega, \partial\tilde{\Omega}) > 0$. *Then there exists a sequence* $\{v_\varepsilon\} \subset C^\infty(\bar{\Omega})$ *such that* $||v_\varepsilon||_{C^1(\bar{\Omega})} \leq C$, $\Delta v_\varepsilon \geq -M$ *in* Ω *and* $||v_\varepsilon - v||_{H^1(\Omega)} \to 0$ *as* $\varepsilon \downarrow 0$.

PROOF. In view of the proof of Lemma A1, it remains to show that $\Delta v_\varepsilon \geq -M$. From (A.1) we deduce that

$$\Delta v_\varepsilon(x) = \varepsilon^{-N} < \Delta v(y) , \rho(\frac{x-y}{\varepsilon}) > \qquad \text{for } x \in \Omega$$

where $<\cdot,\cdot>$ denotes the duality pairing between $H_0^1(\tilde{\Omega})$ and $H^{-1}(\tilde{\Omega})$. In particular, since $\Delta v \geq -M$ we have that for $x \in \Omega$

$$\Delta v_\varepsilon(x) \geq -\varepsilon^{-N} M \int_{\tilde{\Omega}} \frac{\rho(x-y)}{\varepsilon} dy = -\varepsilon^{-N} M \int_{\mathbb{R}} \rho(\frac{u}{\varepsilon}) du = -M$$

which yields the result.

A2. APPROXIMATION OF u_0

LEMMA A3. *Let* $u_0 \in L^\infty(\Omega)$ *with* $u_0 \geq 0$ a.e. *and let* v_ε, $v_j \in C^\infty(\bar{\Omega})$ *be such that* $||v_\varepsilon||_{C^1(\bar{\Omega})}$, $||v_j||_{C^1(\bar{\Omega})} \leq C$. *Then*

(i) *there exists a sequence* $\{u_{0\varepsilon}\} \subset C^\infty(\bar{\Omega})$ *such that* $0 \leq u_{0\varepsilon} \leq ||u_0||_{L^\infty(\Omega)}$, $u_{0\varepsilon}$ *satisfies the compatibility condition*

$$\varphi'_\varepsilon(u_{0\varepsilon}) \frac{\partial u_{0\varepsilon}}{\partial\nu} + u_{0\varepsilon} \frac{\partial v_\varepsilon}{\partial\nu} = 0 \text{ on } \partial\Omega \text{ and } ||u_{0\varepsilon} - u_0||_{L^2(\Omega)} \rightarrow 0 \text{ as } \varepsilon \downarrow 0.$$

(ii) *there exists a sequence* $\{u_{0j}\} \subset C^\infty(\bar{\Omega})$ *such that* $0 \leq u_{0j} \leq C$, u_{0j} *satisfies the compatibility condition*

$$\varphi'(u_{0j} + \frac{1}{n}) \frac{\partial u_{0j}}{\partial\nu} + \frac{\partial v_j}{\partial\nu} u_{0j} + (\frac{\partial v_j}{\partial\nu} - A)/n = 0$$

where A *is a given constant and* $||u_{0j} - u_0||_{L^2(\Omega)} \rightarrow 0$ *as* $j \rightarrow \infty$.

PROOF. Since (i) is practically a special case of (ii) we only prove (ii). We define

$$\tilde{u}_{0j}(x) = j^n \int_\Omega \rho(j(x-y)) u_0(y) dy$$

and note that $0 \leq \tilde{u}_{0j} \leq ||u_0||_{L^\infty(\Omega)}$. Let B be a positive constant. It follows from Friedman [9, p. 39] that one can find a function $w_j \in C^{\infty}(\bar{\Omega})$ such that $w_j\big|_{\partial\Omega} = B$ and $\frac{\partial w_j}{\partial\nu}\Big|_{\partial\Omega} = \frac{-1}{\varphi'(B+n^{-1})}\left(\frac{\partial v_j}{\partial\nu} B + (\frac{\partial v_j}{\partial\nu} - A)/n\right)$. Also, since grad v_j is bounded in $C(\bar{\Omega})$ uniformly in j we have that $||w_j||_{C(\bar{\Omega})} \leq C$. Since $B > 0$, there exists $\tilde{\Omega}_j \subset \Omega$ with $\mathrm{dist}(\tilde{\Omega}_j, \partial\Omega) > 0$ such that $w_j > 0$ on $\Omega \setminus \tilde{\Omega}_j$. Finally we choose $\Omega_{1j} \subset \Omega_{2j} \subset \Omega$, such that $\mathrm{dis}(\Omega_{2j}, \partial\Omega) > 0$, $\mathrm{dist}(\Omega_{1j}, \partial\Omega_{2j}) > 0$, $\Omega_{1j} \supset \tilde{\Omega}_j$ and meas $(\Omega \setminus \Omega_{1j}) \leq 1/j$. We define

$$u_{0j}(x) = \begin{cases} \tilde{u}_{0j}(x) & \text{if } x \in \Omega_{1j} \\ \xi_j(x)\tilde{u}_{0j}(x) + (1-\xi_j(x))w_j(x) & \text{if } x \in \Omega_{2j} \setminus \Omega_{1j} \\ w_j(x) & \text{if } x \in \Omega \setminus \Omega_{2j} \end{cases}$$

where ξ_j is a C^{∞} function such that

$$\xi_j(x) = \begin{cases} 1 & \text{if } x \in \Omega_{1j} \\ [0,1] & \text{if } x \in \Omega_{2j} \setminus \Omega_{1j} \\ 0 & \text{if } x \in \Omega \setminus \Omega_{2j}. \end{cases}$$

We have that $u_{0j} \in C^{\infty}(\bar{\Omega})$. Also

$$||u_{0j} - u_0||_{L^2(\Omega)} \leq ||u_{0j} - \tilde{u}_{0j}||_{L^2(\Omega)} + ||\tilde{u}_{0j} - u_0||_{L^2(\Omega)}.$$

$||\tilde{u}_{0j} - u_0||_{L^2(\Omega)}$ can be made arbitrarily small by choosing j large enough.

The term $||u_{0j} - \tilde{u}_{0j}||_{L^2(\Omega)}$ is bounded by $\frac{1}{j}(||u_0||_{L^{\infty}(\Omega)} + ||w_j||_{L^{\infty}(\Omega)})$ which tends to zero as $j \to \infty$.

APPENDIX B

Here we give the proof of Lemma 9.3. It is a generalization of the proof in the case N = 1, which is given by Diaz and Kersner [6].

PROOF OF LEMMA 9.3. Since $\partial\Omega$ is smooth, Ω satisfies the exterior sphere condition, i.e. there exists a number $R_1 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a point $x_1 \in \mathbb{R}^n \setminus \bar{\Omega}$ such that

$$\overline{B(x_1;R_1)} \cap \bar{\Omega} = \{x_0\}$$

where $B(x_1;R_1) = \{x \in \mathbb{R}^n : |x-x_1| < R_1\}$. Since χ has compact support in Ω, there exists a number $R_2 > R_1$ (which does not depend on x_0) such that

$$\chi = 0 \text{ in } U \equiv \Omega \cap B(x_1;R_2). \tag{B.1}$$

We fix $x_0 \in \partial\Omega$ and we define

$$w(x,\tau) = \psi_{nj}(x,\tau) + \sigma(|x-x_0|) \text{ in } \bar{U} \times [0,t].$$

Here the function $\sigma \in C^2([R_1,R_2])$ will be chosen below such that w attains its maximum on $\bar{U} \times [0,t]$ in $\{x_0\} \times [0,t]$. We assume that

$$\sigma' \leq 0 \text{ and } \sigma'' \geq 0 \text{ in } (R_1, R_2). \tag{B.2}$$

Then w satisfies in $U \times [0,t)$

$$w_t + A_{nj}\Delta w - \operatorname{grad} v_j \operatorname{grad} w \geq$$

$$\geq \varepsilon(n)\sigma''(|x-x_1|) + (K_1 \frac{N-1}{R_1} + K_2)\sigma'(|x-x_1|) = 0$$

if we choose

$$\sigma(r) = C_1\varepsilon(n)K^{-1} e^{-Kr/\varepsilon(n)}, \quad R_1 \leq r \leq R_2,$$

where C_1 is an arbitrary constant and $K = K_1(N-1)R_1^{-1} + K_2$. Note that σ satisfies (B.2). Hence w attains its maximum on the parabolic boundary of $U \times [0,t)$. On this boundary we have

$$\begin{cases} w(x,t) = \sigma(|x-x_0|) + \chi(x) \leq \sigma(R_1), \; x \in U \\ w(x,\tau) = \sigma(|x-x_0|) \leq \sigma(R_1), \; x \in \partial U \cap \partial\Omega, \; \tau \in [0,t] \\ w(x,\tau) = \sigma(|x-x_0|) + \psi_{nj}(x,\tau) \leq \sigma(R_2) + 1, \; x \in \partial U \cap \partial B(x_1;R_2), \tau \in [0,t] \\ w(x_0,\tau) = \sigma(R_1), \; \tau \in [0,t], \end{cases}$$

where we used (B.1) and the facts that $\sigma' \leq 0$ and $0 \leq \psi_{nj} \leq 1$ in Q_t. If we choose

$$C_1 = K\{\varepsilon(n)(e^{-KR_1/\varepsilon(n)} - e^{-KR_2/\varepsilon(n)})\}^{-1}$$

then

$$\sigma(R_2) + 1 = \sigma(R_1).$$

Hence

$$w(x,\tau) \leq w(x_0,\tau) \text{ in } U \times [0,t]$$

and

$$\frac{\partial w}{\partial \nu} \geq 0 \text{ on } \{x_0\} \times [0,t].$$

Thus

$$\frac{\partial \psi_{nj}}{\partial \nu} \geq \sigma'(R_1) = -K\{\varepsilon(n)(1-e^{-K(R_2-R_1)/\varepsilon(n)})\}^{-1} \geq$$

$$\geq -K/\varepsilon(n) \text{ on } \{x_0\} \times [0,t].$$

This completes the proof of Lemma 9.3

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CHAPITRE VIII

ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING : THE EFFECT OF A SEDENTARY COLONY

par

M. Bertsch, M.E. Gurtin, D. Hilhorst et L.A. Peletier.

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On interacting populations that disperse to avoid crowding: The effect of a sedentary colony

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Abstract. An analysis is given of a model for two interacting species, one mobile and the other sedentary, in which the mobile one disperses to avoid crowding. The spatial distribution of the mobile species over the habitat, as it evolves with time, is studied. In particular it is shown that a colony of the sedentary species can form an effective barrier against the spreading mobile species, and prevent it from entering certain parts of the habitat.

Key words: Population dynamics — dispersal — interacting species — spatial segregation

1. Introduction

One of the first persons to propose a continuum model for the dispersal of biological populations was apparently Skellam [14], whose assumption of random dispersal led to the partial differential equation

$$\rho_t = \Delta\rho + \sigma(\rho)$$

with ρ the spatial density and $\sigma(\rho)$ a function which represents the supply of individuals due to births and deaths. (Here the subscript t represents differentiation with respect to time, while Δ is the Laplacian.) There are, however, many biological species for which dispersal — rather than being random — is a response to population pressure[1]. To model this phenomenon Gurney and Nisbet [6] and Gurtin and MacCamy [7] introduced a model based on the equation

$$\rho_t = \Delta(\rho^m) + \sigma(\rho),$$

with $m \geq 2$. The tendency of individuals to avoid crowding is reflected by the nonlinear diffusion term $\Delta(\rho^m)$.

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[1] Field studies and experiments demonstrating the effects of population pressure on dispersal are discussed in detail by Okubo [9] and Shigesada [12]

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Recently, Gurtin and Pipkin [8] extended the theory of [6, 7] to include a finite number of interacting biological groups[2]; these groups might consist of different species or of different age classes of the same species. In the model of [8] the densities ρ_n, $n = 1, 2, \dots, N$, are related through the system

$$\rho_{nt} = k_n \operatorname{div}(\rho_n \nabla U) + \sigma_n(\rho_1, \rho_2, \dots, \rho_N) \tag{1}$$

in which

$$U = \rho_1 + \rho_2 + \cdots + \rho_N$$

is the total density, while the coefficients k_n are nonnegative constants. As is clear from (1), the dispersal velocity for species n is $-k_n \nabla U$, so that each species disperses locally toward lower values of *total* population; in this sense the dispersal is a response to population pressure (cf. the discussion of [8] and the references quoted therein).

In this paper we shall analyze, in some detail, a simple example of (1). We consider two species, one *mobile* ($k_1 > 0$) with density $\rho_1 = u$, the other *sedentary* ($k_2 = 0$) with density $\rho_2 = v$; we neglect births and deaths, so that $\sigma_1 = \sigma_2 = 0$; and we assume a one-dimensional habitat. Under these hypotheses the Eqs. (1) reduce to

$$\begin{aligned} u_t &= [u(u+v)_x]_x, \\ v &= v(x), \end{aligned} \tag{2}$$

where x designates the spatial coordinate.

In biological terms, the assumption underlying (2) is that the dispersal of v-individuals — as well as the supply, due to births and deaths, of both species — takes place on a time scale much longer than that characteristic of the dispersal of u-individuals.

We shall suppose that both groups live in a finite habitat

$$\Omega = (-L, L), \qquad L > 0,$$

that individuals are unable to cross the boundary of Ω,

$$u(u+v)_x = 0 \quad \text{at } x = \pm L, \ t > 0, \tag{3}$$

and that initially the mobile species is distributed according to

$$u(x, 0) = u_0(x), \qquad -L \leq x \leq L. \tag{4}$$

Finally, we shall assume throughout that u_0 and v are continuous nonnegative functions on $\bar{\Omega} = [-L, L]$.

In Sect. 2 we study the set $\mathscr{E}$ of equilibrium solutions of (2) and (3). In particular, we show that $\mathscr{E}$ consists of a continuum of continuous functions: each $q \in \mathscr{E}$ is either a function which is positive on $\bar{\Omega}$ and for which $q + v$ is constant, or it has intervals over which $q = 0$ interspaced by intervals over which $q + v$ is constant (the constant may vary from interval to interval).

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[2] See also Busenberg and Travis [3]. An alternative theory was developed by Shigesada, Kawasaki and Teramoto [13]. A detailed discussion of this theory is given in [8]. See also our remark in Sect. 4.

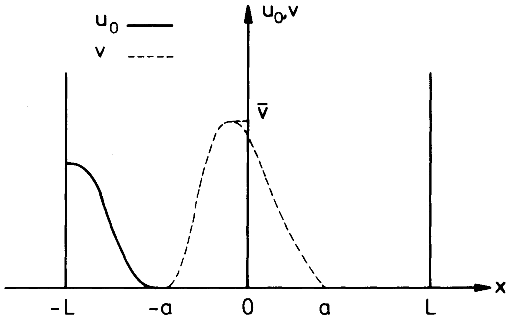

Fig. 1. The initial distributions

In Sect. 3 we turn to the full initial-value problem (2), (3), (4). We define the notion of a weak solution — a notion necessitated by the fact that solutions will generally not be smooth — and we state three theorems: a theorem of existence and uniqueness due to Bertsch and Hilhorst [2], a comparison theorem also due to [2], and a stabilization theorem. The latter theorem shows that the solution $u(x, t)$ of (2), (3), (4) satisfies

$$u(\cdot, t) \to q \in \mathscr{E} \quad \text{as } t \to \infty.$$

As a consequence of this result we show that *when the mobile population is sufficiently large relative to the sedentary population, the mobile species eventually populates the entire habitat.*

In Sect. 4 we consider the converse question: are there circumstances under which the sedentary colony blocks the migration of mobile individuals? Here we shall be interested in the following situation (Fig. 1): the sedentary colony is localized around the center of the habitat, i.e.

$$v(x) = 0 \quad \text{for } 0 < a \leq |x| \leq L;$$

and initially the mobile species lies to one side of this colony, i.e.

$$u_0(x) = 0 \quad \text{for } -a \leq x \leq L.$$

We show that *when*

$$\max_{\bar{\Omega}} u_0 \leq \max_{\bar{\Omega}} v,$$

mobile individuals do not reach the portion of the habitat that lies to the other side of the sedentary colony; that is,

$$u(x, t) = 0 \quad \text{for } a \leq x \leq L,\ t \geq 0.$$

Finally, in Sect. 5 we present a proof of the Stabilization Theorem.

2. Equilibrium solutions

We assume throughout this section that $v \neq 0$ is nonnegative and continuous on $\bar{\Omega}$.

Let $q(x)$ be a time-independent solution of (2) and (3). Then, clearly

$$[q(q+v)']' = 0 \quad \text{on } \Omega,$$

$$q(q+v)' = 0 \quad \text{on } \partial\Omega,$$

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or equivalently (for $q(q+v)'$ absolutely continuous on $\bar{\Omega}$)

$$q(q+v)'=0 \quad \text{on } \bar{\Omega}$$

(prime denotes differentiation). This should motivate the following definition.

An *equilibrium solution* is a nonnegative function q on $\bar{\Omega}$ with:

(i) $q+v$ absolutely continuous on $\bar{\Omega}$;
(ii) $q(q+v)'=0$ on $\bar{\Omega}$.

The next proposition gives a useful decomposition of the set $\mathscr{E}$ of equilibrium solutions.

Proposition 1. *Let*

$$\bar{v}=\max_{\Omega} v.$$

Then

$$\mathscr{E}=\mathscr{E}_1 \cup \mathscr{E}_2$$

with

$$\mathscr{E}_1=\{q \in \mathscr{E}\colon q+v>\bar{v} \text{ on } \bar{\Omega}\},$$

$$\mathscr{E}_2=\{q \in \mathscr{E}\colon q+v \leq \bar{v} \text{ on } \bar{\Omega}\}.$$

Proof: Let $q \in \mathscr{E}$ and let $\mathscr{P}$ be a connected component of the set

$$\{x \in \bar{\Omega}\colon q(x)+v(x)>\bar{v}\}.$$

Then $q>0$ on $\mathscr{P}$, and (ii) yields

$$q(x)+v(x)=c>\bar{v} \quad \text{on } \mathscr{P},$$

with c constant. Thus, since $q+v$ is continuous on $\bar{\Omega}$, $\mathscr{P}$ must coincide with its closure, a possibility only if $\mathscr{P}=\bar{\Omega}$ or $\mathscr{P}=\varnothing$. Since q belongs to $\mathscr{E}_1$ or $\mathscr{E}_2$ according as $\mathscr{P}=\bar{\Omega}$ or $\mathscr{P}=\varnothing$, this completes the proof.

It is clear from the proof above that

$$\mathscr{E}_1=\{c-v\colon c=\text{constant}>\bar{v}\}$$

(cf. Fig. 2a). The specification of $\mathscr{E}_2$ is not so simple; we can, however, give a complete characterization when the set

$$\Omega_c=\{x \in \Omega\colon v(x)>c\}$$

is connected for each $c \in [0, \bar{v})$. In this instance, if we let $v(\bar{x})=\bar{v}$ and define, for $b, c \in [0, \bar{v}]$,

$$\hat{q}(x, b, c)=\begin{cases}\max\{0, b-v(x)\}, & -L \leq x \leq \bar{x},\\ \max\{0, c-v(x)\}, & \bar{x} \leq x \leq L,\end{cases} \tag{5}$$

it follows that (cf. Fig. 2b)

$$\mathscr{E}_2=\{\hat{q}(\cdot, b, c)\colon b, c \in [0, \bar{v}]\}.$$

In the event the set Ω_c consists of two or more disjoint subsets for some values of $c \in [0, \bar{v})$ the set of equilibrium solutions becomes more complicated. An example is given in Fig. 3.

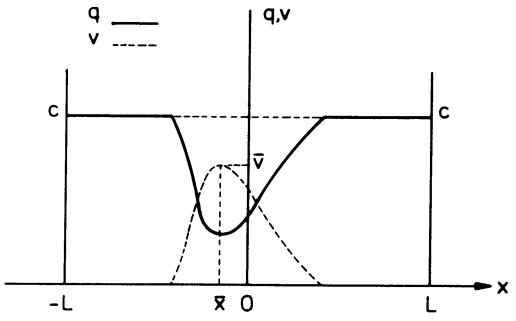
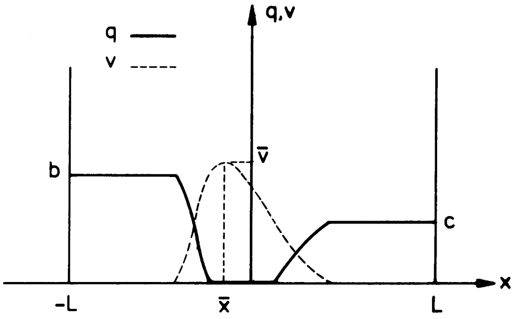


Fig. 2a. $q \in \mathscr{C}_1$



b. $q \in \mathscr{C}_2$; Ω_c connected for every $c \in [0, \bar{v})$

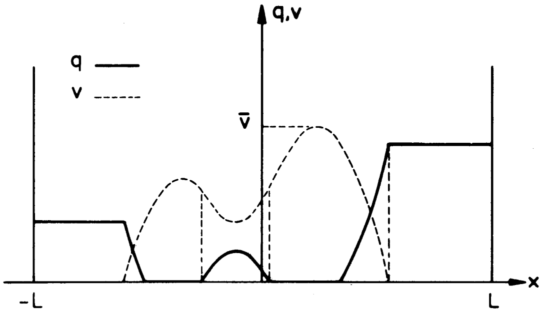


Fig. 3. $q \in \mathscr{C}_2$; Ω_c not connected for some $c \in [0, \bar{v})$

3. General results

Stabilization

We now turn to the evolution of the distribution of mobile individuals and consider the solution $u(x, t)$ of the problem:

$$\text{(I)} \qquad \begin{cases} u_t = [u(u+v)_x]_x & \text{for } x \in \Omega,\ t > 0, & (2) \\ u(u+v)_x = 0 & \text{for } x \in \partial\Omega,\ t > 0, & (3) \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega. & (4) \end{cases}$$

Equation (2) is a *degenerate* diffusion equation. If we write $w = u + v$, then (2) becomes

$$w_t = uw_{xx} + u_x w_x,$$

an equation for w which — for fixed u — is parabolic when $u > 0$, hyperbolic when $u = 0$. Further, for $v = 0$, (2) reduces to the porous media equation

$$u_t = \tfrac{1}{2}(u^2)_{xx},$$

and equation which has been studied extensively in recent years. (We refer to the survey article of Peletier [11].)

Because of the degeneracy of Eq. (2), Problem I may have solutions which are not smooth (cf. the equilibrium solutions sketched in Fig. 2). We therefore introduce the notion of a weak solution, a notion inspired by its analog for the porous media equation (cf. [1]).

Let

$$Q = \Omega \times (0, \infty),$$

and for any function $f(x, t)$ let $f(t) = f(\cdot, t)$.

Definition. A (weak) *solution* $u(x, t)$ of Problem I is a bounded continuous function on $\bar{Q}$ with the property

$$\int_\Omega u(t)\psi(t) = \int_\Omega u_0\psi(0) + \int_0^t \int_\Omega (\tfrac{1}{2}u^2\psi_{xx} + u\psi_t - uv_x\psi_x) \tag{6}$$

for any $t > 0$ and $\psi \in C^2(\bar{Q})$ such that $\psi \geqslant 0$ on Q and $\psi_x(\pm L, t) = 0$ for all $t \geqslant 0$.

If we take $\psi(x, t) \equiv 1$ in (6), we arrive at the *conservation law*

$$\int_\Omega u(t) = \int_\Omega u_0 \quad \text{for all } t \geqslant 0, \tag{7}$$

which asserts that the total number of mobile individuals does not change with time. This is consistent with the assumption that individuals do not cross the boundary of the habitat.

We now state two general results for Problem I. The first states existence and uniqueness, the second gives a comparison principle which we will use repeatedly in what follows. These results, which are due to Bertsch and Hilhorst [2], are based on the following assumptions:

A1. There exists a constant $K > 0$ such that

$$|v(x) - v(y)| \leqslant K|x - y| \quad \text{for all } x, y \in \bar{\Omega}.$$

A2. There exists a constant M such that

$$\frac{v'(x) - v'(y)}{x - y} \geqslant -M \quad \text{for almost all } x, y \in \bar{\Omega}, x \neq y.$$

Proposition 2. *Let v satisfy* A1, *and let $u_0 \in C(\bar{\Omega})$ be nonnegative. Then Problem* I *has a solution u on $\bar{Q}$. If, in addition,* A2 *is satisfied, the solution is unique.*

Proposition 3 (Comparison Principle). *Let v satisfy* A1 *and* A2. *Let u and $\tilde{u}$ be solutions of Problem* I *with initial data $u_0, \tilde{u}_0 \in C(\bar{\Omega})$, $u_0, \tilde{u}_0 \geqslant 0$. Then*

$$\tilde{u}_0 \leqslant u_0 \quad in\ \Omega \Rightarrow \tilde{u} \leqslant u \quad in\ \bar{Q}.$$

A direct consequence of the Comparison Principle is that solutions of Problem I are nonnegative.

The next theorem asserts that solutions stabilize. The proof, which is a bit technical, will be given in Sect. 5.

Stabilization Theorem. *Let v satisfy* A1 *and* A2, *and let $u_0 \in C(\bar{\Omega})$ be nonnegative. Let u be the solution to Problem* I. *Then there exists an equilibrium solution q such that*

$$u(t) \to q \quad in\, C(\bar{\Omega}) \quad as\, t \to \infty. \tag{8}$$

Moreover,

$$\int_\Omega q = \int_\Omega u_0. \tag{9}$$

The following corollary of the Stabilization Theorem shows that if the mobile population is large enough, this species will eventually populate the entire habitat.

Corollary. *Assume, in addition to the hypotheses of the Stabilization Theorem, that*

$$\int_\Omega u_0 > \int_\Omega (\bar{v} - v). \tag{10}$$

Then for some $t_0 > 0$,

$$u(x, t) > 0 \quad for\, all\, x \in \bar{\Omega},\, t \geq t_0. \tag{11}$$

Proof: By the Stability Theorem $u(t)$ tends to a limit $q \in \mathscr{E}$, and by (9) and (10),

$$\int_\Omega (q + v - \bar{v}) > 0.$$

We may therefore conclude from Proposition 1 that $q \in \mathscr{E}_1$, and hence that

$$q > \bar{v} - v \geq 0 \quad \text{on } \bar{\Omega}.$$

Thus, since $u(t) \to q$ in $C(\bar{\Omega})$, there exists a t_0 such that (11) is satisfied.

To conclude this section consider the situation described in Fig. 1. Let

$$\bar{u}_0 = \max_{\bar{\Omega}} u_0, \qquad \bar{v} = \max_{\bar{\Omega}} v \tag{12}$$

and suppose that

$$\bar{u}_0 \leq \bar{v}.$$

Then $\hat{q}(x, \bar{u}_0, 0)$, defined by (5), is an equilibrium solution in $\mathscr{E}_2$, and

$$u_0(x) \leq \hat{q}(x, \bar{u}_0, 0) \quad \text{for } x \in \bar{\Omega}.$$

Thus, by the Comparison Principle,

$$0 \leq u(x, t) \leq \hat{q}(x, \bar{u}_0, 0) \quad \text{for } x \in \bar{\Omega},\, t \geq 0, \tag{13}$$

and hence, since by the Stabilization Theorem $u(t) \to q \in \mathscr{E}$ as $t \to \infty$,

$$0 \leq q(x) \leq \hat{q}(x, \bar{u}_0, 0)$$

(cf. Fig. 2b). Since $q \in \mathscr{C}$, q must belong to $\mathscr{C}_2$; hence

$$q(x) = \hat{q}(x, b, 0)$$

for some $b \in (0, u_0]$, which — using the Stabilization Theorem again — can be determined uniquely from the relation

$$\int_\Omega \hat{q}(x, b, 0)\,dx = \int_\Omega u_0(x)\,dx.$$

In a similar manner, if (10) holds, then $q \in \mathscr{C}_1$, and hence

$$q(x) = c - v(x)$$

for some $c > \bar{v}$, which can be determined uniquely from (9); the result is

$$c = \frac{1}{2L}\int_\Omega [u_0(x) + v(x)]\,dx.$$

4. The Barrier Theorem

In view of the corollary to the Stabilization Theorem, if the mobile population is large enough this species will eventually populate the entire habitat. This motivates our asking whether there are conditions under which the sedentary colony blocks the migration of mobile individuals. The next theorem gives such a condition for the situation sketched in Fig. 1. In particular, it is shown that if $\max u_0 \leq \max v$ then the mobile species never reaches that portion of the habitat which lies to the other side of the sedentary colony.

Barrier Theorem. *Let v satisfy* A1, A2, *and the condition*

$$v(x) = 0 \quad \textit{for } 0 < a \leq |x| \leq L.$$

Let $u_0 \in C(\bar{\Omega})$ be nonnegative and satisfy

$$u_0(x) = 0 \quad \textit{for } -a \leq x \leq L.$$

Then if

$$\max_{\Omega} u_0 \leq \max_{\Omega} v,$$

it follows that

$$u(x, t) = 0 \quad \textit{for } a \leq x \leq L,\ t \geq 0.$$

Proof: Let

$$x_0 = \sup\{x \in \Omega\colon v \leq \bar{u}_0 \text{ on } [-L, x]\}.$$

Then, by (5) and (13),

$$u(x, t) = 0 \quad \text{for } x_0 \leq x \leq L,\ t \geq 0. \tag{14}$$

Since $x_0 < a$, the theorem is proved.

The Barrier Theorem gives a condition on u_0 which insures that mobile individuals do not disperse completely through the sedentary colony. In a similar manner, one can prove that given the right distribution of sedentary individuals — for example the one sketched in Fig. 3 — the mobile species may get trapped *inside* the sedentary colony.

Granted the hypotheses of the Barrier Theorem, it seems reasonable to ask how far into the sedentary colony the mobile species will penetrate. To make this idea precise, note that

$$x_u = \sup\{x \in \Omega\colon\ u(x, t) > 0 \text{ at some } t > 0\}$$

represents the furthest point reached by mobile individuals, while

$$x_v = \sup\{x \in \Omega\colon\ v = 0 \text{ on } [-L, x]\}$$

marks the start of the sedentary colony. We define the ***depth of penetration*** $d(u)$ by

$$d(u) = x_u - x_v.$$

This definition makes sense for $u_0 \not\equiv 0$. Indeed, since each equilibrium solution q is continuous on $\bar{\Omega}$ and constant on $[-L, x_v]$, each nontrivial q satisfies $q \not\equiv 0$ in (x_v, L); it therefore follows from the Stabilization Theorem that $x_u > x_v$, and hence that $d(u) > 0$.

By (14),

$$d(u) \leq x_0 - x_v,$$

and it is clear from Fig. 4 that the steeper the v-distribution to the right of x_v, the smaller the value of this upper bound. With this in mind, consider the situation shown in Fig. 5, in which v has a jump discontinuity of magnitude larger than

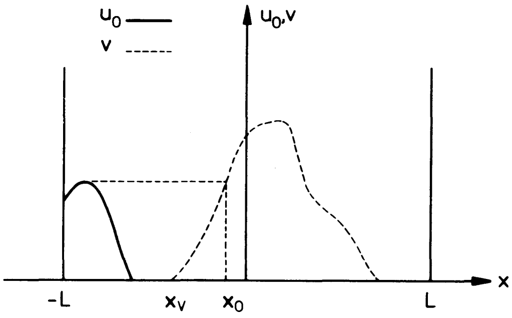


Fig. 4. The points x_0 and x_v

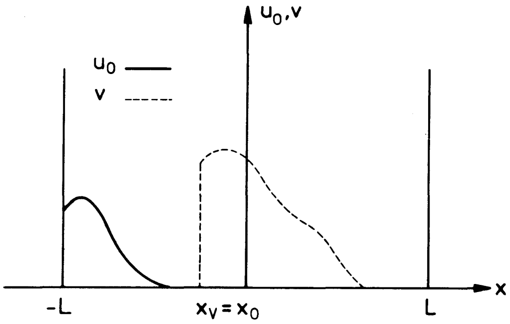


Fig. 5. The case $x_0 = x_v$

$\bar{u}_0$. Of course, discontinuous functions v do not fit into the framework studied here, but an obvious limiting argument leads us to expect that if a solution u exists, its depth of penetration is zero. Such a result would be interesting as it yields *segregation of the two species for all time.*

Remark: An alternative model for species which disperse to avoid crowding is given by Shigesada et al. [13]. (Cf. the discussion of [8].) For two species, one mobile and one sedentary, their model leads to the partial differential equation

$$u_t = [u(u+v)]_{xx}, \qquad x \in \Omega, \qquad t > 0,$$

and the boundary condition

$$[u(u+v)]_x = 0, \qquad x \in \partial\Omega, \qquad t > 0.$$

For this model one can show, using a stabilization theorem analogous to ours, that the mobile species *always* spreads through the entire habitat.

5. Proof of the Stabilization Theorem

We assume throughout this section that v satisfies A1 and A2, and that all initial data referred to are continuous and nonnegative on $\bar{\Omega}$. Further, we will use the notation (12).

Lemma 1. *The solution u of Problem* I *satisfies*

$$u(x, t) \leqslant \bar{c} - v(x) \quad \text{for} - L \leqslant x \leqslant L, t \geqslant 0,$$

where

$$\bar{c} = \bar{u}_0 + \bar{v}.$$

Proof: Let $q(x) = \bar{c} - v(x)$. Clearly, q is an equilibrium solution and

$$q(x) \geqslant \bar{u}_0 \geqslant u_0(x) \quad \text{for } -L \leqslant x \leqslant L.$$

Hence, by the Comparison Principle,

$$u(x, t) \leqslant q(x) \quad \text{for } -L \leqslant x \leqslant L, t \geqslant 0.$$

Our proof of stabilization is inspired by an approach due to Osher and Ralston [10]; it is based on a contraction property which we state as Lemma 2. To state this lemma concisely, we write

$$\|f\| = \int_\Omega |f(x)| \, dx$$

for the $L^1(\Omega)$-norm, and we say that two functions f and g on $\bar{\Omega}$ *intertwine* if there is an interval $I \subset \Omega$ such that

$$f > 0 \quad \text{and} \quad g > 0 \quad \text{on } \bar{I},$$

$$f - g \quad \text{changes sign on } I.$$

Lemma 2. *Let u_1 and u_2 be solutions of Problem* I *with initial data u_{01} and u_{02}. Then*

$$\|u_1(t) - u_2(t)\| \leqslant \|u_{01} - u_{02}\| \quad \text{for } t \geqslant 0.$$

If in addition u_{01} *and* u_{02} *intertwine, then*

$$\|u_1(t) - u_2(t)\| < \|u_{01} - u_{02}\| \quad \textit{for } t > 0.$$

Because the proof of this lemma is quite technical, we shall not give it here; it can be found in [2].

Proof of the Stabilization Theorem: It is useful to view the solution u of Problem I as tracing out a continuous orbit

$$\gamma = \{u(t)\colon t \geq 0\}$$

in the space $C(\bar{\Omega})$. The omega-limit set ω of γ is defined by

$$\omega = \{w \in C(\bar{\Omega})\colon \exists\{t_n\}, \qquad t_n \to \infty \quad \text{as } n \to \infty,$$

such that

$$u(t_n) \to w \quad \text{in } C(\bar{\Omega}) \text{ as } n \to \infty\}.$$

By a result of DiBenedetto [5], the set

$$\gamma_\tau = \{u(t)\colon t \geq \tau > 0\}$$

is precompact in $C(\bar{\Omega})$ for any $\tau > 0$. Hence ω contains at least one element.

Let $w \in \omega$. We shall show that $w \in \mathscr{E}$. Suppose $w \notin \mathscr{E}$. We shall construct an element $\bar{q} \in \mathscr{E}$ which intertwines with w. We choose $\bar{q}$ from the family of functions

$$q_c(x) = \max\{0, c - v(x)\}, \quad c \geq 0.$$

By Lemma 1,

$$0 = q_0(x) \leq w(x) \leq q_{\bar{c}}(x) \quad \text{for } -L \leq x \leq L.$$

Hence there exists a $d \in (0, \bar{c})$ such that q_d and w intertwine. We take $\bar{q} = q_d$.

Consider the function $V\colon C(\bar{\Omega}) \to [0, \infty)$ defined by

$$V(z) = \|z - \bar{q}\|.$$

Because $u \in C(\bar{Q})$, $V(u(t))$ is a continuous function of t which by Lemma 2, is nonincreasing. Thus V is a *Lyapunov function.*

Let $\hat{w}(x, t)$ denote the solution of Problem I with initial data $w(x)$. By the invariance principle, ω is an invariant set and hence $\hat{w}(t) \in \omega$ for all $t \geq 0$. Because V is constant on ω (cf. [4]), it follows that

$$V(\hat{w}(t)) = V(w) \quad \text{for all } t \geq 0.$$

On the other hand, since $\bar{q}$ and w intertwine, we conclude from Lemma 2 that

$$V(\hat{w}(t)) < V(w) \quad \text{for all } t > 0,$$

and we have a contradiction. Thus $w \in \mathscr{E}$, and hence $\omega \subset \mathscr{E}$.

We show next that ω consists of one element only. Suppose that $w_1, w_2 \in \omega$ and $u(t_{in}) \to w_i$ $(i = 1, 2)$ as $n \to \infty$. Suppose further that the sequences $\{t_{1n}\}$ and

$\{t_{2n}\}$ are so chosen that $t_{1n} < t_{2n}$ for all $n \geq 1$. Then, using Lemma 2 we find that

$$\begin{aligned} \|w_2 - w_1\| &= \lim_{n\to\infty} \|u(t_{2n}) - w_1\| \\ &\leq \lim_{n\to\infty} \|u(t_{1n}) - w_1\| \\ &= 0. \end{aligned}$$

Thus $w_1 = w_2$. Hence ω consists of one element.

Writing $\omega = \{q\}$, we are led at once to (8), and this result combined with the conservation law (7) yields (9). This completes the proof.

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CHAPITRE IX

THE EQUATION $C(Z)_T = (|Z_X|^{M-1}Z_X)_X$: THE FREE BOUNDARY INDUCED BY A DISCONTINUITY IN THE DERIVATIVE OF C .

par

M. Bertsch, M.E. Gurtin et D. Hilhorst.

1. Introduction

In this paper we consider the nonlinear diffusion problem

$$(\mathrm{I})\quad \begin{cases} c(z)_t = (|z_x|^{m-1} z_x)_x \quad \text{in } Q := (-L,L) \times \mathbb{R}^+ & (1.1) \\ z(-L,t) = -U \ , \ z(+L,t) = V \quad \text{for } t > 0 \\ z(x,0) = z_0(x) \quad \text{for } x \in (-L,L) \end{cases}$$

where $\mathbb{R}^+ = (0,\infty)$, L, U and V are positive constants , $m > 1$, and

$$c(s) = \begin{cases} c^- s & \text{for } s \leq 0 \\ c^+ s & \text{for } s \geq 0 \ , \end{cases}$$

with $c^{\pm} > 0$. The initial function z_0 satisfies

H1. $z_0 \in C^1([-L,L])$, $z_0(-L) = -U$, $z_0(+L) = V$, $z_0' \geq 0$ on $[-L,L]$,

and the set $\{x \in [-L,L] : z_0'(x) = 0\}$ is empty or consists of a finite number of non-empty closed connected subsets of $[-L,L]$.

Problem I arises in the theory of population dynamics. Let $u(x,t)$ and $v(x,t)$ denote the densities of two populations. Following Gurtin and Pipkin [18] , we assume that the dispersal velocities of u and v are proportional to $-(u+v)_x$, i.e. the dispersal is a response to the population pressure. Neglecting birth and death processes, and assuming that no individuals can leave the habitat $(-L,L)$, we arrive at the problem

$$
(II) \begin{cases} u_t = k_1(u(u+v)_x)_x & \\ & \text{in } Q \\ v_t = k_2(v(u+v)_x)_x & \\ u(u+v)_x = v(u+v)_x = 0 & \text{for } x = \pm L,\ t > 0 \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{for } x \in (-L,L) \ . \end{cases}
$$

Here k_1 and k_2 are positive constants (the case $k_1 > 0$, $k_2 = 0$ was discussed in [7,9]), and the initial distributions u_0 and v_0 are given nonnegative functions.

The connection between the Problems I and II is as follows. Consider the special case of Problem II where initially the two populations are segregated : for some $a \in (-L,L)$

$$u_0(x) \equiv 0 \text{ for } x > a \text{ and } v_0(x) \equiv 0 \text{ for } x < a \ .$$

The question is whether Problem II has a solution pair (u,v) such that $u(.,t)$ and $v(.,t)$ are segregated for all later times. To answer this question we introduce the function

$$z(x,t) \ := -U + \int_{-L}^{x} \{u(s,t) + v(s,t)\}\, ds \quad , \quad (x,t) \in \overline{Q} \ .$$

Using that the total densities of both populations are conserved, i.e.

$$\int_{-L}^{L} u(x,t)dx = U \ := \int_{-L}^{L} u_0(x)dx$$

and

$$\int_{-L}^{L} v(x,t)dx = V \ := \int_{-L}^{L} v_0(x)dx \ ,$$

we find that formally z satisfies Problem I , with

$$m = 2 \ , \quad c^- = 2/k_1 \ , \quad c^+ = 2/k_2 \ ,$$

and

$$z_0(x) = -U + \int_{-L}^{x} \{u_0(s) + v_0(s)\}\, ds \ .$$

For more details about the relation between the Problems I and II , we refer to [8] .

The purpose of the present paper is to study Problem I in detail. The results are used in [8] to construct a pair of segregated solutions of Problem II .

Two mathematical difficulties arise from the differential equation (1.1). First we observe that the equation is of degenerate parabolic type : at points where $z_x = 0$, it looses its parabolicity. In addition, when $c^+ \neq c^-$, the function $c(z)$ is not differentiable at $z = 0$. To prove the existence of a unique solution z of Problem I , there are more or less standard methods to overcome these difficulties. The basic results are given in section 2 ; the proofs are postponed to the appendix.

Since $z_0' \geq 0$ in $(-L,L)$, it follows easily that the solution z satisfies

$$z_x \geq 0 \quad \text{and} \quad -U \leq z \leq V \quad \text{in } \overline{Q} \qquad (1.2\ )$$

The purpose of this paper is to give a detailed description of the sets

$$Q^- = \{(x,t) \in \overline{Q} \ : \ -U \leq z(x,t) < 0 \ \text{ and } \ z_x(x,t) > 0\}$$

and

$$Q^+ = \{(x,t) \in \overline{Q} \ : \ 0 < z(x,t) \leq V \ \text{ and } \ z_x(x,t) > 0\} \ .$$

In view of (1.2) , Q^- and Q^+ are completely determined by the sets

$$N(z) = \{(x,t) \in \overline{Q} \ : \ z(x,t) = 0\}$$

and

$$N(z_x) = \{(x,t) \in \overline{Q} \ : \ z_x(x,t) = 0\} \ .$$

Observe that all these sets are interesting in view of Problem II : Q^- (resp. Q^+) is the set where $u > 0$ and $v = 0$ (resp. $v > 0$ and $u = 0$) , $N(z)$ is the set which separates the regions where $u > 0$ (Q^-) and $v > 0$ (Q^+) , and $N(z_x)$ is the set where both $u = 0$ and $v = 0$. Furthermore we notice that $N(z)$ and $N(z_x)$ are precisely the sets where equation (1.1) is not regular.

In this paper we shall describe $N(z)$ and $N(z_x)$ in detail. For the precise results we refer to section 3 . As an illustrative example we describe here the results in the case that the interval $\{x \in [-L,L] : z_0(x) = 0\}$ has a positive measure, which implies that the sets where $z_0 = 0$ and $z_0' = 0$ overlap. First we observe that, since $z_x \geq 0$, there exist functions $\zeta^{\pm} : [0,\infty) \to (-L,L)$ such that

$$N(z) = \{(x,t) \in \overline{Q} \ : \ \zeta^-(t) \leq x \leq \zeta^+(t) \ , \quad t \geq 0\} \ .$$

In this paper we shall show that ζ^- and ζ^+ are continuous and that there exists a time $T^* > 0$ such that

$$\zeta^-(t) < \zeta^+(t) \qquad \text{for} \quad t \in [0,T^*)$$

and

$$\zeta(t) \ := \zeta^-(t) = \zeta^+(t) \qquad \text{for} \quad t \geq T^* \ .$$

In addition ζ^- is nondecreasing and ζ^+ is nonincreasing on $[0,T^*]$, and

$$z_x(\zeta(t),t) > 0 \qquad \text{for} \quad t > T^*,$$

i.e. $$N(z) \cap N(z_x) = \{(x,t) \in \overline{Q} : \zeta^-(t) \ \leq x \leq \zeta^+(t) \ , \quad 0 \leq t \leq T^*\} \ .$$

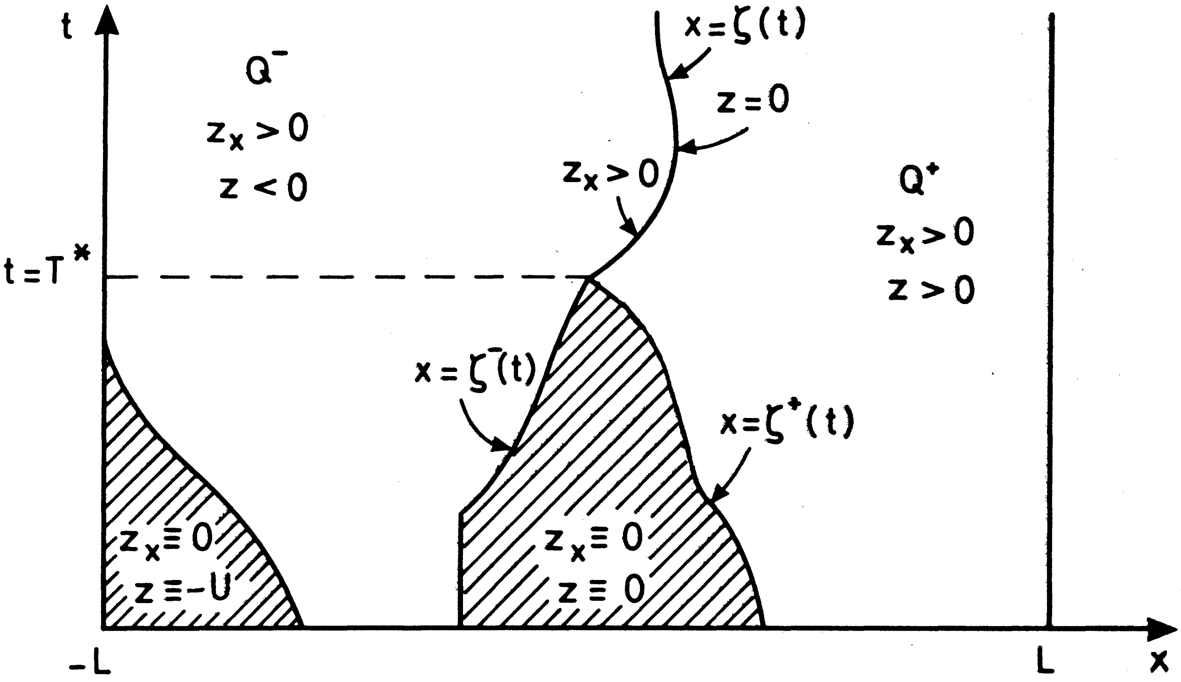

FIGURE 1 .

We prove the main results in the sections 4, 5 and 6 . In section 5 we characterize the sets $N(z_x)$ and $N(z) \cap N(z_x)$. In section 6 we study the level sets of z in the region where $z_x > 0$ and in particular the set $N(z) \setminus N(z_x)$.

In section 4 we introduce the most important tool in the proofs, which is of independent interest and which we discuss here in some detail. We consider an approximating sequence of regularized Problems I_n, whose solutions z_n are such that $z_{nx} \geq 1/n$. Hence the level sets of z_n are actually curves : $x = X_n(p,t)$, defined by

$$z_n(X_n(p,t),t) = z_n(p,0) \qquad \text{for} \quad p \in [-L,L] .$$

In this way we have introduced a coordinate transformation in $\overline{Q}$: $(x,t) \to (p,t)$.

Following an idea by Gurtin, MacCamy and Socolovsky [17] , we show that the function $X_n(p,t)$ satisfies a parabolic differential equation in Q . Using this equation we derive in section 4 the estimate

$$|X_{nt}| \leq \mathscr{C}/t \qquad \text{for } t > 0 , \tag{1.3}$$

or, alternatively, if we assume in addition that $((z_0')^{m-1})'$ is bounded in $(-L,L)$,

$$|X_{nt}| \leq \mathscr{C} \qquad \text{for } t > 0 , \tag{1.4}$$

where the constant $\mathscr{C}$ does not depend on n . These and other estimates for X_n are important for the analysis in the sections 5 and 6 .

In section 7 we discuss the large-time behaviour of the solution z . We show that $z(.,t)$ converges exponentially to the unique steady-state solution of Problem I .

Finally we remark that nearly all the results carry over to the case that the function $s \rightarrow |s|^{m-1} s$ in equation (1.1) is replaced by a more general function φ , which is smooth enough and satisfies $\varphi(0) = \varphi'(0) = 0$, $\varphi'(s) > 0$ for $s \neq 0$, and

$$\int_0^1 \frac{\varphi'(s)}{s} \, ds < \infty .$$

Only the estimate (1.3) cannot be easily generalized to this case, but when we assume that $(\Phi(z_0'))'$ is bounded on $(-L,L)$, where

$$\Phi(s) := \int_0^s \frac{\varphi'(\tau)}{\tau} \, d\tau ,$$

then the estimate (1.4) follows easily.

2 . Preliminaries

First we give the definition of a (weak) solution of Problem I . The unique steady-state solution of Problem I is denoted by $\bar{z}$:

$$\bar{z}(x) = \frac{U + V}{2L} (x + L) - U \ , \quad -L \leq x \leq L \ .$$

We shall use the notation

$$\Omega = (-L,L) \quad \text{and} \quad Q_T = \Omega \times (0,T] \ .$$

Definition 2.1 . *A function* $z \in C([0,\infty) : L^1(\Omega))$ *is a solution of Problem I if*

(i) $z - \bar{z} \in L^\infty((0,\infty) : W^{1,\infty}(\Omega) \cap H_0^1(\Omega))$;

(ii) $z_t \in L^2(Q_T)$ *for all* $T > 0$;

(iii) $z(0) = z_0$;

(iv) *for all* $\psi \in C^1(\bar{Q})$ *with* $\psi(\pm L,t) = 0$ *for* $t > 0$, z *satisfies for all* $T > 0$

$$\iint_{Q_T} \{(c(z))_t \ \psi + |z_x|^{m-1} \ z_x \ \psi_x\} = 0 \ .$$

Observe that if z is a solution of Problem I , then it satisfies equation (1.1) a.e. in Q .

We shall prove the existence of a unique solution of Problem I under slightly more general conditions on the initial function z_0 .

H2 . $z_0 \in W^{1,\infty}(-L,L)$, $z_0(-L) = -U$, $z_0(+L) = V$, $z_0' \geq 0$ a.e. in $(-L,L)$.

Theorem 2.2 . *Let* z_0 *satisfy hypothesis* H2 .

(i) Problem I *has a unique solution* z ;

(ii) $z \in C(\bar{Q})$, $z_x \in C(\bar{\Omega} \times (0,\infty))$, *and* $z_x \geq 0$ *in* Q ;

(iii) the set $\{z(.,t) : t \geq 1\}$ *is precompact in* $C^1(\bar{\Omega})$;

(iv) if $z_0 \in C^1(\bar{\Omega})$, *then* $z_x \in C(\bar{Q})$, *and the set* $\{z(.,t) : t \geq 0\}$ *is precompact in* $C^1(\bar{\Omega})$.

We give the proof of Theorem 2.2 in the appendix. There we shall also show that z can be approximated by solutions z_n of more regular problems. Since we need this approximation in the rest of this paper, we give some details here already. For all proofs we refer to the appendix.

First we approximate the initial function z_0.

Lemma 2.3 . *Let* z_0 *satisfy hypothesis* H2 . *Then there exists a sequence of functions* $\{z_{0n}, n = 1,2,\ldots\} \subset C^\infty(\overline{\Omega})$ *such that*

(i) $z_{0n}(-L) = -U$, $z''_{0n} = 0$ *in a neighbourhood of* $x = \pm L$, *and*

$1/n \leq z'_{0n} \leq \|z'_0\|_\infty + c/n$ *in* Ω *for some* $c > 0$ *;*

(ii) $z_{0n} \to z_0$ *in* $C(\overline{\Omega})$ *as* $n \to \infty$*;*

(iii) *if* $z_0 \in C^1(\overline{\Omega})$, *then* $z_{0n} \to z_0$ *in* $C^1(\overline{\Omega})$ *as* $n \to \infty$;

(iv) *if* $((z'_0)^{m-1})' \in L^\infty(\Omega)$, *then* $\|((z'_{0n})^{m-1})'\|_\infty \leq \|((z'_0)^{m-1})'\|_\infty$.

Here $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ $(1 \leq p \leq \infty)$.

Consider for $T > 0$ the problems

$$(I_n) \begin{cases} c_n(z)_t = (|z_x|^{m-1} z_x)_x & \text{in } Q_T \\ z(-L,t) = -U \ ; \ z(L,t) = V_n & \text{for } 0 < t \leq T \\ z(x,0) = z_{0n}(x) & \text{for } x \in \Omega \ , \end{cases}$$

where z_{0n} is given by Lemma 2.3, $V_n = z_{0n}(L)$, and $c_n \in C^\infty(\mathbb{R})$ satisfies

$c_n \to c$ uniformly on $\mathbb{R}$ as $n \to \infty$,

$c'_n \to c'$ uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$ as $n \to \infty$,

$\min\{c^+,c^-\} \leq c'_n \leq \max\{c^+,c^-\}$ on $\mathbb{R}$.

Lemma 2.4 . *For all* $T > 0$ *Problem* I_n *has a unique (classical) solution* $z_n \in C^{2+\alpha,1+\frac{1}{2}\alpha}(\overline{Q}_T)$ *for each* $\alpha \in (0,1)$. *In addition* z *satisfies*

$$\frac{1}{n} \leq z_{nx} \leq \| z_0' \|_\infty + \frac{c}{n} \text{ in } \overline{Q}_T . \tag{2.1}$$

and

$$\iint_{Q_T} z_{nt}^2 \leq \mathscr{C} , \tag{2.2}$$

where $\mathscr{C}$ *does not depend on* n , *and where* c *is given by Lemma* 2.3 *(i)* .

In the appendix we shall prove that the sequence z_n converges to the unique solution z of Problem I :

$$z_n \to z \text{ in } C(\overline{Q}_T) \text{ as } n \to \infty ,$$

and, for $\tau \in (0,T)$,

$$z_{nx} \to z_x \text{ in } C(\overline{\Omega} x [\tau,T]) \text{ as } n \to \infty . \tag{2.3}$$

Finally, when $z_0 \in C^1(\overline{\Omega})$, then

$$z_{nx} \to z_x \text{ in } C(\overline{Q}_T) \text{ as } n \to \infty . \tag{2.4}$$

Remark. Theorem 2.2 can be proved without the rather restrictive condition $z_0' \geq 0$ in hypothesis H2 . In that case however we need a different approximation of Problem I , since the sequence z_{0n} in Lemma 2.3 no longer exists. Instead one can choose an approximation in which equation (1.1) is replaced by

$$c_n(z)_t = ((|z_x|^{m-1} + \frac{1}{n}) z_x)_x \text{ in } Q .$$

In this paper we have chosen the approximation by Problem I_n , because it will be more convenient to work with in the following sections.

3 . The main results

In this section we describe our main results. Theorem 3.1 characterizes $N(z_x)$, the set where $z_x = 0$. In Theorem 3.2 we describe the level sets of z in the region where $z_x > 0$, in particular the set $N(z) \setminus N(z_x)$. Theorem 3.3 deals with the large time behaviour of z .

First we introduce some notation. We set

$$I = N(z_x) \cap \{t = 0\} \text{ , i.e.}$$

$$I = \{x \in \overline{\Omega} \; : \; z_0'(x) = 0\} \text{ .}$$

If z_0 satisfies hypothesis H1 , we can write for some $\ell \in \{0,1,2,\dots\}$

$$I = \bigcup_{j=1}^{\ell} I_j \tag{3.1}$$

where I_j are nonempty closed connected subsets of $\overline{\Omega}$; if $\ell = 0$, we mean by (3.1) that $I = \emptyset$.

Since $z_0' \geq 0$, we can rearrange the intervals I_j such that for some constants $-U \leq \Theta_1 < \Theta_2 < \dots < \Theta_\ell \leq V$

$$z_0(x) = \Theta_j \qquad \text{for } x \in I_j \text{ .} \tag{3.2}$$

Theorem 3.1 . *Let* z_0 *satisfy hypothesis* H1 , *and let* $z(t)$ *be the solution of Problem* I . *Then there exist constants*

$$0 \leq t_j^{\pm} \leq T_j \text{ , } j = 1,\dots, \ell,$$

and continuous functions

$$\zeta_j^{\pm} : [0,T_j] \to \overline{\Omega} \text{ , } j = 1,2,\dots,\ell \text{ .}$$

such that

$$N(z_x) = \bigcup_{j=1}^{\ell} N_j$$

and

$$z = \Theta_j \quad \text{on} \quad N_j \ , \tag{3.3}$$

where

$$N_j : = \{(x,t) \in \overline{Q} : \zeta_j^-(t) \leq x \leq \zeta_j^+(t) \ , \ 0 \leq t \leq T_j\}$$

for $j = 1,\ldots,\ell$. *Here the functions* $\zeta_j^{\pm}$ *satisfy :*

$\zeta_j^+(T_j) = \zeta_j^-(T_j)$;

ζ_j^-(*resp.* ζ_j^+) *is constant on* $[0,t_j^-]$ (*resp.* $[0,t_j^+]$)

and strictly increasing on (t_j^-, T_j) (*resp. strictly decreasing on* (t_j^+,T_j)) ;

$\zeta_j^{\pm} \in C^1((t_j^{\pm},T_j])$.

Theorem 3.2 . *Let* z_0 *satisfy hypothesis* H1 , *and let* $z(t)$ *be the solution of Problem* I .

(i) *Let* $p \in \overline{\Omega} \setminus I$, *i.e.* $z_0'(p) > 0$. *Then there exists a continuous function* $X(p,.) : [0,\infty) \to \overline{\Omega}$, *such that the curve* $x = X(p,t)$ *is a level set of* z :

$$\{(x,t) \in \overline{Q} : z(x,t) = z_0(p)\} = \{(x,t) : x = X(p,t), \ t \geq 0\} \ .$$

In addition $z_x(X(p,t), t) > 0$ *for* $t > 0$, $X(p,.) \in C^1(\mathbb{R}^+)$, $X(p,0) = p$, *and* $X(p,.)$ *satisfies for* $t > 0$:

$$X_t(p,t) = \begin{cases} -\dfrac{m}{c^-(m-1)} (\text{ƶ}_x{}^{m-1})_x(X(p,t),t) \quad \textit{if} \quad -U \leq \text{ƶ}_0(p) < 0 & (3.4) \\[2ex] -\dfrac{m}{c^+(m-1)} (\text{ƶ}_x{}^{m-1})_x(X(p,t),t) \quad \textit{if} \quad 0 < \text{ƶ}_0(p) \leq V & (3.5) \\[2ex] -\dfrac{m}{c^-(m-1)} \lim\limits_{x\uparrow X(p,t)} (\text{ƶ}_x{}^{m-1})_x(x,t) = & \\ \qquad = -\dfrac{m}{c^+(m-1)} \lim\limits_{x\downarrow X(p,t)} (\text{ƶ}_x{}^{m-1})_x(x,t) \quad \textit{if} \quad \text{ƶ}_0(p)=0. & (3.6) \end{cases}$$

(ii) *Let*

$$p_j = \zeta_j^-(T_j) = \zeta_j^+(T_j) \ , \quad j = 1,\ldots,\ell \ .$$

Then there exists a continuous function $X(p_j,.) : [T_j,\infty) \to \overline{\Omega}$ *such that*

$$\{(x,t) \in \overline{Q} : \text{ƶ}(x,t) = \theta_j, \ t \geq T_j\} = \{(x,t) : x = X(p_j,t), \ t \geq T_j\} \ .$$

In addition $\text{ƶ}_x(X(p_j,t),t) > 0$ *for* $t > T_j$, $X(p_j,.) \in C^1(T_j,\infty)$, $X(p_j,T_j) = p_j$, *and* $X(p_j,.)$ *satisfies the equations* (3.4) , (3.5) *and* (3.6) *for* $t > T_j$, *with* p *replaced by* p_j . *If* $T_j > 0$, *then* $X(p_j,.)$ *is Lipschitz continuous down to* $t = T_j$.

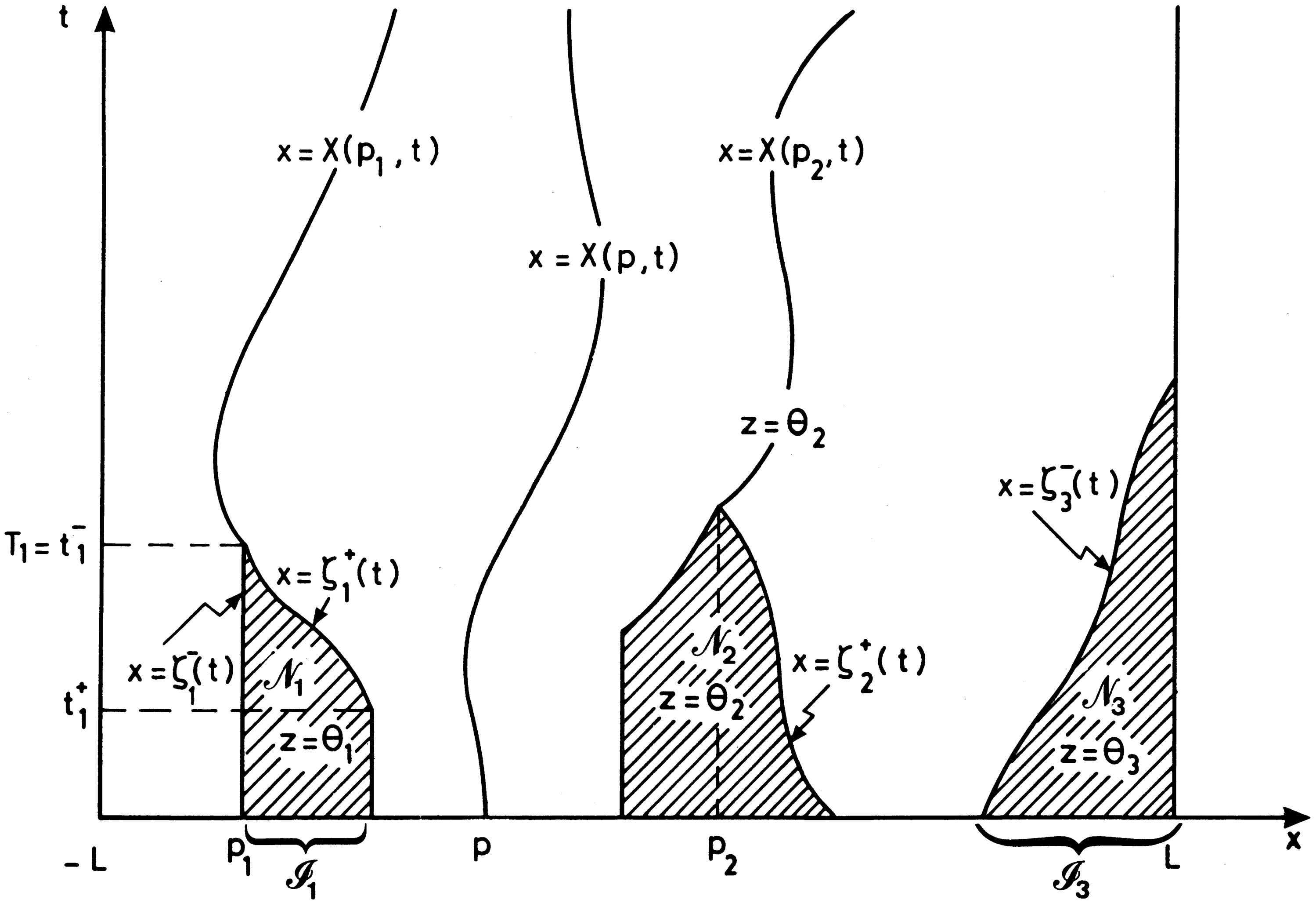

FIGURE 2 . The shaded areas are the sets where $z_x = 0$.

Remark . Since the curves $x = X(p,t)$ $(p \notin I)$ and $x = X(p_j,t)$ are level curves of z , it follows at once that they do not intersect, and that they fill up completely the region where $z_x > 0$.

Theorem 3.3 *(Large time behaviour). Let* $\bar{z}$ *be the unique steady-state solution of Problem* I . *For any initial function which satisfies hypothesis* H2 *(see section* 2 *), there exist constants* $M > 0$ *and* $\gamma > 0$ *such that*

$$\|z(t\ ;\ z_0) - \bar{z}\|_{C^1(\bar{\Omega})} \leq Me^{-\gamma t} \quad \textit{for} \quad t \geq 1$$

4 . A coordinate transformation

In this section we study a coordinate transformation for the approximating Problem I_n , which we introduced in section 2 .

Let $z_n : \overline{Q} \to \mathbb{R}$ be the solution of Problem I_n . Since z_n is a smooth function on $\overline{Q}$, and since, by Lemma 2.4 , $z_{nx} \geq \frac{1}{n}$ in $\overline{Q}$, it follows from the implicit function theorem that there exists a smooth function

$$X_n : \overline{Q} \to \overline{\Omega}$$

such that

$$z_n(X_n(p,t),t) = z_{0n}(p) \quad , \quad p \in \overline{\Omega} \ , \quad t \geq 0, \tag{4.1}$$

i.e. $x = X_n(p,t)$ are the level curves of z_n .

Theorem 4.1 . *Let* X_n *be defined by* (4.1) . *Then there exists a constant* $\mathscr{C} > 0$ *which does not depend on* n , *such that*

$$|X_{nt}| \leq \mathscr{C}/t \qquad \textit{in} \ \overline{\Omega} \times (0,\infty) .$$

Proof . We follow [17] and derive a parabolic equation for X_n . First we observe that it follows from (2.1) that $X_{np} := \frac{\partial}{\partial_p} X_n$ satisfies

$$0 < \gamma(n) \leq X_{np} \leq \Gamma(n) \qquad \text{in } \overline{Q}$$

for some $\gamma(n)$ and $\Gamma(n)$.

In order to simplify the notations we omit the subscripts n from now on.

From the relation

$$\frac{d}{dt} \{z(X(p,t),t)\} = 0$$

and the equation for z , we derive that

$$X_t(p,t) = -\{c'(z(X(p,t),t))\}^{-1} m(m-1)^{-1}((z_x(X(p,t),t))^{m-1})_x \tag{4.2}$$

and thus

$$X_t(p,t) = - (c'(z_0))^{-1} m(m-1)^{-1} \{(z_0')^{m-1}(X_p)^{-(m-1)}\}_p X_p^{-1},$$

where z_0 is considered as a function of p : $z_0 = z_0(p) = z(X(p,t),t)$.

Thus X satisfies the parabolic equation

$$X_t = - (c'(z_0))^{-1} \{(z_0')^{m-1}((X_p)^{-m})_p + m(m-1)^{-1}((z_0')^{m-1})'(X_p)^{-m}\} .$$

Differentiating this equation with respect to t we find that X_t satisfies

$$\begin{aligned}(X_t)_t = {} & (c'(z_0))^{-1} m\{(z_0')^{m-1}((X_p)^{-m-1}(X_t)_p)_p + \\ & + m(m-1)^{-1}((z_0')^{m-1})' (X_p)^{-m-1}(X_t)_p\} .\end{aligned} \tag{4.3}$$

We introduce the auxiliary function

$$q(p,t) = tX_t(p,t) - K X(p,t) \quad , \quad (p,t) \in \overline{Q} ,$$

where the constant K will be determined below. Using (4.3) , we arrive at

$$\begin{aligned} q_t &= (1-K)X_t + t(X_t)_t = \\ &= (1-K)X_t + (c'(z_0))^{-1} m\{(z_0')^{m-1}((X_p)^{-m-1}(q+KX)_p)_p + \\ &+ m(m-1)^{-1}((z_0')^{m-1})' (X_p)^{-m-1}(q+KX)_p\} = \\ &= (c'(z_0))^{-1} m\{(z_0')^{m-1}((X_p)^{-m-1}q_p)_p + \\ &+ m(m-1)^{-1}((z_0')^{m-1})' (X_p)^{-m-1}q_p\} + (1-(m+1)K)X_t .\end{aligned}$$

Choosing $K = (m+1)^{-1}$, i.e. $1 - (m+1)K = 0$, it follows that q attains its maximum and minimum at the parabolic boundary of Q . Since $tX_t = 0$ at the parabolic boundary, Theorem 4.1 follows.

The following theorem shows that when z_0 is more regular, then X_{nt} is uniformly bounded down to $t = 0$.

Theorem 4.2 . *Suppose that* $((z_0')^{m-1})' \in L^\infty(\Omega)$.

Then

$$|X_{nt}| \le \mathscr{C} \qquad \textit{in } \bar{Q}$$

for some constant $\mathscr{C}$ *which does not depend on* n .

Proof . Since X_{nt} satisfies (4.3) , the maximum principle implies that X_{nt} attains its extrema at the parabolic boundary of Q . At the lateral boundaries, $x = \pm L$, $X_{nt} = 0$. By (4.2)

$$\|X_{nt}(\cdot,0)\|_\infty \le C \,\|((z_{0n}')^{m-1})'\|_\infty ,$$

and thus, by Lemma 2.3 (iv) , $X_{nt}(\cdot,0)$ is uniformly bounded on $(-L,L)$. This proves Theorem 4.2 .

5. The set where $z_x = 0$: proof of Theorem 3.1 .

In this section we shall describe the set where $z_x = 0$.

First we remark that the variable $w = z_x$ satisfies porous media equations ([21,11]) in the sets where $z < 0$ and $z > 0$:

$$c^- w_t = (w^m)_{xx} \qquad \text{if } z < 0$$

and

$$c^+ w_t = (w^m)_{xx} \qquad \text{if } z > 0 .$$

For the porous media equation many properties are known about the sets where $w = 0$, which correspond to the sets where $z_x = 0$ [3] . These properties will be frequently used in the proof of Theorem 3.1 .Herefore however, we need first more information about the sets where $z < 0$ and $z > 0$. In this context, the following lemma will be useful : it tells us that the boundaries $z = -\varepsilon$ and $z = \varepsilon$ of the sets where $z < -\varepsilon$ and $z > \varepsilon$, are, for small $\varepsilon > 0$, smooth curves, where $z_x > 0$.

Lemma 5.1 . *Let* $\Theta_j (j=1,\dots,\ell)$ *be defined by* (3.2) , *and let* $\varepsilon_0 > 0$ *be so small that* $\pm\varepsilon \neq \Theta_j (j=1,\dots,\ell)$ *for all* $\varepsilon \in (0,\varepsilon_0)$. *Then for all* $\varepsilon \in (0,\varepsilon_0)$ *there exist continuous functions* $\zeta_{\pm\varepsilon} : [0,\infty) \to \Omega$ *such that*

$$\{(x,t) \in \overline{Q} \ : z(x,t) = \pm\varepsilon\} = \{(x,t) \in \overline{Q} : x = \zeta_{\pm\varepsilon}(t)\} \quad .$$

In addition $\zeta_{\pm\varepsilon} \in C^\infty(0,\infty)$ *and* $z_x(\zeta_{\pm\varepsilon}(t),t) > 0$ *for* $t \geq 0$.

We need two auxiliary lemmas for the proof of Lemma 5.1 .

Lemma 5.2 . *Let* z *be the solution of Problem* I *and suppose that for some* $x_0 \in \overline{\Omega}$ *and* $0 \leq \tau_0 < t_0$

$$z_x(x_0,t) = 0 \qquad \textit{for} \quad t \in (\tau_0,t_0] \ . \tag{5.1}$$

Then

$$z(x_0,t) = z_0(x_0,t_0) \qquad \textit{for} \quad t \in [\tau_0,t_0] \ . \tag{5.2}$$

Proof . Let z_n be the solution of Problem I_n . Then, for $\tau_0 \leq t < t_0$,

$$z_n(x_0,t_0) - z_n(x_0,t) = \int_t^{t_0} z_{nt}(x_0,s)ds =$$

$$= \int_t^{t_0} (c_n'(z_n))^{-1} \left((z_{nx})^m\right)_x ds \,\Big|_{x=x_0} =$$

$$= \frac{m}{m-1} \int_t^{t_0} (c_n'(z_n))^{-1} z_{nx} ((z_{nx})^{m-1})_x \, ds \Big|_{x=x_0} \quad .$$

From Theorem 4.1 and equality (4.2) we find that

$$|z_n(x_0,t_0) - z_n(x_0,t)| \le \frac{\mathscr{C}}{t} \int_t^{t_0} |z_{nx}(x_0,s)| \, ds \ , \ \tau_0 < t < t_0 \qquad (5.3)$$

By (2.3) , $z_{nx} \to z_x$ in $C(\overline{\Omega} x[t,t_0])$ for $\tau_0 < t < t_0$ and thus, by (5.1) and (5.3)

$$z(x_0,t) = z(x_0,t_0) \qquad \text{for} \quad \tau_0 < t < t_0 \ .$$

Finally, by continuity of z, also $z(x_0,\tau_0) = z(x_0,t_0)$.

Lemma 5.3 . *Let* z *be the solution of Problem* I *and let for some* $x_0 \in \overline{\Omega}$ *and* $t_0 > 0$,

$$z_x(x_0,t_0) = 0 \ \textit{and} \quad z(x_0,t_0) \neq 0 \ .$$

Then, for some $j \in \{1,\ldots,\ell\}$,

$$z_x(x_0,t) = 0 \ \textit{and} \quad z(x_0,t) = \Theta_j \ \textit{for} \quad 0 \le t \le t_0 \ ,$$

where Θ_j *is defined by* (3.2) .

Proof . Without loss of generality we may assume that $z(x_0,t_0) < 0$. Then there exists a neighbourhood $N(x_0,t_0)$ of (x_0,t_0) where $z < 0$ and where $w = z_x$ is a (weak) solution of the porous media equation $c^- w_t = (w^m)_{xx}$. It is a well-known property of a solution w of a porous media equation, that the set where $w > 0$ is expanding in the course of time [10,11], i.e. if $(x_0,t) \in N(x_0,t_0)$ for $t_1 \le t \le t_0$, then $w(x_0,t_0) = 0$ implies that

$$z_x(x_0,t) = w(x_0,t) = 0 \qquad \text{for} \quad t \in [t_1,t_0] \ .$$

Hence, if we set

$$\tau_0 = \inf\{t_1 \in [0,t_0] : z_x(x_0,t) = 0 \quad \text{for} \quad t \in (t_1,t_0]\} ,$$

then $\tau_0 < t_0$ and, by Lemma 5.2 ,

$$z(x_0,t) = z(x_0,t_0) < 0 \qquad \text{for} \quad t \in [\tau_0,t_0] .$$

In particular, since $z_x \in C(\overline{Q})$,

$$z_x(x_0,\tau_0) = 0 \quad \text{and} \quad z(x_0,\tau_0) < 0 .$$

Suppose that $\tau_0 > 0$. Then we can repeat the above argument with (x_0,t_0) replaced by (x_0,τ_0) , which contradicts the definition of τ_0 . Hence $\tau_0 = 0$ and $x_0 \in I_j$ and $z_0(x_0) = \Theta_j$ for some $j \in \{1,\dots,\ell\}$. This proves Lemma 5.3 .

Proof of Lemma 5.1 . First we show that $z_x > 0$ on the level sets

$$\{(x,t) \in \overline{Q} : z(x,t) = \varepsilon\} , \quad \varepsilon \in (0,\varepsilon_0) .$$

We argue by contradiction. Suppose there exist an $\varepsilon \in (0,\varepsilon_0)$ and a point $(x_0,t_0) \in \overline{Q}$ such that

$$z_x(x_0,t_0) = 0 \quad \text{and} \quad z(x_0,t_0) = \varepsilon.$$

Then, by Lemma 5.3 ,

$$z_0'(x_0) = 0 \quad \text{and} \quad z_0(x_0) = \varepsilon,$$

which implies the contradiction that $\Theta_j = \varepsilon$ for some $j \in \{1,\dots,\ell\}$.

By the implicit function theorem there exists a function $\zeta_\varepsilon \in C([0,\infty))$ such that $x = \zeta_\varepsilon(t)$ is the level curve $\{z = \varepsilon\}$. To prove that $\zeta_\varepsilon \in C^\infty(0,\infty)$, it is sufficient to show that z is C^∞ in a neighbourhood of any point $(\zeta_\varepsilon(t_0), t_0)$,with $t_0 > 0$. This follows from standard theory, if we choose a neighbourhood where $z_x > 0$ and $z > \frac{1}{2}\varepsilon$, and thus where z is a classical solution of $c^+ z_t = ((z_x)^2)_x$.

The proof for the level sets $\{z = -\varepsilon\}$ with $\varepsilon \in (0,\varepsilon_0)$ is similar.

Now we are ready to prove Theorem 3.1 .

Proof of Theorem 3.1 . First we prove the results in the set where $z > 0$.

Let $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is determined by Lemma 5.1 . The function $w = z_x$ is a weak solution of the porous media equation

$$c^+ w_t = (w^m)_{xx} \quad \text{in} \quad Q_\varepsilon := \{(x,t) : t > 0 \ , \ \zeta_\varepsilon(t) < x < L\} \ ,$$

where ζ_ε is given by Lemma 5.1 . Since ε can be chosen arbitrary small, all the properties which we have to prove about the set where $z_x = 0$, follow, if we restrict ourselves to the set where $z > 0$, from known results about the behaviour of the interfaces of solutions of the porous media equation, which are proved by Knerr [19] and Caffarelli and Friedman [12] (for a survey of the results we refer to [3]) . However, both Knerr and Caffarelli and Friedman consider the Cauchy Problem, and they assume that the set where $w > 0$ is initially connected. But since the main part of their analysis is local, the results carry over to our situation. The main non-local part in their proofs is a lower bound for $(w_n^{m-1})_{xx}$ for $t > 0$, which does not depend on n . Here $w_n := z_{nx}$ and z_n is the approximating sequence which we introduced in section 2 .

The required lower bound is given by the following lemma.

Lemma 5.4 . *For any* $\delta \in (0,T)$ *and* $\varepsilon \in (0,\varepsilon_0)$ *there exists a constant* $k(\delta,\varepsilon)$ *which does not depend on* n , *such that*

$$((z_{nx})^{m-1})_{xx} \geq - k(\delta,\varepsilon) \qquad \text{in} \ Q^\delta_{\varepsilon T} \ ,$$

where

$$Q^\delta_{\varepsilon T} = \{(x,t) \in Q : \zeta_\varepsilon(t) < x < L \ , \ \delta \leq t \leq T\} \ .$$

We postpone the proof of Lemma 5.4 to the end of this section.

In the same way we can prove the required results in the set where $z < 0$. It remains to consider the set where $z = 0$.

First we consider the case that

$$z_0(x) = 0 \quad \text{and} \quad z_0'(x) > 0 \quad \text{for some} \quad x \in \Omega \ .$$

Then, using Lemma 5.1 , there exists an $\varepsilon \in (0,\varepsilon_0]$ such that $z_x > 0$ on the parabolic boundary of the set

$$\{(x,t) \in \overline{Q} : = \zeta_{-\varepsilon}(t) < x < \zeta_\varepsilon(t) \ , \quad t \geq 0\} \ ,$$

and thus, by the maximum principle, $z_x > 0$ in this set. In particular $z_x > 0$ on the level set $\{z = 0\}$.

Finally we consider the case that

$$\Theta_j = 0 \qquad \text{for some} \quad j \in \{1,2,\ldots,\ell\} \ .$$

We define the functions w_{01} and w_{02} on $\overline{\Omega}$ by

$$w_{01}(x) = \begin{cases} z_0'(x) & \text{if } z_0 \leq 0 \\ 0 & \text{if } z_0 \geq 0 \end{cases}$$

and

$$w_{02}(x) = \begin{cases} 0 & \text{if } z_0 \leq 0 \\ z_0'(x) & \text{if } z_0 \geq 0 \ , \end{cases}$$

i.e.

$$z_0(x) = - U + \int_{-L}^{x} (w_{01}(s) + w_{02}(s)) \, ds \ .$$

Let $w_1(t)$ be the solution of the problem

$$(PM_1) \begin{cases} c^- w_t = (w^m)_{xx} \quad \text{in } Q \\ (w^m)_x \, (\pm L,t) = 0 \quad \text{for} \quad t > 0 \\ w(x,0) = w_{01}(x) \quad \text{for} \quad x \in \Omega \ , \end{cases}$$

and $w_2(t)$ the solution of

$$(PM_2)\begin{cases} c^+ w_t = (w^m)_{xx} & \text{in } Q \\ (w^m)_x(\pm L,t) = 0 & \text{for } t > 0 \\ w(x,0) = w_{02}(x) & \text{for } x \in \Omega \end{cases}$$

(a solution of Problem PM_i is again defined in a weak sense , see e.g. [5] for the existence and uniqueness of a solution ; it is known that w_i is continuous on $\bar{\Omega} \times \mathbb{R}^+$)

We define $T_j \geq 0$ by

$$T_j = \sup \{t \geq 0 : P_1(t) \cap P_2(t) = \emptyset\} , \tag{5.4}$$

where

$$P_i(t) = \{x \in [-L,L] : w_i(x,t) > 0\} , \quad t \geq 0, \quad i = 1,2 ,$$

i.e. T_j is the time that the supports of $w_1(t)$ and $w_2(t)$ meet. Observe that $T_j < \infty$.

We use the following lemma, which we shall prove below.

Lemma 5.5 . *Let* $\Theta_j = 0$ *for some* $j \in \{1,...\ell\}$ *and let* T_j *be defined by* (5.4) . *Then, up to time* T_j , *the solution of Problem* I *is given by*

$$z(x,t) = - U + \int_{-L}^{x} (w_1(s,t) + w_2(s,t))\, ds , \quad x \in \bar{\Omega} , \quad t \leq T_j .$$

In particular, if we define for $t \in [0,T_j]$

$$\zeta_j^-(t) = \sup \{x : x \in P_1(t)\} \text{ and } \zeta_j^+(t) = \inf\{ x : x\in P_2(t)\} \tag{5.5}$$

it follows from the results about the interfaces of $w_1(t)$ and $w_2(t)$ that $\zeta_j^{\pm}$ have the required properties.

Finally, the proof of Theorem 3.1 is completed by another lemma, which describes the situation for $t> T_j$.

Lemma 5.6 . *Let* $\Theta_j = 0$ *for some* $j \in \{1,\dots,\ell\}$ *and let* T_j *be defined by* (5.4). *Then*

$$z_x > 0 \quad \text{on } J := \{(x,t) \in \overline{Q} : t > T_j \text{ and } z(x,t) = 0\} .$$

The remainder of this section is devoted to the proofs of the Lemmas 5.5 , 5.6 and 5.4 .

Proof of Lemma 5.5 . Here we shall prove that $z_t \in L^2(Q_{T_j})$ and, for any $\psi \in C^1(\overline{Q}_{T_j})$ with $\psi(\pm L,t) = 0$ for $0 \le t \le T_j$,

$$\iint_{Q_{T_j}} \{c(z)_t \psi + (z_x)^m \psi_x\} = 0 . \tag{5.6}$$

Let the test function ψ be fixed. The functions w_1 and w_2 satisfy, for all $X \in C^{2,1}(\overline{Q}_{T_j})$ with $X_x(\pm L,t) = 0$,

$$c^- \int_\Omega w_1(t)\, X(t) \Bigg|_{t=0}^{t=T_j} = \iint_{Q_{T_j}} \{c^- w_1 X_t + w_1^m X_{xx}\}$$

and

$$c^+ \int_\Omega w_2(t)\, X(t) \Bigg|_{t=0}^{t=T_j} = \iint_{Q_{T_j}} \{c^+ w_2 X_t + w_2^m X_{xx}\} .$$

We substitute $X(x,t) = \int_{-L}^{x} \psi(s,t)ds$ into these equations. Defining $y = \zeta_j^+(T_j) = \zeta_j^-(T_j)$ (see (5.5)) , and using that

$$w_1(x,t) = 0 \quad \text{for } x > y \ ; \quad \int_{-L}^{y} w_1(x,t)dx = U$$

and

$$w_2(x,t) = 0 \quad \text{for } x < y \ ; \quad \int_{y}^{L} w_2(x,t)dx = V ,$$

we obtain after integration by parts

$$- c^- \int_{-L}^{y} z(t)\psi(t) \Big|_{t=0}^{t=T_j} = \int_0^{T_j}\int_{-L}^{y} \{-c^- z\psi_t + (w_1 + w_2)^m \psi_x\} .$$

and

$$- c^+ \int_{y}^{L} z(t)\psi(t) \Big|_{t=0}^{t=T_j} = \int_0^{T_j}\int_{y}^{L} \{-c^+ z\psi_t + (w_1 + w_2)^m \psi_x\} .$$

Adding these equations yields

$$\int_\Omega c(z(t))\psi(t) \Big|_{t=0}^{t=T_j} = \iint_{Q_{T_j}} \{c(z)\psi_t - (z_x)^m \psi_x\} . \qquad (5.7)$$

Now (5.6) follows at once, provided we show that $c(z)_t \in L^2(Q_{T_j})$.

It is standard that $(w_1^m)_x$ and $(w_2^m)_x$ are in $L^2(Q_{T_j})$ and, since $(z_x)^m = w_1^m + w_2^m$ in Q_{T_j} , also $((z_x)^m)_x \in L^2(Q_{T_j})$. Since by (5.7) $c(z)_t = ((z_x)^m)_x$ in the sense of distributions, it follows that $c(z)_t \in L^2(Q_{T_j})$. This completes the proof.

Proof of Lemma 5.6 . If $z_x > 0$ at some point $(x_0,t_0) \in \mathcal{J}$, the maximum principle, applied to the set $\{(x,t) : \zeta_{-\varepsilon}(t) \le x \le \zeta_{+\varepsilon}(t) , t \ge t_0\}$ for some $\varepsilon \in (0,\varepsilon_0)$ (see Lemma 5.1), implies that

$$z_x(x,t) > 0 \quad \text{for} \quad (x,t) \in \mathcal{J} \text{ with } t > t_0 .$$

Hence, arguing by contradiction, we may assume that for some $t_1 > T_j$

$$z_x(x,t) = 0 \quad \text{in } \mathcal{J} \cap \{(x,t) : T_j < x \le t_1\} .$$

We define the functions

$\eta_i : [-L,L] \times [0,t_1] \rightarrow [0,\infty) \quad (i = 1,2\ )$ by

$$\eta_1(x,t) = \begin{cases} z_x(x,t) & \text{if } z(x,t) \leq 0 \\ 0 & \text{if } z(x,t) \geq 0 \end{cases}$$

and

$$\eta_2(x,t) = \begin{cases} 0 & \text{if } z(x,t) \leq 0 \\ z_x(x,t) & \text{if } z(x,t) \geq 0 \ . \end{cases}$$

We claim that η_1 and η_2 are solutions of repectively Problem PM_1 and Problem PM_2 on $[0,t_1]$, i.e. $\eta_1 = w_1$ and $\eta_2 = w_2$ on Q_{t_1} . Accepting this for the moment, it follows that $P_1(t) \cap P_2(t) = \emptyset$ for $0 \leq t \leq t_1$, which is a contradiction with the fact that $t_1 > T_j$.

We show below that η_2 is a solution of Problem PM_2 . The proof that η_1 is a solution of Problem PM_1 is similar. From now on we denote η_2 by η .

Up to $t = T_j$, η is clearly a solution of Problem PM_2 . Thus it is enough to show that

$$c^+ \int_\Omega \eta(t)\ \chi(t) - c^+ \int_\Omega \eta(T_j)\ \chi(T_j) = \\ = \int_{T_j}^t \int_\Omega (c^+ \eta \chi_t + \eta^m \chi_{xx}) \tag{5.8}$$

for all $t \in (T_j,t_1]$ and all functions $\chi \in C^{2,1}(\bar{Q}_{t_1})$ with $\chi_x(\pm L,t) = 0$ for $t \geq 0$.

We fix the test function χ . Let ε_0 be given by Lemma 5.1 and let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$. We write $\zeta_1(t) :\ = \zeta_{\varepsilon 1}(t)$ and $\zeta_2(t) :\ = \zeta_{\varepsilon 2}(t)$.

Then there exist functions X_1 and X_2 in $C^{2,1}(\overline{Q}_{t_1})$ such that

$$\begin{cases} X = X_1 + X_2 & \text{in } Q_{t_1} \\ \operatorname{supp} X_1(t) \subset [-L, \zeta_2(t)) \,, & T_j \leq t \leq t_1 \\ \operatorname{supp} X_2(t) \subset (\zeta_1(t), +L] \,, & T_j \leq t \leq t_1 \,. \end{cases}$$

Since locally in the set $\{(x,t) : z(x,t) > 0\}$, η is a solution of Problem PM_2, (5.8) follows at once for $X = X_2$, and it remains to prove (5.8) for X_1, which we denote by X.

Let us first consider the functions ζ_ε for $\varepsilon > 0$. Since ζ_ε is monotone in ε and bounded from below there exists a function $\overline{\zeta} : [0,\infty) \to \Omega$ such that $\zeta_\varepsilon \downarrow \overline{\zeta}$ as $\varepsilon \downarrow 0$ for $t \geq 0$.
Then $z(\overline{\zeta}(t), t) = \lim\limits_{\varepsilon \downarrow 0} z(\zeta_\varepsilon(t), t) = 0$ and by Theorem 4.1

$\overline{\zeta} \in C^{0,1}([T_j,\infty))$ if $T_j > 0$ and $\overline{\zeta} \in C^{0,1}((0,\infty))$ if $T_j = 0$; in this last case the continuity of z implies that $\overline{\zeta}$ is continuous down to $t = 0$.

Since $\eta \equiv 0$ for $z \leq 0$, it is sufficient to show that for $t \in (T_j, t_1]$

$$c^+ \int_{\overline{\zeta}(t)}^{\zeta_2(t)} \eta(t)\, X(t) - c^+ \int_{\overline{\zeta}(T_j)}^{\zeta_2(T_j)} \eta(T_j)\, X(T_j) - \int_{T_j}^{t} \left\{ \int_{\overline{\zeta}(\tau)}^{\zeta_2(\tau)} (c^+ \eta X_t + \eta^m X_{xx}) \right\} d\tau = 0 \qquad (5.9)$$

Let $T_j < s < t \leq t_1$. Since η is a classical solution of the porous media equation in the set

$\{(x,\tilde{t}) : \zeta_\varepsilon(\tilde{t}) \leq x \leq \zeta_2(\tilde{t}) \,,\ s \leq \tilde{t} \leq t_1\}$ for all $\varepsilon \in (0,\varepsilon_2)$ and since ζ_ε is smooth, we find that

$$c^+ \int_{\zeta_\varepsilon(t)}^{\zeta_2(t)} \eta(t)\, X(t) - c^+ \int_{\zeta_\varepsilon(s)}^{\zeta_2(s)} \eta(s)\, X(s) = c^+ \int_s^t \Big(\frac{d}{d\tau} \int_{\zeta_\varepsilon(\tau)}^{\zeta_2(\tau)} \eta(\tau)\, X(\tau)\Big) d\tau$$

$$= \int_s^t \int_{\zeta_\varepsilon(s)}^{\zeta_2(\tau)} (c^+ \eta\, X_t + (\eta^m)_{xx}\, X) - c^+ \int_s^t \zeta_\varepsilon'(\tau) (\eta\, X)(\zeta_\varepsilon(\tau), \tau))d\tau =$$

$$= \int_s^t \int_{\zeta_\varepsilon(\tau)}^{\zeta_2(\tau)} (c^+\eta\, \chi_t + \eta^m \chi_{xx}) + I_\varepsilon(s,t) \qquad (5.10)$$

for all $\varepsilon \in (0,\varepsilon_2)$ where

$$I_\varepsilon(s,t) = \int_s^t \{(\eta^m \chi_x - (\eta^m)_x \chi)(\zeta_\varepsilon(\tau),\tau) - c^+\zeta_\varepsilon'(\tau)(\eta\chi)(\zeta_\varepsilon(\tau),\tau)\}\, d\tau$$

Since $\eta = z_x$ is a classical solution in

$\{(x,\tau) : \zeta_{\varepsilon/2}(\tau) \leq x \leq \zeta_2(\tau) , s \leq \tau \leq t\}$, it follows as in (4.2) , from the relation $z(\zeta_\varepsilon(\tau),\tau) = \varepsilon$ that

$$c^+\zeta_\varepsilon'(\tau) = -\frac{m}{m-1}((z_x)^{m-1})_x(\zeta_\varepsilon(\tau),\tau) = -\frac{m}{m-1}(\eta^{m-1})_x(\zeta_\varepsilon(\tau),\tau) .$$

Hence

$$I_\varepsilon(s,t) = \int_s^t (\eta^m \chi_x)(\zeta_\varepsilon(\tau),\tau)d\tau$$

Next, using the expression above for I_ε , we let $\varepsilon \downarrow 0$ in (5.10) . Since $z(\bar{\zeta}(\tau),\tau) = 0$, we deduce that $\eta(\bar{\zeta}(\tau),\tau) = 0$. Hence $\eta^m(\zeta_\varepsilon(\tau),\tau) \to 0$ as $\varepsilon \downarrow 0$ for $\tau \in (s,t)$. Then (5.10) becomes

$$c^+ \int_{\bar{\zeta}(t)}^{\zeta_2(t)} \eta(t)\, \chi(t) - c^+ \int_{\bar{\zeta}(s)}^{\zeta_2(s)} \eta(s)\, \chi(s)$$

$$= \int_s^t \int_{\bar{\zeta}(\tau)}^{\zeta_2(\tau)} (c^+\eta\, \chi_t + \eta^m \chi_{xx})$$

Finally letting $s \downarrow T_j$ in the equality above, one obtains (5.9).

<u>Proof of Lemma</u> 5.4 . Let $\tilde{z}_n$ be the extension of z_n on $[-L,3L] \times [0,T]$, defined by

$$\tilde{z}_n(x,t) = 2z_{0n}(L) - z_n(2L - x,t) \quad \text{in } [L,3L] \times [0,T] .$$

Then, restricted to the set $\tilde{Q}_{\varepsilon T}$, defined by

$$\tilde{Q}_{\varepsilon T} = \{(x,t) : \zeta_\varepsilon (t) < x < 2L - \zeta_\varepsilon (t) \ , \ 0 < t \le T\} \ ,$$

$\tilde{z}_n$ is the solution of the problem

$$\left| \begin{array}{ll} c^+ z_t = ((z_x)^m)_x & \text{in } \tilde{Q}_{\varepsilon T} \\ z(\zeta_\varepsilon(t),t) = z_n(\zeta_\varepsilon(t),t) & \\ & \text{for } t \in (0,T] \\ z(2L - \zeta_\varepsilon(t),t) = \tilde{z}_n(2L - \zeta_\varepsilon(t),t) & \\ z(x,0) = \tilde{z}_n(x,0) & \text{for } \zeta_\varepsilon(0) < x < 2L - \zeta_\varepsilon(0) \ . \end{array} \right.$$

To prove Lemma 5.4 , it is enough to show that $\tilde{w}_n = \tilde{z}_{nx}$ satisfies

$$(\tilde{w}_n^{m-1})_{xx} \ge - k(\delta,\varepsilon) \quad \text{in } \tilde{Q}^\delta_{\varepsilon T} \ , \tag{5.11}$$

where

$$\tilde{Q}^\delta_{\varepsilon T} = \tilde{Q}_{\varepsilon T} \cap \{(x,t) : \delta < t \le T\} \ .$$

We define

$$p_n = \frac{m}{m-1} (\tilde{w}_n^{m-1})_{xx} \quad \text{in } \tilde{Q}_{\varepsilon T} \ .$$

Since z is a classical solution on the level line $x = \zeta_\varepsilon(t)$ for $0 < t \le T$,

$$p_n \to m(m-1)^{-1}((z_x)^{m-1})_{xx} \quad \text{on} \quad S_\delta = \{(x,t) : x = \zeta_\varepsilon(t), \tfrac{1}{2}\delta \le t \le T\} \quad \text{as} \quad n \to \infty \ ,$$

and, on the other hand,

$$m(m-1)^{-1} \ ((z_x)^{m-1})_{xx} \ \ge \ - M_1(\delta,\varepsilon) \quad \text{on} \quad S_\delta$$

for some constant $M_1(\delta,\varepsilon)$. Hence there exists a constant $M_2(\delta,\varepsilon)$ which does not depend on n, such that

$$p_n \ \ge \ - M_2(\delta,\varepsilon) \quad \text{on the lateral boundary of } \tilde{Q}^{\frac{1}{2}\delta}_{\varepsilon T} \ . \tag{5.12}$$

We follow Aronson and Bénilan [4] and introduce the differential operator

$$L(q) = q_t - m\,\tilde{w}_n^{m-1}\, q_{xx} - 2m^2\, \tilde{w}_n^{m-2}\, \tilde{w}_{nx}\, q_x - (m+1)q^2 .$$

Then

$$L(p_n) = 0 \quad \text{and} \quad L\left(- \frac{M_3(\delta,\varepsilon)}{t - \frac{1}{2}\delta} \right) \leq 0 \quad \text{in} \quad \tilde{Q}^{\frac{1}{2}\delta}_{\varepsilon T} ,$$

where

$$M_3(\delta,\varepsilon) = \max\left((m+1)^{-1} , M_2(\delta,\varepsilon)\left(T - \frac{1}{2}\delta \right)\right) .$$

By (5.12) , $p_n \geq -M_3(\delta,\varepsilon) / (t - \frac{1}{2}\delta)$ on the parabolic boundary of $\tilde{Q}^{\frac{1}{2}\delta}_{\varepsilon T}$. Thus, by the maximum principle

$$p_n \geq - \frac{M_3(\delta,\varepsilon)}{t - \frac{1}{2}\delta} \quad \text{in} \quad \tilde{Q}^{\frac{1}{2}\delta}_{\varepsilon T} ,$$

and inequality (5.11) follows with $k(\delta,\varepsilon) = 2(m-1)\, M_3\,(\delta,\varepsilon) / \delta m$.

This completes the proof of Lemma 5.4 .

6 . The level curves of z in the set where $z_x > 0$: proof of Theorem 3.2

In this section we shall prove Theorem 3.2. In the set where both $z \neq 0$ and $z_x \neq 0$, equation (1.1) is regular and the results follow easily. The main difficulty is to prove the smoothness of the level curve $z = 0$ in the region where $z_x > 0$.

Proof of Theorem 3.2 .

Theorem 3.1 and the implicit function theorem imply the existence of the continuous functions $X(p,.) : [0,\infty) \to \overline{\Omega}$ for $p \in \overline{\Omega} \setminus I$ and the continuous functions $X(p_j,.) : (T_j,\infty) \to \overline{\Omega}$ for $j = 1,2,\ldots,\ell$. At $t = T_j$, $X(p_j,.)$ is given by $X(p_j,T_j) = p_j$, and the continuity of $X(p_j,.)$ down to T_j follows from the continuity of z .

If $z_0(p) \neq 0$ (resp. $\Theta_j \neq 0$), then z is a classical solution near $x = X(p,t)$ for $t > 0$ (resp. near $x = X(p_j,t)$ for $t > T_j$) . Hence $X(p,.) \in C^\infty(0,\infty)$ (resp. $X(p_j,.) \in C^\infty(T_j,\infty)$) . The expressions (3.4) and (3.5) follow from equation (1.1) and the equality

$$\frac{d}{dt}(z(X(p,t),t)) = 0 .$$

Finally we consider the smoothness of the level curve $z = 0$ in the region where $z_x > 0$, and prove expression (3.6) for X_t . We do this here in the case that $\Theta_j = 0$ for some $j = 1,\ldots,\ell$. In the other case (i.e. $z_0(x) = 0$ for some $x \in [-L,L]$ with $z_0'(x) > 0$) the proofs are similar and we omit them here.

So assume that $\Theta_j = 0$ and let p_j be defined by $z(p_j,T_j) = 0$. Let $\tau > T_j$ be arbitrary . We shall construct neighbourhoods of the point $(X(p_j,\tau),\tau)$ which are rectangles in a new coordinate system, which is a local variant of the coordinate system which we introduced in section 4 .

Let $x = \zeta_{\pm\varepsilon}(t)$ denote the level curves $z = \pm\varepsilon$ for $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is given by Lemma 5.1 . Let $0 < \varepsilon'' < \varepsilon' < \varepsilon_0$ and $T_j < t' < t'' < \tau$ We define the sets $Q'' \subset Q' \subset Q$ by

$$Q' = \{(x,t) : \zeta_{-\varepsilon'}(t) < x < \zeta_{\varepsilon'}(t) \ , \ t' < t \leq \tau\}$$

and

$$Q'' = \{(x,t) : \zeta_{-\varepsilon''}(t) < x < \zeta_{\varepsilon''}(t) \ , \ t'' < t \leq \tau\}.$$

Observe that the point $(X(p_j,\tau),\tau) \in Q'' \subset Q'$.

By Theorem 3.1 and Lemma 5.1 ,

$$z_x > 0 \quad \text{in } \overline{Q}' .$$

Hence we can introduce the coordinate transformation $(x,t) \to (p,t)$ for $(x,t) \in \overline{Q}'$, defined by

$$x = \tilde{X}(\tilde{p},t)$$

and

$$z(\tilde{X}(\tilde{p},t),t) = z(\tilde{p},t') .$$

In the $(\tilde{p},t)$ -plane the sets Q' and Q'' correspond respectively to the rectangles

$$W' = \{(\tilde{p},t) : \zeta_{-\varepsilon'}(t') < \tilde{p} < \zeta_{\varepsilon'}(t') , t' < t \leq \tau\}$$

and

$$W'' = \{(\tilde{p},t) : \zeta_{-\varepsilon''}(t') < \tilde{p} < \zeta_{\varepsilon''}(t') , t'' < t \leq \tau\}.$$

We define the functions $\tilde{X}_n$ on $\overline{W}'$ by

$$\begin{cases} z_n(\tilde{X}_n(\tilde{p},t),t) = z_n(\tilde{p},t') \\ \tilde{X}_n(\tilde{p},t') = \tilde{p} , \end{cases}$$

where z_n is the solution of Problem I_n .

Lemma 6.1 . *(i) The functions* $\tilde{X}_n$ *are uniformly bounded in* $C^1(\overline{W}')$. *(ii) The functions* $\tilde{X}_{nt}$ *are uniformly Hölder continuous in* $\overline{W}''$.

Before proving Lemma 6.1 , we complete the proof of Theorem 3.2 . By Lemma 6.1 (i) , there exist a function X_∞ on $\overline{W}'$ and a subsequence $\tilde{X}_{n_k}$ such that

$$\tilde{X}_{n_k} \to X_\infty \quad \text{in } C(\overline{W}') \text{ as } n_k \to \infty .$$

We claim that

$$X_\infty = \tilde{X} \quad \text{in } W'. \qquad (6.1)$$

Since

$$|z_{n_k}(\tilde{X}_{n_k}(\tilde{p},t),t) - z(\tilde{X}_{\infty}(\tilde{p},t),t)| \leq$$

$$| z_{n_k}(\tilde{X}_{n_k}(\tilde{p},t),t) - z(\tilde{X}_{n_k}(\tilde{p},t),t)| + | z(\tilde{X}_{n_k}(\tilde{p},t),t) - z(X_{\infty}(\tilde{p},t),t)| ,$$

it follows from the uniform convergence of z_n to z in compact subsets of $\overline{Q}$ and the continuity of z in $\overline{Q}$ that

$$z_{n_k}(X_{n_k}(\tilde{p},t),t) \to z(X_{\infty}(\tilde{p},t),t) \quad \text{as} \quad n_k \to \infty .$$

Since $z_x > 0$ in Q' , (6.1) follows, and we obtain that

$$\tilde{X}_n \to \tilde{X} \quad \text{in} \quad C(\overline{W}') \quad \text{as} \quad n \to \infty .$$

Combined with Lemma 6.1 (ii) , this implies that

$$\tilde{X}_{nt} \to \tilde{X}_t \quad \text{in} \quad C(\overline{W}'') \quad \text{as} \quad n \to \infty. \tag{6.2}$$

Observe that, for $(\tilde{p},t) \in \overline{W}''$,

$$X_t(p,t) = \tilde{X}_t(\tilde{p},t) \quad \text{if} \quad X(p,t) = \tilde{X}(\tilde{p},t) . \tag{6.3}$$

Hence, when $\tilde{X}(\tilde{p},t) = X(p,t)$ and $z(\tilde{X}(\tilde{p},t),t) \neq 0$, $\tilde{X}_t(\tilde{p},t)$ is given by (3.4) and (3.5) . Since $\tilde{X}_t$ is continuous on $\overline{W}''$ this implies that when $\tilde{X}(\tilde{p},t) = X(p,t)$ and $z(\tilde{X}(\tilde{p},t),t) = 0$, $\tilde{X}_t(\tilde{p},t)$ is given by (3.6) , and thus, by (6.3) , $X_t(p,t)$ is given by (3.6) .

It remains to prove Lemma 6.1 .

Proof of Lemma 6.1 . (i) Since $z_x > 0$ in $\overline{Q}'$ and $z_{nx} \to z_x$ in $C(\overline{Q}')$ as $n \to \infty$, the functions

$$\tilde{X}_{n\tilde{p}}(\tilde{p},t) = z_{nx}(\tilde{p},t') \ / \ z_{nx}(\tilde{X}_n(\tilde{p},t),t) \tag{6.4}$$

are uniformly bounded with respect to n in W' .

On the other hand, by Theorem 4.1 , $|\tilde{X}_{nt}| \leq C / t'$, and thus $\tilde{X}_n$ is uniformly bounded in $C^1(\overline{W'})$.

(ii) We show below that the functions $\tilde{X}_{ntp}$ are uniformly Hölder continuous in $\overline{W''}$. This property implies the result of Lemma 6.2 (ii) .

$\tilde{X}_{nt}$ satisfies equation (4.3) with p replaced by $\tilde{p}$ and $z_{0n}(p)$ replaced by $z_n(\tilde{p},t')$. Next we differentiate this equation with respect to $\tilde{p}$. Omitting all the tildas again we obtain

$$(X_{ntp})_t = \{a_n(p,t)(X_{ntp})_p + b_n(p,t)\, X_{ntp}\}_p$$

where

$$a_n(p,t) = \frac{m}{c'_n(z_n(p,t'))} (z_{np}(p,t'))^{m-1} (X_{np}(p,t))^{-m-1}$$

and

$$b_n(p,t) = \frac{m}{c'_n(z_n(p,t'))} \{(z_{np}(p,t'))^{m-1} ((X_{np}(p,t))^{-m-1})_p$$

$$+ \frac{m}{m-1} ((z_{np}(p,t'))^{m-1})_p (X_{np}(p,t))^{-m-1}\}$$

Since z_{nx} is uniformly bounded and uniformly bounded away from zero in $\overline{Q}'$, the functions $X_{np}(p,t)$ and $z_{np}(p,t)$ are bounded and uniformly bounded away from zero in $\overline{W}'$ and so is a_n . In order to show that b_n is uniformly bounded in $\overline{W}'$, it remains to prove that $((z_{np}(p,t'))^{m-1})_p$ and $((X_{np}(p,t))^{-m-1})_p$ are uniformly bounded in $\overline{W}'$. This follows easily from Theorem 4.1 , equality (4.2) and (6.4) .

Multiplying the differential equation for X_{nt} by X_{nt} and integrating by parts, we deduce that X_{ntp} is bounded in $L^2(W')$ uniformly in n .

It then follows from [20 , Thm. 8.1 p. 192] and [20, Theorem 10.1 p. 204] that X_{ntp} is uniformly Hölder continuous in $\overline{W}''$. This completes the proof of Lemma 6.1 .

7 . The large time behaviour

In this section we prove that the solution $z(t)$ of Problem I stabilizes as $t \to \infty$.

We define the functional $V : C^1(\overline{\Omega}) \to \mathbb{R}$ by

$$V(q) = (m+1)^{-1} \int_\Omega |q'|^{m+1} .$$

Lemma 7.1 . *Let* $z_0 \in C^1(\overline{\Omega})$ *satisfy hypothesis* H2 , *and let* $z(t)$ *be the solution of Problem* I . *Then*

$$\iint_{Q_t} ((F(z))_t)^2 + V(z(t)) \leq V(z_0) \quad \textit{for } t \geq 0 , \tag{7.1}$$

where

$$F(z) = \int_0^z \sqrt{c'(s)}\, ds .$$

Proof. Let $z_n(t)$ be the solution of Problem I_n . Multiplying the equation for z_n by z_{nt} , we obtain

$$\iint_{Q_t} (c_n(z_n))_t\, z_{nt} = -\frac{1}{m+1} \int_\Omega \{(z_{nx}(t))^{m+1} - (z'_{0n})^{m+1}\} . \tag{7.2}$$

We introduce

$$F_n(z) = \int_0^z \sqrt{c'_n(s)}\, ds ,$$

and we let $n \to \infty$ in (7.2) . Then we arrive at

$$\liminf_{n \to \infty} \iint_{Q_t} ((F_n(z_n))_t)^2 + V(z(t)) \leq V(z_0) . \tag{7.3}$$

Since, by Lemma 2.4 , $(F_n(z_n))_t$ is uniformly bounded in $L^2(Q_t)$ and since $F_n(z_n) \to F(z)$ uniformly in Q_t as $n \to \infty$, we conclude that

$$((F_n(z_n))_t \to (F(z))_t \text{ weakly in } L^2(Q_t) \text{ .}$$

Since the functional $p \to \iint_{Q_t} p^2$ is convex and lower semi continuous in $L^2(Q_t)$, it is also weakly lower semi continuous. Hence

$$\liminf_{\to \infty} \iint_{Q_t} ((F_n(z_n))_t)^2 \geq \iint_{Q_t} ((F(z))_t)^2$$

and (7.1) follows from (7.3) .

Lemma 7.2 . *Let* z_0 *satisfy hypothesis* H2 *and let* $z(t;z_0)$ *be the solution of Problem* I . *Then*

$$z(t;z_0) \to \bar{z} \quad \text{in } C^1(\bar{\Omega}) \quad \text{as} \quad t \to \infty \text{ .}$$

Proof . By Theorem 2.2 (ii) , $z(t;z_0) \in C^1(\bar{\Omega})$ for $t > 0$ and hence we may assume without loss of generality that $z_0 \in C^1(\bar{\Omega})$. By Theorem 2.2 (iv) the orbit $\{z(t;z_0) : t \geq 0\}$ is precompact in $C^1(\bar{\Omega})$ and hence the ω-limit set $\omega(z_0)$, which we define with respect to the topology of $C^1(\bar{\Omega})$, is non-empty. Thus it is sufficient to prove that

$$q \in \omega(z_0) \text{ implies that } q = \bar{z} \text{ .} \tag{7.4}$$

Since $z(t;z_0)$ is continuous in $C^1(\bar{\Omega})$ and V is continuous, it follows from standard stabilization theory (see Dafermos [14]) that :

(i) if $q \in \omega(z_0)$, then $z(t;q) \in \omega(z_0)$ for $t \geq 0$;

(ii) V is constant on $\omega(z_0)$.

This implies that $V(z(t;q)) = V(q)$ for all $q \in \omega(z_0)$, and, using (7.1) , we find that

$$F(z(t;q))_t = 0 \quad \text{a.e. in } Q_t \text{ .}$$

Hence $z(t;q) = q$ for $t \geq 0$, i.e. q is a steady state solution of Problem I. Now (7.4) follows from the uniqueness of $\bar{z}$.

To prove Theorem 3.3 , we have to show that the convergence in Lemma 7.2 is exponential. Our proof follows the same lines as a proof by Alikakos and Rostamian [2] .

Proof of Theorem 3.3 . Again we may assume that $z_0 \in C^1(\bar{\Omega})$. By Lemma 7.2 , $z_x(.,t) > 0$ for t large enough, and hence we may also assume without loss of generality that

$$z_0'(x) \geq \delta > 0 , \quad x \in \Omega .$$

To prove Theorem 3.3 it is sufficient to show that

$$|z_x(x,t) - \frac{U + V}{2L}| \leq Me^{-\gamma t} \quad \text{in } Q . \tag{7.5}$$

for some $M > 0$.

Let z_n be the solution of Problem I_n , where

$$z_{0n}'(x) \geq \delta > 0 , \quad x \in \Omega .$$

The function $w_n = z_{nx}$ satisfies the problem

$$\begin{cases} w_t = \{(c'_n(z_n))^{-1} (w^m)_x\}_x & \text{in } Q & (7.6) \\ w_x(\pm L,t) = 0 & \text{for } t > 0 & (7.7) \\ w(x,0) = z_{0n}'(x) & \text{for } x \in \Omega . & (7.8) \end{cases}$$

By the maximum principle,

$$w_n(x,t) \geq \delta \quad \text{in } Q .$$

If we multiply equation (7.6) by w_n and integrate by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_n^2 = -\int_\Omega \frac{m}{c_n'(z_n)}\, w_n^{m-1}\, w_{nx}^2 \quad ,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_n^2 \leq -\frac{\delta^{m-1} m}{\max(c^+,c^-)}\int_\Omega w_{nx}^2 \quad , \tag{7.9}$$

We define $\tilde{w}_n(x,t) = w_n(x,t) - (U + V_n) / 2L$.

Using that $\int_\Omega \tilde{w}_n(x,t)dx = 0$ for $t \geq 0$, we find from (7.9) that

$$\frac{d}{dt}\int_\Omega \tilde{w}_n^2 \leq - C_1 \int_\Omega \tilde{w}_n^2 \qquad \text{for } t \geq 0$$

where $C_1 = m\, \delta^{m-1} / (2L^2 \max (c^+,c^-))$, and thus

$$\int_\Omega \tilde{w}_n^2 \leq C_2\, e^{-C_1 t} \qquad \text{for } t \geq 0 \, , \tag{7.10}$$

for some constant C_2 . Since (7.10) holds for all n , we arrive at

$$\| z_x(t) - \frac{U + V}{2L} \|_2^2 \leq C_2\, e^{-C_1 t} \qquad \text{for } t \geq 0 \, . \tag{7.11}$$

Following the proof which Alikakos and Rostamian [2] and Alikakos [1] give in similar cases, it follows that (7.11) implies (7.5) .

APPENDIX

In this appendix we prove Theorem 2.2 .

First we show that $c(z)$ satisfies a contraction property in $L^1(\Omega)$, which implies at once that Problem I has at most one solution.

Lemma A.1 . *Let* $z_1(t)$ *and* $z_2(t)$ *be solutions of Problem* I *with initial functions* z_{01} *and* z_{02} *respectively. Then*

$$\|c(z_1(t)) - c(z_2(t))\|_1 \leq \|c(z_{01}) - c(z_{02})\|_1 \quad \textit{for} \quad t \geq 0 .$$

Corollary A.2 . *Problem* I *has at most one solution.*

Proof of Lemma A.1 . We follow the main lines of a proof by Bamberger [6] .

Define the function

$$\operatorname{sgn}(s) = \begin{cases} -1 & \text{if} \quad s < 0 \\ 0 & \text{if} \quad s = 0 \\ +1 & \text{if} \quad s > 0 , \end{cases}$$

and its approximation

$$\operatorname{sgn}_\eta(s) = \begin{cases} -1 & \text{if} \quad s < -\eta \\ s/\eta & \text{if} \quad -\eta \leq s \leq \eta \\ 1 & \text{if} \quad s > \eta. \end{cases}$$

Since $(c(z_1) - c(z_2))_t \in L^2(Q_T)$, we can multiply the difference of the equations for z_1 and z_2 by $\operatorname{sgn}_\eta(z_1 - z_2)$ and integrate by parts. This yields, using Lemma A3 below, that for all $t > 0$

$$\iint_{Q_t} (c(z_1) - c(z_2))_t \ \mathrm{sgn}_\eta(z_1 - z_2) =$$

$$- \iint_{Q_t} (|z_{1x}|^{m-1} z_{1x} - |z_{2x}|^{m-1} z_{2x}) \ \mathrm{sgn}'_\eta (z_1 - z_2)(z_{1x} - z_{2x}) \leq 0$$

and thus, by the dominated convergence theorem,

$$\iint_{Q_t} (c(z_1) - c(z_2))_t \ \mathrm{sgn} \ (z_1 - z_2) \leq 0 \qquad \text{for } t > 0 .$$

Since $\mathrm{sgn}\ (c(z_1) - c(z_2)) = \mathrm{sgn}(z_1 - z_2)$, we find that

$$\iint_{Q_t} (c(z_1) - c(z_2))_t \ \mathrm{sgn}\ (c(z_1) - c(z_2)) \leq 0 \qquad \text{for } t > 0 ,$$

which, again by Lemma A3 , implies that

$$\iint_{Q_t} |c(z_1) - c(z_2)|_t \leq 0 .$$

This completes the proof of Lemma A.1 .

Lemma A.3 . *Let* $G : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function.*

If $w \in W^{1,1} (0,T : L^1(\Omega))$, *then* $G(w) \in W^{1,1}(0,T : L^1(\Omega))$

and

$$\frac{dG(w)}{dt} = G'(w) \frac{dw}{dt} \qquad \textit{a.e.}$$

The proof is given in [13] .

Next we show that Problem I has a solution. We construct this solution as the limit function of the sequence z_n , the solutions of the approximate problems I_n which we introduced in section 2 .

We first prove the lemmas 2.3 and 2.4 of section 2 .

Proof of Lemma 2.3. We define $\tilde{z}_{0n}$ on $\mathbb{R}$ by

$$\tilde{z}_{0n}(x) = \frac{1}{n}(x + L) + \begin{cases} - U & \text{if } x \leq -L + \frac{1}{n} \\ z_0(x) - z_0(-L + \frac{1}{n}) - U & \text{if } x \in [-L+\frac{1}{n} , L - \frac{1}{n}] \\ z_0(L - \frac{1}{n}) - z_0(-L + \frac{1}{n}) - U & \text{if } x \geq L - \frac{1}{n} . \end{cases}$$

Let the function ρ be defined by

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ C \exp\{ 1 / (x^2 - 1)\} & \text{if } x < 1 , \end{cases}$$

where $C > 0$ is a constant such that $||\rho||_1 = 1$, and let

$$z_{0n}(x) = 2n \int_{\mathbb{R}} \rho(2n(x - y)) \tilde{z}_{0n}(y)dy , \quad x \in \mathbb{R} .$$

Then one easily checks that z_{0n} satisfies Lemma 2.3 (i) and (ii) .

In the case that $z_0 \in C^1(\bar{\Omega})$ we construct the sequence z_{0n} in a different way. We set

$$\tilde{v}_{0n}(x) = (z_0'(x))^{m-1} + (\frac{1}{n})^{m-1} , \quad x \in \bar{\Omega} ,$$

and

$$\hat{v}_{0n}(x) = \begin{cases} \tilde{v}_{0n}(L - \frac{1}{n}) & \text{if } x \geq L - \frac{1}{n} \\ \tilde{v}_{0n}(x) & \text{if } x \in [-L + \frac{1}{n} , L - \frac{1}{n}] \\ \tilde{v}_{0n}(-L + \frac{1}{n}) & \text{if } x \leq -L + \frac{1}{n} . \end{cases}$$

Finally we define

$$v_{0n}(x) = 2n \int_{\mathbb{R}} \rho(2n(x - y)) \hat{v}_{0n}(y) dy , \quad x \in \mathbb{R} ,$$

and

$$z_{0n}(x) = - U + \int_{-L}^{x} (v_{0n}(s))^{1/(m-1)} \, ds \ , \quad x \in \mathbb{R} \ .$$

Again one easily checks that z_{0n} satisfies the required properties.

Proof of Lemma 2.4. The function $w_n := z_{nx}$ satisfies (7.6) , (7.7) and (7.8) . It follows from the maximum principle and classical existence and uniqueness theory [20, Th. 5.2 , p. 564] that Problem I_n has a unique solution which satisfies (2.1) . Inequality (2.2) follows at once from (7.2) .

We need two more lemmas for the proof of Theorem 2.2 .

Lemma A.4 . *There exists a constant* $C > 0$ *which does not depend on* n *such that*

$$|z_n(x,t') - z_n(x,t'')| \leq C \ |t' - t''|^{1/2}$$

for all $(x,t') , (x,t'') \in \bar{Q}_T$ *such that* $|t' - t''| \leq 1$.

Proof. This result follows from (2.1) and Gilding [16] .

Lemma A.5 . *(i) For every* $\tau > 0$ *, the functions* $w_n = z_{nx}$ *are equicontinuous on* $\bar{\Omega} \times [\tau,T]$ *, and the modulus of continuity does not depend on* T .
(ii) If $z_0 \in C^1(\bar{\Omega})$ *, the functions* $w_n = z_{nx}$ *are equicontinuous on* $\bar{Q}_T$ *, and the modulus of continuity does not depend on* T .

Proof . The function $p_n = w_n^m$ satisfies the equation

$$(p^{1/m})_t = \{(c_n'(z_n))^{-1} \ p_x\}_x$$

and the boundary conditions $p_x(\pm L,t) = 0$ for $0 \leq t \leq T$.

The results follow at once from DiBenedetto [15] .

Proof of Theorem 2.2 . By the lemmas 2.4 , A.3 and A.4 , there exist a subsequence of $\{z_n\}$ which we denote again by $\{z_n\}$, and a function $z \in C(\overline{Q})$ with $z_x \in C(\overline{\Omega} \times \mathbb{R}^+)$ and $z_x \geq 0$ in Q , such that $z_n \to z$ in $C(\overline{Q}_T)$ as $n \to \infty$ and, for any $\tau \in (0,T)$, $z_{nx} \to z_x$ in $C(\overline{\Omega} \times [\tau, T])$ as $n \to \infty$. If $z_0 \in C^1(\overline{\Omega})$, then $z_{nx} \to z_x$ in $C(\overline{Q}_T)$ as $n \to \infty$.

We claim that z is a solution of Problem I . For all $T > 0$ and $p \in [1,\infty)$

$$z_n \to z \text{ weakly in } L^2(0,T : W^{1,p}(\Omega)) \text{ as } n \to \infty.$$

In addition, it follows from (2.2) and the uniform convergence of $c_n(z_n)$ to $c(z)$ in Q_T as $n \to \infty$ that

$$c_n(z_n)_t \to c(z)_t \text{ weakly in } L^2(Q_T) \text{ as } n \to \infty .$$

Using these properties, it follows that z satisfies the integral equation (iv) of Definition 2.1 , by writing a similar integral equation for z_n in which we let $n \to \infty$.

Finally, because $z \in L^2(0,T : H^1(\Omega))$ and $z_t \in L^2(0,T : L^2(\Omega))$ for $T > 0$, we obtain [22, Lemma 1.2, p. 261] that $z \in C([0,\infty) : L^2(\Omega)) \subset C([0,\infty) : L^1(\Omega))$. Hence, since z satisfies the conditions (i), (ii), (iii) and (iv) of Definition 2.1 , z is a solution of Problem I .

The other properties in Theorem 2.2. follow at once ; the uniqueness of z follows from Corollary A.2 , and the properties (iii) and (iv) of Theorem 2.2 are a consequence of Lemma A.5.

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CHAPITRE X

ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING : PRESERVATION OF SEGREGATION.

par

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1. Introduction

Consider two interacting biological species with populations sufficiently dense that a continuum theory is applicable, and assume that the species are undergoing dispersal on a time scale sufficiently small that births and deaths are negligible. Granted these assumptions, conservation of population requires that

$$\begin{aligned} u_t &= -\mathrm{div}(uq), \\ v_t &= -\mathrm{div}(vw), \end{aligned} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are the spatial densities of the species, while the vector fields $q(x,t)$ and $w(x,t)$ are the corresponding dispersal velocities.

We restrict our attention to situations in which dispersal is a response to population pressure and express this mathematically by requiring that the dispersal of each of the species be driven by the gradient $\nabla(u+v)$ of the total population,[1] $u+v$. We therefore assume that

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[1] For a single species this type of constitutive assumption was introduced by Gurney and Nisbet [1], Gurtin and MacCamy [2].

$$q = -k_1\nabla(u+v), \qquad w = -k_2\nabla(u+v), \tag{1.2}$$

with k_1, k_2 strictly-positive[1] constants, and this leads to the system[2]

$$u_t = k_1 \mathrm{div}[u\nabla(u+v)], \qquad v_t = k_2 \mathrm{div}[v\nabla(u+v)]. \tag{1.3}$$

For convenience, we limit our attention to one space-dimension and we choose the time-scale so that $k_1 = 1$. Then writing $k = k_2$ we have the system

$$u_t = [u(u+v)_x]_x, \qquad v_t = k[v(u+v)_x]_x. \tag{1.4}$$

We shall suppose that the two species live in a finite habitat

$$\Omega = (-L, L), \quad L > 0;$$

that individuals are unable to cross the boundary of Ω,

$$u(u+v)_x = v(u+v)_x = 0 \quad \text{for} \quad x = \pm L, \quad t > 0; \tag{1.5}$$

and that the two populations are prescribed initially,

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[1] The system (1.3) with $k_2 = 0$ was studied by Bertsch and Hilhorst [3] and by Bertsch, Gurtin, Hilhorst, and Peletier [4].

[2] Gurtin and Pipkin [5]. See also Busenberg and Travis [6]. An alternative theory was developed by Shigesada, Kawasaki, and Teramoto [7]. This theory is discussed in [4] and [5].

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{for} \quad x \in \Omega. \tag{1.6}$$

In this paper we shall study the problem (1.4)-(1.6) for <u>initial data which are segregated</u> in the sense that, for some $a \in \Omega$,

$$u_0(x) \equiv 0 \quad \text{for} \quad x > a, \quad v_0(x) \equiv 0 \quad \text{for} \quad x < a. \tag{1.7}$$

As our main result we establish the existence of a solution in which <u>the two species are segregated for all time</u>. This result is quite surprising[1] as it is independent of the initial distributions[2] of the species and of the ratio k of their dispersivities.

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[1] Actually, Gurtin and Pipkin [5] gave a particular solution to (1.2) - corresponding to initial Dirac distributions - in which the two species are segregated for all time. Being a specific solution, it is not clear from this result whether "preservation of segregation" is a generic property of the equations (1.4).

[2] Granted they are segregated.

2. The problem. Results.

We shall use the notation

$$\mathbb{R}^+ = (0,\infty), \quad Q = \Omega \times \mathbb{R}^+, \quad Q_T = \Omega \times (0,T),$$

and, for any function $f: \ Q \to \mathbb{R}$,

$$Q^+(f) = \text{interior}\{(x,t) \in Q: \ f(x,t) > 0\}.$$

Our problem consists in finding functions $u(x,t)$ and $v(x,t)$ on $\bar{Q}$ such that

$$(I) \begin{cases} u_t = [u(u+v)_x]_x \\ v_t = k[v(u+v)_x]_x \end{cases} \text{in } Q, \tag{2.1}$$

$$u(u+v)_x = v(u+v)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \tag{2.2}$$

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x) \quad \text{in } \Omega. \tag{2.3}$$

We shall assume throughout that:

(A1) $k > 0, \quad u_0, v_0 \geq 0, \quad u_0, v_0 \in C(\bar{\Omega})$;

(A2) <u>the initial data are segregated</u>, so that (1.7) holds for some $a \in \Omega$;

(A3)[1] each of the sets $\{x: \ u_0(x) > 0\}$ and $\{x: \ v_0(x) > 0\}$ is connected.

The purpose of this paper is to establish - for such segregated initial data - solutions of (I) which are segregated for all time.

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[1] We make this assumption for convenience only.

Proceeding formally, let (u,v) be a segregated solution. Then the sets $Q^+(u)$ and $Q^+(v)$ are disjoint; hence (assuming $u,v \geq 0$)

$$u \equiv 0 \quad \text{in} \quad Q^+(v), \qquad v \equiv 0 \quad \text{in} \quad Q^+(u),$$

and, by (2.1),

$$u_t = (uu_x)_x \quad \text{in} \quad Q^+(u), \quad v_t = k(vv_x)_x \quad \text{in} \quad Q^+(v). \tag{2.4}$$

Thus where positive u and v obey porous-media equations. As is well known,[1] solutions of the porous-media equation may not be smooth, and for that reason it is advantageous to work with a weak formulation of Problem (I). This is reinforced by the observation that (I) is a free-boundary problem and conditions at the free boundary are generally indigenous to a weak formulation, not required as separate restrictions. (The free boundary is the set

$$\mathfrak{F} = \{\partial Q^+(u) \cup \partial Q^+(v)\} \cap Q \tag{2.5}$$

which separates the region with $u(x,t) > 0$ from that with $u(x,t) = 0$ and separates the region with $v(x,t) > 0$ from that with $v(x,t) = 0$.)

With this in mind, assume for the moment that (u,v) is a smooth solution of (I). If we multiply (2.1) by an <u>arbitrary</u> smooth function $\psi(x,t)$, integrate over Q_T, and use (2.2)

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[1] Cf., e.g., the survey article of Peletier [8].

and (2.3), we arrive at the relations

$$\int_{\Omega} \{u(T)\psi(T) - u_0\psi(0)\} = \int_{Q_T} \{u\psi_t - u(u+v)_x\psi_x\},$$
$$\int_{\Omega} \{v(T)\psi(T) - v_0\psi(0)\} = \int_{Q_T} \{v\psi_t - kv(u+v)_x\psi_x\}, \tag{2.6}$$

where we have used the notation $u(t) = u(\cdot,t)$, etc. We shall use (2.6) as the basis of our definition of a weak solution.

Definition. A (weak) solution of Problem (I) is a pair (u,v) with the following properties:

(i) $u,v \in L^{\infty}(Q_T)$ for $T > 0$; $u(t),v(t) \in L^{\infty}(\Omega)$ for $t \geq 0$; $(u+v)^2 \in L^2(0,T; H^1(\Omega))$ for $T > 0$;

(ii) $u(t),v(t) \geq 0$ almost everywhere in Ω for $t > 0$;

(iii) u and v satisfy (2.6) for all $\psi \in C^1(\bar{Q})$ and $T > 0$.

If, in addition, there is a continuous function ξ: $[0,\infty) \to \Omega$ such that, given any $t > 0$,

$$v(x,t) = 0 \quad \text{for} \quad -L < x < \xi(t),$$
$$u(x,t) = 0 \quad \text{for} \quad \xi(t) < x < L, \tag{2.7}$$

then (u,v) is segregated. We will refer to ξ as a separation curve.

Remarks:

1. The terms $u(u+v)_x$ and $v(u+v)_x$ in (2.6) are defined as follows:

$$u(u+v)_x = \begin{cases} \frac{1}{2}\frac{u}{u+v}[(u+v)^2]_x & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases}$$

and similarly for $v(u+v)_x$. Then, since $u/(u+v) \leq 1$, while $[(u+v)^2]_x \in L^2(Q_T)$, we have

$$u(u+v)_x \in L^2(Q_T) \quad \text{for} \quad T > 0.$$

2. The integral identities (2.6) imply that, as $t \to 0$,

$$u(t) \to u_0, \quad v(t) \to v_0 \quad \text{weakly in} \quad L^2(\Omega);$$

i.e., that

$$\int_\Omega [u(t)-u_0]\psi \to 0, \quad \int_\Omega [v(t)-v_0]\psi \to 0 \tag{2.8}$$

for all $\psi \in L^2(\Omega)$. To verify (2.8) we simply apply (2.6) with $\psi \in C^1(\bar\Omega)$ (independent of time). This yields (2.8) for $\psi \in C^1(\bar\Omega)$ and hence - using a standard argument - for $\psi \in L^2(\Omega)$.

3. The choice $\psi \equiv 1$ in (2.6) leads to the global conservation laws

$$\int_\Omega u(t) = \int_\Omega u_0, \quad \int_\Omega v(t) = \int_\Omega v_0 \tag{2.9}$$

for $t > 0$.

We close this section by stating our main results; the corresponding proofs will be given in Section 3.

Theorem 1. *Problem (I) has exactly one segregated solution.*

Remark. It is important to emphasize that we have established uniqueness only within the class of segregated solutions. Thus we have not ruled out the possibility - for segregated initial data - of solutions which mix. *We conjecture that this cannot happen.*

Theorem 2. *Let (u,v) be the segregated solution of Problem (I). Then:*

(i) $u + v \in C(\bar{Q})$;

(ii) $u_t = (uu_x)_x$ *classically*[1] *in* $Q^+(u)$;

(iii) $v_t = k(vv_x)_x$ *classically in* $Q^+(v)$.

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[1] That is, u is C^∞ on $Q^+(u)$ and there satisfies $u_t = (uu_x)_x$ pointwise.

Our next theorem is concerned with the free boundary $\mathfrak{J}$ (cf. (2.5)). In view of (2.4), the portion of $\mathfrak{J}$ along which the two species are <u>not</u> in contact should have properties similar to those of the free boundary for the porous-media problem:

$$(\mathrm{PM})\begin{cases} \rho_t = (\rho\rho_x)_x & \text{in } Q, \\ \rho\rho_x = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases}$$

As is known,[1] when the initial data have the form

$$\rho_0(x) > 0 \text{ in } (a_1,a_2), \quad \rho_0(x) = 0 \text{ otherwise,}$$

$-L < a_1 < a_2 < L$, the free boundary $Q \cap \partial Q^+(\rho)$ consists of two continuous, time-parametrizable curves, one emanating from a_1, one from a_2. If $b(t)$, $0 \leq t < T_b$, designates the curve from a_1 (resp., a_2), then:

(F_1) $b(t) = b(0)$ on $(0,\tau_b)$ for some $\tau_b \in [0,T_b]$;

(F_2) $b(t)$ is C^1 and strictly decreasing (resp., strictly increasing) on (τ_b,T_b);

(F_3) $b(T_b^-) \in \partial\Omega$.

This discussion should motivate the following definition in which "FB" is shorthand for "free boundary".

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[1] Cf., e.g., the review article by Peletier [8]. See also Aronson and Peletier [9].

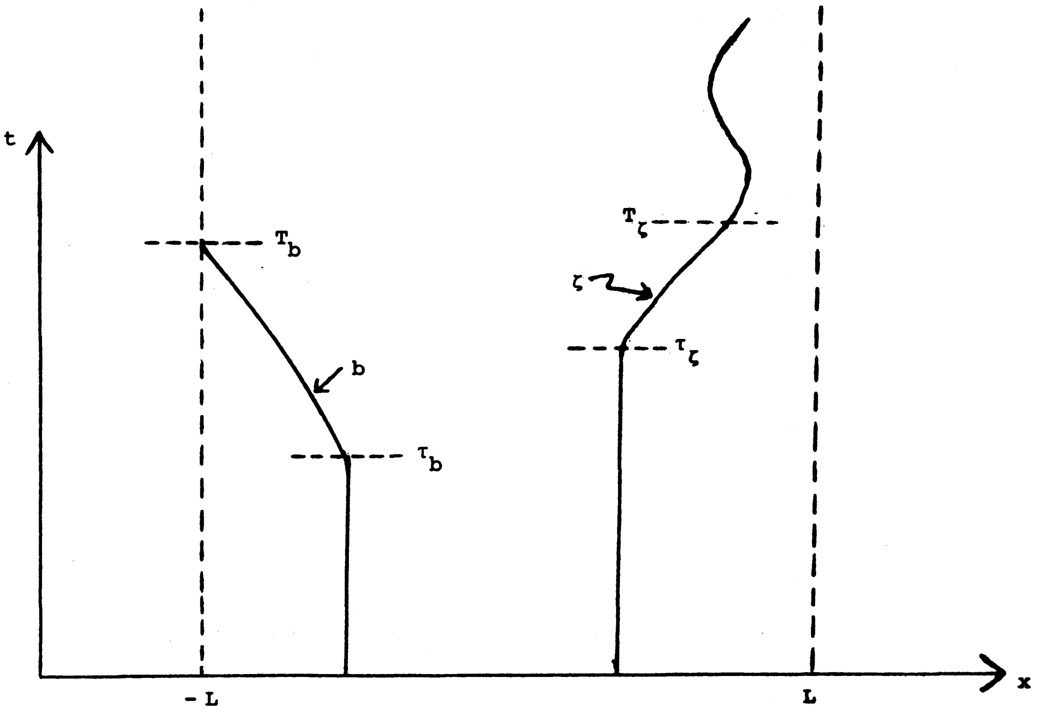

Figure 1. b is a left free-boundary curve extending to $\partial\Omega$. ζ is an internal free-boundary curve that is right up to time T_ζ.

Definition. An FB curve is a continuous function

$$b\colon [0,t_b) \to \Omega$$

(t_b may be ∞). Moreover:

(i) b is internal if $t_b = \infty$;

(ii) b is left (resp., right) up to time $T_b \in [0,t_b]$ if $(F_1),(F_2)$ hold;

(iii) b extends to $\partial\Omega$ if b is right or left up to time t_b with $t_b < \infty$, and $b(t_b^-) \in \partial\Omega$.

Let $b\colon [0,t_b) \to \Omega$ be an FB curve and let $q\colon Q \to \mathbb{R}$. Then FB conditions with velocity q hold from the left (resp., right) on b if given any $t \in (0,t_b)$ at which b is C^1,

$$b'(t) = q(b(t)^+,t) \qquad (\text{resp., } b'(t) = q(b(t)^-,t)).$$

Theorem 3. Let (u,v) be the segregated solution of Problem (I). Then there exist FB curves $b_u, b_v, \zeta_u, \zeta_v$ with the following properties:

(i)[1] $b_u < \zeta_u \le \zeta_v < b_v$ with b_u and ζ_u forming the boundary of $Q^+(u)$ in Q, b_v and ζ_v forming the boundary of $Q^+(v)$ in Q;

(ii) b_u and b_v extend to $\partial\Omega$, with b_u left and b_v right;

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[1] Here each inequality is assumed to hold at those times at which the underlying functions are defined.

(iii) ζ_u *and* ζ_v *are internal, and there is a time* $T \in [0,\infty)$ *such that* ζ_u *is right up to time* T, ζ_v *is left up to time* T, *and*

$$\zeta_u(t) = \zeta_v(t) =: \zeta(t) \quad \text{on } [T,\infty)$$

with $\zeta \in C^1(T,\infty)$; *in addition,*

$$(u+v)(\zeta(t),t) > 0 \quad \text{for} \quad t > T; \tag{2.10}$$

(iv) *FB conditions with velocity* $-u_x$ *hold from the right on* b_u, *from the left on* ζ_u;

(v) *FB conditions with velocity* $-kv_x$ *hold from the right on* ζ_v, *from the left on* b_v.

Remarks.

1. The curve ζ marks that portion of the free boundary on which the two species are in contact. By (2.10), the functions u and v suffer *jump discontinuities* across ζ; more precisely for $t > T$,

$$u(\zeta(t)^-,t) > 0, \quad u(\zeta(t)^+,t) = 0,$$

$$v(\zeta(t)^-,t) = 0, \quad v(\zeta(t)^+,t) > 0.$$

Further, (iii)-(v) of Theorem 3 in conjunction with the continuity of $u+v$ imply that, for $t > T$,

$$u(\zeta(t)^-,\ t) = v(\zeta(t)^+,\ t),$$

$$u_x(\zeta(t)^-,\ t) = kv_x(\zeta(t)^+,\ t) = -\zeta'(t). \tag{2.11}$$

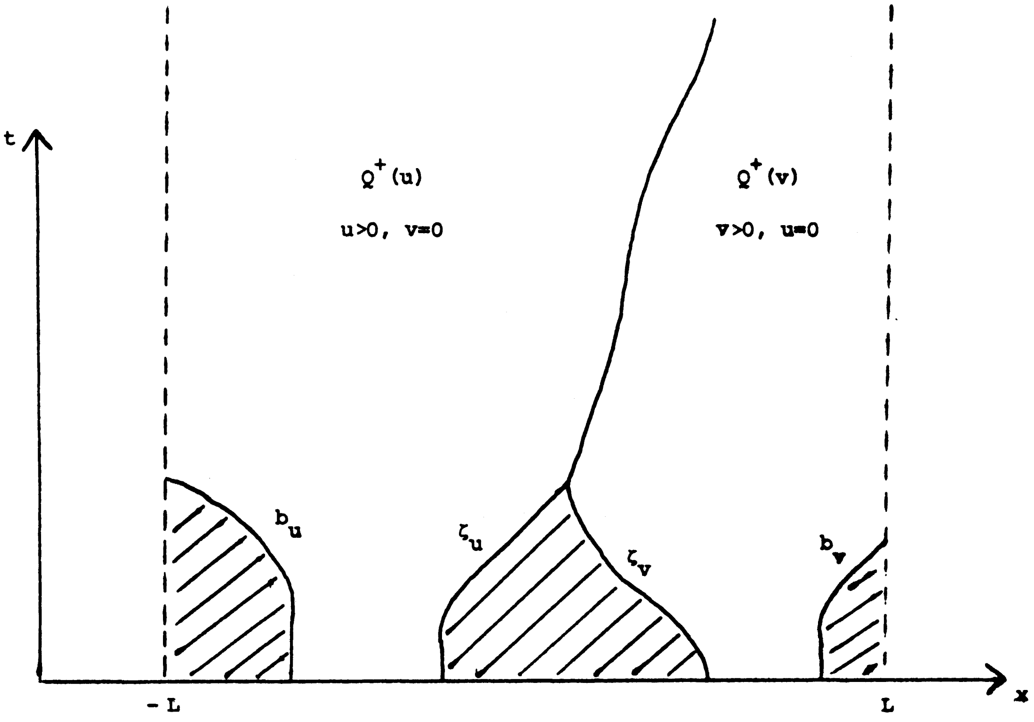

Figure 2. The free boundaries. The shaded areas correspond to u = v = 0.

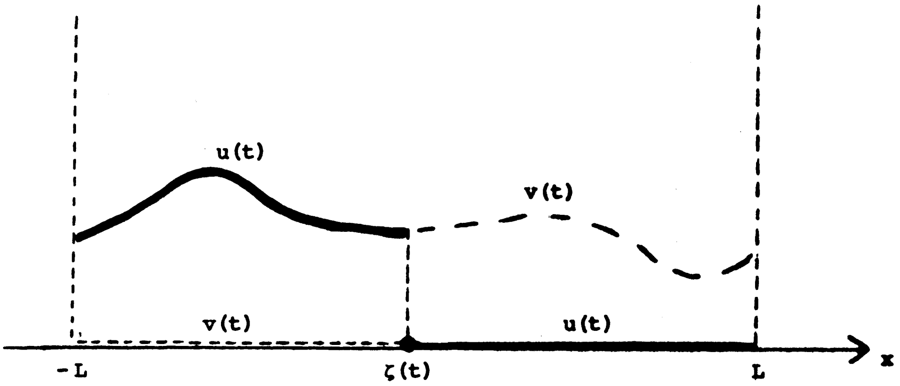

Figure 3. The functions $u(\cdot,t)$ and $v(\cdot,t)$ at a fixed time $t > T$.

2. The results (iv) and (v) assert that each of the "fronts" $b_u(t)$, $\zeta_u(t)$, $\zeta_v(t)$, $b_v(t)$ propagates with the velocity of individuals situated on it; condition (2.11) is the requirement that at the contact front $\zeta(t)$ the two species move together.

Our final result concerns the asymptotic behavior of segregated solutions. Proceeding formally, suppose that $u_\infty(x)$, $v_\infty(x)$ is an equilibrium solution of Problem (I). Then (2.1) and the boundary conditions (2.2) yield

$$u_\infty(u_\infty + v_\infty)' = v_\infty(u_\infty + v_\infty)' = 0 \quad \text{in} \quad \Omega;$$

hence

$$[(u_\infty + v_\infty)^2]' = 0 \quad \text{in} \quad \Omega$$

and

$$u_\infty + v_\infty = \text{constant.}$$

If u_∞ and v_∞ are segregated with habitats in $[-L, x_\infty]$ and $[x_\infty, L]$, respectively, then there exists a constant p such that

$$u_\infty(x) = \begin{cases} p & \text{if} \quad x \in (-L, x_\infty) \\ \frac{1}{2}p & \text{if} \quad x = x_\infty \\ 0 & \text{if} \quad x \in (x_\infty, L) \end{cases} \tag{2.12}$$

$$v_\infty(x) = \begin{cases} 0 & \text{if} \quad x \in (-L, x_\infty) \\ \frac{1}{2}p & \text{if} \quad x = x_\infty \\ p & \text{if} \quad x \in (x_\infty, L). \end{cases}$$

Moreover, if the equilibrium solution (u_∞, v_∞) is reached from the initial data (u_0, v_0), the conservation law (2.9) implies that

$$p = \frac{U + V}{2L}, \qquad x_\infty = \frac{U - pL}{p}, \tag{2.13}$$

where

$$U = \int_\Omega u_0, \qquad V = \int_\Omega v_0. \tag{2.14}$$

That these formal calculations are indeed correct is a consequence of the following theorem.

Theorem 4. *Let* (u,v) *be the segregated solution of* (I). *Then, as* $t \to \infty$,

$$\zeta(t) \to x_\infty, \quad u(t) \to u_\infty, \quad v(t) \to v_\infty,$$

the latter two limits being pointwise in $\Omega \backslash \{x_\infty\}$. *Here* $\zeta(t)$ *is the contact front* (cf. *Theorem* 3), *while* x_∞, u_∞, *and* v_∞ *are defined in* (2.12), (2.13).

3. Reformulation of the problem.

Let (u,v) be a sufficiently smooth segregated solution of Problem (I) and define

$$z(x,t) = -U + \int_{-L}^{x} [u(y,t) + v(y,t)]dy; \tag{3.1}$$

$z(x,t)$ represents the total population, at time t, in the interval $[-L,x]$. In view of the conservation laws (2.9),

$$z(-L,t) = -U, \quad z(\xi(t), t) = 0, \quad z(L,t) = V, \tag{3.2}$$

where U and V are defined in (2.14), so that separation curves $\xi(t)$ (cf. (2.7)) are level curves $z(\xi(t), t) = 0$; in fact,

$$\begin{aligned} z(x,t) &\leq 0 \quad \text{for} \quad x < \xi(t), \\ z(x,t) &\geq 0 \quad \text{for} \quad x > \xi(t). \end{aligned} \tag{3.3}$$

Further, if we differentiate (3.1) with respect to t and use (2.1), (2.2), and (2.7), we find that

$$z_t = \begin{cases} z_x z_{xx} & \text{for} \quad x < \xi(t) \\ k z_x z_{xx} & \text{for} \quad x > \xi(t). \end{cases} \tag{3.4}$$

Thus defining $c\colon \mathbb{R} \to \mathbb{R}$ by

$$c(s) = \begin{cases} s, & s \leq 0 \\ s/k, & s > 0, \end{cases} \tag{3.5}$$

we may use (3.3) to reduce (3.4) to the single equation

$$c(z)_t = z_x z_{xx}$$

on all of Q. Therefore, if we write

$$z_0(x) = -U + \int_{-L}^{x} (u_0+v_0), \tag{3.6}$$

we are led to the following problem for $z(x,t)$:

$$(\mathrm{II}) \begin{cases} c(z)_t = z_x z_{xx} & \text{in } Q, \\ z(-L,t) = -U, \quad z(L,t) = V, & t > 0, \\ z(x,0) = z_0(x) & \text{in } \Omega. \end{cases}$$

We assume, for the remainder of the section, that c <u>and</u> z_0 <u>are defined by</u> (3.5) <u>and</u> (3.6), <u>and that</u> (A1)-(A3) <u>are satisfied</u>.

Problem (II) under hypotheses more general than ours, has been analyzed in [10]. We shall simply state, without proof, a version of the results of [10] appropriate for our use. With this in mind, we first define what we mean by a solution; in that definition, and in what follows, $z_\infty(x)$ designates the unique equilibrium solution of (II):

$$z_\infty(x) = \frac{(U+V)(x+L)}{2L} - U.$$

Definition. A (weak) *solution* of Problem (II) is a function $z \in C([0,\infty); W^{1,\infty}(\Omega))$ with the following properties:

(i) $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$ for $t > 0$;

(ii) $z_t \in L^2(Q_T)$ for $T > 0$;

(iii) for all $\psi \in C^1(\bar{Q})$ with $\psi = 0$ on $\partial\Omega \times (0,\infty)$ and all $T > 0$,

$$\int_\Omega \{c(z(T))\psi(T) - c(z_0)\psi(0)\} = \int_{Q_T} \{c(z)\psi_t - \tfrac{1}{2}(z_x)^2\psi_x\}. \qquad (3.7)$$

Theorem 5 ([10]). *Problem* (II) *has exactly one solution* z. *Moreover*:

(i) $z_x \in C(\bar{Q})$ *with* $z_x \geq 0$;

(ii) $c(z)_t = z_x z_{xx}$ *classically in* $Q^+(z_x)$;

(iii) $Q^+(z_x)$ *is the union of the sets*

$$Q_1 := \{(x,t) \in Q: -U < z(x,t) < 0\},$$

$$Q_2 := \{(x,t) \in Q: 0 < z(x,t) < V\},$$

and there exist free-boundary curves $b_1, b_2, \zeta_1, \zeta_2$ *such that* (i) - (v) *of Theorem* 3 *hold with* $Q^+(u), Q^+(v), b_u, b_v, \zeta_u, \zeta_v$ *replaced by* $Q_1, Q_2, b_1, b_2, \zeta_1, \zeta_2$, *respectively*, *with* $u+v$ *in* (iii) *replaced by* z_x, *and with* u_x *and* v_x *in* (iv) *and* (v) *replaced by* z_{xx};

(iv) $z(t) \to z_\infty$ *in* $C^1(\bar{\Omega})$ *as* $t \to \infty$;

(v) *given any* $x_0 \in \Omega$ *with* $z_0'(x_0) > 0$, *there exists a continuous function* ξ: $[0,\infty) \to \Omega$ *such that*

$$\{(x,t) \in \bar{Q}:\ z(x,t) = z_0(x_0)\} = \{(x,t) \in \bar{Q}:\ x = \xi(t)\};$$

moreover, $\xi \in C^1(0,\infty)$ *and, for* $t > 0$,

$$\xi'(t) = -\kappa z_{xx}(\xi(t),\ t),$$

where $\kappa = 1$ *or* k *according as* $z_0(x_0) < 0$ *or* $z_0(x_0) > 0$.

The next result asserts the equivalence of Problems I and (II) and, when combined with Theorem 5, *yields the validity of Theorems 1-4.*

Theorem 6. *Problems* (I) *and* (II) *are equivalent:*

(i) *Let* z *be a solution of Problem* (II) *and define* u *and* v *on* $\bar{Q}$ *by*

$$\begin{array}{ll} u(x,t) = z_x(x,t),\quad v(x,t) = 0 & \textit{if}\ \ z(x,t) < 0, \\ u(x,t) = 0,\quad v(x,t) = z_x(x,t) & \textit{if}\ \ z(x,t) > 0, \\ u(x,t) = v(x,t) = \frac{1}{2}\, z_x(x,t) & \textit{if}\ \ z(x,t) = 0. \end{array} \tag{3.8}$$

Then (u,v) *is a segregated solution of Problem* (I).

(ii) *Conversely, let* (u,v) *be a segregated solution of Problem* (I) *and define* z *on* $\bar{Q}$ *by* (3.1). *Then* z *solves Problem* (II).

Proof.

(i) Let z be a solution of (II) and define (u,v) through (3.8). By Theorem 5(i), the only nontrivial step in showing that (u,v) solves (I) is proving that u and v satisfy the integral identities (2.6). We shall only verify the first of (2.6); the verification of the second is completely analogous.

For convenience, we write $b = b_1$, $\zeta = \zeta_1$ for the FB curves established in Theorem 5, and we extend $b(t)$ continuously to $[0,\infty)$ by defining $b(t) = -L$ for $t \geq t_b$. By Theorem 5(iii) and (3.8),

$$\operatorname{supp} u(t) = [b(t), \zeta(t)].$$

Choose $\varepsilon > 0$ sufficiently small and let $b_\varepsilon(t)$ and $\zeta_\varepsilon(t)$, respectively, be the level curves $z = -U + \varepsilon$ and $z = -\varepsilon$ (cf. Theorem 5(v)). Then, by Theorem 5(ii) and (3.8), $u_t = (uu_x)_x$ classically and $v \equiv 0$ must both be satisfied in a neighborhood of any (x,t) such that $b_\varepsilon(t) \leq x \leq \zeta_\varepsilon(t)$ and $t > 0$. Further, Theorem 5(v) yields

$$b_\varepsilon'(t) = -u_x(b_\varepsilon(t), t), \quad \zeta_\varepsilon'(t) = -u_x(\zeta_\varepsilon(t), t)$$

for $t > 0$. Thus, choosing $\delta > 0$, the identity

$$\int_\delta^t f'(\tau)d\tau = f(t) - f(\delta)$$

applied to

$$f(\tau) = \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} u(\tau)\psi(\tau)$$

yields, when $\psi \in C^1(\bar{Q})$,

$$\int_{b_\varepsilon(t)}^{\zeta_\varepsilon(t)} u(t)\psi(t) - \int_{b_\varepsilon(\delta)}^{\zeta_\varepsilon(\delta)} u(\delta)\psi(\delta) = \int_\delta^t \left\{ \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\} d\tau. \tag{3.9}$$

Next, since $b_\varepsilon(t) \downarrow b(t)$ and $\zeta_\varepsilon(t) \uparrow \zeta(t)$ as $\varepsilon \downarrow 0$ for each $t \in [0,\infty)$, it follows from Lebesgue's dominated convergence theorem that (3.9) holds with b_ε and ζ_ε replaced by b and ζ. Also, since $z_x \in C(\bar{Q})$, it follows that $z_x(\delta) \to z_0'$ in $C(\bar{\Omega})$ as $\delta \downarrow 0$ and

$$\int_{b(\delta)}^{\zeta(\delta)} u(\delta)\psi(\delta) \to \int_{b(0)}^{\zeta(0)} u_0\psi(0) \quad \text{as } \delta \downarrow 0.$$

Thus a second application of Lebesgue's theorem yields

$$\int_{b(t)}^{\zeta(t)} u(t)\psi(t) - \int_{b(0)}^{\zeta(0)} u_0\psi(0) = \int_0^t \left\{ \int_{b(\tau)}^{\zeta(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\} d\tau,$$

and, since $u(t) \equiv 0$ on $\Omega\backslash(b(t), \zeta(t))$, the first of (2.6) follows.

(ii) Let (u,v) solve (I). We are to prove that z - defined by (3.1) - solves (II). Choose $T > 0$. Then $u,v \in L^\infty(Q_T)$ and hence, by (2.9) and the definition of z_∞, $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$. Note also that, since $z_x = u + v$ and $(u+v)^2 \in L^2(0,T;H^1(\Omega))$, it follows that

$$z_x^2 \in L^2(0,T;H^1(\Omega)), \tag{3.10}$$

where $z_x^2 = (z_x)^2$.

Our next step will be to establish the integral identity (3.7). Thus choose $\chi \in C^1(\bar{Q})$ with $\chi = 0$ on $\partial\Omega \times (0,\infty)$ and take

$$\psi(x,t) = \int_{-L}^{x} \chi(y,t)\,dy$$

in (2.6); in view of (2.7), (3.1), and (3.6), the result is

$$\begin{aligned}
&\int_{-L}^{\xi(T)} z_x(T)\psi(T) - \int_{-L}^{\xi(0)} z_0'\psi(0) = \int_0^T \int_{-L}^{\xi(t)} \{z_x\psi_t - \tfrac{1}{2}(z_x^2)_x\chi\}dxdt,\\
&\int_{\xi(T)}^{L} z_x(T)\psi(T) - \int_{\xi(0)}^{L} z_0'\psi(0) = \int_0^T \int_{\xi(t)}^{L} \{z_x\psi_t - \tfrac{k}{2}(z_x^2)_x\chi\}dxdt.
\end{aligned} \tag{3.11}$$

Since $\psi_x = \chi$, $\psi(-L,t) = 0$, and $\chi(\pm L,t) = 0$, while $z(x,t)$ satisfies (3.2) and (3.3), if we integrate (3.11) by parts and then add the first of the resulting relations to k^{-1} times the second, we arrive at (3.7) (with ψ replaced by χ).

We have only to show that $z_t \in L^2(Q_T)$. But this follows from (3.10) and the fact that, by (3.7), $c(z)_t = \frac{1}{2}(z_x^2)_x$ in the sense of distributions on Q_T. This completes the proof of Theorem 6.

4. Remarks. Open problems. Conjectures.

1. Problem (I) with nonsegregated initial data is open. Here the problem does not reduce to a free boundary problem for a single scalar field z, as one must solve the system (2.1) in regions of interaction (cf. Remark 2).

2. The system (2.1) with $k = 1$ is far simpler to analyze. There the total density $\rho = u + v$ satisfies (PM) with initial data $\rho_0 = u_0 + v_0$, and once ρ is known (2.1) are linear hyperbolic equations for u and v:

$$u_t = (u\rho_x)_x, \quad v_t = (v\rho_x)_x$$

(cf. [5]). Using this reduction one can prove uniqueness within the class of all solutions (as opposed to all segregated solutions), a one can show that solutions which begin mixed remain mixed for all time, including $t = \infty$. (Details will appear elsewhere.)

3. Assume, in place of (1.2), that the dispersal of each of the species is driven by a weighted sum of the densities; i.e., that (in one space-dimension),

$$\begin{aligned} q &= -(k_{11}u + k_{12}v)_x \\ w &= -(k_{21}u + k_{22}v)_x \end{aligned} \qquad (4.1)$$

with all

$$k_{ij} > 0. \qquad (4.2)$$

This constitutive assumption, when combined with the conservation law (1.1) and corresponding zero-flux boundary conditions, leads to the problem

$$(\mathrm{III})\begin{cases} u_t = [u(k_{11}u + k_{12}v)_x]_x \\ \qquad\qquad\qquad\qquad \text{in } Q, \\ v_t = [v(k_{21}u + k_{22}v)_x]_x \\ u(k_{11}u + k_{12}v)_x = v(k_{21}u + k_{22}v)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{4.3}$$

This formulation is greatly simplified if we define new independent variables

$$\alpha(x,t) = k_{11}u(x,t), \quad \beta(x,t) = k_{12}v(x,t)$$

and new constants

$$k = \frac{k_{22}}{k_{12}}, \quad \mu = \frac{k_{12}k_{21}}{k_{11}k_{22}},$$

for then (III) reduces to

$$(\mathrm{IV})\begin{cases} \alpha_t = [\alpha(\alpha+\beta)_x]_x \\ \qquad\qquad\qquad\qquad \text{in } Q, \\ \beta_t = k[\beta(\mu\alpha + \beta)_x]_x \\ \alpha(\alpha + \beta)_x = \beta(\mu\alpha + \beta)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ \alpha(x,0) = \alpha_0(x), \quad \beta(x,0) = \beta_0(x) \quad \text{in} \quad \Omega, \end{cases} \tag{4.4}$$

with

$$\alpha_0 = k_{11}u_0, \quad \beta_0 = k_{12}v_0.$$

4. Consider Problem (III), or equivalently (IV). The terms on the right side of (4.3) involving second derivatives are

$$\begin{pmatrix} uk_{11} & uk_{12} \\ vk_{21} & vk_{22} \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}. \tag{4.5}$$

Writing $A = A(u,v)$ for the coefficient matrix in (4.5) and confining our attention to $u > 0$, $v > 0$, we conclude from (4.2) that there are exactly three possibilities for the eigenvalues $\lambda_1 \leq \lambda_2$ of A, namely:

(i) $\lambda_1 > 0, \lambda_2 > 0$; (ii) $\lambda_1 = 0, \lambda_2 > 0$; (iii) $\lambda_1 < 0, \lambda_2 > 0$.

Moreover, writing K for the matrix

$$K = (k_{ij}),$$

it is not difficult to verify that

(i) $\lambda_1 > 0, \quad \lambda_2 > 0 \iff \det K > 0,$

(ii) $\lambda_1 = 0, \quad \lambda_2 > 0 \iff \det K = 0,$

(iii) $\lambda_1 < 0, \quad \lambda_2 > 0 \iff \det K < 0.$

We consider the three cases separately.

Case (i) ($\det K > 0$). Here the system (4.3) is degenerate parabolic, as it is parabolic when $u > 0$ and $v > 0$, but not when $uv = 0$. Because of this property, we expect that initially-segregated solutions will eventually mix. We also expect them to mix for another reason. Indeed, assume to the contrary that Problem (III) has a segregated solution (u,v). For such a solution we would expect the two populations to spread until they meet, and then to remain in contact along a contact front $\zeta(t)$ (cf. Theorem 3 and Remark 2 following it). From (4.2) one might expect that both $k_{11}u + k_{12}v$ and $k_{21}u + k_{22}v$ would be continuous across ζ, and hence both zero along ζ, a condition which cannot generally be satisfied (cf. Remark 1 following Theorem 3). We are therefore led to the following conjecture: <u>for</u> $\det K > 0$ <u>there are no segregated solutions of Problem</u> (III). In this regard it would be interesting to look at (III) with[1]

$$K = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{pmatrix},$$

$\varepsilon > 0$; in particular, the limit $\varepsilon \to 0$.

Finally, within the context of the biological model, the off-diagonal elements of K drive the segregation of the species, while the diagonal elements, by themselves, result in the usual diffusive behavior. Since $\det K > 0$ yields $k_{11}k_{22} > k_{12}k_{21}$, it would seem reasonable that in this case the two species ultimately mix.

<u>Case</u> (ii) ($\det K = 0$). Here $\mu = 1$ and Problem (IV) is identical to Problem (I). Thus <u>all of our results generalize trivially to populations whose interaction is described by</u> (4.1) <u>with</u> K <u>singular</u>.

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[1] This choice of K arose in discussions with R. Rostamian.

For the case $\det K = 0$ we would like to call the system (4.3) degenerate parabolic-hyperbolic. Indeed, if we set $w = \alpha + \beta$, then, assuming $k \geq 1$, (4.4) can be written as

$$w_t = [\{w + (k-1)\beta\}w_x]_x,$$

$$\beta_t = k(w_x\beta)_x$$

i.e., as a system composed of a degenerate-parabolic equation and a hyperbolic equation. The presence of this last equation makes the discontinuity of u and v at the contact front less surprising.

In this case one can speak of "passive segregation": if the species start segregated, they may remain segregated, as we have seen in the previous sections, and if they start mixed, then, when $k = 1$, they remain mixed for all $t \geq 0$ (see Remark 2 of this section).

Case (iii) ($\det K < 0$). The system (4.3) is now not parabolic, and Problem (II) is probably not well posed. Since the off-diagonal terms in K dominate in this case, one might expect a tendency towards segregation, even in a mixed population.

5. The system (2.1) with $k = 0$ was studied in [3] and [4]. There $v(x) \equiv v_0(x)$ and the problem reduces to solving $(2.1)_1$, $(2.2)_1$, and $(2.3)_1$. In this case, even with segregated initial data, solutions eventually mix, an apparent contradiction in behavior. The limit $k \to 0$ in Problem (I) would therefore be interesting.

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