Astérisque

David Lannes

Space time resonances [after Germain, Masmoudi, Shatah]

Astérisque, tome 352 (2013), Séminaire Bourbaki,
exp. nº 1053, p. 355-388

<http://www.numdam.org/item?id=AST_2013__352__355_0>

Séminaire BOURBAKI Mars 2012
64^{e} année, 2011-2012, n° 1053, p. 355 à 388

SPACE TIME RESONANCES
[after Germain, Masmoudi, Shatah]

by **David LANNES**

INTRODUCTION

An important research program in nonlinear partial differential equations consists in proving the existence of global in time smooth solutions to various nonlinear dispersive equations on $\mathbb{R}^d$ (d integer, $d \geq 1$) with small initial data. This program was initiated about three decades ago and has been the motivation for the development of powerful concepts.

A general feature is that the linear dispersive terms of the equation tend to force the solution to spread and to decay. Various *dispersive estimates* have been derived to provide precise informations on this decay. The contribution of the nonlinear terms is very different. As for ordinary differential equations, they may be responsible for the development of finite time singularities. When dealing with small data, smooth nonlinearities behave roughly as their Taylor expansion at zero. The smaller the homogeneity p of the nonlinearity at the origin, the larger the nonlinear effects. A first class of global existence results can be obtained when dispersive effects dominate nonlinear effects. Since dispersive effects increase with the dimension d, this is the general situation in large dimension and/or large p; in this situation, nonlinearities do not contribute to the large time behavior of the solution (see for instance [**39**]).

In smaller dimension or for lower order nonlinearities, the situation is more complicated and depends on the precise structure of the nonlinearity, not only on its order. For the quadratic wave equation in dimension $d = 3$, Klainerman identified [**25**] the so called *null condition* on the nonlinearities that ensures, with his powerful vector fields method, global existence for small data. This method is very robust and has been used for many other equations; a spectacular illustration is for instance [**4**] for the global nonlinear stability of the Minkowski space (see also [**30**] for a simplified proof using the notion of weak null condition). We also refer for instance to [**20**] for applications to the Schrödinger equation, to **[27]** for a small review, and to **[8]** for a new approach of the vector fields method.

Another powerful technique to obtain global existence of nonlinear dispersive equations in low dimension or lower order nonlinearities is the normal form method popularized by Shatah who used it for the nonlinear Klein-Gordon equation **[34]** (see also **[36]** for a similar approach by Simon). The idea of this method is inspired by the theory of Poincaré's normal forms for dynamical systems; for a quadratic equation for instance, it consists in making a quadratic change of unknown chosen so that the new unknown solves a cubic evolution equation, for which global existence is much easier to establish. In absence of, or with few *time resonances*, this method is very efficient, and has also been used in many works. See for instance **[33, 37, 31]**, as well as **[9]** where the relevance of null conditions for the normal form method is exploited.

In a series of papers **[14, 16, 15, 12, 13]**, Germain, Masmoudi and Shatah introduced a new method to handle situations where the normal form approach cannot be used. The same idea has also been used independently by Gustafson, Nakanishi and Tsai **[17, 18]** for the Gross-Pitaevskii equation. Working on a Duhamel formulation of the equations in Fourier variables, they identify the normal form transform as an integration by parts in time in this formula. Time resonances are the natural obstruction since they create singularities when this integration by parts is performed. Germain, Masmoudi and Shatah propose to complement this approach with an integration by parts in frequency that provides extra time decay, which is helpful to prove global well posedness. The obstructions to this approach are called by the authors *space resonances*; they differ in general from time resonances, which explains why situations that were not covered by the normal form approach can be handled this way. As for the vector fields with the null condition, the structure of the nonlinearities plays an important role for the space time resonance approach; when the nonlinearities cancel some of the singularities created by time or frequency integration by parts, one may expect the normal form method or Germain, Masmoudi and Shatah's more general approach to work even in situations where time and/or space resonances are present. Based on an analogy with optics, we call here these structural conditions *time and space transparency.*

We tried in these notes to distinguish the notion of null condition from those of space and time transparencies; we also relate them to another structural condition on the nonlinearities called *compatibility*, and which is linked to the decay rate of products of solutions of homogeneous linear dispersive equations.

Throughout these notes, we use the following quadratic wave equation as a simple example to explain Klainerman's vector field method, the normal form approach, and

Germain-Masmoudi-Shatah's new method,

$$\partial_t^2 u - \Delta u = Q(\partial u, \partial u), \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)}, \tag{1}$$

where $Q(\cdot,\cdot)$ is a symmetric bilinear form and $\partial u = (\partial_t u, \partial_1 u, \dots, \partial_d u)^T$. When the simplicity of this example hides important phenomena, we also use a system of two coupled such equations. We also comment on the case of general first order symmetric systems because this framework is quite adapted for a comparison of the null condition, the space and time transparencies, and the compatibility condition.

Section 1 is devoted to a general exposure of Klainerman's vector field method, while Section 2 is centered on the normal form approach. These techniques are very classical and our goal is not to review recent results related to them; we just present their basic mechanisms to help understanding the rationale and the interest of the new method of Germain, Masmoudi and Shatah, which is described in Section 3. We also include in this section a description of the authors' global existence result for the water waves equations **[16]**, which is probably the most important example of application of this new method. Finally we point out in Section 4 that the null, transparency and compatibilities conditions play also a role in other contexts than the issue of global existence for small data.

1. KLAINERMAN'S VECTOR FIELDS METHOD

As explained in the introduction, global existence for small initial data is the general scenario for nonlinear dispersive equations when the dimension is large and/or the nonlinearity is of high order at the origin. For the quadratic wave equation (1), global existence is always true when $d \geq 4$. We sketch the proof of this classical result in § 1.1.

1.1. Global existence for the quadratic wave equation (1) in dimension $\mathbf{d \geq 4}$

We prove in this section the following theorem using the vector fields method introduced by Klainerman **[25]**.

THEOREM 1.1 (**[25]**). — *The Cauchy problem* (1) *with smooth compactly supported initial conditions has a smooth solution for all $t \geq 0$ if $d \geq 4$ and ε is small enough.*

For the linear homogeneous wave equation,

$$\partial_t^2 u - \Delta u = 0, \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)}, \tag{2}$$

the energy

$$\mathscr{E}(u) = \frac{1}{2}|\partial_t u|_2^2 + \frac{1}{2}|\nabla u|_2^2$$

is conserved. More generally, one gets the following classical energy inequality after multiplying $\Box u$ by $\partial_t u$ and integrating in space,

$$\mathcal{E}(u)^{1/2}(t) \leq \mathcal{E}(u)^{1/2}(0) + \int_0^t |\Box u(\tau,\cdot)|_2 d\tau. \tag{3}$$

If Z is a vector field that commutes with the operator $\Box = \partial_t^2 - \Delta$, and if u solves (2), one also has

$$(\partial_t^2 - \Delta) Z u = 0, \tag{4}$$

and $\mathcal{E}(Zu)$ is also conserved. More generally, if $Z^1, \dots, Z^n$ is a family of vector fields that commute with the wave operator $\Box$, the quantity $\mathcal{E}(Z^1 \cdots Z^n u)$ is conserved; this yields important information on the regularity and/or decay properties of the solution. For the wave equation, the vector fields that commute with $\Box$ are

$$\partial_\alpha, \qquad Z_{jk} = x_k\partial_j - x_j\partial_k, \qquad Z_j = x_j\partial_t + t\partial_j, \tag{5}$$

where $0 \leq \alpha \leq d$, $1 \leq j,k \leq d$, $(t,x) = (x_0, x_1, \dots, x_d)$ and $\partial_\alpha = \partial_{x_\alpha}$. These vector fields correspond to invariances of the equation, respectively translation and Lorentzian invariances. Another important vector field is given by

$$Z_0 = t\partial_t + \sum_{j=1}^d x_j \partial_j, \tag{6}$$

corresponding to scaling invariance; note that Z_0 does not commute with $\Box = \partial_t^2 - \Delta$ but that $[\Box, Z_0] = 2\Box$, so that the property (4) holds. We call *commuting vector fields* the vector fields (5) and (6).

One can then build Sobolev-type norms based on these vector fields and generalize the standard embedding $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ $(s > d/2)$; more precisely, for all smooth and decaying function v of (t,x), the following Klainerman-Sobolev inequality (due to Klainerman **[25]**, see also **[21, 38]** for a proof) holds,

$$\big(1 + t + |x|\big)^{d-1}\big(1 + |t - |x||\big)|v(t,x)|^2 \leq C \sum_{|I| \leq d/2+1} |Z^I v|_2^2, \tag{7}$$

where Z^I denotes any product of $|I|$ of the above commuting vector fields.

Defining, for all $s \geq 0$, the higher order energy

$$\mathcal{E}^s(v) = \sum_{|I| \leq s} \mathcal{E}(Z^I v),$$

and remarking that for all $0 \leq \alpha \leq d$,

$$Z^I \partial_\alpha = \text{linear combination of vector fields } \partial_\beta Z^J, \text{ with } |J| \leq |I|, \tag{8}$$

the inequality (7) implies that for all product Z^K of $|K|$ commuting vector fields,

$$\big(1 + t + |x|\big)^{d-1}\big(1 + |t - |x||\big)|Z^K \partial v(t,x)|^2 \leq C \mathcal{E}^{d/2+1+|K|}(v). \tag{9}$$

Since $\mathscr{E}^{d/2+1+|K|}(u)$ is a conserved quantity if u solves (2), (9) furnishes decay estimates for the solution of the homogeneous wave equation with smooth and compactly supported initial data. It is also the heart of Klainerman's proof of global existence. Since either $[\Box, Z]U = 0$ or $[\Box, Z]U = 2\Box U = 2Q(\partial u, \partial u)$, we get by applying Z^I to (1) that

$$\begin{aligned}\Box(Z^I u) &= [\Box, Z^I]U + Z^I Q(\partial u, \partial u), \\ &= \sum_{|J|+|K|\leq|I|} Q_{JK}\big(Z^J\partial u, Z^K\partial u\big)\end{aligned} \tag{10}$$

where the Q_{JK} are also bilinear forms on $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$. Therefore from (3),

$$\sum_{|I|\leq s}\mathscr{E}(Z^I u)^{1/2}(t) \leq \sum_{|I|\leq s}\mathscr{E}(Z^I u)^{1/2}(0) + \sum_{|J|+|K|\leq s}\int_0^t |Q_{JK}(Z^J\partial u, Z^K\partial u)(\tau)|_2 d\tau. \tag{11}$$

Defining

$$M_s(t) = \sum_{|I|\leq s}|Z^I\partial u(t,\cdot)|_2, \qquad \text{and} \qquad m_s(t) = \sum_{|I|\leq s}|Z^I\partial u(t,\cdot)|_\infty$$

(so that, thanks to (8), $M_s(t)\sim\mathscr{E}^s(u)^{1/2}$), and remarking that

$$\forall J,K,\quad |J|+|K|\leq s,\qquad |Q_{JK}(Z^J\partial u, Z^K\partial u)(\tau)|_2 \leq C m_{s/2}(\tau)M_s(\tau),$$

we deduce from (11) that

$$M_s(t) \leq C\Big(M_s(0) + \int_0^t m_{s/2}(\tau)M_s(\tau)d\tau\Big). \tag{12}$$

We also deduce from (9) that

$$(1+t^{\frac{d-1}{2}})m_{s/2}(\tau) \leq C M_{s/2+d/2+1}(\tau)$$

and therefore, for all $s \geq d+2$ (so that $s/2 + d/2 + 1 \leq s$),

$$M_s(t) \leq \underline{C}\Big(M_s(0) + \int_0^t (1+\tau^{\frac{d-1}{2}})^{-1}M_s(\tau)^2 d\tau\Big), \tag{13}$$

for some constant $\underline{C} > 0$.

Denoting $c_\infty = \int_0^\infty (1+\tau^{\frac{d-1}{2}})^{-1} < \infty$ (since $d > 3$), we deduce that $M_s(t)$ remains bounded from above by $2\underline{C}M_s(0)$ provided that $1 + 4\underline{C}c_\infty M_s(0) < 2$, which is always possible for small enough ε. Global existence then follows from a standard continuation argument.

1.2. Null forms and global existence in dimension $d=3$

The global existence result proved in the previous section relies on the convergence of the integral $\int_0^\infty (1+\tau^{\frac{d-1}{2}})^{-1}d\tau$ in (13). In dimension $d=3$, this integral diverges logarithmically, and an adaptation of the same arguments yields a lower bound for the existence time, $T_\varepsilon \geq \exp(\frac{c}{\varepsilon})$, for some constant $c>0$. One cannot expect a better result in general; Fritz John **[22]** showed for instance that (radial) solutions to the equation

$$(14)\qquad \partial_t^2 u - \Delta u = (\partial_t u)^2, \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)},$$

blow up at $T_\varepsilon \sim \exp(\frac{c}{\varepsilon})$ (see also **[1]** for other examples of blow up).

On the other hand, as noticed by Nirenberg, it is easy to see that

$$(15)\qquad \partial_t^2 u - \Delta u = -(\partial_t u)^2 + |\nabla u|^2, \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)},$$

admits global solutions for small ε. Let indeed v be the solution of the homogeneous wave Equation (2) with Cauchy data $v_{|t=0} = \exp(\varepsilon u_{(0)}) - 1$ and $\partial_t v_{|t=0} = \varepsilon \exp(\varepsilon u_{(0)}) u_{(1)}$. Setting $\tilde{u} = \ln(1+v)$ (this makes sense for small enough ε), one has $\tilde{u}_{|t=0} = \varepsilon u_{(0)}$, $\partial_t \tilde{u}_{|t=0} = \varepsilon u_{(1)}$ and

$$0 = \partial_t^2 v - \Delta v = (\exp \tilde{u}) \times \big(\partial_t^2 \tilde{u} - \Delta \tilde{u} + (\partial_t \tilde{u})^2 - |\nabla \tilde{u}|^2\big);$$

it follows that $\tilde{u}$ solves the same equation as u, and has the same Cauchy data, so that $u = \tilde{u}$. Since $\tilde{u}$ is obviously globally defined, the result follows. Of course, this method is not robust at all: it does not generalize to systems, or cubic perturbations, or to the quasilinear case treated by Klainerman.

The different behavior observed for (14) and (15) can only come from the structure of the bilinear forms

$$Q(\xi^1,\xi^2) = \tau^1\tau^2 \quad \text{and} \quad Q(\xi^1,\xi^2) = -\tau^1\tau^2 + \eta^1\cdot\eta^2 =: -Q_0(\xi^1,\xi^2)$$

respectively (with $\xi^j = (\tau^j,\eta^j) \in \mathbb{R}^{1+d}$, $j=1,2$)—here, $d=3$, but this discussion holds for all $d \geq 2$. If u and v denote two solutions of the homogeneous wave equation (2) with smooth and compactly supported initial data, then we can deduce from the Klainerman-Sobolev inequality (7) that ∂u and ∂v decay as $O(t^{(-d+1)/2})$ so that $Q(\partial u, \partial v) = O(t^{-d+1})$. However, the decay of ∂u and ∂v is not uniform in all directions. It is of order $O(t^{(-d+1)/2})$ in the direction $\partial_t - \partial_r$ perpendicular to the light cone but of order $O(t^{(-d-1)/2})$ in the tangential directions $\partial_t + \partial_r$ and $\partial_j - \frac{x_j}{r}\partial_r$. The specific property about Q_0 is that it does not contain quadratic terms involving only derivatives in the bad direction; consequently, one has a better decay estimate of the quadratic term

$$(16)\qquad \text{For all } u,v \text{ solving (2)}, \qquad Q_0(\partial u, \partial v) = O(t^{-d}).$$

It can be shown that a quadratic form satisfies this property if and only if it is proportional to Q_0; we then say that it is *compatible* with the wave operator $\Box$ (see §1.3 below for generalizations to first order hyperbolic systems).

For the quadratic wave Equation (1), compatible forms coincide with the bilinear forms that satisfy the following *null condition*, identified by Klainerman (we follow here Klainerman's proof **[26]**),

$$|Q_0(\partial u, \partial v)(t,x)|_\infty \leq \frac{C}{1+t+|x|} \sum_{|I|=1} |Z^I u(t,x)| \sum_{|I|=1} |Z^I v(t,x)|, \tag{17}$$

for all $t \geq 0$, $x \in \mathbb{R}^3$, and all sufficiently smooth functions u and v. They behave therefore like cubic terms for time decay. Such forms are called *null forms.* The fact that Q_0 is a null form follows easily from the double observation that

$$\begin{aligned} Q_0(\partial u, \partial v) &= \frac{1}{t}\big(\partial_t u Z_0 v - \sum_{j=1}^{3} Z_i u \partial_i v\big) \\ &= \frac{1}{|x|}\big(Z_r u \partial_t v - \partial_r u Z_0 v - \sum_{i,j=1}^{3} \partial_i u \frac{x_i}{|x|} Z_{ij} v\big), \end{aligned}$$

where $\partial_r = \sum_{j=1}^3 \frac{x_j}{|x|}\partial_j$ and $Z_r = \sum_{j=1}^3 \frac{x_j}{|x|} Z_j$.

Another important property about the null form Q_0 (and therefore about all null forms for (1)) is that, for all commuting vector fields Z given by (5) or (6),

(18) if Q is a null form for (1), then $[Z,Q]$ is also a null form,

with the notation $[Z,Q](\partial u, \partial v) = ZQ(\partial u, \partial v) - Q(\partial Z u, \partial v) - Q(\partial u, \partial Z v)$. Thanks to (18), all the quadratic forms Q_{JK} in (10) are null forms. Using (17), we can replace the integral in the r.h.s. of the energy estimate (12) by

$$\int_0^t \frac{1}{1+\tau} \sum_{|L|\leq s/2+1} |Z^L u(\tau)|_\infty \sum_{|I|\leq s+1} |Z^I u(\tau,\cdot)|_2 d\tau; \tag{19}$$

according to the discussion above, the previous proof should yield global existence thanks to this extra decay in time; unfortunately, if we set

$$N_s(t) = \sum_{|I|\leq s+1} |Z^I u(t,\cdot)|_2 \quad \text{and} \quad n_s(t,x) = \sum_{|I|\leq s+1} |Z^I u(t,x)|,$$

the quantities $N_s(t)$ and $|n_s(t,\cdot)|_\infty$ are not controlled by $M_s(t)$ and $m_s(t)$ respectively (though the same number of derivatives are involved), and we cannot use the same bootstrap argument.

The reason of this lack of control is that the energy $\mathcal{E}(u)$ controls $|\partial u|_2^2$ but not $\sum_{|I|\leq 1} |Z^I u|_2^2$. One possible way to deal with this obstruction is to look for a new vector field $L(\partial)$ such that $\Box u L(\partial) u$ is a conservation law for a new energy $\mathcal{F}(u)$ that controls more vector fields than $\mathscr{E}(u)$ (which is obtained by taking $L(\partial)=\partial_t$). The vector field

$$L(\partial)u=(1+t^2+|x|^2)\partial_t+2tx\cdot\nabla+(d-1)t,$$

which had been introduced by Morawetz, has this remarkable property (see for instance **[21, 38]** for a proof). The associated energy is then given (in the case $d=3$ we are interested in here) by

$$\mathscr{F}(u)=\mathscr{E}(u)+\frac{1}{2}\sum_{j,k=1}^{3}|Z_{jk}u|_2^2+\frac{1}{2}\sum_{j=1}^{3}|Z_ju|_2^2+\frac{1}{2}|Z_0u+2u|_2^2,$$

and one can then show that this energy provides the additional control we were looking for,

$$\sum_{|I|\leq 1}|Z^Iu|_2\sim\mathscr{F}(u)^{1/2}.$$

Proceeding as in §1.1 but with $\mathscr{F}(u)$ rather than $\mathscr{E}(u)$ (i.e. multiplying $\Box u$ by $L(\partial)u$ instead of $\partial_t u$), the energy inequality (3) becomes

$$\mathscr{F}(u)^{1/2}(t)\leq\mathscr{F}(u)^{1/2}(0)+\int_0^t|(1+\tau+|x|)\Box u(\tau,\cdot)|_2d\tau; \tag{20}$$

similarly, (11) is replaced by

$$N_s(t)\leq C\Big(N_s(0)+\sum_{|J|+|K|\leq s}\int_0^t|(1+\tau+|x|)Q_{JK}(Z^J\partial u,Z^K\partial u)(\tau)|_2d\tau\Big). \tag{21}$$

By the null form property (17), we then get the following estimate instead of (12),

$$N_s(t)\leq C\Big(N_s(0)+\int_0^t|n_{s/2}(\tau,\cdot)|_\infty N_s(\tau)d\tau\Big), \tag{22}$$

and therefore, by Gronwall's lemma,

$$N_s(t)\leq C_1N_s(0)\exp\big(C_1\int_0^t|n_{s/2}(\tau,\cdot)|_\infty d\tau\big) \tag{23}$$

for some constant $C_1>0$.

Using the Klainerman-Sobolev inequality to control $n_{s/2}$ in terms of M_s is clearly not enough to get global existence since we obtain the same logarithmic divergence of the time integral as with the proof of §1.1. We can however use the following L^1-L^∞ decay estimate (see **[21]** for a proof): for all v smooth enough *and with zero Cauchy data* ($v_{|t=0}=\partial_tv_{|t=0}=0$), the following holds

$$(1+t+|x|)|v(t,x)|\leq C\int_{\mathbb{R}^3}\int_0^t\frac{1}{1+\tau+|y|}\sum_{|I|\leq 2}|Z^I\Box v(\tau,y)|d\tau dy. \tag{24}$$

Denoting by $\widetilde{Z^I u}$ the solution to the homogeneous linear wave Equation (2) with same Cauchy data as $Z^I u$, we can use (24) to write

$$\begin{aligned} n_{s/2}(t,x) &\leq \sum_{|I|\leq s/2+1} |(Z^I u - \widetilde{Z^I u})(t,x)| + \sum_{|I|\leq s/2+1} |\widetilde{Z^I u}(t,x)| \\ &\leq \frac{C}{1+t+|x|}\Big(\int_y\int_0^t \frac{1}{1+\tau+|y|} \sum_{|J|+|K|\leq s/2+3} |Q_{JK}(Z^J\partial u, Z^K\partial u)| + \varepsilon\Big). \end{aligned}$$

Since we know by (18) that the Q_{JK} are null forms, we easily get from (17) that

$$n_{s/2}(t,x) \leq \frac{C_2}{1+t+|x|}\Big(\int_0^t \frac{1}{(1+\tau)^2} N_s(\tau)^2 d\tau + \varepsilon\Big) \tag{25}$$

for some constant $C_2 > 0$.

We now prove Klainerman's global existence result by a bootstrap argument on (23) and (25). It is indeed straightforward to prove that the largest time T_* such that

$$\forall t \in [0,T_*), x \in \mathbb{R}^d \qquad N_s(t) \leq \varepsilon C_2 (1+t)^{\varepsilon 2C_1C_2} \quad \text{and} \quad n_{s/2}(t,x) \leq \varepsilon \frac{2C_2}{1+t+|x|}$$

is infinite ($T_* = +\infty$) if C_2 is chosen large enough and if ε is small enough. We have thus proved the following theorem (of which Christodoulou gave independently another proof based on the conformal method **[3]**).

THEOREM 1.2 (**[26, 3]**). — *Let $d = 3$ and Q satisfy the null form conditions* (17) *and* (18). *Then if $u_{(0)}$ and $u_{(1)}$ are smooth and compactly supported, the quadratic wave Equation* (1) *is globally well-posed when ε is small enough.*

Remark 1.3. — For the scalar wave Equation (1) we have seen that the only quadratic forms satisfying (17) and (18) are multiples of Q_0. The same proof works verbatim with, for instance, the system of wave equations introduced in Example 2 below and for which other null forms exist (see Theorem 1.5 below).

Remark 1.4. — The result still holds if cubic terms of the form $F(u, \partial u) = O(|u|^3 + |\partial u|^3)$ are added to the quadratic null forms (treating cubic terms as in Theorem 2.1 below); Nirenberg's trick does not cover this situation. Note also that Klainerman's result also covers the quasilinear case not considered in these notes.

1.3. Generalizations to first order hyperbolic systems

We propose here to discuss some basic generalization of the concepts introduced in the previous section. Sticking to the semilinear case with homogeneous dispersion let us consider here first order hyperbolic systems of the form

$$\partial_t U + A(\partial)U = \mathbf{Q}(U,U), \tag{26}$$

where $U : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^n$, $A(\partial) = \sum_{j=1}^d A_j\partial_j$, the A_j are $n \times n$ real valued symmetric matrices and each of the n components of $\mathbf{Q}$ is a bilinear symmetric form. It follows that for all $\xi \in \mathbb{R}^d$, the eigenvalues of $A(\xi)$ are real and we assume for simplicity that they are of constant multiplicity for $\xi \neq 0$,

$$A(\xi) = \sum_{j=1}^m \lambda_j(\xi)\pi_j(\xi), \tag{27}$$

where $\pi_j(\xi)$ is the eigenprojector associated to the eigenvalue $\lambda_j(\xi)$; these mappings are smooth on $\mathbb{R}^d\backslash\{0\}$ and homogeneous of order 0 and 1 respectively. By convention, we take $\lambda_j(0) = 0$ and $\pi_j(0) = I$. We also assume that

$$\forall j = 1\dots m, \qquad \text{the section } \{\lambda_j(\xi) = 1\} \text{ is strictly convex,} \tag{28}$$

which is equivalent to saying that for all $\xi \neq 0$, the Hessian $\mathrm{Hess}(\lambda_j(\xi))$ is of rank $d-1$. The number of non zero sectional curvatures of the hypersurfaces $\{(\xi, \lambda_j(\xi)), \xi \in \mathbb{R}^d\}$ is indeed related to the L^∞-decay estimates of the homogeneous equation

$$\partial_t U + A(\partial)U = 0, \tag{29}$$

through stationary phase arguments, and (28) implies that this decay is the same $O(t^{-(d-1)/2})$ as for the homogeneous wave equation.

Example 1 (Scalar wave equation). — Writing $v = \partial_t u$, $\mathbf{w} = \nabla u$, the quadratic wave equation (1) can be written under the form (26), with

$$U = \begin{pmatrix} v \\ \mathbf{w} \end{pmatrix}, \qquad A(\xi) = A_\square(\xi) := \begin{pmatrix} 0 & -\xi^T \\ -\xi & 0_{d\times d} \end{pmatrix}, \qquad \mathbf{Q}(U,U) = \begin{pmatrix} Q(U,U) \\ 0_{d\times 1} \end{pmatrix}.$$

For all $\xi \neq 0$, $A(\xi)$ has three eigenvalues $\lambda_0(\xi) = 0$ and $\lambda_\pm(\xi) = \pm|\xi|$; the eigenprojector $\pi_0(\xi)$ is the orthogonal projector onto $(0, \xi^\perp)$ while $\pi_\pm(\xi) = \mathbf{e}_\pm(\xi) \otimes \mathbf{e}_\pm(\xi)$ where the eigenvectors $\mathbf{e}_\pm(\xi)$ are given by $\mathbf{e}_\pm(\xi) = \frac{1}{\sqrt{2}|\xi|}(\mp|\xi|, \xi^T)^T$. This system satisfies (28) except for the identically zero eigenvalue λ_0 which can easily be discarded since $\mathbf{w}$ remains a gradient for all times (so that $\pi_0(\xi)\widehat{U}(t,\xi) = 0$).

Example 2 (System of two wave equations). — Let us consider here a system of two coupled quadratic wave equations of the form (1),

$$\begin{cases} \square u^1 = Q^1_{11}(\partial u^1, \partial u^1) + Q^1_{12}(\partial u^1, \partial u^2) + Q^1_{22}(\partial u^2, \partial u^2), \\ \square u^2 = Q^2_{11}(\partial u^1, \partial u^1) + Q^2_{12}(\partial u^1, \partial u^2) + Q^2_{22}(\partial u^2, \partial u^2), \end{cases} \tag{30}$$

where Q^j_{kl} ($j = 1, 2$, $1 \leq j, k \leq 2$) are bilinear forms on $\mathbb{R}^{d+1}$. Defining U^1 and U^2 as in the previous example, this system can also be put under the form (26), with

$$U = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}, \qquad A(\xi) = \mathrm{diag}(A_\square(\xi), A_\square(\xi)), \qquad \mathbf{Q}(U,U) = \begin{pmatrix} \mathbf{Q}^1(U,U) \\ \mathbf{Q}^2(U,U) \end{pmatrix}$$

with $\mathbf{Q}^j = (Q^j_{11}(U^1,U^1) + Q^j_{12}(U^1,U^2) + Q^j_{22}(U^2,U^2),0,\dots,0)^T$. The eigenvalues are the same as in the previous example, but their multiplicity is twice as large. For the same reasons, we can discard the zero eigenvalue and focus on $\lambda_\pm(\xi) = \pm|\xi|$; the associated eigenprojectors are now of rank 2; they are a 2×2 block diagonal matrix with entries corresponding to the eigenprojectors of the scalar case.

Klainerman's vector fields method can be generalized to systems of the form (26) satisfying (28), and decay estimates in the spirit of (7) can be established **[11]**. One expects therefore global existence in dimension $d \geq 4$ as for the scalar wave equation (note however that since the placeholders for the vector fields are not differential operators anymore, a formula like (10) is in general not true). Let us therefore focus our attention on the case $d = 3$.

We have seen that for the scalar quadratic wave Equation (1), the only compatible quadratic forms Q (those that satisfy the improved decay estimate (16)) are proportional to Q_0. Compatible forms for (26) are defined similarly by the property that

$$(31)\quad \mathbf{Q}(U,V) = O(t^{-d}) \qquad \text{for all } U,V \text{ such that } (\partial_t + A(\partial))U = (\partial_t + A(\partial))V = 0,$$

with smooth and decaying enough initial data $U_{(0)}, V_{(0)}$.

As for the wave operator $\Box$, compatible forms with hyperbolic systems (26) satisfying (28) can be identified **[19]**: they are those for which all the eigenspaces of $A(\xi)$ are isotropic subspaces,

$$(32)\quad \mathbf{Q} \text{ satisfies (31)} \quad \text{iff} \quad \forall \xi \in \mathbb{R}^d\backslash\{0\}, \quad \forall k = 1\dots m, \quad \mathbf{Q}(\pi_k(\xi)\cdot, \pi_k(\xi)\cdot) = 0.$$

This result holds for all $d \geq 2$. In the case $d = 3$ which we are interested in, and for the system of two wave equations considered in Example 2, (32) is equivalent to

$$\forall(\tau,\xi) \in \mathbb{R}^{1+3} \text{ such that } \tau^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \qquad Q^j_{kl}((\tau,\xi)^T, (\tau,\xi)^T) = 0,$$

for $j,k,l = 1,2$. The bilinear forms Q^j_{kl} must therefore be linear combinations of the null form Q_0 for the scalar wave equation, and of the bilinear forms $Q_{\alpha\beta}$ defined as

$$(33)\qquad Q_0(\partial u, \partial v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v, \qquad Q_{\alpha\beta}(\partial u, \partial v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v,$$

with $0 \leq \alpha, \beta \leq 3$. As Q_0, the bilinear forms $Q_{\alpha\beta}$ are null forms in the sense that they satisfy (17) (the commuting vector fields are the same for the wave system (30) as for the scalar wave operator $\Box$). The null forms Q_0 and $Q_{\alpha\beta}$ satisfy moreover the property (8), and with the same proof as in §1.2, we get the following result.

THEOREM 1.5 (**[26, 3]**). — *Let $d = 3$ and assume that the bilinear forms Q^j_{kl} $(1 \leq j,k,l \leq 2)$ are linear combinations of the null forms Q_0 and $Q_{\alpha\beta}$ $(0 \leq \alpha, \beta \leq 3)$. Then for all smooth and compactly supported initial conditions, there exists a unique global solution to the wave system* (30) *if ε is small enough.*

Though this is not apparent in the statement of the theorem, the null forms Q_0 and $Q_{\alpha\beta}$ are of a different nature. This will be clearer with the normal form and space-time resonances approaches developed in the next sections.

2. NORMAL FORMS

The method of normal form was used by Shatah [**34**] to prove global existence for the quadratic nonlinear Klein-Gordon equation in dimension $d = 3$. The idea is to introduce a nonlinear change of variables inspired by the theory of dynamical systems and which reduces the problem to an equation with cubic nonlinearity for which global existence is easier. As a first example, we apply in §2.1 Shatah's method to a system of quadratic wave equations in dimension $d = 3$ and then discuss in §2.2 the case of general first order symmetric systems.

2.1. The method of normal forms for the quadratic wave equation

2.1.1. *Global existence in dimension $d = 3$ for the cubic wave equation.* — Consider the cubic wave equation

$$\Box u = F(u, \partial u), \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)}, \tag{34}$$

where F is smooth and $F(u, \partial u) = O(|u|^3 + |\partial u|^3)$. The following result is the motivation for the normal form method described in the next subsection.

THEOREM 2.1. — *Let $d = 3$. Then for all smooth and compactly supported initial conditions, there exists a unique global solution to the cubic wave Equation* (34) *if ε is small enough.*

Remark 2.2. — The proof works exactly the same for the system of wave equations of Example 2 with cubic instead of quadratic nonlinearities.

The proof of this result follows the same lines as the proof of Theorem 1.2; the extra decay provided by the null form property (17) in the latter case is implied here by the fact the nonlinearity is of higher order. We obtain, instead of (23),

$$N_s(t) \leq C_1 N_s(0) \exp\big(C_1 \int_0^t |(1 + \tau + |\cdot|) n_{s/2}(\tau, \cdot)^2|_\infty d\tau\big), \tag{35}$$

while (25) is replaced by

$$n_{s/2}(t, x) \leq \frac{C_2}{1 + t + |x|}\big(\int_0^t \frac{1}{(1+\tau)} |n_{s/2}(\tau, \cdot)|_\infty N_s(\tau)^2 d\tau + \varepsilon\big). \tag{36}$$

The same bootstrap argument as for Theorem 1.2 then applies.

2.1.2. *Global existence for the quadratic wave equation in dimension $d = 3$ via normal forms.* — Shatah's normal form method works very well for the scalar quadratic wave Equation (1) when the quadratic term satisfies the null condition (17), i.e. when it is proportional to Q_0. It allows in this case to recover Klainerman's result. This example can however be misleading, as we shall see by looking at the system of two wave equations of Example 2. As shown above, the null forms are then linear combinations of Q_0 and the $Q_{\alpha\beta}$ ($0 \leq \alpha, \beta \leq 3$) given by (33). Contrary to the proof of Theorem 1.2 based on the null form property (17) satisfied by both Q_0 and the $Q_{\alpha\beta}$, the normal form approach does not handle the null form $Q_{\alpha\beta}$. We will comment on the structural difference between Q_0 and the $Q_{\alpha\beta}$ in §2.2 below; let us also mention here that the space-time resonance approach of Germain-Masmoudi-Shatah handles such nonlinearities that are beyond the scope of standard normal forms.

For the moment, let us show that Shatah's approach is extremely simple when the nonlinearities are null forms of "Q_0-type", i.e., when the quadratic forms Q^j_{kl} in (30) are of the form

$$Q^j_{kl} = \alpha^j_{kl} Q_0, \tag{37}$$

for some constants α^j_{kl}. Indeed, we can then define

$$v^j = u^j - \frac{\alpha^j_{11}}{2}(u^1)^2 - \frac{\alpha^j_{12}}{2} u^1 u^2 - \frac{\alpha^j_{22}}{2}(u^2)^2, \qquad (j = 1, 2)$$

so that

$$\begin{aligned} \Box v^j &= \Box u^j - \frac{1}{2}\big(2\alpha^j_{11} u^1 \Box u^1 + \alpha^j_{12} u^1 \Box u^2 + \alpha^j_{12} u^2 \Box u^1 + 2\alpha^j_{22} u^2 \Box u^2\big) \\ &\quad - \alpha^j_{11} Q_0(\partial u, \partial u) - \alpha^j_{12} Q_0(\partial u, \partial v) - \alpha^j_{22} Q_0(\partial v, \partial v) \\ &= -\frac{1}{2}\big(2\alpha^j_{11} u^1 \Box u^1 + \alpha^j_{12} u^1 \Box u^2 + \alpha^j_{12} u^2 \Box u^1 + 2\alpha^j_{22} u^2 \Box u^2\big). \end{aligned}$$

We have achieved our normal form transform since the equation for $v = (v^1, v^2)^T$ is cubic in u and can therefore be treated as the cubic terms in the proof of Theorem 2.1. We now briefly sketch how to conclude to global existence in the scalar case (corresponding to $\alpha^j_{kl} = 0$ if $(j, k, l) \neq (1, 1, 1)$), the adaptation to the general case being straightforward..

Defining $N_s(v, t)$ and $n_s(v, t, x)$ as $N_s(t)$ and $n_s(t, x)$ with u replaced by v,

$$N_s(v, t) = \sum_{|I| \leq s+1} |Z^I v|_2 \quad \text{and} \quad n_s(v, t, x) = \sum_{|I| \leq s+1} |Z^I v(t, x)|,$$

and proceeding as for (22), we get

$$\begin{aligned} N_s(v,t) &\le C\Big(N_s(v,0) + \int_0^t \big|(1+\tau+|x|)n_{s/2}(\tau,x)^2\big|_\infty N^s(\tau)d\tau\Big) \\ &\le C\Big(N_s(v,0) + \int_0^t \big|(1+\tau+|x|)n_{s/2}(\tau,x)^2\big|_\infty |n_{s/2}(\tau,\cdot)|_\infty N^s(\tau)d\tau \\ &\qquad + \int_0^t \big|(1+\tau+|x|)n_{s/2}(\tau,x)^2\big|_\infty N^s(v,\tau)d\tau\Big), \end{aligned}$$

where we used that

$$N^s(\tau) \le N^s(v,\tau) + |n_{s/2}(\tau,\cdot)|_\infty N^s(\tau) \tag{38}$$

(this is a simple consequence of the chain rule and the identity $u = v + \frac{1}{2}u^2$) to obtain the second inequality. By Gronwall's lemma, we have therefore the following adaptation of (35),

$$\begin{aligned} N_s(v,t) &\le C\big(N_s(v,0) + \int_0^t |(1+\tau+|\cdot|)n_{s/2}(\tau,\cdot)^2|_\infty |n_{s/2}(\tau,\cdot)|_\infty N^s(\tau)d\tau\big) \\ &\qquad \times \exp\big(\int_0^t |(1+\tau+|\cdot|)n_{s/2}(\tau,\cdot)^2|_\infty d\tau\big). \end{aligned} \tag{39}$$

We finally remark as for (36) that

$$\begin{aligned} n_{s/2}(t,x) &\le n_{s/2}(v,t,x) + n_{s/2}(t,x)^2 \\ &\le \frac{C_2}{1+t+|x|}\Big(\int_0^t \frac{1}{(1+\tau)}|n_{s/2}(\tau,\cdot)|_\infty N_s(\tau)^2 d\tau + \varepsilon\Big) + n_{s/2}(t,x)^2. \end{aligned} \tag{40}$$

Global existence then follows from the same kind of bootstrap argument using (38), (39) and (40).

2.2. Normal form for first order symmetric systems

As in §1.3, consider first order symmetric systems

$$\partial_t U + A(\partial)U = \mathbf{Q}(U,U), \qquad U_{|t=0} = U_{(0)}, \tag{41}$$

satisfying (28). We want to implement the normal form approach by introducing a quadratic perturbation of U of the form

$$V = U + \mathbf{B}(U,U) - e^{-tA(\partial)}\big(U_{(0)} + \mathbf{B}(U_{(0)},U_{(0)})\big), \tag{42}$$

where $\mathbf{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear and chosen in such a way that the equation on V is cubic. The last term in the right hand side of (42) is the solution of the linear homogeneous equation with same Cauchy data as $U + \mathbf{B}(U,U)$. It can be removed for the discussion below, its ony role being to ensure that $V_{|t=0} = 0$, which eases the comparison with the space time resonance approach in §3. One computes

$$\begin{aligned} \partial_t V + A(\partial)V &= (\partial_t + A(\partial))U + (\partial_t + A(\partial))\mathbf{B}(U,U) \\ &= \mathbf{Q}(U,U) + A(\partial)\mathbf{B}(U,U) + \mathbf{B}(\partial_t U, U) + \mathbf{B}(U, \partial_t U). \end{aligned}$$

Replacing $\partial_t U$ by $-A(\partial)U + \mathbf{Q}(U,U)$ in the last two terms, we get

$$\begin{aligned} \partial_t V + A(\partial)V &= \mathbf{B}(\mathbf{Q}(U,U),U) + \mathbf{B}(U,\mathbf{Q}(U,U)), \\ &:= \mathbf{T}(U) \end{aligned} \tag{43}$$

which has cubic nonlinearity $\mathbf{T}$, provided that we are able to find $\mathbf{B}$ such that

$$\mathbf{Q}(U,U) + A(\partial)\mathbf{B}(U,U) - \mathbf{B}(A(\partial)U,U) - \mathbf{B}(U,A(\partial)U) = 0.$$

Taking the Fourier transform with respect to x and projecting onto the eigenspaces of $A(\xi)$, this is equivalent to solving, for all $j = 1 \ldots m$,

$$\sum_{k,l=1}^{m} \int_{\mathbb{R}^d} (\lambda_j(\xi) - \lambda_k(\xi-\eta) - \lambda_l(\eta))\pi_j(\xi)\mathbf{B}(\widehat{U}_k(\xi-\eta), \widehat{U}_l(\eta))d\eta$$
$$= i \sum_{k,l=1}^{m} \int_{\mathbb{R}^d} \pi_j(\xi) Q(\widehat{U}_k(\xi-\eta), \widehat{U}_l(\eta))d\eta,$$

where $\widehat{U}_k(\xi)$ stands for $\pi_k(\xi)\widehat{U}(\xi)$. This leads us to define $\mathbf{B}$ as

$$\mathbf{B}(U^1,U^2) = \sum_{j,k,l=1}^{m} B^j_{kl}(U^1,U^2), \tag{44}$$

and

$$\mathcal{F}[B^j_{kl}(U^1,U^2)](\xi) = i \int_{\mathbb{R}^d} \frac{\pi_j(\xi)Q(\pi_k(\xi-\eta)\widehat{U}^1(\xi-\eta), \pi_l(\eta)\widehat{U}^2(\eta))}{\lambda_j(\xi) - \lambda_k(\xi-\eta) - \lambda_l(\eta)} d\eta. \tag{45}$$

Defining the change of variable (42) requires therefore a close look at the set of *time resonances* $\mathcal{T}$ defined as

$$\mathcal{T} = \bigcup_{j,k,l=1}^{m} \mathcal{T}^j_{kl}, \quad \text{with} \quad \mathcal{T}^j_{kl} = \{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \lambda_j(\xi) - \lambda_k(\xi-\eta) - \lambda_l(\eta) = 0\}. \tag{46}$$

In order to give sense to (45), it is natural to impose the following condition for all $1 \le j,k,l \le m$,

$$\forall (\xi,\eta) \in \mathcal{T}^j_{kl}, \quad \eta \neq 0, \quad \xi - \eta \neq 0, \qquad \pi_j(\xi)\mathbf{Q}(\pi_k(\xi-\eta)\cdot, \pi_l(\eta)\cdot) = 0 \tag{47}$$

(when $\xi = 0$, we set $\pi_j(0) = I$); we will refer to (47) as the *transparency condition*, this terminology being rooted in nonlinear optics, see §4.2. This condition implies in particular that $\mathbf{Q}$ must be a compatible form (i.e., that it must satisfy (32)).

Example 3 (System of two wave equations). — For the system of two wave equations considered in Example 2, we have seen that the identically zero eigenvalue can be discarded, so that we have two relevant eigenvalues $\lambda_\pm(\xi) = \pm|\xi|$. The components of the time resonant set are therefore denoted by $\mathcal{T}^{\pm}_{\pm,\pm}$, and one readily checks that

$$\begin{aligned} &\mathcal{T}^{+}_{--} = \{0,0\}, && \mathcal{T}^{-}_{--} = \{(\xi,\eta), \quad \exists\lambda \geq 1, \quad \xi = \lambda\eta\}, \\ &\mathcal{T}^{-}_{+-} = \{(\xi,\eta), \quad \exists 0 \leq \lambda \leq 1, \quad \xi = \lambda\eta\}, && \mathcal{T}^{-}_{-+} = \{(\xi,\eta), \quad \exists\lambda \leq 0, \quad \xi = -\lambda\eta\}, \end{aligned}$$

and $\mathcal{T}^{\mp}_{\mp,\mp} = \mathcal{T}^{\pm}_{\pm,\pm}$. Since the eigenprojectors $\pi_\pm$ are homogeneous of order zero and $\pi_+(\xi) = \pi_-(-\xi)$, the transparency condition (47) is equivalent to the compatibility condition (32). Note however that this equivalence is not true in general (for a system of three wave equations with different speeds, this would not be the case because $(\lambda_j(\xi),\xi)$ and $(\lambda_l(\eta),\eta)$ are not necessarily colinear at resonances).

The transparency condition (47) is not sufficient to define and derive estimates for the change of variables (42), (44), (45) (this is a major difference with Poincaré's theory of normal form for dynamical systems). Small divisors in (47) may indeed appear in (45) near the resonances and additional assumptions must be made on the order of cancellation of (47). For instance, the following *strong transparency condition* obviously ensures that (47) is well defined,

$$\text{(48)} \qquad \exists C > 0, \qquad \big\|\pi_j(\xi)\mathbf{Q}(\pi_k(\xi-\eta)\cdot,\pi_l(\eta)\cdot)\big\|_{\mathcal{L}^2} \leq C|\lambda_j(\xi) - \lambda_k(\xi-\eta) - \lambda_l(\eta)|$$

for all $(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$, $\xi - \eta \neq 0$, $\eta \neq 0$, and where $\|\cdot\|_{\mathcal{L}^2}$ is the canonical norm for bilinear forms on $\mathbb{R}^{d+1}$. This assumption, however, is very strong and is not satisfied for most applications. In general, when the normal form approach can be implemented, the order of cancellation of the nonlinearity at resonances is intermediate between (47) and (48): the singularities in (45) are not completely removed, but they are controllable.

In order to illustrate these comments, let us go back to the system of two wave equations of Example 2. The difference between the null forms Q_0 and $Q_{\alpha\beta}$ for the normal form approach observed in §2.1.2 can be interpreted in terms of transparency: the former is more transparent than the latter (i.e. it removes more singularities in (45)).

When $\mathbf{Q}$ only involves Q_0 (i.e. when it is as in (37)) one has, with the notations of Examples 1-2, and for all $j,k,l = \pm$ (recall that the identically zero eigenvalue can be discarded),

$$
\begin{aligned}
\|\pi_j(\xi)Q_0(\pi_k(\xi-\eta)\cdot,\pi_l(\eta)\cdot)\|_{\mathscr{L}^2} &\sim \big|Q_0\big(\mathbf{e}_k(\xi-\eta),\mathbf{e}_l(\eta)\big)\big| \\
&= \big|1-kl\frac{(\xi-\eta)\cdot\eta}{|\xi-\eta||\eta|}\big| \\
&= |\varphi^j_{kl}(\xi,\eta)|\frac{j|\xi|+k|\xi-\eta|+l|\eta|}{|\xi-\eta||\eta|}, \qquad (49)
\end{aligned}
$$

where we introduced the notation

$$\varphi^j_{kl}(\xi,\eta)=\lambda_j(\xi)-\lambda_k(\xi-\eta)-\lambda_l(\eta).$$

The fact that the phase $\varphi^j_{kl}(\xi,\eta)$ can be factored out of the right-hand side of (49) reduces considerably the set of singularities. In general, singularities in (45) are located on the set of time-resonances $\mathscr{T}$ given by (46) (see Example 3 for the system of two wave equations), but (49) shows us that thanks to the structure of Q_0, singularities are reduced to the set $\{\eta=0\}\cup\{\eta=\xi\}$.

For the null forms $Q_{\alpha\beta}$, there is no factorization as in (49) and the set of singularities cannot be sufficiently reduced to make the normal form approach operative.

Remark 2.3. — Even in the case where $\mathbf{Q}$ only involves Q_0, singularities in (45) are not entirely removed while we know from §2.1.2 that the normal form transform is extremely simple. This is because this transform is polynomial in u, and therefore singular in ∂u (or equivalently in U if the formulation (41) is used).

To end this section, one can say as a rule of thumb that the normal form approach works if

1. The time resonance set $\mathscr{T}$ is small enough. This is for instance the case in Shatah's paper **[34]** for the Klein-Gordon equation where $\mathscr{T}=\varnothing$; all types of quadratic nonlinearities can then be removed.
2. The time resonance set $\mathscr{T}$ is not necessarily small, but the set of singularities in the normal form transform (45) is considerably reduced by some transparency property of the nonlinearity. For systems of wave equations, the null form Q_0 satisfies such a property, but not the null forms $Q_{\alpha\beta}$.

The space-time resonance approach of Germain-Masmoudi-Shatah is an alternative to Klainerman's vector fields method to handle situations that do not belong to one of the above two cases. As shown in the next section, this method is an elegant and natural extension of Shatah's normal forms.

3. THE SPACE TIME RESONANCE APPROACH

3.1. General description

3.1.1. *The profile formulation.* — The space time resonance approach is based on the Duhamel formulation for the *profile* of the solution of the nonlinear dispersive equation under consideration. For the first order symmetric hyperbolic systems already considered in §1.3 and §2.2, namely,

$$\partial_t U + A(\partial)U = \mathbf{Q}(U,U), \qquad U_{|_{t=0}} = U_{(0)}, \tag{50}$$

the profile of U is defined as

$$W(t) = e^{tA(\partial)}U(t) = \sum_{j=1}^{m} e^{it\lambda_j(D)}\pi_j(D)U(t),$$

where we used the decomposition (27). The Duhamel formulation for (50) takes therefore the form

$$\begin{aligned}\widehat{W}(t,\xi) &= \widehat{U}_{(0)}(\xi) + \sum_{j,k,l=1}^{m}\int_0^t\int_{\mathbb{R}^d} e^{i\tau\varphi^j_{kl}(\xi,\eta)}\pi_j(\xi)\mathbf{Q}(\widehat{W}_k(\tau,\xi-\eta),\widehat{W}_l(\tau,\eta))d\eta d\tau,\\ &:= \widehat{U}_{(0)}(\xi) + \widehat{\mathbf{J}(W,W)}(t,\xi),\end{aligned} \tag{51}$$

where we used the notations

$$\varphi^j_{kl}(\xi,\eta) = \lambda_j(\xi) - \lambda_k(\xi-\eta) - \lambda_l(\eta), \qquad W_j = \pi_j(D)W.$$

The strategy is to construct global solutions to (51) that have the same decay properties as solutions of the homogeneous linear equation. One must therefore find a Banach space X adapted to this behavior; for instance, for systems (50) satisfying (28), one should have $\sup_{t\geq 1} t|u(t)|_\infty \lesssim |u|_X$ to catch the $O(t^{-1})$ time decay of free solutions in dimension $d=3$. One then has to prove that the mapping

$$W \mapsto U_{(0)} + \mathbf{J}(W,W)$$

is a contraction in X in the neighborhood of the origin. Global existence then follows by a standard fixed point theorem.

3.1.2. *Space transparency.* — As said above, the choice of the space X in which we expect the existence of a fixed point for (51) depends on the behavior of the solutions to the homogeneous linear equation, but also on technical estimates on the bilinear form $\mathbf{J}$. These estimates depend on the equation under consideration. Therefore, to make our discussion as general as possible here, we do not give a precise description of the space X here and focus on the strategy proposed by Germain-Masmoudi-Shatah to control the time behavior of $\mathbf{J}$. More details will be given in §3.2 devoted to the application of this method to the water waves equations.

Decomposing $\mathbf{J}$ into $\mathbf{J} = \sum_{j,k,l=1}^{m} J_{kl}^{j}$ with obvious notations, and remarking that

$$\partial_\tau(e^{i\tau\varphi_{kl}^{j}}) = i\varphi_{kl}^{j} e^{i\tau\varphi_{kl}^{j}} \quad \text{and} \quad \partial_\eta(e^{i\tau\varphi_{kl}^{j}}) = i\tau\partial_\eta\varphi_{kl}^{j} e^{i\tau\varphi_{kl}^{j}}$$

(where ∂_η denotes the partial derivative with respect to any of the coordinates of η), we can choose to integrate by parts either with respect to τ or η in the expression for J_{kl}^{j}:

1. Integration by parts with respect to τ. We then obtain (omitting the subscripts k, l for the sake of clarity),

$$\begin{aligned} \widehat{J^j(W,W)}(t,\xi) &= -i\pi_j(\xi)\int_{\mathbb{R}^d} \frac{e^{i\tau\varphi^j}}{\varphi^j}\mathbf{Q}\big(\widehat{W}(\tau,\xi-\eta),\widehat{W}(\tau,\eta)\big)d\eta\Big|_0^t \\ &+ i\pi_j(\xi)\int_0^t\int_{\mathbb{R}^d} \frac{e^{i\tau\varphi^j}}{\varphi^j}\partial_\tau\big[\mathbf{Q}\big(\widehat{W}(\tau,\xi-\eta),\widehat{W}(\tau,\eta)\big)\big]d\eta d\tau. \end{aligned} \tag{52}$$

Remarking that $\partial_\tau W = \sum_j e^{i\tau\lambda_j(D)}\pi_j(D)\mathbf{Q}(U,U)$ and using the notations (43) and (44), we get

$$J^j(W,W) = -e^{it\lambda_j(D)}B^j(U,U) + B^j(U_{(0)},U_{(0)}) + \int_0^t e^{i\tau\lambda^j(\xi)}\pi_j(D)\mathbf{T}(U)d\tau.$$

It follows that we can rewrite (51) under the form

$$U = e^{-tA(\partial)}\big((U_{(0)} + \mathbf{B}(U_{(0)},U_{(0)})\big) - \mathbf{B}(U,U) + V,$$

where V is the solution of $(\partial_t + A(\partial))V = \mathbf{T}(U)$ with zero initial data. This is exactly the normal form transform (42). *Performing a time integration in the integral defining* $\mathbf{J}$ *is therefore equivalent to looking for a standard normal form.*

2. Integration by parts with respect to η. We now obtain (still omitting the subscripts k, l),

$$\widehat{J^j(W,W)}(t,\xi) = i\int_1^t\int_{\mathbb{R}^d} \frac{e^{i\tau\varphi^j}}{\tau}\partial_\eta\big[\frac{1}{\partial_\eta\varphi^j}\pi_j(\xi)\mathbf{Q}\big(\widehat{W}(\tau,\xi-\eta),\widehat{W}(\tau,\eta)\big)\big]d\eta d\tau, \tag{53}$$

where we have changed the lower bound for the time integration in the definition of J to avoid any artificial singularity at $\tau = 0$; since the difficulty in the control of J occurs for large t, this modification does not affect the discussion. The interest of this transformation is twofold: it provides an extra $O(\tau^{-1})$ time decay in the integrand (the same extra decay granted by the reduction to a cubic nonlinearity in the normal form approach when $d = 3$), and it changes the set of singularities which is not given by the time resonant set $\mathscr{T}$ anymore.

Since the first of the two cases discussed above coincides with the normal form approach, we refer therefore to §2.2 for a discussion on the situations where this method can be or not successful. We consequently turn our attention towards the second case, and more particularly to the transformed expression (53).

The set of *time resonances* $\mathcal{T}$ defined in (46) is now irrelevant to describe the singularities of (53), and it must be replaced by the set of *space resonances* $\mathcal{S}$ defined as the set of all (ξ,η) such that $\nabla_\eta\varphi^j_{kl}$ vanishes for some $1\leq j,k,l\leq m$,

$$(54)\qquad \mathcal{S}=\bigcup_{k,l=1}^{m}\mathcal{S}_{kl},\qquad\text{with}\qquad \mathcal{S}_{kl}=\{(\xi,\eta),\quad \nabla\lambda_k(\xi-\eta)=\nabla\lambda_l(\eta)\}.$$

Example 4. — With the notations of Example 3, we get for the system of two wave equations of Example 2,

$$\begin{aligned}\mathcal{S}_{++}=\mathcal{S}_{--}&=\{(\xi,\eta),\quad \exists\lambda>1,\quad \xi=\lambda\eta\},\\ \mathcal{S}_{+-}=\mathcal{S}_{-+}&=\{(\xi,\eta),\quad \exists\lambda<1,\quad \xi=\lambda\eta\}.\end{aligned}$$

We can in particular remark that $\overline{\mathcal{S}_{+-}}=\mathcal{T}^+_{+-}\cup\mathcal{T}^-_{+-}$ and $\overline{\mathcal{S}_{++}}=\mathcal{T}^+_{++}\cup\mathcal{T}^-_{++}$ (the other cases being deduced from these two); therefore, in this particular example, the set of singularities for (53) contains (up to endpoint cases) the set of singularities for the normal form transform (45).

The same discussion as in the end of §2.2 leads to expect two kinds of situations where this new approach may be successful:

1. The space resonance set $\mathcal{S}$ is small enough.
2. The space resonance set $\mathcal{S}$ is not necessarily small, but the set of singularities (53) is considerably reduced by some property of the nonlinearity.

To describe more precisely the second part of this alternative, we now introduce two conditions on the nonlinearity inspired by the transparency and strong transparency conditions (47) and (48). We will call the first one *space transparency condition*,

$$(55)\qquad \forall(\xi,\eta)\in\mathcal{S}_{kl},\quad \eta\neq0,\quad \xi-\eta\neq0,\qquad \pi_j(\xi)\mathbf{Q}(\pi_k(\xi-\eta)\cdot,\pi_l(\eta)\cdot)=0;$$

and the second one, which is a sufficient condition for the transformation (53) to be defined, will be referred to as the *strong space transparency condition*:

$$(56)\quad \exists C>0,\qquad \begin{cases}\mathbf{Q}=\mathbf{Q}_1+\cdots+\mathbf{Q}_d,\\ \big\|\pi_j(\xi)\mathbf{Q}_p(\pi_k(\xi-\eta)\cdot,\pi_l(\eta)\cdot)\big\|_{\mathcal{L}^2}\leq C|\partial_p\lambda_k(\xi-\eta)-\partial_p\lambda_l(\eta)|,\end{cases}$$

for all $p=1\dots d$ and $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$, $\xi-\eta\neq0$, $\eta\neq0$.

Remark 3.1. — Note that the singularities arising when ∂_η hits $(\partial_\eta\varphi)^{-1}$ in (53) are not directly controlled by this space transparency condition. However, these singularities are of the form $|\xi-\eta|^{-1}$ or $|\eta|^{-1}$ and (56) furnishes a control of the corresponding component of (53) in terms of $|\xi|^{-1}\widehat{W}$, or, through Hardy type inequalities, in terms of $\nabla_\xi\widehat{W}$. This is a quantity we already need to handle for other terms coming from the η differentiation in (53). See §3.1.3 for more comments on this point.

Example 5 (System of two wave equations). — We have seen that

$$\overline{\mathscr{S}_{\pm\pm}} = \mathscr{T}^{+}_{\pm\pm} \cup \mathscr{T}^{-}_{\pm\pm},$$

therefore, the fact that all the null forms Q_0 and $Q_{\alpha\beta}$ satisfy the (time) transparency condition (47) implies that they also satisfy the space transparency condition (55).

We have seen in §2.2 that Q_0 does not satisfy the strong transparency condition (48) but that it removes enough singularities to allow the normal form transform (45) to work. We now check wether the situation is similar for $Q_{\alpha\beta}$ and the singular integral (53). One computes that, when $\alpha\beta \neq 0$,

$$\begin{aligned} Q_{\alpha\beta}(\pi_k(\xi-\eta)\cdot, \pi_l(\eta)\cdot) &\sim \frac{(\xi-\eta)_\alpha\eta_\beta - (\xi-\eta)_\beta\eta_\alpha}{|\xi-\eta||\eta|} \\ &= kl\big[\partial_\alpha\lambda_k(\xi-\eta)\partial_\beta\lambda_l(\eta) - \partial_\beta\lambda_k(\xi-\eta)\partial_\alpha\lambda_l(\eta)\big] \\ &= klQ_{\alpha\beta}\big(\nabla\lambda_k(\xi-\eta) - \nabla\lambda_l(\eta), \nabla\lambda_l(\eta)\big), \end{aligned}$$

and (56) is satisfied. It can also be easily checked that $Q_{\alpha 0}$ and $Q_{0\beta}$ also satisfy (56). The transformation (53) can then be well defined.

3.1.3. *Compatibility condition on the phase.* — We will not implement fully the space time resonance approach for the examples used throughout these notes (system of two wave equations for instance). This would raise specific technical details of little interest here; however, we point out here a difficulty inherent to the space resonance method.

If we use the space resonance analysis and integrate by parts with respect to η in (51), we are left with (53) whose integrand contains frequency derivatives of the profiles, e.g. $\partial_\eta \widehat{W}_l(\tau,\eta)$. We need therefore to control $\partial_\xi \widehat{W}(\tau,\xi)$ (taking the inverse Fourier transform, this is equivalent to providing weighted estimates on W). In order to get such controls, let us differentiate (51) with respect to ξ. The most problematic terms are obtained when ∂_ξ hits the exponential terms; they are given by

$$A^j_{kl}(\tau,\xi) = i\int_0^t \int_{\mathbb{R}^d} e^{it\varphi^j_{kl}}\tau\partial_\xi\varphi^j_{kl}\pi_j(\xi)\mathbf{Q}\big(\widehat{W}_k(\tau,\xi-\eta), \widehat{W}(\tau,\eta)\big)d\eta d\tau;$$

frequency differentiation has therefore created a new *time growing* term τ in the integrand. In many situations (e.g. **[15]** for $2d$ quadratic NLS equations and **[16]** for water waves) these additional terms bring some new and helpful (space or time) transparency properties that can be used to get rid of the new time growing term. Such *compatibility conditions* seem to be quite related to the homogeneity of the λ_j; in the present case, the homogeneity is of order one, so that $\xi\cdot\nabla\lambda_j(\xi) = \lambda_j(\xi)$ and we get

$$\begin{aligned}\xi\cdot\nabla_\xi\varphi^j_{kl}(\xi,\eta) &= \xi\cdot(\nabla\lambda_j(\xi)-\nabla\lambda_k(\xi-\eta))\\ &= \lambda_j(\xi)-\lambda_k(\xi-\eta)-\eta\cdot\nabla\lambda_k(\xi-\eta)\\ &= \varphi^j_{kl}(\xi,\eta)+\eta\cdot\nabla\varphi^j_{kl},\end{aligned}$$

which is the sum of a time strong transparent term and of a space strong transparent term.

3.1.4. *Conclusion.* — Combining what we have seen on time and space resonances, we can give another rule of thumb for the full space time resonance approach of Germain-Masmoudi-Shatah: it is likely to work if

1. The set of *space time resonances* $\mathscr{R}=\mathscr{S}\cap\mathscr{T}$ is small enough. This is not the case for the system of two wave equations since $\mathscr{T}=\mathscr{S}$, but for some equations, $\mathscr{R}$ is much smaller than $\mathscr{T}$ (for the quadratic Schrödinger equation for instance [**14**]).
2. The space-time resonance set $\mathscr{R}$ is not necessarily small, but the set of singularities in the normal form transform (45) or the space resonance transform (53) is considerably reduced by some time or space transparency property of the nonlinearity—or by some compatibility condition of the phases. For the system of two waves equations, null forms Q_0 yields time transparency, and the $Q_{\alpha\beta}$ give space transparency.

3.2. An example of application: global existence for $3d$ water waves in surface dimension $d=2$

In their paper [**16**], the authors consider the three-dimensional irrotational water wave problem in presence of gravity. The surface S is parametrized by the graph of a function h, $S=\{(x,h(t,x)),x\in\mathbb{R}^2\}$, and the fluid domain Ω is the region located below this graph. The fluid is assumed to be incompressible and its flow irrotational. The velocity field v inside the fluid can therefore be written as $v=\nabla_{x,z}\Phi$, where Φ is harmonic in Ω. Denoting by ψ the trace of Φ on S, and seeing ψ as a function on $\mathbb{R}^2$ rather than on S, the initial value problem can be written [**7**],

$$\left\{\begin{array}{l}\partial_t h=G(h)\psi,\\ \partial_t\psi=-h-\frac{1}{2}|\nabla\psi|^2+\frac{1}{2(1+|\nabla h|^2)}(G(h)\psi+\nabla h\cdot\nabla\psi)^2,\\ (h,\psi)_{|t=0}=(h_0,\psi_0),\end{array}\right.\tag{57}$$

where $G(h)$ is the Dirichlet-Neumann operator, $G(h)\psi=\sqrt{1+|\nabla h|^2}\partial_n\Phi_{|S}$.

Using the space time resonance approach, Germain-Masmoudi-Shatah proved a global existence result for (57) for small initial data. Writing, with $D = -i\nabla$,

$$\Lambda := |D|, \quad u := h + i\Lambda^{1/2}\psi, \quad f := e^{it\Lambda^{1/2}}u, \quad u_0 := h_0 + i\Lambda^{1/2}\psi_0, \tag{58}$$

their result is the following.

THEOREM 3.2 ([**16**]). — *Let $\delta > 0$, N integer and define*

$$|u|_X := \sup_{t\geq 0} t|u|_{W^{4,\infty}} + (1+t)^{-\delta}|u|_{H^N} + (1+t)^{-\delta}|xf|_2 + |u|_2.$$

If δ is small enough, N large enough, then there exists $\varepsilon > 0$ such that if $|e^{-it\Lambda^{1/2}}u_0|_X < \varepsilon$, then there exists a unique global solution u of (57) *such that $|u|_X < 2\varepsilon$.*

Remark 3.3. — Due to a confusion of notations of the authors in their blow up criterion (see §3.2.1 below), the statement of this theorem should probably be modified; more precisely, one should replace $|u|_{H^N}$ in the definition of X by $|h|_{H^N} + |(\nabla_{X,z}\Phi)_{|S}|_{H^{N-1/2}(S)}$, where Φ is the harmonic extension of ψ in the fluid domain.

Remark 3.4. — For the water waves problem, the eigenvalues of the linear part of the equations are $\lambda_{\pm}(\xi) = \pm|\xi|^{-1/2}$. The Hessian of this matrix has maximal rank, and one expects from the stationary phase theorem a time decay in L^∞ norm of order $O(t^{-d/2})$ (versus $O(t^{-(d-1)/2})$ for the wave equation). Since nonlinearities in (57) are quadratic, this decay is far from enough to conclude to global existence when the horizontal dimension is $d = 1$. If quadratic nonlinearities can be removed (by a normal form) or if an extra $O(t^{-1/2})$ time decay can be gained (by a null form condition), we are in a situation comparable to the wave equation in dimension 3 with quadratic nonlinearities that do *not* satisfy the null condition (see §1.2). One expects in this particular situation an existence time of size $O(e^{ct/\varepsilon^2})$ for initial data of size $O(\varepsilon)$. Such a result was proved by Wu [**43**] with methods combining Klainerman's vector fields and a clever change of variables—note that for technical reasons, the result of [**43**] gives an existence time $O(e^{ct/\varepsilon})$ instead of $O(e^{ct/\varepsilon^2})$.

In horizontal dimension $d = 2$, the L^∞-decay is $O(t^{-1})$, and the situation should be the same as for the quadratic wave equation in dimension 3: a generic existence time of order $O(e^{ct/\varepsilon})$, and global existence if we are able to implement a normal form transform or use a null form condition. This corresponds to Theorem 3.2 above, as well as to another result by Wu [**44**], who extended the tools developed in [**43**] to the case $d = 2$. These two results are not the same though, and their proofs are completely different. A comparison of these results can be found in [**16**] (p. 696).

3.2.1. *Local existence and blow up criterion.* — The first step consists of course in proving a local existence theorem for (57). Since the works of S. Wu [**41, 42**], many local existence theorems have been derived without restrictive condition. The authors use here the approach of [**35**] to prove the following $W^{4,\infty}$-blow up criterion for these local solutions,

$$\forall t \geq 0, \quad E_N(t) \lesssim E_N(0) + \int_0^t |u(\tau)|_{W^{4,\infty}} E_N(\tau) d\tau \tag{59}$$

where u is as in (58) while the energy E_N is

$$E_N \sim |(\nabla_{x,z}\Phi)_{|S}|^2_{H^{N-1/2}(S)} + |h|^2_{H^N(S)},$$

where Φ is the harmonic extension of ψ in the fluid domain. Since (59) can be established by other means and is not related to the space time resonance approach, we do not spend time commenting on it here.

Let us just mention here that due to a confusion of notations, the authors use that $E_N \sim |u|^2_{H^N}$. This does not seem to be true because

$$\nabla\psi = (\nabla\Phi)_{|S} + (\partial_z\Phi)_{|S}\nabla h;$$

controlling $|u|_{H^N}$ requires a control of $|\nabla\psi|_{H^{N-1/2}}$ and therefore of $|h|_{H^{N+1/2}}$, which is not controlled by E_N. This is the reason why we suggested in Remark 3.3 to change the space X in Theorem 3.2.

3.2.2. *The profile formulation.* — To give a formulation of the water waves Equations (57) in terms of profiles (see §3.1.1), we first expand the equations in powers of h and ψ up to quartic terms, using the expansion of the Dirichlet-Neumann operator of [**6**],

$$\begin{cases} \partial_t h = \Lambda\psi - \nabla\cdot(h\nabla\psi) - \Lambda(h\Lambda\psi) - \dfrac{1}{2}\big(\Lambda(h^2\Lambda^2\psi) + \Lambda^2(h^2\Lambda\psi) - 2\Lambda(h\Lambda(h\Lambda\psi)\big) + R_1 \\ \partial_t\psi = -h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}|\Lambda\psi|^2 + \Lambda\psi\big(h\Lambda^2\psi - \Lambda(h\Lambda\psi)\big) + R_2; \end{cases}$$

with the notations (58), this allows us to write

$$\begin{aligned} \widehat{f}(t,\xi) &= \widehat{u}_0(\xi) + \sum_{j=1}^{j} c_{j,\pm,\pm}\int_0^t\int e^{i\tau\varphi_{\pm,\pm}} m_j(\xi,\eta)\widehat{f}_{\mp}(\tau,\eta)\widehat{f}_{\mp}(\tau,\xi-\eta)d\eta d\tau \\ &+\sum_{j=3}^{4} c_{j,\pm,\pm,\pm}\int_0^t\int\int e^{i\tau\varphi_{\pm,\pm,\pm}} m_j(\xi,\eta,\sigma)\widehat{f}_{\mp}(\tau,\eta)\widehat{f}_{\mp}(\tau,\sigma)\widehat{f}_{\mp}(\tau,\xi-\eta-\sigma)d\eta d\sigma d\tau \\ &+\int_0^t e^{is|\xi|^{1/2}}\widehat{R}(s,\xi)d\xi, \end{aligned}$$

$$:= \widehat{u}_0(\xi) + \widehat{\mathbf{J}(f)}, \tag{60}$$

where the $c_{j,\pm,\pm}$ and $c_{j,\pm\pm,\pm}$ are complex coefficients, and $f_+ = f$, $f_- = \bar{f}$, while $R = R_1 + i\Lambda^{1/2}R_2$. Formulas for the phases and the symbols m_j will be given when needed.

As in §3.1.1, the global existence result will follow from a fixed point theorem in X on the mapping $f \mapsto u_0 + \mathbf{J}(f)$; we have therefore to prove that $\mathbf{J}$ is a contraction in X near the origin. The plan of the proof is the following: quadratic terms are first removed with a normal form transform. Controlling the cubic terms is a bit more complicated than for the wave equation (there is no such result as Theorem 2.1 for the water waves equations); this is done by another normal form transform (weakly resonant case) or by an integration by parts in frequency (strongly resonant case). Even though many space time resonances are present, this is made possible by a nonlinear compatibility condition of the phase. Finally, quartic and higher terms decay very fast and can easily be controlled; consequently, we do not comment on them here.

3.2.3. *Quadratic terms.* — We analyze here the quadratic terms in (60), which are of the form

$$\int_0^t \int e^{i\tau\varphi_{\pm,\pm}} m_j(\xi,\eta)\widehat{f_\mp}(\tau,\eta)\widehat{f_\mp}(\tau,\xi-\eta)\,d\eta d\tau \qquad (j=1,2). \tag{61}$$

The quadratic phases $\varphi_{\pm,\pm}$ are given by

$$\varphi_{\pm,\pm}(\xi,\eta) = |\xi|^{1/2} \pm |\eta|^{1/2} \pm |\xi-\eta|^{1/2},$$

and the symbols $m_j(\xi,\eta)$ are

$$m_1(\xi,\eta) = \frac{1}{|\eta|^{1/2}}(\xi\cdot\eta - |\xi||\eta|), \qquad m_2(\xi,\eta) = \frac{1}{2}\frac{|\xi|^{1/2}}{|\eta|^{1/2}|\xi-\eta|^{1/2}}(\eta\cdot(\xi-\eta)+|\eta||\xi-\eta|).$$

Remark that the time resonant set is given by

$$\mathcal{T} = \bigcup_{\pm,\pm} \mathcal{T}_{\pm,\pm}, \quad \text{with} \quad \mathcal{T}_{++} = \{(0,0)\}, \qquad \mathcal{T}_{--} = \{\eta = 0 \text{ or } \xi-\eta = 0\},$$

while $\mathcal{T}_{-+}$ and $\mathcal{T}_{+-}$ are easily deduced from $\mathcal{T}_{--}$ by permutation of the variables. It follows that the symbols m_j satisfy the (time) transparency property (47) but not the strong transparency property (48). There is actually an intermediate transparency property allowing a normal form transform. Before stating this transparency property, let us recall that the outcome of the normal form transformation (52) is a decomposition of (61) of the form

$$(61) = \widehat{g_1}(t,\xi) + \text{ cubic terms} \tag{62}$$

(up to a time independent term that does not raise any difficulty), where $\widehat{g_1}$ is a quadratic term without time integration coming from the integration by parts,

$$\widehat{g_1}(t,\xi) = \int e^{i\tau\varphi_{\pm,\pm}} \mu(\xi,\eta)\widehat{f_\mp}(t,\eta)\widehat{f_\mp}(t,\xi-\eta)\,d\eta,$$

with $\mu = \frac{m_j(\xi,\eta)}{i\varphi_{\pm,\pm}}$. The "transparency" property satisfied by m_1 and m_2 is that $\mu(\xi,\eta)$ belongs to the class of symbols such that

- μ is homogeneous of degree one,
- one has $\mu(\xi,\eta) = \mathcal{C}\big(|\eta|^{1/2}, \frac{\eta}{|\eta|}, \xi\big)$ if $|\eta| \ll |\xi| \sim 1$,
- one has $\mu(\xi,\eta) = \mathcal{C}\big(|\xi-\eta|^{1/2}, \frac{\xi-\eta}{|\xi-\eta|}, \xi\big)$ if $|\xi-\eta| \ll |\xi| \sim 1$,
- one has $\mu(\xi,\eta) = |\xi|^{1/2}\mathcal{C}\big(|\xi|^{1/2}, \frac{\xi}{|\xi|}, \eta\big)$ if $|\xi| \ll |\eta| \sim 1$,

where $\mathcal{C}$ generically denotes a smooth function of its arguments. In particular, μ belongs to the class $\mathcal{B}^1$ of symbols defined below.

Definition 3.5. — *A symbol $m(\xi,\eta)$ belongs to $\mathcal{B}^s$ if*

- *It is homogeneous of degree s,*
- *It is smooth outside $\{\eta = 0\} \cup \{\xi - \eta = 0\} \cup \{\xi = 0\}$,*
- *One has*
$$\text{if} \quad |\xi_1| \ll |\xi_2|, |\xi_3| \sim 1, \qquad m = \mathcal{C}\Big(|\xi_1|, \frac{\xi_1}{|\xi_1|}, \xi_2\Big),$$
where $(\xi_1,\xi_2,\xi_3) = (\xi, \xi-\eta, \eta)$ up to circular permutations.

It is not complicated to show (Theorem C.1 of **[16]**) that the *pseudoproduct operators* associated to such symbols m,

$$B_m(f_1,f_2) := \mathcal{F}^{-1}\int m(\xi,\eta)\widehat{f}_1(\eta)\widehat{d}_2(\xi-\eta)d\eta$$

satisfy standard product estimates. For instance, for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 < p, q < \infty$, and if $s \geq 0$, $k \in \mathbb{N}$ and $m \in \mathcal{B}^s$,

$$|\nabla^k B_m(f_1,f_2)|_{L^r} \lesssim |f_1|_{W^{s+k,p}}|f_2|_{L^q} + |f|_{L^p}|g|_{W^{s+k,q}};$$

the main difference between such pseudoproducts and standards products is that the endpoint cases $p, q = 1, \infty$ are excluded, which induces some slight technical complications.

Using these pseudoproduct estimates one can obtain (see Prop. 4.1 of **[16]**),

$$\text{(63)} \qquad \forall t \geq 0, \qquad t|e^{-it\Lambda^{1/2}}g_1(t,\cdot)|_{W^{4,\infty}} \lesssim |u|_X^2 \quad \text{and} \quad (1+t)^{-\delta}|xg_1|_2 \lesssim |u|_X^2;$$

the second estimate requires more work than the first one; since $|xg_1|_2 = |\nabla_\xi \widehat{g_1}|_2$, it is essential as explained in §3.1.3 that the frequency derivative of the phase, $\partial_\xi\varphi_{\pm,\pm}(\xi,\eta)$, satisfies a compatibility condition allowing time or frequency integration by parts in the corresponding integral. This compatibility condition is given here by

$$\partial_\xi\varphi_{\pm,\pm} = \frac{1}{2}\frac{\xi}{|\xi|^{3/2}} \pm \frac{1}{2}\frac{\xi-\eta}{|\xi-\eta|^{3/2}}.$$

The contribution of the first term is harmless since we recall that we work here with a symbol μ vanishing at order $1/2$ in ξ; the second term can be put with the term $\widehat{f}_{\mp}(\tau,\xi-\eta)$ in (61) and controlled by fractional integration.

3.2.4. *Weakly resonant cubic terms.* — Consider now the cubic terms. They are of the form

$$(64)\qquad \widehat{g}_2(t,\xi)=\int_0^t\int\int e^{i\tau\varphi_{\pm,\pm,\pm}}\mu(\xi,\eta,\sigma)\widehat{f}_{\mp}(\tau,\eta)\widehat{f}_{\mp}(\tau,\eta)\widehat{f}(\tau,\xi-\eta-\sigma)\,d\eta d\sigma d\tau;$$

some of them come from the second line of (60), and others come from the normal form transform of the quadratic terms—see (62). But the relevant distinction we have to make among all the cubic terms deals with their phase, not their provenance. The cubic phases are given by

$$\varphi_{\pm,\pm,\pm}(\xi,\eta,\sigma)=|\xi|^{1/2}\pm|\eta|^{1/2}\pm|\sigma|^{1/2}\pm|\xi-\eta-\sigma|^{1/2}.$$

Some of them have few time resonances (seen now as a subset of $\mathbb{R}^2_\xi\times\mathbb{R}^2_\eta\times\mathbb{R}^2_\sigma$), and we call them *weakly resonant cubic terms.* They correspond to φ_{+++} (for which the time resonant set $\mathscr{T}$ is reduced to a point), and φ_{-++}, φ_{+-+}, φ_{++-} and φ_{---}. Time resonances for φ_{---} (the other cases can be deduced by permutation of the variables) are given by

$$\mathscr{T}_{---}=\{\eta=\sigma=0\quad\text{or}\quad\sigma=\xi-\eta-\sigma=0\quad\text{or}\quad\eta=\xi-\eta-\sigma=0\}.$$

We recall that we need to control $e^{-it\Lambda^{1/2}}g_2$ in $W^{4,\infty}$ and xg_2 in L^2. The difficulties to derive such estimates are essentially the same as those encountered for the weighted estimate of the quadratic terms g_1 in the previous section. We thus get

$$(65)\qquad \forall t\geq 0,\qquad t|e^{-it\Lambda^{1/2}}g_2(t,\cdot)|_{W^{4,\infty}}\lesssim|u|^3_X\quad\text{and}\quad(1+t)^{-\delta}|xg_2|_2\lesssim|u|^3_X.$$

3.2.5. *Strongly resonant cubic terms.* — We are thus left with cubic terms with phases φ_{--+}, φ_{-+-} and φ_{+--}, which are identical up to permutation of the variables. The situation is drastically different than in §3.2.4 because the time resonant set $\mathscr{T}_{--+}$ is now very large (of dimension 5). The time growing terms associated to weighted estimates (see §3.1.3) cannot be controlled by a simple integration in time as for the weakly resonant cubic terms. With the hope of controlling them with an integration by parts in frequency, we look at the space resonant set $\mathscr{S}_{--+}$ (defined as the set of all (ξ,η,σ) such that $\nabla_{\eta,\sigma}\varphi_{--+}(\xi,\eta,\sigma)=0$). According to the analysis of §3.1.4, we would like the space time resonant set $\mathscr{R}_{--+}=\mathscr{T}_{--+}\cap\mathscr{S}_{--+}$ to be small. This set is given by

$$\mathscr{R}_{--+}=\{\xi=\eta=\sigma\},$$

and is therefore much smaller than $\mathscr{T}_{--+}$ (it is of dimension 2), but yet too large to control directly the singularities created by integration by parts in frequency. The key ingredient here is a *bilinear compatibility condition* satisfied by φ_{--+} and bringing more transparency (see §3.1.3 for comments on this point). This compatibility condition can be written

$$\partial_\xi \varphi_{--+} = \mathscr{C}(\xi,\eta,\sigma)\big[\partial_\eta \varphi_{--+}, \partial_\sigma \varphi_{--+}\big],$$

with $\mathscr{C}$ smooth in (ξ,η,σ) and bilinear in the arguments between brackets. This term is strongly space transparent in the sense of (56) (or more exactly, to its obvious generalization to cubic phases). This is the crucial point that allows the authors of [**16**] to conclude, after a delicate technical implementation of these ideas, that strongly resonant terms also satisfy (65).

4. COMPATIBLE FORMS, NULL CONDITION AND TRANSPARENCY IN OTHER CONTEXTS

We have seen that the global existence issue for nonlinear dispersive equation is linked to various conditions on the structure of the nonlinearity, such as compatibility, null condition, or time and space transparency. It is also well known that the null condition plays also a central role for the well posedness issue below the standard regularity threshold $s > d/2$ for semilinear systems, as noticed by Klainerman and Machedon (see for instance [**28**]). We present here two lesser known examples where these notions also play a central role.

4.1. Compensated compactness

Let Ω be an open subset of $\mathbb{R}^N$ and consider a differential operator $\mathbf{A}(\partial)$ and its domain $H(\Omega,\mathbf{A})$,

$$\mathbf{A}(\partial) = \sum_{j=1}^N A_j\partial_j, \qquad H(\Omega,\mathbf{A}) = \{u \in L^2(\Omega)^n, \quad \mathbf{A}(\partial)u \in L^2(\Omega)^n\},$$

where the A_j are $n\times n$ matrices with real constant coefficients. To every continuous function $f:\mathbb{R}^N \to \mathbb{R}$ and every $u \in H(\Omega, A)$, one can associate a function $f(u)$ defined almost everywhere on Ω by $f(u)(x) = f(u(x))$. If we assume that f has at most quadratic growth at infinity, $f(u)$ is a distribution, and it is a typical question of compensated compactness motivated for instance by homogenization problems to answer the question

$$\text{“ When is } f : \begin{array}{ccc} H(\Omega,A) & \to & \mathscr{D}'(\Omega) \\ u & \mapsto & f(u) \end{array} \text{ weakly sequentially continuous? ”;}$$

equivalently, when are we sure that for all $u^\varepsilon \rightharpoonup u^0$ in L^2 such that $\mathbf{A}(\partial)u^\varepsilon \rightharpoonup \mathbf{A}(\partial)u^0$ in L^2, one has $f(u^\varepsilon) \to f(u^0)$ in $\mathcal{D}'(\Omega)$? We refer to the earlier works of Murat and Tartar **[32, 40]** and many subsequent works for a full answer to this question.

Our point here is to make a link between this problem and the different conditions on the nonlinearities seen in these notes (compatibility, null condition, transparency, etc.). Consider therefore $\mathbf{A} = \partial_t + A(\partial)$, with A satisfying (28) (and thus $N = d+1$, $\partial_{d+1} = \partial_t$). It is a classical result of compensated compactness **[32, 40]** that *bilinear forms having the weakly sequential continuity described above coincide with compatible forms* in the sense of (32). We recall that compatible forms coincide with null forms in the sense of (17) for (systems of) wave equations, showing therefore an example of the relevance of null conditions for this problem.

4.2. Diffractive optics

Consider here a first order symmetric system of the form (26)

$$\partial_t U + A(\partial)U = \mathbf{Q}(U,U), \tag{66}$$

with fast oscillating initial conditions

$$U_{|_{t=0}}(x) = \varepsilon^p \sum_{l=1}^{N} u_l^0(x) \exp(\frac{\mathbf{k}_l \cdot x}{\varepsilon}) + \text{c.c.}, \tag{67}$$

where $\mathbf{k}_l \in \mathbb{R}^d \backslash \{0\}$, and *c.c.* stands for "complex conjugate"; when (66) are Maxwell's equation, these initial conditions are sums of wave packets modeling laser pulses. We are interested in the propagation of these N laser pulses (i.e. of the corresponding solution to (66)) for "diffractive" times of order $O(1/\varepsilon)$, for which the propagation is typically described by nonlinear Schrödinger equations. The problem of diffractive optics consists in constructing an approximate solution to (66) and to prove that it remains close to the exact solution over this time scale.

The reason why we are interested in times of order $O(1/\varepsilon)$ is because diffractive effects are of size $O(\varepsilon)$ and their cumulated contribution is of size $O(1)$ for such times. If we want nonlinear effects to be of the same order as the diffractive ones, the amplitude ε^p must be chosen such that the cumulated effects of the nonlinearities are also $O(1)$ for such times. For a quadratic nonlinearity, this corresponds to $p = 1/2$.

Denote by $\lambda_j(\mathbf{k})$ and $\pi_j(\mathbf{k})$ the eigenvalues and eigenprojectors of $A(\mathbf{k})$, and by $\mathbf{c}_j = \nabla\lambda_j(\mathbf{k})$ the group velocity. Assuming that the initial envelopes u_l^0 satisfy $\pi_{j_l}(\mathbf{k}_l)u_l^0 = u_l^0$ for some $1 \le j_l \le m$ (polarization condition), it can be proved **[10, 23, 2]** that consistent approximations to (66) must be of the form

$$U_{app}(t,x) = \sum_{l=1}^{N} u_l(\varepsilon t, x - \mathbf{c}_l t) \exp(\frac{\mathbf{k}_l \cdot x - \lambda_{j_l}(\mathbf{k}_l)t}{\varepsilon}) + c.c. + \varepsilon\langle u\rangle(\varepsilon t, t, x) + \cdots$$

where the sum accounts for the leading order oscillating terms, while $\langle u\rangle$ is the leading order non oscillating term; dots account for lower order terms. Moreover, the $u_l(T,y)$ must satisfy the polarization condition

$$\pi_{j_l}(\mathbf{k}_l)u_l = u_l \tag{68}$$

and the following *linear* Schrödinger equation,

$$\partial_T u_l - \frac{i}{2}\lambda''_{j_l}(\mathbf{k}_l)(\partial_y,\partial_y)u_l = 0, \tag{69}$$

where $\lambda''_{j_l}(\mathbf{k}_l)$ stands for the Hessian of λ_{j_l} at $\mathbf{k}_l$. Nonlinear effects are responsible for the creation of a non oscillating mode $\langle u\rangle$ (even if it is not initially present) by quadratic interaction of the oscillating modes; this effect is called *rectification*; the equation modeling this effect is

$$\partial_T\langle u\rangle + \text{ second order dispersive terms} = \sum_{l=1}^{N}\mathbf{Q}(u_l,\overline{u}_l). \tag{70}$$

When the nonlinearity $\mathbf{Q}$ satisfies the transparency condition (47), nonlinearities disappear from (70) because of (68)—actually, (47) needs only be satisfied for $\xi=0$ and $\eta=\mathbf{k}$, a condition called *weak transparency* in optics. The *nonlinear* Equations (66) with initial conditions (67) are then approximated with an error $O(\varepsilon)$ (e.g. in L^∞ norm) for times of order $O(1/\varepsilon)$ by a system of *linear* equations.

In order to observe the nonlinear effects, it is natural to consider initial conditions of larger amplitude, and therefore take $p=0$ in (67). The weak transparency condition now appears as a compatibility condition to construct an approximate solution in this setting. The set of N linear Schrödinger Equations (69) is now replaced by a set of N *cubic* Schrödinger equations (see [**5**] for $N=1$). Let us make a few comments:

- The cubic nonlinearity for these NLS equations is the same as the one obtained after the normal form transform in §2.2, even though the weak transparent property is not sufficient to implement fully this normal form transform.
- It is possible to have sets of three coupled NLS equations. More precisely, the equations for u_l, $u_{l'}$, $u_{l''}$ are *a priori* coupled if $\mathbf{k}_{l''}=\mathbf{k}_l-\mathbf{k}_{l'}$ and if $(\mathbf{k}_l,\mathbf{k}'_l)$ belongs to the time resonant set of the phase $\varphi(\xi,\eta)=\lambda_{j_l}(\xi)-\lambda_{j'_l}(\xi-\eta)-\lambda_{j''_l}(\eta)$.
- Wave packets travelling at different group speeds do not interact significantly. Therefore, the *a priori* coupling terms identified above are effective if and only if the group speeds of u_l, $u_{l'}$ and $u_{l''}$ are the same [**23, 29**]. Equivalently, $(\mathbf{k}_l,\mathbf{k}'_l)$ must also belong to the space resonant set of the phase φ.

These comments give therefore a physical interpretation in diffractive optics of the space time resonant set (see §3.1.4): *it is a representation of all the possible three wave interactions of diffractive wave packets.*

Note also that the weak transparency condition only allows one to derive a formal model. Stronger notions of transparency, such as (47) or (48) are required to justify this approximation. A very rich discussion of the transparency in optics has been carried out by Joly, Métivier and Rauch [**24**].

Acknowledgments

These notes benefited from precious discussions with J.-M. Delort, J. Rauch and J. Szeftel.

REFERENCES

[1] S. Alinhac – "Blowup of small data solutions for a quasilinear wave equation in two space dimensions", *Ann. of Math.* **149** (1999), no. 1, p. 97–127.

[2] K. Barrailh & D. Lannes – "A general framework for diffractive optics and its applications to lasers with large spectrums and short pulses", *SIAM J. Math. Anal.* **34** (2002), no. 3, p. 636–674.

[3] D. Christodoulou – "Global solutions of nonlinear hyperbolic equations for small initial data", *Comm. Pure Appl. Math.* **39** (1986), no. 2, p. 267–282.

[4] D. Christodoulou & S. Klainerman – *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, vol. 41, Princeton Univ. Press, 1993.

[5] T. Colin – "Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems", *Asymptot. Anal.* **31** (2002), no. 1, p. 69–91.

[6] W. Craig & M. D. Groves – "Hamiltonian long-wave approximations to the water-wave problem", *Wave Motion* **19** (1994), no. 4, p. 367–389.

[7] W. Craig & C. Sulem – "Numerical simulation of gravity waves", *J. Comput. Phys.* **108** (1993), no. 1, p. 73–83.

[8] M. Dafermos & I. Rodnianski – "A new physical-space approach to decay for the wave equation with applications to black hole spacetimes", in *XVIth International Congress on Mathematical Physics*, World Sci. Publ., Hackensack, NJ, 2010, p. 421–432.

[9] J.-M. Delort – "Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1", *Ann. Sci. École Norm. Sup.* **34** (2001), no. 1, p. 1–61; erratum: *Ann. Sci. École Norm. Sup.* **39** (2006), 335-345.

[10] P. Donnat, J.-L. Joly, G. Metivier & J. Rauch – "Diffractive nonlinear geometric optics", in *Séminaire sur les Équations aux Dérivées Partielles, 1995–1996*, Sémin. Équ. Dériv. Partielles, École Polytech., 1996, exp. n° XVII.

[11] V. Georgiev, S. Lucente & G. Ziliotti – "Decay estimates for hyperbolic systems", *Hokkaido Math. J.* **33** (2004), no. 1, p. 83–113.

[12] P. Germain – "Global solutions for coupled Klein-Gordon equations with different speeds", to appear in *Ann. Inst. Fourier.*

[13] P. Germain & N. Masmoudi – "Global existence for the Euler-Maxwell system", submitted.

[14] P. Germain, N. Masmoudi & J. Shatah – "Global solutions for 3D quadratic Schrödinger equations", *Int. Math. Res. Not.* **2009** (2009), no. 3, p. 414–432.

[15] ______, "Global solutions for 2D quadratic Schrödinger equations", *J. Math. Pures Appl.* **97** (2012), no. 5, p. 505–543.

[16] ______, "Global solutions for the gravity water waves equation in dimension 3", *Ann. of Math.* **175** (2012), no. 2, p. 691–754.

[17] S. Gustafson, K. Nakanishi & T.-P. Tsai – "Global dispersive solutions for the Gross-Pitaevskii equation in two and three dimensions", *Ann. Henri Poincaré* **8** (2007), no. 7, p. 1303–1331.

[18] ______, "Scattering theory for the Gross-Pitaevskii equation in three dimensions", *Commun. Contemp. Math.* **11** (2009), no. 4, p. 657–707.

[19] B. Hanouzet & J.-L. Joly – "Applications bilinéaires compatibles avec un opérateur hyperbolique", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987), no. 4, p. 357–376.

[20] N. Hayashi & P. I. Naumkin – "Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations", *Amer. J. Math.* **120** (1998), no. 2, p. 369–389.

[21] L. Hörmander – *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, vol. 26, Springer, 1997.

[22] F. John – "Blow-up for quasilinear wave equations in three space dimensions", *Comm. Pure Appl. Math.* **34** (1981), no. 1, p. 29–51.

[23] J.-L. Joly, G. Metivier & J. Rauch – "Diffractive nonlinear geometric optics with rectification", *Indiana Univ. Math. J.* **47** (1998), no. 4, p. 1167–1241.

[24] ______, "Transparent nonlinear geometric optics and Maxwell-Bloch equations", *J. Differential Equations* **166** (2000), no. 1, p. 175–250.

[25] S. Klainerman – "Uniform decay estimates and the Lorentz invariance of the classical wave equation", *Comm. Pure Appl. Math.* **38** (1985), no. 3, p. 321–332.

[26] ______, "The null condition and global existence to nonlinear wave equations", in *Nonlinear systems of partial differential equations in applied mathematics,*

Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., 1986, p. 293–326.

[27] ———, "Linear stability of black holes (d'après M. Dafermos et I. Rodnianski)", Séminaire Bourbaki, vol. 2009/2010, exposé n° 1015, *Astérisque* **339** (2011), p. 91–135.

[28] S. Klainerman & M. Machedon – "Smoothing estimates for null forms and applications", *Int. Math. Res. Not.* **1994** (1994), no. 9, p. 383–390.

[29] D. Lannes – "Dispersive effects for nonlinear geometrical optics with rectification", *Asymptot. Anal.* **18** (1998), nos. 1-2, p. 111–146.

[30] H. Lindblad & I. Rodnianski – "Global existence for the Einstein vacuum equations in wave coordinates", *Comm. Math. Phys.* **256** (2005), no. 1, p. 43–110.

[31] K. Moriyama, S. Tonegawa & Y. Tsutsumi – "Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension", *Funkcial. Ekvac.* **40** (1997), no. 2, p. 313–333.

[32] F. Murat – "Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **8** (1981), no. 1, p. 69–102.

[33] T. Ozawa, K. Tsutaya & Y. Tsutsumi – "Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions", *Math. Z.* **222** (1996), no. 3, p. 341–362.

[34] J. Shatah – "Normal forms and quadratic nonlinear Klein-Gordon equations", *Comm. Pure Appl. Math.* **38** (1985), no. 5, p. 685–696.

[35] J. Shatah & C. Zeng – "Geometry and a priori estimates for free boundary problems of the Euler equation", *Comm. Pure Appl. Math.* **61** (2008), no. 5, p. 698–744.

[36] J. C. H. Simon – "A wave operator for a nonlinear Klein-Gordon equation", *Lett. Math. Phys.* **7** (1983), no. 5, p. 387–398.

[37] J. C. H. Simon & E. Taflin – "The Cauchy problem for nonlinear Klein-Gordon equations", *Comm. Math. Phys.* **152** (1993), no. 3, p. 433–478.

[38] C. D. Sogge – *Lectures on non-linear wave equations*, International Press, 2008.

[39] W. A. Strauss – "Nonlinear scattering theory at low energy", *J. Funct. Anal.* **41** (1981), no. 1, p. 110–133.

[40] L. Tartar – "Compensated compactness and applications to partial differential equations", in *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, Res. Notes in Math., vol. 39, Pitman, 1979, p. 136–212.

[41] S. Wu – "Well-posedness in Sobolev spaces of the full water wave problem in 2-D", *Invent. Math.* **130** (1997), no. 1, p. 39–72.

[42] ———, "Well-posedness in Sobolev spaces of the full water wave problem in 3-D", *J. Amer. Math. Soc.* **12** (1999), no. 2, p. 445–495.

[43] ———, "Almost global wellposedness of the 2-D full water wave problem", *Invent. Math.* **177** (2009), no. 1, p. 45–135.

[44] ———, "Global wellposedness of the 3-D full water wave problem", *Invent. Math.* **184** (2011), no. 1, p. 125–220.

David LANNES

École normale supérieure
Département de Mathématiques et Applications
45 rue d'Ulm
75230 Paris Cedex 05
E-mail : `David.Lannes@ens.fr`