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Dedicated to my mentors in this study,
Claude Sabbah, Morihiko Saito, and Carlos Simpson,
with appreciation to everything which I learned from them.

WILD HARMONIC BUNDLES AND WILD PURE TWISTOR D-MODULES

Takuro Mochizuki

Abstract. — We study (i) the asymptotic behaviour of wild harmonic bundles, (ii) the relation between semisimple meromorphic flat connections and wild harmonic bundles, (iii) the relation between wild harmonic bundles and polarized wild pure twistor D-modules. As an application, we show the hard Lefschetz theorem for algebraic semisimple holonomic D-modules, conjectured by M. Kashiwara. We also study resolution of turning points for algebraic meromorphic flat bundles.

Résumé **(Fibrés harmoniques sauvages et D-modules sauvages avec structure de twisteur pure)**

Nous étudions (i) le comportement asymptotique d'un fibré harmonique sauvage, (ii) la relation entre connexions méromorphes plates semi-simples et fibrés harmoniques sauvages, (iii) la relation entre fibrés harmoniques sauvages et D-modules sauvages avec structure de twisteur pure. Comme application, nous prouvons le théorème de Lefschetz difficile pour les D-modules holonomes algébriques semi-simples, conjecturé par M. Kashiwara. Nous étudions également la résolution des points tournants pour les connexions méromorphes algébriques plates.

CONTENTS

CHAPTER 1

INTRODUCTION

In our previous work ([**65**], [**66**], [**67**] and [**69**]), we studied tame harmonic bundles after C. Simpson ([**81**], [**82**], [**83**], [**85**], for example), and we established their foundational property. (See also the works of O. Biquard [**9**] and J. Jost and K. Zuo [**41**]). In cooperation with the work of C. Sabbah [**73**] on pure twistor D-modules (based on the theory of pure Hodge modules due to M. Saito [**77**], [**78**]), we obtained some deep results on algebraic regular holonomic D-modules.

In this monograph, we systematically study *wild harmonic bundles* over complex manifolds of arbitrary dimension, and we obtain generalizations of our previous results for tame harmonic bundles in the case of wild harmonic bundles. We also give applications to algebraic meromorphic flat bundles and algebraic holonomic D-modules. Recently, there has been a growing interest in wild harmonic bundles and (not necessarily regular) holonomic D-modules on curves. (See [**10**], [**13**], [**12**], [**18**], [**36**], [**45**], [**74**], [**87**], [**90**], [**95**], for example.) However, in this monograph, we will NOT consider moduli spaces, mirror symmetries, geometric Langlands theory, integrable systems, non-abelian Hodge theory, Painlevé equations, and other fashionable subjects related with harmonic bundles and Higgs bundles. It can be said that our goal is more modest and basic. Nonetheless, the author has been deeply impressed with what he saw in this study, for example an interaction between the theories of harmonic bundles and D-modules.

1.1. Contents of this monograph

Briefly speaking, this study consists of three main bodies and preliminaries for them:

(A) : Asymptotic behaviour of wild harmonic bundles.

(B) : Application to algebraic meromorphic flat bundles.

(C) : Application to wild pure twistor D-modules and algebraic D-modules.

Let us briefly describe each part. We will give some more detailed introductions later (Sections 1.2–1.4).

We have two main issues in Part (A). Let X be a complex manifold, and let D be a normal crossing hypersurface of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on (X, D). Although we will not explain the definition here (see Section 1.2.1, or Section 7.1 for more precision), it means that $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle on $X \setminus D$ satisfying some conditions around each point of D. We would like to prolong it to something on X, that is the first main issue, and fundamental for us. The role of such prolongation may be compared with the nilpotent orbit theorem in Hodge theory, due to W. Schmid [**79**].

Then, we would like to understand more detailed properties. It is achieved by showing that we obtain tame harmonic bundles from wild harmonic bundles as Gr with respect to Stokes filtrations, which is the second main issue in (A). By this reduction, the study of the asymptotic behaviour of wild harmonic bundles is reduced to the tame case investigated in [**67**].

There are two main purposes in Part (B). One is to characterize semisimplicity of algebraic meromorphic flat bundles by the existence of a pluri-harmonic metric with some nice property. The other is to show the existence of resolutions of turning points for algebraic meromorphic flat bundles.

According to K. Corlette [**20**], a flat bundle on a smooth projective variety has a pluri-harmonic metric if and only if it is semisimple, i.e., a direct sum of irreducible ones. This was generalized to the case of meromorphic flat bundles with regular singularity ([**41**] and [**67**]). In this monograph, we will establish such a characterization in the irregular case. We will also study the Kobayashi-Hitchin correspondence for meromorphic flat bundles.

We have an interesting application of such an existence result of pluri-harmonic metric to the resolution of turning points for algebraic meromorphic flat bundles, which is the other main result in Part (B). It seems of foundational importance in the study of algebraic holonomic D-modules, and might be compared with the resolution of singularities for algebraic varieties.

Remark 1.1.1. — Recently, K. Kedlaya established it in a more general situation in a completely different way. See [**46**] and [**47**]. □

In Part (C), we will establish the relation between wild harmonic bundles and polarized wild pure twistor D-modules. Recently, Sabbah introduced the notion of wild pure twistor D-modules [**75**]. Our result roughly says that polarized wild pure twistor D-modules are actually minimal extensions of wild harmonic bundles. Together with the result in Part (B), we obtain the correspondence between semisimple holonomic D-modules and polarizable wild pure twistor D-modules on complex projective varieties. As an application, we will show the Hard Lefschetz theorem for algebraic semisimple (not necessarily regular) holonomic D-modules, conjectured by M. Kashiwara.

We may also say that the result in Part (C) makes us possible to define the push-forward for wild harmonic bundles, which will be useful to produce new wild harmonic bundles, or to enrich some operations for flat bundles by polarized twistor structures.

We need various preliminaries. Let us mention some of the major ones. We will revisit asymptotic analysis for meromorphic flat bundles in a way convenient for us, which was originally studied by H. Majima [**53**] and refined by Sabbah [**72**]. We will put a stress on canonically defined Stokes filtrations. It is fundamental for us to consider Gr with respect to Stokes filtrations, and deformations caused by variation of irregular values. (See Chapters 2–4.)

As another important preliminary, we show that acceptable bundles are naturally prolonged to filtered bundles. (See Section 21.3.) After the work of M. Cornalba-P. Griffiths and Simpson ([**21**], [**81**] and [**82**]) the author studied acceptable bundles, and he obtained such prolongation for acceptable bundles which come from tame harmonic bundles ([**65**], [**66**] and [**67**]). To apply the theory in the wild case, we will establish it for general acceptable bundles. Although only small changes are required, we will give rather detailed arguments in view of its importance. (See Chapter 21.)

1.2. Asymptotic behaviour of wild harmonic bundles

1.2.1. Prolongation

1.2.1.1. Harmonic bundle. — Recall the definition of harmonic bundle [**83**]. Let $(E,\overline{\partial}_E,\theta)$ be a Higgs bundle on a complex manifold. Let h be a Hermitian metric of E. Then, we have the associated unitary connection $\overline{\partial}_E+\partial_E$ and the adjoint $\theta^\dagger$ of θ with respect to h. The metric h is called *pluri-harmonic* if the connection $\mathbb{D}^1:=\overline{\partial}_E+\partial_E+\theta+\theta^\dagger$ is flat. In that case, $(E,\overline{\partial}_E,\theta,h)$ is called a *harmonic bundle.* We also have another equivalent definition. Let (V,∇) be a flat bundle on a complex manifold. Let h be a Hermitian metric of V. Then, we have the decomposition $\nabla=\nabla^u+\Phi$, where ∇^u is a unitary connection and Φ is self-adjoint with respect to h. We have the decompositions $\nabla^u=\partial_V+\overline{\partial}_V$ and $\Phi=\theta+\theta^\dagger$ into the $(1,0)$-part and the $(0,1)$-part. We say that h is a *pluri-harmonic metric* for (V,∇) if $(V,\overline{\partial}_V,\theta)$ is a Higgs bundle. In that case, (V,∇,h) is called a *harmonic bundle.*

Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on a complex manifold Y. For any complex number λ, we have the flat λ-connection $(\mathcal{E}^\lambda,\mathbb{D}^\lambda)$, i.e., the holomorphic vector bundle $\mathcal{E}^\lambda:=(E,\overline{\partial}_E+\lambda\theta^\dagger)$ with the flat λ-connection $\mathbb{D}^\lambda:=\overline{\partial}_E+\lambda\theta^\dagger+\lambda\partial_E+\theta$. We also have the family of λ-flat bundles $(\mathcal{E},\mathbb{D})$ on $\boldsymbol{C}_\lambda\times Y$, i.e., the holomorphic vector bundle $\mathcal{E}:=\big(p_\lambda^{-1}(E),\overline{\partial}_E+\lambda\theta^\dagger+\overline{\partial}_\lambda\big)$ equipped with a family of flat λ-connections $\mathbb{D}:=\overline{\partial}_E+\lambda\theta^\dagger+\lambda\partial_E+\theta$, where $p_\lambda:\boldsymbol{C}_\lambda\times Y\to Y$ denotes the projection.

1.2.1.2. Good wild harmonic bundle. — Let X be a complex manifold with a simple normal crossing hypersurface D. Let $M(X,D)$ be the set of meromorphic functions whose poles are contained in D, and $H(X)$ be the set of holomorphic functions on X.

As mentioned in Section 1.1, it is fundamental for us to prolong a *good wild harmonic bundle* on $X \setminus D$ to something on X. We would like to explain some more details. To begin with, we explain what is a good wild harmonic bundle.

Let us consider the local case, i.e., $X := \Delta^n = \{(z_1, \ldots, z_n) \,\big|\, |z_i| < 1\}$ and $D := \bigcup_{j=1}^{\ell}\{z_j = 0\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X \setminus D$. It is called *strongly unramifiedly good wild harmonic bundle* on (X, D), if there exist a good set of irregular values $\mathrm{Irr}(\theta) \subset M(X, D)/H(X)$, a finite subset $\mathcal{S}p(\theta) \subset \mathbf{C}^{\ell}$ and a decomposition

$$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \bigoplus_{\boldsymbol{\alpha} \in \mathcal{S}p(\theta)} (E_{\mathfrak{a},\boldsymbol{\alpha}}, \theta_{\mathfrak{a},\boldsymbol{\alpha}}), \tag{1}$$

such that the eigenvalues of $\theta_{\mathfrak{a},\boldsymbol{\alpha}}$ are $d\mathfrak{a} + \sum_{i=1}^{\ell} \alpha_i \cdot dz_i/z_i$ modulo multi-valued holomorphic 1-forms on X. (See Definition 2.1.2 for *good set of irregular values.*) In another brief word, for the expression

$$\theta_{\mathfrak{a},\boldsymbol{\alpha}} = d\mathfrak{a} + \sum_{j=1}^{\ell} (\alpha_j + f_{\mathfrak{a},\boldsymbol{\alpha},j}) \cdot \frac{dz_j}{z_j} + \sum_{j=\ell+1}^{n} g_{\mathfrak{a},\boldsymbol{\alpha},j} \cdot dz_j,$$

the characteristic polynomials $\det(T - f_{\mathfrak{a},\boldsymbol{\alpha},j})$ $(j = 1, \ldots, \ell)$ and $\det(T - g_{\mathfrak{a},\boldsymbol{\alpha},j})$ $(j = \ell + 1, \ldots, n)$ are contained in $H(X)[T]$, and $\det(T - f_{\mathfrak{a},\boldsymbol{\alpha},j})_{|\{z_j=0\}} = T^{\mathrm{rank}\, E_{\mathfrak{a},\boldsymbol{\alpha}}}$. We say that $(E, \overline{\partial}_E, \theta, h)$ is a *strongly good wild harmonic bundle* on (X, D), if there exists a ramified covering $\varphi : (X', D') \to (X, D)$ such that $\varphi^*(E, \overline{\partial}_E, \theta, h)$ is a strongly unramifiedly good wild harmonic bundle, i.e., $(E, \overline{\partial}_E, \theta, h)$ is the descent of an unramifiedly good wild harmonic bundle.

The definitions can be easily globalized. Let X be a complex manifold, and let D be a normal crossing hypersurface. A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X \setminus D$ is called a *good wild harmonic bundle* on (X, D), if the following holds:

- For any point $P \in D$, there exists a coordinate neighbourhood X_P around P such that $(E, \overline{\partial}_E, \theta, h)_{|X_P \setminus D}$ is a strongly good wild harmonic bundle on $(X_P, D \cap X_P)$.

We have one more additional notion of *wild harmonic bundle.* Let Y be an irreducible complex (not necessarily smooth) variety. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle defined on the complement of a closed analytic subset Z of Y. We say that $(E, \overline{\partial}_E, \theta, h)$ is wild on (Y, Z) (or sometimes $(Y, Y \setminus Z)$), if there exists a morphism φ of a smooth complex variety Y' to Y such that (i) φ is birational and projective, (ii) $\varphi^{-1}(Z)$ is a normal crossing hypersurface of Y', (iii) $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is a good wild harmonic bundle on $\big(Y', \varphi^{-1}(Z)\big)$.

Essential analysis is done for unramifiedly good wild harmonic bundle. The study of good wild harmonic bundle can be easily reduced to the unramified case. The notion of wild harmonic bundle is rather auxiliary to give statements in the application to wild pure twistor D-modules.

Remark 1.2.1. — In the one dimensional case, any wild harmonic bundle is good, and such an object can be defined in a much simpler way. Namely, we only have to impose the condition that the characteristic polynomial of the associated Higgs field is meromorphic. □

1.2.1.3. Sheaf of holomorphic sections with polynomial growth. — We explain various prolongations of a good wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on (X, D) to some meromorphic objects on X. We consider the local case $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. The constructions and the results can be easily globalized.

For any λ, we would like to prolong the λ-flat bundle $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ on $X \setminus D$ to a meromorphic λ-flat bundle on (X, D). For any $\boldsymbol{a} \in \boldsymbol{R}^\ell$ and any open subset $U \subset X$, we define

$$\text{(2)} \qquad \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda(U) := \Big\{ f \in \mathcal{E}^\lambda(U \setminus D) \,\Big|\, |f|_h = O\Big(\prod_{j=1}^{\ell} |z_j|^{-a_j - \varepsilon}\Big),\ \forall \varepsilon > 0 \Big\}.$$

By taking sheafification, we obtain an increasing sequence of $\mathcal{O}_X$-modules $\mathcal{P}_*\mathcal{E}^\lambda := \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^\ell\big)$ and an $\mathcal{O}_X(*D)$-module $\mathcal{P}\mathcal{E}^\lambda := \bigcup_{\boldsymbol{a} \in \boldsymbol{R}^\ell} \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda$. The following theorem is the starting point of the study on asymptotic behaviour of good wild harmonic bundle.

***Theorem 1.2.2* (Theorem 7.4.3, Theorem 7.4.5).** — *$(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ is a good filtered λ-flat bundle. If $(E, \overline{\partial}_E, \theta, h)$ is unramified with the decomposition in* (1), *the set of irregular values is given by* $\mathrm{Irr}\big(\mathcal{P}\mathcal{E}^\lambda, \mathbb{D}^\lambda\big) = \big\{(1 + |\lambda|^2)\,\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\theta)\big\}$. □

We refer to Section 2.8 for details on the notion of *good filtered λ-flat bundle*, but we briefly explain the meaning of the theorem in the case where $(E, \overline{\partial}_E, \theta, h)$ is *unramified* with a decomposition (1). Let O denote the origin $(0, \ldots, 0)$. Each $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda$ is a locally free $\mathcal{O}_X$-module, and we have the decomposition for the completion at O

$$\text{(3)} \qquad (\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda, \mathbb{D}^\lambda)_{|\widehat{O}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (\mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{E}}^\lambda_{\mathfrak{a}}, \widehat{\mathbb{D}}^\lambda_{\mathfrak{a}}),$$

where $\widehat{\mathbb{D}}^\lambda_{\mathfrak{a}} - (1 + |\lambda|^2)d\mathfrak{a}$ have logarithmic singularities. Moreover, we have such decompositions on the completions at each point of D.

Let us give a remark on the proof. We use the essentially same arguments as those in our previous paper [**67**] to show that $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda$ are locally free. Namely, we give an estimate on the Higgs field θ (Theorem 7.2.1 and Theorem 7.2.4) as the wild version of *Simpson's Main estimate*, which briefly means that the decomposition (1) is asymptotically orthogonal, and that $\theta_{\mathfrak{a},\boldsymbol{\alpha}} - d\mathfrak{a} - \sum \alpha_i\, dz_i/z_i$ are bounded with respect to h and the Poincaré metric of $X \setminus D$. Then, we obtain that $(\mathcal{E}^\lambda, h)$ is *acceptable*, i.e., the curvature of $(\mathcal{E}^\lambda, h)$ is bounded with respect to h and the Poincaré metric (Corollary 7.2.10). We obtain that $\mathcal{P}_*\mathcal{E}^\lambda$ is a filtered bundle by using a general theory of acceptable bundles (Theorem 21.3.1). In the wild case, we need some more arguments to show that $\mathbb{D}^\lambda$ is meromorphic and to obtain the decomposition (3).

1.2.1.4. Family of meromorphic λ-flat bundles. — Next, we would like to consider the prolongation of the family of λ-flat bundles $(\mathcal{E}, \mathbb{D})$ on $\boldsymbol{C}_\lambda \times (X \setminus D)$ to a family of meromorphic λ-flat bundles on $\boldsymbol{C}_\lambda \times (X, D)$. A naive idea is to consider the family $\bigcup_{\lambda \in \boldsymbol{C}_\lambda} \mathcal{P}\mathcal{E}^\lambda$, i.e., the sheaf of holomorphic sections of $\mathcal{E}$ with polynomial growth. In the tame case, it gives a nice *meromorphic* object. (But, note that we need some more consideration for *lattices*.) However, $\bigcup_{\lambda \in \boldsymbol{C}} \mathcal{P}\mathcal{E}^\lambda$ cannot be a good meromorphic prolongment in the wild case, because the irregular values $(1 + |\lambda|^2)\mathfrak{a}$ of $\mathbb{D}^\lambda$ have non-holomorphic dependence on λ, as mentioned in Theorem 1.2.2. Hence, we need the deformations of $\mathcal{P}\mathcal{E}^\lambda$ $(\lambda \neq 0)$ caused by variation of the irregular values (Section 4.5.2).

1.2.1.4.1. We briefly and imprecisely explain such a deformation in the case $\lambda = 1$. First, we can obtain such deformation for an unramifiedly good meromorphic flat bundle, by considering the deformation of Stokes structure. It is easily extended to the (possibly) ramified case, and then the global case. Hence, we explain the essential part, i.e., the local an unramified case. Let $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let $(\mathcal{V}, \nabla)$ be a meromorphic flat bundle on (X, D), with a lattice V which has decompositions as in (3) on the completions at each point of D, for example

$$(V, \nabla)_{|\widehat{O}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{V}_\mathfrak{a}, \widehat{\nabla}_\mathfrak{a}), \qquad (\widehat{\nabla}_\mathfrak{a} - d\mathfrak{a})\widehat{V}_\mathfrak{a} \subset \widehat{V}_\mathfrak{a} \otimes \Omega^1_X(\log D). \tag{4}$$

Such a lattice is called an *unramifiedly good lattice* of $(\mathcal{V}, \nabla)$ (Section 2.3). We should remark that such a lattice may not exist in general. For simplicity, we also assume a non-resonance condition for the lattice, i.e., the difference of the distinct eigenvalues of the residues are not integers. Let S be a small multi-sector in $X \setminus D$ whose closure contains O. Let $\overline{S}$ denote its closure in the real blow up $\widetilde{X}(D)$ of X along D, or more precisely, the fiber product of the real blow up along the irreducible components of D, taken over X. According to the asymptotic analysis for meromorphic flat bundles (see [**53**], [**72**], and Chapter 3 in this monograph), we can lift (4) to a flat decomposition of $V_{|\overline{S}}$:

$$V_{|\overline{S}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} V_{\mathfrak{a}, S}. \tag{5}$$

Namely, the completion of (5) along $\overline{S} \cap \pi^{-1}(O)$ is the same as the pull-back of (4). We consider the order $\leqslant_S$ on $\mathrm{Irr}(\nabla)$ given by $\mathfrak{a} \leqslant_S \mathfrak{b} \iff -\mathrm{Re}(\mathfrak{a})(Q) \leqslant -\mathrm{Re}(\mathfrak{b})(Q)$ at all points Q of S which are sufficiently close to O. Then, it can be shown that

$$\widetilde{\mathcal{F}}^S_\mathfrak{a}(V_{|\overline{S}}) := \bigoplus_{\mathfrak{b} \leqslant_S \mathfrak{a}} V_{\mathfrak{b}, S}$$

is independent of the choice of a lifting (5). Thus, we obtain the filtration $\widetilde{\mathcal{F}}^S$ of $V_{|\overline{S}}$ or $V_{|S}$, which is a ∇-flat filtration indexed by $\big(\mathrm{Irr}(\nabla), \leqslant_S\big)$. It is called the full Stokes filtration. For two small multi-sectors S_i $(i = 1, 2)$, the filtrations $\widetilde{\mathcal{F}}^{S_i}$ satisfy some compatibility on $S_1 \cap S_2$. (See Section 3.1.1.) Such a system of filtrations

$\{\widetilde{\mathcal{F}}^S(V_{|S}) \mid S \subset X \setminus D\}$ is called the Stokes structure of $\mathcal{V}$. It can be shown that the meromorphic flat bundle $(\mathcal{V}, \nabla)$ is recovered from the flat bundle $\mathcal{V}_{|X \setminus D}$ with the Stokes structure.

For any $T > 0$, we set $\mathrm{Irr}(\nabla^{(T)}) := \{T\mathfrak{a} \mid \mathfrak{a} \in \mathrm{Irr}(\nabla)\}$. The natural bijection $\mathrm{Irr}(\nabla) \simeq \mathrm{Irr}(\nabla^{(T)})$ preserves the order $\leqslant_S$. We naturally obtain a system of filtrations $\{\widetilde{\mathcal{F}}^S_{T\mathfrak{a}} \mid \mathfrak{a} \in \mathrm{Irr}(\nabla)\}$ for each small multi-sector indexed by $(\mathrm{Irr}(\nabla^{(T)}), \leqslant_S)$, which gives a new Stokes structure. We obtain the associated meromorphic flat bundle $(\mathcal{V}^{(T)}, \nabla^{(T)})$ on (X, D), which is a deformation caused by variation of irregular values. As mentioned, it is not difficult to extend it to the global and (possibly) ramified case.

Remark 1.2.3. — Such a deformation grew out of the discussion with C. Sabbah. We study it in a more general situation.

After finishing the first version of this monograph, the author found that such a deformation in the curve case appeared in several works such as [**14**], [**22**] and [**91**]. The referee kindly informed that the irregular values were used as parameters for universal deformations in [**55**]. □

1.2.1.4.2. Applying this deformation procedure to $(\mathcal{P}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ with $T = (1 + |\lambda|^2)^{-1}$ for $\lambda \neq 0$, we obtain the meromorphic λ-flat bundle $(\mathcal{Q}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$. We put $\mathcal{Q}\mathcal{E}^0 := \mathcal{P}\mathcal{E}^0$. The next theorem says that the family $\bigcup \mathcal{Q}\mathcal{E}^\lambda$ gives a nice meromorphic object on $\boldsymbol{C}_\lambda \times (X, D)$. For simplicity of the description, let $\mathcal{X}^{(\lambda_0)}$ denote a neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_\lambda \times X$, and $\mathcal{D}^{(\lambda_0)} := \mathcal{X}^{(\lambda_0)} \cap (\boldsymbol{C}_\lambda \times D)$.

***Theorem 1.2.4* (Theorem 11.1.2)**. — *We have a unique family of meromorphic λ-flat bundles $(\mathcal{Q}\mathcal{E}, \mathbb{D})$ on $\boldsymbol{C}_\lambda \times (X, D)$ such that the specialization of $(\mathcal{Q}\mathcal{E}, \mathbb{D})$ to $\{\lambda\} \times X$ is isomorphic to $(\mathcal{Q}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$. Moreover, we have a family of good filtered λ-flat bundles $(\mathcal{Q}^{(\lambda_0)}_* \mathcal{E}, \mathbb{D})$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$ for each $\lambda_0 \in \boldsymbol{C}_\lambda$ with the KMS-structure such that $\mathcal{Q}\mathcal{E}_{|\mathcal{X}^{(\lambda_0)}} = \bigcup \mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}} \mathcal{E}$.* □

We refer to Section 2.8.2 for *KMS-structure.* Although it is quite important in the study of the asymptotic behaviour of wild harmonic bundles, it has already appeared in the tame case [**67**], where we have studied it in detail. Hence, we omit to explain it in this introduction.

Note that we have the uniqueness of the family because $(\mathcal{Q}\mathcal{E}, \mathbb{D})_{\{\lambda\} \times X} \simeq (\mathcal{Q}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$. Hence, to show Theorem 1.2.4, we only have to consider it in the local and unramified case. Then, we shall argue in two steps. We first construct a family of meromorphic λ-flat bundles $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$ by using a deformed metric $\mathcal{P}^{(\lambda_0)}h$ given in (182). Namely, for the metric $\mathcal{P}^{(\lambda_0)}h$, we consider an increasing sequence of $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-modules $\mathcal{P}^{(\lambda_0)}_* \mathcal{E} := (\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}} \mathcal{E} \mid \boldsymbol{a} \in \boldsymbol{R}^\ell)$ given as in (2), and the $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}(*\mathcal{D}^{(\lambda_0)})$-module $\mathcal{P}^{(\lambda_0)}\mathcal{E} := \bigcup_{\boldsymbol{a} \in \boldsymbol{R}^\ell} \mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}} \mathcal{E}$. The restriction of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ to $\{\lambda\} \times X$ is denoted by $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$.

***Theorem 1.2.5* (Theorem 9.1.2, Proposition 9.2.1).** — *$(\mathcal{P}^{(\lambda_0)}_{*}\mathcal{E},\mathbb{D})$ is a family of good filtered λ-flat bundles with the KMS-structure. Moreover, $(\mathcal{P}^{(\lambda_0)}\mathcal{E}^{\lambda},\mathbb{D}^{\lambda})$ is naturally isomorphic to $\big((\mathcal{P}\mathcal{E}^{\lambda})^{T_1(\lambda,\lambda_0)},\mathbb{D}^{\lambda}\big)$, where $T_1(\lambda,\lambda_0):=(1+\lambda\overline{\lambda}_0)(1+|\lambda|^2)^{-1}$.* □

In the second step, we obtain a family of meromorphic λ-flat bundles $(\mathcal{Q}^{(\lambda_0)}\mathcal{E},\mathbb{D})$ as the deformation $(\mathcal{P}^{(\lambda_0)}\mathcal{E},\mathbb{D})^{(T_2(\lambda,\lambda_0))}$, where $T_2(\lambda,\lambda_0):=(1+\lambda\overline{\lambda}_0)^{-1}$. By varying $\lambda_0\in\boldsymbol{C}$ and gluing $\mathcal{Q}^{(\lambda_0)}\mathcal{E}$, we obtain the desired family $(\mathcal{Q}\mathcal{E},\mathbb{D})$.

Remark 1.2.6. — We should emphasize that $\mathcal{Q}\mathcal{E}$ is given on $\boldsymbol{C}_{\lambda}\times X$. Contrastively, the reduction in the next subsection is given locally around any point of D. □

1.2.2. Reduction from wild harmonic bundles to tame harmonic bundles

We would like to analyze more closely the behaviour of good wild harmonic bundle *around a given point of D*. For that purpose, we consider Gr with respect to Stokes structure. The construction in this subsection is given only locally, although the construction in the previous subsection can be easily globalized. We set $X:=\Delta^n$ and $D:=\bigcup_{i=1}^{\ell}\{z_i=0\}$, and we will shrink X around the origin O without mention it.

1.2.2.1. Gr of meromorphic flat bundle associated to the Stokes structure. — Before an explanation of the reduction of (unramifiedly good) wild harmonic bundles, let us explain the procedure to take Gr with respect to Stokes structure for a meromorphic flat bundle $(\mathcal{V},\nabla)$ with an unramifiedly good lattice V. (See Chapter 3 for more details.) For each small multi-sector $S\subset X\setminus D$, we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $V_{|\overline{S}}$ on $\overline{S}$. We obtain a graded bundle $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(V_{|\overline{S}})=\bigoplus\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V_{|\overline{S}})$ on $\overline{S}$ associated to $\widetilde{\mathcal{F}}^S$. Although the filtrations depend on multi-sectors S, they satisfy some compatibility. Hence, we can glue $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V_{|\overline{S}})$ and obtain the bundle $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V_{|\widetilde{X}(D)})$ with the induced meromorphic flat connection $\nabla_{\mathfrak{a}}$ on the real blow up $\widetilde{X}(D)$. It can be shown that it is the pull-back of a meromorphic flat connection $(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V),\nabla_{\mathfrak{a}})$ on (X,D), which is defined to be Gr of (V,∇) with respect to the full Stokes structure. (Such a construction already appeared in [**25**] for meromorphic flat bundles on curves.)

Although it is essentially the same as taking direct summands of the decomposition (4), there are some advantages. The above construction fits to our viewpoint that a meromorphic flat bundles on X is a prolongment of a flat bundle on $X\setminus D$. Moreover, it is suitable for the reduction of a variation of pure twistor structure, explained below.

1.2.2.2. Gr of family of meromorphic λ-flat bundles associated to the Stokes structure

Let $(E,\overline{\partial}_E,\theta,h)$ be an *unramifiedly* good wild harmonic bundle on (X,D) with a decomposition (1). We use the notation in Section 1.2.1. We set $W:=\mathcal{D}\cup(\{0\}\times X)$. Let $\widetilde{\mathcal{X}}(W)$ be the real blow up of $\mathcal{X}$ along W. Let S be a small multi-sector in $\mathcal{X}\setminus W$. As in the case of ordinary meromorphic flat bundles, we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $\mathcal{Q}\mathcal{E}_{|\overline{S}}$. By varying S and gluing $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{Q}\mathcal{E}_{|\overline{S}})$, we obtain a family of

λ-flat bundles on $\widetilde{\mathcal{X}}(W)$. Moreover, as the descent for $\widetilde{\mathcal{X}}(W) \to \mathcal{X}$, we obtain a family of meromorphic λ-flat bundles $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(\mathcal{QE})$ on $\boldsymbol{C}_\lambda \times (X, D)$ for each $\mathfrak{a} \in \mathrm{Irr}(\theta)$. It has the unique irregular value $\mathfrak{a}$. They are called the full reduction of $(\mathcal{QE}, \mathbb{D})$.

1.2.2.3. Gr of variation of pure twistor structure. — From the unramifiedly good wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X \setminus D$, we obtain an unramifiedly good wild harmonic bundle $(E, \partial_E, \theta^\dagger, h)$ on the conjugate $(X^\dagger, D^\dagger)$. We have the associated family of μ-flat bundles $(\mathcal{E}^\dagger, \mathbb{D}^\dagger)$ on $\boldsymbol{C}_\mu \times (X^\dagger \setminus D^\dagger)$, which is prolonged to a family of meromorphic μ-flat bundles $\big(\mathcal{QE}^\dagger, \mathbb{D}^\dagger\big)$ on $\boldsymbol{C}_\mu \times (X^\dagger, D^\dagger)$. We also obtain the full reductions $\mathrm{Gr}_{\overline{\mathfrak{a}}}^{\widetilde{\mathcal{F}}}(\mathcal{QE}^\dagger)$ on $\boldsymbol{C}_\mu \times (X^\dagger, D^\dagger)$ for any $\mathfrak{a} \in \mathrm{Irr}(\theta)$. Note that $\mathrm{Irr}(\theta^\dagger) = \big\{\overline{\mathfrak{a}} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\theta)\big\}$.

Let S be a small multi-sector in $X \setminus D$. Let $U(\lambda_0)$ be a small neighbourhood of $\lambda_0 \neq 0$ in $\boldsymbol{C}_\lambda$. We have the full Stokes filtration $\widetilde{\mathcal{F}}^S\big(\mathcal{QE}_{|U(\lambda_0)\times\overline{S}}\big)$ of $\mathcal{QE}_{|U(\lambda_0)\times\overline{S}}$. Let $U(\mu_0)$ be the neighbourhood of $\mu_0 = \lambda_0^{-1}$ in $\boldsymbol{C}_\mu$, corresponding to $U(\lambda_0)$. We also have the full Stokes filtration $\widetilde{\mathcal{F}}^S\big(\mathcal{QE}^\dagger_{|U(\mu_0)\times\overline{S}^\dagger}\big)$. We can observe that the filtrations are essentially the same on $U(\lambda_0) \times S = U(\mu_0) \times S^\dagger$ (Proposition 11.1.5). Actually, they are characterized by the growth order of the norms of flat sections. Hence, we have a natural isomorphism $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(\mathcal{QE}_{|U(\lambda_0)\times S}) \simeq \mathrm{Gr}_{\overline{\mathfrak{a}}}^{\widetilde{\mathcal{F}}}(\mathcal{QE}^\dagger_{|U(\mu_0)\times S^\dagger})$. By gluing them, we obtain a natural identification:

$$\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(\mathcal{QE})_{|\boldsymbol{C}_\lambda^* \times (X\setminus D)} \simeq \mathrm{Gr}_{\overline{\mathfrak{a}}}^{\widetilde{\mathcal{F}}}(\mathcal{QE}^\dagger)_{|\boldsymbol{C}_\mu^* \times (X\setminus D)}. \tag{6}$$

Recall that the gluing of $(\mathcal{E}, \mathbb{D})$ and $(\mathcal{E}^\dagger, \mathbb{D}^\dagger)$ gives a variation of pure twistor structure $(\mathcal{E}^\triangle, \mathbb{D}^\triangle)$ of weight 0 on $\mathbb{P}^1 \times (X \setminus D)$ with a polarization $\mathcal{S} : \mathcal{E}^\triangle \otimes \sigma^*\mathcal{E}^\triangle \to \mathbb{T}(0)$. (See [**85**] or [**67**]. We will review it in Section 6.1.) Because of the isomorphism (6), we obtain a variation of twistor structure $\big(\mathrm{Gr}_{\mathfrak{a}}(\mathcal{E}^\triangle), \mathbb{D}_{\mathfrak{a}}^\triangle\big)$ for each $\mathfrak{a} \in \mathrm{Irr}(\theta)$, on which we have the induced pairings $\mathcal{S}_{\mathfrak{a}}$. The following theorem is one of the most important results in this paper.

***Theorem 1.2.7* (Theorem 11.2.2).** — $\big(\mathrm{Gr}_{\mathfrak{a}}(\mathcal{E}^\triangle), \mathbb{D}_{\mathfrak{a}}^\triangle, \mathcal{S}_{\mathfrak{a}}\big)$ *is a variation of pure polarized twistor structure of weight* 0*, if we shrink X appropriately. It comes from a harmonic bundle, which is the tensor product of a tame harmonic bundle $(E_{\mathfrak{a}}, \overline{\partial}_{\mathfrak{a}}, \theta_{\mathfrak{a}}, h_{\mathfrak{a}})$ and $L(\mathfrak{a})$.*

Here, $L(\mathfrak{a})$ denotes a harmonic bundle, which consists of a line bundle $\mathcal{O}_{X\setminus D} \cdot e$, the Higgs field $\theta e = e \cdot d\mathfrak{a}$ and the metric $h(e, e) = 1$. □

In some sense, Theorem 1.2.7 reduces the study of the asymptotic behaviour of wild harmonic bundles to the tame case. For example, the completion of $\mathcal{QE}$ along $\boldsymbol{C}_\lambda \times \{O\}$ is naturally isomorphic to the direct sum of the completion of $\mathcal{L}(\mathfrak{a}) \otimes \mathcal{QE}_{\mathfrak{a}}$. With the detailed study on $\mathcal{QE}_{\mathfrak{a}}$ for tame harmonic bundles in [**67**], we can say that we have already understood $\mathcal{L}(\mathfrak{a}) \otimes \mathcal{QE}_{\mathfrak{a}}$ pretty well, and hence $\mathcal{QE}$. Such an observation is very useful when we apply the prolongment of good wild harmonic bundles to the theory of polarized wild pure twistor D-modules. We can also derive the norm estimate.

1.2.2.4. Uniqueness of prolongation. — Recall the uniqueness of prolongation of a flat bundle on $X \setminus D$ to a meromorphic flat bundle on (X, D) with regular singularity, by which we have a very easy characterization of $\mathcal{Q}\mathcal{E}$ in the *tame* case. Namely, assume that we have some family of meromorphic λ-flat bundles $\mathcal{V}$ on $\boldsymbol{C}_\lambda \times (X, D)$ such that (i) the restriction to $\boldsymbol{C}_\lambda \times (X \setminus D)$ is $(\mathcal{E}, \mathbb{D})$, (ii) each restriction $\mathcal{V}^\lambda$ ($\lambda \neq 0$) is regular. Then, we have the natural isomorphism $\mathcal{V} \simeq \mathcal{Q}\mathcal{E}$, if $(E, \overline{\partial}_E, \theta, h)$ is tame. However, in the non-tame case, we do not have such an obvious characterization, which was one of the main obstacles for the author in this study. He has not yet known whether there exists an easy characterization for meromorphic prolongation of a family of λ-flat bundles with good lattices. However, we have a useful characterization of meromorphic prolongation of a variation of polarized pure twistor structure. Let $\big(\widetilde{V}_0, \widetilde{V}_\infty\big)$ be an unramifiedly good meromorphic prolongment of $(\mathcal{E}^\triangle, \mathbb{D}^\triangle, \mathcal{S})$. (See Section 6.2.) We have the variation of $\mathbb{P}^1$-holomorphic bundles $\big(\mathrm{Gr}^{\widetilde{V}}_{\mathfrak{a}}(\mathcal{E}^\triangle), \mathbb{D}^\triangle_{\mathfrak{a}}\big)$ with the pairing $\mathcal{S}_{\mathfrak{a}}$ for each $\mathfrak{a} \in \mathrm{Irr}(\theta)$, obtained as the full reduction with respect to the Stokes structure of $(\widetilde{V}_0, \widetilde{V}_\infty)$.

***Theorem 1.2.8* (Theorem 11.2.2).** — *If $\big(\mathrm{Gr}^{\widetilde{V}}_{\mathfrak{a}}(\mathcal{E}^\triangle), \mathbb{D}^\triangle_{\mathfrak{a}}, \mathcal{S}_{\mathfrak{a}}\big)$ are variations of polarized pure twistor structure of weight* 0 *for any $\mathfrak{a} \in \mathrm{Irr}(\theta)$, the prolongment is canonical, i.e., $\widetilde{V}_0 \simeq \mathcal{Q}\mathcal{E}$ and $\widetilde{V}_\infty \simeq \mathcal{Q}\mathcal{E}^\dagger$.* □

1.3. Application to meromorphic flat bundles

1.3.1. Resolution of turning points of meromorphic flat bundles

We recall the notion of Deligne-Malgrange lattice.

1.3.1.1. Deligne lattice. — Let X be a complex manifold, and let D be a normal crossing hypersurface with the decomposition into irreducible components $D = \bigcup_{i \in I} D_i$. Let $(\mathcal{V}, \nabla)$ be a meromorphic flat connection on (X, D). Namely, $\mathcal{V}$ is an $\mathcal{O}_X(*D)$-module with a flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$. If $(\mathcal{V}, \nabla)$ has regular singularity along D, there exists a lattice $V \subset \mathcal{V}$ with the following properties:

- ∇ is logarithmic with respect to V in the sense $\nabla(V) \subset V \otimes \Omega^1(\log D)$. Note that the residue $\mathrm{Res}_{D_i}(\nabla)$ is given in $\mathrm{End}(V_{|D_i})$ for each irreducible component of D.
- Any eigenvalue α of $\mathrm{Res}_{D_i}(\nabla)$ satisfies $0 \leqslant \mathrm{Re}(\alpha) < 1$.

This lattice V is called the *Deligne lattice* of $(\mathcal{V}, \nabla)$, which plays an important role in the study of meromorphic flat bundles with regular singularities, or more generally, regular holonomic D-modules.

1.3.1.2. Deligne-Malgrange lattice (one dimensional case). — It is natural and important to ask for the existence of such a lattice in the irregular case. If the base manifold is a curve, it is classically well known. Let us consider the case $X := \Delta$ and $D := \{0\}$. According to the Hukuhara-Levelt-Turrittin theorem, there exists an appropriate ramified covering $\varphi : (X', D') \longrightarrow (X, D)$ such that the formal structure of the pull-back $\varphi^*(\mathcal{V}, \nabla)$ is quite simple. Namely, there exists a finite subset $\mathrm{Irr}(\nabla) \subset M(X', D')/H(X')$ and a decomposition

$$\varphi^*(\mathcal{V}, \nabla)_{|\widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{\mathcal{V}}'_{\mathfrak{a}}, \widehat{\nabla}'_{\mathfrak{a}}),$$

such that each $\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}} := \widehat{\nabla}'_{\mathfrak{a}} - d\mathfrak{a}$ has regular singularity. We have the Deligne lattices $\widehat{V}'_{\mathfrak{a}}$ for meromorphic flat bundles with regular singularity $(\widehat{\mathcal{V}}'_{\mathfrak{a}}, \widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}})$, and we obtain the formal lattice

$$\bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} \widehat{V}'_{\mathfrak{a}} \subset \varphi^*(\mathcal{V})_{|\widehat{D}'}.$$

It determines the lattice $V' \subset \varphi^*\mathcal{V}$ with a decomposition

$$(V', \varphi^*\nabla)_{|\widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{V}'_{\mathfrak{a}}, \widehat{\nabla}'_{\mathfrak{a}}),$$

such that (i) $\widehat{\nabla}'_{\mathfrak{a}} - d\mathfrak{a}$ are logarithmic with respect to $\widehat{V}'_{\mathfrak{a}}$ for any $\mathfrak{a}$, (ii) any eigenvalues α of the residue satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$. Since V' is invariant under the action of the Galois group of this ramified covering, we obtain the lattice $V \subset \mathcal{V}$ as the descent of V'. This is the *Deligne-Malgrange lattice* in the one dimensional case.

1.3.1.3. Good Deligne-Malgrange lattice. — In the higher dimensional case, the existence of such a lattice was proved by B. Malgrange [**58**]. But, before recalling his result, we explain what is an ideal generalization in the higher dimensional case. (See Section 2.7.)

Remark 1.3.1. — We also mention the work due to Z. Mebkhout [**59**], [**60**] on the lattice of a meromorphic flat bundle possibly with regular singularity along a hypersurface which is not necessarily normal crossing, by using the results on extension of coherent sheaves ([**28**], [**88**], [**92**]). □

Let X be a complex manifold of arbitrary dimension with a normal crossing hypersurface D. Let V be a lattice of a meromorphic flat bundle $(\mathcal{V}, \nabla)$ on (X, D). We say that V is an *unramifiedly good Deligne-Malgrange lattice* if the following holds at each $P \in D$:

- Let X_P be a small neighbourhood of P in X. Let $I(P) := \{i \,|\, P \in D_i\}$. We set $D_P := X_P \cap D$ and $D_{I(P)} := X_P \cap \bigcap_{i \in I(P)} D_i$. Then, we have a finite subset $\mathrm{Irr}(\nabla, P) \subset M(X_P, D_P)/H(X_P)$ and a decomposition

$$(V, \nabla)_{|\widehat{D}_{I(P)}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla, P)} (\widehat{V}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$$

such that (i) $\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}$ are logarithmic, (ii) the eigenvalues α of the residues satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$. Precisely, we impose the condition that $\mathrm{Irr}(\nabla, P)$ is a good set of irregular values (Section 2.1).

We say that V is a *good Deligne-Malgrange lattice*, if the following holds for each $P \in D$:

- If we take an appropriate ramified covering $\varphi : (X'_P, D'_P) \to (X_P, D_P)$, there exists an unramifiedly good Deligne-Malgrange lattice V' of $\varphi^*(\mathcal{V}, \nabla)$, and $V_{|X_P}$ is the descent of V'.

They are uniquely determined, if they exist. In the one dimensional case, a Deligne-Malgrange lattice is always good in this sense. (See [**64**] for a different but equivalent definition.)

1.3.1.4. Existence theorem of Malgrange. — However, in general, a good Deligne-Malgrange lattice may not exist for a meromorphic flat bundle. Instead, Malgrange proved the following in [**58**]. (See Subsection 2.7.2.1 for a minor complement.)

Proposition 1.3.2. — *There exists an $\mathcal{O}_X$-reflexive lattice $V \subset \mathcal{V}$ and an analytic subset $Z \subset D$ with* $\mathrm{codim}_X(Z) \geqslant 2$ *such that $V_{|X \smallsetminus Z}$ is a good Deligne-Malgrange lattice of $(\mathcal{V}, \nabla)_{|X \smallsetminus Z}$.* □

Although he called it the canonical lattice, we would like to call it *Deligne-Malgrange lattice* in this paper. We have the minimum Z_0 among the closed subset Z as above. Each point of Z_0 is called a *turning point* for $(\mathcal{V}, \nabla)$.

1.3.1.5. Resolution of turning points. — The Deligne-Malgrange lattice is a very nice clue for the study of meromorphic flat bundles. For example, we will use it to obtain a Mehta-Ramanathan type theorem for simple meromorphic flat bundles, i.e., a meromorphic flat bundle is simple if and only if so is its restriction to sufficiently ample generic ample hypersurface. (Section 13.2). However, the existence of turning points is a serious obstacle for an asymptotic analysis of meromorphic flat bundles, as Sabbah observed in his study of Stokes structure of meromorphic flat bundles on complex surfaces. (We have already mentioned asymptotic analysis for meromorphic flat bundles in Section 1.2.1.) He proposed a conjecture to expect the existence of a resolution of turning points. We established it for algebraic meromorphic flat bundles on surfaces [**68**]. We will establish the following theorem in the higher dimensional case.

***Theorem 1.3.3* (Theorem 16.2.1, Corollary 16.4.4).** — *Let X be a smooth proper complex algebraic variety with a normal crossing hypersurface D. Let $(\mathcal{V}, \nabla)$ be a meromorphic flat connection on (X, D). Then, there exists a birational projective morphism $\varphi : X' \to X$ such that (i) $D' := \varphi^{-1}(D)$ is a simple normal crossing hypersurface, (ii) $X' \smallsetminus D' \simeq X \smallsetminus D$, (iii) $\varphi^*(\mathcal{V}, \nabla)$ has no turning points.* □

Theorem 1.3.3 seems of foundational importance in the study of algebraic meromorphic flat bundles or more generally, algebraic holonomic D-modules. Although it is argued in Chapter 16, it can be shown more shortly. Actually, it follows from some of the results in Chapters 2, 7, 8 and Part III. We will briefly discuss ideas of the proof in Subsection 1.3.2.3.

Remark 1.3.4. — Recently, K. Kedlaya showed a generalization with a completely different method in his excellent work [**46**], [**47**]. In particular, he established it for excellent schemes. In the complex analytic situation, he obtained a local result. □

1.3.2. Characterization of semisimplicity of meromorphic flat bundles

According to Corlette [**20**], a flat bundle on a *smooth projective variety* has a pluri-harmonic metric, if and only if it is a semisimple object in the category of flat bundles on X. Such a characterization was established for meromorphic flat bundles with regular singularity, by the work of Jost-Zuo and us ([**41**], [**67**] and [**69**]). We would like to generalize it in the irregular singular case. We need a preparation to state the theorem.

1.3.2.1. $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundles and good Deligne-Malgrange lattice

We explain how good wild harmonic bundles and good Deligne-Malgrange lattices are related. Although we explain the local and unramified case, it is easily generalized in the global and (possibly) ramified case.

Let $X := \Delta^n$ and $D := \bigcup_{j=1}^{\ell}\{z_j = 0\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X \setminus D$ with a decomposition as in (1). We say that $(E, \overline{\partial}_E, \theta, h)$ is an *unramifiedly $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle*, if

$$\mathcal{S}p(\theta) \subset (\sqrt{-1}\boldsymbol{R})^{\ell}.$$

We say that it is a *$\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle*, if it is the descent of an unramifiedly $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle. As mentioned, these definitions can be extended to the global and (possibly) ramified case. Then, good Deligne-Malgrange lattices naturally appear in the study of harmonic bundles by the following proposition, which immediately follows from Proposition 8.2.1.

***Proposition 1.3.5* (Proposition 16.2.6)**. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on (X, D). Then, $(\mathcal{P}_0\mathcal{E}^1, \mathbb{D}^1)$ is the good Deligne-Malgrange lattice of the meromorphic flat bundle $(\mathcal{P}\mathcal{E}^1, \mathbb{D}^1)$.* □

1.3.2.2. Characterization. — Let X be a smooth projective variety, and let D be a normal crossing hypersurface of X. Let $(\mathcal{V}, \nabla)$ be a meromorphic flat bundle on (X, D). Recall that there exists a closed subset $Z \subset D$ with $\operatorname{codim}_X(Z) \geqslant 2$ such that $(\mathcal{V}, \nabla)_{|X \setminus Z}$ has a good Deligne-Malgrange lattice, according to Proposition 1.3.2. The next theorem gives a nice characterization of semisimplicity of $(\mathcal{V}, \nabla)$.

***Theorem 1.3.6* (Theorem 16.2.4).** — *The following conditions are equivalent.*

- $(\mathcal{V}, \nabla)$ *is semisimple in the category of meromorphic flat bundles.*
- *There exists a* $\sqrt{-1}\boldsymbol{R}$*-good wild harmonic bundle* $(E, \overline{\partial}_E, \theta, h)$ *on* $(X \setminus Z, D \setminus Z)$ *such that* $\mathcal{PE}^1_{|X\setminus Z} \simeq \mathcal{V}_{|X\setminus Z}$.

Such a metric is unique up to obvious ambiguity. □

1.3.2.3. Outline of the proof of Theorems 1.3.3 and 1.3.6. — Our proof of the theorems are given by the following flow of the arguments, which is an interesting interaction between the theories of harmonic bundles and meromorphic flat bundles. (But, see Remarks 1.3.9 and 1.3.10 below.)

Step 0 : In the curve case, Theorem 1.3.3 is classical, and Theorem 1.3.6 was known by the work of Biquard-Boalch, Sabbah and Simpson.

Step 1 : We established Theorem 1.3.6 in the case $\dim X = 2$ by using the mod p-reduction method in [**68**].

Step 2 (Theorem 1.3.6 in the case $\dim X = 2$**) :** This step is the motivation for the author to study resolution of turning points. We would like to find a pluri-harmonic metric of $(\mathcal{V}, \nabla)_{|X\setminus D}$, for which there is a standard framework in global analysis. It is briefly and imprecisely as follows: (i) take an initial metric, (ii) deform it along the flow given by a heat equation, (iii) the limit of the heat flow should be a pluri-harmonic metric. Simpson [**81**] essentially established a nice general theory for (ii) and (iii), once an appropriate initial metric is taken in (i). To construct an initial metric, we have to know the local normal form of meromorphic flat bundles. It requires a resolution of turning points in Step 1.

We should remark that we cannot directly use the above framework, even if $(\mathcal{V}, \nabla)$ has no turning points. It will be achieved by the argument in [**66**] and [**69**] prepared for Kobayashi-Hitchin correspondence of meromorphic flat bundles with regular singularities.

Step 3 (Theorem 1.3.3 in the case $\dim X = n$ $(n \geqslant 3)$**) :** This is the easiest part. We have the following Mehta-Ramanathan type theorem.

***Proposition 1.3.7* (Proposition 13.2.1).** — $(\mathcal{V}, \nabla)$ *is simple if and only if* $(\mathcal{V}, \nabla)_{|Y}$ *is simple for an arbitrarily ample generic hypersurface* Y. □

The "if" part is clear, and the other side is non-trivial. This kind of claim is very standard for classical stability conditions in algebraic geometry. The Deligne-Malgrange lattice is equipped with the natural parabolic structure, and Sabbah essentially observed that simplicity and parabolic stability are equivalent. Then, applying the arguments due to Mehta and Ramanathan ([**61**] and [**62**]) we will obtain the desired equivalence.

Then, the inductive argument is easy. For any general and sufficiently ample hypersurface $Y \subset X$, there exists a pluri-harmonic metric h_Y for $(\mathcal{V}, \nabla)_{|Y}$.

There exists a finite subset $Z \subset X$ such that $X \setminus Z$ is covered by such hypersurfaces Y. So, for $P \in X \setminus Z$, take Y such as $P \in Y$, and we would like to define $h_{|P} := h_{Y|P}$. We have to check $h_{Y_1|Y_1 \cap Y_2} = h_{Y_2|Y_1 \cap Y_2}$, but it follows from the uniqueness because $\dim(Y_1 \cap Y_2) \geqslant 1$. Thus, we obtain the desired metric.

Step 4 (Theorem 1.3.8 in the case $\dim X = n$ $(n \geqslant 3)$) : It can be observed that we only have to consider the case where $(\mathcal{V}, \nabla)$ is simple (Corollary 2.7.11). After Step 3, we take a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $(X \setminus Z, D \setminus Z)$ as in Theorem 1.3.6. If $(E, \overline{\partial}_E, \theta, h)$ is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on (X, D), we have the good Deligne-Malgrange lattice and $\mathcal{P}_0\mathcal{E}^1$ of $(\mathcal{P}\mathcal{E}^1, \mathbb{D}^1)$ on X. Because $\operatorname{codim}_X(Z) \geqslant 2$ and $\mathcal{P}\mathcal{E}^1_{|X \setminus Z} \simeq \mathcal{V}_{|X \setminus Z}$, we have the isomorphism $\mathcal{P}\mathcal{E}^1 \simeq \mathcal{V}$ on X, and $\mathcal{P}_0\mathcal{E}^1$ is the good Deligne-Malgrange lattice of $\mathcal{V}$. Hence, if (E, ∇, h) is $\sqrt{-1}\boldsymbol{R}$-good wild on (X, D), we have nothing to do.

Of course, in general, $(E, \overline{\partial}_E, \theta, h)$ is not $\sqrt{-1}\boldsymbol{R}$-good wild on (X, D). However, we have replaced the problem with the control of the eigenvalues of the Higgs field θ, for which we can use classical techniques in algebraic or complex geometry. (See Section 13.5.) It is much easier than the control of irregular values of meromorphic flat bundles, and it can be done. (See Section 15.3.)

1.3.3. Kobayashi-Hitchin correspondence for wild harmonic bundles

We also have a subject related to the characterization of stability of good filtered flat bundles. Let X be a connected smooth projective variety of dimension n with an ample line bundle L, and let D be a simple normal crossing hypersurface. If we are given a good wild harmonic bundle (E, ∇, h) on $X \setminus D$, we obtain the filtered flat bundle $(\mathcal{P}_*\mathcal{E}^1, \mathbb{D}^1)$ as in Theorem 7.4.3. We can show that it is μ_L-polystable, and each stable component has the trivial characteristic numbers. Conversely, we can show the following.

Theorem 1.3.8 **(Theorem 16.1.1).** — *Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (X, D). We put $(E, \nabla) := (\boldsymbol{E}_*, \nabla)_{|X \setminus D}$. If $(\boldsymbol{E}_*, \nabla)$ is a μ_L-stable good filtered flat bundle on (X, D) with trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, there exists a pluri-harmonic metric h of (E, ∇) with the following properties:*

- *(E, ∇, h) is a good wild harmonic bundle on $X \setminus D$.*
- *h is adapted to the parabolic structure of $\boldsymbol{E}_*$.*

Such a pluri-harmonic metric is unique up to obvious ambiguity. □

Remark 1.3.9. — If we know Theorem 1.3.3, it is not difficult to deduce Theorem 1.3.6 from Theorem 1.3.8 as in the tame case, which will be used in the surface case. However, we use Theorem 1.3.6 to show Theorem 1.3.3 in the case $\dim X \geqslant 3$, and so we need some different argument. □

Remark 1.3.10. — As already mentioned several times, after the submission of this monograph, Kedlaya [**47**] obtained the higher dimensional version of the resolution of turning points with a different argument. But, we keep our original flow of the argument. □

1.4. Application to holonomic D-modules and wild pure twistor D-modules

1.4.1. A conjecture of Kashiwara on algebraic holonomic D-modules

Let $\mathcal{M}$ be an algebraic holonomic D_X-module on a smooth complex algebraic variety X. Let us recall some operations on D-modules.

(Push-forward) : Let $F : X \to Y$ be a projective morphism of smooth complex algebraic varieties:

- We have the push-forward $F_\dagger\mathcal{M}$ in the derived category of cohomologically holonomic D_Y-modules. The i-th cohomology sheaves are denoted by $F_\dagger^i\mathcal{M}$, which are algebraic holonomic D_Y-modules.
- We have the Lefschetz morphism $c_1(L) : F_\dagger^i\mathcal{M} \to F_\dagger^{i+2}\mathcal{M}$ for any line bundle L on X.

(Nearby cycle and vanishing cycle functors) : Let $g : X \to \boldsymbol{C}$ be an algebraic function. By applying the nearby cycle functor and the vanishing cycle functor, we obtain algebraic holonomic D_X-modules $\psi_g(\mathcal{M})$ and $\phi_g(\mathcal{M})$. They are equipped with the induced nilpotent maps N. By taking Gr with respect to the weight filtrations $W(N)$, we obtain algebraic holonomic D_X-modules $\mathrm{Gr}^{W(N)}\psi_g(\mathcal{M})$ and $\mathrm{Gr}^{W(N)}\phi_g(\mathcal{M})$.

More generally, according to P. Deligne, for any $n \in \mathbb{Z}_{>0}$ and $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$, we obtain an algebraic holonomic D_X-module $\psi_{g,\mathfrak{a}}(\mathcal{M})$ by applying the nearby cycle functor with ramified exponential twist by $\mathfrak{a}$. (See Section 22.6.3. The author learned this idea from Sabbah.) We also obtain a holonomic D_X-module $\mathrm{Gr}^{W(N)}\psi_{g,\mathfrak{a}}(\mathcal{M})$ by taking Gr with respect to the weight filtration of the induced nilpotent map.

There are several works to find an abelian subcategory $\mathcal{C}$ of the category of algebraic holonomic D-modules with the following properties:

- $\mathcal{O}_X \in \mathcal{C}$ for any smooth quasi-projective variety X.
- $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathcal{C}$ if and only if $\mathcal{M}_i \in \mathcal{C}$.
- $\mathcal{C}$ is stable under push-forward for any projective morphism $F : X \to Y$. Moreover, Hard Lefschetz theorem holds for $\mathcal{C}$ in the sense that $c_1(L)^i : F_\dagger^{-i}\mathcal{M} \to F_\dagger^i\mathcal{M}$ are isomorphisms for any $i \geqslant 0$, any projective morphism F, any relatively ample line bundle L, and any $\mathcal{M} \in \mathcal{C}$.
- $\mathcal{C}$ is stable under the functors $\mathrm{Gr}^W\psi_g$ and $\mathrm{Gr}^W\phi_g$ for any function g.

For example, the category of (regular) holonomic D-modules is stable for the functors $F_\dagger^i$, $\mathrm{Gr}^W\psi_g$ and $\mathrm{Gr}^W\phi_g$. However, the Hard Lefschetz theorem does not hold in general. In their pioneering work [**7**], A. Beilinson, J. Bernstein, P. Deligne and O. Gabber showed the existence of such a subcategory called *geometric origin* by using the technique of the reduction to positive characteristic, which is the minimum among the subcategories with the above property. It is one of the main motivations for this study to show the following theorem.

***Theorem 1.4.1* (A conjecture of Kashiwara, Theorem 19.4.2).** — *The category of algebraic semisimple holonomic D-modules has the above property. Namely, let X be a smooth complex algebraic variety, $\mathcal{M}$ be an algebraic semisimple holonomic D_X-module. Then, the following holds:*

- *Let $F : X \to Y$ be a projective morphism of smooth quasi-projective varieties. Then, $F_\dagger^j(\mathcal{M})$ are also semisimple for any j, and the morphisms $c_1(L)^j : F_\dagger^{-j}\mathcal{M} \to F_\dagger^j\mathcal{M}$ are isomorphisms for any $j \geqslant 0$ and any relatively ample line bundle L. In particular, $F_\dagger\mathcal{M}$ is isomorphic to $\bigoplus F_\dagger^i(\mathcal{M})[-i]$ in the derived category of cohomologically holonomic D_Y-modules.*
- *Let g be an algebraic function on X, and let $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$. Then, $\mathrm{Gr}^W\,\psi_{g,\mathfrak{a}}(\mathcal{M})$ and $\mathrm{Gr}^W\,\phi_g(\mathcal{M})$ are also semisimple.* □

The study of this kind of property of D-modules or perverse sheaves was invented by Beilinson-Bernstein-Deligne-Gabber, which we have already mentioned above. M. Saito proved the property for the category of D-modules underlying polarizable pure Hodge modules in his celebrated work [**77**]. M. Kashiwara conjectured [**42**] that the category of algebraic semisimple holonomic D-modules has the above property. Sabbah proved in [**73**] the property for regular holonomic D-modules underlying $\sqrt{-1}\boldsymbol{R}$-regular polarizable pure twistor D-modules. Simpson [**84**] also suggested such a line of the study. In [**67**], we established the correspondence between algebraic semisimple regular holonomic D-modules and $\sqrt{-1}\boldsymbol{R}$-regular polarizable pure twistor D-modules, and hence the property was proved for algebraic semisimple regular holonomic D-modules. It was also established by the works of V. Drinfeld, G. Boeckle-C. Khare and D. Gaitsgory ([**26**], [**15**], [**31**]) via the method of arithmetic geometry based on the work of L. Lafforgue. And M. de Cataldo-L. Migliorini [**19**] gave another proof of the original result of Beilinson-Bernstein-Deligne-Gabber by using their own Hodge theoretic method but without Saito's method.

Remark 1.4.2. — In contrast to the previous results, regularity is not assumed in Theorem 1.4.1. □

1.4.2. Polarized wild pure twistor D-module.

— Recall that the notion of a *harmonic bundle* is suitable for the study on *semisimplicity* of flat bundles or D-modules from the beginning by Corlette's work. (Recall also Theorem 1.3.6.) Then, a natural strategy to attack Theorem 1.4.1 is the following, which we call Sabbah's program:

- Introduce the category of "holonomic D-modules with pluri-harmonic metrics" which should have the property in Section 1.4.1.
- Show the functorial correspondence between "holonomic D-modules with pluri-harmonic metrics" and algebraic semisimple holonomic D-modules.

Sabbah introduced the notion of polarized wild pure twistor D-modules as "holonomic D-modules with pluri-harmonic metrics". We refer to [**73**] and [**75**] for the precise definition and the basic properties. (We will review it in Section 17.1 with a preparation given in Chapter 22.) Needless to say, it is not obvious at all how to think "pluri-harmonic metrics" for D-modules.

A very important hint was given by Simpson [**85**]. From the beginning of his study [**81**], he was motivated by the similarity between harmonic bundles and variations of polarized pure Hodge structure. In [**85**], he introduced the notion of *mixed twistor structure*, and he gave a new formulation of harmonic bundle as *variation of polarized pure twistor structure*, which is formally parallel to the definition of variation of polarized pure Hodge structure. It makes it possible for us to formulate "the harmonic bundle version" (or "twistor version") of most objects in the theory of variation of Hodge structure. And he proposed a principle, that we called Simpson's Meta-Theorem: the theory of Hodge structure should be generalized to the theory of twistor structure.

In his highly original work ([**77**] and [**78**]), Saito introduced the notion of polarized pure Hodge modules as a vast generalization of variation of polarized pure Hodge structure, and he showed that the category of polarized pure Hodge modules has nice properties such as Hard Lefschetz theorem. It is natural to expect that we can define "holonomic D-modules with pluri-harmonic metrics" as the twistor version of polarized pure Hodge modules.

And it was done by Sabbah. Note that it was still a hard work. We should emphasize that he made various useful innovations and observations such as sesquilinear pairings, their specialization by using Mellin transforms ([**5**] and [**6**]), the nearby cycle functor with ramified exponential twist for $\mathcal{R}$-triples, and so on.

1.4.3. Correspondences. — One of the main purposes of our study is to establish the relation between algebraic semisimple holonomic D-modules and polarizable wild pure twistor D-modules through wild harmonic bundles:

(7)

semisimple algebraic holonomic D-module	$\leftrightarrow$	$\sqrt{-1}\boldsymbol{R}$-wild harmonic bundle	$\leftrightarrow$	polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module

1.4.3.1. Wild harmonic bundle and polarized wild pure twistor D-module. — We said that polarized wild pure twistor D-modules were "holonomic D-modules with pluri-harmonic metrics", as a heuristic explanation. We make it rigorous by the next theorem. For simplicity, we consider the case where X is a smooth projective variety. Let Z be a closed irreducible subvariety of X.

***Theorem 1.4.3* (Theorem 19.1.3).** — *We have a natural equivalence of the categories of the following objects:*

- *Polarized wild pure twistor D-modules whose strict supports are Z.*
- *Wild harmonic bundles defined on Zariski open subsets of Z.* □

Theorem 1.4.3 is not only one of the most important key points in the proof of Theorem 1.4.1 as mentioned in the next subsection, but it also makes it possible for us to consider "push-forward" of wild harmonic bundles in some sense. In other words, the push-forward for holonomic D-modules is enriched by polarized pure twistor structures. It might be useful to investigate the property of morphisms between moduli spaces of flat bundles induced by push-forward. For example, the study of polarized wild pure twistor D-modules is related with Fourier transform ([**57**], [**74**], [**76**]) or Nahm transforms ([**2**], [**40**], [**90**]) for meromorphic flat bundles or wild harmonic bundles on $\mathbb{P}^1$. See also [**87**]. In principle, they should be the specialization of the corresponding transforms of polarizable wild pure twistor D-modules, which could be useful for the study of the corresponding morphisms of the moduli spaces.

The proof of Theorem 1.4.3 briefly consists of three parts.

- We have to prolong wild harmonic bundles $(E, \overline{\partial}_E, \theta, h)$ on a Zariski open subset U of Z to polarized wild pure twistor D-modules on Z. The most essential case is that $X = Z = \Delta^n$, $D := X \setminus U = \bigcup_{i=1}^{\ell}\{z_i = 0\}$, and $(E, \overline{\partial}_E, \theta, h)$ is an unramifiedly good wild harmonic bundle on (X, D). Since the family of meromorphic λ-flat bundles $(\mathcal{QE}, \mathbb{D})$ in Theorem 1.2.4 is too large, we replace it with the "minimal extension". We need the detailed study on the specialization along a function on X. In the tame case, it was done in [**67**]. The wild case is essentially reduced to the tame case by using Theorem 1.2.7.
- For a given polarized wild pure twistor D-module $\mathcal{T}$ whose strict support is Z, it is not difficult to show the existence of a Zariski open subset $U \subset Z$ such that $\mathcal{T}_{|X-(Z\setminus U)}$ comes from a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on U. However, we have to show that $(E, \overline{\partial}_E, \theta, h)$ is a wild harmonic bundle. For that purpose, we need various preliminaries such as resolution of turning points for meromorphic Higgs field (Section 15.3), curve test for wild harmonic bundles (Section 13.5) and so on.
- We have to show the uniqueness of prolongation of wild harmonic bundles to polarized wild pure twistor D-modules. In the tame case, this is rather trivial. (Recall that a flat bundle is uniquely extended to a meromorphic flat bundles with regular singularity.) However, in the wild case, it is not obvious. It essentially follows from Theorem 1.2.8.

1.4.3.2. Semisimple holonomic D-modules and polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules. — Before stating the next theorem, we need some preparations.

- Recall that there exists the subclass of $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundles (Section 1.3.2). We have the corresponding subcategory of *polarized $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules.*

- A polarized wild pure twistor D-module is precisely a *wild pure twistor D-module with a polarization.* There is an obvious ambiguity in the choice of a polarization, as there exists an obvious ambiguity in the choice of a pluri-harmonic metric for a harmonic bundle. A wild pure twistor D-module is called a *polarizable wild pure twistor D-module*, if it has a polarization.

For a polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module $\mathcal{T}$, let $\Xi_{DR}(\mathcal{T})$ denote the underlying holonomic D-module. The next theorem means the correspondence (7). It essentially follows from Theorem 1.4.3 and Theorem 1.3.6.

***Theorem 1.4.4* (Theorem 19.4.1).** — *$\Xi_{DR}(\mathcal{T})$ is semisimple for any polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module $\mathcal{T}$. Moreover, the functor Ξ_{DR} gives an equivalence of the categories of polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules and semisimple holonomic D-modules on X.* □

By transferring the operations for polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules, we obtain that the category of semisimple holonomic D-modules has the desired properties considered in Section 1.4.1. It completes the second part of Sabbah's program.

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I started to study wild harmonic bundles during my stay at Max-Planck Institute for Mathematics in 2005–2006. I spent a hard time of trial and error, and got most of the essential ideas during my stay at Institut des Hautes Études Scientifiques in 2006–2007. Then, I wrote this manuscript at Kyoto University, Department of Mathematics and Research Institute for Mathematical Sciences. The study on wild harmonic bundles is heavily based on the study on tame harmonic bundles, a part of which was done in Osaka City University and Institute for Advance Study. I express my gratitude to the institutions and the colleagues for the excellent mathematical environments and the supports. I also thank the partial financial supports by Sasakawa Foundation, Sumitomo Foundation, and Ministry of Education, Culture, Sports, Science and Technology.

PART I

GOOD MEROMORPHIC ϱ-FLAT BUNDLES

CHAPTER 2

GOOD FORMAL PROPERTY OF A MEROMORPHIC ϱ-FLAT BUNDLE

In this chapter, we shall study the property for a meromorphic ϱ-flat bundle to be formally good.

First two sections 2.1 and 2.2 are preliminary. We recall the notion of good set of irregular values in Section 2.1. We study unramifiedly good lattice for meromorphic formal ϱ-flat bundle in Section 2.2. (See [**46**] for deeper results.) In Section 2.3, we introduce the notion of good lattice of a meromorphic ϱ-flat bundle. It is defined as a lattice whose completions at all points have nice properties. We hope that the completion along a divisor has a nice property, which is studied in Section 2.4. In Section 2.5, we introduce the notion of good filtered ϱ-flat bundle, which will play an important role in the study of wild harmonic bundles. In Section 2.6, we introduce the notion of good lattice at the level $\boldsymbol{m}$. It seems useful for our study of unramifiedly good lattices for which we use inductive arguments on the level. In Section 2.7, we restrict ourselves to ordinary meromorphic flat bundles, and we study good Deligne-Malgrange lattices. In Section 2.8, we prepare some terminology for families of filtered λ-flat bundles, which is significant in our study on wild harmonic bundles.

2.1. Good set of irregular values and truncations

2.1.1. Definition

2.1.1.1. The partial order on $\mathbb{Z}^n$. — We use the partial order $\leqslant_{\mathbb{Z}^n}$ (or simply denoted by $\leqslant$) of $\mathbb{Z}^n$ given by the comparison of each component, i.e., $\boldsymbol{a} \leqslant_{\mathbb{Z}^n} \boldsymbol{b} \Longleftrightarrow a_i \leqslant b_i$, $(\forall\, i)$. Let $\mathbf{0}$ denote the zero in $\mathbb{Z}^n$. It is also denoted by $\mathbf{0}_n$ when we distinguish the dependence on n.

2.1.1.2. Order of poles of meromorphic functions. — Let Δ^ℓ denote the multi-disc

$$\big\{(z_1,\dots,z_\ell)\,\big|\,|z_i|<1,\ i=1,\dots,\ell\big\}.$$

Let Y be a complex manifold. Let $X := \Delta^\ell \times Y$. Let $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^\ell D_i$ be hypersurfaces of X. We also put $D_{\underline{\ell}} = \bigcap_{i=1}^\ell D_i$, which is naturally identified with Y. Let $M(X,D)$ (resp. $H(X)$) denote the space of meromorphic (resp. holomorphic) functions on X whose poles are contained in D. For $\boldsymbol{m} = (m_1,\ldots,m_\ell) \in \mathbb{Z}^\ell$, we put $\boldsymbol{z}^{\boldsymbol{m}} := \prod_{i=1}^\ell z_i^{m_i}$. For any $f \in M(X,D)$, we have the Laurent expansion:

$$f = \sum_{\boldsymbol{m}\in\mathbb{Z}^\ell} f_{\boldsymbol{m}}(y)\,\boldsymbol{z}^{\boldsymbol{m}}.$$

Here $f_{\boldsymbol{m}}$ are holomorphic functions on $D_{\underline{\ell}}$. We will often use the following natural identification without mention:

$$M(X,D)/\boldsymbol{z}^{\boldsymbol{n}}\,H(X) \simeq \Big\{f \in M(X,D)\,\Big|\, f_{\boldsymbol{m}} = 0,\ \forall \boldsymbol{m} \geqslant \boldsymbol{n}\Big\}.$$

Namely, we will often regard an element of $M(X,D)/\boldsymbol{z}^{\boldsymbol{n}}\,H(X)$ as an element of $M(X,D)$ via the above identification.

For $f \in M(X,D)$, let $\mathrm{ord}(f)$ denote the minimum of the set

$$\mathcal{S}(f) := \big\{\boldsymbol{m} \in \mathbb{Z}^\ell \,\big|\, f_{\boldsymbol{m}} \neq 0\big\} \cup \{\boldsymbol{0}_\ell\},$$

if it exists. Note that we are interested in the order of poles, and that $\mathrm{ord}(f)$ is always contained in $\mathbb{Z}^\ell_{\leqslant 0}$ according to this definition (if it exists). We give some examples.

- In the case $f = 0$, we have $\mathcal{S}(f) = \{\boldsymbol{0}\}$, and hence $\mathrm{ord}(f) = \boldsymbol{0}$. More generally, for any holomorphic function f, we have $\mathrm{ord}(f) = \boldsymbol{0}$.
- In the case $f = z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}$, $\mathcal{S}(f)$ is $\{(-1,-1),(-1,0),(0,-1),\boldsymbol{0}\}$, and hence $\mathrm{ord}(f) = (-1,-1)$.
- In the case $f = z_1 z_2^{-1}$, we have $\mathcal{S}(f) = \{\boldsymbol{0},(1,-1)\}$, and hence $\mathrm{ord}(f)$ does not exist.
- In the case $f = z_1^{-1} + z_2^{-1}$, we have $\mathcal{S}(f) = \{(-1,0),(0,-1),\boldsymbol{0}\}$, and hence $\mathrm{ord}(f)$ does not exist.

For any $\mathfrak{a} \in M(X,D)/H(X)$, we take any lift $\widetilde{\mathfrak{a}}$ to $M(X,D)$, and we set $\mathrm{ord}(\mathfrak{a}) := \mathrm{ord}(\widetilde{\mathfrak{a}})$, if the right-hand side exists. Note that it is independent of the choice of a lift $\widetilde{\mathfrak{a}}$. If $\mathrm{ord}(\mathfrak{a}) \neq 0$, $\widetilde{\mathfrak{a}}_{\mathrm{ord}(\mathfrak{a})}$ is independent of the choice of a lift $\widetilde{\mathfrak{a}}$, which is denoted by $\mathfrak{a}_{\mathrm{ord}(\mathfrak{a})}$.

Remark 2.1.1. — Let k be a ring. The above notion of order makes sense for the localization of $k[\![z_1,\ldots,z_n]\!]$ with respect to z_i $(i = 1,\ldots,\ell)$. We will not give this kind of remark in the following. □

2.1.1.3. Good set of irregular values. — We introduce the notion of good set of irregular values, which will be used as index sets of irregular decompositions and Stokes filtrations.

Definition 2.1.2. — A finite subset $\mathcal{I} \subset M(X,D)/H(X)$ is called a good set of irregular values on (X,D), if the following conditions are satisfied:

- $\operatorname{ord}(\mathfrak{a})$ exists for each element $\mathfrak{a} \in \mathcal{I}$. If $\mathfrak{a} \neq 0$ in $M(X,D)/H(X)$, $\mathfrak{a}_{\operatorname{ord}(\mathfrak{a})}$ is invertible on $D_{\underline{\ell}}$.
- $\operatorname{ord}(\mathfrak{a}-\mathfrak{b})$ exists for any two distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, and $(\mathfrak{a}-\mathfrak{b})_{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}$ is invertible on $D_{\underline{\ell}}$.
- The set $\{\operatorname{ord}(\mathfrak{a}-\mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in \mathcal{I}\}$ is totally ordered with respect to the partial order $\leqslant_{\mathbb{Z}^\ell}$ on $\mathbb{Z}^\ell$. □

The third condition is slightly stronger than that considered in [**72**], which seems convenient for our inductive argument on levels.

Remark 2.1.3. — The condition in Definition 2.1.2 does not depend on the choice of a holomorphic coordinate system such that $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. □

Remark 2.1.4. — We will often use a coordinate system such that $\operatorname{ord}(\mathfrak{a}-\mathfrak{b})$ and $\operatorname{ord}(\mathfrak{a})$ are contained in the set $\coprod_{i=0}^{\ell} \mathbb{Z}^i_{<0} \times \mathbf{0}_{\ell-i}$ for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. Such a coordinate system is called admissible for $\mathcal{I}$. □

2.1.1.4. Examples. — The set $\mathcal{I}_0 := \{z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}\}$ is a good set of irregular values. The order of pole is given by $\operatorname{ord}(z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}) = (-1,-1)$. Let us consider the following examples:

$$\mathcal{I}_1 := \{z_1^{-1}z_2^{-1}, z_1^{-1}, 0\}, \quad \mathcal{I}_2 := \{z_1^{-1}z_2^{-1}, z_1^{-1}, z_2^{-1}, 0\}.$$

Then, $\mathcal{I}_1$ is a good set of irregular values. The orders of poles are given as follows:

$$\operatorname{ord}(z_1^{-1}z_2^{-1}) = (-1,-1), \quad \operatorname{ord}(z_1^{-1}) = (-1,0), \quad \operatorname{ord}(0) = (0,0),$$
$$\operatorname{ord}(z_1^{-1} - 0) = (-1,0), \quad \operatorname{ord}(z_1^{-1}z_2^{-1} - 0) = (-1,-1),$$
$$\operatorname{ord}(z_1^{-1}z_2^{-1} - z_1^{-1}) = (-1,-1).$$

However $\mathcal{I}_2$ is not. Actually, $\operatorname{ord}(z_1^{-1} - z_2^{-1})$ does not exist.

We consider the following examples:

$$\mathcal{I}_3 := \{z_1^{-1}z_2^{-1} + z_1^{-1},\ z_1^{-1}z_2^{-1}\}, \quad \mathcal{I}_4 := \{z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1},\ z_1^{-1}z_2^{-1}\}.$$

Then, $\mathcal{I}_3$ is a good set of irregular values. The orders of poles are given as follows:

$$\operatorname{ord}(z_1^{-1}z_2^{-1} + z_1^{-1}) = (-1,-1), \quad \operatorname{ord}(z_1^{-1}z_2^{-1}) = (-1,-1),$$
$$\operatorname{ord}\big((z_1^{-1}z_2^{-1} + z_1^{-1}) - z_1^{-1}z_2^{-1}\big) = (-1,0).$$

However, $\mathcal{I}_4$ is not. Actually, $\operatorname{ord}\big((z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}) - z_1^{-1}z_2^{-1}\big)$ does not exist.

The property for a set of irregular values to be good is not preserved by some canonical constructions. For example, let us consider $\mathcal{I} = \{\mathfrak{a}_i \mid i = 1,2,3,4\}$ given as follows:

$$\mathfrak{a}_1 = z_1^{-1}, \quad \mathfrak{a}_2 = 2z_1^{-1}, \quad \mathfrak{a}_3 = 3z_1^{-1}(1+z_2), \quad \mathfrak{a}_4 = 4z_1^{-1}(1+z_2).$$

Then, $\mathcal{I} \otimes \mathcal{I}_i^\vee := \{\mathfrak{a}_i - \mathfrak{a}_j \mid i,j = 1,2,3,4\}$ is not necessarily good. Actually,

$$(\mathfrak{a}_3 - \mathfrak{a}_4) - (\mathfrak{a}_1 - \mathfrak{a}_2) = z_2 z_1^{-1}.$$

2.1.2. Auxiliary sequence. — Let $\mathcal{I}$ be a good set of irregular values on (X, D). Note that the set $\{\mathrm{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$ is totally ordered, because $\mathrm{ord}(\mathfrak{a}) \not\leqslant \mathrm{ord}(\mathfrak{b})$ and $\mathrm{ord}(\mathfrak{a}) \not\geqslant \mathrm{ord}(\mathfrak{b})$ imply that $\mathrm{ord}(\mathfrak{a} - \mathfrak{b})$ does not exist. We set $\boldsymbol{m}(0) := \min\{\mathrm{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$. We have the set $\mathcal{T}(\mathcal{I}) := \{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in \mathcal{I}\}$ contained in $\mathbb{Z}^{\ell}_{\leqslant 0}$. Note $\boldsymbol{m}(0) \leqslant_{\mathbb{Z}^{\ell}} \boldsymbol{m}$ for any $\boldsymbol{m} \in \mathcal{T}(\mathcal{I})$, because $\mathfrak{a}_{\boldsymbol{m}} \neq 0$ for some $\mathfrak{a} \in \mathcal{I}$. Since $\mathcal{T}(\mathcal{I})$ is assumed to be totally ordered with respect to the partial order $\leqslant_{\mathbb{Z}^{\ell}}$, we can take a sequence $\mathcal{M} := \big(\boldsymbol{m}(0), \boldsymbol{m}(1), \boldsymbol{m}(2), \dots, \boldsymbol{m}(L), \boldsymbol{m}(L+1)\big)$ in $\mathbb{Z}^{\ell}_{\leqslant 0}$ with the following properties:

- $\mathcal{T}(\mathcal{I}) \subset \mathcal{M}$ and $\boldsymbol{m}(L+1) = \boldsymbol{0}_{\ell}$.
- For each $i \leqslant L$, there exists $1 \leqslant \mathfrak{h}(i) \leqslant \ell$ such that $\boldsymbol{m}(i+1) = \boldsymbol{m}(i) + \boldsymbol{\delta}_{\mathfrak{h}(i)}$, where $\boldsymbol{\delta}_j := (\overbrace{0, \dots, 0}^{j-1}, 1, 0, \dots, 0)$,

Such a sequence is called an auxiliary sequence for $\mathcal{I}$. It is not uniquely determined by $\mathcal{I}$. We often omit to mention $\boldsymbol{m}(L+1)$, because it is fixed to be $\boldsymbol{0}$. Auxiliary sequences are convenient for some of our inductive arguments.

Remark 2.1.5. — In the case where D is smooth, i.e., $\ell = 1$, the auxiliary sequence is canonically determined. We have $m(0) := \min\{\mathrm{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$, and $m(j) := m(0) + j$. In this case, we prefer to use the orders $\mathrm{ord}(\mathfrak{a})$ directly. □

2.1.2.1. Example. — In the example in Section 2.1.1,

$$\mathcal{T}(\mathcal{I}_0) = \{\boldsymbol{0}\}, \quad \mathcal{T}(\mathcal{I}_1) = \{(-1,-1), (-1,0), \boldsymbol{0}\}, \quad \mathcal{T}(\mathcal{I}_3) = \{(-1,0), \boldsymbol{0}\}.$$

Hence, $\mathcal{M} = \{(-1,-1), (-1,0), (0,0)\}$ is an auxiliary sequence for them. Note that $\mathcal{M}' = \{(-1,-1), (0,-1), (0,0)\}$ is also an auxiliary sequence for $\mathcal{I}_0$, but not for $\mathcal{I}_i$ $(i = 1, 3)$.

2.1.3. Truncation. — For any $\boldsymbol{m} \in \mathbb{Z}^{\ell}_{\leqslant 0}$, let $\eta_{\boldsymbol{m}} : M(X, D) \to M(X, D)$ be given as follows:

$$\eta_{\boldsymbol{m}}(\mathfrak{a}) := \sum_{\boldsymbol{n} \leqslant \boldsymbol{m}} \mathfrak{a}_{\boldsymbol{n}}\, \boldsymbol{z}^{\boldsymbol{n}}. \tag{8}$$

Let $\xi_{\boldsymbol{m}} : M(X, D) \to M(X, D)$ be given as follows:

$$\xi_{\boldsymbol{m}}(\mathfrak{a}) := \sum_{\boldsymbol{n} \not\geqslant \boldsymbol{m}} \mathfrak{a}_{\boldsymbol{n}}\, \boldsymbol{z}^{\boldsymbol{n}}. \tag{9}$$

The induced maps $M(X, D)/H(X) \to M(X, D)/H(X)$ are also denoted by $\eta_{\boldsymbol{m}}$ and $\xi_{\boldsymbol{m}}$.

Let $\mathcal{I}$ be a good set of irregular values on (X, D). We take an auxiliary sequence $\mathcal{M} = \big(\boldsymbol{m}(0), \boldsymbol{m}(1), \dots, \boldsymbol{m}(L+1)\big)$ for $\mathcal{I}$. The function $\boldsymbol{z}^{-\boldsymbol{m}(i)} \big(\xi_{\boldsymbol{m}(i+1)}(\mathfrak{a}) - \xi_{\boldsymbol{m}(i)}(\mathfrak{a})\big)$ is holomorphic on X, and it is independent of the variable $z_{\mathfrak{h}(i)}$. We define

$$\zeta_{\boldsymbol{m}(i)}(\mathfrak{a}) := \xi_{\boldsymbol{m}(i+1)}(\mathfrak{a}) - \xi_{\boldsymbol{m}(i)}(\mathfrak{a}). \tag{10}$$

By construction, we have $\xi_{\boldsymbol{m}(i)}(\mathfrak{a}) = \sum_{j<i} \zeta_{\boldsymbol{m}(j)}(\mathfrak{a})$.

Lemma 2.1.6. — *For* $\mathfrak{a}_j \in \mathcal{I}$ $(j = 1, 2)$*, the equality* $\eta_{\boldsymbol{m}(i)}(\mathfrak{a}_1) = \eta_{\boldsymbol{m}(i)}(\mathfrak{a}_2)$ *implies* $\xi_{\boldsymbol{m}(i+1)}(\mathfrak{a}_1) = \xi_{\boldsymbol{m}(i+1)}(\mathfrak{a}_2)$*, and hence* $\zeta_{\boldsymbol{m}(i)}(\mathfrak{a}_1) = \zeta_{\boldsymbol{m}(i)}(\mathfrak{a}_2)$*. In particular,* $\xi_{\boldsymbol{m}(i+1)}(\mathfrak{b})$ *and* $\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})$ *are well defined for* $\mathfrak{b} \in \eta_{\boldsymbol{m}(i)}(\mathcal{I})$.

Proof. — Because $\eta_{\boldsymbol{m}(i)}(\mathfrak{a}_1) = \eta_{\boldsymbol{m}(i)}(\mathfrak{a}_2)$, we have $\operatorname{ord}(\mathfrak{a}_1 - \mathfrak{a}_2) \geqslant \boldsymbol{m}(i+1)$. Hence, we have $\mathfrak{a}_{1\,\boldsymbol{n}} = \mathfrak{a}_{2\,\boldsymbol{n}}$ for any $\boldsymbol{n} \not\geqslant \boldsymbol{m}(i+1)$, which implies the claim of the lemma. □

When we are given an auxiliary sequence, it will often be convenient to use the following symbol for $\mathfrak{a} \in \mathcal{I}$:

$$\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{a}) := \xi_{\boldsymbol{m}(i+1)}(\mathfrak{a}). \tag{11}$$

Note $\overline{\eta}_{\boldsymbol{m}(L)}(\mathfrak{a}) = \mathfrak{a}$ in $M(X,D)/H(X)$ for any $\mathfrak{a} \in \mathcal{I}$. We have the decomposition $\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{a}) = \sum_{j \leqslant i} \zeta_{\boldsymbol{m}(j)}(\mathfrak{a})$. The set $\mathcal{I}(\boldsymbol{m}(i)) := \overline{\eta}_{\boldsymbol{m}(i)}(\mathcal{I})$ is called the truncation of $\mathcal{I}$ at the level $\boldsymbol{m}(i)$. It is also a good set of irregular values. We should remark that $\overline{\eta}_{\boldsymbol{m}(i)}$ and the set $\mathcal{I}(\boldsymbol{m}(i))$ depend on the choice of an auxiliary sequence, in general.

Remark 2.1.7. — In the case where D is smooth, we have $\overline{\eta}_p = \eta_p$ for $p \in \mathbb{Z}_{\leqslant 0}$. □

2.1.3.1. Example. — Let us consider $\mathcal{I}_0$ introduced in Section 2.1.1. If we take an auxiliary sequence $\boldsymbol{m}(0) = (-1,-1), \boldsymbol{m}(1) = (-1,0)$, we have

$$\overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}) = z_1^{-1}z_2^{-1} + z_2^{-1}.$$

If we take an auxiliary sequence $\boldsymbol{m}(0) = (-1,-1), \boldsymbol{m}(1) = (0,-1)$, we have

$$\overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1}) = z_1^{-1}z_2^{-1} + z_1^{-1}.$$

Let us consider the example $\mathcal{I}_1$ with the auxiliary sequence $\boldsymbol{m}(0) = (-1,-1)$, $\boldsymbol{m}(1) = (-1,0)$. We have

$$\overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}z_2^{-1}) = z_1^{-1}z_2^{-1}, \quad \overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}) = \overline{\eta}_{\boldsymbol{m}(0)}(0) = 0.$$

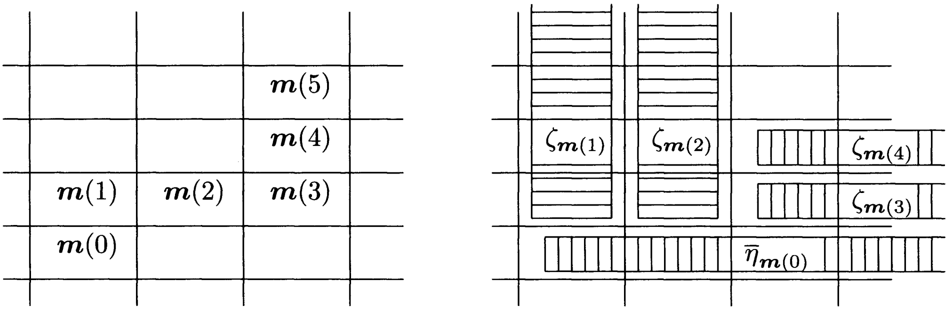
Hence, $\mathcal{I}_1(\boldsymbol{m}(0)) = \{z_1^{-1}z_2^{-1}, 0\}$.

Let us consider the example $\mathcal{I}_3$ with an auxiliary sequence $\boldsymbol{m}(0) = (-1,-1)$, $\boldsymbol{m}(1) = (-1,0)$. We have

$$\overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}z_2^{-1} + z_1^{-1}) = \overline{\eta}_{\boldsymbol{m}(0)}(z_1^{-1}z_2^{-1}) = z_1^{-1}z_2^{-1}.$$

Hence, $\mathcal{I}_3(\boldsymbol{m}(0)) = \{z_1^{-1}z_2^{-1}\}$.

We have the following picture in our mind for truncation.



$L = 4$, $\boldsymbol{m}(0) = (-2,-3)$, $\boldsymbol{m}(1) = (-2,-2)$, $\boldsymbol{m}(2) = (-1,-2)$, $\boldsymbol{m}(3) = (0,-2)$, $\boldsymbol{m}(4) = (0,-1)$, $\boldsymbol{m}(5) = (0,0)$.

2.2. Good lattice in the formal case

2.2.1. Definition. — We recall some definitions related with formal connections for our use. See [**46**] for deeper properties of formal connections.

Let k be an integral domain over $\boldsymbol{C}$. For some $1 \leqslant \ell \leqslant n$, we consider $R_0 := k[\![z_1,\dots,z_n]\!]$ and its localization R with respect to z_i $(i = 1,\dots,\ell)$. Let $\mathcal{X}$ be the formal scheme associated to R_0. Let $\mathcal{D}_i$ denote the formal subscheme of $\mathcal{X}$ corresponding to $z_i = 0$. We put $\mathcal{D} := \bigcup_{i=1}^{\ell} \mathcal{D}_i$. For each $I \subset \underline{\ell}$, we set $\mathcal{D}_I := \bigcap_{i\in I} \mathcal{D}_i$. Let $\mathcal{K} := \operatorname{Spec} k$. We use the natural identifications $R_0 = \mathcal{O}_{\mathcal{X}}$ and $R = \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$.

Let $\mathcal{M}$ be an $\mathcal{O}_{\mathcal{X}}$-module. Let $\varrho \in \mathcal{O}_{\mathcal{K}}$. Recall that a ϱ-connection of $\mathcal{M}$ relative to $\mathcal{K}$ is a k-linear map $\mathbb{D} : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}$ such that $\mathbb{D}(g\cdot s) = (\varrho dg)\cdot s + g\cdot \mathbb{D}s$. A pairing of $\mathbb{D}(s) \in \mathcal{M}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}$ and a vector field v of $\mathcal{X}$ is denoted by $\mathbb{D}(v)s$. It is called flat, if the curvature $\mathbb{D}\circ\mathbb{D} : \mathcal{M} \to \mathcal{M}\otimes\Omega^2_{\mathcal{X}/\mathcal{K}}$ is 0. A meromorphic ϱ-flat bundle on $(\mathcal{X},\mathcal{D})$ is a free $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-module $\mathcal{M}$ equipped with a flat ϱ-connection. If ϱ is nowhere vanishing, we obtain a flat connection $\varrho^{-1}\mathbb{D}$ relative to $\mathcal{K}$. It is often denoted by $\mathbb{D}^f$.

Remark 2.2.1. — We are mainly interested in the cases (i) $k = \boldsymbol{C}$ and $\varrho = 1$ (ordinary flat connection) (ii) $k = \boldsymbol{C}$ and $\varrho = 0$ (Higgs field) (iii) $k = \boldsymbol{C}$ and $\varrho = \lambda \in \boldsymbol{C}$ (flat λ-connection) (iv) k is a $\boldsymbol{C}[\lambda]$-algebra, and $\varrho = \lambda$ (family of flat λ-connections).

We will often omit to say "relative to $\mathcal{K}$", if there is no risk of confusion. □

Let $(\mathcal{M},\mathbb{D})$ be a meromorphic ϱ-flat bundle on $(\mathcal{X},\mathcal{D})$. A coherent $\mathcal{O}_{\mathcal{X}}$-submodule $\mathcal{L} \subset \mathcal{M}$ is called a lattice, if $\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}(*\mathcal{D}) = \mathcal{M}$. The specialization $\mathcal{L}\otimes\mathcal{O}_{\mathcal{D}_I}$ is denoted by $\mathcal{L}_{|\mathcal{D}_I}$.

Definition 2.2.2. — A lattice $\mathcal{L}$ of $\mathcal{M}$ is called $\mathfrak{a}$-logarithmic for $\mathfrak{a} \in \mathcal{O}_{\mathcal{X}}(*\mathcal{D})/\mathcal{O}_{\mathcal{X}}$, if (i) $\mathcal{L}$ is $\mathcal{O}_{\mathcal{X}}$-free, (ii) $\mathbb{D} - d\widetilde{\mathfrak{a}}$ is logarithmic for a lift $\widetilde{\mathfrak{a}}$ of $\mathfrak{a}$ to $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. (We will often use the same symbol $\mathfrak{a}$ to denote a lift to $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$ in the subsequent arguments.)

If $\mathcal{M}$ has an $\mathfrak{a}$-logarithmic lattice, it is called $\mathfrak{a}$-regular. □

Definition 2.2.3. — A lattice $\mathcal{L}$ of $\mathcal{M}$ is called unramifiedly good, if there exist a good set of irregular values $\mathrm{Irr}(\mathbb{D}) \subset \mathcal{O}_{\mathcal{X}}(*\mathcal{D})/\mathcal{O}_{\mathcal{X}}$ and a decomposition

$$(\mathcal{L}, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\mathbb{D})} (\mathcal{L}_{\mathfrak{a}}, \mathbb{D}_{\mathfrak{a}}) \tag{12}$$

such that $\mathbb{D}_{\mathfrak{a}}$ are $\mathfrak{a}$-logarithmic.

If $\mathcal{M}$ has an unramifiedly good lattice, we say that $\mathcal{M}$ is unramifiedly good. □

The decomposition (12) induces

$$\mathcal{M} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\mathbb{D})} \mathcal{L}_{\mathfrak{a}} \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D}). \tag{13}$$

The decompositions (12) and (13) are called irregular decompositions of $\mathcal{L}$ and $\mathcal{M}$, respectively.

Lemma 2.2.4. — *Let $\mathcal{L}$ and $\mathcal{L}'$ be unramifiedly good lattices of $\mathcal{M}$ with irregular decompositions $\mathcal{L} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\mathbb{D})} \mathcal{L}_{\mathfrak{a}}$ and $\mathcal{L}' = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}'(\mathbb{D})} \mathcal{L}'_{\mathfrak{a}}$. Then, we have*

$$\mathcal{L}_{\mathfrak{a}} \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D}) = \mathcal{L}'_{\mathfrak{a}} \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$$

for any $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}) \cup \mathrm{Irr}'(\mathbb{D})$. In particular, the decomposition (12) *is uniquely determined by $\mathcal{L}$, and the decomposition* (13) *is uniquely determined by $\mathcal{M}$.*

Proof. — Take $\mathfrak{a}, \mathfrak{b} \in \mathrm{Irr}(\mathbb{D}) \cup \mathrm{Irr}'(\mathbb{D})$ such that $\mathfrak{a} - \mathfrak{b} \neq 0$ in $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})/\mathcal{O}_{\mathcal{X}}$. We would like to show that the induced morphism $\varphi_{\mathfrak{b},\mathfrak{a}} : \mathcal{L}_{\mathfrak{a}} \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D}) \to \mathcal{L}'_{\mathfrak{b}} \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$ is 0. There exists $1 \leqslant i \leqslant \ell$ such that the order of $\mathfrak{a} - \mathfrak{b}$ with respect to i is strictly smaller than 0. We may assume $i = 1$. Let $\mathcal{A}$ be the localization of $k[\![z_2, \dots, z_n]\!]$ with respect to $\prod_{i=2}^{\ell} z_i$, and let $\mathcal{R} := \mathcal{A}(\!(z_1)\!)$. By a standard result in the one variable case (see Corollary 2.2.18 below), we obtain that the induced morphism $\mathcal{L}_{\mathfrak{a}} \otimes \mathcal{R} \to \mathcal{L}'_{\mathfrak{b}} \otimes \mathcal{R}$ is 0, and hence $\varphi_{\mathfrak{b},\mathfrak{a}} = 0$. Then, the claim of the lemma immediately follows. □

2.2.1.1. Residue and induced $\mathbb{D}$-connections. — If we are given an unramifiedly good lattice $\mathcal{L}$, we obtain an endomorphism $\mathrm{Res}_i(\mathbb{D})$ of $\mathcal{L}_{|\mathcal{D}_i}$ in a standard way. Namely, for any $f \in \mathcal{L}_{\mathfrak{a}|\mathcal{D}_i}$, we take $\widetilde{f} \in \mathcal{L}$ such that $\widetilde{f}_{|\mathcal{D}_i} = f$, and put $\mathrm{Res}_i(\mathbb{D}_{\mathfrak{a}})f := \big(\mathbb{D}_{\mathfrak{a}}^{\mathrm{reg}}(z_i\partial_i)\widetilde{f}\big)_{|\mathcal{D}_i}$, where $\mathbb{D}_{\mathfrak{a}}^{\mathrm{reg}} := \mathbb{D}_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ for a lift $\widetilde{\mathfrak{a}}$ of $\mathfrak{a}$. We set $\mathrm{Res}_i(\mathbb{D}) := \bigoplus \mathrm{Res}_i(\mathbb{D}_{\mathfrak{a}}) \in \mathrm{End}(\mathcal{L}_{|\mathcal{D}_i})$. It is independent of the choice of lifts $\widetilde{f}$ and $\widetilde{\mathfrak{a}}$. It is also independent of the choice of the coordinate functions z_i. For any $I \ni i$, the induced endomorphism of $\mathcal{L}_{|\mathcal{D}_I}$ is also denoted by $\mathrm{Res}_i(\mathbb{D})$.

If $\mathfrak{a}$ does not contain a negative power of z_i, we can define a meromorphic flat ϱ-connection ${}^{i}\mathbb{D}_{\mathfrak{a}}$ on $\mathcal{L}_{\mathfrak{a}|\mathcal{D}_i}$. Let $\mathcal{D}(i^c) := \bigcup_{j\neq i} \mathcal{D}_i$. The section dz_i/z_i of $\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}_i)$ induces a splitting $\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}_i)_{|\mathcal{D}_i} \simeq \Omega^1_{\mathcal{D}_i/\mathcal{K}} \oplus \mathcal{O}_{\mathcal{D}_i}$. Let π denote the projection onto $\Omega^1_{\mathcal{D}_i/\mathcal{K}}$. It induces $\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}_i)(*\mathcal{D}(i^c)) \to \Omega^1_{\mathcal{D}_i/\mathcal{K}}(*\mathcal{D}(i^c))$. Then, for $f \in \mathcal{L}_{\mathfrak{a}|\mathcal{D}_i}$, take $F \in \mathcal{L}$ such that $F_{|\mathcal{D}_i} = f$, and put ${}^{i}\mathbb{D}(f) = \pi\big((\mathbb{D}F)_{|\mathcal{D}_i}\big)$. It is independent of the choice of a lift F. But, it depends on the choice of the function z_i. If we replace z_i

with $\omega\, z_i$ for some invertible ω, the difference between the induced ϱ-connections is $(\omega^{-1}d\omega)_{|\mathcal{D}_i}$.

If we are given lifts $\widetilde{\mathfrak{a}}$ for any $\mathfrak{a} \in \mathcal{I}$, we obtain a flat ϱ-connection ${}^i\mathbb{D}^{\mathrm{reg}}_{\mathfrak{a}}$ of $\mathcal{L}_{\mathfrak{a}|\mathcal{D}_i}$ by the same procedure, and ${}^i\mathbb{D}^{\mathrm{reg}} := \bigoplus_{\mathfrak{a}\in\mathcal{I}} {}^i\mathbb{D}^{\mathrm{reg}}_{\mathfrak{a}}$. It depends on the choice of lifts $\widetilde{\mathfrak{a}}$ and the function z_i.

Lemma 2.2.5. — $\mathrm{Res}_i(\mathbb{D}_{\mathfrak{a}})$ *is* ${}^i\mathbb{D}^{\mathrm{reg}}$*-flat. If* $\rho \neq 0$*, the eigenvalues of* $\mathrm{Res}_i(\mathbb{D})$ *are algebraic over* k.

Proof. — The first claim is clear from the above constructions. The second claim follows from the first one. □

2.2.1.2. Good lattice

Definition 2.2.6. — A lattice $\mathcal{L}$ of $\mathcal{M}$ is called good, if there exists a ramified covering $\varphi : (\mathcal{X}', \mathcal{D}') \to (\mathcal{X}, \mathcal{D})$ and an unramifiedly good lattice $\mathcal{L}'$ of $\mathcal{M}' = \varphi^*\mathcal{M}$ such that $\mathcal{L}$ is the descent of $\mathcal{L}'$. □

If we take an e-th root ζ_i of z_i for appropriate e, we have an extension of rings $R_0 \subset R_0' = k[\![\zeta_1, \dots, \zeta_\ell, z_{\ell+1}, \dots, z_n]\!]$. Let G be the Galois group of the extension. We put $\mathcal{M}' := \mathcal{M} \otimes_{R_0} R_0'$. Then, the above condition says that $\mathcal{M}'$ has a G-equivariant unramifiedly good lattice $\mathcal{L}'$, and $\mathcal{L}$ is the G-invariant part of $\mathcal{L}'$.

Lemma 2.2.7. — *Let* $\mathcal{L}$ *be a good lattice of* $\mathcal{M}$*. Put* $e_1 := (\mathrm{rank}\,\mathcal{M})!$*, and let* $\mathcal{X}_1 \to \mathcal{X}$ *be a ramified covering such that the ramification indices at* $\mathcal{D}_i$ *are* e_1*. Then,* $\mathcal{L}$ *is the descent of an unramifiedly good lattice* $\mathcal{L}_1$ *on* $\mathcal{X}_1$*. In other words, we have an a priori bound on the minimal ramification indices.*

Proof. — We take e, $\mathcal{X}'$, $\mathcal{L}'$ and $\mathcal{M}'$ as above. We may assume that e is divisible by e_1. We have a factorization $\mathcal{X}' \to \mathcal{X}_1 \to \mathcal{X}$. Note that $\mathrm{Irr}(\mathbb{D}) \subset \mathcal{O}_{\mathcal{X}'}(*\mathcal{D}')/\mathcal{O}_{\mathcal{X}'}$ is contained in $\mathcal{O}_{\mathcal{X}_1}(*\mathcal{D}_1)/\mathcal{O}_{\mathcal{X}_1}$. It is well known in the one dimensional case. The higher dimensional case can be reduced to the curve case easily. Then, for the irregular decomposition of $\mathcal{L}'$, each direct summand is stable by the action of the Galois group of $\mathcal{X}'/\mathcal{X}_1$. Then, the descent of $\mathcal{L}'$ to $\mathcal{X}_1$ gives the desired lattice. □

2.2.2. A criterion for a lattice to be good

2.2.2.1. Statement. — Let $\mathcal{X} \to \mathcal{K}$, $\mathcal{D}$ and ϱ be as in Section 2.2.1. For simplicity, we assume that k is a local ring. Then, $\mathcal{X}$ has a unique closed point O. Let $(\mathcal{M}, \mathbb{D})$ be a meromorphic flat bundle on $(\mathcal{X}, \mathcal{D})$. Let $\mathcal{L}$ be a lattice of $\mathcal{M}$. Assume that we are given the following:

- a good set of irregular values $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}(*\mathcal{D})/\mathcal{O}_{\mathcal{X}}$,
- a decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{L}_{\mathfrak{a}}$ as an $\mathcal{O}_{\mathcal{X}}$-module, which is not necessarily compatible with $\mathbb{D}$,

such that, if $p_{\mathfrak{a}}$ denotes the projection onto $\mathcal{L}_{\mathfrak{a}}$, and if we put $\Phi := \sum_{\mathfrak{a}\in\mathcal{I}} d\mathfrak{a} \cdot p_{\mathfrak{a}}$, then $\mathbb{D}^{(0)} := \mathbb{D} - \Phi$ is logarithmic with respect to $\mathcal{L}$. (It is not necessarily flat.)

Proposition 2.2.8. — *$\mathcal{L}$ is an unramifiedly good lattice of $\mathcal{M}$, and we have* $\mathrm{Irr}(\mathbb{D}) = \mathcal{I}$.

2.2.2.2. Preliminary. — Let F be a free $\mathcal{O}_{\mathcal{X}}$-module with a meromorphic flat ϱ-connection $\mathbb{D} : F \to F \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}(*\mathcal{D})$. Let $\boldsymbol{v}$ be a frame of F, and let A be the $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-valued matrix determined by $\mathbb{D}(z_1\partial_1)\boldsymbol{v} = \boldsymbol{v}\,A$. Assume that we have a decomposition $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ such that the corresponding decomposition of A has the following form:

$$A = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

and

- the entries of $A_{p,q}$ are regular, i.e., sections of $\mathcal{O}_{\mathcal{X}}$,
- there exists $\boldsymbol{m} \in \mathbb{Z}^{\ell_1}_{<0}$ for some $1 \leqslant \ell_1 \leqslant \ell$, such that the entries of $\overline{\Omega}_p := \boldsymbol{z}^{-\boldsymbol{m}}\,\Omega_p$ $(p = 1, 2)$ are regular,
- $\overline{\Omega}_{1|\mathcal{O}}$ and $\overline{\Omega}_{2|\mathcal{O}}$ have no common eigenvalues.

Let us consider a change of the base of the following form:

$$\boldsymbol{v}' = \boldsymbol{v}\,G, \quad G = I + \begin{pmatrix} 0 & T_2 \\ T_1 & 0 \end{pmatrix}.$$

Here, the entries of T_p $(p = 1, 2)$ are sections of $z_1\,\mathcal{O}_{\mathcal{X}}$. We would like to take G such that $\mathbb{D}(z_1\partial_1)$ is block-diagonalized with respect to $\boldsymbol{v}'$ as follows:

$$\mathbb{D}(z_1\partial_1)\boldsymbol{v}' = \boldsymbol{v}'\,B, \quad B = \begin{pmatrix} \Omega_1 + Q_1 & 0 \\ 0 & \Omega_2 + Q_2 \end{pmatrix}. \tag{14}$$

Lemma 2.2.9. — *We have regular solutions T_p and Q_p $(p = 1, 2)$ such that* (14) *holds. Moreover, $\boldsymbol{z}^{\boldsymbol{m}} T_p$ are also regular.*

Proof. — The relation of A, G and B are given by $AG + \varrho\, z_1\partial_1 G = GB$. We obtain the equations:

$$A_{11} + A_{12}T_1 + Q_1 = 0, \quad \Omega_2 T_1 + A_{21} + A_{22}T_1 + \varrho\, z_1\partial_1 T_1 = T_1\Omega_1 + T_1 Q_1.$$

By eliminating Q_1, we obtain

$$\Omega_2 T_1 - T_1\Omega_1 + A_{21} + A_{22}T_1 + \varrho\, z_1\partial_1 T_1 + T_1(A_{11} + A_{12}T_1) = 0.$$

We obtain the equation:

$$\overline{\Omega}_2 T_1 - T_1\overline{\Omega}_1 + \boldsymbol{z}^{-\boldsymbol{m}}(A_{21} + A_{22}T_1 + T_1 A_{11} + \varrho\, z_1\partial_1 T_1 + T_1 A_{12} T_1) = 0. \tag{15}$$

We clearly have a regular solution T_1 of (15). Moreover, because $\overline{\Omega}_2 T_1 - T_1\overline{\Omega}_1 \equiv 0$ modulo $\boldsymbol{z}^{-\boldsymbol{m}}$, we obtain that $\boldsymbol{z}^{\boldsymbol{m}} T_1$ is also regular. For such T_1, Q_1 is also regular. Similarly, we have desired T_2 and Q_2, and $\boldsymbol{z}^{\boldsymbol{m}} T_2$ is also regular. □

We put $V_i := z_i\partial_i$ $(i \leqslant \ell)$ and $V_i := \partial_i$ $(i > \ell)$. Let $A^{(i)}$ be the $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-valued matrices determined by $\mathbb{D}(V_i)\boldsymbol{v} = \boldsymbol{v}\,A^{(i)}$. Assume that the decomposition of $A^{(i)}$ corresponding to $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ has the form

$$A^{(i)} = \begin{pmatrix} \Omega_1^{(i)} & 0 \\ 0 & \Omega_2^{(i)} \end{pmatrix} + \begin{pmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{pmatrix},$$

where the entries of $A_{pq}^{(i)}$ are regular, and the entries of $\boldsymbol{z}^{-\boldsymbol{m}}\Omega_p^{(i)}$ are regular. Let $C^{(i)}$ be determined by $\mathbb{D}(V_i)\boldsymbol{v}' = \boldsymbol{v}'\,C^{(i)}$. Since $\boldsymbol{z}^{\boldsymbol{m}}T_p$ $(p = 1, 2)$ are regular, $C^{(i)}$ has the form

$$C^{(i)} = \begin{pmatrix} \Omega_1^{(i)} & 0 \\ 0 & \Omega_2^{(i)} \end{pmatrix} + \begin{pmatrix} C_{11}^{(i)} & C_{12}^{(i)} \\ C_{21}^{(i)} & C_{22}^{(i)} \end{pmatrix},$$

where the entries of $C_{pq}^{(i)}$ are regular.

Lemma 2.2.10. — *We have $C_{12}^{(i)} = 0$ and $C_{21}^{(i)} = 0$.*

Proof. — Because $\big[\mathbb{D}(V_i), \mathbb{D}(z_1\partial_1)\big] = 0$, we have the relation $\varrho\, V_i C^{(1)} + C^{(i)}C^{(1)} = \varrho\, z_1\partial_1 C^{(i)} + C^{(1)}C^{(i)}$, from which we obtain the following equality:

$$C_{12}^{(i)}(\Omega_2 + Q_2) = \varrho\, z_1\partial_1 C_{12}^{(i)} + (\Omega_1 + Q_1)C_{12}^{(i)}.$$

Then, it is easy to obtain $C_{12}^{(i)} = 0$. Similarly, we can obtain $C_{21}^{(i)} = 0$. □

2.2.2.3. Proof of Proposition 2.2.8. — Let us return to the setting in Subsection 2.2.2.1. We use an induction on the number $|\mathcal{I}|$. If $|\mathcal{I}| = 1$, the claim of Proposition 2.2.8 is obvious. Assume that we have already proved the claim of the proposition in the case $|\mathcal{I}| < m_0$, and let us show the case $|\mathcal{I}| = m_0$.

We take an auxiliary sequence $\boldsymbol{m}(0), \ldots, \boldsymbol{m}(L)$ for $\mathcal{I}$. Let $\mathcal{I}(\boldsymbol{m}(0))$ denote the image of $\mathcal{I}$ via $\overline{\eta}_{\boldsymbol{m}(0)}$. It is easy to observe that we only have to consider the case $|\mathcal{I}(\boldsymbol{m}(0))| > 1$.

Lemma 2.2.11. — *We have a flat decomposition*

$$(\mathcal{L}, \mathbb{D}) = \bigoplus_{\mathfrak{b}\in\mathcal{I}(\boldsymbol{m}(0))} (\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)}, \mathbb{D}_{\mathfrak{b}}^{\boldsymbol{m}(0)}),$$

and an $\mathcal{O}_{\mathcal{X}}$-decomposition

$$\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)} = \bigoplus_{\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a})=\mathfrak{b}} (\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)})_{\mathfrak{a}}$$

with the following property:

- *Let $p'_{\mathfrak{a}}$ denote the projection of $\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)}$ onto $(\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)})_{\mathfrak{a}}$, and we put $\Psi_{\mathfrak{b}}^{\boldsymbol{m}(0)} := \sum_{\mathfrak{a}\in\overline{\eta}_{\boldsymbol{m}(0)}^{-1}(\mathfrak{b})} d\mathfrak{a}\, p'_{\mathfrak{a}}$. Then, $\mathbb{D}_{\mathfrak{b}}^{\boldsymbol{m}(0)} - \Psi_{\mathfrak{b}}^{\boldsymbol{m}(0)}$ are logarithmic with respect to $\mathcal{L}_{\mathfrak{b}}^{\boldsymbol{m}(0)}$.*

Proof. — Let $\boldsymbol{v}$ be a frame of $\mathcal{L}$ compatible with the decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{L}_{\mathfrak{a}}$. Let $\mathfrak{a}_i$ be determined by $v_i \in \mathcal{L}_{\mathfrak{a}_i}$. Let Ω be the diagonal matrix valued 1-form whose (i,i)-th entry is $d\mathfrak{a}_i$. By applying Lemma 2.2.10 successively, we obtain a frame $\boldsymbol{w}$ of $\mathcal{L}$ such that $\mathbb{D}\boldsymbol{w} = \boldsymbol{w}\,(\Omega + B)$, where B satisfies the following:

- B is a matrix-valued logarithmic 1-form.
- $B_{i,j} = 0$ unless $\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}_i) = \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}_j)$.

For $\mathfrak{b} \in \mathcal{I}(\boldsymbol{m}(0))$, let $\mathcal{L}^{\boldsymbol{m}(0)}_{\mathfrak{b}}$ be the subbundle generated by $\{w_i \,|\, \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}_i) = \mathfrak{b}\}$. For $\mathfrak{a} \in \mathcal{I}$ with $\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}) = \mathfrak{b}$, let $(\mathcal{L}^{\boldsymbol{m}(0)}_{\mathfrak{b}})_{\mathfrak{a}}$ be the subbundle generated by $\{w_i \,|\, \mathfrak{a}_i = \mathfrak{a}\}$. Then, they have the desired property. □

Because $\mathcal{L}^{\boldsymbol{m}(0)}_{\mathfrak{b}}$ satisfy the assumption in the proposition, we may apply the inductive assumption. Thus, the proof of Proposition 2.2.8 is finished. □

2.2.3. Unramifiedly good Deligne-Malgrange lattice.

— Let us use the setting of Subsection 2.2.1 with $k = \boldsymbol{C}$ and $\varrho = 1$. We use the symbol ∇ instead of $\mathbb{D}$.

Definition 2.2.12. — A lattice $\mathcal{L}$ of a meromorphic flat bundle $(\mathcal{M},\nabla)$ on $(\mathcal{X},\mathcal{D})$ is called an unramifiedly good Deligne-Malgrange lattice, if (i) $\mathcal{L}$ is an unramifiedly good lattice, (ii) the eigenvalues α of $\mathrm{Res}_i(\nabla)$ $(i = 1,\dots,\ell)$ satisfy $0 \leqslant \mathrm{Re}\,\alpha < 1$. □

An unramifiedly good Deligne-Malgrange lattice is uniquely determined, if it exists. If $\mathcal{L} = \bigoplus \mathcal{L}_{\mathfrak{a}}$ is the unramifiedly good Deligne-Malgrange lattice of $\mathcal{M}$, we have frames $\boldsymbol{v}_{\mathfrak{a}}$ of $\mathcal{L}_{\mathfrak{a}}$ such that $(\nabla_{\mathfrak{a}} - d\mathfrak{a})\boldsymbol{v}_{\mathfrak{a}} = \boldsymbol{v}_{\mathfrak{a}}\left(\sum_{j=1}^{\ell} A_j\, dz_j/z_j\right)$, where A_j are constant matrices. They induce a frame $\boldsymbol{v} = (\boldsymbol{v}_{\mathfrak{a}})$ of $\mathcal{L}$. Such a frame is called normalizing frame.

2.2.3.1. Extension. — Let $0 \to (\mathcal{M}^{(1)},\nabla^{(1)}) \to (\mathcal{M}^{(0)},\nabla^{(0)}) \to (\mathcal{M}^{(2)},\nabla^{(2)}) \to 0$ be an exact sequence of meromorphic flat bundles on $(\mathcal{X},\mathcal{D})$. Let us show the following proposition.

Proposition 2.2.13. — *Assume $(\mathcal{M}^{(i)},\nabla^{(i)})$ $(i = 1,2)$ have unramifiedly good Deligne-Malgrange lattices $\mathcal{L}^{(i)}$, and $\mathcal{I} := \mathrm{Irr}(\mathcal{M}^{(1)}) \cup \mathrm{Irr}(\mathcal{M}^{(2)})$ is good. Then, $(\mathcal{M}^{(0)},\nabla^{(0)})$ also has an unramifiedly good Deligne-Malgrange lattice $\mathcal{L}^{(0)}$. We also have the exact sequence $0 \to \mathcal{L}^{(1)} \to \mathcal{L}^{(0)} \to \mathcal{L}^{(2)} \to 0$.*

Although this also seems to follow some deep results in [**46**], we keep our elementary proof. In the following argument, we assume that the coordinate system is admissible for $\mathcal{I}$. An element of $f \in \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$ is called z_j-regular, if f does not contain a negative power of z_j.

2.2.3.1.1. First step. — For $\mathfrak{a},\mathfrak{b} \in \mathcal{I}$, we have the expansion $\mathfrak{a} - \mathfrak{b} = \sum (\mathfrak{a} - \mathfrak{b})_{\ell}\, z_p^{\ell}$. We put $\mathrm{ord}_p(\mathfrak{a}-\mathfrak{b}) := \min\{\ell \,|\, (\mathfrak{a}-\mathfrak{b})_{\ell} \neq 0\}$ and $\mathfrak{s}(\mathfrak{a}-\mathfrak{b}) := \{p \,|\, \mathrm{ord}_p(\mathfrak{a}-\mathfrak{b}) < 0\}$. We put $V_i := z_i\partial_i$ $(i \leqslant \ell)$ and $V_i := \partial_i$ $(i > \ell)$. We take normalizing frames $\boldsymbol{v}^{(i)}$ $(i = 1,2)$ of $\mathcal{L}^{(i)}$ compatible with the irregular decomposition. We have the decomposition of the frames $\boldsymbol{v}^{(i)}=(\boldsymbol{v}^{(i)}_{\mathfrak{a}})$. Let $A^{(i)}_j$ be determined by $\nabla^{(i)}(V_j)\boldsymbol{v}^{(i)}=\boldsymbol{v}^{(i)}\,A^{(i)}_j$. We have the decomposition $A^{(i)}_j=\bigoplus\big(V_j(\mathfrak{a})+\overline{A}^{(i)}_{j,\mathfrak{a}}\big)$, where $\overline{A}^{(i)}_{j,\mathfrak{a}}$ are constant matrices. We also have $\overline{A}^{(i)}_{j,\mathfrak{a}}=0$ for $j>\ell$.

We take a lift $\widetilde{\boldsymbol{v}}^{(2)}$ of $\boldsymbol{v}^{(2)}$ to $\mathcal{M}^{(0)}$. The frame of $\mathcal{M}^{(0)}$ given by $\boldsymbol{v}^{(1)}$ and $\widetilde{\boldsymbol{v}}^{(2)}$ is denoted by $(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)})$. Then, we have

$$\nabla^{(0)}(V_j)\big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)}\big)=\big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)}\big)\begin{pmatrix}A^{(1)}_j & U_j\\ 0 & A^{(2)}_j\end{pmatrix}.$$

Here, the entries of U_j are contained in $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. We have the decomposition $U_j=(U_{j,\mathfrak{a},\mathfrak{b}})$ corresponding to the decompositions of the frames $\boldsymbol{v}^{(i)}=(\boldsymbol{v}^{(i)}_{\mathfrak{a}})$. We will consider transforms of frames of the following form, where the entries of W are contained in $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$:

$$(16)\qquad \big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)\prime}\big)=\big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)}\big)\begin{pmatrix}I & W\\ 0 & I\end{pmatrix},$$

$$\nabla^{(0)}(V_j)\big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)\prime}\big)=\big(\boldsymbol{v}^{(1)},\widetilde{\boldsymbol{v}}^{(2)\prime}\big)\begin{pmatrix}A^{(1)}_j & U'_j\\ 0 & A^{(2)}_j\end{pmatrix}.$$

Lemma 2.2.14. — *We can take a lift $\widetilde{\boldsymbol{v}}^{(2)}$ such that $U_{j,\mathfrak{a},\mathfrak{b}}$ are z_p-regular for any $p\notin\mathfrak{s}(\mathfrak{a}-\mathfrak{b})$ and for any j.*

Proof. — We will inductively take transforms as above, such that the following claims hold:

$P(m)$: Let $p\leqslant m$. If $p\notin\mathfrak{s}(\mathfrak{a}-\mathfrak{b})$, $U_{p,\mathfrak{a},\mathfrak{b}}$ is z_p-regular.

$Q(m)$: Let $p\leqslant m$. If $p\notin\mathfrak{s}(\mathfrak{a}-\mathfrak{b})$, $U_{j,\mathfrak{a},\mathfrak{b}}$ is z_p-regular for any j.

In the conditions, we only have to consider p satisfying $p\leqslant\ell$.

Lemma 2.2.15. — *If $P(m)$ holds, $Q(m)$ also holds.*

Proof. — Due to the commutativity $\big[\nabla^{(0)}(V_j),\nabla^{(0)}(V_p)\big]=0$, we have the following relation:

$$(17)\qquad V_p(U_j)+U_p\,A^{(2)}_j+A^{(1)}_p\,U_j=V_j(U_p)+U_j\,A^{(2)}_p+A^{(1)}_j\,U_p.$$

Hence, we obtain the following equality:

$$(18)\quad z_p\partial_pU_{j,\mathfrak{a},\mathfrak{b}}+V_p(\mathfrak{a}-\mathfrak{b})\,U_{j,\mathfrak{a},\mathfrak{b}}+\overline{A}^{(1)}_{p,\mathfrak{a}}\,U_{j,\mathfrak{a},\mathfrak{b}}-U_{j,\mathfrak{a},\mathfrak{b}}\,\overline{A}^{(2)}_{p,\mathfrak{b}}$$
$$-V_j\big(U_{p,\mathfrak{a},\mathfrak{b}}\big)-V_j(\mathfrak{a}-\mathfrak{b})\,U_{p,\mathfrak{a},\mathfrak{b}}-\overline{A}^{(1)}_{j,\mathfrak{a}}\,U_{p,\mathfrak{a},\mathfrak{b}}+U_{p,\mathfrak{a},\mathfrak{b}}\,\overline{A}^{(2)}_{j,\mathfrak{b}}=0.$$

Assume $U_{j,\mathfrak{a},\mathfrak{b}}\neq0$, and let us consider the expansion $U_{j,\mathfrak{a},\mathfrak{b}}=\sum_{\ell\geqslant N}U_{j,\mathfrak{a},\mathfrak{b},\ell}\,z_p^{\ell}$, where $U_{j,\mathfrak{a},\mathfrak{b},N}\neq0$. Assume $N<0$, and we will derive a contradiction. Because $\operatorname{ord}_p(\mathfrak{a}-\mathfrak{b})\geqslant0$, we obtain the following relation:

$$N\,U_{j,\mathfrak{a},\mathfrak{b},N}+\overline{A}^{(1)}_{p,\mathfrak{a}}\,U_{j,\mathfrak{a},\mathfrak{b},N}-U_{j,\mathfrak{a},\mathfrak{b},N}\,\overline{A}^{(2)}_{p,\mathfrak{b}}=0.$$

Because the eigenvalues α of $\overline{A}^{(1)}_{p,\mathfrak{a}}$ and $\overline{A}^{(2)}_{p,\mathfrak{a}}$ satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$, we obtain $U_{j,\mathfrak{a},\mathfrak{b},N} = 0$, which contradicts with our choice of N. Hence, we obtain $N \geqslant 0$. Thus, the proof of Lemma 2.2.15 is finished. □

Assume $Q(m-1)$ holds for a lift $\widetilde{\boldsymbol{v}}^{(2)}$. We would like to replace it with a lift for which $P(m)$ holds, by successive use of the transforms as in (16). We use the following equality, obtained from the relation $A^{(1)}_j\, W + U_j + V_j(W) = U'_j + W\, A^{(2)}_j$:

$$V_j(\mathfrak{a}-\mathfrak{b})\, W_{\mathfrak{a},\mathfrak{b}} + \overline{A}^{(1)}_{j,\mathfrak{a}}\, W_{\mathfrak{a},\mathfrak{b}} - W_{\mathfrak{a},\mathfrak{b}}\, \overline{A}^{(2)}_{j,\mathfrak{b}} + V_j\big(W_{\mathfrak{a},\mathfrak{b}}\big) + U_{j,\mathfrak{a},\mathfrak{b}} - U'_{j,\mathfrak{a},\mathfrak{b}} = 0. \tag{19}$$

If $m \in \mathfrak{s}(\mathfrak{a}-\mathfrak{b})$, we have nothing to do, and so we assume $m \notin \mathfrak{s}(\mathfrak{a}-\mathfrak{b})$. Let us consider the expansion $U_{m,\mathfrak{a},\mathfrak{b}} = \sum_{\ell \geqslant N} U_{m,\mathfrak{a},\mathfrak{b},\ell}\, z_m^{\ell}$. Assume $N < 0$. Let $W_{\mathfrak{a},\mathfrak{b},N}$ be the unique solution of the following equation:

$$\overline{A}^{(1)}_{m,\mathfrak{a}}\, W_{\mathfrak{a},\mathfrak{b},N} - W_{\mathfrak{a},\mathfrak{b},N}\, \overline{A}^{(2)}_{m,\mathfrak{b}} + N\, W_{\mathfrak{a},\mathfrak{b},N} + U_{m,\mathfrak{a},\mathfrak{b},N} = 0.$$

By the inductive assumption $Q(m-1)$, $U_{m,\mathfrak{a},\mathfrak{b},N}$ is assumed to be z_p-regular for $p < m$ with $p \notin \mathfrak{s}(\mathfrak{a}-\mathfrak{b})$. Hence, $W_{\mathfrak{a},\mathfrak{b},N}$ is also z_p-regular. We put $W_{\mathfrak{a},\mathfrak{b}} = W_{\mathfrak{a},\mathfrak{b},N}\, z_m^N$. Then, because of (19) with $j = m$, the obtained $U'_{m,\mathfrak{a},\mathfrak{b}}$ has the expansion $\sum_{\ell > N} U'_{m,\mathfrak{a},\mathfrak{b},\ell}\, z_m^{\ell}$. Because of (19), $U'_{j,\mathfrak{a},\mathfrak{b}}$ is also z_p-regular for any j and for $p < m$ with $p \notin \mathfrak{s}(\mathfrak{a}-\mathfrak{b})$. Hence, we can eliminate the negative powers in $U_{m,\mathfrak{a},\mathfrak{b}}$ after the finite procedure, preserving the condition $Q(m-1)$, and we can arrive at a lift $\widetilde{\boldsymbol{v}}^{(2)}$ for which $P(m)$ holds.

Therefore, after a finite procedure, we can arrive at a lift $\widetilde{\boldsymbol{v}}^{(2)}$ for which $Q(\ell)$ holds. Thus, the claim of Lemma 2.2.14 is proved. □

2.2.3.1.2. End of the proof of Proposition 2.2.13. — Let $\widetilde{\boldsymbol{v}}^{(2)}$ be a lift as in Lemma 2.2.14. We would like to replace it with a lift for which the U_j-components are contained in $\mathcal{O}_{\mathcal{X}}$, by successive use of (16). We put $F := z_1\partial_1(\mathfrak{a}-\mathfrak{b})$. Note that $F\,\boldsymbol{z}^{-\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}$ is invertible. We put $W_{\mathfrak{a},\mathfrak{b}} := -F^{-1}\, U_{1,\mathfrak{a},\mathfrak{b}}$. Then, we have the following, due to (19):

$$U'_{1,\mathfrak{a},\mathfrak{b}} = F^{-1}\left(U_{1,\mathfrak{a},\mathfrak{b}}\, \overline{A}^{(2)}_{1,\mathfrak{b}} - \overline{A}^{(1)}_{1,\mathfrak{a}}\, U_{1,\mathfrak{a},\mathfrak{b}}\right) - V_1\big(F^{-1}\, U_{1,\mathfrak{a},\mathfrak{b}}\big).$$

Let k be determined by $\operatorname{ord}(\mathfrak{a}-\mathfrak{b}) \in \mathbb{Z}^k_{<0}$, i.e., $\mathfrak{s}(\mathfrak{a}-\mathfrak{b}) = \{1,\ldots,k\}$. We have the subset $\mathcal{S} \subset \mathbb{Z}^k$ and the expansion:

$$U_{1,\mathfrak{a},\mathfrak{b}} = \sum_{\boldsymbol{n}\in\mathcal{S}} U_{1,\mathfrak{a},\mathfrak{b},\boldsymbol{n}}(z_{k+1},\ldots,z_n)\, \boldsymbol{z}^{\boldsymbol{n}}, \quad U_{1,\mathfrak{a},\mathfrak{b},\boldsymbol{n}} \neq 0.$$

Note that $\mathcal{S}$ is bounded below with respect to $\leqslant_{\mathbb{Z}^k}$. Then, the expansion of $U'_{1,\mathfrak{a},\mathfrak{b}}$ is as follows:

$$U'_{1,\mathfrak{a},\mathfrak{b}} = \sum_{\boldsymbol{n}\in\mathcal{S}_1} U'_{1,\mathfrak{a},\mathfrak{b},\boldsymbol{n}}(z_{k+1},\ldots,z_n)\, \boldsymbol{z}^{\boldsymbol{n}}, \quad U'_{1,\mathfrak{a},\mathfrak{b},\boldsymbol{n}} \neq 0.$$

Here $\mathcal{S}_1 = \big\{\boldsymbol{m} - \operatorname{ord}(\mathfrak{a}-\mathfrak{b}) \,\big|\, \boldsymbol{m} \in \mathcal{S}\big\}$. Hence, we can make $\operatorname{ord}_q(U_{1,\mathfrak{a},\mathfrak{b}})$ sufficiently large for any $q = 1,\ldots,k$ after a finite procedure. So, we have arrived at a lift $\widetilde{\boldsymbol{v}}^{(2)}$, for which the entries of $U_{1,\mathfrak{a},\mathfrak{b}}$ are contained in $\mathcal{O}_{\mathcal{X}}$.

Let us show $\mathrm{ord}_q(U_{j,\mathfrak{a},\mathfrak{b}}) \geqslant 0$ for any $q = 1, \ldots, k$ and for any j. We have the subset $\mathcal{S} \subset \mathbb{Z}^k$ bounded below with respect to $\leqslant_{\mathbb{Z}^k}$, and the expansion as follows:

$$U_{j,\mathfrak{a},\mathfrak{b}} = \sum_{\boldsymbol{n} \in \mathcal{S}} U_{j,\mathfrak{a},\mathfrak{b},\boldsymbol{n}}(z_{k+1}, \ldots, z_n)\, \boldsymbol{z}^{\boldsymbol{n}}, \quad U_{j,\mathfrak{a},\mathfrak{b},\boldsymbol{n}} \neq 0.$$

Let $\boldsymbol{n}_0$ be a minimal element of $\mathcal{S}$. Assume $\boldsymbol{n}_0 \notin \mathbb{Z}^k_{\geqslant 0}$. Let us look at the $\boldsymbol{z}^{\boldsymbol{n}_0 + \mathrm{ord}(\mathfrak{a}-\mathfrak{b})}$-term of (18) with $p = 1$. Note that $V_j(\mathfrak{a} - \mathfrak{b})\, U_{1,\mathfrak{a},\mathfrak{b}}$ does not have the $\boldsymbol{z}^{\boldsymbol{n}_0 + \mathrm{ord}(\mathfrak{a}-\mathfrak{b})}$-term, because the entries of $U_{1,\mathfrak{a},\mathfrak{b}}$ are contained in $\mathcal{O}_{\mathcal{X}}$. Hence, we obtain $U_{j,\mathfrak{a},\mathfrak{b},\boldsymbol{n}_0}\, (\mathfrak{a} - \mathfrak{b})_{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})} = 0$, and thus $U_{j,\mathfrak{a},\mathfrak{b},\boldsymbol{n}_0} = 0$ which contradicts with our choice of $\mathcal{S}$. Hence we have $\mathcal{S} \in \mathbb{Z}^k_{\geqslant 0}$.

Therefore, we have arrived at a lift $\widetilde{\boldsymbol{v}}^{(2)}$ for which the entries of U_j are contained in $\mathcal{O}_{\mathcal{X}}$. Let $\mathcal{L}^{(0)}$ be the submodule of $\mathcal{M}^{(0)}$ generated by $(\boldsymbol{v}^{(1)}, \widetilde{\boldsymbol{v}}^{(2)})$ over $\mathcal{O}_{\mathcal{X}}$. By Proposition 2.2.8, $\mathcal{L}^{(0)}$ is an unramifiedly good lattice. We have the exact sequence $0 \to \mathcal{L}^{(1)} \to \mathcal{L}^{(0)} \to \mathcal{L}^{(2)} \to 0$, and we can easily deduce that $\mathcal{L}^{(0)}$ is also Deligne-Malgrange. □

2.2.4. Preliminary from the one variable case (Appendix). — Let k be an integral domain over $\boldsymbol{C}$. We consider $\mathcal{R}_0 := k[\![t]\!]$ and $\mathcal{R} := k(\!(t)\!)$, which are naturally equipped with a derivation ∂_t. An $\mathcal{R}$-module $\mathcal{M}$ is called differential module, if it is equipped with the action of ∂_t such that $\partial_t(f\, s) = \partial_t(f)\, s + f\, \partial_t s$ for $f \in \mathcal{R}$ and $s \in \mathcal{M}$. We recall some basic facts on differential $\mathcal{R}$-modules from [**51**] for reference in our argument.

2.2.4.1. Extension of decomposition. — Let $\mathcal{M}$ be a finitely generated differential $\mathcal{R}$-free module with an $\mathcal{R}_0$-free lattice $\mathcal{L}$ such that $t^{M+1}\partial_t\mathcal{L} \subset \mathcal{L}$ for some $M > 0$. Note that we have an induced endomorphism G of $\mathcal{L} \otimes_{\mathcal{R}_0} k$. Assume that there exists decomposition $(\mathcal{L} \otimes_{\mathcal{R}_0} k, G) = (V_1, G_1) \oplus (V_2, G_2)$. For $i \neq j$, we have the endomorphism $\widetilde{G}_{i,j}$ of $Hom(V_i, V_j)$ given by $\widetilde{G}_{i,j}(f) = f \circ G_i - G_j \circ f$.

Lemma 2.2.16. — *If $\widetilde{G}_{i,j}$ are invertible for $(i,j) = (1,2), (2,1)$, then we have a decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ such that (i) $t^{M+1}\partial_t\mathcal{L}_i \subset \mathcal{L}_i$, (ii) $\mathcal{L}_i \otimes k = V_i$.*

Proof. — We give only a sketch of a proof, by following [**51**]. Let $\boldsymbol{v}$ be a frame of $\mathcal{L}$ with a decomposition $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ such that $\boldsymbol{v}_{i|t=0}$ give frames of V_i. Let A be the $\mathcal{R}_0$-valued matrices determined by $t^{M+1}\partial_t \boldsymbol{v} = \boldsymbol{v}\, A$. Then, A has the following decomposition corresponding to $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$:

$$A = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Here, Ω_i are k-valued matrices determined by $G_i \boldsymbol{v}_i = \boldsymbol{v}_i\, \Omega_i$, and $A_{i,j}$ are $t\mathcal{R}_0$-valued matrices. We consider a change of basis of the following form:

$$\boldsymbol{v}' = \boldsymbol{v}\, G, \quad G = I + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}.$$

Here, the entries of X and Y are contained in $t\mathcal{R}_0$. We would like to take G such that

$$t^{M+1}\partial_t \boldsymbol{v}' = \boldsymbol{v}'\, B, \quad B = \begin{pmatrix} \Omega_1 + Q_1 & 0 \\ 0 & \Omega_2 + Q_2 \end{pmatrix}. \tag{20}$$

The relation between A, G and B are given by $AG + t^{M+1}\partial_t G = GB$. We obtain the equations $A_{11} + A_{12}Y + Q_1 = 0$ and $\Omega_2 Y + A_{21} + A_{22}Y + t^{M+1}\partial_t Y = Y\Omega_1 + YQ_1$. By eliminating Q_1, we obtain the equation

$$\Omega_2 Y - Y\Omega_1 + A_{21} + A_{22}Y + t^{M+1}\partial_t Y + Y(A_{11} + A_{12}Y) = 0. \tag{21}$$

By the assumption, we have the invertibility of the endomorphism on the space of k-valued (r_2, r_1)-matrices, given by $Z \mapsto \Omega_2 Z - Z\Omega_1$, where $r_i := \operatorname{rank} \mathcal{L}_i$ $(i = 1, 2)$. By using a t-expansion, we can find a solution of (21) in the space of $t\mathcal{R}_0$-valued matrices. Similarly, we can find desired X and Q_2. □

2.2.4.2. Uniqueness

Lemma 2.2.17. — *Let $\mathcal{M}$ be an $\mathcal{R}$-free differential module. Assume that there exist an $\mathcal{R}_0$-free lattice $\mathcal{L} \subset \mathcal{M}$ and $\mathfrak{a} \in \mathcal{R} \smallsetminus \mathcal{R}_0$ such that $t\partial_t - t\partial_t\mathfrak{a}$ preserves $\mathcal{L}$. Then, any flat section of $\mathcal{M}$ is* 0.

Proof. — Take $f \in \mathcal{M}$ such that $\partial_t f = 0$. Assume $f \neq 0$, and we will deduce a contradiction. We can take $N \in \mathbb{Z}$ such that $t^N f \in \mathcal{L}$ and the induced element of $\mathcal{L}/t\mathcal{L}$ is non-zero. By the assumption, we have

$$\mathcal{L} \ni (t\partial_t - t\partial_t\mathfrak{a})(t^N f) = (N - t\partial_t\mathfrak{a})t^N f.$$

But, it is easy to see that $(N - t\partial_t\mathfrak{a})t^N f \notin \mathcal{L}$, and thus we have arrived at a contradiction. □

Let $\mathcal{M}_i$ $(i = 1, 2)$ be differential $\mathcal{R}$-free modules with $\mathcal{R}_0$-free lattices $\mathcal{L}_i$ such that $t\partial_t - t\partial_t\mathfrak{a}_i$ preserves $\mathcal{L}_i$.

Corollary 2.2.18. — *Assume $\mathfrak{a}_1 - \mathfrak{a}_2 \neq 0$ in $\mathcal{R}/\mathcal{R}_0$. Then, any flat morphism $\mathcal{M}_1 \to \mathcal{M}_2$ is* 0. □

2.2.4.3. Let $\mathcal{M}$ be a differential $\mathcal{R}$-module. Let E be an $\mathcal{R}_0$-lattice of $\mathcal{M}$ such that $t^{m+1}\partial_t E \subset E$ for some $m > 0$. We have the induced endomorphism G of $E_{|t=0}$.

Lemma 2.2.19. — *Let $s \in \mathcal{M}$. If G is invertible, we have $\partial_t s = 0$ if and only if $s = 0$.* □

Let E_i $(i = 1, 2)$ be lattices of $\mathcal{M}$ such that $t^{m_i+1}\partial_t E_i \subset E_i$ for some $m_i > 0$. Let G_i be the endomorphism of $E_{i|t=0}$ induced by $t^{m_i+1}\partial_t$.

Lemma 2.2.20. — *Assume that G_i are semisimple and non-zero. Let T_i be the set of eigenvalues of G_i. Then, we have $m_1 = m_2$ and $T_1 = T_2$.*

Proof. — By extending k, we may assume that the eigenvalues of G_i are contained in k. We have a ∂_t-decomposition $E_i = \bigoplus_{\mathfrak{b}\in T_i} E_{i,\mathfrak{b}}$ such that $E_{i,\mathfrak{b}|t=0}$ is the eigen space of G_i corresponding to $\mathfrak{b}$. We have the induced map $\varphi_{\mathfrak{c},\mathfrak{b}} : E_{1,\mathfrak{b}} \otimes \mathcal{R} \to E_{2,\mathfrak{c}} \otimes \mathcal{R}$. If $m_1 \neq m_2$ or if $m_1 = m_2$ but $\mathfrak{b} \neq \mathfrak{c}$, we have $\varphi_{\mathfrak{c},\mathfrak{b}} = 0$ by Lemma 2.2.19. Then, the claim of Lemma 2.2.20 follows. □

2.3. Good lattice of meromorphic ϱ-flat bundle

2.3.1. Definition. — Let $\mathcal{X} \to \mathcal{K}$ be a smooth fibration of complex manifolds. Let $\mathcal{D}$ be a simple normal crossing hypersurface of $\mathcal{X}$ such that all intersections of irreducible components are smooth over $\mathcal{K}$. We will implicitly assume that the number of irreducible components of $\mathcal{D}$ is finite. Let ϱ be a holomorphic function on $\mathcal{K}$. For a point P of $\mathcal{X}$, let $\widehat{P}$ denote the completion of $\mathcal{X}$ at P. In the following, for a given $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, let $\mathcal{F}_{|\widehat{P}}$ denote $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\widehat{P}}$.

Let $(\mathcal{E}, \mathbb{D})$ be a meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ relative to $\mathcal{K}$, i.e., $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-coherent sheaf with a flat ϱ-connection $\mathbb{D} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}$ relative to $\mathcal{K}$. (A flat ϱ-connection is defined in a standard way as in the formal case. See Subsection 2.2.1. We will often omit "relative to $\mathcal{K}$" if there is no risk of confusion.)

Definition 2.3.1. — A lattice E of $\mathcal{E}$ is called unramifiedly good at $P \in \mathcal{D}$, if $E_{|\widehat{P}}$ is an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})_{|\widehat{P}}$. If E is unramifiedly good at each point of $\mathcal{D}$, E is called an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})$. □

Notation 2.3.2. — The set of irregular values of $(E, \mathbb{D})_{|\widehat{P}}$ is often denoted by $\mathrm{Irr}(E, \mathbb{D}, P)$, $\mathrm{Irr}(E, P)$ or $\mathrm{Irr}(\mathbb{D}, P)$. □

For $P \in \mathcal{D}$, let $\mathcal{X}_P$ denote a small neighbourhood of P in $\mathcal{X}$, and put $\mathcal{D}_P := \mathcal{X}_P \cap \mathcal{D}$.

Definition 2.3.3

- $(E, \mathbb{D})$ is called good at P, if there exist a small neighbourhood $\mathcal{X}_P$ and a ramified covering $\varphi_P : (\mathcal{X}'_P, \mathcal{D}'_P) \to (\mathcal{X}_P, \mathcal{D}_P)$ such that E is the descent of an unramifiedly good lattice E' of $\varphi_P^*\mathcal{E}$.
- E is called good, if E is good at each point of $\mathcal{D}$. □

In the condition of Definition 2.3.3, such E' is not unique, even if φ_P is fixed. We also remark that $\varphi_P^* E$ is not necessarily unramifiedly good.

Remark 2.3.4. — We will often say that $(E, \mathbb{D})$ is (unramifiedly) good on $(\mathcal{X}, \mathcal{D})$, if E is a (unramifiedly) good lattice of a meromorphic ϱ-flat bundle $\big(E(*\mathcal{D}), \mathbb{D}\big)$ on $(\mathcal{X}, \mathcal{D})$. □

Definition 2.3.5. — A meromorphic ϱ-flat bundle is called (unramifiedly) good, if it locally has an (unramifiedly) good lattice. □

If $\mathcal{K}$ is a point and $\varrho \neq 0$, a good meromorphic ϱ-flat bundle has a global good lattice. Actually, it is given by a Deligne-Malgrange lattice. (See Section 2.7.)

2.3.2. Some functoriality. — Let E be a good lattice of a meromorphic ϱ-flat bundle $(\mathcal{E}, \mathbb{D})$ on $(\mathcal{X}, \mathcal{D})$. Let $E^\vee$ denote the dual of E in the category of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathcal{E}^\vee$ denote the dual of $\mathcal{E}$ in the category of $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-modules. We have $\mathcal{E}^\vee = E^\vee \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. We have the naturally induced flat ϱ-connection $\mathbb{D}$ of $\mathcal{E}^\vee$. We have the following functoriality for taking the dual object.

- $E^\vee$ is a good lattice of $(\mathcal{E}^\vee, \mathbb{D})$. If E is unramified, $E^\vee$ is also unramified. For each $P \in \mathcal{D}$, we have $\mathrm{Irr}(E^\vee, P) = \{-\mathfrak{a} \mid \mathfrak{a} \in \mathrm{Irr}(E, P)\}$.

Let E_i $(i = 1, 2)$ be unramifiedly good lattices of $(\mathcal{E}_i, \mathbb{D}_i)$. We have the following functoriality for tensor product and direct sum.

- If $\mathrm{Irr}(E_1, P) \otimes \mathrm{Irr}(E_2, P) := \{\mathfrak{a}_1 + \mathfrak{a}_2 \mid \mathfrak{a}_i \in \mathrm{Irr}(E_i, P)\}$ is good for any $P \in \mathcal{D}$, then $E_1 \otimes E_2$ is an unramifiedly good lattices of $(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathbb{D})$ with $\mathrm{Irr}(E_1 \otimes E_2, P) = \mathrm{Irr}(E_1, P) \otimes \mathrm{Irr}(E_2, P)$.
- If $\mathrm{Irr}(E_1, P) \oplus \mathrm{Irr}(E_2, P) = \mathrm{Irr}(E_1) \cup \mathrm{Irr}(E_2, P)$ is good for any $P \in \mathcal{D}$, then $E_1 \oplus E_2$ is an unramifiedly good lattices of $(\mathcal{E}_1 \oplus \mathcal{E}_2, \mathbb{D})$ with $\mathrm{Irr}(E_1 \oplus E_2, P) = \mathrm{Irr}(E_1, P) \oplus \mathrm{Irr}(E_2, P)$.

Let $\mathcal{X}_1$ be a complex manifold with a normal crossing hypersurface $\mathcal{D}_1$. Let $F : \mathcal{X}_1 \to \mathcal{X}$ be a morphism such that (i) $F^{-1}(\mathcal{D}) \subset \mathcal{D}_1$, (ii) the induced morphism $\mathcal{X}_1 \to \mathcal{K}$ is a smooth fibration, (iii) any intersection of some irreducible components of $\mathcal{D}_1$ is smooth over $\mathcal{K}$. Let E be a good lattice of $(\mathcal{E}, \mathbb{D})$ on $(\mathcal{X}, \mathcal{D})$. We have the following functoriality for the pull-back.

- F^*E is a good lattice of $F^*(\mathcal{E}, \mathbb{D}) = F^{-1}(\mathcal{E}, \mathbb{D}) \otimes \mathcal{O}_{\mathcal{X}_1}(*\mathcal{D}_1)$. If E is unramifiedly good, F^*E is also unramifiedly good, and we have

$$\mathrm{Irr}(F^*E, P) = \{F^*\mathfrak{a} \mid \mathfrak{a} \in \mathrm{Irr}(E, F(P))\}.$$

2.3.3. A criterion for a lattice to be good. — Let $\mathcal{X}$, $\mathcal{D}$ and $(\mathcal{E}, \mathbb{D})$ be as in Subsection 2.3.1. Let E be a lattice of $\mathcal{E}$. Assume that we are given the following:

- a good set of irregular values $\mathcal{I} \subset M(\mathcal{X}, \mathcal{D})/H(\mathcal{X})$,
- a holomorphic decomposition $E = \bigoplus_{\mathfrak{a} \in \mathcal{I}} E_{\mathfrak{a}}$,

such that, if $p_{\mathfrak{a}}$ denotes the projection onto $E_{\mathfrak{a}}$, and if we put $\Phi := \sum_{\mathfrak{a} \in \mathcal{I}} d\mathfrak{a} \cdot p_{\mathfrak{a}}$, then $\mathbb{D}^{(0)} := \mathbb{D} - \Phi$ is logarithmic with respect to E. (Note that we do not assume $\mathbb{D}^{(0)}$ is flat.)

We obtain the following proposition as a corollary of Proposition 2.2.8. It will be useful in the proof of Theorem 7.4.5.

Proposition 2.3.6. — *E is an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})$. For any $P \in \mathcal{D}$, the set* $\mathrm{Irr}(\mathbb{D}, P)$ *is equal to the image of $\mathcal{I}$ via $M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \to \mathcal{O}_{\widehat{P}}(*\mathcal{D})/\mathcal{O}_{\widehat{P}}$.* □

2.3.4. A family of good lattices is good lattice. — Let $\mathcal{X}$, $\mathcal{D}$, $(\mathcal{E}, \mathbb{D})$ be as in Subsection 2.3.1. For each $y \in \mathcal{K}$, we set $\mathcal{X}^y := \{y\} \times_{\mathcal{K}} \mathcal{X}$ and $\mathcal{D}^y := \{y\} \times_{\mathcal{K}} \mathcal{X}$. We have the induced meromorphic $\varrho(y)$-flat bundle $(\mathcal{E}^y, \mathbb{D}^y)$ on $(\mathcal{X}^y, \mathcal{D}^y)$. Let $\mathcal{I} \subset M(\mathcal{X}, \mathcal{D})/H(\mathcal{X})$ be a good set of irregular values. The image of $\mathcal{I}$ via $M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \to \mathcal{O}_{\widehat{P}}(*\mathcal{D})/\mathcal{O}_{\widehat{P}}$ is denoted by $\mathcal{I}_{\widehat{P}}$. The image via $M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \to M(\mathcal{X}^y, \mathcal{D}^y)/H(\mathcal{X}^y)$ is denoted by $\mathcal{I}^y$. If P is contained in $\mathcal{X}^y$, let $\mathcal{O}^y_{\widehat{P}}$ be the completion of the local ring $\mathcal{O}_{\mathcal{X}^y, P}$, and let $\mathcal{I}^y_{\widehat{P}}$ denote the image of $\mathcal{I}^y$ via $M(\mathcal{X}^y, \mathcal{D}^y)/H(\mathcal{X}^y) \to \mathcal{O}^y_{\widehat{P}}(*\mathcal{D})/\mathcal{O}^y_{\widehat{P}}$.

Proposition 2.3.7. — *Let E be a lattice of $\mathcal{E}$. The following conditions are equivalent.*

- *E is unramifiedly good, and* $\mathrm{Irr}(\mathbb{D}, P) = \mathcal{I}_{\widehat{P}}$ *for any $P \in \mathcal{D}$.*
- *For each $y \in \mathcal{K}$, the specialization $E^y = E \otimes \mathcal{O}_{\mathcal{X}^y}$ is an unramifiedly good lattice of $(\mathcal{E}^y, \mathbb{D}^y)$, and* $\mathrm{Irr}(\mathbb{D}^y, P) = \mathcal{I}^y_{\widehat{P}}$ *$(P \in \mathcal{D}_y)$.*

Proof. — It is easy to see that the first condition implies the second one. We will show the converse. We only have to consider the case $\mathcal{X} = \Delta^n \times \mathcal{K}$ and $\mathcal{D} = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $O = (0, \ldots, 0) \in \Delta^n$. We only have to show that the completion of E along $O \times \mathcal{K}$ has the irregular decomposition with the set of irregular values $\mathcal{I}$.

Let $H(\mathcal{K})$ denote the space of holomorphic functions on $\mathcal{K}$. For each $y \in \mathcal{K}$, we have the specialization $\mathrm{eval}_y : H(\mathcal{K}) \to \boldsymbol{C}$ given by $\mathrm{eval}_y(f) = f(y)$. We put $R_0 := H(\mathcal{K})[\![z_1, \ldots, z_n]\!]$ and $k_0 := \boldsymbol{C}[\![z_1, \ldots, z_n]\!]$, and let R (resp. k) be the localization of R_0 (resp. k_0) with respect to z_i $(i = 1, \ldots, \ell)$. The natural morphism $\mathrm{eval}_y : R_0 \to k_0$ induces a functor from the category of R_0-modules to the category of k_0-modules. The image of an R_0-module E is denoted by E^y. We use the symbol $\mathcal{E}^y$ for an R-module $\mathcal{E}$ in a similar meaning. To show Proposition 2.3.7, we only have to show the following lemma.

Lemma 2.3.8. — *Let $\mathcal{I} \subset R/R_0$ be a good set of irregular values. For any $y \in \mathcal{K}$, let $\mathcal{I}^y \subset k/k_0$ denote the specialization of $\mathcal{I}$ at y.*

Let $(\mathcal{E}, \mathbb{D})$ be a meromorphic ϱ-flat bundle over R. Let E be a free R_0-lattice of $\mathcal{E}$ such that for each $y \in \mathcal{K}$ the restriction E^y is an unramifiedly good lattice of $(\mathcal{E}^y, \mathbb{D}^y)$ with $\mathrm{Irr}(\mathbb{D}^y) = \mathcal{I}^y$*, i.e., we have a decomposition $(E^y, \mathbb{D}^y) = \bigoplus_{\mathfrak{a} \in \mathcal{I}}(E^y_{\mathfrak{a}}, \mathbb{D}^y_{\mathfrak{a}})$ such that $\mathbb{D}^y_{\mathfrak{a}}$ are $\mathfrak{a}^y$-logarithmic.*

Then, E is an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})$ with $\mathrm{Irr}(\mathbb{D}) = \mathcal{I}$.

Proof. — We use an induction on $\bigl|\mathcal{T}(\mathcal{I})\bigr|$ or $|\mathcal{I}|$. (See Section 2.1.2 for $\mathcal{T}(\mathcal{I})$.) By considering the tensor product with a meromorphic ϱ-flat line bundle, we may assume $\min\bigl\{\mathrm{ord}\,\mathfrak{a} \bigm| \mathfrak{a} \in \mathcal{T}(\mathcal{I})\bigr\} = \min\bigl\{\mathrm{ord}\,\mathfrak{a} \bigm| \mathfrak{a} \in \mathcal{I}\bigr\}$. Let us take an auxiliary sequence $\boldsymbol{m}(0), \ldots, \boldsymbol{m}(L)$ for $\mathcal{I}$. (We use $\boldsymbol{m}(0)$ and $\boldsymbol{m}(1)$.) We have $\boldsymbol{m}(1) = \boldsymbol{m}(0) + \boldsymbol{\delta}_j$ for some j. Let F denote the endomorphism of $E_{|z_j=0}$ induced by $\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathbb{D}(z_j\partial_j)$. The eigenvalues of F are given by the set $T = \bigl\{(\boldsymbol{z}^{-\boldsymbol{m}(0)} z_j\partial_j\mathfrak{a})_{|z_j=0} \bigm| \mathfrak{a} \in \mathcal{I}\bigr\}$. Hence, we have the eigen-decomposition:

$$E_{|z_j=0} = \bigoplus_{\mathfrak{b}\in T} \mathbb{E}_{\mathfrak{b}}. \tag{22}$$

Lemma 2.3.9. — *We can take a $\mathbb{D}$-flat decomposition $E = \bigoplus_{\mathfrak{b}\in T} E_{\mathfrak{b}}$ such that $E_{\mathfrak{b}|z_j=0} = \mathbb{E}_{\mathfrak{b}}$.*

Proof. — We only give an outline of the proof. Let $\boldsymbol{v}$ be a frame of E whose restriction to $z_j = 0$ is compatible with (22). We have the decomposition $\boldsymbol{v} = (\boldsymbol{v}_{\mathfrak{b}})$ corresponding to the decomposition (22). We have the following:

$$\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathbb{D}(z_j\partial_j)\boldsymbol{v} = \boldsymbol{v}\left(\bigoplus_{\mathfrak{b}\in T}\Omega_{\mathfrak{b}} + z_j\, B\right).$$

Here, the entries of $\Omega_{\mathfrak{b}}$ and B are regular, and $\Omega_{\mathfrak{b}|O}$ ($\mathfrak{b} \in T$) have no common eigenvalues. Applying an argument in [**51**] (or the argument in Subsection 2.2.2.2), we can take $\boldsymbol{v}$ for which B is block diagonal, i.e., $B = \bigoplus_{\mathfrak{b}\in T} B_{\mathfrak{b}}$. Let $E_{\mathfrak{b}}$ be the R_0-submodule generated by $\boldsymbol{v}_{\mathfrak{b}}$. It can be shown that $E_{\mathfrak{b}}$ is $\mathbb{D}$-flat using the argument in the proof of Lemma 2.2.10. Thus Lemma 2.3.9 is proved. □

Let us return to the proof of Lemma 2.3.8. For $\mathfrak{b} \in T$, let $\mathcal{I}(\mathfrak{b})$ denote the inverse image of $\mathfrak{b}$ by the natural map $\mathcal{I} \to T$. Its specialization at y is denoted by $\mathcal{I}(\mathfrak{b})^y$. We can deduce $E^y_{\mathfrak{b}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}(\mathfrak{b})^y} E^y_{\mathfrak{a}}$. Then, we can apply the inductive assumption to each $(E_{\mathfrak{b}}, \mathbb{D}_{\mathfrak{b}})$. Therefore, we obtain Lemma 2.3.8 and thus Proposition 2.3.7. □

2.4. Decompositions

2.4.1. Openness property. — Let $\mathcal{X}$ and $\mathcal{D}$ be as in Section 2.3.1. Let $(\mathcal{E}, \mathbb{D})$ be a meromorphic flat bundle on $(\mathcal{X}, \mathcal{D})$. Let E be a lattice of $\mathcal{E}$. Assume that it is unramifiedly good at a point $P \in \mathcal{D}$, i.e., there exist a good set of irregular values $\mathrm{Irr}(\mathbb{D}, P) \subset \mathcal{O}_{\widehat{P}}(*\mathcal{D})/\mathcal{O}_{\widehat{P}}$ and a decomposition

$$(E, \mathbb{D})_{|\widehat{P}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P)} ({}^P\widehat{E}_{\mathfrak{a}}, {}^P\widehat{\mathbb{D}}_{\mathfrak{a}})$$

such that ${}^P\widehat{\mathbb{D}}_{\mathfrak{a}} - d\mathfrak{a}$ are logarithmic with respect to $\widehat{E}_{\mathfrak{a}}$. For a small neighbourhood $\mathcal{X}_P$ of P, set $\mathcal{D}_P := \mathcal{D} \cap \mathcal{X}_P$. We will prove the following proposition in Subsection 2.4.6.

Proposition 2.4.1. — *If $\mathcal{X}_P$ is sufficiently small, the following claims hold:*
- $\mathrm{Irr}(\mathbb{D}, P) \subset M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P)$, *i.e., it is contained in the image of the morphism $M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P) \to \mathcal{O}_{\widehat{P}}(*\mathcal{D}_P)/\mathcal{O}_{\widehat{P}}$.*
- *E is unramifiedly good at any point of $P' \in \mathcal{D}_P$.*
- *The good set of irregular values* $\mathrm{Irr}(\mathbb{D}, P')$ *of $(E, \mathbb{D})_{|\widehat{P}'}$ is the image of* $\mathrm{Irr}(\mathbb{D}, P)$ *by $M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}) \to \mathcal{O}_{\widehat{P}'}(*\mathcal{D})/\mathcal{O}_{\widehat{P}'}$.*

2.4.1.1. Good system of irregular values. — Before proceeding, we give a consequence and prepare a terminology. When we are given an unramifiedly good lattice $(E, \mathbb{D})$ of a meromorphic ϱ-flat bundle, we put $\operatorname{Irr}(\mathbb{D}) := \{\operatorname{Irr}(\mathbb{D}, P) \mid P \in \mathcal{D}\}$. Then, Proposition 2.4.1 says that $\operatorname{Irr}(\mathbb{D})$ is a good system of irregular values in the following sense.

Definition 2.4.2. — A system $\mathcal{I}$ of finite subsets $\mathcal{I}_P \subset \mathcal{O}_{\mathcal{X}}(*\mathcal{D})_P/\mathcal{O}_{\mathcal{X},P}$ $(P \in \mathcal{D})$ is called a good system of irregular values, if the following holds for each $P \in \mathcal{D}$:

- Take a neighbourhood $\mathcal{X}_P$ of P such that $\mathcal{I}_P \subset M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P)$, where $\mathcal{D}_P = \mathcal{X}_P \cap \mathcal{D}$. Then, for each $P' \in \mathcal{D}_P$, $\mathcal{I}_{P'}$ is the image of $\mathcal{I}_P$ via $M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P) \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})_{P'}/\mathcal{O}_{\mathcal{X},P'}$. □

Remark 2.4.3. — A good set of irregular values $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}(*\mathcal{D})_P/\mathcal{O}_{\mathcal{X},P}$ naturally induces a good system of irregular values on a neighbourhood of P. In that case, we will not distinguish the induced system and $\mathcal{I}$. □

2.4.2. Decompositions along the intersection of irreducible components

We will also prove a refinement of the second and third claims of Proposition 2.4.1. For simplicity, let us consider the case $\mathcal{X} = \Delta^n \times \mathcal{K}$ and $\mathcal{D} = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. We put $\mathcal{D}_i := \{z_i = 0\}$. For a subset $I \subset \underline{\ell}$, we put $\mathcal{D}_I := \bigcap_{i \in I} \mathcal{D}_i$ and $\mathcal{D}(I) := \bigcup_{i \in I} \mathcal{D}_i$. The complement $\underline{\ell} \setminus I$ is denoted by I^c. Let $\widehat{\mathcal{D}}_I$ and $\widehat{\mathcal{D}}(I)$ denote the completion of $\mathcal{X}$ along $\mathcal{D}_I$ and $\mathcal{D}(I)$, respectively. (See [4], [8] and [**50**]. See also a brief review in Subsection 22.5.1.)

We may assume $P \in \mathcal{D}_{\underline{\ell}}$. For a given small neighbourhood $\mathcal{X}_P$ of P in $\mathcal{X}$, we put $\mathcal{D}_{I,P} := \mathcal{D}_I \cap \mathcal{X}_P$ and $\mathcal{D}(I)_P := \mathcal{D}(I) \cap \mathcal{X}_P$.

Let $(E, \mathbb{D})$ be unramifiedly good at P. Once we know $\operatorname{Irr}(\mathbb{D}, P)$ is contained in $M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P)$, let $\operatorname{Irr}(\mathbb{D}, I)$ denote the image of $\operatorname{Irr}(\mathbb{D}, P)$ via $M(\mathcal{X}_P, \mathcal{D}_P)/H(\mathcal{X}_P) \to M(\mathcal{X}_P, \mathcal{D}(I^c)_P)$. We will prove the following proposition in Subsection 2.4.6.

Proposition 2.4.4. — *Let $\mathcal{X}_P$ be a sufficiently small neighbourhood of P in $\mathcal{X}$. For any subset $I \subset \underline{\ell}$, we have a decomposition*

$$(E, \mathbb{D})_{|\widehat{\mathcal{D}}_{I,P}} = \bigoplus_{\mathfrak{b} \in \operatorname{Irr}(\mathbb{D}, I)} ({}^I\widehat{E}_{\mathfrak{b}}, {}^I\widehat{\mathbb{D}}_{\mathfrak{b}}) \tag{23}$$

such that $({}^I\widehat{\mathbb{D}}_{\mathfrak{b}} - d\mathfrak{b})({}^I\widehat{E}_{\mathfrak{b}}) \subset {}^I\widehat{E}_{\mathfrak{b}} \otimes \Big(\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}(I)) + \Omega^1_{\mathcal{X}/\mathcal{K}}(*\mathcal{D}(I^c))\Big)_{|\mathcal{X}_P}$, where we take a lift of $\mathfrak{b}$ to $M(\mathcal{X}_P, \mathcal{D}_P)$.

The decomposition (23) is called the irregular decomposition of $(E, \mathbb{D})_{|\widehat{\mathcal{D}}_{I,P}}$. It induces the irregular decomposition at any point $P' \in \mathcal{D}_{I,P} \setminus \bigcup_{j \notin I} \mathcal{D}_j$. In that sense, Proposition 2.4.4 refines the second and third claims in Proposition 2.4.1.

Remark 2.4.5. — The property of Proposition 2.4.4 was adopted as definition of "unramifiedly good at P" in the older version of this monograph. □

Remark 2.4.6. — For $\mathfrak{b} \in \mathrm{Irr}(\mathbb{D}, I)$, let $\mathrm{Irr}(\mathbb{D}, P, \mathfrak{b})$ be the inverse image of $\mathfrak{b}$ via the natural map $\mathrm{Irr}(\mathbb{D}, P) \to \mathrm{Irr}(\mathbb{D}, I)$. If we are given the decomposition (23), we have ${}^I\widehat{E}_{\mathfrak{b}|\widehat{P}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P,\mathfrak{b})} {}^P\widehat{E}_{\mathfrak{a}}$. Hence, it is easy to deduce

$$ {}^I\widehat{E}_{\mathfrak{c}|\widehat{\mathcal{D}}_{J,P}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\mathbb{D},J,\mathfrak{c})} {}^J\widehat{E}_{\mathfrak{b}} $$

for $I \subset J$ and $\mathfrak{c} \in \mathrm{Irr}(\mathbb{D}, I)$, where $\mathrm{Irr}(\mathbb{D}, J, \mathfrak{c})$ is the inverse image of $\mathfrak{c}$ via the natural map $\mathrm{Irr}(\mathbb{D}, J) \to \mathrm{Irr}(\mathbb{D}, I)$. □

2.4.3. Decomposition along the union of irreducible components

We continue to use the setting in Subsection 2.4.2. By shrinking $\mathcal{X}$, we assume $\mathcal{X} = \mathcal{X}_P$. We will often need a decomposition on the completion along $\mathcal{D}(I)$ for some $I \subset \underline{\ell}$. For simplicity, let us take an admissible coordinate system (Remark 2.1.4) for the good set $\mathrm{Irr}(\mathbb{D}, P)$, and let us consider decompositions along $\mathcal{D}(\underline{j})$ for $1 \leqslant j \leqslant \ell$, where $\underline{j} := \{1, \dots, j\}$.

Let $\mathrm{Irr}(\mathbb{D}, j)$ and $\mathrm{Irr}'(\mathbb{D}, j)$ denote the images of $\mathrm{Irr}(\mathbb{D}, P)$ via the following natural maps:

$$ M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \longrightarrow M(\mathcal{X}, \mathcal{D})/M\big(\mathcal{X}, \mathcal{D}(\underline{\ell} \setminus \{j\})\big) $$
$$ M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \longrightarrow M(\mathcal{X}, \mathcal{D})/M(\mathcal{X}, \mathcal{D}(\underline{j-1})) $$

Note that the natural map $\mathrm{Irr}'(\mathbb{D}, j) \to \mathrm{Irr}(\mathbb{D}, j)$ is bijective by our choice of the coordinate system, via which we identify them. We have the naturally defined maps $\mathrm{Irr}'(\mathbb{D}, i) \to \mathrm{Irr}'(\mathbb{D}, j)$ for any $i \leqslant j$, which induces $\pi_{j,i} : \mathrm{Irr}(\mathbb{D}, i) \to \mathrm{Irr}(\mathbb{D}, j)$.

Lemma 2.4.7. — *There is the following decomposition:*

$$ (24) \qquad (E, \mathbb{D})_{|\widehat{\mathcal{D}}(\underline{j})} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\mathbb{D},j)} \big(\widehat{E}_{\mathfrak{b},\widehat{\mathcal{D}}(\underline{j})}, \mathbb{D}_{\mathfrak{b}}\big) \quad \text{such that } \widehat{E}_{\mathfrak{b},\widehat{\mathcal{D}}(\underline{j})|\widehat{\mathcal{D}}_i} = \bigoplus_{\substack{\mathfrak{c}\in\mathrm{Irr}(\mathbb{D},i)\\ \pi_{j,i}(\mathfrak{c})=\mathfrak{b}}} {}^i\widehat{E}_{\mathfrak{c}}. $$

Proof. — For $J \subset \underline{\ell}$, let $i(J)$ be the number determined by $\underline{i(J)} \cap J = \varnothing$ and $i(J) + 1 \in J$. We have the maps

$$ M(\mathcal{X}, \mathcal{D})/H(\mathcal{X}) \xrightarrow{p_1} M(\mathcal{X}, \mathcal{D})/M(\mathcal{X}, \mathcal{D}(\underline{i(J)})) \xrightarrow{p_2} M(\mathcal{X}, \mathcal{D})/M(\mathcal{X}, \mathcal{D}(J^c)). $$

Let $\mathrm{Irr}'(\mathbb{D}, J)$ denote the image of $\mathrm{Irr}(\mathbb{D}, P)$ by p_1. Then, p_2 naturally gives a bijection $\mathrm{Irr}'(\mathbb{D}, J) \to \mathrm{Irr}(\mathbb{D}, J)$, by which we identify them. If $j > i(J)$, we have the naturally defined map $\mathrm{Irr}'(\mathbb{D}, J) \to \mathrm{Irr}(\mathbb{D}, j)$, which induces $\pi_{j,J} : \mathrm{Irr}(\mathbb{D}, J) \to \mathrm{Irr}(\mathbb{D}, j)$. For $\mathfrak{b} \in \mathrm{Irr}(\mathbb{D}, j)$, we put ${}^J\widehat{E}_{\mathfrak{b}} := \bigoplus_{\mathfrak{a}\in\pi_{j,J}^{-1}(\mathfrak{b})} {}^J\widehat{E}_{\mathfrak{a}}$. By Remark 2.4.6, we have ${}^I\widehat{E}_{\mathfrak{b}|\widehat{\mathcal{D}}_J} = {}^J\widehat{E}_{\mathfrak{b}}$ for $I \subset J$ and $\mathfrak{b} \in \mathrm{Irr}(\mathbb{D}, j)$. Then, we obtain the decomposition (24) by using a general lemma (Lemma 2.4.12 below). □

2.4.4. Decomposition at the level $\boldsymbol{m}(i)$. — We use the setting in Subsection 2.4.3. Take an auxiliary sequence $\boldsymbol{m}(0),\ldots,\boldsymbol{m}(L),\boldsymbol{m}(L+1)=\boldsymbol{0}$ for $\mathrm{Irr}(\mathbb{D})$. (See Section 2.1.2.) Let $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))$ denote the image of $\mathrm{Irr}(\mathbb{D})$ via $\overline{\eta}_{\boldsymbol{m}(i)}$. Let $k(i)$ denote the number determined by $\boldsymbol{m}(i)\in\mathbb{Z}_{<0}^{k(i)}\times\boldsymbol{0}_{\ell-k(i)}$. Let $\underline{k}(i):=\underline{k(i)}$. We remark that

$$\pi_j:\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))\longrightarrow M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\underline{j-1}))$$

is injective for $j\leqslant k(i)$. We also have the map

$$\overline{\eta}_{\boldsymbol{m}(i),j}:\mathrm{Irr}(\mathbb{D},j)\longrightarrow M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\underline{j-1}))$$

given as follows:

$$\mathrm{Irr}(\mathbb{D},j)\simeq\mathrm{Irr}'(\mathbb{D},j)\subset M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\underline{j-1}))\xrightarrow{\;b\;}M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\underline{j-1})).$$

Here, b is induced by $\overline{\eta}_{\boldsymbol{m}(i)}$. As in Lemma 2.4.7, we obtain the following decomposition:

$$(E,\mathbb{D})_{|\widehat{\mathcal{D}}(\underline{k}(i))}=\bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))}\big(\widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b}},\mathbb{D}_{\mathfrak{b}}\big),\quad\text{where } \widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b}|\widehat{\mathcal{D}}_j}=\bigoplus_{\substack{\mathfrak{c}\in\mathrm{Irr}(\mathbb{D},j)\\ \overline{\eta}_{\boldsymbol{m}(i),j}(\mathfrak{c})=\pi_j(\mathfrak{b})}}{}^j\widehat{E}_{\mathfrak{c}},\quad\big(j\leqslant k(i)\big).\tag{25}$$

The decomposition (25) is called the irregular decomposition at the level $\boldsymbol{m}(i)$.

2.4.5. Zero of ϱ. — We use the setting in Section 2.4.1. It is easy to show the following lemma.

Lemma 2.4.8. — *Assume ϱ is constantly 0. Then, for a sufficiently small neighbourhood $\mathcal{X}_P$ of P, we have $\mathrm{Irr}(\mathbb{D},P)\subset M(\mathcal{X}_P,\mathcal{D}_P)/H(\mathcal{X}_P)$ and a decomposition $(E,\mathbb{D})_{|\mathcal{X}_P}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P)}(E_{\mathfrak{a}},\mathbb{D}_{\mathfrak{a}})$ such that $\mathbb{D}_{\mathfrak{a}}$ are $\mathfrak{a}$-logarithmic.* □

Let us consider the case where ϱ is not constantly 0. For simplicity, we assume that $d\varrho$ is nowhere vanishing on $\mathcal{K}^0:=\varrho^{-1}(0)$. We put $\mathcal{X}^0:=\mathcal{X}\times_{\mathcal{K}}\mathcal{K}^0$ and $\mathcal{D}^0:=\mathcal{D}\times_{\mathcal{K}}\mathcal{K}^0$. We also assume that $P\in\mathcal{D}^0$. As remarked in Lemma 2.4.8, by shrinking $\mathcal{X}$ around P, we have a decomposition

$$(E,\mathbb{D})_{|\mathcal{X}^0}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P)}(E_{\mathfrak{a},\mathcal{X}^0},\mathbb{D}_{\mathfrak{a}})\tag{26}$$

such that $\mathbb{D}_{\mathfrak{a}}-d\mathfrak{a}_{|\mathcal{X}^0}$ are logarithmic. Let $\widehat{\mathcal{X}}^0$ be the completion of $\mathcal{X}$ along $\mathcal{X}^0$. The following lemma can be shown by a standard argument.

Lemma 2.4.9. — *We have a flat decomposition*

$$(E,\mathbb{D})_{|\widehat{\mathcal{X}}^0}=\bigoplus_{\mathfrak{a}\in\mathcal{I}}(E_{\mathfrak{a},\widehat{\mathcal{X}}^0},\mathbb{D}_{\mathfrak{a}})$$

such that (i) its restriction to $\mathcal{X}^0$ is the same as (26), (ii) its restriction to $\widehat{P}$ is the same as the irregular decomposition of $(E,\mathbb{D})_{|\widehat{P}}$. □

We have refinements of Sections 2.4.3 and 2.4.4. We use the setting there. For $\mathfrak{b} \in \operatorname{Irr}(\mathbb{D}, j)$, let $\operatorname{Irr}(\mathbb{D}, P, \mathfrak{b})$ denote the inverse image of $\mathfrak{b}$ by the natural map $\operatorname{Irr}(\mathbb{D}, P) \to \operatorname{Irr}(\mathbb{D}, j)$. We put $\widehat{E}_{\mathfrak{b},\widehat{\mathcal{X}}^0} := \bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\mathbb{D},P,\mathfrak{b})} \widehat{E}_{\mathfrak{a},\widehat{\mathcal{X}}^0}$. We put $W(\underline{j}) := \mathcal{D}(\underline{j}) \cup \mathcal{X}^0$. As in Lemma 2.4.7, we obtain the decomposition

$$(E, \mathbb{D})_{|\widehat{W}(\underline{j})} = \bigoplus_{\mathfrak{b}\in\operatorname{Irr}(\mathbb{D},j)} \big(\widehat{E}^{\boldsymbol{m}(p)}_{\mathfrak{b},\widehat{W}(\underline{j})}, \mathbb{D}_{\mathfrak{b}}\big) \tag{27}$$

such that (i) $\widehat{E}_{\mathfrak{b},\widehat{W}(\underline{j})|\widehat{\mathcal{D}}(\underline{j})} = \widehat{E}_{\mathfrak{b},\widehat{\mathcal{D}}(\underline{j})}$, (ii) $\widehat{E}_{\mathfrak{b},\widehat{W}(\underline{j})|\widehat{\mathcal{X}}^0} = \widehat{E}_{\mathfrak{b},\widehat{\mathcal{X}}^0}$. Similarly, we have the decomposition

$$(E, \mathbb{D})_{|\widehat{W}(\underline{k}(i))} = \bigoplus_{\mathfrak{b}\in\operatorname{Irr}(\mathbb{D},\boldsymbol{m}(i))} \big(\widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},\widehat{W}(\underline{k}(i))}, \mathbb{D}_{\mathfrak{b}}\big) \tag{28}$$

such that (i) $\widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},\widehat{W}(\underline{k}(i))|\widehat{\mathcal{D}}(\underline{k}(i))} = \widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},\widehat{\mathcal{D}}(\underline{k}(i))}$, (ii)$\widehat{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},\widehat{W}(\underline{k}(i))|\widehat{\mathcal{X}}^0} = \bigoplus_{\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{a})=\mathfrak{b}} \widehat{E}_{\mathfrak{a},\widehat{\mathcal{X}}^0}$.

2.4.6. Proof of Proposition 2.4.1 and Proposition 2.4.4. — We only have to show the propositions under the setting of Subsection 2.4.2. In the following, instead of considering a neighbourhood $\mathcal{X}_P$, we will replace $\mathcal{X}$ by a small neighbourhood of P without mention, if it is necessary.

2.4.6.1. Step 1. — We fix $I \subset \underline{\ell}$ for a moment. Let E be a free $\mathcal{O}_{\widehat{\mathcal{D}}_I}$-module with a meromorphic flat connection $\mathbb{D} : E \to E \otimes \Omega^1_{\widehat{\mathcal{D}}_I/\mathcal{K}}(*\mathcal{D})$. Assume that we are given the following:

- $\boldsymbol{m} \in \mathbb{Z}^{\ell}_{\leqslant 0}$ and $i \in I$ such that $m_i < 0$. We set $\boldsymbol{m}' := \boldsymbol{m} + \boldsymbol{\delta}_i$.
- $\mathcal{I} \subset \mathcal{O}_{\widehat{P}}(*\mathcal{D})$ such that, for any $\mathfrak{a} \in \mathcal{I}$, (i) $z_i^{-m_i}\mathfrak{a}$ is independent of the variable z_i, (ii) $\boldsymbol{z}^{-\boldsymbol{m}}\,\mathfrak{a} \in \mathcal{O}_{\widehat{P}}$.
- A decomposition $E_{|\widehat{P}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} {}^PE_{\mathfrak{a}}$ such that $\boldsymbol{z}^{-\boldsymbol{m}'}(\mathbb{D} - d\mathfrak{a})\big({}^PE_{\mathfrak{a}}\big) \subset {}^PE_{\mathfrak{a}} \otimes \Omega^1_{\widehat{P}/\mathcal{K}}(\log \mathcal{D})$.

We set $\mathcal{I}_0 := \big\{(\boldsymbol{z}^{-\boldsymbol{m}}\,\mathfrak{a})(P) \,\big|\, \mathfrak{a} \in \mathcal{I}\big\} \subset \boldsymbol{C}$. We have a naturally defined map $\pi : \mathcal{I} \to \mathcal{I}_0$. We set ${}^PE_{\mathfrak{b}} := \bigoplus_{\pi(\mathfrak{a})=\mathfrak{b}} {}^PE_{\mathfrak{a}}$.

Let $H(\mathcal{D}_I)$ denote the space of holomorphic functions on $\mathcal{D}_I$. Let R denote the localization of $H(\mathcal{D}_I)[\![z_j \,|\, j \in I]\!]$ with respect to $\prod_{j=1}^{\ell} z_j$.

Lemma 2.4.10. — *$\mathcal{I}$ is contained in R, and we have a flat decomposition $E = \bigoplus_{\mathfrak{b}\in\mathcal{I}_0} E_{\mathfrak{b}}$ such that $E_{\mathfrak{b}|\widehat{P}} = {}^PE_{\mathfrak{b}}$.*

Proof. — First, we remark that $\boldsymbol{z}^{-\boldsymbol{m}}\mathbb{D}(z_i\partial_i){}^PE_{\mathfrak{a}} \subset {}^PE_{\mathfrak{a}}$, and thus $\boldsymbol{z}^{-\boldsymbol{m}}\mathbb{D}(z_i\partial_i)E \subset E$. Let F be the endomorphism of $E_{|\widehat{\mathcal{D}}_I\cap\mathcal{D}_i}$ induced by $\boldsymbol{z}^{-\boldsymbol{m}}\mathbb{D}(z_i\partial_i)$. The eigen-decomposition of $F_{|P}$ is given by $E_{|P} = \bigoplus_{\mathfrak{b}\in\mathcal{I}_0} {}^PE_{\mathfrak{b}|P}$. We obtain a decomposition $E_{|\widehat{\mathcal{D}}_I\cap\mathcal{D}_i} = \bigoplus_{\mathfrak{b}\in\mathcal{I}_0} G_{\mathfrak{b}}$ such that (i) $F(G_{\mathfrak{b}}) \subset G_{\mathfrak{b}}$ (ii) $G_{\mathfrak{b}|P} = {}^PE_{\mathfrak{b}|P}$. By comparing F and its completion at P, we obtain that $\mathcal{I} \subset R$. By using a standard argument (see Section 2.2.4), we obtain a decomposition $E = \bigoplus_{\mathfrak{b}\in\mathcal{I}_0} E_{\mathfrak{b}}$ such that (i) $E_{\mathfrak{b}|\widehat{\mathcal{D}}_I\cap\mathcal{D}_i} = G_{\mathfrak{b}}$, (ii) it is preserved by $\boldsymbol{z}^{-\boldsymbol{m}}\mathbb{D}$. By a standard argument as in Corollary 2.2.18, we can show that $E_{\mathfrak{b}|\widehat{P}} = {}^PE_{\mathfrak{b}}$. □

2.4.6.2. Step 2. — For $1 \leqslant p \leqslant \ell$, we put $\underline{p} := \{1, \dots, p\}$. Let E be a free $\mathcal{O}_{\widehat{\mathcal{D}}(\underline{p})}$-module with a meromorphic flat connection $\mathbb{D} : E \to E \otimes \Omega^1_{\widehat{\mathcal{D}}(\underline{p})/\mathcal{K}}(*\mathcal{D})$. Assume that we are given a good set of irregular values $\mathrm{Irr}(\mathbb{D}) \subset \mathcal{O}_{\widehat{P}}(*\mathcal{D}(\underline{p}))/\mathcal{O}_{\widehat{P}}$ and a decomposition

$$(E, \mathbb{D})_{|\widehat{P}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})} ({}^P E_{\mathfrak{a}}, {}^P\mathbb{D}_{\mathfrak{a}})$$

such that ${}^P\mathbb{D}_{\mathfrak{a}}$ are $\mathfrak{a}$-logarithmic. For $I \subset \underline{p}$, let $\mathrm{Irr}(\mathbb{D}, I)$ denote the image of $\mathrm{Irr}(\mathbb{D})$ via the natural map $p_I : \mathcal{O}_{\widehat{P}}(*\mathcal{D}(\underline{p}))/\mathcal{O}_{\widehat{P}} \to \mathcal{O}_{\widehat{P}}(*\mathcal{D}(\underline{p}))/\mathcal{O}_{\widehat{P}}(*\mathcal{D}(I_1))$, where $I_1 := \underline{p} \setminus I$. For each I and $\mathfrak{b} \in \mathrm{Irr}(\mathbb{D}, I)$, we set

$${}^P E_{\mathfrak{b}} := \bigoplus_{\substack{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}) \\ p_I(\mathfrak{a}) = \mathfrak{b}}} {}^P E_{\mathfrak{a}}.$$

Lemma 2.4.11. — *If we shrink $\mathcal{X}$ appropriately,* $\mathrm{Irr}(\mathbb{D})$ *is contained in the image of* $M(\mathcal{X}, \mathcal{D}(\underline{p}))/H(\mathcal{X}) \to \mathcal{O}_{\widehat{P}}(*\mathcal{D}(\underline{p}))/\mathcal{O}_{\widehat{P}}$. *For each $I \subset \underline{p}$, we have a decomposition* $E_{|\widehat{\mathcal{D}}_I} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\mathbb{D}, I)} {}^I E_{\mathfrak{b}}$ *such that* ${}^I E_{\mathfrak{b}|\widehat{P}} = {}^P E_{\mathfrak{b}}$.

Proof. — We use an induction on the rank of E. Assume that the coordinate system is admissible for $\mathrm{Irr}(\mathbb{D})$. Take an auxiliary sequence $\boldsymbol{m}(0), \dots, \boldsymbol{m}(L)$ for $\mathrm{Irr}(\mathbb{D})$. We put $T := \{(\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathfrak{a})(P) \mid \mathfrak{a} \in \mathrm{Irr}(\mathbb{D})\}$. We have the naturally defined map $q : \mathrm{Irr}(\mathbb{D}) \to T$. For each $\alpha \in T$, we put ${}^P E_\alpha = \bigoplus_{q(\mathfrak{a}) = \alpha} {}^P E_{\mathfrak{a}}$. Then, $E_{|\widehat{P}} = \bigoplus {}^P E_\alpha$ is a flat decomposition. It is easy to observe that if we are given a flat decomposition $E_{|\widehat{P}} = \bigoplus_{\alpha \in T} {}^P E'_\alpha$ such that ${}^P E'_{\alpha|P} = {}^P E_{\alpha|P}$, then we have ${}^P E'_\alpha = {}^P E_\alpha$.

Due to Lemma 2.4.10 with $I = \{\mathfrak{h}(0)\}$, $\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a})$ are meromorphic functions for any $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$. Hence, by considering the tensor product with a meromorphic flat line bundle, we only have to consider the case where $|T| \geqslant 2$. Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$.

Take $I \subset \underline{p}$. If $I \cap \underline{k} = \varnothing$, the trivial decomposition is desired one. Let us consider the case $I \cap \underline{k} \neq \varnothing$. We take $i \in I \cap \underline{k}$. Let $m_i(0)$ be the i-th component of $\boldsymbol{m}(0)$. We have $\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathbb{D}(z_i\partial_i)E \subset E$. Let G be the induced endomorphism of $E_{|P}$. It is semisimple, the eigenvalues are given by $\{m_i(0)\alpha \mid \alpha \in T\}$, and the eigendecomposition is given by $E_{|P} = \bigoplus {}^P E_{\alpha|P}$, where $G_{|E_{\alpha|P}}$ is the multiplication of $m_i(0)\alpha$. By applying Lemma 2.4.10, we can extend it to a flat decomposition of $E_{|\widehat{D}_I}$, i.e.,

$$E_{|\widehat{D}_I} = \bigoplus_{\alpha \in T} {}^I E_\alpha, \quad \text{such that} \quad {}^I E_{\alpha|P} = {}^P E_{\alpha|P}.$$

Then, we obtain ${}^I E_{\alpha|\widehat{P}} = {}^P E_\alpha$. For $I \subset J \subset \underline{k}$ as above, we have ${}^I E_{\alpha|\widehat{D}_J} = {}^J E_\alpha$. Due to Lemma 2.4.12 below, we obtain the flat decomposition

$$(E, \mathbb{D})_{|\widehat{D}(\underline{k})} = \bigoplus_{\alpha \in T} (E_\alpha, \mathbb{D}_\alpha), \quad \text{such that } E_{\alpha|\widehat{D}_I} = {}^I E_\alpha.$$

We may apply the inductive assumption to $(E_\alpha, \mathbb{D}_\alpha)$ on $D(\underline{k})$, and we obtain Lemma 2.4.11. □

2.4.6.3. Step 3. — We can complete the proof of Proposition 2.4.1 and Proposition 2.4.4 by applying Lemma 2.4.11 to $(E,\mathbb{D})_{|\widehat{\mathcal{D}}}$.

2.4.6.4. General lemma. — Recall the following general lemma.

Lemma 2.4.12. — *Let $\widehat{V}$ be a free $\mathcal{O}_{\widehat{\mathcal{D}}}$-module on $\mathcal{X}$. Assume that we are given a decomposition $\widehat{V}_{|\widehat{\mathcal{D}}_I}=\bigoplus {}^I\widehat{V}_{\mathfrak{a}}$ for each $I\subset \underline{\ell}$, such that ${}^I\widehat{V}_{\mathfrak{a}|\widehat{\mathcal{D}}_J}={}^J\widehat{V}_{\mathfrak{a}}$ for any $I\subset J$. Then, we have a unique decomposition $\widehat{V}=\bigoplus \widehat{V}_{\mathfrak{a}}$ on $\widehat{\mathcal{D}}$, which induces the decompositions on $\widehat{\mathcal{D}}_I$.*

Proof. — Let ${}^I\pi_{\mathfrak{a}}$ be the projection of $\widehat{V}_{|\widehat{\mathcal{D}}_I}$ onto ${}^IV_{\mathfrak{a}}$. Then, we have ${}^I\pi_{\mathfrak{a}|\widehat{\mathcal{D}}_J}={}^J\pi_{\mathfrak{a}}$. Let $\boldsymbol{v}$ be a frame of V. Let ${}^I\Pi_{\mathfrak{a}}\in M_r(\mathcal{O}_{\widehat{\mathcal{D}}_I})$ be determined by ${}^I\pi_{\mathfrak{a}}(\boldsymbol{v})=\boldsymbol{v}\cdot{}^I\Pi_{\mathfrak{a}}$, where $r=\operatorname{rank}(V)$. Because ${}^I\Pi_{\mathfrak{a}|\widehat{\mathcal{D}}_J}={}^J\Pi_{\mathfrak{a}}$, we have $\Pi_{\mathfrak{a}}\in M_r(\mathcal{O}_{\widehat{\mathcal{D}}})$ such that $\Pi_{\mathfrak{a}|\widehat{\mathcal{D}}_I}={}^I\Pi_{\mathfrak{a}}$. (Use the exact sequence in the proof of Proposition 4.1 [**34**], for example.) Let $\pi_{\mathfrak{a}}$ be the endomorphism of $\widehat{V}$ given by $\pi_{\mathfrak{a}}(\boldsymbol{v}_{|\widehat{\mathcal{D}}})=\boldsymbol{v}_{|\widehat{\mathcal{D}}}\cdot\Pi_{\mathfrak{a}}$, and let $\widehat{V}_{\mathfrak{a}}$ be the image of $\pi_{\mathfrak{a}}$. Then, $\widehat{V}=\bigoplus\widehat{V}_{\mathfrak{a}}$ gives the desired decomposition. □

2.5. Good filtered ϱ-flat bundle

2.5.1. Good filtered ϱ-flat bundle. — Let $\mathcal{X}\to\mathcal{K}$, $\mathcal{D}$ and ϱ be as in Subsection 2.3.1. Let $\mathcal{D}=\bigcup_{i\in\Lambda}\mathcal{D}_i$ be the decomposition into irreducible components. Recall that a filtered ϱ-flat sheaf on $(\mathcal{X},\mathcal{D})$ is defined to be a filtered sheaf $\boldsymbol{E}_*=\big({}_{\boldsymbol{a}}E\,\big|\,\boldsymbol{a}\in\boldsymbol{R}^{\Lambda}\big)$ on $(\mathcal{X},\mathcal{D})$ with a meromorphic flat ϱ-connection $\mathbb{D}$ of the $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-module $\boldsymbol{E}=\bigcup_{\boldsymbol{a}\in\boldsymbol{R}^{\Lambda}}{}_{\boldsymbol{a}}E$. If $\boldsymbol{E}_*$ is a filtered bundle, $(\boldsymbol{E}_*,\mathbb{D})$ is called a filtered ϱ-flat bundle. See Subsection 2.5.3 below for a brief account on filtered sheaf and filtered bundle. We shall use some notation and terminology given there.

Definition 2.5.1. — Let $(\boldsymbol{E}_*,\mathbb{D})$ be a filtered ϱ-flat bundle.
- $(\boldsymbol{E}_*,\mathbb{D})$ is called unramifiedly good, if ${}_{\boldsymbol{c}}E$ are unramifiedly good lattices for any $\boldsymbol{c}\in\boldsymbol{R}^{\Lambda}$.
- $(\boldsymbol{E}_*,\mathbb{D})$ is called good at $P\in\mathcal{D}$, if there exists a ramified covering $\varphi_P:(\mathcal{X}'_P,\mathcal{D}'_P)\to(\mathcal{X}_P,\mathcal{D}_P)$ such that $(\widetilde{\boldsymbol{E}}_*,\varphi_e^*\mathbb{D})$ is unramifiedly good. Here, $(\mathcal{X}_P,\mathcal{D}_P)$, $(\mathcal{X}'_P,\mathcal{D}'_P)$ and φ_P are as in Definition 2.3.3, and $\widetilde{\boldsymbol{E}}_*$ is induced by φ_P and $\boldsymbol{E}_*$ as in Section 2.5.3.3 below.
- $(\boldsymbol{E}_*,\mathbb{D})$ is called good, if it is good at any point of $\mathcal{D}$. In other words, $(\boldsymbol{E}_*,\mathbb{D})$ is good, if it is the descent of an unramifiedly good filtered ϱ-flat bundle around any point of $\mathcal{D}$. □

2.5.2. Residue

2.5.2.1. Unramified case. — Let $\mathcal{X}\to\mathcal{K}$, $\mathcal{D}$, ϱ and $(\mathcal{E},\mathbb{D})$ be as in Section 2.3.1. For simplicity, $\mathcal{K}$ is assumed to be connected. Let E be an unramifiedly good lattice of $(\mathcal{E},\mathbb{D})$. Let $\mathcal{D}_i$ be an irreducible component of $\mathcal{D}$. For each $P\in\mathcal{D}_i$, we have $\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D}_{|\widehat{P}})\in\mathrm{End}(E_{|\widehat{P}\cap\mathcal{D}_i})$. (See Subsection 2.2.1.)

Lemma 2.5.2. — *We have the residue endomorphism* $\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D})\in\mathrm{End}(E_{|\mathcal{D}_i})$ *such that* $\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D})_{|\widehat{P}}=\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D}_{|\widehat{P}})$ *for any* $P\in\mathcal{D}_i$. *The eigenvalues of* $\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D})$ *are the pull-back of possibly multi-valued functions on* $\mathcal{K}$ *if* ϱ *is not constantly* 0 *on* $\mathcal{K}$. *In particular, their restriction to* $\mathcal{D}_i\times_{\mathcal{K}}\{y\}$ *are constant if* $\varrho(y)\neq 0$.

Proof. — The first claim follows from the construction of $\mathrm{Res}_{\mathcal{D}_i}(\mathbb{D}_{|\widehat{P}})$ and Proposition 2.4.4. The second claim follows from the first one and Lemma 2.2.5. □

2.5.2.2. Ramified case. — Let $\mathcal{X}\to\mathcal{K}$, $\mathcal{D}$ and ϱ be as above. If $(\boldsymbol{E}_*,\mathbb{D})$ is unramifiedly good on $(\mathcal{X},\mathcal{D})$, we have the induced endomorphism $\mathrm{Res}_i(\mathbb{D})$ on ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i}$. It preserves the induced filtration ${}^iF$ of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i}$, and hence we have the induced endomorphism $\mathrm{Gr}^F_a\mathrm{Res}_i(\mathbb{D})$ of ${}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$. (See Subsection 2.5.3 for the notation.)

Proposition 2.5.3. — *Even if a good filtered* ϱ*-flat bundle* $(\boldsymbol{E}_*,\mathbb{D})$ *is not necessarily unramified, we have the induced endomorphism* $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D})$ *of* ${}^i\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$ *on* $\mathcal{D}_i$ *for each* $i\in\Lambda$. *It preserves the induced filtrations* ${}^jF$ *of* ${}^i\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)_{|\mathcal{D}_i\cap\mathcal{D}_j}$.

The eigenvalues of $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D})$ *are the pull-back of possibly multi-valued functions on* $\mathcal{K}$ *if* ϱ *is not constantly* 0 *on* $\mathcal{K}$. *In particular, their restriction to* $\mathcal{D}_i\times_{\mathcal{K}}\{y\}$ *are constant if* $\varrho(y)\neq 0$.

Due to Proposition 2.5.3, $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D})$ $(i\in I)$ induce the endomorphisms of ${}^I\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E)$, which are also denoted by $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D})$ or $\mathrm{Res}_i(\mathbb{D})$. In the following, $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D})$ are often denoted by $\mathrm{Res}_i(\mathbb{D})$ for simplicity of the description.

2.5.2.3. Proof of Proposition 2.5.3. — First, we consider the case $\mathcal{X}=\Delta^n$ and $\mathcal{D}=\{z_1=0\}$. We put $\widetilde{\mathcal{X}}:=\mathcal{X}$ and $\widetilde{\mathcal{D}}=\mathcal{D}$, and we have a ramified covering $\varphi_e:(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}})\to(\mathcal{X},\mathcal{D})$ given by $\varphi_e(z_1,\dots,z_n)=(z_1^e,z_2,\dots,z_n)$ such that the induced filtered ϱ-flat bundle $(\widetilde{\boldsymbol{E}}_*,\widetilde{\mathbb{D}})$ on $(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}})$ is unramifiedly good. We take $c\in\boldsymbol{R}$ and we put $\widetilde{c}:=ce$. We have the residue $\mathrm{Res}(\widetilde{\mathbb{D}})$ of ${}_{\widetilde{c}}\widetilde{E}$ on $\widetilde{\mathcal{D}}$. Let $\mu_e:=\{\omega\in\boldsymbol{C}\,|\,\omega^e=1\}$, which naturally acts on $\widetilde{\mathcal{X}}$ by $\omega^*(z_1)=\omega\,z_1$, and $(\widetilde{\boldsymbol{E}}_*,\widetilde{\mathbb{D}})$ is μ_e-equivariant. The endomorphism $\mathrm{Res}(\widetilde{\mathbb{D}})$ is μ_e-equivariant.

We can take a frame $\widetilde{\boldsymbol{v}}$ of ${}_{\widetilde{c}}\widetilde{E}$ such that (i) it is compatible with the induced filtration F of ${}_{\widetilde{c}}\widetilde{E}_{|\widetilde{\mathcal{D}}}$, (ii) for each p we have $b_p\in\mathbb{Z}$ satisfying $0\leqslant b_p<e$ and $\omega^*\widetilde{v}_p=\omega^{-b_p}\,\widetilde{v}_p$. We put $a(\widetilde{v}_p):=\deg^F(\widetilde{v}_p)$. They induce a frame of $\mathrm{Gr}^F\big({}_{\widetilde{c}}\widetilde{E}\big)=\bigoplus_{\widetilde{c}-1<\widetilde{a}\leqslant\widetilde{c}}\mathrm{Gr}^F_{\widetilde{a}}\big({}_{\widetilde{c}}\widetilde{E}\big)$.

We put $v_p:=z^{b_p}\,\widetilde{v}_p$, which is a μ_e-invariant section. The tuple $\boldsymbol{v}=(v_p)$ naturally gives a frame of ${}_cE$ compatible with the parabolic filtration. Hence, it induces a frame of $\mathrm{Gr}^F({}_cE)=\bigoplus_{c-1<a\leqslant c}\mathrm{Gr}^F_a({}_cE)$. The frames give an isomorphism:

$$\mathrm{Gr}^F_{\widetilde{a}}({}_{\widetilde{c}}\widetilde{E})\simeq\bigoplus_{\widetilde{a}-e\,a\in\mathbb{Z}}\mathrm{Gr}^F_a({}_cE).\tag{29}$$

The decomposition of $\mathrm{Gr}^F_{\widetilde{a}}({}_{\widetilde{c}}\widetilde{E})$ corresponding to (29) is given by the eigen-decomposition with respect to the action of ω^*. Since $\mathrm{Res}(\widetilde{\mathbb{D}})$ is μ_m-equivariant, it induces endomorphisms G_a of $\mathrm{Gr}^F_a({}_cE)$. It is easy to check that the isomorphism (29) is independent of the choice of a frame $\widetilde{\boldsymbol{v}}$. It is also independent of the choice of a coordinate chart up to multiplication by a nowhere vanishing function on each direct summand of the right-hand side. Hence, G_a is independent of the choice of frames and coordinate system. For each $c-1 < a \leqslant c$, let $b(a) \in \mathbb{Z}$ be determined by $\widetilde{c}-1 < e\,a + b(a) \leqslant \widetilde{c}$. In this case, we define the endomorphism $\mathrm{Gr}^F_a \mathrm{Res}(\mathbb{D})$ of $\mathrm{Gr}^F_a({}_cE)$ as follows:

$$\mathrm{Gr}^F_a \mathrm{Res}(\mathbb{D}) := e^{-1}\big(G_a + \varrho\, b(a)\big).$$

Lemma 2.5.4. — *If $(\boldsymbol{E}_*, \mathbb{D})$ is unramified, it is the same as the endomorphism induced by the residue* $\mathrm{Res}(\mathbb{D})$.

Proof. — By considering the completion along $\mathcal{D}$, the problem can be reduced to the regular case. Then, the claim can be checked by a direct calculation. □

By using Lemma 2.5.4, we can check that $\mathrm{Gr}^F_a \mathrm{Res}(\mathbb{D})$ is independent of the choice of a ramified covering $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to (\mathcal{X}, \mathcal{D})$. Thus, we obtain the well defined endomorphism $\mathrm{Gr}^F_a \mathrm{Res}(\mathbb{D})$ of $\mathrm{Gr}^F_a({}_cE)$ in the case where $\mathcal{D}$ is smooth.

Let $\mathcal{D}_i^\circ := \mathcal{D}_i \setminus \bigcup_{j\neq i} \mathcal{D}_j$. We have obtained the endomorphism $\mathrm{Gr}^F_a \mathrm{Res}_i(\mathbb{D})$ of $\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)_{|\mathcal{D}_i^\circ}$. Let us show that it can be extended to an endomorphism of $\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$, and that it preserves the parabolic filtrations ${}^{\ell}F$ $(\ell \neq i)$. Since it is a local property, we only have to consider the case $\mathcal{X} := \Delta^n$ and $\mathcal{D} = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We have a ramified covering $\varphi_e : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to (\mathcal{X}, \mathcal{D})$ given by $\varphi_e(z_1, \dots, z_n) = (z_1^e, \dots, z_\ell^e, z_{\ell+1}, \dots, z_n)$, such that the induced filtered ϱ-flat bundle $(\widetilde{\boldsymbol{E}}_*, \widetilde{\mathbb{D}})$ is unramifiedly good. We put $\mathrm{Gal}(\widetilde{\mathcal{X}}/\mathcal{X}) := \big\{\boldsymbol{\omega} = (\omega_1, \dots, \omega_\ell) \,\big|\, \omega_i \in \mu_e\big\}$. We have the natural action of $\mathrm{Gal}(\widetilde{\mathcal{X}}/\mathcal{X})$ on $\widetilde{\mathcal{X}}$ given by $\boldsymbol{\omega}^* z_j = \omega_j\, z_j$ for $j = 1, \dots, \ell$. It is lifted to the action on $(\widetilde{\boldsymbol{E}}_*, \widetilde{\mathbb{D}})$, and $(\boldsymbol{E}_*, \mathbb{D})$ is the descent.

Let $\boldsymbol{c} \in \boldsymbol{R}^{\ell}$ and $\widetilde{\boldsymbol{c}} := e\,\boldsymbol{c}$. Let c_i and $\widetilde{c}_i$ denote the i-th components of $\boldsymbol{c}$ and $\widetilde{\boldsymbol{c}}$, respectively. For any $\widetilde{c}_i - 1 < \widetilde{a} \leqslant \widetilde{c}_i$, we have the endomorphism $\mathrm{Gr}^F_{\widetilde{a}} \mathrm{Res}_{\widetilde{\mathcal{D}}_i}(\widetilde{\mathbb{D}})$ of ${}^i\mathrm{Gr}^F_{\widetilde{a}}({}_{\widetilde{\boldsymbol{c}}}\widetilde{E})$ on $\widetilde{\mathcal{D}}_i$. It is $\mathrm{Gal}(\widetilde{\mathcal{X}}/\mathcal{X})$-equivariant, and the restriction $\mathrm{Gr}^F_{\widetilde{a}} \mathrm{Res}_{\widetilde{\mathcal{D}}_i}(\widetilde{\mathbb{D}})_{|\widetilde{\mathcal{D}}_i \cap \widetilde{\mathcal{D}}_j}$ preserves the induced filtration ${}^jF$ of ${}^i\mathrm{Gr}^F_{\widetilde{a}}({}_{\widetilde{\boldsymbol{c}}}\widetilde{E})_{|\widetilde{\mathcal{D}}_i \cap \widetilde{\mathcal{D}}_j}$.

We can take a frame $\widetilde{\boldsymbol{v}} = (\widetilde{v}_p)$ of ${}_{\widetilde{\boldsymbol{c}}}\widetilde{E}$ such that (i) it is compatible with the filtrations ${}^kF$ $(k = 1, \dots, \ell)$, (ii) there exist tuples of integers $\boldsymbol{b}_p = (b_{p,1}, \dots, b_{p,\ell})$ satisfying $0 \leqslant b_{p,k} < e-1$ and $\boldsymbol{\omega}^* \widetilde{v}_p = \prod_{k=1}^{\ell} \omega_k^{-b_{p,k}}\, \widetilde{v}_p$. (See Section 2.3 of [**67**].) We put $a_k(\widetilde{v}_p) := {}^k\deg^F(\widetilde{v}_p)$. Let $\widetilde{A}^{(i)}$ be the matrix valued holomorphic function on $\widetilde{\mathcal{D}}_i$, determined by $\mathrm{Res}_i(\widetilde{\mathbb{D}})\widetilde{\boldsymbol{v}}_{|\widetilde{\mathcal{D}}_i} = \widetilde{\boldsymbol{v}}_{|\widetilde{\mathcal{D}}_i}\, \widetilde{A}^{(i)}$, i.e., $\mathrm{Res}_i(\widetilde{\mathbb{D}})\widetilde{v}_{q|\widetilde{\mathcal{D}}_i} = \sum_p \widetilde{A}^{(i)}_{p,q}\, \widetilde{v}_{p|\widetilde{\mathcal{D}}_i}$. We have $\widetilde{A}^{(i)}_{p,q} = 0$ unless $b_{p,i} = b_{q,i}$ and $a_i(\widetilde{v}_p) \leqslant a_i(\widetilde{v}_q)$. Due to the $\mathrm{Gal}(\widetilde{\mathcal{X}}/\mathcal{X})$-equivariance of

$\operatorname{Res}_i(\widetilde{\mathbb{D}})$, the following functions are holomorphic on $\widetilde{\mathcal{D}}_i$ and $\operatorname{Gal}(\widetilde{\mathcal{X}}/\mathcal{X})$-invariant:

$$A^{(i)}_{p,q} := \widetilde{A}^{(i)}_{p,q} \prod_{k\neq i} z_k^{b_{p,k}-b_{q,k}}. \tag{30}$$

Moreover, we have $A^{(i)}_{p,q|\widetilde{\mathcal{D}}_\ell\cap\widetilde{\mathcal{D}}_i} = 0$ for $\ell \neq i$, if either one of the following holds:

$$\text{(i) } b_{p,\ell} - b_{q,\ell} > 0, \qquad \text{(ii) } b_{p,\ell} = b_{q,\ell},\ a_\ell(\widetilde{v}_p) > a_\ell(\widetilde{v}_q). \tag{31}$$

We take $c_i - 1 < a \leqslant c_i$, which determines $\widetilde{c}_i - 1 < \widetilde{a} \leqslant \widetilde{c}_i$ such that $b(a) := \widetilde{a} - e\,a \in \mathbb{Z}$. We put $I(a,i) := \{p \,|\, a_i(\widetilde{v}_p) = \widetilde{a},\ b_{p,i} = b(a)\}$. Let $\widetilde{\boldsymbol{u}}_a$ be the tuple $\big(\widetilde{u}_{a,p} := \widetilde{v}_p \,|\, p \in I(a,i)\big)$. We put

$$u_{a,p} := \prod_k z_k^{b_{p,k}}\, \widetilde{u}_{a,p}.$$

Then, $\boldsymbol{u}_a = (u_{a,p} \,|\, p \in I(a,i))$ naturally induces a frame of ${}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$ compatible with the induced filtrations ${}^\ell F$ $(\ell \neq i)$ on $\mathcal{D}_\ell \cap \mathcal{D}_i$. Let $A^{(i)}_a$ be the matrix valued holomorphic function on $\mathcal{D}_i$ given by $\big(A^{(i)}_{p,q} \,|\, p,q \in I(a,i)\big)$. By definition, we have

$$\mathrm{Gr}^F_a \operatorname{Res}_i(\mathbb{D})\boldsymbol{u}_{a|\mathcal{D}_i} = \boldsymbol{u}_{a|\mathcal{D}_i}\, e^{-1}\big(A^{(i)}_a + \varrho\, b(a)\big).$$

It implies that $\mathrm{Gr}^F_a \operatorname{Res}_i(\mathbb{D})$ can be extended to an endomorphism of ${}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)$ on $\mathcal{D}_i$. If ${}^\ell\deg^F(u_p) > {}^\ell\deg^F(u_q)$, one of (31) occurs, and hence $A^{(i)}_{p,q|\mathcal{D}_\ell\cap\mathcal{D}_i} = 0$. It implies that $\mathrm{Gr}^F_a \operatorname{Res}_i(\mathbb{D})_{|\mathcal{D}_\ell\cap\mathcal{D}_i}$ preserves the filtrations ${}^\ell F$ $(\ell \neq i)$. □

2.5.2.4. Some notation. — Let us consider the case $\mathcal{K} = \{y\}$. If $\varrho \neq 0$, the eigenvalues of $\operatorname{Res}_i(\mathbb{D})$ are constant. The endomorphisms $\operatorname{Res}_i(\mathbb{D})$ $(i \in I)$ on ${}^I\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E)$ are commutative. Hence, we have the generalized eigen-decomposition

$${}^I\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E) = \bigoplus_{\boldsymbol{\alpha}} {}^I\mathrm{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E),$$

where the eigenvalues of $\mathrm{Gr}^F \operatorname{Res}_i(\mathbb{D})$ on ${}^I\mathrm{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E)$ are the i-th components of $\boldsymbol{\alpha}$. Recall that we often consider the following sets in this situation:

$$\begin{aligned}
\mathcal{KMS}({}_{\boldsymbol{c}}E,\mathbb{D},I) &:= \big\{(\boldsymbol{a},\boldsymbol{\alpha}) \,\big|\, {}^I\mathrm{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E) \neq 0\big\}\\
\mathcal{KMS}(\boldsymbol{E}_*,\mathbb{D},I) &:= \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^\Lambda} \mathcal{KMS}({}_{\boldsymbol{c}}E,\mathbb{D},I)\\
\mathcal{S}p({}_{\boldsymbol{c}}E,\mathbb{D},I) &:= \big\{\alpha \,\big|\, (\boldsymbol{a},\boldsymbol{\alpha}) \in \mathcal{KMS}({}_{\boldsymbol{c}}E,\mathbb{D},I)\big\}\\
\mathcal{S}p(\boldsymbol{E}_*,\mathbb{D},I) &:= \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^\Lambda} \mathcal{S}p({}_{\boldsymbol{c}}E,\mathbb{D},I).
\end{aligned}$$

Each element of $\mathcal{KMS}(\boldsymbol{E}_*,\mathbb{D},I)$ is called a KMS-spectrum of $(\boldsymbol{E}_*,\mathbb{D})$ at $\mathcal{D}_I$.

Remark 2.5.5. — Even in the case $\varrho = 0$, a similar notion makes sense, if the eigenvalues of $\operatorname{Res}_i(\mathbb{D}^0)$ are assumed to be constant. The condition will be satisfied when we will consider wild harmonic bundles. □

2.5.3. Filtered bundle (Appendix). — Let X be a complex manifold with a simple normal crossing hypersurface $D=\bigcup_{i\in\Lambda}D_i$. A filtered sheaf on (X,D) is a set of data $\boldsymbol{E}_*=\big(\boldsymbol{E},\{{}_{\boldsymbol{c}}E\}\,\big|\,\boldsymbol{c}\in\boldsymbol{R}^{\Lambda}\big)$ as follows:

- $\boldsymbol{E}$ is a torsion-free coherent $\mathcal{O}_X(*D)$-module.
- $\{{}_{\boldsymbol{c}}E\}$ is an increasing filtration by coherent $\mathcal{O}_X$-submodules of $\boldsymbol{E}$ indexed by $\boldsymbol{R}^{\Lambda}$ such that $\boldsymbol{E}_{|X\setminus D}={}_{\boldsymbol{c}}E_{|X\setminus D}$ for any $\boldsymbol{c}$, where the order on $\boldsymbol{R}^{\Lambda}$ is given by $\boldsymbol{a}\leqslant\boldsymbol{b}$ $\iff a_i\leqslant b_i\ (\forall\, i)$. We also have ${}_{\boldsymbol{a}}E=\bigcap_{\boldsymbol{a}<\boldsymbol{b}}{}_{\boldsymbol{b}}E$, where $\boldsymbol{a}<\boldsymbol{b}\iff a_i<b_i\ (\forall\, i)$.
- ${}_{\boldsymbol{a}'}E={}_{\boldsymbol{a}}E\otimes\mathcal{O}_X(-\sum n_j\,D_j)$ as submodules of $\boldsymbol{E}$, where $\boldsymbol{a}'=\boldsymbol{a}-(n_j\,|\,j\in\Lambda)$.
- For any point $P\in X$, let $\Lambda(P):=\{i\in\Lambda\mid P\in D_i\}$. Then, on a small neighbourhood X_P of P, we have a ${}_{\boldsymbol{a}}E_{|X_P}={}_{\boldsymbol{a}'}E_{|X_P}=$ is $a_i=a_i'\ (\forall\, i\in\Lambda(P))$.
- For each $\boldsymbol{c}\in\boldsymbol{R}^{\Lambda}$, let ${}^{i}\mathcal{F}$ be a filtration of ${}_{\boldsymbol{c}}E$ indexed by $]c_i-1,c_i]$ is given as follows:

$$
{}^{i}\mathcal{F}_d({}_{\boldsymbol{c}}E):=\bigcup_{\substack{a_i\leqslant d\\ \boldsymbol{a}\leqslant\boldsymbol{c}}}{}_{\boldsymbol{a}}E. \tag{32}
$$

  Then the tuple ${}_{\boldsymbol{c}}E_*:=\big({}_{\boldsymbol{c}}E,\{{}^{i}\mathcal{F}\,|\,i\in S\}\big)$ is a $\boldsymbol{c}$-parabolic sheaf, i.e., the sets $\big\{a\,\big|\,{}^{i}\mathrm{Gr}^{\mathcal{F}}_a({}_{\boldsymbol{c}}E)\neq 0\big\}$ are finite.

See Subsection 3.2 of [**66**] for some property of filtered sheaf. Each ${}_{\boldsymbol{c}}E_*$ is called the $\boldsymbol{c}$-truncation of $\boldsymbol{E}_*$. We can reconstruct $\boldsymbol{E}_*$ from ${}_{\boldsymbol{c}}E_*$. If each ${}_{\boldsymbol{c}}E$ is locally free, $\boldsymbol{E}_*$ is called a filtered bundle. (See Remark 2.5.6 below.) In the case $\boldsymbol{c}=(0,\dots,0)$, the notation ${}^{\diamond}E$ is also used

2.5.3.1. Induced filtrations. — Let $\boldsymbol{E}_*$ be a filtered bundle on (X,D). For each $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E$, we have a filtration ${}^{i}\mathcal{F}$ given as in (32). Let ${}^{i}F_d({}_{\boldsymbol{c}}E_{|D_i})$ denote the image of the induced map ${}^{i}\mathcal{F}_d({}_{\boldsymbol{c}}E)_{|D_i}\to{}_{\boldsymbol{c}}E_{|D_i}$. It is called the parabolic filtration of ${}_{\boldsymbol{c}}E$. For $I\subset\Lambda$, we have the induced filtrations ${}^{i}F$ $(i\in I)$ of ${}_{\boldsymbol{c}}E_{|D_I}$. It is known ([**16**], [**17**], [**37**]) that they are compatible in the sense that this family locally has a splitting, i.e., for each $P\in D_I$, take a small neighbourhood $D_{I,P}$ of P in D_I, then we have a splitting ${}_{\boldsymbol{c}}E_{|D_I,P}=\bigoplus G_{\boldsymbol{d}}$ such that ${}^{I}F_{\boldsymbol{b}}({}_{\boldsymbol{c}}E_{|D_I,P}):=\bigcap{}^{i}F_{b_i}({}_{\boldsymbol{c}}E_{|D_I,P})=\bigoplus_{\boldsymbol{d}\leqslant\boldsymbol{b}}G_{\boldsymbol{d}}$. It also implies the locally abelian condition in [**39**], i.e., for a small neighbourhood X_P of P in X, we can take a decomposition ${}_{\boldsymbol{c}}E_{|X_P}=\bigoplus H_{\boldsymbol{d}}$ such that ${}^{i}F_b({}_{\boldsymbol{c}}E_{|D_i\cap X_P})=\bigoplus_{d_i\leqslant b}H_{\boldsymbol{d}|D_i}$.

Remark 2.5.6. — The above compatibility condition was imposed in our older but equivalent definition of filtered bundle ([**66**] and [**67**]). □

Let I be a subset of Λ. Let $D_I:=\bigcap_{i\in I}D_i$. For $\boldsymbol{a}\in\boldsymbol{R}^{I}$, we will often consider

$$
{}^{I}F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|D_I}):=\bigcap_{i\in I}{}^{i}F_{a_i}({}_{\boldsymbol{c}}E_{|D_I}),\qquad {}^{I}\mathrm{Gr}^{F}_{\boldsymbol{a}}({}_{\boldsymbol{c}}E):=\frac{{}^{I}F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|D_I})}{\sum_{\boldsymbol{b}\lneq\boldsymbol{a}}{}^{I}F_{\boldsymbol{b}}({}_{\boldsymbol{c}}E_{|D_I})}.
$$

Here, $\boldsymbol{b} \lneq \boldsymbol{a}$ means "$\boldsymbol{b} \leqslant \boldsymbol{a}$ and $\boldsymbol{b} \neq \boldsymbol{a}$". We often consider the following sets in this situation:

$$\mathcal{P}ar({}_{\boldsymbol{c}}E, I) := \left\{\boldsymbol{a} \in \boldsymbol{R}^I \,\middle|\, {}^I\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E) \neq 0\right\}$$
$$\mathcal{P}ar(\boldsymbol{E}_*, I) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^\Lambda} \mathcal{P}ar({}_{\boldsymbol{c}}E, I).$$

2.5.3.2. Compatible frame. — For $P \in X$, let X_P denote a small neighbourhood of P in X, and we put $D_P := D \cap X_P$, and $D_{J,P} := D_J \cap X_P$. Let $\Lambda(P) := \{j \in \Lambda \mid P \in D_j\}$. Let $\boldsymbol{E}_*$ be a filtered bundle on (X, D). We can take a frame $\boldsymbol{v}$ of ${}_{\boldsymbol{c}}E_{|X_P}$ with the following properties:

- For each v_p, the tuple of numbers $\boldsymbol{a}(v_p)$ belong to $\prod_{j\in\Lambda(P)}]c_j - 1, c_j]$.
- For $J \subset \Lambda(P)$, ${}^JF_{\boldsymbol{b}}({}_{\boldsymbol{c}}E_{|D_{J,P}})$ is generated by v_p such that $a_j(v_p) \leqslant b_j$ $(\forall j \in J)$.

Such a frame is called compatible with the parabolic structure of ${}_{\boldsymbol{c}}E$. The number $a_j(v_p)$ is often written as ${}^j\deg^F(v_p)$.

2.5.3.3. Pull-back of filtered bundles. — Let us recall the pull-back of filtered bundles. See [**39**] for a more systematic treatment. Let $X := \Delta_z^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $\widetilde{X} := \Delta_w^m$ and $\widetilde{D} := \bigcup_{j=1}^{k}\{w_j = 0\}$. Let $\varphi : \widetilde{X} \to X$ be a morphism such that $\varphi^{-1}(D) \subset \widetilde{D}$. Then, $\varphi^*(z_i) = \prod_{j=1}^{k} w_j^{\alpha_{j,i}}\, g_i$ for some invertible function g_i $(i = 1, \ldots, \ell)$. Let $\varphi^* : \boldsymbol{R}^\ell \to \boldsymbol{R}^k$ be given by $\varphi_j^*(\boldsymbol{b}) := \sum_{i=1}^{\ell} \alpha_{j,i}\, b_i$. For any $\boldsymbol{b} \in \boldsymbol{R}^k$, we set

$$\mathcal{S}(\boldsymbol{b}) := \left\{(\boldsymbol{a}, \boldsymbol{n}) \in \boldsymbol{R}^\ell \times \mathbb{Z}^k_{\geqslant 0} \,\middle|\, \varphi^*(\boldsymbol{a}) + \boldsymbol{n} \leqslant \boldsymbol{b}\right\}.$$

Let $\boldsymbol{E}_*$ be a filtered sheaf on (X, D). We put

$${}_{\boldsymbol{b}}\widetilde{E} := \sum_{(\boldsymbol{a},\boldsymbol{n})\in\mathcal{S}(\boldsymbol{b})} \boldsymbol{w}^{-\boldsymbol{n}}\, \varphi^*\big({}_{\boldsymbol{a}}E\big).$$

Thus, we obtain a filtered sheaf $\widetilde{\boldsymbol{E}}_*$ on $(\widetilde{X}, \widetilde{D})$. It is independent of the choice of the coordinate systems $\boldsymbol{z}$ and $\boldsymbol{w}$.

Lemma 2.5.7. — *If $\boldsymbol{E}_*$ is a filtered bundle, $\widetilde{\boldsymbol{E}}_*$ is also a filtered bundle.*

Proof. — Let $\boldsymbol{v}$ be a frame of ${}^\diamond E$ compatible with the parabolic filtrations. We put $a_i(v_p) := {}^i\deg^F(v_p)$ and $\boldsymbol{a}(v_p) := \big(a_i(v_p)\big)$. Let $\boldsymbol{c} = (c_j) \in \boldsymbol{R}^k$. Let $n_j(v_i)$ be the integers determined by the condition $c_j - 1 < \varphi_j^*(\boldsymbol{a}(v_p)) + n_j(v_p) \leqslant c_j$. We set

$${}_{\boldsymbol{c}}\widetilde{v}_p := \prod_j w_j^{-n_j(v_p)}\, \varphi^*(v_p).$$

Then, we can check that ${}_{\boldsymbol{c}}\widetilde{\boldsymbol{v}} := \big({}_{\boldsymbol{c}}\widetilde{v}_p\big)$ gives a frame of ${}_{\boldsymbol{c}}\widetilde{E}$ compatible with the parabolic filtrations. □

Let X (resp. $\widetilde{X}$) be a complex manifold with a simple normal crossing hypersurface D (resp. $\widetilde{D}$). Let $\varphi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a morphism. Let $\boldsymbol{E}_*$ be a filtered bundle on (X, D). Applying the above procedure locally, we obtain a filtered bundle $\widetilde{\boldsymbol{E}}_*$ on $(\widetilde{X}, \widetilde{D})$ globally.

2.5.3.4. Descent with respect to a ramified covering. — Let $X := \Delta_z^n$ and $\widetilde{X} := \Delta_w^n$. Let $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$ and $\widetilde{D} := \bigcup_{i=1}^{\ell}\{w_i = 0\}$. Let $\varphi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a ramified covering given by $\varphi(w_1, \dots, w_n) = (w_1^{m_1}, \dots, w_\ell^{m_\ell}, w_{\ell+1}, \dots, w_n)$. Let $\varphi^* : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$ be given by $\varphi^*(a_1, \dots, a_\ell) = (m_1 a_1, \dots, m_\ell a_\ell)$.

Let $\widetilde{\boldsymbol{E}}_* = ({}_{\boldsymbol{b}}\widetilde{E})$ be a filtered sheaf on $(\widetilde{X}, \widetilde{D})$, which is equipped with the $\mathrm{Gal}(\widetilde{X}/X)$-action. Let ${}_{\boldsymbol{a}}E$ be the descent of ${}_{\varphi^*(\boldsymbol{a})}\widetilde{E}$. Thus, we obtain a filtered sheaf $\boldsymbol{E}_*$ on (X, D). It is easy to see that $\boldsymbol{E}_*$ is also a filtered bundle, if $\widetilde{\boldsymbol{E}}_*$ is a filtered bundle. The construction is independent of the choice of a coordinate system.

For any general ramified covering of complex manifolds, we obtain the global descent by applying the above procedure locally.

2.6. Good lattice at the level $\boldsymbol{m}$

We introduce an auxiliary concept of good lattice at the level $\boldsymbol{m}$. It proves useful in the inductive study on Stokes structure. Because we consider only the unramified case, we omit to distinguish it.

2.6.1. Order of the pole. — We introduce an auxiliary notion of "order" of the pole of a 1-form or a meromorphic flat ϱ-connection. Let $\mathcal{X} \to \mathcal{K}$, $\mathcal{D}$, ϱ be as before. Let $\mathcal{D} = \bigcup_{i\in\Lambda} \mathcal{D}_i$ be the decomposition into irreducible components. Let $\boldsymbol{m} \in \mathbb{Z}_{\leqslant 0}^{\Lambda}$. We put $\mathcal{D}^{(1)} := \bigcup_{m_i<0} \mathcal{D}_i$ and $\mathcal{D}^{(2)} := \bigcup_{m_i=0} \mathcal{D}_i$.

Definition 2.6.1. — Let ω be a holomorphic section of $F \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}(*\mathcal{D})$, where F is a locally free $\mathcal{O}_{\mathcal{X}}$-module. We say $\operatorname{ord}\omega \geqslant \boldsymbol{m}$, if it is contained in

$$F \otimes \Big(\boldsymbol{z}^{\boldsymbol{m}}\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(1)}) + \Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(2)})\Big).$$

We have similar conditions for 1-forms on formal complex spaces. □

Let E be a locally free $\mathcal{O}_{\mathcal{X}}$-module with a meromorphic ϱ-flat connection $\mathbb{D}$ of $E(*\mathcal{D})$.

Definition 2.6.2. — We say $\operatorname{ord}(\mathbb{D}) \geqslant \boldsymbol{m}$, if the following holds:

$$\mathbb{D}E \subset E \otimes \big(\boldsymbol{z}^{\boldsymbol{m}}\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(1)}) + \Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(2)})\big). \tag{33}$$

We have a similar condition for the lattice of a meromorphic flat ϱ-connection on formal complex spaces. □

Let $\boldsymbol{v}$ be a frame of E. Let A be determined by $\mathbb{D}\boldsymbol{v} = \boldsymbol{v}\,A$. We have $\operatorname{ord}\mathbb{D} \geqslant \boldsymbol{m}$ if and only if $\operatorname{ord} A \geqslant \boldsymbol{m}$.

Remark 2.6.3. — For any j such that $m_j = 0$, we have the induced endomorphism $\mathrm{Res}_j(\mathbb{D})$ of $E_{|\mathcal{D}_j}$. □

Remark 2.6.4. — The condition (33) implies the following:

$$\mathbb{D}E \subset \boldsymbol{z}^{\boldsymbol{m}} E \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}). \tag{34}$$

It was adopted as the definition of order in the older version of this monograph. The difference is not essential for our purpose. The condition (33) might be more natural, and (34) might be easier to state. □

2.6.2. Good set of irregular values at the level $\boldsymbol{m}$. — This subsection is a complement of Section 2.1. Let Y be a complex manifold. We put $X := \Delta^\ell \times Y$, and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$.

Definition 2.6.5. — Let $\boldsymbol{m} \in \mathbb{Z}^{\ell}_{\leqslant 0} \setminus \{\boldsymbol{0}\}$. A finite set of meromorphic functions $\mathcal{I} = \{\mathfrak{a} = \mathfrak{a}_{\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{m}}\} \subset M(X, D)$ is called a weakly good set of irregular values on (X, D) at the level $\boldsymbol{m}$, if the following holds:

- $\mathfrak{a}_{\boldsymbol{m}} - \mathfrak{b}_{\boldsymbol{m}}$ are invertible holomorphic functions on X for any two distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$.

If moreover the following condition holds for an integer i such that $m_i < 0$, $\mathcal{I}$ is called a weakly good set of irregular values on (X, D) at the level $(\boldsymbol{m}, i)$.

- $\mathfrak{a}_{\boldsymbol{m}} - \mathfrak{b}_{\boldsymbol{m}}$ are independent of the variable z_i.

A weakly good set of irregular values at the level $(\boldsymbol{m}, i)$ is called a good set of irregular values at the level $(\boldsymbol{m}, i)$, if the following condition is satisfied.

- $\mathfrak{a}_{\boldsymbol{m}}$ are holomorphic functions on X, which are independent of z_i. □

Let $\mathcal{I}$ be a weakly good set of irregular values at the level $(\boldsymbol{m}, i)$. We choose any $\mathfrak{a}^{(0)} \in \mathcal{I}$. Then, the set $\mathcal{I}' := \{\mathfrak{a} - \mathfrak{a}^{(0)} \mid \mathfrak{a} \in \mathcal{I}\}$ is a good set of irregular values at the level $(\boldsymbol{m}, i)$.

For a weakly good set of irregular values $\mathcal{I}$ at the level $(\boldsymbol{m}, i)$, we put $\mathcal{I}^{\vee} := \{-\mathfrak{a} \mid \mathfrak{a} \in \mathcal{I}\}$. For $\mathcal{I}_i$ $(i = 1, 2)$, we put $\mathcal{I}_1 \otimes \mathcal{I}_2 := \{\mathfrak{a}_1 + \mathfrak{a}_2 \mid \mathfrak{a}_i \in \mathcal{I}_I\}$ and $\mathcal{I}_1 \oplus \mathcal{I}_2 = \mathcal{I}_1 \cup \mathcal{I}_2$, which are not necessarily weakly good at the level $(\boldsymbol{m}, i)$.

2.6.2.1. Let $\mathcal{J}$ be a good set of irregular values on (X, D). Take an auxiliary sequence $\boldsymbol{m}(0), \boldsymbol{m}(1), \ldots, \boldsymbol{m}(L)$. Let us observe that we have the associated good sets of irregular values on (X, D) at the level $(\boldsymbol{m}(i), \mathfrak{h}(i))$ for $i = 0, 1, \ldots, L$, after shrinking X. Recall that we have the truncations $\mathcal{J}(\boldsymbol{m}(i)) := \overline{\eta}_{\boldsymbol{m}(i)}(\mathcal{J})$. Formally, we set $\mathcal{J}(\boldsymbol{m}(-1)) := \{0\}$. The set $\mathcal{I}_0^{\boldsymbol{m}(0)} := \mathcal{J}(\boldsymbol{m}(0))$ is a good set of irregular values at the level $(\boldsymbol{m}(0), \mathfrak{h}(0))$. We have the naturally induced morphisms $\overline{\eta}_{\boldsymbol{m}(i), \boldsymbol{m}(j)} : \mathcal{J}(\boldsymbol{m}(j)) \to \mathcal{J}(\boldsymbol{m}(i))$ for $j > i$. For any $\mathfrak{a} \in \mathcal{J}(\boldsymbol{m}(i-1))$, we define

$$\mathcal{I}_{\mathfrak{a}}^{\boldsymbol{m}(i)} := \overline{\eta}^{-1}_{\boldsymbol{m}(i-1), \boldsymbol{m}(i)}(\mathfrak{a}), \quad \overline{\mathcal{I}}_{\mathfrak{a}}^{\boldsymbol{m}(i)} := \{\zeta_{\boldsymbol{m}(i)}(\mathfrak{b}) \mid \mathfrak{b} \in \mathcal{I}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\}.$$

Then, $\mathcal{I}_{\mathfrak{a}}^{\boldsymbol{m}(i)}$ are weakly good sets of irregular values at the level $(\boldsymbol{m}(i), \mathfrak{h}(i))$, and $\overline{\mathcal{I}}_{\mathfrak{a}}^{\boldsymbol{m}(i)}$ are good sets of irregular values at the level $(\boldsymbol{m}(i), \mathfrak{h}(i))$.

2.6.3. Good lattice at the level $\boldsymbol{m}$. — Let Y be a complex manifold with a simple normal crossing divisor $\mathcal{D}'_Y$. Let $\mathcal{K}$ be a complex manifold with a holomorphic function ϱ. Let $\mathcal{X} := \Delta_z^k \times Y \times \mathcal{K}$, $\mathcal{D}_{z,i} := \{z_i = 0\}$ and $\mathcal{D}_z := \bigcup_{i=1}^k \mathcal{D}_{z,i}$. We also put $\mathcal{D}_Y := \Delta_z^k \times \mathcal{D}'_Y \times \mathcal{K}$ and $\mathcal{D} := \mathcal{D}_z \cup \mathcal{D}_Y$.

Let $\boldsymbol{m} \in \mathbb{Z}^k_{<0}$, and let $i(0)$ be an integer such that $1 \leqslant i(0) \leqslant k$. We put $\boldsymbol{m}(1) := \boldsymbol{m} + \boldsymbol{\delta}_{i(0)}$. Let E be a locally free $\mathcal{O}_{\mathcal{X}}$-module, and let $\mathbb{D}$ be a meromorphic flat ϱ-connection of $E(*\mathcal{D})$.

Definition 2.6.6. — $(E, \mathbb{D})$ is called a weakly good lattice of a meromorphic ϱ-flat bundle at the level $(\boldsymbol{m}, i(0))$, if there exist a weakly good set of irregular values $\mathcal{I}$ at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$, and a decomposition

$$(E, \mathbb{D})_{|\widehat{\mathcal{D}}_z} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}) \tag{35}$$

such that $\operatorname{ord}(\widehat{\mathbb{D}}_{\mathfrak{a}} - d\mathfrak{a}) \geqslant \boldsymbol{m}(1)$.

If $\mathcal{I}$ is a good set of irregular values at the level $(\boldsymbol{m}, i(0))$, $(E, \mathbb{D}, \mathcal{I})$ is called a good lattice at the level $(\boldsymbol{m}, i(0))$. □

The decomposition (35) is called the irregular decomposition at the level $(\boldsymbol{m}, i(0))$, (or simply $\boldsymbol{m}$). In this situation, we will often say that $(E, \mathbb{D}, \mathcal{I})$ is a (weakly) good lattice at the level $(\boldsymbol{m}, i(0))$. The rank of $\widehat{E}_{\mathfrak{a}}$ will be often denoted by $r(\mathfrak{a})$. The following lemma is clear.

Lemma 2.6.7. — *Let $(E, \mathbb{D}, \mathcal{I})$ be a good lattice at the level $(\boldsymbol{m}, i(0))$. For any $\mathfrak{a} \in M(\mathcal{X}, \mathcal{D})$, we consider the line bundle $\mathcal{L}(\mathfrak{a}) = \mathcal{O}_{\mathcal{X}}\, e$ with the meromorphic ϱ-connection $\mathbb{D}e = e\,(d\mathfrak{a})$. We set $\mathcal{I}' := \{\mathfrak{b} + \mathfrak{a} \mid \mathfrak{b} \in \mathcal{I}\}$ and $(E', \mathbb{D}') := (E, \mathbb{D}) \otimes \mathcal{L}(\mathfrak{a})$. Then, $(E', \mathbb{D}', \mathcal{I}')$ is a weakly good lattice at the level $(\boldsymbol{m}, i(0))$.*

Conversely, let $(E, \mathbb{D}, \mathcal{I})$ be a weakly good lattice at the level $(\boldsymbol{m}, i(0))$. Take any element $\mathfrak{a} \in \mathcal{I}$, and set $\mathcal{I}' := \{\mathfrak{b} - \mathfrak{a} \mid \mathfrak{b} \in \mathcal{I}\}$ and $(E', \mathbb{D}') := (E, \mathbb{D}) \otimes \mathcal{L}(-\mathfrak{a})$. Then, $(E', \mathbb{D}', \mathcal{I}')$ is a good lattice at the level $(\boldsymbol{m}, i(0))$. □

The following lemma can be shown by the argument in the proof of Proposition 2.4.4.

Lemma 2.6.8. — *The condition in Definition* 2.6.6 *is equivalent to the following:*

- *For any $P \in \mathcal{D}_z$, $(E, \mathbb{D})_{|\widehat{P}}$ has a decomposition $(E, \mathbb{D})_{|\widehat{P}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} ({}^P E_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})$ such that* $\operatorname{ord}(\widehat{\mathbb{D}}_{\mathfrak{a}} - d\mathfrak{a}) \geqslant \boldsymbol{m}(1)$. □

We put $\mathcal{K}^0 := \varrho^{-1}(0)$, $\mathcal{X}^0 := \mathcal{X} \times_{\mathcal{K}} \mathcal{K}^0$, etc. For simplicity, we assume that $d\varrho$ is nowhere vanishing on $\mathcal{K}^0$. After shrinking $\mathcal{X}$, we have the irregular decomposition $(E, \mathbb{D})_{|\mathcal{X}^0} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (E_{\mathfrak{a}, \mathcal{X}^0}, \mathbb{D}^0_{\mathfrak{a}})$ such that $\operatorname{ord}(\mathbb{D}^0_{\mathfrak{a}} - d\mathfrak{a}) \geqslant \boldsymbol{m}(1)$. It is uniquely extended to a decomposition $(E, \mathbb{D})_{|\widehat{\mathcal{X}}^0} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}, \widehat{\mathcal{X}}^0}, \widehat{\mathbb{D}}_{\mathfrak{a}})$ on the completion $\widehat{\mathcal{X}}^0$. We put

$W := \mathcal{X}^0 \cup \mathcal{D}_z$. By using Lemma 2.4.12, we obtain a decomposition

$$(E, \mathbb{D})_{|\widehat{W}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} (\widehat{E}_{\mathfrak{a},\widehat{W}}, \widehat{\mathbb{D}}_{\mathfrak{a}}) \tag{36}$$

such that $\operatorname{ord}\big(\widehat{\mathbb{D}}_{\mathfrak{a}} - d\mathfrak{a}\big) \geqslant \boldsymbol{m}(1)$. The decomposition (36) is also called the irregular decomposition at the level $(\boldsymbol{m}, i(0))$.

2.6.4. Residue. — Let $(E, \mathbb{D}, \mathcal{I})$ be a good lattice at the level $(\boldsymbol{m}, i(0))$. Because

$$\mathbb{D}E \subset E \otimes \Big(z^{\boldsymbol{m}}\Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}_z) + \Omega^1_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}_Y)\Big)$$

we obtain the residue $\operatorname{Res}_{Y,j}(\mathbb{D}) \in \operatorname{End}(E_{|\mathcal{D}_{Y,j}})$ for each irreducible component $\mathcal{D}_{Y,j}$ of $\mathcal{D}_Y$ in a standard way. We obtain the residue even in the case where $(E, \mathbb{D}, \mathcal{I})$ is a weakly good lattice at the level $(\boldsymbol{m}, i(0))$ by considering the tensor product with a meromorphic ϱ-flat bundle of rank one.

Lemma 2.6.9. — *Let $(\mathcal{E}, \mathbb{D})$ be a meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ with a good lattice $(E, \mathcal{I})$ at the level $(\boldsymbol{m}, i(0))$. Let P be any point of $\mathcal{D}$ such that $\varrho(P) \neq 0$. Then, we can find a good lattice $(E', \mathcal{I})$ of $E(*\mathcal{D})$ at the level $(\boldsymbol{m}, i(0))$ on a small neighbourhood $\mathcal{X}_P$ with the following non-resonance property:*

- *Let Q be any point of an irreducible component $\mathcal{D}_{Y,j} \cap \mathcal{X}_P$. Then, distinct eigenvalues α, β of $\operatorname{Res}_{Y,j}(\varrho^{-1}\mathbb{D})_{|Q}$ satisfy $\alpha - \beta \notin \mathbb{Z}$.*

Proof. — It can be shown by the standard argument in the proof of Proposition 2.7.5 below. Because we give some more details there, we omit it here. □

2.6.5. Some functoriality. — In general, we use the symbol $V_1^{\perp}$ to denote the subspace $\big\{f \in V_2^{\vee} \mid f(V_1) = 0\big\} \subset V_2^{\vee}$ for given vector spaces $V_1 \subset V_2$, where $V_2^{\vee}$ denotes the dual space of V_2. It is naturally extended in the case of vector bundles. Let $(E, \mathbb{D}, \mathcal{I})$ be a (weakly) good lattice at the level $(\boldsymbol{m}, i(0))$. We set $\mathcal{I}^{\vee} := \big\{-\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathcal{I}\big\}$. Then, the dual $(E^{\vee}, \mathbb{D}^{\vee}, \mathcal{I}^{\vee})$ is also a (weakly) good lattice at the level $(\boldsymbol{m}, i(0))$. The direct summands $\widehat{E}^{\vee}_{\mathfrak{a}}$ $(\mathfrak{a} \in \mathcal{I}^{\vee})$ in the irregular decomposition are given as follows:

$$\widehat{E}^{\vee}_{\mathfrak{a}} = \Bigg(\bigoplus_{\substack{\mathfrak{b}\in\mathcal{I}\\ \mathfrak{b}\neq-\mathfrak{a}}} \widehat{E}_{\mathfrak{b}}\Bigg)^{\perp}.$$

Let $(E_p, \mathbb{D}_p, \mathcal{I}_p)$ $(p = 1, 2)$ be (weakly) good lattices at the level $(\boldsymbol{m}, i(0))$. We put $\mathcal{I}_1 \otimes \mathcal{I}_2 := \big\{\mathfrak{a}_1 + \mathfrak{a}_2 \,\big|\, \mathfrak{a}_p \in \mathcal{I}_p\big\}$. If $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a (weakly) good set of irregular values at the level $(\boldsymbol{m}, i(0))$, then $E_1 \otimes E_2$ is a (weakly) good lattice at the level $(\boldsymbol{m}, i(0))$. The direct summands of the irregular decomposition are given as follows:

$$(\widehat{E_1 \otimes E_2})_{\mathfrak{a}} = \bigoplus_{\substack{(\mathfrak{a}_1,\mathfrak{a}_2)\in\mathcal{I}_1\times\mathcal{I}_2\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}} \widehat{E}_{1,\mathfrak{a}_1} \otimes \widehat{E}_{2,\mathfrak{a}_2}.$$

We put $\mathcal{I}_1 \oplus \mathcal{I}_2 := \mathcal{I}_1 \cup \mathcal{I}_2$. If $\mathcal{I}_1 \oplus \mathcal{I}_2$ is a (weakly) good set of irregular values at the level $(\boldsymbol{m}, i(0))$, the direct sum $E_1 \oplus E_2$ is also a (weakly) good lattice at the level $(\boldsymbol{m}, i(0))$. The direct summands in the irregular decomposition are given as follows:

$$(E_1 \oplus E_2)_{\mathfrak{a}} = E_{1,\mathfrak{a}} \oplus E_{2,\mathfrak{a}}.$$

A morphism $f : (E_1, \mathbb{D}_1) \to (E_2, \mathbb{D}_2)$ of (weakly) good lattices at the level $(\boldsymbol{m}, i(0))$ is defined to be just a flat morphism. Note that the induced morphism $\widehat{f} : (E_1, \mathbb{D}_1)_{|\widehat{W}} \to (E_2, \mathbb{D}_2)_{|\widehat{W}}$ preserves the irregular decomposition.

2.6.6. Remark on growth order of a flat section. — Let $(E, \mathbb{D})$ be a good lattice of $(\mathcal{E}, \mathbb{D})$ at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$. Let $\boldsymbol{v}$ be a frame of E. We have the matrix-valued functions A_i determined by $\mathbb{D}\boldsymbol{v} = \boldsymbol{v}\left(\sum_{i=1}^{n} A_i\, dz_i\right)$. We have the following:

- $A_i = O\big(|z_i^{-1}|\,|\boldsymbol{z}^{\boldsymbol{m}}|\big)$ for $i = 1, \dots, k$.
- $A_i = O\big(|\boldsymbol{z}^{\boldsymbol{m}}|\big) + O(|z_i|^{-1})$ for $i = k+1, \dots, \ell$.
- $A_i = O\big(|\boldsymbol{z}^{\boldsymbol{m}}|\big)$ for $i = \ell+1, \dots, n$.

Assume ϱ is nowhere vanishing on $\mathcal{K}$, for simplicity. Let S be a small multi-sector of $\mathcal{X} \setminus \mathcal{D}_z$, and let f be a $\mathbb{D}$-flat section of $E_{|S}$, which is nowhere vanishing. We have the expression $f = \sum f_i\, v_i$. We obtain a $\boldsymbol{C}^r$-valued function $\boldsymbol{f} = (f_i)$ on S.

Lemma 2.6.10. — *The following holds for some $C > 0$:*

$$\big|\log|\boldsymbol{f}|\big| \leqslant C\,|\boldsymbol{z}^{\boldsymbol{m}}| + C \sum_{i=k+1}^{\ell} \log|z_i|^{-1}.$$

Proof. — It follows from Lemma 20.3.3. □

2.6.7. The induced good lattice at the level $\boldsymbol{m}$. — Let $\mathcal{X} = \Delta^n \times \mathcal{K}$, $\mathcal{D}_i = \{z_i = 0\}$ and $\mathcal{D} = \bigcup_{i=1}^{\ell} \mathcal{D}_i$. Let $(E, \mathbb{D})$ be an unramifiedly good lattice of a meromorphic ϱ-flat bundle $(\mathcal{X}, \mathcal{D})$. For simplicity, we assume that the coordinate system is admissible for the good set $\operatorname{Irr}(\mathbb{D})$. We take an auxiliary sequence $\boldsymbol{m}(0), \boldsymbol{m}(1), \dots, \boldsymbol{m}(L)$ for $\operatorname{Irr}(\mathbb{D})$. Let $\overline{\operatorname{Irr}}(\mathbb{D}, \boldsymbol{m}(i))$ denote the image of $\overline{\eta}_{\boldsymbol{m}(i)} : \operatorname{Irr}(\mathbb{D}) \to M(\mathcal{X}, \mathcal{D})/H(\mathcal{X})$.

Lemma 2.6.11. — $\big(E, \mathbb{D}, \overline{\operatorname{Irr}}(\mathbb{D}, \boldsymbol{m}(0))\big)$ *is a good lattice at the level $(\boldsymbol{m}(0), \mathfrak{h}(0))$. The decomposition is given by the irregular decomposition at the level $(\boldsymbol{m}(0), \mathfrak{h}(0))$:*

$$(E, \mathbb{D})_{|\widehat{\mathcal{D}}(\underline{k}(0))} = \bigoplus_{\mathfrak{a} \in \overline{\operatorname{Irr}}(\mathbb{D}, \boldsymbol{m}(0))} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}).$$

Here $k(0)$ is determined by $\boldsymbol{m}(0) \in \mathbb{Z}_{<0}^{k(0)} \times \boldsymbol{0}_{\ell-k(0)}$, and $\underline{k}(0) = \{1, \dots, k(0)\}$.

Proof. — Let $\mathcal{D}_{\underline{\ell}} := \bigcap_{j=1}^{\ell} \mathcal{D}_j$. Let $\widehat{\boldsymbol{u}}$ be a frame of $E_{|\widehat{\mathcal{D}}_{\underline{\ell}}}$ which is compatible with the irregular decomposition $(E, \mathbb{D})_{|\widehat{\mathcal{D}}_{\underline{\ell}}} = \bigoplus_{\mathfrak{c} \in \operatorname{Irr}(\mathbb{D})} ({}^{\underline{\ell}}\widehat{E}_{\mathfrak{c}}, \mathbb{D}_{\mathfrak{c}})$. The connection 1-form of $\mathbb{D}$ with respect to the frame $\widehat{\boldsymbol{u}}$ is decomposed as $\mathbb{D}\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}\left(\bigoplus_{\mathfrak{c} \in \operatorname{Irr}(\mathbb{D})} \big(d\mathfrak{c}\, I_{\mathfrak{c}} + T_{\mathfrak{c}}\big)\right)$, where $T_{\mathfrak{c}}$ are logarithmic 1-forms, and $I_{\mathfrak{c}}$ are the identity matrices. For each $\mathfrak{a} \in \overline{\operatorname{Irr}}(\mathbb{D}, \boldsymbol{m}(0))$, we put ${}^{\ell}\widehat{E}_{\mathfrak{a}} := \bigoplus_{\mathfrak{c}\in\overline{\eta}^{-1}_{\boldsymbol{m}(0)}(\mathfrak{a})} {}^{\ell}\widehat{E}_{\mathfrak{c}}$. Let $\widehat{\boldsymbol{u}}_{\mathfrak{a}}$ denote the tuple of $\widehat{u}_i \in {}^{\ell}\widehat{E}_{\mathfrak{a}}$, which gives a frame of ${}^{\ell}\widehat{E}_{\mathfrak{a}}$. We put $\mathcal{D}' = \bigcup_{k(0)<i\leqslant\ell}\mathcal{D}_i$. Then, we have $\mathbb{D}\widehat{\boldsymbol{u}}_{\mathfrak{a}} = \widehat{\boldsymbol{u}}_{\mathfrak{a}}\,(d\mathfrak{a}\,I_{\mathfrak{a}} + T'_{\mathfrak{a}})$, where $T'_{\mathfrak{a}} \in \boldsymbol{z}^{\boldsymbol{m}(1)}\Omega^1_{\widehat{\mathcal{D}}_{\underline{\ell}}}(\log\mathcal{D}(\underline{k}(0))) + \Omega^1_{\widehat{\mathcal{D}}_{\underline{\ell}}}(\log\mathcal{D}')$, and $I_{\mathfrak{a}}$ are the identity matrices. Then, the claim of the lemma follows. □

Let $\boldsymbol{m}(i)$ be the minimum of $\mathcal{T}(\mathrm{Irr}(\mathbb{D}))$, i.e., $\left|\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j))\right| = 1$ for $j < i$, and $\left|\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))\right| \geqslant 2$. Note that $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))$ is a weakly good set at the level $(\boldsymbol{m}(i),\mathfrak{h}(i))$.

Lemma 2.6.12. — *$\big(E,\mathbb{D},\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(i))\big)$ is weakly good at the level $(\boldsymbol{m}(i),\mathfrak{h}(i))$.*

Proof. — Take any $\mathfrak{a}\in\mathrm{Irr}(\mathbb{D})$. We consider a line bundle $\mathcal{L}(-\mathfrak{a}) := \mathcal{O}_{\mathcal{X}}\,e$ with the meromorphic ϱ-flat connection $\mathbb{D}e = e\,(-d\mathfrak{a})$. Then, $(E',\mathbb{D}') := (E,\mathbb{D})\otimes\mathcal{L}(-\mathfrak{a})$ is an unramifiedly good lattice with $\mathrm{Irr}(\mathbb{D}') = \big\{\mathfrak{b}-\mathfrak{a}\,\big|\,\mathfrak{b}\in\mathrm{Irr}(\mathbb{D})\big\}$. Note that $\boldsymbol{m}(i),\boldsymbol{m}(i+1),\dots,\boldsymbol{m}(L)$ gives an auxiliary sequence for $\mathrm{Irr}(\mathbb{D}')$. Applying Lemma 2.6.11 to $(E',\mathbb{D}')$, we obtain that $(E',\mathbb{D}')$ is a good lattice at the level $(\boldsymbol{m}(i),\mathfrak{h}(i))$. Then, the claim of the lemma follows. □

2.7. Good Deligne-Malgrange lattice and Deligne-Malgrange lattice

In this section, we consider ordinary flat connections, except the remark in Subsection 2.7.1.1. We use the symbols X, D and ∇ instead of $\mathcal{X}$, $\mathcal{D}$ and $\mathbb{D}$, respectively.

2.7.1. Good Deligne-Malgrange lattice. — Let X be a complex manifold, and let $D=\bigcup_{i\in\Lambda}D_i$ be a simple normal crossing hypersurface.

Definition 2.7.1. — Let E be an unramifiedly good lattice of a meromorphic flat bundle $(\mathcal{E},\nabla)$ on (X,D). It is called unramifiedly good Deligne-Malgrange, if any eigenvalues α of $\mathrm{Res}_{D_i}(\nabla)$ $(i\in\Lambda)$ satisfy $0\leqslant\mathrm{Re}(\alpha)<1$. □

Let E be an unramifiedly good Deligne-Malgrange lattice of a meromorphic flat bundle $(\mathcal{E},\nabla)$ on (X,D). We have the generalized eigen-decomposition $E_{|D_i} = \bigoplus_{\alpha\in\boldsymbol{C}}{}^{i}\mathbb{E}_{\alpha}$ with respect to $\mathrm{Res}_{D_i}(\nabla)$. Then, we have the parabolic filtration ${}^{i}F$ of $E_{|D_i}$ for each irreducible component D_i of D, given in a standard manner:

$$ {}^{i}F_a(E_{|D_i}) := \bigoplus_{-\mathrm{Re}(\alpha)\leqslant a}\mathbb{E}_{\alpha}, \qquad (-1<a\leqslant 0). $$

It is easy to observe that the parabolic filtrations are compatible. The associated filtered bundle is denoted by $\boldsymbol{E}^{DM}_{*}$, which is called the Deligne-Malgrange filtered flat bundle associated to $(\mathcal{E},\nabla)$. We have $E={}^{\diamond}E^{DM}$, i.e., E is the $\boldsymbol{0}$-truncation of $\boldsymbol{E}^{DM}_{*}$.

Lemma 2.7.2. — *Let $\widetilde{X}$ and X be complex manifolds, and let $\widetilde{D}$ and D be simple normal crossing hypersurfaces of $\widetilde{X}$ and X, respectively. Let $\varphi:(\widetilde{X},\widetilde{D})\to(X,D)$ be a ramified covering. Let $(\mathcal{E},\nabla)$ be a meromorphic flat bundle on (X,D), which has an unramifiedly good Deligne-Malgrange lattice E. Then, $\varphi^*(\mathcal{E},\nabla)$ also has an unramifiedly good Deligne-Malgrange lattice $\widetilde{E}$, and E is the descent of $\widetilde{E}$.*

Proof. — We obtain the filtered bundle $\widetilde{\boldsymbol{E}}_*$ on $(\widetilde{X}, \widetilde{D})$ induced by φ and $\boldsymbol{E}_*^{DM}$, as in Section 2.5.3.3. We can observe that ${}^\diamond\widetilde{E}$ is the unramifiedly good Deligne-Malgrange lattice of $\varphi^*\mathcal{E}$. Then, the claims of the lemma are clear. □

Definition 2.7.3. — Let E be a good lattice of a meromorphic flat bundle $(\mathcal{E}, \nabla)$ on (X, D).

- E is called good Deligne-Malgrange at $P \in D$, if there exists a ramified covering $\varphi_P : (X'_P, D'_P) \to (X_P, D_P)$ such that $E_{|X_P}$ is the descent of the unramifiedly good Deligne-Malgrange lattice of $\varphi_P^*\mathcal{E}$. Here, (X_P, D_P), (X'_P, D'_P) and φ_e are as in Definition 2.3.3. Note that we also obtain the natural filtered flat bundle $(\boldsymbol{E}_*^{DM}, \nabla)$ on (X_P, D_P) as the descent of the Deligne-Malgrange filtered flat bundle associated to $\varphi_P^*\mathcal{E}$ in this case.
- E is called good Deligne-Malgrange, if it is good Deligne-Malgrange at any point $P \in D$. In this case, we have the associated filtered flat bundle $(\boldsymbol{E}_*^{DM}, \nabla)$, which is called the Deligne-Malgrange filtered flat bundle associated to $(\mathcal{E}, \nabla)$. □

We should remark that a good Deligne-Malgrange lattice does not necessarily exist for a given meromorphic flat bundle over a higher dimensional variety, in contrast to the one dimensional case. But, it is unique, if it exists, a property which follows from the uniqueness of a (not necessarily good) Deligne-Malgrange lattice (see Section 2.7.2 below) or Lemma 2.7.2.

Proposition 2.7.4. — *Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle. The following conditions are equivalent:*

(a) : *$(\mathcal{E}, \nabla)$ has a good Deligne-Malgrange lattice.*

(b) : *$(\mathcal{E}, \nabla)$ has a good lattice.*

(c) : *For each point $P \in D$, there exists a neighbourhood U of P such that $(\mathcal{E}, \nabla)_{|U}$ has a good lattice.*

(d) : *For each point $P \in D$, there exists a neighbourhood U of P such that $(\mathcal{E}, \nabla)_{|U}$ has a good Deligne-Malgrange lattice.*

Proof. — The implications $(a) \Longrightarrow (b) \Longrightarrow (c)$ are obvious. The implication $(d) \Longrightarrow (a)$ follows from the uniqueness of a Deligne-Malgrange lattice. Let us show the implication $(c) \Longrightarrow (d)$, which can be carried out using a standard successive use of elementary transform. We give only an outline. Due to the uniqueness of a good Deligne-Malgrange lattice, we only have to consider the problem in the unramified case. Let E be an unramifiedly good lattice. Let $\mathcal{S}p(i)$ denote the set of eigenvalues of $\mathrm{Res}_{D_i}(\nabla)$. We have the generalized eigen-decomposition $E_{|D_i} = \bigoplus_{\alpha \in \mathcal{S}p(i)} \mathbb{E}_\alpha$ with respect to $\mathrm{Res}_{D_i}(\nabla)$. We put $a_+ := \max\{\mathrm{Re}(\alpha) \mid \alpha \in \mathcal{S}p(i)\}$ and $a_- := \min\{\mathrm{Re}(\alpha) \mid \alpha \in \mathcal{S}p(i)\}$. We have the subbundles $F_\pm := \bigoplus_{\mathrm{Re}(\alpha) = a_\pm} \mathbb{E}_\alpha$.

Assume $a_- < 0$. We regard F_- as an $\mathcal{O}_X$-module. Let E' denote the kernel of the naturally defined morphism of $\mathcal{O}_X$-modules $E \to F_-$. It is easy to show that E' is also good, the eigenvalues α of $\mathrm{Res}_{D_i}(\nabla')$ satisfy $a_- < \mathrm{Re}(\alpha) \leqslant \max(a_+, 1 + a_-)$. Assume $a_+ > 1$. Let E'' denote the kernel of the naturally defined morphism $E \otimes \mathcal{O}(D_i) \to F_+ \otimes \mathcal{O}(D_i)$. It is easy to show that E'' is also good, and the eigenvalues of $\mathrm{Res}_{D_i}(\nabla'')$ satisfy $\min(a_-, -1 + a_+) \leqslant \mathrm{Re}(\alpha) < a_+$. Hence, by composition of the above procedure, we can construct the unramifiedly good Deligne-Malgrange lattice from an unramifiedly good lattice. □

2.7.1.1. Local existence of non-resonance lattice in the family case. — In this subsection, we consider the case $\mathcal{K}$ is not necessarily a point. We put $\mathcal{X} := X \times \mathcal{K}$, $\mathcal{D} := D \times \mathcal{K}$ and $\mathcal{D}_i := D_i \times \mathcal{K}$. Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ relative to $\mathcal{K}$.

Proposition 2.7.5. — *If $\varrho(P) \neq 0$, there exist a small neighbourhood $\mathcal{X}_P$ of P and a good lattice F of $(\mathcal{E}, \mathbb{D})_{|\mathcal{X}_P}$ with the following property:*
- *Let α and β be distinct eigenvalues of $\mathrm{Res}_i(\varrho^{-1}\mathbb{D})_Q \in \mathrm{End}(F_{|\mathcal{D}_{P,i}})$ for some $Q \in \mathcal{D}_{P,i}$, where $\mathcal{D}_{P,i} = \mathcal{X}_P \cap \mathcal{D}_i$. Then, $\alpha - \beta \notin \mathbb{Z}$.*

Proof. — It can be shown by the argument in the proof of Proposition 2.7.4. We give only an outline. We set $\nabla := \varrho^{-1}\mathbb{D}$. We may assume $\mathcal{X} = \Delta^n \times \mathcal{K}$, $\mathcal{D} = \bigcup_{i=1}^{\ell}\{z_i = 0\}$ and $P = (O, y)$, where O denotes the origin of Δ^n and $y \in \mathcal{K}$. Let E be an unramifiedly good lattice. Let $\mathcal{S}p(i)$ denote the set of the eigenvalues of the endomorphism $\mathrm{Res}_i(\nabla)$ on $E^y_{|\mathcal{D}_i^y}$. We have the generalized eigen-decomposition $E^y_{|\mathcal{D}_i^y} = \bigoplus_{\alpha \in \mathcal{S}p(i)} \mathbb{E}_\alpha$ with respect to $\mathrm{Res}_i(\nabla)$. We put $a_-(E, i) := \min\{\mathrm{Re}(\alpha) \mid \alpha \in \mathcal{S}p(i)\}$ and $a_+(E, i) := \max\{\mathrm{Re}(\alpha) \mid \alpha \in \mathcal{S}p(i)\}$. We have the subbundle $F_- := \bigoplus_{\mathrm{Re}(\alpha) = a_-(E,i)} \mathbb{E}_\alpha$. If $\mathcal{X}_P$ is sufficiently small, we have the subbundle $F'_- \subset E_{|\mathcal{D}_i}$ such that (i) $F'_{-|\mathcal{D}_i^y} = F_-$, (ii) $\mathrm{Res}_{\mathcal{D}_i}(\nabla)(F'_-) \subset F'_-$. Applying the procedure used in the proof of Proposition 2.7.4 to E and F'_-, we obtain a lattice E' such that $a_-(E, i) < a_-(E', i) \leqslant a_-(E, i) + 1$. By successive use of this procedure, we may assume $0 \leqslant a_-(E, i)$. Similarly, we may assume $a_+(E, i) < 1$. Then, if $\mathcal{X}_P$ is sufficiently small, $\alpha - \beta \notin \mathbb{Z}$ for distinct eigenvalues of $\mathrm{Res}_i(\nabla)_Q$ ($Q \in D_{i,P}$). □

2.7.2. Deligne-Malgrange filtered flat sheaf

2.7.2.1. Deligne-Malgrange lattice. — Let X be a complex manifold, and let $D = \bigcup_{i \in \Lambda} D_i$ be a simple normal crossing divisor of X. As already remarked, a meromorphic flat connection $(\mathcal{E}, \nabla)$ does not necessarily have a good lattice. However, according to Malgrange's theorem, it has a lattice which is generically a good Deligne-Malgrange lattice. (See also the work due to Z. Mebkhout [**59**], [**60**] for a construction of lattices of regular singular meromorphic flat bundles whose poles are not necessarily normal crossing divisors.)

***Proposition 2.7.6* (Malgrange [58]).** — *There always exists a unique lattice $E \subset \mathcal{E}$ characterized by the following properties:*
- *E is a coherent reflexive $\mathcal{O}_X$-module.*

- *There exists a Zariski closed subset* Z *of* D *with* $\operatorname{codim}_X(Z) \geqslant 2$*, such that* $E_{|X\setminus Z}$ *is the good Deligne-Malgrange lattice of* $(\mathcal{E},\nabla)_{|X\setminus Z}$*.*

It is called the canonical lattice in [**58**], but we call it Deligne-Malgrange lattice in this paper.

Remark 2.7.7. — In this monograph, a subset $Z\subset X$ is called Zariski closed, if it is a closed complex analytic subset. And a subset $U\subset X$ is called Zariski open, if it is the complement of a Zariski closed set. □

Proof. — We only prove that Z can be Zariski closed in the above sense. Let $N(E)$ denote the closed analytic subset of D such that $Q\in N(E)$ if and only if E_Q is not locally free. Let $D^{[2]}$ denote the set of singular points of D. In [**58**], it is shown that there exists a closed subset $Z\subset D$ in the ordinary topology, such that (i) $D^{[2]}\cup N(E)\subset Z$, (ii) E is good Deligne-Malgrange around any $Q\in D\setminus Z$, (iii) for any $Q\in Z$, there exists a small neighbourhood X_Q of Q in X with a closed analytic subset $\widetilde{Z}_Q$ of $D_Q:=X_Q\cap D$ satisfying $\operatorname{codim}_{X_Q}(\widetilde{Z}_Q)\geqslant 2$ and $X_Q\cap Z\subset \widetilde{Z}_Q$. If the closed subsets Z_i $(i\in\Lambda)$ have the above property, $\bigcap_{i\in\Lambda}Z_i$ also has it. Hence, we have the minimum among the closed subsets with the above property, which will be denoted by Z in the following. Then, $Q\in D$ is contained in Z if and only if one of the following holds: (i) $Q\in D^{[2]}\cup N(E)$, (ii) $Q\notin D^{[2]}\cup N(E)$ and E is not good Deligne-Malgrange at Q.

Let us show that Z is closed analytic. For any $P\in Z$, we can take a small neighbourhood X_P and a closed analytic subset $\widetilde{Z}_P\subset D_P:=D\cap X_P$ satisfying $\operatorname{codim}_{X_P}(\widetilde{Z}_P)\geqslant 2$ and $Z_P=X_P\cap Z\subset\widetilde{Z}_P$. We put $A_P:=\widetilde{Z}_P\setminus Z_P$, which is an open subset of $\widetilde{Z}_P$. We put $D^{[2]}_P:=D^{[2]}\cap X_P$ and $N(E)_P:=N(E)\cap X_P$. Let $\widetilde{Z}_P=\bigcup_{i\in\Gamma}\widetilde{Z}_{P,i}$ be the decomposition into irreducible components. For each $i\in\Gamma$, we put $\widetilde{W}_{P,i}:=D^{[2]}_P\cup N(E)_P\cup\bigcup_{j\neq i}\widetilde{Z}_{P,j}$. If $\widetilde{Z}_{P,i}\setminus A_P\subset\widetilde{W}_{P,i}$, we have $Z_P\subset\widetilde{W}_{P,i}$. Hence, we may and will assume $\widetilde{Z}_{P,i}\setminus(A_P\cup\widetilde{W}_{P,i})\neq\varnothing$ for each $i\in\Gamma$. Then, let us show that $A_P=\varnothing$, i.e., $Z_P=\widetilde{Z}_{P,i}$, which implies that Z is closed analytic subset of X. For that purpose, we only have to show that $\widetilde{Z}_{P,i}\cap A_P=\varnothing$ for each $i\in\Gamma$. Assume the contrary, and we shall deduce a contradiction.

Let $\widetilde{Z}^{\star}_{P,i}$ denote the smooth part of $\widetilde{Z}_{P,i}\setminus\widetilde{W}_{P,i}$. Because $\widetilde{Z}_{P,i}$ is irreducible, $\widetilde{Z}^{\star}_{P,i}$ is connected and non-empty. Because $A_P\cap\widetilde{Z}_{P,i}\neq\varnothing$, we have $A_P\cap\widetilde{Z}^{\star}_{P,i}\neq\varnothing$. We have the two cases: (A) $Z_P\cap\widetilde{Z}^{\star}_{P,i}\neq\varnothing$, (B) $Z_P\cap\widetilde{Z}^{\star}_{P,i}=\varnothing$. In the case (B), note that $Z_P\setminus\widetilde{W}_{P,i}$ is contained in a closed analytic subset whose codimension in X is larger than 3.

We take a point $Q\in Z_P$ as follows. In the case (A), Q is a point in the intersection of $Z_P\cap\widetilde{Z}^{\star}_{P,i}$ and the closure of $A_P\cap\widetilde{Z}^{\star}_{P,i}$ in $\widetilde{Z}^{\star}_{P,i}$. In the case (B), Q is any point of $Z_P\setminus\widetilde{W}_{P,i}$.

We take a small coordinate neighbourhood $(X_Q,z_1,\ldots,z_n)$ around Q such that $D_Q=\{z_1=0\}$. We put $Z_Q:=Z_P\cap X_Q$. In the case (A), we may assume that Z_Q is the complement of a non-empty open subset in $\{z_1 = z_2 = 0\}$. In the case (B), Z_Q is contained in a closed analytic subset Z'_Q with $\operatorname{codim}_{X_Q} Z'_Q \geqslant 3$. By our choice, E is good Deligne-Malgrange around any $Q' \in D_Q \setminus Z_Q$. In this situation, we shall show that E is good Deligne-Malgrange around Q, which contradicts with the choice of Q, and we can conclude that $A_P \cap \widetilde{Z}_{P,i} = \varnothing$.

We only have to consider the case where E is unramifiedly good around any $Q' \in D_Q \setminus Z_Q$. Indeed, after an appropriate ramified covering $\varphi_Q : (X'_Q, D'_Q) \to (X_Q, D_Q)$, the Deligne-Malgrange lattice of $\varphi^* E(*D_Q)$ is unramifiedly good around any point of $\varphi_Q^{-1}(D_Q \setminus Z_Q)$. (See Lemma 2.2.7, for example.) Hence, the claim in the ramified case easily follows from that in the unramified case. So we shall assume that E is unramifiedly good around any $Q' \in D_Q \setminus Z_Q$ in the following argument.

We recall that a holomorphic function on $D_Q \setminus Z_Q$ is naturally extended to a holomorphic function on D_Q. We recall that, for a holomorphic function f on D_Q, if the zero of f is contained in Z_Q, then f is actually nowhere vanishing. We also remark that the fundamental group of $D_Q \setminus Z_Q$ is trivial. Hence, we obtain that there exists a good set of irregular values $\mathcal{I} \subset M(X_Q, D_Q)/H(X_Q)$ such that, for any $Q' \in D_Q \setminus Z_Q$, the restriction of $\mathcal{I}$ to a neighbourhood of Q' is $\operatorname{Irr}(\nabla, Q')$. Then, we can show that $(E, \nabla)_{|\widehat{D}_Q}$ has the irregular decomposition by using a standard argument in [**51**]. We give only an indication.

Let F be a locally free $\mathcal{O}_{\widehat{D}_Q}$-module with a meromorphic connection ∇. Let $\mathcal{I} \subset M(X_Q, D_Q)$ be a good set of irregular values. Assume that, for any $Q' \in D_Q \setminus Z_Q$, we have a decomposition $(F, \nabla)_{|\widehat{D}_{Q'}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (F_{\mathfrak{a},Q'}, \nabla_{\mathfrak{a}})$ such that $\nabla_{\mathfrak{a}}$ are $\mathfrak{a}$-logarithmic, where $D_{Q'} := D \cap X_{Q'}$ for a small neighbourhood $X_{Q'}$ of Q'. Then, we shall show that there exists a decomposition $(F, \nabla) = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (F_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$ such that $\nabla_{\mathfrak{a}}$ are $\mathfrak{a}$-logarithmic. (Then, we obtain the desired decomposition of $(E, \nabla)_{|\widehat{D}_Q}$.) We use an inductive argument.

Let $m(F, \nabla) := \min\{\operatorname{ord}_{z_1} \mathfrak{a} \,|\, \mathfrak{a} \in \mathcal{I}\}$. If $m(F, \nabla) = 0$, there is nothing to do. Let us consider the case $m(F, \nabla) = m$. We set $T := \{-m(z_1^{-m}\mathfrak{a})_{|D_Q} \,\big|\, \mathfrak{a} \in \mathcal{I}\}$. If $|T| = 1$, by considering $\nabla - d\mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{I}$, we can reduce the case $m(F, \nabla) = m + 1$. If $|T| \geqslant 2$, we consider the endomorphism G of $F_{|D_Q}$ induced by $z_1^{-m}\nabla(z_1\partial_1)$. The set of the eigenvalues of $G_{|D_Q \setminus Z_Q}$ is given by T, and we have the eigen-decomposition $F_{|D_Q \setminus Z_Q} = \bigoplus_{\alpha \in T} \mathbb{E}_\alpha$. Then, the set of the eigenvalues of G is T, and we have the eigen-decomposition $F_{|D_Q} = \bigoplus_{\alpha \in T} \mathbb{E}_\alpha$ on D_Q. By a standard argument explained in Lemma 2.2.16, it can be extended to a decomposition $F = \bigoplus_{\alpha \in T} F_\alpha$ such that (i) $z_1^{-m}\nabla(z_1\partial_1)F_\alpha \subset F_\alpha$, (ii) $F_{\alpha|D_Q} = \mathbb{E}_\alpha$. If we restrict it to a small neighbourhood of $Q' \in D_Q \setminus Z_Q$, it is the same as the irregular decomposition around Q'. Hence, it is ∇-flat. We have the natural map $\pi : \mathcal{I} \to T$, and let $\mathcal{I}(\alpha) := \pi^{-1}(\alpha)$. For each $Q' \in D_Q \setminus Z_Q$, we have a decomposition $(F_\alpha, \nabla)_{|\widehat{D}_{Q'}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}(\alpha)} (F_{\mathfrak{a},Q'}, \nabla_{\mathfrak{a}})$. Hence, we can obtain the desired decomposition of (F, ∇) by an easy inductive argument. □

2.7.2.2. Deligne-Malgrange filtered sheaf. — We have the Deligne-Malgrange filtered flat bundle $(\boldsymbol{E}^{DM}_{X\setminus Z\,*},\nabla)$ on $(X\setminus Z, D\setminus Z)$ associated to $(\mathcal{E},\nabla)_{|X\setminus Z}$.

Lemma 2.7.8. — *It can be extended to a filtered flat sheaf on (X,D), i.e., we have the filtered flat sheaf $(\boldsymbol{E}^{DM}_*,\nabla)$ on (X,D) with the following properties:*

- $(\boldsymbol{E}^{DM}_*,\nabla)_{|X\setminus Z}=(\boldsymbol{E}^{DM}_{X\setminus Z\,*},\nabla)$.
- ${}_{\boldsymbol{a}}E^{DM}$ *are coherent reflexive $\mathcal{O}_X$-modules for any $\boldsymbol{a}\in\boldsymbol{R}^{\Lambda}$.*

It is called the Deligne-Malgrange filtered sheaf associated to $(\mathcal{E},\nabla)$.

Proof. — Since we only have to shift the condition on the eigenvalues of the residues, the claim can be shown by repeating the argument of Malgrange. Otherwise, it can be reduced to the existence of Deligne-Malgrange lattice, which is explained in the following. Since the problem is local, we may assume $X=\Delta^n$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. For each $\boldsymbol{a}=(a_i)\in\boldsymbol{R}^{\ell}$, let us consider the line bundle $L_{\boldsymbol{a}}:=\mathcal{O}_X\,e$ with the logarithmic flat connection $\nabla_{\boldsymbol{a}}$ such that $\nabla_{\boldsymbol{a}}e=e\sum_{i=1}^{\ell}a_i\,dz_i/z_i$. We have the Deligne-Malgrange lattice ${}_{\boldsymbol{a}}E'$ of $(\mathcal{E},\nabla)\otimes(L_{\boldsymbol{a}},\nabla_{\boldsymbol{a}})$, and we put ${}_{\boldsymbol{a}}E:={}_{\boldsymbol{a}}E'\otimes L_{-\boldsymbol{a}}$. Then, $\boldsymbol{E}_*=\bigl({}_{\boldsymbol{a}}E\,\big|\,\boldsymbol{a}\in\boldsymbol{R}^{\Lambda}\bigr)$ has the desired property. □

We mention an important property of Deligne-Malgrange filtered sheaf on a projective variety.

Lemma 2.7.9. — *Assume that X is projective, provided with an ample line bundle L. Then, $\mu_L(\boldsymbol{E}^{DM}_*)=0$ always hold. (See Section* 13.1 *for μ_L.)*

Proof. — The claim can be reduced to the one dimensional case, which was shown in [**71**], for example. □

2.7.3. Good formal structure and good lattice. — Let X be a complex manifold with a normal crossing hypersurface D. Let $(\mathcal{E},\nabla)$ be a meromorphic flat bundle on (X,D).

Proposition 2.7.10. — *If $(\mathcal{E},\nabla)_{|\widehat{P}}$ has an (unramifiedly) good Deligne-Malgrange lattice for each $P\in D$, then $(\mathcal{E},\nabla)$ has an (unramifiedly) good Deligne-Malgrange lattice.*

The case D is smooth will be argued in Section 2.7.4, and the normal crossing case will be argued in Section 2.7.5. We remark that we only have to consider the unramified case, according to Lemma 2.2.7.

Before going into the proof of the proposition, we give a consequence, which will be used in Section 16.3.4 for the proof of Theorem 16.2.1.

Corollary 2.7.11. — *Let $0\to\bigl(\mathcal{E}^{(1)},\nabla^{(1)}\bigr)\to\bigl(\mathcal{E}^{(0)},\nabla^{(0)}\bigr)\to\bigl(\mathcal{E}^{(2)},\nabla^{(2)}\bigr)\to 0$ be an exact sequence of meromorphic flat bundles on (X,D). If $(\mathcal{E}^{(i)},\nabla^{(i)})$ $(i=1,2)$ have good Deligne-Malgrange lattices, then $(\mathcal{E}^{(0)},\nabla^{(0)})$ also has a good Deligne-Malgrange lattice.*

Proof. — It immediately follows from Proposition 2.2.13 and Proposition 2.7.10. □

Remark 2.7.12. — In the earlier version of this monograph, the claim of Proposition 2.7.10 was proved in the case $\dim X = 2$. We also proved the claim of Corollary 2.7.11 with a slightly different argument. Proposition 2.7.10 seems useful for simplification of the arguments. Although it is also given in our survey paper [**64**], we include it for the convenience of readers. □

2.7.4. Proof of Proposition 2.7.10 in the smooth divisor case

Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let $\widehat{D}$ denote the completion of X along D. Let $\pi : X \to D$ denote the natural projection.

2.7.4.1. First, let us observe that we can ignore the subsets whose codimension in X is larger than 3. The result in this section is also used in Section 19.3.5. We will use the following easy lemma implicitly.

Lemma 2.7.13. — *Let Z be a closed analytic subset of D with $\operatorname{codim}_D(Z) \geqslant 2$. Let $\mathcal{I}$ be a good set of irregular values on $(X \setminus Z, D \setminus Z)$. Then, it is also a good set of irregular values on (X, D).*

Proof. — Take $\mathfrak{a} \in \mathcal{I}$. By Hartogs property, we obtain that $\mathfrak{a} \in M(X,D)/H(X)$. Because $\mathfrak{a}_{\operatorname{ord}(\mathfrak{a})}$ is nowhere vanishing on $D \setminus Z$, we obtain that it is also nowhere vanishing on D. We can check the other property similarly. □

Let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection on (X, D), i.e, $\mathcal{E}$ is a (not necessarily locally free) coherent $\mathcal{O}_X(*D)$-module with a meromorphic flat connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$. Assume the following:

- There exist a closed analytic subset $Z \subset D$ with $\operatorname{codim}_D(Z) \geqslant 2$, and a good set of irregular values $\mathcal{I} \subset M(X,D)/H(X)$, such that the irregular values of the restrictions $(\mathcal{E}, \nabla)_{|\pi^{-1}(P)}$ are given by $\{\mathfrak{a}_{|\pi^{-1}(P)} \,|\, \mathfrak{a} \in \mathcal{I}\}$ for any $P \in D \setminus Z$.

Lemma 2.7.14. — *If the above condition is satisfied, the Deligne-Malgrange lattice E of $(\mathcal{E}, \nabla)$ is a locally free $\mathcal{O}_X$-module, and unramifiedly good with $\operatorname{Irr}(\nabla, Q) = \mathcal{I}$ for any $Q \in D$.*

Proof. — Since E is reflexive, by extending Z, we may assume that $E_{|X \setminus Z}$ is locally free. By using a well established argument (see [**51**] or [**58**], for example), we can easily obtain the irregular decomposition $E_{|\widehat{D \setminus Z}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} F_{\mathfrak{a}, D \setminus Z}$. Let $\pi_{\mathfrak{a}}$ denote the projection onto $F_{\mathfrak{a}, D \setminus Z}$, which gives a section of $\operatorname{End}(E)_{|\widehat{D \setminus Z}}$.

Let us see that $\pi_{\mathfrak{a}}$ can be extended to a section of $\operatorname{End}(E)_{|\widehat{D}}$. Although it follows from a general result, we show it directly. It is easy to show the following claim by using Hartogs theorem:

- Any section of $\mathcal{O}_{\widehat{D \setminus Z}}$ can be extended to a section of $\mathcal{O}_{\widehat{D}}$.

Since E is reflexive, we can (locally) take an injection $i : E \to \mathcal{O}_X^{\oplus N}$ for some large N such that the cokernel $\operatorname{Cok}(i)$ is torsion-free. We can also take a surjection φ :

$\mathcal{O}_X^{\oplus M} \to E$. The morphisms i, φ and $\pi_{\mathfrak{a}}$ induce a morphism $F_{\mathfrak{a}} : \mathcal{O}^{\oplus M}_{X|\widehat{D\setminus Z}} \to \mathcal{O}^{\oplus N}_{X|\widehat{D\setminus Z}}$. It can be extended to a morphism $\widetilde{F}_{\mathfrak{a}} : \mathcal{O}_{\widehat{D}}^{\oplus M} \to \mathcal{O}_{\widehat{D}}^{\oplus N}$. Since $\mathrm{Cok}(i)$ is torsion free, $\widetilde{F}_{\mathfrak{a}}$ factors through $E_{|\widehat{D}}$. Let $\mathcal{K}$ denote the kernel of $\mathcal{O}_{|\widehat{D}}^{\oplus M} \to E_{|\widehat{D}}$. The restriction of $\widetilde{F}_{\mathfrak{a}}$ to $\mathcal{K}$ on $\widehat{D\setminus Z}$ is 0. Then, we obtain $\widetilde{F}_{\mathfrak{a}|\mathcal{K}} = 0$ because $\mathcal{O}_{\widehat{D}}^{\oplus N}$ is torsion-free. Thus, we obtain the induced maps $\pi_{\mathfrak{a}} : E_{|\widehat{D}} \to E_{|\widehat{D}}$ for $\mathfrak{a} \in \mathcal{I}$, which satisfy $\pi_{\mathfrak{a}} \circ \pi_{\mathfrak{a}} = \pi_{\mathfrak{a}}$, $\pi_{\mathfrak{a}} \circ \pi_{\mathfrak{b}} = 0$ ($\mathfrak{a} \neq \mathfrak{b}$), and $\sum \pi_{\mathfrak{a}} = \mathrm{id}$. They give a decomposition $E = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{E}_{\mathfrak{a}}$. Let us show that $\widehat{E}_{\mathfrak{a}}$ is $\mathfrak{a}$-logarithmic. We only have to consider the case $\mathfrak{a} = 0$.

Take a point $P \in D \setminus Z$. We have the vector space $V := \widehat{E}_{0|P}$. We have the endomorphism f of V induced by the residue. Let $E_0' := V \otimes \mathcal{O}_X$ and $\nabla_0' = d + f \cdot dz_1/z_1$. We have the natural flat isomorphism $(E_0', \nabla_0')_{|\pi^{-1}(P)} \simeq (\widehat{E}_0, \nabla_0)_{|\pi^{-1}(P)}$. Since the codimension of Z in D is larger than 2, we obtain a flat isomorphism $\Phi_{0,D\setminus Z} : (E_0', \nabla_0')_{|\widehat{D\setminus Z}} \simeq (\widehat{E}_0, \nabla_0)_{|\widehat{D\setminus Z}}$. Since E_0' and $\widehat{E}_0$ are both reflexive, the above argument shows that $\Phi_{0,D\setminus Z}$ and $\Phi_{0,D\setminus Z}^{-1}$ can be extended to morphisms on $\widehat{D}$. Thus, we are done. □

2.7.4.2. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on $(\widehat{D}, D)$. Assume the following:

(A) : $(\mathcal{E}, \nabla)_{|\widehat{P}}$ has an unramifiedly good Deligne-Malgrange lattice ${}^{P}E$ for each $P \in D$.

We obtain the following lemma from Proposition 2.4.4.

Lemma 2.7.15. — *Let E be an $\mathcal{O}_{\widehat{D}}$-free lattice of $\mathcal{E}$ such that $E_{|\widehat{P}} = {}^{P}E$ for any $P \in D$. Then, the following holds:*

- *There exists $\mathcal{I} \in z_1^{-1}H(D)[z_1^{-1}]$ such that $\mathcal{I}_{|\widehat{P}} = \mathrm{Irr}(\nabla, P)$ for any $P \in D$.*
- *We have a flat decomposition $E = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a}}$ whose restriction to $\widehat{P}$ is the same as the irregular decomposition of ${}^{P}E$ for any $P \in D$.* □

2.7.4.3. We put $Z := \{z_1 = z_2 = 0\}$. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on $(\widehat{D}, D)$ satisfying the condition (A) in Subsection 2.7.4.2. Assume there exists an $\mathcal{O}_X$-free lattice E of $\mathcal{E}$ such that $E_{|\widehat{P}} = {}^{P}E$ for each $D \setminus Z$. Take any point $P \in D \setminus Z$. We have a naturally induced action of the fundamental group $\pi_1(D \setminus Z, P)$ on $\mathrm{Irr}(\nabla, P)$ by Lemma 2.7.15.

Lemma 2.7.16. — *If the action of $\pi_1(D \setminus Z, P)$ on $\mathrm{Irr}(\nabla, P)$ is trivial, we have $E_{|\widehat{Q}} = {}^{Q}E$ for any $Q \in Z$. In particular, the conclusion of Lemma* 2.7.15 *holds.*

Proof. — Because the action of $\pi_1(D \setminus Z, P)$ on $\mathrm{Irr}(\nabla, P)$ is trivial, we have $\mathcal{I} \subset z_1^{-1}H(D \setminus Z)[z_1^{-1}]$ such that $\mathcal{I}_{|\widehat{P}'} = \mathrm{Irr}(\nabla, P')$ for any $P' \in D \setminus Z$. We set $m := \min\{\mathrm{ord}_{z_1}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$. We use a descending induction on m. If $m = 0$, we can deduce that ∇ is logarithmic, and hence the claim is obvious. Let us consider $m + 1 \Longrightarrow m$. We put

$$T := \{(z_1^{-m} z_1 \partial_1 \mathfrak{a})_{|D} \mid \mathfrak{a} \in \mathcal{I}\} \subset H(D \setminus Z).$$

Because $z_1^{-m}\nabla(z_1\partial_1)({}^PE) \subset {}^PE$ for any $P \in D \setminus Z$, we have $z_1^{-m}\nabla(z_1\partial_1)E_{|D\setminus Z} \subset E_{|D\setminus Z}$, and hence $z_1^{-m}\nabla(z_1\partial_1)E \subset E$. Let G be the endomorphism of $E_{|D}$ induced by $z_1^{-m}\nabla(z_1\partial_1)$. Because the elements of T are the eigenvalues of $G_{|D\setminus Z}$, they are algebraic over $H(D)$. Hence, we obtain $T \subset H(D)$.

Let $Q \in Z$. We will shrink X around Q without mention. Let N be the $H(D)(\!(z_1)\!)$-module corresponding to $\mathcal{E}$, i.e., the space of the global sections of $\mathcal{E}$. Let L be the $H(D)[\![z_1]\!]$-lattice of N corresponding to E. We put $N' := N \otimes M(D,Z)(\!(z_1)\!)$ and $L' := L \otimes M(D,Z)[\![z_1]\!]$. We have the eigen-decomposition of L'/z_1L' with respect to G, which can be extended to a decomposition $L' = \bigoplus_{\mathfrak{b}\in T} L'_{\mathfrak{b}}$ such that $(z_1^{-m+1}\partial_1 - \mathfrak{b})L'_{\mathfrak{b}} \subset L'_{\mathfrak{b}}$ by Lemma 2.2.16. We put $m(Q) := \min\{\operatorname{ord}_{z_1}(\mathfrak{a}) \mid \mathfrak{a} \in \operatorname{Irr}(\nabla, Q)\}$ and

$$T(Q) := \big\{ \big(z_1^{-m(Q)+1}\partial_1\mathfrak{a}\big)_{|D} \,\big|\, \mathfrak{a} \in \operatorname{Irr}(\nabla, Q) \big\}.$$

Lemma 2.7.17. — *We have $m(Q) = m$ and $T(Q) = T$.*

Proof. — We may assume $Q = (0,\dots,0)$. We put $\mathcal{N} := N \otimes \mathcal{O}_{\widehat{Q}}$. It is equipped with an unramifiedly good Deligne-Malgrange lattice ${}^Q\mathcal{L}$ with $z_1\partial_1$-decomposition ${}^Q\mathcal{L} = \bigoplus_{\mathfrak{b}\in T(Q)} {}^Q\mathcal{L}_{\mathfrak{b}}$ such that $(z_1^{-m(Q)+1}\partial_1 - \mathfrak{b}){}^Q\mathcal{L}_{\mathfrak{b}} \subset {}^Q\mathcal{L}_{\mathfrak{b}}$ for any $\mathfrak{b} \in T(Q)$. By considering the extension to the field $\boldsymbol{C}(\!(z_n)\!)\cdots(\!(z_2)\!)(\!(z_1)\!)$, and by using Lemma 2.2.20, we obtain Lemma 2.7.17. □

Let us return to the proof of Lemma 2.7.16. By Lemma 2.7.17, we have the eigen-decomposition of $E_{|D}$ with respect to G. By Lemma 2.2.16, it can be extended to a decomposition $E = \bigoplus_{\mathfrak{b}} E_{\mathfrak{b}}$ such that $(t^{-m+1}\partial_t - \mathfrak{b})E_{\mathfrak{b}} \subset E_{\mathfrak{b}}$. Put $\mathcal{E}_{\mathfrak{b}} = E_{\mathfrak{b}}(*D)$. We can apply the inductive assumption to $\mathcal{E}_{\mathfrak{b}} \otimes L\big(-z_1^{-m}\mathfrak{b}/m\big)$, and the proof of Lemma 2.7.16 is finished. □

Lemma 2.7.18. — *The action of $\pi_1(D \setminus Z)$ on* $\operatorname{Irr}(\nabla, P)$ *is trivial. In particular, the conclusion of Lemma* 2.7.15 *holds.*

Proof. — Because $\operatorname{Irr}(\nabla, P)$ is finite, we can find a ramified covering $\varphi : X' \to X$ given by $\varphi(z_1, \zeta_2, z_3, \dots, z_n) = (z_1, \zeta_2^e, z_3, \dots, z_n)$ such that we can apply Lemma 2.7.16 to $\varphi^*(\mathcal{E}, \nabla)$ and φ^*E. Then, $\varphi^*\operatorname{Irr}(\nabla, P) \subset z_1^{-1}H(X')[z_1^{-1}]$ and $\varphi^*\operatorname{Irr}(\nabla, P)_{|\widehat{Q}} = \varphi^*\operatorname{Irr}(\nabla, Q)$. Hence, we can conclude that the action of $\pi_1(D \setminus Z, P)$ is trivial. □

2.7.4.4. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat sheaf on (X, D) satisfying the condition (A) in Subsection 2.7.4.2.

Lemma 2.7.19. — *The Deligne-Malgrange lattice E of $(\mathcal{E}, \nabla)$ is unramifiedly good Deligne-Malgrange. Namely, the claim of Proposition* 2.7.10 *holds if D is smooth.*

Proof. — There exists a closed analytic subset $Z \subset D$ with $\operatorname{codim}_D(Z) \geqslant 2$ such that $E_{|X\setminus Z}$ is locally free. By Lemma 2.7.16 and Lemma 2.7.18, we obtain that there exists a closed analytic subset $Z' \subset D$ with $\operatorname{codim}_D(Z') \geqslant 2$ such that $E_{|X\setminus Z'}$ is good Deligne-Malgrange. Then, the claim of the lemma follows from Lemma 2.7.14. □

2.7.5. Proof of Proposition 2.7.10 in the normal crossing case

Let $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We put $\partial D_1 := D_1 \cap \bigcup_{2\leqslant j\leqslant \ell} D_j$. We put $D_1^\circ := D_1 \setminus \partial D_1$.

2.7.5.1. We regard $M(D_1, \partial D_1)((z_1))$ as a differential ring equipped with the differential ∂_1. Let $\mathcal{N}$ be a differential $M(D_1, \partial D_1)((z_1))$-module with a $M(D_1, \partial D_1)[\![z_1]\!]$-free lattice $\mathcal{L}$. We put $\mathcal{L}' := \mathcal{L} \otimes H(D_1^\circ)[\![z_1]\!]$. Assume that we have $\mathcal{I} \subset z_1^{-1}H(D_1^\circ)[z_1^{-1}]$ and a decomposition $\mathcal{L}' = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{L}'_{\mathfrak{a}}$ such that (i) $(z_1\partial_1 - z_1\partial_1\mathfrak{a})\mathcal{L}'_{\mathfrak{a}} \subset \mathcal{L}'_{\mathfrak{a}}$, (ii) the eigenvalues α of the induced endomorphism of $\mathcal{L}'_{\mathfrak{a}}/z_1\mathcal{L}'_{\mathfrak{a}}$ satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$, (iii) $\mathfrak{a}_1 - \mathfrak{a}_2$ is invertible in $H(D_1^\circ)((z_1))$ for distinct $\mathfrak{a}_i \in \mathcal{I}$.

Lemma 2.7.20. — *We have $\mathcal{I} \subset z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$ and a decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{L}_{\mathfrak{a}}$ such that (i) $(z_1\partial_1 - z_1\partial_1\mathfrak{a})\mathcal{L}_{\mathfrak{a}} \subset \mathcal{L}_{\mathfrak{a}}$, (ii) the eigenvalues α of the induced endomorphism of $\mathcal{L}_{\mathfrak{a}}/z_1\mathcal{L}_{\mathfrak{a}}$ satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$. Moreover, we have $\mathcal{L}_{\mathfrak{a}} \otimes H(D_1^\circ)[\![z_1]\!] = \mathcal{L}'_{\mathfrak{a}}$.*

Proof. — We use a descending induction on $m(\mathcal{L}) := \min\{\mathrm{ord}_{z_1}(\mathfrak{a}) \,|\, \mathfrak{a} \in \mathcal{I}\}$. If $m(\mathcal{L}) = 0$, there is nothing to prove. Let us consider the case $m(\mathcal{L}) = m < 0$. We put $T(\mathcal{L}) := \{m(z_1^{-m}\mathfrak{a})_{|z_1=0} \,|\, \mathfrak{a} \in \mathcal{I}\}$. Let us consider the endomorphism G of $\mathcal{L}/z_1\mathcal{L}$ induced by $z_1^{-m}\nabla(z_1\partial_1)$. Because the elements of $T(\mathcal{L})$ are the eigenvalues of G, they are algebraic over $M(D_1, \partial D_1)$. Then, we can deduce $T(\mathcal{L}) \subset M(D_1, \partial D_1)$ from $T(\mathcal{L}) \subset H(D_1^\circ)$. If $|T(\mathcal{L})| = 1$, by considering the tensor product with a meromorphic flat bundle of rank one, we can reduce the issue to the case $m(\mathcal{L}) = m + 1$. Let us consider the case $|T(\mathcal{L})| \geqslant 2$. By the assumption, $\mathfrak{b}_1 - \mathfrak{b}_2$ is invertible in $M(D_1, \partial D_1)$ for distinct $\mathfrak{b}_i \in T(\mathcal{L})$. It is standard that the eigen-decomposition of $\mathcal{L}/z_1\mathcal{L}$ can be uniquely extended to a ∇-flat decomposition $\mathcal{L} = \bigoplus_{\mathfrak{b}\in T(\mathcal{L})} \mathcal{L}_{\mathfrak{b}}$. It is easy to observe that $m(\mathcal{L}_{\mathfrak{b}}) \geqslant m$, and $|T(\mathcal{L}_{\mathfrak{b}})| \leqslant 1$ if $m(\mathcal{L}_{\mathfrak{b}}) = m$. Thus, we are done. □

2.7.5.2. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Assume the following:

- $(\mathcal{E}, \nabla)_{|\widehat{P}}$ is unramifiedly good for each $P \in D$.
- The Deligne-Malgrange lattice E of $(\mathcal{E}, \nabla)$ is $\mathcal{O}_X$-locally free.

Let us show that E is unramifiedly good Deligne-Malgrange under the above assumption.

We put $D^{[2]} := \bigcup_{i\neq j}(D_i \cap D_j)$. We can take a ramified covering $\varphi : (X, D) \to (X, D)$ with the following property:

- For each $P \in D_i \setminus D^{[2]}$, the action of $\pi_1(D_i \setminus D^{[2]}, P)$ on $\mathrm{Irr}(\varphi^*\nabla, P)$ is trivial.

By using the argument for Lemma 2.7.18, we may and will assume that the above property holds for $(\mathcal{E}, \nabla)$ from the beginning. We already know that $E_{|X\setminus D^{[2]}}$ is unramifiedly good Deligne-Malgrange. In particular, we have $\mathcal{I} \subset z_1^{-1}H(D_1^\circ)[z_1^{-1}]$ and a decomposition $E_{|\widehat{D}_1^\circ} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{E}_{\mathfrak{a}}$ such that $\big(\nabla(z_1\partial_1) - z_1\partial_1\mathfrak{a}\big)\widehat{E}_{\mathfrak{a}} \subset \widehat{E}_{\mathfrak{a}}$.

Let $\mathcal{M}$ be the differential $M(D_1, \partial D_1)((z_1))$-module corresponding to $\mathcal{E}$, and let $\mathcal{L}$ be the $M(D_1, \partial D_1)[\![z_1]\!]$-lattice induced by E. Applying Lemma 2.7.20, we obtain $\mathcal{I} \subset z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$ and a decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{L}_{\mathfrak{a}}$ such that $(z_1\partial_1 - z_1\partial_1\mathfrak{a})\mathcal{L}_{\mathfrak{a}} \subset \mathcal{L}_{\mathfrak{a}}$. Let $\mathfrak{K} := \boldsymbol{C}((z_n))\cdots((z_2))$. By the natural extension $M(D_1, \partial D_1) \subset \mathfrak{K}$, $\mathcal{L} \otimes \mathfrak{K}[\![z_1]\!]$ is the Deligne-Malgrange lattice of the differential module $\mathfrak{M} := \big(\mathcal{N} \otimes \mathfrak{K}((z_1)), \partial_1\big)$.

Let ${}^{O}E$ be the unramifiedly good Deligne-Malgrange lattice of $\mathcal{E}_{|\widehat{O}}$ with the irregular decomposition ${}^{O}E = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla,O)} {}^{O}E_{\mathfrak{a}}$. Let $\mathrm{Irr}(\nabla, 1)$ be the image of $\mathrm{Irr}(\nabla, O)$ via the map $\mathcal{O}_{\widehat{O}}(*D)/\mathcal{O}_{\widehat{O}} \to \mathcal{O}_{\widehat{O}}(*D)/\mathcal{O}_{\widehat{O}}(*D(\neq 1))$. It is easy to see that $\mathfrak{K}[\![z_1]\!] \otimes {}^{O}E$ is the good Deligne-Malgrange lattice of $(\mathfrak{M}_1, \partial_1)$, and the set of the irregular values is given by $\mathrm{Irr}(\nabla, 1)$. Hence, we obtain $\mathrm{Irr}(\nabla, 1) = \mathcal{I}$ in $z_1^{-1}\mathfrak{K}[z_1^{-1}]$ and ${}^{O}E \otimes \mathfrak{K}[\![z_1]\!] = \mathcal{L}_1 \otimes \mathfrak{K}[\![z_1]\!]$. We can deduce a similar relation for each $i = 2, \ldots, \ell$. Then, we obtain that $\mathrm{Irr}(\nabla) \subset M(X, D)/H(X)$.

We take a frame $\boldsymbol{v}$ of ${}^{O}E$. Let f be a section of E. We have the expression $f = \sum f_p\, v_p$. We obtain $f_p \in \mathfrak{K}[\![z_1]\!]$, and hence f_p is z_1-regular, i.e., f_p does not contain a negative power of z_1. Similarly, we obtain that f_p are z_j-regular for $j = 2, \ldots, \ell$. Thus, we obtain $E_{|\widehat{O}} \subset {}^{O}E$. Similarly, we obtain ${}^{O}E \subset E_{|\widehat{O}}$, and hence $E_{|\widehat{O}} = {}^{O}E$. Thus, we obtain that E is an unramifiedly good Deligne-Malgrange lattice.

2.7.5.3. Let us consider the case where we do not assume that E is locally free. We have a closed analytic subset $Z \subset D$ with $\mathrm{codim}_D(Z) \geqslant 2$ such that $E_{|X\setminus Z}$ is locally free. Then, it is an unramifiedly good Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)_{|X\setminus Z}$, according to the result in Subsection 2.7.5.2. We put $D_1^* := D_1 \setminus Z$ and $\partial D_1^* := \partial D_1 \setminus Z$. We have $\mathcal{I} \subset z_1^{-1}M(D_1^*, \partial D_1^*)[z_1^{-1}]$ and the irregular decomposition $E_{|\widehat{D_1^*}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{E}_{\mathfrak{a},D_1^*}$. By using the Hartogs property and the argument in the proof of Lemma 2.7.14, we obtain $\mathcal{I} \subset z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$ and a decomposition $E_{|\widehat{D}_1} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{E}_{\mathfrak{a}}$ such that (i) $(\nabla(z_1\partial_1) - z_1\partial_1\mathfrak{a})\widehat{E}_{\mathfrak{a}} \subset \widehat{E}_{\mathfrak{a}}$, (ii) the eigenvalues α of the induced endomorphism of $\widehat{E}_{\mathfrak{a}|D_1}$ satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$. Then, as in Subsection 2.7.5.2, we obtain $E \otimes \mathfrak{K}[\![z_1]\!] = {}^{O}E \otimes \mathfrak{K}[\![z_1]\!]$. Let $\boldsymbol{v}$ be a frame of ${}^{O}E$. Let f be a section of E. We have the expression $f = \sum f_p\, v_p$. Then, we obtain that f_p is z_1-regular. Similarly, we obtain that f_p are z_j-regular $(j = 1, \ldots, \ell)$ and hence $E_{|\widehat{O}} \subset {}^{O}E$.

To show ${}^{O}E \subset E_{|\widehat{O}}$, we consider the dual. Put $\mathcal{E}^{\vee} := \mathrm{Hom}_{\mathcal{O}_X(*D)}(\mathcal{E}, \mathcal{O}_X(*D))$, which is equipped with a naturally induced flat connection ∇. Put $E^{\vee} := \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$, which is a lattice of $\mathcal{E}^{\vee}$. It is generically unramifiedly good lattice, and the eigenvalues α of the residue satisfy $-1 < \mathrm{Re}(\alpha) \leqslant 0$. Put ${}^{O}E^{\vee} := \mathrm{Hom}_{\mathcal{O}_{\widehat{O}}}({}^{O}E, \mathcal{O}_{\widehat{O}})$ which is an unramifiedly good lattice of $(\mathcal{E}^{\vee}, \nabla)_{|\widehat{O}}$. The eigenvalues α of the residues satisfy $-1 < \mathrm{Re}(\alpha) \leqslant 0$. Then, we obtain $E^{\vee}_{|\widehat{O}} \subset {}^{O}E^{\vee}$ by the above argument. Note that $E^{\vee}_{|\widehat{O}} = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes \mathcal{O}_{\widehat{O}} \simeq \mathrm{Hom}_{\mathcal{O}_{\widehat{O}}}(E_{|\widehat{O}}, \mathcal{O}_{\widehat{O}})$. Hence, we can conclude that ${}^{O}E = E_{|\widehat{O}}$. Thus, we obtain that E is an unramifiedly good Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)$ at O, and the proof of Proposition 2.7.10 is finished. □

2.8. Family of filtered λ-flat bundles and KMS-structure

2.8.1. Notation. — We will be particularly interested in the case $\mathcal{K} \subset \boldsymbol{C}_\lambda$ and $\varrho(\lambda) = \lambda$. Let X be a complex manifold, and let D be a simple normal crossing hypersurface of X with the irreducible decomposition $D = \bigcup_{i\in\Lambda} D_i$. Let $\mathcal{X}$ be an open subset of $\mathcal{K} \times X$, and $\mathcal{D} := \mathcal{X} \cap (\mathcal{K} \times D)$. If $\mathcal{K}$ is not a point, a flat λ-connection on $(\mathcal{X}, \mathcal{D})$ is called a family of flat λ-connections to emphasize that it does not contain the differential in the λ-direction. We use the words "family of" with similar meanings. For example, "good family of filtered λ-flat bundles" is just a good filtered λ-flat bundle on $(\mathcal{X}, \mathcal{D})$ in the sense of Definition 2.5.1.

For a fixed λ_0, a neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_\lambda \times X$ will often be denoted by $\mathcal{X}^{(\lambda_0)}$. In that case, for any subset Y of X, we will put $\mathcal{Y}^{(\lambda_0)} := (\boldsymbol{C}_\lambda \times Y) \cap \mathcal{X}^{(\lambda_0)}$ and $\mathcal{Y}^\lambda := (\{\lambda\} \times Y) \cap \mathcal{X}^{(\lambda_0)}$. If we are given a family of flat λ-connections on $\mathcal{X}$, its restriction to $\mathcal{X}^\lambda$ is denoted by $\mathbb{D}^\lambda$.

Let $(\boldsymbol{E}_*, \mathbb{D})$ be a good family of filtered λ-flat bundles on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$. For a subset I of Λ, we put $D_I := \bigcap_{i\in I} D_i$. In this case, the induced filtrations of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_I^{(\lambda_0)}}$ are denoted by ${}^iF^{(\lambda_0)}$ for $i \in I$. (See Subsection 2.5.3 for the induced filtration.) For $\boldsymbol{a}_I = (a_i) \in \boldsymbol{R}^I$, we use the following notation:

$$
{}^IF^{(\lambda_0)}_{\boldsymbol{a}_I}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I^{(\lambda_0)}}\big) := \bigcap_{i\in I} {}^iF^{(\lambda_0)}_{a_i}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I^{(\lambda_0)}}\big), \quad {}^I\mathrm{Gr}^{F^{(\lambda_0)}}_{\boldsymbol{a}_I}({}_{\boldsymbol{c}}E) := \frac{{}^IF^{(\lambda_0)}_{\boldsymbol{a}_I}({}_{\boldsymbol{c}}E_{|\mathcal{D}_I^{(\lambda_0)}})}{\sum_{\boldsymbol{b}_I \lneq \boldsymbol{a}_I} {}^IF^{(\lambda_0)}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I^{(\lambda_0)}}\big)}.
$$

We put

$$
\mathcal{P}ar\big({}_{\boldsymbol{c}}E, I\big) := \big\{\boldsymbol{a}_I \in \boldsymbol{R}^I \,\big|\, {}^I\mathrm{Gr}^{F^{(\lambda_0)}}_{\boldsymbol{a}_I}({}_{\boldsymbol{c}}E) \neq 0\big\}, \quad \mathcal{P}ar\big(\boldsymbol{E}_*, I\big) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^\Lambda} \mathcal{P}ar\big({}_{\boldsymbol{c}}E, I\big).
$$

As in Proposition 2.5.3, we have the induced endomorphism $\mathrm{Res}_i(\mathbb{D})$ on ${}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a({}_{\boldsymbol{c}}E)$, which preserves the induced filtrations ${}^kF^{(\lambda_0)}$ of ${}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a({}_{\boldsymbol{c}}E)_{|\mathcal{D}_i^{(\lambda_0)}\cap\mathcal{D}_k^{(\lambda_0)}}$. Hence, we have the well defined endomorphisms $\mathrm{Res}_i(\mathbb{D})$ $(i \in I)$ on ${}^I\mathrm{Gr}^{F^{(\lambda_0)}}_{\boldsymbol{a}_I}\big({}_{\boldsymbol{c}}E\big)$.

2.8.2. KMS-structure. — For simplicity, let us consider the case where $\mathcal{X}^{(\lambda_0)}$ is the product of X and some neighbourhood $U(\lambda_0)$ of λ_0 in $\boldsymbol{C}_\lambda$. Let $\mathfrak{p}(\lambda) : \boldsymbol{R}\times\boldsymbol{C} \to \boldsymbol{R}$ and $\mathfrak{e}(\lambda) : \boldsymbol{R} \times \boldsymbol{C} \to \boldsymbol{C}$ be given as follows:

$$
\mathfrak{p}\big(\lambda, (a,\alpha)\big) = a + 2\,\mathrm{Re}(\lambda\,\overline{\alpha}), \quad \mathfrak{e}\big(\lambda, (a,\alpha)\big) = \alpha - a\,\lambda - \overline{\alpha}\,\lambda^2.
$$

The induced map $\boldsymbol{R}\times\boldsymbol{C} \to \boldsymbol{R}\times\boldsymbol{C}$ is denoted by $\mathfrak{k}(\lambda)$. Let $(\boldsymbol{E}_*, \mathbb{D})$ be a good family of filtered λ-flat bundle on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$.

Definition 2.8.1. — We say that $(\boldsymbol{E}_*, \mathbb{D})$ has the KMS-structure at λ_0 indexed by $T(i) \subset \boldsymbol{R}\times\boldsymbol{C}$ $(i \in \Lambda)$, if the following holds:

- $\mathcal{P}ar(\boldsymbol{E}_*, i)$ is the image of $T(i)$ via the map $\mathfrak{p}(\lambda_0)$.
- For each $a \in \mathcal{P}ar(\boldsymbol{E}_*, i)$, we put $\mathcal{K}(a,i) := \big\{u \in T(i) \,\big|\, \mathfrak{p}(\lambda_0, u) = a\big\}$. Then, the set of the eigenvalues of $\mathrm{Res}_i(\mathbb{D}^\lambda)$ on ${}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a\big({}_{\boldsymbol{c}}E\big)_{|\mathcal{D}_i^\lambda}$ is $\big\{\mathfrak{e}(\lambda,u) \,\big|\, u \in \mathcal{K}(a,i)\big\}$.

In that case, $0 \neq \lambda \in U(\lambda_0)$ is called generic, if $\mathfrak{e}(\lambda) : T(i) \to \boldsymbol{C}$ are injective for any i. □

Assume $(\boldsymbol{E}_*, \mathbb{D})$ has the KMS-structure at λ_0. We have the generalized eigen-decomposition with respect to the induced action of $\mathrm{Res}_i(\mathbb{D})$

$$(37) \qquad {}^i\mathrm{Gr}_a^{F^{(\lambda_0)}}({}_{\boldsymbol{c}}E) = \bigoplus_{u \in \mathcal{K}(a,i)} {}^i\mathcal{G}_u^{(\lambda_0)}({}_{\boldsymbol{c}}E),$$

where $\mathrm{Res}_i(\mathbb{D}) - \mathfrak{e}(\lambda, u)$ are nilpotent on ${}^i\mathcal{G}_u^{(\lambda_0)}({}_{\boldsymbol{c}}E)$.

Remark 2.8.2. — If $(\boldsymbol{E}_*, \mathbb{D})$ has the KMS-structure at λ_0, we have the decomposition ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i^{(\lambda_0)}} = \bigoplus {}^i\mathbb{E}_\alpha^{(\lambda_0)}({}_{\boldsymbol{c}}E_{|\mathcal{D}_i^{(\lambda_0)}})$ indexed by the eigenvalues of $\mathrm{Res}_i(\mathbb{D}^{\lambda_0})$, such that (i) it is preserved by $\mathrm{Res}_i(\mathbb{D})$, (ii) the restriction of the decomposition to $\mathcal{D}_i^{\lambda_0}$ is the generalized eigen-decomposition of $\mathrm{Res}_i(\mathbb{D}^{\lambda_0})$. Then, the decomposition ${}^i\mathbb{E}^{(\lambda_0)}$ and the filtration ${}^iF^{(\lambda_0)}$ are compatible, and ${}^i\mathrm{Gr}_a^{F^{(\lambda_0)}}\, {}^i\mathbb{E}_\alpha^{(\lambda_0)}({}_{\boldsymbol{c}}E_{|\mathcal{D}_i^{(\lambda_0)}})$ is naturally isomorphic to ${}^i\mathcal{G}_u^{(\lambda_0)}({}_{\boldsymbol{c}}E)$, where u is determined by $\mathfrak{k}(\lambda_0, u) = (a, \alpha)$. □

If $(\boldsymbol{E}_*, \mathbb{D})$ is unramified, the KMS-structure is compatible with the irregular decomposition in the following sense. For simplicity we consider the case $X = \Delta^n$, $D_i = \{z_i = 0\}$ and $D = \bigcup_{i=1}^{\ell} D_i$. We have the irregular decomposition ${}_{\boldsymbol{c}}E_{|U(\lambda_0)\times\widehat{O}} = \bigoplus_{\mathfrak{a}\in T} {}_{\boldsymbol{c}}\widehat{E}_{\mathfrak{a}}$. They give a family of filtered λ-flat bundles $\widehat{\boldsymbol{E}}_*$ on $U(\lambda_0) \times \widehat{O}$, with the irregular decomposition $(\widehat{\boldsymbol{E}}_*, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in T}(\widehat{\boldsymbol{E}}_{\mathfrak{a}\,*}, \widehat{\mathbb{D}}_{\mathfrak{a}})$. Each $(\widehat{\boldsymbol{E}}_{\mathfrak{a}*}, \widehat{\mathbb{D}}_{\mathfrak{a}})$ has the KMS-structure at λ_0. By the natural isomorphism ${}_{\boldsymbol{c}}E_{|O} \simeq \bigoplus_{\mathfrak{a}} {}_{\boldsymbol{c}}E_{\mathfrak{a}|O}$, the filtrations $F^{(\lambda_0)}$ and the decompositions $\mathbb{E}^{(\lambda_0)}$ are the same.

2.8.3. Uniqueness of the filtrations. — According to the following lemma, it makes sense to say that $(\boldsymbol{E}, \mathbb{D})$ has the KMS-structure at λ_0.

Lemma 2.8.3. — *Let $(\boldsymbol{E}_{i*}, \mathbb{D}_i)$ $(i = 1, 2)$ be good filtered λ-flat bundles on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$, which have the KMS-structures at λ_0. Assume that we are given an isomorphism $\varphi : (\boldsymbol{E}_1, \mathbb{D}_1) \simeq (\boldsymbol{E}_2, \mathbb{D}_2)$ of families of meromorphic λ-flat bundles. Then, it induces an isomorphism $\varphi : (\boldsymbol{E}_{1\,*}, \mathbb{D}_1) \simeq (\boldsymbol{E}_{2\,*}, \mathbb{D}_2)$ of families of filtered λ-flat bundles.*

Proof. — It can easily be reduced to the case where D is smooth. Let $\widehat{\mathcal{D}}^{(\lambda_0)}$ denote the completion of $\mathcal{X}^{(\lambda_0)}$ along $\mathcal{D}^{(\lambda_0)}$. We are given the induced isomorphism $\boldsymbol{E}_{1|\widehat{\mathcal{D}}^{(\lambda_0)}} \simeq \boldsymbol{E}_{2|\widehat{\mathcal{D}}^{(\lambda_0)}}$. We only have to show that it induces ${}_aE_{1|\widehat{\mathcal{D}}^{(\lambda_0)}} = {}_aE_{2|\widehat{\mathcal{D}}^{(\lambda_0)}}$ for each $a \in \boldsymbol{R}$.

Let us consider the case where $(\boldsymbol{E}_{i*}, \mathbb{D}_i)$ are unramified. We have the irregular decompositions:

$$(\boldsymbol{E}_{i*}, \mathbb{D}_i)_{|\widehat{\mathcal{D}}^{(\lambda_0)}} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\mathbb{D}_i)} (\widehat{\boldsymbol{E}}_{i,\mathfrak{a}\,*}, \widehat{\mathbb{D}}_{i,\mathfrak{a}}).$$

We have $\mathrm{Irr}(\mathbb{D}_1) = \mathrm{Irr}(\mathbb{D}_2)$ and $\widehat{\boldsymbol{E}}_{1,\mathfrak{a}} = \widehat{\boldsymbol{E}}_{2,\mathfrak{a}}$ for each $\mathfrak{a}$. Let T_i $(i = 1, 2)$ denote the index sets of the KMS-structures of $\boldsymbol{E}_{i*}$. Note that there exists a discrete subset Z in $U(\lambda_0) \setminus \{0\}$ such that $\mathfrak{e}(\lambda) : T_1 \cup T_2 \to \boldsymbol{C}$ is injective for any $\lambda \in U(\lambda_0) \setminus Z$. Take any

$\lambda_1 \in U(\lambda_0) \smallsetminus Z$, and a neighbourhood $U(\lambda_1)$ in $U(\lambda_0) \smallsetminus Z$. We set $\mathcal{X}^{(\lambda_1)} := U(\lambda_1) \times X$ and $\mathcal{D}^{(\lambda_1)} := U(\lambda_1) \times D$.

Lemma 2.8.4. — *We have* $\varphi\big({}_aE_{1|\mathcal{X}^{(\lambda_1)}}\big) \subset {}_aE_{2|\mathcal{X}^{(\lambda_1)}}$.

Proof. — Since this is a quite standard claim, we give only an outline. We put

$$\mathcal{L}(a,\lambda_0) := \big\{u \in T_1 \cup T_2 \,\big|\, a-1 < \mathfrak{p}(\lambda_0,u) \leqslant a\big\}.$$

We have the generalized eigen-decomposition

$${}_aE_{i|\mathcal{D}^{(\lambda_1)}} = \bigoplus_{u \in \mathcal{L}(a,\lambda_0)} \mathbb{E}_{\mathfrak{e}(\lambda,u)}\big({}_aE_{i|\mathcal{D}^{(\lambda_1)}}\big),$$

where $\mathrm{Res}(\mathbb{D})$ has a unique eigen value $\mathfrak{e}(\lambda,u)$ on $\mathbb{E}_{\mathfrak{e}(\lambda,u)}\big({}_aE_{i|\mathcal{D}^{(\lambda_1)}}\big)$. It is compatible with the irregular decompositions, i.e.,

$$(38) \qquad {}_a\widehat{E}_{i,\mathfrak{a}|\mathcal{D}^{(\lambda_1)}} = \bigoplus_{u \in \mathcal{L}(a,\lambda_0)} \mathbb{E}_{\mathfrak{e}(\lambda,u)}\big({}_a\widehat{E}_{i,\mathfrak{a}|\mathcal{D}^{(\lambda_1)}}\big).$$

It can be shown that $\mathfrak{e}(\lambda,u_1) - \mathfrak{e}(\lambda,u_2)$ are not contained in $\mathbb{Z} \smallsetminus \{0\}$ for any distinct $u_1, u_2 \in \mathcal{L}(\lambda_0,a)$ and for any $\lambda \in U(\lambda_1)$. Hence, we obtain the flat decomposition

$${}_a\widehat{E}_{i,\mathfrak{a}|\widehat{\mathcal{D}}^{(\lambda_1)}} = \bigoplus_{u \in \mathcal{L}(\lambda_0,a)} {}_a\widehat{E}_{i,\mathfrak{a},u}$$

whose restriction to $\mathcal{D}^{(\lambda_1)}$ is the same as (38). We put ${}_a\widehat{E}_{i,u} := \bigoplus_{\mathfrak{a}} {}_a\widehat{E}_{i,\mathfrak{a},u}$. For each $a \in \boldsymbol{R}$, there exists $b > 0$ such that we have the flat morphism

$$\varphi_{u_1,u_2} : {}_a\widehat{E}_{1,u_1} \longrightarrow {}_{a+b}\widehat{E}_{2,u_2}$$

induced by the flat isomorphism $\boldsymbol{E}_{1|\widehat{\mathcal{D}}^{(\lambda_1)}} \simeq \boldsymbol{E}_{2|\widehat{\mathcal{D}}^{(\lambda_1)}}$. The restriction $\varphi_{u_1,u_2|\mathcal{D}^{(\lambda_1)}}$ has to be compatible with the residues. Hence, it is easy to observe that $\varphi_{u_1,u_2}({}_a\widehat{E}_{1,u_1}) \subset {}_{a+b-N}\widehat{E}_{2,u_2}$ for any $N \geqslant 0$, if $u_1 \neq u_2$. It implies $\varphi_{u_1,u_2} = 0$ unless $u_1 = u_2$. Let us look at $\varphi_{u,u}$. Again, by the comparison of the eigenvalues of the residues, we obtain $\varphi_{u,u}\big({}_a\widehat{E}_{1,u}\big) \subset {}_a\widehat{E}_{2,u}$. Thus, we obtain Lemma 2.8.4. □

Let us return to the proof of Lemma 2.8.3. We have the morphism $\varphi : {}_aE_1 \to {}_{a+b}E_2$ for some b. By Lemma 2.8.4, the induced map ${}_aE_1 \to \mathrm{Gr}^{F^{(\lambda_0)}}_{a+b}(E_{2|U(\lambda_0)\times D})$ is trivial if $b > 0$. Hence, we obtain $\varphi({}_aE_1) \subset {}_aE_2$ in the unramified case.

Let us consider the case where $(\boldsymbol{E}_{i,*}, \mathbb{D}_i)$ are not necessarily unramified. We set $\widetilde{X} := X$ and $\widetilde{D} := D$. We use the notation $\widetilde{\mathcal{X}}^{(\lambda_0)}$ and $\widetilde{\mathcal{D}}^{(\lambda_0)}$ with similar meanings. We take an appropriate ramified covering $\varphi : \widetilde{X} \to X$ such that the induced filtered bundle $(\widetilde{\boldsymbol{E}}_{i,*}, \widetilde{\mathbb{D}}_i)$ via φ and $(\boldsymbol{E}_{i*}, \mathbb{D}_i)$ on $(\widetilde{\mathcal{X}}^{(\lambda_0)}, \widetilde{\mathcal{D}}^{(\lambda_0)})$ is unramifiedly good. It is easy to see that $(\widetilde{\boldsymbol{E}}_{i,*}, \widetilde{\mathbb{D}}_i)$ have the KMS-structure at λ_0. By applying the above result to them, we obtain $\widetilde{\boldsymbol{E}}_{1\,*} = \widetilde{\boldsymbol{E}}_{2\,*}$. Since $\boldsymbol{E}_{i\,*}$ are obtained as the descent of $\widetilde{\boldsymbol{E}}_{i\,*}$, we obtain $\boldsymbol{E}_{1\,*} = \boldsymbol{E}_{2\,*}$. Thus, the proof of Lemma 2.8.3 is finished. □

2.8.4. Openness and invariance. — Pick $\boldsymbol{c} \in \boldsymbol{R}^{\Lambda}$ such that $c_i \notin \mathcal{P}ar\big(\boldsymbol{E}_*, i\big)$ for each $i \in \Lambda$. Let $\pi_{i,a}$ be the projection:

$$ {}^iF_a^{(\lambda_0)}\big({}_{\boldsymbol{c}}E_{|U(\lambda_0)\times D_i}\big) \longrightarrow {}^i\mathrm{Gr}_a^{F^{(\lambda_0)}}\big({}_{\boldsymbol{c}}E\big). $$

Let $\lambda_1 \in U(\lambda_0)$ be sufficiently close to λ_0, and let $U(\lambda_1) \subset U(\lambda_0)$ be a neighbourhood of λ_1. Let $c_i - 1 < b \leqslant c_i$. If $b = \mathfrak{p}(\lambda_1, v)$ for some $v \in \mathcal{K}(a,i)$, we put on $\mathcal{D}_i^{(\lambda_1)}$

$$ {}^iF_b^{(\lambda_1)} := \bigoplus_{\substack{u\in\mathcal{K}(a,i)\\ \mathfrak{p}(\lambda_1,u)\leqslant b}} \pi_{i,a}^{-1}\big({}^i\mathcal{G}_u^{(\lambda_0)}\big). $$

Otherwise, let $b_0 := \max\big\{\mathfrak{p}(\lambda_1, v) < b \,\big|\, v \in \mathcal{K}(a,i)\big\}$, and set ${}^iF_b^{(\lambda_1)} := {}^iF_{b_0}^{(\lambda_1)}$. Thus, we obtain the filtration ${}^iF^{(\lambda_1)}$ of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i^{(\lambda_1)}}$. It induces a family of filtered λ-flat bundles $(\boldsymbol{E}_*^{(\lambda_1)}, \mathbb{D})$ on $(\mathcal{X}^{(\lambda_1)}, \mathcal{D}^{(\lambda_1)})$. It is easy to observe that $\mathrm{Res}_i(\mathbb{D})$ has a unique eigenvalue $\mathfrak{e}(\lambda, u)$ on ${}^i\mathrm{Gr}_{\mathfrak{p}(\lambda_1,u)}^{F^{(\lambda_1)}}\big({}_{\boldsymbol{c}}E\big)_{|\mathcal{D}_1^{(\lambda_1)}}$. Hence, $(\boldsymbol{E}_*^{(\lambda_1)}, \mathbb{D})$ has the KMS-structure at λ_1. The index sets are equal to those for $(\boldsymbol{E}_*^{(\lambda_0)}, \mathbb{D})$.

Remark 2.8.5. — More precisely, there exists an open subset $\lambda_0 \in U'(\lambda_0) \subset U(\lambda_0)$ depending only on the sets $\mathcal{KMS}(\boldsymbol{E}_*^{(\lambda_0)}, i)$ such that the above construction can be applied for any $\lambda_1 \in U'(\lambda_0)$. □

For each $\lambda \in U(\lambda_0)$ sufficiently close to λ_0, we put $\boldsymbol{E}_*^{\lambda} := (\boldsymbol{E}_*^{(\lambda)})_{|\{\lambda\}\times(X,D)}$, which is a good filtered λ-flat bundle. The set $\mathcal{KMS}(\boldsymbol{E}_*^{\lambda}, i)$ is equal to the image of $T(i)$ via the map $\mathfrak{k}(\lambda)$. Note $\mathcal{KMS}(\boldsymbol{E}_*^{0}, i) = T(i)$ in the case $0 \in U(\lambda_0)$. We often identify them.

CHAPTER 3

STOKES STRUCTURE OF A GOOD ϱ-MEROMORPHIC FLAT BUNDLE

In this chapter, we shall study the Stokes structure of an unramifiedly good ϱ-flat bundle. In Section 3.1, we give preliminaries to describe Stokes filtration. In Section 3.2, we state some theorems and propositions for full Stokes filtrations, which will be proved in Section 3.7. We state some theorems and propositions for partial Stokes filtration in Section 3.3, which will be proved in Sections 3.5–3.7.

3.1. Preliminaries

3.1.1. Filtration indexed by a finite ordered set

3.1.1.1. Compatibility. — Let $(I, \leqslant)$ be a finite ordered set. Let V be a vector space. In this section, a filtration F of V indexed by $(I, \leqslant)$ means a family of subspaces $F_a \subset V$ $(a \in I)$ with the following properties:

- $F_a \subset F_b$ if $a \leqslant b$.
- There exists a splitting $V = \bigoplus V_a$ such that $F_a = \bigoplus_{b \leqslant a} V_b$.

We put $F_{<a} := \sum_{b<a} F_b$ and $\mathrm{Gr}^F_a(V) = F_a/F_{<a} \simeq V_a$. For a given subset $S \subset I$, we set $F_S := \sum_{a \in S} F_a$.

Remark 3.1.1. — Note that we have assumed the existence of a splitting, which is unusual. For example, we do not have such a splitting for a filtered flat bundle. We consider the above type of filtration only for Stokes filtrations. If V is finite dimensional, we may replace it with $\sum_{a \in I} \dim \mathrm{Gr}^F_a(V) = \dim V$. □

Let $\varphi : (I, \leqslant) \to (I', \leqslant')$ be a morphism of ordered sets, and let F be a filtration of V indexed by $(I, \leqslant)$. Then, we have the induced filtration F^φ indexed by $(I', \leqslant')$ constructed inductively as follows:

$$F^\varphi_b = F^\varphi_{<b} + \sum_{a \in \varphi^{-1}(b)} F_a.$$

We set $V_b^{\varphi} := \bigoplus_{a\in\varphi^{-1}(b)} V_a$. Then, $V = \bigoplus_{b\in I'} V_b^{\varphi}$ gives a splitting of F^{φ}. We say that F^{φ} is induced by F and φ.

Definition 3.1.2. — Let F and F' be filtrations of V indexed by $(I,\leqslant)$ and $(I',\leqslant')$, respectively. Let $\varphi:(I,\leqslant)\to(I',\leqslant')$ be a morphism of ordered sets. We say that F and F' are compatible over φ, if F' is the same as F^{φ} above.

In the case $I=I'$ (but possibly $(I,\leqslant)\neq(I',\leqslant')$) and $\varphi'=\mathrm{id}$, we just say F and F' are compatible. □

In the case $I=I'$, we have a natural isomorphism $\mathrm{Gr}_a^F(V)\simeq\mathrm{Gr}_a^{F'}(V)$.

Lemma 3.1.3. — *Let F be a filtration of V indexed by $(I,\leqslant)$. Let $\leqslant_i$ $(i\in\Lambda)$ be orders on I such that (i) the identity $\varphi_i:(I,\leqslant)\to(I,\leqslant_i)$ are order preserving, (ii) $a\leqslant b$ if and only if $a\leqslant_i b$ for any $i\in\Lambda$. Then, F can be reconstructed from F^{φ_i} $(i\in\Lambda)$ in the sense $F_a=\bigcap_{i\in\Lambda}F_a^{\varphi_i}$.*

Proof. — We take a splitting $V=\bigoplus_{a\in I}V_a$ of the filtration F. Recall $F_a^{\varphi_i}=\bigoplus_{b\leqslant_i a}V_b$. Then, the claim of the lemma is clear. □

Remark 3.1.4. — It can be generalized for vector bundles appropriately. □

3.1.1.2. Dual. — Let $(I,\leqslant)$ be an ordered set, and let V be a finite dimensional vector space equipped with a filtration F indexed by $(I,\leqslant)$. Let us give an induced filtration $F^{\vee}$ on the dual vector space $V^{\vee}$. We set $I^{\vee}:=I$ and let $\leqslant^{\vee}$ be the order of $I^{\vee}$ defined by $a\leqslant^{\vee}b \iff a\geqslant b$. For distinction, we use the symbol $-a$ if we regard $a\in I$ as an element of $I^{\vee}$. And "$-a\leqslant^{\vee}-b$" is denoted by $-a\leqslant -b$. We hope that there is no risk of confusion.

We take a splitting $V=\bigoplus_{a\in I}V_a$ of the filtration F. In general, for a vector subspace $U\subset V$, let $U^{\perp}\subset V^{\vee}$ be $\{f\in V^{\vee}\,|\,f(v)=0\ \forall v\in U\}$. For each $a\in I$, let $S(a)$ denote the set of $b\in I$ such that $b\not\geqslant a$. We have the subspaces of $V^{\vee}$ given as follows:

$$V_{-a}^{\vee}:=\Big(\bigoplus_{b\neq a}V_b\Big)^{\perp},\quad F_{-a}^{\vee}(V^{\vee}):=\Big(\bigoplus_{b\in S(a)}V_b\Big)^{\perp}.$$

Lemma 3.1.5. — *The subspaces $\{F_{-a}^{\vee}(V^{\vee})\,|\,-a\in I^{\vee}\}$ are well defined, and give a filtration of $V^{\vee}$ indexed by $(I^{\vee},\leqslant)$. The decomposition $V^{\vee}=\bigoplus_{-a\in I^{\vee}}V_{-a}^{\vee}$ gives a splitting of the filtration $F^{\vee}$.*

Proof. — Let us show that $F_{\not\geqslant a}(V):=\bigoplus_{b\in S(a)}V_b$ is independent of the choice of a splitting. Let $V=\bigoplus_{a\in I}V'_a$ be another splitting of F. Note $V'_b\subset F_b(V)=\bigoplus_{c\leqslant b}V_c$. We can observe that $b\in S(a)$ and $c\leqslant b$ imply $c\in S(a)$. Hence, $V'_b\subset F_{\not\geqslant a}(V)$ for any $b\in S(a)$, which implies that $F_{\not\geqslant a}(V)$ is independent of the choice of a splitting. Because $F_{-a}^{\vee}(V^{\vee})=F_{\not\geqslant a}(V)^{\perp}$, we obtain that the subspaces $F_{-a}^{\vee}(V^{\vee})$ are well defined.

If $-b \leqslant -a$, we have $b \geqslant a$, and $F_{\not\geq b}(V) \supset F_{\not\geq a}(V)$. Hence, we obtain $F^{\vee}_{-b}(V^{\vee}) \subset F^{\vee}_{-a}(V^{\vee})$. It is easy to show

$$F^{\vee}_{-a}(V^{\vee}) = \bigoplus_{-b \leqslant -a} V^{\vee}_{-b}.$$

Thus, we obtain the claims of the lemma. □

We give some property of the induced filtration above.

Lemma 3.1.6. — *We have the natural isomorphism* $\mathrm{Gr}^{F^{\vee}}_{-a}(V^{\vee}) \simeq \mathrm{Gr}^{F}_{a}(V)^{\vee}$.

Proof. — The perfect pairing of V and $V^{\vee}$ induces a pairing P of $F_a(V)$ and $F^{\vee}_{-a}(V^{\vee})$. By definition, the restriction of P to $F_{<a}(V) \otimes F^{\vee}_{-a}(V^{\vee})$ is 0. Let $F_{\not\geq a}(V) := \bigoplus_{b \not\geq a} V_b$. Then, we have $F^{\vee}_{<-a}(V^{\vee}) = F_{\not\geq a}(V)^{\perp}$. Hence, the restriction of P to $F_a(V) \otimes F^{\vee}_{<-a}(V^{\vee})$ is 0. Hence, we have the induced pairing P_a of $\mathrm{Gr}^F_a(V)$ and $\mathrm{Gr}^{F^{\vee}}_{-a}(V^{\vee})$. It is easy to check that P_a is perfect by using the induced isomorphisms $\mathrm{Gr}^F_a(V) \simeq V_a$ and $\mathrm{Gr}^{F^{\vee}}_{-a}(V^{\vee}) \simeq V^{\vee}_{-a}$. □

Lemma 3.1.7. — *Let $\varphi : (I_1, \leqslant_1) \to (I_2, \leqslant_2)$ be a morphism of ordered sets. Let F_i $(i = 1, 2)$ be filtrations of a finite dimensional vector space V which are compatible over φ. Then, the induced filtrations $F^{\vee}_i$ $(i = 1, 2)$ are compatible over the induced morphism $\varphi^{\vee} : (I^{\vee}_1, \leqslant_1) \to (I^{\vee}_2, \leqslant_2)$.*

Proof. — It is easy to show the claim by using the induced splitting $V^{\vee} = \bigoplus_{a \in I_1} V^{\vee}_{-a}$. □

Lemma 3.1.8. — *Let V_i be finite dimensional vector spaces with filtrations $F(V_i)$ indexed by an ordered set $(I, \leqslant)$. Let $f : V_1 \to V_2$ be a linear map preserving filtrations. Then, the dual $f^{\vee} : V^{\vee}_2 \to V^{\vee}_1$ preserves the dual filtrations $F^{\vee}(V^{\vee}_i)$.*

Proof. — Because $F^{\vee}_{-a}(V_i) = F_{\not\geq a}(V_i)^{\perp}$, the claim is clear. □

3.1.1.3. Hom-space. — We prepare a notation. Let $(I, \leqslant)$ be an ordered set, and V_i $(i = 1, 2)$ be finite dimensional vector spaces equipped with filtrations indexed by $(I, \leqslant)$. Let $F_0 \mathrm{Hom}(V_1, V_2)$ (resp. $F_{<0} \mathrm{Hom}(V_1, V_2)$) be the vector subspace of $\mathrm{Hom}(V_1, V_2)$ which consists of the linear maps f satisfying the following:

$$f\big(F_a(V_1)\big) \subset F_a(V_2), \quad \text{resp. } f\big(F_a(V_1)\big) \subset F_{<a}(V_2).$$

If we take a splitting $V_i = \bigoplus_{a \in I} V_{i,a}$ of the filtrations, we have the following:
(39)

$$F_0 \mathrm{Hom}(V_1, V_2) = \bigoplus_{a \geqslant b} \mathrm{Hom}(V_{1,a}, V_{2,b}), \quad F_{<0} \mathrm{Hom}(V_1, V_2) = \bigoplus_{a > b} \mathrm{Hom}(V_{1,a}, V_{2,b}).$$

3.1.1.4. Induced filtration. — Let $(I_L, \leqslant) \to (I_{L-1}, \leqslant) \to \cdots \to (I_1, \leqslant)$ be a sequence of surjections of ordered sets. The induced morphism $I_j \to I_k$ is denoted by $\varphi_{j,k}$. Let V be a vector space. Let F^{I_L} be a filtration indexed by I_L. Then, we have the induced filtrations F^{I_j} for $j = 1, \dots, L$ obtained by the procedure explained in Section 3.1.1.1. Moreover, we obtain the following inductive structure. Let $\mathrm{Gr}^{I_j}(V) = \bigoplus_{b \in I_j} \mathrm{Gr}^{I_j}_b(V)$ denote the graded vector space associated to F^{I_j}. For each $b \in I_{j-1}$, we have the induced filtration F^{I_j} on $\mathrm{Gr}^{I_{j-1}}_b(V)$ indexed by the ordered set $\varphi^{-1}_{j,j-1}(b) \subset I_j$, and the associated graded space $\mathrm{Gr}^{I_j} \mathrm{Gr}^{I_{j-1}}(V) = \bigoplus_{b \in I_{j-1}} \bigoplus_{a \in \varphi_{j,j-1}(b)} \mathrm{Gr}^{I_j}_a \mathrm{Gr}^{I_{j-1}}_b(V)$ is naturally isomorphic to $\mathrm{Gr}^{I_j}(V)$.

Conversely, assume that we are given the following inductive data:

- A filtration F^{I_1} of V indexed by I_1. We put $V^{I_1}_a := \mathrm{Gr}^{F^{I_1}}_a(V)$ or $a \in I_1$.
- For each $b \in I_{j-1}$, a filtration F^{I_j} of $V^{I_{j-1}}_b$ indexed by $\varphi^{-1}_{j,j-1}(b)$. We put $V^{I_j}_a := \mathrm{Gr}^{F^{I_j}}_a\big(V^{I_{j-1}}_b\big)$ for $a \in \varphi^{-1}_{j,j-1}(b)$.

Then, we obtain the naturally induced filtration F^{I_j} of V with the following properties:

- F^{I_j} and F^{I_k} are compatible over $\varphi_{j,k}$.
- Let $\mathrm{Gr}^{I_j}(V) = \bigoplus_{a \in I_j} \mathrm{Gr}^{I_j}_a(V)$ denote the graded vector space associated to F^{I_j}. Then, $\mathrm{Gr}^{I_j}_a(V)$ is naturally isomorphic to $V^{I_j}_a$.

The construction is given as follows. Assume that we have already obtained the desired filtration $F^{I_{j-1}}$. Let $a \in I_j$ and $b := \varphi_{j,j-1}(a)$. We have the natural morphism $\pi^{I_{j-1}}_b : F^{I_{j-1}}_b \to \mathrm{Gr}^{I_{j-1}}_b(V) \simeq V^{I_{j-1}}_b$. Then, we put

$$F^{I_j}_a(V) := F^{I_j}_{<a}(V) + \big(\pi^{I_{j-1}}_b\big)^{-1}\big(F^{I_j}_a(V^{I_{j-1}}_b)\big).$$

It is easy to check $\mathrm{Gr}^{I_j}_a(V)$ is isomorphic to $\mathrm{Gr}^{I_j}_a(V^{I_{j-1}}_b) \simeq V^{I_j}_a$.

3.1.2. Holomorphic functions and vector bundles on real blow up

3.1.2.1. Functions and vector bundles. — Let us recall the framework of asymptotic analysis in [72]. See Chapter II.1 of [**72**] for more details and precision. Let X be a complex manifold, and let D be a normal crossing divisor of X. In the following, $\widetilde{X}(D)$ denotes the fiber product, taken over X, of the real blow up of X along the irreducible components of D, which is called the real blow up of X along D in this paper. It is naturally a C^∞-manifold with corners. Hence, C^∞-functions on open subsets of $\widetilde{X}(D)$ make sense. In the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$, we may use the local coordinate $(r_1, \theta_1, \dots, r_\ell, \theta_\ell, z_{\ell+1}, \dots, z_n)$, where (r_i, θ_i) are given by $z_i = r_i\, e^{\sqrt{-1}\theta_i}$.

We have the C^∞-vector bundle $\Omega^{0,1}_X(\log D)$ on X locally generated by $d\overline{z}_i/\overline{z}_i$ $(i = 1, \dots, \ell)$ and $d\overline{z}_i$ $(i = \ell+1, \dots, n)$. The pull-back of $\Omega^{0,1}_X(\log D)$ via π is also denoted by the same symbol. For any open subset $U \subset \widetilde{X}(D)$, we have the well defined differential operator $\overline{\partial} : C^\infty(U) \to C^\infty\big(U, \Omega^{0,1}_X(\log D)\big)$. We say $f \in C^\infty(U)$ is holomorphic, if $\overline{\partial} f = 0$. It is easy to see that $f \in C^\infty(U)$ is holomorphic if and only if $f_{|U \cap (X \smallsetminus D)}$ is holomorphic in the standard sense. The space of holomorphic functions on U is denoted by $\mathcal{A}_{\widetilde{X}(D)}(U)$ and $\mathcal{O}_{\widetilde{X}(D)}(U)$. The sheaf is denoted by $\mathcal{O}_{\widetilde{X}(D)}$.

Let Z be a locally closed subset of $\widetilde{X}(D)$. Let $C^\infty(\widehat{Z})$ denote the space of the C^∞-functions on Z in the sense of Whitney (see Chapters I.2 and I.4 of [**54**]). Namely it is the space of the ∞-jets of X on Z, satisfying some compatibility condition as in Theorem 2.2 of [**54**]. Let U be an open subset of $\widetilde{X}(D)$ such that $Z \subset U$. Then, $C^\infty(\widehat{Z})$ is the same as the limit $\varprojlim C^\infty(U)/I_Z^n$, where I_Z denotes the ideal of $C^\infty(U)$ which consists of the functions f such that $f_{|Z} = 0$. Equivalently, let $I_Z^{(\infty)}$ denote the ideal of $C^\infty(U)$ which consists of the functions f such that $(\partial^J f)_{|Z} = 0$ for any derivation ∂^J with any degree. Then, the natural morphism $C^\infty(U)/I_Z^{(\infty)} \to C^\infty(\widehat{Z})$ is an isomorphism. (See the proof of Theorem 4.1 of [**54**].) We prefer to regard elements of $C^\infty(\widehat{Z})$ as C^∞-functions on the space $\widehat{Z}$ which is the completion of $\widetilde{X}(D)$ along Z. The image of a function $f \in C^\infty(U)$ to $C^\infty(\widehat{Z})$ is denoted by $f_{|\widehat{Z}}$.

The differential operator $\overline{\partial} : C^\infty(U) \to C^\infty\big(U, \Omega_X^{0,1}(\log D)\big)$ induces $\overline{\partial} : C^\infty(\widehat{Z}) \to C^\infty\big(\widehat{Z}, \Omega_X^{0,1}(\log D)\big)$. We say that $f \in C^\infty(\widehat{Z})$ is holomorphic if $\overline{\partial} f = 0$. The space of holomorphic functions on $\widehat{Z}$ is denoted by $\mathcal{A}(\widehat{Z})$ and $\mathcal{O}(\widehat{Z})$. We will use the following fact without mention.

Lemma 3.1.9. — *Let $X = \Delta^n$ and $D = \bigcup_{j=1}^{\ell} \{z_j = 0\}$. Let W be a closed region of $\widetilde{X}(D)$, and let $Z := U \cap \pi^{-1}(D)$. Let f be a holomorphic function on $W \smallsetminus Z$ such that $|f| = O\big(\prod_{j=1}^{\ell} |z_j|^N\big)$ for any N. Then, f extends as a holomorphic function on W such that $f_{|\widehat{Z}} = 0$.*

Conversely, if f is a holomorphic function on W such that $f_{|\widehat{Z}} = 0$, it satisfies $|f| = O\big(\prod_{j=1}^{\ell} |z_j|^N\big)$ for any N.

Proof. — Let f be a function as in the first claim. By using Cauchy's formula, we obtain $\partial^J f = O\big(\prod_{j=1}^{\ell} |z_j|^N\big)$ for any N and for any derivation ∂^J with any degree. (See the proof of Theorem 8.8 of [**94**].) Then, the first claim of the lemma follows. The second claim is obvious. □

We recall Borel-Ritt type theorem, due to Majima [**53**] for his strongly asymptotically developable functions, and due to Sabbah [**72**] in this framework.

Proposition 3.1.10 **(Theorem 2.2 of [53], Proposition 1.1.16 of [72])**

Let $Q \in \pi^{-1}(D)$. Let $f \in \mathcal{A}(\widehat{Z})$, where Z denotes a closed neighbourhood of Q in $\pi^{-1}(D)$. Then, we have a neighbourhood U of Q in $\widetilde{X}(D)$ and a holomorphic function $F \in \mathcal{A}(U)$ such that $F_{|\widehat{Z}_U} = f_{|\widehat{Z}_U}$, where $Z_U := Z \cap U$.

In particular, $\mathcal{A}(\widehat{Z})$ is isomorphic to $\varprojlim \mathcal{A}(U)/\mathcal{I}_Z^n$, where $\mathcal{I}_Z$ denotes the ideal of $\mathcal{A}(U)$ which consists of the holomorphic functions f such that $f_{|Z} = 0$. □

Remark 3.1.11. — Assume that $X = \Delta^n$ and $D = \{z_1 = 0\}$. Let S be a sector in $X \smallsetminus D$ and let $\overline{S}$ be the closure of S in $\widetilde{X}(D)$. We set $Z := \overline{S} \cap \pi^{-1}(D)$. Let $f \in \mathcal{A}(\widehat{Z})$. If we shrink S in the radius direction (Definition 3.4.2), we can take $F \in \mathcal{A}(\overline{S})$ such that $F_{|\widehat{Z}} = f$, which can be shown by the argument in the proof of Theorem 9.3 of [**94**]. □

Let U denote an open subset of $\widetilde{X}(D)$. Since $\widetilde{X}(D)$ is a C^∞-manifold with corner, we have the well defined concept of C^∞-vector bundle on U. Let E be a C^∞-bundle on U. The space of C^∞-sections of E on U is denoted by $C^\infty(U,E)$. Assume we are given a differential operator $\overline{\partial}_E : C^\infty(U,E) \to C^\infty\big(U, E\otimes \Omega_X^{0,1}(\log D)\big)$ such that (i) $\overline{\partial}_E(f\cdot s) = \overline{\partial}(f)\cdot s + f\cdot \overline{\partial}_E(s)$ holds for any $f\in C^\infty(U)$ and $s\in C^\infty(U,E)$, (ii) $\overline{\partial}_E\circ\overline{\partial}_E = 0$. A section s is called holomorphic, if $\overline{\partial}_E(s) = 0$ holds. We always assume the existence of holomorphic local frames of E around each point of $\widetilde{X}(D)$.

3.1.2.2. Some equivalence. — Let $\mathrm{Mod}(\mathcal{O}_X)$ denote the category of $\mathcal{O}_X$-modules. Let $\mathrm{Mod}_{\ell f}(\mathcal{O}_X)$ denote the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ of locally free $\mathcal{O}_X$-sheaves of finite ranks. To state some equivalence, which would be useful for understanding of our later construction, let us introduce a category $\mathcal{C}$.

- Objects of $\mathcal{C}$ are tuples $(\mathcal{F}_{\widetilde{X}(D)}, \mathcal{F}_{\widehat{D}}, \iota)$:
  - $\mathcal{F}_{\widetilde{X}(D)}$ is a locally free $\mathcal{O}_{\widetilde{X}(D)}$-module.
  - $\mathcal{F}_{\widehat{D}}$ is a locally free $\mathcal{O}_{\widehat{D}}$-module.
  - ι is an isomorphism $\pi^*\mathcal{F}_{\widehat{D}}$ and the completion of $\mathcal{F}_{\widetilde{X}(D)}$ along $\pi^{-1}(D)$.
- Morphisms $(\mathcal{F}^{(1)}_{\widetilde{X}(D)}, \mathcal{F}^{(1)}_{\widehat{D}}, \iota^{(1)}) \to (\mathcal{F}^{(2)}_{\widetilde{X}(D)}, \mathcal{F}^{(2)}_{\widehat{D}}, \iota^{(2)})$ are pairs of morphisms $f_{\widetilde{X}(D)} : \mathcal{F}^{(1)}_{\widetilde{X}(D)} \to \mathcal{F}^{(2)}_{\widetilde{X}(D)}$ and $f_{\widehat{D}} : \mathcal{F}^{(1)}_{\widehat{D}} \to \mathcal{F}^{(2)}_{\widehat{D}}$ which are compatible with $\iota^{(1)}$ and $\iota^{(2)}$.

We have the functors $F_1 : \mathrm{Mod}_{\ell f}(\mathcal{O}_X) \to \mathcal{C}$ given by $F_1(\mathcal{F}) := \big(\pi^*\mathcal{F}, \mathcal{F}_{|\widehat{D}}, \iota\big)$, where ι denotes the natural isomorphism and $\pi^*\mathcal{F} = \pi^{-1}\mathcal{F}\otimes_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_{\widetilde{X}(D)}$. We also have the functor $F_2 : \mathcal{C}\to\mathrm{Mod}(\mathcal{O}_X)$ given by $F_2(\mathcal{F}_{\widetilde{X}(D)}, \mathcal{F}_{\widehat{D}}, \iota) = \pi_*(\mathcal{F}_{\widetilde{X}(D)})$, where π_* denotes the push-forward of sheaves.

Proposition 3.1.12. — *For any $(\mathcal{F}_{\widetilde{X}(D)}, \mathcal{F}_{\widehat{D}}, \iota)\in\mathcal{C}$, $\pi_*\mathcal{F}_{\widetilde{X}(D)}$ is $\mathcal{O}_X$-locally free, and hence the functor F_2 factors through* $\mathrm{Mod}_{\ell f}(\mathcal{O}_X)$. *Moreover, we have an isomorphism $\mathcal{F}_{\widehat{D}} \simeq \pi_*(\mathcal{F}_{\widetilde{X}(D)})_{|\widehat{D}}$, which induces ι. (Note that it is unique.)*

Proof. — Let us show the first claim. Since it is a local property, we consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Moreover, we will replace X with a smaller neighbourhood of the origin without mention, if it is necessary.

Let $(\mathcal{F}_{\widetilde{X}(D)}, \mathcal{F}_{\widehat{D}}, \iota)\in\mathcal{C}$. Let $\widehat{\boldsymbol{v}}$ be a frame of $\mathcal{F}_{\widehat{D}}$. Let Q be any point of $\pi^{-1}(D)$. According to Proposition 3.1.10, we can take a small multi-sector S of $X\smallsetminus D$ and a frame $\boldsymbol{v}_S$ of $\mathcal{F}_{\widetilde{X}(D)|\overline{S}}$ such that (i) $\boldsymbol{v}_{S|\widehat{Z}} = \pi^{-1}\widehat{\boldsymbol{v}}$, (ii) Q is contained in the interior part of $\overline{S}$, where $\overline{S}$ denotes the closure of S in $\widetilde{X}(D)$, and $Z := \overline{S}\cap\pi^{-1}(D)$.

We take a covering $X\smallsetminus D = \bigcup_{p=1}^{N} S_p$ by multi-sectors on which we have frames $\boldsymbol{v}_{S_p}$ as above. We take a partition of unity $\big(\chi_p \,\big|\, p = 1,\dots,N\big)$ subordinated to the covering such that $P\chi_p$ are polynomial order in $|z_j^{-1}|$ $(j = 1,\dots,\ell)$ for each differential operator P on X. We obtain a C^∞-frame $\boldsymbol{v}_{C^\infty}$ of $\mathcal{F}_{\widetilde{X}(D)}$ given by $\boldsymbol{v}_{C^\infty} := \sum \boldsymbol{v}_{S_p}\chi_p$, or more precisely, $v_{C^\infty,i} = \sum \chi_p\, v_{S_p,i}$.

Let S be a small multi-sector of $X \setminus D$, and let $\boldsymbol{v}_S$ be as above. Let C_S be a matrix-valued function on $\overline{S}$ determined by $\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_S\,(I + C_S)$, where I is the identity matrix.

Lemma 3.1.13. — $C_{S|\widehat{Z}} = 0$.

Proof. — Let C_p be determined on $S \cap S_p$ by $\boldsymbol{v}_{S_p} = \boldsymbol{v}_S\,(I + C_p)$. Then, $C_{p|\widehat{Z \cap Z_p}} = 0$. By our choice of the partition of unity (χ_p), we obtain $C_{S|\widehat{Z}} = 0$. □

Let A be the matrix valued $(0,1)$-form determined by $\overline{\partial}\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\,A$.

Lemma 3.1.14. — *A gives a matrix-valued C^∞-function on X, and $A_{|\widehat{D}} = 0$.*

Proof. — By Lemma 3.1.13, for each small multi-sector S, $\overline{\partial}\boldsymbol{v}_{C^\infty|\overline{S}} = \boldsymbol{v}_S\,\overline{\partial}C_S = \boldsymbol{v}_{C^\infty|\overline{S}}\,(I + C_S)^{-1}\overline{\partial}C_S$. Hence, $A_{|\widehat{Z}} = 0$ for each S. Then, we obtain the claim of the lemma. □

Let E be the C^∞-vector bundle on X, which is the extension of $\mathcal{F}_{\widetilde{X}(D)|X \setminus D}$ by the frame $\boldsymbol{v}_{C^\infty}$. According to Lemma 3.1.14, the $\overline{\partial}$-operator of $E_{|X \setminus D}$ is naturally extended to a $\overline{\partial}$-operator of E in C^∞, i.e., the holomorphic structure of $E_{|X \setminus D}$ is prolonged to that of E. Let $\mathcal{E}$ denote the sheaf of holomorphic sections of E. By construction, we have a natural isomorphism $\mathcal{E} \simeq \pi_*\mathcal{F}_{\widetilde{X}(D)}$. Thus, the first claim of Proposition 3.1.12 is proved.

By Lemma 3.1.14, $\boldsymbol{v}_{C^\infty|\widehat{D}}$ naturally gives a frame of $\mathcal{E}_{|\widehat{D}}$. By the frames $\boldsymbol{v}_{C^\infty|\widehat{D}}$ and $\widehat{\boldsymbol{v}}$, we obtain an isomorphism $\mathcal{F}_{\widehat{D}} \simeq \mathcal{E}_{|\widehat{D}}$, which induces ι. □

Corollary 3.1.15. — *The functors F_1 and F_2 are equivalence of categories, and they are mutually quasi-inverse.* □

We give some complements.

Lemma 3.1.16. — *F_i $(i = 1, 2)$ preserve dual, tensor product and direct sum.*

Proof. — By construction, the functor F_1 preserves dual, tensor product and direct sum. Then, by Proposition 3.1.12, F_2 also preserves dual, tensor product and direct sum. □

3.1.2.3. Decent of a family of meromorphic ϱ-connections. — We use the setting in Subsection 3.1.3. Let $\mathcal{F}$ be a locally free $\mathcal{O}_{\mathcal{X}}$-module of finite rank. We set $\pi^*\mathcal{F} := \mathcal{O}_{\widetilde{\mathcal{X}}(W)} \otimes_{\pi^{-1}\mathcal{O}_{\mathcal{X}}} \pi^{-1}\mathcal{F}$.

Lemma 3.1.17. — *Let $\mathbb{D}_{\widetilde{\mathcal{X}}(W)} : \pi^*\mathcal{F} \to \pi^*\big(\mathcal{F} \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}(*\mathcal{D})\big)$ be a meromorphic ϱ-connection of $\pi^*\mathcal{F}$. Then, there exists a meromorphic ϱ-connection $\mathbb{D}$ of $\mathcal{F}$ such that $\mathbb{D}_{\widetilde{\mathcal{X}}(W)} = \pi^*\mathbb{D}$.*

Assume moreover that we are given a meromorphic ϱ-connection $\mathbb{D}_{\widehat{W}}$ of $\mathcal{F}_{|\widehat{W}}$ such that $\pi^*\mathbb{D}_{\widehat{W}}$ equals the completion of $\mathbb{D}_{\widetilde{\mathcal{X}}(W)}$ along $\pi^{-1}(W)$. Then, the completion of $\mathbb{D}$ equals $\mathbb{D}_{\widehat{W}}$.

Proof. — Let P be any point of $\mathcal{X}$. Let f be any section of $\mathcal{F}$ on a neighbourhood $\mathcal{X}_P$ of P in $\mathcal{X}$. If $\mathcal{X}_P$ is shrunk appropriately, there exists a number N such that $\mathbb{D}_{\widetilde{\mathcal{X}}(W)}(\pi^*f)$ gives a section of $\pi^*\big(\mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}(N\mathcal{D})\big)$ on $\pi^{-1}(\mathcal{X}_P)$. Because $\pi_*\pi^*\big(\mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}(N\mathcal{D})\big)\simeq\mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}(N\mathcal{D})$, $\mathbb{D}_{\widetilde{\mathcal{X}}(\mathcal{D})}(\pi^*f)$ naturally gives a section of $\mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}(ND)$ on $\mathcal{X}_P$. Thus, we obtain a map of sheaves $\mathbb{D}:\mathcal{F}\to\mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{K}}(*\mathcal{D})$. We can observe that it is a meromorphic ϱ-connection, and satisfies $\pi^*\mathbb{D}=\mathbb{D}_{\widetilde{\mathcal{X}}(D)}$, and thus the first claim is proved.

If we are given a meromorphic ϱ-connection $\mathbb{D}_{\widehat{W}}$ of $\mathcal{F}_{|\widehat{W}}$ as in the second claim, we have $\pi^*(\mathbb{D}_{\widehat{W}})=\mathbb{D}_{\widetilde{\mathcal{X}}(W)|\widehat{\pi^{-1}(W)}}=\pi^*(\mathbb{D}_{|\widehat{W}})$, and hence $\mathbb{D}_{\widehat{W}}=\mathbb{D}_{|\widehat{W}}$. Thus we obtain the second claim. □

3.1.3. Orders on a good set of irregular values. — Let $\mathcal{X}\to\mathcal{K}$ be a smooth fibration of complex manifolds. Let $\mathcal{D}$ be a normal crossing hypersurface of $\mathcal{X}$ such that all intersections of irreducible components are smooth over $\mathcal{K}$. Let ϱ be a holomorphic function on $\mathcal{K}$ such that $d\varrho$ is nowhere vanishing on $\mathcal{K}^0:=\varrho^{-1}(0)$. We put $\mathcal{X}^0:=\mathcal{X}\times_{\mathcal{K}}\mathcal{K}^0$, $\mathcal{D}^0:=\mathcal{D}\times_{\mathcal{K}}\mathcal{K}^0$ and $W:=\mathcal{X}^0\cup\mathcal{D}$. Let $\pi:\widetilde{\mathcal{X}}(W)\to\mathcal{X}$ denote the real blow up.

Let $\mathcal{I}_P\subset\mathcal{O}_{X,P}(*D)/\mathcal{O}_{X,P}$ be a good set of irregular values, where $P\in\mathcal{D}$. For each $Q\in\pi^{-1}(P)$, we shall introduce an order $\leqslant^{\varrho}_Q$ (simply denoted by $\leqslant_Q$) on the set $\mathcal{I}_P$. We can take a coordinate neighbourhood $(\mathcal{X}_P,z_1,\dots,z_n)$ around P such that $\mathcal{D}_P=\mathcal{X}_P\cap\mathcal{D}$ is expressed as $\bigcup_{i=1}^{\ell}\{z_i=0\}$, and that $\mathcal{I}_P\subset M(\mathcal{X}_P,\mathcal{D}_P)/H(\mathcal{X}_P)$. We take a lift $\widetilde{\mathfrak{a}}\in M(\mathcal{X}_P,\mathcal{D}_P)$ for each $\mathfrak{a}\in\mathcal{I}_P$. We put $W_P:=\mathcal{D}_P\cup\mathcal{X}^0_P$. For each distinct $\mathfrak{a},\mathfrak{b}\in\mathcal{I}_P$, we put

$$F_{\mathfrak{a},\mathfrak{b}}:=-\operatorname{Re}\big(\varrho^{-1}(\widetilde{\mathfrak{a}}-\widetilde{\mathfrak{b}})\big)\,\big|\boldsymbol{z}^{-\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\varrho\big|. \tag{40}$$

It naturally induces a C^∞-function on $\widetilde{\mathcal{X}}_P(W_P)$.

Definition 3.1.18. — Let $Q\in\pi^{-1}(P)$. We say $\mathfrak{a}<^{\varrho}_Q\mathfrak{b}$ for distinct $\mathfrak{a},\mathfrak{b}\in\mathcal{I}_P$, if $F_{\mathfrak{a},\mathfrak{b}}(Q)<0$. We say $\mathfrak{a}\leqslant^{\varrho}_Q\mathfrak{b}$ for $\mathfrak{a},\mathfrak{b}\in\mathcal{I}_P$, if $\mathfrak{a}<^{\varrho}_Q\mathfrak{b}$ or $\mathfrak{a}=\mathfrak{b}$. □

It is easy to check that the condition is independent of the choice of a coordinate system and lifts $\widetilde{\mathfrak{a}}$. We will denote it by $\leqslant_Q$, when ϱ is fixed.

Notation 3.1.19. — Later, we will also use the relation $\leqslant^{\varrho}_A$ for a subset $A\subset\widetilde{\mathcal{X}}_P(W_P)$, defined as $\mathfrak{a}<^{\varrho}_A\mathfrak{b}\iff F_{\mathfrak{a},\mathfrak{b}}<0$ on $\overline{A}\cap\pi^{-1}(\mathcal{D}_P)$ for $\mathfrak{a},\mathfrak{b}\in\mathcal{I}_P$, where $\overline{A}$ denotes the closure of A in $\widetilde{\mathcal{X}}_P(W_P)$. □

3.1.3.1. For any given C^∞-manifold (possibly with corners), we mean by a "compact region" the closure of a relatively compact connected open subset which is a C^∞-manifold with corners. Let $\mathfrak{U}(\widetilde{\mathcal{X}}(W))$ denote the set of compact regions in $\widetilde{\mathcal{X}}(W)$. For any point $Q \in \widetilde{\mathcal{X}}(W)$, let $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ denote the set of $\mathcal{U} \in \mathfrak{U}(\widetilde{\mathcal{X}}(W))$ such that Q is contained in the interior part of $\mathcal{U}$.

Let $\boldsymbol{\mathcal{I}} = (\mathcal{I}_P \mid P \in \mathcal{D})$ be a good system of irregular values. For $Q \in \pi^{-1}(P)$, let $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \boldsymbol{\mathcal{I}})$ denote the set of $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ such that, for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P$, we have $\mathfrak{a} \leqslant_Q^\varrho \mathfrak{b} \iff F_{\mathfrak{a},\mathfrak{b}} \leqslant 0$ on $\mathcal{U}$. The following claims are clear.

- If $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ is sufficiently small, it is contained in $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \boldsymbol{\mathcal{I}})$.
- For a given $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \boldsymbol{\mathcal{I}})$, we have $\mathfrak{a} \leqslant_Q^\varrho \mathfrak{b}$ if and only if $F_{\mathfrak{a},\mathfrak{b}} \leqslant 0$ on $\mathcal{U} \smallsetminus \pi^{-1}(W)$.
- Let $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \boldsymbol{\mathcal{I}})$. For $Q' \in \mathcal{U} \cap \pi^{-1}(\mathcal{D})$, the natural map
$$\big(\mathcal{I}_{\pi(Q)}, \leqslant_Q^\varrho\big) \longrightarrow \big(\mathcal{I}_{\pi(Q')}, \leqslant_{Q'}^\varrho\big)$$
is order preserving.

We will often denote $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ and $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \boldsymbol{\mathcal{I}})$ by $\mathfrak{U}(Q)$ and $\mathfrak{U}(Q, \boldsymbol{\mathcal{I}})$ if there is no risk of confusion.

3.2. Good meromorphic ϱ-flat bundle

In this section, we state a theorem and some propositions. They will be proved in Section 3.7. (See Subsection 3.7.8.) We use the following setting unless otherwise specified. Let $\mathcal{X} \to \mathcal{K}$ be a smooth fibration of complex manifolds. Let $\mathcal{D}$ be a normal crossing hypersurface of $\mathcal{X}$ such that any intersections of irreducible components are smooth over $\mathcal{K}$. Let ϱ be a holomorphic function on $\mathcal{K}$ such that $d\varrho$ is nowhere vanishing on $\mathcal{K}^0 := \varrho^{-1}(0)$. We put $\mathcal{X}^0 := \mathcal{X} \times_{\mathcal{K}} \mathcal{K}^0$, $\mathcal{D}^0 := \mathcal{D} \times_{\mathcal{K}} \mathcal{K}^0$ and $W := \mathcal{X}^0 \cup \mathcal{D}$. Let $\pi : \widetilde{\mathcal{X}}(W) \to \mathcal{X}$ denote the real blow up.

3.2.1. Full Stokes filtration. — Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$. We put $\pi^*\mathcal{E} := \pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\widetilde{\mathcal{X}}(W)}$, which is an $\mathcal{O}_{\widetilde{\mathcal{X}}(W)}(*\mathcal{D})$-module. For each $Q \in \pi^{-1}(\mathcal{D})$, let $\pi^*\mathcal{E}_Q$ denote the germ at Q, and $\pi^*\mathcal{E}_{|\widehat{Q}}$ denote the formal completion. In the following, $\mathrm{Irr}(\mathbb{D}, \pi(Q))$ is also denoted by $\mathrm{Irr}(\mathbb{D}, Q)$ for simplicity of the description.

The irregular decomposition of $(\mathcal{E}, \mathbb{D})_{|\widehat{\pi(Q)}}$ induces $\pi^*\mathcal{E}_{|\widehat{Q}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}, Q)} {}^Q\widehat{\mathcal{E}}_{\mathfrak{a}}$. We put
$$\widehat{\mathcal{F}}_{\mathfrak{a}}^Q\big(\pi^*\mathcal{E}_{|\widehat{Q}}\big) := \bigoplus_{\mathfrak{b} \leqslant_Q^\varrho \mathfrak{a}} {}^Q\widehat{\mathcal{E}}_{\mathfrak{a}}.$$

Theorem 3.2.1. — *For any $Q \in \pi^{-1}(\mathcal{D})$, there exists a $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^Q$ of $\pi^*\mathcal{E}_Q$ indexed by $\big(\mathrm{Irr}(\mathbb{D}, Q), \leqslant_Q^\varrho\big)$ with the following properties:*

- $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}^Q}(\pi^*\mathcal{E}_Q)$ *are free* $\mathcal{O}_{\widetilde{\mathcal{X}}(W)}(*\mathcal{D})_Q$*-modules of finite rank, and* $\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_Q)_{|\widehat{Q}} = \widehat{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_{|\widehat{Q}})$.
- *If we take a small* $\mathcal{U}_Q \in \mathfrak{U}(Q, \mathrm{Irr}(\mathbb{D}))$, *(see Subsection* 3.1.3 *for the notation), we have the filtration of* $\pi^*\mathcal{E}_{|\mathcal{U}_Q}$, *which induces* $\widetilde{\mathcal{F}}^Q(\pi^*\mathcal{E}_Q)$. *The filtration is also denoted by* $\widetilde{\mathcal{F}}^Q$. *For any* $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D})$, *we have the induced filtration* $\widetilde{\mathcal{F}}^Q$ *of* $\pi^*\mathcal{E}_{Q'}$. *Then,* $\widetilde{\mathcal{F}}^Q$ *and* $\widetilde{\mathcal{F}}^{Q'}$ *are compatible over* $\big(\mathrm{Irr}(\mathbb{D}, Q), \leqslant^{\varrho}_{Q}\big) \to \big(\mathrm{Irr}(\mathbb{D}, Q'), \leqslant^{\varrho}_{Q'}\big)$.

The conditions characterize the system of filtrations $\big(\widetilde{\mathcal{F}}^Q \,\big|\, Q \in \pi^{-1}(\mathcal{D})\big)$. *If* $\varrho(Q) \neq 0$, *the first property characterizes the filtration* $\widetilde{\mathcal{F}}^Q$.

The filtrations $\widetilde{\mathcal{F}}^Q$ are called the full Stokes filtration of $\mathcal{E}$ at Q. It is also called the Stokes filtration, if there is no risk of confusion.

Remark 3.2.2. — Such a filtration appeared in the classical works on meromorphic flat bundles on curves, for example [**56**] and [**57**]. (See also [**25**].) T. Pantev informed that it is called "Deligne-Malgrange filtration". We keep our terminology "Stokes filtration", partially because we use "Deligne-Malgrange" in a different meaning. □

Let $P \in \mathcal{D}$ with a small neighbourhood $\mathcal{X}_P$. Let $\mathcal{U} \in \mathfrak{U}(\widetilde{\mathcal{X}}_P(W_P))$. Let $\leqslant^{\varrho}_{\mathcal{U}}$ be the order on $\mathrm{Irr}(\mathbb{D}, P)$ defined as follows:

$$\mathfrak{a} \leqslant^{\varrho}_{\mathcal{U}} \mathfrak{b} \Longleftrightarrow \mathfrak{a}_Q \leqslant^{\varrho}_{Q} \mathfrak{b}_Q \ (\forall\, Q \in \mathcal{U} \cap \pi^{-1}(\mathcal{D})).$$

Here, $\mathfrak{a}_Q, \mathfrak{b}_Q \in \mathrm{Irr}(\mathbb{D}, \pi(Q))$ be the induced elements. The use is consistent with Notation 3.1.19.

If $\pi^*\mathcal{E}_{|\mathcal{U}}$ has a $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^{\mathcal{U}}$ indexed by $\big(\mathrm{Irr}(\mathbb{D}, P), \leqslant^{\varrho}_{\mathcal{U}}\big)$ with the following property, which is called a full Stokes filtration on $\mathcal{U}$:

- Let Q be any point of $\mathcal{U} \cap \pi^{-1}(\mathcal{D})$. We have the induced filtration $\widetilde{\mathcal{F}}^{\mathcal{U}}$ of $\pi^*\mathcal{E}_Q$. Then, $\widetilde{\mathcal{F}}^{\mathcal{U}}$ and $\widetilde{\mathcal{F}}^{Q}$ are compatible over $\big(\mathrm{Irr}(\mathbb{D}, P), \leqslant_{\mathcal{U}}\big) \to \big(\mathrm{Irr}(\mathbb{D}, \pi(Q)), \leqslant_Q\big)$.

Such a filtration is uniquely determined if it exists, by Lemma 3.1.3.

For a multi-sector S of $\mathcal{X} \setminus W$, if a full Stokes filtration for $\overline{S}$ exists, it is also called Stokes filtration for S.

3.2.2. Functoriality.

— Let $(\mathcal{E}_i, \mathbb{D}_i)$ $(i = 1, 2)$ be unramifiedly good meromorphic ϱ-flat bundles on $(\mathcal{X}, \mathcal{D})$.

Proposition 3.2.3. — *Let* $F : \mathcal{E}_1 \to \mathcal{E}_2$ *be a* $\mathbb{D}$*-flat morphism. For simplicity, we assume that* $\mathrm{Irr}(\mathbb{D}_1) \cup \mathrm{Irr}(\mathbb{D}_2)$ *is also good, i.e.,* $\mathrm{Irr}(\mathbb{D}_1, P) \cup \mathrm{Irr}(\mathbb{D}_2, P)$ *is good for each* $P \in \mathcal{D}$. *Then, for each* $Q \in \pi^{-1}(\mathcal{D})$, *the induced morphism* $\pi^*\mathcal{E}_{1\,Q} \to \pi^*\mathcal{E}_{2\,Q}$ *is compatible with the full Stokes filtrations.*

Proposition 3.2.4. — *If* $\mathrm{Irr}(\mathbb{D}_1) \otimes \mathrm{Irr}(\mathbb{D}_2)$ *is good, we have*

$$\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}\big(\pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2)_Q\big) = \sum_{\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{a}} \widetilde{\mathcal{F}}^Q_{\mathfrak{a}_1}(\pi^*\mathcal{E}_{1,Q}) \otimes \widetilde{\mathcal{F}}^Q_{\mathfrak{a}_2}(\pi^*\mathcal{E}_{2,Q}).$$

If $\mathrm{Irr}(\mathbb{D}_1) \oplus \mathrm{Irr}(\mathbb{D}_2)$ *is good, we have*

$$\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}\big(\pi^*(\mathcal{E}_1 \oplus \mathcal{E}_2)_Q\big) = \widetilde{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_{1,Q}) \oplus \widetilde{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_{2,Q}).$$

Proposition 3.2.5. — *Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good ϱ-meromorphic flat bundle on $(\mathcal{X}, \mathcal{D})$. The Stokes filtration of $(\mathcal{E}^\vee, \mathbb{D}^\vee)$ at $Q \in \pi^{-1}(\mathcal{D})$ is given as follows:*

$$\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}^\vee_Q) = \Big(\sum_{\mathfrak{b} \not\geq -\mathfrak{a}} \widetilde{\mathcal{F}}^Q_{\mathfrak{b}}(\pi^*\mathcal{E}_Q)\Big)^\perp.$$

3.2.3. Characterization by growth order (the case $\varrho(Q) \neq 0$). — Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$. Let $Q \in \pi^{-1}(\mathcal{D})$. Take a small neighbourhood $\mathcal{U} \in \mathfrak{U}(Q, \mathrm{Irr}(\mathbb{D}))$ and a frame $\boldsymbol{v}$ of $\mathcal{E}_{|\mathcal{U}}$. A $\mathbb{D}$-flat section of $\mathcal{E}_{|\mathcal{U}\setminus\pi^{-1}(W)}$ is expressed as $f = \sum f_j\, v_j$, where $f_j \in \mathcal{O}_{\mathcal{U}\setminus\pi^{-1}(W)}$. Let $\boldsymbol{f}$ denote the tuple (f_j).

Proposition 3.2.6. — *Assume $\varrho(Q) \neq 0$. We have $f \in \widetilde{\mathcal{F}}^Q_{\mathfrak{a}}\mathcal{E}_{|\mathcal{U}\setminus\pi^{-1}(\mathcal{D})}$ if and only if $\big|\exp(\varrho^{-1}\mathfrak{a})\boldsymbol{f}\big|$ is of polynomial order on $\mathcal{U} \setminus \pi^{-1}(\mathcal{D})$.*

Remark 3.2.7. — Let $(z_1, \dots, z_n)$ be a coordinate system around $\pi(Q)$ such that $\mathcal{D}$ is expressed as $\bigcup_{i=1}^{\ell}\{z_i = 0\}$ around $\pi(Q)$. We say that a function F on $\mathcal{U}_Q \setminus \pi^{-1}(\mathcal{D})$ is of polynomial order, if $|F| = O\big(\prod_{i=1}^{\ell}|z_i|^{-N}\big)$ for some $N > 0$. □

3.2.4. The associated graded bundle. — Let $P \in \mathcal{D}$. For each $Q \in \pi^{-1}(P)$, we have $\mathrm{Gr}^{\widetilde{\mathcal{F}}^Q}\big(\pi^*\mathcal{E}_{|\mathcal{U}_Q}, \mathbb{D}\big)$ on a small neighbourhood $\mathcal{U}_Q$ of Q in $\widetilde{\mathcal{X}}(W)$. Although the filtration $\widetilde{\mathcal{F}}^Q$ depends on Q, the system $\big(\widetilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(P)\big)$ satisfies the compatibility condition. Hence, by varying $Q \in \pi^{-1}(P)$ and gluing $\mathrm{Gr}^{\widetilde{\mathcal{F}}^Q}\big(\pi^*\mathcal{E}_{|\mathcal{U}_Q}, \mathbb{D}\big)$, we obtain the associated graded locally free $\mathcal{O}_{\widetilde{\mathcal{X}}(W)}(*\mathcal{D})$-module with a flat ϱ-connection

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P)} \big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}), \mathbb{D}_{\mathfrak{a}}\big)$$

on a neighbourhood of $\pi^{-1}(P)$. By taking the push-forward via π, we obtain an $\mathcal{O}_{\mathcal{X}}$-module with a flat ϱ-connection

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D},P)} \big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{E}_P), \mathbb{D}_{\mathfrak{a}}\big)$$

on a neighbourhood $\mathcal{X}_P$ of P.

Proposition 3.2.8. — $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P, \mathbb{D})$ *is a graded meromorphic ϱ-flat bundle satisfying $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}, \mathbb{D}) \simeq \pi^*\,\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P, \mathbb{D})$. We have a canonical isomorphism $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P, \mathbb{D})_{|\widehat{P}} \simeq (\mathcal{E}_P, \mathbb{D})_{|\widehat{P}}$.*

In particular, each $\big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{E}_P), \mathbb{D}_{\mathfrak{a}}\big) \otimes L(-\mathfrak{a})$ is a regular meromorphic ϱ-flat bundle.

3.2.4.1. Functoriality. — Let $(\mathcal{E}_1, \mathbb{D}_1) \to (\mathcal{E}_2, \mathbb{D}_2)$ be a morphism of unramifiedly good meromorphic ϱ-flat bundles. For simplicity, we assume $\mathrm{Irr}(\mathbb{D}_1, P) \cup \mathrm{Irr}(\mathbb{D}_2, P)$ is good. Then, we have the induced morphism on a neighbourhood of P:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{1,P}) \longrightarrow \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{2,P}).$$

If $\mathrm{Irr}(\mathbb{D}_1, P) \otimes \mathrm{Irr}(\mathbb{D}_2, P)$ is good, we have the following canonical isomorphism on a neighbourhood of P:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}\big((\mathcal{E}_1 \otimes \mathcal{E}_2)_P\big) \simeq \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{1,P}) \otimes \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{2,P}).$$

If $\mathrm{Irr}(\mathbb{D}_1, P) \oplus \mathrm{Irr}(\mathbb{D}_2, P)$ is good, we have the following canonical isomorphism on a neighbourhood of P:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}\big((\mathcal{E}_1 \oplus \mathcal{E}_2)_P\big) \simeq \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{1,P}) \oplus \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_{2,P}).$$

For an unramifiedly good meromorphic ϱ-flat bundle $(\mathcal{E}, \mathbb{D})$, we have the following canonical isomorphism on a neighbourhood of P:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P^\vee) \simeq \mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P)^\vee.$$

3.2.5. Lattice. — Let E be an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})$. We set $\pi^*E := \pi^{-1}(E) \otimes_{\pi^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\widetilde{\mathcal{X}}(W)}$. We have the induced filtration $\widetilde{\mathcal{F}}^Q$ of $\pi^*E_Q \subset \pi^*\mathcal{E}_Q$.

Proposition 3.2.9

- $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\pi^*E_Q)$ *is free as an* $\mathcal{O}_{\widetilde{\mathcal{X}}}$*-module.*
- *We have the induced lattice* $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E_P)$ *of* $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}_P)$. *It is functorial with respect to morphism, tensor product, direct sum and dual.*

3.2.6. Splitting. — We give some statements for splitting of full Stokes filtrations.

3.2.6.1. Flat splitting. — First, we consider $\mathbb{D}$-flat splittings. Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$.

Proposition 3.2.10. — *Assume* $\varrho(Q) \neq 0$. *We can find a* $\mathbb{D}$*-flat splitting of* $\widetilde{\mathcal{F}}^Q$, *i.e., we can find a* $\mathbb{D}$*-flat decomposition* $\pi^*\mathcal{E}_Q = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}, Q)} \pi^*\mathcal{E}_{Q,\mathfrak{a}}$ *such that* $\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_Q) = \bigoplus_{\mathfrak{b} \leqslant^{\varrho}_{Q} \mathfrak{a}} \pi^*\mathcal{E}_{Q,\mathfrak{b}}$.

Let E be an unramifiedly good lattice of $(\mathcal{E}, \mathbb{D})$.

Proposition 3.2.11. — *Assume* $\varrho(Q) \neq 0$ *and that* E *satisfies the non-resonance condition, i.e., for distinct eigenvalues* α *and* β *of* $\mathrm{Res}_{\mathcal{D}_i}(\varrho^{-1}\mathbb{D})$ *we have* $\alpha - \beta \notin \mathbb{Z}$. *Then, we can find a* $\mathbb{D}$*-flat splitting* $\pi^*E_Q = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}, Q)} \pi^*E_{Q,\mathfrak{a}}$ *of* $(\pi^*E_Q, \widetilde{\mathcal{F}}^Q)$.

Note that we have the following natural isomorphisms on a neighbourhood of Q with $\varrho(Q) \neq 0$, for the direct summands of the above decompositions:

$$\pi^*\mathcal{E}_{Q,\mathfrak{a}} \simeq \pi^* \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{E}_{\pi(Q)}), \quad \pi^*E_{Q,\mathfrak{a}} \simeq \pi^* \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(E_{\pi(Q)}).$$

For example, we can take a nice frame of π^*E_Q by lifting a nice frame of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E_{\pi(Q)})$.

If $\mathcal{D}$ is smooth, we do not have to impose additional assumptions such as "non-resonance" and "$\varrho(Q) \neq 0$".

Proposition 3.2.12. — *Assume that $\pi(Q)$ is a smooth point of $\mathcal{D}$. Then, we can find a $\mathbb{D}$-flat splitting of $(\pi^* E_Q, \widetilde{\mathcal{F}}^Q)$ and hence $(\pi^* \mathcal{E}_Q, \widetilde{\mathcal{F}}^Q)$.*

3.2.6.2. Partially flat splitting. — We consider rough splittings in more general situations. Because the claims are local, we use the setting and the notation in Subsection 2.4.2. Let $(\mathcal{E}, \mathbb{D})$ be a good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ with the good set of irregular values $\mathcal{I} \subset M(\mathcal{X}, \mathcal{D})/H(\mathcal{X})$, which are lifted to $M(\mathcal{X}, \mathcal{D})$. Let E be an unramifiedly good lattice. For simplicity, we assume that the coordinate system $(z_1, \dots, z_n)$ is admissible for $\mathcal{I}$. Let p be determined by the following conditions:

$$\mathrm{ord}_p(\mathfrak{a}) < 0 \quad (\forall\, \mathfrak{a} \in \mathcal{I}, \forall\, i \leqslant p), \quad \mathrm{ord}_{p+1}(\mathfrak{a}) = 0\ (\exists\, \mathfrak{a} \in \mathcal{I}).$$

Let $\mathbb{D}_{\leqslant p}$ denote the restriction of $\mathbb{D}$ to the $(z_1, \dots, z_p)$-direction.

Proposition 3.2.13. — *For any $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$, we can find a $\mathbb{D}_{\leqslant p}$-flat splitting of $(\pi^* E_Q, \widetilde{\mathcal{F}}^Q)$. It induces a $\mathbb{D}_{\leqslant p}$-flat splitting of $\widetilde{\mathcal{F}}^{Q'}(\pi^* E_{Q'})$ for Q' contained in a sufficiently small neighbourhood of Q.*

We give a more refined statement. Let ${}^i\mathbb{D}$ $(i \geqslant p+1)$ be the induced flat ϱ-connection on $E_{|\mathcal{D}_i}$ with respect to z_i. (See Subsection 2.2.1.1) Assume that $E_{|\mathcal{D}_i}$ $(i \geqslant p+1)$ are equipped with ${}^i\mathbb{D}$-flat filtrations ${}^iF$.

Proposition 3.2.14. — *For any $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$, we can find a $\mathbb{D}_{\leqslant p}$-flat splitting of $(\pi^* E, \widetilde{\mathcal{F}}^Q)$ on a neighbourhood $\mathcal{U}_Q$ which is compatible with ${}^iF$ and $\mathrm{Res}_i(\mathbb{D})$. Namely we have a $\mathbb{D}_{\leqslant p}$-flat splitting $\pi^* E_{|\mathcal{U}_Q} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \pi^* E_{\mathfrak{a}, \mathcal{U}_Q}$ such that (i) $\mathrm{Res}_i(\mathbb{D})$ preserves $\pi^* E_{\mathfrak{a}, \mathcal{U}_Q | \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_i)}$, (ii) ${}^iF_a = \bigoplus_{\mathfrak{a} \in \mathcal{I}} {}^iF_a \cap \pi^* E_{\mathfrak{a}, \mathcal{U}_Q}$.*

We give some complements.

Proposition 3.2.15. — *Assume $\varrho(Q) \neq 0$. If $\mathcal{U}_Q$ is a sufficiently small neighbourhood of Q, any $\mathbb{D}_{\leqslant p}$-flat splitting of $(E, \widetilde{\mathcal{F}}^Q)$ on $\mathcal{U}_Q \setminus \pi^{-1}(\mathcal{D})$ can be extended to a $\mathbb{D}_{\leqslant p}$-flat splitting on $\mathcal{U}_Q$.*

The following proposition is a refinement of Proposition 3.2.11.

Proposition 3.2.16. — *Assume $\varrho(Q) \neq 0$. If we have $\alpha - \beta \notin \mathbb{Z}$ for distinct eigenvalues α and β of $\mathrm{Res}_{\mathcal{D}_i}(\varrho^{-1}\mathbb{D})$ $(i \geqslant p+1)$, then we can find a $\mathbb{D}$-flat splitting of $\pi^* E_Q$ compatible with ${}^iF$ $(i \geqslant p+1)$.*

3.3. Good meromorphic ϱ-flat bundle at the level $\boldsymbol{m}$

In this section, we state some propositions on Stokes filtrations at the level $\boldsymbol{m}$. The proof will be given in Section 3.6. (See Subsection 3.6.7.) We use the setting in Subsection 2.6.3 unless otherwise specified. Namely, let Y be a complex manifold with a simple normal crossing divisor $\mathcal{D}'_Y$. Let $\mathcal{K}$ be a complex manifold with a holomorphic function ϱ such that $d\varrho$ is nowhere vanishing on $\mathcal{K}^0 := \varrho^{-1}(0)$. Let $\mathcal{X} := \Delta_z^k \times Y \times \mathcal{K}$, $\mathcal{D}_{z,i} := \{z_i = 0\}$ and $\mathcal{D}_z := \bigcup_{i=1}^k \mathcal{D}_{z,i}$. We also put $\mathcal{D}_Y := \Delta_z^k \times \mathcal{D}'_Y \times \mathcal{K}$ and $\mathcal{D} := \mathcal{D}_z \cup \mathcal{D}_Y$. For any subset $I \subset \underline{k}$, we put $\mathcal{D}_{z,I} := \bigcap_{i \in I} \mathcal{D}_{z,i}$. We set $\mathcal{X}^0 := \mathcal{X} \times_{\mathcal{K}} \mathcal{K}^0$ and $W := \mathcal{X}^0 \cup \mathcal{D}_z$. Let $\pi : \widetilde{\mathcal{X}}(W) \to \mathcal{X}$ denote the real blow up of $\mathcal{X}$ along W.

3.3.1. Orders on weakly good sets of irregular values at the level $\boldsymbol{m}$

Let $\boldsymbol{m} \in \mathbb{Z}_{<0}^k$. Let $\mathcal{I}$ be a weakly good set of irregular values on $(\mathcal{X}, \mathcal{D})$ at the level $\boldsymbol{m}$. We put $F_{\mathfrak{a},\mathfrak{b}} := -\operatorname{Re}\big(\varrho^{-1}(\mathfrak{a} - \mathfrak{b})\big)\, |\varrho \boldsymbol{z}^{-\boldsymbol{m}}|$ for any distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. They naturally induce C^∞-functions on $\widetilde{\mathcal{X}}(W)$.

Notation 3.3.1. — Let A be any subset of $\widetilde{\mathcal{X}}(W)$. For distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, we say $\mathfrak{a} <_A^\varrho \mathfrak{b}$ if $F_{\mathfrak{a},\mathfrak{b}}(Q) < 0$ for any $Q \in A$. We say $\mathfrak{a} \leqslant_A^\varrho \mathfrak{b}$ for $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{I}^2$ if either $\mathfrak{a} <_A^\varrho \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{b}$ holds. The relation $\leqslant_A^\varrho$ gives a partial order on $\mathcal{I}$. We use the symbol $\leqslant_P^\varrho$ in the case $A = \{P\}$. (We will use the symbol $\leqslant_P$ instead of $\leqslant_P^\varrho$, if there is no risk of confusion.) □

Let $\mathcal{I}$ be a weakly good set of irregular values at the level $\boldsymbol{m}$. For any $f \in M(\mathcal{X}, \mathcal{D}_z)$, $\mathcal{I}' := \{\mathfrak{a} - f \mid \mathfrak{a} \in \mathcal{I}\}$ is also a weakly good set of irregular values at the level $\boldsymbol{m}$. The natural bijection obviously preserves the orders $\leqslant_A^\varrho$ for any subset A of $\widetilde{\mathcal{X}}(W)$.

For $Q \in \pi^{-1}(\mathcal{D})$, let $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \mathcal{I})$ or $\mathfrak{U}(Q, \mathcal{I})$ denote the set of $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ such that $\leqslant_Q^\varrho = \leqslant_{\mathcal{U}}^\varrho$. If $\mathcal{U} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W))$ is sufficiently small, it is contained in $\mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \mathcal{I})$. For any point $Q' \in \pi^{-1}(\mathcal{D}) \cap \mathcal{U}$, the map $\big(\mathcal{I}, \leqslant_Q^\varrho\big) \to \big(\mathcal{I}, \leqslant_{Q'}^\varrho\big)$ preserves the orders.

3.3.2. Stokes filtration at the level $\boldsymbol{m}$. — Let $(\mathcal{E}, \mathbb{D})$ be a meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ with a weakly good lattice $(E, \mathcal{I})$ at the level $(\boldsymbol{m}, i(0))$. The restriction of $\mathbb{D}$ to the Δ_z^k-direction is denoted by $\mathbb{D}_z$. Let $\pi : \widetilde{\mathcal{X}}(W) \to \mathcal{X}$ be the real blow up. We have the decomposition induced by (36):

$$\pi^* E_{|\widehat{\pi^{-1}(W)}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} E_{\widehat{\pi^{-1}(W)},\mathfrak{a}}.$$

Proposition 3.3.2. — *For any $Q \in \pi^{-1}(\mathcal{D}_{z,\underline{k}})$, there exists a unique $\mathbb{D}$-flat filtration $\mathcal{F}^Q$ of $\pi^*\mathcal{E}_Q$ indexed by $\big(\mathcal{I}, \leqslant_Q^\varrho\big)$ with the following properties:*

- *It comes from a filtration of $\pi^* E_Q$ such that $\operatorname{Gr}^{\mathcal{F}^Q}(\pi^* E_Q)$ is a locally free $\mathcal{O}_{\widetilde{\mathcal{X}}(\mathcal{D})}$-module.*

- *Take $\mathcal{U} \in \mathfrak{U}(Q, \mathcal{I})$ on which $\mathcal{F}^Q$ is given. Then we have*
$$\mathcal{F}^Q_{\mathfrak{a}}(\pi^* E)_{|\mathcal{U}\cap\widehat{\pi^{-1}(W)}} = \bigoplus_{\mathfrak{b}\leqslant^{\varrho}_{Q}\mathfrak{a}} E_{\widehat{\pi^{-1}(W)},\mathfrak{a}|\mathcal{U}}.$$

The filtration $\mathcal{F}^Q$ is called a Stokes filtration at the level $(\boldsymbol{m}, i(0))$ (or simply at the level $\boldsymbol{m}$).

Proposition 3.3.3. — *Take $\mathcal{U} \in \mathfrak{U}(Q, \mathcal{I})$ on which $\mathcal{F}^Q$ is given. For any $Q' \in \mathcal{U} \cap \pi^{-1}(\mathcal{D}_{z,\underline{k}})$, the natural morphism $\big(\pi^*\mathcal{E}_{Q'}, \mathcal{F}^Q\big) \to \big(\pi^*\mathcal{E}_{Q'}, \mathcal{F}^{Q'}\big)$ is compatible.*

Remark 3.3.4. — Let $\mathcal{U} \in \mathfrak{U}(\widetilde{\mathcal{X}}(W))$. If $\pi^*\mathcal{E}_{|\mathcal{U}}$ has a $\mathbb{D}$-flat filtration $\mathcal{F}^{\mathcal{U}}$ indexed by $\big(\mathcal{I}, \leqslant^{\varrho}_{\mathcal{U}}\big)$ with the following property, which is also called a Stokes filtration on $\mathcal{U}$:

- Let Q be any point of $\mathcal{U}\cap\pi^{-1}(\mathcal{D}_{z,\underline{k}})$. We have the induced filtration $\mathcal{F}^{\mathcal{U}}$ of π^*E_Q. Then, $\mathcal{F}^{\mathcal{U}}$ and $\mathcal{F}^Q$ are compatible over $\big(\mathcal{I}, \leqslant^{\varrho}_{\mathcal{U}}\big) \to \big(\mathcal{I}, \leqslant^{\varrho}_{Q}\big)$.

Such a filtration is uniquely determined if it exists, by Lemma 3.1.3. For a multi-sector S of $\mathcal{X} \setminus W$, if the Stokes filtration for $\overline{S}$ exists, it is also called Stokes filtration for S. □

3.3.3. Functoriality. — Let $(\mathcal{E}_i, \mathbb{D}_i)$ $(i = 1, 2)$ be meromorphic ϱ-flat bundles on $(\mathcal{X}, \mathcal{D})$ with weakly good lattices $(E_i, \mathcal{I}_i)$.

Proposition 3.3.5. — *Let $F : \mathcal{E}_1 \to \mathcal{E}_2$ be a $\mathbb{D}$-flat morphism. For simplicity, we assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$. Then, for each $Q \in \pi^{-1}(\mathcal{D}_{z,\underline{k}})$, the induced morphism $\pi^*\mathcal{E}_{1\,Q} \to \pi^*\mathcal{E}_{2\,Q}$ is compatible with the Stokes filtrations at the level $\boldsymbol{m}$.*

Proposition 3.3.6. — *If $\mathcal{I}_1 \otimes \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, we have*
$$\mathcal{F}^Q_{\mathfrak{a}}\big(\pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2)_Q\big) = \sum_{\mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}} \mathcal{F}^Q_{\mathfrak{a}_1}(\pi^*\mathcal{E}_{1,Q}) \otimes \mathcal{F}^Q_{\mathfrak{a}_2}(\pi^*\mathcal{E}_{2,Q}).$$
If $\mathcal{I}_1 \oplus \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, we have
$$\mathcal{F}^Q_{\mathfrak{a}}\big(\pi^*(\mathcal{E}_1 \oplus \mathcal{E}_2)_Q\big) = \mathcal{F}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_{1,Q}) \oplus \mathcal{F}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}_{2,Q}).$$

Proposition 3.3.7. — *Let $(\mathcal{E}, \mathbb{D})$ be a ϱ-meromorphic flat bundle on $(\mathcal{X}, \mathcal{D})$ with a weakly good lattice $(E, \mathcal{I})$ at the level $(\boldsymbol{m}, i(0))$. The Stokes filtration of $(\mathcal{E}^{\vee}, \mathbb{D}^{\vee})$ at the level $\boldsymbol{m}$ is given as follows:*
$$\mathcal{F}^Q_{\mathfrak{a}}(\pi^*\mathcal{E}^{\vee}_Q) = \Big(\sum_{\mathfrak{b}\not\geqslant-\mathfrak{a}} \mathcal{F}^Q_{\mathfrak{b}}(\pi^*\mathcal{E}_Q)\Big)^{\perp}.$$

3.3.4. Characterization by growth order. — Let $(\mathcal{E}, \mathbb{D})$ and E be as above. Let $\boldsymbol{v}$ be a frame of E. Let $\mathcal{U}_Q \in \mathfrak{U}(Q, \mathcal{I})$ be sufficiently small. For any $\mathbb{D}_z$-flat section f of $E_{|\mathcal{U}_Q \setminus \pi^{-1}(W)}$, we have the expression $f = \sum f_j v_j$. Put $\boldsymbol{f} := (f_j)$.

Proposition 3.3.8. — *Assume $\varrho(Q) \neq 0$. We have $f \in \mathcal{F}^Q_{\mathfrak{a}}$, if and only if the following holds for some $C > 0$:*

$$\bigl|\boldsymbol{f} \exp(\varrho^{-1}\mathfrak{a})\bigr| = O\bigl(\exp(C|\boldsymbol{z}^{\boldsymbol{m}(1)}|)\,|z_{i(0)}|^{-C}\bigr).$$

3.3.5. The associated graded bundle. — For any $P \in \mathcal{D}_{z,\underline{k}}$, we have the associated graded sheaf with an induced flat ϱ-connection on a neighbourhood of $\pi^{-1}(P)$:

$$\operatorname{Gr}^{\mathcal{F}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \bigl(\operatorname{Gr}^{\mathcal{F}}_{\mathfrak{a}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}), \mathbb{D}_{\mathfrak{a}}\bigr).$$

By taking the push-forward via π, we obtain a graded $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-module with a flat ϱ-connection on a neighbourhood of P:

$$\operatorname{Gr}^{\mathcal{F}}(\mathcal{E}_P, \mathbb{D}) = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \bigl(\operatorname{Gr}^{\mathcal{F}}_{\mathfrak{a}}(\mathcal{E}_P), \mathbb{D}_{\mathfrak{a}}\bigr).$$

Similarly, we have $\operatorname{Gr}^{\mathcal{F}}(\pi^* E_{\pi^{-1}(P)}, \mathbb{D})$ and $\operatorname{Gr}^{\mathcal{F}}(E_P, \mathbb{D})$ on neighbourhoods of $\pi^{-1}(P)$ and P, respectively.

Proposition 3.3.9. — $\operatorname{Gr}^{\mathcal{F}}(\mathcal{E}_P, \mathbb{D})$ *is a graded meromorphic ϱ-flat bundle on a neighbourhood of P such that $\operatorname{Gr}^{\mathcal{F}}(\pi^*\mathcal{E}_{\pi^{-1}(P)}, \mathbb{D}) \simeq \pi^* \operatorname{Gr}^{\mathcal{F}}(\mathcal{E}_P, \mathbb{D})$. We also have $\operatorname{Gr}^{\mathcal{F}}(\pi^* E_{\pi^{-1}(P)}, \mathbb{D}) \simeq \pi^* \operatorname{Gr}^{\mathcal{F}}(E_P, \mathbb{D})$.*

For each $\mathfrak{a}$, we have a canonical isomorphism $\operatorname{Gr}^{\mathcal{F}}_{\mathfrak{a}}(E, \mathbb{D})_{|\widehat{W}} \simeq (\widehat{E}_{\mathfrak{a}\widehat{W}}, \widehat{\mathbb{D}}^{\varrho}_{\mathfrak{a}})$, where the right-hand side is the direct summand in (36). In particular, $\bigl(\operatorname{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{E}), \mathbb{D}_{\mathfrak{a}}\bigr)$ has a weakly good lattice $\bigl(\operatorname{Gr}^{\mathcal{F}}_{\mathfrak{a}}(E), \{\mathfrak{a}\}\bigr)$ at the level $(\boldsymbol{m}, i(0))$.

3.3.5.1. Functoriality. — If we are given a morphism of meromorphic ϱ-flat bundles $F : (\mathcal{E}_1, \mathbb{D}_1) \to (\mathcal{E}_2, \mathbb{D}_2)$ with weakly good lattices $(E_i, \mathcal{I}_i)$ at the level $(\boldsymbol{m}, i(0))$, we have the induced morphism

$$\operatorname{Gr}^{\mathcal{F}}(F) : \operatorname{Gr}^{\mathcal{F}}(E_1, \mathbb{D}_1) \longrightarrow \operatorname{Gr}^{\mathcal{F}}(E_2, \mathbb{D}_2).$$

In particular, we have $\operatorname{Gr}^{\mathcal{F}}(\mathcal{E}_1, \mathbb{D}_1) \to \operatorname{Gr}^{\mathcal{F}}(\mathcal{E}_2, \mathbb{D}_2)$.

If $\mathcal{I}_1 \otimes \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, we have the following canonical isomorphism on a neighbourhood of P:

$$\operatorname{Gr}^{\mathcal{F}}(E_1 \otimes E_2) \simeq \operatorname{Gr}^{\mathcal{F}}(E_1) \otimes \operatorname{Gr}^{\mathcal{F}}(E_2).$$

If $\mathcal{I}_1 \oplus \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, we have the following canonical isomorphism on a neighbourhood of P:

$$\operatorname{Gr}^{\mathcal{F}}(E_1 \oplus E_2) \simeq \operatorname{Gr}^{\mathcal{F}}(E_1) \oplus \operatorname{Gr}^{\mathcal{F}}(E_2).$$

Let $(E, \mathbb{D}, \mathcal{I})$ be a weakly good lattice at the level $(\boldsymbol{m}, i(0))$. We have a canonical isomorphism on a neighbourhood of P:

$$\mathrm{Gr}^{\mathcal{F}}(E^{\vee}) \simeq \mathrm{Gr}^{\mathcal{F}}(E)^{\vee}.$$

3.3.6. Splitting. — Let $\mathcal{D}_Y = \bigcup_{j\in\Lambda} \mathcal{D}_{Y,j}$ be the irreducible decomposition. For each $j \in \Lambda$, we have the residue $\mathrm{Res}_{Y,j}(\mathbb{D})$ on $E_{|\mathcal{D}_{Y,j}}$. Assume that we are given a filtration ${}^{j}F$ of $E_{|\mathcal{D}_{Y,j}}$ which is flat with respect to the locally induced ϱ-connection ${}^{j}\mathbb{D}$.

Proposition 3.3.10. — *We can take a $\mathbb{D}_z$-flat splitting $\pi^* E_{|\mathcal{U}_Q} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a},\mathcal{U}_Q}$ which is compatible with $\mathrm{Res}_{Y,j}(\mathbb{D})$ and ${}^{j}F$ $(j \in \Lambda)$.*

Proposition 3.3.11. — *Assume $\varrho(Q) \neq 0$. If the non-resonance condition is satisfied for each $\mathrm{Res}_{Y,j}(\varrho^{-1}\mathbb{D})$, we can take a $\mathbb{D}$-flat splitting compatible with ${}^{j}F$ $(j \in \Lambda)$.*

Proposition 3.3.12. — *Assume $\varrho(Q) \neq 0$. If $\mathcal{U}_Q$ is a sufficiently small neighbourhood of Q, any $\mathbb{D}_z$-flat splitting of $(\mathcal{E}, \mathcal{F}^Q)$ on $\mathcal{U}_Q \setminus \pi^{-1}(\mathcal{D}_z)$ can be extended to a $\mathbb{D}_z$-flat splitting on $\mathcal{U}_Q$.*

Proposition 3.3.13. — *If $\mathcal{D}_z$ is smooth and if $\mathcal{D}_Y = \varnothing$, we can find a $\mathbb{D}$-flat splitting of $\mathcal{F}^Q$.*

3.4. Some notation

3.4.1. We prepare some notation for the proof of the claims in Sections 3.2 and 3.3. Let $\mathrm{Sec}[\delta, \theta^{(0)}, \theta^{(1)}]$ denote the following closed sector in Δ^*:

$$\mathrm{Sec}[\delta, \theta^{(0)}, \theta^{(1)}] := \big\{ z \in \Delta^* \,\big|\, 0 < |z| \leqslant \delta,\ \theta^{(0)} \leqslant \arg(z) \leqslant \theta^{(1)} \big\}.$$

Let $\pi : \widetilde{\Delta}(0) \to \Delta$ denote the real blow up along 0. Let $\overline{\mathrm{Sec}}\big[\delta, \theta^{(0)}, \theta^{(1)}\big]$ denote the closure of $\mathrm{Sec}\big[\delta, \theta^{(0)}, \theta^{(1)}\big]$ in the real blow up $\widetilde{\Delta}(0)$ along 0:

$$\overline{\mathrm{Sec}}\big[\delta, \theta^{(0)}, \theta^{(1)}\big] := \big\{ (t, \theta) \,\big|\, 0 \leqslant t \leqslant \delta,\ \theta^{(0)} \leqslant \theta \leqslant \theta^{(1)} \big\}.$$

We put $\mathrm{Sec}\big[\theta^{(0)}, \theta^{(1)}\big] := \pi^{-1}(0) \cap \overline{\mathrm{Sec}}\big[\delta, \theta^{(0)}, \theta^{(1)}\big]$.

Let $X := \Delta^k \times Y$ and $D := \bigcup_{i=1}^{k} \{z_i = 0\}$. In this paper, a multi-sector (or sector, for simplicity) of $X \setminus D$ means a subset S of the following form

$$\prod_{j\in I} \mathrm{Sec}[\delta_j, \theta_j^{(0)}, \theta_j^{(1)}] \times U \subset (\Delta^*)^I \times \big((\Delta^*)^{I^c} \times Y\big), \tag{41}$$

where $I \subset \underline{k}$, $I^c := \underline{k} \setminus I$, $\mathrm{Sec}[\delta_j, \theta_j^{(0)}, \theta_j^{(1)}] \subset \widetilde{\Delta}_{z_j}(0)$, and U denotes a compact region in $(\Delta^*)^{I^c} \times Y$. (We admit the case $I = \varnothing$.) Let $\pi : \widetilde{X}(D) \to X$ denote the real blow up of X along D. The closure of S in $\widetilde{X}(D)$ is denoted by $\overline{S}$.

Notation 3.4.1. — Let $\mathcal{MS}(X \smallsetminus D)$ denote the set of multi-sectors in $X \smallsetminus D$. For a point $Q \in \pi^{-1}(D)$, let $\mathcal{MS}(Q, X \smallsetminus D)$ denote the set of multi-sectors S in $X \smallsetminus D$ such that Q is contained in the interior part of $\overline{S}$. □

Definition 3.4.2. — We say that we shrink $S = \prod_{j\in I} \mathrm{Sec}[\delta_j, \theta_j^{(0)}, \theta_j^{(1)}] \times U$ in the radius direction, when we replace it with $S' = \prod_{j\in I} \mathrm{Sec}[\delta'_j, \theta_j^{(0)}, \theta_j^{(1)}] \times U$ for some $\delta'_j \leqslant \delta_j$. □

3.4.2. We use the setting in Subsection 3.3. If $\mathcal{K}^0 \neq \varnothing$, we implicitly assume that $\mathcal{K}$ is a product $\Delta_\varrho \times \mathcal{K}'$ to consider a sector of $\mathcal{K}^* := \mathcal{K} \smallsetminus \mathcal{K}^0$. Because we are interested in local properties, this is not essential.

Let $\mathcal{I}$ be a weakly good set of irregular values at the level $(\boldsymbol{m}, i(0))$. Let $\mathrm{Sep}(\mathfrak{a}, \mathfrak{b})$ denote the subset $(F_{\mathfrak{a},\mathfrak{b}})^{-1}(0) \subset \widetilde{\mathcal{X}}(W)$, and let $\mathrm{Sep}(\mathcal{I})$ denote the union of $\mathrm{Sep}(\mathfrak{a}, \mathfrak{b})$ for pairs $(\mathfrak{a}, \mathfrak{b})$ of distinct elements of $\mathcal{I}$.

Notation 3.4.3. — Let $\mathcal{MS}(\mathcal{X} \smallsetminus W, \mathcal{I})$ denote the set of the multi-sectors S of $\mathcal{X} \smallsetminus W$ which is the product of

$$\mathrm{Sec}[\delta_{i(0)}, \theta_{i(0)}^{(0)}, \theta_{i(0)}^{(1)}] \subset \Delta^*_{z_{i(0)}}, \quad U_1 \subset \prod_{p\neq i(0)} \Delta^*_{z_p}, \quad U_2 \subset Y \times \mathcal{K}^*,$$

where U_1 (resp. U_2) is a compact region or a multi-sector of $\prod_{p\neq i(0)} \Delta^*_{z_p}$ (resp. $Y \times \mathcal{K}^*$). We assume $|m_{i(0)}| \cdot |\theta_{i(0)}^{(0)} - \theta_{i(0)}^{(1)}| < \pi$, and moreover, there exist $\theta'^{(0)}_{i(0)} < \theta'^{(1)}_{i(0)}$ in the open interval $]\theta_{i(0)}^{(0)}, \theta_{i(0)}^{(1)}[$ such that $\overline{S} \cap \mathrm{Sep}(\mathcal{I}) \subset \mathrm{Sec}[\delta_{i(0)}, \theta'^{(0)}_{i(0)}, \theta'^{(1)}_{i(0)}] \times U_1 \times U_2$. □

The following lemma is clear.

Lemma 3.4.4. — *We put $\mathcal{D}^0_{z,\underline{k}} := \mathcal{D}_{z,\underline{k}} \times_{\mathcal{K}} \mathcal{K}^0$.*

- *Let $Z_0 \subset \pi^{-1}(\mathcal{D}_{z,\underline{k}} \smallsetminus \mathcal{D}^0_{z,\underline{k}})$ be a product of $\prod_{p=1}^k \mathrm{Sec}[\theta_p^{(0)}, \theta_p^{(1)}]$ and a compact region U of $Y \times \mathcal{K}^*$ such that*

$$Z_0 \cap \mathrm{Sep}(\mathcal{I}) \subset \prod_{p\neq i(0)} \mathrm{Sec}[\theta_p^{(0)}, \theta_p^{(1)}] \times \mathrm{Sec}[\theta'^{(0)}_{i(0)}, \theta'^{(1)}_{i(0)}] \times U$$

for some $\theta'^{(0)}_{i(0)} < \theta'^{(1)}_{i(0)}$ in $]\theta_{i(0)}^{(0)}, \theta_{i(0)}^{(1)}[$, where $\mathrm{Sec}[\theta_p^{(0)}, \theta_p^{(1)}] \subset \widetilde{\Delta}_{z_p}(0)$. Take sufficiently small $\delta_p > 0$ $(p = 1, \dots, k)$, and put $S := \prod_{p=1}^k \mathrm{Sec}[\delta_p, \theta_p^{(0)}, \theta_p^{(1)}] \times U$. Then, $S \in \mathcal{MS}(\mathcal{X} \smallsetminus W, \mathcal{I})$.

- *Assume $\mathcal{K} = \Delta_\varrho \times \mathcal{K}'$. Let $Z_0 \subset \pi^{-1}(\mathcal{D}^0_{z,\underline{k}})$ be the product of $\mathrm{Sec}[\theta_\varrho^{(0)}, \theta_\varrho^{(1)}] \times \prod_{p=1}^k \mathrm{Sec}[\theta_p^{(0)}, \theta_p^{(1)}]$ and a compact region U of $Y \times \mathcal{K}'$ such that*

$$Z_0 \cap \mathrm{Sep}(\mathcal{I}) \subset \mathrm{Sec}[\theta_\varrho^{(0)}, \theta_\varrho^{(1)}] \times \prod_{p\neq i(0)} \mathrm{Sec}[\theta_p^{(0)}, \theta_p^{(1)}] \times \mathrm{Sec}[\theta'^{(0)}_{i(0)}, \theta'^{(1)}_{i(0)}] \times U$$

for some $\theta'^{(0)}_{i(0)} < \theta'^{(1)}_{i(0)}$ in $]\theta_{i(0)}^{(0)}, \theta_{i(0)}^{(1)}[$, where $\mathrm{Sec}[\theta_\varrho^{(0)}, \theta_\varrho^{(1)}] \subset \widetilde{\Delta}_\varrho(0)$. Take sufficiently small $\delta_\lambda > 0$ and $\delta_p > 0$ $(p = 1, \dots, k)$. We put $S := \mathrm{Sec}[\delta_\lambda, \theta_\lambda^{(0)}, \theta_\lambda^{(1)}] \times \prod_{p=1}^\ell \mathrm{Sec}[\delta_p, \theta_p^{(0)}, \theta_p^{(1)}] \times U$. Then, $S \in \mathcal{MS}(\mathcal{X} \smallsetminus W, \mathcal{I})$. □

Notation 3.4.5. — For $Q \in \pi^{-1}(\mathcal{D}_{z,\underline{k}})$, let $\mathcal{MS}(Q, \mathcal{X} \setminus W, \mathcal{I})$ denote the set of multi-sectors $S \in \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I}) \cap \mathcal{MS}(Q, \mathcal{X} \setminus W)$ such that $\overline{S} \in \mathfrak{U}(Q, \widetilde{\mathcal{X}}(W), \mathcal{I})$. □

We obtain the following lemma from Lemma 3.4.4, which will be used implicitly.

Lemma 3.4.6. — *Let Q be any point of $\pi^{-1}(\mathcal{D}_{z,\underline{k}})$. Let $\mathcal{U}$ be any neighbourhood of Q in $\widetilde{\mathcal{X}}(W)$. Then, there exists $S \in \mathcal{MS}(Q, \mathcal{X} \setminus W, \mathcal{I})$ such that $\overline{S} \subset \mathcal{U}$.* □

The following lemma is clear by the condition, which will be also used implicitly.

Lemma 3.4.7. — *For any $S \in \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I})$, there exists $S' \subset S$ such that (i) $S' \in \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I})$, (ii) $\overline{S}' \cap \pi^{-1}(\mathcal{D}_{z,j}) = \varnothing$ $(j \neq i(0))$ and $\overline{S}' \cap \pi^{-1}(\mathcal{X}^0) = \varnothing$, (iii) $\leqslant^{\varrho}_{S} = \leqslant^{\varrho}_{S'}$.* □

3.5. Preliminary in the smooth divisor case

In the following sections, π^*E and $\pi^*\mathcal{E}$ are denoted by E and $\mathcal{E}$, if there is no risk of confusion. We use the setting in Section 3.3 with $k = 1$ and $\mathcal{D}_Y = \varnothing$, i.e., $\mathcal{X} = \Delta_{z_1} \times Y \times \mathcal{K}$ and $\mathcal{D} = \{0\} \times Y \times \mathcal{K}$. We also assume that ϱ is nowhere vanishing on $\mathcal{K}$. Let $\pi : \widetilde{\mathcal{X}}(\mathcal{D}) \to \mathcal{X}$ denote the real blow up along $\mathcal{D}$. For a multi-sector S in $\mathcal{X} \setminus \mathcal{D}$, let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(\mathcal{D})$, and the intersection $\overline{S} \cap \pi^{-1}(\mathcal{D})$ is denoted by Z. In the following, $\leqslant^{\varrho}_{S}$ and $<^{\varrho}_{S}$ are denoted by $\leqslant_S$ and $<_S$, respectively.

3.5.1. Existence of a decomposition.

— Let $m \in \mathbb{Z}_{<0}$. Let $\mathcal{I}$ be a good set of irregular values on $(\mathcal{X}, \mathcal{D})$ at the level m. Let $(E, \mathbb{D}, \mathcal{I})$ be a good lattice of a meromorphic ϱ-flat bundle at the level m. We have the irregular decomposition at the level m:

$$(E, \mathbb{D})_{|\widehat{\mathcal{D}}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}). \tag{42}$$

Lemma 3.5.1. — *For any $S \in \mathcal{MS}\big(\mathcal{X} \setminus \mathcal{D}, \mathcal{I}\big)$, we have a decomposition of $E_{|\overline{S}}$*

$$E_{|\overline{S}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} E_{\mathfrak{a},S} \tag{43}$$

such that (i) flat with respect to ∂_{z_1}, (ii) the restriction to $\widehat{Z}$ is equal to the pull-back of the irregular decomposition (42) *on $\widehat{\mathcal{D}}$.*

Proof. — We closely follow the standard argument (Chapter 12 of [**94**], for example). We give only an outline. Let $\widehat{\boldsymbol{w}} = (\widehat{w}_i)$ be a frame of $E_{|\widehat{\mathcal{D}}}$ which is compatible with the decomposition (42). When $\widehat{w}_i \in \widehat{E}_{\mathfrak{a}}$, we put $\mathfrak{a}(i) := \mathfrak{a}$. Let $M_r(\boldsymbol{C})$ denote the space of r-th square matrices, where $r = \operatorname{rank} E$. Let $M_{\mathfrak{a},\mathfrak{b}}$ denote the set of $A \in M_r(\boldsymbol{C})$ such that $A_{i,j} = 0$ unless $\mathfrak{a}(i) = \mathfrak{a}$ and $\mathfrak{a}(j) = \mathfrak{b}$. Thus, we obtain the decomposition:

$$M_r(\boldsymbol{C}) = \bigoplus_{\mathfrak{a},\mathfrak{b}} M_{\mathfrak{a},\mathfrak{b}}. \tag{44}$$

We have the natural isomorphism $M_{\mathfrak{a},\mathfrak{a}} \simeq M_{\operatorname{rank} \widehat{E}_{\mathfrak{a}}}(\boldsymbol{C})$. The element corresponding to the identity matrix is denoted by $I_{\mathfrak{a}}$. Any element $A \in M_r(\boldsymbol{C})$ has the decomposition $A = \sum A_{\mathfrak{a},\mathfrak{b}}$ corresponding to (44). Let $M_r(\boldsymbol{C})^d$ (resp. $M_r(\boldsymbol{C})^r$) denote the subspace of $M_r(\boldsymbol{C})$ which consists of A such that $A_{\mathfrak{a},\mathfrak{b}} = 0$ unless (resp. if) $\mathfrak{a} = \mathfrak{b}$. Any element $A \in M_r(\boldsymbol{C})$ is decomposed into $A = A^d + A^r$, where $A^d \in M_r(\boldsymbol{C})^d$ and $A^r \in M_r(\boldsymbol{C})^r$. We have the corresponding decomposition for $M_r(\boldsymbol{C})$-valued functions.

By Proposition 3.1.10, we can take a holomorphic frame $\boldsymbol{w}$ of $\pi^* E_{|\overline{S}}$ such that $\boldsymbol{w}_{|\widehat{Z}} = \pi^{-1}(\widehat{\boldsymbol{w}})_{|\widehat{Z}}$. Let A be determined by $\mathbb{D}^f(z_1\partial_{z_1})\boldsymbol{w} = \boldsymbol{w}\cdot A$. By construction, A^d is of the form $\sum (m\varrho^{-1}\mathfrak{a})\, I_{\mathfrak{a}} + A_0^d$, where $|A_0^d| = O\big(|z_1|^{m+1}\big)$. We also have $A^r_{|\widehat{Z}} = 0$.

We consider the change of the frame of the form $\boldsymbol{w}' = \boldsymbol{w}\,(I + B)$ such that $\mathbb{D}^f(z_1\partial_{z_1})\boldsymbol{w}' = \boldsymbol{w}'\big(A^d + R\big)$, where $B = B^r$ and $R = R^d$ and $B_{|\widehat{Z}} = R_{|\widehat{Z}} = 0$. The condition is as follows:

$$R = (A^r\,B)^d, \quad A^r + A^d B + (A^r\,B)^r + z_1\partial_{z_1}B = B\,(A^d + R). \tag{45}$$

By eliminating R, we obtain the following equation for $M_r(\boldsymbol{C})^r$-valued function B:

$$z_1\partial_{z_1}B = B\,A^d - A^d B - (A^r B)^r - A^r + B\,(A^r B)^d. \tag{46}$$

Note $A^r = O(|z_1|^N)$ for any N. By changing the variable $x = z_1^{-1}$, we can apply Proposition 20.1.1 to (46). Recall that S is of the form $\operatorname{Sec}[\delta, \theta^{(0)}, \theta^{(1)}] \times U$, where U denotes a compact region in $Y \times \mathcal{K}$. We can find solutions B and R of (45) such that $B = O(|z_1|^N)$ and $R = O(|z_1|^N)$ for any N on $\operatorname{Sec}[\delta', \theta^{(0)}, \theta^{(1)}] \times U$. Since B and R are holomorphic, we also obtain $B_{|\widehat{Z}} = 0$ and $R_{|\widehat{Z}} = 0$. Hence, we obtain a decomposition like (43). We can extend it to a decomposition on $\operatorname{Sec}[\delta, \theta^{(0)}, \theta^{(1)}] \times U$ by using $\mathbb{D}^f$. Thus, we are done. □

3.5.2. Stokes filtration at the level m. — Let $S \in \mathcal{MS}(\mathcal{X} \setminus \mathcal{D}, \mathcal{I})$ and a decomposition $E_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a},S}$ as in Lemma 3.5.1. We define

$$\mathcal{F}^S_{\mathfrak{a}} := \bigoplus_{\mathfrak{b}\leqslant_S \mathfrak{a}} E_{\mathfrak{b},S}, \quad \mathcal{F}^S_{<\mathfrak{a}} := \sum_{\mathfrak{b}<_S\mathfrak{a}} \mathcal{F}^S_{\mathfrak{b}} = \bigoplus_{\mathfrak{b}<_S\mathfrak{a}} E_{\mathfrak{b},S}.$$

Lemma 3.5.2. — *$\mathcal{F}^S_{\mathfrak{a}}$ is independent of the choice of a splitting* (43), *and it is $\mathbb{D}$-flat.*

Proof. — Let $E_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E'_{\mathfrak{a},S}$ be another splitting. The inclusion $i_{\mathfrak{a}} : E_{\mathfrak{a},S} \subset E_{|\overline{S}}$ and the projection $p'_{\mathfrak{b}} : E_{|\overline{S}} \to E'_{\mathfrak{b},S}$ give ∂_{z_1}-flat morphisms $E_{\mathfrak{a},S} \to E'_{\mathfrak{b},S}$ for any $\mathfrak{a},\mathfrak{b}$. Let $f_{\mathfrak{a},\mathfrak{b}}$ denote the composite morphism. We have $f_{\mathfrak{a},\mathfrak{b}|\widehat{Z}} = 0$ for $\mathfrak{a} \neq \mathfrak{b}$ by construction, and $f_{\mathfrak{a},\mathfrak{a}|\widehat{Z}} = \mathrm{id}$. In particular, $f_{\mathfrak{a},\mathfrak{b}}$ are bounded. Hence, we obtain $f_{\mathfrak{a},\mathfrak{b}} = 0$ unless $\mathfrak{b} \leqslant_S \mathfrak{a}$ due to Corollary 20.3.7.

Let V be a holomorphic vector field in the Y-direction on U. We have the frames $\boldsymbol{v}_{\mathfrak{a}}$ of $E_{\mathfrak{a},S}$ on S. Let $p_{\mathfrak{a}}$ denote the projection $E_{|\overline{S}} \to E_{\mathfrak{a},S}$. Let $A_{\mathfrak{a},\mathfrak{b}}(V)$ ($\mathfrak{a} \neq \mathfrak{b}$) be determined by $\big(p_{\mathfrak{b}} \circ \mathbb{D}^f(V) \circ i_{\mathfrak{a}}\big)\boldsymbol{v}_{\mathfrak{a}} = \boldsymbol{v}_{\mathfrak{b}} \cdot A_{\mathfrak{a},\mathfrak{b}}(V)$. Then, $A_{\mathfrak{a},\mathfrak{b}}(V)_{|\widehat{Z}} = 0$ and ∂_{z_1}-flat. Thus, we obtain $A_{\mathfrak{a},\mathfrak{b}}(V) = 0$ unless $\mathfrak{b} \leqslant_S \mathfrak{a}$ due to Corollary 20.3.7. □

Lemma 3.5.3. — *The filtration $\mathcal{F}^S$ indexed by $(\mathcal{I}, \leqslant_S)$ is characterized by the following properties:*

- *$\mathcal{F}^S$ is flat with respect to ∂_{z_1}.*
- *$\mathcal{F}^S_{\mathfrak{a}|\widehat{Z}} = \bigoplus_{\mathfrak{b}\leqslant_S \mathfrak{a}} \widehat{E}_{\mathfrak{b}}$.*

We call $\mathcal{F}^S$ Stokes filtration at the level m.

Proof. — Let $\mathrm{Gr}_{\mathfrak{a}} := \mathcal{F}^S_{\mathfrak{a}}/\mathcal{F}^S_{<\mathfrak{a}}$, and let $r(\mathfrak{a}) := \operatorname{rank} \mathrm{Gr}_{\mathfrak{a}}$. For a frame $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ of $\mathrm{Gr}_{\mathfrak{a}}$ on $\overline{S}$, let $A_{\mathfrak{a}}$ be determined by $\mathbb{D}^f_{\mathfrak{a}}(\partial_{z_1})\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{a}} \cdot A_{\mathfrak{a}}$, where $\mathbb{D}^f_{\mathfrak{a}}$ denotes the induced family of the flat connections of $\mathrm{Gr}_{\mathfrak{a}}$. Then, $A_{\mathfrak{a}}$ is of the form $\varrho^{-1}\partial_{z_1}\mathfrak{a} + A^{\circ}_{\mathfrak{a}}$ where $A^{\circ}_{\mathfrak{a}}$ is the holomorphic section of $\varrho^{-1} M_{r(\mathfrak{a})}(\boldsymbol{C}) \otimes \mathcal{O}_{\mathcal{X}}(m\mathcal{D})$.

Let $\mathcal{F}'^S$ be a filtration of $E_{|\overline{S}}$ on $\overline{S}$, which has the above property. We set $\mathrm{Gr}'_{\mathfrak{a}} := \mathcal{F}'^S_{\mathfrak{a}}/\mathcal{F}'^S_{<\mathfrak{a}}$, which is equipped with the family of connections $\mathbb{D}'^f_{\mathfrak{a}}$ along the Δ_{z_1}-direction. For any frame $\overline{\boldsymbol{v}}'_{\mathfrak{a}}$ of $\mathrm{Gr}'_{\mathfrak{a}}$, let $A'_{\mathfrak{a}}$ be determined by $\mathbb{D}'^f_{\mathfrak{a}}(\partial_{z_1})\overline{\boldsymbol{v}}'_{\mathfrak{a}} = \overline{\boldsymbol{v}}'_{\mathfrak{a}}\cdot A'_{\mathfrak{a}}$, and then $A'_{\mathfrak{a}}$ is of a form similar to $A_{\mathfrak{a}}$.

We use an induction on the order $\leqslant_S$. We put $\mathcal{F}^S_{\mathcal{B}} := \sum_{\mathfrak{a}\in\mathcal{B}} \mathcal{F}^S_{\mathfrak{a}}$ for any subset $\mathcal{B} \subset \mathcal{I}$. Assume that we already know $\mathcal{F}^S_{\mathfrak{b}} = \mathcal{F}'^S_{\mathfrak{b}}$ for any $\mathfrak{b} <_S \mathfrak{a}$, and we will show $\mathcal{F}^S_{\mathfrak{a}} = \mathcal{F}'^S_{\mathfrak{a}}$. Let $\mathcal{B}$ be the subset of $\mathcal{I}$ such that $\mathcal{F}'^S_{\mathfrak{a}} \subset \mathcal{F}^S_{\mathcal{B}}$ and $\mathcal{F}'^S_{\mathfrak{a}} \not\subset \mathcal{F}^S_{\mathcal{B}'}$ for any $\mathcal{B}' \subsetneq \mathcal{B}$. Let $\mathfrak{c}$ be any maximal element of $\mathcal{B}$. Then, we obtain the flat morphism $\phi_{\mathfrak{a},\mathfrak{c}} : \mathrm{Gr}'_{\mathfrak{a}} \to \mathrm{Gr}_{\mathfrak{c}}$. Due to Corollary 20.3.7, we obtain $\phi_{\mathfrak{a},\mathfrak{c}} = 0$ unless $\mathfrak{a} \geqslant_S \mathfrak{c}$. Therefore, we obtain $\mathcal{F}'^S_{\mathfrak{a}} \subset \mathcal{F}^S_{\mathcal{B}} \subset \mathcal{F}^S_{\mathfrak{a}}$. By comparison of the ranks, we obtain $\mathcal{F}^S_{\mathfrak{a}} = \mathcal{F}'^S_{\mathfrak{a}}$. Thus, we are done. □

3.6. Proof of the statements in Section 3.3

We use the setting in Subsection 3.3. Since we are interested in a local theory, we put $Y := \Delta^n_{\zeta}$, $\mathcal{D}_{Y,j} := \{\zeta_j = 0\}$ and $\mathcal{D}_Y := \bigcup_{j=1}^{\ell} \mathcal{D}_{Y,j}$ for some $1 \leqslant \ell \leqslant n$, although it does not matter until Subsection 3.6.3. We put $\mathfrak{Z} := \pi^{-1}(\mathcal{D}_{z,\underline{k}})$. In this section, P denotes a point in the real blow up. For any multi-sector S in $\mathcal{X} \smallsetminus W$, let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(W)$, and let Z denote $\overline{S} \cap \pi^{-1}(W)$. The orders $\leqslant^{\varrho}_S$ and $<^{\varrho}_S$ are denoted by $\leqslant_S$ and $<_S$, respectively.

3.6.1. Existence of the Stokes filtration. — Let $\boldsymbol{m} \in \mathbb{Z}^k_{<0}$. Let $(E, \mathbb{D}, \mathcal{I})$ be a good lattice of a meromorphic ϱ-flat bundle at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$. The irregular decomposition (35) induces the following on $\widehat{Z}$:

$$(E, \mathbb{D})_{|\widehat{Z}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})_{|\widehat{Z}}. \tag{47}$$

We put $\mathcal{F}^Z_{\mathfrak{a}} := \bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}} \widehat{E}_{\mathfrak{b}|\widehat{Z}}$, and thus we obtain a filtration $\mathcal{F}^Z$ indexed by $(\mathcal{I}, \leqslant_S)$. Let $\mathbb{D}_z$ denote the restriction of $\mathbb{D}$ to the Δ^k_z-direction.

Proposition 3.6.1. — *Take $S \in \mathcal{MS}(\mathcal{X} \smallsetminus W, \mathcal{I})$ such that $\overline{S} \cap \mathfrak{Z} \neq \varnothing$. If we sufficiently shrink S in the radius direction, the following holds:*

- *There exists a unique* $\mathbb{D}$*-flat filtration* $\mathcal{F}^S$ *of* $E_{|\overline{S}}$ *indexed by* $(\mathcal{I}, \leqslant_S)$ *such that* $\mathcal{F}^S_{|\widehat{Z}} = \mathcal{F}^Z$*. Moreover, if a* $\mathbb{D}_z$*-flat filtration* $\mathcal{F}'^S$ *of* $E_{|\overline{S}}$ *indexed by* $(\mathcal{I}, \leqslant_S)$ *satisfies* $\mathcal{F}'^S_{|\widehat{Z}} = \mathcal{F}^Z$*, then* $\mathcal{F}'^S = \mathcal{F}^S$*.*
- *There exists a* $\mathbb{D}_z$*-flat splitting of the filtration* $\mathcal{F}^S$ *on* $\overline{S}$*. Note that the restriction of such a splitting to* $\widehat{Z}$ *is equal to* (47)*.*
- *If* ϱ *is nowhere vanishing, any* $\mathbb{D}_z$*-flat splitting of* $\mathcal{F}^S$ *on* S *can be extended to a splitting on* $\overline{S}$*.*

We call $\mathcal{F}^S$ *the Stokes filtration of* $(E, \mathbb{D})$ *on* S *at the level* $\boldsymbol{m}$*.*

Proof. — We may assume $i(0) = 1$. We show the following lemma analogous to Lemma 3.5.3.

Lemma 3.6.2. — *If we sufficiently shrink* S *in the radius direction, there exists a decomposition on* $\overline{S}$

$$E_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} {}^1E_{\mathfrak{a},S} \tag{48}$$

such that (i) it is flat with respect to ∂_{z_1}*, (ii) its restriction to* $\widehat{Z}$ *is the same as* (47)*, where* $Z := \overline{S} \cap \pi^{-1}(W)$*.*

Proof. — We closely follow the standard argument as in the proof of Lemma 3.5.1. Hence, we give only an outline. We will shrink S in the radius direction without mention. We take a frame $\widehat{\boldsymbol{w}} = (\widehat{\boldsymbol{w}}_{\mathfrak{a}})$ of $E_{|\widehat{W}}$ compatible with the irregular decomposition (35) at the level $\boldsymbol{m}$. We use the decomposition of matrices as in the proof of Lemma 3.5.1.

By Proposition 3.1.10, we can take a holomorphic frame $\boldsymbol{w}$ of $\pi^* E_{|\overline{S}}$ such that $\boldsymbol{w}_{|\widehat{Z}} = \pi^{-1}(\widehat{\boldsymbol{w}})$. Let A be determined by $\mathbb{D}^f(z_1\partial_1)\boldsymbol{w} = \boldsymbol{w}\, A$. By construction, A^d is of the form $\sum_{\mathfrak{a}}(\varrho^{-1}z_1\partial_1\mathfrak{a})\, I_{\mathfrak{a}} + A^d_0$, where $|A^d_0| = O(|\boldsymbol{z}^{\boldsymbol{m}(1)}|)$. We also have $A^r_{|\widehat{Z}} = 0$.

We consider a change of frames of the form $\boldsymbol{w}' = \boldsymbol{w}\,(I+B)$ such that $\mathbb{D}^f(z_1\partial_1)\boldsymbol{w}' = \boldsymbol{w}'(A^d + R)$, where $B = B^r$, $R = R^d$, and $B_{|\widehat{Z}} = R_{|\widehat{Z}} = 0$. Then, we obtain the equation (45) for B and R, and (46) for B by eliminating R.

We use the change of variables $x = z_1^{-1}$ and $y_i = z_{i+1}^{-1}$ $(i = 1, \dots, k-1)$, and $y_k = \varrho^{-1}$ if $\mathcal{K}^0 \neq \varnothing$. By applying Proposition 20.1.1 to (46), we can find solutions B and R of (45) such that $B = O\Big(\prod_{i=1}^k |z_i|^N |\varrho|^N\Big)$ and $R = O\Big(\prod_{i=1}^k |z_i|^N |\varrho|^N\Big)$ for any N. Since B and R are holomorphic, we also obtain $B_{|\widehat{Z}} = 0$ and $R_{|\widehat{Z}} = 0$. Thus, the proof of Lemma 3.6.2 is finished. □

Let S be as in Lemma 3.6.2. We put $\mathcal{F}^S_{\mathfrak{a}} := \bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}} {}^1E_{\mathfrak{a},S}$.

Lemma 3.6.3. — *They are independent of the choice of the decomposition* (48)*, and they are* $\mathbb{D}_z$*-flat.*

Proof. — We can take a multi-sector $S' \subset S$ such that (i) $S' \in \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I})$, (ii) $\overline{S}' \cap \pi^{-1}(\mathcal{D}_{z,j}) = \varnothing$ for $j \neq i(0)$ and $\overline{S}' \cap \pi^{-1}(\mathcal{X}^0) = \varnothing$, (iii) $\leqslant_{S'} \; = \; \leqslant_S$. Then, we can show the claim of the lemma by considering the restriction to $\overline{S}'$ and by using Lemma 3.5.2. □

Let us consider the second condition. Let $\mathrm{Gr}_{\mathfrak{a}}$ denote the induced flat bundle on $\overline{S}$. The frame $\boldsymbol{w}'$ in the proof of Lemma 3.6.2 induces a frame $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ of $\mathrm{Gr}_{\mathfrak{a}}$. Let $B_{\mathfrak{a}}$ be determined by $\mathbb{D}_{z,\mathfrak{a}}\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{a}}\bigl(d_z\mathfrak{a} + B_{\mathfrak{a}}\bigr)$. By shrinking S in the radius direction, we may assume that $|B_{\mathfrak{b}}|$ and $|B_{\mathfrak{c}}|$ are sufficiently smaller than $\bigl|\mathrm{Re}\bigl(\varrho^{-1}(\mathfrak{b}-\mathfrak{c})\bigr)\bigr|$ on S for any $\mathfrak{b} >_S \mathfrak{c}$. Let us consider the following claim:

$A(\mathfrak{a})$: There exists a $\mathbb{D}_z$-flat splitting $\mathcal{F}^S_{\mathfrak{a}} = \bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}} E_{\mathfrak{b},S}$ on $\overline{S}$.

We have a similar claim $A(<_S \mathfrak{a})$ on the existence of a $\mathbb{D}_z$-flat splitting of $\mathcal{F}^S_{<\mathfrak{a}}$.

We show $A(\mathfrak{a})$ by induction on the order $\leqslant_S$. If $\mathfrak{a}$ is minimal, the claim $A(\mathfrak{a})$ is clear. The claim $A(<_S \mathfrak{a})$ follows from $A(\mathfrak{b})$ for any $\mathfrak{b} <_S \mathfrak{a}$. Let us show $A(\mathfrak{a})$ by assuming $A(<_S \mathfrak{a})$. Let $f_{\mathfrak{a}}$ be the morphism $\mathrm{Gr}_{\mathfrak{a}} \simeq {}^1E_{\mathfrak{a},S} \subset \mathcal{F}^S_{\mathfrak{a}}$. Then, we obtain the following morphism

$$\mathbb{D}_z(f_{\mathfrak{a}}) : \mathrm{Gr}_{\mathfrak{a}} \longrightarrow \mathcal{F}^S_{<\mathfrak{a}} \otimes \Omega^1_z = \bigoplus_{\mathfrak{b}<\mathfrak{a}} E_{\mathfrak{b},S} \otimes \Omega^1_z.$$

We have $\mathbb{D}_z(f_{\mathfrak{a}})_{|\widehat{Z}} = 0$ by construction of $\mathcal{F}^S$.

Note that S is the product of a multi-sector $S_{\boldsymbol{z}} \subset (\Delta^*)^k$ and $U \subset Y \times \mathcal{K}^*$ which is a sector or a compact region. The closure of $S_{\boldsymbol{z}}$ in Δ^k contains 0. Take a point Q of $S_{\boldsymbol{z}}$. We take $g_{\mathfrak{a},\mathfrak{b}} : \mathrm{Gr}_{\mathfrak{a}} \to E_{\mathfrak{b},S}$ such that $\mathbb{D}_z(g_{\mathfrak{a},\mathfrak{b}}) = (\mathbb{D}_z f_{\mathfrak{a}})_{\mathfrak{b}}$, and $g_{\mathfrak{a},\mathfrak{b}|Q\times U} = 0$. By sufficiently shrinking S in the radius direction, we can apply Lemma 20.3.1 in this situation, and we obtain $g_{\mathfrak{a},\mathfrak{b}|\widehat{Z}} = 0$. We put $\overline{f}_{\mathfrak{a}} := f_{\mathfrak{a}} - \sum_{\mathfrak{b}<\mathfrak{a}} g_{\mathfrak{a},\mathfrak{b}}$, then $\overline{f}_{\mathfrak{a}} : \mathrm{Gr}_{\mathfrak{a}} \to \mathcal{F}^S_{\mathfrak{a}}$ is $\mathbb{D}_z$-flat and $\overline{f}_{\mathfrak{a}}(\overline{\boldsymbol{v}}_{\mathfrak{a}})_{|\widehat{Z}} = \widehat{\boldsymbol{v}}_{\mathfrak{a}|\widehat{Z}}$. Thus, we obtain $A(\mathfrak{a})$, and the second condition in Proposition 3.6.1 is satisfied.

Assume ϱ is nowhere vanishing. If $\mathfrak{b} >_S \mathfrak{c}$, any $\mathbb{D}_z$-flat morphism $\mathrm{Gr}^{\mathcal{F}}_{\mathfrak{b}}(E_{|S}) \to \mathrm{Gr}^{\mathcal{F}}_{\mathfrak{c}}(E_{|S})$ has the order $O\bigl(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\bigr)$ due to Corollary 20.3.6. Then, the third condition in Proposition 3.6.1 is satisfied. □

Let $\mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$ denote the set of $S \in \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I})$ such that $E_{|\overline{S}}$ has a $\mathbb{D}$-flat filtration $\mathcal{F}^S$ as in Proposition 3.6.1. If ϱ is invertible and $\mathcal{D}$ is smooth, we have $\mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I}) = \mathcal{MS}(\mathcal{X} \setminus W, \mathcal{I})$.

Corollary 3.6.4. — *For any point $P \in \mathfrak{Z}$, there exists $\mathcal{U}_P \in \mathfrak{U}\bigl(P, \widetilde{\mathcal{X}}(W), \mathcal{I}\bigr)$ such that, for any $S \in \mathcal{MS}(P, \mathcal{X} \setminus W, \mathcal{I})$ with $\overline{S} \subset \mathcal{U}_P$, there exists a unique $\mathbb{D}$-flat filtration $\mathcal{F}^S$ of $E_{|\overline{S}}$ indexed by $(\mathcal{I}, \leqslant_S)$ satisfying the conditions in Proposition* 3.6.1.

Let $\mathcal{MS}^*(P, \mathcal{X} \setminus W, \mathcal{I})$ denote the set of such multi-sectors. □

Remark 3.6.5. — Even if $(E, \mathbb{D})$ is a weakly good lattice at the level $(\boldsymbol{m}, i(0))$, $(\mathrm{End}(E), \mathbb{D})$ is not necessarily a weakly good lattice at the level $(\boldsymbol{m}, i(0))$. We may not have a Stokes filtration of $\mathrm{End}(E)_{|\overline{S}}$ for $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$. However, as remarked in Section 3.1.1.3, the $\mathbb{D}$-flat subbundles $\mathcal{F}_0 \operatorname{End}(E)_{|\overline{S}}$ and $\mathcal{F}_{<0} \operatorname{End}(E)_{|\overline{S}}$ are well defined, and this will be implicitly used. □

3.6.2. Compatibility

Lemma 3.6.6. — *Let $S, S' \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$ such that $S' \subset S$.*

- *The filtrations $\mathcal{F}^S$ and $\mathcal{F}^{S'}$ are compatible over $(\mathcal{I}, \leqslant_S) \to (\mathcal{I}, \leqslant_{S'})$ in the sense of Definition* 3.1.2. *In particular, we have $\mathcal{F}^S = \mathcal{F}^{S'}$ if $(\mathcal{I}, \leqslant_S)$ is isomorphic to $(\mathcal{I}, \leqslant_{S'})$.*
- *The decomposition $E_{|\overline{S}'} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a},S|\overline{S}'}$ gives a $\mathbb{D}_z$-flat splitting of the filtration $\mathcal{F}^{S'}$.*

Proof. — It follows from the characterization of the Stokes filtrations (Proposition 3.6.1). □

3.6.3. Splitting with nice property. — We have the induced ϱ-flat connection ${}^j\mathbb{D}$ of $E_{|\mathcal{D}_{Y,j}}$. Since $\mathcal{F}^S$ is $\mathbb{D}$-flat, $\operatorname{Res}_{Y,j}(\mathbb{D})$ and ${}^j\mathbb{D}$ preserve the filtration $\mathcal{F}^S_{|\mathcal{D}_{Y,j}}$. Assume that we are given filtrations ${}^jF$ $(j = 1, \ldots, \ell)$ of $E_{|\mathcal{D}_{Y,j}}$ which are preserved by $\operatorname{Res}_{Y,j}(\mathbb{D})$ and ${}^j\mathbb{D}$.

Proposition 3.6.7. — *Let $S \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$. We have a $\mathbb{D}_z$-flat splitting of the filtration $\mathcal{F}^S$, whose restriction to $\overline{S} \cap \mathcal{D}_{Y,j}$ is compatible with the residues* $\operatorname{Res}_{Y,j}(\mathbb{D})$ *and the filtrations ${}^jF$ for $j = 1, \ldots, \ell$, after we shrink S in the radius direction appropriately.*

Proof. — Take a large number N. Let $\widehat{W}^{(N)}$ denote the N-th infinitesimal neighbourhood of W. By Lemma 3.6.30 below, we can take a decomposition $E = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a},N}$ such that (i) it is the same as the irregular decomposition at the level $\boldsymbol{m}$ on $\widehat{W}^{(N)}$, (ii) the restriction to $\mathcal{D}_{Y,j}$ is compatible with $\operatorname{Res}_j(\mathbb{D})$ and ${}^jF$ for each $j = 1, \ldots, \ell$.

Recall that S is the product of a multi-sector $S_{\boldsymbol{z}} \subset (\Delta^*)^k$ and $S_0 \subset Y \times \mathcal{K}^*$, where S_0 is a sector or a compact region. The closure of $S_{\boldsymbol{z}}$ in Δ^k_z contains the origin O_z. Let $\overline{S}_0$ denote the closure of S_0 in the real blow up of $Y \times \mathcal{K}$ along $Y \times \mathcal{K}^0$. Let Q be any point of $S_{\boldsymbol{z}}$. We have the following morphisms:

$$\mathcal{F}^S_{\mathfrak{a}|Q\times\overline{S}_0} \longrightarrow E_{|Q\times\overline{S}_0} \longrightarrow \bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}} E_{\mathfrak{b},N|Q\times\overline{S}_0}. \tag{49}$$

If Q is sufficiently close to O_z, the composite of the morphisms in (49) is an isomorphism. Let $\mathcal{G}^Q_{\mathfrak{a}} \subset \mathcal{F}^S_{\mathfrak{a}|Q\times\overline{S}_0}$ denote the inverse image of $E_{\mathfrak{a},N|Q\times\overline{S}_0}$. We may assume $\mathcal{F}^S_{<\mathfrak{a}|Q\times\overline{S}_0} \cap \mathcal{G}^Q_{\mathfrak{a}} = \{0\}$. Then, $\mathcal{G}^Q_{\mathfrak{a}}$ can be extended to a $\mathbb{D}_z$-flat subbundle $\mathcal{G}^S_{\mathfrak{a}}$ of $E_{|S}$. If S is shrunk in the radius direction appropriately, they give subbundles of $E_{|\overline{S}}$, due to Corollary 20.3.9. By construction, it gives a splitting with the desired property. □

3.6.3.1. Special case 1. — We consider a $\mathbb{D}$-flat splitting in the non-resonant case. We assume that ϱ is nowhere vanishing. Let O_ζ denote the origin in $Y = \Delta_\zeta^n$. Take $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$ of the form $S_1 \times U_1 \times U_2$ where S_1 is a multi-sector in $(\Delta^*)^k$ whose closure contains the origin, U_1 is a neighbourhood of O_ζ in Y, and U_2 is a small compact region in $\mathcal{K}$.

Proposition 3.6.8. — *Assume $\alpha - \beta \notin \mathbb{Z}$ for distinct eigenvalues α, β of $\mathrm{Res}_{Y,j}(\mathbb{D}^f)_{|\mathcal{D}_{Y,j}}$ ($j = 1, \dots, \ell$, $y \in U_2$). Then, we have a $\mathbb{D}$-flat splitting of $\mathcal{F}^S$, whose restriction to $S \cap \mathcal{D}_{Y,j}$ is compatible with ${}^jF$ for each $j = 1, \dots, \ell$.*

Proof. — Let Q be as in the proof of Proposition 3.6.7, where we construct the $\mathbb{D}_z$-flat splitting of the filtration $\mathcal{F}^S$ on $Q \times U_1 \times U_2$. In particular, we have the splitting

$$E_{|Q\times O_\zeta \times U_2} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a}}^{Q\times O_\zeta\times U_2}, \tag{50}$$

which is compatible with the endomorphisms $\mathrm{Res}_j(\mathbb{D})$ and the filtrations ${}^jF$ ($j = 1, \dots, \ell$). By the assumption on the eigenvalues of $\mathrm{Res}_j(\mathbb{D})$, (50) can be extended to a $\mathbb{D}$-flat splitting of $\mathcal{F}^S$ on $Q \times U_1 \times U_2$. By extending it to a flat splitting of $\mathcal{F}^S$ on S, we obtain the desired splitting. □

3.6.3.2. Special case 2. — We consider the case $\mathcal{D}$ is smooth, i.e., $k = 1$ and $\mathcal{D}_Y = \varnothing$.

Lemma 3.6.9. — *For any $P \in \mathfrak{Z}$, there exist $S_P \in \mathcal{MS}^*(P, \mathcal{X} \setminus W, \mathcal{I})$ and a $\mathbb{D}$-flat splitting of the filtration $\mathcal{F}^{S_P}$ on $\overline{S}_P$.*

Proof. — According to Proposition 3.6.8, we only have to consider the case $\varrho(P) = 0$. We use the symbol m instead of $\boldsymbol{m}$. We would like to take a $\mathbb{D}$-flat morphism $\phi_{\mathfrak{a}}$: $\mathrm{Gr}_{\mathfrak{a}}(E_{|\overline{S}}) \to E_{|\overline{S}}$ for some $S \in \mathcal{MS}^*(P, \mathcal{X} \setminus W, \mathcal{I})$. We construct such a morphism inductively with respect to the order $\leqslant_P$. If $\mathfrak{a}$ is minimal with respect to $\leqslant_P$, we have nothing to do. Assume that we have already taken such morphisms for any $\mathfrak{b} <_S \mathfrak{a}$.

We have a $\mathbb{D}_z$-flat splitting $E_{|\overline{S}} = \bigoplus E_{\mathfrak{c},S}$ of the filtration $\mathcal{F}^S$. By the inductive assumption, we may assume that $\mathcal{F}^S_{<\mathfrak{a}}(E_{|\overline{S}}) = \bigoplus_{\mathfrak{b}<_P\mathfrak{a}} E_{\mathfrak{b}}$ is $\mathbb{D}$-flat. Let $f_{\mathfrak{a}}$ be the morphism $\mathrm{Gr}_{\mathfrak{a}}(E_{|\overline{S}}) \simeq E_{\mathfrak{a},S} \to E_{|\overline{S}} \simeq \bigoplus \mathrm{Gr}_{\mathfrak{b}}(E_{|\overline{S}})$. We remark dz_1-component of $\mathbb{D}(f_{\mathfrak{a}})$ is 0. We have the decomposition $\mathbb{D}(f_{\mathfrak{a}}) = \sum_{\mathfrak{b}<_S\mathfrak{a}} \mathbb{D}(f_{\mathfrak{a}})_{\mathfrak{b}}$ corresponding to the $\mathbb{D}$-flat decomposition $\mathcal{F}^S_{<\mathfrak{a}} = \bigoplus_{\mathfrak{b}<_S\mathfrak{a}} E_{\mathfrak{b},S}$:

$$\mathbb{D}(f_{\mathfrak{a}})_{\mathfrak{b}} : \mathrm{Gr}_{\mathfrak{a}}(E_{|\overline{S}}) \longrightarrow \mathrm{Gr}_{\mathfrak{b}}(E_{|\overline{S}}) \otimes \Omega^1_{\mathcal{X}/\mathcal{K}} \otimes \mathcal{O}(m\,D).$$

Let $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ be a holomorphic frame of $\mathrm{Gr}_{\mathfrak{a}}(E_{|\overline{S}})$ for each $\mathfrak{a} \in \mathcal{I}$. Let $R_{\mathfrak{a}}$ be determined by $\mathbb{D}_{\mathfrak{a}}\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{a}}\,(d\mathfrak{a} + R_{\mathfrak{a}})$. We have $R_{\mathfrak{a}} = O\big(|z_1^{m+1}|\big)$. Let A be determined by $\mathbb{D}(f_{\mathfrak{a}})_{\mathfrak{b}}\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{b}}\,A$. Since $\mathbb{D}(f_{\mathfrak{a}})_{\mathfrak{b}}$ is $\mathbb{D}_z$-flat, we have the following estimate for some $C > 0$:

$$A\,\exp\big(\varrho^{-1}(\mathfrak{b} - \mathfrak{a})\big) = \begin{cases} O\big(\exp(C\,|\varrho^{-1}\,z_1^{m+1}|)\big) & (m < -1) \\[2ex] O\big(\exp(C\,|\varrho^{-1}|\,\log|z_1^{-1}|)\big) & (m = -1) \end{cases}$$

By shrinking S, we obtain the estimate $A = O\big(\exp(-\varepsilon|\varrho^{-1} z_1^m|)\big)$ for some $\varepsilon > 0$. Let $P_1 := \pi(P) \in \mathcal{D}^0$. Recall that we have assumed $\mathcal{K} = \Delta_\varrho^1 \times \mathcal{K}'$, which induces $\mathcal{K} \to \mathcal{K}^0$ and hence $q : \mathcal{X} \to \mathcal{D}^0$. If we shrink S, we can take a section $g_{\mathfrak{a},\mathfrak{b}} : \mathrm{Gr}_{\mathfrak{a}}(E_{|S}) \to \mathrm{Gr}_{\mathfrak{b}}(E_{|S})$ satisfying the following conditions:

- $\mathbb{D}(g_{\mathfrak{a},\mathfrak{b}}) = \mathbb{D}(f_{\mathfrak{a}})_{\mathfrak{b}}$ and $g_{\mathfrak{a},\mathfrak{b}|q^{-1}(P_1)\cap S} = 0$.
- $g_{\mathfrak{a},\mathfrak{b}} = O\big(\exp(-\varepsilon|\varrho^{-1} z_1^m|/2)\big)$.

We put $\phi_{\mathfrak{a}} := f_{\mathfrak{a}} - \sum g_{\mathfrak{a},\mathfrak{b}}$. Because of the estimate for $g_{\mathfrak{a},\mathfrak{b}}$, the $\mathbb{D}$-flat morphism $\phi_{\mathfrak{a}}$ can be extended on $\overline{S}$. Thus, the inductive argument can proceed. □

3.6.4. Functoriality

3.6.4.1. Dual. — Let $(E, \mathbb{D}, \mathcal{I})$ be a weakly good lattice at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$. Let $S \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$. Let $E_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E_{\mathfrak{a},S}$ be a $\mathbb{D}_z$-flat splitting of the filtration $\mathcal{F}^S(E_{|\overline{S}})$ whose restriction to $\mathcal{D}_{Y,j}$ is compatible with $\mathrm{Res}_{Y,j}(\mathbb{D})$ for each $j = 1, \dots, \ell$. For any $\mathfrak{a} \in \mathcal{I}$, we put

$$E^\vee_{-\mathfrak{a},S} := \Big(\bigoplus_{\substack{\mathfrak{b}\in\mathcal{I}\\ \mathfrak{b}\neq\mathfrak{a}}} E_{\mathfrak{b},S}\Big)^\perp, \qquad \mathcal{F}^S_{\not\geq\mathfrak{a}}(E_{|\overline{S}}) := \sum_{\substack{\mathfrak{b}\in\mathcal{I}\\ \mathfrak{b}\not\geq_S\mathfrak{a}}} \mathcal{F}^S_{\mathfrak{b}}(E_{|\overline{S}}). \tag{51}$$

Lemma 3.6.10

- *The decomposition* $E^\vee_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E^\vee_{-\mathfrak{a},S}$ *gives a* $\mathbb{D}_z^\vee$*-flat splitting of the Stokes filtration* $\mathcal{F}^S(E^\vee_{|\overline{S}})$*, whose restriction to* $\mathcal{D}_{Y,j}$ *is compatible with* $\mathrm{Res}_{Y,j}(\mathbb{D}^\vee)$ *for each* $j = 1, \dots, \ell$.
- *In particular,* $\mathcal{F}^S_{-\mathfrak{a}}(E^\vee_{|\overline{S}}) = \mathcal{F}^S_{\not\geq\mathfrak{a}}(E_{|\overline{S}})^\perp$ *for any* $\mathfrak{a} \in \mathcal{I}$.

Proof. — The second claim follows from the first claim. We put

$$\mathcal{F}'^S_{-\mathfrak{a}}(E^\vee_{|\overline{S}}) := \bigoplus_{\substack{\mathfrak{b}\in\mathcal{I}\\ -\mathfrak{b}\leqslant_S-\mathfrak{a}}} E^\vee_{-\mathfrak{b},S}.$$

For the first claim, we only have to show $\mathcal{F}'^S_{-\mathfrak{a}}(E^\vee_{|\overline{S}}) = \mathcal{F}^S_{-\mathfrak{a}}(E^\vee_{|\overline{S}})$. It is easy to check $\mathcal{F}^S_{-\mathfrak{a}}(E^\vee_{|\overline{S}})_{|\widehat{Z}} = \mathcal{F}^Z_{-\mathfrak{a}}(E^\vee_{|\widehat{Z}})$. Then, the first claim follows from the uniqueness in Proposition 3.6.1. □

3.6.4.2. Tensor product and direct sum. — Let $(E_p, \mathbb{D}_p, \mathcal{I}_p)$ $(p = 1, 2)$ be weakly good lattices at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$. Let us consider the case where $\mathcal{I}_1 \otimes \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$. We put $(\widetilde{E}, \widetilde{\mathbb{D}}) := (E_1, \mathbb{D}_1) \otimes (E_2, \mathbb{D}_2)$. For a multisector $S \in \bigcap_{p=1,2} \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I}_p)$, we take $\mathbb{D}_z$-flat splittings $E_{p|\overline{S}} = \bigoplus_{\mathfrak{a}_p\in\mathcal{I}_p} E_{p,\mathfrak{a}_p,S}$ whose restrictions to $\mathcal{D}_{Y,j}$ are compatible with $\mathrm{Res}_{Y,j}(\mathbb{D}_p)$. We put

$$\widetilde{E}_{\mathfrak{a},S} := \bigoplus_{\substack{(\mathfrak{a}_1,\mathfrak{a}_2)\in\mathcal{I}_1\times\mathcal{I}_2\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}} E_{1,\mathfrak{a}_1,S} \otimes E_{2,\mathfrak{a}_2,S}. \tag{52}$$

The following lemma can be shown by the argument used in the proof of Lemma 3.6.10.

Lemma 3.6.11

- *The decomposition* $\widetilde{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}_1\otimes\mathcal{I}_2} \widetilde{E}_{\mathfrak{a},S}$ *gives a* $\mathbb{D}_z$*-flat splitting of the Stokes filtration of* $\widetilde{E}_{|\overline{S}}$*, whose restriction to* $\mathcal{D}_{Y,j}$ *is compatible with* $\mathrm{Res}_{Y,j}(\widetilde{\mathbb{D}})$ *for each* $j = 1, \dots, \ell$.
- *In particular,* $\mathcal{F}^S_{\mathfrak{a}}(\widetilde{E}_{|\overline{S}})$ *is equal to* $\sum_{\mathfrak{a}_1+\mathfrak{a}_2\leqslant_S \mathfrak{a}} \mathcal{F}^S_{\mathfrak{a}_1}(E_{1|\overline{S}}) \otimes \mathcal{F}^S_{\mathfrak{a}_2}(E_{2|\overline{S}})$. □

Let us consider the case where $\mathcal{I}_1 \oplus \mathcal{I}_2 := \mathcal{I}_1 \cup \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$. For a multi-sector $S \in \bigcap_{p=1,2} \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I}_p)$, we take $\mathbb{D}_z$-flat splittings of $E_{p|\overline{S}} = \bigoplus E_{p,\mathfrak{a},S}$ whose restriction to $\mathcal{D}_{Y,j}$ is compatible with $\mathrm{Res}_{Y,j}(\mathbb{D})$. We put

$$(E_1 \oplus E_2)_{\mathfrak{a},S} := E_{1,\mathfrak{a},S} \oplus E_{2,\mathfrak{a},S}. \tag{53}$$

The following lemma is obvious.

Lemma 3.6.12

- *The decomposition* $(E_1 \oplus E_2)_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}_1\oplus\mathcal{I}_2} (E_1 \oplus E_2)_{\mathfrak{a},S}$ *gives a* $\mathbb{D}_z$*-flat splitting of the Stokes filtration of* $(E_1 \oplus E_2)_{|\overline{S}}$*, whose restriction to* $\mathcal{D}_{Y,j}$ *is compatible with* $\mathrm{Res}_{Y,j}(\mathbb{D})$.
- *In particular,* $\mathcal{F}^S_{\mathfrak{a}}\big((E_1 \oplus E_2)_{|\overline{S}}\big)$ *is equal to* $\mathcal{F}^S_{\mathfrak{a}}(E_{1|\overline{S}}) \oplus \mathcal{F}^S_{\mathfrak{a}}(E_{2|\overline{S}})$. □

3.6.4.3. Morphism. — Let $(E_p, \mathbb{D}_p, \mathcal{I}_p)$ $(p = 1, 2)$ be as above. Let $F : (E_1, \mathbb{D}_1) \to (E_2, \mathbb{D}_2)$ be a flat morphism. We assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$.

Lemma 3.6.13. — *Let* $S \in \bigcap_{p=1,2} \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I}_p)$*. The restriction* $F_{|\overline{S}}$ *preserves the Stokes filtrations.*

Proof. — Let $\widehat{\boldsymbol{v}}_p$ be frames of $E_{p|\widehat{W}}$ compatible with the decompositions $E_{p|\widehat{W}} = \bigoplus \widehat{E}_{p,\mathfrak{a}|\widehat{W}}$. Let $\widehat{A}$ be determined by $F(\widehat{\boldsymbol{v}}_1) = \widehat{\boldsymbol{v}}_2\, \widehat{A}$. We have the decomposition $\widehat{A} = \bigoplus \widehat{A}_{\mathfrak{a},\mathfrak{a}}$. Let $S \in \bigcap_{p=1,2} \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I}_p)$. We have the Stokes filtrations $\mathcal{F}^S(E_{p|\overline{S}})$ $(p = 1, 2)$. We take $\mathbb{D}_z$-flat splittings $E_{p|\overline{S}} = \bigoplus E_{p,\mathfrak{a},S}$ of the filtrations $\mathcal{F}^S(E_{p|\overline{S}})$, and let $\boldsymbol{v}_{p,S}$ be lifts of $\widehat{\boldsymbol{v}}_p$ to $E_{p|\overline{S}}$ compatible with the splittings. Let $F_{\mathfrak{b},\mathfrak{a}}$ be the $\mathbb{D}_z$-flat morphism $E_{1,\mathfrak{a},S} \to E_{2,\mathfrak{b},S}$ induced by $F_{|\overline{S}}$. Let $A_{\mathfrak{b},\mathfrak{a}}$ be determined by $F_{\mathfrak{b},\mathfrak{a}}(\boldsymbol{v}_{1,\mathfrak{a},S}) = \boldsymbol{v}_{2,\mathfrak{b},S}\, A_{\mathfrak{b},\mathfrak{a}}$. Lemma 3.6.13 can be reduced to the following lemma.

Lemma 3.6.14

- $A_{\mathfrak{b},\mathfrak{a}} = 0$ *unless* $\mathfrak{a} \geqslant_S \mathfrak{b}$.
- *In the case* $\mathfrak{a} >_S \mathfrak{b}$*, we have the following estimate for some* $C > 0$*:*

$$A_{\mathfrak{b},\mathfrak{a}} \exp\big(\varrho^{-1}(\mathfrak{b} - \mathfrak{a})\big) = O\big(\exp(C\, |\varrho^{-1} \boldsymbol{z}^{\boldsymbol{m}(1)}|\, \log |z_{i(0)}^{-1}|)\big),$$

- $A_{\mathfrak{a},\mathfrak{a}|\widehat{Z}} = \widehat{A}_{\mathfrak{a},\mathfrak{a}}$. *In particular,* $\big|A_{\mathfrak{a},\mathfrak{a}}\big|$ *is bounded.*

Proof. — The first two claims follow from Corollary 20.3.7 and Corollary 20.3.6. The third claim is clear. Thus, we obtain Lemma 3.6.14 and Lemma 3.6.13. □

Corollary 3.6.15

- *If the restriction of F to $\mathcal{X} \setminus \mathcal{D}$ is an isomorphism, we have $\mathcal{I}_1 = \mathcal{I}_2$ and $\mathcal{F}^S_{\mathfrak{a}}(E_{1|S\setminus\mathcal{D}}) = \mathcal{F}^S_{\mathfrak{a}}(E_{2|S\setminus\mathcal{D}})$.*
- *In particular, the Stokes filtration $\mathcal{F}^S$ at the level $(\boldsymbol{m}, i(0))$ depends only on the meromorphic ϱ-flat bundle $\big(E(*\mathcal{D}), \mathbb{D}\big)$, in the sense that it is independent of the choice of a weakly good lattice $E \subset E(*\mathcal{D})$ at the level $(\boldsymbol{m}, i(0))$.*

Proof. — F induces an isomorphism $E_1(*\mathcal{D}) \simeq E_2(*\mathcal{D})$, and hence $E_1(*\mathcal{D})_{|\widehat{W}_z} \simeq E_2(*\mathcal{D})_{|\widehat{W}_z}$. Then, we obtain an isomorphism $\widehat{E}_{1,\mathfrak{a}}(*\mathcal{D}) \simeq \widehat{E}_{2,\mathfrak{a}}(*\mathcal{D})$ for $\mathfrak{a} \in \mathcal{I}_1 \cup \mathcal{I}_2$. Hence, we have $\mathcal{I}_1 = \mathcal{I}_2$. Since we have the inclusion $\mathcal{F}^S_{\mathfrak{a}}(E_{1|S}) \subset \mathcal{F}^S_{\mathfrak{a}}(E_{2|S})$ by Lemma 3.6.13, we obtain $\mathcal{F}^S_{\mathfrak{a}}(E_{1|S}) = \mathcal{F}^S_{\mathfrak{a}}(E_{2|S})$ by the comparison of the ranks. Thus, the first claim is proved. The second claim follows from the first claim. □

3.6.5. The associated graded bundle. — For any $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$ and $\mathfrak{a} \in \mathcal{I}$, we obtain a bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\overline{S}})$ with a meromorphic flat ϱ-connection $\mathbb{D}_{\mathfrak{a},\overline{S}}$ on $\overline{S}$, by taking Gr with respect to the filtration $\mathcal{F}^S$. By definition of Stokes filtrations, we have a natural isomorphism

$$\big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\overline{S}}), \mathbb{D}_{\mathfrak{a},\overline{S}}\big)_{|\widehat{Z}} \simeq (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})_{|\widehat{Z}}.$$

When S is varied, we can glue them and obtain a bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)})$ with a meromorphic flat ϱ-connection $\mathbb{D}_{\mathfrak{a},\widetilde{\mathcal{V}}(W)}$ on $\widetilde{\mathcal{V}}(W)$, where $\mathcal{V}$ denotes some neighbourhood of $\mathcal{D}_{z,\underline{k}}$, and $\widetilde{\mathcal{V}}(W)$ denotes the real blow up of $\mathcal{V}$ along $W' := \mathcal{V} \cap W$. By the construction, we are given an isomorphism

$$\big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)}), \mathbb{D}_{\mathfrak{a},\widetilde{\mathcal{V}}(W)}\big)_{|\widehat{\pi^{-1}(W')}} \simeq (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})_{|\widehat{\pi^{-1}(W')}}.$$

According to Proposition 3.1.12, Corollary 3.1.15 and Lemma 3.1.17, there exists a holomorphic vector bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$ with a meromorphic flat ϱ-connections $\mathbb{D}_{\mathfrak{a}}$ on $(\mathcal{V}, \mathcal{D} \cap \mathcal{V})$ such that

$$\begin{aligned} \pi^*\big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \mathbb{D}_{\mathfrak{a}}\big)_{|\widetilde{\mathcal{V}}(W)} &\simeq \big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)}), \mathbb{D}_{\mathfrak{a},\widetilde{\mathcal{V}}(W)}\big), \\ \big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \mathbb{D}_{\mathfrak{a}}\big)_{|\widehat{W}'} &\simeq (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})_{|\widehat{W}'}. \end{aligned} \tag{54}$$

It is well defined on the germ of neighbourhoods of $\mathcal{D}_{z,\underline{k}}$ in $\mathcal{X}$.

Corollary 3.6.16. — *If we shrink X, we can take a frame $\widehat{\boldsymbol{v}} = (\widehat{\boldsymbol{v}}_{\mathfrak{a}})$ of $E_{|\widehat{W}}$ compatible with the irregular decomposition at the level $\boldsymbol{m}$, such that the power series $\widehat{R}_{\mathfrak{a}}$ is convergent, where $\mathbb{D}_{\mathfrak{a}}\widehat{\boldsymbol{v}}_{\mathfrak{a}} = \widehat{\boldsymbol{v}}_{\mathfrak{a}}\,\widehat{R}_{\mathfrak{a}}$.*

Proof. — We take a holomorphic frame $\boldsymbol{w}_{\mathfrak{a}}$ of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$ on $\mathcal{V}$. It induces a frame $\boldsymbol{w}_{\mathfrak{a}|\widehat{W}}$ of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)_{|\widehat{W}} \simeq \widehat{E}_{\mathfrak{a}}$. We only have to put $\widehat{\boldsymbol{v}}_{\mathfrak{a}} := \boldsymbol{w}_{\mathfrak{a}|\widehat{W}}$. □

Let $\boldsymbol{w}_{\mathfrak{a}}$ be a frame of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$. Let $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$, and let $E_{|\overline{S}} = \bigoplus E_{\mathfrak{a},S}$ be a $\mathbb{D}_z$-flat splitting of the Stokes filtration $\mathcal{F}^S$. By the natural isomorphism $E_{\mathfrak{a},S} \simeq$

$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)_{|\overline{S}}$, we take a lift $\boldsymbol{w}_{\mathfrak{a},S}$ of $\boldsymbol{w}_{\mathfrak{a}}$. Thus, we obtain a frame $\boldsymbol{w}_S = \big(\boldsymbol{w}_{\mathfrak{a},S}\big)$. The following corollary is clear from the construction.

Corollary 3.6.17. — *Let $\boldsymbol{v}$ be a frame of E, and let G_S be the matrix by $\boldsymbol{v} = \boldsymbol{w}_S\, G_S$. Then, G_S and G_S^{-1} are bounded on S.* □

Remark 3.6.18. — If $k = 1$ and if ϱ is nowhere vanishing, we obtain $\big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \mathbb{D}_{\mathfrak{a}}\big)$ on $(\mathcal{X}, \mathcal{D})$, not only on $(\mathcal{V}, \mathcal{D}\cap\mathcal{V})$. Note that we can always extend a good lattice of a meromorphic ϱ-flat bundle on $(\mathcal{V}, \mathcal{D}\cap\mathcal{V})$ to that on $(\mathcal{X}, \mathcal{D})$. □

3.6.5.1. Functoriality. — Let $(E, \mathbb{D}, \mathcal{I})$ be a good lattice at the level $(\boldsymbol{m}, i(0))$. From $(E^\vee, \mathbb{D}^\vee, \mathcal{I}^\vee)$, we obtain the associated graded bundle $\mathrm{Gr}^{\boldsymbol{m}}(E^\vee) = \bigoplus_{\mathfrak{a}\in\mathcal{I}^\vee} \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E^\vee)$ with an induced meromorphic flat ϱ-connection.

Lemma 3.6.19. — *We have a natural flat isomorphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E^\vee) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{-\mathfrak{a}}(E)^\vee$.*

Proof. — By Lemma 3.6.10, we have the natural isomorphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E^\vee)_{|\widetilde{\mathcal{V}}(W)} \simeq \mathrm{Gr}^{\boldsymbol{m}}_{-\mathfrak{a}}(E)^\vee_{|\mathcal{V}(W)}$. It induces the desired isomorphism on $\mathcal{V}$. □

Let $(E_p, \nabla_p, \mathcal{I}_p)$ $(p = 1, 2)$ be weakly good lattices at the level $(\boldsymbol{m}, i(0))$. The following lemma can be shown similarly.

Lemma 3.6.20. — *Assume $\mathcal{I}_1 \otimes \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$. Let $(\widetilde{E}, \widetilde{\mathbb{D}}) := (E_p, \mathbb{D}_1) \otimes (E_p, \mathbb{D}_2)$. Then, we have the following natural isomorphism for each $\mathfrak{a} \in \mathcal{I}_1 \otimes \mathcal{I}_2$:*

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(\widetilde{E}) \simeq \bigoplus_{\substack{(\mathfrak{a}_1,\mathfrak{a}_2)\in\mathcal{I}_1\times\mathcal{I}_2\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}} \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}_1}(E_1) \otimes \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}_2}(E_2). \tag{55}$$

If $\mathcal{I}_1 \oplus \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, then we have $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1 \oplus E_2) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \oplus \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)$. □

Lemma 3.6.21. — *Let $F : (E_1, \mathbb{D}_1) \to (E_2, \mathbb{D}_2)$ be a flat morphism. Assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is good, for simplicity. Then, we have the naturally induced morphism*

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(F) : \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2).$$

Proof. — We have the induced morphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1)_{|\widetilde{\mathcal{V}}(W)} \to \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)_{|\widetilde{\mathcal{V}}(W)}$ by Lemma 3.6.13. It induces the desired morphism on $\mathcal{V}$. □

Corollary 3.6.22. — *In the situation of Lemma 3.6.21, if $E_{1|\mathcal{X}\setminus\mathcal{D}} \to E_{2|\mathcal{X}\setminus\mathcal{D}}$ is an isomorphism, we have induced isomorphisms:*

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \otimes \mathcal{O}(*\mathcal{D}) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2) \otimes \mathcal{O}(*\mathcal{D}) \quad (\mathfrak{a} \in \mathcal{I}).$$

Hence, the graded meromorphic ϱ-flat bundle

$$\bigoplus_{\mathfrak{a}} \big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E) \otimes \mathcal{O}(*\mathcal{D}), \mathbb{D}_{\mathfrak{a}}\big)$$

is well defined for the meromorphic ϱ-flat bundle $\big(E(*\mathcal{D}), \mathbb{D}\big)$.

Proof. — By Corollary 3.6.15, the restriction $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1)_{|\mathcal{V}\setminus\mathcal{D}} \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)_{|\mathcal{V}\setminus\mathcal{D}}$ is an isomorphism. Hence, the induced morphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1)\otimes\mathcal{O}(*\mathcal{D}) \to \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)\otimes\mathcal{O}(*\mathcal{D})$ is an isomorphism. □

3.6.6. A characterization by the growth order. — Assume that ϱ is nowhere vanishing. Let $(E,\mathbb{D},\mathcal{I})$ be a good lattice at the level $(\boldsymbol{m}, i(0))$. Take any frame $\boldsymbol{v}$ of E. Let $S\in\mathcal{MS}^*(\mathcal{X}\setminus\mathcal{D}_z,\mathcal{I})$. Let f be a $\mathbb{D}_z$-flat section of $E_{|S}$. We have the expression $f=\sum f_j\, v_j$, and obtain $\boldsymbol{f}=(f_j)$.

Lemma 3.6.23. — *We have $f\in\mathcal{F}^S_{\mathfrak{b}}$ if and only if the following holds for some $C>0$:*

$$\bigl|\boldsymbol{f}\,\exp(\varrho^{-1}\,\mathfrak{b})\bigr| = O\bigl(\exp(C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}|)|z_{i(0)}|^{-C}\bigr).$$

Recall $\boldsymbol{m}(1) := \boldsymbol{m}+\boldsymbol{\delta}_{i(0)}$.

Proof. — We take a $\mathbb{D}_z$-flat splitting $E_{|\overline{S}}=\bigoplus E_{\mathfrak{a},S}$ of $\mathcal{F}^S$, and take a frame $\boldsymbol{v}_S=(\boldsymbol{v}_{\mathfrak{a},S})$ of $E_{|\overline{S}}$ compatible with the splitting. Let $r(\mathfrak{a}) := \operatorname{rank} E_{\mathfrak{a},S}$. We have the expression $f=\sum_{\mathfrak{a}}\sum_{j=1}^{r(\mathfrak{a})} f_{\mathfrak{a},S,j}\,v_{\mathfrak{a},S,j}$, and obtain $\boldsymbol{f}_{\mathfrak{a}}=(f_{\mathfrak{a},S,j}\,|\,j=1,\dots,r(\mathfrak{a}))$. Note $|\boldsymbol{f}|$ and $\sum|\boldsymbol{f}_{\mathfrak{a}}|$ are mutually bounded.

Let $R_{\mathfrak{a}}$ be determined by $\mathbb{D}^f_z\boldsymbol{v}_{\mathfrak{a}\,S} = \boldsymbol{v}_{\mathfrak{a}\,S}\Bigl(\bigoplus\bigl(d_z(\varrho^{-1}\,\mathfrak{a})+R_{\mathfrak{a}}\bigr)\Bigr)$. Then, it is a holomorphic section of the following:

$$\sum_{i=1}^{k} M_{r(\mathfrak{a})}(\boldsymbol{C})\otimes\boldsymbol{z}^{\boldsymbol{m}(1)}\,\mathcal{O}_{\overline{S}}\,dz_i/z_i\quad(\text{if } m_{i(0)}<-1)$$

$$\sum_{i\neq i(0)} M_{r(\mathfrak{a})}(\boldsymbol{C})\otimes\boldsymbol{z}^{\boldsymbol{m}(1)}\,\mathcal{O}_{\overline{S}}\,dz_i/z_i + M_{r(\mathfrak{a})}(\boldsymbol{C})\otimes\mathcal{O}_{\overline{S}}\,dz_i/z_i\quad(\text{if } m_{i(0)}=-1).$$

Since each $f_{\mathfrak{a}}$ is $\mathbb{D}_z$-flat, we obtain the following estimate in the case $f_{\mathfrak{a}}\neq 0$, by using Lemma 20.3.3:

$$\Bigl|\log\bigl|\boldsymbol{f}_{\mathfrak{a}}\exp(\varrho^{-1}\mathfrak{a})\bigr|\Bigr| \leqslant C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}| + C\log|z_{i(0)}^{-1}|.$$

Then, the claim of the lemma follows. □

Let us consider the case $Y=\Delta^n_\zeta$ and $\mathcal{D}_Y=\bigcup_{i=1}^{\ell}\{\zeta_i=0\}$. Let f be a $\mathbb{D}$-flat section of $E_{|S\setminus\mathcal{D}_Y}$. We obtain the following lemma from Lemma 2.6.10 by the argument in the proof of Lemma 3.6.23.

Lemma 3.6.24. — *We have $f\in\mathcal{F}^S_{\mathfrak{b}}$ if and only if the following holds for some $C>0$:*

$$\bigl|\boldsymbol{f}\,\exp(\varrho^{-1}\,\mathfrak{b})\bigr| = O\Bigl(\exp(C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}|)|z_{i(0)}|^{-C}\prod_{j=1}^{\ell}|\zeta_j|^{-C}\Bigr).$$ □

3.6.7. Proof of the claims in Section 3.3. — Corollary 3.6.4 implies Proposition 3.3.2. Lemma 3.6.6 implies Proposition 3.3.3. The functoriality in Subsection 3.3.3 follows from those in Subsection 3.6.4. The growth estimate in Proposition 3.3.8 is implied by that in Lemma 3.6.23. The associated graded meromorphic ϱ-flat bundle and its functoriality are studied in Subsection 3.6.5. Proposition 3.3.10 is implied by Proposition 3.6.7. Proposition 3.3.11 follows from Proposition 3.6.8. Proposition 3.3.12 follows from Proposition 3.3.11 and Lemma 2.6.9. Proposition 3.3.13 is implied by Lemma 3.6.9

3.6.8. Appendix (Lifting of formal frames). — We discuss liftings of frames. Although we will use such concepts in our later argument, readers can skip this part.

3.6.8.1. Holomorphic lift on small sectors. — We take a frame $\widehat{\boldsymbol{v}} = (\widehat{\boldsymbol{v}}_{\mathfrak{a}})$ of $E_{|\widehat{W}}$ compatible with the irregular decomposition. Let $\widehat{R}_{\mathfrak{a}}$ be determined by $\mathbb{D}_{\mathfrak{a}}\widehat{\boldsymbol{v}}_{\mathfrak{a}} = \widehat{\boldsymbol{v}}_{\mathfrak{a}}\left(d\mathfrak{a} + \widehat{R}_{\mathfrak{a}}\right)$.

Let $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$. We take a $\mathbb{D}_z$-flat decomposition $E_{|\overline{S}} = \bigoplus_{\mathfrak{a}} E_{\mathfrak{a},S}$ which gives a splitting of $\mathcal{F}^S$ as in Proposition 3.6.1. We can take a frame $\boldsymbol{v}_{\mathfrak{a},S}$ of $E_{\mathfrak{a},S}$ such that $\boldsymbol{v}_{\mathfrak{a},S|\widehat{Z}} = \widehat{\boldsymbol{v}}_{\mathfrak{a}|\widehat{Z}}$, and we put $\boldsymbol{v}_S := (\boldsymbol{v}_{\mathfrak{a},S})$, which is called a holomorphic lift of $\widehat{\boldsymbol{v}}$ on $\overline{S}$.

Let $E_{|\overline{S}} = \bigoplus_{\mathfrak{a}} E'_{\mathfrak{a},S}$ be another $\mathbb{D}_z$-flat splitting of $\mathcal{F}^S$, and let $\boldsymbol{v}'_S = \left(\boldsymbol{v}'_{\mathfrak{a},S}\right)$ be a holomorphic lift compatible with the splitting. Let $C = (C_{\mathfrak{a}',\mathfrak{a}})$ be determined by $\boldsymbol{v}_S = \boldsymbol{v}'_S\,(I + C)$, where I denotes the identity matrix.

Lemma 3.6.25

- *We have $C_{\mathfrak{a}',\mathfrak{a}|\widehat{Z}} = 0$ and $C_{\mathfrak{a}',\mathfrak{a}} = 0$ unless $\mathfrak{a}' \leqslant_S \mathfrak{a}$.*
- *If $\mathfrak{a}' <_S \mathfrak{a}$, we have the following for some $C > 0$:*

$$C_{\mathfrak{a}',\mathfrak{a}} \exp\big(\varrho^{-1}(\mathfrak{a}' - \mathfrak{a})\big) = O\Big(\exp\big(C\,|\varrho^{-1}|(|\boldsymbol{z}^{\boldsymbol{m}(1)}| + \log|z_{i(0)}^{-1}|)\big)\Big).$$

Proof. — We have $C_{\mathfrak{a}',\mathfrak{a}|\widehat{Z}} = 0$ by construction. Since $\boldsymbol{v}_S$ and $\boldsymbol{v}'_S$ are compatible with the filtration $\mathcal{F}^S$ by construction, we have $C_{\mathfrak{a}',\mathfrak{a}} = 0$ unless $\mathfrak{a}' \leqslant_S \mathfrak{a}$. The other property follows from the estimate of the norm of $\mathbb{D}_z$-flat sections (Lemma 20.3.5). □

3.6.8.2. C^∞-lift on X

Lemma 3.6.26. — *We have a local C^∞-frame $\boldsymbol{v}_{C^\infty} = \left(\boldsymbol{v}_{\mathfrak{a},C^\infty}\right)$ of E on some neighbourhood $\mathcal{V}$ of $\mathcal{D}_{z,\underline{k}}$ with the following properties:*

- $\boldsymbol{v}_{C^\infty|\widehat{\mathcal{D}}_z} = \widehat{\boldsymbol{v}}$.
- *Let $S \in \mathcal{MS}^*(\mathcal{X} \setminus W, \mathcal{I})$, and let $\boldsymbol{v}_S$ be a holomorphic lift of $\widehat{\boldsymbol{v}}$ on $\overline{S}$ as in Section* 3.6.8.1. *Let B_S be determined by $\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_S\,(I + B_S)$ on $\pi^{-1}(\mathcal{V}) \cap \overline{S}$. Then, the following holds:*
  - $B_{S|\widehat{Z}} = 0$.
  - *Let $B_S = \left(B_{S,\mathfrak{a},\mathfrak{b}}\right)$ be the decomposition, corresponding to the decomposition of the frame $\widehat{\boldsymbol{v}} = (\widehat{\boldsymbol{v}}_{\mathfrak{a}})$. Then, we have $B_{S,\mathfrak{a},\mathfrak{b}} = 0$ unless $\mathfrak{a} \leqslant_S \mathfrak{b}$.*

- *In the case* $\mathfrak{a} <_S \mathfrak{b}$, *we have the following estimate of the* C^q*-derivatives of* $B_{S,\mathfrak{a},\mathfrak{b}} \exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\big)$ *for some* $C>0$ *and* $N(q) \geqslant 0$ $(q \in \mathbb{Z}_{\geqslant 0})$*:*

$$O\Big(\exp\Big(C\,|\varrho^{-1}|\big(|\boldsymbol{z}^{\boldsymbol{m}(1)}| + \log\big|z_{i(0)}^{-1}\big|\big)\Big) \prod_{j=1}^{k} |z_j|^{-N(q)} |\varrho^{-1}|^{-N(q)}\Big).$$

In particular, the frame $\boldsymbol{v}_{C^\infty|S}$ *is compatible with the Stokes filtration* $\mathcal{F}^S$ *for* $S \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$ *in the sense that* $\big(\boldsymbol{v}_{\mathfrak{b},C^\infty} \bigm| \mathfrak{b} \leqslant_S \mathfrak{a}\big)$ *gives a frame of* $\mathcal{F}^S_{\mathfrak{a}}$ *for each* $\mathfrak{a}$.

Such a frame $\boldsymbol{v}_{C^\infty}$ *is called a* C^∞*-lift of* $\widehat{\boldsymbol{v}}$.

Proof. — In the following, for a given multi-sector S, let S° denote its interior part. We take multi-sectors $S^{(j)} \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$ $(j = 1, \dots, N)$ of $\mathcal{X} \smallsetminus W$ such that $\bigcup S^{(j)\circ} = \mathcal{V} \smallsetminus W$, where $\mathcal{V}$ is some open neighbourhood of $\mathcal{D}_{z,\underline{k}}$ in $\mathcal{X}$. We take holomorphic lifts $\boldsymbol{v}_{S^{(j)}} = \big(v_{S^{(j)},i}\big)$ of $\widehat{\boldsymbol{v}}$ on $\overline{S}^{(j)}$. We only have to glue them in C^∞-sense as follows, for example. We take small sectors $S_3^{(j)} \subset S_2^{(j)} \subset S_1^{(j)} = S^{(j)}$ such that (i) $\bigcup_{j=1}^N S_3^{(j)\circ} = \mathcal{V} \smallsetminus W$, (ii) $S_a^{(j)}$ in $\mathcal{V} \smallsetminus W$ is contained in $S_{a-1}^{(j)\circ}$ for $a = 2, 3$. We take C^∞-functions χ_j on $\mathcal{V} \smallsetminus W$ such that (i) $\chi_j \geqslant 0$, (ii) $\chi_j > 0$ on $S_3^{(j)}$, and $\chi_j = 0$ outside of $S_2^{(j)}$, (iii) each $(\partial_\varrho^m \prod \partial_i^{m_i})\chi_j$ is polynomial order in $|\varrho^{-1}|$ and $|z_i^{-1}|$ $(i = 1, \dots, k)$, (iv) $\sum_{j=1}^N \chi_j = 1$. We put $\boldsymbol{v}_{C^\infty} := \sum_{j=1}^N \chi_j\, \boldsymbol{v}_{S^{(j)}}$, or more precisely, $v_{C^\infty,i} := \sum_{j=1}^N \chi_j\, v_{S^{(j)},i}$. Then, $\boldsymbol{v}_{C^\infty} := (v_{C^\infty,i})$ gives a C^∞-frame on $\pi^{-1}(\mathcal{V})$.

Let $S \in \mathcal{MS}^*(\mathcal{X} \smallsetminus W, \mathcal{I})$. Let $C^{(j)}$ be determined by $\boldsymbol{v}_{S^{(j)}} = \boldsymbol{v}_S\,(I + C^{(j)})$ on $\overline{S} \cap \overline{S}^{(j)}$, where I denotes the identity matrix. Let $Z(S, S^{(j)}) = \overline{S}^{(j)} \cap \overline{S} \cap \pi^{-1}(\mathcal{D}_z)$. Due to Lemma 3.6.25, we have (i) $C^{(j)}_{|\widehat{Z}(S,S^{(j)})} = 0$, (ii) $C^{(j)}_{\mathfrak{a},\mathfrak{b}} = 0$ unless $\mathfrak{a} \leqslant_{S^{(j)} \cap S} \mathfrak{b}$, (iii) $C^{(j)}_{\mathfrak{a},\mathfrak{b}} \exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\big) = O\big(\exp\big(C'|\varrho^{-1}|(|\boldsymbol{z}^{\boldsymbol{m}(1)}| + \log|z_{i(0)}^{-1}|)\big)\big)$ if $\mathfrak{a} <_{S^{(j)} \cap S} \mathfrak{b}$. By construction, $\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_S\,\big(I + \sum \chi_j\, C^{(j)}\big)$ holds. Hence, $\boldsymbol{v}_{C^\infty}$ satisfies the desired estimate on S. This also implies that $\boldsymbol{v}_{C^\infty}$ gives a C^∞-frame of E on $\mathcal{V}$. □

Let us look at the connection form of $\mathbb{D}$ with respect to $\boldsymbol{v}_{C^\infty}$. Let $I_{\mathfrak{a}}$ denote the identity matrix whose size is rank $\widehat{E}_{\mathfrak{a}}$. Then, we have the following:

$$(\mathbb{D} + \overline{\partial}_\lambda)\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\Big(\Big(\bigoplus_{\mathfrak{a} \in \mathcal{I}} d\mathfrak{a}\, I_{\mathfrak{a}}\Big) + R\Big).$$

We put $\mathcal{D}^{(1)} := \mathcal{D}_z$, $\mathcal{D}^{(2)} = \mathcal{D}_Y$ if $m_{i(0)} < -1$, or $\mathcal{D}^{(1)} := \bigcup_{j \neq i(0)} \mathcal{D}_{z,j}$, $\mathcal{D}^{(2)} := \mathcal{D}_{z,i(0)} \cup \mathcal{D}_Y$ if $m_{i(0)} = -1$. We can deduce the following from the property of $\boldsymbol{v}_{C^\infty}$.

Lemma 3.6.27. — *R is a C^∞-section of*

$$M_r(\boldsymbol{C}) \otimes \Big(\boldsymbol{z}^{\boldsymbol{m}(1)}\Omega^{1,0}_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(1)}) + \Omega^{1,0}_{\mathcal{X}/\mathcal{K}}(\log \mathcal{D}^{(2)}) + \Omega^{0,1}_{\mathcal{X}}\Big),$$

and we have $R_{\mathfrak{a},\mathfrak{a}|\widehat{W}} = \widehat{R}_{\mathfrak{a}}$ *and* $R_{\mathfrak{a},\mathfrak{b}|\widehat{W}} = 0$ *for* $\mathfrak{a} \neq \mathfrak{b}$. *For each sufficiently small sector S, the following holds:*

1. $R_{\mathfrak{a},\mathfrak{b}|S} = 0$ *unless* $\mathfrak{a} \leqslant_S \mathfrak{b}$.

2. *If $\mathfrak{a} <_S \mathfrak{b}$, the C^q-derivatives of $R_{\mathfrak{a},\mathfrak{b}} \exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\big)$ is*

$$O\Big(\exp\Big(C\,|\varrho^{-1}|\big(|\boldsymbol{z}^{\boldsymbol{m}(1)}| + \log|z_{i(0)}^{-1}|\big)\Big)|\varrho|^{-N(q)}\prod_{j=1}^{k}|z_j|^{-N(q)}\Big)$$

for some $C > 0$ and $N(q) \geqslant 0$ ($q \in \mathbb{Z}_{\geqslant 0}$). □

Let $\boldsymbol{v}'_{C^\infty}$ be another C^∞-lift of $\widehat{\boldsymbol{v}}$. Let B be determined by $\boldsymbol{v}'_{C^\infty} = \boldsymbol{v}_{C^\infty}(I+B)$. It is easy to deduce the following.

Lemma 3.6.28. — *We have $B_{|\widehat{W}} = 0$. On each sufficiently small sector S, we have $B_{\mathfrak{a},\mathfrak{b}|S} = 0$ unless $\mathfrak{a} \leqslant_S \mathfrak{b}$. If $\mathfrak{a} <_S \mathfrak{b}$, the C^q-derivatives of $B_{\mathfrak{a},\mathfrak{b}} \exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\big)$ is*

$$O\Big(\exp\Big(C|\varrho^{-1}|\big(|\boldsymbol{z}^{\boldsymbol{m}(1)}| + \log|z_{i(0)}^{-1}|\big)\Big)|\varrho|^{-N(q)}\prod_{j=1}^{k}|z_j|^{-N(q)}\Big)$$

for some $C > 0$ and $N(q) \geqslant 0$ ($q \in \mathbb{Z}_{\geqslant 0}$). □

3.6.9. Approximation of formal decompositions (Appendix)

Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$ for some $\ell \leqslant n$. In the following argument, N will denote a large integer, and we will shrink X around the origin $(0,\dots,0)$ without mention. Let $\widehat{D}$ denote the completion of X along D. (See [**4**], [**8**] and [**50**]. See also a brief review in Subsection 22.5.1.) Let $\widehat{D}^{(N)}$ denote the N-th infinitesimal neighbourhood of D in X. Let $\iota : \widehat{D} \to X$ denote the natural morphism of ringed spaces. For any $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{F}_{|\widehat{D}}$ denote $\iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{\widehat{D}}$. We use the symbol $\mathcal{F}_{|\widehat{D}^{(N)}}$ in a similar meaning.

Let V be a free $\mathcal{O}_X$-module of finite rank, equipped with the following data:

- Sections $f_k^{(i)} \in \mathrm{End}(V_{|D_k})$ for $k = \ell+1,\dots,n$ and $i = 1,\dots,M$.
- Filtrations ${}^k\mathcal{F}$ of $V_{|D_k}$ for $k = \ell+1,\dots,n$.

Here, M denotes some positive integer.

3.6.9.1. Approximation of an endomorphism. — Let $\widehat{F} \in \mathrm{End}(V)_{|\widehat{D}}$ satisfy the following conditions:

- $\big[f^{(i)}_{k|\widehat{D}\cap D_k}, \widehat{F}_{|\widehat{D}\cap D_k}\big] = 0$ for $k = \ell+1,\dots,n$ and $i = 1,\dots,M$, where $[\cdot,\cdot]$ denotes the commutator.
- $\widehat{F}_{|\widehat{D}\cap D_k}$ preserves ${}^k\mathcal{F}_{|\widehat{D}\cap D_k}$ for $k = \ell+1,\dots,n$.

Here, $\widehat{D}\cap D_k$ means the completion of D_k along $D \cap D_k$.

Lemma 3.6.29. — *For any large N, we can take a section $F^{(N)} \in \mathrm{End}(V)$ such that $F^{(N)}_{|\widehat{D}^{(N)}} = \widehat{F}_{|\widehat{D}^{(N)}}$, with the following properties:*

- *$\big[f_k^{(i)}, F_{|D_k}\big] = 0$ for $k = \ell+1,\dots,n$ and $i = 1,\dots,M$.*
- *$F^{(N)}_{|D_k}$ preserves the filtrations ${}^k\mathcal{F}$ for $k = \ell+1,\dots,n$.*

Proof. — Let $\mathrm{Cok}\big(\mathrm{ad}(f_k^{(i)})\big)$ denote the cokernel of the morphism

$$\mathrm{ad}(f_k^{(i)}) : \mathrm{End}(V)_{|D_k} \longrightarrow \mathrm{End}(V)_{|D_k}$$

given by $\mathrm{ad}(f_k^{(i)})(g) = [f_k^{(i)}, g]$. We put $\widehat{f}_k^{(i)} := f^{(i)}_{k|D_k\cap\widehat{D}}$. Let $\mathrm{Cok}\big(\mathrm{ad}(\widehat{f}_k^{(i)})\big)$ denote the cokernel of the morphism $\mathrm{ad}(\widehat{f}_k^{(i)}) : \mathrm{End}(V)_{|\widehat{D}\cap D_k} \to \mathrm{End}(V)_{|\widehat{D}\cap D_k}$.

For $k = \ell+1, \dots, n$, let $\mathrm{End}'(V_{|D_k})$ denote the sheaf of sections of $\mathrm{End}(V_{|D_k})$ preserving ${}^k\mathcal{F}$. Similarly, let $\mathrm{End}'(\widehat{V}_{|\widehat{D}\cap D_k})$ denote the sheaf of sections of $\mathrm{End}(\widehat{V}_{|\widehat{D}\cap D_k})$ preserving ${}^k\mathcal{F}$. We put

$$A_k := \mathrm{End}(V_{|D_k}) / \mathrm{End}'(V_{|D_k}), \quad \widehat{A}_k := \mathrm{End}(\widehat{V}_{|D_k\cap\widehat{D}}) / \mathrm{End}'(\widehat{V}_{|D_k\cap\widehat{D}}).$$

Let $\iota_k : D_k \to X$ and $\widehat{\iota}_k : D_k \cap \widehat{D} \to \widehat{D}$ denote the inclusions. We set

$$\mathcal{F} := \mathrm{Ker}\Big(\mathrm{End}(V) \longrightarrow \bigoplus_{k=\ell+1}^{n} \iota_{k*}A_k \oplus \bigoplus_{k=\ell+1}^{n}\bigoplus_{i=1}^{M} \iota_{k*}\,\mathrm{Cok}\big(\mathrm{ad}(f_k^{(i)})\big)\Big),$$

$$\widehat{\mathcal{F}} := \mathrm{Ker}\Big(\mathrm{End}(V)_{|\widehat{D}} \longrightarrow \bigoplus_{k=\ell+1}^{n} \iota_{k*}\widehat{A}_k \oplus \bigoplus_{k=\ell+1}^{n}\bigoplus_{i=1}^{M} \widehat{\iota}_{k*}\,\mathrm{Cok}\big(\mathrm{ad}(\widehat{f}_k^{(i)})\big)\Big).$$

We only have to show that $\widehat{\mathcal{F}}$ is the completion of $\mathcal{F}$ along D, which implies the claim of the lemma. Note that $\mathrm{Cok}\big(\mathrm{ad}(\widehat{f}_k^{(i)})\big)$ and $\widehat{A}_k$ are the completions of $\mathrm{Cok}\big(\mathrm{ad}(f_k^{(i)})\big)$ and A_k, respectively. Since $\mathcal{O}_{\widehat{D}}$ is faithfully flat over $\mathcal{O}_X$ ([8]), we obtain $\widehat{\mathcal{F}} \simeq \mathcal{F}_{|\widehat{D}}$. Thus Lemma 3.6.29 is finished. □

3.6.9.2. Approximation of a decomposition. — Assume that we are given a formal decomposition $V_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{V}_{\mathfrak{b}}$ satisfying the following condition:

- $\widehat{V}_{\mathfrak{a}|\widehat{D}\cap D_k}$ $(\mathfrak{a}\in\mathcal{I})$ are preserved by $f^{(i)}_{k|\widehat{D}\cap D_k}$ for $k = \ell+1, \dots, n$ and $i = 1, \dots, M$.
- ${}^k\mathcal{F} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} {}^k\mathcal{F} \cap V_{\mathfrak{a}|\widehat{D}\cap D_k}$.

Lemma 3.6.30. — *For any large N, we can take a decomposition $V = \bigoplus_{\mathfrak{a}\in\mathcal{I}} V_{\mathfrak{a}}^{(N)}$ such that $V^{(N)}_{\mathfrak{a}|\widehat{D}^{(N)}} = \widehat{V}_{\mathfrak{a}|\widehat{D}^{(N)}}$, with the following properties:*

- *$V^{(N)}_{\mathfrak{a}|D_k}$ $(\mathfrak{a}\in\mathcal{I})$ are preserved by $f_k^{(i)}$ for $k = \ell+1, \dots, n$ and $i = 1, \dots, M$.*
- *${}^k\mathcal{F}_{|\widehat{D}\cap D_k} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} {}^k\mathcal{F} \cap V^{(N)}_{\mathfrak{a}|D_k}$.*

Proof. — Let $\widehat{\pi}_{\mathfrak{a}}$ $(\mathfrak{a}\in\mathcal{I})$ be the projection of $V_{|\widehat{D}}$ onto $\widehat{V}_{\mathfrak{a}}$. We take an injection $\psi : \mathcal{I} \to \mathbb{Z}$, and we put $\widehat{F} := \sum_{\mathfrak{a}\in\mathcal{I}} \psi(\mathfrak{a}) \cdot \widehat{\pi}_{\mathfrak{a}}$. We take $F^{(N)}$ for $\widehat{F}$ as in Lemma 3.6.29. After shrinking X, we have the decomposition $V = \bigoplus V_{\mathfrak{a},N}$ such that (i) $F^{(N)}(V_{\mathfrak{a},N}) \subset V_{\mathfrak{a},N}$, (ii) the eigenvalues of $F^{(N)}_{|V_{\mathfrak{a},N}}$ are close to $\psi(\mathfrak{a})$. Then, it gives the desired decomposition. □

3.7. Proof of the statements in Section 3.2

3.7.1. Preliminary. — We use the setting and the notation in Subsection 2.4.2. Let $\mathcal{J} \subset M(\mathcal{X},\mathcal{D})/H(\mathcal{X})$ be a good set of irregular values. We assume that the coordinate system $(z_1,\ldots,z_n)$ is admissible for $\mathcal{J}$. We take an auxiliary sequence $\boldsymbol{m}(0),\ldots,\boldsymbol{m}(L),\boldsymbol{m}(L+1)=\mathbf{0}$ (Section 2.1). Let $k(p)$ be determined by $\boldsymbol{m}(p) \in \mathbb{Z}_{<0}^{k(p)} \times \mathbf{0}$. Let $\mathcal{J}(\boldsymbol{m}(p))$ and $\mathcal{I}_{\mathfrak{c}}^{\boldsymbol{m}(p)}$ be as in Subsection 2.6.2. Let $\overline{\eta}_{\boldsymbol{m}(p)} : \mathcal{J} \to \mathcal{J}(\boldsymbol{m}(p))$ be the induced map. For each $\mathfrak{c} \in \mathcal{J}(\boldsymbol{m}(p))$, we put $\mathcal{J}_{\mathfrak{c}} := \overline{\eta}_{\boldsymbol{m}(p)}^{-1}(\mathfrak{c})$, which is a good set of irregular values. We also have the naturally induced map $\overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)} : \mathcal{J}(\boldsymbol{m}(p)) \to \mathcal{J}(\boldsymbol{m}(p-1))$.

Let $W(\underline{k}) := \mathcal{X}^0 \cup \mathcal{D}(\underline{k})$. Let $\pi_k : \widetilde{\mathcal{X}}(W(\underline{k})) \to \mathcal{X}$ be the real blow up of $\mathcal{X}$ at $W(\underline{k})$. In particular, $\pi_\ell =: \pi$. We have the naturally induced maps $\varpi_{k,m} : \widetilde{\mathcal{X}}(W(\underline{m})) \to \widetilde{\mathcal{X}}(W(\underline{k}))$ for $m \geqslant k$. The map $\widetilde{\mathcal{X}}(W) \to \widetilde{\mathcal{X}}(W(\underline{k}))$ is denoted by ϖ_k. We will use the following obvious lemma implicitly.

Lemma 3.7.1. — *Let $P \in \mathcal{D}_{\underline{\ell}}$ and $Q \in \pi^{-1}(P)$. For $\mathfrak{a},\mathfrak{b} \in \mathcal{J}$, we have $\mathfrak{a} \leqslant_Q^{\varrho} \mathfrak{b}$, if and only if $\overline{\eta}_{\boldsymbol{m}(p)}(\mathfrak{a}) \leqslant^{\varrho}_{\varpi_{\underline{k}(p)}(Q)} \overline{\eta}_{\boldsymbol{m}(p)}(\mathfrak{b})$, where $\boldsymbol{m}(p) = \mathrm{ord}(\mathfrak{a}-\mathfrak{b})$.* □

For any $P \in \mathcal{D}$, let $\mathcal{J}_P$ denote the image of $\mathcal{J}$ by $M(\mathcal{X},\mathcal{D})/H(\mathcal{X}) \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})_P/\mathcal{O}_{\mathcal{X},P}$. For any i, we put $\mathcal{D}_i^* := \mathcal{D}_i \setminus \Big(\mathcal{X}^0 \cup \bigcup_{j\neq i} \mathcal{D}_j\Big)$. Note that the natural map $\mathcal{J} \to \mathcal{J}_P$ is bijective for any $P \in \mathcal{D}_1^*$.

Lemma 3.7.2. — *Let $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$. We take a small $\mathcal{U}_Q \in \mathfrak{U}(Q,\mathcal{J})$ and $\mathfrak{a},\mathfrak{b} \in \mathcal{J}$. Then, we have $\mathfrak{a} \leqslant_Q^{\varrho} \mathfrak{b}$, if and only if $\mathfrak{a} \leqslant_{Q'}^{\varrho} \mathfrak{b}$ for any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_1^*)$. More strongly, we have $\mathfrak{a} \leqslant_Q^{\varrho} \mathfrak{b}$, if and only if there is a dense subset B of $\mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_1^*)$ such that $\mathfrak{a} \leqslant_{Q'}^{\varrho} \mathfrak{b}$ for any $Q' \in B$.*

Proof. — We have $\mathfrak{a} <_{Q'}^{\varrho} \mathfrak{b}$ if and only if $F_{\mathfrak{a},\mathfrak{b}}(Q') < 0$. For fixed $\mathfrak{a},\mathfrak{b}$, after an appropriate coordinate change, we may assume $\mathfrak{a}-\mathfrak{b} = \boldsymbol{z}^{\boldsymbol{m}}$. Then, the claim of the lemma is clear. □

We state it in a slightly generalized form.

Lemma 3.7.3. — *Let $I \subset \underline{\ell}$.*

- *Take $P \in \mathcal{D}_I \setminus \bigcup_{j\notin I} \mathcal{D}_j$. We have the naturally induced bijective map $\mathcal{J}_P \to \mathcal{J}_{P'}$ for any $P' \in \mathcal{D}^*_{\min I}$.*
- *Let $Q \in \pi^{-1}(P)$. We take a small $\mathcal{U}_Q \in \mathfrak{U}(Q,\mathcal{J}_P)$ and $\mathfrak{a},\mathfrak{b} \in \mathcal{J}_P$. Then, we have $\mathfrak{a} \leqslant_Q^{\varrho} \mathfrak{b}$ if and only if $\mathfrak{a} \leqslant_{Q'}^{\varrho} \mathfrak{b}$ for any $Q' \in \mathcal{U}_P \cap \pi^{-1}(\mathcal{D}^*_{\min I})$. More strongly, we have $\mathfrak{a} \leqslant_Q^{\varrho} \mathfrak{b}$ if and only if there exists a dense subset $B \subset \mathcal{U}_P \cap \pi^{-1}(\mathcal{D}^*_{\min I})$ such that $\mathfrak{a} \leqslant_{Q'}^{\varrho} \mathfrak{b}$ for any $Q' \in B$.* □

3.7.2. Reduction. — Let $(\mathcal{E},\mathbb{D})$ be an unramifiedly good ϱ-meromorphic flat bundle on $(\mathcal{X},\mathcal{D})$ with a good lattice E and a good set of irregular values $\mathcal{J}$. We assume that the coordinate system is admissible for $\mathcal{J}$. We use the notation in Subsection 3.7.1. We shall construct the associated graded meromorphic ϱ-flat bundle $\mathrm{Gr}^{\boldsymbol{m}(p)}(\mathcal{E},\mathbb{D})$ with an unramifiedly good lattice $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)$ for any p, defined on a neighbourhood of $\mathcal{D}_{\underline{k}(0)}$. We remark $\mathcal{D}_{\underline{k}(0)}\subset\mathcal{D}_{\underline{k}(p)}$, which we will implicitly use.

3.7.2.1. One step reduction. — Let us consider the case where $\mathcal{J}(\boldsymbol{m}(p-1))$ consists of a unique element $\mathfrak{a}$. Then, $(E,\mathbb{D})$ is a weakly good lattice at the level $(\boldsymbol{m}(p),\mathfrak{h}(p))$. By the procedure in Subsection 3.6.5, after shrinking $\mathcal{X}$ around $\mathcal{D}_{\underline{k}(p)}$, we obtain a graded holomorphic bundle $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)=\bigoplus_{\mathfrak{a}\in\mathcal{J}(\boldsymbol{m}(p))}\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E)$ with a meromorphic flat ϱ-connection $\mathbb{D}^{\boldsymbol{m}(p)}=\bigoplus\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}}$ on $(\mathcal{X},\mathcal{D})$. Due to (54), the completion of $\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E,\mathbb{D}):=(\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E),\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}})$ along $W(\underline{k}(p))$ is naturally isomorphic to $\big(\widehat{E}^{\boldsymbol{m}(p)}_{\mathfrak{a},\widehat{W}(\underline{k}(p))},\mathbb{D}_{\mathfrak{a}}\big)$ in (28):

$$(\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E),\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}})_{|\widehat{W}(\underline{k}(p))}\simeq\big(\widehat{E}^{\boldsymbol{m}(p)}_{\mathfrak{a},\widehat{W}(\underline{k}(p))},\mathbb{D}_{\mathfrak{a}}\big).$$

In particular, $\big(\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E),\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}}\big)$ is also an unramifiedly good lattice. We have $\mathrm{Irr}(\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}})=\mathcal{J}_{\mathfrak{a}}$. In particular, its image by $\overline{\eta}_{\boldsymbol{m}(p)}$ consists of one element.

3.7.2.2. Reduction at the level $\boldsymbol{m}(p)$. — By shrinking $\mathcal{X}$ around $\mathcal{D}(\underline{k}(0))$, we shall inductively construct the unramifiedly good lattices

$$\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E,\mathbb{D})=\big(\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E),\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{b}}\big)$$

on $(\mathcal{X},\mathcal{D})$ for $\mathfrak{b}\in\mathcal{J}(\boldsymbol{m}(j))$ $(j=0,\dots,L)$ with the following property:

- The completion of $\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E,\mathbb{D})$ along $W(\underline{k}(p))$ is naturally isomorphic to $(\widehat{E}^{\boldsymbol{m}(p)}_{\mathfrak{b}},\widehat{\mathbb{D}}_{\mathfrak{b}})$ in (28):

$$\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E,\mathbb{D})_{|\widehat{W}(\underline{k}(p))}\simeq(\widehat{E}^{\boldsymbol{m}(p)}_{\mathfrak{b},\widehat{W}(\underline{k}(p))},\widehat{\mathbb{D}}_{\mathfrak{b}}).\tag{56}$$

Namely, for any $\mathfrak{b}\in\mathcal{J}(\boldsymbol{m}(p))$, we put $\mathfrak{a}:=\overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)}(\mathfrak{b})\in\mathcal{J}(\boldsymbol{m}(p-1))$, and we define

$$\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E,\mathbb{D}):=\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}\,\mathrm{Gr}^{\boldsymbol{m}(p-1)}_{\mathfrak{a}}(E,\mathbb{D}).$$

By (56), the following holds:

- We have $\mathrm{Irr}(\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{b}})=\overline{\eta}^{-1}_{\boldsymbol{m}(p)}(\mathfrak{b})$ for any $\mathfrak{b}\in\mathcal{J}(\boldsymbol{m}(p))$. In particular, its image by $\overline{\eta}_{\boldsymbol{m}(p)}$ consists of one element.
- We have the following natural isomorphism for each p:

$$(E,\mathbb{D})_{|\widehat{W}(\underline{k}(p))}\simeq\bigoplus_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(p))}\big(\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}(E),\mathbb{D}^{\boldsymbol{m}(p)}_{\mathfrak{a}}\big)_{|\widehat{W}(\underline{k}(p))}.$$

In particular, $\mathrm{Gr}^{\boldsymbol{m}(L)}_{\mathfrak{a}}(E,\mathbb{D})$ are $\mathfrak{a}$-logarithmic.

Remark 3.7.4. — In the following, we often formally put $\mathrm{Irr}(\mathbb{D},\boldsymbol{m}(-1)) := \{0\}$, $\mathrm{Gr}_0^{\boldsymbol{m}(-1)}(E) = E$ and $\mathbb{D}_0^{\boldsymbol{m}(-1)} := \mathbb{D}$. We also often use the symbol $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(E)$ instead of $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(L)}(E)$, which is called the full reduction of $(E,\mathbb{D})$. □

3.7.2.3. Functoriality of the associated graded bundle. — Let $(E_r,\mathbb{D}_r)$ $(r=1,2)$ be unramifiedly good such that $\mathrm{Irr}(\mathbb{D}_r) = \mathcal{J}_r$. By using Lemma 3.6.21 inductively, we obtain the following lemma.

Lemma 3.7.5. — *Let $F : (E_1,\mathbb{D}_1) \to (E_2,\mathbb{D}_2)$ be a morphism. Assume $\mathcal{J}_1 \cup \mathcal{J}_2$ is also good. We have the naturally induced flat morphisms $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(F) : \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_1) \to \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_2)$ for any $\mathfrak{a}$.* □

Corollary 3.7.6. — *If the restriction of F to $\mathcal{X} \setminus \mathcal{D}$ is an isomorphism, we obtain naturally induced isomorphisms for any $\mathfrak{a} \in \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(p))$:*

$$\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(F) : \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_1)(*\mathcal{D}) \longrightarrow \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_2)(*\mathcal{D}).$$

In particular, the graded meromorphic ϱ-flat bundle $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)(*\mathcal{D})$ is well defined for $\big(E(*\mathcal{D}),\mathbb{D}\big)$, in the sense that it is independent of the choice of an unramifiedly good lattice E. □

If $\mathcal{J}_1 \otimes \mathcal{J}_2$ is good, we obtain the following natural isomorphism for any $\mathfrak{a} \in \overline{\mathrm{Irr}}(\widetilde{\mathbb{D}},\boldsymbol{m}(p))$ by using Lemma 3.6.20 inductively:

$$\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}\big(E_1 \otimes E_2,\widetilde{\mathbb{D}}\big) \simeq \bigoplus_{\substack{\mathfrak{a}_p \in \overline{\mathrm{Irr}}(\mathbb{D}_p,\boldsymbol{m}(p))\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}} \mathrm{Gr}_{\mathfrak{a}_1}^{\boldsymbol{m}(p)}\big(E_1,\mathbb{D}_1\big) \otimes \mathrm{Gr}_{\mathfrak{a}_2}^{\boldsymbol{m}(p)}\big(E_2,\mathbb{D}_2\big).$$

If $\mathcal{J}_1 \oplus \mathcal{J}_2$ is good, we have

$$\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_1 \oplus E_2) \simeq \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_1) \oplus \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E_2).$$

If $(E,\mathbb{D})$ is unramifiedly good, the dual $(E^\vee,\mathbb{D}^\vee)$ is also unramifiedly good. By using Lemma 3.6.19 inductively, we obtain the following natural isomorphism for any $\mathfrak{a} \in \mathcal{J}$:

$$\mathrm{Gr}_{-\mathfrak{a}}^{\boldsymbol{m}(p)}(E^\vee,\mathbb{D}^\vee) \simeq \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E,\mathbb{D})^\vee.$$

3.7.3. Full and partial Stokes filtrations. — Let $(\mathcal{E},\mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X},\mathcal{D})$, with a good lattice E and a good set of irregular values $\mathcal{J}$. We shall explain the construction of full and partial Stokes filtrations of the stalks $\mathcal{E}_{|Q}$ for $Q \in \pi^{-1}(\mathcal{D})$.

As explained in Subsection 3.7.2, after shrinking $\mathcal{X}$ around $\mathcal{D}(\underline{k}(0))$, we may have the graded meromorphic ϱ-flat bundle $\mathrm{Gr}^{\boldsymbol{m}(p)}(\mathcal{E},\mathbb{D})$ with the unramifiedly good lattice $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)$ on $(\mathcal{X},\mathcal{D})$. For $k(0) \leqslant k \leqslant \ell$, let us consider the real blow up $\pi_k : \widetilde{\mathcal{X}}(W(\underline{k})) \to \mathcal{X}$. Let Q be any point of $\pi^{-1}(\mathcal{D}_{\underline{k}})$. The image of Q by $\varpi_{k(p),k} : \widetilde{\mathcal{X}}(W(\underline{k})) \to \widetilde{\mathcal{X}}(W(\underline{k}(p)))$ is denoted by Q_p. We have a small neighbourhood $\mathcal{U}_{Q_p}$

of Q_p in $\widetilde{\mathcal{X}}(W(\underline{k}(p)))$ and the Stokes filtration $\mathcal{F}^{Q_p}$ of $\mathrm{Gr}^{\boldsymbol{m}(p-1)}_{\mathfrak{a}}(E)_{|\mathcal{U}_{Q_p}}$ indexed by $\mathcal{J}(\boldsymbol{m}(p),\mathfrak{a})$ with $\leqslant_Q$, where $\mathfrak{a}\in\mathcal{J}(\boldsymbol{m}(p-1))$ and

$$\mathcal{J}(\boldsymbol{m}(p),\mathfrak{a}) := \big\{\mathfrak{b}\in\mathcal{J}(\boldsymbol{m}(p)), \overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)}(\mathfrak{b})=\mathfrak{a}\big\}.$$

We take a small neighbourhood $\mathcal{U}_Q$ of Q in $\widetilde{\mathcal{X}}(W(\underline{k}))$ such that $\varpi_{k(p),k}(\mathcal{U}_Q)\subset\mathcal{U}_{Q_p}$. We obtain the filtered bundle $\big(\mathrm{Gr}^{\boldsymbol{m}(p-1)}_{\mathfrak{a}}(E)_{\mathcal{U}_Q},\mathcal{F}^{Q_p}\big)$ for each $\mathfrak{a}\in\mathcal{J}(\boldsymbol{m}(p-1))$, and the associated graded bundle is naturally isomorphic to

$$\bigoplus_{\mathfrak{b}\in\mathcal{J}(\boldsymbol{m}(p),\mathfrak{a})}\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(E)_{|\mathcal{U}_Q},$$

By applying the inductive procedure in Section 3.1.1.4, we obtain the $\mathbb{D}$-flat filtration $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ of $E_{|\mathcal{U}_Q}$ indexed by the ordered set $\big(\mathcal{J}(\boldsymbol{m}(p)),\leqslant^{\varrho}_{Q}\big)$. It is called the partial Stokes filtration of $E_{|\mathcal{U}_Q}$ at the level $\boldsymbol{m}(p)$. In particular, $\mathcal{F}^{\boldsymbol{m}(L)\,Q}$ is called the full Stokes filtration, and denoted by $\widetilde{\mathcal{F}}^{Q}$. We have the induced filtrations of the stalks of E and $\mathcal{E}$ at Q, which are denoted by the same symbols.

The following compatibility is clear by construction.

Lemma 3.7.7. — *Let $Q\in\pi^{-1}(\mathcal{D}_{\underline{k}})$. We take neighbourhoods $\mathcal{U}_Q$ as above. Let $Q_1\in\pi^{-1}(\mathcal{D}_{\underline{k}})\cap\mathcal{U}_Q$. We take $\mathcal{U}_{Q_1}\subset\mathcal{U}_Q$. Then, the filtrations $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ and $\mathcal{F}^{Q_1\,\boldsymbol{m}(p)}$ of $E_{|\mathcal{U}_{Q_1}}$ are compatible over $\big(\mathcal{J}(\boldsymbol{m}(p)),\leqslant^{\varrho}_{Q}\big)\to\big(\mathcal{J}(\boldsymbol{m}(p)),\leqslant^{\varrho}_{Q_1}\big)$.* □

3.7.3.1. Functoriality. — Let $(E_r,\mathbb{D}_r)$ $(r=1,2)$ be unramifiedly good such that $\mathrm{Irr}(\mathbb{D}_r)=\mathcal{J}_r$. By using Proposition 3.3.5, we obtain the following lemma.

Lemma 3.7.8. — *Let $F:(E_1,\mathbb{D}_1)\to(E_2,\mathbb{D}_2)$ be a morphism. Assume $\mathcal{J}_1\cup\mathcal{J}_2$ is also good. Let $Q\in\pi^{-1}(\mathcal{D}_{\underline{k}})$, where $k(0)\leqslant k\leqslant\ell$. The morphisms of stalks $F_Q: E_{1,Q}\to E_{2,Q}$ preserve the Stokes filtrations at the level $\boldsymbol{m}(p)$. In particular, the filtrations $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ of the stalk $\mathcal{E}_Q$ are well defined for $(\mathcal{E},\mathbb{D})$ in the sense that they are independent of the choice of an unramifiedly good lattice E.* □

By using the propositions in Subsection 3.3.3 inductively, we also obtain the functoriality of the full and partial Stokes filtrations for dual, tensor product and direct sum as in Subsection 3.2.2. We have a similar functoriality for the partial Stokes filtrations.

3.7.4. Compatibility and characterization. — By applying the procedure explained in Subsection 3.7.3 to the restriction of $(\mathcal{E},\mathbb{D})$ to a small neighbourhood of any point of $\mathcal{D}$, we obtain the full Stokes filtration of the stalk of E at any point of $\pi^{-1}(\mathcal{D})$. We shall argue the comparison of the filtrations.

3.7.4.1. Preliminary. — We consider the case $\mathcal{D} = \mathcal{D}_1$. For simplicity, we assume that ϱ is nowhere vanishing. Let $(\mathcal{E}, \mathbb{D})$ be an unramifiedly good meromorphic ϱ-flat bundle on $(\mathcal{X}, \mathcal{D})$ with a good lattice E and a good set of irregular values $\mathcal{J}$. We can use the order of the poles of elements of $\mathcal{J}$ as the auxiliary sequence. For $Q \in \pi^{-1}(\mathcal{D})$, we have the partial Stokes filtrations $\mathcal{F}^{(m)}$ of $\mathcal{E}_Q$ and E_Q.

Lemma 3.7.9. — *Let $\overline{\mathcal{F}}^{(m)}$ be a $\mathbb{D}$-flat filtration of E_Q indexed by $(\mathcal{J}(m), \leqslant_Q^{\varrho})$ such that $\overline{\mathcal{F}}^{(m)}_{|\widehat{Q}} = \mathcal{F}^{(m)}_{|\widehat{Q}}$. Then, we obtain $\overline{\mathcal{F}}^{(m)} = \mathcal{F}^{(m)}$.*

Proof. — If we take a sufficiently small neighbourhood $\mathcal{U}_Q$ of Q in $\widetilde{\mathcal{X}}(\mathcal{D})$, we obtain $\overline{\mathcal{F}}^{(m)}_{|\widehat{\pi^{-1}(\mathcal{D})\cap\mathcal{U}_Q}} = \mathcal{F}^{(m)}_{|\widehat{\pi^{-1}(\mathcal{D})\cap\mathcal{U}_Q}}$. Then, we obtain $\overline{\mathcal{F}}^{(m)} = \mathcal{F}^{(m)}$ by using the argument in the proof of Lemma 3.5.3. □

3.7.4.2. Let us return to the original setting. Let $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$. We take a small $\mathcal{U}_Q \in \mathfrak{U}(Q, \mathcal{J})$. We set $\mathcal{D}_i^* := \mathcal{D}_i \setminus \big(W \cup \bigcup_{j\neq i} \mathcal{D}_j\big)$.

Lemma 3.7.10. — *Take any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_i^*)$ for some $1 \leqslant i \leqslant \ell$. Then, the filtrations $\widetilde{\mathcal{F}}^Q$ and $\widetilde{\mathcal{F}}^{Q'}$ are compatible over $\big(\mathcal{J}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{J}_{Q'}, \leqslant_{Q'}^{\varrho}\big)$.*

Proof. — We construct a filtration $\overline{\mathcal{F}}$ of $E_{|\mathcal{U}_Q}$ from $\widetilde{\mathcal{F}}^Q$ and $\big(\mathcal{J}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{J}_{Q'}, \leqslant_{Q'}^{\varrho}\big)$. By construction of $\widetilde{\mathcal{F}}^Q$, we can easily check that $\overline{\mathcal{F}}_{|\widehat{Q}'} = \widetilde{\mathcal{F}}^{Q'}$. Then, the claim follows from Lemma 3.7.9. □

Let us note that $\widetilde{\mathcal{F}}^Q$ can be reconstructed from the filtrations $\widetilde{\mathcal{F}}^{Q'}(Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_1^*))$ in the following sense.

Lemma 3.7.11. — *Let $\overline{\mathcal{F}}$ be a filtration of $E_{|\mathcal{U}_Q}$ such that $\overline{\mathcal{F}}_{|\widehat{Q}'}$ and $\widetilde{\mathcal{F}}^{Q'}_{|\widehat{Q}'}$ are compatible over $\big(\mathcal{J}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{J}_{\pi(Q')}, \leqslant_{Q'}^{\varrho}\big)$ for any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}_1^*)$. Then, we have $\widetilde{\mathcal{F}}^Q = \overline{\mathcal{F}}$.*

Proof. — It follows from Lemma 3.1.3 and Lemma 3.7.2. □

We state it in a slightly generalized form.

Lemma 3.7.12. — *Let $Q \in \pi^{-1}(\mathcal{D})$. Let $i(Q) := \min\{i \,|\, \pi(Q) \in \mathcal{D}_i\}$. Take a sufficiently small neighbourhood $\mathcal{U}_Q$ of Q. Let $\overline{\mathcal{F}}$ be a filtration of $E_{|\mathcal{U}_Q}$ indexed by $\big(\mathcal{J}_{\pi(Q)}, \leqslant_Q^{\varrho}\big)$ with the following property:*

- *For any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D}^*_{i(Q)})$, the filtrations $\overline{\mathcal{F}}_{|\widehat{Q}'}$ and $\widetilde{\mathcal{F}}^{Q'}_{|\widehat{Q}'}$ are compatible over $\big(\mathcal{J}_{\pi(Q)}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{J}_{\pi(Q')}, \leqslant_{Q'}^{\varrho}\big)$.*

Then, $\overline{\mathcal{F}} = \widetilde{\mathcal{F}}^Q$. □

Lemma 3.7.13. — *Let $Q \in \pi^{-1}(\mathcal{D})$. We take a sufficiently small neighbourhood $\mathcal{U}_Q$ of Q in $\widetilde{\mathcal{X}}(W)$. Then, for any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D})$, the filtrations $\widetilde{\mathcal{F}}^Q$ and $\widetilde{\mathcal{F}}^{Q'}$ are compatible over $\big(\mathcal{J}_{\pi(Q)}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{J}_{\pi(Q')}, \leqslant_{Q'}^{\varrho}\big)$.*

Proof. — We construct $\overline{\mathcal{F}}$ from $\widetilde{\mathcal{F}}^{Q}$ by $\big(\mathcal{J}_{\pi(Q)}, \leqslant^{\varrho}_{Q}\big) \to \big(\mathcal{J}_{\pi(Q')}, \leqslant^{\varrho}_{Q'}\big)$. By Lemma 3.7.10, $\overline{\mathcal{F}}$ satisfies the condition in Lemma 3.7.12 for $\widetilde{\mathcal{F}}^{Q'}$. Hence, we obtain $\overline{\mathcal{F}} = \widetilde{\mathcal{F}}^{Q'}$. □

Let us compare the Stokes filtrations of $(E, \mathbb{D})$ and $\mathrm{Gr}^{\boldsymbol{m}(p)}(E, \mathbb{D})$. Let $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$. Let $\mathcal{U}_Q$ be a small neighbourhood of Q in $\widetilde{\mathcal{X}}(W)$. We have the full Stokes filtration $\widetilde{\mathcal{F}}^{Q}$ and the partial Stokes filtration $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ of $E_{|\mathcal{U}_Q}$. By construction, we have a natural isomorphism

$$\mathrm{Gr}^{\mathcal{F}^{Q\,\boldsymbol{m}(p)}}(E_{|\mathcal{U}_Q}) \simeq \mathrm{Gr}^{\boldsymbol{m}(p)}(E)_{|\mathcal{U}_Q}. \tag{57}$$

Let $R \in \mathcal{U}_Q$. Take a small neighbourhood $\mathcal{U}_R \subset \mathcal{U}_Q$. We have the full Stokes filtration $\widetilde{\mathcal{F}}^{R}(E_{\mathcal{U}_R})$ which induces a filtration on the left-hand side of (57). We also have the full Stokes filtration $\widetilde{\mathcal{F}}^{R}(\mathrm{Gr}^{\boldsymbol{m}(p)}(E)_{|\mathcal{U}_R})$, i.e., the right-hand side of (57).

Corollary 3.7.14. — (57) *is an isomorphism of filtered bundles.*

Proof. — By Lemma 3.7.13, both filtrations are induced by the full Stokes filtration $\widetilde{\mathcal{F}}$ of $E_{|\mathcal{U}_Q}$. □

Corollary 3.7.15. — *Let $P \in \mathcal{D}_i$ be a smooth point of $\mathcal{D}$. Assume that the i-th component of $\boldsymbol{m}(p)$ is negative. Then, we have a natural isomorphism $\mathrm{Gr}^{(-1)}(E_P) \simeq \mathrm{Gr}^{(-1)}\mathrm{Gr}^{\boldsymbol{m}(p)}(E_P)$.* □

3.7.4.3. Let us show a refinement of Lemma 3.7.9 in the normal crossing case.

Lemma 3.7.16. — *Let $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$ such that $\varrho(Q) \neq 0$. Let $\overline{\mathcal{F}}$ be a $\mathbb{D}$-flat filtration of E_Q indexed by $\big(\mathcal{J}, \leqslant^{\varrho}_{Q}\big)$ such that $\overline{\mathcal{F}}_{|\widehat{Q}} = \widetilde{\mathcal{F}}^{Q}_{|\widehat{Q}}$. Then, we have $\overline{\mathcal{F}} = \widetilde{\mathcal{F}}^{Q}$.*

Proof. — We take a small $\mathcal{U}_Q \in \mathfrak{U}(Q, \mathcal{J})$ on which $E_{|\mathcal{U}_Q}$ has the full Stokes filtration $\widetilde{\mathcal{F}}^{Q}$ and the filtration $\overline{\mathcal{F}}$. We can take a linear map $\varphi : \Delta \to \Delta^n$ such that (i) the image of the induced map $\varphi_{\mathcal{K}} : \Delta \times \mathcal{K} \to \mathcal{X}$ is not contained in $\mathcal{D}$, (ii) Q is contained in the image of the induced map $\widetilde{\varphi}_{\mathcal{K}} : \widetilde{\Delta}(0) \times \mathcal{K} \to \widetilde{\mathcal{X}}(\mathcal{D})$. Let R be the inverse image of Q via $\widetilde{\varphi}_{\mathcal{K}}$. We take a small neighbourhood $\mathcal{U}_R$ of R in $\widetilde{\Delta}(0) \times \mathcal{K}$. By the $\mathbb{D}$-flatness, we only have to show $\overline{\mathcal{F}}_{|\varphi_{\mathcal{K}}(\mathcal{U}_R)} = \mathcal{F}^{Q}_{|\varphi_{\mathcal{K}}(\mathcal{U}_R)}$.

The pull-back $\varphi_{\mathcal{K}}^{*}(\mathcal{E}, \mathbb{D})$ has the unramifiedly good lattice $\varphi_{\mathcal{K}}^{*}E$, and the set of irregular values is given by $\mathcal{J}_1 := \{\varphi^{*}\mathfrak{a} \mid \mathfrak{a} \in \mathcal{J}\}$. We remark that the natural map $\mathcal{J} \to \mathcal{J}_1$ is bijective, and the orders $\leqslant^{\varrho}_{Q}$ and $\leqslant^{\varrho}_{R}$ are the same. Then, by Lemma 3.7.9, we obtain that the restrictions of $\widetilde{\mathcal{F}}^{Q}$ and $\overline{\mathcal{F}}$ to $\varphi(\mathcal{U}_R)$ are equal to the full Stokes filtration of $\varphi^{*}(E)_{|\mathcal{U}_R}$. □

3.7.5. Splitting of the full Stokes filtration

3.7.5.1. Flat splitting. — Let $(E, \mathbb{D})$ be an unramifiedly good lattice on $(\mathcal{X}, \mathcal{D})$ with $\mathrm{Irr}(\mathbb{D}) = \mathcal{J}$. First, we consider the non-resonant case.

Condition 3.7.17. — We have $\alpha - \beta \notin \mathbb{Z}$ for any distinct eigenvalues α, β of $\mathrm{Res}_j(\varrho^{-1}\mathbb{D})_{|\mathcal{D}_j}$ $(j = 2, \ldots, \ell)$. □

If $(E, \mathbb{D})$ satisfies Condition 3.7.17, the induced lattices $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{m}(p)}(E, \mathbb{D})$ also satisfy Condition 3.7.17 for any $\mathfrak{a} \in \mathcal{J}(\boldsymbol{m}(p))$, which follows from (56).

Proposition 3.7.18. — *Assume that $(E, \mathbb{D})$ satisfies Condition 3.7.17. Let k satisfy $k(0) \leqslant k \leqslant \ell$. Take $Q \in \pi_k^{-1}(\mathcal{D}) \subset \widetilde{\mathcal{X}}(W(\underline{k}))$ such that $\varrho(Q) \neq 0$. Then, there exists a small neighbourhood $\mathcal{U}_Q$ on which we can take a $\mathbb{D}$-flat splitting $E_{|\mathcal{U}_Q} = \bigoplus_{\mathfrak{a} \in \mathcal{J}} E_{\mathfrak{a}, \mathcal{U}_Q}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^S$.*

Proof. — We only have to consider the case $Q \in \pi_k^{-1}(\mathcal{D}_{\underline{\ell}})$. We take $\mathcal{U}_{Q_p}$ as in Subsection 3.7.3. By Proposition 3.6.8, we can find a $\mathbb{D}$-flat splitting of the Stokes filtration $\mathcal{F}^{Q_p\, \boldsymbol{m}(p)}$, i.e., $\mathrm{Gr}^{\boldsymbol{m}(p-1)}(E)_{|\mathcal{U}_{Q_p}} \simeq \mathrm{Gr}^{\boldsymbol{m}(p)}(E)_{|\mathcal{U}_{Q_p}}$. Then, we can construct a desired splitting by lifting the splittings inductively. □

Proposition 3.7.19. — *Assume that $\mathcal{D}$ is smooth. For any $Q \in \pi^{-1}(\mathcal{D})$ there exists a small neighbourhood $\mathcal{U}_Q$ on which we can take a $\mathbb{D}$-flat splitting $E_{|\overline{S}} = \bigoplus_{\mathfrak{a} \in \mathcal{J}} E_{\mathfrak{a}, S}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $E_{|\overline{S}}$.*

Proof. — We only have to apply Lemma 3.6.9 inductively. □

3.7.5.2. Partially flat splitting. — Even in the general case, we have partially flat splittings, a fact which can be shown by the argument in the proof of Proposition 3.7.18.

Lemma 3.7.20. — *Let $k(0) \leqslant k \leqslant \ell$, and let $Q \in \pi_k^{-1}(\mathcal{D}) \subset \widetilde{\mathcal{X}}(W(\underline{k}))$. There exists a neighbourhood $\mathcal{U}_Q$ on which we can take a $\mathbb{D}_{\leqslant k(p)}$-flat splitting of the partial Stokes filtration $\mathcal{F}^{Q\, \boldsymbol{m}(p)}$ at the level $\boldsymbol{m}(p)$.* □

3.7.6. Characterization of holomorphic sections of E. — Let $k(0) \leqslant k \leqslant \ell$. Let $\boldsymbol{v}$ be a frame of E, and $\boldsymbol{u}$ be a frame of $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)$. Take $Q \in \pi_k^{-1}(\mathcal{D}_{\underline{\ell}})$ and a small neighbourhood $\mathcal{U}_Q$ on which we have the Stokes filtrations $\mathcal{F}^{Q\, \boldsymbol{m}(p)}$ and its splitting $E_{|\mathcal{U}_Q} \simeq \bigoplus_{\mathfrak{b} \in \mathcal{J}(\boldsymbol{m}(p))} \mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{m}(p)}(E)_{|\mathcal{U}_Q}$. By using the splitting, we obtain a frame $\boldsymbol{u}_Q$ of $E_{|\mathcal{U}_Q}$.

Lemma 3.7.21. — *Let G_Q be the matrix-valued function determined by $\boldsymbol{v}_{|\mathcal{U}_Q} = \boldsymbol{u}_Q\, G_Q$. Then, G_Q and G_Q^{-1} are bounded on $\mathcal{U}_Q$.* □

We take $Q_j \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$ $(j = 1, \dots, N)$ such that $\bigcup_{i=1}^{N} \mathcal{U}_{Q_i}$ contains $\pi^{-1}(\mathcal{D}_{\underline{\ell}})$. We take $\boldsymbol{u}_{Q_i}$ as above. Let f be a holomorphic section of $E_{|\mathcal{X}\setminus\mathcal{D}}$. We have the expressions $f_{|\mathcal{U}_{Q_i}\setminus\pi^{-1}(W)} = \sum f_{Q_i,j}\, u_{Q_i,j}$, where $f_{Q_i,j}$ are holomorphic functions on $\mathcal{U}_{Q_i} \setminus \pi^{-1}(W)$. We obtain the following lemma from Lemma 3.7.21.

Lemma 3.7.22. — *f is a section of E if and only if all $f_{Q_i,j}$ are bounded.* □

3.7.7. Characterization by growth order. — Assume that ϱ is nowhere vanishing. Let $\boldsymbol{v}$ be a holomorphic frame of E on $\mathcal{X}$. Let $Q \in \pi^{-1}(\mathcal{D}_{\underline{\ell}})$, and let $\mathcal{U}_Q$ be a small neighbourhood of Q. Let f be a flat section of $E_{|\mathcal{U}_Q\setminus\pi^{-1}(\mathcal{D})}$. We have the expression $f = \sum f_j\, v_j$, and obtain $\boldsymbol{f} = (f_j)$.

Proposition 3.7.23. — *f is contained in $\mathcal{F}^{Q,\boldsymbol{m}(p)}_{\mathfrak{a}}(E_{|\mathcal{U}_Q\setminus\pi^{-1}(\mathcal{D})})$ if and only if the following estimate holds for some $C > 0$ and $N > 0$:*

$$\Big|\boldsymbol{f}\,\exp\big(\varrho^{-1}\overline{\eta}_{\boldsymbol{m}(p)}(\mathfrak{a})\big)\Big| = O\Big(\exp\big(C|\boldsymbol{z}^{\boldsymbol{m}(p+1)}|\big) \prod_{k(p+1)<j\leqslant \ell} |z_j|^{-N}\Big).$$

Proof. — We may replace E with a lattice satisfying Condition 3.7.17 by a meromorphic transform. Hence, we may and will assume that the condition is satisfied from the beginning. Due to Proposition 3.7.18, we can take a flat splitting $E_{|\mathcal{U}_Q} = \bigoplus E_{\mathfrak{a},Q}$ of the partial Stokes filtration $\mathcal{F}^{Q,\boldsymbol{m}(p)}$. Let $\boldsymbol{u}$ be a frame of $\mathrm{Gr}^{\boldsymbol{m}(p)}(E)$ compatible with the grading, and let $\boldsymbol{u}_Q$ be the lift of $\boldsymbol{u}$ to $E_{\mathfrak{a},Q}$ on $\mathcal{U}_Q$ via the splitting.

We have the expression $f = \sum f_{Q,j} \cdot u_{Q,j}$. Let $\boldsymbol{f}_Q := (f_{Q,j})$. Corresponding to the grading, we have the decomposition $\boldsymbol{f}_Q = (\boldsymbol{f}_{Q,\mathfrak{b}} \,|\, \mathfrak{b} \in \mathcal{J}(\boldsymbol{m}(p)))$. By Lemma 3.7.21, $\sum |\boldsymbol{f}_{Q,\mathfrak{b}}|$ and $|\boldsymbol{f}|$ are mutually bounded. Then, the claim follows from Lemma 3.6.24. □

3.7.8. Proof of the claims in Section 3.2. — The filtrations in Theorem 3.2.1 was constructed in Subsection 3.7.3. It clearly satisfies the first claim in the theorem. The compatibility is given in Lemma 3.7.13. By Lemma 3.7.9 and Lemma 3.7.12, the conditions characterize the filtrations. If $\varrho(Q) \neq 0$, the first property suffices for characterization according to Lemma 3.7.16. Thus, Theorem 3.2.1 is proved.

As remarked in Subsection 3.7.3.1, the functoriality of the full Stokes filtration follows from the inductive construction of the full Stokes filtration and the functoriality in Stokes filtration of weakly good lattices in Subsection 3.3.3. Proposition 3.2.6 is Proposition 3.7.23. Proposition 3.2.8 and the functoriality of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}$ is clear from our construction of the full Stokes filtration. Proposition 3.2.9 is also clear.

According to Proposition 2.7.5, we can locally take a non-resonant lattice. Hence, we obtain Proposition 3.2.10 from Proposition 3.7.18. Proposition 3.2.11 also follows from Proposition 3.7.18. Proposition 3.2.12 is Proposition 3.7.19. As remarked in Lemma 3.7.20, we can obtain Proposition 3.2.13 and Proposition 3.2.14 by an inductive use of Proposition 3.3.10. We obtain Proposition 3.2.15 by successive use of Proposition 3.6.1. We also obtain Proposition 3.2.16 by successive use of Proposition 3.3.11.

CHAPTER 4

FULL STOKES DATA AND RIEMANN-HILBERT-BIRKHOFF CORRESPONDENCE

In this chapter, we study the Stokes structure with more details, assuming that ϱ is nowhere vanishing. It is our purpose to describe an irregular singularity in terms of more (though not completely) topological data, called full Stokes data.

In Section 4.1, we will introduce the notion of full pre-Stokes data, which is a system of filtrations of a ϱ-flat bundle on the real blow up. If we are interested in only ordinary meromorphic flat bundles, we only have to consider pre-Stokes data. But, we are interested in families and lattices, too. So we shall introduce the notion of Stokes data in Section 4.2, which is a set of full pre-Stokes data with lattices. Then, in Section 4.3, we will establish the correspondence between Stokes data and unramifiedly good lattices of meromorphic ϱ-flat bundles, which is called Riemann-Hilbert-Birkhoff correspondence in this monograph. (We follow [**53**], where this kind of problem is called "the Riemann-Hilbert-Birkhoff problem of weak sense".)

As an application, we study the extension of a ϱ-meromorphic flat bundle in Section 4.4. The special case given in Section 4.5 will play an important role in our study of wild harmonic bundles.

4.1. Full pre-Stokes data

4.1.1. Definition. — Let $\mathcal{X} \to \mathcal{K}$ be a smooth fibration of complex manifolds. Let $\mathcal{D}$ be a normal crossing hypersurface of $\mathcal{X}$ such that all intersections of irreducible components are smooth over $\mathcal{K}$. Let ϱ be a nowhere vanishing holomorphic function on $\mathcal{X}$. Let $\pi : \widetilde{\mathcal{X}}(\mathcal{D}) \to \mathcal{X}$ be the real blow up. The pull-back of $\mathcal{O}_{\mathcal{K}}$ via the projection $\widetilde{\mathcal{X}}(\mathcal{D}) \to \mathcal{K}$ is also denoted by $\mathcal{O}_{\mathcal{K}}$. Its restriction to a subset of $\widetilde{\mathcal{X}}(\mathcal{D})$ is also denoted by $\mathcal{O}_{\mathcal{K}}$.

Definition 4.1.1. — Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values on $(\mathcal{X}, \mathcal{D})$. Let $\mathcal{U}$ be a locally simply connected subset of $\widetilde{\mathcal{X}}(\mathcal{D})$. Let $\mathfrak{V}$ be a locally free $\mathcal{O}_{\mathcal{K}}$-module on $\mathcal{U}$. Full pre-Stokes data of $\mathfrak{V}$ over $\boldsymbol{\mathcal{I}}$ is a system $\widetilde{\boldsymbol{\mathcal{F}}}$ of filtrations $\widetilde{\mathcal{F}}^Q$ of germs $\mathfrak{V}_Q$

$(Q \in \mathcal{U} \cap \pi^{-1}(\mathcal{D}))$ indexed by $\big(\mathcal{I}_{\pi(Q)}, \leqslant_Q^{\varrho}\big)$ satisfying the following compatibility condition:

- Let $Q \in \mathcal{U} \cap \pi^{-1}(\mathcal{D})$. Take a small neighbourhood $\mathcal{U}_Q$ in $\mathcal{U}$ on which the filtration $\widetilde{\mathcal{F}}^Q$ is given. Note that when $\mathcal{U}_Q$ is sufficiently small, we have $\mathfrak{a} \leqslant_Q \mathfrak{b}$ if and only if $\mathfrak{a} \leqslant_{Q'} \mathfrak{b}$ for any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D})$. Then, for any $Q' \in \mathcal{U}_Q \cap \pi^{-1}(\mathcal{D})$, $\big(\mathfrak{V}_{Q'}, \widetilde{\mathcal{F}}^Q\big) \to \big(\mathfrak{V}_{Q'}, \widetilde{\mathcal{F}}^{Q'}\big)$ is compatible over $\big(\mathcal{I}_{\pi(Q)}, \leqslant_Q^{\varrho}\big) \to \big(\mathcal{I}_{\pi(Q')}, \leqslant_{Q'}^{\varrho}\big)$. □

By the compatibility, we have the associated graded $\mathcal{O}_{\mathcal{K}}$-module on $\pi^{-1}(P) \cap \mathcal{U}$ $(P \in \mathcal{D})$, which is denoted by $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathfrak{V}_{\pi^{-1}(P)\cap\mathcal{U}})$. Note that $\mathfrak{V}$ can be extended to a locally free $\mathcal{O}_{\mathcal{K}}$-module on a neighbourhood of $\mathcal{U}$ in $\widetilde{\mathcal{X}}(\mathcal{D})$, and $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathfrak{V}_{\pi^{-1}(P)\cap\mathcal{U}})$ is naturally extended to a locally free $\mathcal{O}_{\mathcal{K}}$-module on a neighbourhood of $\pi^{-1}(P) \cap \mathcal{U}$.

4.1.1.1. Let $\mathfrak{V}_i$ $(i = 1, 2)$ be locally free $\mathcal{O}_{\mathcal{K}}$-modules on $U \subset \widetilde{\mathcal{X}}(\mathcal{D})$ equipped with full pre-Stokes data $\widetilde{\boldsymbol{\mathcal{F}}}_i$ over $\boldsymbol{\mathcal{I}}$. A morphism $F : (\mathfrak{V}_1, \widetilde{\boldsymbol{\mathcal{F}}}_1) \to (\mathfrak{V}_2, \widetilde{\boldsymbol{\mathcal{F}}}_2)$ is defined to be a morphism of $\mathcal{O}_{\mathcal{K}}$-modules such that the induced morphisms $\mathfrak{V}_{1\,Q} \to \mathfrak{V}_{2\,Q}$ preserve filtrations for any $Q \in \pi^{-1}(\mathcal{D})$. If $\mathfrak{V}$ is equipped with a set of full pre-Stokes data over $\boldsymbol{\mathcal{I}}$, then the dual $\mathfrak{V}^{\vee}$ is equipped with induced full pre-Stokes data over $\boldsymbol{\mathcal{I}}^{\vee}$. Let $\mathfrak{V}_i$ be equipped with full pre-Stokes data over $\boldsymbol{\mathcal{I}}_i$. If $\boldsymbol{\mathcal{I}}_1 \otimes \boldsymbol{\mathcal{I}}_2$ is good, then $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ is equipped with induced full pre-Stokes data over $\boldsymbol{\mathcal{I}}_1 \otimes \boldsymbol{\mathcal{I}}_2$. If $\boldsymbol{\mathcal{I}}_1 \oplus \boldsymbol{\mathcal{I}}_2$ is good, then $\mathfrak{V}_1 \oplus \mathfrak{V}_2$ is equipped with induced full pre-Stokes data over $\boldsymbol{\mathcal{I}}_1 \oplus \boldsymbol{\mathcal{I}}_2$.

4.1.1.2. Uniqueness. — Let U be an open subset of $\pi^{-1}(\mathcal{D})$. Let $\mathfrak{V}$ be a locally free $\mathcal{O}_{\mathcal{K}}$-module on U.

Lemma 4.1.2. — *Let $\widetilde{\boldsymbol{\mathcal{F}}}_i$ $(i = 1, 2)$ be full pre-Stokes data of $\mathfrak{V}$.*

- *If there exists a dense subset $U' \subset U$ such that $\widetilde{\mathcal{F}}_1^Q = \widetilde{\mathcal{F}}_2^Q$ for $Q \in U'$. Then, we have $\widetilde{\boldsymbol{\mathcal{F}}}_1 = \widetilde{\boldsymbol{\mathcal{F}}}_2$.*
- *Let Z be any subset of U. If $\widetilde{\mathcal{F}}_1^Q = \widetilde{\mathcal{F}}_2^Q$ for any $Q \in Z$, there exists a neighbourhood Z' of Z such that $\widetilde{\mathcal{F}}_1^Q = \widetilde{\mathcal{F}}_2^Q$ for any $Q \in Z'$.*

Proof. — The first claim easily follows from Lemma 3.7.3 and Lemma 3.1.3. The second claim is clear from the compatibility condition. □

Let $\mathfrak{V}_i$ $(i = 1, 2)$ be $\mathcal{O}_{\mathcal{K}}$-modules on U with a morphism $F : \mathfrak{V}_1 \to \mathfrak{V}_2$. It is easy to deduce the following corollary.

Corollary 4.1.3. — *Let $\widetilde{\boldsymbol{\mathcal{F}}}_i$ $(i = 1, 2)$ be full pre-Stokes structures of $\mathfrak{V}_i$.*

- *If there exists a dense subset $U' \subset U$ such that F preserves $\widetilde{\mathcal{F}}^Q$ for $Q \in U'$. Then, F preserves $\widetilde{\boldsymbol{\mathcal{F}}}$.*
- *Let Z be any subset of U. If F preserves $\widetilde{\mathcal{F}}^Q$ for any $Q \in Z$, there exists a neighbourhood Z' of Z such that F preserves $\widetilde{\mathcal{F}}^Q$ for any $Q \in Z'$.* □

4.1.2. Filtration on a small convex set. — We put $\mathcal{X} := \Delta^n \times \mathcal{K}$, $\mathcal{D}_i = \{z_i = 0\}$, $\mathcal{D} := \bigcup_{i=1}^{\ell} \mathcal{D}_i$ and $\mathcal{D}_{\underline{\ell}} := \bigcap_{i=1}^{\ell} \mathcal{D}_i$. Let $\pi : \widetilde{X}(D) \to X$ be the real blow up. We have the natural identification $\pi^{-1}(\mathcal{D}_{\underline{\ell}}) = (S^1)^\ell \times \mathcal{D}_{\underline{\ell}}$ by the coordinate $(z_1, \dots, z_n)$. We use the polar coordinate $(\theta_1, \dots, \theta_\ell)$ for $(S^1)^\ell$, induced by $(z_1, \dots, z_\ell)$. Let $\mathcal{I} \subset M(X, D)/H(X)$ be a good set of irregular values.

Condition 4.1.4. — Let $P \in \mathcal{D}_{\underline{\ell}}$. Let $\mathcal{C}$ be a closed convex subset of $(S^1)^\ell$ satisfying the following:

- There exist $(\theta_1^{(0)}, \dots, \theta_\ell^{(0)})$ such that $\mathcal{C} \subset \{(\theta_1, \dots, \theta_\ell) \,|\, |\theta_i - \theta_i^{(0)}| < \pi/2\}$. In particular, we can identify $\mathcal{C}$ with a closed region in $\boldsymbol{R}^\ell$.
- We naturally regard $\mathcal{C}(P) := \mathcal{C} \times \{P\}$ as a subset of $\pi^{-1}(D_{\underline{\ell}})$. Then, for each pair $(\mathfrak{a}, \mathfrak{b})$ of distinct elements of $\mathcal{I}$, if $\mathcal{C}(P) \cap F_{\mathfrak{a},\mathfrak{b}}^{-1}(0) \neq \varnothing$, it divides $\mathcal{C}(P)$ into two closed regions. □

Proposition 4.1.5. — *Let $\mathfrak{V}$ be a locally free $\mathcal{O}_{\mathcal{K}}$-module on $\mathcal{C}(P)$ with a set of full pre-Stokes data $(\widetilde{\mathcal{F}}^Q \,|\, Q \in \mathcal{C}(P))$. Then, there exists a unique global filtration $\widetilde{\mathcal{F}}^{\mathcal{C}(P)}$ indexed by $(\mathcal{I}_P, \leqslant^{\varrho}_{\mathcal{C}(P)})$ such that, for any $Q \in \mathcal{C}(P)$, the filtrations $\widetilde{\mathcal{F}}^{\mathcal{C}(P)}$ and $\widetilde{\mathcal{F}}^Q$ are compatible over $(\mathcal{I}_P, \leqslant^{\varrho}_{\mathcal{C}(P)}) \to (\mathcal{I}_P, \leqslant^{\varrho}_Q)$. In other words, there exists a decomposition $\mathfrak{V} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \mathfrak{V}_{\mathfrak{a}}$, which gives a splitting of $\widetilde{\mathcal{F}}^Q$ for any $Q \in \mathcal{C}(P)$.*

Proof. — In the proof, $\mathcal{C}(P)$ is denoted by $\mathcal{C}$ for simplicity of the description. Let $\boldsymbol{\mathfrak{V}}$ be the space of global sections of $\mathfrak{V}$. We have a natural isomorphism $\boldsymbol{\mathfrak{V}} \simeq \mathfrak{V}_Q$ for every $Q \in \mathcal{C}$, from which we obtain a filtration $\widetilde{\mathcal{F}}^Q$ $(Q \in \mathcal{C})$ on $\boldsymbol{\mathfrak{V}}$. We shall show that there exists a unique filtration $\widetilde{\mathcal{F}}^{\mathcal{C}}$ of $\boldsymbol{\mathfrak{V}}$ such that for any $Q \in \mathcal{C}$, the filtrations $\widetilde{\mathcal{F}}^{\mathcal{C}}$ and $\widetilde{\mathcal{F}}^Q$ are compatible over $(\mathcal{I}_P, \leqslant^{\varrho}_{\mathcal{C}}) \to (\mathcal{I}_P, \leqslant^{\varrho}_Q)$.

For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P$, we have $\mathfrak{a} \leqslant^{\varrho}_{\mathcal{C}} \mathfrak{b}$ if and only if $\mathfrak{a} \leqslant^{\varrho}_Q \mathfrak{b}$ for any $Q \in \mathcal{C}$. Hence, the uniqueness of such a filtration follows from Lemma 3.1.3.

Put $H_{\mathfrak{a},\mathfrak{b}} := F_{\mathfrak{a},\mathfrak{b}}^{-1}(0)$ for distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. A connected component of $\mathcal{C} \setminus \bigcup H_{\mathfrak{a},\mathfrak{b}}$ is called a chamber. If Q is contained in a chamber, then $\leqslant^{\varrho}_Q$ is totally ordered. If Q and Q' are contained in the same chamber, we have $\leqslant^{\varrho}_Q \;=\; \leqslant^{\varrho}_{Q'}$.

Take Q_0 in a chamber, and let $\mathfrak{a}$ be the minimum with respect to $\leqslant^{\varrho}_{Q_0}$. Note that $\mathfrak{a}$ is also minimal with respect to $\leqslant^{\varrho}_{\mathcal{C}}$. Let us observe that $\widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}$ is contained in $\widetilde{\mathcal{F}}^Q_{\mathfrak{a}}$ for any $Q \in \mathcal{C}$. We take the interval I connecting Q and Q_0. We take points $R_0 = Q_0, R_1, R_2, \dots, R_{N-1}, R_N = Q$ in I such that the open interval (R_i, R_{i+1}) is contained in a chamber. For $R', R'' \in (R_i, R_{i+1})$, we have $\widetilde{\mathcal{F}}^{R'}_{\mathfrak{a}} = \widetilde{\mathcal{F}}^{R''}_{\mathfrak{a}}$ by the compatibility condition for full pre-Stokes data. For $R \in (R_{i-1}, R_{i+1})$, we have $\widetilde{\mathcal{F}}^{R_i}_{\mathfrak{a}} \subset \widetilde{\mathcal{F}}^R_{\mathfrak{a}}$. Let $\mathfrak{b} \in \mathcal{I} \setminus \{\mathfrak{a}\}$. If $F_{\mathfrak{a},\mathfrak{b}}(R_i) \neq 0$, we have $F_{\mathfrak{a},\mathfrak{b}}(R) \neq 0$ for $R \in (R_{i-1}, R_i)$. If $F_{\mathfrak{a},\mathfrak{b}}(R_i) = 0$, the function $F_{\mathfrak{a},\mathfrak{b}}$ is monotonously increasing along I around R_i. Hence, $F_{\mathfrak{a},\mathfrak{b}}(R_i) > 0$ if and only if $F_{\mathfrak{a},\mathfrak{b}}(R) > 0$ for $R \in (R_{i-1}, R_i)$. It implies $\widetilde{\mathcal{F}}^R_{\mathfrak{a}} \subset \widetilde{\mathcal{F}}^{R_i}_{\mathfrak{a}}$ for $R \in (R_{i-1}, R_i)$. Therefore, we obtain $\widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}} \subset \widetilde{\mathcal{F}}^Q_{\mathfrak{a}}$. We can also deduce that $\widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}} \to \mathrm{Gr}^{\widetilde{\mathcal{F}}^Q}_{\mathfrak{a}}$ is an isomorphism for any Q. Hence, in particular, if $\mathfrak{b} \neq \mathfrak{a}$ is minimal with respect to $\leqslant_Q$, we have $\widetilde{\mathcal{F}}^Q_{\mathfrak{b}} \cap \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}} = 0$.

We put $\mathfrak{V}_0 := \mathfrak{V}/\widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}$. For any $Q \in \mathcal{C}$ and $\mathfrak{b} \in \mathcal{I}$, let $\widetilde{\mathcal{F}}^{Q}_{\mathfrak{b}}(\mathfrak{V}_0)$ be the image of $\widetilde{\mathcal{F}}^{Q}_{\mathfrak{b}}(\mathfrak{V})$ to $\mathfrak{V}_0$. Let $\mathfrak{V} = \bigoplus \mathfrak{V}_{\mathfrak{b},Q}$ be a splitting of $\widetilde{\mathcal{F}}^{Q}$. We remark that we may assume $\mathfrak{V}_{\mathfrak{a},Q} = \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}$. Then, it is easy to see that the images of $\mathfrak{V}_{\mathfrak{b},Q}$ give a splitting of the filtration $\widetilde{\mathcal{F}}^{Q}(\mathfrak{V}_0)$. We can also easily observe that the system of filtrations $\big(\widetilde{\mathcal{F}}^{Q}(\mathfrak{V}_0) \,\big|\, Q \in \mathcal{C}\big)$ gives a set of full pre-Stokes data of the locally free $\mathcal{O}_{\mathcal{K}}$-module $\mathfrak{V}_0$ on $\mathcal{C}$ associated to $\mathfrak{V}_0$.

Assume that we have filtrations $\widetilde{\mathcal{F}}^{\mathcal{C}}$ for $\mathfrak{V}_0$ and $\mathfrak{V}$ with the desired property. Then, $\widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{b}}(\mathfrak{V}_0)$ is obtained as the image of $\widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{b}}(\mathfrak{V})$. Actually, let $\mathfrak{V} = \bigoplus_{\mathfrak{b}\in\mathcal{I}} \mathfrak{V}_{\mathfrak{b}}$ be a splitting of $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V})$. We may assume $\mathfrak{V}_{\mathfrak{a}} = \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}$. The decomposition also gives a splitting of $\widetilde{\mathcal{F}}^{Q}(\mathfrak{V})$ for each $Q \in \mathcal{C}$. We have the induced decomposition $\mathfrak{V}_0 = \bigoplus \mathfrak{V}_{0,\mathfrak{b}}$, which gives a splitting of $\widetilde{\mathcal{F}}^{Q}(\mathfrak{V}_0)$ for each $Q \in \mathcal{C}$. It implies that the decomposition gives a splitting of $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_0)$ by the uniqueness, and we can conclude that $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_0)$ is obtained as the image of $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V})$.

Let us show the claim of the proposition by using an induction on $|\mathcal{I}|$. The case $|\mathcal{I}| = 1$ is obvious. Let Q_0 be a point in a chamber, and let $\mathfrak{a}$ be the minimal with respect to $\leqslant_{Q_0}$. If $\mathfrak{a}$ is the minimum with respect to $\leqslant_{\mathcal{C}}$, we can construct the desired filtration of $\mathfrak{V}$ as the pull-back via $\mathfrak{V} \to \mathfrak{V}_0$. Assume that $\mathfrak{a}$ is not the minimum. We can find a point Q_1 in a chamber such that $\mathfrak{a}$ is not minimal with respect to $\leqslant_{Q_1}$. Let $\mathfrak{b} \in \mathcal{I}$ be minimal with respect to $\leqslant_{Q_1}$. We remark $\widetilde{\mathcal{F}}^{Q_1}_{\mathfrak{b}} \cap \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}} = 0$. We put $\mathfrak{V}_1 := \mathfrak{V}/\widetilde{\mathcal{F}}^{Q_1}_{\mathfrak{b}}$ and $\mathfrak{V}_2 := \mathfrak{V}/\big(\widetilde{\mathcal{F}}^{Q_1}_{\mathfrak{b}} \oplus \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}\big)$. As remarked above, the associated locally free $\mathcal{O}_{\mathcal{K}}$-modules $\mathfrak{V}_i$ $(i = 0, 1, 2)$ are equipped with the induced full pre-Stokes structure. By construction, we have $\widetilde{\mathcal{F}}^{Q}_{\mathfrak{c}}(\mathfrak{V}) = \widetilde{\mathcal{F}}^{Q}_{\mathfrak{c}}(\mathfrak{V}_1) \times_{\widetilde{\mathcal{F}}^{Q}_{\mathfrak{c}}(\mathfrak{V}_2)} \widetilde{\mathcal{F}}^{Q}_{\mathfrak{c}}(\mathfrak{V}_0)$ for any $Q \in \mathcal{C}$ and $\mathfrak{c} \in \mathcal{I} \smallsetminus \{\mathfrak{a}, \mathfrak{b}\}$.

By the inductive assumption, $\mathfrak{V}_i$ are equipped with the filtration $\widetilde{\mathcal{F}}^{\mathcal{C}}$ with the desired property. Note that $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_2)$ is obtained as the image of $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_i)$ $(i = 0, 1)$. We put $\widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{c}}(\mathfrak{V}) := \widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{c}}(\mathfrak{V}_0) \times_{\widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{c}}(\mathfrak{V}_2)} \widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{c}}(\mathfrak{V}_1)$. Let us check that $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V})$ has the desired property. Let $\mathfrak{V}_2 = \bigoplus \mathfrak{V}_{2,\mathfrak{c}}$ be a splitting of $\widetilde{\mathcal{F}}^{\mathcal{C}}$. Let $\mathfrak{V}_{\mathfrak{c}} \subset \widetilde{\mathcal{F}}^{\mathcal{C}}_{\mathfrak{c}}(\mathfrak{V})$ be a lift of $\mathfrak{V}_{2,\mathfrak{c}}$. We put $\mathfrak{V}_{\mathfrak{a}} := \widetilde{\mathcal{F}}^{Q_0}_{\mathfrak{a}}$ and $\mathfrak{V}_{\mathfrak{b}} := \widetilde{\mathcal{F}}^{Q_1}_{\mathfrak{b}}$. By using that $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_2)$ is obtained as the image of $\widetilde{\mathcal{F}}^{\mathcal{C}}(\mathfrak{V}_i)$ $(i = 0, 1)$, we can check that $\mathfrak{V} = \bigoplus \mathfrak{V}_{\mathfrak{c}}$ is a splitting of the filtration $\widetilde{\mathcal{F}}^{\mathcal{C}}$. Similarly, we can check that it gives a splitting of each $\widetilde{\mathcal{F}}^{Q}(\mathfrak{V})$. Hence, $\widetilde{\mathcal{F}}^{\mathcal{C}}$ is compatible with $\widetilde{\mathcal{F}}^{Q}$ for each $Q \in \mathcal{C}$. □

It seems useful to consider the following type of covering of $\pi^{-1}(P)$ $(P \in \mathcal{D})$.

Definition 4.1.6. — A finite covering $\{\mathcal{C}_i \,|\, i \in \Gamma\}$ of $\pi^{-1}(P)$ is called good for $\mathcal{I}$, if any intersection $\mathcal{C}_I := \bigcap_{i\in I} \mathcal{C}_i$ $(I \subset \Gamma)$ satisfy Condition 4.1.4. □

Lemma 4.1.7. — *For a given $\mathcal{I}$, there exists a good cover of $\pi^{-1}(P)$.*

Proof. — For example, we can construct such a finite covering by using polytopes surrounded by hypersurfaces, which transversally intersect $H_{\mathfrak{a},\mathfrak{b}}$ for any distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P$. (See the proof of Proposition 4.1.5 for $H_{\mathfrak{a},\mathfrak{b}}$.) □

4.1.3. Full pre-Stokes data for ϱ-flat bundle. — Let $(V, \mathbb{D})$ be a ϱ-flat bundle on $\mathcal{X} \setminus \mathcal{D}$. By considering $\mathbb{D}$-flat and $\mathcal{K}$-holomorphic sections of V, we obtain an $\mathcal{O}_{\mathcal{K}}$-module $\mathfrak{V}'$ on $\mathcal{X} \setminus \mathcal{D}$. By taking the push-forward via $\iota : \mathcal{X} \setminus \mathcal{D} \subset \widetilde{\mathcal{X}}(\mathcal{D})$, we obtain an $\mathcal{O}_{\mathcal{K}}$-module $\mathfrak{V}$ on $\widetilde{\mathcal{X}}(\mathcal{D})$, which is our main concern. Let A be any locally simply connected subset of $\widetilde{\mathcal{X}}(\mathcal{D})$. If $\mathfrak{V}$ is given in this way, a set of full pre-Stokes data of $\mathfrak{V}_{|A}$ is equivalent to a system of $\mathbb{D}$-flat filtrations $\widetilde{\mathcal{F}}^Q$ of germs $\iota_*(V)_Q$ $(Q \in A \cap \pi^{-1}(\mathcal{D}))$ such that (i) they satisfy the compatibility condition as in Definition 4.1.1, (ii) each $\widetilde{\mathcal{F}}^Q$ has a $\mathbb{D}$-flat splitting.

We immediately obtain the following lemma from Proposition 4.1.5.

Proposition 4.1.8. — *Let $P \in \mathcal{D}$. If $U_1 \subset \pi^{-1}(P)$ is sufficiently small so that there exists $C(P) \supset U_1$ satisfying Condition 4.1.4, then there exists a unique filtration $\widetilde{\mathcal{F}}^{U_1}$ of $\iota_*(V)_{|U_1}$ such that $\widetilde{\mathcal{F}}^{U_1}$ and $\widetilde{\mathcal{F}}^Q$ $(Q \in U_1)$ are compatible over $(\mathcal{I}_P, \leqslant^{\varrho}_{U_1}) \to (\mathcal{I}_P, \leqslant^{\varrho}_Q$ $)$.* □

Lemma 4.1.9. — *Let $(V, \mathbb{D})$ be as above. Let $P \in \mathcal{D}$. If we are given a set of full pre-Stokes data of $\iota_*(V)_{|\pi^{-1}(P)}$, they can be uniquely extended to a set of full pre-Stokes data on a small neighbourhood of $\pi^{-1}(P)$.*

Proof[1]. — For any $Q \in \pi^{-1}(P)$, we take a small neighbourhood $\mathcal{U}_Q$ in $\widetilde{\mathcal{X}}(\mathcal{D})$ such that $\leqslant_Q = \leqslant_{\mathcal{U}_Q}$. We can find $Q_1, \ldots, Q_N \in \pi^{-1}(P)$ such that $\pi^{-1}(P) \subset \bigcup \mathcal{U}_{Q_i}$. We may assume $\mathcal{U}_{Q_i}$ are the product $C_i \times B$ where B is a neighbourhood of P in $[0,1[^{\ell} \times \mathcal{D}_{\underline{\ell}}$, and $C_i \subset (S^1)^{\ell}$. (We use the natural identification $\widetilde{\mathcal{X}}(D) \simeq ([0,1[\times S^1)^{\ell} \times \mathcal{D}_{\underline{\ell}}$.) As remarked in the second claim of Lemma 4.1.2, for $Q' \in \mathcal{U}_{Q_i}$, we have the induced filtration $\widetilde{\mathcal{F}}^{Q',Q_i}$ of $\iota_*(V)_{Q'}$, induced by $\widetilde{\mathcal{F}}^{Q_i}$ and $(\mathcal{I}_P, \leqslant_{Q_i}) \to (\mathcal{I}_{\pi(Q')}, \leqslant_{Q'})$. For any $R \in P \times (C_i \cap C_j)$, there exists a neighbourhood $\mathcal{U}_R \subset \mathcal{U}_{Q_i} \cap \mathcal{U}_{Q_j}$ such that, for any $Q' \in \mathcal{U}_R$, both $\widetilde{\mathcal{F}}^{Q',Q_i}$ and $\widetilde{\mathcal{F}}^{Q',Q_j}$ are induced by $\widetilde{\mathcal{F}}^R$ and $(\mathcal{I}_P, \leqslant_R) \to (\mathcal{I}_{\pi(Q')}, \leqslant_{Q'})$, and hence they are the same. Therefore, by shrinking B, we obtain a Stokes structure $\iota_*(V)_{\pi^{-1}(B)}$ whose restriction to $\pi^{-1}(P)$ is the same as the given one. The uniqueness follows from the first claim of Lemma 4.1.2. □

4.2. Full Stokes data

4.2.1. The associated graded bundles. — Let $(V, \mathbb{D})$ be a ϱ-flat bundle on $\mathcal{X} \setminus \mathcal{D}$. For each $P \in \mathcal{D}$, $\mathcal{X}_P$ denotes a small neighbourhood of P. We put $\mathcal{D}_P := \mathcal{X}_P \cap \mathcal{D}$. We will shrink it without mention. Let $\iota : \mathcal{X} \setminus \mathcal{D} \subset \widetilde{\mathcal{X}}(\mathcal{D})$ be the inclusion. Let $\widetilde{\boldsymbol{\mathcal{F}}}$ be a set of full pre-Stokes data of $\iota_* V$ over a good system of irregular values $\boldsymbol{\mathcal{I}}$. For any point $P \in \mathcal{D}$, we obtain a graded sheaf $\mathrm{Gr}^{\widetilde{\boldsymbol{\mathcal{F}}}}(\iota_* V_{|\pi^{-1}(\mathcal{X}_P)})$ with a ϱ-flat connection $\mathbb{D}$.

1. The author thanks the referee of [**64**] for this simplified proof.

In particular, we obtain a graded ϱ-flat bundle on $\mathcal{X}_P^*$, where $\mathcal{X}_P^* = \mathcal{X}_P \setminus \mathcal{D}_P$:

$$V_{P,\mathfrak{a}} := \mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\mathcal{F}}}}\big(V_{|\mathcal{X}_P^*}\big), \quad V_P := \bigoplus_{\mathfrak{a}\in\mathcal{I}_P} V_{P,\mathfrak{a}}.$$

4.2.1.1. Variant. — Let $\mathcal{D} = \bigcup_{i\in\Lambda}\mathcal{D}_i$ be the irreducible decomposition. Let $I(P) := \{i\in\Lambda \mid P\in\mathcal{D}_i\}$. For any $J\subset I(P)$, we put $J^c := I(P)\setminus J$. Let $\mathcal{I}_P^J$ be the image of $\overline{\eta}_J : \mathcal{I}_P \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})_P/\mathcal{O}_{\mathcal{X}}(*\mathcal{D}(J^c))_P$. We obtain a filtration $\mathcal{F}^{Q,J}$ of $\iota_*(V)_Q$ indexed by $\big(\mathcal{I}_P^J, \leqslant_Q^{\varrho}\big)$ for $Q\in\pi^{-1}(P)$, and the induced filtration $\widetilde{\mathcal{F}}^Q$ on $\mathrm{Gr}_{\mathfrak{b}}^{\mathcal{F}^{Q,J}}\big(\iota_*(V)_Q\big)$ indexed by $\big(\overline{\eta}_J^{-1}(\mathfrak{b}), \leqslant_Q^{\varrho}\big)$. In particular, we obtain a system of filtrations $\boldsymbol{\mathcal{F}}^J := \big(\mathcal{F}^{Q,J} \mid Q\in\pi^{-1}(P)\big)$. It satisfies the compatibility condition in the following sense.

- Take a small $\mathcal{U}_Q \in \mathfrak{U}(Q,\mathcal{I}_P^J)$ on which $\mathcal{F}^{Q,J}$ is given. (Recall the notation in Section 3.1.3.1.) For $Q'\in\mathcal{U}_Q\cap\pi^{-1}(P)$, we have the induced filtration $\mathcal{F}^{Q,J}$ and $\mathcal{F}^{Q',J}$ of $\iota_*(V)_{Q'}$. Then, they are compatible over $\big(\mathcal{I}_P^J,\leqslant_Q^{\varrho}\big)\to\big(\mathcal{I}_P^J,\leqslant_{Q'}^{\varrho}\big)$.

For $\mathfrak{b}\in\mathcal{I}_P^J$, we obtain $\mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{\mathcal{F}}^J}\big(\iota_*(V)_{\pi^{-1}(\mathcal{X}_P)}\big)$ with an induced ϱ-flat connection on $\pi^{-1}(\mathcal{X}_P)$. In particular, we obtain a graded ϱ-flat bundle on $\mathcal{X}_P^*$:

$$V_{P,\mathfrak{b}}^J := \mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{\mathcal{F}}^J}\big(V_{|\mathcal{X}_P^*}\big), \quad V_P^J := \bigoplus_{\mathfrak{b}\in\mathcal{I}_P^J} V_{P,\mathfrak{b}}^J.$$

We have induced full pre-Stokes data of $\iota_*(V_{P,\mathfrak{b}}^J)_{|\pi^{-1}(P)}$ over the good set $\overline{\eta}_J^{-1}(\mathfrak{b})$. By construction, we have a natural isomorphism of graded ϱ-flat bundle

$$\mathrm{Gr}^{\widetilde{\boldsymbol{\mathcal{F}}}} V_P^J \simeq V_P. \tag{58}$$

Let $P'\in\mathcal{X}_P\cap\mathcal{D}_J$ with $I(P')=J$. By the compatibility of the given $\widetilde{\boldsymbol{\mathcal{F}}}$, we have a natural isomorphism

$$V^J_{P|\mathcal{X}^*_{P'}} \simeq V_{P'}. \tag{59}$$

4.2.2. Full Stokes data. — We continue to use the notation in Subsection 4.2.1. A graded extension of a set of full pre-Stokes data $\widetilde{\boldsymbol{\mathcal{F}}}$ is a datum $\boldsymbol{\mathcal{G}} = \big(\mathcal{G}_P \mid P\in\mathcal{D}\big)$ as follows:

- $\mathcal{G}_P = \bigoplus_{\mathfrak{a}\in\mathcal{I}_P}\mathcal{G}_{P,\mathfrak{a}}$ is a graded locally free $\mathcal{O}_{\mathcal{X}_P}$-module with an isomorphism $\mathcal{G}_{P,\mathfrak{a}|\mathcal{X}_P^*}\simeq V_{P,\mathfrak{a}}$ such that $\mathbb{D}_{\mathfrak{a}}$ is $\mathfrak{a}$-logarithmic with respect to $\mathcal{G}_{P,\mathfrak{a}}$.

Note it induces a subsheaf

$$\pi_P^*\mathcal{G}_P := \pi_P^{-1}\mathcal{G}_P\otimes\mathcal{O}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_P)} \subset \iota_{P*}V_P,$$

where $\pi_P : \widetilde{\mathcal{X}}_P(\mathcal{D}_P)\to\mathcal{X}_P$ and $\iota_P : \mathcal{X}_P\setminus\mathcal{D}_P\subset\mathcal{X}_P$ are natural maps. Namely, for each $P\in\mathcal{D}$, we are considering an $\mathcal{O}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_P)}$-lattice of $\iota_{P*}V_P$ whose push-forward to $\mathcal{X}_P$ is a locally free $\mathcal{O}_{\mathcal{X}_P}$-module. We remark that $\mathcal{G}_P$ is recovered as $\pi_{P*}\pi_P^*\mathcal{G}_P$, which will be implicitly used.

We introduce a compatibility condition for the graded extension $\boldsymbol{\mathcal{G}}$. First, we impose the following compatibility.

- If $P'\in\mathcal{D}_{P,I(P)}$, we impose $\mathcal{G}_{P',\mathfrak{a}} = \mathcal{G}_{P,\mathfrak{a}|\mathcal{X}_{P'}}$.

Then, for each $J \subset I(P)$ and $\mathfrak{b} \in \mathcal{I}_P^J$, we obtain a locally free $\mathcal{O}_X$-module $\mathcal{G}_{\mathfrak{b}}^J$ on $\mathcal{V} \setminus \mathcal{D}_P(J^c)$, where $\mathcal{V}$ is a neighbourhood of $\mathcal{D}_{P,J}$. Actually, it is obtained as the gluing of $V_{P,\mathfrak{b}}^J$ and $\mathcal{G}_{P',\mathfrak{b}}$ for $P' \in \mathcal{D}_{P,J} \setminus \bigcup_{i \notin J} \mathcal{D}_{P,i}$. We remark (59). By shrinking $\mathcal{X}_P$, we may assume $\mathcal{X}_P = \mathcal{V}$. Let $\pi_{P,J}$ be the restriction of π_P to $\widetilde{\mathcal{X}}_P(\mathcal{D}_P) \setminus \pi_P^{-1}(\mathcal{D}_P(J^c))$. Let $\iota_{P,J} : \widetilde{\mathcal{X}}_P(\mathcal{D}_P) \setminus \pi_P^{-1}(\mathcal{D}_P(J^c)) \subset \widetilde{\mathcal{X}}_P(\mathcal{D}_P)$. Then, we obtain a subsheaf $\iota_{P,J*}\pi_{P,J}^*\mathcal{G}_{\mathfrak{b}}^J \subset \iota_{P*}V_{P,\mathfrak{b}}^J$. By construction, it induces an isomorphism

$$\iota_{P,J*}\pi_{P,J}^*\mathcal{G}_{\mathfrak{b}}^J \otimes \iota_{P*}\mathcal{O}_{\mathcal{X}_P^*} \simeq \iota_{P*}V_{P,\mathfrak{b}}^J. \tag{60}$$

Then, the compatibility condition is given as follows:

- For any $Q \in \pi^{-1}(P)$, the filtration $\widetilde{\mathcal{F}}^Q$ of $\iota_{P*}V_{P,\mathfrak{b}}^J$ is induced by a filtration of $\iota_{P,J*}\pi_{P,J}^*\mathcal{G}_{\mathfrak{b}}^J$ and (60).
- The restrictions of $\pi_P^*\mathcal{G}_P$ and $\mathrm{Gr}^{\widetilde{\mathcal{F}}}\, \iota_{P,J*}\pi_{P,J}^*\mathcal{G}_{\mathfrak{b}}^J$ to $\widetilde{\mathcal{X}}_P(\mathcal{D}_P) \setminus \pi^{-1}(\mathcal{D}_P(J^c))$ are isomorphic, extending (58).

If the compatibility condition is satisfied, $(\widetilde{\boldsymbol{\mathcal{F}}}, \boldsymbol{\mathcal{G}})$ is called a set of full Stokes data.

4.2.2.1. Let $(V_i, \mathbb{D}_i)$ be ϱ-flat bundles on $\mathcal{X} \setminus \mathcal{D}$ equipped with full Stokes data $\widetilde{\mathcal{SD}}_i = (\widetilde{\boldsymbol{\mathcal{F}}}_i, \boldsymbol{\mathcal{G}}_i)$. A morphism $F : (V_1, \mathbb{D}_1, \widetilde{\mathcal{SD}}_1) \to (V_2, \mathbb{D}_2, \widetilde{\mathcal{SD}}_2)$ is defined to be a morphism $(V_1, \mathbb{D}_1, \widetilde{\boldsymbol{\mathcal{F}}}_1) \to (V_2, \mathbb{D}_2, \widetilde{\boldsymbol{\mathcal{F}}}_2)$ such that, for any $P \in \mathcal{D}$ and $\mathfrak{a} \in \mathcal{I}_P$, the induced morphisms $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(V_{1|\mathcal{X}_P^*}) \to \mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(V_{2|\mathcal{X}_P^*})$ can be extended to $\mathcal{G}_{1,P,\mathfrak{a}} \to \mathcal{G}_{2,P,\mathfrak{a}}$. Full Stokes data have obvious functoriality for dual, tensor product and direct sum.

4.2.2.2. Meromorphic Stokes data. — A meromorphic graded extension of a set of full pre-Stokes data $\widetilde{\boldsymbol{\mathcal{F}}}$ is a tuple $\boldsymbol{\mathcal{G}} = \big(\mathcal{G}_P \,\big|\, P \in \mathcal{D}\big)$, where each $\mathcal{G}_P = \bigoplus_{\mathfrak{a} \in \mathcal{I}_P} \mathcal{G}_{P,\mathfrak{a}}$ is a locally free $\mathcal{O}_{\mathcal{X}_P}(*\mathcal{D}_P)$-module with isomorphisms $\mathcal{G}_{P,\mathfrak{a}|\mathcal{X}_P^*} \simeq \mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}(V_{|\mathcal{X}_P^*})$ such that $\mathbb{D}_{\mathfrak{a}}$ are $\mathfrak{a}$-regular with respect to $\mathcal{G}_{P,\mathfrak{a}}$. We can consider a compatibility condition similar to the above. If the compatibility condition is satisfied, $(\widetilde{\boldsymbol{\mathcal{F}}}, \boldsymbol{\mathcal{G}})$ is called a set of meromorphic full Stokes data.

4.2.2.3. Another formulation of compatibility. — We give another formulation of compatibility condition for $\boldsymbol{\mathcal{G}}$. Let us consider $\widetilde{\mathcal{X}}_P(\mathcal{D}_P) \xrightarrow{\varpi} \widetilde{\mathcal{X}}_P(\mathcal{D}_P(J^c)) \xrightarrow{\pi_1} \mathcal{X}_P$. Take $Q_1 \in \pi_1^{-1}(P)$. We remark that the filtrations $\widetilde{\mathcal{F}}^Q$ $(Q \in \varpi^{-1}(Q_1))$ on $\iota_* V_{P,\mathfrak{b}}^J$ are constant, because the orders $\leqslant_Q^{\varrho}$ on $\overline{\eta}_J^{-1}(\mathfrak{b})$ are independent of $Q \in \varpi^{-1}(Q_1)$. Let $\iota_J : \mathcal{X}_P^* \subset \widetilde{\mathcal{X}}_P(\mathcal{D}_P(J^c))$. By the above consideration, we obtain an induced $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^{Q_1}$ on the stalk $\iota_{J*}(V_{P,\mathfrak{b}}^J)_{Q_1}$. The system of filtrations $\big(\widetilde{\mathcal{F}}^{Q_1} \,\big|\, Q_1 \in \pi_1^{-1}(P)\big)$ satisfies the standard compatibility condition.

Let ι'_J denote the inclusions of $\mathcal{X}_P \setminus \mathcal{D}_P(J^c)$ into $\widetilde{\mathcal{X}}_P(\mathcal{D}_P(J^c))$. We have the subsheaf $\iota'_{J*}\big({}^J\mathcal{G}_{\mathfrak{a}}\big) \subset \iota_{J*}V_{P,\mathfrak{b}}^J$. Now, we can state the compatibility condition.

- For any $Q_1 \in \pi_1^{-1}(P)$, the filtration $\widetilde{\mathcal{F}}^{Q_1}$ of $\iota_{J*}(V_{P,\mathfrak{b}}^J)_{Q_1}$ comes from a filtration of $\iota'_{J*}\big({}^J\mathcal{G}_{\mathfrak{a}}\big)_{Q_1}$. We obtain $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}\big({}^J\mathcal{G}_{\mathfrak{b}}\big)$ $(\mathfrak{a} \in \overline{\eta}_J^{-1}(\mathfrak{b}))$ on $\mathcal{X}_P \setminus \mathcal{D}_P(J^c)$ in a standard way.
- $\mathrm{Gr}_{\mathfrak{a}}^{\widetilde{\mathcal{F}}}\big({}^J\mathcal{G}_{\mathfrak{b}}\big)$ is isomorphic to $\mathcal{G}_{P,\mathfrak{a}|\mathcal{X}_P \setminus \mathcal{D}_P(J^c)}$, extending (58).

4.2.3. Stokes data associated to an unramifiedly good lattice. — Let $(\mathcal{E},\mathbb{D})$ be a meromorphic ϱ-flat bundle with an unramifiedly good lattice E on $(\mathcal{X},\mathcal{D})$. Let $(V,\mathbb{D}) := (\mathcal{E},\mathbb{D})_{|\mathcal{X}\setminus\mathcal{D}}$. According to Theorem 3.2.1, we have induced full pre-Stokes data $\widetilde{\boldsymbol{\mathcal{F}}}$ of $(V,\mathbb{D})$. According to Proposition 3.2.9, we have a graded extension $\boldsymbol{\mathcal{G}}$ of the full pre-Stokes data.

Lemma 4.2.1. — *$\boldsymbol{\mathcal{G}}$ satisfies the compatibility condition in Subsection* 4.2.2.

Proof. — The full Stokes filtrations $\widetilde{\mathcal{F}}^Q$ $(Q\in\pi^{-1}(P))$ are induced by filtrations of π^*E. Hence, the induced filtration $\mathcal{F}^{Q,J}$ $(Q\in\pi^{-1}(P))$ are so. We obtain $\mathrm{Gr}^{\mathcal{F}^J}_{\mathfrak{b}}(E)$ on $\mathcal{X}_P$. It is easy to see that ${}^J\mathcal{G}_{\mathfrak{b}}$ is naturally isomorphic to the restriction of $\mathrm{Gr}^{\mathcal{F}^J}_{\mathfrak{b}}(E)$ to $\mathcal{X}_P\setminus\mathcal{D}_P(J^c)$. We also have $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}\mathrm{Gr}^{\mathcal{F}^J}_{\mathfrak{b}}(E)\simeq\mathcal{G}_{P,\mathfrak{a}}$. Then, the claim of the lemma is clear. □

Namely, $(V,\mathbb{D})$ is equipped with a naturally induced set of Stokes data, which is functorial, according to the results in Subsections 3.2.2 and 3.2.5.

4.2.3.1. Complement on splitting of Stokes filtrations. — We give a complement on a splitting of Stokes filtration of a good meromorphic ϱ-flat bundle. We set $\mathcal{X}:=\Delta^n\times\mathcal{K}$ and $\mathcal{D}=\bigcup_{i=1}^{\ell}\{z_i=0\}$. We assume that the coordinate system is admissible. We use the notation in Subsections 2.4.2 and 3.7.1.

Let $\mathcal{C}(P)\subset\pi^{-1}(P)$ be as in Condition 4.1.4. We denote it by $\mathcal{C}$. Let $\mathcal{U}$ be a small neighbourhood of $\mathcal{C}$ in $\widetilde{\mathcal{X}}(\mathcal{D})$, which will be shrunk. As in Proposition 4.1.8, we have a $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^{\mathcal{C}}$ of $(\iota_*E)_{|\mathcal{C}}$, which can be extended to a $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^{\mathcal{U}}$ of $\iota_*E_{|\mathcal{U}}$. We can take a splitting $E_{|\mathcal{U}}=\bigoplus_{\mathfrak{a}\in\mathcal{I}_P}E_{\mathcal{U},\mathfrak{a}}$ of $\widetilde{\mathcal{F}}^{\mathcal{U}}$ with the following property:

- For $\mathfrak{b}\in\mathcal{I}(\boldsymbol{m}(p))$, we put

$$E^{\boldsymbol{m}(p)}_{\mathcal{U},\mathfrak{b}} := \bigoplus_{\mathfrak{a}\in\overline{\eta}^{-1}_{\boldsymbol{m}(p)}(\mathfrak{b})} E_{\mathcal{U},\mathfrak{a}}.$$

  Then, the decomposition is $\mathbb{D}_{\leqslant k(p)}$-flat.

Actually, we successively apply the third claim of Proposition 3.6.1.

Let $\boldsymbol{v}=(\boldsymbol{v}_{\mathfrak{a}})$ be a frame of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E)$ compatible with the grading. By the natural isomorphism $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E)_{|\mathcal{U}}\simeq E_{|\mathcal{U}}$ given by the above splitting, we make a frame $\boldsymbol{v}_{\mathcal{U}}$.

Let $E_{|\mathcal{U}}=\bigoplus E'_{\mathcal{U},\mathfrak{a}}$ be another decomposition with the above property. We obtain another frame $\boldsymbol{v}'_{\mathcal{U}}$. Let $C=(C_{\mathfrak{a},\mathfrak{b}})$ be the matrix determined by $\boldsymbol{v}_{\mathcal{U}}=\boldsymbol{v}'_{\mathcal{U}}(I+C)$, where I denotes the identity matrix.

Lemma 4.2.2. — *We have $C_{\mathfrak{a},\mathfrak{b}}=0$ unless $\mathfrak{a}<_{\mathcal{U}}\mathfrak{b}$. If $\mathfrak{a}<_{\mathcal{U}}\mathfrak{b}$, we have the estimate*

$$C_{\mathfrak{a},\mathfrak{b}}\exp\bigl(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\bigr)=O\Bigl(\prod_{i=1}^{k(\mathfrak{a},\mathfrak{b})}|z_i|^{-N}\Bigr).$$

Here, $k(\mathfrak{a},\mathfrak{b})$ is determined by $\mathrm{ord}(\mathfrak{a},\mathfrak{b})\in\mathbb{Z}^{k(\mathfrak{a},\mathfrak{b})}_{<0}\times\boldsymbol{0}$, *and N is a large number.*

Proof. — It follows from that the induced morphism $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{b}}(E)_{|\mathcal{U}}\to\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(E)_{|\mathcal{U}}$ is $\mathbb{D}_{\leqslant k(\mathfrak{a},\mathfrak{b})}$-flat. □

4.3. Riemann-Hilbert-Birkhoff correspondence

4.3.1. Statement. — Let $\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ be the category of unramifiedly good lattices $(E,\mathbb{D})$ of meromorphic ϱ-flat bundles on $(\mathcal{X},\mathcal{D})$ whose good system of irregular values is contained in $\boldsymbol{\mathcal{I}}$, i.e., $\mathrm{Irr}(\nabla,P)\subset\mathcal{I}_P$ for any $P\in\mathcal{D}$. Let $\mathrm{SDL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ be the category of ϱ-flat bundles with full Stokes data over $\boldsymbol{\mathcal{I}}$. As explained in Subsection 4.2.3, we have a functor

$$\mathrm{RHB}:\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})\longrightarrow\mathrm{SDL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}}).$$

We will prove the following theorem in Subsections 4.3.2–4.3.5.

Theorem 4.3.1. — *The functor* RHB *is an equivalence.*

Let $\mathrm{MF}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ be the category of unramifiedly good meromorphic ϱ-flat bundles on $(\mathcal{X},\mathcal{D})$ whose good system of irregular values is contained in $\boldsymbol{\mathcal{I}}$. Let $\mathrm{SD}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ be the category of ϱ-flat bundle with a set of full meromorphic Stokes data over $\boldsymbol{\mathcal{I}}$.

Corollary 4.3.2. — *The naturally defined functor*

$$\mathrm{RHB}:\mathrm{MF}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})\longrightarrow\mathrm{SD}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$$

is an equivalence. □

Let G be a finite group acting on $(\mathcal{X},\mathcal{D})$ over $\mathcal{K}$. Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values such that $\mathcal{I}_P=g^*\mathcal{I}_{g(P)}$ for any $g\in G$ and $P\in\mathcal{D}$. Let $(V,\mathbb{D})$ be a $\mathbb{D}$-flat bundle on $\mathcal{X}\setminus\mathcal{D}$ with a G-action, i.e., for each $g\in G$, we are given an isomorphism $g^*(V,\mathbb{D})\simeq(V,\mathbb{D})$ compatible with the group law. Let $\widetilde{\boldsymbol{\mathcal{F}}}$ be a set of full pre-Stokes data of $(V,\mathbb{D})$. For each $g\in G$, we have the induced full pre-Stokes data $g^*\widetilde{\boldsymbol{\mathcal{F}}}$ of $(V,\mathbb{D})\simeq g^*(V,\mathbb{D})$. The set of full pre-Stokes data is called G-equivariant if $g^*\widetilde{\boldsymbol{\mathcal{F}}}=\widetilde{\boldsymbol{\mathcal{F}}}$ for any $g\in G$. Similarly the G-equivariance of a set of Stokes data is defined. The category of G-equivariant full Stokes data is denoted by $\mathrm{SDL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$. Let $\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$ denote the category of G-equivariant unramifiedly good lattices of meromorphic ϱ-flat bundles over $\boldsymbol{\mathcal{I}}$. We use the symbols $\mathrm{SD}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$ and $\mathrm{MF}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$ with similar meanings. It is easy to deduce the following as a corollary of Theorem 4.3.1.

Corollary 4.3.3. — *The functors* $\mathrm{RHB}:\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G\to\mathrm{SDL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$ *and* $\mathrm{RHB}:\mathrm{MF}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G\to\mathrm{SD}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})^G$ *are equivalences.* □

4.3.1.1. Descent. — Let $\varphi:(\mathcal{X}',\mathcal{D}')\to(\mathcal{X},\mathcal{D})$ be a ramified Galois covering with the Galois group G. Let $\boldsymbol{\mathcal{I}}':=\varphi^*\boldsymbol{\mathcal{I}}$. We have naturally defined descent functors $\mathrm{Des}:\mathrm{MFL}(\mathcal{X}',\mathcal{D}',\boldsymbol{\mathcal{I}}')^G\to\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ and $\mathrm{Des}:\mathrm{SDL}(\mathcal{X}',\mathcal{D}',\boldsymbol{\mathcal{I}}')^G\to\mathrm{SDL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$.

Proposition 4.3.4. — *We have a natural isomorphism* $\mathrm{Des}\circ\mathrm{RHB}\simeq\mathrm{RHB}\circ\mathrm{Des}$.

Proof. — Let $(E',\mathbb{D}')\in\mathrm{MFL}(\mathcal{X}',\mathcal{D}',\boldsymbol{\mathcal{I}}')^G$. We set $(E,\mathbb{D}):=\mathrm{Des}(E',\mathbb{D}')$, $(V',\mathbb{D}',(\widetilde{\boldsymbol{\mathcal{F}}}',\boldsymbol{\mathcal{G}}')):=\mathrm{RHB}(E',\mathbb{D}')$ and $(V,\mathbb{D},(\widetilde{\boldsymbol{\mathcal{F}}},\boldsymbol{\mathcal{G}})):=\mathrm{RHB}(E,\mathbb{D})$. By using the characterization of full Stokes filtration in Theorem 3.2.1, we obtain that $\widetilde{\boldsymbol{\mathcal{F}}}$ is the descent of $\widetilde{\boldsymbol{\mathcal{F}}}'$.

Let $\pi : \widetilde{\mathcal{X}}(\mathcal{D}) \to \mathcal{X}$ and $\pi' : \widetilde{\mathcal{X}}'(\mathcal{D}') \to \mathcal{X}'$ be blow up. Let $P \in \mathcal{D}$. We take a small neighbourhood $\mathcal{X}_P$ of P in $\mathcal{X}$. We put $\mathcal{X}'_P := \varphi^{-1}(\mathcal{X}_P)$, which is a neighbourhood of the finite set $\varphi^{-1}(P)$. We have $\mathrm{Gr}^{\widetilde{\mathcal{F}}'}_{\mathfrak{a}}\big(\pi'^{*}E'_{|\pi'^{-1}(\mathcal{X}'_P)}\big)$ on $\pi'^{-1}(\mathcal{X}'_P)$. We can easily compare $\mathcal{G}_{P,\mathfrak{a}}$ and the descent of $\bigoplus_{P'\in\pi^{-1}(P)}\mathcal{G}_{P',\mathfrak{a}}$, because both of them are induced by $\mathrm{Gr}^{\widetilde{\mathcal{F}}'}_{\mathfrak{a}}\big(\pi'^{*}E'_{|\pi'^{-1}(\mathcal{X}'_P)}\big)$. □

4.3.1.2. Classification of unramifiedly good meromorphic flat bundles. — By setting (i) $\mathcal{K}$ is a point, (ii) $\varrho = 1$ in Corollary 4.3.2, we obtain a classification of unramifiedly good meromorphic flat bundles in terms of full pre-Stokes data. Note that we need only full pre-Stokes data, because the unramifiedly good Deligne-Malgrange lattices are canonically associated.

In the one dimensional case, Malgrange [**56**] showed the correspondence between meromorphic flat bundles and flat bundles with pre-Stokes data. See also the work due to Sibuya [**80**] on the classification of meromorphic flat bundles on curves.

In the higher dimensional case, such a classification was studied in [**53**] and [**72**], from different viewpoints. If we are interested in deformations by a variation of irregular values, the classification according to pre-Stokes data is useful to the author.

4.3.2. Fully faithfulness. — Let us show that RHB is fully faithful. Let $(E_i, \mathbb{D}_i) \in \mathrm{MFL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$ $(i = 1, 2)$.

Lemma 4.3.5. — *The natural map*

$$\mathrm{Hom}\big((E_1,\mathbb{D}_1),(E_2,\mathbb{D}_2)\big) \longrightarrow \mathrm{Hom}\big(\mathrm{RHB}(E_1,\mathbb{D}_1),\mathrm{RHB}(E_2,\mathbb{D}_2)\big)$$

is bijective.

Proof. — It is clearly injective. Let us show the surjectivity. Let

$$F : \mathrm{RHB}(E_1,\mathbb{D}_1) \longrightarrow \mathrm{RHB}(E_2,\mathbb{D}_2)$$

be a morphism. We only have to show that the underlying morphism $F : (E_1,\mathbb{D}_1)_{|\mathcal{X}\setminus\mathcal{D}} \to (E_2,\mathbb{D}_2)_{|\mathcal{X}\setminus\mathcal{D}}$ can be extended to a morphism $E_1 \to E_2$. By Hartogs property, we may assume that $\mathcal{D}$ is smooth. Let $Q \in \pi^{-1}(P)$. We take a $\mathbb{D}$-flat splitting $E_{i|\mathcal{U}_Q} \simeq \bigoplus E_{i,Q,\mathfrak{a}}$ of the full Stokes filtration. The flat morphism F induces $F_{\mathfrak{b},\mathfrak{a}} : E_{i,\mathfrak{a}|\mathcal{U}^*_Q} \to E_{i,\mathfrak{b}|\mathcal{U}^*_Q}$. Because the full Stokes filtrations are preserved by F, we have $F_{\mathfrak{b},\mathfrak{a}} = 0$ unless $\mathfrak{a} \geqslant^{\varrho}_{Q} \mathfrak{b}$. By construction $F_{\mathfrak{a},\mathfrak{a}}$ is bounded. Because $F_{\mathfrak{b},\mathfrak{a}}$ is $\mathbb{D}$-flat, $F_{\mathfrak{b},\mathfrak{a}}\exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{b})\big)$ is of polynomial order in $|z_1^{-1}|$. Hence, we obtain that F can be extended to a morphism $E_{1|\mathcal{U}_Q} \to E_{2|\mathcal{U}_Q}$ and, by varying Q, we obtain $E_{1|\widetilde{\mathcal{X}}(\mathcal{D})} \to E_{2|\widetilde{\mathcal{X}}(\mathcal{D})}$, which induces $E_1 \to E_2$. □

It remains to show that RHB is essentially surjective. Namely, we have to give a construction of an unramifiedly good lattice from a set of full Stokes data. By

Lemma 4.3.5, we only have to give it locally. We will give an inductive construction with respect to levels.

4.3.3. Pre-Stokes data and Stokes data at the level $\boldsymbol{m}$. — We use the setting in Section 3.3. For a subset $\mathcal{A} \subset \mathcal{X}$, we put $\mathcal{A}^\star := \mathcal{A} \setminus \mathcal{D}_z$. Let $P \in \mathcal{D}_{z,\underline{k}}$. Let $\mathcal{X}_P$ denote a small neighbourhood of P in $\mathcal{X}$. It will be shrunk if it is necessary.

Let V be a locally free $\mathcal{O}_{\mathcal{X}^\star}$-module with a meromorphic ϱ-flat connection $\mathbb{D}$ whose pole is contained in $\mathcal{D}^\star = \mathcal{D}_Y^\star$. Assume that $\mathbb{D}$ is $\mathfrak{b}$-logarithmic for some $\mathfrak{b} \in M(\mathcal{X}, \mathcal{D})$. Let $\iota : \mathcal{X}^\star \subset \widetilde{\mathcal{X}}(\mathcal{D}_z)$ and $\pi : \widetilde{\mathcal{X}}(\mathcal{D}_z) \to \mathcal{X}$ be natural maps. Let $\mathcal{I} \subset M(\mathcal{X}, \mathcal{D})$ be a weakly good set of irregular values at the level $(\boldsymbol{m}, i(0))$ such that $\{\mathfrak{a} - \mathfrak{b} \mid \mathfrak{a} \in \mathcal{I}\}$ is good at the level $(\boldsymbol{m}, i(0))$.

Definition 4.3.6. — A set of pre-Stokes data $\boldsymbol{\mathcal{F}}$ of $(V, \mathbb{D})$ at the level $(\boldsymbol{m}, i(0))$ at P is a system of $\mathbb{D}$-flat filtrations $\mathcal{F}^Q$ $(Q \in \pi^{-1}(P))$ of stalks $\iota_*(V)_Q$ indexed by $(\mathcal{I}, \leqslant_Q^\varrho)$, satisfying the compatibility condition. (We assume the existence of a $\mathbb{D}_z$-flat splitting, instead of a $\mathbb{D}$-flat splitting.) We will often omit to distinguish "P" and "at the level $(\boldsymbol{m}, i(0))$" if there is no risk of confusion. □

Varying $Q \in \pi^{-1}(P)$, for each $\mathfrak{a} \in \mathcal{I}$, we obtain a locally free $\mathcal{O}_{\mathcal{X}_P^\star}$-module $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_{|\mathcal{X}_P^\star})$ with $\mathfrak{b}$-logarithmic flat ϱ-connection $\mathbb{D}_{\mathfrak{a}}$ on $(\mathcal{X}_P^\star, \mathcal{D}_P^\star)$. (Note that $\mathfrak{a} - \mathfrak{b}$ is holomorphic on $\mathcal{X}_P^\star$.)

Definition 4.3.7. — A graded extension of $\boldsymbol{\mathcal{F}}$ is a tuple of locally free $\mathcal{O}_{\mathcal{X}_P}$-modules $E_{\mathfrak{a}}$ $(\mathfrak{a} \in \mathcal{I})$ such that (i) $E_{\mathfrak{a}|\mathcal{X}_P^\star} \simeq \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_{\mathcal{X}_P^\star})$, (ii) $\mathrm{ord}(\mathbb{D}_{\mathfrak{a}} - d\mathfrak{a}) \geqslant \boldsymbol{m}(1) := \boldsymbol{m} + \boldsymbol{\delta}_{i(0)}$ for each $\mathfrak{a} \in \mathcal{I}$. The tuple $\mathcal{SD} = (\mathcal{I}, \boldsymbol{\mathcal{F}}, \{E_{\mathfrak{a}}\})$ is called a set of Stokes data at the level $(\boldsymbol{m}, i(0))$ at P. □

Let $(V_p, \mathbb{D}_p)$ $(p = 1, 2)$ be $\mathfrak{b}$-logarithmic ϱ-flat bundles $\mathbb{D}_p$ with pre-Stokes data $\boldsymbol{\mathcal{F}}_p$ $(p = 1, 2)$ at the level $(\boldsymbol{m}, i(0))$ over $\mathcal{I}_p$. A morphism $(V_1, \mathbb{D}_1, \boldsymbol{\mathcal{F}}_1) \to (V_2, \mathbb{D}_2, \boldsymbol{\mathcal{F}}_2)$ is defined to be a flat morphism which preserves Stokes filtrations at any $Q \in \pi^{-1}(P)$. Note that we obtain a naturally induced morphism of $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_1) \to \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_2)$. If moreover they are equipped with graded extensions, a morphism $(V_1, \mathbb{D}_1, \mathcal{SD}_1) \to (V_2, \mathbb{D}_2, \mathcal{SD}_2)$ is defined to be a morphism $(V_1, \mathbb{D}_1, \boldsymbol{\mathcal{F}}_1) \to (V_2, \mathbb{D}_2, \boldsymbol{\mathcal{F}}_2)$ such that the induced morphisms $\mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_1) \to \mathrm{Gr}_{\mathfrak{a}}^{\boldsymbol{\mathcal{F}}}(V_2)$ can be extended to $E_{1,\mathfrak{a}} \to E_{2,\mathfrak{a}}$.

4.3.4. Good lattice and Stokes data at the level $\boldsymbol{m}$. — We continue to use the notation in Subsection 4.3.3. In terms of Stokes data, we can summarize the results in Section 3.3 for a weakly good lattice at the level $\boldsymbol{m}$.

Proposition 4.3.8. — *Let $(E, \mathbb{D}, \mathcal{I})$ be a weakly good lattice of a meromorphic ϱ-flat bundle at the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D})$. We put $(V, \mathbb{D}) := (E, \mathbb{D})_{|\mathcal{X} \setminus \mathcal{D}_z}$. Then, at each point $P \in \mathcal{D}_{z,\underline{k}}$, we have the Stokes data $\mathcal{SD}(E, \mathbb{D})$ at the level $(\boldsymbol{m}, i(0))$ for $(V, \mathbb{D})$ associated to $(E, \mathbb{D})$. The correspondence is functorial, and it preserves direct sum, tensor product and dual.* □

We study the converse. Namely, for a given Stokes data at the level $(\boldsymbol{m}, i(0))$ at $P \in \mathcal{D}_{z,\underline{k}}$, we shall construct a good lattice at the level $(\boldsymbol{m}, i(0))$ on a neighbourhood of P.

4.3.4.1. Construction. — Let $(V, \mathbb{D})$ be a $\mathfrak{b}$-logarithmic ϱ-flat bundle on $(\mathcal{X}^\star, \mathcal{D}^\star)$ with a set of Stokes data at P. We use a subscript "z" to indicate that we consider differentials relative to $Y \times \mathcal{K}$. Because $\mathbb{D} - d\mathfrak{b}$ is logarithmic, $\mathbb{D}_z - d_z\mathfrak{b}$ gives a flat ϱ-connection relative to $Y \times \mathcal{K}$.

We take a holomorphic frame $\overline{\boldsymbol{v}}_\mathfrak{a}$ of $E_\mathfrak{a}$ on $\mathcal{X}_P$. Let $R_\mathfrak{a}$ be the connection 1-form of $\mathbb{D}_\mathfrak{a}$ with respect to $\overline{\boldsymbol{v}}_\mathfrak{a}$, i.e., $\mathbb{D}_\mathfrak{a}\overline{\boldsymbol{v}}_\mathfrak{a} = \overline{\boldsymbol{v}}_\mathfrak{a} R_\mathfrak{a}$. Take $Q \in \pi^{-1}(P)$ and a small neighbourhood $\mathcal{U}_Q$. We put $\mathcal{U}_Q^\star = \mathcal{U}_Q \setminus \pi^{-1}(\mathcal{D}_z)$. We can take a $\mathbb{D}_z$-flat splitting $V_{|\mathcal{U}_Q} = \bigoplus V_{\mathfrak{a},Q}$ of $\mathcal{F}^Q$. Let $\boldsymbol{v}_{\mathfrak{a},Q}$ be the lift of $\overline{\boldsymbol{v}}_{\mathfrak{a}|\mathcal{U}_Q^\star}$ to $V_{\mathfrak{a},Q}$. Then, $\boldsymbol{v}_Q = (\boldsymbol{v}_{\mathfrak{a},Q})$ gives a frame of $V_{|\mathcal{U}_Q^\star}$. Let R_Q be determined by $\mathbb{D}\boldsymbol{v}_Q = \boldsymbol{v}_Q R_Q$. We have the decomposition $R_Q = (R_{\mathfrak{a},\mathfrak{c},Q})$ corresponding to $\boldsymbol{v}_Q = (\boldsymbol{v}_{\mathfrak{a},Q})$.

Lemma 4.3.9. — *We have the following:*

- $R_{\mathfrak{a},\mathfrak{c},Q} = 0$ *unless* $\mathfrak{a} \leqslant_Q^\varrho \mathfrak{c}$.
- $R_{\mathfrak{a},\mathfrak{a},Q} = R_\mathfrak{a}$.
- *For* $\mathfrak{a} <_Q^\varrho \mathfrak{c}$, *there exists* $C > 0$ *such that*

$$R_{\mathfrak{a},\mathfrak{c},Q}\, \exp\bigl(\varrho^{-1}(\mathfrak{a} - \mathfrak{c})\bigr) = O\Bigl(\exp\bigl(C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}|\bigr)\,|z_{i(0)}|^{-C}\Bigr).$$

Proof. — Since the filtration $\mathcal{F}^Q$ is $\mathbb{D}$-flat, we obtain the first claim. The second claim is clear by construction. Since the lift is taken for a $\mathbb{D}_z$-flat splitting, the dz_i-components of $R_{\mathfrak{a},\mathfrak{c},Q}$ are 0. We have the expression $R_{\mathfrak{a},\mathfrak{c},Q} = \sum_{j=1}^n R_{\mathfrak{a},\mathfrak{c},Q,j}\, d\zeta_j$. Let $F_{\mathfrak{a},\mathfrak{c},j} : E_{\mathfrak{c}|\mathcal{U}_Q} \to E_{\mathfrak{a}|\mathcal{U}_Q}$ be determined by $F_{\mathfrak{a},\mathfrak{c},j}\overline{\boldsymbol{v}}_\mathfrak{c} = \overline{\boldsymbol{v}}_\mathfrak{a} R_{\mathfrak{a},\mathfrak{c},Q,j}$. They are $\mathbb{D}_z$-flat. Hence, we obtain the desired estimate for $R_{\mathfrak{a},\mathfrak{c},Q,j}$ by using Lemma 20.3.5. □

Lemma 4.3.10. — *Let* $\boldsymbol{v}'_Q$ *be a frame of* $V_{|\mathcal{U}_Q^\star}$ *induced by another splitting* $V_{|\mathcal{U}_Q^\star} = \bigoplus V'_{\mathfrak{a},Q}$. *Let* C *be determined by* $\boldsymbol{v}_Q = \boldsymbol{v}'_Q\,(I + C)$. *We have (i)* $C_{|\widehat{Z}} = 0$, *(ii)* $C_{\mathfrak{a},\mathfrak{c}} = 0$ *unless* $\mathfrak{a} <_Q \mathfrak{c}$, *(iii)* $C_{\mathfrak{a},\mathfrak{c}}\exp\bigl(\varrho^{-1}(\mathfrak{a} - \mathfrak{c})\bigr) = O\bigl(\exp\bigl(C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}|\bigr)\,|z_{i(0)}|^{-C}\bigr)$ *for some* $C > 0$.

Proof. — Two $\mathbb{D}_z$-flat splittings induce a $\mathbb{D}_z$-flat map $\Phi_{\mathfrak{a},\mathfrak{c}} : E_{\mathfrak{c}|\mathcal{U}_Q} \to E_{\mathfrak{a}|\mathcal{U}_Q}$ for $\mathfrak{a} <_Q \mathfrak{c}$. It can be shown that $\Phi_{\mathfrak{a},\mathfrak{c}}\exp\bigl(\varrho^{-1}(\mathfrak{a} - \mathfrak{c})\bigr) = O\bigl(\exp\bigl(C\,|\boldsymbol{z}^{\boldsymbol{m}(1)}|\bigr)|z_{i(0)}|^{-C}\bigr)$ for some $C > 0$ with respect to the frames $\overline{\boldsymbol{v}}_\mathfrak{a}$ and $\overline{\boldsymbol{v}}_\mathfrak{c}$ by Lemma 20.3.5. Then, the claim of Lemma 4.3.10 follows. □

By using the natural isomorphisms of holomorphic bundles

$$V_{|\mathcal{U}_Q^\star} = \bigoplus_\mathfrak{a} V_{\mathfrak{a},Q} \simeq \bigoplus_\mathfrak{a} E_{\mathfrak{a}|\mathcal{U}_Q^\star},$$

we extend $V_{|\mathcal{U}_Q^\star}$ to a holomorphic vector bundle $\widetilde{V}_{\mathcal{U}_Q}$ on $\mathcal{U}_Q$. By Lemma 4.3.9, $\mathbb{D}_{|\mathcal{U}_Q^\star}$ can be extended to a meromorphic flat ϱ-connection $\mathbb{D}_Q$ of $\widetilde{V}_{\mathcal{U}_Q}$ on $\mathcal{U}_Q$. Moreover, we have the following isomorphism:

$$(61) \qquad (\widetilde{V}_{\mathcal{U}_Q}, \mathbb{D}_{\mathcal{U}_Q})_{|\widehat{\pi^{-1}(\mathcal{D}_z)}\cap\mathcal{U}_Q} \simeq \bigoplus_{\mathfrak{a}} (E_{\mathfrak{a}}, \mathbb{D}_{\mathfrak{a}})_{|\widehat{\pi^{-1}(\mathcal{D}_z)}\cap\mathcal{U}_Q}.$$

By Lemma 4.3.10, $(\widetilde{V}_{\mathcal{U}_Q}, \mathbb{D}_{\mathcal{U}_Q})$ and the isomorphism (61) are independent of the choice of a $\mathbb{D}_z$-flat splitting $V = \bigoplus V_{\mathfrak{a},Q}$. If $Q' \in \mathcal{U}_{Q'} \subset \mathcal{U}_Q$, we have $(\widetilde{V}_{\mathcal{U}_Q}, \mathbb{D}_{\mathcal{U}_Q})_{|\mathcal{U}_{Q'}} = (\widetilde{V}_{\mathcal{U}_{Q'}}, \mathbb{D}_{\mathcal{U}_{Q'}})$. By varying Q and gluing them, we obtain a holomorphic vector bundle $\widetilde{V}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}$ with a meromorphic flat ϱ-connection $\mathbb{D}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}$. Moreover, we have an isomorphism

$$(62) \qquad (\widetilde{V}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}, \mathbb{D}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})})_{|\widehat{\pi^{-1}(\mathcal{D}_{P,z})}} \simeq \bigoplus_{\mathfrak{a}} (E_{\mathfrak{a}}, \mathbb{D}_{\mathfrak{a}})_{|\widehat{\pi^{-1}(\mathcal{D}'_z)}}.$$

According to Proposition 3.1.12, Corollary 3.1.15 and Lemma 3.1.17, there exists a holomorphic vector bundle $\widetilde{V}$ with a meromorphic flat ϱ-connection $\mathbb{D}$ on $(\mathcal{X}_P, \mathcal{D}_{P,z})$ such that

$$(63) \qquad \pi^*(\widetilde{V}, \mathbb{D}) \simeq (\widetilde{V}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}, \mathbb{D}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}), \quad (\widetilde{V}, \mathbb{D})_{|\widehat{\mathcal{D}}_{P,z}} \simeq \bigoplus_{\mathfrak{a}} (E_{\mathfrak{a}}, \mathbb{D}_{\mathfrak{a}})_{|\widehat{\mathcal{D}}_{P,z}}.$$

4.3.4.2. Functoriality. — Let $(V, \mathbb{D})$ be a $\mathfrak{b}$-logarithmic ϱ-flat bundle on $(\mathcal{X}^\star, \mathcal{D}^\star)$ with a set of Stokes data at the level $(\boldsymbol{m}, i(0))$. Then, $(V^\vee, \mathbb{D}^\vee)$ is also equipped with an induced Stokes data at the level $(\boldsymbol{m}, i(0))$.

Lemma 4.3.11. — *The associated extension of $(V^\vee, \mathbb{D}^\vee)$ is naturally isomorphic to the dual of that of $(V, \mathbb{D})$.*

Proof. — Let $V_{|\mathcal{U}_Q^\star} = \bigoplus_{\mathfrak{a}} V_{\mathfrak{a},Q}$ be a $\mathbb{D}_z$-flat splitting. It induces a $\mathbb{D}_z$-flat splitting $V^\vee_{|\mathcal{U}_Q^\star} = \bigoplus V^\vee_{-\mathfrak{a},Q}$. We extend $V^\vee_{|\mathcal{U}_Q^\star}$ to $V^\vee_{\mathcal{U}_Q}$ by using the splitting. Then, we have a natural isomorphism $(\widetilde{V}_{\mathcal{U}_Q})^\vee \simeq \widetilde{V}^\vee_{\mathcal{U}_Q}$. Hence, we obtain $(\widetilde{V}_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})})^\vee \simeq \widetilde{V}^\vee_{\widetilde{\mathcal{X}}_P(\mathcal{D}_{P,z})}$. It induces the desired isomorphism. □

Lemma 4.3.12. — *Let $(V_i, \mathbb{D}_i)$ $(i = 1, 2)$ be $\mathfrak{b}$-logarithmic ϱ-flat bundles on $(\mathcal{X}^\star, \mathcal{D}^\star)$ with Stokes data at the level $(\boldsymbol{m}, i(0))$ at P.*

- *If $\mathcal{I}_1 \otimes \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, we have the induced Stokes data of $(V_1, \mathbb{D}_1) \otimes (V_2, \mathbb{D}_2)$ at the level $(\boldsymbol{m}, i(0))$ at P, and the associated extension of $(V_1, \mathbb{D}_1) \otimes (V_2, \mathbb{D}_2)$ is naturally isomorphic to $\widetilde{V}_1 \otimes \widetilde{V}_2$.*
- *If $\mathcal{I}_1 \oplus \mathcal{I}_2$ is weakly good at the level $(\boldsymbol{m}, i(0))$, then we have the induced Stokes data of $(V_1, \mathbb{D}_1) \oplus (V_2, \mathbb{D}_2)$ at the level $(\boldsymbol{m}, i(0))$ at P, and the associated extension is naturally isomorphic to $\widetilde{V}_1 \oplus \widetilde{V}_2$.* □

Let $(V_i, \mathbb{D}_i)$ $(i = 1, 2)$ be $\mathfrak{b}$-logarithmic ϱ-flat bundles on $(\mathcal{X}^\star, \mathcal{D}^\star)$ equipped with Stokes data $\mathcal{SD}_i$ at the level $(\boldsymbol{m}, i(0))$ at P. For simplicity, we assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is also weakly good at the level $(\boldsymbol{m}, i(0))$.

Lemma 4.3.13. — *Let $F : (V_1, \mathbb{D}_1, \mathcal{SD}_1) \to (V_2, \mathbb{D}_2, \mathcal{SD}_2)$ be a morphism. We have the naturally induced morphism $\widetilde{F} : \widetilde{V}_1 \to \widetilde{V}_2$ on $\mathcal{X}_P$.*

Proof. — We take $\mathbb{D}_z$-flat splittings $V_{i|\mathcal{U}_Q} = \bigoplus V_{i,\mathfrak{a},Q}$ of the filtrations $\mathcal{F}^Q$. Let $\overline{\boldsymbol{v}}_{i,\mathfrak{a}}$ be holomorphic frames of $E_{i,\mathfrak{a}}$. Let $\boldsymbol{v}_{i,\mathfrak{a},Q}$ denote the lifts of $\overline{\boldsymbol{v}}_{i,\mathfrak{a},Q}$ to $V_{i,\mathfrak{a},Q}$. They give frames $\boldsymbol{v}_{i,Q}$ of $V_{i|\mathcal{U}_Q}$.

Let A_Q be determined by $F(\boldsymbol{v}_{1,Q}) = \boldsymbol{v}_{2,Q}\, A_Q$. We have the decomposition $A_Q = (A_{\mathfrak{a},\mathfrak{c},Q})$ corresponding to the decomposition $\boldsymbol{v}_{i,Q} = (\boldsymbol{v}_{i,\mathfrak{a},Q})$ $(i = 1, 2)$. Since F preserves the filtrations $\mathcal{F}^Q$, we have $A_{\mathfrak{a},\mathfrak{c}} = 0$ unless $\mathfrak{a} \leqslant_Q \mathfrak{c}$. Since $A_{\mathfrak{a},\mathfrak{a},Q}$ satisfy $F_{\mathfrak{a}}(\overline{\boldsymbol{v}}_{1,\mathfrak{a}}) = \overline{\boldsymbol{v}}_{2,\mathfrak{a}}\, A_{\mathfrak{a},\mathfrak{a},Q}$, they are holomorphic on $\mathcal{U}_Q$. In the case $\mathfrak{a} <_Q \mathfrak{c}$, we obtain the estimate

$$A_{\mathfrak{a},\mathfrak{c},Q} \exp\big(\varrho^{-1}(\mathfrak{a}-\mathfrak{c})\big) = O\Big(\exp\big(C|\boldsymbol{z}^{\boldsymbol{m}(1)}|\big)\, |z_{i(0)}|^{-C}\Big)$$

for some positive constant C, by using Lemma 20.3.5. Hence, F can be extended to a morphism on $\pi^{-1}(\mathcal{X}_P)$, and the claim of Lemma 4.3.13 follows. □

Corollary 4.3.14. — *If the restriction of $F_{|\mathcal{X}\smallsetminus\mathcal{D}}$ is an isomorphism, the induced morphism $\widetilde{F} : \widetilde{V}_1(*\mathcal{D}) \to \widetilde{V}_2(*\mathcal{D})$ is an isomorphism.* □

4.3.5. Proof of Theorem 4.3.1. — Let us show that RHB is essentially surjective. We only have to argue it locally. We put $\mathcal{X} := \Delta^n \times \mathcal{K}$ and $\mathcal{D} := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $\mathcal{I} \subset M(\mathcal{X},\mathcal{D})/H(\mathcal{X})$ be a good set of irregular values. We assume that the coordinate system is admissible for $\mathcal{I}$, and we take an auxiliary sequence $\boldsymbol{m}(p)$ for $\mathcal{I}$. We shall construct an unramifiedly good meromorphic flat bundle around $P \in \mathcal{D}_{\underline{\ell}}$, from a flat bundle with a set of full Stokes data.

4.3.5.1. Graded bundles associated to full pre-Stokes data. — We have a refinement of the construction in Subsection 4.2.1.1. We use the notation there. Let $(V, \mathbb{D})$ be a ϱ-flat bundle on $\mathcal{X} \smallsetminus \mathcal{D}$ with a set of full pre-Stokes data $\widetilde{\boldsymbol{\mathcal{F}}}$ over a good set of irregular values $\mathcal{I}$. For each $Q \in \pi^{-1}(P)$, we obtain the induced filtration $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ of $\iota_*(V)_Q$ indexed by $(\mathcal{I}(\boldsymbol{m}(p)), \leqslant_Q^{\varrho})$. (See Subsection 2.6.2.1 for $\mathcal{I}(\boldsymbol{m}(p))$.) On the associated graded sheaf $\mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{m}(p)}(\iota_*(V)_Q)$, we have the induced filtration $\widetilde{\mathcal{F}}^Q$ indexed by $(\overline{\eta}_{\boldsymbol{m}(p)}^{-1}(\mathfrak{b}), \leqslant_Q^{\varrho})$. By varying $Q \in \pi^{-1}(P)$, we obtain $\mathrm{Gr}^{\boldsymbol{m}(p)}(\iota_*(V,\mathbb{D})_{|\pi^{-1}(\mathcal{X}_P)})$, and a $\mathbb{D}$-flat bundle $\mathrm{Gr}^{\boldsymbol{m}(p)}(V_{|\mathcal{X}_P^*})$ on $\mathcal{X}_P^*$. Let $\pi_{\boldsymbol{m}(p)} : \widetilde{\mathcal{X}}(\mathcal{D}(\underline{k}(p))) \to \mathcal{X}$ be the real blow up, and let $\iota_{\boldsymbol{m}(p)} : \mathcal{X}_P^* \subset \widetilde{\mathcal{X}}(\mathcal{D}(\underline{k}(p)))$. For each $Q_1 \in \pi_{\boldsymbol{m}(p)}^{-1}(P)$, we have the $\mathbb{D}$-flat induced filtration $\widetilde{\mathcal{F}}^{Q_1}$ of $\iota_{\boldsymbol{m}(p)*}\,\mathrm{Gr}^{\boldsymbol{m}(p)}(V_{|\mathcal{X}_P^*})_{Q_1}$.

For a given $J \subset \underline{\ell}$, let $\boldsymbol{m}(p_J)$ be determined by $m_i(p_J + 1) = 0$ for any $i \in J$ and $m_i(p_J) \neq 0$ for some $i \in J$. Note that the the image of $\mathcal{I}(\boldsymbol{m}(p_J))$ by $M(\mathcal{X},\mathcal{D})/H(\mathcal{X}) \to M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(J^c))$ coincides with $\mathcal{I}^J$. We have $\boldsymbol{\mathcal{F}}^J = \boldsymbol{\mathcal{F}}^{\boldsymbol{m}(p_J)}$.

4.3.5.2. Construction. — Let $(V, \mathbb{D}, \widetilde{\boldsymbol{\mathcal{F}}}, \boldsymbol{\mathcal{G}}) \in \mathrm{SD}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$. Take $P \in \mathcal{D}_{\underline{\ell}}$. In the following, we will replace $\mathcal{X}$ with a small neighbourhood of P if it is necessary. We have $\mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{m}(p)}(V)$ on $\mathcal{X} \smallsetminus \mathcal{D}$ for $\mathfrak{b} \in \mathcal{I}(\boldsymbol{m}(p))$. We shall construct a $\mathfrak{b}$-logarithmic extension $E_{\mathfrak{b}}^{\boldsymbol{m}(p)}$ of $\mathrm{Gr}_{\mathfrak{b}}^{\boldsymbol{m}(p)} V$ on $\mathcal{X} \smallsetminus \mathcal{D}(\underline{k}(p+1))$.

For $J(k) := \{k+1,\dots,\ell\}$, we take $\boldsymbol{m}(p_{J(k)})$ as in Subsection 4.3.5.1. Then, for $\mathfrak{b}\in\mathcal{I}(\boldsymbol{m}(p_{J(k)}))$, we have a locally free $\mathcal{O}_{\mathcal{X}\setminus\mathcal{D}(\underline{k})}$-module $\mathcal{G}^{J(k)}_{\mathfrak{b}}$. Let $\pi_k:\widetilde{\mathcal{X}}(\mathcal{D}(\underline{k}))\to\mathcal{X}$ and $\iota_k:\mathcal{X}\setminus\mathcal{D}(\underline{k})\to\widetilde{\mathcal{X}}(\mathcal{D}(\underline{k}))$ be natural maps. For each $Q\in\pi_k^{-1}(P)$, we have the $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^Q$ of $\iota_{k*}(\mathcal{G}^{J(k)}_{\mathfrak{b}})_Q$ indexed by $\big(\overline{\eta}^{-1}_{\boldsymbol{m}(p_{J(k)})}(\mathfrak{b}),\leqslant^{\varrho}_Q\big)$. (See the argument in Subsection 4.2.2.3.) We have the induced filtrations $\mathcal{F}^{Q\,\boldsymbol{m}(p)}$ for any $p\geqslant p_{J(k)}$. Since they satisfy a compatibility condition, we obtain $\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{a}}\big(\mathcal{G}^{J(k)}_{\mathfrak{b}}\big)$ on $\mathcal{X}\setminus\mathcal{D}(\underline{k})$, for any $p\geqslant p_{J(k)}$ and $\mathfrak{a}\in\overline{\eta}^{-1}_{\boldsymbol{m}(p_{J(k)})}(\mathfrak{b})$. By using the compatibility condition of lattices in Stokes data, we obtain

$$\mathrm{Gr}^{\boldsymbol{m}(p_{J(k-1)})}_{\mathfrak{a}}\big(\mathcal{G}^{J(k)}_{\mathfrak{b}}\big)\simeq\mathcal{G}^{J(k-1)}_{\mathfrak{a}|\mathcal{X}\setminus\mathcal{D}(\underline{k})}\tag{64}$$

for any $\mathfrak{a}\in\mathcal{I}(\boldsymbol{m}(p_{J(k-1)}))$.

For p, we take k such that $p_{J(k)}\leqslant p<p_{J(k+1)}$. For any $\mathfrak{c}\in\mathcal{I}(\boldsymbol{m}(p))$, we put $E^p_{\mathfrak{c}}:=\mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{c}}\big(\mathcal{G}^{J(k)}_{\mathfrak{b}}\big)$, which is equipped with the induced filtrations $\mathcal{F}^{Q\,\boldsymbol{m}(p')}$ for any $Q\in\pi^{-1}(P)$ and $p'\geqslant p$. The system of the filtrations satisfies a compatibility condition. By (64), we have a natural isomorphism $\mathrm{Gr}^{\boldsymbol{m}(q)}_{\mathfrak{c}}E^p_{\mathfrak{b}}\simeq E^q_{\mathfrak{c}|\mathcal{X}\setminus\mathcal{D}(\underline{k}(p-1))}$.

Then, by a successive use of the construction in Subsection 4.3.4.1, we can construct a locally free $\mathcal{O}_{\mathcal{X}}$-module E_P with an isomorphism $E_{|\mathcal{X}\setminus\mathcal{D}}\simeq V_{|\mathcal{X}\setminus\mathcal{D}}$ and $(E,\mathbb{D})_{|\widehat{P}}\simeq\bigoplus_{\mathfrak{a}\in\mathcal{I}_P}\mathcal{G}_{P,\mathfrak{a}|\widehat{P}}$. In particular, E is an unramifiedly good lattice. By construction, for each $Q\in\pi^{-1}(P)$, the full Stokes filtration $\widetilde{\mathcal{F}}^Q$ of E at Q is the same as that in the given Stokes data $\mathcal{SD}$. It implies that, for any $Q\in\pi^{-1}(\mathcal{D})$, the full Stokes filtration of E at Q is the same as that in the given Stokes data, according to Lemma 4.1.2. (Note that we have shrunk $\mathcal{X}$ around P.)

Let $J\subset\underline{\ell}$. Take $P'\in\mathcal{D}_J$ with $I(P')=J$. It remains to show that we have a natural isomorphism

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E_{P'})\simeq\bigoplus_{\mathfrak{a}\in\mathcal{I}_{P'}}\mathcal{G}_{P',\mathfrak{a}}\tag{65}$$

on a small neighbourhood $\mathcal{X}_{P'}$ of P'. We have $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E_{P'})\simeq\mathrm{Gr}^{\mathcal{F}^J}(E)_{|\mathcal{X}_{P'}}$. If $J=J(k)$ for some k, (65) is clear by our construction of E. In the general case, we put $k:=\min J$. Because we naturally have $\mathcal{G}^{J(k)}_{\mathfrak{a}|\mathcal{X}_{P'}}\simeq\mathcal{G}_{P',\mathfrak{a}}$ and $\mathrm{Gr}^{\mathcal{F}^J}(E)\simeq\mathrm{Gr}^{\mathcal{F}^{J(k)}}(E)$, we have the desired isomorphism. Thus the proof of Theorem 4.3.1 is finished. □

4.4. Extension of Stokes data

4.4.1. Statement. — Let $\mathcal{X}\to\mathcal{B}\to\mathcal{K}$ be smooth fibrations of complex manifolds. Let $\mathcal{D}$ be a normal crossing hypersurface of $\mathcal{X}$ such that each intersection of irreducible components is smooth over $\mathcal{B}$. For simplicity, we assume the following:

- $a:\mathcal{B}\to\mathcal{K}$ is equipped with a section $b:\mathcal{K}\to\mathcal{B}$, and each fiber of a is simply connected.

- We put $\mathcal{X}^b := \mathcal{X} \times_{\mathcal{B}} b$ and $\mathcal{D}^b := \mathcal{D} \times_{\mathcal{B}} b$. Then, $(\mathcal{X}, \mathcal{D})$ is topologically a product of $(\mathcal{X}^b, \mathcal{D}^b)$ and $\mathcal{B}$.

For example, we would like to consider the case $\mathcal{B} = \mathcal{K} \times B$ and $(\mathcal{X}, \mathcal{D}) = (\mathcal{X}^b, \mathcal{D}^b) \times \mathcal{B}$ as complex manifolds.

Let ϱ be a nowhere vanishing holomorphic function on $\mathcal{K}$. Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values on $(\mathcal{X}, \mathcal{D})$. Its restriction to $\mathcal{X}^b$ is denoted by $\boldsymbol{\mathcal{I}}^b$.

Theorem 4.4.1. — *The restriction* $\mathrm{SDL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}}) \to \mathrm{SDL}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)$ *is an equivalence.*

Note that, under the assumption, the restriction induces an equivalence between ϱ-flat bundles over $\mathcal{X} \setminus \mathcal{D}$ and $\mathcal{X}^b \setminus \mathcal{D}^b$. For a ϱ-flat bundle $(V, \mathbb{D})$ on $\mathcal{X} \setminus \mathcal{D}$, let $(V^b, \mathbb{D}^b)$ denote its restriction to $\mathcal{X}^b \setminus \mathcal{D}^b$. Theorem 4.4.1 says that a set of Stokes data of $(V^b, \mathbb{D}^b)$ over $\boldsymbol{\mathcal{I}}^b$ can be uniquely extended to a set of Stokes data of $(V, \mathbb{D})$ in a functorial way. By using the uniqueness, we easily obtain the following:

Corollary 4.4.2. — *The above extension is functorial with respect to dual, tensor product, direct sum in an obvious sense.*

Corollary 4.4.3. — *The restriction* $\mathrm{MFL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}}) \to \mathrm{MFL}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)$ *is an equivalence.* □

4.4.1.1. Variants. — We immediately obtain the meromorphic variant.

Corollary 4.4.4. — *The restrictions*

$$\mathrm{SD}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}}) \longrightarrow \mathrm{SD}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b), \quad \mathrm{MF}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}}) \longrightarrow \mathrm{MF}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)$$

are equivalences. □

Let G be a finite group acting on $(\mathcal{X}, \mathcal{D})$ over $\mathcal{B}$. Let $\mathcal{C}(\mathcal{X} \setminus \mathcal{D})^G$ be the category of G-equivariant ϱ-flat bundles on $\mathcal{X} \setminus \mathcal{D}$. By using the uniqueness, we easily obtain the following.

Corollary 4.4.5. — *The restrictions*

$$\mathrm{SDL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})^G \longrightarrow \mathrm{SDL}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)^G, \quad \mathrm{MFL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})^G \longrightarrow \mathrm{MFL}(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)^G$$

are equivalences. We have the meromorphic variant. □

4.4.2. Extension of full pre-Stokes data. — Let us consider the claim for full pre-Stokes data.

Proposition 4.4.6

- *Let $(V, \mathbb{D})$ be a ϱ-flat bundle on $\mathcal{X} \setminus \mathcal{D}$. If we are given a set of full pre-Stokes data of $(V^b, \mathbb{D}^b)$ over $(\mathcal{X}^b, \mathcal{D}^b, \boldsymbol{\mathcal{I}}^b)$, it is uniquely extended to a set of full pre-Stokes data of $(V, \mathbb{D})$ over $(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$.*

- *Let $(V_i, \mathbb{D}_i)$ $(i = 1, 2)$ be ϱ-flat bundles on $\mathcal{X} \setminus \mathcal{D}$ equipped with full pre-Stokes structures $\widetilde{\boldsymbol{\mathcal{F}}}_i$ over $(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$. Let $F : (V_1, \mathbb{D}_1) \to (V_2, \mathbb{D}_2)$ be a morphism. If its restriction F^b preserves full Stokes filtrations, then F does so.*

We only give the proof of the first claim. The second claim can be shown similarly.

4.4.2.1. We put $\mathcal{X}_0 := \Delta^n \times Y \times \mathcal{K}$ and $\mathcal{D}_0 := \bigcup_{i=1}^n \{z_i = 0\}$. We consider the case $\mathcal{X} := B \times \mathcal{X}_0$ and $\mathcal{D} := B \times \mathcal{D}_0$ for some simply connected complex manifold B. Let $\pi : \widetilde{\mathcal{X}}(\mathcal{D}) \to \mathcal{X}$ and $\pi_0 : \widetilde{\mathcal{X}}_0(\mathcal{D}_0) \to \mathcal{X}_0$ be the real blow up. Let $\iota : \mathcal{X} \setminus \mathcal{D} \subset \widetilde{\mathcal{X}}(\mathcal{D})$ and $\iota_0 : \mathcal{X}_0 \setminus \mathcal{D}_0 \subset \widetilde{\mathcal{X}}_0(\mathcal{D}_0)$ be natural maps. Let $(V, \mathbb{D})$ be a ϱ-flat bundle on $\mathcal{X} \setminus \mathcal{D}$. Let P be any point of $\mathcal{D}_0$. Note that we have natural identifications $\mathcal{I}_{(b,P)} \simeq \mathcal{I}_{(b',P)}$ for $b, b' \in B$. In the following, for $Q \in \pi_0^{-1}(P)$, let U_Q denote a small neighbourhood of Q in $\pi_0^{-1}(P)$.

4.4.2.2. Let $b_0 \in B$. Assume that we are given a full pre-Stokes structure of $\iota_*(V, \mathbb{D})_{|\pi^{-1}(b_0,P)}$. According to Lemma 4.1.9, there exists a neighbourhood B_0 of b_0 such that $\iota_*(V)_{|B_0 \times \pi_0^{-1}(P)}$ has a full pre-Stokes structure whose restriction to $b \times \pi_0^{-1}(P)$ is equal to the given one. Assume that we are given two such full pre-Stokes structures $\boldsymbol{\mathcal{F}}_i$ of $\iota_*(V)_{|B_0 \times \pi_0^{-1}(P)}$. For any $Q \in b \times \pi_0^{-1}(P)$, there exists a small neighbourhood $\mathcal{U}_Q$ in $\widetilde{\mathcal{X}}(\mathcal{D})$ such that $\boldsymbol{\mathcal{F}}_{1|\mathcal{U}_Q} = \boldsymbol{\mathcal{F}}_{2|\mathcal{U}_Q}$. Hence, there exists a small neighbourhood $B_0' \subset B_0$ such that $\boldsymbol{\mathcal{F}}_{1|B_0' \times \pi_0^{-1}(P)} = \boldsymbol{\mathcal{F}}_{2|B_0' \times \pi_0^{-1}(P)}$.

4.4.2.3. Let $b_1 \in B$ with a neighbourhood B_1. Assume that there is an open subset $B_1' \subset B_1$ such that (i) b_1 is contained in the closure of B_1' in B_1, (ii) for any small ball B_P around P, $B_P \cap B_1'$ is connected, (iii) a full pre-Stokes structure of $\iota_*(V)_{|B_1' \times \pi_0^{-1}(P)}$ is given. Let $Q \in \pi_0^{-1}(P)$. We can take a small neighbourhood $U(b_1, Q) = B_2 \times U_Q \subset B_1 \times \pi_0^{-1}(P)$ such that $\leqslant_{(b_1,Q)} \;=\; \leqslant_{U(b_1,Q)}$.

Lemma 4.4.7. — *If $U(b_1, Q)$ is sufficiently small, we have $\leqslant_{(b_1,Q)} = \leqslant_{U'(b_1,Q)}$, where $U'(b_1, Q) := U_1(b_1, Q) \times_B B_1'$. Moreover, for any $b \in B_2$, we have $\leqslant_{(b_1,Q)} = \leqslant_{U^b(b_1,Q)}$, where $U^b(b_1, Q) = b \times_B U(b_1, Q)$.*

Proof. — For any fixed pair $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_{(b,P)}$, after appropriate coordinate change, $F_{\mathfrak{a},\mathfrak{b}} = -\operatorname{Re}\big(|\boldsymbol{z}^{-\boldsymbol{m}}|\, \boldsymbol{z}^{\boldsymbol{m}}\big)$. Hence, $F_{\mathfrak{a},\mathfrak{b}}^{-1}(0) \cap \pi_0^{-1}(B_2 \times P) \to B_2 \times P$ is a smooth fibration, if B_2 is sufficiently small. Then, the claim is clear. □

Take $b_2 \in B_2 \cap B_1'$. By using Proposition 4.1.5, we have a $\mathbb{D}$-flat filtration of $\iota_*(V)_{|b_2 \times U_Q}$. It can be extended to a $\mathbb{D}$-flat filtration on $U(b_1, Q)$. It is independent of the choice of b_2. By varying $Q \in \pi_0^{-1}(P)$, we obtain a neighbourhood B_4 of b_1 and full pre-Stokes structure of $\iota_*(V)_{|B_4 \times \pi_0^{-1}(P)}$. Note that the uniqueness is also obtained.

4.4.2.4. Assume that we are given a full pre-Stokes structure of $\iota_*(V)_{|b_3 \times \pi_0^{-1}(\mathcal{D}_0)}$. Take any $b_4 \in B$ and $P \in \mathcal{D}_0$. We take a path γ connecting b_3 and b_4. By a continuity method, we can show that there exists a neighbourhood $\mathcal{V}$ of γ and a unique full pre-Stokes structure of $\iota_*(V)_{|\mathcal{V}\times\pi_0^{-1}(P)}$ whose restriction to $b_3 \times \pi_0^{-1}(P)$ is equal to the given one. In particular, we obtain a unique full pre-Stokes structure of $\iota_*(V)_{|b_4\times\pi_0^{-1}(P)}$. It is easy to show that the filtrations are independent of the choice of a path, and that the compatibility condition is satisfied. Thus, the first claim of Proposition 4.4.6 is proved under the setting of Subsection 4.4.2.1.

4.4.2.5. Let us return to the setting in Subsection 4.4.1. Let $\mathcal{D} = \bigcup_{i\in\Lambda}\mathcal{D}_i$ be the decomposition into irreducible components. For $I \subset \Lambda$, we put $\mathcal{D}_I^* := \bigcap_{i\in I}\mathcal{D}_i \setminus \bigcup_{i\notin I}\mathcal{D}_i$. Let $\mathcal{X} \xrightarrow{c} \mathcal{B} \xrightarrow{a} \mathcal{K}$. Take $P \in \mathcal{D}$. Let $I(P) := \{i \in \Lambda \,|\, P \in \mathcal{D}_i\}$. We put $y := a\circ c(P)$. We can take a path γ in $(a\circ c)^{-1}(y)\cap\mathcal{D}^*_{I(P)}$ connecting P and $c^{-1}(b(y))\cap\mathcal{D}^*_{I(P)}$. Note that such a path is unique up to homotopy. By a continuity method, we can show that there exists a neighbourhood $\mathcal{V}$ of γ and a unique full pre-Stokes structure of $\iota_*(V)_{|\pi^{-1}(\gamma)}$ whose restriction to the intersection with $\pi^{-1}(\mathcal{D}^b)$ is equal to the given one. In particular, we obtain a full pre-Stokes structure of $\iota_*(V)_{|\pi_0^{-1}(P)}$. It is easy to show that the filtrations are independent of the choice of a path, and that the compatibility condition is satisfied. Thus, the first claim of Proposition 4.4.6 is proved.

4.4.3. Extension of $\mathfrak{b}$-logarithmic bundle. — Let $(V,\mathbb{D})$ be a $\mathbb{D}$-flat bundle on $\mathcal{X}\setminus\mathcal{D}$. Let $\mathfrak{b}\in M(\mathcal{X},\mathcal{D})$.

Lemma 4.4.8. — *Assume that the restriction $(V^b,\mathbb{D}^b)$ is extended to a $\mathfrak{b}^b$-logarithmic ϱ-flat bundle E_0. Then, $(V,\mathbb{D})$ is uniquely extended to a $\mathfrak{b}$-logarithmic ϱ-flat bundle E such that $E_{|\mathcal{X}\times b} = E_0$.*

Proof. — We only have to consider the case $\mathfrak{b}=0$. By the uniqueness, the claim is a local property. Hence, we may assume $(\mathcal{X},\mathcal{D}) = B\times(\mathcal{X}_0,\mathcal{D}_0)$. Then the existence is clear, because $(V,\mathbb{D})$ is isomorphic to the pull-back of $(V,\mathbb{D})_{|b\times(\mathcal{X}_0\setminus\mathcal{D}_0)}$. Let us show the uniqueness. Let E be such an extension. If we restrict $\mathbb{D}$ to the B-direction, there is no pole, i.e., we obtain a ϱ-flat connection relative to $\mathcal{X}_0$ without any pole. Then, the claim is clear. □

We mention a consequence of this lemma for the proof of Theorem 4.4.1. Let $(\boldsymbol{\mathcal{F}}^b,\boldsymbol{\mathcal{G}}^b)$ be a set of full Stokes data of $(V^b,\mathbb{D}^b)$. According to Proposition 4.4.6, $\boldsymbol{\mathcal{F}}^b$ is uniquely extended to a set of full pre-Stokes data of $(V,\mathbb{D})$. By Lemma 4.4.8, we obtain that $\boldsymbol{\mathcal{G}}^b$ is also uniquely extended to a graded extension of $\boldsymbol{\mathcal{F}}^b$. It remains to show the compatibility condition for $\boldsymbol{\mathcal{G}}$.

4.4.4. Compatibility. — We give a preparation. We put $\mathcal{X}_0 := \Delta^n\times Y\times\mathcal{K}$ and $\mathcal{D}_{0,z} := \bigcup_{i=1}^n\{z_i=0\}$. Let $\mathcal{D}_{0,Y}$ be a hypersurface obtained as the pull-back of a normal crossing hypersurface of Y. We put $\mathcal{D}_0 := \mathcal{D}_{0,z}\cup\mathcal{D}_{0,Y}$. Let B be a complex manifold. We put $\mathcal{X} := B\times\mathcal{X}_0$, $\mathcal{D}_z := B\times\mathcal{D}_{0,z}$, and $\mathcal{D} := B\times\mathcal{D}_0$. Let $\mathcal{I}$ be a good set of irregular values on $(\mathcal{X},\mathcal{D}_z)$.

We have the following blow up:

$$\pi_{0,z}:\widetilde{\mathcal{X}}_0(\mathcal{D}_{0,z})\longrightarrow\widetilde{\mathcal{X}}_0,\qquad \pi_0:\widetilde{\mathcal{X}}_0(\mathcal{D}_0)\longrightarrow\widetilde{\mathcal{X}}_0,$$
$$\pi_z:\widetilde{\mathcal{X}}(\mathcal{D}_z)\longrightarrow\widetilde{\mathcal{X}},\qquad \pi:\widetilde{\mathcal{X}}(\mathcal{D})\longrightarrow\widetilde{\mathcal{X}}.$$

We have the following inclusions:

$$\iota_0:\mathcal{X}_0\setminus\mathcal{D}_0\longrightarrow\widetilde{\mathcal{X}}_0(\mathcal{D}_{0,z}),\qquad \iota_0':\mathcal{X}_0\setminus\mathcal{D}_{0,z}\longrightarrow\widetilde{\mathcal{X}}_0(\mathcal{D}_{0,z}),$$
$$\iota:\mathcal{X}\setminus\mathcal{D}\longrightarrow\widetilde{\mathcal{X}}(\mathcal{D}_z),\qquad \iota':\mathcal{X}\setminus\mathcal{D}_z\longrightarrow\widetilde{\mathcal{X}}(\mathcal{D}_z).$$

Let $(E,\mathbb{D})$ be a logarithmic ϱ-flat bundle on $(\mathcal{X}\setminus\mathcal{D}_z,\mathcal{D}\setminus\mathcal{D}_z)$. We put $(V,\mathbb{D}):=(E,\mathbb{D})_{|\mathcal{X}\setminus\mathcal{D}}$. We have a natural inclusion $\iota'_*E\subset\iota_*V$.

Let $\widetilde{\boldsymbol{\mathcal{F}}}$ be a full pre-Stokes structure of $(V,\mathbb{D})$ over $\mathcal{I}$. Because $\mathcal{I}$ is contained in $M(\mathcal{X},\mathcal{D}_z)/H(\mathcal{X})$, we have the induced filtrations $\widetilde{\mathcal{F}}^{b,Q}$ of the stalks $\iota_*(V)_{b,Q}$ for any $(b,Q)\in\widetilde{\mathcal{X}}(\mathcal{D}_z)$.

Let $b_0\in B$. We have the restriction $(V^{b_0},\mathbb{D}^{b_0})$ on $\{b_0\}\times(\mathcal{X}_0\setminus\mathcal{D}_0)$ and $(E^{b_0},\mathbb{D}^{b_0})$ on $\{b_0\}\times(\mathcal{X}_0\setminus\mathcal{D}_{0,z})$. We have the natural inclusion $\iota'_{0*}E^{b_0}\subset\iota_{0*}V^{b_0}$. We have the filtrations $\widetilde{\mathcal{F}}^Q$ of $(\iota_{0*}V)_Q$ for any $Q\in\pi_{0,z}^{-1}(\mathcal{D}_{0,z})$.

Lemma 4.4.9. — *Assume the following:*

- *Take any $Q\in\pi_{0,z}^{-1}(\mathcal{D}_{0,z})$, then $\widetilde{\mathcal{F}}^Q$ of $\iota_{0*}(V^{b_0})_Q$ is induced by a filtration of $\iota'_{0*}(E^{b_0})_Q$.*

Then, for any $b\in B$ and $Q\in\pi_0^{-1}(\mathcal{D}_z)$, $\widetilde{\mathcal{F}}^{(b,Q)}$ of $\iota_*(V)_{(b,Q)}$ is induced by a filtration of $\iota'_*(E)_{(b,Q)}$.

Note that the assumption and the claim are trivial if Q is not contained in the inverse image of $\mathcal{D}_z\cap\mathcal{D}_Y$.

Proof. — By the assumption, we have the filtration $\widetilde{\mathcal{F}}_1^Q$ of the stalk $(\iota'_{0*}E^{b_0})_Q$ for each $Q\in\pi_{0,z}^{-1}(\mathcal{D}_{0,z})$. The system is denoted by $\widetilde{\boldsymbol{\mathcal{F}}}_1$. It satisfies the compatibility condition. Because $(E,\mathbb{D})$ is logarithmic, by forgetting the differentials in the Y-direction, we obtain a flat ϱ-connection $\mathbb{D}_1$ relative to $Y\times\mathcal{K}$. The system $\widetilde{\boldsymbol{\mathcal{F}}}_1$ gives a full pre-Stokes structure of $\iota'_{0*}(E^{b_0},\mathbb{D}_1^{b_0})$. According to Proposition 4.4.6, it is uniquely extended to a full pre-Stokes structure $\widetilde{\boldsymbol{\mathcal{F}}}_2$ of $\iota'_*(E,\mathbb{D}_1)$.

The restriction of $\widetilde{\boldsymbol{\mathcal{F}}}$ to $\widetilde{\mathcal{X}}(\mathcal{D}_z)\setminus\pi_z^{-1}(\mathcal{D}_Y)$ gives a full pre-Stokes structure of $\iota_*(V,\mathbb{D}_1)_{|\widetilde{\mathcal{X}}(\mathcal{D}_z)\setminus\pi_z^{-1}(\mathcal{D}_Y)}$. By the uniqueness, we obtain the coincidence of the restrictions of $\widetilde{\boldsymbol{\mathcal{F}}}$ and $\widetilde{\boldsymbol{\mathcal{F}}}_2$ to $\widetilde{\mathcal{X}}(\mathcal{D}_z)\setminus\pi_z^{-1}(\mathcal{D}_Y)$. Then, we can deduce that $\widetilde{\boldsymbol{\mathcal{F}}}$ is induced by $\widetilde{\boldsymbol{\mathcal{F}}}_2$ on $\widetilde{\mathcal{X}}(\mathcal{D}_z)$. □

4.4.5. End of the proof of Theorem 4.4.1. — Let us finish the proof of Theorem 4.4.1. It remains to check the compatibility condition for $\boldsymbol{\mathcal{G}}$. We use the setting in Subsection 4.4.2.1. Let $P\in\mathcal{D}_0$. In the following, $\mathcal{X}_P$ denotes a small neighbourhood of $B\times\{P\}$. We have a natural bijection $\mathcal{I}_{(b,P)}\simeq\mathcal{I}_{(b',P)}$ for any $b,b'\in B$. We will identify them naturally, and denoted by $\widetilde{\mathcal{I}}_P$. From a full pre-Stokes structure of V, we have $\mathrm{Gr}^{\mathcal{F}^J}_{\mathfrak{b}}(V)$ on $\mathcal{X}^*_P$ for $\mathfrak{b} \in \widetilde{\mathcal{I}}^J_P$. The filtrations $\widetilde{\mathcal{F}}^Q$ $(Q \in \pi^{-1}(B \times P))$ induce a full pre-Stokes structure $\widetilde{\boldsymbol{\mathcal{F}}}$ of $\mathrm{Gr}^{\mathcal{F}^J}_{\mathfrak{b}}(V)$, by Lemma 4.1.9. We have ${}^J\mathcal{G}_{\mathfrak{a}}$ on $\mathcal{X}_P \setminus \mathcal{D}_P(J^c)$. Assume the following:

- The filtrations $\widetilde{\mathcal{F}}^Q$ of $\iota_{J*}\mathrm{Gr}^{\mathcal{F}}_{\mathfrak{b}}(V^{b_0})_Q$ $(Q \in \pi_0^{-1}(P))$ are induced by filtrations of $\iota'_{J*}\big({}^J\mathcal{G}^{b_0}_{\mathfrak{b}}\big)_Q$.
- $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}\big({}^J\mathcal{G}^{b_0}_{\mathfrak{b}}\big)$ is isomorphic to $\mathcal{G}^{b_0}_{P,\mathfrak{b}|\mathcal{X}_P \setminus \mathcal{D}_P(J^c)}$.

By Lemma 4.4.9, we obtain that the filtrations $\widetilde{\mathcal{F}}^{(b,Q)}$ of $\iota_{J*}\mathrm{Gr}^{\mathcal{F}}_{\mathfrak{b}}(V)_{(b,Q)}$ $(Q \in \pi_0^{-1}(P),\ b \in B)$ are induced by filtrations of $\iota'_{J*}\big({}^J\mathcal{G}_{\mathfrak{b}}\big)_{(b,Q)}$. By using Lemma 4.4.8, we obtain that $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}\big({}^J\mathcal{G}_{\mathfrak{b}}\big)$ is isomorphic to $\mathcal{G}_{P,\mathfrak{b}|\mathcal{X}_P \setminus \mathcal{D}_P(J^c)}$. Thus, we obtain Theorem 4.4.1. □

4.5. Deformation

4.5.1. Deformation $E^{(\mathcal{T})}$

4.5.1.1. Unramified case. — Let $\mathcal{C}$ be a simply connected compact region in $\boldsymbol{C}^m$. We put $(\mathcal{X}^\circ, \mathcal{D}^\circ) := (\mathcal{X}, \mathcal{D}) \times \mathcal{C}$. Let $\mathcal{T}$ be a holomorphic function on $\mathcal{K} \times \mathcal{C}$, which is nowhere vanishing. Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values on $(\mathcal{X}, \mathcal{D})$. For each $(P, c) \in \mathcal{D}^\circ$, we put $\mathcal{I}^{(\mathcal{T})}_{(P,c)} := \big\{\mathcal{T}\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathcal{I}_P\big\}$. Thus, we obtain a good system of irregular values $\boldsymbol{\mathcal{I}}^{(\mathcal{T})}$.

We have an obvious deformation of a ϱ-flat bundle with a set of Stokes data on $(\mathcal{X}, \mathcal{D})$ over $\boldsymbol{\mathcal{I}}$. Let $(V, \mathbb{D}, \mathcal{SD}) \in \mathrm{SD}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$. Let $(V^\circ, \mathbb{D}^\circ)$ be the ϱ-flat bundle on $\mathcal{X}^\circ \setminus \mathcal{D}^\circ$ obtained as the pull-back via the projection to $\mathcal{X} \setminus \mathcal{D}$. Applying Theorem 4.3.1, we obtain $(V^\circ, \mathbb{D}^\circ, \mathcal{SD}^{(\mathcal{T})}) \in \mathrm{SD}(\mathcal{X}^\circ, \mathcal{D}^\circ, \boldsymbol{\mathcal{I}}^{(\mathcal{T})})$.

We have the corresponding deformation for unramifiedly good lattice of a meromorphic ϱ-flat bundle. Namely, for $(E, \mathbb{D}) \in \mathrm{MFL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$, we have $(E^{(\mathcal{T})}, \mathbb{D}^{(\mathcal{T})}) \in \mathrm{MFL}(\mathcal{X}^\circ, \mathcal{D}^\circ, \boldsymbol{\mathcal{I}}^{(\mathcal{T})})$, corresponding to the obvious deformation of the Stokes data as above. It is unique up to canonical isomorphisms.

4.5.1.2. Remark on descent. — Let $\varphi : (\mathcal{X}', \mathcal{D}') \to (\mathcal{X}, \mathcal{D})$ be a ramified Galois covering over $\mathcal{K}$ with the Galois group G. We put $\boldsymbol{\mathcal{I}}' := \varphi^*\boldsymbol{\mathcal{I}}$. Take $(E', \mathbb{D}') \in \mathrm{MFL}\big(\mathcal{X}', \mathcal{D}', \boldsymbol{\mathcal{I}}'\big)^G$. Let $(E, \mathbb{D}) \in \mathrm{MFL}(\mathcal{X}, \mathcal{D}, \boldsymbol{\mathcal{I}})$ be the descent of $(E', \mathbb{D}')$. According to Corollary 4.4.5, $(E', \mathbb{D}')^{(\mathcal{T})}$ is also G-equivariant.

Lemma 4.5.1. — *$(E, \mathbb{D})^{(\mathcal{T})}$ is the descent of $(E', \mathbb{D}')^{(\mathcal{T})}$.*

Proof. — Let $(E_1, \mathbb{D}_1)$ be the descent of $(E', \mathbb{D}')^{(\mathcal{T})}$. By construction, the restrictions of $(E, \mathbb{D})^{(\mathcal{T})}$ and $(E_1, \mathbb{D}_1)$ to $\mathcal{X}^b$ are naturally isomorphic. By Corollary 4.4.3, they are isomorphic on $\mathcal{X}$. □

Let us consider the case $\boldsymbol{\mathcal{I}}'$ is not necessarily the pull-back of a good system of irregular values on $(\mathcal{X}, \mathcal{D})$. Let $(E'_i, \mathbb{D}'_i) \in \mathrm{MFL}(\mathcal{X}', \mathcal{D}', \boldsymbol{\mathcal{I}}')$ $(i = 1, 2)$. Their descent $(E_i, \mathbb{D}_i)$ are not necessarily unramified.

Lemma 4.5.2. — *If* $(E_1,\mathbb{D}_1)\simeq(E_2,\mathbb{D}_2)$, *then* $(E_1,\mathbb{D}_1)^{(\mathcal{T})}\simeq(E_2,\mathbb{D}_2)^{(\mathcal{T})}$.

Proof. — It is easy to reduce the issue to the case where $\mathcal{D}$ is smooth. Moreover, we only have to consider the case $\dim\mathcal{X}=1$. Because $E_1'\cap E_2'$ is also an unramifiedly good lattice, we may assume $E_1'\subset E_2'$. We have $E_1'^{(\mathcal{T})}\subset E_2'^{(\mathcal{T})}$, and we only have to compare their sections which are invariant with respect to the Galois actions. Then, the claim is obvious. □

4.5.1.3. General case. — Let $(E,\mathbb{D})$ be a good lattice of a meromorphic ϱ-flat bundle on $(\mathcal{X},\mathcal{D})$, which is not necessarily unramified. For any $P\in\mathcal{D}$, we can take a small neighbourhood $\mathcal{X}_P$ and a ramified covering $\varphi_P:(\mathcal{X}_P',\mathcal{D}_P')\longrightarrow(\mathcal{X}_P,\mathcal{D}_P)$ such that $(E,\mathbb{D})$ is the descent of an unramifiedly good lattice $(E',\mathbb{D}')$ on $(\mathcal{X}_P',\mathcal{D}_P')$. By applying the procedure in the unramified case, we obtain the deformation $\big(E',\mathbb{D}'\big)^{(\mathcal{T})}$ on $(\mathcal{X}_P'^{\circ},\mathcal{D}_P'^{\circ})$. By taking the descent, we obtain $(E,\mathbb{D})_P^{(\mathcal{T})}$ on $(\mathcal{X}_P^{\circ},\mathcal{D}_P^{\circ})$. It is well defined up to canonical isomorphisms as a germ of good lattice of a meromorphic ϱ-flat bundle at P, according to Lemma 4.5.1 and Lemma 4.5.2. By gluing, we can globalize and obtain a good lattice $(E,\mathbb{D})^{(\mathcal{T})}$ of a meromorphic ϱ-flat bundle on $(\mathcal{X}^{\circ},\mathcal{D}^{\circ})$.

If we are given a good filtered ϱ-flat bundle $(\boldsymbol{E}_*,\mathbb{D})$ on $(\mathcal{X},\mathcal{D})$, we obtain a good filtered ϱ-flat bundle $(\boldsymbol{E}_*^{(\mathcal{T})},\mathbb{D}^{(\mathcal{T})})$ by applying the above procedure.

4.5.1.4. Functoriality. — The deformation is compatible with dual, tensor product and direct sum, under the appropriate assumption on the irregular values. Namely, we have the following natural isomorphisms:

$$(E_1\oplus E_2)^{(\mathcal{T})}\simeq E_1^{(\mathcal{T})}\oplus E_2^{(\mathcal{T})},\quad (E_1\otimes E_2)^{(\mathcal{T})}\simeq E_1^{(\mathcal{T})}\otimes E_2^{(\mathcal{T})},\quad \big(E^{\vee}\big)^{(\mathcal{T})}\simeq\big(E^{(\mathcal{T})}\big)^{\vee}.$$

Let $(E_p,\mathbb{D}_p)\in\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})$ $(p=1,2)$ with a morphism $f:(E_1,\mathbb{D}_1)\longrightarrow(E_2,\mathbb{D}_2)$. Then, we have the induced morphism $f^{(\mathcal{T})}:(E_1^{(\mathcal{T})},\mathbb{D}_1^{(\mathcal{T})})\longrightarrow(E_2^{(\mathcal{T})},\mathbb{D}_2^{(\mathcal{T})})$.

4.5.1.5. Let $\mathcal{X}_1$ be a complex manifold with a normal crossing hypersurface $\mathcal{D}_1$. Let $F:\mathcal{X}_1\longrightarrow\mathcal{X}$ be a morphism such that (i) $F^{-1}(\mathcal{D})\subset\mathcal{D}_1$, (ii) the induced morphism $\mathcal{X}_1\longrightarrow\mathcal{K}$ is a smooth fibration, (iii) each intersection of some irreducible components of $\mathcal{D}_1$ is smooth over $\mathcal{K}$. Let E be a good lattice of $(\mathcal{E},\mathbb{D})$ on $(\mathcal{X},\mathcal{D})$. We obtain a good lattice $E_1:=F^*E$ of $(\mathcal{E}_1,\mathbb{D}_1):=F^*(\mathcal{E},\mathbb{D})\otimes\mathcal{O}_{\mathcal{X}_1}(*\mathcal{D}_1)$.

Lemma 4.5.3. — *Let $F_{\mathcal{C}}$ be the induced morphism $\mathcal{X}_1^{\circ}\longrightarrow\mathcal{X}^{\circ}$. We have natural isomorphisms*

$$E_1^{(\mathcal{T})}\simeq F_{\mathcal{C}}^*E^{(\mathcal{T})},\quad(\mathcal{E}_1,\mathbb{D})^{(\mathcal{T})}\simeq F_{\mathcal{C}}^*\big(\mathcal{E},\mathbb{D}\big)^{(\mathcal{T})}\otimes\mathcal{O}_{\mathcal{X}_1^{\circ}}(*\mathcal{D}_1^{\circ}).$$

Proof. — We only have to consider the unramified case, in which the claim follows from Theorem 4.4.1. □

4.5.2. Deformation $E^{(T)}$. — Let T be a nowhere vanishing holomorphic function on $\mathcal{K}$ such that $|\arg(T)| < \pi/2$. For a given good lattice $(E,\mathbb{D})$ on $(\mathcal{X},\mathcal{D})$, we shall construct a good lattice $(E^{(T)},\mathbb{D}^{(T)})$ on $(\mathcal{X},\mathcal{D})$. We take a compact region $\mathcal{C}\subset \boldsymbol{C}$ which contains 0 and 1, and take a nowhere vanishing holomorphic function $\mathcal{T}:\mathcal{K}\times\mathcal{C}\to\boldsymbol{C}$ such that (i) $\mathcal{T}_{|\mathcal{K}\times\{0\}}=1$, (ii) $\mathcal{T}_{|\mathcal{K}\times\{1\}}=T$, (iii) $|\arg(\mathcal{T})|<\pi/2$. Then, we obtain the deformation $(E^{(\mathcal{T})},\mathbb{D}^{(\mathcal{T})})$ on $(\mathcal{X}^{\circ},\mathcal{D}^{\circ})$. By taking the specialization at $c=1$, we obtain the desired $(E^{(T)},\mathbb{D}^{(T)})$.

Lemma 4.5.4. — *$(E^{(T)},\mathbb{D}^{(T)})$ is independent of the choice of $(\mathcal{C},\mathcal{T})$ up to canonical isomorphisms.*

Proof. — We only have to consider the local and unramified case. Let $\mathcal{T}_i$ $(i=0,1)$ be functions as above. We take a small neighbourhood $\mathcal{C}_2$ of $\{0\leqslant c_2\leqslant 1\}$ in $\boldsymbol{C}$. We consider a holomorphic function $\mathcal{T}_2:=(1-c_2)\mathcal{T}_0+c_2\mathcal{T}_1$ on $\mathcal{K}\times\mathcal{C}\times\mathcal{C}_2$. We obtain $(E,\mathbb{D})^{(\mathcal{T}_2)}$ on $(\mathcal{X},\mathcal{D})\times(\mathcal{C}\times\mathcal{C}_2)$. The specializations at $(c,c_2)=(1,i)$ $(i=0,1)$ correspond to $(E,\mathbb{D})^{(\mathcal{T}_i)}$. Let $p:\mathcal{X}\times\mathcal{C}_2\to\mathcal{X}$ be the projection. By Theorem 4.4.1, we have a natural isomorphism $(E,\mathbb{D})^{(\mathcal{T}_2)}_{|\mathcal{X}\times\{1\}\times\mathcal{C}_2}\simeq p^*(E,\mathbb{D})^{(\mathcal{T}_0)}$. Hence, $(E,\mathbb{D})^{(\mathcal{T}_i)}$ $(i=0,1)$ are naturally isomorphic. □

Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values on $(\mathcal{X},\mathcal{D})$. For each $P\in\mathcal{D}$, we put $\mathcal{I}_P^{(T)}:=\{T\mathfrak{a}\mid\mathfrak{a}\in\mathcal{I}_P\}$, and we obtain a good system of irregular values $\boldsymbol{\mathcal{I}}^{(T)}$. The above construction gives $\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}})\to\mathrm{MFL}(\mathcal{X},\mathcal{D},\boldsymbol{\mathcal{I}}^{(T)})$, in the unramified case.

Lemma 4.5.5. — *Let T_i $(i=1,2)$ be holomorphic functions on $\mathcal{K}$ such that $\big|\arg(T_i)\big|<\pi/2$ and $\big|\arg(T_1T_2)\big|<\pi/2$. We have a canonical isomorphism $E^{(T_1T_2)}\simeq(E^{(T_1)})^{(T_2)}$.*

Proof. — We only have to check the claim in the local and unramified case. We take a small neighbourhood $\mathcal{C}_i$ of $\{0\leqslant c_i\leqslant 1\}$ in $\boldsymbol{C}$. Let us consider the function $\mathcal{T}(c_1,c_2)=(1-c_1+c_1T_1)(1-c_2+c_2T_2)$ on $\mathcal{K}\times\mathcal{C}_1\times\mathcal{C}_2$. We have the deformation $(E,\mathbb{D})^{(\mathcal{T})}$ on $\mathcal{X}\times\mathcal{C}_1\times\mathcal{C}_2$. We can easily show that both $(E,\mathbb{D})^{(T_1T_2)}$ and $\big((E,\mathbb{D})^{(T_1)}\big)^{(T_2)}$ are naturally isomorphic to the specialization of $(E,\mathbb{D})^{(\mathcal{T})}$ at $(1,1)$. □

Lemma 4.5.6. — *Under the appropriate assumptions on the irregular values, the following holds:*

- *The deformation is compatible with dual, tensor product, and direct sum.*
- *Let $(E_1,\mathbb{D}_1)\to(E_2,\mathbb{D}_2)$ be a flat morphism. Then, we have an induced flat morphism $(E_1^{(T)},\mathbb{D}_1^{(T)})\to(E_2^{(T)},\mathbb{D}_2^{(T)})$.* □

Let $\mathcal{X}_1$ be a complex manifold with a normal crossing hypersurface $\mathcal{D}_1$. Let $F:\mathcal{X}_1\to\mathcal{X}$ be a morphism such that (i) $F^{-1}(\mathcal{D})\subset\mathcal{D}_1$, (ii) the induced morphism $\mathcal{X}_1\to\mathcal{K}$ is a smooth fibration, (iii) any intersection of some irreducible components of $\mathcal{D}_1$ is smooth over $\mathcal{K}$. Let E be a good lattice of $(\mathcal{E},\mathbb{D})$ on $(\mathcal{X},\mathcal{D})$. We obtain a good lattice $E_1:=F^*E$ of $(\mathcal{E}_1,\mathbb{D}_1):=F^*(\mathcal{E},\mathbb{D})$. We obtain the following lemma from Lemma 4.5.3.

Lemma 4.5.7. — *We have a natural isomorphism $E_1^{(T)}\simeq F^*E^{(T)}$.* □

4.5.3. Deformation in the case that $|\arg(T)|$ is small. — We give a characterization of holomorphic sections of $(E^{(T)}, \mathbb{D}^{(T)})$ when $|\arg(T)|$ is sufficiently small. We explain it in the case where $(E, \mathbb{D})$ is unramified. We put $\mathcal{X} = \Delta^n \times \mathcal{K}$ and $\mathcal{D} = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let $\boldsymbol{\mathcal{I}}$ be a good system of irregular values on $(\mathcal{X}, \mathcal{D})$.

Let $P \in \mathcal{D}_{\underline{\ell}}$. We take a covering $\{U_i \mid i \in \Gamma\}$ of $\pi^{-1}(P)$, which is good for $\mathcal{I}_P$. (See Definition 4.1.6.) We take neighbourhoods $\mathcal{U}_i$ of U_i in $\widetilde{\mathcal{X}}(\mathcal{D})$. Assume the following for T:

- $\{U_i\}$ is good for $\mathcal{I}_P^{(T_c)}$ for $0 \leqslant c \leqslant 1$, where $T_c := 1 + c(T-1)$.

This is satisfied if $\arg(T)$ is sufficiently small for fixed $\{U_i\}$.

Let $\boldsymbol{v} = (\boldsymbol{v}_{\mathfrak{a}})$ be a frame of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E)$ compatible with the grading. Let $\mathcal{U}_I := \bigcap_{i \in I} \mathcal{U}_i$ for some $I \subset \Gamma$. As in Subsection 4.2.3.1, we take a splitting

$$E_{|\mathcal{U}_I} = \bigoplus E_{\mathcal{U}_I, \mathfrak{a}}. \tag{66}$$

We have the induced frame $\boldsymbol{v}_I$ of $E_{|\mathcal{U}_I}$. We put

$$\boldsymbol{v}_{I,\mathfrak{a}}^{(T)} := \boldsymbol{v}_{I,\mathfrak{a}} \exp\big((T-1)\varrho^{-1}\mathfrak{a}\big), \quad \boldsymbol{v}_I^{(T)} := \big(\boldsymbol{v}_{I,\mathfrak{a}}^{(T)}\big).$$

Lemma 4.5.8. — *Let $\boldsymbol{w}$ be a frame of $E^{(T)}$. Let G_I be determined by*

$$\boldsymbol{w}_{|\mathcal{U}_I \smallsetminus \pi^{-1}(\mathcal{D})} = \boldsymbol{v}^{(T)}_{I|\mathcal{U}_I \smallsetminus \pi^{-1}(\mathcal{D})} G_I.$$

Then, the entries of G_I and G_I^{-1} are holomorphic on $\mathcal{U}_I$.

The splitting $E_{|\mathcal{U}_I \smallsetminus \pi^{-1}(\mathcal{D})} = \bigoplus E_{\mathcal{U}_I, \mathfrak{a}|\mathcal{U}_I \smallsetminus \pi^{-1}(\mathcal{D})}$ gives a splitting of the full Stokes filtration of $E^{(T)}$.

Proof. — Let $\mathcal{C}$ be a small neighbourhood of $\{0 \leqslant c \leqslant 1\} \subset \boldsymbol{C}$. We consider the function $\mathcal{T} = 1 + c(T-1)$. We have an unramifiedly good lattice $(E, \mathbb{D})^{(\mathcal{T})}$ over $\mathcal{C} \times (\mathcal{X}, \mathcal{D})$. By the assumption on $\arg(T)$, the natural map $\mathcal{I}_P^{(\mathcal{T})} \to \mathcal{I}_P$ induces isomorphisms of the ordered sets:

$$\big(\mathcal{I}_P^{(\mathcal{T})}, \leqslant_{\mathcal{C} \times \mathcal{U}_I}\big) \longrightarrow \big(\mathcal{I}_P, \leqslant_{\mathcal{U}_I}\big).$$

By using the flat ϱ-connection, we obtain a filtration $\widetilde{\mathcal{F}}^{\mathcal{C} \times \mathcal{U}_I}$ of $E^{(\mathcal{T})}_{|\mathcal{C} \times \mathcal{U}_I}$ from $\widetilde{\mathcal{F}}^{\mathcal{U}_I}$ of $E_{|\mathcal{U}_I}$. We also obtain a splitting of $\widetilde{\mathcal{F}}^{\mathcal{C} \times \mathcal{U}_I}$ from the splitting (66) of $\widetilde{\mathcal{F}}^{\mathcal{U}_I}$. We remark that, for any $(c, Q) \in \mathcal{C} \times \mathcal{U}_I$ the filtrations $\widetilde{\mathcal{F}}^{\mathcal{U}_I}$ and $\widetilde{\mathcal{F}}^{(c,Q)}$ are compatible over $\big(\mathcal{I}_P^{(\mathcal{T})}, \leqslant_{\mathcal{C} \times \mathcal{U}_I}\big) \to \big(\mathcal{I}_P^{(\mathcal{T})}, \leqslant_{(c,Q)}\big)$, a property which follows from the characterization of the full Stokes filtration in Theorem 3.2.1. In particular, the restriction of the splitting to $\{1\} \times \big(\mathcal{U}_I \smallsetminus \pi^{-1}(\mathcal{D})\big)$ gives a splitting of the filtration $\widetilde{\mathcal{F}}^{\mathcal{U}_I}$ of $E^{(T)}$. Note that $\boldsymbol{v}_I^{(T)}$ naturally gives a frame of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(E) \otimes \mathcal{L}\big((T-1)\mathfrak{a}\big)$, where $\mathcal{L}\big((T-1)\mathfrak{a}\big)$ denotes $\mathcal{O}_{\mathcal{X}}\, e$ with a flat ϱ-connection $\mathbb{D}e = e\,(T-1)d\mathfrak{a}$. Then, we obtain the first claim of the lemma from Lemma 3.7.21. The second claim follows from the first one. □

Let f be a holomorphic section of $E_{|\mathcal{X}\setminus\mathcal{D}}$. We have the corresponding decomposition $f_{|\mathcal{U}_I} = \sum f_{\mathfrak{a},I}$. We have the expression $f_{\mathfrak{a},I} = \sum f^{(T)}_{\mathfrak{a},I,j}\, v^{(T)}_{\mathfrak{a},I,j}$. We put $\boldsymbol{f}_{\mathfrak{a},I} := \big(f^{(T)}_{\mathfrak{a},I,j}\big)$.

Corollary 4.5.9. — *f gives a section of $E^{(T)}$ if and only if $\boldsymbol{f}^{(T)}_{\mathfrak{a},I}$ is bounded for any $\mathfrak{a}$ and I.* □

CHAPTER 5

L^2-COHOMOLOGY OF FILTERED λ-FLAT BUNDLE ON CURVES

Our goal is to compare various cohomology groups associated to a filtered λ-flat bundle induced by a harmonic bundle on a curve. It will be achieved in Section 18.2. This chapter is a preparation for local comparisons. In Sections 5.1–5.2, we consider the case where λ is fixed. In Sections 5.3–5.4, we study the family version. We separate them although they are essentially the same. The statements are given in Sections 5.1 and 5.3, respectively.

5.1. Local quasi-isomorphisms for fixed λ

We put $X := \Delta_z$. For any subset $Y \subset X$, we put $Y^* := Y \setminus \{O\}$. Let $(V_*, \mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X, O).

5.1.1. Sheaves of L^2-sections and holomorphic L^2-sections

5.1.1.1. Preliminaries for metric. — Let $\boldsymbol{v}$ be a frame of ${}^\diamond V$ compatible with the parabolic filtration F and the weight filtration W on Gr^F associated to the nilpotent part of the residue $\mathrm{Res}(\mathbb{D}^\lambda)$. We put $a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. Note $-1 < a(v_i) \leqslant 0$ and $k(v_i) \in \mathbb{Z}$. Let h be the Hermitian metric given as follows:

$$h(v_i, v_j) := \delta_{i,j}\, |z|^{-2a(v_i)} \,(-\log|z|)^{k(v_i)}.$$

If a metric h' comes from another choice of a frame $\boldsymbol{v}'$ compatible with F and W, the metrics h and h' are mutually bounded. Let g_{p} denote the Poincaré metric of X^*.

We recall the basic property of the metric h as above. Let $f = \sum f_j v_j$ be a holomorphic section of $V_{|X^*}$. It is L^2 with respect to h and g_{p}, if and only if the following holds for each f_i:

- f_i is holomorphic at O, if (i) $-1 < a(v_i) < 0$, or (ii) $a(v_i) = 0$ and $k(v_i) \leqslant 0$.
- f_i is holomorphic at O and $f_i(O) = 0$, if $a(v_i) = 0$ and $k(v_i) > 0$.

Let $\omega = \sum f_i\, v_i\, dz$ be a holomorphic section of $V \otimes \Omega_X^{1,0}$ on X^*. It is L^2 with respect to h and g_{p} if and only if the following holds:

- f_i is holomorphic at O, if (i) $-1 < a(v_i) < 0$, or (ii) $a(v_i) = 0$ and $k(v_i) \leqslant -2$.
- f_i is holomorphic at O and $f_i(O) = 0$, if $a(v_i) = 0$ and $k(v_i) > -2$.

We can check these claims by direct computations.

5.1.1.2. Sheaf of L^2-sections. — Note that the λ-connection $\mathbb{D}^\lambda$ of V and the derivation $\lambda\partial_X + \overline{\partial}_X$ induce a derivation of $V \otimes \Omega_X^{\bullet,\bullet}$, which is also denoted by $\mathbb{D}^\lambda$. For an open subset $U \subset X$, let $\mathcal{L}^p(V_*, \mathbb{D}^\lambda)(U)$ be the space of sections τ of $V \otimes \Omega_X^p$ on U^* with the following property:

- τ and $\mathbb{D}^\lambda\tau$ are L^2 locally on U, with respect to h and $g_{\mathbf{p}}$.

Let $\mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)(U)$ be the space of C^∞-sections τ of $V \otimes \Omega_X^p$ on U^* with the following property:

- τ and $\mathbb{D}^\lambda\tau$ are L^2 and of polynomial order in $|z^{-1}|$ locally on U with respect to h and $g_{\mathbf{p}}$.

Remark 5.1.1. — In the following, we say just "polynomial order" instead of "polynomial order in $|z^{-1}|$". □

Thus, we obtain complexes of sheaves $\mathcal{L}^\bullet(V_*, \mathbb{D}^\lambda)$ and $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. Let $\mathcal{L}^p_{\mathrm{hol}}(V_*, \mathbb{D}^\lambda)$ $(p = 0, 1)$ be the subsheaves $\mathcal{L}^p(V_*, \mathbb{D}^\lambda)$, which consists of holomorphic p-forms. By the general theory of holomorphic functions, we have $\mathcal{L}^p_{\mathrm{hol}}(V_*, \mathbb{D}^\lambda) \subset \mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$.

We shall prove the following proposition in Sections 5.2.1–5.2.5. The arguments are a minor modification of those in [**73**] and [**96**].

Proposition 5.1.2. — *The naturally defined morphisms*

$$\mathcal{L}^\bullet_{\mathrm{hol}}(V_*, \mathbb{D}^\lambda) \xrightarrow{\ \varphi_0\ } \mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda) \xrightarrow{\ \psi_1\ } \mathcal{L}^\bullet(V_*, \mathbb{D}^\lambda)$$

are quasi-isomorphisms.

5.1.1.3. Algebraically determined sheaf. — Let $X' = \Delta_{z'}$, and let φ_n denote the ramified covering $X' \to X$ given by $\varphi_n(z') = z'^n$. Recall that we have the induced good filtered λ-flat bundle on (X', O') as in Section 2.5.3.3, which is denoted by $(V'_*, \mathbb{D}'^\lambda)$. If we choose n appropriately, $(V'_*, \mathbb{D}'^\lambda)$ is unramified, and we have the irregular decomposition:

$$(V'_*, \mathbb{D}'^\lambda)_{|\widehat{O}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}'^\lambda)} (\widehat{V}'_{\mathfrak{a}*}, \mathbb{D}'^\lambda_{\mathfrak{a}}). \tag{67}$$

Since $\widehat{V}'_{0*}$ and $\bigoplus_{\mathfrak{a}\neq 0} \widehat{V}'_{\mathfrak{a}*}$ are $\mathrm{Gal}(X'/X)$-equivariant, we have the descent to $\widehat{O}$ which are denoted by $\widehat{V}_{\mathrm{reg}\,*}$ and $\widehat{V}_{\mathrm{irr}\,*}$, respectively.

Let $a \in \boldsymbol{R}$ and $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}'^\lambda)$. We have the weight filtration W of the nilpotent part of the residue $\mathrm{Gr}^F_a \mathrm{Res}(\mathbb{D}'^\lambda)$ on $\mathrm{Gr}^F_a(\widehat{V}'_{\mathfrak{a}})$. Let $W_k({}_a\widehat{V}'_{\mathfrak{a}})$ denote the pull-back of $W_k \mathrm{Gr}^F_a(\widehat{V}'_{\mathfrak{a}})$ via the natural projection ${}_a\widehat{V}'_{\mathfrak{a}} \to \mathrm{Gr}^F_a(\widehat{V}'_{\mathfrak{a}})$. For each $\mathfrak{a} \neq 0$, we put

$$\mathcal{S}(\widehat{V}'_{\mathfrak{a}*} \otimes \Omega^{0,0}_{X'}) := W_{-2}({}_{\mathrm{ord}(\mathfrak{a})}\widehat{V}'_{\mathfrak{a}}), \quad \mathcal{S}(\widehat{V}'_{\mathfrak{a}*} \otimes \Omega^{1,0}_{X'}) := W_{-2}({}^{\diamond}\widehat{V}'_{\mathfrak{a}})\,\frac{dz'}{z'}.$$

We have the descent of $\bigoplus_{\mathfrak{a}\neq 0}\mathcal{S}\big(\widehat{V}'_{\mathfrak{a}*}\otimes\Omega_X^{p,0}\big)$, which is denoted by $\mathcal{S}(\widehat{V}_{\mathrm{irr}\,*}\otimes\Omega_X^{p,0})$.

We have the generalized eigen-decomposition with respect to the residue

$$\mathrm{Gr}_0^F(\widehat{V}_0)=\bigoplus_{\alpha}\mathbb{E}_\alpha\,\mathrm{Gr}_0^F(\widehat{V}_0),$$

where the restriction of $\mathrm{Res}(\mathbb{D}^\lambda)$ to $\mathbb{E}_\alpha\,\mathrm{Gr}_0^F(\widehat{V}_0)$ has the unique eigenvalue α. We have the weight filtration W of the nilpotent part of the residue $\mathrm{Res}(\mathbb{D}^\lambda)$ on each $\mathbb{E}_\alpha\,\mathrm{Gr}_0^F(\widehat{V}_0)$ and $\mathrm{Gr}_0^F(\widehat{V}_0)$. Let $\mathcal{S}(\widehat{V}_{0*}\otimes\Omega_X^{0,0})$ denote the inverse image of $\bigoplus_{\alpha\neq 0}W_{-2}\mathbb{E}_\alpha\,\mathrm{Gr}_0^F({}^\diamond\widehat{V}_0)\oplus W_0\mathbb{E}_0\,\mathrm{Gr}_0^F({}^\diamond\widehat{V}_0)$ via the projection ${}^\diamond\widehat{V}_0\to\mathrm{Gr}_0^F(\widehat{V}_0)$. Let $\mathcal{S}(\widehat{V}_{0*}\otimes\Omega_X^{1,0})$ denote the inverse image of $W_{-2}\,\mathrm{Gr}_0^F(\widehat{V}_0)$ via the projection ${}^\diamond\widehat{V}_0\,dz/z\to\mathrm{Gr}_0^F(\widehat{V}_0)$. Thus, we obtain lattices

$$\mathcal{S}(\widehat{V}_*\otimes\Omega_X^{p,0}):=\mathcal{S}(\widehat{V}_{0*}\otimes\Omega_X^{p,0})\oplus\mathcal{S}(\widehat{V}_{\mathrm{irr}\,*}\otimes\Omega_X^{p,0})$$

of $\big(V\otimes\Omega_X^{p,0}(*O)\big)_{|\widehat{O}}$. They induce lattices $\mathcal{S}(V_*\otimes\Omega_X^{p,0})$ of $V\otimes\Omega_X^{p,0}(*O)$. The λ-connection $\mathbb{D}^\lambda$ on V_* and the differential $\lambda\,d_X$ on $\Omega_X^{\bullet,0}$ induce $\mathbb{D}^\lambda:\mathcal{S}(V_*\otimes\Omega_X^{0,0})\to\mathcal{S}(V_*\otimes\Omega_X^{1,0})$. Thus, we obtain a complex of sheaves $\mathcal{S}(V_*\otimes\Omega^{0,0})\xrightarrow{\mathbb{D}^\lambda}\mathcal{S}(V_*\otimes\Omega^{1,0})$.

Lemma 5.1.3. — *We have a natural inclusion $\mathcal{S}(V_*\otimes\Omega^{\bullet,0})\to\mathcal{L}^\bullet_{\mathrm{hol}}(V_*,\mathbb{D}^\lambda)$, which is an isomorphism.*

Proof. — We only have to compare the stalks at O. By the condition in Section 5.1.1.1, it is easy to check that sections of $\mathcal{S}(V_*\otimes\Omega^{p,0})$ are L^2. It also implies that $\mathbb{D}f$ is L^2 for a section f of $\mathcal{S}(V_*)$. Hence, we obtain $\mathcal{S}(V_*\otimes\Omega^{p,0})\subset\mathcal{L}^p_{\mathrm{hol}}(V_*,\mathbb{D}^\lambda)$ naturally. In the case $p=1$, it is clearly an isomorphism. Let $f\in\mathcal{L}^0_{\mathrm{hol}}(V_*,\mathbb{D}^\lambda)$. Because f is L^2, each f_i is holomorphic. Then, we obtain $f\in\mathcal{S}(V_*)$ from $\mathbb{D}^\lambda f\in\mathcal{S}(V_*\otimes\Omega^{1,0})$. □

5.1.1.4. Remark. — This kind of theorems, such as Proposition 5.1.2 and Lemma 5.1.3, was first proved by S. Zucker [**96**] in his study on singular variation of Hodge structure. Namely, he used the quasi-isomorphism to obtain a Hodge structure on the intersection cohomology of a variation of polarized pure Hodge structure. If h comes from a variation of polarized pure Hodge structure, it is easy to see that $\mathcal{S}(V_*\otimes\Omega^{\bullet,0})$ with $\lambda=1$ is naturally quasi-isomorphic to the de Rham complex of the minimal extension of $V_{|X^*}$ on X. The quasi-isomorphism with $\lambda=1$ plays the role connecting the intersection cohomology and the L^2-cohomology. Moreover, he obtained a Hodge structure on L^2-cohomology by using the quasi-isomorphism with $\lambda=0$ with some global harmonic analysis.

For regular filtered λ-flat bundles, it was proved in [**73**] and [**67**] for the study on tame harmonic bundles on curves. In [**67**], we used Zucker's method in a straightforward way. Sabbah [**73**] introduced an improvement to argue it in a unified way for various λ. He also studied the irregular singular case in [**71**]. We will use a different argument to deal with Stokes structure.

5.1.2. Variants in the case $\lambda \neq 0$. — We shall introduce complexes of sheaves $\widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ and $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ on X, whose restrictions to X^* are the same as $\mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{|X^*}$. Let S be a small sector in X^*, and let $\overline{S}$ denote its closure in the real blow up $\widetilde{X}(O)$. We can take a lift of $\overline{S}$ in the real blow up $\widetilde{X}'(O')$ via the covering map $\widetilde{X}'(O') \to \widetilde{X}(O)$, which is denoted by the same notation $\overline{S}$. If $\lambda \neq 0$, we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $V'_{|\overline{S}}$. We can take a flat splitting ${}_aV'_{|\overline{S}} = \bigoplus_{\mathfrak{a}} {}_aV'_{\mathfrak{a},S}$. We can naturally identify $V'_{|S}$ and $V_{|S}$. Each section f of $V \otimes \Omega^p_X$ on S has the corresponding decomposition $f = \sum f_{\mathfrak{a},S}$.

5.1.2.1. Variant 1. — For an open set $U \subset X$ with $O \in U$, let $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)(U)$ be the space of C^∞-sections τ of $V \otimes \Omega^p_X$ on U^*, such that the following estimate holds with respect to h and g_{p} on each small sector S:

(a1) : $\tau_{\mathfrak{a},S}$ and $\mathbb{D}^\lambda \tau_{\mathfrak{a},S}$ $(\mathfrak{a} \neq 0)$ are of polynomial order.

(a2) : $\tau_{0,S}$ and $\mathbb{D}^\lambda \tau_{0,S}$ are L^2 and of polynomial order.

The conditions are independent of the choice of a flat splitting and a lift of sector to $\widetilde{X}'(O')$. Then, we obtain the complex of sheaves $\widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. We will prove the following proposition in Section 5.2.6.

Proposition 5.1.4. — *The naturally defined morphisms*

$$\mathcal{S}(V_* \otimes \Omega^{\bullet,0}_X) \xrightarrow{\varphi_0} \mathcal{L}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda) \xrightarrow{\psi_2} \widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$$

are quasi-isomorphisms.

5.1.2.2. Variant 2. — For an open set $U \subset X$ with $O \in U$, let $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)(U)$ be the space of C^∞-sections τ of $V \otimes \Omega^p_X$ on U^*, such that the following estimate holds with respect to h and g_{p} on each small sector S:

(b1) : $\tau_{\mathfrak{a},S}$ and $\mathbb{D}^\lambda \tau_{\mathfrak{a},S}$ $(\mathfrak{a} \neq 0)$ are $O(|z|^N)$ for any $N > 0$ with respect to h and g_{p}.

(b2) : $\tau_{0,S}$ and $\mathbb{D}^\lambda \tau_{0,S}$ satisfy (a2).

Then, we obtain the complex of sheaves $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. We will prove the following proposition in Section 5.2.7.

Proposition 5.1.5. — *The naturally defined morphisms*

$$\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda) \xrightarrow{\varphi_1} \mathcal{L}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda) \xrightarrow{\psi_2} \widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$$

are quasi-isomorphisms.

5.1.3. Deformation of the Stokes structure ($\lambda \neq 0$). — Let $T > 0$. We have the deformation $\big(V^{(T)}_*, \mathbb{D}^\lambda\big)$ as in Section 4.5.2. To compare $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(V^{(T)}_*, \mathbb{D}^\lambda\big)$ and $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(V_*, \mathbb{D}^\lambda\big)$, we shall introduce a complex of sheaves $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(V, h, h^{(T)}_{C^\infty}\big)$.

5.1.3.1. Preliminary for metrics. — On each small sector S in X^*, we take a flat splitting $V_{|\overline{S}} = \bigoplus V_{\mathfrak{a},S}$ as in Section 5.1.2. Let $G_S^{(T)}$ be the endomorphism given by $\bigoplus_{\mathfrak{a}} \exp\big((1-T)\,\lambda^{-1}\,\mathfrak{a}\big)\,\mathrm{id}_{V_{\mathfrak{a},S}}$. Let $h_S^{(T)}$ be the Hermitian metric of $V_{|S}$ given by $h_S^{(T)}(u,v) := h\big(G_S^{(T)}(u), G_S^{(T)}(v)\big)$. If we construct $h_S'^{(T)}$ from other h' and $G_S'^{(T)}$, $h_S'^{(T)}$ and $h_S^{(T)}$ are mutually bounded. By varying S and gluing $h_S^{(T)}$ in C^∞, we obtain a C^∞-metric $h_{C^\infty}^{(T)}$.

We can construct a metric $h^{(T)}$ for $V_*^{(T)}$ as in Section 5.1.1.2, by taking a frame $\boldsymbol{v}^{(T)}$ of ${}^\diamond V^{(T)}$ compatible with the parabolic filtration F and the weight filtration W on Gr^F.

Lemma 5.1.6. — *The metrics $h_{C^\infty}^{(T)}$ and $h^{(T)}$ are mutually bounded.*

Proof. — We may assume that V_* is unramified. Let $\widehat{\boldsymbol{w}} = (\widehat{\boldsymbol{w}}_{\mathfrak{a}})$ be a frame of $V_{|\widehat{O}}$, such that (i) it is compatible with the irregular decomposition and the parabolic filtration, (ii) the induced frame of $\mathrm{Gr}^F({}^\diamond V)$ is compatible with the weight filtration W. We put $a(w_i) := \deg^F(w_i)$ and $k(w_i) := \deg^W(w_i)$. We take a lift $\boldsymbol{w}_S = (\boldsymbol{w}_{\mathfrak{a},S})$ of $\widehat{\boldsymbol{w}}$ to ${}^\diamond V_{|\overline{S}}$, compatible with a $\mathbb{D}^\lambda$-flat splitting of the full Stokes filtration. Let $\widetilde{h}_S$ be the Hermitian metric of $V_{|S}$ determined by $\widetilde{h}_S(w_i,w_j) := \delta_{i,j}\,|z|^{-2a(w_i)}\,\big(-\log|z|\big)^{k(w_i)}$. Then, it is mutually bounded with $h_{|S}$.

We have the frame $\widehat{\boldsymbol{w}}^{(T)}$ of ${}^\diamond V_{|\widehat{O}}^{(T)}$, which is obtained from $\widehat{\boldsymbol{w}}$ by the natural (non-flat) isomorphism $V_{*|\widehat{O}} \simeq V_{*|\widehat{O}}^{(T)}$. We put $\boldsymbol{w}_{\mathfrak{a},S}^{(T)} := \exp\big((T-1)\,\lambda^{-1}\,\mathfrak{a}\big)\,\boldsymbol{w}_{\mathfrak{a},S}$, and $\boldsymbol{w}_S^{(T)} := (\boldsymbol{w}_{\mathfrak{a},S}^{(T)})$. Then $\boldsymbol{w}_S^{(T)}$ is a lift of $\widehat{\boldsymbol{w}}^{(T)}$ to $V_{|\overline{S}}^{(T)}$ compatible with a $\mathbb{D}^\lambda$-flat splitting of the full Stokes filtration. Let $\widetilde{h}_S^{(T)}$ be the metric of $V_{|S}^{(T)}$ given by $\widetilde{h}_S^{(T)}(w_i^{(T)}, w_j^{(T)}) = \delta_{i,j}\,|z|^{-2a(w_i^{(T)})}\,\big(-\log|z|\big)^{k(w_i^{(T)})}$. Then, $\widetilde{h}_S^{(T)}$ and $(h^{(T)})_{|S}$ are mutually bounded.

By the construction, we have $\widetilde{h}_S^{(T)}(u,v) = \widetilde{h}_S\big(G_S^{(T)}(u), G_S^{(T)}(v)\big)$. Then, the claim of the lemma follows. □

5.1.3.2. A complex of sheaves. — We shall introduce the complex of sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}\big(V,h,h_{C^\infty}^{(T)}\big)$ on X. We set $\overline{\mathcal{L}}^p_{\mathrm{poly}}\big(V,h,h_{C^\infty}^{(T)}\big)_{|X^*} = \overline{\mathcal{L}}^p_{\mathrm{poly}}\big(V_*,\mathbb{D}^\lambda\big)_{|X^*}$. For an open set $U \subset X$ with $O \in U$, let $\overline{\mathcal{L}}^p_{\mathrm{poly}}\big(V,h,h_{C^\infty}^{(T)}\big)(U)$ be the space of C^∞-sections τ of $V \otimes \Omega^p$ on U^* such that the following estimate holds on each small sector S:

- **(c1)** : $\tau_{\mathfrak{a},S}$ and $\mathbb{D}^\lambda\tau_{\mathfrak{a},S}$ ($\mathfrak{a} \neq 0$) are $O\big(|z|^N\big)$ for any N with respect to both $(h,g_{\mathbf{p}})$ and $(h_{C^\infty}^{(T)}, g_{\mathbf{p}})$.
- **(c2)** : $\tau_{0,S}$ and $\mathbb{D}^\lambda\tau_{0,S}$ are L^2 and of polynomial order with respect to $(h,g_{\mathbf{p}})$. In other words, they satisfy the condition (a2). Note that the restrictions of h and $h_{C^\infty}^{(T)}$ to $V_{0,S}$ are mutually bounded.

Thus, we obtain the complex of sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}\big(V,h,h_{C^\infty}^{(T)}\big)$.

5.1.3.3. Statement. — By construction of the complexes and Lemma 5.1.6, we have the following natural morphisms:

$$\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda) \longleftarrow \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V, h, h^{(T)}_{C^\infty}) \longrightarrow \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V^{(T)}_*, \mathbb{D}^\lambda). \tag{68}$$

We will prove the following proposition in Section 5.2.8.

Proposition 5.1.7. — *The morphisms in* (68) *are quasi-isomorphisms.*

5.1.3.4. Remark. — We give a consequence of Proposition 5.1.7 for holonomic D-modules on *projective curves.* Let C be a smooth projective curve. Let V be a meromorphic flat bundle on C. For a given $T_i > 0$ $(i = 1, 2)$, we have the deformation $V^{(T_i)}$. Let $M^{(T_i)}$ be the minimal extension of $V^{(T_i)}$. We can deduce a natural isomorphism of the cohomology of D-modules $H^*_{DR}(C, M^{(T_1)}) \simeq H^*_{DR}(C, M^{(T_2)})$ by using the above quasi-isomorphisms. Actually, let $V^{(T_i)}_*$ be the Deligne-Malgrange filtered bundle associated to $V^{(T_i)}$. (See Section 2.7.) It is standard that $\mathcal{S}(V^{(T_i)}_* \otimes \Omega^{\bullet,0})$ is naturally quasi-isomorphic to the de Rham complex of $M^{(T_i)}$. Hence, the quasi-isomorphisms in Propositions 5.1.4, 5.1.5 and 5.1.7 induce the desired isomorphism $H^*_{DR}(C, M^{(T_1)}) \simeq H^*_{DR}(C, M^{(T_2)})$.

We can also obtain such an isomorphism directly from the construction of $M^{(T_i)}$. Recall that $V^{(T_i)}$ are obtained as the specialization of a meromorphic flat bundle $V^{(\mathcal{T})}$ on $\mathcal{C} \times C$ for some appropriate complex manifold $\mathcal{C}$ with a function $\mathcal{T}$. Let $M^{(\mathcal{T})}$ be the minimal extension of $V^{(\mathcal{T})}$. Then, $M^{(T_i)}$ are obtained as the specialization of $M^{(\mathcal{T})}$. By taking push-forward of $M^{(\mathcal{T})}$ to $\mathcal{C}$, we obtain a flat bundle whose fiber over $c \in \mathcal{C}$ is naturally quasi-isomorphic to $H^*_{DR}(\{c\} \times C, M^{(\mathcal{T}(c))})$. Hence, the parallel transport induces the desired isomorphism. It seems possible to check that the two isomorphisms are the same by using the family version of the quasi-isomorphisms, which might simplify our argument. We would like to give more details somewhere.

5.2. Proof for fixed λ

5.2.1. An estimate in [96]. — We recall a result due to Zucker [**96**]. We use the Poincaré metric g_{P} and the associated volume form $\mathrm{dvol}_{g_{\mathrm{P}}}$ of X^* around O. We use the polar coordinate $z = r\,e^{\sqrt{-1}\theta}$. Let $\mathcal{L}$ be a holomorphic line bundle on X^* with a holomorphic frame σ and a metric h such that $|\sigma|_h \sim r^{-a}\,|\log r|^{k/2}$, where $-1 < a \leqslant 1/2$ and $k \in \mathbb{Z}$. Let $\|\omega\|_{h,g_{\mathrm{P}}}$ denote the L^2-norm of a section ω of $\mathcal{L} \otimes \Omega^p$ with respect to h and g_{P}.

Let $0 < R < 1/2$. We put $X(R) := \{z \in \boldsymbol{C} \mid |z| \leqslant R\}$ and $X^*(R) = X(R) \setminus \{O\}$. Let $\omega = g\,\sigma\,d\overline{z}/\overline{z}$ be a C^∞-section of $\mathcal{L} \otimes \Omega^{0,1}_X$ on $X^*(R)$ with compact support. We have the Fourier expansion $g = \sum_{m\in\mathbb{Z}} g_m(r)\,e^{\sqrt{-1}m\theta}$. We put $g^{(1)} := \sum_{m\neq 0} g_m(r)\,e^{\sqrt{-1}m\theta}$, and thus we have the decompositions $g = g_0 + g^{(1)}$ and

$\omega=\omega_0+\omega^{(1)}$. In the cases $(n<0)$ or $(n=0, a=0, k>1)$, we put

$$u_n := 2r^n \int_0^r \rho^{-n-1}\, g_n(\rho)\, d\rho. \tag{69}$$

In the cases $(n>0)$, $(n=0,-1<a<0)$ or $(n=0, a=0, k<1)$, we put

$$u_n := -2r^n \int_r^R \rho^{-n-1}\, g_n(\rho)\, d\rho. \tag{70}$$

We will not consider the cases $(n=0, a>0)$ and $(n=0, a=0, k=1)$. Then, we set

$$\Phi^{(1)}(\omega) := \sum_{n\in\mathbb{Z}, n\neq 0} u_n(r)\, e^{\sqrt{-1}n\theta}. \tag{71}$$

When $a\leqslant 0$ and $(a,k)\neq(0,1)$ are satisfied, we also put

$$\Phi(\omega) := \sum_{n\in\mathbb{Z}} u_n(r)\, e^{\sqrt{-1}n\theta}. \tag{72}$$

The following proposition is proved in the proof of Proposition 6.4 and Proposition 11.5 of [**96**].

Proposition 5.2.1. — *Assume $R>0$ is sufficiently small. Let ω be as above.*

- *We have $\overline{\partial}\Phi^{(1)}(\omega)=\omega^{(1)}$. There exists a positive constant C_1, such that $\|\Phi^{(1)}(\omega)\|_{h,g_{\mathbf{p}}}\leqslant C_1\,\|\omega^{(1)}\|_{h,g_{\mathbf{p}}}$. If we fix a compact subset K_1 of $\{-1<t\leqslant 1/2\}\times\mathbb{Z}$, the constant C_1 can be taken independently from $(a,k)\in K_1$. (But it may depend on K_1.)*
- *Assume $a\leqslant 0$ and $(a,k)\neq(0,1)$. We have $\overline{\partial}\Phi(\omega)=\omega$. We also have a constant C_2 such that $\|\Phi(\omega)\|_{h,g_{\mathbf{p}}}\leqslant C_2\,\|\omega\|_{h,g_{\mathbf{p}}}$. If we fix a compact subset K_2 of $\{-1<t<0\}\times\mathbb{Z}$, the constant C_2 can be taken independently from $(a,k)\in K_2$.* □

Let $L^2(X^*(R),\mathcal{L})$ (resp. $L^2(X^*(R),\mathcal{L}\otimes\Omega^{0,1})$) denote the space of L^2-sections of $\mathcal{L}$ (resp. $\mathcal{L}\otimes\Omega^{0,1}$) on $X^*(R)$, which are L^2 with respect to h and $g_{\mathbf{p}}$. We obtain the following corollary.

Corollary 5.2.2

- *$\Phi^{(1)}$ induces a bounded linear map $L^2\big(X^*(R),\mathcal{L}\otimes\Omega^{0,1}\big)\to L^2\big(X^*(R),\mathcal{L}\big)$. The range of $\Phi^{(1)}$ is contained in the domain of $\overline{\partial}$, and $\overline{\partial}\circ\Phi^{(1)}(\omega)=\omega^{(1)}$ holds for any $\omega\in L^2\big(X^*(R),\mathcal{L}\otimes\Omega^{0,1}\big)$. If we fix a compact subset K_1 of $\{-1<t\leqslant 1/2\}\times\mathbb{Z}$, the norm of $\Phi^{(1)}$ is uniformly bounded for $(a,k)\in K_1$.*
- *If $a\leqslant 0$ and $(a,k)\neq(0,1)$, Φ induces a bounded linear map $L^2\big(X^*(R),\mathcal{L}\otimes\Omega^{0,1}\big)\to L^2\big(X^*(R),\mathcal{L}\big)$. The range of Φ is contained in the domain of $\overline{\partial}$, and $\overline{\partial}\circ\Phi$ is the identity of $L^2\big(X^*(R),\mathcal{L}\otimes\Omega^{0,1}\big)$. If we fix a compact subset K_2 of the set $\{-1<t<0\}\times\mathbb{Z}$, the norm of Φ is uniformly bounded for $(a,k)\in K_2$.* □

5.2.2. An estimate in [82]. — Let $f(z)\,d\bar{z}$ be a $(0,1)$-form on Δ^* such that $|f(z)| \leqslant |z|^a(-\log|z|)^k$ and that the support of f is compact in Δ. We often need a solution g of the equation $\overline{\partial} g = f\,d\bar{z}$ satisfying some growth estimate around the origin. For that purpose, we put

$$H(f)(z) := \int \frac{f(w)}{z-w}\frac{\sqrt{-1}}{2\pi}dw\,d\overline{w}.$$

Lemma 5.2.3

- *In the case $-2 < a < -1$, we have $|H(f)| = O\big(|z|^{a+1}(-\log|z|)^k\big)$.*
- *In the case ($a = -2$, $k < -1$) or ($a = -1$, $k > -1$), we have $|H(f)| = O\big(|z|^{a+1}(-\log|z|)^{k+1}\big)$.*
- *In the case $a = -1$ and $k = -1$, $|H(f)| = O\big(|z|^{a+1}(-\log|z|)^{k+1}\log(-\log|z|)\big)$.*

Proof. — See Page 759–760 of [82]. □

5.2.3. Preliminaries. — Let us start the proof of the propositions in Section 5.1. By an easy argument to use the descent, we can reduce the problem to the unramified case. Therefore, we may and will assume that $(V_*, \mathbb{D}^\lambda)$ is unramified. We use the polar coordinate $z = r\,e^{\sqrt{-1}\theta}$. We may assume the following for the frame $\boldsymbol{v}$, moreover:

1. $\boldsymbol{v}$ is compatible with the irregular decomposition in N-th order for some large N, i.e., $\boldsymbol{v}_{|\widehat{O}^{(N)}}$ is compatible with the decomposition of $V_{*|\widehat{O}^{(N)}}$ induced by (67), where $\widehat{O}^{(N)}$ denotes the N-th infinitesimal neighbourhood of O.
2. $\boldsymbol{v}_{|O}$ is compatible with the generalized eigen-decomposition of $\mathrm{Res}(\mathbb{D}^\lambda)$.
3. Let $N_{a,\alpha,\mathfrak{a}}$ denote the nilpotent part of the endomorphisms on $\mathrm{Gr}^F_a\,\mathbb{E}_\alpha\big(V_{\mathfrak{a}|O}\big)$ induced by $\mathrm{Res}(\mathbb{D}^\lambda)$. Then, $N_{a,\alpha,\mathfrak{a}}$ are represented by Jordan matrices with respect to the induced frames.

We have the irregular value $\mathfrak{a}(v_i)$, and the eigenvalue $\alpha(v_i)$ of $\mathrm{Res}(\mathbb{D}^\lambda)$ corresponding to v_i. We also put $a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. We define

$$\mathcal{B}(k) := \big\{v_i \,\big|\, a(v_i) = \mathfrak{a}(v_i) = \alpha(v_i) = 0,\ k(v_i) = k\big\} \\ \cup \big\{v_i \,\big|\, a(v_i) = 0,\ \big(\mathfrak{a}(v_i), \alpha(v_i)\big) \neq (0,0)\big\}.$$

Let A be determined by $\mathbb{D}^\lambda \boldsymbol{v} = \boldsymbol{v}\,A$. Let Γ be the diagonal matrix whose (i,i)-entries are $\alpha(v_i)\,dz/z + d\mathfrak{a}(v_i)$. We put $A_0 := A - \Gamma$. We use the symbol F_A to denote the section of $\mathrm{End}(V)\otimes\Omega^{1,0}$ determined by $F_A(\boldsymbol{v}) = \boldsymbol{v}\,A$. We use the symbol F_{A_0} in a similar meaning. Then, F_{A_0} is bounded with respect to h and $g_{\mathbf{p}}$. We have the following decomposition:

$$A_0 = \bigoplus_{a,\alpha,\mathfrak{a}} J_{a,\alpha,\mathfrak{a}}\frac{dz}{z} + A_0'\frac{dz}{z}.$$

Here A_0' is holomorphic and $F_{A_0'|O}$ strictly decreases the parabolic filtration. And $J_{a,\alpha,\mathfrak{a}}$ are constant Jordan matrices and represent $N_{a,\alpha,\mathfrak{a}}$ with respect to the induced frames of $\mathrm{Gr}^F_a\,\mathbb{E}_\alpha(V_{\mathfrak{a}|O})$.

The $(1,0)$-operator ∂ is defined by $\partial\big(\sum f_i\, v_i\big) = \sum \partial f_i \cdot v_i$. Then, we have $\mathbb{D}^\lambda = \overline{\partial} + \lambda\partial + F_A$.

Let $\mathrm{dvol}_{g_{\mathrm{P}}}$ denote the volume form of the Poincaré metric. Recall that a section $\sum f_i\, v_i$ is L^2 if and only if the following holds:

$$\sum \int |f_i|^2\, |z|^{-2a(v_i)} (-\log|z|^2)^{k(v_i)}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty.$$

A section $\sum f_i\, v_i\, dz/z + \sum g_j\, v_j\, d\overline{z}/\overline{z}$ is L^2 if and only if the following holds:

$$\sum \int |f_i|^2\, |z|^{-2a(v_i)} (-\log|z|^2)^{k(v_i)+2}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty,$$
$$\sum \int |g_j|^2\, |z|^{-2a(v_j)} (-\log|z|^2)^{k(v_j)+2}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty.$$

A section $\sum f_i\, v_i\, dz\, d\overline{z}/|z|^2$ is L^2, if and only if the following holds:

$$\sum \int |f_i|^2\, |z|^{-2a(v_i)} (-\log|z|^2)^{k(v_i)+4}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty.$$

5.2.4. Vanishing of $\mathcal{H}^2$ of $\mathcal{L}^\bullet(V_*, \mathbb{D}^\lambda)$ and $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. — Let us consider Proposition 5.1.2. Let ω be an L^2-section of $V \otimes \Omega^2$. We have the expression:

$$\omega = f\,\frac{dz\, d\overline{z}}{|z|^2}, \quad f = \sum f_i\, v_i.$$

Each f_i has the Fourier expansion $f_i = \sum_{m\in\mathbb{Z}} f_{i,m}(r)\, e^{\sqrt{-1}m\theta}$. We set

$$\mathcal{A}^{(0)}(f) := \sum_{v_i \in \mathcal{B}(-1)} f_{i,0}(r)\, v_i, \quad \mathcal{A}^{(1)}(f) := f - \mathcal{A}^{(0)}(f).$$

We have the decomposition $f = \mathcal{A}^{(0)}(f) + \mathcal{A}^{(1)}(f)$. We have the corresponding decomposition $\omega = \mathcal{A}^{(0)}(\omega) + \mathcal{A}^{(1)}(\omega)$. Recall we have the following equalities:

$$\partial\big(f_{i,0}(r)\big) = \frac{1}{2} r \frac{\partial f_{i,0}}{\partial r} \frac{dz}{z}, \quad \overline{\partial}\big(f_{i,0}(r)\big) = \frac{1}{2} r \frac{\partial f_{i,0}}{\partial r} \frac{d\overline{z}}{\overline{z}}. \tag{73}$$

We show the following lemma based on Sabbah's idea contained in [**73**].

Lemma 5.2.4

- *We have an L^2-section $\tau^{(1)}$ of $V \otimes \Omega^{1,0}$ such that $\overline{\partial}\tau^{(1)} = \mathcal{A}^{(1)}(\omega)$.*
- *We have an L^2-section $\tau^{(0)}$ of $V \otimes \Omega^1$ such that*
  (i) *$\mathbb{D}^\lambda \tau^{(0)}$ is also L^2,*
  (ii) *$\mathcal{A}^{(0)}\big(\omega - \mathbb{D}^\lambda\tau^{(0)}\big) = 0$.*
- *In particular, we can take an L^2-section τ of $V \otimes \Omega^1$ such that $\mathbb{D}^\lambda\tau = \omega$.*

If ω is C^∞ and of polynomial order, $\tau^{(i)}$ $(i = 0, 1)$, $\mathbb{D}^\lambda\tau^{(0)}$ and τ are also C^∞ and of polynomial order.

Proof. — The first claim follows from Corollary 5.2.2. Let us show the second claim. The proof that we give also works in the family case. We give preliminary arguments.

5.2.4.1. (A). — In the case $a(v_i)=0$, $\mathfrak{a}(v_i)=0$ and $\alpha(v_i)\neq 0$, we put

$$\tau_1 := f_{i,0}\, v_i\, \frac{d\bar{z}}{\bar{z}} + \lambda f_{i,0}\, v_i\, \frac{dz}{z}. \tag{74}$$

Due to (73), we have $\mathbb{D}^{\lambda}\tau_1 = F_A\big(f_{i,0}\, v_i\big)$. Hence, we have the following:

$$\begin{aligned} f_{i,0}\, v_i\, \frac{dz\, d\bar{z}}{|z|^2} - \alpha(v_i)^{-1}\mathbb{D}^{\lambda}\tau_1 &= f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2} - \alpha(v_i)^{-1} F_A\big(f_{i,0}\, v_i\big)\frac{d\bar{z}}{\bar{z}} \\ &= \alpha(v_i)^{-1}F_{A_0}\big(f_{i,0}\, v_i\big)\frac{d\bar{z}}{\bar{z}} =: \sum B_j\, v_j \frac{dz\, d\bar{z}}{|z|^2}. \end{aligned} \tag{75}$$

Because $f_{i,0}\, v_i\, (dz/z)\, (d\bar{z}/\bar{z})$ is L^2, the sections

$$F_A(f_{i,0}\, v_i)\, d\bar{z}/\bar{z}, \qquad f_{i,0}\, v_i\, dz/z, \qquad f_{i,0}\, v_i\, d\bar{z}/\bar{z}$$

are also L^2. In particular, τ_1 and $\mathbb{D}^{\lambda}\tau_1$ are L^2. Because F_{A_0} is bounded, the right-hand side of (75) is also L^2. Let us look at B_j more closely. Because A_0 is holomorphic, we have $B_j = \sum_{m\geqslant 0} B_{j,m}(r)\, e^{\sqrt{-1}m\theta}$. If $a(v_j)=0$, we have $B_{j,0}(r)=0$ unless $\big(\mathfrak{a}(v_j),\alpha(v_j)\big) = \big(\mathfrak{a}(v_i),\alpha(v_i)\big)$ and $N_{a,\alpha,\mathfrak{a}} v_{i|O} = v_{j|O}$. Note $\deg^W(v_j) < \deg^W(v_i)$ for such v_j.

5.2.4.2. (B). — Let us consider the case $a(v_i)=0$ and $\mathfrak{a}(v_i)\neq 0$. Let k be determined by $\mathfrak{a}(v_i) = \sum_{j=1}^{k} \mathfrak{a}_j(v_i) z^{-j}$ and $\mathfrak{a}_k(v_i)\neq 0$. Recall we have the following:

$$\begin{aligned} \partial\Big(z^k f_{i,0}\, v_i\, \frac{d\bar{z}}{\bar{z}}\Big) &= z^k\partial\big(f_{i,0}\, v_i\big)\frac{d\bar{z}}{\bar{z}} + k\, z^k\, f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2}, \\ \overline{\partial}\Big(z^k\, f_{0,i}\, v_i\, \frac{dz}{z}\Big) &= z^k\overline{\partial}\big(f_{i,0}\, v_i\big)\frac{dz}{z}. \end{aligned} \tag{76}$$

We consider the following:

$$\tau_1 := z^k\, f_{i,0}\, v_i\, \frac{d\bar{z}}{\bar{z}} + \lambda\, z^k\, f_{i,0}\, v_i\, \frac{dz}{z}. \tag{77}$$

It is L^2, and we have the following:

$$\begin{aligned} \mathbb{D}^{\lambda}(\tau_1) &= F_A\Big(z^k\, f_{i,0}\, v_i\, \frac{d\bar{z}}{\bar{z}}\Big) + \lambda\, k\, z^k\, f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2} \\ &= \Big(z\frac{\partial\mathfrak{a}(v_i)}{\partial z} + \alpha(v_i) + k\,\lambda\Big)\, z^k\, f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2} + z^k\, F_{A_0}\big(f_{i,0}\, v_i\big)\, \frac{d\bar{z}}{\bar{z}}. \end{aligned} \tag{78}$$

Hence, $\mathbb{D}^{\lambda}(\tau_1)$ is also L^2. Let B_j be determined by the following:

$$f_{i,0}\, v_i\, \frac{dz\, d\bar{z}}{|z|^2} - \frac{1}{-k\,\mathfrak{a}_k(v_i)}\mathbb{D}^{\lambda}\tau_1 =: \sum B_j\, v_j\, \frac{dz\, d\bar{z}}{|z|^2}.$$

We have $B_j = \sum_{m>0} B_{j,m}(r)\, e^{\sqrt{-1}m\theta}$. It means

$$\mathcal{A}^{(0)}\Big(f_{i,0}\, v_i\, \frac{dz\, d\bar{z}}{|z|^2} - \frac{1}{-k\,\mathfrak{a}_k(v_i)}\mathbb{D}^{\lambda}\tau_1\Big) = 0.$$

5.2.4.3. (C). — Let us consider the case $a(v_i)=0$, $\mathfrak{a}(v_i)=0$, $\alpha(v_i)=0$ and $k(v_i)=-1$. Let $i(1)$ be determined by $N_{0,0,0}v_{i(1)|O}=v_{i|O}$ in $\mathrm{Gr}_0^F\,\mathbb{E}_0(V_{0|O})$. We put

$$\tau_1 := f_{i,0}\,v_{i(1)}\,\frac{d\overline{z}}{\overline{z}}+\lambda\,f_{i,0}\,v_{i(1)}\,\frac{dz}{z}. \tag{79}$$

It is L^2, and we have the following:

$$\mathbb{D}^{\lambda}(\tau_1)=f_{i,0}\,v_i\,\frac{dz\,d\overline{z}}{|z|^2}+F_{A_0'}(\tau_1).$$

Hence, $\mathbb{D}^{\lambda}(\tau_1)$ is also L^2. Let B_j be determined by the following:

$$f_{i,0}\,v_i\,\frac{dz\,d\overline{z}}{|z|^2}-\mathbb{D}^{\lambda}(\tau_1)=\sum B_j\,v_j\,\frac{dz\,d\overline{z}}{|z|^2}.$$

If $a(v_j)=0$, we have $B_j=\sum_{m>0}B_{j,m}(r)\,e^{\sqrt{-1}m\theta}$. In particular, we have the following:

$$\mathcal{A}^{(0)}\Big(f_{i,0}\,v_i\,\frac{dz\,d\overline{z}}{|z|^2}-\mathbb{D}^{\lambda}(\tau_1)\Big)=0.$$

5.2.4.4. Let us show the second claim of Lemma 5.2.4. We have

$$\mathcal{A}^{(0)}(\omega)=\sum_{v_i\in\mathcal{B}(-1)}f_{i,0}\,v_i\frac{dz\,d\overline{z}}{|z|^2}.$$

Applying the procedure in (B) and (C), we may and will assume that $f_{i,0}=0$ unless $a(v_i)=0$, $\mathfrak{a}(v_i)=0$ and $\alpha(v_i)\neq 0$. Let $k_0=\max\big\{k(v_i)\,\big|\,a(v_i)=0,\mathfrak{a}(v_i)=0,\alpha(v_i)\neq 0\big\}$. Applying the procedure in (A), we can kill the coefficients in $\mathcal{A}^{(0)}(\omega)$, of v_i with $a(v_i)=0$, $\mathfrak{a}(v_i)=0$ and $k(v_i)=k_0$. By using an easy descending induction, we can kill $\mathcal{A}^{(0)}(\omega)$. Thus, we obtain the second claim of Lemma 5.2.4.

Let us finish the proof of Lemma 5.2.4. Assume that ω is C^∞ and of polynomial order. By construction, $\tau^{(0)}$ is C^∞ and of polynomial order by construction. We can check that $\mathbb{D}^\lambda\tau^{(0)}$ is also C^∞ and of polynomial order from its explicit description. Because ω is C^∞ and of polynomial order by construction, $\mathcal{A}^{(1)}(\omega)$ is C^∞ and of polynomial order. Let $\mathcal{A}^{(1)}_j(\omega)$ and $\tau^{(1)}_j$ denote the coefficients of v_j in $\mathcal{A}^{(1)}$ and $\tau^{(1)}$, respectively. We obtain that $\tau^{(1)}_j$ is C^∞ on X^* by the equation $\overline{\partial}\tau^{(1)}_j=\mathcal{A}^{(1)}_j(\omega)$ and the elliptic regularity of $\overline{\partial}$. If M is sufficiently large, we have (i) $\tau^{(1)}_j$ is L^2 on X, (ii) $z^M\mathcal{A}^{(1)}_j(\omega)$ is L^∞ on X, (iii) $\overline{\partial}(z^M\tau^{(1)}_j)=z^M\mathcal{A}^{(1)}_j(\omega)$ as a distribution on X. By Sobolev's embedding, we obtain that $z^M\tau^{(1)}_j$ is L^p for some $p>2$. Then, by using Sobolev's embedding again, we obtain that $z^M\tau^{(1)}_j$ is L^∞ on X. Namely $\tau^{(1)}_j$ is of polynomial order. Thus, the proof of Lemma 5.2.4 is finished. □

5.2.5. $\mathcal{H}^j(\psi_1\circ\varphi_0)$ and $\mathcal{H}^j(\varphi_0)$ for $j=0,1$. — Let us prove Proposition 5.1.2. Let ω be an L^2-section of $V\otimes\Omega^1$ such that $\mathbb{D}^\lambda\omega=0$. We have the expression

$\omega = f^{1,0}\, dz/z + f^{0,1}\, d\bar{z}/\bar{z}$. We set

$$\mathcal{A}^{(0)}(f^{0,1}) = \sum_{v_i \in \mathcal{B}(1)} f^{(0,1)}_{i,0}(r)\, v_i, \quad \mathcal{A}^{(1)}(f^{0,1}) := f^{0,1} - \mathcal{A}^{(0)}(f^{0,1}).$$

We have the decomposition $f^{0,1} = \mathcal{A}^{(0)}(f^{0,1}) + \mathcal{A}^{(1)}(f^{0,1})$. We have the corresponding decomposition $\omega^{0,1} = \mathcal{A}^{(0)}(\omega^{0,1}) + \mathcal{A}^{(1)}(\omega^{0,1})$.

Lemma 5.2.5

- *We have an L^2-section $\tau^{(1)}$ of V such that $\overline{\partial}\tau^{(1)} = \mathcal{A}^{(1)}(\omega^{0,1})$.*
- *We have an L^2-section $\tau^{(0)}$ of V such that*
  - (i) *$\overline{\partial}\tau^{(0)}$ is also L^2,*
  - (ii) $\mathcal{A}^{(0)}(\omega^{0,1} - \overline{\partial}\tau^{(0)}) = 0$.

As a result, we can take an L^2-section τ of V such that $\overline{\partial}\tau = \omega^{0,1}$. If $\omega^{0,1}$ is C^∞ and of polynomial order, $\tau^{(i)}$ $(i = 0, 1)$, $\overline{\partial}\tau^{(0)}$ and τ are also C^∞ and of polynomial order.

Proof. — The first claim follows from Corollary 5.2.2. Let C_j be the functions determined by the following:

$$F_{A_0}\Big(f^{0,1} d\bar{z}/\bar{z}\Big) = \sum C_j\, v_j\, \frac{dz\, d\bar{z}}{|z|^2}. \tag{80}$$

From $\mathbb{D}^\lambda \omega = 0$, we obtain the following relation by considering the v_i-component:

$$\lambda \partial f_i^{0,1}\, v_i \frac{d\bar{z}}{\bar{z}} + f_i^{0,1}\Big(d\mathfrak{a}(v_i) + \alpha(v_i)\frac{dz}{z}\Big)\, v_i\, \frac{d\bar{z}}{\bar{z}} + \overline{\partial} f_i^{1,0}\, v_i\, \frac{dz}{z} + C_i\, v_i\, \frac{dz\, d\bar{z}}{|z|^2} = 0. \tag{81}$$

We use the Fourier expansion $C_j = \sum C_{j,m}\, e^{\sqrt{-1}m\theta}$. We give some preliminary arguments.

(A) Let us consider the case $a(v_i) = 0$ and $\mathfrak{a}(v_i) \neq 0$. Let k be determined by $\mathfrak{a}(v_i) = \sum_{j=1}^{k} \mathfrak{a}_j(v_i)\, z^{-j}$ with $\mathfrak{a}_k(v_i) \neq 0$. Let us look at the $e^{-\sqrt{-1}k\theta}$-component of (81). Multiplying it by r^k, we obtain the following, where we omit to write $dz\, d\bar{z}/|z|^2$:

$$\begin{aligned} -k\, f_{i,0}^{0,1}\, \mathfrak{a}_k(v_i) - \sum_{\substack{0<m<k\\ m+j=k}} f_{i,-m}^{0,1}\, j\, \mathfrak{a}_j(v_i)\, r^{k-j} + r^k \alpha(v_i)\, f_{i,-k}^{0,1} + \frac{1}{2} r \frac{\partial}{\partial r}(\lambda r^k\, f_{i,-k}^{0,1}) \\ - k\lambda f_{i,-k}^{0,1}\, r^k - \frac{1}{2} r\frac{\partial}{\partial r}(r^k\, f_{i,-k}^{1,0}) + r^k\, C_{i,-k} = 0. \end{aligned} \tag{82}$$

We consider the following:

$$\rho := \Big(- \sum_{\substack{0<m<k\\ m+j=k}} f_{i,-m}^{0,1}\, j\, \mathfrak{a}_j(v_i)\, r^{k-j} + r^k\, (\alpha(v_i) - k\lambda)\, f_{i,-k}^{0,1} + r^k C_{i,-k}\Big)\, v_i\, \frac{d\bar{z}}{\bar{z}}. \tag{83}$$

Then, we have $\int |\rho|_h^2\, r^{-2\varepsilon}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty$ for some $\varepsilon > 0$. By Corollary 5.2.2, we can take ρ_1 such that $\int |\rho_1|_h^2\, r^{-2\varepsilon}\, \mathrm{dvol}_{g_{\mathrm{P}}} < \infty$ and $\overline{\partial}\rho_1 = \rho$. Note that we have the following:

$$\overline{\partial}\Big((r^k \lambda f_{i,-k}^{0,1} - r^k f_{i,-k}^{1,0})\, v_i\Big) = \frac{1}{2} r \frac{\partial}{\partial r}(r^k \lambda f_{i,-k}^{0,1} - r^k f_{i,-k}^{1,0})\, v_i\, \frac{d\bar{z}}{\bar{z}}. \tag{84}$$

Hence, we have an L^2-section τ_2 such that $f_{i,0}^{0,1} v_i\, d\bar{z}/\bar{z} = \overline{\partial}\tau_2$.

(B) Let us consider the case $a(v_i)=0$, $\mathfrak{a}(v_i)=0$ and $\alpha(v_i)\neq 0$. Let us look at the $e^{\sqrt{-1}0\theta}$-component of (81). We have the following:

$$\frac{\lambda}{2}r\frac{\partial f^{0,1}_{i,0}}{\partial r}\frac{dz\,d\overline{z}}{|z|^2}+\alpha(v_i)\,f^{0,1}_{i,0}\frac{dz\,d\overline{z}}{|z|^2}-\frac{1}{2}r\frac{\partial f^{1,0}_{i,0}}{\partial r}\frac{dz\,d\overline{z}}{|z|^2}+C_{i,0}\frac{dz\,d\overline{z}}{|z|^2}=0.$$

Hence, we have the following:

$$\frac{1}{2}r\frac{\partial}{\partial r}\big(\lambda f^{0,1}_{i,0}-f^{1,0}_{i,0}\big)+\alpha(v_i)\,f^{0,1}_{i,0}=\begin{cases}-f^{0,1}_{i(1),0}+R & (\text{if }\exists\, v_{i(1)},\ N_{\mathfrak{a},\alpha,a}v_{i(1)}=v_i),\\ R & \text{otherwise.}\end{cases}$$

Here, $\int|R\,v_i|^2_h\,r^{-\varepsilon}\,\mathrm{dvol}_{g_{\mathrm{p}}}<\infty$. Then, by using an easy inductive argument, we can show that there exists τ_2 such that $\overline{\partial}\tau_2=f^{0,1}_{i,0}\,v_i\,d\overline{z}/\overline{z}$.

(C) Let us consider the case $a(v_i)=0$, $\mathfrak{a}(v_i)=0$, $\alpha(v_i)=0$ and $k(v_i)=1$. Let $i(-1)$ be determined by $Nv_{i|O}=v_{i(-1)|O}$. Let us look at the $e^{\sqrt{-1}0\theta}$-component of (81) for $v_{i(-1)}$. We have the following:

$$\lambda\frac{r}{2}\frac{\partial f^{0,1}_{i(-1),0}}{\partial r}\frac{dz\,d\overline{z}}{|z|^2}-\frac{r}{2}\frac{\partial}{\partial r}f^{1,0}_{i(-1),0}\frac{dz\,d\overline{z}}{|z|^2}+f^{0,1}_{i,0}\frac{dz\,d\overline{z}}{|z|^2}+\big(C_{i(-1),0}-f^{0,1}_{i,0}\big)\frac{dz\,d\overline{z}}{|z|^2}=0.$$

Hence, we have the following:

$$f^{0,1}_{i,0}=\frac{1}{2}r\frac{\partial}{\partial r}\big(-\lambda f^{0,1}_{i(-1),0}+f^{1,0}_{i(-1),0}\big)+R.$$

Here, $\int|R\,v_i|^2\,|z|^{-\varepsilon}\,\mathrm{dvol}_{g_{\mathrm{p}}}<\infty$ for some $\varepsilon>0$. We also have the following:

$$\big|f^{0,1}_{i(-1),0}\,v_i\big|^2_h\sim\big|f^{0,1}_{i(-1),0}\,v_{i(-1)}\,d\overline{z}/\overline{z}\big|^2_{h,g_{\mathrm{p}}}.$$

Hence, we obtain that $f^{0,1}_{i(-1),0}\,v_i$ is L^2. Similarly, $f^{1,0}_{i(-1),0}\,v_i$ is also L^2. Thus, there exists an L^2-section τ_2 such that $\overline{\partial}\tau_2=f^{0,1}_{i,0}\,v_i\,d\overline{z}/\overline{z}$.

The second claim of Lemma 5.2.5 follows from the above considerations (A), (B), (C). Assume that ω is C^∞ and of polynomial order. By the argument in the proof of Lemma 5.2.4, we can show that $\tau^{(1)}$ is C^∞ and of polynomial order. In (A), if ω is C^∞ and of polynomial order, ρ, $f^{0,1}_{i,-k}$, $f^{1,0}_{i,-k}$ are C^∞ and of polynomial order, and we can show that ρ_1 is C^∞ and of polynomial order by the argument in the proof of Lemma 5.2.4. Hence, τ_2 in (A) is C^∞ and of polynomial order. We can show that τ_2 in (B), (C) are C^∞ and of polynomial order. Then, $\tau^{(0)}$ and $\overline{\partial}\tau^{(0)}$ in Lemma 5.2.5 are C^∞ and of polynomial order. □

We put $\rho:=\omega-\mathbb{D}^\lambda\tau$ which is a holomorphic section of $V\otimes\Omega^{1,0}$ on X^*. We have the decomposition $\rho=\sum\rho_i$, where each ρ_i is the product of v_i and a holomorphic $(1,0)$-form on X^*.

Lemma 5.2.6. — *Let $\ell(v_i)\in\mathbb{Z}_{\geqslant 0}$ be determined as follows:*

- *We put $\ell(v_i):=-\mathrm{ord}(\mathfrak{a}(v_i))+1$ in the case $\mathfrak{a}(v_i)\neq 0$.*
- *We put $\ell(v_i):=1$ in the case $\mathfrak{a}(v_i)=0$ and $\alpha(v_i)+\lambda a(v_i)\neq 0$.*
- *We put $\ell(v_i):=0$ otherwise.*

Then, $z^{\ell(v_i)}\,\rho_i$ is L^2 with respect to h and $g_{\mathbf{p}}$. In the second case, $(-\log|z|)^{-1}\rho_i$ is L^2, a property which is stronger than the previous one.

In particular, ρ is C^∞ and of polynomial order.

Proof. — Let δ' denote the $(1,0)$-operator determined by h and $\overline{\partial}$. Let B be determined by $\delta'\boldsymbol{v} = \boldsymbol{v}\,B$. Then, B is diagonal, and the (i,i)-entry is as follows:

$$-a(v_i)\frac{dz}{z} + \frac{k(v_i)}{-2\log|z|}\,\frac{dz}{z}.$$

The curvature $R(h)$ of $\overline{\partial} + \delta'$ is bounded with respect to h and $g_{\mathbf{p}}$. Hence, $\delta'\tau$ is also L^2. (See the argument in the proof of Lemma 7.4.11, for example.)

Let $\mathbb{D}^{\lambda(1,0)}$ denote the $(1,0)$-part of $\mathbb{D}^\lambda$. We put $G := \mathbb{D}^{\lambda(1,0)} - \lambda\delta'$, which is a section of $\mathrm{End}(V)\otimes\Omega^{1,0}$. Let A_1 be determined by $G\boldsymbol{v} = \boldsymbol{v}\,A_1$. Then, we have the decomposition $A_1 = \Gamma' + C$, where F_C is bounded with respect to h and $g_{\mathbf{p}}$, and Γ' is the diagonal matrix whose (i,i)-entry is as follows:

$$d\mathfrak{a}(v_i) + \big(\alpha(v_i) + \lambda a(v_i)\big)\frac{dz}{z}.$$

We have the decomposition $\rho = \omega^{1,0} - \lambda\delta'\tau - \lambda F_C(\tau) - \lambda F_{\Gamma'}(\tau)$. Note $\omega^{1,0} - \lambda\delta'\tau - \lambda F_C(\tau)$ is L^2. Then, the claim of the lemma follows. □

Let A_0 and Γ be as in Section 5.2.3. We put $\mathbb{D}^\lambda_0 := \mathbb{D}^\lambda - F_\Gamma$. We have $\mathbb{D}^\lambda_0\boldsymbol{v} = \boldsymbol{v}\,A_0$. Recall F_{A_0} is bounded with respect to h and $g_{\mathbf{p}}$.

- In the case $\mathfrak{a}(v_i) \neq 0$, we have the L^2-holomorphic section κ_i such that $\rho_i = \big(d\mathfrak{a}(v_i) + \alpha(v_i)\,dz/z\big)\,\kappa_i$. Note $\mathbb{D}^\lambda_0(\kappa_i)$ is also L^2.
- In the case $\mathfrak{a}(v_i) = 0$ and $a(v_i) < 0$, we have $(-\log|z|)^{-1}\,\rho_i$ is L^2. Then, we obtain the L^2-property of ρ_i. See Section 5.1.1.1.
- In the case $\mathfrak{a}(v_i) = 0$, $a(v_i) = 0$ and $\alpha(v_i) \neq 0$, we have $z\,\rho_i$ is L^2. Hence, we have the L^2-holomorphic section κ_i such that $\alpha(v_i)\,\kappa_i\,dz/z = \rho_i$. Note $\mathbb{D}^\lambda_0(\kappa_i)$ is also L^2.
- In the case $\mathfrak{a}(v_i) = 0$, $a(v_i) = 0$ and $\alpha(v_i) = 0$, we have ρ_i is L^2.

Hence, we obtain the following lemma.

Lemma 5.2.7. — *There exists an L^2-section ν of V such that (i) $\mathbb{D}^\lambda\nu$ is also L^2, (ii) $\omega - \mathbb{D}^\lambda\nu$ is a holomorphic $(1,0)$-form. If $\omega^{0,1}$ is C^∞ and of polynomial order, ν is also C^∞ and of polynomial order.* □

Let us finish the proof of Proposition 5.1.2. Lemma 5.2.5 and Lemma 5.2.7 imply that $\psi_1\circ\varphi_0$ is a quasi-isomorphism.

Let $\omega \in \mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. We take ν as in Lemma 5.2.7. Because $\omega - \mathbb{D}^\lambda\nu$ is C^∞ and of polynomial order, we obtain that $\mathbb{D}^\lambda\nu$ is also C^∞ and of polynomial order. Hence, $\nu \in \mathcal{L}^0_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. It implies that $\mathcal{H}^1(\varphi_0)$ is surjective. The injectivity of $\mathcal{H}^1(\varphi_0)$ follows from the injectivity of $\mathcal{H}^1(\psi_1\circ\varphi_0)$. It is easy to see $\mathcal{H}^0(\varphi_0)$ is an isomorphism by Lemma 5.1.3. Thus Proposition 5.1.2 is proved. □

5.2.6. Proof of Proposition 5.1.4. — By an easy descent argument with respect to the ramified covering $X' \to X$, we may and will assume that $(V_*, \mathbb{D}^\lambda)$ is unramified. We have the full reduction $\big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V), \mathbb{D}^\lambda_{\mathfrak{a}}\big)$ for each $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)$. (See Section 3.2.4.) Let $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ be a holomorphic frame of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V)$ such that (i) compatible with the induced parabolic structure, the generalized eigen-decomposition of $\mathrm{Res}(\mathbb{D}^\lambda_{\mathfrak{a}})$, (ii) the induced frame of $\mathrm{Gr}^F \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V)$ is compatible with the weight filtration of the nilpotent part of $\mathrm{Res}(\mathbb{D}^\lambda)$. Let $R_{\mathfrak{a}}$ be determined by $\mathbb{D}^\lambda_{\mathfrak{a}} \overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{a}}\,(d\mathfrak{a} + R_{\mathfrak{a}})$.

Let $X^* = \bigcup_{j=1}^M S_j$ be a covering by small sectors. We have the full Stokes filtration $\widetilde{\mathcal{F}}^{S_j}$ and a flat splitting $V_{|\overline{S}_j} = \bigoplus_{\mathfrak{a}} V_{\mathfrak{a},S_j}$. We take the lift of $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ to $V_{\mathfrak{a},S_j}$. By gluing them as in Section 3.6.8.2, we obtain a C^∞-frame $\boldsymbol{v}_{C^\infty} = \big(\boldsymbol{v}_{\mathfrak{a},C^\infty}\big)$. Let $V_{\mathfrak{a},C^\infty}$ denote the subbundle generated by $\boldsymbol{v}_{\mathfrak{a},C^\infty}$. We obtain a decomposition $V_{|X^*} = \bigoplus V_{\mathfrak{a},C^\infty}$. Let τ be a local C^∞-section of $V \otimes \Omega^p$, which has the corresponding decomposition $\tau = \sum \tau_{\mathfrak{a}}$. Then, τ is a section of $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$, if and only if the following estimate holds for h and $g_{\mathbf{p}}$:

- τ and $\mathbb{D}^\lambda \tau$ are of polynomial order.
- τ_{0,C^∞} and $(\mathbb{D}^\lambda \tau)_{0,C^\infty}$ are L^2.

Let C be determined by the following:

$$\mathbb{D}^\lambda \boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\,\Big(\bigoplus(d\mathfrak{a} + R_{\mathfrak{a}}) + C\Big).$$

Then, we have $C = O\big(|z|^N\big)$ for any $N > 0$. Let $\mathbb{D}^{\lambda\prime}$ be the flat λ-connection determined by the following:

$$\mathbb{D}^{\lambda\prime} \boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\,\Big(\bigoplus(d\mathfrak{a} + R_{\mathfrak{a}})\Big).$$

The $(0,1)$-part of $\mathbb{D}^{\lambda\prime}$ is denoted by $\overline{\partial}'$. We put $F := \mathbb{D}^\lambda - \mathbb{D}^{\lambda\prime}$, and then we have $|F|_h = O\big(|z|^N\big)$ for any $N > 0$. We obtain a complex $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^{\lambda\prime})$ from $\mathbb{D}^{\lambda\prime}$ as in Section 5.1.1. As a sheaf, we have $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^{\lambda\prime}) = \widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ for $p = 0, 1, 2$.

Lemma 5.2.8. — *For any $\omega \in \widetilde{\mathcal{L}}^2_{\mathrm{poly}}(V_*, \mathbb{D}^{\lambda\prime}) = \widetilde{\mathcal{L}}^2_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$, we can take $\tau \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ such that $\mathbb{D}^{\lambda\prime}\tau = \omega$. Note that $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^{\lambda\prime}) = \widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ as remarked above.*

Proof. — We only have to consider the case where $\mathbb{D}^\lambda$ has a unique irregular value $\mathfrak{a}$. (Note that $\mathbb{D}^{\lambda\prime}$ has the same irregular value.) In the case $\mathfrak{a} = 0$, we may apply the results in Section 5.2.4. In the case $\mathfrak{a} \neq 0$, we only have to use Lemma 5.2.3, for example. □

Lemma 5.2.9. — *For any $\omega \in \widetilde{\mathcal{L}}^2_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$, we can take $\widetilde{\tau} \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ such that $\mathbb{D}^\lambda \widetilde{\tau} = \omega$.*

In particular, we obtain the vanishing of $\mathcal{H}^2$ of the complex of sheaves $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ at O.

Proof. — Take τ as in Lemma 5.2.8. We have $\omega - \mathbb{D}^\lambda\tau = O(|z|^N)$ for any N. Take some large M. According to Lemma 5.2.3, we can take a section κ of $V \otimes \Omega^{1,0}$ such that (i) $\overline{\partial}\kappa = \omega - \mathbb{D}^\lambda\tau$, (ii) $|\kappa| = O(|z|^M)$. We have $\kappa \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$, and $\mathbb{D}^\lambda(\tau + \kappa) = \omega$. Thus, we obtain Lemma 5.2.9 □

Let $\omega \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ such that $\mathbb{D}^\lambda\omega = 0$. We have $\mathbb{D}^{\lambda\prime}\omega = -F\omega = O(|z|^N)$ for any N. Hence, we can take a large $M > 0$ and a C^∞-section κ of $V \otimes \Omega^{1,0}$ such that $\overline{\partial}'\kappa = \mathbb{D}^{\lambda\prime}\omega$ and $|\kappa| = O(|z|^M)$. We put $\omega' := \omega - \kappa$, and then $\mathbb{D}^{\lambda\prime}\omega' = 0$. Note that the $(0,1)$-part of ω' and ω are equal.

Lemma 5.2.10. — *There exists a local section $\tau \in \widetilde{\mathcal{L}}^0(V_*, \mathbb{D}^\lambda)$ around O such that $\overline{\partial}'\tau = \omega^{0,1}$.*

Proof. — We may assume that $\mathbb{D}^\lambda$ (and hence $\mathbb{D}^{\lambda\prime}$) has a unique irregular value $\mathfrak{a}$. In the case $\mathfrak{a} = 0$, we can apply the result in Section 5.2.5. In the case $\mathfrak{a} \neq 0$, we can take τ such that (i) $\overline{\partial}'\tau = \omega^{0,1}$, (ii) $|\tau| = O(|z|^{-M})$ for some large M. Let us show that $\mathbb{D}^{\lambda\prime}\tau$ is of polynomial order. Let h' be the C^∞-Hermitian metric of $V_{|X-O}$ such that $h'(v_{C^\infty,i}, v_{C^\infty,j}) = \delta_{i,j}$. Note that h' and h are mutually bounded up to polynomial order. Let δ_1' be the $(1,0)$-operator determined by $\overline{\partial}'$ and h'. Note that $z^{M_1}\omega^{0,1}$ and $z^{M_1}\tau$ are bounded for some large M_1, and that $\overline{\partial}'(z^{M_1}\tau) = z^{M_1}\omega^{0,1}$. Then, it can be shown that $\delta_1'(z^{M_1}\tau)$ is L^2. (See the argument in the proof of Lemma 7.4.11, for example.) Thus, we obtain that $z^{M_1}\delta_1'\tau$ is L^2. Taking large M_2, we obtain $z^{M_2}(\delta_1'\tau - \omega^{1,0})$ is also L^2.

Since $\omega^{1,0} - \delta_1'\tau$ is holomorphic with respect to $\overline{\partial}'$, we obtain $\delta_1'\tau - \omega^{1,0} = O(|z|^{-M_3})$ for some large M_3. Then, we obtain the desired estimate for $\delta_1'\tau$. □

Lemma 5.2.11. — *We can take a section $\widetilde{\tau} \in \widetilde{\mathcal{L}}^0(V_*, \mathbb{D}^\lambda)$ such that $\overline{\partial}\widetilde{\tau} = \omega^{0,1}$.*

Proof. — Let τ be as in Lemma 5.2.10. We have $\mathbb{D}^\lambda\tau - \mathbb{D}^{\lambda\prime}\tau = F\tau = O(|z|^N)$ for any $N > 0$. According to Lemma 5.2.3, we can take a section ν of V such that (i) $|\nu| = O(|z|^M)$ for some large $M > 0$, (ii) $\overline{\partial}\nu = F^{0,1}\tau$. Let δ' be the $(1,0)$-operator determined by h and $\overline{\partial}$. Then, $\delta'\nu$ is L^2. If M is sufficiently large, $(\mathbb{D}^{\lambda(1,0)} - \lambda\delta')\nu$ is $O(|z|^{M/2})$. We put $\rho = \sum \rho_{\mathfrak{a}} := \omega - \kappa - \mathbb{D}^\lambda\tau + \mathbb{D}^\lambda\nu$. Then, ρ is a holomorphic section of $V \otimes \Omega^{1,0}$, $z^L\rho_{\mathfrak{a}}$ is L^2 for some L for any $\mathfrak{a}$, and ρ_0 is L^2. Hence, we obtain $\widetilde{\tau} := \tau + \nu \in \widetilde{\mathcal{L}}^0_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$. Thus, Lemma 5.2.11 is proved. □

Let $\widetilde{\mathcal{S}}(V_* \otimes \Omega^{p,0})$ be the sheaf of meromorphic sections τ of $V \otimes \Omega^{p,0}$ with the following property:

- Let $\tau_{|\widehat{O}} = \widehat{\tau}_{\mathrm{reg}} + \widehat{\tau}_{\mathrm{irr}}$ be the decomposition corresponding to the irregular decomposition. Then, $\widehat{\tau}_{\mathrm{reg}}$ is contained in $\mathcal{S}(\widehat{V}_{\mathrm{reg}\,*} \otimes \Omega^{p,0})$.

By using Lemma 5.2.9 and Lemma 5.2.11, it is easy to show that the natural inclusion $\widetilde{\mathcal{S}}(V_* \otimes \Omega^{\bullet,0}) \to \widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ is a quasi-isomorphism. It is also standard and easy to show that the natural inclusion $\mathcal{S}(V_* \otimes \Omega^{\bullet,0}) \to \widetilde{\mathcal{S}}(V_* \otimes \Omega^{\bullet,0})$ is a quasi-isomorphism. Hence, we obtain that $\psi_2 \circ \varphi_0$ is a quasi-isomorphism. Thus, the proof of Proposition 5.1.4 is finished. □

5.2.7. Proof of Proposition 5.1.5. — Let $\pi : \widetilde{X}(O) \to X$ denote the projection. For an open subset U of $\widetilde{X}(O)$, let $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}(U)$ be the space of C^∞-sections τ of $V \otimes \Omega^p_X$ on $U \setminus \pi^{-1}(O)$, such that $\tau_{|U\cap S}$ and $\mathbb{D}^\lambda \tau_{|U\cap S}$ satisfy the conditions **(a1)** and **(a2)** with respect to h and $g_{\mathbf{p}}$ on each small sector $S \cap U$. By taking sheafification, we obtain a complex of sheaves $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ on $\widetilde{X}(O)$. Similarly, we obtain a complex of sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ on $\widetilde{X}(O)$ from **(b1)** and **(b2)**.

Lemma 5.2.12. — *The natural inclusion*

$$(\psi_2 \circ \varphi_1)_{\widetilde{X}(O)} : \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)} \longrightarrow \widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$$

is a quasi-isomorphism.

Proof. — By choosing a flat splitting, we may assume that $(V_*, \mathbb{D}^\lambda)$ has a unique irregular value $\mathfrak{a}$. In the case $\mathfrak{a} = 0$, the claim is trivial. By using the results in Section 20.2.2, we can show the vanishing of the higher cohomology sheaves of both $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h)$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h)$ on $\widetilde{X}(O)$. The comparison of the 0-th cohomology sheaves is easy. □

Note that $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ and $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ are c-soft in the sense of Definition 2.5.5 of [**44**], and that $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ and $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)$ are obtained as their push-forward. Hence, we obtain Proposition 5.1.5 from Lemma 5.2.12. □

5.2.8. Proof of Proposition 5.1.7. — We only have to consider the first morphism. We use the notation in Section 5.2.7. We obtain a complex of sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h, h^{(T)})_{\widetilde{X}(O)}$ on $\widetilde{X}(O)$ from the conditions **(c1)** and **(c2)**.

Lemma 5.2.13. — *The inclusion* $\iota : \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h, h^{(T)})_{\widetilde{X}(O)} \to \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ *is a quasi-isomorphism.*

Proof. — We may assume that $(V_*, \mathbb{D}^\lambda)$ has the unique irregular value $\mathfrak{a}$. In the case $\mathfrak{a} = 0$, the claim is trivial. By using Lemmas 20.2.2 and 20.2.3, we can show the vanishing of the higher cohomology sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h)$ on $\widetilde{X}(O)$. By the same argument as that in the proof of Lemmas 20.2.2 and 20.2.3, we can also show the vanishing of the higher cohomology sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h, h^{(T)})$ on $\widetilde{X}(O)$. The comparison of the 0-th cohomology sheaves is easy. Thus, we obtain Lemma 5.2.13. □

Because $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, h, h^{(T)})_{\widetilde{X}(O)}$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}^\lambda)_{\widetilde{X}(O)}$ are c-soft, we obtain Proposition 5.1.7 as the push-forward of Lemma 5.2.13. □

5.3. Local quasi-isomorphisms in family

Let $\lambda_0 \in \boldsymbol{C}_\lambda$. Let $\mathcal{K}$ be a neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$, which will be shrunk if it is necessary. We put $\mathcal{X} := \mathcal{K} \times X$ and $\mathcal{D} := \mathcal{K} \times D$, where $D = \{O\}$. Let p_λ denote the projection forgetting the $\mathcal{K}$-component. For any subset $\mathcal{U}$ of $\mathcal{X}$, we use the symbol $\mathcal{U}^*$ to denote $\mathcal{U} \setminus \mathcal{D}$. Let $(V_*, \mathbb{D})$ be a good family of filtered λ-flat bundles on $(\mathcal{X}, \mathcal{D})$ with KMS-structure at λ_0 indexed by T. We shall consider the family versions of the complexes in Section 5.1.1.

We have the induced endomorphism $\mathrm{Gr}_a^{F^{(\lambda_0)}} \mathrm{Res}(\mathbb{D})$ on $\mathrm{Gr}_a^{F^{(\lambda_0)}}(V)$. We assume that the conjugacy classes of the nilpotent part of $\mathrm{Gr}_a^{F^{(\lambda_0)}} \mathrm{Res}(\mathbb{D})$ are independent of $\lambda \in \mathcal{K}$. The weight filtration is denoted by W.

5.3.1. Preliminary for metrics. — We have the filtration $F^{(\lambda_0)}$ and the decomposition $\mathbb{E}^{(\lambda_0)}$ of ${}^\diamond V$ as in Section 2.8.2. Let $\boldsymbol{v}$ be a frame of ${}^\diamond V$ compatible with $F^{(\lambda_0)}$, $\mathbb{E}^{(\lambda_0)}$ and the weight filtration W on $\mathrm{Gr}^{F^{(\lambda_0)},\mathbb{E}^{(\lambda_0)}}({}^\diamond V)$. We have $u(v_i) \in \boldsymbol{R} \times \boldsymbol{C}$ such that $\mathfrak{k}(\lambda_0, u(v_i)) = \deg^{F^{(\lambda_0)},\mathbb{E}^{(\lambda_0)}}(v_i)$. We put $k(v_i) := \deg^W(v_i)$. Let h be the Hermitian metric given as follows:

$$h(v_i, v_j) := \delta_{i,j}\, |z|^{-2\mathfrak{p}(\lambda, u(v_i))} \left(-\log|z|\right)^{k(v_i)}.$$

Note $-1 < \mathfrak{p}(\lambda_0, u(v_i)) \leqslant 0$ for each i. Recall $\mathfrak{p}(\lambda, u(v_i)) = a(v_i) + \mathrm{Re}(\overline{\lambda}\alpha(v_i))$, where $u(v_i) = (a(v_i), \alpha(v_i)) \in \boldsymbol{R}\times\boldsymbol{C}$. Hence, if $\mathfrak{p}(\lambda_0, u(v_i)) < 0$, we may and will assume that $\mathfrak{p}(\lambda, u(v_i)) < 0$ on $\mathcal{K}$. If $\mathfrak{p}(\lambda_0, u(v_i)) = 0$ and $u(v_i) \neq (0,0)$, we have $\mathfrak{p}(\lambda, u(v_i)) > 0$ on an open subset whose closure contains λ_0. If $u(v_i) = (0,0)$, $\mathfrak{p}(\lambda, u(v_i))$ is constantly 0.

Note if h' comes from another choice of $\boldsymbol{v}'$, the metrics h and h' are mutually bounded. In the following, we will use the metric $\widetilde{g}_{\mathbf{p}} := g_{\mathbf{p}} + d\lambda\, d\overline{\lambda}$ and the induced volume form $\mathrm{dvol}_{\widetilde{g}}$ for the base space $\mathcal{X} \setminus \mathcal{D}$.

5.3.1.1. Condition for a holomorphic section to be L^2. — Let $f = \sum f_i\, v_i$ be a holomorphic section on an open subset U of $\mathcal{K} \times X^*$. Let us describe the condition for f to be L^2 on a neighbourhood of $(\lambda, O) \in \mathcal{D}$. By the orthogonality, we only have to consider each $f_i\, v_i$.

$\mathfrak{p}(\lambda, u(v_i)) < 0$: f_i is holomorphic at (λ, O).

$u(v_i) = (0,0)$, $k(v_i) \leqslant 0$: f_i is holomorphic at (λ, O).

$u(v_i) = (0,0)$, $k(v_i) > 0$: f_i is holomorphic at (λ, O) and $f_i(\lambda, O) = 0$.

$\mathfrak{p}(\lambda, u(v_i)) = 0$, $u(v_i) \neq (0,0)$: f_i is holomorphic at (λ, O), and $f_i(\lambda, O) = 0$.

$\mathfrak{p}(\lambda, u(v_i)) > 0$: f_i is holomorphic at (λ, O), and $f_i(\lambda, O) = 0$.

We have similar conditions for a holomorphic section $f_i\, v_i\, dz/z$ of $V \otimes \Omega^{1,0}$ to be L^2 with respect to $(h, g_{\mathbf{p}})$.

$\mathfrak{p}(\lambda, u(v_i)) < 0$: f_i is holomorphic at (λ, O).

$u(v_i) = (0,0)$, $k(v_i) \leqslant -2$: f_i is holomorphic at (λ, O).

$u(v_i) = (0,0)$, $k(v_i) > -2$: f_i is holomorphic at (λ, O) and $f_i(\lambda, O) = 0$.

$\mathfrak{p}(\lambda, u(v_i)) = 0$, $u(v_i) \neq (0,0)$: f_i is holomorphic at (λ, O), and $f_i(\lambda, O) = 0$.

$\mathfrak{p}(\lambda, u(v_i)) > 0$: f_i is holomorphic at (λ, O), and $f_i(\lambda, O) = 0$.

5.3.1.2. Remark. — Let U be a subset of $\mathcal{K} \times X^*$. Let f be a C^∞-section of $V \otimes p_\lambda^* \Omega^p$ on U. We say that f is λ-holomorphic, if $\overline{\partial}_\lambda f = 0$. Assume $U = \mathcal{K} \times X^*(R)$, for simplicity. Assume that f is λ-holomorphic. We have the expression $f = \sum f_i \, v_i$. We have $-\overline{\partial}_\lambda \partial_\lambda \log |f_i \, v_i|^2_{h, \widetilde{g}_{\mathbf{p}}} = 0$. Hence, $|f|^2_{h,\widetilde{g}_{\mathbf{p}}}$ is subharmonic with respect to λ. It implies the following, for any λ-holomorphic C^∞-section f which is L^2 with respect to h and $\widetilde{g}_{\mathbf{p}}$:

1. The restrictions $f_{|\{\lambda\} \times X^*(R)}$ are also L^2. Moreover, when we fix a compact subset $\mathcal{K}_1$ of the interior of $\mathcal{K}$, there exists a constant $C_1 > 0$ such that $\|f_{|\{\lambda\}\times X^*(R)}\|_{h,g_{\mathbf{p}}} \leqslant C_1 \, \|f\|_{h,\widetilde{g}_{\mathbf{p}}}$ for any $\lambda \in \mathcal{K}_1$. The constant C_1 is independent of R.
2. For any $\varepsilon > 0$, we take a small R' such that $\big\|f_{|\mathcal{K}\times X^*(R')}\big\|_{h,\widetilde{g}_{\mathbf{p}}} < \varepsilon$. Then, for any $\lambda \in \mathcal{K}_1$, we obtain $\|f_{|\{\lambda\}\times X^*(R')}\|_{h,g_{\mathbf{p}}} \leqslant C_1 \, \varepsilon$. Hence, for the expression $f = \sum f_i \, v_i$, we can show the continuity of the function from $\mathcal{K}$ to the space of L^2-forms on $X^*(R)$ given by $\lambda \mapsto f_{i|\{\lambda\}\times X^*(R)} \, |v_i|_h$, for example. (The claim 2 follows from 1.)

Note if a Hermitian metric h' is mutually bounded with h, then the claims 1 and 2 above also hold for h', although $|f|^2_{h',\widetilde{g}_{\mathbf{p}}}$ is not necessarily subharmonic with respect to λ.

5.3.2. Sheaves of L^2-sections

5.3.2.1. λ-holomorphic L^2-sections and holomorphic L^2-sections. — The family of λ-connections $\mathbb{D}$ of V and the differential $\lambda\partial_X + \overline{\partial}_X$ of $\Omega^\bullet_X$ induce the differential of the λ-holomorphic C^∞-sections of $V \otimes p_\lambda^* \Omega^\bullet_X$ on open subsets of $\mathcal{X}^*$, which is also denoted by $\mathbb{D}$. We shall introduce a complex of sheaves $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})(\mathcal{U})$ on $\mathcal{X}$, which is an extension of the sheaf of λ-holomorphic C^∞-sections of $V \otimes \Omega^\bullet$.

For any open subset $\mathcal{U}$ of $\mathcal{X}$, let $\mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D})(\mathcal{U})$ be the space of λ-holomorphic C^∞-sections τ of $V \otimes \Omega^p$ on open subsets $\mathcal{U}^*$ with the following property:

- τ and $\mathbb{D}\tau$ are L^2 and of polynomial order locally on $\mathcal{U}$ with respect to h and $\widetilde{g}_{\mathbf{p}}$.

Thus, we obtain a complex of sheaves $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})$ on $\mathcal{X}^{(\lambda_0)}$.

Let $\mathcal{L}^p_{\mathrm{hol}}(V_*, \mathbb{D})$ $(p = 1, 2)$ be the subsheaf of $\mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D})$, which consists of holomorphic $(p, 0)$-forms. We will show the following proposition in Section 5.4.2–5.4.5.

Proposition 5.3.1. — *The natural inclusion $\varphi_0 : \mathcal{L}^p_{\mathrm{hol}}(V_*, \mathbb{D}) \to \mathcal{L}^p_{\mathrm{poly}}(V_*, \mathbb{D})$ is a quasi-isomorphism.*

5.3.2.2. Algebraically determined sheaf. — We shall give a rather algebraic description of the complexes up to quasi-isomorphisms. Let X' and φ_n be as in Section 5.1.1.

We put $D' := \{O'\}$ and $\mathcal{X}' := \mathcal{K} \times X'$ and $\mathcal{D}' := \mathcal{K} \times D'$. Recall that we have the induced good family of filtered λ-flat bundles on $(\mathcal{X}', \mathcal{D}')$ as in Section 2.5.3.3, which is denoted by $(V'_*, \mathbb{D}')$. It is easy to see that $(V'_*, \mathbb{D}')$ also has the KMS-structure at λ_0 if we shrink $\mathcal{K}$ appropriately. If we choose n appropriately, we have the irregular decomposition:

$$(V'_*, \mathbb{D}')_{|\widehat{\mathcal{D}}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}')} (\widehat{V}'_{\mathfrak{a}*}, \mathbb{D}'_{\mathfrak{a}}).$$

Since $\widehat{V}'_{0*}$ and $\bigoplus_{\mathfrak{a}\neq 0} \widehat{V}'_{\mathfrak{a}*}$ are $\mathrm{Gal}(X'/X)$-equivariant, we have the descent to $\widehat{\mathcal{D}}$ which are denoted by $\widehat{V}_{\mathrm{reg}}$ and $\widehat{V}_{\mathrm{irr}}$, respectively.

Let $a \in \boldsymbol{R}$ and $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}')$. We use the notation in Section 2.8.2. We have the generalized eigen-decomposition

$$\mathrm{Gr}^{F^{(\lambda_0)}}_a (\widehat{V}'_{\mathfrak{a}}) = \bigoplus_{u \in \mathcal{K}(a)} \mathbb{E}_{\mathfrak{e}(\lambda,u)} \mathrm{Gr}^{F^{(\lambda_0)}}_a (\widehat{V}'_{\mathfrak{a}}),$$

where the restriction of $\mathrm{Res}(\mathbb{D}) - \mathfrak{e}(\lambda, u)$ to $\mathbb{E}_{\mathfrak{e}(\lambda,u)} \mathrm{Gr}^{F^{(\lambda_0)}}_a (\widehat{V}'_{\mathfrak{a}})$ is nilpotent. We have the weight filtration W of the nilpotent part of $\mathrm{Gr}^{F^{(\lambda_0)}}_a \mathrm{Res}(\mathbb{D}')$ on $\mathbb{E}_{\mathfrak{e}(\lambda,u)} \mathrm{Gr}^{F^{(\lambda_0)}}_a (\widehat{V}'_{\mathfrak{a}})$ and $\mathrm{Gr}^{F^{(\lambda_0)}}_a (\widehat{V}'_{\mathfrak{a}})$.

For any $\mathfrak{a} \neq 0$, let $\mathcal{S}(\widehat{V}'_{\mathfrak{a}*} \otimes \Omega^{0,0}_{X'})$ denote the pull-back of $W_{-2}\mathbb{E}_{-\mathrm{ord}(\mathfrak{a})\lambda} \mathrm{Gr}^{F^{(\lambda_0)}}_{\mathrm{ord}(\mathfrak{a})} (\widehat{V}'_{\mathfrak{a}})$ via the projection ${}_{\mathrm{ord}(\mathfrak{a})}\widehat{V}'_{\mathfrak{a}} \to \mathrm{Gr}^{F^{(\lambda_0)}}_{\mathrm{ord}(\mathfrak{a})} (\widehat{V}'_{\mathfrak{a}})$. Let $\mathcal{S}(\widehat{V}'_{\mathfrak{a}*} \otimes \Omega^{1,0}_{X'})$ denote the pull-back of $W_{-2}\mathbb{E}_0 \mathrm{Gr}^{F^{(\lambda_0)}}_0 (\widehat{V}'_{\mathfrak{a}})$ via the projection ${}^\diamond\widehat{V}'_{\mathfrak{a}} dz'/z' \to \mathrm{Gr}^{F^{(\lambda_0)}}_0 (\widehat{V}'_{\mathfrak{a}})$. The descent of $\bigoplus_{\mathfrak{a}\neq 0} \mathcal{S}(\widehat{V}'_{\mathfrak{a}*} \otimes \Omega^{p,0}_{X'})$ is denoted by $\mathcal{S}(\widehat{V}_{\mathrm{irr}\,*} \otimes \Omega^{p,0}_X)$.

Let $\mathcal{S}(\widehat{V}_{0*} \otimes \Omega^{0,0}_X)$ denote the inverse image of $W_0\mathbb{E}_0 \mathrm{Gr}^{F^{(\lambda_0)}}_0 (\widehat{V}_0)$ via the projection ${}^\diamond\widehat{V}_0 \to \mathrm{Gr}^{F^{(\lambda_0)}}_0 (\widehat{V}_0)$. Let $\mathcal{S}(\widehat{V}_{0*} \otimes \Omega^{1,0}_X)$ denote the inverse image of $W_{-2}\mathbb{E}_0 \mathrm{Gr}^{F^{(\lambda_0)}}_0 \widehat{V}_0$ via the projection ${}^\diamond\widehat{V}_0 dz/z \to \mathrm{Gr}^{F^{(\lambda_0)}}_0 \widehat{V}_0$.

Thus, we obtain lattices $\mathcal{S}(\widehat{V}_* \otimes \Omega^{p,0}_X) = \mathcal{S}(\widehat{V}_{0*} \otimes \Omega^{p,0}_X) \oplus \mathcal{S}(\widehat{V}_{\mathrm{irr}\,*} \otimes \Omega^{p,0}_X)$ of $\big(V \otimes p_\lambda^* \Omega^{p,0}_X(*D)\big)_{|\widehat{\mathcal{D}}}$. They induce lattices $\mathcal{S}(V_* \otimes \Omega^{p,0}_X)$ of $V \otimes p_\lambda^*\Omega^{p,0}_X(*D)$. The family of λ-connections $\mathbb{D}$ on V_* and the differential $\lambda\, d_X$ on $\Omega^{\bullet,0}_X$ induce $\mathbb{D} : \mathcal{S}(V_* \otimes \Omega^{0,0}_X) \to \mathcal{S}(V_* \otimes \Omega^{1,0}_X)$. Thus, we obtain a complex of sheaves $\mathcal{S}(V_* \otimes \Omega^{0,0}_X) \xrightarrow{\mathbb{D}} \mathcal{S}(V_* \otimes \Omega^{1,0}_X)$. We will show the following lemma in Section 5.4.3.

Lemma 5.3.2. — *We have a natural inclusion $\mathcal{S}(V_* \otimes \Omega^{\bullet,0}_X) \to \mathcal{L}^{\bullet}_{\mathrm{hol}}(V_*, \mathbb{D})$. It is a quasi-isomorphism. If λ_0 is generic, moreover, it is an isomorphism.*

5.3.3. Variants in the case $\lambda_0 \neq 0$. — We shall introduce complexes of sheaves $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D})$ and $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V_*, \mathbb{D})$ on $\mathcal{X}$, which are extensions of the sheaf of λ-holomorphic C^∞-sections of $V \otimes \Omega^\bullet$ on $\mathcal{X}^*$. Although they can be given as in Section 5.1.2, we define them in a slightly different but equivalent way. We assume $\lambda_0 \neq 0$.

5.3.3.1. Decomposition. — We have the full reduction $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V')$ for $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$. We take a frame $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V')$ compatible with the induced decomposition $\mathbb{E}^{(\lambda_0)}$, the induced filtration $F^{(\lambda_0)}$, and the weight filtration W on $\mathrm{Gr}^{\mathbb{E}^{(\lambda_0)},F^{(\lambda_0)}}\big(^\diamond V'\big)$. For each $g \in \mathrm{Gal}(X'/X)$, we have a naturally induced isomorphism $g : \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(^\diamond V') \to \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{g\,\mathfrak{a}}(^\diamond V')$. We assume $g^*\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{g^*\mathfrak{a}} \cdot \boldsymbol{\omega}_{g,\mathfrak{a},i}$ for any $g \in \mathrm{Gal}(X'/X)$, where $\boldsymbol{\omega}_{g,\mathfrak{a}}$ is a tuple of $\omega_{g,\mathfrak{a},i} \in \boldsymbol{C}$ with $|\omega_{g,\mathfrak{a},i}| = 1$. (More precisely, we assume $g^*\overline{v}_{\mathfrak{a},i} = \omega_{g,\mathfrak{a},i} \cdot \overline{v}_{g\,\mathfrak{a},i}$.)

Let S be a small sector in $\mathcal{X}' \smallsetminus \mathcal{D}'$, and let $\overline{S}$ denote its closure in the real blow up $\widetilde{\mathcal{X}}'(\mathcal{D}')$. When S is small, we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $V'_{|\overline{S}}$, and we can take a $\mathbb{D}$-flat splitting $V'_{|\overline{S}} = \bigoplus_{\mathfrak{a}} V'_{\mathfrak{a},S}$. We have the holomorphic lift $\boldsymbol{v}_{\mathfrak{a},S}$ of $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ to $V'_{\mathfrak{a},S}$.

By shrinking $\mathcal{K}$, we take a covering $\mathcal{X}' \smallsetminus \mathcal{D}' = \bigcup_{j=1}^M S_j$ by small sectors with the following properties:

- Each S_j is the product of a small sector of $X^*(R)$ and $\mathcal{K}$.
- We have the full Stokes filtrations $\widetilde{\mathcal{F}}^{S_j}$, flat splittings $V'_{|\overline{S}_j} = \bigoplus V_{\mathfrak{a},S_j}$, and lifts $\boldsymbol{v}_{\mathfrak{a},S_j}$ of $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ to $V_{\mathfrak{a},S_j}$.
- We may assume that $g^{-1}(S_j)$ is the same as some $S_{i(j,g)}$ for any $g \in \mathrm{Gal}(X'/X)$ and j, and $g^*\,\boldsymbol{v}_{\mathfrak{a},S_j} = \boldsymbol{v}_{g^*\mathfrak{a},S_{i(g,j)}} \cdot \boldsymbol{\omega}_{g,\mathfrak{a}}$.

By gluing them as in Section 3.6.8.2, we obtain a C^∞-frame $\boldsymbol{v}_{C^\infty} = \big(\boldsymbol{v}_{\mathfrak{a},C^\infty}\big)$ of $^\diamond V'$ on $\mathcal{X}$. We may assume the following:

- $g^*\boldsymbol{v}_{\mathfrak{a},C^\infty} = \boldsymbol{v}_{g^*\mathfrak{a},C^\infty} \cdot \boldsymbol{\omega}_{g,\mathfrak{a}}$ for $g \in \mathrm{Gal}(X'/X)$.
- $\boldsymbol{v}_{\mathfrak{a},C^\infty}$ are λ-holomorphic.

Let $V'_{\mathrm{irr},C^\infty}$ be the C^∞-subbundle of $V'_{|\mathcal{X}'\smallsetminus\mathcal{D}'}$ generated by $\boldsymbol{v}_{\mathfrak{a},C^\infty}$ $(\mathfrak{a} \neq 0)$, and let $V'_{\mathrm{reg},C^\infty}$ be the C^∞-subbundle of $V'_{|\mathcal{X}'\smallsetminus\mathcal{D}'}$ generated by $\boldsymbol{v}_{0,C^\infty}$. Since they are $\mathrm{Gal}(X'/X)$-equivariant, they induce a decomposition $V_{|\mathcal{X}\smallsetminus\mathcal{D}} = V_{\mathrm{reg},C^\infty} \oplus V_{\mathrm{irr},C^\infty}$. The decomposition is λ-holomorphic. Note that it is not canonical.

Lemma 5.3.3. — *Let S be any small sector of $\mathcal{X}' \smallsetminus \mathcal{D}'$ such that we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $V'_{|\overline{S}}$. Let $Z := \overline{S} \cap \pi^{-1}(\mathcal{D})$, where π denotes the projection of $\widetilde{\mathcal{X}}'(\mathcal{D}')$ to $\mathcal{X}'$. Let $V' = \bigoplus_{\mathfrak{a}} V_{\mathfrak{a},S}$ be any splitting of $\widetilde{\mathcal{F}}^S$. Let $\boldsymbol{v}_{\mathfrak{a},S}$ be any holomorphic lift of $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ to $V_{\mathfrak{a},S}$, and let $\boldsymbol{v}_S = (\boldsymbol{v}_{\mathfrak{a},S})$.*

Let C be determined by $\boldsymbol{v}_S = \boldsymbol{v}_{C^\infty}(I + C)$. Corresponding to the decomposition of the frames, we have the decomposition $C = \big(C_{\mathfrak{a},\mathfrak{b}}\big)$. Then, the following holds:

- $C_{\mathfrak{a},\mathfrak{b}|\widehat{Z}} = 0$.
- $C_{\mathfrak{a},\mathfrak{b}} = 0$ *unless* $\mathfrak{a} \leqslant_S \mathfrak{b}$.
- $C_{\mathfrak{a},\mathfrak{b}} \exp\big(\lambda^{-1}(\mathfrak{a} - \mathfrak{b})\big)$ *is* $O\big(|z|^{-N}\big)$ *for some N in the case* $\mathfrak{a} <_S \mathfrak{b}$.

Proof. — It can be shown using the same argument as that in the proof of Lemma 3.6.26. □

For any local C^∞-section τ of $V \otimes \Omega^p$, we have the corresponding decompositions $\tau = \tau_{\mathrm{reg},C^\infty} + \tau_{\mathrm{irr},C^\infty}$ and $\mathbb{D}\tau = (\mathbb{D}\tau)_{\mathrm{reg},C^\infty} + (\mathbb{D}\tau)_{\mathrm{irr},C^\infty}$.

5.3.3.2. Complexes. — For any open set $\mathcal{U}\subset\mathcal{X}$, let $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*,\mathbb{D})(\mathcal{U})$ denote the space of λ-holomorphic C^∞-sections τ of $V\otimes\Omega^p$ on $\mathcal{U}^*$ such that the following estimates hold for $\widetilde{g}_{\mathbf{p}}$ and h:

(a1) : $\tau_{\mathrm{irr},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{irr},C^\infty}$ are of polynomial order locally on $\mathcal{U}$.

(a2) : $\tau_{\mathrm{reg},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{reg},C^\infty}$ are L^2 and of polynomial order locally on $\mathcal{U}$.

Let $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V_*,\mathbb{D})(\mathcal{U})$ denote the space of λ-holomorphic C^∞-sections τ of $V\otimes\Omega^p$ on $\mathcal{U}^*$ such that the following estimates hold for h and $\widetilde{g}_{\mathbf{p}}$:

(b1) : $\tau_{\mathrm{irr},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{irr},C^\infty}$ are $O\big(|z|^N\big)$ for any N locally on $\mathcal{U}$.

(b2) : $\tau_{\mathrm{reg},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{reg},C^\infty}$ are L^2 and of polynomial order locally on $\mathcal{U}$.

Thus, we obtain the complexes of sheaves $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})$. The following lemma is clear from Lemma 5.3.3.

Lemma 5.3.4. — *On the germ of neighbourhoods of (λ_0,O) in $\mathcal{X}$, the complexes of sheaves $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})$ are well defined, i.e., they are independent of the choices of covering $\mathcal{X}\setminus\mathcal{K}=\bigcup S_i$, flat splittings on $\overline{S}_i$ of $\mathcal{F}^{S_i}$ and C^∞-gluings of them.* □

We will show the following proposition in Sections 5.4.6 and 5.4.7.

Proposition 5.3.5. — *The natural morphisms*

$$\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})\xrightarrow{\varphi_2}\mathcal{L}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})\xrightarrow{\varphi_1}\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*,\mathbb{D})$$

are quasi-isomorphisms.

5.3.4. Deformation of the Stokes structure. — Assume $\lambda_0\neq 0$. We consider family versions of the complexes in Section 5.1.3.

5.3.4.1. Construction of a metric. — Let us take a finite covering $\mathcal{X}\setminus\mathcal{D}=\bigcup_{j=1}^M S_j$ as in Section 5.3.3. Let $T=T(\lambda):=1+|\lambda|^2$, and $F^{(T)}_{S_j}:=\bigoplus_{\mathfrak{a}}\exp\big((1-T)\lambda^{-1}\mathfrak{a}\big)\,\mathrm{id}_{V_{\mathfrak{a},S_j}}$. We consider the metric $h^{(T)}_{S_j}(u,v):=h\big(F^{(T)}_{S_j}(u),F^{(T)}_{S_j}(v)\big)$. By gluing $h^{(T)}_{S_j}$ in C^∞, we obtain a C^∞-metric $h^{(T)}$ of $V'_{|\mathcal{X}'\setminus\mathcal{D}'}$. We may assume that it is $\mathrm{Gal}(X'/X)$-equivariant. The induced metric of $V_{|\mathcal{X}\setminus\mathcal{D}}$ is also denoted by $h^{(T)}$. Note that the restrictions of the metrics $h^{(T)}$ and h to $V_{\mathrm{reg},C^\infty}$ are mutually bounded.

Remark 5.3.6. — $h^{(T)}$ is not obtained as a metric for a good family of filtered λ-flat bundles as in Section 5.3.1. The specialization of $h^{(T)}$ to $\{\lambda\}\times X$ is obtained as a metric for the good filtered λ-flat bundle $(V^\lambda_*,\mathbb{D}^\lambda)^{(T(\lambda))}$. (See Section 5.3.5.2 for V^λ_*.) □

5.3.4.2. Complexes. — For any open set $\mathcal{U} \subset \mathcal{X}$, let $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V, h^{(T)})(\mathcal{U})$ be the space of λ-holomorphic C^∞-sections τ of $V \otimes \Omega^p$ on $\mathcal{U}^*$ with the following growth estimate with respect to $h^{(T)}$ and $\widetilde{g}_{\mathbf{p}}$ locally on $\mathcal{U}$:

(c1) : $\tau_{\mathrm{irr},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{irr},C^\infty}$ are $O(|z|^N)$ for any $N > 0$.

(c2) : $\tau_{\mathrm{reg},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{reg},C^\infty}$ are L^2 and of polynomial order. (Equivalently, they are L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$.)

Let $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V, h, h^{(T)})(\mathcal{U})$ be the space of λ-holomorphic C^∞-sections τ of $V \otimes \Omega^p$ with the following properties:

(d1) : $\tau_{\mathrm{irr},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{irr},C^\infty}$ are $O(|z|^N)$ for any $N > 0$ locally on $\mathcal{U}$, with respect to both $(h, \widetilde{g}_{\mathbf{p}})$ and $(h^{(T)}, \widetilde{g}_{\mathbf{p}})$.

(d2) : $\tau_{\mathrm{reg},C^\infty}$ and $(\mathbb{D}\tau)_{\mathrm{reg},C^\infty}$ satisfy the conditions (c2).

Thus, we obtain complexes of sheaves $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h^{(T)})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h, h^{(T)})$. The following lemma is clear from Lemma 5.3.3.

Lemma 5.3.7. — *$\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h^{(T)})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h, h^{(T)})$ are well defined for $(V_*, \mathbb{D})$ on the germ of neighbourhoods of (λ_0, O) in $\mathcal{X}$.* □

5.3.4.3. Statement. — We will prove the following lemma in Section 5.4.7.

Proposition 5.3.8. — *The naturally defined morphisms*

$$\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}) \longleftarrow \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h, h^{(T)}) \longrightarrow \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V, h^{(T)})$$

are quasi-isomorphisms.

5.3.5. Complement for varying λ. — Assume $\lambda_0 \neq 0$. Let $\lambda_1 \in \mathcal{K}$. We take a small neighbourhood $\mathcal{K}_1$ of λ_1 in $\mathcal{K}$, and we put $\mathcal{X}^{(\lambda_1)} := \mathcal{K}_1 \times X$. We have the good family of filtered λ-flat bundles $V_*^{(\lambda_1)}$ obtained from V_*, as explained in Section 2.8.2. If we construct h_1 for $V_*^{(\lambda_1)}$ as above, h_1 and $h_{|\mathcal{X}^{(\lambda_1)}}$ are mutually bounded.

5.3.5.1. By the previous procedures, we obtain complexes of sheaves on $\mathcal{X}^{(\lambda_1)}$:

$$\mathcal{S}\big(V_*^{(\lambda_1)} \otimes \Omega^{\bullet,0}\big), \quad \mathcal{L}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D}), \quad \widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D}), \quad \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D}).$$

By construction, $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D})$, $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D})$ are the same as the restrictions of $\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})$, $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})$ and $\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})$ to $\mathcal{X}^{(\lambda_1)}$, respectively. The sheaves for $(V_*^{(\lambda_1)}, \mathbb{D})$ as in Section 5.3.4 are also the same as the restriction of the sheaves for $(V_*, \mathbb{D})$.

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}\big(V_* \otimes \Omega^{\bullet,0}\big)_{|\mathcal{X}^{(\lambda_1)}} & \longrightarrow & \mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})_{|\mathcal{X}^{(\lambda_1)}} \\ {\scriptstyle \rho_1}\big\downarrow & & {\scriptstyle =}\big\downarrow \\ \mathcal{S}\big(V_*^{(\lambda_1)} \otimes \Omega^{\bullet,0}\big) & \longrightarrow & \mathcal{L}^\bullet_{\mathrm{poly}}(V_*^{(\lambda_1)}, \mathbb{D}) \end{array}$$

It is easy to see that ρ_1 is a quasi-isomorphism. (See the proof of Lemma 5.3.2.2.)

5.3.5.2. Let $\mathcal{X}^{\lambda_1} := \{\lambda_1\} \times X$. By considering the specialization at $\mathcal{X}^{\lambda_1}$, i.e., taking the cokernel of the multiplication of $\lambda - \lambda_1$, we obtain a good filtered λ_1-flat bundle $(\mathcal{V}_*^{\lambda_1}, \mathbb{D}^{\lambda_1}) := (\mathcal{V}_*^{(\lambda_1)}, \mathbb{D})_{|\mathcal{X}^{\lambda_1}}$. We obtain the complex of sheaves, as in Section 5.1:

$$\mathcal{S}(\mathcal{V}_*^{\lambda_1} \otimes \Omega^{\bullet,0}), \quad \mathcal{L}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, \mathbb{D}^{\lambda_1}), \quad \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, \mathbb{D}^{\lambda_1}).$$

The restriction of h to $\mathcal{X}^{\lambda_1}$ is denoted by h_{λ_1}. We also obtain the following complexes of sheaves:

$$\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V^{\lambda_1}, h_{\lambda_1}, h^{(T(\lambda_1))}_{\lambda_1\, C^\infty}), \quad \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V^{\lambda_1}, h^{(T(\lambda_1))}_{\lambda_1, C^\infty}).$$

By taking the specialization at λ_1, we obtain the following commutative diagrams, which will be used in Section 18.2.

$$\begin{array}{ccccc}
\mathcal{S}(\mathcal{V}_* \otimes \Omega^{\bullet,0})_{|\mathcal{X}^{\lambda_1}} & \longrightarrow & \mathcal{L}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*, \mathbb{D})_{|\mathcal{X}^{\lambda_1}} & \longleftarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*, \mathbb{D})_{|\mathcal{X}^{\lambda_1}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}(\mathcal{V}_*^{\lambda_1} \otimes \Omega^{\bullet,0}) & \longrightarrow & \mathcal{L}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, \mathbb{D}) & \longleftarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, \mathbb{D})
\end{array} \tag{85}$$

$$\begin{array}{ccccc}
\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*, \mathbb{D})_{|\mathcal{X}^{\lambda_1}} & \longleftarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*, h, h^{(T)})_{|\mathcal{X}^{\lambda_1}} & \longrightarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*, h^{(T)})_{|\mathcal{X}^{\lambda_1}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, \mathbb{D}) & \longleftarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, h_{\lambda_1}, h^{(T(\lambda_1))}_{\lambda_1, C^\infty}) & \longrightarrow & \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(\mathcal{V}_*^{\lambda_1}, h^{(T(\lambda_1))}_{\lambda_1, C^\infty})
\end{array} \tag{86}$$

5.4. Proof in the family case

Because we will use the almost same arguments as those in Section 5.2, we will omit some details.

5.4.1. Variant of an estimate in [96]. — Let $u = (a, \alpha)$ such that $-1 < \mathfrak{p}(\lambda_0, u) \leqslant 0$, and let $k \in \mathbb{Z}$. Let $\mathcal{L}$ be a holomorphic line bundle on $\mathcal{K} \times X^*$ with the holomorphic frame σ and the metric h such that $h(\sigma, \sigma) = |z|^{-2\mathfrak{p}(\lambda, u)} \big|\log |z|\big|^k$. Let $\omega = g\, \sigma\, d\overline{z}/\overline{z}$ be a λ-holomorphic C^∞-section of $\mathcal{L} \otimes p_\lambda^* \Omega_X^{0,1}$ on $\mathcal{K} \times X^*(R)$, which is L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$. We have the Fourier decomposition $g = \sum g_n(r, \lambda)\, e^{\sqrt{-1}n\theta}$. As in Section 5.2.1, we put $g^{(1)} := \sum_{m \neq 0} g_m(r)\, e^{\sqrt{-1}m\theta}$, and thus we have the decompositions $g = g_0 + g^{(1)}$ and $\omega = \omega_0 + \omega^{(1)}$. Let $\overline{\partial}_z$ and $\overline{\partial}_\lambda$ denote the natural $(0,1)$-operator along X-direction and $\mathcal{K}$-direction, respectively. We have a direct consequence of the estimate in [**96**], which we formulate for the reference in the subsequent argument.

Lemma 5.4.1. — *We have a λ-holomorphic C^∞-section τ, which is L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$, with the following properties:*

- *We have $\overline{\partial}_z \tau = \omega$ in the case where one of $(\mathfrak{p}(\lambda_0, u) < 0)$ or $(u = (0,0), k \neq 1)$ holds.*
- *We have $\overline{\partial}_z \tau = \omega^{(1)}$ otherwise.*

Proof. — Note a remark in Section 5.3.1.2. Let ω_λ denote the restriction of ω to $\{\lambda\}\times X^*(R)$. Let us consider the case where one of $(\mathfrak{p}(\lambda_0,u)<0)$ or $(u=(0,0),k\neq 1)$ holds. Applying Φ to each ω_λ, we obtain the L^2-section $\Phi(\omega)$ of $\mathcal{L}$. By construction, it satisfies $\overline{\partial}_z\Phi(\omega)=\omega$ and $\overline{\partial}_\lambda\Phi(\omega)=0$ as a distribution. Then, it follows that $\Phi(\omega)$ is C^∞ and λ-holomorphic due to standard ellipticity. By using Sobolev embedding, we obtain $\Phi(\omega)$ is of polynomial order. (See the last part of the proof of Lemma 5.2.4.)

If neither $(\mathfrak{p}(\lambda_0,u)<0)$ or $(u=(0,0),k\neq 1)$ are satisfied, we obtain the desired section by applying $\Phi^{(1)}$. □

5.4.2. Preliminaries. — Let us start the proof of the propositions in Section 5.3. By an easy argument of descent, we can reduce the problem to the unramified case. Therefore, we may and will assume that $(V_*,\mathbb{D})$ is unramified. We use the polar coordinate $z=r\,e^{\sqrt{-1}\theta}$. We may assume the following for the frame $\boldsymbol{v}$:

1. $\boldsymbol{v}$ is compatible with the irregular decomposition in N-th order for some large N. (See Section 5.2.3.)
2. Let $N_{a,\alpha,\mathfrak{a}}$ be the nilpotent part of the endomorphisms on $\mathrm{Gr}^{F^{(\lambda_0)}}_a\mathbb{E}^{(\lambda_0)}_\alpha(V_{\mathfrak{a}|\mathcal{D}})$ induced by $\mathrm{Res}(\mathbb{D})$. Then, $N_{a,\alpha,\mathfrak{a}}$ are represented by Jordan matrices with respect to the induced frames.

We have the irregular value $\mathfrak{a}(v_i)$. We also put $a(v_i):=\deg^{F^{(\lambda_0)}}(v_i)$, $\alpha(v_i):=\deg^{\mathbb{E}^{(\lambda_0)}}(v_i)$ and $k(v_i):=\deg^W(v_i)$. Let $u(v_i)\in \boldsymbol{R}\times\boldsymbol{C}$ be determined by $\mathfrak{k}(\lambda_0,u(v_i))=\big(a(v_i),\alpha(v_i)\big)$. We define

$$\mathcal{B}(k):=\big\{v_i\,\big|\,\mathfrak{a}(v_i)=0,\ u(v_i)=(0,0),\ k(v_i)=k\big\}\cup \\ \big\{v_i\,\big|\,a(v_i)=0,\ \big(\mathfrak{a}(v_i),\alpha(v_i)\big)\neq(0,0)\big\}.$$

Let A be determined by $\mathbb{D}\boldsymbol{v}=\boldsymbol{v}\,A$. Let Γ be the diagonal matrix whose (i,i)-entries are

$$\mathfrak{e}\big(\lambda,u(v_i)\big)\,dz/z+d\mathfrak{a}(v_i).$$

We put $A_0:=A-\Gamma$. We use the symbol F_A to denote the section of $\mathrm{End}(V)\otimes\Omega^{1,0}$ determined by $F_A(\boldsymbol{v})=\boldsymbol{v}\,A$. We use the symbol F_{A_0} in a similar meaning. Then, F_{A_0} is bounded with respect to h and $\widetilde{g}_{\mathbf{p}}$. We have the following decomposition:

$$A_0=\bigoplus_{a,\alpha,\mathfrak{a}}J_{a,\alpha,\mathfrak{a}}\,\frac{dz}{z}+A_0'\,\frac{dz}{z}.$$

Here A_0' is holomorphic and $F_{A_0'|\mathcal{D}}$ strictly decreases the filtration $F^{(\lambda_0)}$. And $J_{a,\alpha,\mathfrak{a}}$ are constant Jordan matrices and represent $N_{a,\alpha,\mathfrak{a}}$ with respect to the induced frames of $\mathrm{Gr}^{F^{(\lambda_0)}\,\mathbb{E}^{(\lambda_0)}}_{a,\alpha}(V_{\mathfrak{a}|\mathcal{D}})$.

The $(1,0)$-operator ∂ is defined by $\partial\big(\sum f_i\,v_i\big)=\sum\partial_X f_i\,v_i$. Then, we have $\mathbb{D}=\overline{\partial}+\lambda\partial+F_A$.

5.4.3. Proof of Lemma 5.3.2.2. — Let $f = \sum f_i\, v_i\, dz/z$ be a section of $\mathcal{S}(V_* \otimes \Omega^{1,0})$ on an open subset U of $\mathcal{X}$. Let us show that it is L^2 around $(\lambda, O) \in U \cap \mathcal{D}$. By construction, f_i is holomorphic. If $\mathfrak{p}(\lambda, u(v_i)) > 0$, we have $\mathfrak{p}(\lambda_0, u(v_i)) = 0$ and $u(v_i) \neq (0,0)$. Hence, we have $f_i(\lambda, O) = 0$. If $\mathfrak{p}(\lambda, u(v_i)) = 0$ and $u(v_i) \neq (0,0)$, we have $\mathfrak{p}(\lambda_0, u(v_i)) = 0$, and $f_i(\lambda, O) = 0$ by construction of $\mathcal{S}(V_* \otimes \Omega^{1,0})$. If $u(v_i) = (0,0)$ and $k(v_i) > -2$, we have $f_i(\lambda, O) = 0$ by construction. Hence, f is L^2 by the condition described in Section 5.3.1.1, i.e., $\mathcal{S}(V_* \otimes \Omega^{1,0}) \subset \mathcal{L}^1_{\mathrm{hol}}(V_*, \mathbb{D})$. Similarly and more easily, we can check that a section of $\mathcal{S}(V_*)$ is L^2. Because $\mathbb{D}f \in \mathcal{S}(V_* \otimes \Omega^{1,0}) \subset \mathcal{L}^1_{\mathrm{hol}}(V_*, \mathbb{D})$, we obtain $f \in \mathcal{L}^0_{\mathrm{hol}}(V_*, \mathbb{D})$. Hence, we obtain a natural inclusion $\mathcal{S}(V_* \otimes \Omega^{\bullet,0}) \to \mathcal{L}^\bullet_{\mathrm{hol}}(V_*, \mathbb{D}^\lambda)$.

Assume λ_0 satisfies the following condition for the index set T of the KMS-structure:

(A) : If $u \in T$ satisfies $\mathfrak{p}(\lambda_0, u) = 0$, then $u = (0,0)$.

If $\mathcal{K}$ is small, we have $\mathfrak{p}(\lambda, u(v_i)) \leqslant 0$ for any $\lambda \in \mathcal{K}$, and $\mathfrak{p}(\lambda, u(v_i)) = 0 \iff \mathfrak{p}(\lambda_0, u(v_i)) = 0 \iff u(v_i) = (0,0)$. Hence, we obtain that $\mathcal{S}(V_* \otimes \Omega^{1,0}) \subset \mathcal{L}^1_{\mathrm{hol}}(V_*, \mathbb{D})$ is an isomorphism. Let $f \in \mathcal{L}^0_{\mathrm{hol}}(V_*, \mathbb{D})$. We have a description $f = \sum f_i\, v_i$. Because it is L^2, each f_i are holomorphic. Then, we can deduce $f \in \mathcal{S}(V_*)$ from $\mathbb{D}f \in \mathcal{L}^1_{\mathrm{hol}}(V_*, \mathbb{D}) = \mathcal{S}(V_* \otimes \Omega^{1,0})$. Even if λ_0 does not satisfy (A), we obtain that the inclusion of the germs at (λ_0, O) is an isomorphism by the same argument.

Let us show that the inclusion of the germs $\mathcal{S}(V_* \otimes \Omega^{\bullet,0})_{(\lambda_1, O)} \to \mathcal{L}^\bullet_{\mathrm{hol}}(V_*, \mathbb{D})_{(\lambda_1, O)}$ is a quasi-isomorphism at each $(\lambda_1, O) \in \mathcal{D} \setminus \{(\lambda_0, O)\}$. We may assume that (λ_1, O) satisfies (A).

Let $\mathcal{K}_1 \subset \mathcal{K}$ be a small neighbourhood of λ_1. We put $\mathcal{X}^{(\lambda_1)} := \mathcal{K}_1 \times X$. We have the good family of filtered λ-flat bundles $V_*^{(\lambda_1)}$ obtained from V_*, as explained in Section 2.8.2. Then, we obtain complexes of sheaves $\mathcal{S}\big(V_*^{(\lambda_1)} \otimes \Omega^{\bullet,0}\big)$ and $\mathcal{L}^\bullet_{\mathrm{hol}}(V_*^{(\lambda_1)}, \mathbb{D})$ on $\mathcal{X}^{(\lambda_1)}$, and the following morphisms:

$$\mathcal{S}\big(V_* \otimes \Omega^{\bullet,0}\big)_{|\mathcal{X}^{(\lambda_1)}} \xrightarrow{a} \mathcal{S}\big(V_*^{(\lambda_1)} \otimes \Omega^{\bullet,0}\big) \xrightarrow{b} \mathcal{L}^\bullet_{\mathrm{hol}}(V_*^{(\lambda_1)}, \mathbb{D}) = \mathcal{L}^\bullet_{\mathrm{hol}}(V_*, \mathbb{D})_{|\mathcal{X}^{(\lambda_1)}}.$$

By the previous consideration, we already know that (b) is an isomorphism. It is easy to show a is a quasi-isomorphism by a direct computation. Thus, the proof of Lemma 5.3.2.2 is finished. □

5.4.4. Vanishing of $\mathcal{H}^2\big(\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})\big)$. — Let us show $\mathcal{H}^2\big(\mathcal{L}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D})\big) = 0$. We only have to show such a claim for the germ at (λ_0, O). (See a remark in Section 5.3.5.1. We will omit similar remarks in the following.) Let ω be a λ-holomorphic C^∞-section of $V \otimes \Omega^2$, which is L^2 with respect to h and $\widetilde{g}_{\mathrm{p}}$. We have the expression:

$$\omega = f\, \frac{dz}{z}\, \frac{d\overline{z}}{\overline{z}}, \quad f = \sum f_i\, v_i$$

Each f_i has the Fourier expansion $f_i = \sum_{m\in\mathbb{Z}} f_{i,m}(r)\, e^{\sqrt{-1}m\theta}$. We set

$$\mathcal{A}^{(0)}(f) := \sum_{v_i\in\mathcal{B}(-1)} f_{i,0}(r)\, v_i, \quad \mathcal{A}^{(1)}(f) := f - \mathcal{A}^{(0)}(f).$$

We have the decomposition $f = \mathcal{A}^{(0)}(f) + \mathcal{A}^{(1)}(f)$. We have the corresponding decomposition $\omega = \mathcal{A}^{(0)}(\omega) + \mathcal{A}^{(1)}(\omega)$. We show the following lemma based on an idea of Sabbah contained in [**73**].

Lemma 5.4.2

- *We have a section $\tau^{(1)}$ of $\mathcal{L}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathbb{D}\tau^{(1)} = \mathcal{A}^{(1)}(\omega)$.*
- *We have a section $\tau^{(0)}$ of $\mathcal{L}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathcal{A}^{(0)}\big(\omega - \mathbb{D}\tau^{(0)}\big) = 0$.*
- *In particular, we can take a section τ of $\mathcal{L}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathbb{D}\tau = \omega$.*

Proof. — The argument is almost the same as the proof of Lemma 5.2.4. Briefly, we only have to replace $\alpha(v_i)$ with $\mathfrak{e}(\lambda, u(v_i))$. We give only an indication. The first claim follows from Lemma 5.4.1. Let us show the second claim. We give preliminary arguments.

(A) If $a(v_i) = 0$, $\mathfrak{a}(v_i) = 0$ and $\alpha(v_i) \neq 0$ hold, let τ_1 be given by (74). We have $\mathbb{D}\tau_1 = F_A(f_{i,0}\, v_i)\, d\bar{z}/\bar{z}$. We obtain that τ_1 and $\mathbb{D}\tau_1$ are L^2 and of polynomial order, by using the estimate for $f_{i,0}\, v_i\, (dz\, d\bar{z}/|z|^2)$. Moreover, we have the following formula:

$$(87)\quad f_{i,0}\, v_i \cdot \frac{dz\, d\bar{z}}{|z|^2} - \mathfrak{e}\big(\lambda, u(v_i)\big)^{-1}\, \mathbb{D}\tau_1 = f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2} - \mathfrak{e}\big(\lambda, u(v_i)\big)^{-1} F_A\big(f_{i,0}\, v_i\big) \frac{d\bar{z}}{\bar{z}}$$
$$= \mathfrak{e}\big(\lambda, u(v_i)\big)^{-1} F_{A_0}\big(f_{i,0}\, v_i\big) \frac{d\bar{z}}{\bar{z}} =: \sum B_j\, v_j.$$

Because F_{A_0} is bounded, the right-hand side of (87) is also L^2 and of polynomial order. Let us see B_j more closely. If $a(v_j) = 0$, we have the Fourier expansion $B_j = \sum_{m\geqslant 0} B_{j,m}(r, \lambda) e^{\sqrt{-1}m\theta}$, and $B_{j,0}(r, \lambda) = 0$ unless $\big(\mathfrak{a}(v_j), \alpha(v_j)\big) = \big(\mathfrak{a}(v_i), \alpha(v_i)\big)$ and $N_{a,\alpha,\mathfrak{a}} v_{i|\mathcal{D}} = v_{j|\mathcal{D}}$. Note $\deg^W(v_j) < \deg^W(v_i)$ for such v_j.

(B) Let us consider the case where $a(v_i) = 0$ and $\mathfrak{a}(v_i) \neq 0$ hold. Let k be determined by $\mathfrak{a}(v_i) = \sum_{j=1}^{k} \mathfrak{a}_j(v_i) z^{-j}$ and $\mathfrak{a}_k(v_i) \neq 0$. Let τ_1 be given as in (77), which is L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$. And we have the following equality, as in the proof of Lemma 5.2.4:

$$(88)\quad \mathbb{D}(\tau_1) = F_A\Big(z^k\, f_{i,0}\, v_i\, \frac{d\bar{z}}{\bar{z}}\Big) + \lambda\, k\, z^k\, f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2}$$
$$= \Big(z \frac{\partial \mathfrak{a}(v_i)}{\partial z} + \mathfrak{e}\big(\lambda, u(v_i)\big) + k\,\lambda\Big)\, z^k\, f_{i,0}\, v_i \frac{dz\, d\bar{z}}{|z|^2} + z^k\, F_{A_0}\big(f_{i,0}\, v_i\big)\, \frac{d\bar{z}}{\bar{z}}.$$

Hence, $\mathbb{D}(\tau_1)$ is also L^2 and of polynomial order. Let B_j be determined by the following:

$$f_{i,0}\, v_i\, \frac{dz\, d\bar{z}}{|z|^2} - \frac{1}{-k\, \mathfrak{a}_k(v_i)} \mathbb{D}\tau_1 =: \sum B_j\, v_j\, \frac{dz\, d\bar{z}}{|z|^2}.$$

If $a(v_j) = 0$, we have $B_j = \sum_{m>0} B_{j,m}(r)\, e^{\sqrt{-1}m\theta}$.

(C) Let us consider the case where $a(v_i)=0$, $\mathfrak{a}(v_i)=0$, $\alpha(v_i)=0$ and $k(v_i)=-1$ hold. Let $i(1)$ be determined by $N_{0,0,0}v_{i(1)|\mathcal{D}}=v_{i|\mathcal{D}}$ in $\mathrm{Gr}^{F^{(\lambda_0)},\mathbb{E}^{(\lambda_0)}}_{(0,0)}(V)$. Let τ_1 be given by (79). Then, it is L^2 and of polynomial order, and we have the following:

$$\mathbb{D}(\tau_1)=f_{i,0}\,v_i\,\frac{dz\,d\overline{z}}{|z|^2}+F_{A_0'}(\tau_1).$$

Hence, $\mathbb{D}(\tau_1)$ is also L^2 and of polynomial order. Let B_j be determined by the following:

$$f_{i,0}\,v_i\,\frac{dz\,d\overline{z}}{|z|^2}-\mathbb{D}(\tau_1)=\sum B_j\,v_j\,\frac{dz\,d\overline{z}}{|z|^2}.$$

If $a(v_j)=0$, we have $B_j=\sum_{m>0}B_{j,m}(r)\,e^{\sqrt{-1}m\theta}$.

By using the above preliminary arguments (A), (B) and (C) with an easy induction, we can show the second claim of Lemma 5.4.2. □

5.4.5. Morphisms $\mathcal{H}^j(\varphi_0)$ for $j=0,1$. — Let us show that $\mathcal{H}^j(\varphi_0)$ $(j=0,1)$ are isomorphisms. Let ω be a section of $\mathcal{L}^1_{\mathrm{poly}}(V_*,\mathbb{D})$ such that $\mathbb{D}\omega=0$. We have the expression $\omega=f^{1,0}\,dz/z+f^{0,1}\,d\overline{z}/\overline{z}$. We set

$$\mathcal{A}^{(0)}(f^{0,1})=\sum_{v_i\in\mathcal{B}(1)}f^{(0,1)}_{i,0}(r)\,v_i,\quad \mathcal{A}^{(1)}(f^{0,1}):=f^{0,1}-\mathcal{A}^{(0)}(f^{0,1}).$$

We have the decomposition $f^{0,1}=\mathcal{A}^{(0)}(f^{0,1})+\mathcal{A}^{(1)}(f^{0,1})$. We have the corresponding decomposition $\omega^{0,1}=\mathcal{A}^{(0)}(\omega^{0,1})+\mathcal{A}^{(1)}(\omega^{0,1})$.

Lemma 5.4.3

- *There exists a λ-holomorphic C^∞-section $\tau^{(1)}$ of V such that (i) L^2 and of polynomial order, (ii) $\overline{\partial}\tau^{(1)}=\mathcal{A}^{(1)}(\omega^{0,1})$.*
- *We have a λ-holomorphic C^∞-section $\tau^{(0)}$ of V such that (i) L^2 and of polynomial order, (ii) $\mathcal{A}^{(0)}(\omega^{0,1}-\overline{\partial}\tau^{(0)})=0$.*

As a result, we can take a λ-holomorphic C^∞-section τ of V such that (i) L^2 and of polynomial order, (ii) $\overline{\partial}\tau=\omega^{0,1}$.

Proof. — The argument is almost the same as in the proof of Lemma 5.2.5. We only have to replace $\alpha(v_i)$ with $\mathfrak{e}(\lambda,u(v_i))$. The first claim follows from Lemma 5.4.1. Let C_j be the functions determined by the following:

$$F_{A_0}\Big(f^{0,1}d\overline{z}/\overline{z}\Big)=\sum C_j\,v_j\,\frac{dz\,d\overline{z}}{|z|^2}.$$

From $\mathbb{D}\omega=0$, we obtain the following relation by taking the v_i-component:

$$\lambda\partial f_i^{0,1}v_i\frac{d\overline{z}}{\overline{z}}+f_i^{0,1}\Big(d\mathfrak{a}(v_i)+\mathfrak{e}\big(\lambda,u(v_i)\big)\frac{dz}{z}\Big)v_i\frac{d\overline{z}}{\overline{z}}+\overline{\partial}f_i^{1,0}v_i\frac{dz}{z}+C_iv_i\frac{dz\,d\overline{z}}{|z|^2}=0 \tag{89}$$

We use the Fourier expansion $C_j=\sum C_{j,m}\,e^{\sqrt{-1}m\theta}$. We give some preliminary arguments.

(A) Let us consider the case where $a(v_i) = 0$ and $\mathfrak{a}(v_i) \neq 0$ hold. Let k be determined by $\mathfrak{a}(v_i) = \sum_{j=1}^{k} \mathfrak{a}_j(v_i)\, z^{-j}$ and $\mathfrak{a}_k(v_i) \neq 0$. By looking at the $e^{-\sqrt{-1}k\theta}$-component of (89), we obtain (82), with $\mathfrak{e}(\lambda, u(v_i))$ instead of $\alpha(v_i)$, as in the proof of Lemma 5.2.5. Let ρ be given by (83). If $\mathcal{K}$ is sufficiently small, we have $\int |\rho|_h^2 r^{-2\varepsilon}\, \mathrm{dvol}_{\widetilde{g}_{\mathbf{p}}} < \infty$ for some $\varepsilon > 0$. By Lemma 5.4.1, we can take a λ-holomorphic C^∞-section ρ_1 such that (i) $\rho_1\, |z|^{-\varepsilon}$ is L^2 and of polynomial order, (ii) $\overline{\partial}\rho_1 = \rho$. Note (84). Hence, we have a λ-holomorphic C^∞-section τ_2 such that (i) τ_2 is L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$, (ii) $f_{i,0}^{0,1} v_i\, d\overline{z}/\overline{z} = \overline{\partial}\tau_2$.

(B) We can argue the case where $a(v_i) = 0$, $\mathfrak{a}(v_i) = 0$ and $\alpha(v_i) \neq 0$ hold, by using the method in the part (B) of the proof of Lemma 5.2.5. We only have to replace $\alpha(v_i)$ with $\mathfrak{e}(\lambda, u(v_i))$.

(C) We can argue the case where $a(v_i) = 0$, $\mathfrak{a}(v_i) = 0$, $\alpha(v_i) = 0$ and $k(v_i) = 1$ hold, by using the method in the proof of Lemma 5.2.5.

It is easy to obtain Lemma 5.4.3 by using the above considerations. □

We put $\rho := \omega - \mathbb{D}\tau$ on $\mathcal{X} \setminus \mathcal{D}$, which gives a holomorphic section of $V \otimes \Omega^{1,0}$ on $\mathcal{X} \setminus \mathcal{D}$. We have the decomposition $\rho = \sum \rho_i$, where each ρ_i is the product of v_i and a holomorphic $(1,0)$-form on $\mathcal{X} \setminus \mathcal{D}$.

Lemma 5.4.4. — *Let $\ell(v_i) \in \mathbb{Z}_{\geqslant 0}$ be determined as follows:*

- *We put $\ell(v_i) := -\operatorname{ord}(\mathfrak{a}(v_i)) + 1$ in the case $\mathfrak{a}(v_i) \neq 0$.*
- *We put $\ell(v_i) := 1$ in the case where $\mathfrak{a}(v_i) = 0$ and $u(v_i) \neq (0,0)$ hold.*
- *We put $\ell(v_i) := 0$ otherwise.*

Then, $z^{\ell(v_i)}\rho_i$ is L^2 with respect to h and $\widetilde{g}_{\mathbf{p}}$. In the second case, $(-\log|z|)^{-1}\rho_i$ is L^2, more strongly.

Proof. — We consider differentials only along the direction of X in the following argument. Let δ' denote the $(1,0)$-operator determined by h and $\overline{\partial}$. Let B be determined by $\delta' \boldsymbol{v} = \boldsymbol{v}\, B$. Then, B is diagonal, and the (i,i)-entries are as follows:

$$-\mathfrak{p}\big(\lambda, u(v_i)\big)\frac{dz}{z} + \frac{k(v_i)}{-2\log|z|}\,\frac{dz}{z}.$$

The curvature $R(h)$ of $\overline{\partial} + \delta'$ is expressed by the diagonal matrix with respect to the frame $\boldsymbol{v}$, whose (i,i)-entries are $-k(v_i)\,|z|^{-2}(-\log|z|^2)^{-2}dz\,d\overline{z}$. Hence, $\delta'\tau$ is also C^∞ and of polynomial order. (See the argument in the proof of Lemma 7.4.11.)

Let $\mathbb{D}^{(1,0)}$ denote the $(1,0)$-part of $\mathbb{D}$. We put $G := \mathbb{D}^{(1,0)} - \lambda\delta'$, which is a section of $\operatorname{End}(V) \otimes \Omega^{1,0}$. Let A_1 be determined by $G\,\boldsymbol{v} = \boldsymbol{v}\,A_1$. Then, we have the decomposition $A_1 = \Gamma' + C$, where F_C is bounded with respect to h and $\widetilde{g}_{\mathbf{p}}$, and Γ' is the diagonal matrix whose (i,i)-entries are as follows:

$$d\mathfrak{a}(v_i) + \Big(\mathfrak{e}\big(\lambda, u(v_i)\big) + \lambda\,\mathfrak{p}\big(\lambda, u(v_i)\big)\Big)\frac{dz}{z}.$$

We have the decomposition $\rho = \omega^{1,0} - \lambda\delta'\tau - \lambda F_C(\tau) - \lambda F_{\Gamma'}(\tau)$. Note that $\omega^{1,0} - \lambda\delta'\tau - \lambda F_C(\tau)$ is L^2 and of polynomial order with respect to h and $g_{\mathbf{p}}$. Then, the claim of Lemma 5.4.4 follows. □

Let Γ and A_0 be as in Section 5.4.2. We put $\mathbb{D}_0 := \mathbb{D} - F_\Gamma$. We have $\mathbb{D}_0 \boldsymbol{v} = \boldsymbol{v} A_0$. Recall F_{A_0} is bounded with respect to h and $\widetilde{g}_{\mathbf{p}}$.

- In the case $\mathfrak{a}(v_i) \neq 0$, we have a holomorphic section κ_i (i) which is L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$, (ii) such that $\rho_i = \big(d\mathfrak{a}(v_i) + \mathfrak{e}(\lambda, u(v_i))dz/z\big)\kappa_i$. Note that $\mathbb{D}_0(\kappa_i)$ is also L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$.
- If $\mathfrak{a}(v_i) = 0$ and $a(v_i) < 0$ hold, we have $(-\log|z|^2)^{-1}\rho_i$ is L^2. Then, we obtain that ρ_i is a section of $\mathcal{S}\big(V_* \otimes \Omega^{1,0}, \mathbb{D}\big)$ from the holomorphic property.
- If $\mathfrak{a}(v_i) = 0$, $a(v_i) = 0$ and $\alpha(v_i) \neq 0$ hold, we have $z\,\rho_i$ is L^2. Hence, we have the L^2-holomorphic section κ_i such that $\mathfrak{e}(\lambda, u(v_i))\kappa_i dz/z = \rho_i$. Note $\mathbb{D}_0(\kappa_i)$ is also L^2.
- If $\mathfrak{a}(v_i) = 0$, $a(v_i) = 0$ and $\alpha(v_i) = 0$ hold, we have ρ_i is contained in $\mathcal{S}\big(V_* \otimes \Omega^{1,0}, \mathbb{D}\big)$.

Hence, we obtain the following lemma.

Lemma 5.4.5. — *There exists a section ν of $\mathcal{L}^0_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\omega - \mathbb{D}\nu$ is a holomorphic $(1,0)$-form.* □

From Lemma 5.4.5, it is easy to obtain that $\mathcal{H}^1(\varphi_0)$ is an isomorphism. It is easy to show that $\mathcal{H}^0(\varphi_0)$ is an isomorphism. Thus the proof of Proposition 5.3.1 is finished. □

5.4.6. The morphism φ_1. — Let us show that φ_1 in Proposition 5.3.5 is a quasi-isomorphism. We only have to show that the induced morphism $\mathcal{S}(V_* \otimes \Omega^{\bullet,0}) \to \widetilde{\mathcal{L}}_{\mathrm{poly}}(V_*, \mathbb{D})$ is a quasi-isomorphism. We only have to show it for the germ at (λ_0, O). By an easy argument of descent with respect to the ramified covering $X' \to X$, we may and will assume that $(V_*, \mathbb{D})$ is unramified. Let $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ and $\boldsymbol{v}_{\mathfrak{a},C^\infty}$ be as in Section 5.3.3. Let $R_{\mathfrak{a}}$ be determined by $\mathbb{D}\overline{\boldsymbol{v}}_{\mathfrak{a}} = \overline{\boldsymbol{v}}_{\mathfrak{a}}\,(d\mathfrak{a} + R_{\mathfrak{a}})$. Let C be determined by the following:

$$\mathbb{D}\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\,\Big(\bigoplus(d\mathfrak{a} + R_{\mathfrak{a}}) + C\Big).$$

Then, C is λ-holomorphic, and it satisfies $C = O\big(|z|^N\big)$ for any $N > 0$. Let $\mathbb{D}'$ be the family of flat λ-connections determined by the following:

$$\mathbb{D}'\boldsymbol{v}_{C^\infty} = \boldsymbol{v}_{C^\infty}\,\Big(\bigoplus(d\mathfrak{a} + R_{\mathfrak{a}})\Big).$$

The $(0,1)$-part of $\mathbb{D}'$ is denoted by $\overline{\partial}'$. We put $F := \mathbb{D} - \mathbb{D}'$, and then we have $|F|_h = O\big(|z|^N\big)$ for any $N > 0$. We obtain a complex of sheaves $\widetilde{\mathcal{L}}^\bullet_{\mathrm{poly}}(V_*, \mathbb{D}')$ from $\mathbb{D}'$ as in Section 5.3.3. As sheaves, we have $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}\big(V_*, \mathbb{D}'\big) = \widetilde{\mathcal{L}}^p_{\mathrm{poly}}\big(V_*, \mathbb{D}\big)$ for $p = 0, 1, 2$.

Lemma 5.4.6. — *For any $\omega \in \widetilde{\mathcal{L}}^2_{\mathrm{poly}}(V_*, \mathbb{D})$, we can take $\tau \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathbb{D}'\tau = \omega$.*

Proof. — We only have to consider the case where $\mathbb{D}$ has a unique irregular value $\mathfrak{a}$. In the case $\mathfrak{a} = 0$, we may apply the results in Section 5.4.4. In the case $\mathfrak{a} \neq 0$, we need only use Lemma 5.2.3, for example. □

Lemma 5.4.7. — *For any $\omega \in \widetilde{\mathcal{L}}^2_{\mathrm{poly}}(V_*, \mathbb{D})$, we can take $\widetilde{\tau} \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathbb{D}\widetilde{\tau} = \omega$.*

In particular, we obtain the vanishing of $\mathcal{H}^2$ of $\widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*, \mathbb{D})$.

Proof. — Take τ as in Lemma 5.4.6. We have $\omega - \mathbb{D}\tau = O(|z|^N)$ for any N. Take some large M. According to Lemma 5.2.3, we can take a section κ of $V \otimes \Omega^{1,0}$ such that (i) $\overline{\partial}\kappa = \omega - \mathbb{D}\tau$, (ii) $|\kappa| = O(|z|^M)$. We have $\kappa \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D})$, and $\mathbb{D}(\tau + \kappa) = \omega$. Thus, we obtain Lemma 5.4.7 □

Let $\omega \in \widetilde{\mathcal{L}}^1_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\mathbb{D}\omega = 0$. We have $\mathbb{D}'\omega = -F\omega = O(|z|^N)$ for any N. Hence, we can take a large $M > 0$ and a λ-holomorphic C^∞-section κ of $V \otimes \Omega^{1,0}$ such that $\overline{\partial}'\kappa = \mathbb{D}'\omega$ and $|\kappa| = O(|z|^M)$. We put $\omega' := \omega - \kappa$, and then $\mathbb{D}'\omega' = 0$. Note that the $(0,1)$-parts of ω and ω' are equal.

Lemma 5.4.8. — *There exists a local section $\tau \in \widetilde{\mathcal{L}}^0(V_*, \mathbb{D})$ around (λ_0, O) such that $\overline{\partial}'\tau = \omega^{0,1}$.*

Proof. — We may assume that $\mathbb{D}$ has a unique irregular value $\mathfrak{a}$. In the case $\mathfrak{a} = 0$, we can apply the result in Section 5.4.5. Let us consider the case $\mathfrak{a} \neq 0$. We can take τ such that (i) $\overline{\partial}'\tau = \omega^{0,1}$, (ii) $|\tau| = O(|z|^{-M})$ for some large M. Let us show $\mathbb{D}'\tau$ is of polynomial order. Let h' be the C^∞-Hermitian metric of $V_{|\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}}$ such that $h'(v_{C^\infty,i}, v_{C^\infty,j}) = \delta_{i,j}$. Note that h' and h are mutually bounded up to polynomial order. Let δ'_1 be the $(1,0)$-operator determined by $\overline{\partial}'$ and h'. We consider differentials only along the direction of X. Note that $z^{M_1}\,\omega^{0,1}$ and $z^{M_1}\,\tau$ are bounded for some large M_1. We also have $\overline{\partial}'(z^{M_1}\,\tau) = z^{M_1}\,\omega^{0,1}$. Since the curvature of $R(h, \overline{\partial}')$ is 0, it can be shown that $\delta'_1(z^{M_1}\,\tau)$ is L^2 uniformly for λ. (See the proof of Lemma 7.4.11, for example.) Thus, we obtain that $z^{M_1}\,\delta'_1\tau$ is L^2 uniformly for λ. Taking large M_2, we obtain $z^{M_2}\,(\delta'_1\tau - \omega^{1,0})$ is also L^2 uniformly for λ.

Since $\omega^{1,0} - \delta'_1\tau$ is holomorphic with respect to $\overline{\partial}'$, we obtain $\delta'_1\tau - \omega^{1,0} = O(|z|^{-M_3})$ for some large M_3. Then, we obtain the desired estimate for $\delta'_1\tau$. □

Lemma 5.4.9. — *We can take a section $\widetilde{\tau} \in \widetilde{\mathcal{L}}^0_{\mathrm{poly}}(V_*, \mathbb{D})$ such that $\overline{\partial}\widetilde{\tau} = \omega^{0,1}$*

Proof. — Let τ be as in Lemma 5.4.8. We have $\mathbb{D}\tau - \mathbb{D}'\tau = F\tau = O(|z|^N)$ for any $N > 0$. Because of Lemma 5.2.3, we can take a section ν of V such that (i) $|\nu| = O(|z|^M)$ for some large $M > 0$, (ii) $\overline{\partial}\nu = F^{0,1}\,\tau$. Let δ' be the $(1,0)$-operator determined by h and $\overline{\partial}$. We consider the differentials only in the direction of X. Since the curvature $R(h,\overline{\partial})$ is uniformly bounded with respect to h and $g_{\mathbf{p}}$, it can be shown that $\delta'\nu$ is L^2 uniformly for λ. If M is sufficiently large, $(\mathbb{D}^{(1,0)}-\lambda\delta')\nu$ is $O(|z|^{M/2})$. We put $\rho=\sum\rho_{\mathfrak{a}}:=\omega-\kappa-\mathbb{D}\tau+\mathbb{D}\nu$. Then, we obtain that (i) ρ is a holomorphic section of $V\otimes\Omega^{1,0}$, (ii) $z^{M_1}\,\rho_{\mathfrak{a}}$ ($\mathfrak{a}\in\mathrm{Irr}(\mathbb{D})$) are L^2 for some large M_1 with respect to $\widetilde{g}_{\mathbf{p}}$ and h uniformly for λ, (iii) ρ_0 is L^2 with respect to h and $\widetilde{g}_{\mathbf{p}}$ uniformly for λ. Hence, we obtain $\nu\in\widetilde{\mathcal{L}}^0_{\mathrm{poly}}(V_*,\mathbb{D})$. Thus, Lemma 5.4.9 is proved. □

Let $\widetilde{\mathcal{S}}(V_*\otimes\Omega^{p,0})$ be the sheaf of meromorphic sections τ of $V\otimes\Omega^{p,0}$ with the following property:

- Let $\tau_{|\widehat{\mathcal{D}}}=\widehat{\tau}_{\mathrm{reg}}+\widehat{\tau}_{\mathrm{irr}}$ be the decomposition corresponding to the irregular decomposition. Then, $\widehat{\tau}_{\mathrm{reg}}$ is contained in $\mathcal{S}(\widehat{V}_{\mathrm{reg}\,*}\otimes\Omega^{p,0})$.

By using Lemma 5.4.7 and Lemma 5.4.9, it is easy to show that the natural inclusion $\widetilde{\mathcal{S}}(V_*\otimes\Omega^{\bullet,0})\to\widetilde{\mathcal{L}}_{\mathrm{poly}}(V_*,\mathbb{D})$ is a quasi-isomorphism. It is also standard and easy to show that the natural inclusion $\mathcal{S}(V_*\otimes\Omega^{\bullet,0})\to\widetilde{\mathcal{S}}(V_*\otimes\Omega^{\bullet,0})$ is a quasi-isomorphism. Hence, we obtain that φ_1 is a quasi-isomorphism. Thus, a half of Proposition 5.3.5 is finished.

5.4.7. Proof of Propositions 5.3.5 and 5.3.8. — Let $\pi:\widetilde{\mathcal{X}}(\mathcal{D})\to\mathcal{X}$ denote the projection. For any open subset $\mathcal{U}$ of $\widetilde{\mathcal{X}}(\mathcal{D})$, let $\widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}(\mathcal{U})$, denote the space of λ-holomorphic C^∞-sections τ of $V\otimes\Omega^p$ on $\mathcal{U}\setminus\pi^{-1}(\mathcal{D})$ such that the conditions **(a1)** and **(a2)** are satisfied. By taking sheafification, we obtain a complex of sheaves $\widetilde{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ on $\widetilde{\mathcal{X}}(\mathcal{D})$. Similarly, we obtain a complex of sheaves $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ and $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V,h,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ on the real blow up $\widetilde{\mathcal{X}}(\mathcal{D})$, corresponding to the sheaves $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V_*,\mathbb{D})$, $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V,h^{(T)})$ and $\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}(V,h,h^{(T)})$.

Let S be a small sector in $\mathcal{X}\setminus\mathcal{D}$ such that we have the full Stokes filtration $\widetilde{\mathcal{F}}^S$ of $V_{|\overline{S}}$. Let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(\mathcal{D})$. We can take a flat splitting $V_{|S}=\bigoplus V_{\mathfrak{a},S}$. For a λ-holomorphic C^∞-section τ of $V\otimes\Omega^p$ on S, we have the decomposition $\tau=\sum\tau_{\mathfrak{a},S}$ corresponding to $V_{|S}=\bigoplus V_{\mathfrak{a},S}$.

Lemma 5.4.10. — *τ is a section of $\widetilde{\mathcal{L}}^p_{\mathrm{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ on $\overline{S}$, if and only if the following estimate holds locally on $\overline{S}$ with respect to h and $\widetilde{g}_{\mathbf{p}}$:*

(a1') : *$\tau_{\mathfrak{a},S}$ and $\mathbb{D}\tau_{\mathfrak{a},S}$ are of polynomial order for $\mathfrak{a}\neq 0$.*

(a2') : *$\tau_{0,S}$ and $\mathbb{D}\tau_{0,S}$ are L^2 and of polynomial order.*

τ is a section of $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ on $\overline{S}$, if and only if the following estimate holds locally on $\overline{S}$ with respect to h and $\widetilde{g}_{\mathbf{p}}$:

(b1') : *$\tau_{\mathfrak{a},S}$ and $\mathbb{D}\tau_{\mathfrak{a},S}$ are $O(|z|^N)$ for any $N>0$.*

(a2') : *$\tau_{0,S}$ and $\mathbb{D}\tau_{0,S}$ are L^2 and of polynomial order.*

τ is a section of $\overline{\mathcal{L}}^p_{\mathrm{poly}}(V,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, if and only if the following holds locally on $\overline{S}$:

(c1') : *$\tau_{\mathfrak{a},S}$ and $\mathbb{D}\tau_{\mathfrak{a},S}$ are $O(|z|^N)$ for any $N>0$ with respect to $h^{(T)}$ and $\widetilde{g}_{\mathbf{p}}$.*

(a2') : *$\tau_{0,S}$ and $\mathbb{D}\tau_{0,S}$ are L^2 and of polynomial order with respect to h and $\widetilde{g}_{\mathbf{p}}$.*

τ is a section of $\overline{\mathcal{L}}^p_{\text{poly}}(V,h,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, if and only if the following holds locally on $\overline{S}$:

(d1') : *$\tau_{\mathfrak{a},S}$ and $\mathbb{D}\tau_{\mathfrak{a},S}$ are $O(|z|^N)$ for any $N>0$ with respect to both $(h^{(T)},\widetilde{g}_{\mathbf{p}})$ and $(h,\widetilde{g}_{\mathbf{p}})$.*

(a2') : *$\tau_{0,S}$ and $\mathbb{D}\tau_{0,S}$ are L^2 and of polynomial order with respect to $(h,\widetilde{g}_{\mathbf{p}})$.*

Proof. — It follows from Lemma 5.3.3. □

Lemma 5.4.11. — *The following natural morphisms are quasi-isomorphisms.*

$$\widetilde{\mathcal{L}}^{\bullet}_{\text{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})} \longleftarrow \overline{\mathcal{L}}^{\bullet}_{\text{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})} \longleftarrow \overline{\mathcal{L}}^{\bullet}_{\text{poly}}(V,h,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})} \longrightarrow \overline{\mathcal{L}}^{\bullet}_{\text{poly}}(V,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}.$$

Proof. — We only have to consider the case where $\mathbb{D}$ has a unique irregular value $\mathfrak{a}$. If $\mathfrak{a}=0$, the sheaves are the same. Let us consider the case $\mathfrak{a}\neq 0$. Let $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ be as in Section 5.3.3. We have the lift $\boldsymbol{v}_{\mathfrak{a},S}$ of $\overline{\boldsymbol{v}}_{\mathfrak{a}}$ to $V_{\mathfrak{a},S}$. We may take a $\mathbb{D}$-flat frame $\boldsymbol{u}_{\mathfrak{a},S}$ of $V_{\mathfrak{a},S}$. Let $G_{\mathfrak{a}}$ be determined by $\boldsymbol{u}_{\mathfrak{a},S}=\boldsymbol{v}_{\mathfrak{a},S}\,G_{\mathfrak{a}}$. Then, $G_{\mathfrak{a}}$ and $G_{\mathfrak{a}}^{-1}$ are bounded up to polynomial order, uniformly for λ (See Lemma 20.3.3, for example.) Then, we can show that the vanishing of the higher cohomology sheaves of $\widetilde{\mathcal{L}}^p_{\text{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, $\overline{\mathcal{L}}^p_{\text{poly}}(V_*,\mathbb{D})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, $\overline{\mathcal{L}}^p_{\text{poly}}(V,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$ and $\overline{\mathcal{L}}^p_{\text{poly}}(V,h,h^{(T)})_{\widetilde{\mathcal{X}}(\mathcal{D})}$, by using the results in Section 20.2.2. The comparison of the 0-th cohomology sheaves are easy. Thus, we obtain Lemma 5.4.11. □

By applying the push-forward to the quasi-isomorphisms in Lemma 5.4.11, we obtain the rest of Proposition 5.3.5 and Proposition 5.3.8. □

CHAPTER 6

MEROMORPHIC VARIATION OF TWISTOR STRUCTURE

One of the main results in this monograph is the reduction from unramifiedly good wild harmonic bundle to tame harmonic bundle (Theorem 11.2.2). It is convenient to prepare the procedure for reduction with respect to Stokes filtrations in a more general situation. That is the main purpose in Section 6.2. We introduce the notion of meromorphic prolongment of a variation of twistor structure with a symmetric pairing (Definition 6.2.6), and we explain the procedure to take Gr with respect to Stokes structure in Section 6.2.5.

In Section 6.1, we give a review on the notion of variation of polarized pure twistor structure due to Simpson [**85**] (see also [**73**], [**65**] and [**67**]).

6.1. Variation of polarized pure twistor structure

We recall the notion of twistor structure introduced by Simpson in [**85**], in a way convenient for our purpose. See also [**35**], [**73**], [**65**] and [**67**].

6.1.1. Some sheaves and differential operators on $\mathbb{P}^1 \times X$. — Let $\mathbb{P}^1$ denote a one dimensional complex projective space. We regard it as the gluing of two complex lines $\boldsymbol{C}_\lambda$ and $\boldsymbol{C}_\mu$ by $\lambda = \mu^{-1}$. We set $\boldsymbol{C}_\lambda^* := \boldsymbol{C}_\lambda \setminus \{0\}$.

Let X be a complex manifold. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{X}^0 := \{0\} \times X$. Let $\widetilde{\Omega}^{1,0}_{\mathcal{X}}$ be the C^∞-bundle associated to $\Omega^{1,0}_{\mathcal{X}}(\log \mathcal{X}^0) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{X}^0)$. We put $\widetilde{\Omega}^{0,1}_{\mathcal{X}} := \Omega^{0,1}_{\mathcal{X}}$, and we define

$$\widetilde{\Omega}^1_{\mathcal{X}} := \widetilde{\Omega}^{1,0}_{\mathcal{X}} \oplus \widetilde{\Omega}^{0,1}_{\mathcal{X}}, \quad \widetilde{\Omega}^\bullet_{\mathcal{X}} := \bigwedge{}^\bullet \widetilde{\Omega}^1_{\mathcal{X}}.$$

The associated sheaves of C^∞-sections are denoted by the same symbols. Let $\widetilde{\mathbb{D}}^f_X : \widetilde{\Omega}^\bullet_{\mathcal{X}} \to \widetilde{\Omega}^{\bullet+1}_{\mathcal{X}}$ denote the differential operator induced by the exterior differential d of $\mathcal{X}$.

Let $X^\dagger$ denote the conjugate of X. We set $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$. By the same procedure, we obtain the C^∞-bundles $\widetilde{\Omega}^\bullet_{\mathcal{X}^\dagger}$ with the differential operator $\widetilde{\mathbb{D}}^{\dagger f}_X$.

Their restrictions to $\boldsymbol{C}_\lambda^* \times X = \boldsymbol{C}_\mu^* \times X^\dagger$ are naturally isomorphic:

$$\big(\widetilde{\Omega}^\bullet_{\mathcal{X}}, \widetilde{\mathbb{D}}^f_X\big)_{|\boldsymbol{C}_\lambda^*\times X} = \big(\Omega^\bullet_{\boldsymbol{C}_\lambda^*\times X}, d\big) = \big(\widetilde{\Omega}^\bullet_{\mathcal{X}^\dagger}, \widetilde{\mathbb{D}}^{\dagger f}_X\big)_{|\boldsymbol{C}_\mu^*\times X^\dagger}$$

By gluing them, we obtain a graded C^∞-bundle $\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}$ with a differential operator $\widetilde{\mathbb{D}}^\triangle_X$.

Remark 6.1.1. — $\widetilde{\mathbb{D}}^f_X$ and $\widetilde{\mathbb{D}}^{\dagger f}_X$ are denoted also by d, if there is no risk of confusion. □

We have the decomposition $\widetilde{\Omega}^1_{\mathbb{P}^1\times X} = \xi\Omega^1_X \oplus \widetilde{\Omega}^1_{\mathbb{P}^1}$ into the X-direction and the $\mathbb{P}^1$-direction. The restriction of $\widetilde{\mathbb{D}}^\triangle_X$ to the X-direction is denoted by $\mathbb{D}^\triangle_X$. The restriction to the $\mathbb{P}^1$-direction is denoted by $d_{\mathbb{P}^1}$. We have the decomposition

$$\widetilde{\Omega}^1_{\mathbb{P}^1} = \pi^*\Omega^{1,0}_{\mathbb{P}^1}(2\cdot\{0,\infty\}) \oplus \pi^*\Omega^{0,1}_{\mathbb{P}^1},$$

into the $(1,0)$-part and the $(0,1)$-part, where π denotes the projection $\mathbb{P}^1\times X \to \mathbb{P}^1$. We have the corresponding decomposition $d_{\mathbb{P}^1} = \partial_{\mathbb{P}^1} + \overline{\partial}_{\mathbb{P}^1}$.

Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be the anti-holomorphic involution given by $\sigma([z_0 : z_1]) = [-\overline{z}_1 : \overline{z}_0]$. The induced diffeomorphism $\mathbb{P}^1\times X \to \mathbb{P}^1\times X$ is also denoted by σ. The multiplication on $\sigma^*\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}$ is twisted as $g\cdot\sigma^*(\omega) = \sigma^*\big(\overline{\sigma^*(g)}\cdot\omega\big)$ for a function g and a section ω of $\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}$. Then, we have the C^∞-isomorphism $\Phi_\sigma : \sigma^*\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X} \simeq \widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}$ given by the complex conjugate and the ordinary pull-back

$$\Phi_\sigma(\sigma^*\omega) = \overline{\sigma^*(\omega)}.$$

It is easy to check that $\Phi_\sigma\circ\sigma^*(\widetilde{\mathbb{D}}^\triangle_X) = \widetilde{\mathbb{D}}^\triangle_X\circ\Phi_\sigma$. Similar relations hold for $\mathbb{D}^\triangle_X$ and $d_{\mathbb{P}^1}$. If we are given an additional bundle $\mathcal{F}$, the induced isomorphism $\mathcal{F}\otimes\sigma^*\big(\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}\big) \simeq \mathcal{F}\otimes\widetilde{\Omega}^\bullet_{\mathbb{P}^1\times X}$ is also denoted by Φ_σ.

6.1.2. Definitions

6.1.2.1. Variation of twistor structure (variation of $\mathbb{P}^1$-holomorphic bundle)

Let V be a C^∞-vector bundle on $\mathbb{P}^1\times X$. We use the same symbol to denote the associated sheaf of C^∞-sections. A $\mathbb{P}^1$-holomorphic structure of V is defined to be a differential operator

$$d''_{\mathbb{P}^1,V} : V \longrightarrow V\otimes\pi^*\Omega^{0,1}_{\mathbb{P}^1}$$

satisfying (i) $d''_{\mathbb{P}^1,V}(f\cdot s) = f\cdot d''_{\mathbb{P}^1,V}(s) + \overline{\partial}_{\mathbb{P}^1}(f)\cdot s$ for a C^∞-function f and a section s of V, (ii) $d''_{\mathbb{P}^1,V}\circ d''_{\mathbb{P}^1,V} = 0$. Such a tuple $(V, d''_{\mathbb{P}^1,V})$ is called a $\mathbb{P}^1$-holomorphic vector bundle.

A $T\widetilde{T}$-structure of $(V, d''_{\mathbb{P}^1,V})$ is a differential operator

$$\mathbb{D}^\triangle_V : V \longrightarrow V\otimes\xi\Omega^1_X$$

such that (i) $\mathbb{D}^\triangle_V(f\cdot s) = f\cdot\mathbb{D}^\triangle_V(s) + \mathbb{D}^\triangle_X(f)\cdot s$ for a C^∞-function f and a section s of V, (ii) $(d''_{\mathbb{P}^1,V} + \mathbb{D}^\triangle_V)^2 = 0$. Such a tuple $(V, d''_{\mathbb{P}^1,V}, \mathbb{D}^\triangle_V)$ is called *a $T\widetilde{T}$-structure* in [**35**], or *a variation of* $\mathbb{P}^1$*-holomorphic vector bundle* in [**67**]. In this section, we prefer to call it *variation of twistor structure.* We will not distinguish them.

If X is a point, it is just a holomorphic vector bundle on $\mathbb{P}^1$.

Remark 6.1.2. — We will often omit to specify $d''_{\mathbb{P}^1,V}$ when we consider $\mathbb{P}^1$-holomorphic bundles or variations of twistor structure (variations of $\mathbb{P}^1$-holomorphic bundle). □

A morphism of variation of twistor structure

$$F:(V_1,d''_{\mathbb{P}^1,V_1},\mathbb{D}^{\triangle}_{V_1})\longrightarrow(V_2,d''_{\mathbb{P}^1,V_2},\mathbb{D}^{\triangle}_{V_2})$$

is defined to be a morphism of the associated sheaves of C^∞-sections, compatible with the differential operators. If X is a point, it is equivalent to an $\mathcal{O}_{\mathbb{P}^1}$-morphism.

6.1.2.2. Some functoriality. — Let $(V,\mathbb{D}^{\triangle}_V)$ be a variation of twistor structure. Let $f:Y\to X$ be a holomorphic map of complex manifolds. Then, we have the naturally induced variation of twistor structure $f^*(V,\mathbb{D}^{\triangle}_V)$ as in the case of ordinary connections.

Let $\sigma:\mathbb{P}^1\to\mathbb{P}^1$ be as above. Then, σ^*V is naturally equipped with a $\mathbb{P}^1$-holomorphic structure and a $T\widetilde{T}$-structure $\mathbb{D}^{\triangle}_{\sigma^*V}$ given as follows:

$$\big(\mathbb{D}^{\triangle}_{\sigma^*V}+d''_{\sigma^*V}\big)\big(\Phi_\sigma(\sigma^*s)\big)=\Phi_\sigma\Big(\sigma^*\big((\mathbb{D}^{\triangle}_V+d''_V)s\big)\Big).$$

Here, s denotes a section of $V\otimes\xi\Omega^\bullet_X$. Thus, we obtain the pull-back of variation of twistor structure by σ.

Direct sum, tensor product, and dual for variation of twistor structure are defined in obvious manners.

6.1.2.3. Variation of pure twistor structure. — Let $(V,d''_{\mathbb{P}^1,V})$ be a $\mathbb{P}^1$-holomorphic vector bundle on $\mathbb{P}^1\times X$. It is called pure of weight w if the restrictions $V_P:=(V,d''_{\mathbb{P}^1,V})_{|\mathbb{P}^1\times\{P\}}$ are pure twistor structure of weight w for any $P\in X$, i.e., V_P are isomorphic to direct sums of $\mathcal{O}_{\mathbb{P}^1}(w)$. A variation of twistor structure is called pure of weight w, if the underlying $\mathbb{P}^1$-holomorphic vector bundle is pure of weight w.

6.1.2.4. Example (Tate objects). — Let $\mathbb{T}(w)$ be a Tate object in the theory of twistor structure. (See [**85**] and Section 3.3.1 of [**67**].) It is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2w)$, and equipped with the distinguished frames

$$\mathbb{T}(w)_{|\boldsymbol{C}_\lambda}=\mathcal{O}_{\boldsymbol{C}_\lambda}\,t_0^{(w)},\quad \mathbb{T}(w)_{|\boldsymbol{C}_\mu}=\mathcal{O}_{\boldsymbol{C}_\mu}\,t_\infty^{(w)},\quad \mathbb{T}(w)_{|\boldsymbol{C}^*_\lambda}=\mathcal{O}_{\boldsymbol{C}^*_\lambda}\,t_1^{(w)}.$$

The transformation is given by

$$t_0^{(w)}=(\sqrt{-1}\lambda)^w\,t_1^{(w)},\quad t_\infty^{(w)}=(-\sqrt{-1}\mu)^w\,t_1^{(w)}.$$

In particular, $(\sqrt{-1}\lambda)^{-2w}t_0^{(w)}=t_\infty^{(w)}$.

We may identify $\mathbb{T}(w)$ with $\mathcal{O}_{\mathbb{P}^1}\big(-w\cdot 0-w\cdot\infty\big)$ by the correspondence $t_1^{(w)}\longleftrightarrow 1$, up to constant multiplication. In particular, we will implicitly use the identification of $\mathbb{T}(0)$ with $\mathcal{O}_{\mathbb{P}^1}$ by $t_1^{(0)}\longleftrightarrow 1$. We will also implicitly use the identification $\mathbb{T}(m)\otimes\mathbb{T}(n)\simeq\mathbb{T}(m+n)$ given by $t_a^{(m)}\otimes t_a^{(n)}\longleftrightarrow t_a^{(m+n)}$.

6.1.2.5. Example. — In Section 3.3.2 of [**67**], we considered a line bundle $\mathcal{O}(p,q)$ on $\mathbb{P}^1$ with a natural $\boldsymbol{C}^*$, which is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(p+q)$ and equipped with the distinguished frames:

$$\mathcal{O}(p,q)_{|\boldsymbol{C}_\lambda} = \mathcal{O}_{\boldsymbol{C}_\lambda}\, f_0^{(p,q)}, \quad \mathcal{O}(p,q)_{|\boldsymbol{C}_\mu} = \mathcal{O}_{\boldsymbol{C}_\mu}\, f_\infty^{(p,q)}, \quad \mathcal{O}(p,q)_{|\boldsymbol{C}_\lambda^*} = \mathcal{O}_{\boldsymbol{C}_\lambda^*}\, f_1^{(p,q)}.$$

The transformation is given by

$$f_0^{(p,q)} = (\sqrt{-1}\lambda)^{-p}\, f_1^{(p,q)}, \quad f_\infty^{(p,q)} = (-\sqrt{-1}\mu)^{-q}\, f_1^{(p,q)}.$$

In particular, $(\sqrt{-1}\lambda)^{p+q} f_0^{(p,q)} = f_\infty^{(p,q)}$.

We may identify $\mathcal{O}(p,q)$ with $\mathcal{O}_{\mathbb{P}^1}(p\cdot 0 + q\cdot\infty)$ by the correspondence $f_1^{(p,q)} \longleftrightarrow 1$, up to constant multiplication. We will implicitly use the identification $\mathcal{O}(p,q)\otimes\mathcal{O}(p',q') \simeq \mathcal{O}(p+p',q+q')$ given by $f_a^{(p,q)}\otimes f_a^{(p',q')} \longleftrightarrow f_a^{(p+p',q+q')}$. We will also implicitly identify $\mathbb{T}(w)$ with $\mathcal{O}(-w,-w)$ by $t_a^{(w)} = f_a^{(-w,-w)}$ for $a = 0, 1, \infty$.

If we forget the natural $\boldsymbol{C}^*$-actions, $\mathcal{O}(p,q)$ and $\mathcal{O}(p+r,q-r)$ are identified by $f_\kappa^{(p,q)} \longleftrightarrow f_\kappa^{(p+r,q-r)}$ for $\kappa = 0,\infty$. In that case, $f_\kappa^{(p,q)}$ are denoted by $f_\kappa^{(p+q)}$.

Let X be a complex manifold. We have the pull-back of $\mathbb{T}(w)$ and $\mathcal{O}(p,q)$ via the map from X to a point. They are denoted by $\mathbb{T}(w)_X$ and $\mathcal{O}(p,q)_X$, respectively. We will often omit the subscript X, if there is no risk of confusion. For a variation of twistor structure $(V,\mathbb{D}^\triangle)$, the tensor product $(V,\mathbb{D}^\triangle)\otimes\mathbb{T}(w)$ is called the w-th Tate twist of $(V,\mathbb{D}^\triangle)$.

6.1.2.6. Polarization. — Recall that we have the isomorphism ([**67**])

$$\iota_{\mathbb{T}(w)} : \sigma^*\mathbb{T}(w) \simeq \mathbb{T}(w),$$

given by the natural identification $\sigma^*\mathcal{O}(-w\cdot 0 - w\cdot\infty) \simeq \mathcal{O}(-w\cdot 0 - w\cdot\infty)$ via $\sigma^*(1) \longleftrightarrow 1$, or equivalently,

$$\sigma^* t_1^{(w)} \longleftrightarrow t_1^{(w)}, \quad \sigma^* t_0^{(w)} \longleftrightarrow (-1)^w\, t_0^{(w)}, \quad \sigma^* t_\infty^{(w)} \longleftrightarrow (-1)^w\, t_\infty^{(w)}.$$

For a variation of twistor structure $(V,\mathbb{D}_V^\triangle)$ on $\mathbb{P}^1\times X$, a morphism

$$\mathcal{S} : (V,\mathbb{D}_V^\triangle)\otimes\sigma^*(V,\mathbb{D}_V^\triangle) \longrightarrow \mathbb{T}(-w)_X$$

is called a pairing of weight w, if it is $(-1)^w$-symmetric in the following sense:

$$\iota_{\mathbb{T}(-w)}\circ\sigma^*\mathcal{S} = (-1)^w\mathcal{S}\circ\text{exchange} : \sigma^*V\otimes V \longrightarrow \mathbb{T}(-w)_X.$$

Here, exchange denotes the natural morphism $\sigma^*V\otimes V \to V\otimes\sigma^*V$ induced by the exchange of the components. It is also called $(-1)^w$-symmetric pairing, if we would like to emphasize $(-1)^w$-symmetric property.

Let $(V,\mathbb{D}_V^\triangle)$ be a variation of *pure* twistor structure of weight w on $\mathbb{P}^1\times X$. Let $\mathcal{S} : (V,\mathbb{D}_V^\triangle)\otimes\sigma^*(V,\mathbb{D}_V^\triangle) \to \mathbb{T}(-w)_X$ be a pairing of weight w. We say that $\mathcal{S}$ is a polarization of $(V,\mathbb{D}_V^\triangle)$, if $\mathcal{S}_P := \mathcal{S}_{|\mathbb{P}^1\times\{P\}}$ is a polarization of $V_P := (V, d''_{\mathbb{P}^1})_{|\mathbb{P}^1\times\{P\}}$ for each $P\in X$. Namely, the following holds:

- If $w=0$, the induced Hermitian pairing $H^0(\mathcal{S}_P)$ of $H^0(\mathbb{P}^1, V_P)$ is positive definite.

- In the general case, the induced pairing $\mathcal{S}_P \otimes \mathcal{S}_{0,-w}$ of $V_P \otimes \mathcal{O}(0,-w)$ is a polarization of the pure twistor structure. (See Example 2 below for $\mathcal{S}_{0,-w}$.)

When S is a polarization of a pure twistor structure V of weight n, the induced pairing $\sigma(S) : \sigma^*(V) \otimes V \to \mathbb{T}(-n)$ and $S^\vee : V^\vee \otimes \sigma^*(V^\vee) \to \mathbb{T}(n)$ are also polarizations. (See Lemma 3.38 of [**67**].)

6.1.2.7. Example 1. — The identification $\iota_{\mathbb{T}(w)}$ induces a flat morphism $\mathcal{S}_{\mathbb{T}(w)}$: $\mathbb{T}(w) \otimes \sigma^*\mathbb{T}(w) \to \mathbb{T}(2w)$. It is a polarization of $\mathbb{T}(w)$ of weight $-2w$.

6.1.2.8. Example 2. — The flat isomorphism $\iota_{(p,q)} : \sigma^*\mathcal{O}(p,q) \simeq \mathcal{O}(q,p)$ in [**67**] is given by $\sigma^* f_0^{(p,q)} \mapsto (\sqrt{-1})^{p+q} f_\infty^{(q,p)}$, $\sigma^* f_\infty^{(p,q)} \mapsto (-\sqrt{-1})^{p+q} f_0^{(q,p)}$, and $\sigma^* f_1^{(p,q)} \mapsto (\sqrt{-1})^{q-p} f_1^{(q,p)}$. Hence, we obtain the morphism

$$\mathcal{S}_{p,q} : \mathcal{O}(p,q) \otimes \sigma^*\mathcal{O}(p,q) \longrightarrow \mathbb{T}(-p-q).$$

It is a polarization of weight $p+q$.

Remark 6.1.3. — It is essential to fix an isomorphism $\iota : \sigma^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ such that $\sigma^*\iota \circ \iota = -1$. It is unique up to conjugacy. There could be a choice of a frame to reduce signatures. □

6.1.2.9. Relation with harmonic bundles. — Simpson observed the equivalence between the notions of variation of polarized pure twistor structure and harmonic bundles. (See [**85**]. See also [**67**] and [**73**].) Let $p : \mathbb{P}^1 \times X \to X$ be the projection. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on X. We set $\mathcal{E}^\triangle := p^*E$, which is naturally a $\mathbb{P}^1$-holomorphic bundle. It is equipped with the differential operator $\mathbb{D}^\triangle := \overline{\partial}_E + \lambda\theta^\dagger + \partial_E + \lambda^{-1}\theta$, and $(\mathcal{E}^\triangle, \mathbb{D}^\triangle)$ is a variation of pure twistor structure of weight 0. The polarization $\mathcal{S}$ is given by $\mathcal{S}(u \otimes \sigma^* v) := p^*(h)(u, \sigma^* v)$.

6.1.3. Gluing construction. — Recall the gluing construction of variation of pure twistor structure in [**85**]. See also [**67**]. We have the decomposition $\widetilde{\Omega}^1_{\mathcal{X}} = \xi\widetilde{\Omega}^1_{X|\mathcal{X}} \oplus \widetilde{\Omega}^1_{\boldsymbol{C}_\lambda}$ into the X-direction and the $\boldsymbol{C}_\lambda$-direction. Let d_X denote the restriction of the exterior differential to the X-direction. Similarly, we have the decomposition $\widetilde{\Omega}^1_{\mathcal{X}^\dagger} = \xi\widetilde{\Omega}^1_{X|\mathcal{X}^\dagger} \oplus \widetilde{\Omega}^1_{\boldsymbol{C}_\mu}$, and the restriction of $\widetilde{\mathbb{D}}^{\dagger f}_X$ to the X-direction is denoted by $d_{X^\dagger}$. The notions of $\boldsymbol{C}_\lambda$-holomorphic bundles or $\boldsymbol{C}_\mu$-holomorphic bundles are defined as in the case of $\mathbb{P}^1$-holomorphic bundles.

Remark 6.1.4. — d_X and $d_{X^\dagger}$ are also denoted just by d, if there is no risk of confusion. □

Let $(V_0, d''_{\boldsymbol{C}_\lambda, V_0})$ be a $\boldsymbol{C}_\lambda$-holomorphic bundle on $\mathcal{X}$. A T-structure [**35**] of V_0 is a differential operator

$$\mathbb{D}^f_{V_0} : V_0 \longrightarrow V_0 \otimes \xi\Omega^1_{X|\mathcal{X}}$$

satisfying (i) $\mathbb{D}^f_{V_0}(f \cdot s) = d_X f \cdot s + f \cdot \mathbb{D}^f_{V_0}(s)$ for a function f and a section s of V, (ii) $\big(d''_{\boldsymbol{C}_\lambda, V_0} + \mathbb{D}^f_{V_0}\big)^2 = 0$.

Let $(V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty})$ be a $\boldsymbol{C}_\mu$-holomorphic vector bundle on $\mathcal{X}^\dagger$. A $\widetilde{T}$-structure [**35**] is defined to be a differential operator

$$\mathbb{D}^{\dagger f}_{V_\infty} : V_\infty \longrightarrow V_\infty \otimes \xi\Omega^1_{X|\mathcal{X}^\dagger}$$

satisfying conditions similar to (i) and (ii) above.

Assume that we are given an isomorphism Φ:

$$\Phi : (V_0, d''_{\boldsymbol{C}_\lambda, V_0}, \mathbb{D}^f_{V_0})_{|\boldsymbol{C}^*_\lambda \times X} \simeq (V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty}, \mathbb{D}^{\dagger f}_{V_\infty})_{|\boldsymbol{C}^*_\mu \times X^\dagger}. \tag{90}$$

We obtain a C^∞-vector bundle V on $\mathbb{P}^1 \times X$ by gluing V_0 and V_∞ via Φ. By the condition (90), $d''_{\boldsymbol{C}_\lambda, V_0}$ and $d''_{\boldsymbol{C}_\mu, V_\infty}$ give $\mathbb{P}^1$-holomorphic structure $d''_{\mathbb{P}^1, V}$, and $\mathbb{D}^f_{V_0}$ and $\mathbb{D}^{\dagger f}_{V_\infty}$ induce the $T\widetilde{T}$-structure $\mathbb{D}^\triangle_V$. Thus, we obtain a variation of twistor structure $(V, d''_{\mathbb{P}^1, V}, \mathbb{D}^\triangle_V)$.

Conversely, we naturally obtain such $(V_0, d''_{\boldsymbol{C}_\lambda, V_0}, \mathbb{D}^f_{V_0})$, $(V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty}, \mathbb{D}^{\dagger f}_{V_\infty})$ and Φ from a variation of twistor structure $(V, d''_{\mathbb{P}^1, V}, \mathbb{D}^\triangle_V)$ as the restriction to $\mathcal{X}$ and $\mathcal{X}^\dagger$, respectively.

Under the natural isomorphism

$$\xi\Omega^1_{X|\mathcal{X}} = \lambda^{-1} \cdot \Omega^{1,0}_{\mathcal{X}/\boldsymbol{C}_\lambda} \oplus \Omega^{0,1}_{\mathcal{X}/\boldsymbol{C}_\lambda} \simeq \Omega^{1,0}_{\mathcal{X}/\boldsymbol{C}_\lambda} \oplus \Omega^{0,1}_{\mathcal{X}/\boldsymbol{C}_\lambda} = \Omega^1_{\mathcal{X}/\boldsymbol{C}_\lambda}$$

a T-structure $\mathbb{D}^f_{V_0}$ induces a holomorphic family of flat λ-connections $\mathbb{D}_{V_0}$. Similarly, a $\widetilde{T}$-structure of $\mathbb{D}^{\dagger f}_{V_\infty}$ naturally induces a holomorphic family of flat μ-connections $\mathbb{D}^\dagger_{V_\infty}$. Hence, a variation of twistor structure is regarded as the gluing of families of λ-flat bundles and μ-flat bundles.

6.1.3.1. Let $(V, \mathbb{D}^\triangle_V)$ be a variation of twistor structure on $\mathbb{P}^1 \times (X \setminus D)$. Let $\mathbb{D}_{\sigma^* V_\infty}$ (resp. $\mathbb{D}^\dagger_{\sigma^* V_0}$) denote the associated family of flat λ-connections (resp. μ-connections) on $\sigma^* V_\infty$ (resp. $\sigma^* V_0$). The following lemma can be checked by an easy and direct calculation. We remark the signature.

Lemma 6.1.5. — *Let f be a local section of V_∞, and let A_i and B_i $(i = 1, \ldots, n)$ be determined by $\mathbb{D}^\dagger f = \sum A_i \cdot dz_i + \sum B_i \cdot d\overline{z}_i$, where A_i and B_i are local sections of V_∞. Then, we have*

$$\mathbb{D}_{\sigma^* V_\infty}(\sigma^* f) = \sum \sigma^*(A_i) \cdot d\overline{z}_i - \sum \sigma^*(B_i) \cdot dz_i.$$

Similarly, we have

$$\mathbb{D}^\dagger_{\sigma^* V_0}(\sigma^* g) = -\sum \sigma^*(A_i) \cdot d\overline{z}_i + \sum \sigma^*(B_i) \cdot dz_i$$

for a local section g of V_0 with $\mathbb{D} g = \sum A_i \cdot dz_i + \sum B_i \cdot d\overline{z}_i$. □

6.2. Good meromorphic prolongment of variation of twistor structure

We shall introduce the notion of unramifiedly good meromorphic prolongment for variation of twistor structure. We shall also observe that a graded variation of twistor structure is obtained as the graduation with respect to Stokes structure.

6.2.1. Unramifiedly good meromorphic prolongment.

— Let X be a complex manifold with a normal crossing hypersurface D. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{X}^\sharp := \boldsymbol{C}_\lambda^* \times X$. Let p_λ be the projection forgetting the λ-component. For any subset $\mathcal{K}$ of $\boldsymbol{C}_\lambda$, we put $\mathcal{X}_\mathcal{K} := \mathcal{K} \times X$. We use the symbols $\mathcal{D}$, $\mathcal{D}^\sharp$, and $\mathcal{D}_\mathcal{K}$ with similar meanings.

Definition 6.2.1. — Let $(V_0, \mathbb{D})$ be a family of λ-flat bundles on $\mathcal{X} \setminus \mathcal{D}$. A family of meromorphic λ-flat bundles $(\widetilde{V}_0, \mathbb{D})$ on $(\mathcal{X}_\mathcal{K}, \mathcal{D}_\mathcal{K})$ is called an unramifiedly good meromorphic prolongment of $(V_0, \mathbb{D})$, if the following holds:

- The restriction of $(\widetilde{V}_0, \mathbb{D})$ to $\mathcal{X}_\mathcal{K} \setminus \mathcal{D}_\mathcal{K}$ is $(V_0, \mathbb{D})_{|\mathcal{X}_\mathcal{K} \setminus \mathcal{D}_\mathcal{K}}$.
- $(\widetilde{V}_0, \mathbb{D})$ locally has an unramifiedly good lattice, i.e., for each $P \in \mathcal{D}_\mathcal{K}$, there exists a small neighbourhood $\mathcal{X}_P$ such that $(\widetilde{V}_0, \mathbb{D})_{|\mathcal{X}_P}$ has an unramifiedly good lattice.
- $\mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, P) \subset \mathcal{O}_X(*D)_{p_\lambda(P)} / \mathcal{O}_{X, p_\lambda(P)}$, i.e., the elements of $\mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, P)$ are independent of the variable λ. □

Under the third condition, we have $\mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, P) = \mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, P')$ for any $p_\lambda(P) = p_\lambda(P')$, if $\mathcal{K}$ is connected. In that case, for $R \in D$, we take $P \in \mathcal{D}$ such that $p_\lambda(P) = R$, and put $\mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, R) := \mathrm{Irr}(\widetilde{V}_0, \mathbb{D}, P)$. They will be denoted also by $\mathrm{Irr}(\widetilde{V}_0, R)$ or $\mathrm{Irr}(\mathbb{D}, R)$.

If we are interested only on good family of meromorphic λ-flat bundles, it is too strong to impose the independence from λ for irregular values. However, it seems appropriate to impose it when we consider meromorphic prolongment of a variation of twistor structure. (See Subsection 6.2.2. In this case, we should have comparison of the Stokes structures of $(\widetilde{V}_0, \mathbb{D}_{V_0})$ and $(\widetilde{V}_\infty, \mathbb{D}^\dagger_{\widetilde{V}_\infty})$. It seems to lead us to the property that the irregular values are independent of λ and μ.) Indeed, we will study meromorphic prolongment of variation of twistor structure associated to unramifiedly good wild harmonic bundles in Chapter 11, and their irregular values are independent of λ and μ.

6.2.2. Meromorphic prolongment of a variation of twistor structure

Let $X^\dagger$ denote the conjugate of X. We put $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$ and $\mathcal{X}^{\dagger\sharp} := \boldsymbol{C}_\mu^* \times X^\dagger$. Let p_μ denote the projection forgetting the μ-component. For any subset $\mathcal{H}$ of $\boldsymbol{C}_\mu$, we put $\mathcal{X}^\dagger_\mathcal{H} := \mathcal{H} \times X^\dagger$. We use the symbols $D^\dagger$, $\mathcal{D}^\dagger$, $\mathcal{D}^{\dagger\sharp}$, and $\mathcal{D}^\dagger_\mathcal{H}$ with similar meanings. We use the C^∞-identification $\mathcal{X}^\sharp = \mathcal{X}^{\dagger\sharp}$ given by $\lambda = \mu^{-1}$, which preserves the $\boldsymbol{C}_\lambda^*$-holomorphic structure.

Let $(V, \mathbb{D}^{\triangle})$ be a variation of twistor structure on $(X \setminus D) \times \mathbb{P}^1$. We have the associated family of λ-flat bundles $(V_0, \mathbb{D})$ on $\mathcal{X} \setminus \mathcal{D}$, and the associated family of μ-flat bundles $(V_\infty, \mathbb{D}^\dagger)$ on $\mathcal{X}^\dagger \setminus \mathcal{D}^\dagger$. Let $\mathbb{D}^f$ and $\mathbb{D}^{\dagger f}$ denote the associated families of flat connections. We have the isomorphism $V_{0|\mathcal{X}^\sharp \setminus \mathcal{D}^\sharp} \simeq V_{\infty|\mathcal{X}^{\dagger\sharp} \setminus \mathcal{D}^{\dagger\sharp}}$ preserving the families of the flat connections and the holomorphic structures along the $\boldsymbol{C}^*_\lambda$-direction.

Let $\mathcal{K} \subset \boldsymbol{C}_\lambda$ and $\mathcal{H} \subset \boldsymbol{C}_\mu$ be connected compact regions such that the union of the interior points of $\mathcal{K}$ and $\mathcal{H}$ is $\mathbb{P}^1$. Assume that we are given the following:

- An unramifiedly good meromorphic prolongment $(\widetilde{V}_0, \mathbb{D})$ on $(\mathcal{X}_\mathcal{K}, \mathcal{D}_\mathcal{K})$ of $(V_0, \mathbb{D})$.
- An unramifiedly good meromorphic prolongment $(\widetilde{V}_\infty, \mathbb{D}^\dagger)$ on $(\mathcal{X}^\dagger_\mathcal{H}, \mathcal{D}^\dagger_\mathcal{H})$ of $(V_\infty, \mathbb{D}^\dagger)$.
- For any $R \in D = D^\dagger$, the sets $\operatorname{Irr}(\widetilde{V}_\infty, \mathbb{D}^\dagger, R)$ and $\operatorname{Irr}(\widetilde{V}_0, \mathbb{D}, R)$ are related as

$$\operatorname{Irr}(\widetilde{V}_\infty, \mathbb{D}^\dagger, R) = \big\{\overline{\mathfrak{a}} \,\big|\, \mathfrak{a} \in \operatorname{Irr}(\widetilde{V}_0, \mathbb{D}, R)\big\}.$$

Let $\pi : \widetilde{\mathcal{X}}^\sharp(\mathcal{D}^\sharp) \to \mathcal{X}^\sharp$ denote the real blow up of $\mathcal{X}^\sharp$ along $\mathcal{D}^\sharp$. We have the $\mathcal{O}_{\boldsymbol{C}^*_\lambda}$-module $\mathfrak{V}_0$ on $\widetilde{\mathcal{X}}^\sharp(\mathcal{D}^\sharp)$ associated to $(V_0, \mathbb{D})_{|\mathcal{X}^\sharp \setminus \mathcal{D}^\sharp}$ as in Subsection 4.1.3. For any $Q \in \pi^{-1}(\mathcal{D}^\sharp_\mathcal{K})$, we have the full Stokes filtration $\widetilde{\mathcal{F}}^Q(\mathfrak{V}_{0,Q})$ of the stalk of $\mathfrak{V}_0$ at Q for the meromorphic extension $\widetilde{V}_0$. Similarly, let $\pi : \widetilde{\mathcal{X}}^{\dagger\sharp}(\mathcal{D}^{\dagger\sharp}) \to \mathcal{X}^\dagger$ denote the real blow up of $\mathcal{X}^{\dagger\sharp}$ along $\mathcal{D}^{\dagger\sharp}$, and let $\mathfrak{V}_\infty$ denote the sheaf on $\widetilde{\mathcal{X}}^{\dagger\sharp}(\mathcal{D}^{\dagger\sharp})$ associated to V_∞. For any point $Q \in \pi^{-1}(\mathcal{D}^{\dagger\sharp}_\mathcal{H})$, we have the full Stokes filtration $\widetilde{\mathcal{F}}^Q(\mathfrak{V}_{\infty,Q})$.

Note the natural identifications $\widetilde{\mathcal{X}}^\sharp(\mathcal{D}^\sharp) = \widetilde{\mathcal{X}}^{\dagger\sharp}(\mathcal{D}^{\dagger\sharp})$ and $\mathfrak{V}_0 = \mathfrak{V}_\infty$, and hence $\mathfrak{V}_{0,P} = \mathfrak{V}_{\infty,P}$ for any $P \in \pi^{-1}(\mathcal{D}_\mathcal{K} \cap \mathcal{D}^\dagger_\mathcal{H})$. We also remark

$$\operatorname{Re}(\mu^{-1}\overline{\mathfrak{a}}) = |\mu|^{-2} \operatorname{Re}(\lambda^{-1}\mathfrak{a})$$

for $\lambda = \mu^{-1}$. Hence, the natural bijection $\operatorname{Irr}(\widetilde{V}_0, P) \to \operatorname{Irr}(\widetilde{V}_\infty, P)$ induces an isomorphism of ordered sets $(\operatorname{Irr}(\widetilde{V}_0, P), \leqslant^\lambda_Q)$ and $(\operatorname{Irr}(\widetilde{V}_\infty, P), \leqslant^\mu_Q)$ for any $Q \in \pi^{-1}(P)$.

Definition 6.2.2

- We say that the Stokes structure of $\widetilde{V}_0$ and $\widetilde{V}_\infty$ are the same, if the filtrations $\widetilde{\mathcal{F}}^Q(\mathfrak{V}_{0,Q})$ and $\widetilde{\mathcal{F}}^Q(\mathfrak{V}_{\infty,Q})$ are the same for any $Q \in \pi^{-1}(\mathcal{D}_\mathcal{K} \cap \mathcal{D}^\dagger_\mathcal{H})$, under the above identification of the index sets.
- If the Stokes structures of $\widetilde{V}_0$ and $\widetilde{V}_\infty$ are the same, $(\widetilde{V}_0, \widetilde{V}_\infty)$ is called an unramifiedly good meromorphic prolongment of the variation of twistor structure $(V, \mathbb{D}^\triangle)$. □

6.2.3. Meromorphic prolongment of the conjugate. — Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be the anti-holomorphic involution given by $\sigma([z_0 : z_\infty]) = [-\overline{z}_\infty : \overline{z}_0]$. If we regard $\mathbb{P}^1 = \boldsymbol{C}_\lambda \cup \boldsymbol{C}_\mu$ by $\lambda = z_0/z_\infty$ and $\mu = z_\infty/z_0$, we have the induced map $\sigma : \boldsymbol{C}_\lambda \to \boldsymbol{C}_\mu$ given by $\sigma(\lambda) = -\overline{\lambda}$. We have the naturally induced maps such as $\mathcal{X} \to \mathcal{X}^\dagger$ and $\mathcal{X}^\sharp \to \mathcal{X}^{\dagger\sharp}$, which are also denoted by σ.

Let $(V, \mathbb{D}^\triangle_V)$ be a variation of twistor structure on $\mathbb{P}^1 \times (X \setminus D)$. Let $\mathbb{D}_{\sigma^* V_\infty}$ (resp. $\mathbb{D}^\dagger_{\sigma^* V_0}$) denote the associated family of flat λ-connections (resp. μ-connections) on $\sigma^* V_\infty$ (resp. $\sigma^* V_0$). Let $(\widetilde{V}_\infty, \mathbb{D}^\dagger)$ be an unramifiedly good meromorphic prolongment of $(V_\infty, \mathbb{D}^\dagger)$ on $(\mathcal{X}^\dagger_{\sigma(\mathcal{K})}, \mathcal{D}^\dagger_{\sigma(\mathcal{K})})$. Let $\sigma^*\widetilde{V}_\infty$ be the sheaf on $\mathcal{X}_\mathcal{K}$, given by $\sigma^*(\widetilde{V}_\infty)(\mathcal{U}) := \widetilde{V}_\infty(\sigma(\mathcal{U}))$ for any open subset $\mathcal{U}$ of $\mathcal{X}_\mathcal{K}$. We have the natural $\mathcal{O}_\mathcal{X}(*\mathcal{D})$-module structure on $\sigma^*(\widetilde{V}_\infty)(\mathcal{U})$ given by $f \cdot \sigma^*(s) := \sigma^*\big(\overline{\sigma^*(f)} \cdot s\big)$. The family of λ-flat connections $\mathbb{D}_{\sigma^* V_\infty}$ naturally gives a family of meromorphic flat λ-connections on $\sigma^*\widetilde{V}_\infty$. Although the following lemma is clear, we remark the signature (Lemma 6.1.5).

Lemma 6.2.3

- *Let $(\sigma^*\widetilde{V}_\infty, \mathbb{D}_{\sigma^* V_\infty})$ be as above. Then, it gives an unramifiedly good meromorphic prolongment of $(\sigma^* V_\infty, \mathbb{D}_{\sigma^* V_\infty})$. For each $R \in D$, we have*
$$\mathrm{Irr}(\sigma^* V_\infty, R) = \big\{-\overline{\mathfrak{a}} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(V_\infty, R)\big\}.$$
- *Similarly, let $(\widetilde{V}_0, \mathbb{D})$ be an unramifiedly good meromorphic prolongment of $(V_0, \mathbb{D})$. Then, $\big(\sigma^*\widetilde{V}_0, \mathbb{D}^\dagger_{\sigma^* V_0}\big)$ on $(\mathcal{X}^\dagger, \mathcal{D}^\dagger)$ gives an unramifiedly good meromorphic prolongment of $(\sigma^* V_0, \mathbb{D}^\dagger_{\sigma^* V_0})$. For each $R \in D$, we have*
$$\mathrm{Irr}(\sigma^* V_0, R) = \big\{-\overline{\mathfrak{a}} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(V_0, R)\big\}.$$
- *If $(\widetilde{V}_0, \widetilde{V}_\infty)$ is an unramifiedly good meromorphic prolongment of $(V, \mathbb{D}^\triangle)$, then $(\sigma^*\widetilde{V}_0, \sigma^*\widetilde{V}_\infty)$ is an unramifiedly good meromorphic prolongment of $\sigma^*(V, \mathbb{D}^\triangle)$.* □

6.2.4. Meromorphic prolongment of the pairing

Let $\mathcal{S} : (V, \mathbb{D}^\triangle) \otimes \sigma^*(V, \mathbb{D}^\triangle) \to \mathbb{T}(0)$ be a pairing of variation of twistor structure of weight 0. It consists of morphisms $\mathcal{S}_0 : V_0 \otimes \sigma^* V_\infty \to \mathcal{O}_{\mathcal{X} \setminus \mathcal{D}}$ and $\mathcal{S}_\infty : V_\infty \otimes \sigma^* V_0 \to \mathcal{O}_{\mathcal{X}^\dagger \setminus \mathcal{D}^\dagger}$ which are compatible with (i) the families of λ-connections or μ-connections, (ii) the gluing on $\mathcal{X} \setminus W = \mathcal{X}^\dagger \setminus W^\dagger$.

Let $(\widetilde{V}_0, \widetilde{V}_\infty)$ be an unramifiedly good meromorphic prolongment of $(V, \mathbb{D})$. For simplicity, we assume that $\widetilde{V}_0$ and $\widetilde{V}_\infty$ are given on $\mathcal{X}_\mathcal{K}$ and $\mathcal{X}^\dagger_{\sigma(\mathcal{K})}$.

Definition 6.2.4. — A pairing $\widetilde{\mathcal{S}}_0 : \widetilde{V}_0 \otimes \sigma^*\widetilde{V}_\infty \to \mathcal{O}_{\mathcal{X}_\mathcal{K}}(*\mathcal{D}_\mathcal{K})$ is called a meromorphic prolongment of $\mathcal{S}_0$ if $\widetilde{\mathcal{S}}_{0|\mathcal{X}_\mathcal{K} \setminus \mathcal{D}_\mathcal{K}} = \mathcal{S}_{0|\mathcal{X}_\mathcal{K} \setminus \mathcal{D}_\mathcal{K}}$. Similarly, a pairing $\widetilde{\mathcal{S}}_\infty : \widetilde{V}_\infty \otimes \sigma^*\widetilde{V}_0 \to \mathcal{O}_{\mathcal{X}^\dagger_{\sigma(\mathcal{K})}}(*\mathcal{D}^\dagger_{\sigma(\mathcal{K})})$ is called a meromorphic prolongment of $\mathcal{S}_\infty$, if $\widetilde{\mathcal{S}}_{\infty|\mathcal{X}^\dagger_{\sigma(\mathcal{K})} \setminus \mathcal{D}^\dagger_{\sigma(\mathcal{K})}} = \mathcal{S}_{\infty|\mathcal{X}^\dagger_{\sigma(\mathcal{K})} \setminus \mathcal{D}^\dagger_{\sigma(\mathcal{K})}}$ □

The following lemma is clear.

Lemma 6.2.5. — *Let $\mathcal{S} : V \otimes \sigma^* V \to \mathbb{T}(0)$ be a pairing of weight* 0.

- *A meromorphic prolongment of $\mathcal{S}_0$ is unique, if it exists. Similarly, a meromorphic prolongment of $\mathcal{S}_\infty$ is unique, if it exists.*
- *$\mathcal{S}_0$ has a meromorphic prolongment, if and only if $\mathcal{S}_\infty$ has a meromorphic prolongment.*

- *$\mathcal{S}_0$ has a meromorphic prolongment, if and only if the induced morphism $V^\vee_{0|\mathcal{X}_\mathcal{K}\setminus\mathcal{D}_\mathcal{K}} \simeq \sigma^* V_{\infty|\mathcal{X}^\dagger_{\sigma(\mathcal{K})}\setminus\mathcal{D}^\dagger_{\sigma(\mathcal{K})}}$ can be extended to $\widetilde{V}_0^\vee \simeq \sigma^*(\widetilde{V}_\infty)$.* □

Definition 6.2.6. — Let $(V, \mathbb{D}^\triangle)$ be a variation of twistor structure with a pairing $\mathcal{S}$ of weight 0. Let $(\widetilde{V}_0, \widetilde{V}_\infty)$ be an unramifiedly good meromorphic prolongment of $(V, \mathbb{D}^\triangle)$. We say that $(\widetilde{V}_0, \widetilde{V}_\infty)$ is an unramifiedly good meromorphic prolongment of $(V, \mathbb{D}^\triangle, \mathcal{S})$, if $\mathcal{S}_0$ has the meromorphic prolongment. □

6.2.5. Reduction of meromorphic variation of twistor structure. — Let $(V, \mathbb{D}^\triangle)$ be a variation of twistor structure on $\mathbb{P}^1 \times (X \setminus D)$ with a pairing $\mathcal{S}$ of weight 0. Let $(\widetilde{V}_0, \widetilde{V}_\infty)$ be an unramifiedly good meromorphic prolongment of $(V, \mathbb{D}^\triangle, \mathcal{S})$. For simplicity, we assume that $\widetilde{V}_0$ and $\widetilde{V}_\infty$ are given on $\mathcal{X}_\mathcal{K}$ and $\mathcal{X}^\dagger_{\sigma(\mathcal{K})}$, respectively.

6.2.5.1. Full reduction. — Let $R \in D$. By taking Gr with respect to full Stokes filtrations on a small neighbourhood $\mathcal{X}_R$ of $p_\lambda^{-1}(R) \cap \mathcal{D}_\mathcal{K}$, we obtain a graded meromorphic family of λ-flat bundles

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\widetilde{V}_{0,R}) = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\widetilde{V}_0,R)} \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\widetilde{V}_{0,R})$$

on $\mathcal{X}_R$. Similarly, we obtain a graded meromorphic family of μ-flat bundles on a small neighbourhood $\mathcal{X}^\dagger_R$ of $p_\mu^{-1}(R) \cap \mathcal{D}^\dagger_{\sigma(\mathcal{K})}$:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\widetilde{V}_{\infty,R}) = \bigoplus_{\mathfrak{b}\in \mathrm{Irr}(\widetilde{V}_\infty,R)} \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{b}}(\widetilde{V}_{\infty,R}).$$

We may assume that $\mathcal{X}_R$ and $\mathcal{X}^\dagger_R$ are of the form $\mathcal{K} \times X_R$ and $\sigma(\mathcal{K}) \times X^\dagger_R$. We set $D_R := X_R \cap D$. Because of the coincidence of the Stokes filtrations, we have a natural isomorphism for each $\mathfrak{a} \in \mathrm{Irr}(\widetilde{V}_0, R)$:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\widetilde{V}_0)_{|(\mathcal{K}\cap\sigma^*(\mathcal{K}))\times(X_R\setminus D_R)} \simeq \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\widetilde{V}_\infty)_{|(\mathcal{K}\cap\sigma^*(\mathcal{K}))\times(X_R\setminus D_R)}.$$

By gluing, we obtain a variation of twistor structure on $\mathbb{P}^1 \times (X_R \setminus D_R)$, which is denoted by $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V_R, \mathbb{D}^\triangle_R)$. It is equipped with an induced unramifiedly good meromorphic prolongment $\big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\widetilde{V}_{0,R}), \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\widetilde{V}_{\infty,R})\big)$.

From the isomorphism $\widetilde{V}_0^\vee \simeq \sigma^*(\widetilde{V}_\infty)$ on $\mathcal{X}_\mathcal{K}$ induced by $\mathcal{S}$, we obtain an isomorphism $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{-\mathfrak{a}}(\widetilde{V}^\vee_{0,R}) \simeq \sigma^* \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\widetilde{V}_{\infty,R})$ on $\mathcal{X}_{R,\mathcal{K}}$. Namely, we have an induced pairing:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{S}_{0,R}) : \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\widetilde{V}_{0,R}) \otimes \sigma^* \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\widetilde{V}_{\infty,R}) \longrightarrow \mathcal{O}_{\mathcal{X}_{R,\mathcal{K}}}(*\mathcal{D}_{R,\mathcal{K}}).$$

Similarly, we have an induced pairing:

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\mathcal{S}_{\infty,R}) : \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\overline{\mathfrak{a}}}(\widetilde{V}_{\infty,R}) \otimes \sigma^* \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\widetilde{V}_{0,R}) \longrightarrow \mathcal{O}_{\mathcal{X}^\dagger_{R,\mathcal{K}}}(*\mathcal{D}^\dagger_{R,\mathcal{K}}).$$

It is easy to observe that their restrictions to $(\mathcal{K} \cap \sigma(\mathcal{K})) \times (X_R \setminus D_R)$ are the same. Hence, we have an induced symmetric pairing $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{S}_R)$ of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V_R, \mathbb{D}^\triangle_R)$ equipped with a meromorphic prolongment. The tuple is denoted by $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V,\mathbb{D}^{\triangle},\mathcal{S})$. We obtain

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}(V,\mathbb{D}^{\triangle},\mathcal{S}) := \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\widetilde{V}_0,R)} \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(V,\mathbb{D}^{\triangle},\mathcal{S}).$$

It is called the full reduction of $(V,\mathbb{D}^{\triangle},\mathcal{S})$, and it is equipped with an unramifiedly good meromorphic prolongment $\big(\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\widetilde{V}_{0,R}),\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\widetilde{V}_{\infty,R})\big)$.

6.2.5.2. Refinement. — Let us consider the case $X=\Delta^n$, $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$ and $R\in\bigcap_{i=1}^{\ell}\{z_i=0\}$. We take an auxiliary sequence $\boldsymbol{m}(0),\dots,\boldsymbol{m}(L),\boldsymbol{m}(L+1)=\boldsymbol{0}$ for the good set $\mathcal{I}:=\mathrm{Irr}(\widetilde{V}_0,P)$. As in the case of full reduction, we obtain graded variation of twistor structure

$$\mathrm{Gr}^{\boldsymbol{m}(p)}(V_R,\mathbb{D}^{\triangle}_R,\mathcal{S}) = \bigoplus_{\mathfrak{b}\in\overline{\eta}_{\boldsymbol{m}(p)}(\mathcal{I})} \mathrm{Gr}^{\boldsymbol{m}(p)}_{\mathfrak{b}}(V_R,\mathbb{D}^{\triangle}_R,\mathcal{S}).$$

It is naturally equipped with an unramifiedly good meromorphic prolongment $\big(\mathrm{Gr}^{\boldsymbol{m}(p)}(\widetilde{V}_{0,R}),\mathrm{Gr}^{\boldsymbol{m}(p)}(\widetilde{V}_{\infty,R})\big)$. In particular, $\big(\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_{0,R}),\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_{\infty,R})\big)$ is called the one step reduction.

Remark 6.2.7. — Let us consider the case $X:=\Delta^n$ and $D:=\{z_1=0\}$. Assume that we are given an unramifiedly good meromorphic prolongment $(\widetilde{V}_0,\widetilde{V}_\infty)$ of $(V,\mathbb{D}^{\triangle})$ on $(\mathcal{X},\mathcal{X}^{\dagger})$. If D is smooth, the reductions $\mathrm{Gr}^{(j)}(V,\mathbb{D}^{\triangle},\mathcal{S})$ are equipped with unramifiedly good meromorphic prolongments $\big(\mathrm{Gr}^{(j)}(\widetilde{V}_0),\mathrm{Gr}^{(j)}(\widetilde{V}_\infty)\big)$ given on $(\mathcal{X}',\mathcal{X}'^{\dagger})$. (Namely, we do not have to consider the restriction to a compact region in $\boldsymbol{C}_\lambda$, in this case.) □

6.2.5.3. Compatibility. — We give a remark on compatibility. We assume that the coordinate system is admissible for a good set $\mathcal{I}$ (Remark 2.1.4). Let k be determined by $\boldsymbol{m}(0)\in\mathbb{Z}^k_{<0}\times\boldsymbol{0}_{\ell-k}$.

Take $1\leqslant j\leqslant k$ and $R_1\in D_j\cap X_R$, which is not a singular point of $D(\underline{k})$. Let X_{R_1} be a small neighbourhood of R_1 in X. We set $X^*_{R_1}:=X_{R_1}\smallsetminus D$. We put

$$(V_1,\mathbb{D}^{\triangle}_1,\mathcal{S}_1):=(V,\mathbb{D}^{\triangle},\mathcal{S})_{|\mathbb{P}^1\times X^*_Q},\quad (V_2,\mathbb{D}^{\triangle}_2,\mathcal{S}_2):=\mathrm{Gr}^{\boldsymbol{m}(0)}(V_R,\mathbb{D}^{\triangle}_R,\mathcal{S})_{|\mathbb{P}^1\times X^*_Q}.$$

They are equipped with the induced unramifiedly meromorphic prolongments.

Lemma 6.2.8. — *We have a natural isomorphism:*

$$\mathrm{Gr}^{\widetilde{\mathcal{F}}}\big(V_1,\mathbb{D}^{\triangle}_1,\mathcal{S}_1\big)\simeq\mathrm{Gr}^{\widetilde{\mathcal{F}}}\big(V_2,\mathbb{D}^{\triangle}_2,\mathcal{S}_2\big).$$

Proof. — It follows from Corollary 3.7.14. □

PART II

PROLONGATION OF WILD HARMONIC BUNDLE

CHAPTER 7

PROLONGMENTS $\mathcal{P}\mathcal{E}^{\lambda}$ FOR UNRAMIFIEDLY GOOD WILD HARMONIC BUNDLES

We start to study wild harmonic bundle. In Section 7.1, we state the definition of wild harmonic bundles and some related conditions for Higgs fields.

In Section 7.2, we state some estimates related with the Higgs field of an unramifiedly good wild harmonic bundle (the wild version of Simpson's main estimate), which will be proved in Section 7.3. These estimates are the most foundational.

In Section 7.4, we consider the sheaves of holomorphic sections whose norms are of polynomial orders, and we show that they form a good filtered λ-flat bundle (Theorem 7.4.3). We also obtain a characterization of the Stokes filtrations in terms of the growth order of the norms of flat sections (Proposition 7.4.4).

In Section 7.5, we study the comparison of the irregular decompositions for $(\mathcal{P}\mathcal{E}^0, \mathbb{D}^0)$ and $(\mathcal{P}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$. We note that the family version will be studied in Section 9.4. In the proof, we give an estimate for the connection form of the unitary connection associated to $(\mathcal{E}^{\lambda}, h)$ (Lemma 7.5.5). It will also be useful for other purposes.

We would like to compare the deformations caused by variation of irregular values (Section 4.5.2) and by modification of the Hermitian metrics. It will be achieved in Proposition 9.2.1. We make a preparation in Section 7.6.

In Section 7.7, we give a criterion for a holomorphic section to be bounded with respect to h.

7.1. Definition of wild harmonic bundles

7.1.1. Local condition for Higgs fields. — Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X \setminus D$, where X is a complex manifold, and D is a normal crossing divisor of X. We would like to state some conditions for the Higgs field θ. First, let us consider the case $X = \Delta^n = \{\boldsymbol{z} = (z_1, \dots, z_n) \,\big|\, |z_i| < 1\}$, $D_i = \{z_i = 0\}$ and $D = \bigcup_{i=1}^{\ell} D_i$. In that case, we have the expression:

$$\theta = \sum_{j=1}^{\ell} F_j \frac{dz_j}{z_j} + \sum_{j=\ell+1}^{n} G_j\, dz_j.$$

We have the characteristic polynomials

$$\det\big(T - F_j(\boldsymbol{z})\big) = \sum A_{j,k}(\boldsymbol{z})\, T^k, \quad \det\big(T - G_j(\boldsymbol{z})\big) = \sum B_{j,k}(\boldsymbol{z})\, T^k.$$

The coefficients $A_{j,k}$ and $B_{j,k}$ are holomorphic on $X \setminus D$.

Definition 7.1.1. — We say that θ is tame, if $A_{j,k}$ and $B_{j,k}$ are holomorphic on X for any k, and moreover, if the restriction of $A_{j,k}$ to D_j are constant for any $j = 1, \dots, \ell$ and any k. □

The condition is independent of the choice of a coordinate system.

Remark 7.1.2. — If θ comes from a tame harmonic bundle, θ is tame in the above sense. We do not have to assume that the $A_{j,k|D_j}$ are constant for the definition of tame harmonic bundle. It is automatically satisfied. □

Let $\mathcal{A}$ be a $\boldsymbol{Q}$-vector subspace of $\boldsymbol{C}$.

Definition 7.1.3. — We say that θ is $\mathcal{A}$-tame, if θ is tame and the roots of the polynomials $\det(T - F_j(\boldsymbol{z}))_{|D_j}$ are contained in $\mathcal{A}$, for any j. □

Definition 7.1.4

- We say that θ is strongly unramifiedly ($\mathcal{A}$-)good on (X, D), if we have a good set of irregular values $\mathrm{Irr}(\theta) \subset M(X,D)/H(X)$ and a decomposition

(91) $$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}),$$

  such that $\theta_{\mathfrak{a}} - d\mathfrak{a}\, \mathrm{id}_{E_{\mathfrak{a}}}$ are ($\mathcal{A}$-)tame.
- We say that θ is strongly ($\mathcal{A}$-)good on (X, D), if $\varphi_e^*(\theta)$ is unramifiedly good for some $e \in \mathbb{Z}_{>0}$, where φ_e is the covering given by $\varphi_e(z_1, \dots, z_n) = (z_1^e, \dots, z_\ell^e, z_{\ell+1}, \dots, z_n)$. □

The condition is independent of the choice of a coordinate system. The adjective "strongly" means the existence of the global decomposition (91). (Compare it with Definition 2.3.1. See also Definition 7.1.5 below.) But, we will often omit "strongly", if there is no risk of confusion.

7.1.2. Global condition for Higgs fields. — Definition 7.1.4 can be globalized easily.

Definition 7.1.5. — Let X be a general complex manifold. Let D be a normal crossing hypersurface of X, and let (E, θ) be a Higgs bundle on $X \setminus D$.

- θ is called (unramifiedly, $\mathcal{A}$-)good at $P \in D$, if it is strongly (unramifiedly, $\mathcal{A}$-)good on a coordinate neighbourhood of P.
- θ is called (unramifiedly, $\mathcal{A}$-)good on (X, D), if it is (unramifiedly, $\mathcal{A}$-)good at each point $P \in D$. □

We also introduce the more general condition.

Definition 7.1.6. — Let X be a complex analytic space, and let Z be a closed analytic subset of X such that $X \setminus Z$ is smooth. Let (E, θ) be a Higgs bundle on $X \setminus Z$. The Higgs field θ is called ($\mathcal{A}$-)wild on (X, Z), if there exists a complex manifold X' with a projective birational map $\varphi : X' \to X$ such that (i) $\varphi^{-1}(Z)$ is normal crossing, (ii) $\varphi^{-1}\theta$ is ($\mathcal{A}$-)good on $(X', \varphi^{-1}(Z))$. □

7.1.3. Condition for harmonic bundles. — We introduce the notion of (unramifiedly) good wild harmonic bundles, which is one of the main subjects in this monograph.

Definition 7.1.7. — Let X be a complex manifold, and let D be a normal crossing hypersurface of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X \setminus D$.
- It is called $\mathcal{A}$-tame on (X, D), if θ is $\mathcal{A}$-tame on (X, D).
- It is called (unramifiedly, $\mathcal{A}$-)good wild harmonic bundle on (X, D), if θ is (unramifiedly, $\mathcal{A}$-)good on (X, D). □

In the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$, we will often implicitly assume that θ is *strongly* (unramifiedly $\mathcal{A}$-)good. We will often say that $(E, \overline{\partial}_E, \theta, h)$ is a (unramifiedly $\mathcal{A}$-)good wild harmonic bundle on $X \setminus D$ instead of (X, D). In our previous paper [**67**], a $\sqrt{-1}\boldsymbol{R}$-tame harmonic bundle is called a tame pure imaginary harmonic bundle.

We introduce a more general notion.

Definition 7.1.8. — Let Z be a closed analytic subset of X. A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X \setminus Z$ is called ($\mathcal{A}$-)wild on (X, Z) if θ is ($\mathcal{A}$-)wild on (X, Z). (We will also say that $(E, \overline{\partial}_E, \theta, h)$ is ($\mathcal{A}$-)wild on $(X, X \setminus Z)$, instead of (X, Z).) □

Analysis will be mainly done for (unramifiedly) good wild harmonic bundles. In the curve case, any wild harmonic bundle is good. We remark that even if $(E, \overline{\partial}_E, \theta, h)$ is a wild harmonic bundle on (X, D), where D is normal crossing, it is not necessarily good. In Chapter 15, we will study when a harmonic bundle is wild.

7.2. Simpson's main estimate and acceptability of the associated bundles

7.2.1. Setting. — Let $X = \Delta^n$, $D_i = \{z_i = 0\}$ and $D = \bigcup_{i=1}^{\ell} D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on (X, D). By shrinking X, we assume to have the irregular decomposition:

$$(E,\theta)=\bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\theta)}(E_{\mathfrak{a}},\theta_{\mathfrak{a}}). \tag{92}$$

Here, $\theta_{\mathfrak{a}}-d\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}$ are tame. We have the expression:

$$\theta=\sum_{j=1}^{n}f_j\,dz_j.$$

We put $f_j^{\mathrm{reg}}:=f_j-\sum\partial_j\mathfrak{a}\,\pi_{\mathfrak{a}}$ for $j=1,\dots,n$, where $\pi_{\mathfrak{a}}$ denotes the projection onto $E_{\mathfrak{a}}$. By the assumption, $\det\big(T-z_jf_j^{\mathrm{reg}}\big)_{|D_j}$ $(j=1,\dots,\ell)$ are polynomials in a formal variable T with constant coefficients. Let $\mathcal{S}p(\theta,j)\subset \boldsymbol{C}$ denote the set of the solutions of $\det\big(T-z_jf_j^{\mathrm{reg}}\big)_{|D_j}=0$. For each $j=1,\dots,\ell$, we assume to have the decomposition

$$(E,f_j^{\mathrm{reg}})=\bigoplus_{\alpha\in\mathcal{S}p(\theta,j)}(E_{j,\alpha},f_{j,\alpha}^{\mathrm{reg}}) \tag{93}$$

such that the eigenvalues β of $(z_j\,f_{j,\alpha}^{\mathrm{reg}})_{|Q}$ satisfies $|\beta-\alpha|\leqslant C_0\,|z_j(Q)|^{\varepsilon_0}$ for some $C_0,\varepsilon_0>0$.

We put $X(R):=\big\{(z_1,\dots,z_n)\in X\,\big|\,|z_i|<R\big\}$ for $R<1$ and $X^*(R):=(X\setminus D)\cap X(R)$.

7.2.2. Main estimate for the irregular part. — We take an auxiliary sequence $\mathcal{M}:=\big(\boldsymbol{m}(0),\boldsymbol{m}(1),\boldsymbol{m}(2),\dots,\boldsymbol{m}(L)\big)\subset\mathbb{Z}^{\ell}_{\leqslant 0}\setminus\{\boldsymbol{0}_{\ell}\}$ for the good set of the irregular values $\mathrm{Irr}(\theta)$, i.e., (i) $\boldsymbol{m}(0)=\min\{\mathrm{ord}(\mathfrak{a})\,|\,\mathfrak{a}\in\mathrm{Irr}(\theta)\}$, (ii) we have $\mathfrak{h}(i)$ such that $\boldsymbol{m}(i+1)=\boldsymbol{m}(i)+\boldsymbol{\delta}_{\mathfrak{h}(i)}$ for each $i\leqslant L-1$ and $\boldsymbol{m}(L)+\boldsymbol{\delta}_{\mathfrak{h}(L)}=\boldsymbol{0}_{\ell}$, (iii) $\mathcal{T}(\theta):=\big\{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})\,|\,\mathfrak{a},\mathfrak{b}\in\mathrm{Irr}(\theta)\big\}\subset\mathcal{M}$.

Let $\overline{\eta}_{\boldsymbol{m}}:\mathrm{Irr}(\theta)\to M(X,D)$ be given as in (11). Let $\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})$ denote the image of $\mathrm{Irr}(\theta)$ via $\overline{\eta}_{\boldsymbol{m}}$ for $\boldsymbol{m}\in\mathcal{M}$. We may and will fix auxiliary total orders $\leqslant$ on the finite sets $\mathrm{Irr}(\theta)$ and $\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})$ for any $\boldsymbol{m}\in\mathcal{M}$ such that $\overline{\eta}_{\boldsymbol{m}}$ are order preserving.

For each $\boldsymbol{m}\in\mathcal{M}$ and $\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})$, we define

$$E^{\boldsymbol{m}}_{\mathfrak{b}}:=\bigoplus_{\overline{\eta}_{\boldsymbol{m}}(\mathfrak{a})=\mathfrak{b}}E_{\mathfrak{a}},\quad F^{\boldsymbol{m}}_{\mathfrak{b}}(E):=\bigoplus_{\substack{\mathfrak{c}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})\\ \mathfrak{c}\leqslant\mathfrak{b}}}E^{\boldsymbol{m}}_{\mathfrak{c}},\quad F^{\boldsymbol{m}}_{<\mathfrak{b}}(E):=\bigoplus_{\substack{\mathfrak{c}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})\\ \mathfrak{c}<\mathfrak{b}}}E^{\boldsymbol{m}}_{\mathfrak{c}}.$$

Let $E^{\boldsymbol{m}\prime}_{\mathfrak{b}}$ denote the orthogonal complement of $F^{\boldsymbol{m}}_{<\mathfrak{b}}(E)$ in $F^{\boldsymbol{m}}_{\mathfrak{b}}(E)$. Let $\pi^{\boldsymbol{m}}_{\mathfrak{b}}$ denote the projection of E onto $E^{\boldsymbol{m}}_{\mathfrak{b}}$ with respect to the decomposition $E=\bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})}E^{\boldsymbol{m}}_{\mathfrak{b}}$. Let $\pi^{\boldsymbol{m}\prime}_{\mathfrak{b}}$ denote the orthogonal projection onto $E^{\boldsymbol{m}\prime}_{\mathfrak{b}}$. We put $\mathcal{R}^{\boldsymbol{m}}_{\mathfrak{a}}:=\pi^{\boldsymbol{m}}_{\mathfrak{a}}-\pi^{\boldsymbol{m}\prime}_{\mathfrak{a}}$.

We will prove the following theorem in Sections 7.3.1–7.3.4.

Theorem 7.2.1. — *There exist positive constants R_1, ε_1 and A_1 such that the following holds on $X^*(R_1)$ for any $\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})$:*

$$\big|\mathcal{R}^{\boldsymbol{m}}_{\mathfrak{b}}\big|_h\leqslant A_1\,\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}}|\big).$$

In particular, the decomposition $E = \bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})} E^{\boldsymbol{m}}_{\mathfrak{b}}$ *is* $\exp\bigl(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}}|\bigr)$*-asymptotically orthogonal, in the sense that there exists* $A_1' > 0$ *such that the following holds for any* $u_i \in E^{\boldsymbol{m}}_{\mathfrak{b}_i|Q}$ *with* $\mathfrak{b}_1 \neq \mathfrak{b}_2$*:*

$$|h(u_1,u_2)| \leqslant A_1'\,|u_1|_h\,|u_2|_h\,\exp\bigl(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}}(Q)|\bigr)$$

The constants A_1, A_1', ε_1 *and* R_1 *may depend only on* $\mathrm{rank}(E)$, C_0, ε_0, $\mathrm{Irr}(\theta)$ *and* $\mathcal{S}p(\theta,j)$ $(j=1,\dots,\ell)$ *in Section* 7.2.1.

Corollary 7.2.2. — *We have the estimate* $\bigl|\pi^{\boldsymbol{m}}_{\mathfrak{a}} - \pi^{\boldsymbol{m}\dagger}_{\mathfrak{a}}\bigr|_h \leqslant 2A_1\exp\bigl(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}}|\bigr)$, *where* $\pi^{\boldsymbol{m}\dagger}_{\mathfrak{a}}$ *denotes the adjoint of* $\pi^{\boldsymbol{m}}_{\mathfrak{a}}$ *with respect to* h. □

Remark 7.2.3. — The main part of Theorem 7.2.1 is the claim for $\boldsymbol{m}\in\mathcal{T}(\theta)$, but it is convenient to take an auxiliary sequence $\mathcal{M}$ for inductive arguments in both the proof and the use. □

7.2.3. Main estimate for the regular part. — We take an order $\leqslant$ on $\mathcal{S}p(\theta,j)$ for each j. We put $F_{j,\alpha}(E) := \bigoplus_{\beta\leqslant\alpha} E_{j,\alpha}$. Let $E'_{j,\alpha}$ denote the orthogonal complement of $F_{j,<\alpha}$ in $F_{j,\alpha}$. Let $\pi_{j,\alpha}$ denote the projection of E onto $E_{j,\alpha}$ with respect to the decomposition $E=\bigoplus E_{j,\alpha}$. Let $\pi'_{j,\alpha}$ denote the orthogonal projection onto $E'_{j,\alpha}$. For any $j=1,\dots,\ell$, we set

$$q_j := f_j^{\mathrm{reg}} - \sum_{\alpha\in\mathcal{S}p(\theta,j)} \frac{\alpha}{z_j}\,\pi'_{j,\alpha}.$$

We put $\mathcal{R}^{\mathrm{reg}}_{j,\alpha} := \pi_{j,\alpha} - \pi'_{j,\alpha}$. We will show the following theorem in Sections 7.3.5–7.3.7.

Theorem 7.2.4. — *We have the following estimates:*

- $P(\mathrm{reg})$: *We have* $\bigl|z_j\,f_j^{\mathrm{reg}}\bigr|_h \leqslant A_2$ $(j=1,\dots,\ell)$ *and* $\bigl|f_j^{\mathrm{reg}}\bigr|_h \leqslant A_2$ $(j=\ell+1,\dots,n)$ *on* $X^*(R_2)$.
- $Q(\mathrm{reg})$: $|q_j|_h \leqslant A_2\,|z_j|^{-1}\bigl(-\log|z_j|\bigr)^{-1}$ *for* $j=1,\dots,\ell$ *on* $X^*(R_2)$.
- $R(\mathrm{reg})$: $\bigl|\mathcal{R}^{\mathrm{reg}}_{j,\alpha}\bigr|_h \leqslant A_2\,|z_j|^{\varepsilon_2}$ *on* $X^*(R_2)$ *for* $j=1,\dots,\ell$. *In particular, the decomposition* $E=\bigoplus E_{j,\alpha}$ *is* $O(|z_j|^{\varepsilon_2})$*-asymptotically orthogonal.*

The positive constants A_2, R_2 *and* ε_2 *may depend only on* $\mathrm{rank}(E)$, C_0, ε_0, $\mathrm{Irr}(\theta)$ *and* $\mathcal{S}p(\theta,j)$ *in Section* 7.2.1.

Corollary 7.2.5. — *We have the estimate* $\bigl|\pi_{j,\alpha}-\pi^{\dagger}_{j,\alpha}\bigr|_h \leqslant 2A_2\,|z_j|^{\varepsilon_2}$, *where* $\pi^{\dagger}_{j,\alpha}$ *denotes the adjoint of* $\pi_{j,\alpha}$ *with respect to* h. □

For $j=1,\dots,\ell$, we consider the following:

$$f_j^{\mathrm{nil}} := f_j^{\mathrm{reg}} - \sum_{\alpha\in\mathcal{S}p(\theta,j)} \frac{\alpha}{z_j}\,\pi_{j,\alpha}.$$

Corollary 7.2.6. — *We have* $\bigl|f_j^{\mathrm{nil}}\bigr|_h \leqslant A_2'\,|z_j|^{-1}\bigl(-\log|z_j|\bigr)^{-1}$. □

For any $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{S}p(\theta):=\prod_{j=1}^{\ell}\mathcal{S}p(\theta,j)$, we set $\operatorname{Diff}(\boldsymbol{\alpha},\boldsymbol{\beta}):=\{j\mid\alpha_j\neq\beta_j\}$ and

$$\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\beta}):=\prod_{j\in\operatorname{Diff}(\boldsymbol{\alpha},\boldsymbol{\beta})}|z_j|^{\varepsilon}.$$

We put $E_{\mathfrak{a},\boldsymbol{\alpha}}:=E_{\mathfrak{a}}\cap\bigcap_{j=1}^{\ell}E_{j,\alpha_j}$. We have the following immediate corollary of Theorem 7.2.1 and Theorem 7.2.4.

Corollary 7.2.7. — *In the case $(\mathfrak{a},\boldsymbol{\alpha})\neq(\mathfrak{b},\boldsymbol{\beta})$, the subbundles $E_{\mathfrak{a},\boldsymbol{\alpha}}$ and $E_{\mathfrak{b},\boldsymbol{\beta}}$ are $\exp\bigl(-\varepsilon|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}|\bigr)\,\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\beta})$-asymptotically orthogonal for some $\varepsilon>0$.* □

7.2.4. Complementary estimates for the Higgs field. — We give some refinements, which immediately follow from the theorems. The proof will be given in Section 7.3.8. All constants and estimates may depend only on $\operatorname{rank}(E)$, C_0, ε_0, $\operatorname{Irr}(\theta)$ and $\mathcal{S}p(\theta,j)$ in Section 7.2.1. We have the decomposition:

$$\operatorname{End}(E)=\bigoplus_{\mathfrak{a},\mathfrak{a}'\in\operatorname{Irr}(\theta)}\ \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\mathcal{S}p(\theta)}\operatorname{Hom}\bigl(E_{\mathfrak{a},\boldsymbol{\alpha}},E_{\mathfrak{a}',\boldsymbol{\alpha}'}\bigr).\tag{94}$$

For a section F of $\operatorname{End}(E)$, we have the corresponding decomposition:

$$F=\sum_{\mathfrak{a},\mathfrak{a}'\in\operatorname{Irr}(\theta)}\ \sum_{\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\mathcal{S}p(\theta)}F_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}.\tag{95}$$

Let $\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\alpha}')$ be as in Section 7.2.3.

Proposition 7.2.8. — *We have the following estimates on $X^*(R_3)$:*

$$\Bigl|\bigl(\pi_{\mathfrak{b}}^{\boldsymbol{m}(p)\dagger}-\pi_{\mathfrak{b}}^{\boldsymbol{m}(p)}\bigr)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Bigr|_h\leqslant A_3\exp\bigl(-\varepsilon_3|\boldsymbol{z}^{\boldsymbol{m}(p)}|-\varepsilon_3|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\bigr)\,\mathcal{Q}_{\varepsilon_3}(\boldsymbol{\alpha},\boldsymbol{\alpha}').$$

For $j=1,\dots,\ell$,

$$\Bigl|\bigl(\pi_{j,\gamma}^{\dagger}-\pi_{j,\gamma}\bigr)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Bigr|_h\leqslant A_3|z_j|^{\varepsilon_3}\exp\bigl(-\varepsilon_3|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\bigr)\,\mathcal{Q}_{\varepsilon_3}(\boldsymbol{\alpha},\boldsymbol{\alpha}'),$$

$$\Bigl|\bigl(f_j^{\operatorname{nil}\dagger}\bigr)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Bigr|_h\leqslant A_3\,|z_j|^{-1}(-\log|z_j|)^{-1}\exp\bigl(-\varepsilon_3|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\bigr)\,\mathcal{Q}_{\varepsilon_3}(\boldsymbol{\alpha},\boldsymbol{\alpha}').$$

For $j=\ell+1,\dots,n$,

$$\Bigl|\bigl(f_j^{\operatorname{reg}\dagger}\bigr)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Bigr|_h\leqslant A_3\exp\bigl(-\varepsilon_3|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\bigr)\,\mathcal{Q}_{\varepsilon_3}(\boldsymbol{\alpha},\boldsymbol{\alpha}').$$

7.2.5. Estimate of the curvature. — Let g_{p} denote the Poincaré metric of $X\smallsetminus D$. For any section F of $\operatorname{End}(E)\otimes\Omega^p$, we have the decomposition $F=\sum F_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}$ corresponding to the decomposition (94). We will prove the following proposition in Section 7.3.9.

Proposition 7.2.9. — *We have the following estimates on $X^*(R_4)$ with respect to h and g_{p}:*

$$\Bigl|\bigl[\theta,\theta^\dagger\bigr]_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Bigr|_{h,g_{\mathrm{p}}}\leqslant A_4\exp\bigl(-\varepsilon_4|\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\bigr)\,\mathcal{Q}_{\varepsilon_4}(\boldsymbol{\alpha},\boldsymbol{\alpha}')$$

In particular, $\bigl|[\theta,\theta^\dagger]\bigr|_{h,g_{\mathrm{p}}}\leqslant A_5$ on $X^*(R_4)$. Here the constants $\varepsilon_4,R_4,A_4,A_5$ may depend only on $\operatorname{rank}(E)$, C_0, ε_0, $\operatorname{Irr}(\theta)$ *and* $\mathcal{S}p(\theta,j)$ *in Section* 7.2.1.

Let $d''_\lambda := \overline{\partial}_E + \lambda\theta^\dagger$. The holomorphic bundle (E, d''_λ) is denoted by $\mathcal{E}^\lambda$. The curvature of the unitary connection associated to h and d''_λ is denoted by $R(h, d''_\lambda)$. Recall the relation $R(h, d''_\lambda) = -(1+|\lambda|^2)\,[\theta, \theta^\dagger]$. Hence, we obtain the following direct corollary of Proposition 7.2.9.

Corollary 7.2.10. — *We have the following estimates on $X^*(R_4)$ with respect to h and g_{p}:*

$$\Big|R(h, d''_\lambda)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\Big|_{h,g_{\mathrm{p}}} \leqslant (1+|\lambda|^2)\, A_4 \exp\big(-\varepsilon_4 |\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}|\big)\, \mathcal{Q}_{\varepsilon_4}(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$$

In particular, $\big|R(h, d''_\lambda)\big|_{h,g_{\mathrm{p}}} \leqslant (1+|\lambda|^2)\, A_5$ on $X^*(R_4)$. Therefore, $(\mathcal{E}^\lambda, h)$ is acceptable. In particular, $\big(\operatorname{End}(\mathcal{E}^\lambda), h\big)$ is also acceptable. □

7.3. Proof of the estimates

7.3.1. Inductive statement for the irregular part. — In the following argument, all constants and estimates may depend only on $\operatorname{rank}(E)$, C_0, ε_0, $\operatorname{Irr}(\theta)$ and $\mathcal{S}p(\theta, j)$ in Section 7.2.1. Let f and g denote some functions on X^*. We say $f = O(g)$ if $|f| \leqslant A \cdot |g|$ holds on $X^*(R)$, and we say $f \sim g$ if $A^{-1} \cdot |g| \leqslant |f| \leqslant A \cdot |g|$ holds on $X^*(R)$, for some positive constants $A, R > 0$.

For $\mathfrak{a} \in \operatorname{Irr}(\theta)$, let $\zeta_{\boldsymbol{m}(i)}(\mathfrak{a})$ be as in (10). It is also well defined for $\mathfrak{a} \in \overline{\operatorname{Irr}}(\theta, \boldsymbol{m}(i))$. Note $\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{a}) = \sum_{j\leqslant i} \zeta_{\boldsymbol{m}(j)}(\mathfrak{a})$ for $\mathfrak{a} \in \operatorname{Irr}(\theta)$. We define $f_j^{\boldsymbol{m}(i)}$ and $\mu_j^{\boldsymbol{m}(i)}$ as follows:

$$(96)\quad f_j^{\boldsymbol{m}(i)} := f_j - \sum_{\mathfrak{a}\in\operatorname{Irr}(\theta)} \partial_j \overline{\eta}_{\boldsymbol{m}(i-1)}(\mathfrak{a}) \cdot \pi_{\mathfrak{a}} = f_j^{\boldsymbol{m}(i-1)} - \sum_{\mathfrak{a}\in\operatorname{Irr}(\theta)} \partial_j \zeta_{\boldsymbol{m}(i-1)}(\mathfrak{a}) \cdot \pi_{\mathfrak{a}}$$
$$= f_j^{\boldsymbol{m}(i-1)} - \sum_{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i-1))} \partial_j \zeta_{\boldsymbol{m}(i-1)}(\mathfrak{b}) \cdot \pi_{\mathfrak{b}}^{\boldsymbol{m}(i-1)}.$$

$$(97)\quad \mu_j^{\boldsymbol{m}(i)} := f_j^{\boldsymbol{m}(i)} - \sum_{\mathfrak{a}\in\operatorname{Irr}(\theta)} \partial_j \zeta_{\boldsymbol{m}(i)}(\mathfrak{a}) \cdot \pi'_{\mathfrak{a}} = f_j^{\boldsymbol{m}(i)} - \sum_{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i))} \partial_j \zeta_{\boldsymbol{m}(i)}(\mathfrak{b}) \cdot \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}.$$

For the proof of Theorem 7.2.1, we show the following claims inductively on i.

$P(i, \mathrm{irr})$: $|f_j^{\boldsymbol{m}(i')}|_h = O\big(|\boldsymbol{z}^{\boldsymbol{m}(i')-\boldsymbol{\delta}_j}|\big)$ for $j = 1, \dots, \ell$ and for any $i' \leqslant i$.

$Q(i, \mathrm{irr})$: $|\mu_{\mathfrak{b}(i')}^{\boldsymbol{m}(i')}|_h = O\big(|\boldsymbol{z}^{\boldsymbol{m}(i')}|\big)$ for any $i' \leqslant i$.

$R(i, \mathrm{irr})$: $|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i')}|_h = O\big(\exp(-\varepsilon(3, i') \cdot |\boldsymbol{z}^{\boldsymbol{m}(i')}|)\big)$ for any $\mathfrak{b} \in \overline{\operatorname{Irr}}\big(\theta, \boldsymbol{m}(i')\big)$ and for any $i' \leqslant i$. In particular, the decomposition $E = \bigoplus_{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i'))} E_{\mathfrak{b}}^{\boldsymbol{m}(i')}$ is $\exp\big(-\varepsilon(3, i') \cdot |\boldsymbol{z}^{\boldsymbol{m}(i')}|\big)$-asymptotically orthogonal for any $i' \leqslant i$.

Remark 7.3.1. — By considering the pull-back of $(E, \overline{\partial}_E, \theta, h)$ via a ramified covering $\varphi : X \to X$ given by $\varphi(z_1, \dots, z_n) = (z_1^d, \dots, z_\ell^d, z_{\ell+1}, \dots, z_n)$, and by shrinking X, we may and will assume that there exists a constant C_{10}, which is independent of the choice of $Q \in X \setminus D$, such that

$$\left|\beta - \frac{\partial \mathfrak{a}}{\partial z_j}(Q) - \frac{\alpha}{z_j(Q)}\right| \leqslant C_{10}$$

for any eigenvalue β of $f_{j,(\mathfrak{a},\alpha)|Q}$, where $f_{j,(\mathfrak{a},\alpha)}$ denotes the restriction of f_j to $E_{j,\alpha} \cap E_{\mathfrak{a}}$. □

Remark 7.3.2. — By tensoring with a rank one meromorphic connection, we may and will assume that, for any $\boldsymbol{m} \in \mathcal{M}$, there exists $\mathfrak{a} \in \operatorname{Irr}(\theta)$ such that $\mathfrak{a}_{\boldsymbol{m}} \neq 0$. □

7.3.2. $P(i-1,\mathrm{irr}) + Q(i-1,\mathrm{irr}) + R(i-1,\mathrm{irr}) \Longrightarrow P(i,\mathrm{irr})$**.** — Let Δ_j denote the Laplacian with respect to the variable z_j, i.e., $\Delta_j := -\partial^2/\partial z_j \partial \overline{z}_j$. Because of the commutativity $\left[f_j, f_j^{\boldsymbol{m}(i)}\right] = 0$, we have the following inequality:

$$\Delta_j \log\left|f_j^{\boldsymbol{m}(i)}\right|_h^2 \leqslant -\frac{\left|[f_j^{\dagger}, f_j^{\boldsymbol{m}(i)}]\right|_h^2}{\left|f_j^{\boldsymbol{m}(i)}\right|_h^2}. \tag{98}$$

(See Page 731 of [**82**].) Recall $f_j = f_j^{\boldsymbol{m}(i)} + \sum \partial_j \overline{\eta}_{\boldsymbol{m}(i-1)}(\mathfrak{a}) \cdot \pi_{\mathfrak{a}}$. We have the following equality:

$$\sum_{\mathfrak{a}} \partial_j \overline{\eta}_{\boldsymbol{m}(i-1)}(\mathfrak{a}) \cdot \pi_{\mathfrak{a}} = \sum_{\mathfrak{a}} \sum_{i'<i} \partial_j \zeta_{\boldsymbol{m}(i')}(\mathfrak{a}) \cdot \pi_{\mathfrak{a}} = \sum_{i'<i} \sum_{\mathfrak{b} \in \overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i'))} \partial_j \zeta_{\boldsymbol{m}(i')}(\mathfrak{b}) \cdot \pi_{\mathfrak{b}}^{\boldsymbol{m}(i')}.$$

For any section F of $\operatorname{End}(E)$, we have the decomposition $F = \mathcal{C}^{(i')}(F) + \mathcal{D}^{(i')}(F)$ as follows:

$$\mathcal{C}^{(i')}(F) \in \bigoplus_{\substack{\mathfrak{b},\mathfrak{b}' \in \overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i')) \\ \mathfrak{b} \neq \mathfrak{b}'}} \operatorname{Hom}\big(E_{\mathfrak{b}}^{\boldsymbol{m}(i')\prime}, E_{\mathfrak{b}'}^{\boldsymbol{m}(i')\prime}\big), \quad \mathcal{D}^{(i')}(F) \in \bigoplus_{\mathfrak{b} \in \overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i'))} \operatorname{End}\big(E_{\mathfrak{b}}^{\boldsymbol{m}(i')\prime}\big).$$

Then, we have $\left[\pi_{\mathfrak{b}}^{\boldsymbol{m}(i')\dagger}, f_j^{\boldsymbol{m}(i)}\right] = \left[\mathcal{C}^{(i')}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i')\dagger}), f_j^{\boldsymbol{m}(i)}\right] + \left[\mathcal{D}^{(i')}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i')\dagger}), \mathcal{C}^{(i')}(f_j^{\boldsymbol{m}(i)})\right]$ for any $i' < i$. Both the first and second terms are $O\big(\exp(-\varepsilon(4,i')|\boldsymbol{z}^{\boldsymbol{m}(i')}|)\big) \cdot |f_j^{\boldsymbol{m}(i)}|$ because of $R(i-1,\mathrm{irr})$. Note $\left|\partial_j \zeta_{\boldsymbol{m}(i')}(\mathfrak{b})\right| = O\big(|\boldsymbol{z}^{\boldsymbol{m}(i')}|\big)$ if the j-th component of $\boldsymbol{m}(i')$ is 0. Hence, we have the estimates

$$|\partial_j \zeta_{\boldsymbol{m}(i')}(\mathfrak{b})| \cdot \exp\big(-\varepsilon(5,i')|\boldsymbol{z}^{\boldsymbol{m}(i')}|\big) = O\Big(\exp\big(-\varepsilon(6,i')|\boldsymbol{z}^{\boldsymbol{m}(i')}|\big)\Big).$$

Therefore, we obtain the following estimate:

$$\Big|\big[\big(\partial_j \zeta_{\boldsymbol{m}(i')}(\mathfrak{b}) \cdot \pi_{\mathfrak{b}}^{\boldsymbol{m}(i')}\big)^{\dagger},\ f_j^{\boldsymbol{m}(i)}\big]\Big|_h = O\Big(\exp\big(-\varepsilon(7,i') \cdot |\boldsymbol{z}^{\boldsymbol{m}(i')}|\big)\Big) \cdot |f_j^{\boldsymbol{m}(i)}|_h.$$

Hence, we obtain the following inequality on $X^*(R_{4,i})$ from (98) for some constant $C_{20,i}$:

$$\Delta_j \log\left|f_j^{\boldsymbol{m}(i)}\right|_h^2 \leqslant -\frac{\left|[f_j^{\boldsymbol{m}(i)\dagger}, f_j^{\boldsymbol{m}(i)}]\right|_h^2}{\left|f_j^{\boldsymbol{m}(i)}\right|_h^2} + C_{20,i}. \tag{99}$$

All eigenvalues of $\big(z_j \cdot f_j^{\boldsymbol{m}(i)}\big)_{|Q}$ for $Q \in X^*(R_{5,i})$ are dominated by $C_{21,i} \cdot \big|\boldsymbol{z}^{\boldsymbol{m}(i)}(Q)\big|$. Hence, we can show $P(i)$ by a standard argument. We give only an outline of the argument. Let $\pi_j : X \longrightarrow D_j$ denote the natural projection. Let P be any point of $D_j \setminus \bigcup_{1 \leqslant k \leqslant \ell, k \neq j} D_k$. Let $\mathbb{H}$ denote the upper half plane. We have the universal covering $\varphi : \mathbb{H} \longrightarrow \pi_j^{-1}(P) \setminus \{P\}$ given by $z_j = \exp\big(\sqrt{-1}\zeta\big)$. Let $Y(R_{5,i}) := \varphi^{-1}\big(X(R_{5,i})\big)$. We put $F := \varphi^{-1}\big(z_j \cdot f_j^{\boldsymbol{m}(i)}\big)$. Let Δ_ζ denote the Laplacian $-\partial^2/\partial\zeta\partial\overline{\zeta}$. Because of (99), we have the following inequality on $Y(R_{5,i})$:

$$\Delta_\zeta \log |F|_h^2 \leqslant -\frac{\big|[F, F^\dagger]\big|_h^2}{|F|_h^2} + C_{22,i}.$$

By using the argument for the proof of Proposition 2.10 in [**66**] (based on [**1**] and [**82**]), we obtain that $|F|_h^2$ is dominated by the sum of the square of the absolute values of the eigenvalues of F on $Y(R_{5,i})$, and the estimate may depend only on C_{22}. Thus, we obtain $P(i, \mathrm{irr})$. (Actually, we do not use $Q(i-1, \mathrm{irr})$.)

7.3.3. $P(i, \mathrm{irr}) + Q(i-1, \mathrm{irr}) + R(i-1, \mathrm{irr}) \Longrightarrow Q(i, \mathrm{irr})$. — We put $j := \mathfrak{h}(i)$ for simplicity of the description. We remark the j-th component of $\boldsymbol{m}(i)$ is negative, a property we will use implicitly. For any point $Q \in X \setminus D$, let $G_j^{\boldsymbol{m}(i)}(Q) := \sum \mathfrak{m}(\beta) \cdot |\beta|^2$, where β runs through the eigenvalues of $f_{j|Q}^{\boldsymbol{m}(i)}$, and $\mathfrak{m}(\beta)$ denotes the multiplicity of β. We put $H_j^{\boldsymbol{m}(i)} := \big|f_{j|Q}^{\boldsymbol{m}(i)}\big|^2 - G_j^{\boldsymbol{m}(i)}(Q)$. We only have to show $H_j^{\boldsymbol{m}(i)} = O\big(\big|\boldsymbol{z}^{\boldsymbol{m}(i)}\big|^2\big)$. We consider the following:

$$\tau_j^{\boldsymbol{m}(i)} := \sum_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \partial_j\big(\mathfrak{a} - \overline{\eta}_{\boldsymbol{m}(i-1)}(\mathfrak{a})\big) \cdot \pi'_{\mathfrak{a}} + \sum_{\alpha \in \mathcal{S}p(\theta, j)} \frac{\alpha}{z_j} \cdot \pi'_{j,\alpha}.$$

We have $\big|\tau_j^{\boldsymbol{m}(i)}\big|_h \sim \big|\boldsymbol{z}^{\boldsymbol{m}(i) - \boldsymbol{\delta}_j}\big|$, and $\Delta_j \log\big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2$ is C^∞. We set

$$k_j^{\boldsymbol{m}(i)} := \log\left(\frac{\big|f_j^{\boldsymbol{m}(i)}\big|_h^2}{\big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2}\right) = \log\left(1 + \frac{H_j^{\boldsymbol{m}(i)} + G_j^{\boldsymbol{m}(i)} - \big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2}{\big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2}\right).$$

Lemma 7.3.3. — $\big|G_j^{\boldsymbol{m}(i)} - |\tau_j^{\boldsymbol{m}(i)}|^2\big| = O\big(\boldsymbol{z}^{\boldsymbol{m}(i) - \boldsymbol{\delta}_j}\big)$.

Proof. — Let $f_{j,(\mathfrak{a},\alpha)}^{\boldsymbol{m}(i)}$ denote the restriction of $f_j^{\boldsymbol{m}(i)}$ to $E_{\mathfrak{a}} \cap E_{j,\alpha}$, and let β be any eigenvalue of $f_{j,(\mathfrak{a},\alpha)|Q}^{\boldsymbol{m}(i)}$. We have the following estimate (see Remark 7.3.1):

$$\left|\frac{\partial \mathfrak{a}}{\partial z_j}(Q) + \frac{\alpha}{z_j(Q)} - \beta\right| \leqslant C_{30}$$

Hence, we have the following:

$$|\beta|^2 - \left|\frac{\partial \mathfrak{a}}{\partial z_j}(Q) + \frac{\alpha}{z_j(Q)}\right|^2 = O\big(\big|\boldsymbol{z}^{\boldsymbol{m}(i) - \boldsymbol{\delta}_j}\big|\big)$$

Then, the claim of the lemma follows. □

By using a standard argument using elementary linear algebra (see Page 729 of [**82**]), we can show that there exists a constant $C_{31} > 0$ such that $\big|[f_j^{\boldsymbol{m}(i)\dagger}, f_j^{\boldsymbol{m}(i)}]\big|_h \geqslant C_{31}\cdot H_j^{\boldsymbol{m}(i)}$. Recall (99), where we have used $R(i-1, \mathrm{irr})$. Hence, we have the following estimate on $X^*(R_{30,i})$ for some constants C_i:

$$\Delta_j k_j^{\boldsymbol{m}(i)} \leqslant -C_{32}\cdot \frac{\big(H_j^{\boldsymbol{m}(i)}\big)^2}{|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2} + C_{33} \leqslant -C_{34}|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2\left(\frac{H_j^{\boldsymbol{m}(i)}}{\big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2}\right)^2 + C_{35}.$$

We already know $H_j^{\boldsymbol{m}(i)} = O\big(\big|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}\big|^2\big)$ due to $P(i, \mathrm{irr})$. Hence, we have the following estimate:

$$(100)\qquad k_j^{\boldsymbol{m}(i)} \sim \frac{\big|H_j^{\boldsymbol{m}(i)} + G_j^{\boldsymbol{m}(i)} - |\tau_j^{\boldsymbol{m}(i)}|_h^2\big|}{\big|\tau_j^{\boldsymbol{m}(i)}\big|_h^2}.$$

Put $\mathcal{Z}(L) := \big\{Q \in X^*(R_{30,i}) \,\big|\, H_j^{\boldsymbol{m}(i)} \geqslant L\cdot \big|\boldsymbol{z}^{\boldsymbol{m}(i)}\big|^2\big\}$ for some large $L > 0$. Note that $2\boldsymbol{m}(i) \leqslant \boldsymbol{m}(i) - \boldsymbol{\delta}_j$. Hence, $H_j^{\boldsymbol{m}(i)}$ is sufficiently larger than $\big|G_j^{\boldsymbol{m}(i)} - |\tau_j^{\boldsymbol{m}(i)}|^2\big|$ on $\mathcal{Z}(L)$, and we obtain the following on $\mathcal{Z}(L) \cap X^*(R_{31,i})$:

$$(101)\qquad C_{36}^{-1}\frac{H_j^{\boldsymbol{m}(i)}}{\big|\tau_j^{\boldsymbol{m}(i)}\big|^2} \leqslant k_j^{\boldsymbol{m}(i)} \leqslant C_{36}\frac{H_j^{\boldsymbol{m}(i)}}{\big|\tau_j^{\boldsymbol{m}(i)}\big|^2}.$$

We have $2\big(\boldsymbol{m}(i) - \boldsymbol{\delta}_j\big) + 4\boldsymbol{m}(i) - 4\big(\boldsymbol{m}(i) - \boldsymbol{\delta}_j\big) = 2\boldsymbol{m}(i) + 2\boldsymbol{\delta}_j \leqslant \boldsymbol{0}$. Hence, we have the following inequality on $\mathcal{Z}(L) \cap X^*(R_{31,i})$:

$$(102)\qquad \Delta_j k_j^{\boldsymbol{m}(i)} \leqslant -C_{37}\cdot \big|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}\big|^2\big(k_j^{\boldsymbol{m}(i)}\big)^2 \leqslant -C_{38}|z_j|^{-4}\big(k_j^{\boldsymbol{m}(i)}\big)^2$$

We would like to compare the functions $|z_j|^2$ and $k_j^{\boldsymbol{m}(i)}$.

Lemma 7.3.4. — *Let $R_{32,i} < R_{31,i}$. We can take C_{39} with the following properties:*
- $C_{39}\cdot |z_j|^2 > k_j^{\boldsymbol{m}(i)}$ *on* $\big\{|z_j| = R_{32,i}\big\}$.
- *If* $k_j^{\boldsymbol{m}(i)}(Q) \geqslant C_{39}\cdot |z_j(Q)|^2$ *for* $Q \in X^*(R_{32,i})$, *we have* $Q \in \mathcal{Z}(L)$.

Proof. — Due to (100) and $H_j^{\boldsymbol{m}(i)} = O(|z^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2)$, we have the boundedness of $k_j^{\boldsymbol{m}(i)}$. Hence, the first condition is satisfied for sufficiently large C_{39}. Due to (101), the second condition is satisfied for sufficiently large C_{39}. □

Take $C_{40} > \max\big\{C_{38}^{-1}, C_{39}\big\}$. Note the following inequality for any $\eta \geqslant 0$:

$$\Delta_j\Big(C_{40}\cdot |z_j|^2 - \eta\log|z_j|\Big) > -C_{38}|z_j|^{-4}\big(C_{40}\cdot|z_j|^2 - \eta\log|z_j|\big)^2.$$

Let P be any point of $D_j \setminus \bigcup_{1\leqslant k\leqslant \ell, k\neq j} D_k$. Let us consider the following set:

$$\mathcal{Z}_P(C_{40}, \eta) := \big\{Q \in \pi_j^{-1}(P) \cap X^*(R_{32,i}) \,\big|\, k_j^{\boldsymbol{m}(i)}(Q) > C_{40}|z_j(Q)|^2 - \eta\log|z_j(Q)|^2\big\}$$

Since $k_j^{\boldsymbol{m}(i)}$ is bounded, the closure of $\mathcal{Z}_P(C_{40},\eta)$ in $\pi^{-1}(P)\setminus\{P\}$ is compact. By the choice of C_{39}, the closure of $\mathcal{Z}_P(C_{40},\eta)$ has no intersection with $\{|z_j|=R_{32,i}\}$. Because $\mathcal{Z}_P(C_{40},\eta)\subset\mathcal{Z}(L)\cap X^*(R_{31,i})$, we have the following on $\mathcal{Z}_P(C_{40},\eta)$:

$$\Delta_j\big(k_j^{\boldsymbol{m}(i)}-\big(C_{40}|z_j|^2-\eta\log|z_j|\big)\big)<-C_{38}|z_j|^{-4}\Big(\big(k_j^{\boldsymbol{m}(i)}\big)^2-\big(C_{40}|z_j|^2-\eta\log|z_j|\big)^2\Big)\leqslant 0.$$

Therefore, $k_j^{\boldsymbol{m}(i)}-\big(C_{40}|z_j|^2-\eta\log|z_j|\big)$ takes the maximum at the boundary of $\mathcal{Z}_P(C_{40},\eta)$, which has to be 0. Thus, we have arrived at the contradiction, and we can conclude $\mathcal{Z}_P(C_{40},\eta)=\varnothing$. By taking $\eta\to 0$, we obtain $k_j^{\boldsymbol{m}(i)}\leqslant C_{40}|z_j|^2$ on $\pi_j^{-1}(P)\cap X^*(R_{32,i})$. Due to (101), we obtain $H_j^{\boldsymbol{m}(i)}=O\big(|\boldsymbol{z}^{\boldsymbol{m}(i)}|^2\big)$ and thus $Q(i,\mathrm{irr})$. (Actually, we do not use $Q(i-1,\mathrm{irr})$.)

7.3.4. $P(i,\mathrm{irr})+Q(i,\mathrm{irr})+R(i-1,\mathrm{irr})\Longrightarrow R(i,\mathrm{irr})$. — We continue to put $j:=\mathfrak{h}(i)$. Because of the commutativity $[f_j,\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]=0$, we have the following inequality:

$$\Delta_j\log\big|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|_h^2\leqslant-\frac{\big|[f_j^\dagger,\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]\big|_h^2}{|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}|_h^2}\tag{103}$$

. We consider the following function:

$$k_{\mathfrak{b}}^{\boldsymbol{m}(i)}:=\log\left(\frac{|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}|_h^2}{|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}|_h^2}\right)=\log\left(1+\frac{|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}|_h^2}{|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}|_h^2}\right).$$

Because $[f_j^{\boldsymbol{m}(i)},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]=0$, we have the following:

$$\begin{aligned}0&=[f_j^{\boldsymbol{m}(i)},\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}]+[f_j^{\boldsymbol{m}(i)},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}]\\&=\Big[\sum\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})\cdot\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}+\mu_j^{\boldsymbol{m}(i)},\ \mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}\Big]+[f_j^{\boldsymbol{m}(i)},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}].\end{aligned}$$

We have $[\mu_j^{\boldsymbol{m}(i)},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}]=[f_j^{\boldsymbol{m}(i)},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}]=O\big(|\boldsymbol{z}^{\boldsymbol{m}(i)}|\big)$ due to $Q(i,\mathrm{irr})$, and hence $\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}=O\big(|z_j|\big)$. In particular, we have the following estimate:

$$k_{\mathfrak{b}}^{\boldsymbol{m}(i)}\sim\big|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|_h^2.\tag{104}$$

From (103), we have the following:

$$\Delta_jk_{\mathfrak{b}}^{\boldsymbol{m}(i)}\leqslant-\frac{\big|[f_j^\dagger,\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]\big|_h^2}{\big|\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|_h^2}.$$

Note the following:

$$f_j^\dagger=\sum_{i'\leqslant i}\sum_{\mathfrak{c}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i'))}\overline{\partial_j\zeta_{\boldsymbol{m}(i')}(\mathfrak{c})}\cdot\pi_{\mathfrak{c}}^{\boldsymbol{m}(i')\prime}+\sum_{i'<i}\sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i'))}\overline{\partial_j\zeta_{\boldsymbol{m}(i')}(\mathfrak{c})}\cdot\mathcal{R}_{\mathfrak{c}}^{\boldsymbol{m}(i')\dagger}+\mu_j^{\boldsymbol{m}(i)\dagger}.$$

Because of $Q(i,\mathrm{irr})$ and $R(i-1,\mathrm{irr})$, the second and third terms in the right-hand side are assumed to be much smaller than the first term on $X^*(R_{40,i})$. Let $p^{\boldsymbol{m}(i)}$ denote the projection of $\mathrm{End}(E)$ onto the direct summand $\bigoplus_{\mathfrak{b}>\mathfrak{b}'}Hom\big(E_{\mathfrak{b}}^{\boldsymbol{m}(i)},E_{\mathfrak{b}'}^{\boldsymbol{m}(i)}\big)$. From the equality $[f_j^{\dagger},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]=[f_j^{\dagger},\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}]+[f_j^{\dagger},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime}]$, we have the following inequality on $X^*(R_{42,i})$:

$$\big|[f_j^{\dagger},\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}]\big|_h^2\geqslant\big|p^{\boldsymbol{m}(i)}[f_j^{\dagger},\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}]\big|_h^2\geqslant C_{41}\cdot|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2\cdot\big|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|_h^2.$$

Hence, we obtain the following on $X^*(R_{42,i})$:

$$\Delta_j k_{\mathfrak{b}}^{\boldsymbol{m}(i)}\leqslant -C_{42}\cdot|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2\cdot\big|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|^2\leqslant -C_{43}\cdot|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}|^2\cdot k_{\mathfrak{b}}^{\boldsymbol{m}(i)}.$$

We take small $\varepsilon(10,i)>0$ such that the following holds:

$$\text{(105)}\qquad \Big(\frac{m_j(i)}{2}\Big)^2\cdot\varepsilon(10,i)^2\leqslant C_{43}.$$

We have the following inequality for any $\eta\geqslant 0$:

$$\text{(106)}\quad \Delta_j\Big(\exp\big(-\varepsilon(10,i)|\boldsymbol{z}^{\boldsymbol{m}(i)}|\big)-\eta\log|z_j|\Big)\geqslant$$
$$-\Big(\frac{m_j(i)}{2}\Big)^2\cdot\varepsilon(10,i)^2\cdot\big|\boldsymbol{z}^{\boldsymbol{m}(i)-\boldsymbol{\delta}_j}\big|^2\cdot\Big(\exp\big(-\varepsilon(10,i)|\boldsymbol{z}^{\boldsymbol{m}(i)}|\big)-\eta\log|z_j|\Big).$$

For $\eta>0$ and $P\in D_j\setminus\bigcup_{1\leqslant k\leqslant\ell,k\neq j}D_k$, we put

$$\Psi_\eta(P,z_j):=\exp\Big(\varepsilon(10,i)\prod_{p\neq j}\big|z_p^{m_p(i)}(P)\big|\cdot\big(R_{42,i}^{m_j(i)}-\big|z_j^{m_j(i)}\big|\big)\Big)-\eta\cdot\log|z_j|.$$

Because of (106), we obtain the following:

$$\Delta_j\big(\Psi_\eta(P,z_j)\big)\geqslant-\Big(\frac{m_j(i)}{2}\Big)^2\varepsilon(10,i)^2\prod_{p\neq j}\big|z_p^{2m_p(i)}(P)\big|\cdot\big|z_j^{2m_j(i)-2}\big|\times\Psi_\eta(P,z_j).$$

We have $\Psi_\eta(P,z_j)=1-\eta\log R_{42,i}$ for $|z_j|=R_{42,i}$, which is larger than $1/2$, if η is sufficiently small. We already know that $k_{\mathfrak{b}}^{\boldsymbol{m}(i)}$ is bounded. Hence, we can take a constant C_{44} such that $k_{\mathfrak{b}|\pi_j^{-1}(P)}^{\boldsymbol{m}(i)}(z_j)<C_{44}\cdot\Psi_\eta(P,z_j)$ on $\{|z_j|=R_{42,i}\}$. Let us consider the following set:

$$\mathcal{Z}(P,\eta):=\Big\{z_j\in\pi_j^{-1}(P)\cap X^*(R_{42,i})\,\big|\,k_{\mathfrak{b}|\pi_j^{-1}(P)}^{\boldsymbol{m}(i)}(z_j)>C_{44}\cdot\Psi_\eta(P,z_j)\Big\}.$$

Since $k_{\mathfrak{b}}^{\boldsymbol{m}(i)}$ is bounded, the closure of $\mathcal{Z}(P,\eta)$ in $\{0<|z_j|\leqslant 1\}$ is compact. By our choice of C_{44}, the closure of $\mathcal{Z}(P,\eta)$ has no intersection with $\big\{|z_j|=R_{42,i}\big\}$. On $\mathcal{Z}(P,\eta)$, we have the following inequality by our choice of $\varepsilon(10,i)$ as in (105):

$$\Delta_j\Big(k_{\mathfrak{b}|\pi_j^{-1}(P)}^{\boldsymbol{m}(i)}-C_{44}\cdot\Psi_\eta(P,z_j)\Big)\leqslant 0.$$

Hence, the values of $k_{\mathfrak{b}|\pi_j^{-1}(P)}^{\boldsymbol{m}(i)}-C_{44}\Psi_\eta(P,z_j)$ on $\mathcal{Z}(P,\eta)$ is not larger than the boundary values, which is 0. Thus, we have arrived at a contradiction, and we obtain $\mathcal{Z}(P,\eta)=\varnothing$. By taking $\eta\to 0$, we obtain the following inequality on $\pi^{-1}(P)\cap X^*(R_{42,i})$:

$$k_{\mathfrak{b}|\pi_j^{-1}(P)}^{\boldsymbol{m}(i)}\leqslant C_{44}\cdot\exp\Big(\varepsilon(10,i)\cdot\prod_{p\neq j}\big|z_p^{m_p(i)}(P)\big|\cdot\big(R_{42,i}^{m_j(i)}-\big|z_j^{m_j(i)}\big|\big)\Big).$$

Let $R_{43,i} := R_{42,i}/2$. Then, the following holds on $\pi_j^{-1}(P) \cap X^*(R_{43,i})$:

$$k^{\boldsymbol{m}(i)}_{\mathfrak{b}|\pi_j^{-1}(P)} \leqslant C_{45} \cdot \exp\Big(-\varepsilon(11,i) \cdot \prod_{p\neq j} \big|z_p^{m_p(i)}(P)\big| \cdot \big|z_j^{m_j(i)}\big|\Big).$$

Hence, we obtain $\big|\mathcal{R}_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big|_h = O\big(\exp\big(-\varepsilon(11,i)\cdot\big|\boldsymbol{z}^{\boldsymbol{m}(i)}\big|\big)\big)$.

Thus the proof of Theorem 7.2.1 is finished. □

7.3.5. $P(\mathrm{reg})$. — Let us begin the proof of Theorem 7.2.4. We have the following:

$$f_j^{\dagger} = f_j^{\mathrm{reg}\,\dagger} + \sum_i \sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \overline{\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}\cdot\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)} + \sum_i \sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \overline{\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}\cdot\big(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big).$$

We have the estimate $\big|\pi_{\mathfrak{b}}^{\boldsymbol{m}\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}}\big|_h = O\big(\exp(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$. We note that $|\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})| = O(|\boldsymbol{z}^{\boldsymbol{m}(i)}|)$ in the case where the j-th component of $\boldsymbol{m}(i)$ is 0. Therefore, we have the following inequality on $X^*(R_{60})$:

$$\Delta_j \log|f_j^{\mathrm{reg}}|^2 \leqslant -\frac{\big|[f_j^{\dagger}, f_j^{\mathrm{reg}}]\big|_h^2}{|f_j^{\mathrm{reg}}|^2} \leqslant -\frac{\big|[f_j^{\mathrm{reg}\,\dagger}, f_j^{\mathrm{reg}}]\big|_h^2}{|f_j^{\mathrm{reg}}|_h^2} + C_{60}. \tag{107}$$

Since the eigenvalues of f_j^{reg} (resp. $z_j\cdot f_j^{\mathrm{reg}}$) are bounded on $X\setminus D$ in the case $j>\ell$ (resp. $j\leqslant\ell$), we obtain the desired estimate by using the argument in Section 7.3.2 and the inequality (107).

7.3.6. $Q(\mathrm{reg})$. — We put $\rho_j := \sum_{\alpha\in\mathcal{S}p(j)}\mathfrak{m}(\alpha)\cdot|\alpha|^2$, where $\mathfrak{m}(\alpha)$ denotes the multiplicity of α. By considering the tensor products with the rank one Higgs bundle, we may assume $\rho_j\neq 0$. Similarly, we also put $G_j^{\mathrm{reg}}(Q) := \sum\mathfrak{m}(\beta)\cdot|\beta|^2$, where β runs through the eigenvalues of $f^{\mathrm{reg}}_{j|Q}$. We put $H_j^{\mathrm{reg}} := \big|f^{\mathrm{reg}}_{j|Q}\big|_h^2 - G_j^{\mathrm{reg}}(Q)$. By Remark 7.3.1, we have $|q_j|^2 - H_j^{\mathrm{reg}} = O(1)$ and $\rho_j|z_j|^{-2} - G_j^{\mathrm{reg}} = O(|z_j|^{-1})$. We only have to show $H_j^{\mathrm{reg}} \leqslant C\big(|z_j|^{-2}(-\log|z_j|)^{-2}\big)$. We put

$$k_j^{\mathrm{reg}} := \log\Big(\frac{|f_j^{\mathrm{reg}}|^2}{\rho_j|z_j|^{-2}}\Big) = \log\Big(1 + \frac{H_j^{\mathrm{reg}} + (G_j^{\mathrm{reg}} - \rho_j|z_j|^{-2})}{\rho_j|z_j|^{-2}}\Big).$$

Let us consider the set

$$\mathcal{Z}(L) := \big\{Q\in X^*(R_{60}) \,\big|\, H_j^{\mathrm{reg}}(Q) > L|z_j(Q)|^{-2}(-\log|z_j(Q)|)^{-2}\big\}$$

for some large $L>0$. On $\mathcal{Z}(L)$, we have $k_j^{\mathrm{reg}} \sim H_j^{\mathrm{reg}}\cdot|z_j|^{-2}$. From (107), we have the following inequality on $\mathcal{Z}(L)$:

$$\Delta_j k_j^{\mathrm{reg}} \leqslant -C_{61}\frac{(H_j^{\mathrm{reg}})^2}{|z_j|^{-2}} + C_{62} \leqslant \frac{-C_{63}}{|z_j|^2}\Big(\frac{H_j^{\mathrm{reg}}}{\rho_j|z_j|^{-2}}\Big)^2 + C_{64} \leqslant -C_{65}\frac{\big(k_j^{\mathrm{reg}}\big)^2}{|z_j|^2} + C_{66}.$$

We have $k_j^{\mathrm{reg}} \geqslant C_{70}(-\log|z_j|)^{-2}$ on $\mathcal{Z}(L)$. Hence, we have the following on $\mathcal{Z}(L)$, if L is sufficiently large:

$$\Delta_j(k_j^{\mathrm{reg}}) \leqslant -C_{67}\frac{\big(k_j^{\mathrm{reg}}\big)^2}{|z_j|^2}. \tag{108}$$

Lemma 7.3.5. — *We can take* C_{71} *and* R_{61} *with the following properties:*

- $\Delta_j\big(C_{71}(-\log|z_j|)^{-2}\big) \geqslant -C_{67}\cdot|z_j|^{-2}\big(C_{71}(-\log|z_j|)^{-2}\big)^2$.
- $k_j^{\mathrm{reg}} < C_{71}(-\log|z_j|)^{-2}$ *on* $\{|z_j| = R_{61}\}$.
- *If* $k_j^{\mathrm{reg}}(Q) > C_{71}(-\log|z_j(Q)|)^{-2}$ *for* $Q \in X^*(R_{61})$, *we have* $Q \in \mathcal{Z}(L)$.

Proof. — The first condition can be checked by a direct calculation as in [**82**] or [**67**]. Since we already know that k_j^{reg} is bounded, the second condition can be satisfied. Since we have $k_j^{\mathrm{reg}} \sim \big(H_j^{\mathrm{reg}} + (G_j^{\mathrm{reg}} - \rho_j|z_j|^{-2})\big)\cdot|z_j|^2$, the third condition can be satisfied. □

Let P be any point of $D_j \setminus \bigcup_{1\leqslant k\leqslant \ell, k\neq j} D_k$, and let us consider the following set:

$$\mathcal{Z}(\eta) := \big\{Q \in \pi_j^{-1}(P)\cap X^*(R_{60}) \,\big|\, k_j^{\mathrm{reg}}(Q) > C_{71}(-\log|z_j|)^{-2} - \eta\cdot\log|z_j|\big\}.$$

Then, we can show $\mathcal{Z}(\eta) = \varnothing$ by using a standard argument as in Section 7.3.3, with (108) and Lemma 7.3.5. (See [**82**] or [**67**].) By taking $\eta \to 0$, we obtain the estimate $k_j^{\mathrm{reg}} \leqslant C_{71}\big(-\log|z_j|\big)^{-2}$, which implies $\big|H_j^{\mathrm{reg}}\big| = O\big(|z_j|^{-2}(-\log|z_j|)^{-2}\big)$. Therefore, we obtain $Q(j,\mathrm{reg})$.

7.3.7. $R(\mathrm{reg})$. — We have $0 = \big[f_j^{\mathrm{reg}}, \pi_{j,\alpha}\big] = \big[f_j^{\mathrm{reg}}, \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\big] + \big[f_j^{\mathrm{reg}}, \pi'_{j,\alpha}\big]$. We also have $\big[f_j^{\mathrm{reg}}, \pi'_{j,\alpha}\big] = O\left(|z_j|^{-1}(-\log|z_j|)^{-1}\right)$ by $Q(\mathrm{reg})$. By using it, we obtain the preliminary estimate $\mathcal{R}_{j,\alpha}^{\mathrm{reg}} = O\big((-\log|z_j|)^{-1}\big)$. We have the following:

$$(109)\quad f_j^{\dagger} = \sum_i \sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \overline{\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}\cdot\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)} + \sum_{\alpha\in\mathcal{S}p(\theta,j)} \frac{\overline{\alpha}}{\overline{z}_j}\cdot\pi'_{j,\alpha} + \sum_i \sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \overline{\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}\cdot\big(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}\big) + q_j^{\dagger}.$$

The last two terms are much smaller than the first two terms on $X^*(R_{80})$. We have the following:

$$(110)\quad \big[f_j^{\dagger}, \pi_{j,\alpha}\big] = \Big[\sum_i\sum_{\mathfrak{b}} \overline{\partial_j\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}), \pi'_{j,\alpha}\Big] + [q_j^{\dagger}, \pi'_{j,\alpha}] + \Big[\sum_{\beta}\frac{\overline{\beta}}{\overline{z}_j}\cdot\pi'_{j,\beta}, \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\Big] + \Big[\sum_i\sum_{\mathfrak{b}}\overline{\partial_j\zeta(\mathfrak{b})}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}), \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\Big] + [q_j^{\dagger}, \mathcal{R}_{j,\alpha}^{\mathrm{reg}}].$$

By using an argument in Section 7.3.4, we obtain the following on $X^*(R_{80})$:

$$(111)\quad \big|[f_j^{\dagger}, \pi_{j,\alpha}]\big|_h \geqslant \Big|\Big[\sum\frac{\overline{\beta}}{\overline{z}_j}\pi'_{j,\beta},\ \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\Big]\Big|_h - \Big|\Big[\sum_i\sum_{\mathfrak{b}}\overline{\partial_j\zeta(\mathfrak{b})}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}), \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\Big]\Big|_h - \Big|\big[q^{\dagger}, \mathcal{R}_{j,\alpha}^{\mathrm{reg}}\big]\Big|_h - \Big|\Big[\sum_i\sum_{\mathfrak{b}}\overline{\partial_j\zeta(\mathfrak{b})}(\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger} - \pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}), \pi'_{j,\alpha}\Big]\Big|_h \geqslant C_{80}|z_j|^{-1}\big|\mathcal{R}_{j,\alpha}^{\mathrm{reg}}\big|_h - C_{81}.$$

We consider the following function:

$$k_{j,\alpha} := \log\Big(\frac{|\pi_{j,\alpha}|_h^2}{|\pi'_{j,\alpha}|_h^2}\Big) = \log\Big(1+\frac{|\mathcal{R}^{\mathrm{reg}}_{j,\alpha}|_h^2}{|\pi'_{j,\alpha}|_h^2}\Big).$$

Recall that we already know that $k_{j,\alpha}$ is bounded. Hence, we have the following inequality on $X^*(R_{80})$:

$$\Delta_j k_{j,\alpha} \leqslant -\frac{\big|[f_j^\dagger, \pi_{j,\alpha}]\big|_h^2}{|\pi_{j,\alpha}|^2} \leqslant -\frac{C_{82}}{|z_j|^2}\frac{|\mathcal{R}^{\mathrm{reg}}_{j,\alpha}|_h^2}{|\pi'_{j,\alpha}|_h^2} + C_{83} \leqslant -\frac{C_{84}}{|z_j|^2}\cdot k_{j,\alpha} + C_{85}.$$

By using an argument as in Section 7.3.3 (see also [**82**] or [**67**]), we obtain $k_{j,\alpha} = O\big(|z_j|^{\varepsilon(20)}\big)$, and hence $\big|\mathcal{R}^{\mathrm{reg}}_{j,\alpha}\big|_h = O\big(|z_j|^{\varepsilon(20)}\big)$. Thus, the proof of Theorem 7.2.4 is accomplished. □

7.3.8. Proof of Proposition 7.2.8

Lemma 7.3.6. — *We have the following estimates:*

$$\Big|\big[\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}, \pi_{\mathfrak{b}}^{\boldsymbol{m}(q)}\big]\Big|_h = O\Big(\exp\big(-\varepsilon_{10}|\boldsymbol{z}^{\boldsymbol{m}(p)}| - \varepsilon_{10}|\boldsymbol{z}^{\boldsymbol{m}(q)}|\big)\Big), \tag{112}$$

$$\Big|\big[\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}, \pi_{j,\alpha}\big]\Big|_h = \Big|\big[\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)}, \pi_{j,\alpha}^\dagger\big]\Big| = O\Big(\exp\big(-\varepsilon_{10}\cdot|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\cdot|z_j|^{\varepsilon_{10}}\Big), \tag{113}$$

$$\Big|\big[\pi_{i,\alpha}, \pi_{j,\alpha}^\dagger\big]\Big|_h = O\Big(|z_i|^{\varepsilon_{10}}\cdot|z_j|^{\varepsilon_{10}}\Big). \tag{114}$$

Proof. — Due to Corollary 7.2.2, we obtain the following:

$$\big[\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}, \pi_{\mathfrak{b}}^{\boldsymbol{m}(q)}\big] = \big[(\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger} - \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)}), \pi_{\mathfrak{b}}^{\boldsymbol{m}(q)}\big] = O\Big(\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\Big).$$

Similarly, we obtain $\big[\pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}, \pi_{\mathfrak{b}}^{\boldsymbol{m}(q)}\big] = O\big(\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(q)}|\big)\big)$. Then, we obtain (112). The estimates (113) and (114) can be shown using a similar argument together with Corollary 7.2.2 and Corollary 7.2.5. □

Lemma 7.3.7. — *We have the following estimates for $j = 1, \dots, \ell$:*

$$\begin{aligned} \big[f_j^{\mathrm{nil}}, \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}\big] &= O\Big(|z_j|^{-1}\big(-\log|z_j|\big)^{-1}\cdot\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\Big), \\ \big[f_j^{\mathrm{nil}}, \pi_{i,\alpha}^\dagger\big] &= O\Big(|z_j|^{-1}\big(-\log|z_j|\big)^{-1}\cdot|z_i|^{\varepsilon_2}\Big). \end{aligned} \tag{115}$$

We have the following estimates for $j = \ell+1, \dots, n$:

$$\big[f_j^{\mathrm{reg}}, \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}\big] = O\Big(\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\Big), \quad \big[f_j^{\mathrm{reg}}, \pi_{i,\alpha}^\dagger\big] = O\big(|z_i|^{\varepsilon_2}\big). \tag{116}$$

Proof. — We have the following equalities:

$$\big[f_j^{\mathrm{nil}}, \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger}\big] = \big[f_j^{\mathrm{nil}}, \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)\dagger} - \pi_{\mathfrak{a}}^{\boldsymbol{m}(p)}\big], \quad \big[f_j^{\mathrm{nil}}, \pi_{i,\alpha}^\dagger\big] = \big[f_j^{\mathrm{nil}}, \pi_{i,\alpha}^\dagger - \pi_{i,\alpha}\big].$$

Then, the estimate (115) follows. The estimate (116) can be shown similarly. □

The naturally defined map $\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(p)) \longrightarrow \overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(p-1))$ is denoted by $\overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)}$. We set

$$\mathfrak{U}(p) := \Big\{(\mathfrak{c},\mathfrak{c}')\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(p))^2 \,\Big|\, \mathfrak{c}\neq\mathfrak{c}',\ \overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)}(\mathfrak{c})=\overline{\eta}_{\boldsymbol{m}(p-1),\boldsymbol{m}(p)}(\mathfrak{c}')\Big\}.$$

We have the decomposition:

$$\mathrm{End}(E)=\bigoplus_{p}\ \bigoplus_{(\mathfrak{c},\mathfrak{c}')\in\mathfrak{U}(p)}\mathrm{Hom}\big(E^{\boldsymbol{m}(p)}_{\mathfrak{c}},E^{\boldsymbol{m}(p)}_{\mathfrak{c}'}\big)\oplus\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\mathrm{End}(E_{\mathfrak{a}},E_{\mathfrak{a}}).$$

For any section F of $\mathrm{End}(E)$, we have the corresponding decomposition:

$$F=\sum_{p}\ \sum_{(\mathfrak{c},\mathfrak{c}')\in\mathfrak{U}(p)}F^{\boldsymbol{m}(p)}_{\mathfrak{c},\mathfrak{c}'}+\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}F_{\mathfrak{a}}.$$

Lemma 7.3.8. — *We have the following estimates:*

$$(117)\qquad \big(\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}-\pi^{\boldsymbol{m}(p)}_{\mathfrak{b}}\big)^{\boldsymbol{m}(q)}_{\mathfrak{a},\mathfrak{a}'}=O\Big(\exp\big(-\varepsilon_{10}|\boldsymbol{z}^{\boldsymbol{m}(p)}|-\varepsilon_{10}|\boldsymbol{z}^{\boldsymbol{m}(q)}|\big)\Big),$$

$$(118)\qquad \big(\pi^{\dagger}_{j,\alpha}-\pi_{j,\alpha}\big)^{\boldsymbol{m}(q)}_{\mathfrak{a},\mathfrak{a}'}=O\Big(\exp\big(-\varepsilon_{10}|\boldsymbol{z}^{\boldsymbol{m}(q)}|\big)\cdot|z_j|^{\varepsilon_{10}}\Big),\quad (j=1,\dots,\ell),$$

$$(119)\qquad \big(f^{\mathrm{nil}\,\dagger}_{j}\big)^{\boldsymbol{m}(p)}_{\mathfrak{a},\mathfrak{b}}=O\Big(|z_j|^{-1}(-\log|z_j|)^{-1}\cdot\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\Big),\quad (j=1,\dots,\ell),$$

$$(120)\qquad \big(f^{\mathrm{reg}\,\dagger}_{j}\big)^{\boldsymbol{m}(p)}_{\mathfrak{a},\mathfrak{b}}=O\Big(\exp\big(-\varepsilon_1|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\Big),\quad (j=\ell+1,\dots,n).$$

Proof. — We have the following equalities for $\mathfrak{a}\neq\mathfrak{a}'$:

$$(121)\qquad \big(\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}-\pi^{\boldsymbol{m}(p)}_{\mathfrak{b}}\big)^{\boldsymbol{m}(q)}_{\mathfrak{a},\mathfrak{a}'}=\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}'}\circ\big(\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}-\pi^{\boldsymbol{m}(p)}_{\mathfrak{b}}\big)\circ\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}}$$
$$=\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}'}\circ\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}\circ\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}}=\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}'}\circ\Big(\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}\circ\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}}-\pi^{\boldsymbol{m}(q)}_{\mathfrak{a}}\circ\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{b}}\Big).$$

Then, the estimate (117) follows from (112). The other estimates can be shown similarly. □

For each $j=1,\dots,\ell$, we have the decomposition:

$$\mathrm{End}(E)=\bigoplus_{\alpha,\beta\in\mathcal{S}p(\theta,j)}\mathrm{Hom}(E_{j,\alpha},E_{j,\beta}).$$

For a section F of $\mathrm{End}(E)$, we have the corresponding decomposition $F=\sum F_{j,\alpha,\beta}$.

Lemma 7.3.9. — *We have the following estimates:*

$$\big(\pi^{\boldsymbol{m}(p)\dagger}_{\mathfrak{a}}-\pi^{\boldsymbol{m}(p)}_{\mathfrak{a}}\big)_{j,\alpha,\beta}=O\Big(\exp\big(-\varepsilon_{11}|\boldsymbol{z}^{\boldsymbol{m}(p)}|\big)\cdot|z_j|^{\varepsilon_{11}}\Big).$$

For $p=1,\dots,\ell$ and $\gamma\in\mathcal{S}p(\theta,\gamma)$,

$$\big(\pi^{\dagger}_{p,\gamma}-\pi_{p,\gamma}\big)_{j,\alpha,\beta}=O\big(|z_p|^{\varepsilon_{11}}\cdot|z_j|^{\varepsilon_{11}}\big).$$

For $p=1,\dots,\ell$,

$$\big(f^{\mathrm{nil}\,\dagger}_{p}\big)_{j,\alpha,\beta}=O\big(|z_j|^{\varepsilon_{11}}|z_p|^{-1}(-\log|z_p|)^{-1}\big).$$

For $p=\ell+1,\dots,n$,

$$\big(f^{\mathrm{reg}\,\dagger}_{p}\big)_{j,\alpha,\beta}=O\big(|z_j|^{\varepsilon_{11}}\big).$$

Proof. — We can show them using the argument in the proof of Lemma 7.3.8. □

Proposition 7.2.8 follows from Lemma 7.3.8 and Lemma 7.3.9. □

7.3.9. Proof of Proposition 7.2.9. — Let us consider the following:

$$\overline{\Phi} := \sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)} d\overline{\mathfrak{a}}\cdot\pi_{\mathfrak{a}} + \sum_{j=1}^{\ell}\sum_{\alpha\in\mathcal{S}p(\theta,j)} \alpha\cdot\frac{d\overline{z}_j}{\overline{z}_j}\cdot\pi_{j,\alpha}. \tag{122}$$

We have the following:

$$\begin{aligned}\theta^{\dagger}-\overline{\Phi} &= \sum_{p}\sum_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \overline{d\zeta_{\boldsymbol{m}(i)}(\mathfrak{a})}\cdot\big(\pi_{\mathfrak{a}}^{\boldsymbol{m}(i)\dagger}-\pi_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)\\ &+\sum_{j=1}^{\ell}\sum_{\alpha\in\mathcal{S}p(\theta,j)}\alpha\cdot\frac{d\overline{z}_j}{\overline{z}_j}\cdot\big(\pi_{j,\alpha}^{\dagger}-\pi_{j,\alpha}\big)+\sum_{j=1}^{\ell} f_j^{\mathrm{nil}\,\dagger}\cdot d\overline{z}_j+\sum_{j=\ell+1}^{n} f_j^{\mathrm{reg}\,\dagger}\cdot d\overline{z}_j.\end{aligned} \tag{123}$$

From Proposition 7.2.8, we obtain the following estimate with respect to h and $g_{\mathbf{p}}$:

$$\big(\theta^{\dagger}-\overline{\Phi}\big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} = O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}|\big)\cdot\mathcal{Q}_{\varepsilon}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\Big). \tag{124}$$

Then, we obtain the following estimate with respect to h and $g_{\mathbf{p}}$:

$$\begin{aligned}\big[\theta,\theta^{\dagger}\big]_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} &= \big[\theta,\theta^{\dagger}-\overline{\Phi}\big]_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\\ &=\Big(d(\mathfrak{a}-\mathfrak{a}')+\sum_{j=1}^{\ell}(\alpha_j-\alpha_j')\frac{Oz_j}{z_j}+O(1)\Big)\cdot\big(\theta^{\dagger}-\overline{\Phi}\big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')}\\ &= O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}|\big)\cdot\mathcal{Q}_{\varepsilon}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\Big).\end{aligned} \tag{125}$$

Thus, the proof of Proposition 7.2.9 is accomplished. □

7.4. The associated good filtered λ-flat bundle

7.4.1. Statements and some notation. — Let X be a complex manifold, and let D be a simple normal crossing hypersurface of X with the decomposition into irreducible components $D=\bigcup_{i\in\Lambda}D_i$. Let $(E,\overline{\partial}_E,\theta,h)$ be a good wild harmonic bundle on $X\setminus D$. Let $\mathcal{E}^{\lambda}$ denote the holomorphic vector bundle $(E,\overline{\partial}_E+\lambda\theta^{\dagger})$ on $X\setminus D$.

Notation 7.4.1. — Let $\boldsymbol{a}=(a_i\,|\,i\in\Lambda)\in\boldsymbol{R}^{\Lambda}$. Let U be an open subset of X with a holomorphic coordinate $(z_1,\ldots,z_n)$ such that $U\cap D=\bigcup_{j=1}^{\ell}\{z_j=0\}$. For each $j=1,\ldots,\ell$, we have some $i(j)\in\Lambda$ such that $D_{i(j)}\cap U=\{z_j=0\}$. Let $b_j:=a_{i(j)}$. We define

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}(U):=\Big\{f\in\mathcal{E}^{\lambda}(U\setminus D)\,\Big|\,|f|_h=O\Big(\prod_{j=1}^{\ell}|z_j|^{-b_j-\varepsilon}\Big),\ \forall\varepsilon>0\Big\}. \tag{126}$$

Taking the sheafification, we obtain the $\mathcal{O}_X$-module $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$. We also obtain the $\mathcal{O}_X(*D)$-module $\mathcal{P}\mathcal{E}^{\lambda} := \bigcup_{\boldsymbol{a}} \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$. The filtered sheaf $\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda} \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^{\ell}\big)$ is denoted by $\mathcal{P}_*\mathcal{E}^{\lambda}$. □

Remark 7.4.2. — In our previous papers (for example [**67**]), we used the symbol ${}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ to denote $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$. Since we will consider several kinds of prolongation in the wild case, we prefer the symbol $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ for distinction. □

Theorem 7.4.3. — *$\mathcal{P}_*\mathcal{E}^{\lambda}$ is a filtered bundle on (X, D). The weak norm estimate up to log order holds in the sense of Theorem* 21.3.2.

Proof. — Due to Corollary 7.2.10, $(\mathcal{E}^{\lambda}, h)$ is acceptable. Hence, the claim follows from Theorem 21.3.1. □

We use the symbol ${}^{i}F$ to denote the induced filtration of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}_{|D_i}$ for $i \in \Lambda$.

We have the induced Higgs field and the induced Hermitian metric of $\mathrm{End}(E)$, which are denoted by the same symbols θ and h, respectively. Note that the harmonic bundle $\big(\mathrm{End}(E), \overline{\partial}_{\mathrm{End}(E)}, h, \theta\big)$ is a wild harmonic bundle on $X \setminus D$, but not necessarily good. (See an example in Section 2.1.1.4.) We use the symbol $\big(\mathrm{End}(\mathcal{E}^{\lambda}), \mathbb{D}^{\lambda}\big)$ to denote the associated λ-flat bundle. Although $\big(\mathrm{End}(E), \overline{\partial}_{\mathrm{End}(E)}, h, \theta\big)$ is not necessarily good, $\big(\mathrm{End}(\mathcal{E}^{\lambda}), h\big)$ is acceptable, as remarked in Corollary 7.2.10. The prolongment corresponding to $\boldsymbol{a}$ is denoted by $\mathcal{P}_{\boldsymbol{a}}\,\mathrm{End}(\mathcal{E}^{\lambda})$. We will implicitly use the following proposition in the argument below, which immediately follows from Proposition 21.3.3.

Proposition 7.4.4. — *$\mathcal{P}_0\,\mathrm{End}(\mathcal{E}^{\lambda})$ is naturally isomorphic to the sheaf of local endomorphisms f of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ such that $f_{|D_i}$ preserve the parabolic filtrations ${}^{i}F$ for $i \in \Lambda$.* □

We will prove the following theorem in Section 7.4.3.

Theorem 7.4.5

- *$\mathbb{D}^{\lambda}$ is a meromorphic flat λ-connection with respect to $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$.*
- *$(\mathcal{P}_*\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$ is a good filtered λ-flat bundle. If θ is unramifiedly good, $(\mathcal{P}_*\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$ is an unramified good filtered λ-flat bundle.*
- *Under the setting in Section* 7.2.1, *the set of the irregular values is given as follows:*

$$\mathrm{Irr}(\mathcal{P}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda}) = \big\{(1+|\lambda|^2)\cdot \mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\theta)\big\}.$$

The claims are local, and can be easily reduced to the unramified case. Hence, we may and will give the proof under the setting in Section 7.2.1. Then, the claims in the case $\lambda = 0$ are a direct consequence of Simpson's Main estimate: The decomposition (92) is prolonged to $\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{0}, \mathbb{D}^{0}\big) = \bigoplus\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{0}_{\mathfrak{a}}, \mathbb{D}^{0}_{\mathfrak{a}}\big)$ on X due to the asymptotic orthogonality in Theorem 7.2.1, and we obtain that $\mathbb{D}^{0}_{\mathfrak{a}} - d\mathfrak{a}\cdot \mathrm{id}_{\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{0}_{\mathfrak{a}}}$ are logarithmic, due to the estimate of the norm of the Higgs field in Theorem 7.2.4.

7.4.1.1. Characterization of the Stokes filtration. — Before going to the proof for the case $\lambda \neq 0$, let us state the characterization of the Stokes filtration of the meromorphic λ-flat bundle $(\mathcal{PE}^\lambda, \mathbb{D}^\lambda)$ $(\lambda \neq 0)$ in terms of the growth order of the norms of the flat sections with respect to h. Since the property is local, we give the statement under the setting in Section 7.2.1. Let S be a small multi-sector in $X \setminus D$. We have the partial Stokes filtration $\mathcal{F}^{S,\boldsymbol{m}(i)}$ of $\mathcal{PE}^\lambda_{|\overline{S}}$ at the level $\boldsymbol{m}(i)$ (Section 3.7.3) indexed by the ordered set $\big(\overline{\mathrm{Irr}}(\theta, \boldsymbol{m}(i)), \leqslant^\lambda_S\big)$. It is flat with respect to $\mathbb{D}^\lambda$, and it is characterized by the growth order of the flat sections with respect to h.

Proposition 7.4.6. — *Let f be a flat section of $\mathcal{E}^\lambda_{|S}$. We have $f \in \mathcal{F}^{S,\boldsymbol{m}(i)}_{\mathfrak{b}}$ if and only if the following estimate holds for some $C > 0$ and $M > 0$:*

$$\big|f \cdot \exp\big((\lambda^{-1} + \overline{\lambda}) \cdot \overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{b})\big)\big|_h = O\Big(\exp\big(C|\boldsymbol{z}^{\boldsymbol{m}(i+1)}|\big) \cdot \prod_{k(i+1)<j\leqslant \ell} |z_j|^{-M}\Big).$$

Here, $k(i)$ are determined by $\boldsymbol{m}(i) \in \mathbb{Z}^{k(i)}_{<0} \times \boldsymbol{0}$.

Proof. — It follows from Proposition 3.7.23 and the weak norm estimate for the acceptable bundles. (See Theorems 7.4.3 and 21.3.2). □

7.4.1.2. Example 1. — Let $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Take $(\boldsymbol{a}, \boldsymbol{\alpha}) \in \boldsymbol{R}^\ell \times \boldsymbol{C}^\ell$ and $\mathfrak{a} \in M(X, D)$. We assume that $\boldsymbol{z}^{-\boldsymbol{m}}\,\mathfrak{a}$ is nowhere vanishing holomorphic function on X for some $\boldsymbol{m} \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$. Let $L(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a})$ be the harmonic bundle on $X \setminus D$ given as follows:

$$L(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a}) = \mathcal{O}_{X\setminus D} \cdot e, \quad \theta = d\mathfrak{a} + \sum_{i=1}^{\ell} \alpha_i \frac{dz_i}{z_i}, \quad h(e, e) = \prod_{i=1}^{\ell} |z_i|^{-2a_i}.$$

The associated operators ∂ and $\theta^\dagger$ are as follows:

$$\partial\, e = e\Big(-\sum_{i=1}^{\ell} a_i \frac{dz_i}{z_i}\Big), \quad \theta^\dagger = d\overline{\mathfrak{a}} + \sum_{i=1}^{\ell} \overline{\alpha}_i \frac{d\overline{z}_i}{\overline{z}_i}.$$

Let $\mathcal{L}^\lambda(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a})$ be the associated λ-flat bundle. We have the holomorphic section u^λ of $\mathcal{L}^\lambda(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a})$ given as follows:

$$u^\lambda = \exp\Big(-\lambda\overline{\mathfrak{a}} + \overline{\lambda}\mathfrak{a} - \sum_{i=1}^{\ell} \lambda\, \overline{\alpha}_i \log|z_i|^2\Big)\, e.$$

We can easily check the following:

$$|u^\lambda|_h = \prod_{i=1}^{\ell} |z_i|^{-\mathfrak{p}(\lambda, a_i, \alpha_i)}, \quad \mathbb{D}^\lambda u^\lambda = u^\lambda\Big((1 + |\lambda|^2)\, d\mathfrak{a} + \sum_{i=1}^{\ell} \mathfrak{e}(\lambda, a_i, \alpha_i) \frac{dz_i}{z_i}\Big).$$

See Subsection 2.8.2 for the maps $\mathfrak{p}(\lambda)$ and $\mathfrak{e}(\lambda)$. We put $\mathfrak{p}(\lambda, \boldsymbol{a}, \boldsymbol{\alpha}) = \big(\mathfrak{p}(\lambda, a_i, \alpha_i)\big) \in \boldsymbol{R}^\ell$. We have a natural isomorphism $\mathcal{P}_{\mathfrak{p}(\lambda,\boldsymbol{a},\boldsymbol{\alpha})}\mathcal{L}^\lambda(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a}) \simeq \mathcal{O}_X \cdot u^\lambda$. The residues $\mathrm{Res}_i(\mathbb{D}^\lambda)$ on $\mathcal{P}_{\mathfrak{p}(\lambda,\boldsymbol{u})}\mathcal{L}^\lambda(\boldsymbol{a}, \boldsymbol{\alpha}, \mathfrak{a})_{|D_i}$ are given by the multiplication of $\mathfrak{e}(\lambda, a_i, \alpha_i)$.

Note that if $\mathfrak{a} \neq 0$, u^λ depends on λ in a non-holomorphic way.

7.4.1.3. Example 2. — Let $X := \Delta$ and $D := \{O\}$. Let V be a finite dimensional $\boldsymbol{C}$-vector space with a nilpotent map N. Recall that we have a tame harmonic bundle $E(V,N)$ on $X \setminus D$ with the following property:

- Let $\mathcal{E}^\lambda(V,N)$ be the associated λ-flat bundle. The parabolic structure of $\mathcal{P}_0\mathcal{E}^\lambda(V,N)$ is trivial, and we have an isomorphism:

$$(\mathcal{P}_0\mathcal{E}^\lambda(V,N)_{|O}, \mathrm{Res}(\mathbb{D}^\lambda)) \simeq (V,N)$$

For example, it can be constructed as follows. We put $y := -\log|z|^2$. Recall that we have the harmonic bundle $\mathrm{Mod}(2) := (E, \overline{\partial}_E, \theta, h)$ given as follows:

$$E := \mathcal{O}_{X\setminus D}\, e_1 \oplus \mathcal{O}_{X\setminus D}\, e_{-1}, \quad H(h,\boldsymbol{e}) = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \quad \theta\, \boldsymbol{e} = \boldsymbol{e} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dz}{z}.$$

Here, $H(h,\boldsymbol{e})$ is the matrix whose (i,j)-entries are $h(e_i,e_j)$. We take a frame $\boldsymbol{v}^\lambda$ of the holomorphic bundle $\mathcal{E}^\lambda$ given as follows:

$$\boldsymbol{v}^\lambda = \boldsymbol{e} \begin{pmatrix} 1 & -\lambda y^{-1} \\ 0 & 1 \end{pmatrix}. \tag{127}$$

Then $\boldsymbol{v}^\lambda$ gives a frame of $\mathcal{P}_0\mathcal{E}$, and the following holds:

$$\mathbb{D}^\lambda \boldsymbol{v}^\lambda = \boldsymbol{v}^\lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dz}{z}, \quad \mathrm{Res}(\mathbb{D}^\lambda)\boldsymbol{v}^\lambda_{|O} = \boldsymbol{v}^\lambda_{|O} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, we can construct $E(V,N)$ as a direct sum of some symmetric tensor products of $\mathrm{Mod}(2)$.

7.4.2. Construction of a decomposition. — We use the setting in Section 7.2.1. Let $\lambda \neq 0$. For $\boldsymbol{m}(i) = (m_1(i), \dots, m_\ell(i)) \in \mathcal{M}$, we put

$$\mathfrak{s}(i) := \{j \mid m_j(i) \neq 0\}.$$

Lemma 7.4.7. — *Let ε be any small positive number, and let N be any large number. We can take holomorphic sections $p^{\boldsymbol{m}(i)}_{\mathfrak{b}}$ of $\mathcal{P}_0\,\mathrm{End}(\mathcal{E}^\lambda)$ for $\boldsymbol{m}(i) \in \mathcal{M}$ and $\mathfrak{b} \in \overline{\mathrm{Irr}}(\theta, \boldsymbol{m}(i))$, such that the following holds:*

$$\left|p^{\boldsymbol{m}(i)}_{\mathfrak{b}} - \pi^{\boldsymbol{m}(i)}_{\mathfrak{b}}\right|_h = O\Big(\prod_{j\in\mathfrak{s}(i)} |z_j|^{2N} \cdot \prod_{j\notin\mathfrak{s}(i)} |z_j|^{-\varepsilon}\Big),$$

$$\big(p^{\boldsymbol{m}(i)}_{\mathfrak{b}}\big)^2 = p^{\boldsymbol{m}(i)}_{\mathfrak{b}}, \quad [p^{\boldsymbol{m}(i)}_{\mathfrak{b}_1}, p^{\boldsymbol{m}(i)}_{\mathfrak{b}_2}] = 0, \quad p^{\boldsymbol{m}(i-1)}_{\mathfrak{a}} = \sum_{\substack{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i)) \\ \overline{\eta}_{i-1,i}(\mathfrak{b})=\mathfrak{a}}} p^{\boldsymbol{m}(i)}_{\mathfrak{b}}.$$

Here, $\overline{\eta}_{i-1,i} : \overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i)) \to \overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i-1))$ denotes the naturally defined map.

Proof. — Let $d''_\lambda := \overline{\partial}_E + \lambda\theta^\dagger$. We have $d''_\lambda \pi^{\boldsymbol{m}(i)}_{\mathfrak{b}} = \lambda\cdot\big[\theta^\dagger, \pi^{\boldsymbol{m}(i)}_{\mathfrak{b}}\big] = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(i)}|)\big)$ with respect to h and g_{p}, due to Proposition 7.2.8. Let ε_{11} be sufficiently smaller than ε, and let N_{11} and N_{12} be sufficiently larger than N. By Lemma 21.2.3, we can take sections $s^{\boldsymbol{m}(i)}_{\mathfrak{b}}$ of $\operatorname{End}(E)$ on $X \setminus D$ such that the following holds:

$$d''_\lambda s^{\boldsymbol{m}(i)}_{\mathfrak{b}} = d''_\lambda \pi^{\boldsymbol{m}(i)}_{\mathfrak{b}},$$

$$\int \big|s^{\boldsymbol{m}(i)}_{\mathfrak{b}}\big|^2_h \prod_{j\in\mathfrak{s}(i)} |z_j|^{-10N_{11}} \prod_{j\notin\mathfrak{s}(i)} |z_j|^{\varepsilon_{11}} \Big(-\sum_{j=1}^{\ell} \log|z_j|\Big)^{N_{12}} \operatorname{dvol}_{g_{\mathrm{p}}} < \infty.$$

By Lemma 21.9.1, we obtain the following estimate:

$$\big|s^{\boldsymbol{m}(i)}_{\mathfrak{b}}\big|_h = O\Big(\prod_{j\in\mathfrak{s}(i)} |z_j|^{4N_{11}} \prod_{j\notin\mathfrak{s}(i)} |z_j|^{-2\varepsilon_{11}}\Big).$$

We put $\overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}} := \pi^{\boldsymbol{m}(i)}_{\mathfrak{b}} - s^{\boldsymbol{m}(i)}_{\mathfrak{b}}$. If ε_{11} is sufficiently small, $\overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}}$ gives a section of $\mathcal{P}_0 \operatorname{End}(\mathcal{E}^\lambda)$. Moreover, we have the following:

$$\Big(\big(\overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}}\big)^2 - \overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}}\Big)\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(i)}\mathcal{E}^\lambda,$$

$$\big[\overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}_1}, \overline{p}^{\boldsymbol{m}(j)}_{\mathfrak{b}_2}\big]\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda\big) \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(j)}\mathcal{E}^\lambda \quad (i \leqslant j),$$

$$\Big(\overline{p}^{\boldsymbol{m}(i-1)}_{\mathfrak{a}} - \sum_{\substack{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i))\\ \overline{\eta}_{i-1,i}(\mathfrak{b})=\mathfrak{a}}} \overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}}\Big)\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(i)}\mathcal{E}^\lambda.$$

Here, $\boldsymbol{\delta}(i)$ denote the elements of $\mathbb{Z}^\ell$ such that the j-th components $\delta_j(i)$ are given by

$$\delta_j(i) := \begin{cases} 1 & (j \in \mathfrak{s}(i)),\\ 0 & (j \notin \mathfrak{s}(i)).\end{cases}$$

We would like to modify $\overline{p}^{\boldsymbol{m}(i)}_{\mathfrak{b}}$ inductively on i so that the desired conditions are satisfied. Consider the following state $P(i)$:

$P(i)$: We have $p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}} \in \mathcal{P}_0 \operatorname{End}(\mathcal{E}^\lambda)$ for any $\ell < i$ and for $\mathfrak{b} \in \overline{\operatorname{Irr}}(\theta, \boldsymbol{m}(\ell))$, such that the following holds:

$$\big(p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}} - \overline{p}^{\boldsymbol{m}(\ell)}_{\mathfrak{b}}\big)\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda \subset \mathcal{P}_{\boldsymbol{a}-2N_{11}\boldsymbol{\delta}(\ell)}\mathcal{E}^\lambda,$$

$$\big(p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}}\big)^2 = p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}}, \quad \big[p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}_1}, p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}_2}\big] = 0, \quad p^{\boldsymbol{m}(\ell-1)}_{\mathfrak{a}} = \sum_{\substack{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(\ell))\\ \overline{\eta}_{\ell-1,\ell}(\mathfrak{b})=\mathfrak{a}}} p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}},$$

$$\big[p^{\boldsymbol{m}(\ell)}_{\mathfrak{b}_1}, \overline{p}^{\boldsymbol{m}(j)}_{\mathfrak{b}_2}\big] = 0 \quad (\ell < i \leqslant j).$$

Let us give a procedure from $P(i-1)$ to $P(i)$.

First, we give a procedure $P(0) \Longrightarrow P(1)$. Take an injection $\varphi : \overline{\operatorname{Irr}}(\theta, \boldsymbol{m}(0)) \to \mathbb{Z}$. We consider the following:

$$\overline{\Phi}_0 := \sum_{\mathfrak{a}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(0))} \varphi(\mathfrak{a}) \cdot \overline{p}^{\boldsymbol{m}(0)}_{\mathfrak{a}}.$$

Note that $\overline{\Phi}_0$ gives an endomorphism of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ for each $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$, which preserves the filtrations ${}^jF$ $(j = 1, \ldots, \ell)$. We have the decomposition $\mathcal{P}_0\mathcal{E}^{\lambda} = \bigoplus_{m \in \mathbb{Z}} V_m$ such that (i) $\overline{\Phi}_0(V_m) \subset V_m$, (ii) the eigenvalues of $\overline{\Phi}_0$ on V_m are close to m, (iii) it is compatible with the filtrations ${}^jF$ $(j = 1, \ldots, \ell)$. Let $p_{\mathfrak{a}}^{\boldsymbol{m}(0)}$ denote the projection onto $V_{\varphi(\mathfrak{a})}$. Then, we have $\big(p_{\mathfrak{a}}^{\boldsymbol{m}(0)} - \overline{p}_{\mathfrak{a}}^{\boldsymbol{m}(0)}\big)\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}\big) \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(0)}\mathcal{E}^{\lambda}$. We also have (i) $\big[p_{\mathfrak{a}}^{\boldsymbol{m}(0)}, p_{\mathfrak{b}}^{\boldsymbol{m}(0)}\big] = 0$, (ii) $(p_{\mathfrak{a}}^{\boldsymbol{m}(0)})^2 = p_{\mathfrak{a}}^{\boldsymbol{m}(0)}$, (iii) $\sum p_{\mathfrak{a}}^{\boldsymbol{m}(0)} = \mathrm{id}$.

For $i > 0$ and $\mathfrak{b} \in \mathrm{Irr}(\theta, \boldsymbol{m}(i))$, we have the decomposition

$$\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)} = \sum (\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)})_{m,n}, \quad (\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)})_{m,n} \in \mathrm{Hom}(V_n, V_m).$$

We put $\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime} := \sum_n (\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)})_{n,n}$.

Lemma 7.4.8. — *We have* $(\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)\prime} - \overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)})(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}) \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(i)}\mathcal{E}^{\lambda}$.

Proof. — We put $\Phi_0 := \sum \varphi(\mathfrak{a}) p_{\mathfrak{a}}^{\boldsymbol{m}(0)}$. We have $(\Phi_0 - \overline{\Phi}_0)\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda} \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(i)}\mathcal{E}^{\lambda}$. For any positive integer M, let Φ_0^M denote the M-iteration of Φ_0. Then, we have the following:

$$[\Phi_0^M, \overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)}]\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda} \subset \mathcal{P}_{\boldsymbol{a}-3N_{11}\boldsymbol{\delta}(i)}\mathcal{E}^{\lambda}, \quad [\Phi_0^M, \overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)}] = \sum (m^M - n^M)(\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(i)})_{m,n}.$$

Then, we can easily derive the claim of Lemma 7.4.8. □

We replace $\overline{p}_{\mathfrak{a}}^{\boldsymbol{m}(i)}$ with $\overline{p}_{\mathfrak{a}}^{\boldsymbol{m}(i)\prime}$, and then we arrive at the state $P(1)$. Assume we are in the state $P(i-1)$. We apply the above argument for $P(0) \Longrightarrow P(1)$ to each $\mathrm{Im}\, p_{\mathfrak{a}}^{\boldsymbol{m}(i-1)}$ with the endomorphisms $\overline{p}_{\mathfrak{b}}^{\boldsymbol{m}(j)}$ $(j \geqslant i)$. Then, we can arrive at $P(i)$. When we arrive at $P(L)$, the proof of Lemma 7.4.7 is finished. □

7.4.3. Proof of Theorem 7.4.5. — Let δ'_{λ} be the $(1,0)$-operator determined by the condition $d''_{\lambda} + \delta'_{\lambda}$ is unitary, i.e., $\delta'_{\lambda} = \partial - \overline{\lambda}\theta$. Let $R(h)$ denote the curvature of $d''_{\lambda} + \delta'_{\lambda}$. We have $\mathbb{D}^{\lambda} = \overline{\partial}_E + \lambda\theta^{\dagger} + \lambda\partial_E + \theta = \overline{\partial}_E + \lambda\theta^{\dagger} + \lambda\delta'_{\lambda} + (1 + |\lambda|^2)\theta$. Let $p_{\mathfrak{b}}^{\boldsymbol{m}(i)}$ be as in Lemma 7.4.7. We put

$$\mathbb{D}_0^{\lambda} := \overline{\partial}_E + \lambda\theta^{\dagger} + \lambda\delta'_{\lambda} + (1 + |\lambda|^2)\Big(\theta - \sum_{i=1}^{L}\sum_{\mathfrak{b}} d\zeta_{\boldsymbol{m}(i)}(\mathfrak{b}) \cdot p_{\mathfrak{b}}^{\boldsymbol{m}(i)}\Big).$$

It gives a holomorphic connection of $\mathcal{E}^{\lambda}$ on $X \setminus D$, which is not necessarily flat. We have $\mathbb{D}^{\lambda} = \mathbb{D}_0^{\lambda} + \sum_i \sum_{\mathfrak{b}} d\zeta_{\boldsymbol{m}(i)}(\mathfrak{b}) \cdot p_{\mathfrak{b}}^{\boldsymbol{m}(i)}$.

Proposition 7.4.9. — $\mathbb{D}_0^{\lambda}$ *is logarithmic with respect to* $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$.

Proof. — By considering the tensor product with the rank one harmonic bundle, we may and will assume $\boldsymbol{a} = (0, \ldots, 0)$. First, we consider the case where D is smooth, say $D = D_1$. Let $\pi_j : X \setminus D \to D_j$ denote the natural projection for $j = 1, \ldots, n$. Let $d''_{\lambda,j}$, $\delta'_{\lambda,j}$ and $R_j(h)$ denote the restriction of d''_{λ}, δ'_{λ} and $R(h)$ to the curves $\pi_j^{-1}(Q)$ $(Q \in D_j)$, respectively. Let f be a holomorphic section of $\mathcal{P}_0\mathcal{E}^{\lambda}$. Because of the acceptability of $(\mathcal{E}^\lambda, h)$, we have $|f|_h \leqslant C_{20} \cdot \big(-\log|z_1|\big)^{N_{20}}$ (Proposition 21.2.8). In the following estimate, we do not have to be concerned with the signature.

Lemma 7.4.10. — *Let $\chi(z_j)$ be any test function on $\big\{z_j \in \boldsymbol{C} \,\big|\, |z_j| < 1\big\}$. Let $j \neq 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds for any $Q \in D_j \setminus D_1$:*

$$\int_{\pi_j^{-1}(Q)} \big(\delta'_{\lambda,j}(\chi\cdot f),\, \delta'_{\lambda,j}(\chi\cdot f)\big)_h < C_\varepsilon \cdot |z_1(Q)|^{-2\varepsilon}. \tag{128}$$

Proof. — The left-hand side of (128) can be rewritten as follows:

$$\begin{aligned}(129)\quad &\pm\int_{\pi_j^{-1}(Q)} \big(\chi\cdot f,\, d''_{\lambda,j}\delta'_{\lambda,j}(\chi\cdot f)\big)_h = \\ &\pm\int_{\pi_j^{-1}(Q)} \big(\chi\cdot f,\, R_j(h)(\chi\cdot f)\big)_h \pm \int_{\pi_j^{-1}(Q)} \big(\chi\cdot f,\, \delta'_{\lambda,j}d''_{\lambda,j}(\chi\cdot f)\big)_h = \\ &\qquad\pm\int_{\pi_j^{-1}(Q)} \big(\chi\cdot f,\, R_j(h)(\chi\cdot f)\big)_h \pm \int_{\pi_j^{-1}(Q)} \big(d''_{\lambda,j}(\chi\cdot f),\, d''_{\lambda,j}(\chi\cdot f)\big)_h.\end{aligned}$$

Then, the claim of Lemma 7.4.10 easily follows. □

Lemma 7.4.11. — *Fix a small $\varepsilon > 0$. There exists a constant C such that the following holds for any $Q \in D_1$:*

$$\int_{\pi_1^{-1}(Q)} \big(\delta'_{\lambda,1}f,\, \delta'_{\lambda,1}f\big)_h \cdot |z_1|^{2\varepsilon} < C.$$

Proof. — Let ρ be a non-negative C^∞-function on $\boldsymbol{R}$ such that $\rho(t) = 1$ for $t \leqslant 1/2$ and $\rho(t) = 0$ for $t \geqslant 2/3$. We put $\chi_M(z_1) := \rho\big(-M^{-1}\log|z_1|\big)$ for any $M > 1$. We only have to show that there exists a constant $C > 0$, which is independent of $Q \in D_1$ and $M > 1$, such that the following holds:

$$\int_{\pi_1^{-1}(Q)} \big(\delta'_{\lambda,1}(\chi_M f),\, \delta'_{\lambda,1}(\chi_M f)\big)_h |z_1|^{2\varepsilon} < C. \tag{130}$$

In the following argument, we will ignore the contribution to the Stokes formula from the integral over $\partial\pi_1^{-1}(Q)$, because they are uniformly dominated. (Recall that D is assumed to be smooth.) The left-hand side of (130) can be rewritten as follows, up to the contribution from $\partial\pi_1^{-1}(Q)$:

(131)
$$\pm\int_{\pi_1^{-1}(Q)} \big(\chi_M f,\, d''_{\lambda,1}\delta'_{\lambda,1}(\chi_M f)\big)_h |z_1|^{2\varepsilon} \pm \int_{\pi_1^{-1}(Q)} \big(\chi_M f,\, \delta'_{\lambda,1}(\chi_M f)\big)_h\, \varepsilon\, |z_1|^{2\varepsilon}\frac{dz_1}{z_1}.$$

The first term can be rewritten as follows, up to the contribution of $\partial\pi_1^{-1}(Q)$:

(132)

$$\pm\int_{\pi_1^{-1}(Q)}\big(\chi_M f,\, R_1(h)(\chi_M f)\big)\cdot|z_1|^{2\varepsilon}\pm\int_{\pi_1^{-1}(Q)}\big(\chi_M f,\, \delta'_{\lambda,1}d''_{\lambda,1}(\chi_M f)\big)\cdot|z_1|^{2\varepsilon}$$
$$=\pm\int_{\pi_1^{-1}(Q)}\big(\chi_M f,\, R_1(h)(\chi_M f)\big)\cdot|z_1|^{2\varepsilon}\pm\int_{\pi_1^{-1}(Q)}\big(d''_{\lambda,1}(\chi_M f),\, d''_{\lambda,1}(\chi_M f)\big)\cdot|z_1|^{2\varepsilon}$$
$$\pm\int_{\pi_1^{-1}(Q)}\big(\chi_M f,\, d''_{\lambda,1}(\chi_M f)\big)\cdot\varepsilon\cdot|z_1|^{2\varepsilon}\frac{dz_1}{z_1}.$$

The right-hand side of (132) is uniformly dominated. The second term in (131) is as follows, up to the contribution of $\partial\pi_1^{-1}(Q)$:

$$\pm\int_{\pi_1^{-1}(Q)}(\chi_M f,\,\chi_M f)\,\varepsilon^2\,|z_1|^{2\varepsilon}\frac{dz_1\,d\overline{z}_1}{|z_1|^2}\pm\int_{\pi_1^{-1}(Q)}\big(d''_{\lambda,1}(\chi_M f),\,\chi_M f\big)\,\varepsilon\,|z_1|^{2\varepsilon}\frac{dz_1}{z_1}.$$

It is bounded uniformly for $Q\in D_1$. Then, it is easy to show the existence of a constant C, which is independent of N and P such that (130) holds. Thus, the proof of Lemma 7.4.11 is finished. □

Lemma 7.4.12. — *We put $\Psi:=\theta-\sum_i\sum_{\mathfrak{b}}d\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})\cdot p_{\mathfrak{b}}^{\boldsymbol{m}(i)}$. Then we have the estimates $\Psi(\partial_j)=O(1)$ $(j\neq 1)$ and $\Psi(\partial_1)=O(|z_1|^{-1})$ with respect to h.*

Proof. — We have the boundedness of $\partial_j\zeta_{\boldsymbol{m}}(\mathfrak{b})\cdot\big(\pi_{\mathfrak{b}}^{\boldsymbol{m}}-p_{\mathfrak{b}}^{\boldsymbol{m}}\big)$ by construction of $p_{\mathfrak{b}}^{\boldsymbol{m}}$. According to Theorem 7.2.1 and Theorem 7.2.4, the dz_j-components $(j\neq 1)$ of $\theta-\sum_i\sum_{\mathfrak{b}}d\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})\cdot\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)}$ are $O(1)$, and the dz_1-component is $O\big(|z_1|^{-1}\big)$. Then, the claim of Lemma 7.4.12 follows. □

Let $j\neq 1$. According to Lemma 7.4.10 and Lemma 7.4.12, there exists a constant $C_{10}>0$ such that the following holds for any $Q\in D_j\setminus D_1$:

$$\int_{\substack{\pi_j^{-1}(Q)\\|z_j|<1/2}}\big|\mathbb{D}_0^{\lambda}(\partial_j)f_{|\pi_j^{-1}(Q)}\big|^2\cdot|dz_j\cdot d\overline{z}_j|<C_{10}\big|z_1(Q)\big|^{-2\varepsilon}.$$

Let $\boldsymbol{v}=(v_p)$ be a frame of $\mathcal{P}_0\mathcal{E}^{\lambda}$ compatible with the parabolic structure. We put $a(v_p):=\deg^F(v_p)$. We have the expression:

$$\mathbb{D}_0^{\lambda}(\partial_j)f=\sum_p A_p^j\cdot v_p.$$

Then, A_p^j are holomorphic functions on $X\setminus D$. We also have the following:

$$\int_{\substack{\pi_j^{-1}(Q)\\|z_j|<1/2}}\big|A^j_{p|\pi_i^{-1}(Q)}\big|^2\cdot|z_1(Q)|^{2\varepsilon-2a(v_p)}<C_{11}.$$

By using Fubini's theorem, we have

$$\int_{\pi_1^{-1}(P)}\big|A^j_{p|\pi_1^{-1}(P)}\big|^2\cdot|z_1|^{2\varepsilon'-2a(v_p)-2}\cdot|dz_1\cdot d\overline{z}_1|<\infty$$

for almost all $P \in D_1$ such that $|z_j(P)| < 1/2$ and for some $0 < \varepsilon' < \varepsilon$. Hence A_p^j are holomorphic around the origin O, and we obtain $\mathbb{D}_0^\lambda(\partial_j)f \in \mathcal{P}_0\mathcal{E}^\lambda$.

According to Lemma 7.4.11 and Lemma 7.4.12, there exists a constant $C_{12} > 0$ such that the following holds for any $Q \in D_1$:

$$\int_{\pi_1^{-1}(Q)} \left|\mathbb{D}_0^\lambda(z_1\partial_1)f\right|_h^2 \cdot |z_1|^{2\varepsilon-2} \cdot |dz_1 \cdot d\bar{z}_1| < C_{12}.$$

We can deduce $\mathbb{D}_0^\lambda(z_1\partial_1)f \in \mathcal{P}_0\mathcal{E}^\lambda$ as in the case $j \neq 1$. Thus, we obtain that $\mathbb{D}_0^\lambda$ is logarithmic, when D is smooth.

The general case can easily be reduced to the case where D is smooth, due to the theorem of Hartogs. Thus, the proof of Proposition 7.4.9 is finished. □

Now, the first claim of Theorem 7.4.5 immediately follows from Proposition 7.4.9. The second and the third claims of Theorem 7.4.5 follows from Proposition 2.3.6 and Proposition 7.4.9. □

7.5. Comparison of the irregular decompositions in the case $\boldsymbol{m} < \boldsymbol{0}_\ell$

7.5.1. Statements. — We use the setting in Section 7.2.1. Let $\lambda \neq 0$. Assume there exists $\boldsymbol{m} \in \mathcal{M}$ such that $\boldsymbol{m} < \boldsymbol{0}_\ell$. We have the irregular decomposition on $\widehat{D}$ as in (25):

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{|\widehat{D}} = \bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})} \mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{E}}^\lambda_{\mathfrak{b}}. \tag{133}$$

In the following, N will denote a large integer. Let $\widehat{D}^{(N)}$ denote the N-th infinitesimal neighbourhood of D. Let $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda = \bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})} \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{\mathfrak{b},N}$ be a decomposition whose restriction to $\widehat{D}^{(N)}$ is the same as the restriction of (133) to $\widehat{D}^{(N)}$. Let $q^{\boldsymbol{m}}_{\mathfrak{b},N}$ be the projection onto $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{\mathfrak{b},N}$ with respect to the decomposition. We will prove the following proposition in Sections 7.5.2–7.5.4.

Proposition 7.5.1. — *We have the estimate* $\left|q^{\boldsymbol{m}}_{\mathfrak{b},N} - \pi^{\boldsymbol{m}}_{\mathfrak{b}}\right|_h = O\bigl(\prod_{i=1}^\ell |z_i|^N\bigr)$. *In particular, the decomposition* $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda = \bigoplus \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{\mathfrak{b},N}$ *is* $\prod_{i=1}^\ell |z_i|^N$*-asymptotically orthogonal with respect to* h.

Remark 7.5.2. — See Section 7.5.5 for the refinement in the general case. □

Before going into the proof, we give some consequences. Let S be a small multisector in $X \setminus D$, and let $\overline{S}$ denote the closure of S in the real blow up $\widetilde{X}(D)$. As explained in Section 3.7.3, we have the partial Stokes filtration $\mathcal{F}^{S,\boldsymbol{m}}$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{|\overline{S}}$ $(\lambda \neq 0)$ at the level $\boldsymbol{m}$, indexed by the ordered set $\bigl(\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}), \leqslant^\lambda_S\bigr)$. Because $\boldsymbol{m} < \boldsymbol{0}_\ell$, we can take a $\mathbb{D}^\lambda$-flat splitting:

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda{}_{|\overline{S}} = \bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})} \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{\mathfrak{b},S}. \tag{134}$$

Let $p^{\boldsymbol{m}}_{\mathfrak{b},S}$ be the projection onto $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{\mathfrak{b},S}$ with respect to the decomposition (134).

Corollary 7.5.3. — *We have* $\bigl|p^{\boldsymbol{m}}_{\mathfrak{b},S} - \pi^{\boldsymbol{m}}_{\mathfrak{b}}\bigr|_h \leqslant C_N \prod_{i=1}^{\ell} |z_i|^N$ *for any* $N > 0$. *In particular, the decomposition* (134) *is* $\prod_{i=1}^{\ell} |z_i|^N$*-asymptotically orthogonal for any* $N > 0$.

Proof. — For any N, we take $q^{\boldsymbol{m}}_{\mathfrak{b},N}$ as above. We have $\bigl|q^{\boldsymbol{m}}_{\mathfrak{b},N} - p^{\boldsymbol{m}}_{\mathfrak{b},S}\bigr|_h \leqslant C''_N \prod_{i=1}^{\ell} |z_i|^N$. Hence, the claim follows from Proposition 7.5.1. □

By varying S and gluing $p^{\boldsymbol{m}}_{\mathfrak{b},S}$ in C^∞ as in Section 3.6.8.2, we obtain $p^{\boldsymbol{m}}_{\mathfrak{b},C^\infty}$.

Corollary 7.5.4. — *We have* $\bigl|p^{\boldsymbol{m}}_{\mathfrak{b},C^\infty} - \pi^{\boldsymbol{m}}_{\mathfrak{b}}\bigr|_h \leqslant C_N \prod_{i=1}^{\ell} |z_i|^N$ *for any* $N > 0$.

Proof. — It follows from Corollary 7.5.3. □

7.5.2. Estimate of $\partial_j \pi^{\boldsymbol{m}}_{\mathfrak{a}}$ in the case $\boldsymbol{m} < \boldsymbol{0}_\ell$. — Recall $\mathcal{E}^0 = E$ as holomorphic bundles. We use the symbol E in this section. For a section f of E with $\partial f = \sum_{j=1}^{n} A_j \, dz_j$, we put $\partial_j f := A_j \, dz_j$. We take a holomorphic frame $\boldsymbol{v}$ of ${}^\diamond E$ compatible with the parabolic structure and the decompositions (92) and (93). For $j = 1, \ldots, n$, let F_j be the section of $\mathrm{End}(E) \otimes \Omega^{1,0}_{X \setminus D}$ determined by $F_j \boldsymbol{v} = \partial_j \boldsymbol{v}$. Then, we have the following estimate for some $M > 0$, due to Lemma 21.9.3 below:

$$|F_j|_h = O\Bigl(\Bigl(\sum_{i=1}^{\ell}(-\log|z_i|)\Bigr)^M\Bigr). \tag{135}$$

We have the decomposition:

$$\begin{aligned} \mathrm{End}(E) &= \mathcal{D}^{\boldsymbol{m}}(\mathrm{End}(E)) \oplus \mathcal{C}^{\boldsymbol{m}}(\mathrm{End}(E)) \\ \mathcal{D}^{\boldsymbol{m}}(\mathrm{End}(E)) &:= \bigoplus_{\mathfrak{b} \in \overline{\mathrm{Irr}}(\theta,\boldsymbol{m})} \mathrm{End}(E^{\boldsymbol{m}}_{\mathfrak{b}}), \\ \mathcal{C}^{\boldsymbol{m}}(\mathrm{End}(E)) &:= \bigoplus_{\substack{\mathfrak{b},\mathfrak{b}' \in \overline{\mathrm{Irr}}(\theta,\boldsymbol{m}),\\ \mathfrak{b} \neq \mathfrak{b}'}} \mathrm{Hom}(E^{\boldsymbol{m}}_{\mathfrak{b}}, E^{\boldsymbol{m}}_{\mathfrak{b}'}). \end{aligned}$$

According to Theorem 7.2.1, $\mathcal{D}^{\boldsymbol{m}}\bigl(\mathrm{End}(E)\bigr)$ and $\mathcal{C}^{\boldsymbol{m}}\bigl(\mathrm{End}(E)\bigr)$ are $\exp\bigl(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|\bigr)$-asymptotically orthogonal. For a section g of $\mathrm{End}(E) \otimes \Omega^{1,0}$, we have the corresponding decomposition $g = \mathcal{D}^{\boldsymbol{m}}(g) + \mathcal{C}^{\boldsymbol{m}}(g)$.

Lemma 7.5.5. — *Around the origin $O \in X$, we have the following estimates for some* $\varepsilon > 0$:

$$\partial_j \pi^{\boldsymbol{m}}_{\mathfrak{a}} = O\bigl(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\bigr) \quad \bigl(\mathfrak{a} \in \overline{\mathrm{Irr}}(\theta,\boldsymbol{m})\bigr), \qquad \mathcal{C}^{\boldsymbol{m}}(F_j) = O\Bigl(\exp\bigl(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|\bigr)\Bigr).$$

Proof. — We have $\partial_j \pi^{\boldsymbol{m}}_{\mathfrak{a}} = \bigl[\pi^{\boldsymbol{m}}_{\mathfrak{a}}, F_j\bigr] \in \mathcal{C}^{\boldsymbol{m}}(\mathrm{End}(E)) \otimes \Omega^{1,0}$. Due to the estimate (135), $\partial_j \pi^{\boldsymbol{m}}_{\mathfrak{a}}$ is bounded up to log order, with respect to h and the Poincaré metric. Since $\mathcal{D}^{\boldsymbol{m}}(\mathrm{End}(E))$ and $\mathcal{C}^{\boldsymbol{m}}(\mathrm{End}(E))$ are $\exp\bigl(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|\bigr)$-asymptotically orthogonal, we obtain the following estimate:

$$(\pi^{\boldsymbol{m}}_{\mathfrak{a}}, \partial_j \pi^{\boldsymbol{m}}_{\mathfrak{a}})_h = O\bigl(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\bigr).$$

Let $R_j(h)$ denote the $dz_j\, d\bar{z}_j$-component of $R(h)$. We have $\overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}} = \big[\pi_{\mathfrak{a}}^{\boldsymbol{m}}, R_j(h)\big] = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$ due to Corollary 7.2.10. Hence, we obtain the following:

$$\big(\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h = O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|\big)\Big).$$

Let π_j denote the projection $X \setminus D \to D_j$. Let us consider the case $j \leqslant \ell$. We put $S_j := \{1,\dots,\ell\} \setminus \{j\}$. Let P be any point of D_j°. Note the following equality on $\pi_j^{-1}(P)$, for any $\eta > 0$:

$$(136)\quad \big(\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \exp\Big(\eta\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big) = \\ \partial_j\bigg(\big(\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \exp\Big(\eta\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big)\bigg) \\ + \big(\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \exp\Big(\eta\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big)\frac{\eta\, m_j}{2}|z_j|^{m_j}\frac{dz_j}{z_j} \\ - \big(\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \exp\Big(\eta\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big).$$

Hence, we obtain the following finiteness for some $\eta > 0$ which is sufficiently smaller than ε:

$$\int_{\pi_j^{-1}(P)}\big(\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \exp\Big(\eta\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big) < C.$$

Let $\varphi_P : \mathbb{H} \to \pi_j^{-1}(P)$ be the covering given by $\zeta \mapsto \exp(2\pi\sqrt{-1}\zeta)$. We put

$$K_n := \big\{(\xi,\eta)\in\mathbb{H}\;\big|\; -1<\xi<1, n-1<\eta<n+1\big\}.$$

We have $e^{2\pi m_j}\cdot e^{-2\pi m_j n} < |z_j|^{m_j} < e^{-2\pi m_j}\cdot e^{-2\pi m_j n}$ on K_n. (Note $m_j < 0$.) Hence we have the following:

$$(137)\qquad \int_{K_n}\big(\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}, \partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h \leqslant C\exp\Big(-\eta\, e^{2\pi m_j}\, e^{2\pi m_j n}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big).$$

Since $(E,\overline{\partial}_E,h)$ is acceptable, there exists an n_0 such that we can apply Uhlenbeck's theorem [**93**] to the pull-back $\varphi_P^{-1}(E,\overline{\partial}_E,h)_{|K_n}$ for any $n \geqslant n_0$ and any $P \in D_j$. Hence, we can take an orthonormal frame $\boldsymbol{e}$ of $\varphi_P^{-1}\operatorname{End}(E)$ on K_n ($n \geqslant n_0$) such that A is small, where A is determined by $(\partial_j + \overline{\partial}_j)\boldsymbol{e} = \boldsymbol{e}\,A$. We have the expression $\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}} = \sum_i \rho_i\, e_i\, dz_j$. We put $\boldsymbol{\rho} := (\rho_i)$. The inequality (137) gives the estimate of the L^2-norm of $\boldsymbol{\rho}$. From the estimate of $\overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}$, we obtain the following:

$$(138)\qquad \big|\overline{\partial}_j\boldsymbol{\rho} + A^{0,1}\boldsymbol{\rho}\big| \leqslant C_2\exp\Big(-\eta_2\, e^{2\pi n m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big).$$

By using a standard argument, we obtain the estimate of the sup norm of $\boldsymbol{\rho}$ on K_n from (138) and (137). Then we obtain the following estimate on K_n:
(139)
$$\big|\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big|_h \leqslant C_3\exp\Big(-\eta_3\,e^{2\pi n m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big) \leqslant C_4\exp\Big(-\eta_4\,|z_j|^{m_j}\prod_{i\in S_j}|z_i(P)|^{m_i}\Big).$$

Thus, we are done in the case $j\leqslant\ell$.

Let us consider the case $j>\ell$. We have the following:

$$\int_{\pi_j^{-1}(Q)}\big(\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}},\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h=\int_{\substack{\pi_j^{-1}(Q)\\|z_j|=1}}\big(\pi_{\mathfrak{a}}^{\boldsymbol{m}},\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h-\int_{\pi_j^{-1}(Q)}\big(\pi_{\mathfrak{a}}^{\boldsymbol{m}},\overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\big)_h.$$

Thus, we obtain $\|\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}\|_{L^2}\leqslant C\exp\big(-\varepsilon\prod_{p=1}^{\ell}|z_p(Q)|^{m_p}\big)$. We can obtain the estimate for the sup norm of $\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}$ from the L^2-estimate of $\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}$ and the estimate for the sup norm of $\overline{\partial}_j\partial_j\pi_{\mathfrak{a}}^{\boldsymbol{m}}$. (See the argument to obtain (139) above.) Thus, the proof of Lemma 7.5.5 is finished. □

7.5.3. Estimate of $\mathbb{D}^\lambda p_{\mathfrak{a},N}^{\boldsymbol{m}}$ in the case $\boldsymbol{m}<\boldsymbol{0}_\ell$. — Let $p_{\mathfrak{b},N}^{\boldsymbol{m}}$ ($\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})$) be as in Lemma 7.4.7. We have the decomposition:

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda=\bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m})}\mathrm{Im}\,p_{\mathfrak{b},N}^{\boldsymbol{m}}.\tag{140}$$

Lemma 7.5.6. — $\mathbb{D}^\lambda(p_{\mathfrak{b},N}^{\boldsymbol{m}})=O\big(\prod_{j=1}^{\ell}|z_j|^N\big)$.

Proof. — We have $\partial_E\pi_{\mathfrak{b}}^{\boldsymbol{m}}=O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$ due to Lemma 7.5.5. We also have $[\theta,\pi_{\mathfrak{b}}^{\boldsymbol{m}}]=0$ and $[\theta^\dagger,\pi_{\mathfrak{b}}^{\boldsymbol{m}}]=O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$. Hence, $\mathbb{D}^\lambda\pi_{\mathfrak{b}}^{\boldsymbol{m}}=O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$.

We put $s_{\mathfrak{b}}^{\boldsymbol{m}}:=\pi_{\mathfrak{b}}^{\boldsymbol{m}}-p_{\mathfrak{b}}^{\boldsymbol{m}}$. We have $\big|s_{\mathfrak{b}}^{\boldsymbol{m}}\big|_h=O\big(\prod_{j=1}^{\ell}|z_j|^{2N}\big)$. We have the following:

$$\mathbb{D}^\lambda s_{\mathfrak{b}}^{\boldsymbol{m}}=d_\lambda''\pi_{\mathfrak{b}}^{\boldsymbol{m}}+\lambda\delta_\lambda's_{\mathfrak{b}}^{\boldsymbol{m}}+(1+|\lambda|^2)\theta s_{\mathfrak{b}}^{\boldsymbol{m}}.$$

We have $d_\lambda''\pi_{\mathfrak{b}}^{\boldsymbol{m}}=O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$. We have $\theta s_{\mathfrak{b}}^{\boldsymbol{m}}=O\big(\prod_{i=1}^{\ell}|z_i|^{3N/2}\big)$. Let us look at $\delta_\lambda's_{\mathfrak{b}}^{\boldsymbol{m}}$. Let π_j denote the projection $X\smallsetminus D\to D_j$.

Lemma 7.5.7. — *We have the following estimates independently from $j=1,\dots,n$ and $Q\in\pi_j^{-1}(D)$.*
- $\big\||z_j|^{-3N/2}\delta_{\lambda,j}'s^{\boldsymbol{m}}_{\mathfrak{b}|\pi_j^{-1}(Q)}\big\|_{L^2}\leqslant C\prod_{i\in S_j}\big|z_i(Q)\big|^N$ *for $j=1,\dots,\ell$, where $S_j:=\{1,\dots,\ell\}\smallsetminus\{j\}$.*
- $\big\|\delta_{\lambda,j}'s^{\boldsymbol{m}}_{\mathfrak{b}|\pi_j^{-1}(Q)}\big\|_{L^2}\leqslant C\prod_{i=1}^{\ell}|z_i(Q)|^N$ *for $j=\ell+1,\dots,n$.*

Proof. — Let us consider the case $j\leqslant\ell$. We put $L:=3N/2$, for simplicity of the notation. Let ρ_1 be a non-negative C^∞-function on $\boldsymbol{R}$ such that $\rho_1(t)=1$ for $t\leqslant 1/2$ and $\rho_1(t)=0$ for $t\geqslant 2/3$. Let κ be a non-negative C^∞-function on Δ such that $\kappa(z)=1$ for $|z|\leqslant 1/2$ and $\kappa(z)=0$ for $|z|\geqslant 2/3$. We put $\chi_M(z_j):=$

$\rho_1\bigl(-M^{-1}\log|z_j|\bigr)\,\kappa(z_j)$. In the following estimate, we do not have to be careful on the signature. We have the following:

$$
\begin{aligned}
(141)\quad &\int_{\pi_j^{-1}(Q)} \bigl(\delta'_{\lambda,j}(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}), \delta'_{\lambda,j}(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h\,|z_j|^{-2L}\\
&\qquad = \pm\int_{\pi_j^{-1}(Q)} \bigl(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}, d''_{\lambda,j}\delta'_{\lambda,j}(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h\,|z_j|^{-2L}\\
&\qquad\qquad \pm\int_{\pi_j^{-1}(Q)} \bigl(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}, \delta'_{\lambda,j}(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h\,L\,|z_j|^{-2L}\frac{dz_j}{z_j}.
\end{aligned}
$$

The first term of the right-hand side is as follows:

$$
\begin{aligned}
&\pm\int_{\pi_j^{-1}(Q)} \bigl(\chi s_{\mathfrak{b}}^{\boldsymbol{m}}, R(h,d''_\lambda)(\chi s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h|z_j|^{-2L} \pm\int_{\pi_j^{-1}(Q)} \bigl(\chi s_{\mathfrak{b}}^{\boldsymbol{m}}, \delta'_{\lambda,j}d''_{\lambda,j}(\chi s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h|z_j|^{-2L}\\
&= \pm\int_{\pi_j^{-1}(Q)} \bigl(\chi s_{\mathfrak{b}}^{\boldsymbol{m}}, R(h,d''_\lambda)(\chi s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h|z_j|^{-2L} \pm\int_{\pi_j^{-1}(Q)} \bigl(d''_{\lambda,j}(\chi s_{\mathfrak{b}}^{\boldsymbol{m}}), d''_{\lambda,j}(\chi s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)|z_j|^{-2L}\\
&\qquad\qquad \pm\int \bigl(\chi s_{\mathfrak{b}}^{\boldsymbol{m}}, d''_{\lambda,j}(\chi s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr)_h\,L|z_j|^{-2L}\frac{d\overline{z}_j}{\overline{z}_j}.
\end{aligned}
$$

(M is omitted.) The second term is as follows:

$$
\begin{aligned}
&\pm\int_{\pi_j^{-1}(Q)} \bigl(d''_\lambda(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}), \chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}\bigr)_h\,L\,|z_j|^{-2L}\frac{dz_j}{z_j}\\
&\qquad\qquad \pm\int_{\pi_j^{-1}(Q)} \bigl(\chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}, \chi_M s_{\mathfrak{b}}^{\boldsymbol{m}}\bigr)\,L^2|z_j|^{-2L}\frac{d\overline{z}_j\,dz_j}{|z_j|^2}.
\end{aligned}
$$

Then, the claim in the case $j \leqslant \ell$ follows from the estimate for $|s_{\mathfrak{b}}^{\boldsymbol{m}}|_h$ and $|d''_\lambda s_{\mathfrak{b}}^{\boldsymbol{m}}|_h$ in the limit $M \to \infty$.

Let us consider the case $j > \ell$. We have the following:

$$
\begin{aligned}
&\int_{\pi_j^{-1}(Q)} \bigl(\delta'_{\lambda,j}(\kappa s_{\mathfrak{b}}^{\boldsymbol{m}}), \delta'_{\lambda,j}(\kappa s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr) =\\
&\qquad \pm\int_{\pi_j^{-1}(Q)} \bigl(d''_{\lambda,j}(\kappa s_{\mathfrak{b}}^{\boldsymbol{m}}), d''_{\lambda,j}(\kappa s_{\mathfrak{b}}^{\boldsymbol{m}})\bigr) \pm\int_{\pi_j^{-1}(Q)} \bigl(\kappa s_{\mathfrak{b}}^{\boldsymbol{m}}, R_j(d''_\lambda, h)\kappa s_{\mathfrak{b}}^{\boldsymbol{m}}\bigr).
\end{aligned}
$$

Hence, the claim in the case $j > \ell$ also follows from the estimate of $|s_{\mathfrak{b}}^{\boldsymbol{m}}|_h$ and $|d''_\lambda s_{\mathfrak{b}}^{\boldsymbol{m}}|_h$. Thus, the proof of Lemma 7.5.7 is finished. □

Let us finish the proof of Lemma 7.5.6. From the estimate of $\mathbb{D}^\lambda \pi_{\mathfrak{b}}^{\boldsymbol{m}}$ and $\mathbb{D}^\lambda s_{\mathfrak{b}}^{\boldsymbol{m}}$, we obtain the following:

$$
\Bigl\|\bigl(|z_j|^{-L}\,\mathbb{D}_j^\lambda p_{\mathfrak{a},N}^{\boldsymbol{m}}\bigr)_{|\pi_j^{-1}(Q)}\Bigr\|_{L^2} \leqslant C\prod_{i\in S_j}|z_i(Q)|^{4N/3},\quad (j=1,\dots,\ell),
$$

$$
\Bigl\|\mathbb{D}_j^\lambda p_{\mathfrak{a},N|\pi_j^{-1}(Q)}^{\boldsymbol{m}}\Bigr\|_{L^2} \leqslant C\prod_{i=1}^{\ell}|z_i(Q)|^{4N/3},\quad (j=\ell+1,\dots,n).
$$

Then, by the holomorphic property, we obtain $\mathbb{D}^{\lambda}p^{\boldsymbol{m}}_{\mathfrak{a},N} = O\big(\prod_{j=1}^{\ell}|z_j|^N\big)$. Thus the proof of Lemma 7.5.6 is accomplished. □

7.5.4. End of Proof of Proposition 7.5.1

Lemma 7.5.8. — *The restrictions of the decompositions* (133) *and* (140) *to* $\widehat{D}^{(N)}$ *are the same.*

Proof. — Let $\boldsymbol{v}_N = (\boldsymbol{v}_{\mathfrak{a},N})$ and $\boldsymbol{w}_N = (\boldsymbol{w}_{\mathfrak{a},N})$ be frames of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ whose restrictions to $\widehat{D}^{(N)}$ are compatible with the decompositions (133) and (140), respectively. Namely, $\boldsymbol{v}_{\mathfrak{a},N|\widehat{D}^{(N)}}$ and $\boldsymbol{w}_{\mathfrak{a},N|\widehat{D}^{(N)}}$ give frames of $\mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|\widehat{D}^{(N)}}$ and $\big(\operatorname{Im} p^{\boldsymbol{m}}_{\mathfrak{a},N}\big)_{|\widehat{D}^{(N)}}$, respectively. Let A and B be determined by the following:

$$\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathbb{D}^{\lambda}\boldsymbol{v}_N = \boldsymbol{v}_N\,A, \quad \boldsymbol{z}^{-\boldsymbol{m}(0)}\mathbb{D}^{\lambda}\boldsymbol{w}_N = \boldsymbol{w}_N\,B.$$

We have the decompositions $A = (A_{\mathfrak{a},\mathfrak{b}})$ and $B = (B_{\mathfrak{a},\mathfrak{b}})$, corresponding to the decompositions of the frames. By our choice, we have $A_{\mathfrak{a},\mathfrak{b}} \equiv B_{\mathfrak{a},\mathfrak{b}} \equiv 0$ for $\mathfrak{a} \neq \mathfrak{b}$ modulo $\boldsymbol{z}^{-\boldsymbol{m}(0)}\prod_{j=1}^{\ell} z_j^N$. Let C be determined by $\boldsymbol{v} = \boldsymbol{w}\,C$, which has the decomposition $C = (C_{\mathfrak{a},\mathfrak{b}})$. We obtain the following modulo $\boldsymbol{z}^{-\boldsymbol{m}(0)}\prod_{j=1}^{\ell} z_j^N$:

$$B_{\mathfrak{a},\mathfrak{a}}\,C_{\mathfrak{a},\mathfrak{b}} - C_{\mathfrak{a},\mathfrak{b}}\,A_{\mathfrak{b},\mathfrak{b}} + \boldsymbol{z}^{-\boldsymbol{m}(0)}C_{\mathfrak{a},\mathfrak{b}} \equiv 0. \tag{142}$$

Assume $C_{\mathfrak{a},\mathfrak{b}} \not\equiv 0$ modulo $\prod_{j=1}^{\ell} z_j^N$. We have the expansion $C_{\mathfrak{a},\mathfrak{b}} = \sum_{\boldsymbol{n}\in\mathbb{Z}^{\ell}_{\geqslant 0}} C_{\mathfrak{a},\mathfrak{b};\boldsymbol{n}}\boldsymbol{z}^{\boldsymbol{n}}$. We set $\boldsymbol{\delta} := (1,\dots,1)$. Let $\boldsymbol{n}_0 \not\geqslant N\,\boldsymbol{\delta}$ be a minimal among $\boldsymbol{n}$ such that $C_{\mathfrak{a},\mathfrak{b};\boldsymbol{n}} \neq 0$. Then, we obtain $(\mathfrak{a}-\mathfrak{b})_{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\,C_{\mathfrak{a},\mathfrak{b};\boldsymbol{n}_0} = 0$ from (142). Note $-\boldsymbol{m}(0)+\operatorname{ord}(\mathfrak{a}-\mathfrak{b})+\boldsymbol{n}_0 \not\geqslant -\boldsymbol{m}(0) + N\,\boldsymbol{\delta}$. Hence, we obtain $C_{\mathfrak{a},\mathfrak{b};\boldsymbol{n}_0} = 0$ which contradicts with our choice of $\boldsymbol{n}_0$. Hence, we obtain $C_{\mathfrak{a},\mathfrak{b}} \equiv 0$ modulo $\prod_{j=1}^{\ell} z_j^N$. This implies the claim of Lemma 7.5.8. □

Since we have $q^{\boldsymbol{m}}_{\mathfrak{b},N} - p^{\boldsymbol{m}}_{\mathfrak{b},N} \equiv 0$ modulo $\prod_{i=1}^{\ell} z_i^N$, the claim of Proposition 7.5.1 is obtained. □

7.5.5. Complement. — We use the setting in Section 7.2.1. Moreover, we assume that the coordinate system is admissible for the good set $\operatorname{Irr}(\theta)$, for simplicity. Let k be the integer determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0}\times\mathbf{0}_{\ell-k}$. Let $\boldsymbol{m} \in \mathcal{M}$ such that $\boldsymbol{m} \in \mathbb{Z}^k_{<0}\times\mathbf{0}_{\ell-k}$. We put $D(\leqslant k) := \bigcup_{j=1}^{k} D_j$. We have the irregular decomposition as in (25):

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}{}_{|\widehat{D}(\leqslant k)} = \bigoplus_{\mathfrak{b}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m})} (\mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{E}}^{\lambda})^{\boldsymbol{m}}_{\mathfrak{b}}.$$

The projection onto $(\mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{E}}^{\lambda})^{\boldsymbol{m}}_{\mathfrak{b}}$ is denoted by $\widehat{p}^{\boldsymbol{m}}_{\mathfrak{b}}$.

By Lemma 3.6.30, we can take endomorphisms $p^{\boldsymbol{m}}_{\mathfrak{b},N} \in \mathcal{P}_0\operatorname{End}(\mathcal{E}^{\lambda})$ such that (i) $p^{\boldsymbol{m}}_{\mathfrak{b},N|\widehat{D}^{(N)}(\leqslant k)} = \widehat{p}^{\boldsymbol{m}}_{\mathfrak{b}|\widehat{D}^{(N)}(\leqslant k)}$, (ii) $\big[p^{\boldsymbol{m}}_{\mathfrak{b},N|D_i}, \operatorname{Res}_i(\mathbb{D}^{\lambda})\big] = 0$ for $i = k+1,\dots,\ell$. According to the norm estimate of tame harmonic bundles [**67**], $\big|p^{\boldsymbol{m}}_{\mathfrak{b},N}\big|_h$ is bounded on $\prod_{i=1}^{k}\{1/2 \leqslant |z_i| \leqslant 2/3\} \times (\Delta^*)^{\ell-k} \times \Delta^{n-\ell}$. (Otherwise, we can use Lemma 7.7.1 below.)

Recall that we have the projection $\pi_{\mathfrak{b}}^{\boldsymbol{m}}$ in the Higgs side, as given in Section 7.2.2.

Lemma 7.5.9. — $p_{\mathfrak{b},N}^{\boldsymbol{m}} - \pi_{\mathfrak{b}}^{\boldsymbol{m}} = O\big(\prod_{i=1}^{k}|z_i|^N\big)$.

Proof. — Let $q : X\setminus D \to (\Delta^*)^{\ell-k}\times\Delta^{n-\ell}$ be given by $q(z_1,\dots,z_n) = (z_{k+1},\dots,z_n)$. Let $d''_{\lambda} := \overline{\partial}_E + \lambda\theta^{\dagger}$. The restriction to $q^{-1}(Q)$ is also denoted by the same notation, where $Q \in (\Delta^*)^{\ell-k}\times\Delta^{n-\ell}$. By using Proposition 7.2.8, we obtain the following estimate on $q^{-1}(Q)$, which is uniform for $Q\in(\Delta^*)^{\ell-k}\times\Delta^{n-\ell}$:

$$d''_{\lambda}\big(p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)}\big) = -d''_{\lambda}\big(\pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)}\big) = O\Big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\Big).$$

The restrictions $(\mathcal{E}^{\lambda},h)_{|q^{-1}(Q)}$ are acceptable, and the curvatures are uniformly bounded for $Q\in(\Delta^*)^{\ell-k}\times\Delta^{n-\ell}$. By using Lemma 21.2.3 and Lemma 21.9.1, we can take a section t_Q of $\operatorname{End}(\mathcal{E}^{\lambda})_{|q^{-1}(Q)}$ such that (i) $d''_{\lambda}t_Q = d''_{\lambda}\big(p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)}\big)$, (ii) $|t_Q|_h \leqslant C_1\prod_{i=1}^{k}|z_i|^N$, where the constant C_1 is independent of Q.

By construction, we have $d''_{\lambda}\big(p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)} - t_Q\big) = 0$ and

$$\big|p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)} - t_Q\big|_h \leqslant C_2$$

on $\prod_{i=1}^{k}\{1/2\leqslant|z_i|\leqslant 2/3\}$ independently from Q. From the estimate for t_Q and Proposition 7.5.1, we have the estimate

$$\big|p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)} - t_Q\big|_h \leqslant C_{Q,N}\prod_{i=1}^{k}|z_i|^N$$

depending on Q. Then, by Proposition 21.2.8, we obtain the estimate

$$\big|p^{\boldsymbol{m}}_{\mathfrak{b},N|q^{-1}(Q)} - \pi^{\boldsymbol{m}}_{\mathfrak{b}|q^{-1}(Q)} - t_Q\big|_h \leqslant C_N\prod_{i=1}^{k}|z_i|^N$$

independently from Q. Thus, we are done. □

7.6. Small deformation of $\mathcal{PE}^{\lambda}$ in the smooth divisor case

7.6.1. Characterization of the deformation $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda})^{(T)}$ via the metric

We use the setting in Section 7.2.1 with $\ell = 1$, i.e., D is the smooth divisor D_1. We use a slightly different notation. We use the symbol j instead of $\boldsymbol{m}(i)$ to denote an element of $\mathcal{M}$. Let $m(0)$ denote the minimum of the numbers j such that $\operatorname{Irr}(\theta,j)\neq\{0\}$. For any $\mathfrak{b}\in\operatorname{Irr}(\theta,j)$, we put

$$E^{(j)}_{\mathfrak{b}} := \bigoplus_{\substack{\mathfrak{a}\in\operatorname{Irr}(\theta)\\ \eta_j(\mathfrak{a})=\mathfrak{b}}} E_{\mathfrak{a}}.$$

Let $\pi^{(j)}_{\mathfrak{b}}$ denote the projection of E onto $E^{(j)}_{\mathfrak{b}}$. Recall we have the following estimate with respect to h for some $\varepsilon_1>0$, due to Theorems 7.2.1, 7.2.4 and Lemma 7.5.5:

$$\mathbb{D}^{\lambda}\pi^{(j)}_{\mathfrak{b}} = O\big(\exp(-\varepsilon_1|z_1|^{j})\big). \qquad (143)$$

For $w \in \boldsymbol{C}$, we consider the following:

$$g_{\mathrm{irr}}(w) := \sum_{\mathfrak{a}\in \mathrm{Irr}(\theta)} \exp(w\cdot\overline{\mathfrak{a}})\cdot\pi_{\mathfrak{a}}.$$

Let $g_{\mathrm{irr}}(w)^*h$ be the metric given by $g_{\mathrm{irr}}(w)^*h(u,v) = h\bigl(g_{\mathrm{irr}}(w)u, g_{\mathrm{irr}}(w)v\bigr)$. We put

$$T_1(w) := 1 - \frac{\lambda\cdot\overline{w}}{1+|\lambda|^2}. \tag{144}$$

For any j and any $\mathfrak{a} \in \mathrm{Irr}(\theta, j-1)$, we put $\mathcal{I}^{(j)}_{\mathfrak{a}} := \eta^{-1}_{j,j-1}(\mathfrak{a})$. Formally, we put $\mathcal{I}_0^{(m(0))} := \mathrm{Irr}(\theta, m(0))$. Let $\pi : \widetilde{X}(D) \to X$ denote the real blow up of X along D. Fix $\lambda \neq 0$. We regard X and D as $\{\lambda\}\times X$ and $\{\lambda\}\times D$. For any j and any distinct $\mathfrak{b}_1, \mathfrak{b}_2 \in \eta^{-1}_{j,j-1}(\mathfrak{a})$, we put $F_{\mathfrak{b}_1,\mathfrak{b}_2} := -|z_1^{-j}|\,\mathrm{Re}\bigl(\lambda^{-1}(\mathfrak{b}_1-\mathfrak{b}_2)\bigr)$, which are C^∞-functions on $\widetilde{X}(D)$. Let $S_1,\dots,S_N$ be small multi-sectors of $X\setminus D$, such that (i) the union of their interior parts is $X\setminus D$, (ii) $S_i \in \mathcal{MS}(X\setminus D, \mathcal{I}^{(j)}_{\mathfrak{a}})$ for any j and any $\mathfrak{a}\in \mathrm{Irr}(\theta, j-1)$. Let $\overline{S}_i$ denote the closure of S_i in $\widetilde{X}(D)$, and let Z_i denote $\overline{S}_i \cap \pi^{-1}(D)$. We may assume the following for each S_i, each j, each $\mathfrak{a}\in\mathrm{Irr}(\theta,j-1)$ and each pair $(\mathfrak{b}_1,\mathfrak{b}_2)$ of distinct elements in $\mathcal{I}^{(j)}_{\mathfrak{a}}$:

(A1) : If the intersection $Z_i \cap \{F_{\mathfrak{b}_1,\mathfrak{b}_2} = 0\}$ is not empty, $F_{\mathfrak{b}_1,\mathfrak{b}_2}$ is monotone with respect to $\arg(z_1)$.

If we choose sufficiently small $0 < \varepsilon_2 \leqslant \varepsilon_1$, either one of the following holds for each S_i, each j, each $\mathfrak{a}\in\mathrm{Irr}(\theta,j-1)$ and each pair $(\mathfrak{b}_1,\mathfrak{b}_2)$ of distinct elements in $\mathcal{I}^{(j)}_{\mathfrak{a}}$:

(A2) : If the intersection $Z_i \cap \{F_{\mathfrak{b}_1,\mathfrak{b}_2} = 0\}$ is empty, $\bigl|F_{\mathfrak{b}_1,\mathfrak{b}_2}\bigr| \geqslant \varepsilon_2/2$ holds on Z_i.

We will prove the following proposition in Sections 7.6.2–7.6.5.

Proposition 7.6.1. — *Assume the following:*
- $|w| < \varepsilon_2/100$.
- $\bigl|\arg(T_1(w))\bigr|$ *is small such that the natural bijection*

$$\bigl\{\bigl(t+(1-t)T_1(w)\bigr)\cdot\mathfrak{a}\bigm|\mathfrak{a}\in\mathrm{Irr}(\theta)\bigr\} \longrightarrow \mathrm{Irr}(\theta)$$

preserves the orders $\leqslant^{\lambda}_{S_i}$ $(i=1,\dots,N)$ *for any* $0\leqslant t\leqslant 1$.

Then, the following holds:
- *Let f be a holomorphic section of $\mathcal{E}^\lambda$ on $X\setminus D$. It gives a section of the sheaf $(\mathcal{P}_a\mathcal{E}^\lambda)^{(T_1(w))}$, if and only if it satisfies the following growth condition:*

$$|f|_{g_{\mathrm{irr}}(w)^*h} = O\bigl(|z_1|^{-a-\varepsilon}\bigr) \quad (\forall\,\varepsilon>0).$$

7.6.2. Comparison of the irregular decompositions on small multi-sectors

Let $\pi : \widetilde{X}(D)\to X$ denote the real blow up of X along D.

Proposition 7.6.2. — *For any point $P\in\pi^{-1}(D)$, there exist a multi-sector $S_P \in \mathcal{MS}(P, X\setminus D)$ and a $\mathbb{D}^\lambda$-flat splitting $\mathcal{P}_a\mathcal{E}^\lambda_{|\overline{S}_P} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,j)}\mathcal{P}_a\mathcal{E}^{\lambda\natural}_{\mathfrak{a},S_P}$ of the Stokes filtration $\mathcal{F}^{S_P(j)}$ on $\overline{S}_P$ at the level j with the following property:*

- *Let $p^{(j)\natural}_{\mathfrak{a},S_P}$ denote the projection onto $\mathcal{P}_a\mathcal{E}^{\lambda\natural}_{\mathfrak{a},S_P}$ with respect to the decomposition. Then, $\pi^{(j)}_{\mathfrak{a}} - p^{(j)\natural}_{\mathfrak{a},S_P} = O\big(\exp(-\varepsilon_1|z_1^j|/2)\big)$ with respect to h.*

In particular, the decomposition $\mathcal{P}_a\mathcal{E}^{\lambda}_{|\overline{S}_P} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,j)} \mathcal{P}_a\mathcal{E}^{\lambda\natural}_{\mathfrak{a},S_P}$ is $O\big(\exp(-\varepsilon_1|z_1^j|/2)\big)$-asymptotically orthogonal.

Proof. — By using Lemma 20.2.1, we can take $\mathcal{Q}^{(j)}_{\mathfrak{b},S}$ such that $\mathbb{D}^{\lambda}\mathcal{Q}^{(j)}_{\mathfrak{b},S} = \mathbb{D}^{\lambda}\pi^{(j)}_{\mathfrak{b}}$ and $\mathcal{Q}^{(j)}_{\mathfrak{b},S} = O\big(\exp(-\varepsilon_1|z_1|^j/2)\big)$. We put $p^{(j)\natural}_{\mathfrak{b},S} := \pi^{(j)}_{\mathfrak{b}} - \mathcal{Q}^{(j)}_{\mathfrak{b},S}$, which is $\mathbb{D}^{\lambda}$-flat. By applying the modification as in the proof of Lemma 7.4.7, we may and will assume that $\big[p^{(j)\natural}_{\mathfrak{b},S}, p^{(j)\natural}_{\mathfrak{c},S}\big] = 0$ and $p^{(j)\natural}_{\mathfrak{b},S} \circ p^{(j)\natural}_{\mathfrak{b},S} = p^{(j)\natural}_{\mathfrak{b},S}$.

We put $\mathcal{F}^{S\,(j)\natural}_{\mathfrak{b}} := \bigoplus_{\mathfrak{c}\leqslant_S^{\lambda}\mathfrak{b}} \operatorname{Im} p^{(j)\natural}_{\mathfrak{c},S}$. Let us compare the filtrations $\mathcal{F}^{S\,(j)\natural}$ and $\mathcal{F}^{S\,(j)}$. We take a splitting $\mathcal{P}_a\mathcal{E}^{\lambda}_{|\overline{S}} = \bigoplus \mathcal{P}_a\mathcal{E}^{\lambda}_{\mathfrak{a},S}$, and let $p^{(j)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}_a\mathcal{E}^{\lambda}_{\mathfrak{a},S}$ with respect to the decomposition. Since we already know that $p^{(j)}_{\mathfrak{a},S} - \pi^{(j)}_{\mathfrak{a}} = O(|z_1|^N)$ for any N (Corollary 7.5.3), we have $p^{(j)\natural}_{\mathfrak{a},S} - p^{(j)}_{\mathfrak{a},S} = O\big(|z_1|^N\big)$ for any $N > 0$. Hence, both $\mathcal{F}^{S\,(j)}_{\mathfrak{b}|\widehat{Z}}$ and $\mathcal{F}^{S\,(j)\natural}_{\mathfrak{b}|\widehat{Z}}$ are the same as $\mathcal{F}^{Z\,(j)}_{|\mathfrak{b}}$. Hence, we obtain $\mathcal{F}^{S\,(j)}_{\mathfrak{b}} = \mathcal{F}^{S\,(j)\natural}_{\mathfrak{b}}$. (Use the uniqueness in Proposition 3.6.1 successively.) In other words, the decomposition $\bigoplus \operatorname{Im} p^{(j)\natural}_{\mathfrak{b},S}$ gives a splitting of the filtration $\mathcal{F}^{S\,(j)}$. Thus, Proposition 7.6.2 is proved. □

We have the following corollaries.

Corollary 7.6.3. — *Let S be a small multi-sector in $X \setminus D$, and let $\mathcal{P}_a\mathcal{E}^{\lambda}_{|\overline{S}} = \bigoplus_{\mathfrak{c}\in\mathrm{Irr}(\theta,j)} \mathcal{P}_a\mathcal{E}^{\lambda}_{\mathfrak{c},S}$ be any $\mathbb{D}^{\lambda}$-flat splitting of the filtration $\mathcal{F}^{S\,(j)}$. Let $p^{(j)}_{\mathfrak{c},S}$ denote the projection onto $\mathcal{PE}^{\lambda}_{\mathfrak{c},S}$. Then, we have $\pi^{(j)}_{\mathfrak{c}} - p^{(j)}_{\mathfrak{c},S} = O\big(\exp(-\varepsilon'|z_1|^j)\big)$ for some $\varepsilon' > 0$.*

Proof. — Take a finite covering of S by multi-sectors S_P as in Proposition 7.6.2, and compare $p^{(j)}_{\mathfrak{c},S}$, $p^{(j)\natural}_{\mathfrak{c},S_P}$ and $\pi^{(j)}_{\mathfrak{c}}$ on each S_P. □

7.6.3. Comparison of the irregular decompositions on S_i.

— Let $\mathcal{PE}^{\lambda}_{|\overline{S}_i} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{PE}^{\lambda}_{\mathfrak{a},S_i}$ be a $\mathbb{D}$-flat splitting of the full Stokes filtration $\widetilde{\mathcal{F}}^{S_i}$. For any $\mathfrak{b} \in \mathrm{Irr}(\theta,j)$, we put

$$\mathcal{PE}^{\lambda\,(j)}_{\mathfrak{b},S_i} = \bigoplus_{\substack{\mathfrak{a}\in\mathrm{Irr}(\theta)\\ \eta_j(\mathfrak{a})=\mathfrak{b}}} \mathcal{PE}^{\lambda}_{\mathfrak{a},S_i}.$$

Let $p^{(j)}_{\mathfrak{b},S_i}$ denote the projection onto $\mathcal{PE}^{\lambda\,(j)}_{\mathfrak{b},S_i}$.

Proposition 7.6.4. — *We have the following estimate with respect to h:*

$$p^{(j)}_{\mathfrak{b},S_i} - \pi^{(j)}_{\mathfrak{b}} = O\Big(\exp\big(-\varepsilon_2|z_1^j|/10\big)\Big).$$

Proof. — For simplicity of the description, we set $S := S_i$, $\overline{S} := \overline{S}_i$ and $Z := Z_i$. We have the decomposition:

$$\operatorname{End}(\mathcal{PE}^{\lambda}_{|\overline{S}}) = \bigoplus_{(\mathfrak{b}_1,\mathfrak{b}_2)\in \operatorname{Irr}(\theta,j)^2} \operatorname{Hom}\big(\mathcal{PE}^{\lambda\,(j)}_{\mathfrak{b}_1,S}, \mathcal{PE}^{\lambda\,(j)}_{\mathfrak{b}_2,S}\big).$$

For any point $P \in Z$, we take a small multi-sector S_P as in Proposition 7.6.2. Let $\overline{S}_P$ denote the closure of S_P in $\widetilde{X}(D)$, and let $Z_P := \overline{S}_P \cap \pi^{-1}(D)$. If S_P is sufficiently small, we may assume that one of the following holds on Z_P, for each distinct $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{I}^{(j)}_{\mathfrak{a}}$ ($\mathfrak{a} \in \operatorname{Irr}(\theta, j)$):

- $F_{\mathfrak{b}_1,\mathfrak{b}_2} \leqslant -\varepsilon_2/8$.
- $F_{\mathfrak{b}_1,\mathfrak{b}_2} \geqslant -\varepsilon_2/4$.

We have the decomposition:

$$p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S|S_P} = \sum_{\substack{(\mathfrak{c}_1,\mathfrak{c}_2)\in \operatorname{Irr}(\theta,j)^2\\ \mathfrak{c}_1<_{S_P}\mathfrak{c}_2}} \big(p^{(j)\natural}_{\mathfrak{b},S_P}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} \in \bigoplus_{\substack{(\mathfrak{c}_1,\mathfrak{c}_2)\in \operatorname{Irr}(\theta,j)^2\\ \mathfrak{c}_1<_{S_P}\mathfrak{c}_2}} \operatorname{Hom}\big(\mathcal{PE}^{\lambda\,(j)}_{\mathfrak{c}_1,S}, \mathcal{PE}^{\lambda\,(j)}_{\mathfrak{c}_2,S}\big)_{|S_P}.$$

Lemma 7.6.5. — *In the case $j = m(0)$, we have*

$$p^{(m(0))\natural}_{\mathfrak{b},S_P} - p^{(m(0))}_{\mathfrak{b},S|S_P} = O\Big(\exp\big(-\varepsilon_2|z_1^{m(0)}|/10\big)\Big)$$

with respect to h for any $\mathfrak{b} \in \operatorname{Irr}(\theta, m(0))$.

Proof. — We only have to have estimates of $\big(p^{(m(0))\natural}_{\mathfrak{b},S_P} - p^{(m(0))}_{\mathfrak{b},S|S_P}\big)_{\mathfrak{c}_1,\mathfrak{c}_2}$ for $\mathfrak{c}_1 <_{S_P} \mathfrak{c}_2$. In the case $F_{\mathfrak{c}_1,\mathfrak{c}_2} \leqslant -\varepsilon_2/8$ on S_P, we have the following by flatness:

$$\big(p^{(m(0))\natural}_{\mathfrak{b},S_P} - p^{(m(0))}_{\mathfrak{b},S|S_P}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = O\Big(\exp\big(-\varepsilon_2|z_1^{m(0)}|/10\big)\Big).$$

In the case $F_{\mathfrak{c}_1,\mathfrak{c}_2} \geqslant -\varepsilon_2/4$ on S_P, (A2) cannot happen for the $\mathfrak{c}_1, \mathfrak{c}_2$ and $S = S_i$. Hence, we can take a sequence $P_1 = P, P_2, \dots, P_t \in Z$ with the following properties:

- $Z_{P_i} \cap Z_{P_{i+1}} \neq \varnothing$.
- Z_{P_t} intersects with $\{F_{\mathfrak{c}_1,\mathfrak{c}_2} > 0\}$.
- $F_{\mathfrak{c}_1,\mathfrak{c}_2} \geqslant -\varepsilon_2/4$ on each Z_{P_i}.

By the second condition, we have $\big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_t}} - p^{(m(0))}_{\mathfrak{b},S|S_{P_t}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = 0$ on S_{P_t}. On $S_{P_i} \cap S_{P_{i+1}}$, we have

$$\begin{aligned}(145)\quad p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}} &= \big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - \pi^{(m(0))}_{\mathfrak{b}}\big) - \big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}} - \pi^{(m(0))}_{\mathfrak{b}}\big)\\ &= O\Big(\exp\big(-\varepsilon_1|z_1^{m(0)}|/2\big)\Big).\end{aligned}$$

If $p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}}$ is not 0, we have the following for some $B, C, N > 0$, because of Lemma 20.3.2:

$$\Big|\big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} \exp\big((\mathfrak{c}_1 - \mathfrak{c}_2)/\lambda\big)\Big| \geqslant B \cdot \exp\big(-C|z_1^{m(0)+1}|\big) \cdot |z_1|^N.$$

Hence, we obtain the following for some $B' > 0$:

$$(146)\qquad \big|\big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2}\big| \geqslant B' \cdot \exp\big(-\varepsilon_2|z_1^{m(0)}|/3\big).$$

From (145) and (146), we obtain a contradiction. Hence, we have

$$\big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S_{P_{i+1}}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = 0$$

on $S_{P_i} \cap S_{P_{i+1}}$. Then, we can show $\big(p^{(m(0))\natural}_{\mathfrak{b},S_{P_i}} - p^{(m(0))\natural}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = 0$ on S_{P_i} by using an inductive argument. In particular, we obtain $\big(p^{(m(0))\natural}_{\mathfrak{b},S_P} - p^{(m(0))}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = 0$ on S_P. Thus, we obtain the desired estimate, and the proof of Lemma 7.6.5 is finished. □

Lemma 7.6.6. — *We have $p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S|S_P} = O\big(\exp(-\varepsilon_2|z_1^j|/10)\big)$ with respect to h for any j and any $\mathfrak{b} \in \mathrm{Irr}(\theta, j)$.*

Proof. — We omit to denote $|S_P$ for simplicity of the description. We use an induction on j. The case $j = m(0)$ is already done (Lemma 7.6.5). Let us look at $\big(p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2}$ for $(\mathfrak{c}_1, \mathfrak{c}_2) \in \mathrm{Irr}(\theta, j)^2$. Let us consider the case $\mathrm{ord}(\mathfrak{c}_1 - \mathfrak{c}_2) < j$. We have the following:

$$(147)\qquad \big(p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = p^{(j)}_{\mathfrak{c}_2,S} \circ \big(p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S}\big) \circ p^{(j)}_{\mathfrak{c}_1,S} = p^{(j)}_{\mathfrak{c}_2,S} \circ p^{(j)\natural}_{\mathfrak{b},S_P} \circ p^{(j)}_{\mathfrak{c}_1,S}.$$

Because $\mathrm{ord}(\mathfrak{c}_1 - \mathfrak{c}_2) < j$, one of $\mathrm{ord}(\mathfrak{c}_1 - \mathfrak{b})$ or $\mathrm{ord}(\mathfrak{c}_2 - \mathfrak{b})$ is strictly smaller than j. In the case $q := \mathrm{ord}(\mathfrak{c}_1 - \mathfrak{b}) < j$, we have the following:

$$(148)\qquad p^{(j)}_{\mathfrak{c}_2,S} \circ p^{(j)\natural}_{\mathfrak{b},S_P} \circ p^{(j)}_{\mathfrak{c}_1,S} = p^{(j)}_{\mathfrak{c}_2,S} \circ p^{(j)\natural}_{\mathfrak{b},S_P} \circ p^{(q)\natural}_{\eta_q(\mathfrak{b}),S_P} \circ p^{(q)}_{\eta_q(\mathfrak{c}_1),S} \circ p^{(j)}_{\mathfrak{c}_1,S}.$$

By the inductive assumption, we have the following:

$$(149)\ p^{(q)\natural}_{\eta_q(\mathfrak{b}),S_P} \circ p^{(q)}_{\eta_q(\mathfrak{c}_1),S} = \big(p^{(q)\natural}_{\eta_q(\mathfrak{b}),S_P} - p^{(q)}_{\eta_q(\mathfrak{b}),S}\big) \circ p^{(q)}_{\eta_q(\mathfrak{c}_1),S} = O\Big(\exp\big(-\varepsilon_2|z_1^q|/10\big)\Big).$$

From (147), (148) and (149), we obtain the desired estimate for $\big(p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2}$ in the case $\mathrm{ord}(\mathfrak{c}_1 - \mathfrak{b}) < j$. We can obtain a similar estimate in the case $\mathrm{ord}(\mathfrak{c}_2 - \mathfrak{b}) < j$.

Let us consider the case $\mathrm{ord}(\mathfrak{c}_1 - \mathfrak{c}_2) = j$. Note we have $\eta_{j-1}(\mathfrak{c}_1) = \eta_{j-1}(\mathfrak{c}_2)$. In the case $F_{\mathfrak{c}_1,\mathfrak{c}_2} \leqslant -\varepsilon_2/8$ on S_P, we have the following by flatness:

$$\big(p^{(j)\natural}_{\mathfrak{b},S_P} - p^{(j)}_{\mathfrak{b},S}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = O\Big(\exp\big(-\varepsilon_2|z_1^j|/10\big)\Big).$$

In the case $F_{\mathfrak{c}_1,\mathfrak{c}_2} \geqslant -\varepsilon_2/4$ on S_P, we can take a sequence $P_1 = P, P_2, \ldots, P_t \in Z$ with the following properties:

- $Z_{P_i} \cap Z_{P_{i+1}} \neq \varnothing$.
- Z_{P_t} intersects with $\{F_{\mathfrak{c}_1,\mathfrak{c}_2} > 0\}$.
- $F_{\mathfrak{c}_1,\mathfrak{c}_2} \geqslant -\varepsilon_2/4$ on each Z_{P_i}.

By the second condition, we have $\big(p^{(j)\natural}_{\mathfrak{b},S_{P_t}} - p^{(j)}_{\mathfrak{b},S|S_{P_t}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} = 0$ on S_{P_t}. On $S_{P_i} \cap S_{P_{i+1}}$, we have

$$(150)\ p^{(j)\natural}_{\mathfrak{b},S_{P_i}} - p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}} = \big(p^{(j)\natural}_{\mathfrak{b},S_{P_i}} - \pi^{(j)}_{\mathfrak{b}}\big) - \big(p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}} - \pi^{(j)}_{\mathfrak{b}}\big) = O\Big(\exp\big(-\varepsilon_1|z_1^j|/2\big)\Big).$$

If $p^{(j)\natural}_{\mathfrak{b},S_{P_i}} - p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}}$ is not 0, we have the following for some $B, C, N > 0$, due to Lemma 20.3.2:

$$\big|\big(p^{(j)\natural}_{\mathfrak{b},S_{P_i}} - p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}}\big)_{\mathfrak{c}_1,\mathfrak{c}_2} \exp\big((\mathfrak{c}_1 - \mathfrak{c}_2)/\lambda\big)\big| \geqslant B \cdot \exp\big(-C|z_1^{j+1}|\big) \cdot |z_1|^N.$$

Hence, we obtain the following for some $B'>0$:

$$(151)\qquad \left|\left(p^{(j)\natural}_{\mathfrak{b},S_{P_i}}-p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}}\right)_{\mathfrak{c}_1,\mathfrak{c}_2}\right|\geqslant B'\cdot\exp\left(-\varepsilon_2|z_1^j|/3\right).$$

From (150) and (151) we obtain a contradiction. Hence we have

$$\left(p^{(j)\natural}_{\mathfrak{b},S_{P_i}}-p^{(j)\natural}_{\mathfrak{b},S_{P_{i+1}}}\right)_{\mathfrak{c}_1,\mathfrak{c}_2}=0$$

on $S_{P_i}\cap S_{P_{i+1}}$. We obtain $\left(p^{(j)\natural}_{\mathfrak{b},S_{P_i}}-p^{(j)\natural}_{\mathfrak{b},S}\right)_{\mathfrak{c}_1,\mathfrak{c}_2}=0$ on S_{P_i} by an inductive argument. In particular, $\left(p^{(j)\natural}_{\mathfrak{b},S_P}-p^{(j)}_{\mathfrak{b},S}\right)_{\mathfrak{c}_1,\mathfrak{c}_2}=0$ on S_P. Thus, we obtain the desired estimate, and the proof of Lemma 7.6.6 is finished. □

Proposition 7.6.4 immediately follows from Proposition 7.6.2 and Lemma 7.6.6. □

7.6.4. Comparison of some metrics. — We consider the following:

$$(152)\qquad F(w):=\exp\left(w\cdot\mathcal{B}\right),\quad \mathcal{B}:=\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\mathfrak{a}\cdot\pi_{\mathfrak{a}}=\sum_{m(0)\leqslant j\leqslant -1}\ \sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)}\zeta_j(\mathfrak{b})\cdot\pi^{(j)}_{\mathfrak{b}}.$$

Let $F(w)^*h$ be the metric given by $F(w)^*h(u,v)=h\left(F(w)u,F(w)v\right)$.

Lemma 7.6.7. — *$F(\overline{w})^*h$ and $g_{\mathrm{irr}}(w)^*h$ are mutually bounded.*

Proof. — Because $g_{\mathrm{irr}}(w)\circ F(\overline{w})^{-1}=\sum\exp\left(2\sqrt{-1}\,\mathrm{Im}(w\cdot\overline{\mathfrak{a}})\right)\cdot\pi_{\mathfrak{a}}$, we have the boundedness of $\left|g_{\mathrm{irr}}(w)\circ F(\overline{w})^{-1}\right|_h$. Similarly, $\left|F(\overline{w})\circ g_{\mathrm{irr}}(w)^{-1}\right|_h$ is bounded. Then, the claim of Lemma 7.6.7 follows. □

For each $S:=S_i$, we put

$$(153)\qquad \mathcal{B}_S:=\sum_j\ \sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)}\zeta_j(\mathfrak{b})\cdot p^{(j)}_{\mathfrak{b},S},\qquad F_S(w):=\exp\left(w\cdot\mathcal{B}_S\right).$$

Lemma 7.6.8. — *For $|w|<\varepsilon_2/100$, we have the following estimate:*

$$\left|F(w)\circ F_S(w)^{-1}-1\right|_h=O\Big(\exp\left(-\varepsilon_2|z_1|^{-1}/100\right)\Big)$$
$$\left|F_S(w)\circ F(w)^{-1}-1\right|_h=O\Big(\exp\left(-\varepsilon_2|z_1|^{-1}/100\right)\Big).$$

Proof. — We have $F(w)=\prod F^{(j)}(w)$ and $F_S(w)=\prod F^{(j)}_S(w)$, where $F^{(j)}$ and $F^{(j)}_S$ are given as follows:

$$F^{(j)}(w):=\exp\Big(\sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)}w\cdot\zeta_j(\mathfrak{b})\cdot\pi^{(j)}_{\mathfrak{b}}\Big),\quad F^{(j)}_S(w):=\exp\Big(\sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)}w\cdot\zeta_j(\mathfrak{b})\cdot p^{(j)}_{\mathfrak{b},S}\Big).$$

We consider $G^{(j)}:=F^{(j)}-F^{(j)}_S$. Because $\pi^{(j)}_{\mathfrak{b}}-p^{(j)}_{\mathfrak{b},S}=O\left(\exp(-\varepsilon_2|z_1|^j/10)\right)$, we have $G^{(j)}=O\left(\exp(-\varepsilon_2|z_1|^j/15)\right)$. We set

$$\widetilde{G}^{(j)}:=\prod_{i>j}F^{(i)}_S\circ G^{(j)}\circ(F^{(j)}_S)^{-1}\circ\prod_{i>j}\left(F^{(i)}_S\right)^{-1}.$$

We have $|\widetilde{G}^{(j)}|_h = O\big(\exp(-\varepsilon_2|z_1|^j/20)\big)$. We have the following equality:

$$F\circ F_S^{-1} = (1+\widetilde{G}^{(-1)})\circ\cdots\circ(1+\widetilde{G}^{(m(0)+1)})\circ(1+\widetilde{G}^{(m(0))}).$$

Then, the claim of Lemma 7.6.8 follows. □

Let $F_S(w)^*h$ be the metric given by $F_S(w)^*h(u,v) = h\big(F_S(w)u, F_S(w)v\big)$.

Lemma 7.6.9. — *For $|w| < \varepsilon_2/100$, the metrics $F(w)^*h$ and $F_S(w)^*h$ are mutually bounded.*

Proof. — We have the following:

$$|v|_{F(w)^*h} \leqslant |v|_{F_S(w)^*h}\cdot\big|F_S(w)^{-1}\circ F(w)\big|_{F_S(w)^*h} = |v|_{F_S(w)^*h}\cdot\big|F(w)\circ F_S(w)^{-1}\big|_h.$$

Hence, we obtain $|v|_{F(w)^*h} \leqslant C\cdot|v|_{F_S(w)^*h}$ due to Lemma 7.6.8. We obtain $|v|_{F_S(w)^*h} \leqslant C\cdot|v|_{F(w)^*h}$ by a similar argument. Thus, the proof of Lemma 7.6.9 is finished. □

7.6.5. End of proof of Proposition 7.6.1. — According to Corollary 4.5.9, f gives a section of $\big(\mathcal{P}_a\mathcal{E}^\lambda\big)^{(T_1(w))}$ if and only if $|f|_{F_{S_i}(w)^*h}$ is bounded on S_i for $i = 1,\ldots,N$. In Lemma 7.6.9, we have obtained that $F(w)^*h$ and $F_{S_i}(w)^*h$ are mutually bounded on S_i $(i=1,\ldots,N)$. Hence, the claim of Proposition 7.6.1 follows. □

7.7. Boundedness of some section

We use the setting in Section 7.2.1. For simplicity, we assume that the coordinate system is admissible for the good set $\mathrm{Irr}(\theta)$. Let k be determined by $\boldsymbol{m}(0)\in\mathbb{Z}^k_{<0}\times\mathbf{0}_{\ell-k}$. We put $D(\leqslant k) := \bigcup_{j=1}^k D_j$. Let M be a sufficiently large integer, say $M > 100\cdot\ell\cdot|\boldsymbol{m}(0)|$, where $|\boldsymbol{m}(0)| = \sum_{i=1}^k |m(0)_i|$.

Lemma 7.7.1. — *Let f be a section of $\mathcal{P}_0\,\mathrm{End}(\mathcal{E}^\lambda)$ with the following properties:*
- *$\mathbb{D}^\lambda f = 0$ on $\widehat{D}^{(M)}(\leqslant k)$.*
- *$\big[\mathrm{Res}_i(\mathbb{D}^\lambda), f_{|D_i}\big] = 0$ for $i = k+1,\ldots,\ell$.*

Then, $|f|_h$ is bounded.

Proof. — We use an induction on ℓ. In the case $\ell = 0$, the claim is trivial. In the following argument, we will assume that the claim of the lemma holds in the case $\ell-1$.

Let us show the following claims for $m = 1,\ldots,\ell$ by a descending induction on m:

A(m) : Let f be a section of $\mathcal{P}_0\,\mathrm{End}(\mathcal{E}^\lambda)$ with the following properties:
- $\mathbb{D}^\lambda f = 0$ on $\widehat{D}^{(M_m)}(\leqslant k)$, where $M_m = M - (\ell - m)\cdot|\boldsymbol{m}(0)|$.
- $\big[\mathrm{Res}_i(\mathbb{D}^\lambda), f_{|D_i}\big] = 0$ for $i = m+1,\ldots,\ell$.

Then, we have the following estimate for some $C > 0$ and $N > 0$:

$$|f|_h \leqslant C \cdot \Big(- \sum_{i=1}^{m} \log |z_i| \Big)^N.$$

The claim $A(\ell)$ holds due to the general result of the acceptable bundles (Theorem 21.3.2). Let us show $A(m-1)$ by assuming $A(m)$. Let $f \in \mathcal{P}_0 \operatorname{End}(\mathcal{E}^\lambda)$ such that (i) $\mathbb{D}^\lambda f_{|\widehat{D}^{(M_{m-1})}(\leqslant k)} = 0$, (ii) $\big[\operatorname{Res}_i(\mathbb{D}^\lambda), f_{|D_i}\big] = 0$ for $i = m, \dots, \ell$. We would like to obtain the estimate:

$$|f|_h \leqslant C \cdot \Big(- \sum_{i=1}^{m-1} \log |z_i| \Big)^N.$$

We put $g := \mathbb{D}^\lambda(\partial_m) f \in \operatorname{End}(\mathcal{P}_0\mathcal{E}^\lambda)$. We have (i) $\mathbb{D}^\lambda g_{|\widehat{D}^{(M_m)}(\leqslant k)} = 0$ and, for $i = m+1, \dots, \ell$, (ii) $\big[\operatorname{Res}_i(\mathbb{D}^\lambda), g_{|D_i}\big] = 0$. Hence, we can apply $A(m)$ to g, and we obtain the estimate for some $C > 0$ and $N > 0$:

$$|g|_h \leqslant C \cdot \Big(- \sum_{i=1}^{m} \log |z_i| \Big)^N.$$

Let $\pi_m : X \to D_m$ denote the projection. We put $\pi_m^{-1}(Q)^* := \pi_m^{-1}(Q) \setminus \{Q\}$ for $Q \in D_m^\circ$. Then, we obtain the following:

$$\int_{\pi_m^{-1}(Q)^*} \big|g_{|\pi_m^{-1}(Q)^*}\big|_h^2 \cdot \big|dz_m \cdot d\overline{z}_m\big| \leqslant C_1 \cdot \Big(- \sum_{j=1}^{m-1} \log |z_j(Q)| \Big)^{N_1}.$$

Let Δ_m denote the Laplacian $-\partial_{z_m}\partial_{\overline{z}_m}$. By Corollary 7.7.5 below, we have the following inequality on $\pi_m^{-1}(Q)^*$:

$$\Delta_m\big(|f_{|\pi_m^{-1}(Q)^*}|_h^2\big) \leqslant \big|g_{|\pi_m^{-1}(Q)^*}\big|_h^2.$$

We can take $G_Q(z_m)$ satisfying the following:

$$\Delta_m G_Q = |g_{|\pi_m^{-1}(Q)}|^2, \quad \sup\big|G_Q\big| \leqslant C' \cdot \Big(- \sum_{j=1}^{m-1} \log |z_j|(Q) \Big)^N$$

Note $\big|f_{|\pi_m^{-1}(Q)}\big|_h^2 - G_Q$ is bounded on $\pi_m^{-1}(Q)^*$ for each Q, a property which follows from the norm estimate in the curve case (Proposition 8.1.1, below). Therefore, we have $\Delta_m\big(\big|f_{|\pi_m^{-1}(Q)}\big|_h^2 - G_Q\big) \leqslant 0$ as distributions on $\pi_m^{-1}(Q)$ (Lemma 2.2 of [**82**]). Hence, we obtain the following:

$$(154) \quad \sup\big|f_{|\pi_m^{-1}(Q)}\big|_h^2 \leqslant \max_{|z_m'|=1/2} \big|f_{|\pi_m^{-1}(Q)}(z_m')\big|_h^2 + C''\Big(- \sum_{i=1}^{m-1} \log |z_i|^2(Q) \Big)^N \leqslant C'''\Big(- \sum_{i=1}^{m-1} \log |z_i|^2(Q) \Big)^N.$$

For the estimate of $\max_{|z'_m|=1/2}\big|f_{|\pi_m^{-1}(Q)}(z'_m)\big|_h^2$ we have used here the inductive assumption on ℓ. Thus, we obtain $A(m-1)$, and the descending induction on m can proceed. The claim $A(0)$ means Lemma 7.7.1. □

We can show the following lemma by the same argument.

Lemma 7.7.2. — *Let f be a section of $\mathcal{P}_0\mathcal{E}^\lambda$ with the following properties:*
- $\mathbb{D}^\lambda f = 0$ *on* $\widehat{D}^{(M)}(\leqslant k)$.
- $\mathrm{Res}_i(\mathbb{D}^\lambda)(f) = 0$ *for* $i = k+1,\dots,\ell$.

Then, $|f|_h$ is bounded. □

Let $(E_i, \overline{\partial}_i, \theta_i, h_i)$ $(i=1,2)$ be unramifiedly good wild harmonic bundles as in Section 7.2.1, with the good set of irregular values $\mathrm{Irr}(\theta_i)$. For simplicity, we assume that $\mathrm{Irr}(\theta_1) = \mathrm{Irr}(\theta_2)$, and that the coordinate system is admissible for $\mathrm{Irr}(\theta_1)$. Let $\boldsymbol{m}(0)$ be the minimum of $\mathrm{Irr}(\theta_1)$, and let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$. Let M be a sufficiently large integer, say $M > 100\cdot\ell\cdot|\boldsymbol{m}(0)|$, where $|\boldsymbol{m}(0)| = \sum_{i=1}^k |m(0)_i|$. Note that $\mathrm{Hom}(\mathcal{E}_1^\lambda, \mathcal{E}_2^\lambda)$ with the naturally induced metric is acceptable. The following lemma can be shown by the same argument.

Lemma 7.7.3. — *Let f be a section of $\mathcal{P}_0\,\mathrm{Hom}(\mathcal{E}_1^\lambda, \mathcal{E}_2^\lambda)$ with the following properties:*
- $\mathbb{D}^\lambda f = 0$ *on* $\widehat{D}^{(M)}(\leqslant k)$.
- $\big[\mathrm{Res}_i(\mathbb{D}^\lambda), f_{|D_i}\big] = 0$ *for* $i = k+1,\dots,\ell$.

Then, $|f|_h$ is bounded. □

7.7.1. Weitzenböck formula (Appendix). — Let us recall a variant of Weitzenböck formula for harmonic bundles due to Simpson with a slightly refined form. It will be used in Section 10.4. The original one (Corollary 7.7.5) was used in the proof of Lemma 7.7.1. It will be also useful in Sections 9.3 and 10.3.

Let z be a coordinate of $\boldsymbol{C}$, and let U be an open subset of $\boldsymbol{C}$. We use the Euclidean metric $g = dz\cdot d\overline{z}$. Let Δ'' denote the Laplacian $-\partial_z\partial_{\overline{z}}$.

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on U. For any complex number λ, let $d''_\lambda := \overline{\partial}_E + \lambda\theta^\dagger$, $\delta'_\lambda := \partial_E - \overline{\lambda}\theta$, and $\mathbb{D}^\lambda := d''_\lambda + \lambda\partial_E + \theta$.

Lemma 7.7.4. — *Let s be a C^∞-section of E. We have the following inequality:*

$$\Delta''|s|_h^2 \leqslant \big|\mathbb{D}^\lambda s\big|_{h,g}^2 + 2|s|_h\cdot\big|\delta'_h d''_\lambda s\big|_{h,g}.$$

Proof. — We use the pairing $(E\otimes\Omega^{p,q})\otimes(E\otimes\Omega^{r,s}) \to \Omega^{p+s,q+r}$ induced by h, which is denoted by $(\cdot,\cdot)_h$. Let $R(h)$ denote the curvature of the unitary connection $d''_\lambda + \delta'_\lambda$. We have the following equality:

$$(155)\qquad \begin{aligned}\partial\overline{\partial}|s|_h^2 &= \big(\delta'_\lambda d''_\lambda s, s\big)_h - \big(d''_\lambda s, d''_\lambda s\big)_h + \big(\delta'_\lambda s, \delta'_\lambda s\big)_h + \big(s, d''_\lambda\delta'_\lambda s\big)_h\\ &= \big(\delta'_\lambda d''_\lambda s, s\big)_h - \big(s, \delta'_\lambda d''_\lambda s\big)_h + \big(s, R(h)s\big)_h - \big(d''_\lambda s, d''_\lambda s\big)_h + \big(\delta'_\lambda s, \delta'_\lambda s\big)_h.\end{aligned}$$

We have the following equality:

$$\big(s, R(h)s\big)_h = -(1+|\lambda|^2)\cdot\big(\theta\cdot s, \theta\cdot s\big)_h - (1+|\lambda|^2)\cdot\big(\theta^\dagger\cdot s, \theta^\dagger\cdot s\big)_h.$$

Let $\mathbb{D}^{\lambda\prime} := \lambda\partial_E + \theta$. Then, we have the following:

$$\big(\delta'_{\lambda}s, \delta'_{\lambda}s\big)_h - (1+|\lambda|^2)\cdot\big(\theta\cdot s, \theta\cdot s\big)_h = (1+|\lambda|^2)\cdot\big(\partial_E s, \partial_E s\big)_h - \big(\mathbb{D}^{\lambda\prime}s, \mathbb{D}^{\lambda\prime}s\big)_h.$$

Hence, we obtain the following:

$$\partial\overline{\partial}|s|_h^2 = \big(\delta'_{\lambda}d''_{\lambda}s,\, s\big)_h - \big(s, \delta'_{\lambda}d''_{\lambda}s\big)_h - (1+|\lambda|^2)\cdot\big(\theta^{\dagger}s, \theta^{\dagger}s\big)_h - \big(d''_{\lambda}s, d''_{\lambda}s\big)_h$$
$$+ (1+|\lambda|^2)\cdot\big(\partial_E s, \partial_E s\big)_h - \big(\mathbb{D}^{\lambda\prime}s, \mathbb{D}^{\lambda\prime}s\big)_h.$$

Then, the claim of the lemma follows. □

***Corollary 7.7.5* (Lemma 4.18 of [65]).** — *Let s be a holomorphic section of $\mathcal{E}^{\lambda}$. Then, we have the inequality $\Delta''|s|_h^2 \leqslant \big|\mathbb{D}^{\lambda}s\big|_{h,g}^2$. (Lemma* 4.18 *of* [**65**] *should be corrected.)* □

CHAPTER 8

SOME BASIC RESULTS IN THE CURVE CASE

In this chapter, we study the one dimensional case. In Section 8.1, we show the norm estimate for holomorphic sections of $\mathcal{P}\mathcal{E}^{\lambda}$. In Section 8.2, we show the correspondence between the parabolic weights and the residues for $(\mathcal{P}\mathcal{E}^0, \mathbb{D}^0)$ and $(\mathcal{P}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$. These are natural generalizations of Simpson's results in the tame case. The arguments are also essentially the same. In Section 8.3, we give a characterization of the lattice $\mathcal{P}_a\mathcal{E}^{\lambda}$ by the eigenvalues of the residues in the case where λ is generic. In Section 8.4, we argue some basic property of harmonic forms for wild harmonic bundles on quasiprojective curves, which will be used in the proof of the Hard Lefschetz Theorem for polarized wild pure twistor D-modules. (See Section 18.2.)

8.1. Norm estimate for holomorphic sections of $\mathcal{P}_c\mathcal{E}^{\lambda}$

8.1.1. Statement. — We put $X = \Delta$ and $D = \{O\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X \setminus D$. We assume to have the decomposition on X:

$$(E, \theta) = \bigoplus_{(\mathfrak{a},\alpha)\in \operatorname{Irr}(\theta)\times \mathcal{S}p(\theta)} (E_{\mathfrak{a},\alpha}, \theta_{\mathfrak{a},\alpha}). \tag{156}$$

We have the expression $\theta = f \cdot dz$.

As explained in Chapter 7, we have the prolongment $\mathcal{P}_c\mathcal{E}^{\lambda}$ for each $\lambda \in \boldsymbol{C}$ and $c \in \boldsymbol{R}$. Let F denote the parabolic filtration of $\mathcal{P}_c\mathcal{E}^{\lambda}_{|O}$. We have the endomorphism $\operatorname{Res}(\mathbb{D}^{\lambda})$ of $\mathcal{P}_c\mathcal{E}^{\lambda}_{|O}$, which preserves the parabolic filtration. The induced endomorphism of $\operatorname{Gr}^F(\mathcal{P}_c\mathcal{E}^{\lambda}_{|O})$ is also denoted by $\operatorname{Res}(\mathbb{D}^{\lambda})$. Let W denote the weight filtration of $\operatorname{Gr}^F(\mathcal{P}_c\mathcal{E}^{\lambda}_{|O})$ associated to the nilpotent part of $\operatorname{Res}(\mathbb{D}^{\lambda})$.

Let $\boldsymbol{v}$ be a holomorphic frame of $\mathcal{P}_c\mathcal{E}^{\lambda}$ such that (i) it is compatible with the parabolic filtration F, (ii) the induced frame on $\operatorname{Gr}^F(\mathcal{P}_c\mathcal{E}^{\lambda}_{|O})$ is compatible with the weight filtration W. We put $a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. Let h_0 be the

C^∞-metric of E given as follows:

$$h_0(v_i, v_j) := \delta_{i,j} \cdot |z|^{-2a(v_i)} \cdot (-\log |z|^2)^{k(v_i)}. \tag{157}$$

Proposition 8.1.1. — *The metrics h and h_0 are mutually bounded. In other words, the standard norm estimate holds.*

Since it can be shown by the argument in [**82**], we give only an indication.

8.1.2. The case $\lambda = 0$. — First let us consider the case $\lambda = 0$. Let $\pi_{\mathfrak{a},\alpha}$ be the projection onto $E_{\mathfrak{a},\alpha}$ in the decomposition (156). According to Theorems 7.2.1 and 7.2.4, $\pi_{\mathfrak{a},\alpha}$ are bounded. Hence, we have the following decomposition as the prolongment of the decomposition (156):

$$(\mathcal{P}_c\mathcal{E}^0, \mathbb{D}^0) = \bigoplus_{(\mathfrak{a},\alpha)\in \mathrm{Irr}(\theta)\times \mathcal{S}p(\theta)} (\mathcal{P}_c\mathcal{E}^0_{\mathfrak{a},\alpha}, \mathbb{D}^0_{\mathfrak{a},\alpha}). \tag{158}$$

This decomposition is compatible with the parabolic filtration and the residue $\mathrm{Res}(\theta)$. We may assume $\boldsymbol{v}$ is compatible with the decomposition (158).

For each $(a, \alpha, \mathfrak{a})$, we have the endomorphism of $V_{a,\alpha,\mathfrak{a}} := \mathrm{Gr}^F_a(\mathcal{P}_c\mathcal{E}^0_{\alpha,\mathfrak{a}})$ induced by $\mathrm{Res}(\theta)$. The nilpotent part is denoted by $N_{a,\alpha,\mathfrak{a}}$. We have the model harmonic bundle on $X \setminus D$ obtained from $(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}})$ denoted by $E(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}})$. (See Subsection 7.4.1.3). We have the rank one harmonic bundle $L(a, \alpha, \mathfrak{a}) = (\mathcal{O}\cdot e, \theta^L_{a,\alpha,\mathfrak{a}}, h^L_{a,\alpha,\mathfrak{a}})$ as in Subsection 7.4.1.2. Then, we obtain the wild harmonic bundle $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{h}, \widetilde{\theta}) := \bigoplus_{(a,\alpha,\mathfrak{a})} E(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}}) \otimes L(a, \alpha, \mathfrak{a})$. According to Theorem 7.4.3, we obtain $\mathcal{P}_c\widetilde{\mathcal{E}}^0$ for each $c \in \boldsymbol{R}$, and the decomposition:

$$(\mathcal{P}_c\widetilde{\mathcal{E}}^0, \widetilde{\theta}) = \bigoplus_{(\alpha,\mathfrak{a})} \bigoplus_{a} (\mathcal{P}_c\widetilde{\mathcal{E}}^0_{a,\alpha,\mathfrak{a}}, \widetilde{\theta}_{a,\alpha,\mathfrak{a}}). \tag{159}$$

We can take a holomorphic isomorphism $\Phi : \mathcal{P}_c\widetilde{\mathcal{E}}^0 \to \mathcal{P}_c\mathcal{E}^0$ such that (i) it preserves the decompositions (158) and (159), (ii) it preserves the parabolic filtrations, (iii) $\mathrm{Gr}^F(\Phi_{|O})$ is compatible with the residues. It can be checked by a direct calculation that $\Phi(\widetilde{h})$ and h_0 are mutually bounded. Recall that h_0 and h are mutually bounded up to log order, and hence $\Phi(\widetilde{h})$ and h are mutually bounded up to log order. Due to a general result of Simpson (Corollary 4.3 of [**82**]), we have the following:

- Let K be a Hermitian metric of $E = \mathcal{E}^0$ with the following properties:
  - (i) It is adapted to the filtered bundle $\mathcal{P}_*\mathcal{E}^0$. The induced metric of the dual $\mathcal{E}^{0\vee}$ is also adapted to the induced parabolic filtration of $\mathcal{P}_*\mathcal{E}^{0\vee}$.
  - (ii) Let $\theta^\dagger_K$ denote the adjoint of θ with respect to K, and let ∂_K be the $(1,0)$-operator determined by $\overline{\partial}_E$ and K. Let F_K denote the curvature of the connection $\overline{\partial}_E + \partial_K + \theta + \theta^\dagger_K$. Then, F_K is L^p with respect to K and the Euclid metric $dz \cdot d\overline{z}$.

  Then, the metrics K and h are mutually bounded.

It is easy to check that the conditions (i) and (ii) hold for $\Phi(\widetilde{h})$. Thus, we can conclude that h and h_0 are mutually bounded in the case $\lambda = 0$.

8.1.3. The case $\lambda \neq 0$. — Let us consider the case $\lambda \neq 0$. We have the irregular decomposition:

$$(\mathcal{P}_c\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\theta)} (\mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}, \mathbb{D}^{\lambda}_{\mathfrak{a}}). \tag{160}$$

We have the parabolic filtration F and the endomorphism $\operatorname{Res}(\mathbb{D}^{\lambda})$ of $\mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|O}$. They are compatible. Let $\mathbb{E}_{\alpha}(\mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|O})$ denote the generalized eigen space of $\operatorname{Res}(\mathbb{D}^{\lambda})$ corresponding to the eigenvalue α. We obtain a vector space $V_{a,\alpha,\mathfrak{a}} := \operatorname{Gr}^F_a \mathbb{E}_{\alpha}(\mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|O})$. Let $N_{a,\alpha,\mathfrak{a}}$ denote the nilpotent part of the endomorphism of $V_{a,\alpha,\mathfrak{a}}$ induced by $\operatorname{Res}(\mathbb{D}^{\lambda})$. We have the model bundle $E(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}})$ obtained from $(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}})$. Let $(b,\beta) \in \boldsymbol{R} \times \boldsymbol{C}$ be given by the condition $\mathfrak{k}(\lambda, b, \beta) = (a, \alpha)$. (See Subsection 2.8.2 for $\mathfrak{k}(\lambda)$.) Let $L(b,\beta,\mathfrak{a})$ be the harmonic bundle of rank one as in Subsection 7.4.1.2. We obtain a wild harmonic bundle

$$(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{h}, \widetilde{\theta}) := \bigoplus_{(a,\alpha,\mathfrak{a})} E(V_{a,\alpha,\mathfrak{a}}, N_{a,\alpha,\mathfrak{a}}) \otimes L(b,\beta,\mathfrak{a}).$$

We have the associated meromorphic λ-flat bundle $(\mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}, \widetilde{\mathbb{D}}^{\lambda})$ which has the decomposition:

$$\mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda} = \bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\theta)} \Big(\bigoplus_{a,\alpha} \mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}}\Big). \tag{161}$$

By construction, we have an isomorphism

$$\Phi_{O,a,\alpha,\mathfrak{a}} : (\operatorname{Gr}^F_a \mathbb{E}_{\alpha}\mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}}, \operatorname{Res}\widetilde{\mathbb{D}}^{\lambda}) \simeq (\operatorname{Gr}^F_a \mathbb{E}_{\alpha}\mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}, \operatorname{Res}\mathbb{D}^{\lambda}_{\mathfrak{a}}).$$

We can take an isomorphism $\Phi_{O,\mathfrak{a}} : \bigoplus_{a,\alpha} \mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}|O} \simeq \mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|O}$, which preserves the parabolic filtration, and induces $\bigoplus_{a,\alpha} \Phi_{O,a,\alpha,\mathfrak{a}}$. Let $\widehat{D}^{(N)}$ denote the N-th infinitesimal neighbourhood of D for a large integer N. We can take an isomorphism

$$\Phi_{\widehat{D}^{(N)},\mathfrak{a}} : \bigoplus_{a,\alpha} \mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}|\widehat{D}^{(N)}} \simeq \mathcal{P}_c\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}|\widehat{D}^{(N)}}$$

such that $\Phi_{\widehat{D}^{(N)},\mathfrak{a}|O} = \Phi_{O,\mathfrak{a}}$. By a general theory, we can take a holomorphic decomposition of $\mathcal{P}_c\mathcal{E}^{\lambda}$ whose restriction to $\widehat{D}^{(N)}$ is the same as (160):

$$\mathcal{P}_c\mathcal{E}^{\lambda} = \bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\theta)} \mathcal{P}_c\mathcal{E}^{\lambda}_{\mathfrak{a},N}. \tag{162}$$

Let $q_{\mathfrak{a},N}$ denote the projection onto $\mathcal{P}_c\mathcal{E}^{\lambda}_{\mathfrak{a},N}$. We can take a holomorphic isomorphism $\bigoplus_{a,\alpha} \mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}} \simeq \mathcal{P}_c\mathcal{E}^{\lambda}_{\mathfrak{a},N}$ whose restriction to $\widehat{D}^{(N)}$ is equal to $\Phi_{\widehat{D}^{(N)},\mathfrak{a}}$. In particular, we obtain $\Phi : \mathcal{P}_c\widetilde{\mathcal{E}}^{\lambda} \to \mathcal{P}_c\mathcal{E}^{\lambda}$ such that (i) it preserves the decompositions (161) and (162), (ii) $\Phi_{|O}$ preserves the parabolic filtrations, (iii) the induced map $\operatorname{Gr}^F(\Phi)$ is compatible with the residues. We identify $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ in the following argument by Φ. By a direct calculation, we can show that $\widetilde{h}$ and h_0 are mutually bounded. Hence, we only have to show $\widetilde{h}$ and h are mutually bounded. We remark that we already know that $\widetilde{h}$ and h are mutually bounded up to log order by the weak norm estimate (Theorem 7.4.3), which we will use implicitly.

Let $\mathfrak{F}$ be the endomorphism of $\mathcal{P}_c\mathcal{E}^\lambda$ determined by $\mathfrak{F}\cdot dz/z = \mathbb{D}^\lambda - \widetilde{\mathbb{D}}^\lambda$. By construction, we have $\mathfrak{F}(\mathcal{P}_c\mathcal{E}^\lambda) \subset \mathcal{P}_{c-\varepsilon}\mathcal{E}^\lambda$ for some $\varepsilon > 0$. We have the estimate $[\mathfrak{F}, q_{\mathfrak{a},N}] = [\mathbb{D}^\lambda, q_{\mathfrak{a},N}] = O(|z|^{N/2})$ by construction of $q_{\mathfrak{a},N}$. (Note that we already know that $\widetilde{h}$ and h are mutually bounded, this estimate does not depend on the choice of a metric. We will often omit this type of remark.) Recall that $\mathbb{D}^\lambda$ and $\widetilde{h}$ determine the operators $\theta_{\widetilde{h}} \in \operatorname{End}(E)\otimes\Omega^{1,0}$, $\theta^\dagger_{\widetilde{h}} \in \operatorname{End}(E)\otimes\Omega^{0,1}$ and the pseudo-curvature

$$G(\mathbb{D}^\lambda, \widetilde{h}) := -\lambda^{-1}(1+|\lambda|^2)^2\cdot(\overline{\partial}_{\widetilde{h}} + \theta_{\widetilde{h}})^2 = -\lambda^{-1}(1+|\lambda|^2)^2\cdot\overline{\partial}_{\widetilde{h}}\theta_{\widetilde{h}}.$$

(See [**69**].) As in the case $\lambda = 0$, we obtain that h and $\widetilde{h}$ are mutually bounded, once we show that $G(\mathbb{D}^\lambda, \widetilde{h})$ is L^p for some $p > 1$, thanks to a general result of Simpson. (Corollary 4.3 of [**82**]. See also Section 4.3 of [**65**]. Note, in [**65**], the pseudo-curvature is considered as $G(\mathbb{D}^\lambda, \widetilde{h}) = \overline{\partial}_{\widetilde{h}}\theta_{\widetilde{h}}$, which does not make any essential difference in the conclusion.)

Let $d''_\lambda := \overline{\partial}_E + \lambda\theta^\dagger$, which is the holomorphic structure of $\mathcal{E}^\lambda$. We have the equalities:

$$0 = (1+|\lambda|^2)^{-1}\cdot G(\widetilde{\mathbb{D}}^\lambda, \widetilde{h}) = R(d''_\lambda, \widetilde{h}) + (1+|\lambda|^2)\big[\widetilde{\theta}, \widetilde{\theta}^\dagger_{\widetilde{h}}\big],$$

$$(1+|\lambda|^2)^{-1}G(\mathbb{D}^\lambda, \widetilde{h}) = R(d''_\lambda, \widetilde{h}) + (1+|\lambda|^2)\big[\theta_{\widetilde{h}}, \theta^\dagger_{\widetilde{h}}\big] = -(1+|\lambda|^2)\big([\widetilde{\theta}, \widetilde{\theta}^\dagger_{\widetilde{h}}] - [\theta_{\widetilde{h}}, \theta^\dagger_{\widetilde{h}}]\big).$$

Recall the following equalities (Section 2.2 of [**69**]):

$$\theta_{\widetilde{h}} = \frac{\mathfrak{F}}{1+|\lambda|^2} + \widetilde{\theta}, \quad \theta^\dagger_{\widetilde{h}} = \frac{\mathfrak{F}^\dagger_{\widetilde{h}}}{1+|\lambda|^2} + \widetilde{\theta}^\dagger_{\widetilde{h}}.$$

We have $[\mathfrak{F}, \mathfrak{F}^\dagger_{\widetilde{h}}] = O(|z|^\varepsilon)$. Let us estimate $\big[\mathfrak{F}, \widetilde{\theta}^\dagger_{\widetilde{h}}\big]$. We have the decomposition:

$$\operatorname{End}(\mathcal{P}_c\mathcal{E}^\lambda) = \mathcal{C}(\mathcal{P}_c\mathcal{E}^\lambda)\oplus\mathcal{D}(\mathcal{P}_c\mathcal{E}^\lambda),$$

$$\mathcal{C}(\mathcal{P}_c\mathcal{E}^\lambda) = \bigoplus_{\mathfrak{a}\neq\mathfrak{b}}\operatorname{Hom}\big(\mathcal{P}_c\mathcal{E}^\lambda_{\mathfrak{a},N}, \mathcal{P}_c\mathcal{E}^\lambda_{\mathfrak{b},N}\big), \quad \mathcal{D}(\mathcal{P}_c\mathcal{E}^\lambda) = \bigoplus_{\mathfrak{a}}\operatorname{End}\big(\mathcal{P}_c\mathcal{E}^\lambda_{\mathfrak{a},N}\big).$$

We have the corresponding decomposition $\mathfrak{F} = \mathcal{C}(\mathfrak{F}) + \mathcal{D}(\mathfrak{F})$. By construction, we have $\mathcal{C}(\mathfrak{F}) = O(|z|^N)$ with respect to $\widetilde{h}$, and thus $\big[\mathcal{C}(\mathfrak{F}), \widetilde{\theta}^\dagger_{\widetilde{h}}\big] = O\big(|z|^{N/2}\big)$. We have the decomposition:

$$\widetilde{\theta} = \bigoplus_{a,\alpha,\mathfrak{a}}\Big(\big(d\mathfrak{a} + \beta\cdot dz/z\big)\cdot\operatorname{id}_{\mathcal{P}_c\widetilde{\mathcal{E}}_{a,\alpha,\mathfrak{a}}} + \widetilde{\theta}'_{a,\alpha,\mathfrak{a}}\Big).$$

The terms $(\widetilde{\theta}'_{a,\alpha,\mathfrak{a}})^\dagger_{\widetilde{h}}$ are $O\big(|z|^{-1}(-\log|z|)^{-1}\big)$ with respect to $\widetilde{h}$, and the terms $\big(d\overline{\mathfrak{a}} + \overline{\alpha}\cdot d\overline{z}/\overline{z}^{-1}\big)\cdot\operatorname{id}_{\mathcal{P}_c\widetilde{\mathcal{E}}_{a,\alpha,\mathfrak{a}}}$ do not contribute to $\big[\mathcal{D}(\mathfrak{F}), \widetilde{\theta}^\dagger_{\widetilde{h}}\big]$. Hence, we obtain $\big[\mathcal{D}(\mathfrak{F}), \widetilde{\theta}^\dagger_{\widetilde{h}}\big] = O\big(|z|^{\varepsilon-2}dzd\overline{z}\big)$, and $G(\mathbb{D}^\lambda, \widetilde{h})$ is L^p for some $p > 1$. Thus the proof of Proposition 8.1.1 is finished. □

8.2. Comparison of the data at O

8.2.1. Statement. — Let X, D and $(E,\overline{\partial}_E,\theta,h)$ be as in Subsection 8.1.1. Let us compare the data at the origin O of $\mathcal{PE}^{\lambda}$ and $\mathcal{PE}^0$. For $\lambda \neq 0$, we have the vector space $\mathrm{Gr}^F_a(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$ with the endomorphism $\mathrm{Res}(\widehat{\mathbb{D}}^{\lambda}_{\mathfrak{a}})$. We have the generalized eigen-decomposition:

$$\mathrm{Gr}^F_a(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}) = \bigoplus_{\alpha\in \boldsymbol{C}} \mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}).$$

The residue $\mathrm{Res}(\mathbb{D}^{\lambda})$ induces an endomorphism, whose nilpotent part is denoted by $N^{\lambda}_{\mathfrak{a},a,\alpha}$. Similarly, we have the vector spaces $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{PE}^0_{\mathfrak{a}})$ with the nilpotent endomorphism $N^0_{\mathfrak{a},a,\alpha}$. We consider the following sets:

$$\mathcal{KMS}(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}) := \big\{(a,\alpha)\in \boldsymbol{R}\times\boldsymbol{C}\,\big|\, \mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}) \neq 0\big\} \quad (\lambda\neq 0),$$
$$\mathcal{KMS}(\mathcal{PE}^0_{\mathfrak{a}}) := \big\{(a,\alpha)\in \boldsymbol{R}\times\boldsymbol{C}\,\big|\, \mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{PE}^0_{\mathfrak{a}}) \neq 0\big\}.$$

The dimension of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$ is called the multiplicity of (a,α), and denoted by $\mathfrak{m}(\lambda,a,\alpha)$. The following proposition was observed by Biquard-Boalch ([**10**]), at least if the wild harmonic bundle is given on a quasiprojective curve.

Proposition 8.2.1. — *$\mathfrak{k}(\lambda)$ gives a bijective map $\mathcal{KMS}(\mathcal{PE}^0_{\mathfrak{a}}) \to \mathcal{KMS}(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$ preserving the multiplicities. The conjugacy classes of $N^0_{\mathfrak{a},a,\alpha}$ and $N^{\lambda}_{\mathfrak{a},\mathfrak{k}(\lambda,a,\alpha)}$ are the same.*

The claims can be shown using the essentially same argument as that in [**82**]. We indicate only an outline.

8.2.2. An estimate in [82]. — Let E be a holomorphic vector bundle on a punctured disc Δ^* with a frame $\boldsymbol{v}$. Let K be a Hermitian metric of E for which $\boldsymbol{v}$ is orthogonal and $|v_i|_K = |z|^{-a_i}(-\log|z|)^{-k_i/2}$ for some $a_i\in\boldsymbol{R}$ and $k_i\in\mathbb{Z}$.

Let F be a C^{∞}-section of E such that $|F|_K$ is bounded. For the expression $F = \sum F_i\cdot v_i$, we know $|F_i|\cdot|z|^{a_i}(-\log|z|)^{-k_i/2}$ are bounded.

Proposition 8.2.2. — *Let M be the section of E determined by $\overline{\partial}F = M\cdot d\overline{z}$. Assume the following conditions:*

$$|M|_K = O\big(|z|^{-1}(-\log|z|)^{-1}\big),\qquad \int |M|^2_K\cdot(-\log|z|)\cdot\big|dz\cdot d\overline{z}\big| < \infty.$$

Then, the following holds:

- *In the case $a_i\neq 0$, we have $|F_i|\cdot|z|^{a_i}(-\log|z|)^{-k_i/2} = O\big((-\log|z|)^{-1}\big)$.*
- *In the case $a_i = 0$ and $k_i\neq 0$, we have $\big\|F_i\cdot|z|^{a_i}(-\log|z|)^{-k_i/2}\big\|_W<\infty$, where $\big\|\cdot\big\|_W$ is given as follows:*

$$\|G\|^2_W := \int |G|^2_K \frac{\big|dz\cdot d\overline{z}\big|}{|z|^2(-\log|z|)^2\log(-\log|z|)}. \tag{163}$$

Proof. — See the argument in Page 765–767 of [**82**]. □

8.2.3. Decompositions. — As the special case of Lemma 7.4.7, we have the following lemma.

Lemma 8.2.3. — *Take any large number N. We can take holomorphic sections $p_{\mathfrak{a},N}$ of* $\mathrm{End}(\mathcal{P}_c\mathcal{E}^{\lambda})$ *for* $\mathfrak{a}\in\mathrm{Irr}(\theta)$ *such that the following holds:*

$$p_{\mathfrak{a},N}-\pi_{\mathfrak{a}}=O\big(|z|^{2N}\big),\quad \big(p_{\mathfrak{a},N}\big)^2=p_{\mathfrak{a},N},\quad \big[p_{\mathfrak{a}_1,N},p_{\mathfrak{a}_2,N}\big]=0,\qquad \sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}p_{\mathfrak{a},N}=\mathrm{id}.$$

They preserve the parabolic structure. □

We take a refinement of the decomposition:

Lemma 8.2.4. — *We can take holomorphic sections $p_{\mathfrak{a},\alpha}$ of* $\mathrm{End}(^{\diamond}\mathcal{E}^{\lambda})$ *for* $(\mathfrak{a},\alpha)\in\mathrm{Irr}(\theta)\times\mathcal{S}p(\theta)$ *such that the following holds:*

$$p_{\mathfrak{a},\alpha}-\pi_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon}\big),\quad \big[p_{\mathfrak{a},\alpha},p_{\mathfrak{b},\beta}\big]=0,\quad \big(p_{\mathfrak{a},\alpha}\big)^2=p_{\mathfrak{a},\alpha},\quad \sum_{\alpha}p_{\mathfrak{a},\alpha}=p_{\mathfrak{a},N}.$$

Proof. — The argument is essentially the same as that in the proof of Lemma 7.4.7. We give only an outline. Let $d''_{\lambda}:=\overline{\partial}_E+\lambda\theta^{\dagger}$, which is the holomorphic structure of $\mathcal{E}^{\lambda}$. We have $d''_{\lambda}\pi_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon}\big)$ with respect to h and the Poincaré metric. By Lemma 21.2.3, we can take sections $s_{\mathfrak{a},\alpha}$ of $\mathrm{End}(E)$ satisfying the following:

$$d''_{\lambda}s_{\mathfrak{a},\alpha}=d_{\lambda''}\pi_{\mathfrak{a},\alpha},\qquad \int\big|s_{\mathfrak{a},\alpha}\big|_h^2\cdot|z|^{-2\varepsilon'}\big(-\log|z|\big)^N\cdot\mathrm{dvol}_{g_{\mathrm{p}}}.$$

By Lemma 21.9.1, we obtain $s_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon''}\big)$ for some $\varepsilon''>0$.

We put $\overline{p}_{\mathfrak{a},\alpha}:=\pi_{\mathfrak{a},\alpha}-s_{\mathfrak{a},\alpha}$. Then, we have the following:

$$\big(\overline{p}_{\mathfrak{a},\alpha}\big)^2-\overline{p}_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon}\big),\quad \big[\overline{p}_{\mathfrak{a},\alpha},\overline{p}_{\mathfrak{b},\beta}\big]=O\big(|z|^{\varepsilon}\big),\quad \sum_{\alpha}\overline{p}_{\mathfrak{a},\alpha}-\overline{p}_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon}\big).$$

By modifying $\overline{p}_{\mathfrak{a},\alpha}$ with order $|z|^{\varepsilon}$ as in the proof of Lemma 7.4.7, we obtain the desired $p_{\mathfrak{a},\alpha}$. □

8.2.4. The eigenvalues of $\mathrm{Res}(\mathbb{D}^{\lambda})$**.** — We put $\Phi:=\sum_{\mathfrak{a},\alpha}\big(d\mathfrak{a}+\alpha\cdot dz/z\big)\cdot p_{\mathfrak{a},\alpha}$.

Lemma 8.2.5. — *Let v be a holomorphic section of $\mathcal{E}^{\lambda}$ such that*

$$|v|_h\sim|z|^{-a}(-\log|z|)^k.$$

Then, the following estimate holds:

$$\Big|\mathbb{D}^{\lambda}v-(1+|\lambda|^2)\Phi(v)+\lambda\cdot a\cdot v\cdot\frac{dz}{z}\Big|_h=O\big(|z|^{-a}(-\log|z|)^{k-1}\big)\frac{dz}{z}.$$

Proof. — We only have to consider the case $a=0$. Let δ'_{λ} be determined by d''_{λ} and h. Then the following holds:

$$\mathbb{D}^{\lambda}v-(1+|\lambda|^2)\Phi(v)=\lambda\delta'_{\lambda}v-(1+|\lambda|^2)(\theta-\Phi)(v)$$

Because $(\theta-\Phi)(v)=O((-\log|z|)^{k-1})dz/z$, we only have to show

$$\int|\delta'_{\lambda}v|^2(-\log|z|)^{1-\varepsilon-2k}|dz\cdot d\overline{z}|<\infty$$

for some $\varepsilon>0$. This can be shown using the argument in Page 761–762 in [**82**]. □

We take a holomorphic frame $\boldsymbol{v}$ of $\mathcal{P}_a\mathcal{E}^{\lambda}$ such that (i) it is compatible with the decomposition $\mathcal{P}_a\mathcal{E}^{\lambda}=\bigoplus\operatorname{Im}p_{\mathfrak{a},\alpha}$, (ii) it is compatible with the parabolic filtration, (iii) the induced frame of $\operatorname{Gr}^F_a(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$ is compatible with the weight filtration. We put $a(v_i):=\deg^F(v_i)$. Let $(\mathfrak{a}(v_i),\alpha(v_i))$ be determined by the condition that $v_i\in\operatorname{Im}p_{\mathfrak{a}(v_i),\alpha(v_i)}$.

We consider the following:

$$\mathbb{D}^{\lambda}_0:=\mathbb{D}^{\lambda}-(1+|\lambda|^2)\sum_{\mathfrak{a}\in\operatorname{Irr}(\theta)}d\mathfrak{a}\cdot p_{\mathfrak{a}}.$$

Lemma 8.2.6. — *Let A be the matrix determined by $\operatorname{Res}(\mathbb{D}^{\lambda}_0)\boldsymbol{v}_{|O}=\boldsymbol{v}_{|O}\cdot A$. If the order of $v_1,\dots,v_r$ is compatible with the parabolic filtration and the weight filtration, A is triangular, and the i-th diagonal entries are*

$$(1+|\lambda|^2)\alpha(v_i)-\lambda\cdot a(v_i).$$

Proof. — It follows from Lemma 8.2.5. □

8.2.5. Comparison map. — We put $V_{a,\alpha,\mathfrak{a}}:=\operatorname{Gr}^F_a(\mathcal{P}_c\mathcal{E}^0_{\mathfrak{a},\alpha})$ on which we have the nilpotent endomorphism $N^0_{a,\alpha,\mathfrak{a}}$. We take model bundles:

$$(\widetilde{E}_{a,\alpha,\mathfrak{a}},\widetilde{\theta}_{a,\alpha,\mathfrak{a}},\widetilde{h}_{a,\alpha,\mathfrak{a}})=\big(V_{a,\alpha,\mathfrak{a}}\otimes\mathcal{O}_{\Delta^*},N^0_{a,\alpha,\mathfrak{a}}dz/z,h_{a,\alpha,\mathfrak{a}}\big)\otimes L\big(a,\alpha,\mathfrak{a}\big),$$
$$(\widetilde{E},\widetilde{\theta},\widetilde{h}):=\bigoplus_{(a,\alpha,\mathfrak{a})}\big(\widetilde{E}_{a,\alpha,\mathfrak{a}},\widetilde{\theta}_{a,\alpha,\mathfrak{a}},\widetilde{h}_{a,\alpha,\mathfrak{a}}\big).$$

Let $\Psi:{}^{\diamond}\widetilde{E}\to{}^{\diamond}E$ be a holomorphic isomorphism such that (i) it preserves the decompositions, (ii) it preserves the parabolic filtration, (iii) $\operatorname{Gr}^F(\operatorname{Res}(G))=0$ where $G:=\Psi\circ\widetilde{\theta}-\theta\circ\Psi\in Hom({}^{\diamond}\widetilde{E},{}^{\diamond}E)\otimes\Omega^{1,0}(\log D)$. Note that Ψ and Ψ^{-1} are bounded, due to Proposition 8.1.1. We identify E and $\widetilde{E}$ via Ψ as C^{∞}-bundles.

By construction, we have $|\theta-\widetilde{\theta}|_{\widetilde{h}}=O(|z|^{\varepsilon})\cdot dz/z$. Due to the asymptotic orthogonality (Theorems 7.2.1 and 7.2.4), we have the following estimate:

$$|\widetilde{\theta}^{\dagger}_{\widetilde{h}}-\theta^{\dagger}_h|_{\widetilde{h}}=O\Big(\frac{dz}{|z|\big(-\log|z|\big)}\Big).$$

The following lemma can be shown by the same argument as that in the proof of Lemma 7.7 of [**82**].

Lemma 8.2.7. — *We have the finiteness $\int|M|^2_h(-\log|z|)\cdot\operatorname{dvol}_{g_{\mathrm{p}}}<\infty$, where M is determined by $\widetilde{\theta}^{\dagger}_{\widetilde{h}}-\theta^{\dagger}_h=M\cdot d\overline{z}$.* □

8.2.6. End of the proof of Proposition 8.2.1. — We have the induced λ-connection $(\widetilde{\mathcal{E}}^{\lambda},\widetilde{\mathbb{D}}^{\lambda})=\bigoplus(\widetilde{\mathcal{E}}^{\lambda}_{a,\alpha,\mathfrak{a}},\widetilde{\mathbb{D}}^{\lambda}_{a,\alpha,\mathfrak{a}})$, with the canonical frame $\widetilde{\boldsymbol{v}}=(\widetilde{\boldsymbol{v}}_{a,\alpha,\mathfrak{a}})$. We put $\widetilde{p}_{\mathfrak{a},\alpha}:=\widetilde{\pi}_{\mathfrak{a},\alpha}=\pi_{\mathfrak{a},\alpha}$. We have the decomposition $\widetilde{\mathcal{E}}^{\lambda}=\bigoplus\operatorname{Im}\widetilde{p}_{\mathfrak{a},\alpha}$.

Let $\boldsymbol{v}$ be as in Subsection 8.2.4. Let $\deg^F(v_i)$ and $\deg^W(v_i)$ denote the degree of v_i with respect to the parabolic filtration and the weight filtration. Let $(\mathfrak{a}(v_i),\alpha(v_i))$ be determined by the condition that $v_i\in\operatorname{Im}p_{\mathfrak{a}(v_i),\alpha(v_i)}$. We use the symbols $\deg^F(\widetilde{v}_i)$, $\deg^W(\widetilde{v}_i)$ and $(\mathfrak{a}(\widetilde{v}_i),\alpha(\widetilde{v}_i))$, with similar meanings. Let $I=(I_{j,i})$ be determined by the relation $\widetilde{v}_i=\sum_j I_{j,i}\cdot v_j$.

Lemma 8.2.8. — *We put $\mathcal{B}_{j,i}:=I_{j,i}|z|^{-\deg^F(v_j)+\deg^F(\widetilde{v}_i)+(\deg^W(v_j)-\deg^W(\widetilde{v}_i))/2}$.*
- *In the case $(\mathfrak{a}(v_j),\alpha(v_j))\neq(\mathfrak{a}(\widetilde{v}_i),\alpha(\widetilde{v}_i))$, we have $\left|\mathcal{B}_{j,i}\right|\leqslant C\cdot|z|^{\varepsilon}$ for some $\varepsilon>0$ and $C>0$.*
- *In the case $\deg^F(v_j)\neq\deg^F(\widetilde{v}_i)$, we have $\left|\mathcal{B}_{j,i}\right|\leqslant C(-\log|z|)^{-1}$ for some $C>0$.*
- *In the case $\deg^W(v_j)\neq\deg^W(\widetilde{v}_i)$, we have $\|\mathcal{B}_{j,i}(-\log|z|)\|_W<\infty$, where $\|\cdot\|_W$ is given as in* (163).

Proof. — We have the estimates $\widetilde{p}_{\mathfrak{a},\alpha}-p_{\mathfrak{a},\alpha}=O\big(|z|^{\varepsilon}\big)$ with respect to h. Hence, the first claim follows. The other claims can be shown using the argument in [**82**], using Lemma 8.2.7 and Proposition 8.2.2. (See also [**67**].) □

Then, we can show the following equality by the argument in the proof of Proposition 7.6 and Theorem 7 of [**82**], using Lemma 8.2.6 and Lemma 8.2.8:

$$\dim\operatorname{Gr}^W_k\mathbb{E}_\beta\operatorname{Gr}^F_b(\mathcal{E}^{\lambda}_{\mathfrak{b}})=\dim\operatorname{Gr}^W_k\mathbb{E}_\beta\operatorname{Gr}^F_b(\widetilde{\mathcal{E}}^{\lambda}_{\mathfrak{b}}).\tag{164}$$

We can check the claim of Proposition 8.2.1 for $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h})$ by a direct calculation. Together with (164), we obtain the claims of Proposition 8.2.1 for $(E,\overline{\partial}_E,\theta,h)$. □

8.3. A characterization of the lattices for generic λ

Let X, D and $(E,\overline{\partial}_E,\theta,h)$ be as in Subsection 8.1.1. We have the set $\mathcal{KMS}(E)\subset\boldsymbol{R}\times\boldsymbol{C}$ of the KMS-spectra at $\lambda=0$. A complex number λ is called generic with respect to $\mathcal{KMS}(E)$, if $\mathfrak{e}(\lambda):\mathcal{KMS}(E)\to\boldsymbol{C}$ is injective. For generic λ, we have the following characterization of $\mathcal{P}_a\mathcal{E}^{\lambda}$ $(a\in\boldsymbol{R})$.

Proposition 8.3.1. — *Let λ be generic with respect to $\mathcal{KMS}(E)$. Let V be a good lattice of the meromorphic λ-flat bundle $(\mathcal{P}\mathcal{E}^{\lambda},\mathbb{D}^{\lambda})$ with the following property:*
- *The set of the eigenvalues of* $\operatorname{Res}(\mathbb{D}^{\lambda})$ *on $V_{|D}$ is the same as the following:*

$$\big\{\mathfrak{e}(\lambda,u)\,\big|\,u\in\mathcal{KMS}(E),\ a-1<\mathfrak{p}(\lambda,u)\leqslant a\big\}.\tag{165}$$

Then, $V=\mathcal{P}_a\mathcal{E}^{\lambda}$.

Proof. — Let S denote the set (165). Note that $\lambda^{-1}(\alpha-\beta)\notin\mathbb{Z}$ for any distinct $\alpha,\beta\in S$. Hence, we have the flat decompositions:

$$(\mathcal{P}_a\mathcal{E}^\lambda,\mathbb{D}^\lambda)_{|\widehat{D}}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\bigoplus_{\alpha\in S}(\mathcal{P}_a\mathcal{E}^\lambda_{\mathfrak{a},\alpha},\mathbb{D}^\lambda_{\mathfrak{a},\alpha})\qquad(V,\mathbb{D}^\lambda)_{|\widehat{D}}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\bigoplus_{\alpha\in S}(\widehat{V}_{\mathfrak{a},\alpha},\mathbb{D}^\lambda_{\mathfrak{a},\alpha}).$$

Here, $\mathbb{D}^\lambda_{\mathfrak{a},\alpha}-d\mathfrak{a}-\alpha\cdot dz/z$ are logarithmic with respect to $\mathcal{P}_a\mathcal{E}^\lambda_{\mathfrak{a},\alpha}$ or $\widehat{V}_{\mathfrak{a},\alpha}$, and the residues are nilpotent. Then, we obtain $\mathcal{P}_a\mathcal{E}^\lambda_{\mathfrak{a},\alpha}=\widehat{V}_{\mathfrak{a},\alpha}$ using a standard and classical argument: (i) We can show $\bigoplus_\alpha\mathcal{P}_a\mathcal{E}^\lambda_{\mathfrak{a},\alpha}\otimes\mathcal{O}(*D)=\bigoplus_\alpha\widehat{V}_{\mathfrak{a},\alpha}\otimes\mathcal{O}(*D)$ by using Corollary 2.2.18. (ii) By the assumption for S, we can also obtain $\mathcal{P}_a\mathcal{E}^\lambda_{\mathfrak{a},\alpha}\otimes\mathcal{O}(*D)=\widehat{V}_{\mathfrak{a},\alpha}\otimes\mathcal{O}(*D)$ by a similar argument with a minor modification. (iii) In the regular case, this kind of claim is well known. (See Proposition II. 5.4 of [**24**], for example.) □

Remark 8.3.2. — The proposition can easily be generalized to the higher dimensional case. □

8.4. Harmonic forms

8.4.1. The space of harmonic forms of $\mathcal{E}^\lambda$. — Let Y be a smooth projective curve, and let D be a finite subset of Y. Let $(E,\overline{\partial}_E,\theta,h)$ be a wild harmonic bundle on $Y\setminus D$. Let ω be a Kähler form of $Y\setminus D$, which is Poincaré-like around D. We reformulate the result in Section 8.4.3.4 below. Let $\mathbb{D}^{\lambda *}$ denote the formal adjoint of $\mathbb{D}^\lambda$ with respect h and ω. We put $\Delta^\lambda:=\mathbb{D}^\lambda\circ\mathbb{D}^{\lambda *}+\mathbb{D}^{\lambda *}\circ\mathbb{D}^\lambda$. Recall $\Delta^\lambda=(1+|\lambda|^2)\cdot\Delta^0$.

Proposition 8.4.1. — *Let ϕ be an L^2-section of E on $Y\setminus D$. We have the following equivalence:*

$$\Delta^\lambda\phi=0\text{ for some }\lambda\iff\Delta^\lambda\phi=0\text{ for any }\lambda$$
$$\iff\mathbb{D}^\lambda\phi=0\text{ for some }\lambda\iff\mathbb{D}^\lambda\phi=0\textit{ for any }\lambda.$$

Let ϕ be an L^2-section of $E\otimes\Omega^{1,1}$ on $Y\setminus D$. We have the following equivalence:

$$\Delta^\lambda\phi=0\textit{ for some }\lambda\iff\Delta^\lambda\phi=0\textit{ for any }\lambda$$
$$\iff\mathbb{D}^{\lambda *}\phi=0\textit{ for some }\lambda\iff\mathbb{D}^{\lambda *}\phi=0\textit{ for any }\lambda.$$

Proof. — Let us show the claim for 0-forms. According to Proposition 8.4.18 below, we have the equivalence $\Delta^\lambda\phi=0\iff\mathbb{D}^\lambda\phi=0$ for a fixed λ. Then, the desired equivalence follows from $\Delta^\lambda=(1+|\lambda|^2)\Delta^0$. The claim for 2-forms can be reduced to the case of 0-forms. □

Proposition 8.4.2. — *Let ϕ be an L^2-section of $E\otimes\Omega^1$ on $Y\setminus D$. The following conditions are equivalent.*

(a) : $\Delta^\lambda\phi=0$ *for some* λ.

(b) : $\Delta^{\lambda}\phi = 0$ *for any* λ.

(c) : $\mathbb{D}^{\lambda}\phi = \mathbb{D}^{\lambda *}\phi = 0$ *for some* λ.

(d) : $\mathbb{D}^{\lambda}\phi = \mathbb{D}^{\lambda *}\phi = 0$ *for any* λ.

(e) : $(\overline{\partial} + \theta)\phi = (\partial + \theta^{\dagger})\phi = 0$.

Proof. — The equivalence of the conditions (a)–(d) can be shown by using the same argument as in the proof of Proposition 8.4.1. Note $\mathbb{D}^{0*} = \sqrt{-1}\Lambda_{\omega}\big(\partial + \theta^{\dagger}\big)$ on the 1-forms. Hence, the condition (e) is equivalent to (c) with $\lambda = 0$. □

Remark 8.4.3. — The equivalence of (c) and (d) in Proposition 8.4.2 can be shown by a direct calculation. Note that $\mathbb{D}^{\lambda *}\phi = 0$ is equivalent to $\mathbb{D}^{\lambda \star}\phi = 0$ for a 1-form ϕ. (See [**69**] for the notation $\mathbb{D}^{\lambda\star}$.) Let $A := \overline{\partial} + \theta$ and $B := \partial + \theta^{\dagger}$. Then, we have the equalities $(1 + |\lambda|^2)A = \mathbb{D}^{\lambda} - \lambda\mathbb{D}^{\lambda\star}$ and $(1 + |\lambda|^2)B = \mathbb{D}^{\lambda\star} + \lambda\mathbb{D}^{\lambda}$. Hence, $\mathbb{D}^{\lambda_1}$ and $\mathbb{D}^{\lambda_1\star}$ can be expressed as the linear combinations of $\mathbb{D}^{\lambda}$ and $\mathbb{D}^{\lambda\star}$ for any λ_1. □

The following notation will be used in Section 18.2.

Notation 8.4.4. — Let Harm^i denote the space of the L^2-sections of $E \otimes \Omega^i$ satisfying the conditions in Propositions 8.4.1 and 8.4.2. □

Needless to say, the equivalence in the propositions may not hold if Y is not projective. Hence, we have to distinguish the conditions.

8.4.2. Decay of L^2-harmonic 1-forms around the singularity. — Let us study the behaviour of harmonic 1-forms around the singularity with more details. Let X, D and $(E, \overline{\partial}_E, \theta, h)$ be as in Subsection 8.1.1. We use the Poincaré metric of $X \setminus D$. Let τ be an L^2-section of $E \otimes \Omega^1$ on X^* such that $(\overline{\partial}_E + \theta)\tau = (\partial_E + \theta^{\dagger})\tau = 0$. We have the decomposition $\tau = \sum_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \tau_{\mathfrak{a}}$ corresponding to the decomposition $E = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} E_{\mathfrak{a}}$. We will use the following proposition in Section 18.2.

Proposition 8.4.5. — *τ is of polynomial order with respect to h and the Poincaré metric. For $\mathfrak{a} \neq 0$, we have the estimate $\tau_{\mathfrak{a}} = O\big(\exp(-\varepsilon|z|^{\mathrm{ord}(\mathfrak{a})})\big)$.*

Proof. — For simplicity, we use the symbol $\partial A/\partial z$ to denote the coupling of ∂A and ∂_z for a section A of E. For the expression $\tau = A \cdot dz + B \cdot d\overline{z}$, we have the following equalities:

$$\frac{\partial A}{\partial \overline{z}} - f(B) = 0, \quad \frac{\partial B}{\partial z} - f^{\dagger}(A) = 0. \tag{166}$$

Lemma 8.4.6. — *A and B are of polynomial order with respect to h.*

Proof. — Let $\boldsymbol{v}$ be a frame of ${}^{\diamond}E$ compatible with the decomposition (156), the parabolic filtration and the weight filtration. Let Θ, $\Theta^{\dagger}$ and C be the matrix-valued functions determined by the conditions:

$$\theta\boldsymbol{v} = \boldsymbol{v} \cdot \Theta \cdot dz, \quad \theta^{\dagger}\boldsymbol{v} = \boldsymbol{v} \cdot \Theta^{\dagger} d\overline{z}, \quad \partial\boldsymbol{v} = \boldsymbol{v} \cdot C \cdot dz.$$

Then, Θ, $\Theta^\dagger$ and C are of polynomial order, a property which follows from the estimate for the Higgs field (Theorems 7.2.1 and 7.2.4) and the acceptability of $(E, \overline{\partial}_E, h)$ (Theorem 21.3.2 and Lemma 21.9.3). Let $\boldsymbol{A} = (A_i)$ and $\boldsymbol{B} = (B_i)$ be $\boldsymbol{C}^{\operatorname{rank} E}$-valued functions determined by $A = \sum A_i \cdot v_i$ and $B = \sum B_i \cdot v_i$. Due to (166), the following equalities hold on Δ^*:

$$\frac{\partial \boldsymbol{A}}{\partial \overline{z}} - \Theta(\boldsymbol{B}) = 0, \quad \frac{\partial \boldsymbol{B}}{\partial z} + [C, \boldsymbol{B}] - \Theta^\dagger(\boldsymbol{A}) = 0.$$

Hence, we have the following for any non-negative integer $\ell \geqslant 0$:

$$\frac{\partial(z^\ell \boldsymbol{A})}{\partial \overline{z}} - \Theta(z^\ell \boldsymbol{B}) = 0, \quad \frac{\partial(z^\ell \boldsymbol{B})}{\partial z} - \ell z^{\ell-1}\boldsymbol{B} + z^\ell[C, \boldsymbol{B}] - z^\ell \Theta^\dagger(\boldsymbol{A}) = 0. \tag{167}$$

If ℓ is sufficiently large, (167) holds on X as distributions. For large N, $z^N \boldsymbol{A}$ and $\Theta(z^N \boldsymbol{B})$ are L^p for some $p > 0$. Then $z^N \boldsymbol{A}$ is L_1^p for some $p > 2$ because of the first equality in (167), and hence $z^N \boldsymbol{A}$ is bounded. Thus, A is shown to be of polynomial order. By applying a similar argument to $(E, \partial_E, \theta^\dagger, h)$, it can be shown that B is also of polynomial order. □

We give a refinement. We put $\mathcal{S}_1(j) := \{\mathfrak{a} \in \operatorname{Irr}(\theta) \mid \operatorname{ord}(\mathfrak{a}) \leqslant j\}$ and $\mathcal{S}_0(j) := \{\mathfrak{a} \in \operatorname{Irr}(\theta) \mid \operatorname{ord}(\mathfrak{a}) > j\}$. We put $E_a^{(j)} := \bigoplus_{\mathfrak{a} \in \mathcal{S}_a(j)} E_{\mathfrak{a}}$ for $a = 0, 1$. Let $B = B_1^{(j)} + B_0^{(j)}$ and $A = A_1^{(j)} + A_0^{(j)}$ be the decompositions corresponding to $E = E_1^{(j)} \oplus E_0^{(j)}$. For a differential operator $\mathfrak{D} : E \to E$, we have the decomposition $\mathfrak{D} = \sum_{a,b=1,2} \mathfrak{D}_{a,b}$, where $\mathfrak{D}_{a,b}^{(j)} : E_a^{(j)} \to E_b^{(j)}$. We put $\mathcal{D}^{(j)}(\mathfrak{D}) := \mathfrak{D}_{1,1}^{(j)} + \mathfrak{D}_{0,0}^{(j)}$ and $\mathcal{C}^{(j)}(\mathfrak{D}) := \mathfrak{D}_{1,0}^{(j)} + \mathfrak{D}_{0,1}^{(j)}$.

Lemma 8.4.7. — *$A_1^{(j)} = O\big(\exp(-\varepsilon|z|^j)\big)$ for some $\varepsilon > 0$, with respect to h.*

Proof. — Let us look at the $E_1^{(j)}$-component of the equality $\partial B/\partial z - f^\dagger(A) = 0$. Due to Lemma 7.5.5 and the $\exp(-\varepsilon|z|^j)$-asymptotic orthogonality of the decomposition $E = \bigoplus_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} E_{\mathfrak{a}}^{(j)}$ (Theorem 7.2.1), we obtain the following:

$$\begin{aligned} 0 &= \mathcal{D}^{(j)}(\partial_z)B_1^{(j)} + \mathcal{C}^{(j)}(\partial_z)B_0^{(j)} - \mathcal{D}^{(j)}(f^\dagger)A_1^{(j)} - \mathcal{C}^{(j)}(f^\dagger)A_0^{(j)} \\ &= \mathcal{D}^{(j)}(\partial_z)B_1^{(j)} - \mathcal{D}^{(j)}(f^\dagger)A_1^{(j)} + O\big(\exp(-C|z|^j)\big). \end{aligned} \tag{168}$$

We have the unique L^2-section $g_1^{(j)}$ of $E_1^{(j)}$ such that $\theta\big(g_1^{(j)}\big) = A_1^{(j)} \cdot dz$. Then, we have $(\overline{\partial} + \theta)g_1^{(j)} = A_1^{(j)}dz + B_1^{(j)}d\overline{z}$ on $X \smallsetminus D$, i.e., $\partial g_1^{(j)}/\partial \overline{z} = B_1^{(j)}$ and $f(g_1^{(j)}) = A_1^{(j)}$. Since $A_1^{(j)}$ is of polynomial order, we obtain that $g_1^{(j)}$ is of polynomial order. We have the following equality on $X \smallsetminus D$:

$$-\frac{\partial^2}{\partial z \partial \overline{z}}|g_1^{(j)}|_h^2 = -\Big(\frac{\partial^2 g_1^{(j)}}{\partial z \partial \overline{z}}, g_1^{(j)}\Big)_h - \Big|\frac{\partial g_1^{(j)}}{\partial z}\Big|_h^2 - \Big|\frac{\partial g_1^{(j)}}{\partial \overline{z}}\Big|_h^2 - \Big(g_1^{(j)}, \frac{\partial^2 g_1^{(j)}}{\partial \overline{z} \partial z}\Big)_h.$$

Since $A_1^{(j)}$ and $B_1^{(j)}$ are of polynomial order, we obtain the following:

$$(169)\quad \Big(\frac{\partial^2 g_1^{(j)}}{\partial z\partial\overline{z}}, g_1^{(j)}\Big)_h = \Big(\mathcal{D}^{(j)}(\partial_z)\frac{\partial g_1^{(j)}}{\partial\overline{z}}, g_1^{(j)}\Big)_h + O\big(\exp(-C|z|^j)\big)$$
$$= \big(\mathcal{D}^{(j)}(f^\dagger)f(g_1^{(j)}), g_1^{(j)}\big)_h + O\big(\exp(-C|z|^j)\big) = \big(f^\dagger f(g_1^{(j)}), g_1^{(j)}\big)_h + O\big(\exp(-C|z|^j)\big)$$
$$= \big(f(g_1^{(j)}), f(g_1^{(j)})\big)_h + O\big(\exp(-C|z|^j)\big).$$

We also have the following:

$$\Big(\frac{\partial^2 g_1^{(j)}}{\partial z\partial\overline{z}} - \frac{\partial^2 g_1^{(j)}}{\partial\overline{z}\partial z}\Big) dz\cdot d\overline{z} = (\partial\overline{\partial}+\overline{\partial}\partial)g_1^{(j)} = -(\theta\theta^\dagger+\theta^\dagger\theta)g_1^{(j)} = -(ff^\dagger - f^\dagger f)g_1^{(j)}\cdot dz\cdot d\overline{z}.$$

Therefore, we obtain the following:

$$(170)\quad -\frac{\partial^2}{\partial z\partial\overline{z}}|g_1^{(j)}|_h^2 = -\Big(\frac{\partial^2 g_1^{(j)}}{\partial z\partial\overline{z}}, g_1^{(j)}\Big)_h - \Big(g_1^{(j)}, \frac{\partial^2 g_1^{(j)}}{\partial z\partial\overline{z}}\Big)_h$$
$$- \big(g_1^{(j)}, (ff^\dagger - f^\dagger f)g_1^{(j)}\big)_h - \Big|\frac{\partial g_1^{(j)}}{\partial z}\Big|_h^2 - \Big|\frac{\partial g_1^{(j)}}{\partial\overline{z}}\Big|_h^2$$
$$= -\big|f(g_1^{(j)})\big|_h^2 - \big|f^\dagger(g_1^{(j)})\big|_h^2 - \Big|\frac{\partial g_1^{(j)}}{\partial z}\Big|_h^2 - \Big|\frac{\partial g_1^{(j)}}{\partial\overline{z}}\Big|_h^2 + O\big(\exp(-C|z|^j)\big)$$
$$\leqslant -C_1|z|^{2(j-1)}\cdot|g_1^{(j)}|_h^2 + C_2\exp(-C_3|z|^j).$$

Lemma 8.4.8. — *The following inequality holds on X as distributions:*

$$(171)\qquad -\frac{\partial^2}{\partial z\partial\overline{z}}|g_1^{(j)}|_h^2 \leqslant -C_1\cdot|z|^{2(j-1)}\cdot|g_1^{(j)}|_h^2 + C_2\cdot\exp\big(-C_3|z|^j\big).$$

Proof. — We already know that the inequality holds on $X\setminus D$. Let φ be a test function. We have the following:

$$\int_{|z|\geqslant\delta}|g_1^{(j)}|_h^2\frac{\partial^2\varphi}{\partial z\partial\overline{z}}dz\cdot d\overline{z} =$$
$$\pm\int_{|z|=\delta}|g_1^{(j)}|_h^2\frac{\partial\varphi}{\partial\overline{z}}d\overline{z} \pm \int_{|z|=\delta}\frac{\partial|g_1^{(j)}|^2}{\partial z}\varphi\cdot dz + \int_{|z|\geqslant\delta}\frac{\partial^2|g_1^{(j)}|_h^2}{\partial z\partial\overline{z}}\varphi\cdot dz\cdot d\overline{z}.$$

Hence, we only have to show the existence of a sequence $\{\delta_i\}$ with $\delta_i\to 0$ such that the following holds:

$$\lim\int_{|z|=\delta_i}|g_1^{(j)}|_h^2\cdot\frac{\partial\varphi}{\partial\overline{z}}\cdot d\overline{z} = 0,\quad \lim\int_{|z|=\delta_i}\frac{\partial|g_1^{(j)}|_h^2}{\partial z}\varphi\cdot dz = 0.$$

Let us show the second convergence. The first one can be shown by a similar argument. By construction, we have the following finiteness:

$$\int|g_1^{(j)}|_h^2\cdot|z|^{2(j-1)}\cdot|dz\cdot d\overline{z}| < \infty,\quad \int\big(\overline{\partial}g_1^{(j)}, \overline{\partial}g_1^{(j)}\big)_h < \infty.$$

Let ρ be a non-negative C^∞-function on $\boldsymbol{R}$ such that $\rho(t) = 1$ for $t\leqslant 1/2$ and $\rho(t) = 0$ for $t\geqslant 2/3$. We put $\chi_N(z) := \rho\big(-N^{-1}\log|z|\big)$. Note $\partial\chi_N(z)$ and $\overline{\partial}\chi_N(z)$

are uniformly bounded with respect to the Poincaré metric. We have the following estimate, which is independent of N:

$$\int\big(\partial(\chi_N\cdot g_1^{(j)}),\partial(\chi_N\cdot g_1^{(j)})\big)=\\ \pm\int\big(\chi_N\cdot g_1^{(j)},\,R(h)\cdot\chi_N\cdot g_1^{(j)}\big)\pm\int\big(\overline{\partial}(\chi_N\cdot g_1^{(j)}),\overline{\partial}(\chi_N\cdot g_1^{(j)})\big)<C.$$

Thus, we obtain the finiteness $\int\big(\partial g_1^{(j)},\partial g_1^{(j)}\big)<\infty$. Then, we obtain the following finiteness:

$$\int r^{j-1}dr\Big(\int\Big|\frac{\partial|g_1^{(j)}|_h^2}{\partial z}\varphi\Big|\cdot r\cdot d\theta\Big)\leqslant\\ C\Big(\int\big(\partial g_1^{(j)},\partial g_1^{(j)}\big)_h\Big)^{1/2}\Big(\int|g_1^{(j)}|_h^2|z|^{2(j-1)}\cdot|dz\cdot d\overline{z}|\Big)^{1/2}<\infty. \tag{172}$$

The existence of the desired sequence $\{\delta_i\}$ follows from (172). Thus, the proof of Lemma 8.4.8 is finished. □

Let us return to the proof of Lemma 8.4.7. In general, we have the following inequality for $\ell>0$:

$$-\frac{\partial^2}{\partial z\partial\overline{z}}\exp\big(-C|z|^{-\ell}\big)=\exp\big(-C|z|^{-\ell}\big)\frac{\ell^2\cdot C}{4}|z|^{-\ell-2}-\exp\big(-C|z|^{-\ell}\big)\frac{(\ell\cdot C)^2}{4}|z|^{-2\ell-2}.$$

Hence, we obtain the following inequality for some appropriate constant $G>0$:

$$-\frac{\partial^2}{\partial z\partial\overline{z}}\exp\big(-G|z|^{j}\big)\geqslant -C_1\cdot|z|^{2(j-1)}\exp\big(-G|z|^{j}\big)+C_2\cdot\exp\big(-C_3|z|^{j}\big). \tag{173}$$

We obtain $|g_1^{(j)}|_h\leqslant C_4\exp(-G|z|^j)$, by using (173), Lemma 8.4.8 and the standard argument as in [**1**] and [**82**]. (See also the proof of Theorem 7.2.1 and Theorem 7.2.4.) Because $\big|A_1^{(j)}\big|_h\leqslant C_6|z|^{j-1}\cdot|g_1^{(j)}|_h$, the proof of Lemma 8.4.7 is accomplished. □

Lemma 8.4.9. — *$B_1^{(j)}=O\big(\exp(-C|z|^j)\big)$ with respect to h.*

Proof. — We can apply a similar argument to a harmonic bundle $(E,\overline{\partial}_E,h,\theta^\dagger)$ on $(X\setminus D)^\dagger$. We have the decomposition similar to (156):

$$(E,f^\dagger)=\bigoplus_{(\mathfrak{a},\alpha)\in\mathrm{Irr}(\theta^\dagger)\times\mathcal{S}p(\theta^\dagger)}\big(E^\dagger_{\overline{\mathfrak{a}},\overline{\alpha}},f^\dagger_{\overline{\mathfrak{a}},\overline{\alpha}}\big)=\bigoplus_{\overline{\mathfrak{a}}\in\mathrm{Irr}(\theta^\dagger)}\big(E^\dagger_{\overline{\mathfrak{a}}},f^\dagger_{\overline{\mathfrak{a}}}\big).$$

We put $\mathcal{S}_1^\dagger(j):=\big\{\overline{\mathfrak{a}}\in\mathrm{Irr}(\theta^\dagger)\,\big|\,\mathrm{ord}(\overline{\mathfrak{a}})\leqslant j\big\}$ and $\mathcal{S}_0^\dagger(j):=\big\{\overline{\mathfrak{a}}\in\mathrm{Irr}(\theta^\dagger)\,\big|\,\mathrm{ord}(\overline{\mathfrak{a}})>j\big\}$. As in the previous argument, we put $E_a^{\dagger(j)}:=\bigoplus_{\overline{\mathfrak{a}}\in\mathcal{S}_a^\dagger(j)}E^\dagger_{\overline{\mathfrak{a}}}$. By Lemma 8.4.7, we have the corresponding decomposition $B=B_0^{\dagger(j)}+B_1^{\dagger(j)}$, and we obtain $B_1^{\dagger(j)}=O(\exp(-C|z|^j))$ with respect to h.

Let $\pi_a^{\dagger(j)}$ denote the projection onto $E_a^{\dagger(j)}$ with respect to the decomposition $E=E_0^{\dagger(j)}\oplus E_1^{\dagger(j)}$. Let $\pi_a^{(j)}$ denote the projection onto $E_a^{(j)}$ with respect to the decomposition $E = E_0^{(j)} \oplus E_1^{(j)}$. According to Theorem 7.2.1, we have the estimate $\pi_a^{(j)} - \pi_a^{(j)\dagger} = O\big(\exp(-\varepsilon|z|^j)\big)$ with respect to h. We have the following:

$$B_1^{(j)} = \pi_1^{(j)}(B) = B_1^{\dagger(j)} + \big(\pi_1^{(j)} - \pi_1^{\dagger(j)}\big)(B).$$

Since B is of polynomial order, we obtain the desired estimate for $B_1^{(j)}$. Thus, the proof of Lemma 8.4.9 is finished □

Now, the claim of Proposition 8.4.5 immediately follows from Lemmas 8.4.6, 8.4.7 and 8.4.9. □

Remark 8.4.10. — Let Y, D, ω and $(E, \overline{\partial}_E, \theta, h)$ be as in Subsection 8.4.1. Due to Proposition 8.4.2, we can apply Proposition 8.4.5 to study the behaviour of any L^2-harmonic 1-form of $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ around D, after taking an appropriate ramified covering. □

Remark 8.4.11. — Let τ be an L^2-section of E such that $(\overline{\partial} + \theta)\tau = 0$. For the decomposition $\tau = \sum \tau_{\mathfrak{a}}$ corresponding to $E = \bigoplus E_{\mathfrak{a}}$, we obviously have the vanishings $\tau_{\mathfrak{a}} = 0$ for $\mathfrak{a} \neq 0$. We also have a similar claim for 2-forms. □

Remark 8.4.12. — In [**90**], the exponential decay of harmonic forms is shown in a special case with a different method. □

8.4.3. General remarks on L^2-harmonic forms (Appendix)

8.4.3.1. L^2-cohomology and harmonic forms. — We recall a general remark on the L^2-cohomology, following [**96**]. Let (Y, g) be a complete Kähler manifold. Let dvol_g denote the volume form associated to g. Let $(V, \mathbb{D}^\lambda)$ be a λ-flat bundle on Y with a Hermitian metric h. (See [**86**] and [**69**].) Let $\mathcal{L}^j(V)$ be the space of sections f of $V \otimes \Omega^j$ on Y, such that f and $\mathbb{D}^\lambda f$ are L^2 with respect to h and g. (Here, $\mathbb{D}^\lambda f$ is taken in the sense of distributions. But, we do not have to be concerned with it, because (Y, g) is complete. See [**3**].) Thus, we obtain a complex $(\mathcal{L}^\bullet(V), \mathbb{D}^\lambda)$. Let $H^j\big(\mathcal{L}^\bullet(V)\big)$ denote the j-th cohomology group of the complex.

Let $\mathbb{D}^{\lambda *}$ denote the formal adjoint of $\mathbb{D}^\lambda$ with respect to h and g. Let $\boldsymbol{H}^j$ denote the space of L^2-sections f of $V \otimes \Omega^j$ satisfying $\mathbb{D}^\lambda f = 0$ and $\mathbb{D}^{\lambda *} f = 0$.

Lemma 8.4.13. — *Assume that $\bigoplus_j H^j\big(\mathcal{L}^\bullet(V)\big)$ is finite dimensional. Then, the natural map $\bigoplus_j \boldsymbol{H}^j \to \bigoplus_j H^j\big(\mathcal{L}^\bullet(V)\big)$ is an isomorphism.*

Proof. — We use the inner product $(f, g) := \int h(f, g) \cdot \mathrm{dvol}_g + \int h(\mathbb{D}^\lambda f, \mathbb{D}^\lambda g) \cdot \mathrm{dvol}_g$, via which $\mathcal{L}^j(V)$ is the Hilbert space. Let Z^j denote the kernel of $\mathbb{D}^\lambda : \mathcal{L}^j(V) \to \mathcal{L}^{j+1}(V)$, which is the closed subspace of $\mathcal{L}^j(V)$. Let $Z^{j\perp}$ denote the orthogonal complement of Z^j in $\mathcal{L}^j(V)$. Let us see the continuous operator $\mathbb{D}^\lambda : \mathcal{L}^{j-1}(V) \to Z^j$. Because we have assumed $\dim H^j\big(\mathcal{L}^\bullet(V)\big) < \infty$, the image R^j of $\mathbb{D}^\lambda$ is closed. Let $\boldsymbol{H}_1^j$ denote the orthogonal complement of R^j in Z^j. Then, we have the orthogonal decomposition $\mathcal{L}^j(V) = R^j \oplus \boldsymbol{H}_1^j \oplus Z^j$, and $\boldsymbol{H}_1$ is naturally isomorphic to $H^j\big(\mathcal{L}^\bullet(V)\big)$.

Let us show $\boldsymbol{H}_1^j = \boldsymbol{H}^j$. Let $f \in \boldsymbol{H}_1^j$. For any C^∞-section φ of $V \otimes \Omega^{j-1}$ with compact support, we have $\int \big(\mathbb{D}^\lambda \varphi, f\big)_{h,\omega} \,\mathrm{dvol}_\omega = 0$. Then, we obtain $\mathbb{D}^{\lambda *} f = 0$ in the sense of distributions, i.e., $f \in \boldsymbol{H}^j$. For any $f \in \boldsymbol{H}_1^j$ and for any φ as above, we have $\int \big(\mathbb{D}^\lambda \varphi, f\big)_{h,\omega} = \int \big(\varphi, \mathbb{D}^{\lambda *} f\big)_{h,\omega} = 0$, and hence $f \in \boldsymbol{H}_1^j$. Thus, we obtain Lemma 8.4.13. □

8.4.3.2. Harmonic forms on a complete Kähler manifold. — Let (Y, g) and $(V, \mathbb{D}^\lambda)$ be as in Section 8.4.3.1. Let $\Delta^\lambda := \mathbb{D}^\lambda \circ \mathbb{D}^{\lambda *} + \mathbb{D}^{\lambda *} \circ \mathbb{D}^\lambda$. We recall the following general remark.

Lemma 8.4.14. — *Let ϕ be an L^2-section of $V \otimes \Omega^j$ such that $\mathbb{D}^\lambda \phi$ and $\mathbb{D}^{\lambda *} \phi$ are L^2. Then, $\Delta^\lambda \phi = 0$ if and only if $\mathbb{D}^\lambda \phi = \mathbb{D}^{\lambda *} \phi = 0$.*

Proof. — We only have to show the "only if" part. We fix a base point $x_0 \in X$. Let $d(x, y)$ denote the distance of $x, y \in X$ induced by g. Let $B(R)$ denote the set $\big\{x \in X \,\big|\, d(x, x_0) \leqslant R\big\}$. As in Page 90–91 of [**3**], we can take a sequence of Lipschitz functions χ_n $(n = 1, 2, \dots)$ such that (i) $0 \leqslant \chi_n \leqslant 1$, (ii) $\chi_n(x) = 1$ on $B(n)$ and $\chi_n(x) = 0$ on $X - B(2n)$, (iii) $|d\chi_n| \leqslant C$ for some fixed C and for almost every $x \in X$. Assume ϕ satisfies $\Delta^\lambda \phi = 0$. We have the following:

$$0 = \int \big(\Delta^\lambda \phi,\, \chi_n \phi\big)_{h,g} \,\mathrm{dvol}_g - \int \big(\mathbb{D}^\lambda \phi,\, \mathbb{D}^\lambda(\chi_n \phi)\big)_{h,g} \,\mathrm{dvol}_g \\ - \int \big(\mathbb{D}^{\lambda *} \phi,\, \mathbb{D}^{\lambda *}(\chi_n \phi)\big)_{h,g} \,\mathrm{dvol}_g \,.$$

Then, we obtain the vanishing $\big\|\mathbb{D}^\lambda \phi\big\|_{h,\omega}^2 + \big\|\mathbb{D}^{\lambda *} \phi\big\|_{h,\omega}^2 = 0$ due to the theorem of Lebesgue. □

8.4.3.3. A general remark on L^2-conditions. — Let X denote the closed disc $\big\{z \in \boldsymbol{C} \,\big|\, |z| \leqslant 1\big\}$, and we put $D = \{O\}$. Let ω be a Poincaré-like Kähler form of $X \setminus D$. Let dvol_ω denote the volume form associated to ω. Let $(V, \mathbb{D}^\lambda)$ be a λ-flat bundle on $X \setminus D$ with a Hermitian metric h. We recall a general remark.

Lemma 8.4.15. — *Let ϕ be a C^∞-section of $V \otimes \Omega^j$ on $X \setminus D$. Assume that ϕ and $\Delta^\lambda \phi$ are L^2 with respect to h and ω. Then, $\mathbb{D}^\lambda \phi$ and $\mathbb{D}^{\lambda *} \phi$ are also L^2 with respect to h and ω.*

Proof. — For any C^∞-section f of $V \otimes \Omega^j$ on $X \setminus D$, let $\|f\|_{h,\omega}$ denote the L^2-norm with respect to h and ω. If the support of f is compact, we have the following, for some number B_0 which is independent of f:

$$\big\|\mathbb{D}^\lambda f\big\|_{h,\omega}^2 + \big\|\mathbb{D}^{\lambda *} f\big\|_{h,\omega}^2 = B_0 \int_{X \setminus D} \big(f,\, \Delta^\lambda(f)\big)_{h,\omega} \,\mathrm{dvol}_\omega \,. \tag{174}$$

Let ϕ be a C^∞-section of V on $X \setminus D$, such that ϕ and $\Delta^\lambda \phi$ are L^2. To show the L^2-property of $\mathbb{D}^\lambda \phi$, we only have to be concerned with the behaviour around D. Hence, we may and will assume that the support of ϕ is contained in $\{|z| \leqslant 1/2\}$, i.e, there are no contribution of the boundary $\{|z|=1\}$ to the Stokes formula. Let χ be any test function on $X\setminus D$. Let $\mathbb{D}^{\lambda\star}_h$ be the operator given as in Section 2.2 of [**69**]. (We will use the symbol $\mathbb{D}^{\lambda\star}$ for simplicity.) Recall $\mathbb{D}^{\lambda *}=-\sqrt{-1}\big[\Lambda_\omega,\,\mathbb{D}^{\lambda\star}\big]$. We have the following equality:

$$(175)\quad \Delta^\lambda(\chi\phi)=\mathbb{D}^{\lambda *}\mathbb{D}^\lambda(\chi\phi)=\mathbb{D}^{\lambda *}\big(\chi\mathbb{D}^\lambda\phi\big)+\mathbb{D}^{\lambda *}\big(\mathbb{D}^\lambda(\chi)\cdot\phi\big)\\ =\chi\Delta^\lambda(\phi)-\sqrt{-1}\Lambda_\omega\big(\mathbb{D}^{\lambda\star}(\chi)\cdot\mathbb{D}^\lambda\phi\big)+\mathbb{D}^{\lambda *}\big(\mathbb{D}^\lambda(\chi)\cdot\phi\big).$$

We obtain the following:

$$(176)\quad \int_{X\setminus D}\big(\Delta^\lambda(\chi\phi),\,\chi\phi\big)_h\,\mathrm{dvol}_\omega=\int_{X\setminus D}\big(\chi\Delta^\lambda\phi,\,\chi\phi\big)_h\,\mathrm{dvol}_\omega\\ -\sqrt{-1}\int_{X\setminus D}\Big(\Lambda_\omega\big(\mathbb{D}^{\lambda\star}(\chi)\cdot\mathbb{D}^\lambda\phi\big),\,\chi\phi\Big)_h\,\mathrm{dvol}_\omega+\int_{X\setminus D}\big(\mathbb{D}^\lambda(\chi)\cdot\phi,\,\mathbb{D}^\lambda(\chi\phi)\big)_h\,\mathrm{dvol}_\omega\,.$$

The second term in the right-hand side of (176) can be rewritten as follows, up to constant multiplication:

$$(177)\quad \int_{X\setminus D}\Big(\Lambda_\omega\big(\mathbb{D}^{\lambda\star}(\chi)\cdot\chi\mathbb{D}^\lambda\phi\big),\,\phi\Big)_h\,\mathrm{dvol}_\omega=\\ \int_{X\setminus D}\Big(\Lambda_\omega\big(\mathbb{D}^{\lambda\star}(\chi)\cdot\mathbb{D}^\lambda(\chi\phi)-\mathbb{D}^{\lambda\star}(\chi)\cdot\mathbb{D}^\lambda(\chi)\cdot\phi\big),\,\phi\Big)_h\,\mathrm{dvol}_\omega\,.$$

Let ρ be an $\boldsymbol{R}_{\geqslant 0}$-valued C^∞-function on $\boldsymbol{R}$ such that $\rho(t)=1$ for $t\leqslant 1/2$ and $\rho(t)=0$ for $t\geqslant 1$. Let $\chi_N:=\rho\big(-N^{-1}\log|z|\big)$. Note $\partial\chi_N$, $\overline{\partial}\chi_N$ and $\overline{\partial}\partial\chi_N$ are uniformly bounded with respect to ω. We obtain the following inequality for some constants $B>0$, from (174), (175), (176) and (177):

$$\big\|\mathbb{D}^\lambda(\chi_N\cdot\phi)\big\|_{h,\omega}^2\leqslant B\cdot\|\phi\|_{h,\omega}\cdot\Big(\|\phi\|_{h,\omega}+\|\Delta^\lambda\phi\|_{h,\omega}+\|\mathbb{D}^\lambda(\chi_N\phi)\|_{h,\omega}\Big).$$

We obtain the uniform boundedness of $\big\|\mathbb{D}^\lambda(\chi_N\cdot\phi)\big\|_{h,\omega}$, and hence $\|\mathbb{D}^\lambda(\phi)\|_{h,\omega}<\infty$.

Let ϕ be a C^∞-section of $V\otimes\Omega^1$ on $X\setminus D$ such that ϕ and $\Delta^\lambda\phi$ are L^2. As in the case of 0-form, we may assume that the support of ϕ is contained in $\{|z|\leqslant 1/2\}$. Let χ be any test function on $X\setminus D$. We have the following equalities:

$$\mathbb{D}^{\lambda *}\mathbb{D}^\lambda(\chi\phi)=\mathbb{D}^{\lambda *}\big(\mathbb{D}^\lambda(\chi)\cdot\phi+\chi\cdot\mathbb{D}^\lambda(\phi)\big)=\\ \mathbb{D}^{\lambda *}\big(\mathbb{D}^\lambda(\chi)\cdot\phi\big)+\chi\cdot\mathbb{D}^{\lambda *}\mathbb{D}^\lambda(\phi)+\sqrt{-1}\mathbb{D}^{\lambda\star}(\chi)\cdot\Lambda_\omega(\mathbb{D}^\lambda\phi).$$

Similarly, we have the following:

$$\mathbb{D}^\lambda\mathbb{D}^{\lambda *}(\chi\phi)=\mathbb{D}^\lambda\big(-\sqrt{-1}\Lambda_\omega(\mathbb{D}^{\lambda\star}(\chi)\cdot\phi)\big)+\chi\cdot\mathbb{D}^\lambda\mathbb{D}^{\lambda *}\phi+\mathbb{D}^\lambda\chi\cdot\mathbb{D}^{\lambda *}\phi.$$

Hence, we have the following:

$$(178)\quad \int \big(\Delta^{\lambda}(\chi\phi),\ \chi\phi\big)_{h,\omega}\,\mathrm{dvol}_{\omega} = \int_{X\smallsetminus D}\Big(\mathbb{D}^{\lambda\,*}(\mathbb{D}^{\lambda}(\chi)\cdot\phi) - \sqrt{-1}\mathbb{D}^{\lambda}\big(\Lambda_{\omega}(\mathbb{D}^{\lambda\,*}\chi\cdot\phi)\big),\ \chi\phi\Big)_{h,\omega}\mathrm{dvol}_{\omega} + \int_{X\smallsetminus D}\Big(\sqrt{-1}\mathbb{D}^{\lambda\,*}\chi\cdot\Lambda_{\omega}\mathbb{D}^{\lambda}\phi + \mathbb{D}^{\lambda}\chi\cdot\mathbb{D}^{\lambda\,*}\phi,\ \chi\phi\Big)_{h,\omega}\mathrm{dvol}_{\omega} + \int_{X\smallsetminus D}\big(\chi\Delta^{\lambda}(\phi),\ \chi\phi\big)_{h,\omega}\mathrm{dvol}_{\omega}.$$

The first term in the right-hand side can be rewritten as follows:

$$(179)\quad -\int_{X\smallsetminus D}\big(\mathbb{D}^{\lambda}(\chi)\cdot\phi,\ \mathbb{D}^{\lambda}(\chi\phi)\big)_{h,\omega}\mathrm{dvol}_{\omega} + \int_{X\smallsetminus D}\big(\sqrt{-1}\Lambda_{\omega}(\mathbb{D}^{\lambda\,*}\chi\cdot\phi),\ \mathbb{D}^{\lambda\,*}(\chi\phi)\big)_{h,\omega}\mathrm{dvol}_{\omega}.$$

The second term in the right-hand side of (178) can be rewritten as follows:

$$(180)\quad \int_{X\smallsetminus D}\Big(\sqrt{-1}\mathbb{D}^{\lambda\,*}\chi\cdot\Lambda_{\omega}\mathbb{D}^{\lambda}(\chi\phi) + \mathbb{D}^{\lambda}\chi\cdot\mathbb{D}^{\lambda\,*}(\chi\phi),\ \phi\Big)_{h,\omega}\mathrm{dvol}_{\omega} - \int_{X\smallsetminus D}\Big(\sqrt{-1}\mathbb{D}^{\lambda\,*}\chi\cdot\Lambda_{\omega}\big(\mathbb{D}^{\lambda}(\chi)\cdot\phi\big) + \sqrt{-1}\mathbb{D}^{\lambda}\chi\cdot\Lambda_{\omega}\big(\mathbb{D}^{\lambda\,*}(\chi)\cdot\phi\big),\ \phi\Big)_{h,\omega}\mathrm{dvol}_{\omega}.$$

Let χ_N be as above. Due to (178), (179) and (180), we obtain the following, for some $B>0$:

$$(181)\quad \big\|\mathbb{D}^{\lambda}(\chi_N\cdot\phi)\big\|^2_{h,\omega} + \big\|\mathbb{D}^{\lambda\,*}(\chi_N\cdot\phi)\big\|^2_{h,\omega} \leqslant B\|\phi\|_{h,\omega}\Big(\|\phi\|_{h,\omega} + \big\|\mathbb{D}^{\lambda}(\chi_N\cdot\phi)\big\|_{h,\omega} + \big\|\mathbb{D}^{\lambda\,*}(\chi_N\cdot\phi)\big\|_{h,\omega} + \big\|\Delta^{\lambda}\phi\big\|_{h,\omega}\Big).$$

Then, we obtain the uniform boundedness of $\big\|\mathbb{D}^{\lambda}(\chi_N\cdot\phi)\big\|_{h,\omega} + \big\|\mathbb{D}^{\lambda\,*}(\chi_N\cdot\phi)\big\|_{h,\omega}$, and hence $\big\|\mathbb{D}^{\lambda}\phi\big\|_{h,\omega}<\infty$ and $\big\|\mathbb{D}^{\lambda\,*}\phi\big\|_{h,\omega}<\infty$.

The claim for 2-forms can be reduced to that for 0-forms. Thus, the proof of Lemma 8.4.15 is finished. □

A section ϕ of $V\otimes\Omega^j$ is called a harmonic i-form of V, if $\Delta^{\lambda}\phi=0$. We obtain the following corollary.

Corollary 8.4.16. — *Let ϕ be a harmonic i-form of V which is L^2 with respect to h and ω. Then, $\mathbb{D}^{\lambda}\phi$ and $\mathbb{D}^{\lambda\,*}\phi$ are also L^2 with respect to ω and h.* □

Remark 8.4.17. — The claims of Lemma 8.4.15 and Corollary 8.4.16 should hold in the case of complete Kähler manifold with an appropriate exhaustion function. We omit the details. □

8.4.3.4. Refinement of Lemma 8.4.14. — Let C be a smooth projective curve, and let Z be a finite subset of Z. Let $(V, \mathbb{D}^\lambda)$ be a λ-flat bundle on $C \setminus Z$ with a Hermitian metric h. Let ω be a Kähler form of $C \setminus D$ which is Poincaré-like around D.

Proposition 8.4.18. — *Let ϕ be an L^2-section of $V \otimes \Omega^i$ on $C \setminus D$. Then, $\Delta^\lambda \phi = 0$ if and only if $\mathbb{D}^\lambda \phi = \mathbb{D}^{\lambda *} \phi = 0$.*

Proof. — We only have to show the "only if" part. Assume $\Delta^\lambda \phi = 0$. Because Corollary 8.4.16, we obtain that $\mathbb{D}^\lambda \phi$ and $\mathbb{D}^{\lambda *} \phi$ are L^2. Then, we obtain $\mathbb{D}^\lambda \phi = 0$ and $\mathbb{D}^{\lambda *} \phi = 0$ due to Lemma 8.4.14. □

Remark 8.4.19. — Proposition 8.4.18 should hold for a λ-flat bundle on a complete Kähler manifold with appropriate exhaustion functions. We omit the details. □

CHAPTER 9

ASSOCIATED FAMILY OF MEROMORPHIC λ-FLAT BUNDLES

Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X \setminus D$, where X is a complex manifold and D is a simple normal crossing hypersurface. We have the family of λ-flat bundles $(\mathcal{E}, \mathbb{D})$ on $\boldsymbol{C}_\lambda \times (X \setminus D)$ associated to $(E, \overline{\partial}_E, \theta, h)$. We would like to extend it on $\boldsymbol{C}_\lambda \times X$ in a meromorphic way. We have already obtained a meromorphic prolongment $\mathcal{P}\mathcal{E}^\lambda$ of $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ for each fixed λ in Chapter 7. However, as was mentioned in the *Introduction*, the family $\bigcup \mathcal{P}\mathcal{E}^\lambda$ cannot be a nice meromorphic object unless the harmonic bundle is tame.

In this chapter, for a given complex number λ_0, we study a preliminary prolongment $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ on a neighbourhood of $\{\lambda_0\} \times X$ obtained as the sheaf of holomorphic sections whose norms are of polynomial growth with respect to a modified metric $\mathcal{P}^{(\lambda_0)}h$. It will be deformed to $\mathcal{Q}^{(\lambda_0)}\mathcal{E}$ in Section 11.1, that is the desired family.

In Section 9.1, we construct a filtered bundle $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$. In Section 9.2, we show that $(\mathcal{P}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D})$ is a good family of filtered λ-flat bundles. We also show that the specializations $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$ are obtained as the deformation of $\mathcal{P}\mathcal{E}^\lambda$ caused by a variation of irregular values (Section 4.5.2).

In Section 9.3, we give a remark on the growth order of the norms of partially flat sections uniformly for λ. This is a preparation for the proof of Theorem 11.2.2. (See Section 11.4.)

We study a locally uniform comparison of irregular decompositions of $(\mathcal{P}\mathcal{E}^0, \mathbb{D}^0)$ and $(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ in Section 9.4. Because we will not use it in the other part of this monograph, the reader can skip it.

9.1. Filtered bundle $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$

9.1.1. Local construction of $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}^\lambda$ and $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$ in the unramified case

We use the setting and the notation in Section 7.2. We put $g(\lambda) := g_{\mathrm{irr}}(\lambda) \cdot g_{\mathrm{reg}}(\lambda)$, where g_{irr} and g_{reg} are given as follows:

$$g_{\mathrm{irr}}(\lambda) := \exp\Big(\sum_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \lambda \overline{\mathfrak{a}} \cdot \pi_{\mathfrak{a}}\Big), \quad g_{\mathrm{reg}}(\lambda) := \prod_{j=1}^{\ell} \exp\Big(\sum_{\alpha \in \mathcal{S}p(\theta, j)} \lambda \cdot \overline{\alpha} \cdot \log|z_j|^2 \cdot \pi_{j,\alpha}\Big).$$

Here, $\pi_{\mathfrak{a}}$ denote the projections onto $E_{\mathfrak{a}}$ in the decomposition (92), and $\pi_{j,\alpha}$ denote the projections onto $E_{j,\alpha}$ in the decomposition (93).

Let $U(\lambda_0)$ denote a neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$. We set $\mathcal{X}^{(\lambda_0)} := U(\lambda_0) \times X$. We use the symbols $\mathcal{D}^{(\lambda_0)}$ and $\mathcal{D}_i^{(\lambda_0)}$ with similar meanings. We have the following Hermitian metrics of $\mathcal{E}_{|\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}}$:

$$
\begin{aligned}
\mathcal{P}^{(\lambda_0)}h(u,v) &:= h\big(g(\lambda-\lambda_0)u, g(\lambda-\lambda_0)v\big) \\
\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h(u,v) &:= h\big(g_{\mathrm{irr}}(\lambda-\lambda_0)u, g_{\mathrm{irr}}(\lambda-\lambda_0)v\big).
\end{aligned}
\tag{182}
$$

The naturally induced metric of $\mathcal{E}^\lambda$ ($\lambda \in U(\lambda_0)$) is also denoted by the same symbols.

Notation 9.1.1. — Let $\boldsymbol{a} \in \boldsymbol{R}^\ell$. Let V be an open subset of $\mathcal{X}^{(\lambda_0)}$. We set

$$
\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}(V) := \Big\{ f \in \mathcal{E}(V^*) \,\Big|\, |f|_{\mathcal{P}^{(\lambda_0)}h} = O\Big(\prod_{j=1}^{\ell} |z_j|^{-a_j-\varepsilon}\Big),\ \forall \varepsilon > 0 \Big\},
$$

where $V^* := V \setminus \mathcal{D}^{(\lambda_0)}$. By taking sheafification, we obtain an $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-module $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$. We put $\mathcal{P}^{(\lambda_0)}\mathcal{E} := \bigcup_{\boldsymbol{a}\in \boldsymbol{R}^\ell} \mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$. The filtered sheaf on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$ is denoted by $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$.

The specializations of $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$ and $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ to $\{\lambda\} \times X$ are denoted by $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}^\lambda$ and $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$, respectively. The specialization of $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$ to $\{\lambda\} \times (X, D)$ is denoted by $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}^\lambda$. □

9.1.2. Global construction of $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}^\lambda$ and $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$. — The construction of the filtered sheaf $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$ in Section 9.1.1 can be obviously globalized and extended to the ramified case. Let X be a general complex manifold, and let D be a simple normal crossing hypersurface with the decomposition into irreducible components $D = \bigcup_{i\in\Lambda} D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X \setminus D$, which is not necessarily unramified. Let U be an open subset of X with a holomorphic coordinate $(z_1, \dots, z_n)$ such that $U \cap D = \bigcup_{j=1}^{\ell}\{z_j = 0\}$. We take a ramified covering $\varphi : U' \to U$ given by $\varphi(\zeta_1, \dots, \zeta_n) = (\zeta_1^{m_1}, \dots, \zeta_\ell^{m_\ell}, \zeta_{\ell+1}, \dots, \zeta_n)$ such that $\varphi^*(E, \overline{\partial}_E, \theta, h)$ is unramified. Then, we obtain the Hermitian metrics as in (182). Since it is equivariant with respect to $\mathrm{Gal}(U'/U)$, we obtain the Hermitian metrics $\mathcal{P}^{(\lambda_0)}h_U$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_U$ of $\mathcal{E}_{|U(\lambda_0)\times(U\setminus D)}$. By the same procedure, we obtain the filtered sheaf $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}_U$ on $U(\lambda_0) \times (U, D \cap U)$. When we are given two such open sets U_i ($i = 1, 2$) of X, the Hermitian metrics $\mathcal{P}^{(\lambda_0)}h_{U_i}$ ($i = 1, 2$) are mutually bounded. Hence, the restrictions of the filtered sheaves $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}_{U_i}$ to $U(\lambda_0) \times \big(U_1 \cap U_2, D \cap (U_1 \cap U_2)\big)$ are the same. By varying U and gluing $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}_U$, we obtain the filtered sheaf $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$, where $\mathcal{X}^{(\lambda_0)}$ denotes a neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_\lambda \times X$ and $\mathcal{D}^{(\lambda_0)} := \mathcal{X}^{(\lambda_0)} \cap (\boldsymbol{C}_\lambda \times D)$.

We will show the following theorem in Section 9.1.3.

Theorem 9.1.2. — *$\mathcal{P}_*^{(\lambda_0)}\mathcal{E}$ is a filtered vector bundle on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$, if $\mathcal{X}^{(\lambda_0)}$ is sufficiently small.*

We have the induced filtration ${}^iF^{(\lambda_0)}$ of $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}_{|\mathcal{D}_i^{(\lambda_0)}}$. The tuple $\big({}^iF^{(\lambda_0)} \,\big|\, i \in \Lambda\big)$ is denoted by $\boldsymbol{F}^{(\lambda_0)}$. We have the weak norm estimate up to small polynomial order in the following sense. For simplicity, we consider the case $X = \Delta^n$ and $D = \bigcup_{j=1}^{\ell}\{z_j = 0\}$. Assume that $U(\lambda_0)$ is sufficiently small. Let $\boldsymbol{v}$ be a frame of $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$ compatible with $\boldsymbol{F}^{(\lambda_0)}$. Let $a_j(v_i) := {}^j\deg^{F^{(\lambda_0)}}(v_i)$. We put $v_i' := v_i \prod_{j=1}^{\ell} |z_j|^{a_j(v_i)}$. Let $H(\mathcal{P}^{(\lambda_0)}h, \boldsymbol{v}')$ denote the Hermitian matrix-valued function whose (i,j)-entries are given by $\mathcal{P}^{(\lambda_0)}h(v_i', v_j')$. We have the weak norm estimate up to small polynomial order, which will also be proved in Section 9.1.3.

Proposition 9.1.3. — *For any $\varepsilon > 0$, there exist a positive constant C_ε such that*

$$C_\varepsilon^{-1} \prod_{j=1}^{\ell} |z_j|^{\varepsilon} \leqslant H(\mathcal{P}^{(\lambda_0)}h, \boldsymbol{v}') \leqslant C_\varepsilon \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}.$$

9.1.3. Prolongment $\mathcal{T}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$.

— Let us return to the setting in Section 9.1.1. We put $\mathcal{T}^{(\lambda_0)}d''_\lambda := g(\lambda - \lambda_0) \circ \big(\overline{\partial}_E + \lambda\theta^\dagger\big) \circ g(\lambda - \lambda_0)^{-1}$. Let $\mathcal{T}^{(\lambda_0)}\mathcal{E}$ denote the following holomorphic bundle on $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$:

$$\big(p_\lambda^{-1}E, \mathcal{T}^{(\lambda_0)}d''_\lambda + \overline{\partial}_\lambda\big).$$

Let $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$. For any open subset $V \subset \mathcal{X}^{(\lambda_0)}$, we define

$$\mathcal{T}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}(V) := \Big\{ f \in \mathcal{T}^{(\lambda_0)}\mathcal{E}(V^*) \,\Big|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-a_i-\varepsilon}\Big),\ \forall \varepsilon > 0 \Big\},$$

where $V^* := V \setminus \mathcal{D}^{(\lambda_0)}$. By taking sheafification, we obtain a filtered sheaf $\mathcal{T}_*^{(\lambda_0)}\mathcal{E}$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$. The following lemma is clear from the construction.

Lemma 9.1.4. — *The multiplication of $g(\lambda - \lambda_0)$ induces the holomorphic isomorphisms $\mathcal{E} \simeq \mathcal{T}^{(\lambda_0)}\mathcal{E}$ and $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E} \simeq \mathcal{T}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$.* □

Therefore, we obtain Theorem 9.1.2 and Proposition 9.1.3 from the following proposition.

Proposition 9.1.5. — *If $\mathcal{X}^{(\lambda_0)}$ is sufficiently small, $\mathcal{T}_*^{(\lambda_0)}\mathcal{E}$ is a filtered bundle on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$. The weak norm estimate up to small polynomial order holds for $(\mathcal{T}_*^{(\lambda_0)}\mathcal{E}, h)$. (See Proposition* 9.1.3 *for weak norm estimate up to small polynomial order.)*

Proof. — We put $\Lambda(\lambda - \lambda_0) := \mathcal{T}^{(\lambda_0)}d''_\lambda - d''_{\lambda_0}$. According to Theorem 21.8.1, Proposition 9.1.5 follows from the following lemma.

Lemma 9.1.6. — *If $U(\lambda_0)$ is sufficiently small, we have $\big|\Lambda(\lambda-\lambda_0)\big|_{h,g_{\mathbf{p}}} \leqslant C|\lambda-\lambda_0|$ on $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$, where $g_{\mathbf{p}}$ denotes the Poincaré metric for $X \setminus D$.*

Proof. — Let $\overline{\Phi}$ be given by (122). We have the following equality:

$$\Lambda(\lambda-\lambda_0) = \lambda_0 \cdot \Big(g(\lambda-\lambda_0)\circ\big(\theta^\dagger-\overline{\Phi}\big)\circ g(\lambda-\lambda_0)^{-1} - \big(\theta^\dagger-\overline{\Phi}\big)\Big) + (\lambda-\lambda_0)\cdot g(\lambda-\lambda_0)\circ\Big(\theta^\dagger-\overline{\Phi}\Big)\circ g(\lambda-\lambda_0)^{-1}. \tag{183}$$

Let us look at the first term of (183). We use the decomposition as in (95). If $(\mathfrak{a},\boldsymbol{\alpha}) \neq (\mathfrak{a}',\boldsymbol{\alpha}')$, we obtain the following from (124):

$$\begin{aligned}
&\Big(g(\lambda-\lambda_0)\circ\big(\theta^\dagger-\overline{\Phi}\big)\circ g(\lambda-\lambda_0)^{-1} - \big(\theta^\dagger-\overline{\Phi}\big)\Big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} \\
&= \Big(\exp\big((\lambda-\lambda_0)(\overline{\mathfrak{a}}'-\overline{\mathfrak{a}})\big)\cdot\prod_{j=1}^{\ell}|z_j|^{2(\lambda-\lambda_0)(\overline{\alpha}'_j-\overline{\alpha}_j)} - 1\Big)\cdot\big(\theta^\dagger-\overline{\Phi}\big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} \\
&= |\lambda-\lambda_0|\cdot O\Big(\exp\big((\lambda-\lambda_0)(\overline{\mathfrak{a}}'-\overline{\mathfrak{a}})\big)\cdot\prod_{j=1}^{\ell}|z_j|^{2(\lambda-\lambda_0)(\overline{\alpha}'_j-\overline{\alpha}_j)}\Big)\times \\
&\qquad O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}|\big)\cdot\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\alpha}')\Big).
\end{aligned} \tag{184}$$

If $|\lambda-\lambda_0|$ is sufficiently small, we have

$$\big|(\lambda-\lambda_0)(\overline{\mathfrak{a}}'-\overline{\mathfrak{a}})\big| - \varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}| \leqslant -\varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}|/2$$

$$\prod_{j=1}^{\ell}|z_j|^{2(\lambda-\lambda_0)(\overline{\alpha}'_j-\overline{\alpha}_j)}\times\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\alpha}') = O\Big(\mathcal{Q}_{\varepsilon/2}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\Big)$$

We also have $\big(g(\lambda-\lambda_0)\circ\big(\theta^\dagger-\overline{\Phi}\big)\circ g(\lambda-\lambda_0)^{-1} - \big(\theta^\dagger-\overline{\Phi}\big)\big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a},\boldsymbol{\alpha})} = 0$. Hence, we obtain the desired estimate for the first term.

For the second term, we have the following in the case $(\mathfrak{a},\boldsymbol{\alpha}) \neq (\mathfrak{a}',\boldsymbol{\alpha}')$:

$$\begin{aligned}
&\Big(g(\lambda-\lambda_0)\circ\big(\theta^\dagger-\overline{\Phi}\big)\circ g(\lambda-\lambda_0)^{-1}\Big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} \\
&= \exp\big((\lambda-\lambda_0)(\overline{\mathfrak{a}}'-\overline{\mathfrak{a}})\big)\cdot\prod_{j=1}^{\ell}|z_j|^{2(\lambda-\lambda_0)(\overline{\alpha}'_j-\overline{\alpha}_j)}\cdot\big(\theta^\dagger-\overline{\Phi}\big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a}',\boldsymbol{\alpha}')} \\
&= O\Big(\exp\big((\lambda-\lambda_0)(\overline{\mathfrak{a}}'-\overline{\mathfrak{a}})\big)\prod_{j=1}^{\ell}|z_j|^{2(\lambda-\lambda_0)(\overline{\alpha}'_j-\overline{\alpha}_j)}\Big)\cdot O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}|\big)\mathcal{Q}_\varepsilon(\boldsymbol{\alpha},\boldsymbol{\alpha}')\Big).
\end{aligned} \tag{185}$$

We also have the following:

$$\Big(g(\lambda-\lambda_0)\circ\big(\theta^\dagger-\overline{\Phi}\big)\circ g(\lambda-\lambda_0)^{-1}\Big)_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{a},\boldsymbol{\alpha})} = \Big(\sum_{j=1}^{\ell} f_j^{\mathrm{nil}}\cdot dz_j + \sum_{j=\ell+1}^{n} f_j^{\mathrm{reg}}\cdot dz_j\Big).$$

Therefore, we obtain the desired estimate in Lemma 9.1.6. The proof of the propositions is also finished. □

9.2. Family of meromorphic flat λ-connections on $\mathcal{P}_*^{(\lambda_0)}\mathcal{E}$

9.2.1. Comparison of $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$ and $\mathcal{P}\mathcal{E}^\lambda$. — We continue to use the setting in Section 9.1.2. We would like to give another interpretation of the specialization $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$, as the deformation of meromorphic λ-flat bundles caused by variation of the irregular values, explained in Section 4.5.2. For any complex number λ, we set

$$T(\lambda) := \frac{1+\lambda\overline{\lambda}_0}{1+|\lambda|^2}. \tag{186}$$

We take $\boldsymbol{a} = (a_i) \in \boldsymbol{R}^\Lambda$ such that $a_i \notin \mathcal{P}ar(\mathcal{P}\mathcal{E}^{\lambda_0}, i)$ for each i.

Proposition 9.2.1. — *Let P be any point of X. There exist a neighbourhood X_P of P in X and a neighbourhood $U_P(\lambda_0)$ of λ_0 in $\boldsymbol{C}_\lambda$, such that the following holds for any $\lambda \in U_P(\lambda_0)$:*

- *We have the natural isomorphism $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}^\lambda_{|X_P} \simeq (\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^\lambda_{|X_P})^{(T(\lambda))}$ of $\mathcal{O}_{X_P}$-modules, which is the extension of the identity on $X_P \setminus D$. (We put $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^0)^{(1)} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^0$ formally, in the case $\lambda_0 = 0$.)*
- *In particular, we have the isomorphism $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda_{|X_P} \simeq (\mathcal{P}\mathcal{E}^\lambda_{|X_P})^{(T(\lambda))}$ of $\mathcal{O}_{X_P}(*D)$-modules, which is the extension of the identity on $X_P \setminus D$.*

Proof. — The claims are local properties. Therefore, we may and will use the setting in Section 9.1.1. We only have to show the first claim. Let $U(\lambda_0)$ denote a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$. Let $\boldsymbol{a} \in \boldsymbol{R}^\ell$ be as above. We consider the prolongment for the metric $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$, i.e., for any open subset $V \subset \mathcal{X}^{(\lambda_0)}$, we set

$$\mathcal{P}_{\mathrm{irr}\,\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}(V) := \Big\{ f \in \mathcal{E}(V^*) \,\Big|\, |f|_{\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h} = O\Big(\prod_{j=1}^{\ell} |z_j|^{-a_j-\varepsilon}\Big),\ \forall \varepsilon > 0 \Big\},$$

where $V^* := V \setminus \mathcal{D}^{(\lambda_0)}$. Thus, we obtain an $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-module $\mathcal{P}_{\mathrm{irr}\,\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$.

Lemma 9.2.2. — *We have $\mathcal{P}_{\mathrm{irr}\,\boldsymbol{a}}^{(\lambda_0)}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$ for $\boldsymbol{a}$ as in this subsection, if $U(\lambda_0)$ is sufficiently small.*

Proof. — Note that $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$ and $\mathcal{P}^{(\lambda_0)}h$ are mutually bounded up to $|z|^{-\eta|\lambda-\lambda_0|}$-order for some $\eta > 0$. Then, the claim of Lemma 9.2.2 follows from Proposition 9.1.3. □

We remark $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h = g_{\mathrm{irr}}(\lambda-\lambda_0)^*h$. (See Section 7.6.1 for $g_{\mathrm{irr}}(w)^*h$.)

Lemma 9.2.3. — *If the divisor D is smooth, the claim of Proposition* 9.2.1 *holds.*

Proof. — We would like to apply Proposition 7.6.1. We use the notation in Section 7.6.1. Note $T(\lambda) = T_1(\lambda - \lambda_0)$.

Let us consider the case $\lambda_0 = 0$. Let $\mathcal{X}^{(0)} := U(0) \times X$ and $W := (U(0) \times D) \cup (\{0\} \times X)$. We take a finite covering $\mathcal{X}^{(0)} \setminus W = \bigcup S_i$ satisfying the following:

$$S_i \in \bigcap_j \bigcap_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} \mathcal{MS}(\mathcal{X}^{(0)} \setminus W, \mathcal{I}_{\mathfrak{a}}^{(j)}).$$

For each $\lambda \in U(0) \setminus \{0\}$, let $S_i^{\lambda} := S_i \cap (\{\lambda\} \times X)$. Then, we have

$$S_i^{\lambda} \in \bigcap_j \bigcap_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} \mathcal{MS}(\{\lambda\} \times (X \setminus D), \mathcal{I}_{\mathfrak{a}}^{(j)}).$$

If $U(0)$ is sufficiently small, we may assume $\varepsilon_2 = \varepsilon_1$ in the condition (A2). We have $T(\lambda) = (1 + |\lambda|^2)^{-1} > 0$ for any λ, and hence the second assumption in Proposition 7.6.1 is trivial. Therefore, the claim of Lemma 9.2.3 in the case $\lambda_0 = 0$ immediately follows from Proposition 7.6.1.

Let us consider the case $\lambda_0 \neq 0$. We can take a finite covering $X \setminus D = \bigcup_{i=1}^{N} S_i$, where S_i are multi-sectors satisfying the following:

$$\{\lambda_0\} \times S_i \in \bigcap_j \bigcap_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} \mathcal{MS}(\{\lambda_0\} \times (X \setminus D), \mathcal{I}_{\mathfrak{a}}^{(j)}).$$

If $U(\lambda_0)$ is sufficiently small, we have the following for any $\lambda \in U(\lambda_0)$:

$$\{\lambda\} \times S_i \in \bigcap_j \bigcap_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} \mathcal{MS}(\{\lambda\} \times (X \setminus D), \mathcal{I}_{\mathfrak{a}}^{(j)}).$$

We may assume that the conditions (A1) and (A2) are also satisfied for each $\lambda \in U(\lambda_0)$. We put $T_t(\lambda) := t + (1-t)T(\lambda)$. Let $(\mathcal{I}_{\mathfrak{a}}^{(j)})^{(T_t(\lambda))} := \{T_t(\lambda) \cdot \mathfrak{c} \,|\, \mathfrak{c} \in \mathcal{I}_{\mathfrak{a}}^{(j)}\}$. If $U(\lambda_0)$ is sufficiently small, we may also have the following:

$$\{\lambda\} \times S_i \in \bigcap_j \bigcap_{\mathfrak{a} \in \operatorname{Irr}(\theta, j)} \mathcal{MS}(\{\lambda\} \times (X \setminus D), (\mathcal{I}_{\mathfrak{a}}^{(j)})^{(T_t(\lambda))}).$$

Then, the second assumption of Proposition 7.6.1 is satisfied, and the claim of Lemma 9.2.3 follows. □

Due to Theorem 9.1.2, $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}^{\lambda}$ is locally free. We also know that $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda})^{(T(\lambda))}$ is also locally free. Let $D^{[2]} := \bigcup_{i \neq j} D_i \cap D_j$. By using Lemma 9.2.3 and Lemma 4.5.7, we obtain a natural isomorphism of $\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}^{\lambda}$ and $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda})^{(T(\lambda))}$ on $X \setminus D^{[2]}$. By Hartogs theorem, it can be extended to the isomorphism on X. Thus, the proof of Proposition 9.2.1 is finished. □

We have immediate consequences of Proposition 9.2.1.

Corollary 9.2.4. — *Let P be any point of X. Let X_P and $U_P(\lambda_0)$ be as in Proposition 9.2.1.*

- *For any $\lambda \in U_P(\lambda_0)$, the flat λ-connection $\mathbb{D}^{\lambda}$ of $\mathcal{P}^{(\lambda_0)}\mathcal{E}^{\lambda}_{|X_P}$ is meromorphic.*
- *$(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})_{|X_P}$ is good.*

- *The set of the irregular values of* $\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda, \mathbb{D}^\lambda\big)$ *is given by*

$$\operatorname{Irr}(\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda, \mathbb{D}^\lambda) := \big\{(1+\lambda\overline{\lambda}_0)\cdot\mathfrak{a}\,\big|\,\mathfrak{a}\in\operatorname{Irr}(\theta)\big\}$$

under the setting in Section 7.2. □

9.2.2. The family of flat λ-connections of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ and the KMS-structure at λ_0

We continue to use the setting in Section 9.1.2.

Corollary 9.2.5. — *Let P be any point of X. Let X_P and $U_P(\lambda_0)$ be as in Proposition* 9.2.1.

- *The family of flat λ-connections $\mathbb{D}$ of $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|U_P(\lambda_0)\times X_P}$ is meromorphic.*
- *The family of filtered λ-flat bundles $(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ is good on $U(\lambda_0)\times\big(X_P, X_P\cap D\big)$.*
- *The irregular values of $\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D}\big)$ is given by*

$$\operatorname{Irr}(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D}) := \big\{(1+\lambda\overline{\lambda}_0)\cdot\mathfrak{a}\,\big|\,\mathfrak{a}\in\operatorname{Irr}(\theta)\big\}$$

under the setting in Section 7.2.

Proof. — It follows from Corollary 9.2.4 and Proposition 2.3.7. □

Proposition 9.2.6. — *$(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ has the KMS-structure at λ_0 in the sense of Definition* 2.8.1.

Proof. — We may assume that D is smooth. We consider $\mathcal{P}_a^{(\lambda_0)}\mathcal{E}/\mathcal{P}_{<a}^{(\lambda_0)}\mathcal{E}$ on D. Take $\varepsilon>0$. If $U(\lambda_0)$ is sufficiently small, we have $\big(\mathcal{P}_a^{(\lambda_0)}\mathcal{E}/\mathcal{P}_{<a}^{(\lambda_0)}\mathcal{E}\big)_{|D\times\{\lambda\}}\simeq\big(\mathcal{P}_{a+\varepsilon}\mathcal{E}^\lambda\big)^{T(\lambda)}/\big(\mathcal{P}_{a-\varepsilon}\mathcal{E}^\lambda\big)^{T(\lambda)}$ for any $\lambda\in U(\lambda_0)$. Hence, we can conclude that the set of the eigenvalues of $\operatorname{Res}(\mathbb{D}^\lambda)$ on $\big(\mathcal{P}_a^{(\lambda_0)}\mathcal{E}/\mathcal{P}_{<a}^{(\lambda_0)}\mathcal{E}\big)_{|D_i\times\{\lambda\}}$ is given by the following, according to Proposition 8.2.1:

$$\big\{\mathfrak{e}(\lambda,u)\,\big|\,u\in\mathcal{KMS}(\mathcal{E}^0),\ \mathfrak{p}(\lambda_0,u)=a\big\}.$$

Thus, we are done. □

From the proof of Proposition 9.2.6, we also obtain the following corollary.

Corollary 9.2.7. — *Let P be any point of X, and let X_P and $U_P(\lambda_0)$ be as in Proposition* 9.2.1. *For each $\lambda\in U_P(\lambda_0)$, we have the induced filtered bundle $(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|X_P})_*^\lambda$ as given in Section* 2.8.2. *If $U_P(\lambda_0)$ is sufficiently small, we have the natural isomorphism $(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|X_P})_*^\lambda\simeq(\mathcal{P}_*\mathcal{E}_{|X_P}^\lambda)^{(T(\lambda))}$ of the family of λ-flat bundles for each $\lambda\in U_P(\lambda_0)$, where $T(\lambda)$ is given as in* (186). *In particular, we have the isomorphism of meromorphic λ-flat bundles $(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|X_P})^\lambda\simeq(\mathcal{P}\mathcal{E}_{|X_P}^\lambda)^{(T(\lambda))}$ for each $\lambda\in U_P(\lambda_0)$.* □

9.2.3. Comparison of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ and $\mathcal{P}^{(\lambda_1)}\mathcal{E}$. — We continue to use the setting in Section 9.1.2. Let $\boldsymbol{a} \in \boldsymbol{R}^{\Lambda}$ be as in Subsection 9.2.1. Let P be any point of X, and let X_P and $U_P(\lambda_0)$ be as in Proposition 9.2.1. We take $\lambda_1 \in U_P(\lambda_0)$. By shrinking X_P, we may also assume to have $(\mathcal{P}^{(\lambda_1)}\mathcal{E}, \mathbb{D})$ on $U_P(\lambda_1) \times (X_P, D \cap X_P)$ for some neighbourhood $U_P(\lambda_1) \subset U_P(\lambda_0)$ of λ_1. We assume $0 \notin U_P(\lambda_1)$, although λ_0 may be 0.

Proposition 9.2.8

- *The family of meromorphic λ-flat bundles* $\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D}\big)_{|U_P(\lambda_1)\times X_P}$ *is naturally isomorphic to the deformation* $\big(\mathcal{P}^{(\lambda_1)}\mathcal{E}, \mathbb{D}\big)^{(T')}_{|U_P(\lambda_1)\times X_P}$ *with*
$$T' = (1 + \lambda\overline{\lambda}_0) \cdot (1 + \lambda\overline{\lambda}_1)^{-1}.$$
- *The family of meromorphic λ-flat bundles* $\big(\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}, \mathbb{D}\big)_{|U_P(\lambda_1)\times X_P}$ *is naturally isomorphic to the deformation* $\big(\mathcal{P}^{(\lambda_1)}_{\boldsymbol{a}}\mathcal{E}, \mathbb{D}\big)^{(T')}_{|U_P(\lambda_1)\times X_P}$.

Proof. — We only have to show the second claim. According to Lemma 4.5.5 and Proposition 9.2.1, both the restrictions $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}_{|\{\lambda\}\times X_P}$ and $\big(\mathcal{P}^{(\lambda_1)}_{\boldsymbol{a}}\mathcal{E}\big)^{(T')}_{|\{\lambda\}\times X_P}$ are naturally isomorphic to $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda})^{(T_1)}_{|X_P}$, where $T_1 = (1+\lambda\overline{\lambda}_0)(1+|\lambda|^2)^{-1}$. Then, it is easy to see that the natural isomorphism of $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}_{|U(\lambda_0)\times(X_P\setminus D)}$ and $\big(\mathcal{P}^{(\lambda_1)}_{\boldsymbol{a}}\mathcal{E}\big)^{(T')}_{|U(\lambda_0)\times(X_P\setminus D)}$ can be extended to the isomorphism on X_P. □

From the family of λ-flat bundles $(\mathcal{P}^{(\lambda_0)}_{*}\mathcal{E}, \mathbb{D})$ on $U_P(\lambda_0) \times (X_P, X_P \cap D)$ with the KMS-structure at λ_0, we obtain the family of λ-flat bundles $\big((\mathcal{P}^{(\lambda_0)}\mathcal{E})^{(\lambda_1)}_{*}, \mathbb{D}\big)$ on $U_P(\lambda_1) \times (X_P, X_P \cap D)$ as in Section 2.8.2.

Proposition 9.2.9. — *We have the natural isomorphism*
$$\big((\mathcal{P}^{(\lambda_0)}\mathcal{E})^{(\lambda_1)}_{*}, \mathbb{D}\big) \simeq \big(\mathcal{P}^{(\lambda_1)}_{*}\mathcal{E}, \mathbb{D}\big)^{(T')},$$
where T' is given in Proposition 9.2.8.

Proof. — We have the isomorphism $\big((\mathcal{P}^{(\lambda_0)}\mathcal{E})^{(\lambda_1)}, \mathbb{D}\big) \simeq \big(\mathcal{P}^{(\lambda_1)}\mathcal{E}, \mathbb{D}\big)^{(T')}$ due to Proposition 9.2.8. Then, the claim follows from Lemma 2.8.3. □

9.3. Estimate of the norms of partially flat sections

We use the setting in Section 7.2.1. For simplicity, we assume that the coordinate is admissible for the good set $\mathrm{Irr}(\theta)$. Let k be determined by the condition $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$. Let $\lambda_0 \in \boldsymbol{C}_{\lambda}$. Let $\mathcal{X}^{(\lambda_0)}$ denote a neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_{\lambda} \times X$. We use the symbols like $\mathcal{D}^{(\lambda_0)}(\leqslant k)$ with similar meanings. In the case $\lambda_0 = 0$, we put $W(\leqslant k) := \mathcal{D}^{(\lambda_0)}(\leqslant k) \cup \big(\{0\} \times X\big)$. Otherwise, we put $W(\leqslant k) := \mathcal{D}^{(\lambda_0)}(\leqslant k)$. Let $\pi : \widetilde{\mathcal{X}}^{(\lambda_0)}(W(\leqslant k)) \to \mathcal{X}^{(\lambda_0)}$ denote the real blow up of $\mathcal{X}^{(\lambda_0)}$ along $W(\leqslant k)$.

Let $\mathcal{I}^{(\lambda_0)}$ denote the image of $\operatorname{Irr}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E},\mathbb{D}\big)$ by $\overline{\eta}_{\boldsymbol{m}(0)}$. Let $\mathbb{D}_{\leqslant k}$ denote the restriction of $\mathbb{D}$ to derivations along the $(z_1,\dots,z_k)$-direction.

Let S be a multi-sector in $\mathcal{X}^{(\lambda_0)}\setminus W(\leqslant k)$, and let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}^{(\lambda_0)}(W(\leqslant k))$. If S is sufficiently small, we have the partial Stokes filtration $\mathcal{F}^S$ of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ on $\overline{S}$ at the level $\boldsymbol{m}(0)$ indexed by the ordered set $(\mathcal{I}^{(\lambda_0)},\leqslant_S)$ due to Proposition 3.3.2.

Lemma 9.3.1. — *Let f be a $\mathbb{D}_{\leqslant k}$-flat section of* $\operatorname{End}(\mathcal{P}^{(\lambda_0)}\mathcal{E})_{|S}$ *such that*

- $\big[\operatorname{Res}_i(\mathbb{D}),f_{|\mathcal{D}_i^{(\lambda_0)}\cap S}\big]=0$ *for* $i=k+1,\dots,\ell$.
- $f_{|\mathcal{D}_i^{(\lambda_0)}\cap S}$ *preserves the filtrations* ${}^iF^{(\lambda_0)}$ *for* $i=k+1,\dots,\ell$.
- $f\big(\mathcal{F}^S_{\mathfrak{a}}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}\big)\subset\mathcal{F}^S_{<\mathfrak{a}}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}$ *for any* $\mathfrak{a}\in\operatorname{Irr}(\mathcal{P}^{(\lambda_0)}\mathcal{E},\mathbb{D})$.

If we shrink S in the radius direction, we have the estimate

$$|f|_h\leqslant C\cdot\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)$$

for some $C>0$ and $\varepsilon>0$.

Proof. — We use an induction on ℓ. We assume that the claim of the lemma holds for any unramifiedly good wild harmonic bundles on $X'\setminus D'$, where $X'=\Delta^n$, $D'_i=\{z_i=0\}$ and $D'=\bigcup_{i=1}^{\ell'}D'_i$ for $\ell'<\ell$.

By shrinking S in the radius direction, we can take a $\mathbb{D}_{\leqslant k}$-flat splitting of $\mathcal{F}^S$:

$$\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{|\overline{S}}=\bigoplus_{\mathfrak{a}\in\mathcal{I}^{(\lambda_0)}}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}. \tag{187}$$

Let $\boldsymbol{u}_S=(\boldsymbol{u}_{\mathfrak{a},S})$ be a frame of $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$ compatible with the decomposition (187). Let $B_{\mathfrak{a},S}$ be the matrix valued function determined by $\mathbb{D}_{\leqslant k}\boldsymbol{u}_{\mathfrak{a},S}=\boldsymbol{u}_{\mathfrak{a},S}\cdot\big(d_{\leqslant k}\mathfrak{a}+B_{\mathfrak{a},S}\big)$, where $d_{\leqslant k}$ denote the restriction of the exterior differential along the $(z_1,\dots,z_k)$-direction. Shrinking S in the radius direction, we may assume that $\big|\lambda^{-1}\cdot B_{\mathfrak{a},S}\big|$ are sufficiently smaller than $\big|\operatorname{Re}(\lambda^{-1}(\mathfrak{a}-\mathfrak{b}))\big|$ on S for any $\mathfrak{a},\mathfrak{b}\in\mathcal{I}^{(\lambda_0)}$ such that $\mathfrak{a}<_S\mathfrak{b}$. By shrinking X, we may assume that S is of the form $\prod_{i=1}^k\operatorname{Sec}[1,\theta_i^{(0)},\theta_i^{(1)}]\times\Delta^{n-k}\times U(\lambda_0)$ in the following argument, where $U(\lambda_0)$ is a neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$ in the case $\lambda_0\neq0$, or a small sector $\operatorname{Sec}[\delta_\lambda,\theta_\lambda^{(0)},\theta_\lambda^{(1)}]\subset\Delta_\lambda^*$ in the case $\lambda_0=0$.

Let us show the following statement by a descending induction on $m\geqslant k$:

A(m) : Let f be a $\mathbb{D}_{\leqslant k}$-flat section of $\mathcal{P}_0^{(\lambda_0)}\operatorname{End}\mathcal{E}_{|S}$ such that

- $\operatorname{Res}_i(\mathbb{D})f_{|\mathcal{D}_i^{(\lambda_0)}\cap S}=0$ for $i=m+1,\dots,\ell$,
- $f\big(\mathcal{F}^S_{\mathfrak{a}}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}\big)\subset\mathcal{F}^S_{<\mathfrak{a}}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}$ for any $\mathfrak{a}$.

If we shrink S in the radius direction, we have the following estimate for some $C>0$, $\varepsilon>0$ and $N>0$:

$$|f|_h\leqslant C\cdot\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\cdot\Big(-\sum_{i=k+1}^{m}\log|z_i|\Big)^N.$$

First, let us show $A(\ell)$. We have the expression $f=\sum_{\mathfrak{a}<_S\mathfrak{b}} f_{\mathfrak{a},\mathfrak{b},i,j}\cdot u_{\mathfrak{a},i}\otimes u^{\vee}_{\mathfrak{b},j}$. Because f is $\mathbb{D}_{\leqslant k}$-flat, we have $\bigl|f_{\mathfrak{a},\mathfrak{b},i,j}\bigr| = O\bigl(\exp(-\varepsilon_1|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr)$ due to Corollary 20.3.6 and Corollary 20.3.9. Let $\boldsymbol{v}$ be a frame of $\mathcal{P}^{(\lambda_0)}_0\mathcal{E}$ for which we have the expression $f=\sum \widetilde{f}_{i,j}\cdot v_i\otimes v_j^{\vee}$. Let B be determined by $\boldsymbol{u}_S=\boldsymbol{v}\cdot B$, and then B and B^{-1} are bounded. Hence, we have $\bigl|\widetilde{f}_{i,j}\bigr| = O\bigl(\exp(-\varepsilon_2|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr)$. We put

$$\mathcal{Z} := \bigl\{(\lambda,z_1,\dots,z_n)\in S\,\big|\,1/2\leqslant |z_i|\leqslant 1,\ i=k+1,\dots,\ell\bigr\}.$$

Lemma 9.3.2. — *We have the estimate* $|f|_h = O\bigl(\exp(-\varepsilon_3|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr)$ *on* $\mathcal{Z}$.

Proof. — Let $\mathcal{P}^{(\lambda_0)}h$ be the metric as in Section 9.1.1. By construction of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$, we have $|v_i|_{\mathcal{P}^{(\lambda_0)}h}\leqslant C\cdot\prod_{i=1}^k|z_i|^{-\delta}$ on $\mathcal{Z}$ for some $\delta>0$. Hence, we have $|f|_{\mathcal{P}^{(\lambda_0)}h} = O\bigl(\exp(-\varepsilon_4|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr)$ on $\mathcal{Z}$ for some $\varepsilon_4>0$. The metrics $\mathcal{P}^{(\lambda_0)}h$ and h are mutually bounded up to $\exp\bigl(C'|\lambda-\lambda_0|\cdot|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)$-order. Hence, we obtain the estimate with respect to h by shrinking S appropriately. □

Let us return to the proof of Lemma 9.3.1. Let $\mathcal{Z}_1:=\bigcap_{i=k+1}^{\ell}(\mathcal{D}_i\cap S)$. Let $\pi_{\mathcal{Z}_1}$ denote the projection $S\longrightarrow\mathcal{Z}_1$. Let us consider the restriction to $\pi_{\mathcal{Z}_1}^{-1}(\lambda,Q)$ for $(\lambda,Q)\in\mathcal{Z}_1$. The metrized holomorphic bundles $(\mathrm{End}(\mathcal{E}),h)_{|\pi_{\mathcal{Z}_1}^{-1}(\lambda,Q)}$ are acceptable, whose curvatures are dominated uniformly for $(\lambda,Q)\in\mathcal{Z}_1$. Hence, we obtain the following estimate due to Proposition 21.2.8:

$$\bigl|f_{|\pi^{-1}(\lambda,Q)}(z_{k+1},\dots,z_\ell)\bigr|_h\leqslant C\max_{\substack{|z_i'|=1/2\\ i=k+1,\dots,\ell}}\bigl|f_{|\pi^{-1}(\lambda,Q)}(z'_{k+1},\dots,z'_\ell)\bigr|_h\Bigl(-\sum_{i=k+1}^{\ell}\log|z_i|\Bigr)^N.$$

Here, the constant C is independent of λ and Q. Thus, we obtain $A(\ell)$.

Let us show $A(m-1)$, by assuming $A(m)$. We put $g:=\mathbb{D}(\partial_m)f$. Since g satisfies the assumption of $A(m)$, we have the following estimate:

$$|g|_h\leqslant C\cdot\exp\bigl(-\varepsilon_5\cdot|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\cdot\Bigl(-\sum_{i=k+1}^{m}\log|z_i|\Bigr)^N.$$

Let $\pi_m:S\longrightarrow\mathcal{D}_m^{(\lambda_0)}$ denote the projection. We put $\pi_m^{-1}(\lambda,Q)^*:=\pi_m^{-1}(\lambda,Q)\setminus\{(\lambda,Q)\}$ for $(\lambda,Q)\in\pi_m(S)$. Let Δ_m denote the Laplacian $-\partial_{z_m}\cdot\partial_{\overline{z}_m}$. We have $\Delta_m\bigl|f_{|\pi_m^{-1}(\lambda,Q)}\bigr|_h^2\leqslant\bigl|g_{|\pi_m^{-1}(\lambda,Q)}\bigr|_h^2$ by Corollary 7.7.5 on $\pi_m^{-1}(\lambda,Q)^*$. Because $\mathrm{Res}_m(\mathbb{D})(f_{|(\lambda,Q)})=0$, we obtain the boundedness of the section $f_{|\pi_m^{-1}(\lambda,Q)}$ by the norm estimate in the curve case. (See Proposition 8.1.1.) We can take $G_{\lambda,Q}$ with the following property:

$$\Delta_m G_{\lambda,Q}=\bigl|g_{|\pi_m^{-1}(\lambda,Q)}\bigr|_h^2,\qquad |G_{\lambda,Q}|\leqslant C\cdot\exp\bigl(-\varepsilon_6|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigl(-\sum_{i=k+1}^{m-1}\log|z_i|\Bigr)^N.$$

Here, the constant C is independent of the choice of Q and λ. Then, we have $\Delta_m\big(|f_{|\pi_m^{-1}(\lambda,Q)}|_h^2 - G_{\lambda,Q}\big) \leqslant 0$, and hence we obtain the following:

$$\big|f_{|\pi_m^{-1}(\lambda,Q)}(z_m)\big|_h^2 \leqslant \max_{|z'_m|=1/2}\big|f_{|\pi_m^{-1}(\lambda,Q)}(z'_m)\big|_h^2 + 2C\exp(-\varepsilon_7|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\Big(-\sum_{i=k+1}^{m-1}\log|z_i|\Big)^N.$$

By the inductive assumption on ℓ, we may assume to have the desired estimate for the restriction of f to $\big\{\boldsymbol{z}\in X\smallsetminus D\,\big|\,1/3<|z_m|<2/3\big\}\times U(\lambda_0)$. Thus, we obtain $A(m-1)$, and $A(k)$ means the claim of Lemma 9.3.1. □

We can show the following lemmas by using the same argument as in the proof of Lemma 9.3.1.

Lemma 9.3.3. — *Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)}\smallsetminus W(\leqslant k)$. Let f be a $\mathbb{D}_{\leqslant k}$-flat section of $\mathcal{F}^S_{<0}\mathcal{P}^{(\lambda_0)}_0\mathcal{E}_{|S}$ such that $\mathrm{Res}_i(\mathbb{D})f_{|\mathcal{D}_i^{(\lambda_0)}\cap S}=0$ for $i=k+1,\dots,\ell$. When we shrink S in the radius direction, we have the estimate $|f|_h\leqslant C\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)$ for some $C>0$ and $\varepsilon>0$.* □

Lemma 9.3.4. — *Let $\lambda\neq 0$. Let S be a small multi-sector of $X\smallsetminus D(\leqslant k)$, such that we have the partial Stokes filtration $\mathcal{F}^S$ of $\mathcal{P}\mathcal{E}^\lambda_{|\overline{S}}$ at the level $\boldsymbol{m}(0)$. Let f be a $\mathbb{D}^\lambda_{\leqslant k}$-flat section of $\mathcal{F}^S_{<0}\mathcal{P}\mathcal{E}^\lambda_{|\overline{S}}$ such that $\mathrm{Res}_i(\mathbb{D}^\lambda)f_{|D_i\cap\overline{S}}=0$ for $i=k+1,\dots,\ell$. When we shrink S in the radius direction, we have $|f|_h\leqslant C\exp\big(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)$ for some $C>0$ and $\varepsilon>0$.* □

9.4. Locally uniform comparison of the irregular decompositions

We will compare the irregular decompositions of $(\mathcal{P}\mathcal{E}^0,\mathbb{D}^0)$ and $(\mathcal{P}^{(\lambda_0)}\mathcal{E},\mathbb{D})$. The main results of this section are Lemma 9.4.4, Corollary 9.4.5, Lemma 9.4.8 and Corollary 9.4.9. Because we will not use them in the rest of this paper, the reader can skip here.

9.4.1. Around $\lambda_0\neq 0$. — We continue to use the setting in Section 9.3. Let us consider the case $\lambda_0\neq 0$.

Lemma 9.4.1. — *Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)}\smallsetminus\mathcal{D}^{(\lambda_0)}(\leqslant k)$.*

- *We can take a $\mathbb{D}_{\leqslant k}$-flat splitting $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}=\bigoplus\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ of the Stokes filtration $\mathcal{F}^S$ at the level $\boldsymbol{m}(0)$, whose restriction to $S\cap\mathcal{D}_i^{(\lambda_0)}$ is compatible with the residues $\mathrm{Res}_i(\mathbb{D})$ and the filtrations ${}^iF^{(\lambda_0)}$ for $i=k+1,\dots,\ell$.*
- *If λ_0 is generic, then we can take a $\mathbb{D}$-flat splitting with the above property.*

Proof. — It follows from Proposition 3.6.7 and Proposition 3.6.8. □

Let $\mathcal{X}^{\lambda} := \{\lambda\} \times X$. We use the symbols like $\mathcal{D}^{\lambda}(\leqslant k)$ with similar meanings. We also use the symbol S^{λ} to denote $S \cap \mathcal{X}^{\lambda}$ for a multi-sector S in $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}(\leqslant k)$.

Lemma 9.4.2. — *Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}(\leqslant k)$. Let $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus \mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ be a $\mathbb{D}_{\leqslant k}$-flat splitting as in Lemma 9.4.1. If $|\lambda-\lambda_0|$ is sufficiently small, the restriction of the splitting to $\mathcal{X}^{\lambda}$ gives a splitting of the Stokes filtration $\mathcal{F}^{S^{\lambda}}$ of $\mathcal{P}\mathcal{E}^{\lambda}_{|S^{\lambda}}$.*

Proof. — The restriction $\mathcal{P}^{(\lambda_0)}\mathcal{E}^{\lambda}_{|\overline{S}^{\lambda}} = \bigoplus \mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S|\overline{S}^{\lambda}}$ gives a splitting of the Stokes filtrations $\mathcal{F}^{S^{\lambda}}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}^{\lambda}_{|\overline{S}^{\lambda}}\big)$ of $\mathcal{P}^{(\lambda_0)}\mathcal{E}^{\lambda}_{|\overline{S}^{\lambda}}$. Then, the claim follows from Lemma 4.5.8. □

Let $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$. Let $p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}$ come from another decomposition with the first property in Lemma 9.4.1.

Lemma 9.4.3. — *When we shrink S in the radius direction, we have $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S} - p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S} = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to h on $S \setminus \mathcal{D}^{(\lambda_0)}$ for some $\varepsilon > 0$.*

Proof. — We have (i) $\big(p^{\boldsymbol{m}(0)}_{\mathfrak{a},S} - p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}\big)\mathcal{F}^S_{\mathfrak{a}} \subset \mathcal{F}^S_{<\mathfrak{a}}$, (ii) $\big[p^{\boldsymbol{m}(0)}_{\mathfrak{a},S} - p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}, \mathrm{Res}_i(\mathbb{D})\big] = 0$ on $S \cap \mathcal{D}^{(\lambda_0)}_i$ for each $i = k+1,\dots,\ell$, (iii) the restriction of $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S} - p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}$ to $S \cap \mathcal{D}^{(\lambda_0)}_i$ preserves the filtration for each $i = k+1,\dots,\ell$. Hence, the claim follows from Lemma 9.3.1. □

We take small multi-sectors S_j $(j = 1,\dots,N)$ such that the union of the interior points of S_j is $\mathcal{V}_0 \setminus \mathcal{D}^{(\lambda_0)}(\leqslant k)$, where $\mathcal{V}_0$ denotes a neighbourhood of $\mathcal{D}^{(\lambda_0)}_{\underline{k}}$. By gluing $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S_j}$ in C^{∞} as in Section 3.6.8.2, we construct a C^{∞}-map $p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^{\infty}}$. Due to Lemma 9.4.3, we have $(\overline{\partial}_E + \lambda\theta^{\dagger})p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^{\infty}} = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to h and the Poincaré metric g_{p} on $\mathcal{V} \setminus \mathcal{D}^{(\lambda_0)}$, where $\mathcal{V}$ denotes some neighbourhood of $\mathcal{D}^{(\lambda_0)}_{\underline{k}}$.

Lemma 9.4.4. — *We have the estimate $\big|\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^{\infty}}\big|_h \leqslant C_N \prod_{i=1}^{k} |z_i|^N$ for any $N > 0$ on $\mathcal{V}_1 \setminus \mathcal{D}^{(\lambda_0)}$, where $\mathcal{V}_1$ denotes some neighbourhood of $\mathcal{D}^{(\lambda_0)}_{\underline{k}}$.*

Proof. — By shrinking X, we may assume $(\overline{\partial}_E + \lambda\theta^{\dagger})p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^{\infty}} = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to h and g_{p} on $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$. Let $\pi : \mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)} \to \mathcal{D}^{(\lambda_0)}_{\underline{k}}$ denote the natural projection. Then, the restrictions $(\mathcal{E},h)_{|\pi^{-1}(\lambda,Q)}$ are acceptable, and the curvatures are dominated uniformly for $(\lambda,Q) \in \pi(\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)})$. We also have $\big((\overline{\partial}_E + \lambda\theta^{\dagger})\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big)_{|\pi^{-1}(\lambda,Q)} = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to (h, g_{p}), which is uniform for (λ,Q). Thus, we obtain the following estimate uniformly for (λ,Q):

$$\Big(\big(\overline{\partial}_E + \lambda\theta^{\dagger}\big)\big(p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^{\infty}} - \pi^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big)\Big)_{|\pi^{-1}(\lambda,Q)} = O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big).$$

Let $\mathcal{Z} := \big\{(\lambda, z_1,\dots,z_n) \,\big|\, 1/2 < |z_i| < 1 \ (i = 1,\dots,k)\big\}$. We obviously have the boundedness of $\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}}$ on $\mathcal{Z}$. Due to Lemma 9.3.1 with $k = 0$, we also have the boundedness of $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}$ on $S\cap\mathcal{Z}$, and hence we obtain the boundedness of $p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}$ on $\mathcal{Z}$. Moreover, we have $\big|p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}-\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big|_{h|\pi^{-1}(\lambda,Q)}\leqslant C_{(\lambda,Q),N}\prod_{i=1}^k|z_i|^N$ for any $(\lambda,Q)\in\mathcal{D}^{(\lambda_0)}_{\underline{k}}$ and any $N>0$, due to Corollary 7.5.4 and Lemma 9.4.2. Then, the claim of the lemma follows from Lemma 21.9.2. □

Corollary 9.4.5. — *Let S be a small sector in $\mathcal{X}^{(\lambda_0)}\smallsetminus\mathcal{D}^{(\lambda_0)}(\leqslant k)$. Let $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}=\bigoplus\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ be a splitting of the Stokes filtration $\mathcal{F}^S$ at the level $\boldsymbol{m}(0)$ as in Lemma 9.4.1. We have the estimate $\big|\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}}-p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}\big|_h\leqslant C_N\prod_{i=1}^k|z_i|^N$ for any $N>0$ on $(\mathcal{V}_1\cap S)\smallsetminus\mathcal{D}^{(\lambda_0)}$.* □

9.4.2. Around $\lambda_0=0$. — Let us consider the case $\lambda_0=0$.

Lemma 9.4.6. — *Let $S=S_\lambda\times S_{\boldsymbol{z}}$ be a sufficiently small multi-sector in $\mathcal{X}^{(0)}\smallsetminus W(\leqslant k)$, where S_λ and $S_{\boldsymbol{z}}$ denote sectors in $U(0)\smallsetminus\{0\}$ and $X\smallsetminus D$, respectively.*

- *We have a $\mathbb{D}_{\leqslant k}$-flat splitting $\mathcal{P}^{(0)}\mathcal{E}_{|\overline{S}}=\bigoplus\mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{a},S}$ of $\mathcal{F}^S$ whose restriction to $S\cap\mathcal{D}^{(0)}_i$ is compatible with the residues $\mathrm{Res}_i(\mathbb{D})$ and the filtrations ${}^iF^{(\lambda_0)}$ for $i=k+1,\dots,\ell$.*
- *For any $\lambda\in S_\lambda$, the restriction $\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|S^\lambda}=\bigoplus\mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{a},S|S^\lambda}$ gives a splitting of the Stokes filtration of $\mathcal{P}\mathcal{E}^\lambda_{|S^\lambda}$ with the above property.*

Proof. — The first claim follows from Proposition 3.6.7. The restriction $\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|\overline{S}^\lambda}=\bigoplus\mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{a},S|\overline{S}^\lambda}$ gives a splitting of the Stokes filtration $\mathcal{F}^{S^\lambda}\big(\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|\overline{S}^\lambda}\big)$ of $\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|\overline{S}^\lambda}$. The filtration $\mathcal{F}^{S^\lambda}\big(\mathcal{P}\mathcal{E}^\lambda_{|S^\lambda}\big)$ given by

$$\mathcal{F}^{S^\lambda}_{(1+|\lambda|^2)\mathfrak{a}}\big(\mathcal{P}\mathcal{E}^\lambda_{|S^\lambda}\big):=\mathcal{F}^{S^\lambda}_{\mathfrak{a}}\big(\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|S^\lambda}\big).$$

is the same as the Stokes filtration of $\mathcal{P}\mathcal{E}^\lambda_{|S^\lambda}$. Hence, the second claim follows. □

Let $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{a},S}$. Let $p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}$ come from another decomposition with the property in Lemma 9.4.6.

Lemma 9.4.7. — *When we shrink S in the radius direction, we have $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}-p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}=O\big(\exp(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to h on $S\smallsetminus\mathcal{D}^{(0)}$.*

Proof. — We have (i) $\big(p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}-p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}\big)\mathcal{F}^S_{\mathfrak{a}}\subset\mathcal{F}^S_{<\mathfrak{a}}$, (ii) $\big[p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}-p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S},\mathrm{Res}_i(\mathbb{D})\big]=0$ on $S\cap\mathcal{D}^{(0)}_i$ for $i=k+1,\dots,\ell$, (iii) the restriction of $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}-p^{\boldsymbol{m}(0)\prime}_{\mathfrak{a},S}$ to $S\cap\mathcal{D}^{(0)}_i$ preserves the filtration ${}^iF^{(\lambda_0)}$ for each $i=k+1,\dots,\ell$. Then, the claim follows from Lemma 9.3.1. □

We take small multi-sectors S_j $(j=1,\dots,N)$ such that the union of the interior points of S_j is $\mathcal{V}_0\smallsetminus W(\leqslant k)$, where $\mathcal{V}_0$ denotes a neighbourhood of $\{0\}\times D_{\underline{k}}$. By gluing $p^{\boldsymbol{m}(0)}_{\mathfrak{a},S_j}$ in C^∞ as in Section 3.6.8.2, we construct the C^∞-map $p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}$ on $\mathcal{V}_0\smallsetminus W(\leqslant k)$, which can be extended to a C^∞-map on $\mathcal{V}_0\smallsetminus\mathcal{D}^{(0)}$. Shrinking $\mathcal{V}_0$, we have the estimate

$(\overline{\partial}_E + \lambda\theta^\dagger + \overline{\partial}_\lambda)p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty} = O\big(\exp(-\varepsilon_1|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ with respect to h and $g_{\mathbf{p}}$, due to Lemma 9.4.7.

Lemma 9.4.8. — *We have the estimate* $\big|\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}\big|_h \leqslant C_N \cdot |\lambda| \cdot \prod_{i=1}^k |z_i|^N$ *for any* $N > 0$ *on* $\mathcal{V}_1 \setminus \mathcal{D}^{(0)}(\leqslant k)$, *where* $\mathcal{V}_1$ *denotes some neighbourhood of* $\{0\} \times D_{\underline{k}}$.

Proof. — Take small $0 < \delta_1 < \delta_2$. Due to Lemma 9.4.4, we may assume to have $\big|\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}\big|_h \leqslant C'_N \cdot \prod_{i=1}^k |z_i|^N$ with respect to h on $\big\{\lambda \,\big|\, \delta_1 \leqslant |\lambda| \leqslant \delta_2\big\} \times (X \setminus D)$ for any $N > 0$ by shrinking X. We may assume to have the following estimate on $\{0 < |\lambda| < \delta_2\} \times (X \setminus D)$, with respect to h and the Euclidean metric $d\lambda \cdot d\overline{\lambda}$:

$$g(\boldsymbol{z},\lambda) := \overline{\partial}_\lambda\big(\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty}\big) = -\overline{\partial}_\lambda p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty} = O\Big(\exp\big(-\varepsilon_2|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).$$

Let $\boldsymbol{e}$ be an orthonormal frame of $\mathrm{End}(E)$ on $X \setminus D$ with respect to the metric induced by h. We have the expression $g = \sum g_i \cdot e_i$. We have the estimate $\big|g_i\big| \leqslant C \cdot \exp\big(-\varepsilon_3|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)$. Let χ be a positive valued C^∞-function on $\boldsymbol{C}_\lambda$ such that $\chi(\lambda) = 1$ for $|\lambda| \leqslant (\delta_1 + \delta_2)/2$ and $\chi(\lambda) = 0$ for $|\lambda| \geqslant \delta_2$. We put

$$G_i(\boldsymbol{z},\lambda) := \lambda \int_{\boldsymbol{C}} \frac{g_i(\boldsymbol{z},\mu)\cdot\mu^{-1}\cdot\chi(\mu)}{\mu - \lambda}\frac{\sqrt{-1}}{2\pi}d\mu \cdot d\overline{\mu}.$$

We put $G := \sum G_i \cdot e_i$. Then, we have $\overline{\partial}_\lambda G = g$ on $\big\{|\lambda| < (\delta_1 + \delta_2)/2\big\} \times (X \setminus D)$. We also have the following estimate:

$$\big|G(\boldsymbol{z},\lambda)\big|_h \leqslant C \cdot |\lambda| \cdot \exp\big(-\varepsilon_4 \cdot |\boldsymbol{z}^{\boldsymbol{m}(0)}|\big).$$

Let us see $H := \pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},C^\infty} - G$. We have $\overline{\partial}_\lambda H = 0$ and $H(\boldsymbol{z},0) = 0$. By the Schwarz lemma, we obtain $\big|H(\boldsymbol{z},\lambda)\big|_h \leqslant C_N \cdot |\lambda| \cdot \prod_{i=1}^k |z_i|^N$ for any N. Then, we obtain the desired estimate. □

Corollary 9.4.9. — *We have the estimate* $\big|\pi^{\boldsymbol{m}(0)}_{\mathfrak{a}} - p^{\boldsymbol{m}(0)}_{\mathfrak{a},S}\big|_h \leqslant C_N \cdot |\lambda| \cdot \prod_{i=1}^k |z_i|^N$ *for any* $N > 0$. □

CHAPTER 10

SMOOTH DIVISOR CASE

Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X \setminus D$, where X is a complex manifold and D is a simple normal crossing hypersurface. In Chapter 9, we studied the prolongment $(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ of $(\mathcal{E}, \mathbb{D})$, i.e., the sheaf of holomorphic sections whose norms are of polynomial growth with respect to the Hermitian metric $\mathcal{P}^{(\lambda_0)}h$. In this chapter, we restrict ourselves to the case where D is smooth, and we will do more refined analysis. The results in this chapter are rather technical, and preliminary for the later sections.

In Section 10.1, we show that $(\mathcal{E}, \mathcal{P}^{(\lambda_0)}h)$ is acceptable (Proposition 10.1.1), which is new even in the tame case. For the proof, we obtain a complementary estimate of the connection form (Lemma 10.1.3 and Corollary 10.1.4) which is also useful for other purposes.

Recall we have studied in Section 9.4 the comparison of irregular decompositions for $(\mathcal{P}\mathcal{E}^0, \mathbb{D}^0)$ and $(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ at the level $\boldsymbol{m}(0)$. In Section 10.2, we compare irregular decompositions at any level under the assumption that D is smooth (Proposition 10.2.2 and Proposition 10.2.3). This comparison will be used in the proof of Theorem 11.2.2 in one way. We also use it for the comparison of the Hermitian metrics with some twist (Corollary 10.2.6), which will be used for the family version of norm estimate (Section 10.4).

In Section 10.3, we show a standard norm estimate (Proposition 10.3.2) for $(\mathcal{P}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ under the assumption that D is smooth, which is a generalization of Proposition 8.1.1, and preliminary for the family version of norm estimate (Section 10.4). We also show an estimate of a connection form (Proposition 10.3.3), which can be skipped.

In Section 10.4, we show the family version of the norm estimate (Proposition 10.4.2). This is preliminary for Step 1 in the proof of Theorem 18.1.1 (Section 18.2).

Throughout this chapter, we use the setting in Section 7.2.1 *with* $\ell = 1$*, i.e., D is assumed to be smooth.*

10.1. Acceptability of $\mathcal{P}^{(\lambda_0)}h$

10.1.1. Statement. — Let $U(\lambda_0)$ denote a small neighbourhood of λ_0. We will prove the following proposition in Section 10.1.3.

Proposition 10.1.1. — *$(\mathcal{E}, \mathcal{P}^{(\lambda_0)}h)$ is acceptable on $(X \setminus D) \times U(\lambda_0)$, if $U(\lambda_0)$ is sufficiently small.*

In particular, $\mathcal{P}^{(\lambda_0)}_{*}\mathcal{E} = \big(\mathcal{P}^{(\lambda_0)}_{a}\mathcal{E} \,\big|\, a \in \boldsymbol{R}\big)$ is a filtered bundle. We have the induced filtration $F^{(\lambda_0)}$ of $\mathcal{P}^{(\lambda_0)}_{a}\mathcal{E}_{|U(\lambda_0)\times D}$. Let $\boldsymbol{v}$ be a frame of $\mathcal{P}^{(\lambda_0)}_{a}\mathcal{E}$ which is compatible with the filtrations $F^{(\lambda_0)}$. We put

$$v_i' := v_i \cdot |z_1|^{\deg^{F^{(\lambda_0)}}(v_i)}.$$

Let $H(\mathcal{P}^{(\lambda_0)}h, \boldsymbol{v}')$ denote the Hermitian matrix-valued function whose (i,j)-th entries are $\mathcal{P}^{(\lambda_0)}h(v_i', v_j')$.

Corollary 10.1.2. — *We have the weak norm estimate, i.e., $C_1 \cdot (-\log|z_1|)^{-N} \leqslant H(\mathcal{P}^{(\lambda_0)}h, \boldsymbol{v}') \leqslant C_2 \cdot \big(-\log|z_1|\big)^{N}$ for some $C_1, C_2, N > 0$.*

Proof. — It follows from the general result for the acceptable bundles (Theorem 21.3.2). □

10.1.2. Estimate of the connection form. — Let $\eta_m : M(X,D) \to M(X,D)$ be given by $\eta_m(\mathfrak{a}) = \sum_{m(0)\leqslant j\leqslant m} \mathfrak{a}_j \cdot z_1^j$. The image $\eta_m(\mathrm{Irr}(\theta))$ is denoted by $\mathrm{Irr}(\theta, m)$. We take total orders $\leqslant$ on $\mathrm{Irr}(\theta)$ and $\mathrm{Irr}(\theta,m)$ such that η_m are order preserving. We put

$$E^{(m)}_{\mathfrak{b}} = \bigoplus_{\substack{\mathfrak{a}\in \mathrm{Irr}(\theta)\\ \eta_m(\mathfrak{a})=\mathfrak{b}}} E_{\mathfrak{a}}, \quad F^{(m)}_{\mathfrak{b}} := \bigoplus_{\substack{\mathfrak{c}\in \mathrm{Irr}(\theta,m)\\ \mathfrak{c}\leqslant \mathfrak{b}}} E^{(m)}_{\mathfrak{b}}, \quad F^{(m)}_{<\mathfrak{b}} := \bigoplus_{\substack{\mathfrak{c}\in \mathrm{Irr}(\theta,m)\\ \mathfrak{c}< \mathfrak{b}}} E^{(m)}_{\mathfrak{b}}.$$

Let $\pi^{(m)}_{\mathfrak{b}}$ denote the projection onto $E^{(m)}_{\mathfrak{b}}$ with respect to the decomposition $E = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,m)} E^{(m)}_{\mathfrak{b}}$. We also put $E^{(0)}_{(\mathfrak{a},\alpha)} := E_{(\mathfrak{a},\alpha)}$. Let $\pi^{(0)}_{\mathfrak{a},\alpha}$ denote the projection onto $E^{(0)}_{(\mathfrak{a},\alpha)}$ with respect to the decomposition (156).

For simplicity, we use the symbol γ to denote an element of $\mathrm{Irr}(\theta) \times \mathcal{S}p(\theta)$. We have the decomposition $\mathrm{End}(E) = \mathcal{C}^{(0)}\big(\mathrm{End}(E)\big) \oplus \mathcal{D}^{(0)}\big(\mathrm{End}(E)\big)$:

$$\mathcal{D}^{(0)}\big(\mathrm{End}(E)\big) := \bigoplus_{\gamma\in\mathrm{Irr}(\theta)\times\mathcal{S}p(\theta)} \mathrm{End}\big(E^{(0)}_{\gamma}\big),$$

$$\mathcal{C}^{(0)}\big(\mathrm{End}(E)\big) := \bigoplus_{\substack{\gamma,\gamma'\in\mathrm{Irr}(\theta)\times\mathcal{S}p(\theta)\\ \gamma\neq\gamma'}} \mathrm{Hom}\big(E^{(0)}_{\gamma}, E^{(0)}_{\gamma'}\big)$$

We use the symbols $\mathcal{D}^{(m)}\big(\mathrm{End}(E)\big)$ and $\mathcal{C}^{(m)}\big(\mathrm{End}(E)\big)$ $(m \leqslant -1)$ with similar meanings. For any section $G \in \mathrm{End}(E) \otimes \Omega^{p,q}$, we have the corresponding decomposition $G = \mathcal{D}^{(m)}(G) + \mathcal{C}^{(m)}(G)$.

Let ∂_E denote the $(1,0)$-operator associated to $\overline{\partial}_E$ and h. Let $\boldsymbol{v}$ be a frame of ${}^\diamond E$ compatible with the decomposition (156) and the parabolic filtration. Let $F \in \mathrm{End}(E) \otimes \Omega^{1,0}$ determined by $F(\boldsymbol{v}) = \partial_E \boldsymbol{v}$.

Lemma 10.1.3. — *We have the following estimates on X^* for some $\varepsilon > 0$:*
(The case $j < 0$)

$$\mathcal{C}^{(j)}(F) = O\big(\exp(-\varepsilon|z_1|^j)\big), \quad \partial_E \pi_{\mathfrak{b}}^{(j)} = O\big(\exp(-\varepsilon|z_1|^j)\big). \tag{188}$$

(The case $j = 0$)

$$\mathcal{C}^{(0)}(F) = O\Big(|z_1|^{\varepsilon}\frac{dz_1}{z_1}\Big), \quad \partial_E \pi_{\mathfrak{a},\alpha}^{(0)} = O\big(|z_1|^{\varepsilon}\big)\frac{dz_1}{z_1}. \tag{189}$$

Proof. — The estimate (188) is a special case of Lemma 7.5.5. Let us consider the case $j = 0$. Since the argument is essentially the same, we give only an outline. In this proof, ε will denote a positive constant, and we will make it smaller without mention. We have $\partial_E \pi_\gamma^{(0)} = [F, \pi_\gamma^{(0)}] \in \mathcal{C}^{(0)}(\mathrm{End}(E)) \otimes \Omega^{1,0}$ and the estimate

$$\partial_E \pi_\gamma^{(0)} = O\Big(\big(-\log|z_1|\big)^N\Big)$$

for some N with respect to h and $g_{\mathbf{p}}$. Because $\mathcal{C}^{(0)}(\mathrm{End}(E))$ and $\mathcal{D}^{(0)}(\mathrm{End}(E))$ are $|z_1|^{\varepsilon}$-asymptotically orthogonal (Theorem 7.2.1 and Theorem 7.2.4), we obtain the following estimate with respect to h and $g_{\mathbf{p}}$:

$$\big(\pi_\gamma^{(0)}, \partial_E \pi_\gamma^{(0)}\big) = O(|z_1|^{\varepsilon}).$$

We also have the following estimate with respect to h and $g_{\mathbf{p}}$, due to Corollary 7.2.10:

$$\overline{\partial}_E \partial_E \pi_\gamma^{(0)} = \big[R(h), \pi_\gamma^{(0)}\big] = O\big(|z_1|^{\varepsilon}\big). \tag{190}$$

We have the following equality:

$$\begin{aligned}\big(\partial_E \pi_\gamma^{(0)}, \partial_E \pi_\gamma^{(0)}\big)_h |z_1|^{-\varepsilon} &= \partial\Big(\big(\pi_\gamma^{(0)}, \partial_E \pi_\gamma^{(0)}\big)_h |z_1|^{-\varepsilon}\Big) \\ &\quad + \big(\pi_\gamma^{(0)}, \partial_E \pi_\gamma^{(0)}\big)_h \cdot \frac{\varepsilon}{2} \cdot |z_1|^{-\varepsilon}\frac{dz_1}{z_1} - \big(\pi_\gamma^{(0)}, \overline{\partial}_E \partial_E \pi_\gamma^{(0)}\big)_h |z_1|^{-\varepsilon}.\end{aligned}$$

Hence, we obtain the following finiteness:

$$\int \big(\partial_E \pi_\gamma^{(0)}, \partial_E \pi_\gamma^{(0)}\big)_h \cdot |z_1|^{-\varepsilon} < \infty. \tag{191}$$

From (190) and (191), we obtain the desired estimate. (See the argument in the proof of Lemma 7.5.5.) □

Corollary 10.1.4. — *For $p < j$, we have $\mathcal{C}^{(p)}(\partial_E \pi_{\mathfrak{a}}^{(j)}) = O\big(\exp(-\varepsilon|z_1|^p)\big)$.*

Proof. — We have the following:

$$[F, \pi_{\mathfrak{a}}^{(j)}] = \big[\mathcal{C}^{(p)}(F) + \mathcal{D}^{(p)}(F),\ \mathcal{D}^{(p)}(\pi_{\mathfrak{a}}^{(j)})\big] = O\big(\exp(-\varepsilon|z_1|^p)\big) + \big[\mathcal{D}^{(p)}(F), \mathcal{D}^{(p)}(\pi_{\mathfrak{a}}^{(j)})\big].$$

Hence, $\mathcal{C}^{(p)}\big([F, \pi_{\mathfrak{a}}^{(j)}]\big) = O\big(\exp(-\varepsilon|z_1|^p)\big)$, and the claim of the corollary follows. □

10.1.3. Proof of Proposition 10.1.1. — We use the notation in Section 9.1.3. We set $\widetilde{g}_{\mathbf{p}} := g_{\mathbf{p}} + d\lambda \cdot d\overline{\lambda}$. We only have to show that the curvature of $(\mathcal{T}^{(\lambda_0)}\mathcal{E}, h)$ is bounded with respect to h and $\widetilde{g}_{\mathbf{p}}$. Let $\delta'_{\lambda_0} = \partial_E - \overline{\lambda}_0\theta$ denote the $(1,0)$-operator determined by d''_{λ_0} and h. In the following, $g(\lambda - \lambda_0)$ is also denoted by g.

Lemma 10.1.5. — *If $U(\lambda_0)$ is sufficiently small, $\delta'_{\lambda_0}\Lambda(\lambda-\lambda_0)$ is bounded with respect to h and $g_{\mathbf{p}}$ uniformly for $\lambda \in U(\lambda_0)$.*

Proof. — In the proof, we will make $U(\lambda_0)$ smaller without mention. Let $\overline{\Phi}$ be given by (122). We have

$$\Lambda(\lambda-\lambda_0) = -(\lambda-\lambda_0)\cdot\overline{\Phi} + \lambda \cdot g(\lambda-\lambda_0)\cdot\theta^{\dagger}\cdot g(\lambda-\lambda_0)^{-1} - \lambda_0\cdot\theta^{\dagger}.$$

We have $\delta'_{\lambda_0}\theta^{\dagger} = -\overline{\lambda}_0\big[\theta,\theta^{\dagger}\big]$, which is bounded (Proposition 7.2.9). Due to Lemma 10.1.3, we have the following estimate of $\delta'_{\lambda_0}\overline{\Phi}$ with respect to h and $g_{\mathbf{p}}$:

$$\big[\partial_E - \overline{\lambda}_0\theta, \overline{\Phi}\big] = \sum_{m(0)\leqslant j\leqslant -1}\sum_{\mathfrak{b}} \overline{d\zeta_j(\mathfrak{b})}\cdot\partial_E\pi^{(j)}_{\mathfrak{b}} + \sum_{(\mathfrak{a},\alpha)\in\operatorname{Irr}(\theta)\times\mathcal{S}p(\theta)} \overline{\alpha}\frac{d\overline{z}_1}{\overline{z}_1}\cdot\partial_E\pi^{(0)}_{\mathfrak{a},\alpha} = O\big(|z_1|^{\varepsilon}\big).$$

It remains to estimate $\delta'_{\lambda_0}\big(g(\lambda-\lambda_0)\cdot\theta^{\dagger}\cdot g(\lambda-\lambda_0)^{-1}\big)$, which can be rewritten as follows:

$$\text{(192)}\qquad \Big[\partial_E - \overline{\lambda}_0\theta,\ g\cdot\theta^{\dagger}\cdot g^{-1}\Big] = g\cdot\Big[g^{-1}\partial_E g,\ \theta^{\dagger}\Big]\cdot g^{-1} - \overline{\lambda}_0\cdot g\cdot\big[\theta,\theta^{\dagger}\big]\cdot g^{-1}.$$

Due to Proposition 7.2.9, the second term in the right-hand side of (192) is bounded with respect to h and $g_{\mathbf{p}}$ uniformly for $\lambda\in U(\lambda_0)$. Let F be as in Section 10.1.2. We have the following equality:

$$g^{-1}\partial_E g = g^{-1}\big[F, g\big] + \sum_{\mathfrak{a},\alpha}\overline{\alpha}\cdot\frac{dz_1}{z_1}\pi^{(0)}_{\mathfrak{a},\alpha}.$$

Hence, we have the following estimate with respect to h and $g_{\mathbf{p}}$:

$$\text{(193)}\quad \delta'_{\lambda_0}\big(g\cdot\theta^{\dagger}\cdot g^{-1}\big) = g\big[g^{-1}\cdot F\cdot g - F,\ \theta^{\dagger}\big]\cdot g^{-1} + \sum_{\mathfrak{a},\alpha} g\cdot\overline{\alpha}\cdot\frac{dz_1}{z_1}\big[\pi^{(0)}_{\mathfrak{a},\alpha},\ \theta^{\dagger}\big]\cdot g^{-1} + O(1).$$

The second term of the right-hand side of (193) is bounded uniformly for λ, due to Lemma 7.3.6 and Lemma 7.3.7. Let us look at the first term in (193). For the decomposition as in (95), we obtain the following estimate from Lemma 10.1.3:

$$\big(g^{-1}\cdot F\cdot g - F\big)_{(\mathfrak{a},\alpha),(\mathfrak{a}',\alpha')} = \begin{cases} O\big(\exp(-\varepsilon|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')})\big) & (\mathfrak{a}\neq\mathfrak{a}') \\ O\big(|z_1|^{\varepsilon}\big) & (\mathfrak{a}=\mathfrak{a}', \alpha\neq\alpha') \end{cases}$$

Then, we obtain the estimate for the first term from Proposition 7.2.8. Thus, we obtain Lemma 10.1.5. □

Let $R\big(\mathcal{T}^{(\lambda_0)}d''_{\lambda}, h\big)$ be the curvature of the unitary connection associated to $\mathcal{T}^{(\lambda_0)}d''_{\lambda}$ and h.

Lemma 10.1.6. — *If $U(\lambda_0)$ is sufficiently small, $R\big(\mathcal{T}^{(\lambda_0)}d''_{\lambda}, h\big)$ is bounded with respect to h and $g_{\mathbf{p}}$ uniformly for $\lambda\in U(\lambda_0)$.*

Proof. — Let $\Lambda^{\dagger}(\lambda-\lambda_0)$ denote the adjoint of $\Lambda(\lambda-\lambda_0)$ with respect to h. Then, $R(\mathcal{T}^{(\lambda_0)}d''_{\lambda},h)$ can be rewritten as follows:

$$[d''_{\lambda_0},\delta'_{\lambda_0}]+\delta'_{\lambda_0}\Lambda(\lambda-\lambda_0)-d''_{\lambda_0}\Lambda^{\dagger}(\lambda-\lambda_0)-\big[\Lambda(\lambda-\lambda_0),\Lambda^{\dagger}(\lambda-\lambda_0)\big]. \tag{194}$$

If $U(\lambda_0)$ is sufficiently small, the first and fourth terms are bounded uniformly for $\lambda\in U(\lambda_0)$ with respect to h and $g_{\mathbf{p}}$, due to Corollary 7.2.10 and Lemma 9.1.6. We have already obtained the estimate for the second term in Lemma 10.1.5. Since the third term is adjoint to the second term up to signature, we obtain Lemma 10.1.6. □

The curvature of $(\mathcal{T}^{(\lambda_0)}\mathcal{E},h)$ is rewritten as follows:

$$\big[\overline{\partial}_{\lambda}+d''_{\lambda_0}+\Lambda,\partial_{\lambda}+\delta'_{\lambda_0}-\Lambda^{\dagger}\big]=R(\mathcal{T}^{(\lambda_0)}d''_{\lambda},h)-\overline{\partial}_{\lambda}\Lambda^{\dagger}+\partial_{\lambda}\Lambda.$$

We only have to show the uniform boundedness of $\partial_{\lambda}\Lambda$ with respect to h and $\widetilde{g}_{\mathbf{p}}$, which can be rewritten as follows:

$$\partial_{\lambda}\Lambda=\lambda_0\cdot g\cdot\big[g^{-1}\partial_{\lambda}g,\ \theta^{\dagger}\big]\cdot g^{-1}+d\lambda\cdot g\cdot(\theta^{\dagger}-\overline{\Phi})\cdot g^{-1}+\lambda\cdot g\cdot\big[g^{-1}\partial_{\lambda}g,\theta^{\dagger}-\overline{\Phi}\big]g^{-1}. \tag{195}$$

We have the following equality:

$$g^{-1}\partial_{\lambda}g=\sum_{\mathfrak{a},\alpha}\big(\overline{\mathfrak{a}}+\overline{\alpha}\log|z_1|^2\big)\cdot d\lambda\cdot\pi^{(0)}_{\mathfrak{a},\alpha}.$$

Then, it is easy to show that the right-hand side of (195) is bounded uniformly for λ with respect to h and $\widetilde{g}_{\mathbf{p}}$, by using Proposition 7.2.8. Thus, Proposition 10.1.1 is proved. □

10.2. Locally uniform comparison of the irregular decompositions

10.2.1. Statements. — Let $U(\lambda_0)$ denote a neighbourhood of λ_0 in $\boldsymbol{C}_{\lambda}$. Let $\mathcal{X}^{(\lambda_0)}:=U(\lambda_0)\times X$ and $\mathcal{D}^{(\lambda_0)}:=U(\lambda_0)\times D$. We have two metrics of $\mathcal{E}_{|\mathcal{X}^{(\lambda_0)}\setminus\mathcal{D}^{(\lambda_0)}}$. One is h, and the other is $\mathcal{P}^{(\lambda_0)}h$ given in (182).

We put $W:=\mathcal{D}^{(\lambda_0)}$ in the case $\lambda_0\neq 0$, and $W:=\mathcal{D}^{(0)}\cup\big(\{0\}\times X\big)$ in the case $\lambda_0=0$. Let $\pi:\widetilde{\mathcal{X}}^{(\lambda_0)}(W)\to\mathcal{X}^{(\lambda_0)}$ denote the real blow up along W. Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)}\setminus W$. Let $\overline{S}$ denote the closure of S in the real blow up $\widetilde{\mathcal{X}}^{(\lambda_0)}(W)$. We have the Stokes filtrations $\mathcal{F}^{(j)}$ of $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$ at the level j. We will prove the estimate with respect to h in the following proposition in Section 10.2.6.

Proposition 10.2.1. — *Let f be a $\mathbb{D}$-flat section of $\mathrm{End}(\mathcal{E})_{|\overline{S}\setminus\pi^{-1}\mathcal{D}^{(\lambda_0)}}$ such that $f\big(\mathcal{F}^{(j)}_{\mathfrak{a}}{}^{S}\mathcal{P}^{(\lambda_0)}_{0}\mathcal{E}\big)\subset\mathcal{F}^{(j)}_{<\mathfrak{a}}{}^{S}\mathcal{P}^{(\lambda_0)}_{0}\mathcal{E}$ for any $\mathfrak{a}\in\mathrm{Irr}(\theta)$. If we shrink S in the radius direction, we have the following estimates for some $\varepsilon>0$:*

$$|f|_{\mathcal{P}^{(\lambda_0)}h}=O\Big(\exp\big(-\varepsilon|\lambda^{-1}z_1^{j}|\big)\Big),\quad |f|_{h}=O\Big(\exp\big(-\varepsilon|\lambda^{-1}z_1^{j}|\big)\Big). \tag{196}$$

Similarly, let f be a $\mathbb{D}$-flat section of $\mathcal{F}^{(j)\,S}_{<0}\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}\setminus\pi^{-1}(\mathcal{D}^{(\lambda_0)})}$. If we shrink S in the radius direction, we have the following estimates for some $\varepsilon>0$:

$$(197)\qquad \big|f\big|_{\mathcal{P}^{(\lambda_0)}h}=O\Big(\exp\big(-\varepsilon|\lambda^{-1}z_1^j|\big)\Big),\quad \big|f\big|_h=O\Big(\exp\big(-\varepsilon|\lambda^{-1}z_1^j|\big)\Big).$$

Note that the estimate for $\mathcal{P}^{(\lambda_0)}h$ clearly follows from Corollary 20.3.6 and Corollary 10.1.2, which we will use implicitly.

Let $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|\overline{S}}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,j)}\mathcal{P}^{(\lambda_0)}_a\mathcal{E}^{(j)}_{\mathfrak{a},S}$ be any $\mathbb{D}$-flat splitting of the filtration $\mathcal{F}^{S\,(j)}$, and let $p^{(j)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}^{(\lambda_0)}\mathcal{E}^{(j)}_{\mathfrak{a},S}$. We will prove the following proposition in Sections 10.2.3–10.2.5.

Proposition 10.2.2. — *If we shrink S in the radius direction, we have the estimate*

$$\pi^{(j)}_{\mathfrak{a}}-p^{(j)}_{\mathfrak{a},S}=O\big(|\lambda|\cdot\exp(-\varepsilon|z_1^j|)\big)$$

for some $\varepsilon>0$ with respect to both the metrics $\mathcal{P}^{(\lambda_0)}h$ and h.

We take a small multi-sectors S_i $(i=1,\dots,N)$ of $\mathcal{X}^{(\lambda_0)}\setminus W$ such that the union of the interior points of S_i is $\mathcal{V}\setminus W$, where $\mathcal{V}$ denotes a neighbourhood of $\mathcal{D}^{(\lambda_0)}$. If S_i are sufficiently small, we have the partial Stokes filtrations $\mathcal{F}^{(j)\,S_i}$ of $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|\overline{S}_i}$. We can take $\mathbb{D}$-flat splittings $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|\overline{S}_i}=\bigoplus\mathcal{P}^{(\lambda_0)}_a\mathcal{E}^{(j)}_{\mathfrak{a},S_i}$. Let $p^{(j)}_{\mathfrak{a},S_i}$ denote the projection onto $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}^{(j)}_{\mathfrak{a},S_i}$. By gluing them in C^∞ as in Section 3.6.8.2, we construct the C^∞-section $p^{(j)}_{\mathfrak{a},C^\infty}$ of $\mathrm{End}(\mathcal{E})$ on $\mathcal{V}\setminus W$, which can be extended to the C^∞-section on $\mathcal{V}\setminus\mathcal{D}^{(\lambda_0)}$. We will prove the following proposition in Sections 10.2.3–10.2.5.

Proposition 10.2.3. — *If we shrink $\mathcal{X}^{(\lambda_0)}$, we have the estimate*

$$\pi^{(j)}_{\mathfrak{a}}-p^{(j)}_{\mathfrak{a},C^\infty}=O\Big(|\lambda|\cdot\exp\big(-\varepsilon|z_1^j|\big)\Big)$$

for some $\varepsilon>0$ with respect to both the metrics $\mathcal{P}^{(\lambda_0)}h$ and h.

Although the estimate with respect to $\mathcal{P}^{(\lambda_0)}h$ in Propositions 10.2.2 and 10.2.3 are equivalent, we state them separately for our convenience in the proof.

We have the following corollary as a consequence of Proposition 10.2.1.

Corollary 10.2.4. — *Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)}\setminus W$, and let*

$$\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|\overline{S}}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,j)}\mathcal{P}^{(\lambda_0)}_a\mathcal{E}^{(j)}_{\mathfrak{a},S}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,j)}\mathcal{P}^{(\lambda_0)}_a\mathcal{E}^{(j)\prime}_{\mathfrak{a},S}$$

be two $\mathbb{D}$-flat splittings of the filtration $\mathcal{F}^{S\,(j)}$. Let $p^{(j)}_{\mathfrak{a},S}$ and $p^{(j)\prime}_{\mathfrak{a},S}$ denote the projections onto $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ and $\mathcal{P}^{(\lambda_0)}\mathcal{E}'_{\mathfrak{a},S}$, respectively. Then, $p^{(j)\prime}_{\mathfrak{a},S}-p^{(j)}_{\mathfrak{a},S}=O\big(\exp(-\varepsilon|\lambda|^{-1}|z_1|^j)\big)$ for some $\varepsilon>0$ with respect to both the metrics $\mathcal{P}^{(\lambda_0)}h$ and h. □

10.2.2. Locally uniform comparison of the twisted metrics. — Let S be a small multi-sector in $\mathcal{X}^{(\lambda_0)} \setminus W$. We take a $\mathbb{D}$-flat splitting $\mathcal{P}_a^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{P}_a^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^S$. We have the induced splittings $\mathcal{P}_a^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)} \mathcal{P}_a^{(\lambda_0)}\mathcal{E}_{\mathfrak{b},S}^{(j)}$ of the partial Stokes filtration $\mathcal{F}^{S\,(j)}$. We consider the following for $w \in \boldsymbol{C}$, as in (153):

$$F_S(w) := \exp\bigl(w \cdot \mathcal{B}_S\bigr), \quad \mathcal{B}_S := \sum_{m(0)\leqslant j\leqslant -1} \sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)} \zeta_j(\mathfrak{b}) \cdot p_{\mathfrak{b},S}^{(j)}. \tag{198}$$

We also consider the following as in (152):

$$F(w) := \exp\bigl(w \cdot \mathcal{B}\bigr), \quad \mathcal{B} := \sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathfrak{a} \cdot \pi_{\mathfrak{a}} = \sum_{m(0)\leqslant j\leqslant -1} \sum_{\mathfrak{b}\in\mathrm{Irr}(\theta,j)} \zeta_j(\mathfrak{b}) \cdot \pi_{\mathfrak{b}}^{(j)}.$$

Corollary 10.2.5. — *Take $\eta > 0$. If we shrink $U(\lambda_0)$, there exist some constants $\varepsilon > 0$ and $C > 0$ such that the following estimate holds for any $|w| < \eta$ and for both the metrics h and $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$:*

$$F_S(w) \circ F(w)^{-1} - 1 = O\Bigl(\exp\bigl(-\varepsilon|z_1^{-1}|\bigr)\Bigr), \quad F(w) \circ F_S(w)^{-1} - 1 = O\Bigl(\exp\bigl(-\varepsilon|z_1^{-1}|\bigr)\Bigr).$$

Proof. — It can be shown by using the argument in the proof of Lemma 7.6.8, together with Proposition 10.2.2. □

Corollary 10.2.6. — *If we shrink $U(\lambda_0)$, the following holds:*

- *$F_S(\overline{\lambda} - \overline{\lambda}_0)^*h$ and $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$ are mutually bounded, uniformly for λ.*
- *$F_S(-\overline{\lambda} + \overline{\lambda}_0)^*\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$ and h are mutually bounded, uniformly for λ.*

Proof. — We have $\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h = F(\overline{\lambda} - \overline{\lambda}_0)^*h$ and $h = F(-\overline{\lambda} + \overline{\lambda}_0)^*\mathcal{P}_{\mathrm{irr}}^{(\lambda_0)}h$. Hence, the claim of Corollary 10.2.6 can be shown using the same argument as that in the proof of Lemma 7.6.9, together with Corollary 10.2.5. □

10.2.3. Preliminary. — We have the decomposition:

$$\mathrm{End}(E) = \bigoplus_{\mathfrak{a},\mathfrak{b}\in\mathrm{Irr}(\theta)} \mathrm{Hom}(E_{\mathfrak{a}}, E_{\mathfrak{b}}). \tag{199}$$

For any section G of $\mathrm{End}(E)$, we have the corresponding decomposition $G = \sum G_{\mathfrak{a},\mathfrak{b}}$. We will use the following lemma implicitly, which is obvious from the construction of $\mathcal{P}^{(\lambda_0)}h$.

Lemma 10.2.7. — *Let Y be a subset of $\mathcal{X}^{(\lambda_0)} \setminus W$. Let f be a section of $\mathrm{End}(E)$ on Y. Assume that the following holds for some $\varepsilon > 0$:*

$$|f_{\mathfrak{a},\mathfrak{b}}|_h = O\bigl(\exp\bigl(-\varepsilon|\lambda^{-1}z_1^q| - \varepsilon|z_1^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}|\bigr)\bigr). \tag{200}$$

Then, if we shrink $U(\lambda_0)$, the following holds for some $\varepsilon' > 0$:

$$|f_{\mathfrak{a},\mathfrak{b}}|_{\mathcal{P}^{(\lambda_0)}h} = O\Bigl(\exp\bigl(-\varepsilon'|\lambda^{-1}z_1^q| - \varepsilon'|z_1^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}|\bigr)\Bigr). \tag{201}$$

Conversely, if (201) *holds for some* $\varepsilon' > 0$, *we obtain* (200) *for some* $\varepsilon > 0$ *by shrinking* $U(\lambda_0)$. □

10.2.4. Comparison around $\lambda_0 \neq 0$. — We use the notation in Section 10.1.2. We have the following estimates for some $\varepsilon_1 > 0$ with respect to h and $g_{\mathbf{p}}$, due to Theorem 7.2.1, Theorem 7.2.4, Lemma 10.1.3 and Corollary 10.1.4:

$$\mathbb{D}\pi_{\mathfrak{b}}^{(j)} = O\big(\exp(-\varepsilon_1|z_1|^j)\big), \quad \mathcal{C}^{(p)}\big(\mathbb{D}\pi_{\mathfrak{b}}^{(j)}\big) = O\big(\exp(-\varepsilon_1|z_1|^p)\big), \quad (p < j).$$

According to Lemma 10.2.7, we may assume to have the following estimate on $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$ with respect to $\mathcal{P}^{(\lambda_0)}h$:

$$\mathbb{D}\pi_{\mathfrak{b}}^{(j)} = O\big(\exp(-\varepsilon_1|z_1|^j/2)\big).$$

Let $\pi : \widetilde{\mathcal{X}}^{(\lambda_0)}(\mathcal{D}^{(\lambda_0)}) \to \mathcal{X}^{(\lambda_0)}$ denote the real blow up of $\mathcal{X}^{(\lambda_0)}$ along $\mathcal{D}^{(\lambda_0)}$.

Lemma 10.2.8. — *For any point* $P \in \pi^{-1}(\mathcal{D}^{(\lambda_0)})$, *there exist a multi-sector* $S \in \mathcal{MS}(P, \mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)})$ *and a* $\mathbb{D}$*-flat splitting* $\mathcal{P}_{a}^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\theta,j)} \mathcal{P}_{a}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}^{(j)\natural}$ *of the Stokes filtration* $\mathcal{F}^{S\,(j)}$ *on* $\overline{S}$ *at the level* j *with the following property:*

- *Let* $p_{\mathfrak{a},S}^{(j)\natural}$ *denote the projection onto* $\mathcal{P}_{a}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}^{(j)\natural}$ *with respect to the decomposition. Then,*
$$\pi_{\mathfrak{a}}^{(j)} - p_{\mathfrak{a},S}^{(j)\natural} = O\big(\exp(-\varepsilon_1|z_1|^j/10)\big)$$
with respect to $\mathcal{P}^{(\lambda_0)}h$

Proof. — The argument is essentially the same as in the proof of Proposition 7.6.2. By using Lemma 20.2.1, we can take $\mathcal{Q}_{\mathfrak{b},S}^{(j)}$ such that $\mathbb{D}\mathcal{Q}_{\mathfrak{b},S}^{(j)} = \mathbb{D}\pi_{\mathfrak{b}}^{(j)}$ and $\mathcal{Q}_{\mathfrak{b},S}^{(j)} = O\big(\exp(-\varepsilon_1|z_1|^j)\big)$ with respect to $\mathcal{P}^{(\lambda_0)}h$. We put $p_{\mathfrak{b},S}^{(j)\natural} := \pi_{\mathfrak{b}}^{(j)} - \mathcal{Q}_{\mathfrak{b},S}^{(j)}$, which is $\mathbb{D}$-flat. By applying the modification as in the proof of Lemma 7.4.7, we may and will assume $\big[p_{\mathfrak{b},S}^{(j)\natural}, p_{\mathfrak{c},S}^{(j)\natural}\big] = 0$ and $p_{\mathfrak{b},S}^{(j)\natural} \circ p_{\mathfrak{b},S}^{(j)\natural} = p_{\mathfrak{b},S}^{(j)\natural}$.

We put $\mathcal{F}_{\mathfrak{b}}^{S\,(j)\natural} := \bigoplus_{\mathfrak{c}\leqslant_S \mathfrak{b}} \mathrm{Im}\, p_{\mathfrak{c},S}^{(j)\natural}$. Let us compare the filtrations $\mathcal{F}^{S\,(j)\natural}$ and $\mathcal{F}^{S\,(j)}$. Let S^{λ} denote $S \cap \mathcal{X}^{\lambda}$. Let us consider the filtrations of $\mathcal{E}_{|S^{\lambda}}^{\lambda}$ given as follows:

$$\overline{\mathcal{F}}^{S^{\lambda}\,(j)}_{(1+|\lambda|^2)\mathfrak{a}}\big(\mathcal{E}_{|S^{\lambda}}^{\lambda}\big) := \mathcal{F}^{S^{\lambda}\,(j)}_{(1+\lambda\overline{\lambda}_0)\mathfrak{a}}\big(\mathcal{E}_{|\overline{S}}\big)_{|S^{\lambda}}, \quad \overline{\mathcal{F}}^{S^{\lambda}\,(j)\natural}_{(1+|\lambda|^2)\mathfrak{a}}\big(\mathcal{E}_{|S^{\lambda}}^{\lambda}\big) := \mathcal{F}^{S^{\lambda}\,(j)\natural}_{(1+\lambda\overline{\lambda}_0)\mathfrak{a}}\big(\mathcal{E}_{|\overline{S}}\big)_{|S^{\lambda}}.$$

If $|\lambda - \lambda_0|$ is sufficiently small, $\overline{\mathcal{F}}^{S^{\lambda}\,(j)}(\mathcal{E}_{|S^{\lambda}}^{\lambda})$ is the same as the Stokes filtration $\mathcal{F}^{S^{\lambda}\,(j)}\big(\mathcal{P}\mathcal{E}_{|S^{\lambda}}^{\lambda}\big)$ of $\mathcal{P}\mathcal{E}_{|S^{\lambda}}^{\lambda}$. By construction, $\overline{\mathcal{F}}^{S^{\lambda}\,(j)\natural}\big(\mathcal{E}_{|S^{\lambda}}^{\lambda}\big)$ is the same as the filtration $\mathcal{F}^{S^{\lambda}\,(j)\natural}$ of $\mathcal{P}\mathcal{E}_{|S^{\lambda}}^{\lambda}$ in the proof of Proposition 7.6.2. Hence, we obtain that the specializations of $\mathcal{F}^{S\,(j)\natural}$ and $\mathcal{F}^{S\,(j)}$ to S^{λ} are the same. As a result, we obtain that the filtrations $\mathcal{F}^{S\,(j)\natural}$ and $\mathcal{F}^{S\,(j)}$ are the same. Thus, we obtain Lemma 10.2.8. □

Let us show Proposition 10.2.2 in the case $\lambda_0 \neq 0$. We take a finite covering $S \subset \bigcup S_{P_i}$, where S_{P_i} are as in Lemma 10.2.8. We have $p_{\mathfrak{a},S}^{(j)} - p_{\mathfrak{a},S_{P_i}}^{(j)\natural} = O\big(\exp(-\varepsilon|z_1|^j)\big)$ with respect to $\mathcal{P}^{(\lambda_0)}h$ on S_{P_i} for some $\varepsilon > 0$ by using the estimate for $\mathcal{P}^{(\lambda_0)}h$ in

Proposition 10.2.1. Then, we obtain the estimate of $|p^{(j)}_{\mathfrak{a},S} - \pi^{(j)}_{\mathfrak{a}}|_{\mathcal{P}^{(\lambda_0)}h}$ by Lemma 10.2.8.

Let us show the estimate for h. We may assume to have $[p^{(j)}_{\mathfrak{a}\,S}, p^{(i)}_{\mathfrak{b}\,S}] = 0$. We also have $[\pi^{(j)}_{\mathfrak{a}}, \pi^{(i)}_{\mathfrak{b}}] = 0$. Then, we obtain

$$[p^{(j)}_{\mathfrak{a}\,S}, \pi^{(i)}_{\mathfrak{b}}] = O\Big(\exp\big(-\varepsilon_1|z_1|^{\imath} - \varepsilon_1|z_1|^{\jmath}\big)\Big)$$

with respect to $\mathcal{P}^{(\lambda_0)}h$ from the previous estimate. (See the proof of Lemma 7.3.6, for example.) By the same argument as in the proof of Lemma 7.3.8, we obtain the following:

$$\big|\big(p^{(j)}_{\mathfrak{b},S} - \pi^{(j)}_{\mathfrak{b}}\big)_{\mathfrak{a},\mathfrak{a}'}\big|_{\mathcal{P}^{(\lambda_0)}h} = O\Big(\exp\big(-\varepsilon_2|z_1|^{\jmath} - \varepsilon_2|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{a}')}\big)\Big). \tag{202}$$

By Lemma 10.2.7, we obtain the desired estimate for $\pi^{(j)}_{\mathfrak{a}} - p^{(j)}_{\mathfrak{a},S} = O\big(\exp(-\varepsilon|z_1|^{\jmath})\big)$ with respect to h. Thus, the proof of Proposition 10.2.2 in the case $\lambda_0 \neq 0$ is finished. Proposition 10.2.3 in the case $\lambda_0 \neq 0$ immediately follows.

10.2.5. Comparison around $\lambda_0 = 0$. — Let us show Proposition 10.2.3 in the case $\lambda_0 = 0$. Let S be a small multi-sector in $\mathcal{X}^{(0)} \setminus W$. Let $p^{(j)}_{\mathfrak{a},S}$ and $p^{(j)\prime}_{\mathfrak{a},S}$ come from $\mathbb{D}$-flat splittings of the filtration $\mathcal{F}^{S(j)}$. If we shrink S in the radius direction, we have the following estimate for some $\varepsilon_1 > 0$ with respect to the metric $\mathcal{P}^{(0)}h$:

$$p^{(j)}_{\mathfrak{a},S} - p^{(j)\prime}_{\mathfrak{a},S} = O\Big(\exp\big(-\varepsilon_1|\lambda^{-1}z_1^{\jmath}|\big)\Big). \tag{203}$$

Let $p^{(j)}_{\mathfrak{a},C^\infty}$ be as in Subsection 10.2.1. By shrinking $\mathcal{V}$, we obtain the following estimate on $\mathcal{V} \setminus \mathcal{D}^{(0)}$ with respect to $\mathcal{P}^{(0)}h$ by using (203) and $\overline{\partial}_\lambda \pi^{(j)}_{\mathfrak{a}} = 0$:

$$g(\lambda, \boldsymbol{z}) \cdot d\overline{\lambda} := \overline{\partial}_\lambda\big(p^{(j)}_{\mathfrak{a},C^\infty} - \pi^{(j)}_{\mathfrak{a}}\big) = O\Big(\exp\big(-\varepsilon_2|\lambda^{-1}z_1^{\jmath}|\big)\cdot d\overline{\lambda}\Big). \tag{204}$$

Take small $0<\delta_1<\delta_2$. Let us estimate $\pi^{(j)}_{\mathfrak{a}} - p^{(j)}_{\mathfrak{a},C^\infty}$ on $\mathcal{Z} := \{\lambda \mid \delta_1 \leqslant |\lambda| \leqslant \delta_2\} \times (X \setminus D)$.

Lemma 10.2.9. — *Let S be a small multi-sector in $\mathcal{Z} - (\mathcal{Z} \cap \mathcal{D}^{(0)})$. Let $\mathcal{P}^{(0)}\mathcal{E}_{|\overline{S}} = \bigoplus \mathcal{P}^{(0)}\mathcal{E}^{(j)}_{\mathfrak{a},S}$ be a $\mathbb{D}$-flat splitting of the Stokes filtration $\mathcal{F}^{S\,(j)}$, and let $p^{(j)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}^{(0)}\mathcal{E}^{(j)}_{\mathfrak{a},S}$. Then, $\pi^{(j)}_{\mathfrak{a}} - p^{(j)}_{\mathfrak{a},S} = O\big(\exp(-\varepsilon_3|z_1|^{\jmath})\big)$ with respect to $\mathcal{P}^{(0)}h$ for some $\varepsilon_3 > 0$. As a result, $\pi^{(j)}_{\mathfrak{a}} - p^{(j)}_{\mathfrak{a},C^\infty} = O\big(\exp(-\varepsilon_4|z_1^{\jmath}|)\big)$ for some $\varepsilon_4 > 0$ with respect to $\mathcal{P}^{(0)}h$ on $\mathcal{Z} - (\mathcal{Z} \cap \mathcal{D}^{(0)})$.*

Proof. — It can be shown by the arguments used in the proof of Lemma 10.2.8 and Proposition 10.2.2 in the case $\lambda_0 \neq 0$. We remark that the Stokes filtrations of $\mathcal{P}^{(0)}\mathcal{E}^\lambda$ and $\mathcal{P}\mathcal{E}^\lambda$ are essentially the same, i.e., $\mathcal{F}^{S^\lambda}_{\mathfrak{a}}\big(\mathcal{P}^{(0)}\mathcal{E}^\lambda_{|S^\lambda}\big) = \mathcal{F}^{S^\lambda}_{(1+|\lambda|^2)\mathfrak{a}}\big(\mathcal{P}\mathcal{E}^\lambda_{|S^\lambda}\big)$. □

Let us continue the proof of Proposition 10.2.3. Let $\boldsymbol{v}$ be a holomorphic frame of $\mathcal{P}^{(0)}_0 \operatorname{End}\mathcal{E}$ on $\mathcal{X}^{(0)}$. We have the expression $g = \sum g_i \cdot v_i$. We have the estimate $|g_i| \leqslant C \cdot \exp\big(-\varepsilon_5|\lambda^{-1}z_1^{\jmath}|\big)$. Let χ be a non-negative valued C^∞-function on $\boldsymbol{C}_\lambda$ such that $\chi(\lambda) = 1$ for $|\lambda| \leqslant (\delta_1+\delta_2)/2$ and $\chi(\lambda) = 0$ for $|\lambda| \geqslant \delta_2$. We consider the following:

$$G_i(\boldsymbol{z},\lambda) := \lambda \cdot \int \frac{g_i(\boldsymbol{z},\mu)\cdot\mu^{-1}\cdot\chi(\mu)}{\mu-\lambda}\cdot\frac{\sqrt{-1}}{2\pi}\cdot d\mu\cdot d\overline{\mu}.$$

We put $G := \sum G_i \cdot v_i$. Then, we have $\overline{\partial}_\lambda G = g$ on $\big\{|\lambda| < (\delta_1+\delta_2)/2\big\} \times (X \setminus D)$. We also have the following estimate:

$$\big|G(\boldsymbol{z},\lambda)\big|_{\mathcal{P}^{(0)}h} \leqslant C\cdot|\lambda|\cdot\exp\big(-\varepsilon_6\cdot|z_1^j|\big).$$

Let us look at $H := \pi_{\mathfrak{a}}^{(j)} - p_{\mathfrak{a},C^\infty}^{(j)} - G$. We have $\overline{\partial}_\lambda H = 0$ and $H(\boldsymbol{z},0) = 0$. By a lemma of Schwarz and Lemma 10.2.9, we obtain $\big|H(\boldsymbol{z},\lambda)\big|_{\mathcal{P}^{(0)}h} = O\big(|\lambda|\cdot\exp(-\varepsilon_7\cdot|z_1|^j)\big)$. Then, we obtain

$$\big|\pi_{\mathfrak{a}}^{(j)} - p_{\mathfrak{a},C^\infty}^{(j)}\big|_{\mathcal{P}^{(\lambda_0)}h} = O\Big(|\lambda|\cdot\exp\big(-\varepsilon_8\cdot|z_1|^j\big)\Big), \tag{205}$$

i.e., the estimate with respect to $\mathcal{P}^{(0)}h$ in Proposition 10.2.3.

Let us show the estimate with respect to h. By construction, we have the following estimate:

$$\Big|\big[p_{\mathfrak{a},C^\infty}^{(j)}, p_{\mathfrak{b},C^\infty}^{(i)}\big]\Big|_{\mathcal{P}^{(\lambda_0)}h} \leqslant C_{21}\cdot\exp\big(-\varepsilon_{21}|\lambda^{-1}z_1^i| - \varepsilon_{21}|\lambda^{-1}z_1^j|\big). \tag{206}$$

We also have $[\pi_{\mathfrak{a}}^{(j)}, \pi_{\mathfrak{b}}^{(i)}] = 0$. From (205) and (206), we obtain the following estimate:

$$\Big|\big[p_{\mathfrak{a},C^\infty}^{(j)}, \pi_{\mathfrak{b}}^{(i)}\big]\Big|_{\mathcal{P}^{(\lambda_0)}h} \leqslant C_{22}\cdot|\lambda|\cdot\exp\big(-\varepsilon_{22}|z_1|^j - \varepsilon_{22}|z_1|^i\big). \tag{207}$$

We have the decomposition $p_{\mathfrak{a},C^\infty}^{(j)} - \pi_{\mathfrak{a}}^{(j)} = \sum_{\mathfrak{b},\mathfrak{b}'}\big(p_{\mathfrak{a},C^\infty}^{(j)} - \pi_{\mathfrak{a}}^{(j)}\big)_{\mathfrak{b},\mathfrak{b}'}$ corresponding to (199). By the same argument as that in the proof of Lemma 7.3.8, we obtain the following estimate from (207):

$$\Big|\big(p_{\mathfrak{b},C^\infty}^{(j)} - \pi_{\mathfrak{b}}^{(j)}\big)_{\mathfrak{a},\mathfrak{a}'}\Big|_{\mathcal{P}^{(\lambda_0)}h} \leqslant C_{23}\cdot|\lambda|\cdot\exp\big(-\varepsilon_{23}|z_1|^j - \varepsilon_{23}|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{a}')}\big). \tag{208}$$

Then the estimate with respect to h in Proposition 10.2.3 follows from Lemma 10.2.7.

The estimate with respect to $\mathcal{P}^{(0)}h$ in Proposition 10.2.2 immediately follows from Proposition 10.2.3. The estimate with respect to h follows from the estimate with respect to $\mathcal{P}^{(0)}h$ as above.

10.2.6. Proof of Proposition 10.2.1. — Let us consider the first claim. We take a $\mathbb{D}$-flat splitting $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\theta)}\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^S$. We only have to consider the case $f \in \operatorname{Hom}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}, \mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{b},S}\big)$ for some $\mathfrak{a} >_S \mathfrak{b}$. We have the following:

$$|f|_{\mathcal{P}^{(\lambda_0)}h} = O\Big(\exp\big(-\varepsilon|\lambda^{-1}z_1^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}|\big)\Big).$$

We would like to obtain the estimate with respect to h, by using Lemma 10.2.7. For any ℓ and $\mathfrak{c} \in \operatorname{Irr}(\theta,\ell)$ such that $\mathfrak{c} \neq \eta_\ell(\mathfrak{a})$, we have the following estimate with respect to $\mathcal{P}^{(\lambda_0)}h$:

$$f\circ\pi_{\mathfrak{c}}^{(\ell)}=f\circ(\pi_{\mathfrak{c}}^{(\ell)}-p_{\mathfrak{c}}^{(\ell)})=O\Big(\exp\big(-\varepsilon|z_1^{\ell}|-\varepsilon|\lambda^{-1}z_1^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}|\big)\Big).\tag{209}$$

We have similar estimate for $\pi_{\mathfrak{d}}^{(\ell)}\circ f$ for $\mathfrak{d}\neq\eta_\ell(\mathfrak{b})$. Then, we obtain the desired estimate (196) by using Lemma 10.2.7.

Let us consider the second claim. We only have to consider the case $f\in\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ for some $\mathfrak{a}<_S 0$. We have the decomposition $f=\sum f_{\mathfrak{b}}$ corresponding to $E=\bigoplus E_{\mathfrak{b}}$. We have the following estimate for $\mathfrak{c}\neq\eta_\ell(\mathfrak{a})$ with respect to $\mathcal{P}^{(\lambda_0)}h$:

$$f_{\mathfrak{c}}=\pi_{\mathfrak{c}}^{(\ell)}(f)=\big(\pi_{\mathfrak{c}}^{(\ell)}-p_{\mathfrak{c}}^{(\ell)}\big)(f)=O\Big(\exp\big(-\varepsilon|z_1^{\ell}|-\varepsilon|\lambda^{-1}z_1^{\mathrm{ord}(\mathfrak{a})}|\big)\Big).$$

Then, it is easy to obtain (197). □

10.3. Norm estimate for a fixed λ

10.3.1. Norm estimate for $\mathcal{P}\mathcal{E}^{\lambda}$. — Let F denote the parabolic filtration of $\mathcal{P}_a\mathcal{E}^{\lambda}_{|D}$. The residue $\mathrm{Res}(\mathbb{D}^{\lambda})_{|Q}$ $(Q\in D)$ induces the endomorphism of $\mathrm{Gr}^F_a(\mathcal{P}\mathcal{E}^{\lambda}_{|Q})$. Let $N^{\lambda}_{a|Q}$ denote the nilpotent part.

Lemma 10.3.1. — *The conjugacy classes of $N^{\lambda}_{a|Q}$ are independent of the choice of $Q\in D$. In particular, the weight filtration W gives the filtration in the category of vector bundles.*

Proof. — Let us consider the case where λ is generic (Definition 2.8.1). We have the irregular decomposition

$$(\mathcal{P}_a\mathcal{E}^{\lambda},\mathbb{D}^{\lambda})_{|\widehat{D}}=\bigoplus(\mathcal{P}_a\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}},\widehat{\mathbb{D}}^{\lambda}_{\mathfrak{a}}),$$

which is compatible with the parabolic filtrations F. We have the flat λ-connection $\mathbb{D}^{\lambda}_{\mathfrak{a},a}$ on $\mathrm{Gr}^F_a(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$ given by $\mathbb{D}^{\lambda}_{\mathfrak{a},a}(f)=p_a\widehat{\mathbb{D}}^{\lambda}_{\mathfrak{a}}(F)$, where p_a is the naturally induced map $\mathcal{P}_a\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}\to\mathrm{Gr}^F_a(\mathcal{P}\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}})$, and F is a section of $\mathcal{P}_a\widehat{\mathcal{E}}^{\lambda}_{\mathfrak{a}}$ such that $p_a(F)=f$. Since the residue is flat with respect to $\mathbb{D}^{\lambda}_{\mathfrak{a},a}$, the conjugacy classes are independent of $Q\in D$.

Let us consider the general case. Let $\pi:X\setminus D\to D$ denote the projection. Let $(E_Q,\overline{\partial}_{E_Q},\theta_Q,h_Q)$ denote the restriction of $(E,\overline{\partial}_E,\theta,h)$ to $\pi^{-1}(Q)$. We have the λ-connections $(\mathcal{P}\mathcal{E}^{\lambda}_Q,\mathbb{D}^{\lambda}_Q)$. We have the correspondence of the conjugacy classes of the nilpotent part of the residues of $\mathrm{Res}(\mathbb{D}^{\lambda}_Q)$ and $\mathrm{Res}(\mathbb{D}^{\lambda_1}_Q)$ for any λ_1 in Proposition 8.2.1. Hence, the claim holds in the general case. □

Let $\boldsymbol{v}$ be a holomorphic frame of $\mathcal{P}_c\mathcal{E}^{\lambda}$ such that (i) it is compatible with the parabolic filtration, (ii) the induced frame on $\mathrm{Gr}^F(\mathcal{P}_c\mathcal{E}^{\lambda})$ is compatible with the weight filtration W. We put $a(v_i):=\deg^F(v_i)$ and $k(v_i):=\deg^W(v_i)$. Let h_0 be the C^{∞}-metric of E given by $h_0(v_i,v_j):=\delta_{i,j}\cdot|z_1|^{-2a(v_i)}\cdot(-\log|z_1|^2)^{k(v_i)}$.

Proposition 10.3.2. — *The metrics h and h_0 are mutually bounded.*

Proof. — We give only an outline. We have the irregular decomposition:

$$(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\theta)} (\mathcal{P}_*\widehat{\mathcal{E}}^\lambda_{\mathfrak{a}}, \mathbb{D}^\lambda_{\mathfrak{a}}). \tag{210}$$

Here, $\mathbb{D}^\lambda_{\mathfrak{a}} - (1+|\lambda|^2)d\mathfrak{a}$ are logarithmic with respect to $\mathcal{P}_*\widehat{\mathcal{E}}^\lambda_{\mathfrak{a}}$. By taking Gr associated to the generalized eigen-decomposition $\mathbb{E}$ and the parabolic filtration F, we obtain vector spaces

$$\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\widehat{\mathcal{E}}^\lambda_{\mathfrak{a}|O}).$$

For each $(a,\alpha,\mathfrak{a})$, we have the model bundle on Δ^* obtained from the vector space $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\mathcal{E}^\lambda_{\mathfrak{a}|O})$ with the nilpotent part $N_{a,\alpha\mathfrak{a}|O}$ of the residue (Section 7.4.1.3), which is denoted by $(E^1_{a,\alpha,\mathfrak{a}}, \overline{\partial}_{E^1_{a,\alpha,\mathfrak{a}}}, h^1_{a,\alpha,\mathfrak{a}}, \theta^1_{a,\alpha,\mathfrak{a}})$. Let $u=(b,\beta)\in \boldsymbol{R}\times\boldsymbol{C}$ be determined by $\mathfrak{k}(\lambda,u)=(a,\alpha)$. We have the rank one harmonic bundle $L(u,\mathfrak{a}) = \big(\mathcal{O}\cdot e, \theta^2_{u,\mathfrak{a}}, h^2_{u,\mathfrak{a}}\big)$ on $X\setminus D$ given as follows:

$$\theta^2_{u,\mathfrak{a}} = d\mathfrak{a} + \beta\cdot\frac{dz_1}{z_1}, \quad h^2_{u,\mathfrak{a}}(e,e) = |z_1|^{-2b}.$$

Let $q_1 : X\setminus D \to \Delta^*$ be the projection $q_1(z_1,\dots,z_n)=z_1$. We set

$$\big(E_{a,\alpha,\mathfrak{a}}, \overline{\partial}_{E_{a,\alpha,\mathfrak{a}}}, \theta_{a,\alpha,\mathfrak{a}}, h_{a,\alpha,\mathfrak{a}}\big) := q_1^*\big(E^1_{a,\alpha,\mathfrak{a}}, \overline{\partial}_{E^1_{a,\alpha,\mathfrak{a}}}, \theta^1_{a,\alpha,\mathfrak{a}}, h^1_{a,\alpha,\mathfrak{a}}\big)\otimes L(u,\mathfrak{a}).$$

Let $(E_1, \overline{\partial}_{E_1}, h_1, \theta_1)$ denote their direct sum.

For a large N, let $\widehat{D}^{(N)}$ denote the N-th infinitesimal neighbourhood of D. We take a holomorphic decomposition

$$\mathcal{P}_c\mathcal{E}^\lambda = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{P}_c\mathcal{E}^\lambda_{\mathfrak{a},N}, \tag{211}$$

whose restriction to $\widehat{D}^{(N)}$ is the same as the restriction of (210). We have the natural decomposition

$$\mathcal{P}_c\mathcal{E}^\lambda_1 = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{P}_c\mathcal{E}^\lambda_{1,\mathfrak{a}}, \tag{212}$$

where $\mathcal{P}_c\mathcal{E}^\lambda_{1,\mathfrak{a}}$ are induced by $\bigoplus_{(a,\alpha)}\big(E_{a,\alpha,\mathfrak{a}}, \overline{\partial}_{E_{a,\alpha,\mathfrak{a}}}, \theta_{a,\alpha,\mathfrak{a}}, h_{a,\alpha,\mathfrak{a}}\big)$.

We can take a holomorphic isomorphism $\Phi : \mathcal{P}_c\mathcal{E}^\lambda_1 \to \mathcal{P}_c\mathcal{E}^\lambda$ such that (i) it preserves the decompositions (211) and (212), (ii) it preserves the parabolic filtration, (iii) $\mathrm{Gr}^F(\Phi_{|D})$ is compatible with the residues. It is easy to show that $\Phi(h_1)$ is mutually bounded with h_0.

We have the harmonic bundle $E_2 := \mathrm{Hom}(E_1,E)$ with the induced Higgs field θ_2 and the induced pluri-harmonic metric h_2. We can regard Φ as a section of $\mathcal{P}_0\mathcal{E}^\lambda_2$. For each $Q\in D$, we have $g_Q := \mathbb{D}^\lambda_Q(\Phi_{|\pi^{-1}(Q)}) \in \mathcal{P}_{-\varepsilon+1}\mathcal{E}^\lambda_{2|Q}$ for some $\varepsilon>0$. Therefore, it is L^p for some $p>2$, and the L^p-norm is bounded uniformly for Q. We have $\Delta\big|\Phi_{|\pi_1^{-1}(Q)}\big|^2_{h_2} \leqslant |g_Q|^2$ on $\pi^{-1}(Q)\setminus\{Q\}$, due to Corollary 7.7.5. We already know $\big|\Phi_{|\pi_1^{-1}(Q)}\big|_{h_2}$ is bounded (Proposition 8.1.1). Hence, the inequality holds on $\pi_1^{-1}(Q)$ as distributions. Therefore, $\Phi_{|\pi^{-1}(Q)}$ can be estimated by the values on $\partial\pi^{-1}(Q)$ and the L^p-norm of $|g_Q|$. Thus, we obtain the claim of Proposition 10.3.2. □

10.3.2. Estimate of the connection form (Appendix). — We have a complement on the estimate of the connection form in Lemma 21.9.3 and Lemma 10.1.3. Since we will not use it below, the reader can skip this subsection. Let $\boldsymbol{v}$ be a frame of ${}^{\diamond}E$ compatible with the decomposition $E = \bigoplus_{(\mathfrak{a},\alpha)\in \mathrm{Irr}(\theta)\times \mathcal{S}p(\theta)} E_{\mathfrak{a},\alpha}$, the parabolic filtration, and the weight filtration W. Let $F \in \mathrm{End}(E)\otimes\Omega^{1,0}$ be determined by $F(\boldsymbol{v}) = \partial\boldsymbol{v}$. Let $F_0 \in \mathrm{End}(E)\otimes\Omega^{1,0}$ be determined by $F_0(v_i) = -a(v_i)\cdot v_i \cdot dz_1/z_1$.

Proposition 10.3.3. — *We have the following estimate:*

$$F_1 = F_0 + O\left(\frac{\log(-\log|z_1|)}{-\log|z_1|}\right)\frac{dz_1}{z_1} + \sum_{j=2}^{n} O(1)\cdot dz_j.$$

Proof. — We give only an outline. Let h_0 be the metric as in Subsection 10.3.1, and let s be determined by $h = h_0\cdot s$. The connection form for h_0 satisfies the above estimate. We have $\partial_h = \partial_{h_0} + s^{-1}\partial_{h_0}s$. We only have to estimate $s^{-1}\partial_{h_0}s$. By the estimate of the curvatures $R(h)$ and $R(h_0)$, we obtain the following estimate with respect to h_0:

$$\overline{\partial}(s^{-1}\partial_{h_0}s) = O\left(\frac{dz\cdot d\overline{z}}{|z|^2\cdot\left(-\log|z|^2\right)^2}\right).$$

Let $\boldsymbol{v}^{\vee} = (v_i^{\vee})$ denote the dual frame of $\boldsymbol{v}$. We have the expressions

$$s^{-1}\partial_{h_0}s = \sum A_{i,j}^{(p)}\cdot v_i\cdot v_j^{\vee}\cdot dz_p \quad\text{and}\quad \overline{\partial}(s^{-1}\partial_{h_0}s) = \sum B_{i,j}^{(p,q)} v_i\cdot v_j^{\vee}\cdot dz_p\cdot d\overline{z}_q.$$

We have $\partial A_{i,j}^{(p)}/\partial\overline{z}_p = B_{i,j}^{(p,p)}$. Let $\partial_{h_0,p}$ denote the restriction of ∂_{h_0} to the z_p-direction. Let $\pi_p : X\smallsetminus D \to D_p$ denote the projection.

By using Lemma 5.2.3, we obtain the following expression:

$$s^{-1}\partial_{h_0,1}s = O\left(\frac{\log(-\log|z|)}{-\log|z|}\right)\frac{dz_1}{z_1} + \Phi\cdot dz_1. \tag{213}$$

Here, $\overline{\partial}_1\Phi = 0$. We also have the finiteness $\int_{\pi_1^{-1}(Q)} |s^{-1}\partial_{h_0,1}s|_{h_0}^2 < \infty$ for any $Q\in D_1$. (See Lemma 5.5 of [**66**], for example.) Hence, $\Phi_{|\pi_1^{-1}(Q)}$ is also L^2, with respect to h_0. Then, it is easy to obtain the desired estimate for the dz_1-component by using the maximum principle for holomorphic functions.

Let us look at the dz_p-component ($p>1$). Let $\chi(z_p)$ be a non-negative C^∞-function on Δ^* such that $\chi(z_p) = 1$ for $|z_p|\leqslant 1/2$ and $\chi(z_p) = 0$ for $|z_p|\geqslant 2/3$. We can show $\int_{\pi_p^{-1}(Q)}\big(s^{-1}\partial_{h_0,p}(\chi s), \partial_{h_0,p}(\chi s)\big)$ is estimated uniformly for $Q\in D_p\smallsetminus D_1$. (See the argument in the proof of Lemma 21.9.3.) We also have the uniform estimate of the sup norm of $\overline{\partial}_p(s^{-1}\partial_{h_0,p}s)_{|\pi_p^{-1}(Q)}$. Hence, we obtain the uniform estimate of the sup norm of $s^{-1}\partial_p s_{|\pi_p^{-1}(Q)}$. □

10.4. Norm estimate in family

10.4.1. Statement. — We have the good family of filtered λ-flat bundles $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$. We have the filtration $F^{(\lambda_0)}$ and the decomposition $\mathbb{E}^{(\lambda_0)}$ of $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|U(\lambda_0)\times D}$. (See Remark 2.8.2 for the decomposition $\mathbb{E}^{(\lambda_0)}$.)

Lemma 10.4.1. — *The conjugacy class of the nilpotent part of* $\mathrm{Res}(\mathbb{D})_{|(\lambda,P)}$ *is independent of the choice of* $(\lambda, P) \in U(\lambda_0) \times D$.

Proof. — It follows from Corollary 9.2.7 and Lemma 10.3.1. □

Let $\boldsymbol{v}$ be a frame of $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}$ compatible with $\big(\mathbb{E}^{(\lambda_0)}, F^{(\lambda_0)}, W\big)$. Let $u(v_i) \in \boldsymbol{R}\times\boldsymbol{C}$ be determined by $\mathfrak{k}(\lambda_0, u(v_i)) = \big(\deg^{F^{(\lambda_0)}}(v_i), \deg^{\mathbb{E}^{(\lambda_0)}}(v_i)\big)$. We put $k(v_i) := \deg^W(v_i)$. Let h_0 be the metric given as follows:

$$h_0(v_i, v_j) := \delta_{i,j}\cdot |z_1|^{-2\mathfrak{p}(\lambda,u(v_i))}\cdot\big(-\log|z_1|\big)^{k(v_i)}.$$

We will prove the following proposition in Sections 10.4.3–10.4.6.

Proposition 10.4.2. — *If we shrink* $U(\lambda_0)$, $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ *and* h_0 *are mutually bounded on* $U(\lambda_0)\times(X\setminus D)$. *(See Section* 9.1.1 *for the Hermitian metric* $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$.*)*

Before going into the proof of the proposition, we give a corollary. Let S and F_S be as in Section 10.2.2.

Corollary 10.4.3. — *If we shrink* $U(\lambda_0)$, *the metrics* h *and* $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*h_0$ *are mutually bounded on* S.

Proof. — It follows from Proposition 10.4.2 and Corollary 10.2.6. □

10.4.2. Preliminary comparison. — As a preparation, we give a weak comparison up to small polynomial order.

Lemma 10.4.4. — *The metrics* $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ *and* h_0 *are uniformly mutually bounded up to* $|z_1|^{-\eta|\lambda-\lambda_0|}(-\log|z_1|)^N$*-order for some* $\eta > 0$.

Proof. — Let $a(v_i) := \deg^{F^{(\lambda_0)}}(v_i)$, and let h_0' be the Hermitian metric given by

$$h_0'(v_i, v_j) := \delta_{i,j}\cdot|z_1|^{-2a(v_i)}.$$

We already know that $\mathcal{P}^{(\lambda_0)}h$ and h_0' are uniformly mutually bounded up to log order because of the acceptability of $\mathcal{P}^{(\lambda_0)}h$ (Proposition 10.1.1). Then, the claim of Lemma 10.4.4 follows. □

10.4.3. Comparison map for the associated graded bundle. — We have the decomposition:

$$(\mathcal{P}_*^{(\lambda_0)}\mathcal{E},\mathbb{D})_{|U(\lambda_0)\times\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \big(\mathcal{P}_*^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}},\widehat{\mathbb{D}}_{\mathfrak{a}}\big)_{|U(\lambda_0)\times\widehat{D}}.$$

Here, $\widehat{\mathbb{D}}_{\mathfrak{a}} - (1+\lambda\overline{\lambda}_0)\cdot d\mathfrak{a}$ are logarithmic with respect to $\mathcal{P}_*^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}$. Take any point $P\in D$. We have the vector space $V_{\mathfrak{a},u} := \mathrm{Gr}^{F^{(\lambda_0)},\mathbb{E}^{(\lambda_0)}}_{\mathfrak{k}(\lambda_0,u)}\big(\mathcal{P}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}\big)_{|(\lambda_0,P)}$. We have the endomorphism $N_{\mathfrak{a},u}$, which is the nilpotent part of $\mathrm{Res}(\mathbb{D})_{|(\lambda_0,P)}$. Let $\big(E'_{1,\mathfrak{a},u},\overline{\partial}'_{1,\mathfrak{a},u},\theta'_{1,\mathfrak{a},u},h'_{1,\mathfrak{a},u}\big)$ be the model bundle associated to $(V_{\mathfrak{a},u},N_{\mathfrak{a},u})$ (Section 7.4.1.3). Let $L(u,\mathfrak{a})$ be the harmonic bundle of rank one as in the proof of Proposition 10.3.2. We put

$$(E_{1,\mathfrak{a}},\overline{\partial}_{1,\mathfrak{a}},\theta_{1,\mathfrak{a}},h_{1,\mathfrak{a}}) := \bigoplus_u \big(E'_{1,\mathfrak{a},u},\overline{\partial}'_{1,\mathfrak{a},u},\theta'_{1,\mathfrak{a},u},h'_{1,\mathfrak{a},u}\big)\otimes L(u,\mathfrak{a}),$$
$$(E_1,\overline{\partial}_1,\theta_1,h_1) := \bigoplus_{\mathfrak{a}} (E_{1,\mathfrak{a}},\overline{\partial}_{1,\mathfrak{a}},\theta_{1,\mathfrak{a}},h_{1,\mathfrak{a}}).$$

Let $(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}_1,\mathbb{D}_1) = \bigoplus\big(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}_{1,\mathfrak{a}},\mathbb{D}_{1,\mathfrak{a}}\big)$ denote the associated family of filtered λ-flat bundles, equipped with the Hermitian metric $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1 = \bigoplus \mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_{1,\mathfrak{a}}$.

Taking the associated graded bundle of $(\mathcal{P}_*^{(\lambda_0)}\mathcal{E},\mathbb{D})$ with respect to the Stokes structure, we obtain a family of filtered λ-flat bundles $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}\big(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}\big)$ with $\mathbb{D}_{\mathfrak{a}}$. Note that $\mathbb{D}_{\mathfrak{a}}-(1+\lambda\overline{\lambda}_0)\cdot d\mathfrak{a}$ and $\mathbb{D}_{1,\mathfrak{a}}-(1+\lambda\overline{\lambda}_0)\cdot d\mathfrak{a}$ are logarithmic. We can take a holomorphic isomorphism $\Phi_{\mathfrak{a}}:\mathcal{P}_*^{(\lambda_0)}\mathcal{E}_{1,\mathfrak{a}}\to\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(\mathcal{P}_*^{(\lambda_0)}\mathcal{E})$ with the following properties:

- It preserves the filtration $F^{(\lambda_0)}$,
- $\mathrm{Gr}^F(\Phi_{\mathfrak{a}|U(\lambda_0)\times D})$ is compatible with $\mathrm{Res}(\mathbb{D}_{\mathfrak{a}})$ and $\mathrm{Res}(\mathbb{D}_{1,\mathfrak{a}})$.

10.4.4. Comparison map around $\lambda_0\neq 0$. — Let us consider the case $\lambda_0\neq 0$. We take a finite covering $U(\lambda_0)\times(X\setminus D)=\bigcup S_j$ by small multi-sectors. We would like to construct a C^∞-map $\Phi:\mathcal{P}^{(\lambda_0)}\mathcal{E}_1\to\mathcal{P}^{(\lambda_0)}\mathcal{E}$ such that (i) $\Phi_{|S_j}$ are compatible with the full Stokes filtrations $\widetilde{\mathcal{F}}^{S_j}$ on S_j, (ii) $\mathrm{Gr}^{\widetilde{\mathcal{F}}^{S_j}}_{\mathfrak{a}}(\Phi_{|S_j})=\Phi_{\mathfrak{a}|S_j}$. We take a flat splitting $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}_j}=\bigoplus\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S_j}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^{S_j}$. Then, we obtain a holomorphic isomorphism $\Phi_{S_j}:\mathcal{P}^{(\lambda_0)}\mathcal{E}_{1|\overline{S}_j}\simeq\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}_j}$ preserving the decomposition. By gluing them in C^∞, we obtain the desired map.

Lemma 10.4.5. — *$\Phi_{|\lambda\times(X\setminus D)}$ is bounded with respect to h and h_1 for each λ.*

Proof. — $\Phi_{|\lambda\times(X\setminus D)}$ can be extended to a C^∞-isomorphism $\mathcal{P}\mathcal{E}_1^\lambda\to\mathcal{P}\mathcal{E}^\lambda$ such that (i) it preserves the parabolic structure, (ii) $\mathrm{Gr}^F(\Phi_{|\{\lambda\}\times D})$ is compatible with the residues. Then, the claim of the lemma follows from the norm estimate (Proposition 10.3.2). □

On a small sector $S=S_j$, we have the decomposition:

$$\mathrm{Hom}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{1|\overline{S}},\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}\big)=\bigoplus_{\mathfrak{a},\mathfrak{b}}\mathrm{Hom}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{1,\mathfrak{a},S},\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{b},S}\big). \tag{214}$$

We give a general lemma.

Lemma 10.4.6. — *Let s be a section of* $\mathrm{Hom}\big(\mathcal{P}^{(\lambda_0)}\mathcal{E}_{1,\mathfrak{a},S},\mathcal{P}^{(\lambda_0)}\mathcal{E}_{\mathfrak{b},S}\big)$ *with the following estimate with respect to $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$:*

$$s=\begin{cases}O\big(\exp(-\varepsilon_1|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})\big) & (\mathfrak{a}\neq\mathfrak{b})\\ O\big(|z_1|^{c}\big) & (\mathfrak{a}=\mathfrak{b})\end{cases}$$

Here, $c\in\boldsymbol{R}$ and $\varepsilon_1>0$. If we shrink $U(\lambda_0)$, we also have the following estimate with respect to h and h_1 for some $\varepsilon_2>0$:

$$s=\begin{cases}O\big(\exp(-\varepsilon_2|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})\big) & (\mathfrak{a}\neq\mathfrak{b})\\ O\big(|z_1|^{c}\big) & (\mathfrak{a}=\mathfrak{b})\end{cases}\tag{215}$$

Proof. — Let F_S be as in Section 10.2.2. By the assumption, the estimate for s like (215) holds with respect to the metrics $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$. We may assume to have $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1=h_1$. By Corollary 10.2.6, the metrics $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and h are mutually bounded. Hence, we obtain Lemma 10.4.6. □

We have the decomposition $\Phi=\sum\Phi_{\mathfrak{a},\mathfrak{b},S}$ corresponding to (214).

Lemma 10.4.7. — *If $\mathfrak{a}\neq\mathfrak{b}$, we have the following estimate with respect to h, h_1 and the Euclid metric g_X of X, uniformly for λ:*

$$\Phi_{\mathfrak{a},\mathfrak{b},S}=O\Big(\exp\big(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)\tag{216}$$

$$\mathbb{D}(\Phi)_{\mathfrak{a},\mathfrak{b},S}=O\Big(\exp\big(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)\tag{217}$$

$$\mathbb{D}^{1,0}d''\Phi_{\mathfrak{a},\mathfrak{b},S}=O\Big(\exp\big(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)\tag{218}$$

$$d''\Phi_{\mathfrak{a},\mathfrak{b},S}=O\Big(\exp\big(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)\tag{219}$$

Here, $d''=\overline{\partial}_E+\lambda\theta^\dagger$ denotes the holomorphic structure of $\mathcal{E}$ along the X-direction, and $\mathbb{D}^{1,0}$ denotes the $(1,0)$-part of $\mathbb{D}$, i.e., $\mathbb{D}^{1,0}:=\lambda\partial_E+\theta$.

Proof. — It is clear that $\Phi_{\mathfrak{a},\mathfrak{b}}=O\big(\exp(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})\big)$ for h_0 and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$. Since h_0 and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ are uniformly mutually bounded up to small polynomial order, we obtain $\Phi_{\mathfrak{a},\mathfrak{b}}=O\big(\exp(-\varepsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})\big)$ for $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$. Then, we obtain (216) due to Lemma 10.4.6. The others can be shown similarly. □

We have $d''\Phi_{\mathfrak{a},\mathfrak{a},S}=0$ by construction. Hence, we obtain the following uniform estimate with respect to h, h_1 and g_X from (219):

$$d''\Phi=O\big(\exp(-\varepsilon|z_1|^{-1})\big).$$

Then, we have the following uniform estimates with respect to h, h_1 and g_X:

$$(\theta \circ d''\Phi - d''\Phi \circ \theta_1) = O\big(\exp(-\varepsilon|z_1|^{-1})\big). \tag{220}$$

Let $\delta'_\lambda := \partial_E - \overline{\lambda}\theta$ and $\delta'_{1,\lambda} := \partial_{E_1} - \overline{\lambda}\theta_1$. We have the relation $\mathbb{D}^{(1,0)} = \lambda\delta'_\lambda + (1+|\lambda|^2)\theta$. and $\mathbb{D}_1^{(1,0)} = \lambda\delta'_{1,\lambda} + (1+|\lambda|^2)\theta_1$. From (218) and (220), we obtain the following uniform estimate with respect to h, h_1 and g_X:

$$(\delta'_\lambda \circ d''\Phi - d''\Phi \circ \delta'_{1,\lambda}) = O\Big(\exp(-\varepsilon|z_1|^{-1})\Big). \tag{221}$$

We prepare other estimates.

Lemma 10.4.8. — *Φ is uniformly bounded up to small polynomial order with respect to h and h_1.*

Proof. — Φ is uniformly bounded with respect to h_0 and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h_1$. Hence, it is uniformly bounded up to small polynomial order with respect to $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h_1$ (Lemma 10.4.4). Since the components $\Phi_{\mathfrak{a},\mathfrak{b}}$ ($\mathfrak{a} \neq \mathfrak{b}$) are estimated as in (216), we obtain Lemma 10.4.8 by using Lemma 10.4.6. □

Lemma 10.4.9. — *We have the uniform estimate $\mathbb{D}\Phi_{\mathfrak{a},\mathfrak{a},S} = O\big(|z_1|^{-1+\varepsilon}\big)$ for some $\varepsilon > 0$, with respect to h, h_1 and g_X.*

Proof. — By construction, we have the uniform estimate $\mathbb{D}\Phi_{\mathfrak{a},\mathfrak{a},S} = O\big(|z_1|^{-1+\varepsilon_1}\big)$ for some $\varepsilon_1 > 0$, with respect to h_0, $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h_1$ and g_X. Since h_0 and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$ are uniformly mutually bounded up to $|z_1|^{-\eta|\lambda-\lambda_0|}(-\log|z_1|)^N$-order (Lemma 10.4.4), we obtain the uniform estimate $\mathbb{D}\Phi_{\mathfrak{a},\mathfrak{a},S} = O\big(|z_1|^{-1+\varepsilon_2}\big)$ for some $\varepsilon_2 > 0$ with respect to $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$, $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h_1$ and g_X. Then, we obtain the claim of Lemma 10.4.9 by using Lemma 10.4.6. □

From Lemma 10.4.9 and (217), we obtain the following estimates with respect to h, h_1 and g_X uniformly for λ:

$$\mathbb{D}\Phi = O\big(|z_1|^{-1+\varepsilon}\big). \tag{222}$$

10.4.5. Comparison map around $\lambda_0 = 0$. — We almost repeat the construction in Section 10.4.4. We take a finite covering $(U(0) \setminus \{0\}) \times (X \setminus D) = \bigcup S_j$ by small multi-sectors. We would like to construct a C^∞-map $\Phi : \mathcal{P}^{(0)}\mathcal{E}_1 \to \mathcal{P}^{(0)}\mathcal{E}$ such that (i) $\Phi_{|S_j}$ are compatible with the full Stokes filtrations $\widetilde{\mathcal{F}}^{S_j}$ on S_j, (ii) $\mathrm{Gr}^{\widetilde{\mathcal{F}}^{S_j}}_{\mathfrak{a}}(\Phi_{|S_j}) = \Phi_{\mathfrak{a}|S_j}$. We take a flat splitting $\mathcal{P}^{(0)}\mathcal{E}_{|\overline{S}_j} = \bigoplus \mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{a},S_j}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^{S_j}$. Then, we obtain the isomorphism $\Phi_{S_j} : \mathcal{P}^{(0)}\mathcal{E}_{1|\overline{S}_j} = \mathcal{P}^{(0)}\mathcal{E}_{|\overline{S}_j}$ preserving the decomposition. By gluing them in C^∞, we obtain the desired map. The following lemma can be shown by the same argument as in the proof of Lemma 10.4.5.

Lemma 10.4.10. — *$\Phi_{|\lambda\times(X\setminus D)}$ is bounded with respect to h and h_1 for each λ.* □

On a small sector $S = S_j$, we have the decomposition:

$$\operatorname{Hom}\bigl(\mathcal{P}^{(0)}\mathcal{E}_{1|\overline{S}}, \mathcal{P}^{(0)}\mathcal{E}_{|\overline{S}}\bigr) = \bigoplus_{\mathfrak{a},\mathfrak{b}} \operatorname{Hom}\bigl(\mathcal{P}^{(0)}\mathcal{E}_{1,\mathfrak{a},S}, \mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{b},S}\bigr). \tag{223}$$

Lemma 10.4.11. — *Let s be a section of $\operatorname{Hom}\bigl(\mathcal{P}^{(0)}\mathcal{E}_{1,\mathfrak{a},S}, \mathcal{P}^{(0)}\mathcal{E}_{\mathfrak{b},S}\bigr)$ with the following estimate with respect to $\mathcal{P}^{(0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(0)}_{\mathrm{irr}}h_1$:*

$$s = \begin{cases} O\bigl(\exp(-\varepsilon_1 \cdot |\lambda|^{-1} \cdot |z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})\bigr) & (\mathfrak{a} \neq \mathfrak{b}) \\ O\bigl(|z_1|^c\bigr) & (\mathfrak{a} = \mathfrak{b}) \end{cases}$$

Here, $c \in \boldsymbol{R}$ and $\varepsilon_1 > 0$. If we shrink $U(0)$, we also have the following estimate with respect to h and h_1 for some $\varepsilon_2 > 0$:

$$s = \begin{cases} O\bigl(\exp(-\varepsilon_2 \cdot |\lambda|^{-1} \cdot |z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})\bigr) & (\mathfrak{a} \neq \mathfrak{b}) \\ O\bigl(|z_1|^c\bigr) & (\mathfrak{a} = \mathfrak{b}) \end{cases}$$

Proof. — It can be shown using the same argument as that in the proof of Lemma 10.4.6. □

We have the decomposition $\Phi = \sum \Phi_{\mathfrak{a},\mathfrak{b},S}$ corresponding to (223).

Lemma 10.4.12. — *If $\mathfrak{a} \neq \mathfrak{b}$, we have the following estimate with respect to h and h_1, uniformly for λ:*

$$\Phi_{\mathfrak{a},\mathfrak{b},S} = O\Bigl(\exp\bigl(-\varepsilon|\lambda|^{-1}|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\Bigr) \tag{224}$$

$$\mathbb{D}(\Phi)_{\mathfrak{a},\mathfrak{b},S} = O\Bigl(\exp\bigl(-\varepsilon|\lambda|^{-1}|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\Bigr) \tag{225}$$

$$\mathbb{D}^{1,0}d''\Phi_{\mathfrak{a},\mathfrak{b},S} = O\Bigl(\exp\bigl(-\varepsilon|\lambda|^{-1}|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\Bigr) \tag{226}$$

$$d''\Phi_{\mathfrak{a},\mathfrak{b},S} = O\Bigl(\exp\bigl(-\varepsilon|\lambda|^{-1}|z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\Bigr) \tag{227}$$

Proof. — We have $\Phi_{\mathfrak{a},\mathfrak{b},S} = O\bigl(\exp(-\varepsilon|\lambda|^{-1} \cdot |z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})\bigr)$ with respect to h_0 and $\mathcal{P}^{(0)}_{\mathrm{irr}}h_1$. Since $\mathcal{P}^{(0)}_{\mathrm{irr}}h$ and h_0 are uniformly mutually bounded up to small polynomial order, we obtain $\Phi_{\mathfrak{a},\mathfrak{b},S} = O\bigl(\exp(-\varepsilon|\lambda|^{-1} \cdot |z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})\bigr)$ with respect to $\mathcal{P}^{(0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(0)}_{\mathrm{irr}}h_1$. Then, we obtain (224) with respect to h and h_1 by Lemma 10.4.11. The others can be shown similarly. □

By construction, we have $d''\Phi_{\mathfrak{a},\mathfrak{a}} = 0$. Hence, we obtain the following estimate with respect to h and h_1, from (227):

$$d''\Phi = O\bigl(\exp(-\varepsilon|\lambda|^{-1}|z_1|^{-1})\bigr).$$

Then, we obtain the following estimate with respect to h, h_1 and g_X:

$$\bigl(\theta \circ d''\Phi - d''\Phi \circ \theta_1\bigr) = O\Bigl(\exp\bigl(-\varepsilon|\lambda|^{-1}|z_1|^{-1}\bigr)\Bigr). \tag{228}$$

Thus, we obtain the following estimate from (226) and (228):

$$(229)\qquad \big(\delta'_{\lambda}\circ d''\Phi - d''\Phi\circ\delta'_{1,\lambda}\big) = O\Big(\exp\big(-\varepsilon|\lambda|^{-1}|z_1|^{-1}\big)\Big).$$

We have the counterparts of Lemma 10.4.8 and Lemma 10.4.9, which can be shown in similar ways.

Lemma 10.4.13. — *Φ is uniformly bounded up to small polynomial order, with respect to h, h_1 and g_X.* □

Lemma 10.4.14. — *We have the estimate $\mathbb{D}\Phi_{\mathfrak{a},\mathfrak{a},S} = O(|z_1|^{-1+\varepsilon})$ for some $\varepsilon > 0$ with respect to h, h_1 and g_X.* □

From Lemma 10.4.14 and (225), we obtain the following estimate with respect to h, h_1 and g_X:

$$(230)\qquad \mathbb{D}\Phi = O\big(|z_1|^{-1+\varepsilon}\big).$$

10.4.6. End of proof of Proposition 10.4.2. — By construction, it is easy to see that $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$ and h_0 are mutually bounded. Hence, we only have to show that $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$ are mutually bounded uniformly for λ.

Lemma 10.4.15. — *Once we show that h and h_1 are mutually bounded, we obtain $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$ are mutually bounded.*

Proof. — Let S and F_S be as in Section 10.2.2. If h and h_1 are mutually bounded, $F_S(\overline{\lambda}-\overline{\lambda}_0)^*h$ and $F_S(\overline{\lambda}-\overline{\lambda}_0)^*h_1$ are mutually bounded. We may assume to have $F_S(\overline{\lambda}-\overline{\lambda}_0)^*h_1 = \mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h_1$. By Corollary 10.2.6, $F_S(\overline{\lambda}-\overline{\lambda}_0)^*h$ and $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ are mutually bounded. Thus, Lemma 10.4.15 is proved. □

Let us show that h and h_1 are mutually bounded. We only have to show that Φ and Φ^{-1} are bounded uniformly for λ with respect to h and h_1.

Let $\Phi_\lambda := \Phi_{|\{\lambda\}\times(X\setminus D)}$. We regard it as a C^∞-section of the harmonic bundle $Hom(\mathcal{E}^\lambda_1,\mathcal{E}^\lambda)$ with the induced λ-connection $\mathbb{D}^\lambda_2$ and the pluri-harmonic metric h_2. Let $\pi : X \to D$ be the projection. Let Q be any point of D. We put $\pi^{-1}(Q)^* := \pi^{-1}(Q)\setminus\{Q\}$. Applying Lemma 7.7.4, we obtain the following inequality on each $\pi^{-1}(Q)^*$:

$$(231)\qquad -\frac{\partial^2}{\partial z_1\partial\overline{z}_1}\big|\Phi_\lambda\big|^2_{h_2} \leqslant \big|\mathbb{D}^\lambda_2\Phi_\lambda\big|^2_{h_2} + 2\big|\Phi_\lambda\big|_{h_2}\cdot\big|\delta'_{2,\lambda}\circ d''_{2,\lambda}(\Phi_\lambda)\big|_{h_2}.$$

From (231), (221), (222), (229), (230), Lemma 10.4.8 and Lemma 10.4.13, we obtain the inequality on $\pi^{-1}(Q)^*$

$$-\frac{\partial^2}{\partial z_1\partial\overline{z}_1}|\Phi_\lambda|^2 \leqslant C|z_1|^{-2+\varepsilon},$$

where the constant C is independent of λ and Q. Since we already know the boundedness of $|\Phi_\lambda|^2_{h_2}$ on $\pi^{-1}(Q)^*$ (Lemma 10.4.5), the inequality holds on $\pi^{-1}(Q)$ as distributions. Hence, the values $\left|\Phi\right|_{h_2}$ on $\pi^{-1}(Q)^*$ is dominated by the values on the boundary $\partial\pi^{-1}(Q)$. Thus, we obtain the uniform boundedness of Φ. Similarly, we obtain the uniform boundedness for Φ^{-1}. Thus, we obtain that h and h_1 are mutually bounded, and the proof of Proposition 10.4.2 is finished. □

CHAPTER 11

PROLONGATION AND REDUCTION OF VARIATIONS OF POLARIZED PURE TWISTOR STRUCTURES

In this chapter, we study the reduction with respect to the Stokes structure of unramifiedly good wild harmonic bundles (Theorem 11.2.2). This is one of the main results in the study of wild harmonic bundles. Recall that we obtain a polarized mixed twistor structure from a tame harmonic bundle by taking Gr with respect to KMS-structure, which is one of the most important achievements in the study of tame harmonic bundles. We also have the reduction from polarized mixed twistor structure to polarized mixed Hodge structure by taking Gr with respect to the weight filtration. (See [**67**] for these reductions. See also a survey in [**63**].) By these reductions, the study of the asymptotic behaviour of wild harmonic bundles is reduced to that of variation of Hodge structures.

In Section 11.1, we define the canonical prolongment of a variation of pure twistor structures whose underlying harmonic bundle is good wild. In Section 11.1.1, we first construct a family of meromorphic λ-flat bundles $(\mathcal{QE}, \mathbb{D}^{\lambda})$ on $\boldsymbol{C}_{\lambda} \times (X, D)$ associated to a good wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on (X, D). This family of meromorphic flat bundles will be also useful in Chapter 12. This subsection is a continuation of Section 9.2. Applying the construction to the harmonic bundle $(E, \partial_E, \theta^{\dagger}, h)$ on the conjugate complex manifolds $(X^{\dagger}, D^{\dagger})$, we obtain the prolongment $(\mathcal{QE}^{\dagger}, \mathbb{D}^{\dagger})$ on $\boldsymbol{C}_{\mu} \times (X^{\dagger}, D^{\dagger})$ in Section 11.1.2. Then, we show in Section 11.1.3 that $(\mathcal{QE}, \mathcal{QE}^{\dagger})$ gives a meromorphic prolongment (Section 6.2) of the variation of polarized pure twistor structure $(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$ associated to $(E, \overline{\partial}_E, \theta, h)$, which is called the canonical prolongment.

In Section 11.2, we give the statement of the reduction theorem (Theorem 11.2.2). Note that it also implies a characterization of the canonical prolongment $(\mathcal{QE}, \mathcal{QE}^{\dagger})$, which is non-trivial because of the absence of the uniqueness of an unramifiedly good meromorphic prolongment with irregular singularity. We explain a plan of the proof in Section 11.2.2.

In Section 11.3, we give a sufficient condition for a holomorphic vector bundle on $\mathbb{P}^1$ with a symmetric pairing to be a polarized pure twistor structure. Intuitively, the twistor property and the positive definiteness should be open conditions. We can expect that a holomorphic vector bundle with a symmetric pairing on $\mathbb{P}^1$ is a polarized pure twistor structure, if it is "close to" a polarized pure twistor structure. We give a condition to describe "close to", which is convenient for our purpose. This is one of the main tools in the proof.

We study the one step reduction in Section 11.4. Briefly speaking, we show that the one step reduction is a variation of polarized pure twistor structure if and only if the original one is so. The main tools are Lemma 11.3.6 and Lemma 9.3.1. However, we cannot conclude anything about prolongments in this stage, and this will be postponed until Section 11.6.1.

We study in Section 11.5 the full reduction in the case where D is smooth. As in Section 11.4, we show that the full reduction is a variation of polarized pure twistor structure if and only if the original one is so. Moreover, we can obtain a characterization of the canonical prolongment. The argument is essentially the same as that in Section 11.4, except that we use Proposition 10.2.1 instead of Lemma 9.3.1.

The proof of Theorem 11.2.2 is finished in Section 11.6.

In Section 11.7, we argue the norm estimate for holomorphic sections of $\mathcal{P}\mathcal{E}^{\lambda}$. Briefly speaking, our result says that the pluri-harmonic metric h is determined up to boundedness by the parabolic structure and the residues, as in the case of tame harmonic bundles. In Sections 11.7.1–11.7.2, we construct comparison maps between the reductions and the original one, and we show that they are bounded (Theorem 11.7.2). Since we can apply the norm estimate for a tame harmonic bundle to a wild harmonic bundle with a unique irregular value, Theorem 11.7.2 implies the norm estimate for unramifiedly good wild harmonic bundles. The norm estimate for good wild harmonic bundles can be easily reduced to the unramified case.

In Sections 11.7.3–11.7.4, we give a rather detailed description of the norm estimate in the surface case, which is essentially the same as those in Section 2.5 of [**66**]. The result will be used in Proposition 13.6.4 to show the vanishing of some characteristic numbers of the filtered bundle associated to a good wild harmonic bundle.

Section 11.8 is an appendix in which we consider the reduction with respect to KMS structure for a *regular* meromorphic variation of pure twistor structure with a pairing on $\mathbb{P}^1 \times \Delta^*$, and we shall give a characterization of purity and polarizability in terms of the limit twistor structure (Lemma 11.8.6). For that purpose, in Subsection 11.8.1–11.8.3, we shall review the construction of a limit vector bundle on $\mathbb{P}^1$ from a variation of twistor structure on a punctured disc with regular meromorphic extension, called the limit twistor structure. When it comes from a tame harmonic bundle, the construction is explained in [**67**]. We need only a minor change. Then, we show the characterization. Although a similar result was obtained in [**73**] for $\mathcal{R}$-triples with a different method, we would like to understand it from our viewpoint. Although we restrict ourselves to the one dimensional case, the arguments can be generalized to the higher dimensional case.

11.1. Canonical prolongation

11.1.1. Prolongment $\mathcal{QE}$. — Let X be a complex manifold with a simple normal crossing hypersurface $D = \bigcup_{i\in\Lambda} D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X \setminus D$. Let $\mathcal{X}^{(\lambda_0)}$ denote a neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_\lambda \times X$. We use the symbol $\mathcal{D}^{(\lambda_0)}$ in a similar meaning. We have the associated family of λ-flat bundles $(\mathcal{P}_*^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$. (See Section 9.1.) In the case $\lambda_0 \neq 0$, we have the following deformation in Section 4.5.2:

$$\mathcal{Q}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E} := \big(\mathcal{P}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}\big)^{(T(\lambda))}, \quad T(\lambda) = \frac{1}{1+\lambda\overline{\lambda}_0}.$$

In the case $\lambda_0 = 0$, we define $\mathcal{Q}_{\boldsymbol{a}}^{(0)}\mathcal{E} := \mathcal{P}_{\boldsymbol{a}}^{(0)}\mathcal{E}$. Thus, we obtain a global filtered bundle

$$\mathcal{Q}_*^{(\lambda_0)}\mathcal{E} = \big(\mathcal{Q}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E} \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^\Lambda\big)$$

on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$ with the family of meromorphic flat λ-connections $\mathbb{D}$. We put $\mathcal{Q}^{(\lambda_0)}\mathcal{E} := \bigcup_{\boldsymbol{a}\in\boldsymbol{R}^\Lambda} \mathcal{Q}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}$, which is an $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}(*\mathcal{D}^{(\lambda_0)})$-locally free sheaf.

Lemma 11.1.1. — *Let P be any point of X. Let X_P, $U_P(\lambda_0)$ and $U_P(\lambda_1)$ be as in Section* 9.2.3. *Then, we have a natural isomorphism $\mathcal{Q}^{(\lambda_0)}\mathcal{E}_{|U_P(\lambda_1)\times X_P} \simeq \mathcal{Q}^{(\lambda_1)}\mathcal{E}_{|U_P(\lambda_1)\times X_P}$. We mean by "natural" that the restriction to $X \setminus D$ is the identity.*

Proof. — It follows from Proposition 9.2.8 and Lemma 4.5.5. □

Thus, we obtain a family of meromorphic λ-flat bundles $(\mathcal{QE}, \mathbb{D})$ on $\boldsymbol{C}_\lambda \times (X, D)$. The restriction of $(\mathcal{QE}, \mathbb{D})$ to $\mathcal{X}^\lambda$ is denoted by $(\mathcal{QE}^\lambda, \mathbb{D}^\lambda)$. The following theorem is clear from the construction.

Theorem 11.1.2

- *We have a natural isomorphism of the meromorphic λ-flat bundles $(\mathcal{QE}^\lambda, \mathbb{D}^\lambda) \simeq (\mathcal{PE}^\lambda, \mathbb{D}^\lambda)^{(T)}$ where $T := (1+|\lambda|^2)^{-1}$. We mean by "natural" that the restriction to $X \setminus D$ is the identity. Moreover, we have an isomorphism of good filtered λ-flat bundles $(\mathcal{Q}_*^{(\lambda)}\mathcal{E}^\lambda, \mathbb{D}^\lambda) \simeq (\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)^{(T)}$.*
- *Under the setting in Section* 7.2.1, *the set of the irregular values* $\mathrm{Irr}(\mathcal{Q}_{\boldsymbol{a}}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ *is* $\mathrm{Irr}(\theta)$.
- *$(\mathcal{Q}_*^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ is a good family of filtered λ-flat bundles with the KMS-structure at λ_0. If θ is unramified, $(\mathcal{Q}_*^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ is also unramified.*
- *Under the setting of Section* 7.2.1, *the set of the irregular values* $\mathrm{Irr}(\mathcal{Q}_{\boldsymbol{a}}^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ *is* $\mathrm{Irr}(\theta)$. □

Let us look at the Stokes filtrations under the setting in Section 7.2.1. We take an auxiliary sequence for $\mathrm{Irr}(\theta)$. Let S be a small multi-sector in $X \setminus D$. If $\lambda \neq 0$, we have the partial Stokes filtration $\mathcal{F}^{S,\boldsymbol{m}(i)}(\mathcal{QE}^\lambda_{|\overline{S}})$ of $\mathcal{QE}^\lambda_{|\overline{S}}$ at the level $\boldsymbol{m}(i)$. (See Section 3.7.3.)

Proposition 11.1.3. — *Let f be a flat section of $\mathcal{E}^{\lambda}_{|S}$. We have $f \in \mathcal{F}^{S,\boldsymbol{m}(i)}_{\mathfrak{b}}(\mathcal{QE}^{\lambda})_{|S}$, if and only if the following estimate holds for some $C>0$ and $N>0$:*

$$\big|f\,\exp\big((\lambda^{-1}+\overline{\lambda})\,\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{b})\big)\big|_h = O\Big(\exp\big(C|\boldsymbol{z}^{\boldsymbol{m}(i+1)}|\big)\prod_{k(i+1)<j\leqslant \ell}|z_j|^{-N}\Big).$$

Proof. — It follows from Proposition 7.4.6 and Theorem 11.1.2. □

11.1.2. Prolongments $\mathcal{PE}^{\dagger\mu}$, $\mathcal{P}^{(\mu_0)}_{*}\mathcal{E}^{\dagger}$ and $\mathcal{QE}^{\dagger}$. — For simplicity, we restrict ourselves to the setting in Section 7.2.1. We obtain an unramifiedly good wild harmonic bundle $(E,\partial_E,\theta^{\dagger},h)$ on $X^{\dagger}\setminus D^{\dagger}$. We have the family of μ-flat bundles $(\mathcal{E}^{\dagger},\mathbb{D}^{\dagger})$ on $\boldsymbol{C}_{\mu}\times(X^{\dagger}\setminus D^{\dagger})$, and the restriction $(\mathcal{E}^{\dagger\mu},\mathbb{D}^{\dagger\mu})$ to $\{\mu\}\times(X^{\dagger}\setminus D^{\dagger})$. As in Section 7.4.1, we obtain an unramifiedly good filtered μ-flat bundle $\big(\mathcal{P}_*\mathcal{E}^{\dagger\mu},\mathbb{D}^{\dagger\mu}\big)$ on $(X^{\dagger},D^{\dagger})$ for each μ. The set of the irregular values of $\mathbb{D}^{\dagger\mu}$ is given as follows:

$$\operatorname{Irr}\big(\mathcal{PE}^{\dagger\mu},\mathbb{D}^{\dagger\mu}\big)=\big\{(1+|\mu|^2)\,\overline{\mathfrak{a}}\;\big|\;\mathfrak{a}\in\operatorname{Irr}(\theta)\big\}.$$

Let $U(\mu_0)$ denote a small neighbourhood of μ_0. Applying the construction in Chapter 9 to $(E,\partial_E,\theta^{\dagger},h)$, we obtain a family of filtered μ-flat bundles $(\mathcal{P}^{(\mu_0)}_{*}\mathcal{E}^{\dagger},\mathbb{D}^{\dagger})$ with the KMS-structure at μ_0. Applying the deformation as in Section 11.1.1, we obtain a family of filtered μ-flat bundles $(\mathcal{Q}^{(\mu_0)}_{*}\mathcal{E}^{\dagger},\mathbb{D}^{\dagger})$ on $U(\mu_0)\times(X^{\dagger},D^{\dagger})$ and $\mathcal{O}_{U(\mu_0)\times X^{\dagger}}\big(*(U(\mu_0)\times D^{\dagger})\big)$-free module $\mathcal{Q}^{(\mu_0)}\mathcal{E}^{\dagger}$. By gluing them, we obtain the family of meromorphic μ-flat bundles $(\mathcal{QE}^{\dagger},\mathbb{D}^{\dagger})$ on $\boldsymbol{C}_{\mu}\times(X^{\dagger},D^{\dagger})$. The restriction of $\mathcal{QE}^{\dagger}$ to $\{\mu\}\times X$ is denoted by $\mathcal{QE}^{\dagger\mu}$.

We have a characterization of the Stokes filtration of the meromorphic μ-flat bundle $\big(\mathcal{QE}^{\dagger\mu},\mathbb{D}^{\dagger\mu}\big)$ in the case $\mu\neq 0$ in terms of the growth order with respect to h. Let $S^{\dagger}$ be a small multi-sector in $X^{\dagger}\setminus D^{\dagger}$. We have the partial Stokes filtration $\mathcal{F}^{S^{\dagger},\boldsymbol{m}(i)}(\mathcal{QE}^{\dagger\mu}_{|\overline{S}^{\dagger}})$ of $\mathcal{QE}^{\dagger\mu}_{|\overline{S}^{\dagger}}$ at the level $\boldsymbol{m}(i)$ indexed by the ordered set $\big(\operatorname{Irr}(\theta^{\dagger},\boldsymbol{m}(i)),\leqslant^{\mu}_{S^{\dagger}}\big)$.

Proposition 11.1.4. — *Let f be a flat section of $\mathcal{E}^{\dagger\mu}_{|S^{\dagger}}$. We have $f\in\mathcal{F}^{S^{\dagger},\boldsymbol{m}(i)}_{\overline{\mathfrak{b}}}$, if and only if the following estimate holds for some $C>0$ and $N>0$:*

$$\Big|f\,\exp\big((\mu^{-1}+\overline{\mu})\,\overline{\eta}_{\boldsymbol{m}(i)}(\overline{\mathfrak{b}})\big)\Big|_h = O\Big(\exp\big(C|\boldsymbol{z}^{\boldsymbol{m}(i+1)}|\big)\prod_{k(i+1)<j\leqslant \ell}|z_j|^{-N}\Big).\qquad\square$$

11.1.3. Canonical prolongation of a variation of polarized pure twistor structures. — We continue to use the setting in Section 7.2.1. We have the variation of polarized pure twistor structures $(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S})$ of weight 0 associated to $(E,\overline{\partial}_E,\theta,h)$. (See [**85**]. See also [**67**].) In particular, it gives a variation of twistor structure with a symmetric pairing.

Proposition 11.1.5. — *The pair $(\mathcal{QE},\mathcal{QE}^{\dagger})$ is an unramifiedly good meromorphic prolongment of $\big(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S}\big)$ (Section 6.2).*

Proof. — Let $\lambda = \mu^{-1}$. Let us show that the Stokes filtrations $\mathcal{F}^{S,\boldsymbol{m}(i)}$ of $(\mathcal{QE}^{\lambda}, \mathbb{D}^{\lambda,f})$ and $\mathcal{F}^{\dagger\, S,\boldsymbol{m}(i)}$ of $(\mathcal{QE}^{\dagger\,\mu}, \mathbb{D}^{\dagger\,\mu\, f})$ are the same. Note

$$\mathrm{Re}\big((\lambda^{-1}+\overline{\lambda})\,\mathfrak{b}\big) = \mathrm{Re}\big((\mu^{-1}+\overline{\mu})\,\overline{\mathfrak{b}}\big)$$

for $\mathfrak{b} \in \overline{\mathrm{Irr}}(\theta, \boldsymbol{m}(i))$ and $\lambda = \mu^{-1}$. Since both the flat filtrations $\mathcal{F}^{S,\boldsymbol{m}(i)}$ and $\mathcal{F}^{\dagger\, S,\boldsymbol{m}(i)}$ are characterized by the same condition on the growth order of the norms of flat sections with respect to h (Proposition 11.1.3 and Proposition 11.1.4), they are the same.

Let us show that the pairing $\mathcal{S}_0 : \mathcal{E} \otimes \sigma^*\mathcal{E}^{\dagger} \to \mathcal{O}_{\mathcal{X}\smallsetminus\mathcal{D}}$ can be extended to $\mathcal{QS}_0$: $\mathcal{QE} \otimes \sigma^*\mathcal{QE}^{\dagger} \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. Take $\lambda_0 \in \boldsymbol{C}_{\lambda}$, and let $U(\lambda_0)$ denote a small neighbourhood of λ_0. Let $\mathcal{X}^{(\lambda_0)} := U(\lambda_0) \times X$ and $\mathcal{D}^{(\lambda_0)} := U(\lambda_0) \times D$.

Lemma 11.1.6. — *$\mathcal{S}_0$ is prolonged to a pairing*

$$\mathcal{P}^{(\lambda_0)}\mathcal{S}_0 : \mathcal{P}^{(\lambda_0)}\mathcal{E} \otimes \sigma^*\mathcal{P}^{(\sigma(\lambda_0))}\mathcal{E}^{\dagger} \longrightarrow \mathcal{O}_{\mathcal{X}^{(\lambda_0)}}(*\mathcal{D}^{(\lambda_0)}).$$

Proof. — Let f_1 be a section of $\mathcal{P}^{(\lambda_0)}(\mathcal{E})$ on $\mathcal{X}^{(\lambda_0)}$. Let $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$ be the Hermitian metric given in (182). By construction, we have the estimates $|f_1|_{\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h} = O\Big(\prod_{j=1}^{\ell}|z_j|^{-N}\Big)$ for some N. Let f_2 be a section of $\mathcal{P}^{(\sigma(\lambda_0))}\mathcal{E}$ on $\sigma\big(\mathcal{X}^{(\lambda_0)}\big)$. Let $\mathcal{P}^{\dagger(\mu_0)}_{\mathrm{irr}}h$ of $p_{\mu}^{-1}E$ on $U(\mu_0) \times (X \smallsetminus D)$ given as follows:

$$\mathcal{P}^{\dagger(\mu_0)}_{\mathrm{irr}}h(u,v) = h\Big(g^{\dagger}_{\mathrm{irr}}(\mu-\mu_0)u, g^{\dagger}_{\mathrm{irr}}(\mu-\mu_0)v\Big), \quad g^{\dagger}_{\mathrm{irr}}(\mu) := \exp\Big(\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mu\,\mathfrak{a}\,\pi^{\dagger}_{\mathfrak{a}}\Big).$$

We have the estimate $|f_2|_{\mathcal{P}^{\dagger(\sigma(\lambda_0))}_{\mathrm{irr}}h} = O\Big(\prod_{j=1}^{\ell}|z_j|^{-N}\Big)$ for some N. Recall $\sigma(\lambda) = -\overline{\lambda}$, if we regard σ as a morphism $\boldsymbol{C}_{\lambda} \to \boldsymbol{C}_{\mu}$. (See Subsection 6.2.3.) Then, we have

$$g^{\dagger}_{\mathrm{irr}}(\sigma(\lambda)-\sigma(\lambda_0)) = \big(g_{\mathrm{irr}}(\lambda-\lambda_0)^{-1}\big)^{\dagger}.$$

We obtain the following:

$$\mathcal{S}_0(f_1, \sigma^* f_2) = h\big(f_1, \sigma^* f_2\big) = h\Big(g_{\mathrm{irr}}(\lambda-\lambda_0)f_1, \sigma^*\big(g^{\dagger}_{\mathrm{irr}}(\mu-\mu_0)f_2\big)\Big) = O\Big(\prod_{j=1}^{\ell}|z_j|^{-2N}\Big).$$

Hence, we obtain Lemma 11.1.6. □

By the functoriality of the deformation in Lemma 4.5.6, we obtain a pairing

$$\mathcal{Q}^{(\lambda_0)}\mathcal{S}_0 : \mathcal{Q}^{(\lambda_0)}\mathcal{E} \otimes \sigma^*\mathcal{Q}^{(\sigma(\lambda_0))}\mathcal{E}^{\dagger} \longrightarrow \mathcal{O}_{\mathcal{X}^{(\lambda_0)}}\big(*\mathcal{D}(\lambda_0)\big)$$

for $\lambda_0 \neq 0$, which is a prolongment of $\mathcal{S}_0$. Note that such a prolongment is unique if it exists. Hence, we have $\mathcal{Q}^{(\lambda_0)}\mathcal{S}_{0|\mathcal{X}^{(\lambda_1)}} = \mathcal{Q}^{(\lambda_1)}\mathcal{S}_0$ if $\mathcal{X}^{(\lambda_1)} \subset \mathcal{X}^{(\lambda_0)}$. By varying λ_0 and gluing them, we obtain the desired prolongment $\mathcal{QS}_0$ of $\mathcal{S}_0$. Thus we obtain Proposition 11.1.5. □

Definition 11.1.7. — The prolongment given by $\big(\mathcal{QE}, \mathcal{QE}^{\dagger}\big)$ is called the canonical prolongment of $(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$. □

11.2. Reduction and uniqueness

11.2.1. Statement. — We set $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. For any subset $\mathcal{K} \subset \boldsymbol{C}_\lambda$, we put $\mathcal{X}_{\mathcal{K}} := \mathcal{K} \times X$ and $\mathcal{X}^\dagger_{\sigma(\mathcal{K})} := \sigma(\mathcal{K}) \times X^\dagger$. For $0 < R < 1$, we put

$$X(R) := \big\{(z_1, \dots, z_n) \in X \,\big|\, |z_i| < R\big\}, \quad X^*(R) := (X \smallsetminus D) \cap X(R).$$

Let $(V, \mathbb{D}^\triangle)$ be a variation of twistor structure of weight 0 on $\mathbb{P}^1 \times (X \smallsetminus D)$ equipped with a symmetric pairing $\mathcal{S} : (V, \mathbb{D}^\triangle) \otimes \sigma^*(V, \mathbb{D}^\triangle) \to \mathbb{T}(0)$. Assume the following:

- We are given an unramifiedly good meromorphic prolongment $\widetilde{\boldsymbol{V}} := (\widetilde{V}_0, \widetilde{V}_\infty)$ of $(V, \mathbb{D}^\triangle, \mathcal{S})$ on $\big(\mathcal{X}_{\mathcal{K}}, \mathcal{X}^\dagger_{\sigma(\mathcal{K})}\big)$. (See Section 6.2.) Let T denote the good set of irregular values of the prolongment $(\widetilde{V}_0, \mathbb{D})$.
- $(\widetilde{V}_0, \mathbb{D})$ has the KMS-structure at each $\lambda_0 \in \mathcal{K}$, i.e., there exists an increasing sequence of lattices $\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}} \widetilde{V}_0 \subset \widetilde{V}_0$ on a neighbourhood of $\{\lambda_0\} \times X$ indexed by $\boldsymbol{a} \in \boldsymbol{R}^\ell$ such that $(\mathcal{Q}^{(\lambda_0)}_* \widetilde{V}_0, \mathbb{D})$ is a family of good filtered λ-flat bundles with KMS-structure.
- Similarly, $(\widetilde{V}_\infty, \mathbb{D})$ has the KMS-structure at each $\mu_0 \in \sigma(\mathcal{K})$. The corresponding family of filtered μ-flat bundle is denoted by $\mathcal{Q}^{(\mu_0)}_* \widetilde{V}_\infty$.

Recall that we have the reduction $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V, \mathbb{D}^\triangle, \mathcal{S})$ associated to the full Stokes structure. (See Subsection 6.2.5. Although it is denoted by $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(V, \mathbb{D}^\triangle, \mathcal{S})$ there, we use the symbol $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V, \mathbb{D}^\triangle, \mathcal{S})$ to emphasize the dependence on the choice of $\widetilde{\boldsymbol{V}}$.)

Remark 11.2.1. — If $(V, \mathbb{D}^\triangle, \mathcal{S})$ as above is a variation of polarized pure twistor structure of weight 0, the underlying harmonic bundle is unramifiedly good wild by the above assumption, because the Higgs field is given by the restriction of $\mathbb{D}^\triangle$ to $\{0\} \times (X \smallsetminus D)$. We will use this property implicitly in the subsequent argument. □

We will prove the following theorem in Sections 11.4–11.6.

Theorem 11.2.2. — *The following conditions are equivalent:*

(P1) : $(V, \mathbb{D}^\triangle, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$ *is a variation of polarized pure twistor structure of weight* 0 *for some* $0 < R < 1$, *and the prolongment* $\widetilde{\boldsymbol{V}}$ *is canonical (Definition* 11.1.7*).*

(P2) : *The full reduction* $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V, \mathbb{D}^\triangle, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$ *is a variation of polarized pure twistor structure of weight* 0 *for some* $0 < R < 1$.

We give a consequence. We take an auxiliary sequence $\boldsymbol{m}(0), \dots, \boldsymbol{m}(L)$ for the good set of irregular values T. Recall that we have the reduction $\mathrm{Gr}^{\boldsymbol{m}(p)}(V, \mathbb{D}^\triangle, \mathcal{S})$ associated to the partial Stokes structure at the level $\boldsymbol{m}(p)$.

Corollary 11.2.3. — *The following conditions are equivalent:*

- $(V, \mathbb{D}^\triangle, \mathcal{S})$ *is a variation of polarized pure twistor structure of weight* 0 *for some* $0 < R < 1$, *and the prolongment* $\widetilde{\boldsymbol{V}}$ *is canonical.*

- *The one step reduction* $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})_{|\mathbb{P}^1\times X^*(R)}$ *with respect to* $\widetilde{\boldsymbol{V}}$ *is a variation of polarized pure twistor structure of weight* 0 *for some* $0<R<1$*, and the induced prolongment* $\mathrm{Gr}^{\boldsymbol{m}(0)}\widetilde{\boldsymbol{V}} := (\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0),\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_\infty))$ *is canonical.*

Proof. — According to Theorem 11.2.2, both conditions are equivalent to the second condition in Theorem 11.2.2. □

11.2.2. Plan of the proof. — Before going into the proof, we give a rough sketch of the argument. In Section 11.4, we study the one step reduction, and we will prove the following proposition, which is a part of the claim of Corollary 11.2.3.

Proposition 11.2.4. — *The following holds:*

(A) : *Assume that* $(V,\mathbb{D}^{\triangle},\mathcal{S})$ *is a variation of polarized pure twistor structure of weight* 0*, and that* $(\widetilde{V}_0,\widetilde{V}_\infty)$ *is canonical. Then, there exists* $0<R<1$ *such that* $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})_{|\mathbb{P}^1\times X^*(R)}$ *is a variation of polarized pure twistor structure of weight* 0.

(B) : *If* $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})$ *is a variation of polarized pure twistor structure of weight* 0*, there exists* $0<R<1$ *such that* $(V,\mathbb{D}^{\triangle},\mathcal{S})_{|\mathbb{P}^1\times X^*(R)}$ *is a variation of polarized pure twistor structure of weight* 0.

We remark that it is not so easy to deduce a conclusion on prolongments. For example, in the claim (A), $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})$ comes from a harmonic bundle, say $(E^{(1)},\overline{\partial}^{(1)},\theta^{(1)},h^{(1)})$ which is easily shown to be unramifiedly good wild. In this stage, $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})$ has two prolongments. One is the induced prolongment $\big(\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0),\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_\infty)\big)$. The other one is the canonical prolongment $\big(\mathcal{QE}^{(1)},\mathcal{QE}^{(1)\dagger}\big)$ associated to $(E^{(1)},\overline{\partial}^{(1)},\theta^{(1)},h^{(1)})$. For an inductive argument, we would like to show that they are the same. We will show it eventually, but it does not seem easy to show it directly. Hence, we will postpone it.

In Section 11.5, we will study the full reduction in the case where D is smooth, and we will show the following proposition.

Proposition 11.2.5. — *If D is smooth, the claim of Theorem* 11.2.2 *holds.*

Note that the prolongment of the full reduction is easily controlled. As in the previous case, we have two prolongments. One is the induced prolongment, and the other is canonical. But, the full reduction is graded by the good set T, and each graded piece has a unique irregular value for both the prolongments. Hence, the two prolongments have to be the same by the uniqueness of the Deligne extension of flat bundles.

Once we have established Theorem 11.2.2 in the smooth divisor case, it is not difficult to compare the prolongments $\mathcal{QE}^{(1)}$ and $\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0)$ for the one step reduction in the normal crossing divisor case. Then, we obtain Theorem 11.2.2 by an inductive argument, which will be done in Section 11.6.

11.3. Preliminary for convergence

We will give preliminaries on the convergence of a sequence of twistor structures.

11.3.1. Preliminaries. — Let V be an n-dimensional vector space with basis $e_1, \ldots, e_n$. Let h be a Hermitian pairing of V. There exists a positive number ε_0, depending only on n, with the following property:

- If $\left|h(e_i, e_j) - \delta_{i,j}\right| < \varepsilon_0$, then h is positive definite. Here $\delta_{i,j}$ denotes 1 $(i = j)$ or 0 $(i \neq j)$.

Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be given by $\sigma\bigl([z_0 : z_1]\bigr) = [-\overline{z}_1 : \overline{z}_0]$.

Lemma 11.3.1. — *Let V be a pure twistor structure of weight* 0 *with a symmetric pairing $S : V \otimes \sigma^* V \to \mathbb{T}(0)$. Let λ be any point of $\mathbb{P}^1$. Assume the following :*

- *There exists a holomorphic frame $w_1, \ldots, w_n$ of V on $\mathbb{P}^1$ such that*

$$\left|S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - \delta_{i,j}\right| < \varepsilon_0.$$

Then, S is a polarization of V.

Proof. — Since any global section of $\mathbb{T}(0) = \mathscr{O}_{\mathbb{P}^1}$ is constant, we have

$$S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) = S(w_i, \sigma^* w_j).$$

Hence, the claim is clear by the choice of ε_0. □

11.3.2. Estimate on a bundle. — In this section, we use the standard Fubini-Study metric of $\mathbb{P}^1$. Let E be a holomorphic vector bundle on $\mathbb{P}^1$ with a holomorphic basis $\boldsymbol{v} = (v_1, \ldots, v_r)$, i.e., $E \simeq \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^1} v_i$. Let h be the Hermitian metric given by $h(v_i, v_j) = \delta_{i,j}$. Let $L_1^2(E)$ be the space of the sections f of E such that f and $\overline{\partial}_E f$ are L^2 on $\mathbb{P}^1$ with respect to h and the Fubini-Study metric. It is the L^2-space with the Hermitian product $(f, g)_{L_1^2} := \int h(f, g) + \int h(\overline{\partial}_E f, \overline{\partial}_E g)$. Let $L^2(E \otimes \Omega^{0,1})$ denote the space of L^2-sections of $E \otimes \Omega^{0,1}$. We have the surjectivity of $\overline{\partial}_E : L_1^2(E) \to L^2(E \otimes \Omega^{0,1})$. Let H denote the orthogonal complement of $\operatorname{Ker}(\overline{\partial}_E)$ in $L_1^2(E)$. Then, $\overline{\partial}_E : H \to L^2(E \otimes \Omega^{0,1})$ is homeomorphic. There exists a constant C_1 such that $C_1^{-1} \|u\|_{L^2} \leqslant \|\overline{\partial}_E u\|_{L^2} \leqslant C_1 \|u\|_{L^2}$ for $u \in H$.

Let A be a C^∞-section of $\operatorname{End}(E) \otimes \Omega^{0,1}$ such that $\sup |A|_h \leqslant \delta$. Let E_A denote the holomorphic bundle $(E, \overline{\partial}_E + A)$.

Lemma 11.3.2. — *If δ is sufficiently small, we have $2^{-1} \|\overline{\partial}_E u\|_{L^2} \leqslant \|(\overline{\partial}_E + A)u\|_{L^2} \leqslant 2\|\overline{\partial}_E u\|_{L^2}$ for any $u \in H$. Hence, $\overline{\partial}_E + A : H \to L^2(E \otimes \Omega^{0,1})$ is a homeomorphism. In particular, E_A is a pure twistor structure of weight* 0.

Proof. — If δ is sufficiently small, we have the following for any $u \in H$:

$$\|(\overline{\partial}_E + A)u\|_{L^2} \leqslant \|\overline{\partial}_E u\|_{L^2} + \|Au\|_{L^2} \leqslant 2\|\overline{\partial}_E u\|_{L^2},$$

$$\|(\overline{\partial}_E + A)u\|_{L^2} \geqslant \|\overline{\partial}_E u\|_{L^2} - \|Au\|_{L^2} \geqslant \frac{1}{2}\|\overline{\partial}_E u\|_{L^2}.$$

Then, the claim of the lemma immediately follows. □

Lemma 11.3.3. — *There exist constants $\delta > 0$ and $C_{10} > 0$ with the following property:*

- *If* $\sup |A|_h < \delta$, *there exists* $u_i \in H$ *such that* $(\overline{\partial}_E + A)u_i = (\overline{\partial}_E + A)v_i$ *and* $\sup |u_i|_h < C_{10} \sup |A|_h$. *Note that* $|v_i|_h = 1$.

Proof. — It follows from Lemma 11.3.2 and a standard bootstrapping argument. □

Let $S_h : E \otimes \sigma^* E \to \mathbb{T}(0)$ be the polarization of the pure twistor structure E corresponding to h. Let d_{Herm} denote the natural distance of the space of the Hermitian metrics of a vector space.

Lemma 11.3.4. — *Let $S : E_A \otimes \sigma^* E_A \to \mathbb{T}(0)$ be a symmetric pairing. Fix a point λ of $\mathbb{P}^1$, and assume moreover the following for some $\eta > 0$:*

- $\bigl|S_{|\lambda}(u, \sigma^* v) - S_{h|\lambda}(u, \sigma^* v)\bigr| < \eta\, |u|_h\, |v|_h$ *for any* $u \in E_{|\lambda}$ *and* $v \in E_{|\sigma(\lambda)}$.

If δ and η are sufficiently small, (E_A, S) is a polarized pure twistor structure of weight 0. Moreover, the following holds for the metric h_S corresponding to S:

$$\sup_{\mu \in \mathbb{P}^1} d_{\mathrm{Herm}}(h_{S|\mu}, h_{|\mu}) \leqslant C_{12}\, (\delta + \eta).$$

Here, C_{12} is a constant depending only on $\operatorname{rank} E$.

Proof. — We may assume that E_A is pure twistor structure of weight 0, due to Lemma 11.3.2. Let ε_0 be as in Section 11.3.1. If δ is sufficiently small, we can take sections $u_i \in H$ such that $(\overline{\partial}_E + A)u_i = (\overline{\partial}_E + A)v_i$ and $\sup |u_i|_h \leqslant C_{10}\, \delta < \varepsilon_0/100$, as remarked in Lemma 11.3.3. We put $w_i := v_i - u_i$. We may assume that $\boldsymbol{w} = (w_1 \ldots, w_n)$ gives a holomorphic frame of E_A. We have the following:

$$\begin{aligned}(232)\quad S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - \delta_{i,j} &= S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - S_{h|\lambda}(v_{i|\lambda}, \sigma^* v_{j|\sigma(\lambda)}) \\ &= \Bigl(S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - S_{h|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr)\Bigr) \\ &\qquad + \Bigl(S_{h|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - S_{h|\lambda}(v_{i|\lambda}, \sigma^* v_{j|\sigma(\lambda)})\Bigr).\end{aligned}$$

We have

$$\Bigl|S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - S_{h|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr)\Bigr| \leqslant 4\eta.$$

We also have

$$\Bigl|S_{h|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - S_{h|\lambda}\bigl(v_{i|\lambda}, \sigma^* v_{j|\sigma(\lambda)}\bigr)\Bigr| \leqslant 4C_{10}\, \delta.$$

Hence, we obtain $\Bigl|S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - \delta_{i,j}\Bigr| \leqslant \varepsilon_0$, and the first claim of Lemma 11.3.4 follows from Lemma 11.3.1.

Let us show the second claim. We have $\Bigl|S_{|\lambda}\bigl(w_{i|\lambda}, \sigma^* w_{j|\sigma(\lambda)}\bigr) - \delta_{i,j}\Bigr| \leqslant C_{20}\, (\delta + \eta)$ due to the above argument. Hence, there exists a constant matrix B_1 with the following properties:

- The frame $\widetilde{\boldsymbol{w}} = \boldsymbol{w}(I + B_1)$ is orthonormal with respect to h_S.
- $|B_1| < C_{21}\, (\delta + \eta)$.

Let B_0 be determined by $\boldsymbol{w} = \boldsymbol{v}\,(I + B_0)$. By construction, we have $\sup|B_0| \leqslant C_{20}\,\delta$. Therefore, we have $\sup|B_2| \leqslant C_{22}\,(\delta+\eta)$, where B_2 is determined by $\widetilde{\boldsymbol{w}} = \boldsymbol{v}\,(I+B_2)$. Since $\widetilde{\boldsymbol{w}}$ and $\boldsymbol{v}$ are orthonormal frames with respect to h_S and h respectively, we obtain the second claim of Lemma 11.3.4. □

11.3.3. Estimate through a C^∞-map. — Let $(V^{(1)}, S^{(1)})$ be a polarized pure twistor structure. Let $h^{(1)}$ be the corresponding metric of $H^0(\mathbb{P}^1, V^{(1)})$. It induces a Hermitian metric of $V^{(1)}$, which is also denoted by $h^{(1)}$.

Let $V^{(2)}$ be a holomorphic vector bundle on $\mathbb{P}^1$. Let $F : V^{(1)} \to V^{(2)}$ be a C^∞-isomorphism. Assume the following for some $\delta_1 > 0$:

$$\sup_{\mathbb{P}^1}\bigl|F^{-1}\circ\overline{\partial}_{V^{(2)}}\circ F - \overline{\partial}_{V^{(1)}}\bigr|_{h^{(1)}} \leqslant \delta_1.$$

Lemma 11.3.5. — *If δ_1 is sufficiently small, $V^{(2)}$ is also a pure twistor structure.*

Proof. — It follows from Lemma 11.3.2. □

Let $S^{(2)}$ be a symmetric pairing of $V^{(2)}$, and let F be as above. Fix a point λ of $\mathbb{P}^1$. Assume moreover that there exists a $\delta_2 > 0$ such that the following holds for any $u \in V^{(1)}_{|\lambda}$ and $v \in V^{(1)}_{|\sigma(\lambda)}$:

$$\bigl|S^{(2)}\bigl(F(u), \sigma^*(F(v))\bigr) - S^{(1)}(u, \sigma^* v)\bigr| \leqslant \delta_2\,|u|_{h^{(1)}}\,|v|_{h^{(1)}}.$$

Lemma 11.3.6. — *If δ_i $(i = 1, 2)$ are sufficiently small, the following holds:*
- *$(V^{(2)}, S^{(2)})$ is a polarized pure twistor structure. Let $h^{(2)}$ denote the corresponding Hermitian metric.*
- *We have $\sup_{\mathbb{P}^1} d_{\mathrm{Herm}}\bigl(h^{(1)}, F^* h^{(2)}\bigr) \leqslant C\,(\delta_1+\delta_2)$, where C is a constant depending only on the rank.*

Proof. — It follows from Lemma 11.3.4. □

11.4. One step reduction

11.4.1. Preliminary. — For simplicity, we assume that the coordinate system is admissible for the good set T. Let k be the number determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$. By shrinking X, we may assume that $\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0)$ and $\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_\infty)$ are given on $\mathcal{X}_{\mathcal{K}}$ and $\mathcal{X}^\dagger_{\sigma(\mathcal{K})}$, respectively. Let $d''_{\mathbb{P}^1,V}$ and $d''_{\mathbb{P}^1,\mathrm{Gr}^{\boldsymbol{m}(0)}(V)}$ denote the $\mathbb{P}^1$-holomorphic structures of V and $\mathrm{Gr}^{\boldsymbol{m}(0)}(V)$, respectively.

In the following, we assume one of the following:

(Case A) : $(V, \mathbb{D}^\triangle, \mathcal{S})$ is a variation of polarized pure twistor structure of weight 0, and the prolongment $\widetilde{\boldsymbol{V}}$ is canonical. The underlying harmonic bundle is denoted by $(E, \overline{\partial}_E, \theta, h)$.

(Case B) : $\mathrm{Gr}^{\boldsymbol{m}(0)}(V, \mathbb{D}^{\triangle}, \mathcal{S})$ is a variation of polarized pure twistor structure of weight 0. The underlying harmonic bundle is denoted by $(E^{(1)}, \overline{\partial}^{(1)}, \theta^{(1)}, h^{(1)})$. It is graded as

$$(E^{(1)}, \overline{\partial}^{(1)}, \theta^{(1)}, h^{(1)}) = \bigoplus_{\mathfrak{a} \in T(\boldsymbol{m}(0))} (E^{(1)}_{\mathfrak{a}}, \overline{\partial}^{(1)}_{\mathfrak{a}}, \theta^{(1)}_{\mathfrak{a}}, h^{(1)}_{\mathfrak{a}}),$$

corresponding to the decomposition $\mathrm{Gr}^{\boldsymbol{m}(0)}(V, \mathbb{D}^{\triangle}, \mathcal{S}) = \bigoplus \mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(V, \mathbb{D}^{\triangle}, \mathcal{S})$.

We have some remarks in Case (B), which will be implicitly used in the subsequent argument. We have the canonical prolongment $(\mathcal{Q}\mathcal{E}^{(1)}, \mathbb{D}^{(1)}) = \bigoplus \big(\mathcal{Q}\mathcal{E}^{(1)}_{\mathfrak{a}}, \mathbb{D}^{(1)}_{\mathfrak{a}}\big)$. Note that we should distinguish $\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0)$ and $\mathcal{Q}\mathcal{E}^{(1)}$. However, we have a natural isomorphism

$$\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0)_{|\boldsymbol{C}_\lambda \times (X \setminus D(\leqslant k))} \simeq \mathcal{Q}\mathcal{E}^{(1)}_{|\boldsymbol{C}_\lambda \times (X \setminus D(\leqslant k))},$$

because both of them are regular along D_i for $i > k$. The induced KMS-structures at any λ_0 are also the same due to the uniqueness of a KMS-structure. We also remark that the grading naturally gives a splitting of the Stokes filtration at the level $\boldsymbol{m}(0)$ for each small multi-sector.

11.4.2. Preliminary estimate. — Let $\lambda_0 \in \mathcal{K}$, and let $U_\lambda(\lambda_0)$ denote a small neighbourhood of λ_0. If $\lambda_0 \neq 0$, we assume $0 \notin U_\lambda(\lambda_0)$. We will shrink it in the following argument without mention. We set $W := U_\lambda(\lambda_0) \times D(\leqslant k)$ if $\lambda_0 \neq 0$, and $W := (U_\lambda(\lambda_0) \times D(\leqslant k)) \cup (\{0\} \times X)$ if $\lambda_0 = 0$. Let S be a multi-sector in $(U_\lambda(\lambda_0) \times X) \setminus W$, and let $\overline{S}$ denote the closure of S in the real blow up of $U_\lambda(\lambda_0) \times X$ along W. If S is sufficiently small, we can take a $\mathbb{D}_{\leqslant k}$-flat splitting $\mathcal{Q}^{(\lambda_0)}_0 \widetilde{V}_{0|\overline{S}} = \bigoplus_{\mathfrak{a}} \mathcal{Q}^{(\lambda_0)}_0 \widetilde{V}_{0,\mathfrak{a},S}$ of the partial Stokes filtration $\mathcal{F}^S$ at the level $\boldsymbol{m}(0)$, whose restrictions to $S \cap \big(U_\lambda(\lambda_0) \times D_i\big)$ is compatible with the endomorphism $\mathrm{Res}_i(\mathbb{D})$ and the filtration ${}^{i}F^{(\lambda_0)}$ for each $i = k+1, \dots, \ell$. (See Proposition 3.6.7.) The splitting induces a $\mathbb{D}_{\leqslant k}$-flat isomorphism on S:

$$g_S : \mathrm{Gr}^{\boldsymbol{m}(0)}(\mathcal{Q}^{(\lambda_0)}_0 \widetilde{V}_0)_{|S} \simeq \mathcal{Q}^{(\lambda_0)}_0 \widetilde{V}_{0|S}. \tag{233}$$

Its inverse is denoted by f_S.

Lemma 11.4.1. — *Let g'_S and f'_S be obtained from another $\mathbb{D}_{\leqslant k}$-flat splitting of $\mathcal{F}^S$. If S is shrunk in the radius direction, the following holds:*

- **(Case A) :** $\mathrm{id} - f_S^{-1} \circ f'_S = O\big(\exp(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ *for some $\varepsilon > 0$ with respect to h.*
- **(Case B) :** $\mathrm{id} - g_S^{-1} \circ g'_S = O\big(\exp(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\big)$ *for some $\varepsilon > 0$ with respect to $h^{(1)}$.*

Proof. — Let us consider Case (A). Note $\mathcal{Q}^{(\lambda_0)}_0 \widetilde{V}_0 = \mathcal{Q}^{(\lambda_0)}_0 \mathcal{E}$ in this case. The $\mathbb{D}_{\leqslant k}$-flat endomorphism $\mathfrak{F} := \mathrm{id} - f_S^{-1} \circ f'_S$ strictly decreases the Stokes filtration of $\mathcal{Q}^{(\lambda_0)}_0 \mathcal{E}_{|\overline{S}}$ at the level $\boldsymbol{m}(0)$. It is compatible with the filtrations ${}^{i}F^{(\lambda_0)}$ and the endomorphisms

$\mathrm{Res}_i(\mathbb{D})$ on $S \cap \big(U_\lambda(\lambda_0) \times D_i\big)$ $(i = k+1, \dots, \ell)$. If $U_\lambda(\lambda_0)$ is shrunk, the Stokes filtrations of $\mathcal{Q}_0^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$ and $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$ are the same. By shrinking S in the radius direction we obtain the desired estimate due to Lemma 9.3.1. We can argue Case (B) in a similar way. □

Let $g_S^{(p)}$ $(p = 1, \dots, L)$ be $\mathbb{D}_{\leqslant k}$-flat isomorphisms as in (233). Let $f_S^{(p)}$ denote their inverses. Let χ_p $(p = 1, \dots, L)$ be non-negative valued C^∞-functions on S such that (i) $\sum \chi_p = 1$, (ii) $\overline{\partial}\chi_p$ are of polynomial order in $|\lambda^{-1}|$ and $|z_i^{-1}|$ for $i = 1, \dots, k$. We set $g := \sum \chi_p\, g_p$ and $f := \sum \chi_p\, f_p$.

Lemma 11.4.2. — *In Case (A), we have the following estimate with respect to h:*

$$f^{-1} \circ d''_{\mathbb{P}^1, \mathrm{Gr}^{\boldsymbol{m}(0)}(V)} \circ f - d''_{\mathbb{P}^1, V} = O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big). \tag{234}$$

In Case (B), we have the following estimate with respect to $h^{(1)}$:

$$g^{-1} \circ d''_{\mathbb{P}^1, V} \circ g - d''_{\mathbb{P}^1, \mathrm{Gr}^{\boldsymbol{m}(0)}(V)} = O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big). \tag{235}$$

Proof. — Let us show (234) in Case (A). We remark the following estimate with respect to h, due to Lemma 11.4.1:

$$(f_S^{(p)})^{-1} \circ f - \mathrm{id} = \sum \chi_q \left((f_S^{(p)})^{-1} \circ f_S^{(q)} - \mathrm{id}\right) = O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).$$

The left-hand side of (234) can be rewritten as follows:

$$f^{-1} \circ \big(\overline{\partial}_{\mathbb{P}^1, \mathrm{Gr}^{\boldsymbol{m}(0)}(V)} \circ f - f \circ \overline{\partial}_{\mathbb{P}^1, V}\big) = f^{-1} \circ \Big(\sum \overline{\partial}\chi_p\, f_S^{(p)}\Big) = \sum \overline{\partial}\chi_p \left(f^{-1}\, f_S^{(p)} - \mathrm{id}\right).$$

Hence, we obtain the desired estimate (234). We can show (235) in Case (B) by the same argument. □

11.4.2.1. The other side. — Let $U_\mu(\mu_0)$ denote a small neighbourhood of $\mu_0 \in \sigma(\mathcal{K})$. We set $W^\dagger := \big(\{0\} \times X^\dagger\big) \cup \big(U_\mu(0) \times D^\dagger(\leqslant k)\big)$. If we are given a small multi-sector S of $\big(U_\mu(\mu_0) \times X^\dagger\big) \setminus W^\dagger$, we can take a $\mathbb{D}^\dagger_{\leqslant k}$-flat isomorphism

$$g_S^\dagger : \mathrm{Gr}^{\boldsymbol{m}(0)}(\mathcal{Q}_0^{(\mu_0)}\widetilde{V}_\infty)_{|S} \simeq \mathcal{Q}_0^{(\mu_0)}\widetilde{V}_{\infty|S} \tag{236}$$

in a similar way, by taking a $\mathbb{D}^\dagger_{\leqslant k}$-flat splitting of the Stokes filtration at the level $\boldsymbol{m}(0)$. Its inverse is denoted by $f_S^\dagger$. We have estimates for them similar to Lemma 11.4.1 and Lemma 11.4.2.

11.4.2.2. Remark for gluing. — Let $\lambda_0 \neq 0$ and $\mu_0 = \lambda_0^{-1}$. Let $U_\lambda(\lambda_0)$ be a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$, and let $U_\mu(\mu_0)$ be the corresponding neighbourhood of μ_0 in $\boldsymbol{C}_\mu$. Assume that λ_0 is generic with respect to the KMS-structure of $\widetilde{V}_0$. If S is a sufficiently small multi-sector in $U_\lambda(\lambda_0) \times \big(X \setminus D(\leqslant k)\big)$, we can take a $\mathbb{D}$-flat splitting $\mathcal{Q}_0^{(\lambda_0)}\widetilde{V}_{0|\overline{S}} = \bigoplus_{\mathfrak{a}} \mathcal{Q}_0^{(\lambda_0)}\widetilde{V}_{0,\mathfrak{a},S}$. (See Proposition 3.6.8.) Note that S naturally gives a small multi-sector of $U_\mu(\mu_0) \times \big(X^\dagger \setminus D^\dagger(\leqslant k)\big)$, and that the splitting naturally induces a $\mathbb{D}^\dagger$-splitting $\mathcal{Q}_0^{(\mu_0)}\widetilde{V}_{\infty|\overline{S}} = \bigoplus_{\mathfrak{a}} \mathcal{Q}_0^{(\mu_0)}\widetilde{V}_{\infty,\mathfrak{a},S}$.

11.4.2.3. Estimate for pairing. — Let S be a sufficiently small multi-sector of $\big(U_\lambda(\lambda_0)\times X\big)\setminus W$. Let g_S be as in (233), and let f_S denote its inverse. We obtain the small multi-sector $\sigma(S)$ of $\sigma\big(U_\lambda(\lambda_0)\big)\times\big(X^\dagger\setminus D^\dagger(\leqslant k)\big)$. Let $g^\dagger_{\sigma(S)}$ be as in (236) for $\sigma(S)$, and let $f^\dagger_{\sigma(S)}$ denote its inverse.

Lemma 11.4.3. — *In Case (A), we have the following estimate with respect to h:*

$$\mathcal{S}-\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}\circ\big(f_S\otimes\sigma^*f^\dagger_{\sigma(S)}\big)=O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).\tag{237}$$

In Case (B), we have the following estimate with respect to $h^{(1)}$:

$$\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}-\mathcal{S}\circ\big(g_S\otimes\sigma^*g^\dagger_{\sigma(S)}\big)=O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).\tag{238}$$

Proof. — By the perfect pairings, we have the natural isomorphisms

$$\widetilde{V}_\infty\simeq\sigma^*(\widetilde{V}_0^\vee),\quad \mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_\infty)\simeq\sigma^*\big(\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0)^\vee\big).$$

Hence, g_S induces $\sigma^*(g_S^\vee)$ as in (236) for $\sigma(S)$. If $g^\dagger_{\sigma(S)}=\sigma^*(g_S^\vee)$, both (237) and (238) vanish.

Let us show (237) in Case (A). We set $\mathfrak{H}:=\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}\circ\big(f_S\otimes(\sigma^*f^\dagger_{\sigma(S)}-f_S^\vee)\big)$. Note that $\mathcal{S}$ gives an identification $\sigma^*\mathcal{QE}^\dagger\simeq\mathcal{QE}^\vee$, and we can regard $\mathfrak{H}$ as a $\mathbb{D}_{\leqslant k}$-flat section of $\mathcal{P}^{(\lambda_0)}\operatorname{End}(\mathcal{E}^\lambda)$ such that (i) it is compatible with the filtrations ${}^iF$ and the endomorphisms $\mathrm{Res}_i(\mathbb{D})$ on $S\cap\big(U_\lambda(\lambda_0)\times D_i\big)$ $(i=k+1,\dots,\ell)$, (ii) $\mathfrak{H}$ strictly decreases the Stokes filtration of $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$. Hence, we obtain the desired estimate from Lemma 9.3.1. We can show (238) in Case (B) by a similar argument. □

Let $g_S^{(p)}$ $(p=1,\dots,L)$ be as in (233), and $f_S^{(p)}$ denote their inverses. Let $\chi_{S,p}$ $(p=1,\dots,L)$ be non-negative valued functions such that (i) $\sum\chi_{S,p}=1$, (ii) $\overline{\partial}\chi_{S,p}$ are of polynomial order in $|\lambda^{-1}|$ and $|z_i^{-1}|$ for $i=1,\dots,k$. We set $g:=\sum\chi_p\,g_S^{(p)}$ and $f:=\sum\chi_p\,f_S^{(p)}$. Let $g^{\dagger\,(q)}_{\sigma(S)}$ $(q=1,\dots,M)$ be as in (236) for $\sigma(S)$, and $f^{\dagger\,(q)}_{\sigma(S)}$ denote their inverses. Let $\chi_{\sigma(S),q}$ $(q=1,\dots,M)$ be non-negative valued functions such that (i) $\sum\chi_{\sigma(S),q}=1$, (ii) $\overline{\partial}\chi_{\sigma(S),q}$ are of polynomial order in $|\mu^{-1}|$ and $|z_i^{-1}|$ for $i=1,\dots,k$. We set $g^\dagger:=\sum\chi_{\sigma(S),q}\,g^{\dagger(q)}_{\sigma(S)}$ and $f^\dagger:=\sum\chi_{\sigma(S),q}\,f^{\dagger(q)}_{\sigma(S)}$.

Lemma 11.4.4. — *In Case (A), we have the following estimate with respect to h:*

$$\mathcal{S}-\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}\circ(f\otimes\sigma^*f^\dagger)=O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).\tag{239}$$

In Case (B), we have the following estimate with respect to $h^{(1)}$:

$$\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}-\mathcal{S}\circ(g\otimes\sigma^*g^\dagger)=O\Big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).\tag{240}$$

Proof. — It follows from Lemma 11.4.3. □

11.4.3. Proof of Proposition 11.2.4. — By shrinking $\mathcal{K}$, we may assume that every $\lambda\in\sigma(\mathcal{K})\cap\mathcal{K}$ is generic with respect to the KMS-structure of $\widetilde{V}_0$ and $\widetilde{V}_\infty$. We set $W_{\mathcal{K}}:=\big(\mathcal{K}\times D(\leqslant k)\big)\cup\big(\{0\}\times X\big)$. We take a finite covering $\mathcal{X}_{\mathcal{K}}\setminus W_{\mathcal{K}}\subset\bigcup_{p=1}^{L}S_p$ by small multi-sectors S_p satisfying the following properties:

- On each S_p, we can take a $\mathbb{D}_{\leqslant k}$-flat morphism g_{S_p} as in (233). Its inverse is denoted by f_{S_p}.
- On each $\sigma(S_p)$, we can take a $\mathbb{D}^{\dagger}_{\leqslant k}$-flat morphism $g^{\dagger}_{\sigma(S_p)}$ as in (236). Its inverse is denoted by $f^{\dagger}_{\sigma(S_p)}$.
- If $S_p\cap\mathcal{K}\cap\sigma(\mathcal{K})\neq\varnothing$, we assume that g_{S_p} comes from a $\mathbb{D}$-flat splitting. We assume a similar condition for $g^{\dagger}_{\sigma(S_p)}$.

We take a partition of unity $\big(\chi_{S_p},\chi_{\sigma(S_p)}\,\big|\,p=1,\dots,L\big)$ subordinated to the covering $(S_p,\sigma(S_p)\,|\,p=1,\dots,L)$ such that $\partial\chi_{S_p}$ (resp. $\partial\chi_{\sigma(S_p)}$) are of polynomial order in $|\lambda|^{-1}$ (resp. $|\lambda|$) and $|z_i|^{-1}$ for $i=1,\dots,k$. We set

$$g:=\sum_p\chi_{S_p}\,g_{S_p}+\sum_p\chi_{\sigma(S_p)}\,g^{\dagger}_{\sigma(S_p)},\qquad f:=\sum_p\chi_{S_p}\,f_{S_p}+\sum_p\chi_{\sigma(S_p)}\,f^{\dagger}_{\sigma(S_p)}.$$

Let us consider the case (A). By Lemma 11.4.2 and its analogue for the μ-side, we obtain the following with respect to h:

$$f^{-1}\circ d''_{\mathbb{P}^1,\mathrm{Gr}^{\boldsymbol{m}(0)}(V)}\circ f-d''_{\mathbb{P}^1,V}=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).$$

By Lemma 11.4.4, we have the following estimate with respect to h:

$$\mathcal{S}-\mathrm{Gr}^{\boldsymbol{m}(0)}\mathcal{S}\circ(f\otimes\sigma^*f^{\dagger})=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big).$$

Then, there exists $0<R<1$ such that $\mathrm{Gr}^{\boldsymbol{m}(0)}(V,\mathbb{D}^{\triangle},\mathcal{S})$ is a variation of polarized pure twistor structure, due to Lemma 11.3.5 and Lemma 11.3.6. Hence, the claim (A) of Proposition 11.2.4 is proved. The claim (B) can be shown in a similar way. □

11.5. Full reduction in the smooth divisor case

In this section, we assume $D=D_1$, and we will prove Proposition 11.2.5. The argument is essentially the same as that in the proof of Proposition 11.2.4. We almost repeat it by changing Lemma 9.3.1 with Proposition 10.2.1.

Let $d''_{\mathbb{P}^1,V}$ and $d''_{\mathbb{P}^1,\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)}$ denote the $\mathbb{P}^1$-holomorphic structure of V and $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)$. In the following, we assume one of the following:

(Case A) : $(V,\mathbb{D}^{\triangle},\mathcal{S})$ is a variation of polarized pure twistor structure of weight 0, and the prolongment $\widetilde{\boldsymbol{V}}$ is canonical. The underlying harmonic bundle is denoted by $(E,\overline{\partial}_E,\theta,h)$.

(Case C) : $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V,\mathbb{D}^{\triangle},\mathcal{S})$ is a variation of polarized pure twistor structure of weight 0. The underlying harmonic bundles are denoted by $(E^{(0)},\overline{\partial}^{(0)},\theta^{(0)},h^{(0)})$.

It is graded as

$$(E^{(0)},\overline{\partial}^{(0)},\theta^{(0)},h^{(0)})=\bigoplus_{\mathfrak{a}\in T}(E^{(0)}_{\mathfrak{a}},\overline{\partial}^{(0)}_{\mathfrak{a}},\theta^{(0)}_{\mathfrak{a}},h^{(0)}_{\mathfrak{a}}),$$

corresponding to $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V,\mathbb{D}^{\triangle},\mathcal{S})=\bigoplus\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}_{\mathfrak{a}}(V,\mathbb{D}^{\triangle},\mathcal{S})$.

We have some remarks in Case (C), which we will implicitly use in the subsequent argument. We have the canonical prolongment $(\mathcal{QE}^{(0)},\mathbb{D}^{(0)})=\bigoplus\big(\mathcal{QE}^{(0)}_{\mathfrak{a}},\mathbb{D}^{(0)}_{\mathfrak{a}}\big)$. Since $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}_{\mathfrak{a}}(\widetilde{V}_0)$ and $\mathcal{QE}^{(0)}_{\mathfrak{a}}$ have the unique irregular value $\mathfrak{a}$, the natural isomorphism on $\boldsymbol{C}_\lambda\times(X\setminus D)$ can be extended to the isomorphism on $\boldsymbol{C}_\lambda\times X$ by the uniqueness of the Deligne extension of flat bundles. We also remark that the grading gives a splitting of the full Stokes filtration for each small multi-sector.

11.5.1. Preliminary estimate. — Let $\lambda_0\in\mathcal{K}$, and let $U_\lambda(\lambda_0)$ denote a small neighbourhood of λ_0. We will shrink it in the following argument without mention. We set $W:=U_\lambda(\lambda_0)\times D$ if $\lambda_0\neq 0$, and $W:=(\{0\}\times X)\cup\big(U_\lambda(0)\times D\big)$ if $\lambda_0=0$. Let S be a multi-sector in $\big(U_\lambda(\lambda_0)\times X\big)\setminus W$, and let $\overline{S}$ denote the closure of S in the real blow up of $U_\lambda(\lambda_0)\times X$ along W. If S is sufficiently small, we can take a $\mathbb{D}$-flat splitting $\mathcal{Q}^{(\lambda_0)}_0\widetilde{V}_{0|\overline{S}}=\bigoplus_{\mathfrak{a}\in T}\mathcal{Q}^{(\lambda_0)}_0\widetilde{V}_{0,\mathfrak{a},S}$ of the full Stokes filtration $\widetilde{\mathcal{F}}^S$. The splitting induces a $\mathbb{D}$-flat isomorphism

$$g_S:\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{Q}^{(\lambda_0)}_0\widetilde{V}_0,\mathbb{D})_{|S}\simeq(\mathcal{Q}^{(\lambda_0)}_0\widetilde{V}_0,\mathbb{D})_{|S}. \tag{241}$$

Its inverse is denoted by f_S. We can show the following lemma by using Proposition 10.2.1 and the argument in the proof of Lemma 11.4.1.

Lemma 11.5.1. — *Let g'_S and f'_S be obtained from another $\mathbb{D}$-flat splitting of $\mathcal{F}^S$. If S is shrunk in the radius direction, the following holds:*

- **(Case A)** : $\mathrm{id}-f_S^{-1}\circ f'_S=O\big(\exp(-\varepsilon|\lambda^{-1}z_1^{-1}|)\big)$ *for some $\varepsilon>0$ with respect to h.*
- **(Case C)** : $\mathrm{id}-g_S^{-1}\circ g'_S=O\big(\exp(-\varepsilon|\lambda^{-1}z_1^{-1}|)\big)$ *for some $\varepsilon>0$ with respect to $h^{(0)}$.* □

Let $g^{(p)}_S$ $(p=1,\dots,L)$ be $\mathbb{D}$-flat isomorphisms as in (241). Let $f^{(p)}_S$ denote their inverses. Let χ_p $(p=1,\dots,L)$ be non-negative valued C^∞-functions on S such that (i) $\sum\chi_p=1$, (ii) $\overline{\partial}\chi_p$ are of polynomial order in $|\lambda^{-1}|$ and $|z_1^{-1}|$. We set $g:=\sum\chi_p\,g_p$ and $f:=\sum\chi_p\,f_p$.

Lemma 11.5.2. — *In Case (A), we have the following estimate with respect to h:*

$$f^{-1}\circ d''_{\mathbb{P}^1,\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)}\circ f-d''_{\mathbb{P}^1,V}=O\Big(\exp\big(-\varepsilon|z_1^{-1}|\big)\Big). \tag{242}$$

In Case (C), we have the following estimate with respect to $h^{(0)}$:

$$g^{-1}\circ d''_{\mathbb{P}^1,V}\circ g-d''_{\mathbb{P}^1,\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)}=O\Big(\exp\big(-\varepsilon|z_1^{-1}|\big)\Big). \tag{243}$$

Proof. — It follows from Lemma 11.5.1. □

11.5.1.1. The other side. — Let $\mu_0 \in \sigma(\mathcal{K}) \setminus \{0\}$, and let $U_\mu(\mu_0)$ denote a small neighbourhood of μ_0. If we are given a small multi-sector S of $U_\mu(\mu_0) \times \big(X^\dagger \setminus D^\dagger\big)$, we can take a $\mathbb{D}^\dagger$-flat isomorphism

$$g_S^\dagger : \operatorname{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{Q}_0^{(\mu_0)}\widetilde{V}_\infty, \mathbb{D}^\dagger)_{|S} \simeq (\mathcal{Q}_0^{(\mu_0)}\widetilde{V}_\infty, \mathbb{D}^\dagger)_{|S} \tag{244}$$

in a similar way, by taking a $\mathbb{D}^\dagger$-flat splitting of the Stokes filtration. Its inverse is denoted by $f_S^\dagger$. We have estimates for them similar to Lemma 11.5.1 and Lemma 11.5.2.

11.5.1.2. Remark for gluing. — Let $\lambda_0 \neq 0$ and $\mu_0 = \lambda_0^{-1}$. Let $U_\lambda(\lambda_0)$ be a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$, and let $U_\mu(\mu_0)$ be the corresponding neighbourhood of μ_0 in $\boldsymbol{C}_\mu$. Let S be a sufficiently small sector in $U_\lambda(\lambda_0) \times (X \setminus D)$, and let $\mathcal{Q}_0^{(\lambda_0)}\widetilde{V}_{0|\overline{S}} = \bigoplus_{\mathfrak{a}} \mathcal{Q}_0^{(\lambda_0)}\widetilde{V}_{0,\mathfrak{a},S}$ be a $\mathbb{D}$-flat splitting. Note that S naturally gives a small multi-sector of $U_\mu(\mu_0) \times \big(X^\dagger \setminus D^\dagger\big)$, and that the splitting naturally induces a $\mathbb{D}^\dagger$-splitting $\mathcal{Q}_0^{(\mu_0)}\widetilde{V}_{\infty|\overline{S}} = \bigoplus_{\mathfrak{a}} \mathcal{Q}_0^{(\mu_0)}\widetilde{V}_{\infty,\mathfrak{a},S}$.

11.5.1.3. Estimate for pairing. — Let S be a sufficiently small multi-sector of $U_\lambda(\lambda_0) \times \big(X \setminus D\big)$. Let g_S be as in (241), and let f_S denote its inverse. We obtain the small multi-sector $\sigma(S)$ of $\sigma\big(U_\lambda(\lambda_0)\big) \times \big(X^\dagger \setminus D^\dagger\big)$. Let $g^\dagger_{\sigma(S)}$ be as in (244) for $\sigma(S)$, and let $f^\dagger_{\sigma(S)}$ denote its inverse. We can show the following lemma by using Proposition 10.2.1 and the argument in the proof of Lemma 11.4.3.

Lemma 11.5.3. — *In Case (A), we have the following estimate with respect to h:*

$$\mathcal{S} - \operatorname{Gr}^{\widetilde{\boldsymbol{V}}} \mathcal{S} \circ (f_S \otimes \sigma^* f^\dagger_{\sigma(S)}) = O\Big(\exp\big(-\varepsilon|z_1^{-1}|\big)\Big). \tag{245}$$

In Case (C), we have the following estimate with respect to $h^{(0)}$:

$$\operatorname{Gr}^{\widetilde{\boldsymbol{V}}} \mathcal{S} - \mathcal{S} \circ (g_S \otimes \sigma^* g^\dagger_{\sigma(S)}) = O\Big(\exp\big(-\varepsilon|z_1^{-1}|\big)\Big). \tag{246}$$

□

Let $g_S^{(p)}$ $(p = 1, \ldots, L)$ be as in (241), and $f_S^{(p)}$ denote their inverses. Let $\chi_{S,p}$ $(p = 1, \ldots, L)$ be non-negative valued functions such that (i) $\sum \chi_{S,p} = 1$, (ii) $\overline{\partial}\chi_{S,p}$ are of polynomial order in $|\lambda^{-1}|$ and $|z_1^{-1}|$. We set $g = \sum \chi_p\, g_S^{(p)}$ and $f = \sum \chi_p\, f_S^{(p)}$. Let $g^{\dagger\,(q)}_{\sigma(S)}$ $(q = 1, \ldots, M)$ be as in (244) for $\sigma(S)$, and $f^{\dagger\,(q)}_{\sigma(S)}$ denote their inverses. Let $\chi_{\sigma(S),q}$ $(q = 1, \ldots, M)$ be non-negative valued functions such that (i) $\sum \chi_{\sigma(S),q} = 1$, (ii) $\overline{\partial}\chi_{\sigma(S),q}$ are of polynomial order in $|\mu^{-1}|$ and $|z_1^{-1}|$. We set $g^\dagger = \sum \chi_{\sigma(S),q}\, g^{\dagger(q)}_{\sigma(S)}$ and $f^\dagger = \sum \chi_{\sigma(S),q}\, f^{\dagger(q)}_{\sigma(S)}$.

Lemma 11.5.4. — *In Case (A), we have the following estimate with respect to h:*

$$\mathcal{S} - \operatorname{Gr}^{\widetilde{\boldsymbol{V}}} \mathcal{S} \circ (f \otimes \sigma^* f^\dagger) = O\Big(\exp\big(-\varepsilon|\boldsymbol{z}^{-1}|\big)\Big). \tag{247}$$

In Case (C), we have the following estimate with respect to $h^{(0)}$:

$$\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}\mathcal{S}-\mathcal{S}\circ(g\otimes\sigma^*g^\dagger)=O\Big(\exp\big(-\varepsilon|z^{-1}|\big)\Big). \tag{248}$$

Proof. — It follows from Lemma 11.5.3. □

11.5.2. Proof of Proposition 11.2.5. — We take a compact region $\mathcal{K}\subset \boldsymbol{C}_\lambda$ such that $\mathcal{K}\cup\sigma(\mathcal{K})=\mathbb{P}^1$. We set $W_{\mathcal{K}}:=\big(\mathcal{K}\times D\big)\cup\big(\{0\}\times X\big)$. We take a finite covering $\mathcal{X}_{\mathcal{K}}\setminus W_{\mathcal{K}}\subset\bigcup_{p=1}^L S_p$ by small multi-sectors S_p satisfying the following:

- On each S_p, we can take a $\mathbb{D}$-flat morphism g_{S_p} as in (241). Its inverse is denoted by f_{S_p}.
- On each $\sigma(S_p)$, we can take a $\mathbb{D}^\dagger$-flat morphism $g^\dagger_{\sigma(S_p)}$ as in (244). Its inverse is denoted by $f^\dagger_{\sigma(S_p)}$.

We take a partition of unity $\big(\chi_{S_p},\chi_{\sigma(S_p)}\,\big|\,p=1,\dots,L\big)$ subordinated to the covering $(S_p,\sigma(S_p)\,|\,p=1,\dots,L)$ such that $\partial\chi_{S_p}$ (resp. $\partial\chi_{\sigma(S_p)}$) are of polynomial order in $|\lambda|^{-1}$ (resp. $|\lambda|$) and $|z_1|^{-1}$. We set

$$g:=\sum_p\chi_{S_p}\,g_{S_p}+\sum_p\chi_{\sigma(S_p)}\,g^\dagger_{\sigma(S_p)},\qquad f:=\sum_p\chi_{S_p}\,f_{S_p}+\sum_p\chi_{\sigma(S_p)}\,f^\dagger_{\sigma(S_p)}.$$

Let us consider the case (A). By Lemma 11.5.2 and its analogue in the μ-side, we obtain the following with respect to h:

$$f^{-1}\circ d''_{\mathbb{P}^1,\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)}\circ f-d''_{\mathbb{P}^1,V}=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|z_1^{-1}|\big)\Big).$$

By Lemma 11.5.4, we have the following estimate with respect to h:

$$\mathcal{S}-\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}\mathcal{S}\circ(f_S\otimes\sigma^*f^\dagger_{\sigma(S)})=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|z_1^{-1}|\big)\Big).$$

Then, there exists $0<R<1$ such that $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V,\mathbb{D}^\triangle,\mathcal{S})_{|\mathbb{P}^1\times X^*(R)}$ is a variation of polarized pure twistor structure, due to Lemma 11.3.5 and Lemma 11.3.6.

Let us consider the case (C). By a similar argument, we obtain that $(V,\mathbb{D}^\triangle,\mathcal{S})$ is a variation of polarized pure twistor structure. Namely, by Lemma 11.5.2, we obtain the following with respect to $h^{(0)}$:

$$g^{-1}\circ d''_{\mathbb{P}^1,V}\circ g-d''_{\mathbb{P}^1,\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}(V)}=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|z_1^{-1}|\big)\Big).$$

By Lemma 11.5.4, we have the following estimate with respect to h:

$$\mathrm{Gr}^{\widetilde{\boldsymbol{V}}}\mathcal{S}-\mathcal{S}\circ(g\otimes\sigma^*g^\dagger)=O\Big(\exp\big(-\varepsilon\,(|\lambda^{-1}|+|\lambda|)\,|z_1^{-1}|\big)\Big).$$

Then, there exists $0<R<1$ such that $(V,\mathbb{D}^\triangle,\mathcal{S})_{|\mathbb{P}^1\times X^*(R)}$ is a variation of polarized pure twistor structure, due to Lemma 11.3.5 and Lemma 11.3.6. Let $(E,\overline{\partial}_E,\theta,h)$ be the underlying harmonic bundle. Let d_{Herm} denote the natural distance on the symmetric space of Hermitian metrics. Due to Lemma 11.3.6, we have

$$d_{\mathrm{Herm}}(g^*h,h^{(0)})=O\Big(\exp\big(-\varepsilon|z_1^{-1}|\big)\Big).$$

In particular, the Hermitian metrics g^*h and $h^{(0)}$ are mutually bounded. It means that the full Stokes filtrations $\widetilde{\mathcal{F}}^S$ of the family $\widetilde{V}_0$ of meromorphic λ-flat bundles can be characterized by the growth orders of the norms of the flat sections with respect to h, as in Proposition 11.1.3. Therefore, there exists a natural isomorphism between the meromorphic prolongments $\widetilde{V}_0 \simeq \mathcal{QE}$. Similarly, we obtain $\widetilde{V}_\infty \simeq \mathcal{QE}^\dagger$. Hence $(\widetilde{V}_0, \widetilde{V}_\infty)$ is canonical. □

11.6. End of Proof of Theorem 11.2.2

11.6.1. From (P1) to (P2). — Assume that $(V, \mathbb{D}^\triangle, \mathcal{S})$ is a variation of polarized pure twistor structure of weight 0, and that the prolongment $\widetilde{\boldsymbol{V}}$ is canonical. If we shrink X, we have the harmonic bundle $\big(E^{(1)}, \overline{\partial}^{(1)}, \theta^{(1)}, h^{(1)}\big)$ on $X \setminus D$ which induces $\mathrm{Gr}^{\boldsymbol{m}(0)}(V^\triangle, \mathbb{D}^\triangle, \mathcal{S})$ by Proposition 11.2.4.

Lemma 11.6.1. — $\mathrm{Gr}^{\boldsymbol{m}(0)}\widetilde{V} := \big(\mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_0), \mathrm{Gr}^{\boldsymbol{m}(0)}(\widetilde{V}_\infty)\big)$ *is canonical.*

Proof. — Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{\ell-k}$. We take $1 \leqslant j \leqslant k$. Let $Q \in D_j$ be sufficiently close to $D_{\underline{k}}$, which is not singular point of $D(\leqslant k)$. We take a small neighbourhood X_Q of Q in X. We set $X_Q^* := X_Q \setminus D$. We put

$$(V_1, \mathbb{D}_1^\triangle, \mathcal{S}_1) := (V, \mathbb{D}^\triangle, \mathcal{S})_{|\mathbb{P}^1 \times X_Q^*}, \quad (V_2, \mathbb{D}_2^\triangle, \mathcal{S}_2) := \Big(\mathrm{Gr}^{\boldsymbol{m}(0)}(V, \mathbb{D}^\triangle, \mathcal{S})\Big)_{|\mathbb{P}^1 \times X_Q^*}.$$

They are equipped with the meromorphic prolongments induced by $\widetilde{\boldsymbol{V}}$ and $\mathrm{Gr}^{\boldsymbol{m}(0)}\widetilde{\boldsymbol{V}}$, respectively. They are denoted by $\widetilde{\boldsymbol{V}}_1$ and $\widetilde{\boldsymbol{V}}_2$, respectively. As remarked in Lemma 6.2.8, we have a natural isomorphism:

$$\mathrm{Gr}^{\widetilde{\boldsymbol{V}}_1}(V_1, \mathbb{D}_1^\triangle, \mathcal{S}_1) \simeq \mathrm{Gr}^{\widetilde{\boldsymbol{V}}_2}(V_2, \mathbb{D}_2^\triangle, \mathcal{S}_2).$$

By Proposition 11.2.5, $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}_1}(V_1, \mathbb{D}_1^\triangle, \mathcal{S}_1)$ is a variation of polarized pure twistor structure. Again, according to Proposition 11.2.5, $\widetilde{\boldsymbol{V}}_2$ is the canonical prolongment of $(V_2, \mathbb{D}_2^\triangle, \mathcal{S}_2)$. Then, it is easy to conclude that $\mathrm{Gr}^{\boldsymbol{m}(0)}\widetilde{\boldsymbol{V}}$ is the canonical prolongment of $\mathrm{Gr}^{\boldsymbol{m}(0)}(V, \mathbb{D}^\triangle, \mathcal{S})$. □

Now the claim $(P1) \Longrightarrow (P2)$ can be shown by an easy induction.

11.6.2. From (P2) to (P1). — In the following, we will shrink X without mention. By using Proposition 11.2.4 in a descending inductive way, we obtain that $\mathrm{Gr}^{\boldsymbol{m}(i)}(V, \mathbb{D}^\triangle, \mathcal{S})$ are variations of polarized pure twistor structure for any i. In particular, $(V, \mathbb{D}^\triangle, \mathcal{S})$ is a variation of polarized pure twistor structure. Let $(E, \overline{\partial}_E, \theta, h)$ be the underlying harmonic bundle, which is unramifiedly good wild with the good set T. For each $j = 1, \dots, \ell$, let $T(j)$ denote the image of T via the natural map

$$M(X, D)/H(X) \longrightarrow M(X, D)/M(X, D(\neq j)),$$

where $D(\neq j) := \bigcup_{i \neq j, 1 \leqslant i \leqslant \ell} D_i$. We put $D_j^{sm} := D_j \setminus D(\neq j)$. We take an auxiliary sequence $\boldsymbol{m}(0), \dots, \boldsymbol{m}(L)$ for T. There exists $q(j)$ such that $m_j(q(j)) = -1$ and

$m_j(q(j)+1) = 0$, where $m_j(i)$ denotes the j-th component of $\boldsymbol{m}(i)$. Note we have the natural bijection $\overline{\eta}_{\boldsymbol{m}(q(j))}(T) \simeq T(j)$.

Let $Q \in D_j^{sm}$. Let X_Q be a small neighbourhood of Q. We put $D_Q := X_Q \cap D$ and $X_Q^* := X_Q \setminus D_Q$. We set $(V_Q, \mathbb{D}_Q^{\triangle}, \mathcal{S}_Q) := (V, \mathbb{D}^{\triangle}, \mathcal{S})_{|\mathbb{P}^1 \times X_Q^*}$. It is equipped with a prolongment $\widetilde{\boldsymbol{V}}_Q$ induced by $\widetilde{\boldsymbol{V}}$. By the choice of $q(j)$,

$$\mathrm{Gr}^{\widetilde{\boldsymbol{V}}_Q}(V_Q, \mathbb{D}_Q^{\triangle}, \mathcal{S}_Q) \simeq \mathrm{Gr}^{\boldsymbol{m}(q(j))}(V, \mathbb{D}^{\triangle}, \mathcal{S})_{|\mathbb{P}^1 \times X_Q^*}.$$

Since $\mathrm{Gr}^{\widetilde{\boldsymbol{V}}_Q}(V_Q, \mathbb{D}_Q^{\triangle}, \mathcal{S}_Q)$ is a variation of polarized pure twistor structure, the prolongment $\widetilde{\boldsymbol{V}}_Q$ is canonical due to Proposition 11.2.5. Hence, we obtain that $\widetilde{\boldsymbol{V}}$ is canonical, and thus the proof of Theorem 11.2.2 is finished. □

11.7. Norm estimate

11.7.1. One step reduction. — We use the setting in Section 7.2.1. We have the variation of polarized pure twistor structure $(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$ associated to $(E, \overline{\partial}_E, \theta, h)$. For simplicity, we assume that the coordinate system is admissible for the good set $\mathrm{Irr}(\theta)$. In the following argument, we will shrink X without mention, if it is necessary.

We take an auxiliary sequence $\boldsymbol{m}(0), \boldsymbol{m}(1), \ldots, \boldsymbol{m}(L)$ for $\mathrm{Irr}(\theta)$. Let $(E_0, \overline{\partial}_0, \theta_0, h_0)$ be the unramifiedly good wild harmonic bundle underlying $\mathrm{Gr}^{\boldsymbol{m}(0)}(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$. We have the associated meromorphic λ-flat bundles $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$, and $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_0^{\lambda}, \mathbb{D}_0^{\lambda})$ on (X, D). Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}_{<0}^k \times \boldsymbol{0}_{\ell-k}$. By construction, we have the following natural isomorphism:

$$\widehat{\Phi} : \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_0^{\lambda}, \mathbb{D}_0^{\lambda}\big)_{|\widehat{D}(\leqslant k)} \simeq \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda}\big)_{|\widehat{D}(\leqslant k)}.$$

We have the harmonic bundles $E_1 := \mathrm{Hom}(E_0, E)$ and $E_2 := \mathrm{Hom}(E, E_0)$ with the naturally induced Higgs fields θ_i and pluri-harmonic metrics h_i $(i = 1, 2)$. Note that $(\mathcal{E}_i^{\lambda}, h_i)$ $(i = 1, 2)$ are acceptable. Let $\mathcal{P}\mathcal{E}_i^{\lambda}$ denote the associated meromorphic λ-bundle. We can regard $\widehat{\Phi}$ and $\widehat{\Phi}^{-1}$ as sections of $\mathcal{P}_0\mathcal{E}^{\lambda}_{1|\widehat{D}(\leqslant k)}$ and $\mathcal{P}_0\mathcal{E}^{\lambda}_{2|\widehat{D}(\leqslant k)}$, respectively.

Let N be a sufficiently large number. According to Lemma 3.6.29, we can take a section Φ_N of $\mathcal{P}_0\mathcal{E}^{\lambda}_{1|\widehat{D}(\leqslant k)}$ with the following properties:

- $\widehat{\Phi}_{|\widehat{D}^{(N)}(\leqslant k)} = \Phi_{N|\widehat{D}^{(N)}(\leqslant k)}$, where $\widehat{D}^{(N)}(\leqslant k)$ denotes the N-th infinitesimal neighbourhood of $D(\leqslant k)$.
- $\mathrm{Res}_i(\mathbb{D}_1^{\lambda})(\Phi_{N|D_i}) = 0$ for $i = k+1, \ldots, \ell$.

We can regard Φ_N as an isomorphism of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_0^{\lambda}$ and $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ preserving the parabolic filtration. If we shrink X appropriately, Φ_N is an isomorphism.

Proposition 11.7.1. — *$\Phi_{N|X\setminus D}$ and $\Phi_{N|X\setminus D}^{-1}$ are bounded with respect to h and h_0.*

Proof. — If we regard Φ_N as a section of $\mathcal{P}_0\mathcal{E}_1^{\lambda}$, we obtain the boundedness with respect to h_1 due to Lemma 7.7.3. Thus, Φ_N is bounded with respect to h and h_0.

We can regard Φ_N^{-1} as a section of $\mathcal{P}_0\mathcal{E}_2^{\lambda}$ satisfying $\widehat{\Phi}^{-1}_{|\widehat{D}^{(N)}(\leqslant k)} = \Phi^{-1}_{N|\widehat{D}^{(N)}(\leqslant k)}$ and $\mathrm{Res}_i(\mathbb{D}_2^{\lambda})(\Phi^{-1}_{N|D_i}) = 0$ for $i = k+1, \ldots, \ell$. Then, Φ_N^{-1} is bounded with respect to h_2 due to Lemma 7.7.3. Thus, Φ_N^{-1} is also bounded with respect to h and h_0. □

11.7.2. Full reduction. — By using an inductive argument, we can reduce the norm estimate for unramifiedly good wild harmonic bundle to that for tame harmonic bundles studied in [**67**]. Let $(E_3, \overline{\partial}_3, \theta_3, h_3)$ be the harmonic bundle underlying $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$. It is graded

$$(249)\qquad \big(E_3, \overline{\partial}_{E_3}, \theta_3, h_3\big) := \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \overline{\partial}_{E_{\mathfrak{a}}}, \theta_{\mathfrak{a}}, h_{\mathfrak{a}}),$$

corresponding to the decomposition $\mathrm{Gr}^{\widetilde{\mathcal{F}}}(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S}) = \bigoplus \mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$. Each $\theta_{\mathfrak{a}} - d\mathfrak{a}$ is tame. We have the associated meromorphic λ-flat bundles $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$ and $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_3^{\lambda}, \mathbb{D}_3^{\lambda})$.

Theorem 11.7.2. — *There exists a holomorphic isomorphism $\Phi : \mathcal{P}_{\boldsymbol{a}}\mathcal{E}_3^{\lambda} \longrightarrow \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$ with the following properties:*

- *It preserves the parabolic structures.*
- $\mathrm{Res}_i(\mathbb{D}^{\lambda}) \circ \Phi_{|D_i} - \Phi_{|D_i} \circ \mathrm{Res}_i(\mathbb{D}_3^{\lambda}) = 0.$

Moreover, Φ and Φ^{-1} are bounded with respect to h and h_3.

Proof. — We only have to use Proposition 11.7.1 inductively. □

Remark 11.7.3. — We obtained the norm estimate for holomorphic sections of λ-flat bundles associated to tame harmonic bundles, in terms of the parabolic filtration and the weight filtration. (See Theorem 13.29 of [**67**].) Since each $(E_{\mathfrak{a}}, \overline{\partial}_{\mathfrak{a}}, \theta_{\mathfrak{a}} - d\mathfrak{a}, h_{\mathfrak{a}})$ is tame, such an estimate can be applied to $(\mathcal{P}\mathcal{E}_3^{\lambda}, \mathbb{D}_3^{\lambda}, h_3)$, and transferred to $(\mathcal{P}\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda}, h)$ via the morphism Φ in Theorem 11.7.2. Hence, we obtain a satisfactory norm estimate for holomorphic sections of good wild harmonic bundles. □

11.7.3. Surface case. — We give a rather detailed description of the norm estimate for holomorphic sections of $\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}$ in the case $\dim X = 2$ and $D = D_1 \cup D_2$. In this subsection, $(E, \overline{\partial}_E, \theta, h)$ is assumed to be good wild, but not necessarily unramified.

We have the parabolic filtrations ${}^iF$ of $\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|D_i}$ $(i = 1, 2)$. Let ${}^i\mathrm{Gr}^F_a\big(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}\big) := {}^iF_a\big(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}\big)/{}^iF_{<a}\big(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}\big)$, which are equipped with the induced endomorphisms ${}^i\mathrm{Gr}_a(\mathrm{Res}_i(\mathbb{D}^{\lambda}))$. The eigenvalues of ${}^i\mathrm{Gr}_a(\mathrm{Res}_i(\mathbb{D}^{\lambda}))_{|Q}$ are independent of the choice of $Q \in D_i$. (This property is clear in the case $\lambda \neq 0$. It follows from Proposition 8.2.1 or the definition of good wild harmonic bundles, in the case $\lambda = 0$.) Hence, we have the well defined nilpotent part $N_{i,a}$ of $\mathrm{Gr}_a(\mathrm{Res}_i(\mathbb{D}^{\lambda}))$. Let $\mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, i)$ denote the set of $a \in \boldsymbol{R}$ such that ${}^i\mathrm{Gr}^F_a\big(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}\big) \neq 0$.

We put ${}^{2}\!F_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|O}) := {}^{1}F_{a_1}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|O}) \cap {}^{2}F_{a_2}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|O})$ for $\boldsymbol{a} = (a_1, a_2) \in \boldsymbol{R}^2$. Then, we put

$$ {}^{2}\mathrm{Gr}^{F}_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}) := \frac{{}^{2}\!F_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|O})}{\sum_{\boldsymbol{b}\lneq\boldsymbol{a}} {}^{2}\!F_{\boldsymbol{b}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}_{|O})}. $$

Let $\mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, O)$ denote the set of $\boldsymbol{a} \in \boldsymbol{R}^2$ such that ${}^{2}\mathrm{Gr}^{F}_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}) \neq 0$. On ${}^{2}\mathrm{Gr}^{F}_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda})$, we have the induced endomorphisms ${}^{2}\mathrm{Gr}_{\boldsymbol{a}}(\mathrm{Res}_i(\mathbb{D}^{\lambda}))$ $(i = 1, 2)$. The nilpotent parts are denoted by $N_{i,\boldsymbol{a}}$.

Lemma 11.7.4

- *The conjugacy classes of $N_{i,a|Q}$ are independent of the choice of $Q \in D_i$.*
- *Let $q_i : \mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, O) \to \mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, i)$ denote the projection. The conjugacy classes of $N_{i,a|O}$ and $\bigoplus_{\boldsymbol{a}\in q_i^{-1}(a)} N_{i,\boldsymbol{a}}$ are the same.*

Proof. — We take a ramified covering $\varphi : \widetilde{X} \to X$ given by $\varphi(\zeta_1, \zeta_2) = (\zeta_1^e, \zeta_2^e)$ such that $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is unramified. Let $(\mathcal{P}_*\widetilde{\mathcal{E}}^{\lambda}, \widetilde{\mathbb{D}}^{\lambda})$ denote the associated filtered λ-flat bundle. Take $\widetilde{Q} \in \widetilde{D}_i \setminus \{\widetilde{O}\}$ and $Q = \varphi(\widetilde{Q}) \in D_i \setminus \{O\}$. We have the following isomorphisms for any $-1 < b \leqslant 0$ which preserve the conjugacy classes of the nilpotent parts of the residues:

$$ {}^{i}\mathrm{Gr}^{F}_{b}(\mathcal{P}_0\widetilde{\mathcal{E}}^{\lambda})_{|\widetilde{Q}} \simeq \bigoplus_{\substack{a\in\mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, i)\\ ae-b\in\mathbb{Z}}} {}^{i}\mathrm{Gr}^{F}_{a}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda})_{|Q}. $$

For $\boldsymbol{b} \in \,]-1, 0]^2$, we have the following isomorphisms preserving the conjugacy classes of the nilpotent parts of the residues:

$$ {}^{2}\mathrm{Gr}^{F}_{\boldsymbol{b}}(\mathcal{P}_0\widetilde{\mathcal{E}}^{\lambda})_{|\widetilde{O}} \simeq \bigoplus_{\substack{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda}, O)\\ e\boldsymbol{a}-\boldsymbol{b}\in\mathbb{Z}^2}} {}^{2}\mathrm{Gr}^{F}_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda})_{|O}. $$

Hence, we only have to show the claim for $(\mathcal{P}_*\widetilde{\mathcal{E}}^{\lambda}, \widetilde{\mathbb{D}}^{\lambda})$, i.e., in the unramified case.

Let $(\widetilde{E}_0, \overline{\partial}_{\widetilde{E}_0}, \widetilde{\theta}_0, \widetilde{h}_0)$ be obtained as the full reduction from $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$. We can take an isomorphism $\widetilde{\Phi} : \mathcal{P}_0\widetilde{\mathcal{E}}^{\lambda}_0 \to \mathcal{P}_0\widetilde{\mathcal{E}}^{\lambda}$ which preserves the parabolic filtrations and the residues, as in Theorem 11.7.2. Hence, we only have to show the claim for tame harmonic bundles.

In the tame case, the claim is proven in [**6**′]. We indicate an outline. If λ is generic, i.e., the maps $\mathfrak{e}(\lambda) : \mathcal{KMS}(\mathcal{E}^0, i) \to \boldsymbol{C}$ are injective for any i, the claim is clear, because the conjugacy classes of the nilpotent parts of $\lambda^{-1}\,\mathrm{Res}_i(\mathbb{D}^{\lambda})_{|Q}$ for any $Q \in D_i$ are equal to those of the logarithm of the unipotent part of the monodromy around D_i. By using Corollary 12.43 of [**67**], we obtain that the conjugacy classes of the nilpotent parts of ${}^{2}\mathrm{Gr}^{F}\,\mathrm{Res}_i(\mathbb{D}^{\lambda})_{|O}$ and ${}^{i}\mathrm{Gr}^{F}\,\mathrm{Res}_i(\mathbb{D}^{\lambda})_{|Q}$ for $Q \in D_i \setminus \{O\}$ are independent of the choice of λ. Thus, we are done. □

We put $N_1 := \bigoplus N_{1,a}$. Let $W(N_1)$ denote the weight filtration of N_1 on ${}^{1}\mathrm{Gr}^{F}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda})$. In particular, we obtain the filtration $W(N_1)_{|O}$ of ${}^{1}\mathrm{Gr}^{F}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^{\lambda})_{|O}$. It induces the filtration on ${}^{\underline{2}}\mathrm{Gr}^F(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)$, which is denoted by $W(N_1)^{(1)}$. We also put $N_1^{(1)} := \bigoplus N_{1,\boldsymbol{a}}$, which is an endomorphism of ${}^{\underline{2}}\mathrm{Gr}^F(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)$. We have the weight filtration $W(N_1^{(1)})$ of ${}^{\underline{2}}\mathrm{Gr}^F(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)$.

Lemma 11.7.5. — *$W(N_1)^{(1)}$ and $W(N_1^{(1)})$ are the same.*

Proof. — Although this claim is proved in [**67**], we give an outline. We have the induced filtration ${}^2F$ on ${}^1\mathrm{Gr}^F(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)_{|O}$. Let $\mathcal{V}$ be the vector bundle on $\operatorname{Spec}\boldsymbol{C}[t]$ obtained as the Rees bundle associated to ${}^2F$. The endomorphism $N_{1|O}$ naturally induces $\mathcal{N}_1$ on $\mathcal{V}$. The restriction to $t=0$ is $N_{1|O}^{(1)}$. Since the degeneration of the conjugacy classes does not happen, the weight filtration of $W(\mathcal{N}_1)$ is the filtration in the category of the vector bundles on $\operatorname{Spec}\boldsymbol{C}[t]$. The specialization at $t=0$ is equal to $W(N_1^{(1)})$, and the specialization at $t\neq 0$ is equal to $W(N)$.

Let $\mathcal{W}$ denote the filtration naturally induced by $W(N_1)$. The specialization of $\mathcal{W}$ at $t=0$ is equal to $W(N_1)^{(1)}$. We also have $\mathcal{W}=W(\mathcal{N}_1)$. Hence, we obtain $W(N_1^{(1)})=W(N_1)^{(1)}$. □

We have the nilpotent endomorphism $N_{\boldsymbol{a}}(\underline{2})=N_{1,\boldsymbol{a}}+N_{2,\boldsymbol{a}}$.

Lemma 11.7.6. — *There exists a decomposition*

$$\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda=\bigoplus_{(\boldsymbol{a},\boldsymbol{k})\in\mathcal{P}ar(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda,O)\times\mathbb{Z}^2}U_{\boldsymbol{a},\boldsymbol{k}}$$

with the following properties:

- *It gives a splitting of the parabolic filtrations:*

$$\bigoplus_{q_i(\boldsymbol{b})\leqslant a}\bigoplus_{\boldsymbol{k}}U_{\boldsymbol{b},\boldsymbol{k}|D_i}={}^iF_a(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda_{|D_i}),$$
$$\bigoplus_{\boldsymbol{b}\leqslant\boldsymbol{a}}\bigoplus_{\boldsymbol{k}}U_{\boldsymbol{b},\boldsymbol{k}|O}={}^{\underline{2}}F_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda_{|O}).$$

- *Under the isomorphism ${}^1\mathrm{Gr}^F_a(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)\simeq\bigoplus_{\boldsymbol{a}\in q_1^{-1}(a)}\bigoplus_{\boldsymbol{k}}U_{\boldsymbol{a},\boldsymbol{k}|D_1}$, the following holds:*

$$\bigoplus_{\boldsymbol{a}\in q_1^{-1}(a)}\bigoplus_{k_1\leqslant\ell_1}U_{\boldsymbol{a},\boldsymbol{k}|D_1}=W_{\ell_1}(N_1)\big({}^1\mathrm{Gr}^F_a(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)\big).$$

Under the isomorphism ${}^{\underline{2}}\mathrm{Gr}^F_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)\simeq\bigoplus_{\boldsymbol{k}}U_{\boldsymbol{a},\boldsymbol{k}|O}$, the following holds:

$$\bigoplus_{\boldsymbol{k}\leqslant\boldsymbol{\ell}}U_{\boldsymbol{a},\boldsymbol{k}|O}=W_{\ell_1}(N_1)\cap W_{\ell_2}(N(\underline{2}))\Big({}^{\underline{2}}\mathrm{Gr}^F_{\boldsymbol{a}}(\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda)\Big).$$

Proof. — It follows from Lemma 11.7.5. (We can use a more general result, for example Corollary 4.47 of [**67**].) □

We take a decomposition as in Lemma 11.7.6. Let $\boldsymbol{v}$ be a frame of $\mathcal{P}_{\boldsymbol{c}}\mathcal{E}^\lambda$ compatible with the decomposition. When $v_i\in U_{\boldsymbol{a},\boldsymbol{k}}$, we put $\boldsymbol{a}(v_i):=\boldsymbol{a}$ and $\boldsymbol{k}(v_i):=\boldsymbol{k}$.

Let h_1 be the Hermitian metric given as follows:

$$h_1(v_i, v_j) := \delta_{i,j} \, |z_1|^{-2a_1(v_i)} \, |z_2|^{-2a_2(v_i)} \big(-\log|z_1|\big)^{k_1(v_i)} \big(-\log|z_2|\big)^{k_2(v_i)-k_1(v_i)}.$$

Let $Z := \big\{(z_1, z_2) \,\big|\, |z_1| < C|z_2|\big\}$.

Proposition 11.7.7. — *h and h_1 are mutually bounded on Z.*

Proof. — The problem can be reduced to the unramified case. It follows from Theorem 11.7.2 and the norm estimate in the tame case (Theorem 13.29 of [**67**]). □

11.7.4. Blow up. — Let $\widetilde{X} := \Delta^2 = \{(\zeta_1, \zeta_2)\}$. Let $\pi : \widetilde{X} \to X$ be given by $\pi(\zeta_1, \zeta_2) = (\zeta_1\zeta_2, \zeta_2)$. Let $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \pi^{-1}(E, \overline{\partial}_E, \theta, h)$. We have the associated filtered bundle $\mathcal{P}_*\widetilde{\mathcal{E}}^\lambda$.

Lemma 11.7.8. — *$\mathcal{P}_*\widetilde{\mathcal{E}}^\lambda$ is obtained from $\mathcal{P}_*\mathcal{E}^\lambda$ by the procedure in Section* 2.5.3.3.

Proof. — It follows from the weak norm estimate for the acceptable bundles. □

For simplicity, we consider the case $\boldsymbol{c} = (0,0)$. In the case $a_1(v_i) + a_2(v_i) > -1$, we put $\widetilde{v}_i := \pi^* v_i$. In the case $a_1(v_i) + a_2(v_i) \leqslant -1$, we put $\widetilde{v}_i := \pi^* v_i \, \zeta_2^{-1}$. Then, $\widetilde{\boldsymbol{v}} = (\widetilde{v}_i)$ gives a frame of $\mathcal{P}_0\widetilde{\mathcal{E}}^\lambda$ compatible with the parabolic structure. We put $a_j(\widetilde{v}_i) := {}^j\deg^F(\widetilde{v}_i)$. We also put $k_j(\widetilde{v}_i) := k_j(v_i)$. Let $\widetilde{h}_0$ be the Hermitian metric given as follows:

$$\widetilde{h}_0(\widetilde{v}_i, \widetilde{v}_j) = \delta_{i,j} \, |\zeta_1|^{-2a_1(\widetilde{v}_i)} \, |\zeta_2|^{-2a_2(\widetilde{v}_i)}.$$

Let χ be a non-negative valued function on $\boldsymbol{R}$ such that $\chi(t) = 1$ $(t \leqslant 1/2)$ and $\chi(t) = 0$ $(t \geqslant 2/3)$. Let $\rho(\zeta) : \boldsymbol{C}^* \to \boldsymbol{R}$ be the function given by $\rho(\zeta) = -\chi(|\zeta|) \log|\zeta|^2$. We set

$$\widetilde{h}_1(\widetilde{v}_i, \widetilde{v}_j) := \widetilde{h}_0(\widetilde{v}_i, \widetilde{v}_j) \, (1 + \rho(\zeta_1) + \rho(\zeta_2))^{k_1(\widetilde{v}_i)} \, (1 + \rho(\zeta_2))^{k_2(\widetilde{v}_i) - k_1(\widetilde{v}_i)}.$$

Lemma 11.7.9. — *$\pi^* h$ and $\widetilde{h}_1$ are mutually bounded. The curvature $R(\widetilde{h}_0)$ is* 0. *Moreover, $R(\widetilde{h}_1)$ and $\partial_{\widetilde{h}_1} - \partial_{\widetilde{h}_0}$ are bounded with respect to both $(\widetilde{\omega}, \widetilde{h}_i)$ $(i = 0, 1)$, where $\widetilde{\omega}$ denotes the Poincaré metric of $\widetilde{X} - \widetilde{D}$.*

Proof. — The first claim follows from the norm estimate. The other claim can be shown by direct calculations. □

11.8. Regular meromorphic variation of twistor structure on a disc (Appendix)

Let X be the disc $\{z \in \boldsymbol{C} \,|\, |z| < 1\}$, and let $D = \{0\}$. We consider a regular meromorphic extension of a variation of twistor structure on $\mathbb{P}^1 \times (X \setminus D)$. In [**67**], we showed that if the variation of twistor structure is pure and polarized, then the limit twistor structure is mixed polarized. We shall study the converse (Lemma 11.8.6).

Although a similar result is given in [**73**], we would like to understand it from our viewpoint.

We shall review the construction of the limit twistor structure (Subsections 11.8.1–11.8.3). Then, we study the characterization of pure and polarized property.

11.8.1. Preliminary. — Let $\lambda_0 \in \boldsymbol{C}$, and let $\mathcal{K}$ be a neighbourhood of λ_0 in $\boldsymbol{C}$. We put $\mathcal{X} := \mathcal{K} \times X$ and $\mathcal{D} := \mathcal{K} \times D$. In this case, $\mathcal{D}$ can be identified with $\mathcal{K}$ naturally.

Let $(\boldsymbol{V}_*, \mathbb{D})$ be a good family of filtered λ-flat bundles, which is regular in the sense $\mathbb{D}({}_a V) \subset {}_a V \otimes \Omega^{1,0}(\log D)$. The restriction to $\mathcal{X} \setminus \mathcal{D}$ is denoted by V. Assume that $\boldsymbol{V}_*$ has the KMS-structure at λ_0 indexed by $T \subset \boldsymbol{R} \times \boldsymbol{C}$, i.e., $\mathcal{KMS}(\boldsymbol{V}_*) = \big\{\mathfrak{k}(\lambda_0, u) \,\big|\, u \in T\big\}$. (See Section 2.8.) We have the natural $\mathbb{Z}$-action on T given by $n \cdot u = u + (n, 0)$. We recall some objects induced by the KMS structure.

11.8.1.1. The induced bundle $\mathcal{G}_u(V)$. — Recall that we have the induced filtration $F^{(\lambda_0)}$ of ${}_a V$ and the generalized eigen-decomposition

$$\operatorname{Gr}_a^{F^{(\lambda_0)}}(V) = \bigoplus_{\substack{u \in T, \\ \mathfrak{k}(\lambda_0, u) = a}} \mathcal{G}_u^{(\lambda_0)}(V)$$

on $\mathcal{K}$, as in Section 2.8.2. Let $\mathcal{N}_u$ denote the nilpotent part of the residue on $\mathcal{G}_u^{(\lambda_0)}(V)$.

11.8.1.2. The KMS-structure of the space of multi-valued flat sections. — Assume $\lambda_0 \neq 0$. The restriction of $(V, \mathbb{D})$ to $\{\lambda\} \times (X \setminus D)$ is denoted by $(V^\lambda, \mathbb{D}^\lambda)$. Let $\mathcal{H}(V)$ denote the holomorphic vector bundle on $\mathcal{K}$, whose fiber over λ is the space $H(V^\lambda)$ of the multi-valued flat sections of $(V^\lambda, \mathbb{D}^{\lambda,f})$. We have the monodromy automorphism M along the loop with the counter-clockwise direction. The restriction to λ is denoted by M^λ. The set of the eigenvalues of M^λ is given by $\mathcal{S}p^f(V^\lambda) := \big\{\mathfrak{e}^f(\lambda, u) \,\big|\, u \in T/\mathbb{Z}\big\}$. (Recall $\mathfrak{e}^f(\lambda, u) = \exp\big(-2\pi\sqrt{-1}\lambda^{-1}\mathfrak{e}(\lambda, u)\big)$.) We have the unique monodromy invariant decomposition

$$\mathcal{H}(V) = \bigoplus_{\omega \in \mathcal{S}p^f(V^{\lambda_0})} \mathbb{E}_\omega^{(\lambda_0)} \mathcal{H}(V) \tag{250}$$

whose restriction to λ_0 is the same as the generalized eigen-decomposition of M^{λ_0}.

Let $\mathcal{K}^* := \mathcal{K} \setminus \{\lambda_0\}$. We may assume that any $\lambda \in \mathcal{K}^*$ is generic. We have the generalized eigen-decomposition:

$$\mathcal{H}(V)_{|\mathcal{K}^*} = \bigoplus_{u \in T/\mathbb{Z}} \mathbb{E}_{\mathfrak{e}^f(\lambda,u)} \mathcal{H}(V)_{|\mathcal{K}^*}.$$

Here, the fiber of $\mathbb{E}_{\mathfrak{e}^f(\lambda,u)} \mathcal{H}(V)_{|\mathcal{K}^*}$ over λ is the generalized eigen space of M^λ corresponding to $\mathfrak{e}^f(\lambda, u)$. We put

$$\mathcal{F}_b^{(\lambda_0)} \mathcal{H}(V)_{|\mathcal{K}^*} := \bigoplus_{\substack{u \in T/\mathbb{Z} \\ \mathfrak{p}^f(\lambda_0, u) \leqslant b}} \mathbb{E}_{\mathfrak{e}(\lambda,u)} \mathcal{H}(V)_{|\mathcal{K}^*},$$

where $\mathfrak{p}^f(\lambda,u) := \mathrm{Re}\big(\lambda\overline{\alpha}+\lambda^{-1}\alpha\big) = \mathfrak{p}(\lambda,u)+\mathrm{Re}\big(\lambda^{-1}\,\mathfrak{e}(\lambda,u)\big)$. Thus, we obtain a filtration indexed by $\big\{\mathfrak{p}^f(\lambda,u)\,\big|\,u\in T/\mathbb{Z}\big\}\subset \boldsymbol{R}$. It can be extended to a filtration of $\mathcal{H}(V)$ on $\mathcal{K}$, denoted by $\mathcal{F}^{(\lambda_0)}$. It is monodromy invariant, and compatible with the decomposition $\mathbb{E}^{(\lambda_0)}$.

We put $\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V) := \mathrm{Gr}^{\mathcal{F}^{(\lambda_0)},\mathbb{E}^{(\lambda_0)}}_{\mathfrak{k}^f(\lambda_0,u)}\mathcal{H}(V) = \mathrm{Gr}^{\mathcal{F}^{(\lambda_0)}}_{\mathfrak{p}^f(\lambda_0,u)}\,\mathbb{E}^{(\lambda_0)}_{\mathfrak{e}^f(\lambda_0,u)}\mathcal{H}(V)$ on $\mathcal{K}$. We have the automorphism M_u induced by M, whose unipotent part is denoted by M_u^{uni}. We put $N_u := (-2\pi\sqrt{-1})^{-1}\log M_u^{\mathrm{uni}}$.

Remark 11.8.1. — $\mathcal{F}^{(\lambda_0)}$ is not related with Stoke filtrations. Because we consider this kind of filtration only in the regular case, there is no risk of confusion. □

11.8.1.3. The decomposition and the filtration of ${}_aV$. — The decomposition (250) induces a $\mathbb{D}$-flat decomposition $\boldsymbol{V}_* = \bigoplus_{\omega\in\mathcal{S}p^f(V^{\lambda_0})}\boldsymbol{V}_{\omega,*}$. The filtration $\mathcal{F}^{(\lambda_0)}$ of $\mathcal{H}(V)$ naturally induces a $\mathbb{D}$-flat filtration of V on $\mathcal{X}\setminus\mathcal{D}$, which is also denoted by $\mathcal{F}^{(\lambda_0)}$. By construction, the subbundles $\mathcal{F}_b^{(\lambda_0)}$ of V are naturally extended to those of ${}_aV_{|\mathcal{X}\setminus\{(\lambda_0,O)\}}$. Moreover, we have the following lemma.

Lemma 11.8.2. — *$\mathcal{F}_b^{(\lambda_0)}$ are naturally extended to subbundles of ${}_aV$ on $\mathcal{X}$. Namely, we obtain an induced filtration $\mathcal{F}^{(\lambda_0)}$ of ${}_aV$ in the category of vector bundles on $\mathcal{X}$.*

Proof. — It is easy to reduce the problem to the case $\mathcal{S}p^f(V^{\lambda_0})=\{1\}$. First, let us consider the case where (i) b is the minimum among the numbers c such that $\mathrm{Gr}_c^{\mathcal{F}^{(\lambda_0)}}\mathcal{H}(V)\neq 0$, (ii) $\mathrm{rank}\,\mathcal{F}_b^{(\lambda_0)}\mathcal{H}(V)=1$. Let s be a frame of $\mathcal{F}_b^{(\lambda_0)}\mathcal{H}(V)$. We have the element $u_0\in T$ such that $\mathfrak{p}^f(\lambda_0,u_0)=b$ and $a-1<\mathfrak{p}(\lambda_0,u_0)\leqslant a$. Then, $t := s\cdot\exp\big(\lambda^{-1}\mathfrak{e}(\lambda,u_0)\log z\big)$ naturally gives a single-valued holomorphic section of $\mathcal{F}_b^{(\lambda_0)}\big({}_aV_{|\mathcal{X}\setminus\{(\lambda_0,O)\}}\big)$. It can be extended to a section of ${}_aV$. Let us show $t_{|(\lambda_0,O)}\neq 0$ in ${}_aV^{\lambda_0}{}_{|O}$. Let t^{λ_0} denote the restriction of t to $\{\lambda_0\}\times X$. We have the relation $\mathbb{D}^{\lambda_0}t^{\lambda_0} = t^{\lambda_0}\cdot\mathfrak{e}(\lambda_0,u_0)\,dz/z$. If $t^{\lambda_0}_{|O}=0$, there exists $\big(c,\mathfrak{e}(\lambda_0,u_0)\big)\in\mathcal{KMS}(\boldsymbol{V}^{\lambda_0}_*)$ such that $c<a$. However, it is easy to show $\mathfrak{p}(\lambda_0,u)\geqslant\mathfrak{p}(\lambda_0,u_0)=a$ for any $u\in T$ such that $\mathfrak{e}(\lambda,u)=\mathfrak{e}(\lambda_0,u_0)$, by using the relation $\mathfrak{p}(\lambda_0,u)-\mathfrak{p}(\lambda_0,u_0) = \mathfrak{p}^f(\lambda_0,u)-\mathfrak{p}^f(\lambda_0,u_0)$. Hence, we obtain $t^{\lambda_0}_{|(\lambda_0,O)}\neq 0$ in ${}_aV_{|(\lambda,O)}$. Thus, $\mathcal{F}_b^{(\lambda_0)}$ gives a subbundle of ${}_aV$ on $\mathcal{X}$ in this case.

We can reduce the general case to the above case, by using the exterior product. □

By construction, the filtration $\mathcal{F}^{(\lambda_0)}$ is compatible with the decomposition ${}_aV = \bigoplus{}_aV_\omega$. Let us look at the restriction of the decomposition and the filtration to $\mathcal{D}$. Clearly, we have

$${}_aV_{\omega|\mathcal{D}} = \bigoplus_{\exp(-2\pi\sqrt{-1}(\alpha/\lambda))=\omega}\mathbb{E}_\alpha^{(\lambda_0)}({}_aV_{|\mathcal{D}}).$$

(See Remark 2.8.2 for $\mathbb{E}_\alpha^{(\lambda_0)}$.) We take $u\in T$ such that $a-1<\mathfrak{p}(\lambda_0,u)\leqslant a$ and $\mathfrak{e}^f(\lambda_0,u)=\omega$. It is easy to show

$$\mathcal{F}^{(\lambda_0)}_{\mathfrak{p}(\lambda_0,u)}\big({}_aV_\omega\big)_{|\mathcal{D}}' = F^{(\lambda_0)}_{\mathfrak{p}(\lambda_0,u)}\mathbb{E}^{(\lambda_0)}_{\mathfrak{e}(\lambda_0,u)}({}_aV_{|\mathcal{D}})\oplus\bigoplus_{\alpha\in S_1}\mathbb{E}_\alpha^{(\lambda_0)}({}_aV_{|\mathcal{D}}). \tag{251}$$

Here, $S_1 := \{\exp(-2\pi\sqrt{-1}(\alpha/\lambda_0)) = \omega,\ \lambda_0^{-1}(\alpha - \mathfrak{e}(\lambda_0,u)) \in \mathbb{Z}_{<0}\}$.

11.8.1.4. The induced bundle $\mathcal{G}_u\mathcal{V}$ and the isomorphism. — For any $u \in T$, we set

$$\mathcal{G}_u^{(\lambda_0)}\mathcal{V} := \frac{\mathcal{F}^{(\lambda_0)}_{\mathfrak{p}^f(\lambda_0,u)}\Big({}_{\mathfrak{p}(\lambda_0,u)}V_{\mathfrak{e}^f(\lambda_0,u)}\Big)}{\mathcal{F}^{(\lambda_0)}_{<\mathfrak{p}^f(\lambda_0,u)}\Big({}_{\mathfrak{p}(\lambda_0,u)}V_{\mathfrak{e}^f(\lambda_0,u)}\Big)}.$$

We have an induced family of flat λ-connections $\mathbb{D}_u$ on $\mathcal{G}_u^{(\lambda_0)}\mathcal{V}$, which is logarithmic. The eigenvalue of the residue $\mathrm{Res}(\mathbb{D}_u)$ is $\mathfrak{e}(\lambda,u)$. The nilpotent part is denoted by $\mathcal{N}_u$.

By (251), we have a natural isomorphism $(\mathcal{G}_u^{(\lambda_0)}\mathcal{V}_{|\mathcal{D}}, \mathcal{N}_u) \simeq (\mathcal{G}_u^{(\lambda_0)}(V), \mathcal{N}_u)$. The space of the multi-valued flat sections $\mathcal{H}(\mathcal{G}_u^{(\lambda_0)}\mathcal{V})$ with the monodromy is naturally isomorphic to $\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V)$ with M_u. In this situation, we have an isomorphism Φ^{can} : $(\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V), \lambda\, N_u) \to (\mathcal{G}_u^{(\lambda_0)}(V), \mathcal{N}_u)$ which is given as follows. (See Section 10.4.1 of [**67**], for example.) Let F be a section of $\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V)$. We can regard it as a multi-valued flat section of $\mathcal{G}_u\mathcal{V}$ on $\mathcal{X} \setminus \mathcal{D}$. Then, we put $\Phi_u^{\mathrm{can}}(F) := F_0$ for the expansion

$$F = \sum_{j=0}^{m} F_j\, \exp\big(-\lambda^{-1}\mathfrak{e}(\lambda,u)\log z\big)\, (\log z)^j,$$

where F_j are holomorphic sections of $\mathcal{G}_u\mathcal{V}$.

11.8.2. Globalization. — Let $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{D} := \boldsymbol{C}_\lambda \times D$. If $(V,\mathbb{D})$ is a family of meromorphic λ-flat bundles on $(\mathcal{X},\mathcal{D})$, which has the KMS-structure at each $\lambda_0 \in \boldsymbol{C}_\lambda$ indexed by T. For each λ_0, we have a neighbourhood $U(\lambda_0)$ of λ_0 in $\boldsymbol{C}_\lambda$, and we obtain $(\mathcal{G}_u^{(\lambda_0)}(V), \mathcal{N}_u)$ on $U(\lambda_0)$. In the case $\lambda_0 \neq 0$, we also have $(\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V),\, N_u)$ and the isomorphism $\Phi^{\mathrm{can}} : (\mathcal{G}_u^{(\lambda_0)}\mathcal{H}(V),\, \lambda\, N_u) \to (\mathcal{G}_u^{(\lambda_0)}(V), \mathcal{N}_u)$. By using the uniqueness of KMS-structure (see Lemma 2.8.3, for example), we can glue them for various λ_0, and so we obtain the following:

- A bundle $\mathcal{G}_u(V)$ with the nilpotent endomorphism $\mathcal{N}_u$ on $\boldsymbol{C}_\lambda$.
- A bundle $\mathcal{G}_u\mathcal{H}(V)$ with the nilpotent endomorphism N_u on $\boldsymbol{C}_\lambda^*$.
- An isomorphism $\Phi^{\mathrm{can}} : (\mathcal{G}_u\mathcal{H}(V), \lambda\, N_u) \to (\mathcal{G}_u(V), \mathcal{N}_u)_{|\boldsymbol{C}_\lambda^*}$.

We also obtain a vector bundle $\mathcal{G}_u\mathcal{V}$ on $\boldsymbol{C}_\lambda^* \times X$ with a family of logarithmic flat λ-connections $\mathbb{D}_u$, which is locally constructed as above.

Remark 11.8.3. — The construction is functorial and compatible with dual, tensor product, and direct sum. □

11.8.3. Gluing. — Let $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$. We use the symbols $\mathcal{D}$ and $\mathcal{D}^\dagger$ with similar meanings. For $\lambda_0 \in \boldsymbol{C}_\lambda$, let $\mathcal{X}^{(\lambda_0)}$ denote a product of X and a neighbourhood $U(\lambda_0)$ of λ_0. We use the symbols $\mathcal{D}^{(\lambda_0)}$, $\mathcal{X}^{\dagger(\mu_0)}$, and $\mathcal{D}^{\dagger(\mu_0)}$ with similar meanings.

Let $(V,\mathbb{D}^\triangle)$ be a variation of twistor structure on $\mathbb{P}^1 \times (X \setminus D)$ with an unramifiedly good meromorphic prolongment $(\widetilde{V}_0, \widetilde{V}_\infty)$. Moreover, we assume the following:

- $\widetilde{V}_0$ has the KMS-structure at each λ_0. Namely, we have locally free $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-lattices $\mathcal{P}^{(\lambda_0)}_a(V) \subset \widetilde{V}_0$ ($a \in \boldsymbol{R}$) such that $(\mathcal{P}^{(\lambda_0)}_*(V_0), \mathbb{D}_{V_0})$ is a regular family of filtered λ-flat bundles with KMS-structure indexed by T_0.
- Similarly, for each μ_0, we have locally free $\mathcal{O}_{\mathcal{X}^{\dagger(\mu_0)}}$-lattices $\mathcal{P}^{(\mu_0)}_a(V_\infty) \subset \widetilde{V}_\infty$ ($a \in \boldsymbol{R}$) such that $(\mathcal{P}^{(\mu_0)}_*(V_\infty), \mathbb{D}^\dagger_{V_\infty})$ is a regular family of filtered μ-flat bundles with KMS-structure indexed by T_∞.

In that case, we say that $(\widetilde{V}_0, \widetilde{V}_\infty)$ is a regular meromorphic extension of $(V, \mathbb{D}^\triangle)$.

Lemma 11.8.4

- *The map $u = (a, \alpha) \mapsto u^\dagger = (-a, \overline{\alpha})$ induces a bijection $T_0 \simeq T_\infty$.*
- *We have a natural isomorphism $\big(\mathcal{G}_u\mathcal{H}(V_0), N_u\big) \simeq \big(\mathcal{G}_{u^\dagger}\mathcal{H}(V_\infty), -N^\dagger_{u^\dagger}\big)$.*

Proof. — Let $(V_0^\lambda, \mathbb{D}^\lambda)$ denote the restriction of $(V_0, \mathbb{D}_{V_0})$ to $\{\lambda\} \times X$. Let $\mathcal{S}p\big(\mathcal{P}_a V_0^\lambda\big)$ denote the set of the eigenvalues of $\mathrm{Res}(\mathbb{D}^\lambda)$, and $\mathcal{S}p(V_0^\lambda) := \bigcup_{a \in \boldsymbol{R}} \mathcal{S}p\big(\mathcal{P}_a V_0^\lambda\big)$. Let $\mathcal{S}p^f\big(V_0^\lambda\big)$ denote the set of the eigenvalues of the monodromy on the space of multi-valued flat sections of V_0^λ, where the monodromy is taken along the loop with the counter-clockwise direction. Then, we have the bijective correspondence between $\mathcal{S}p(V_0^\lambda)/\mathbb{Z}$ and $\mathcal{S}p^f(V_0^\lambda)$ given by $\alpha \longleftrightarrow \exp\big(-2\pi\sqrt{-1}(\alpha/\lambda)\big)$. (The action of $\mathbb{Z}$ on $\mathcal{S}p(V^\lambda)$ is given by $n \cdot \alpha = \alpha + n\lambda$.) Hence, the set $\mathcal{S}p(V_0^\lambda)$ is determined by $\mathcal{S}p^f(V_0^\lambda)$.

Note that $\mathcal{S}p(V_0^\lambda)$ is the image of T_0 via the map $\mathfrak{e}(\lambda)$. There exists a discrete subset Z_0 of $\boldsymbol{C}^*_\lambda$ such that $\mathfrak{e}(\lambda) : T_0 \to \boldsymbol{C}$ is injective for any $\lambda \in \boldsymbol{C}^*_\lambda \setminus Z_0$. Hence, T_0 is determined by the family of sets $\big\{\mathcal{S}p^f(V_0^\lambda) \,\big|\, \lambda \in \boldsymbol{C}^*_\lambda \setminus Z_0\big\}$.

Similarly, let V_∞^μ denote the restriction of V_∞ to $\{\mu\} \times (X^\dagger \setminus D^\dagger)$, and let $\mathcal{S}p^f(V_\infty^\mu)$ ($\mu \neq 0$) denote the set of the eigenvalues of the monodromy on the space of the multi-valued flat sections of V_∞^μ, where the monodromy is taken along the loop with the clockwise direction. There exists a discrete subset $Z_\infty \subset \boldsymbol{C}^*_\mu$ such that $\mathfrak{e}(\mu) : T_\infty \to \boldsymbol{C}$ is injective for any $\mu \in \boldsymbol{C}_\mu \setminus Z_\infty$, and the set T_∞ is determined by the family of the sets $\big\{\mathcal{S}p^f(V_\infty^\mu) \,\big|\, \mu \in \boldsymbol{C}^*_\mu \setminus Z_\infty\big\}$.

For $\lambda = \mu^{-1}$, we have a natural bijection $\mathcal{S}p^f(V_0^\lambda) \simeq \mathcal{S}p^f(V_\infty^\mu)$ given by $\omega \leftrightarrow \omega^{-1}$. Then, we obtain the desired bijection $T_0 \simeq T_\infty$ by formal calculation. Note that $\mathfrak{e}(\lambda, u) = \mathfrak{e}(\lambda^{-1}, u^\dagger)$. Hence, we obtain the first claim.

We have the natural identification $\mathcal{H}(V_0) \simeq \mathcal{H}(V_\infty)$ on $\boldsymbol{C}^*_\lambda = \boldsymbol{C}^*_\mu$. By using $\mathfrak{p}^f(\lambda, u) = \mathfrak{p}^f(\lambda^{-1}, u^\dagger)$ and $\mathfrak{e}^f(\lambda, u) = \mathfrak{e}^f(\lambda^{-1}, u^\dagger)^{-1}$, we can show the second claim. □

Then, we obtain the vector bundle $S_u(V)$ on $\mathbb{P}^1$ by gluing $\mathcal{G}_u(V_0)$ and $\mathcal{G}_{u^\dagger}(V_\infty)$. We also obtain the nilpotent map $\mathcal{N}^\triangle : S_u(V) \to S_u(V) \otimes \mathbb{T}(-1)$, where $\mathcal{N}^\triangle_{|\boldsymbol{C}_\lambda} := \mathcal{N}_u \cdot t_0^{(-1)}$ and $\mathcal{N}^\triangle_{|\boldsymbol{C}_\mu} = \mathcal{N}^\dagger_{u^\dagger} \cdot t_\infty^{(-1)}$.

Remark 11.8.5. — The construction is functorial and compatible with dual, tensor product, and direct sum. □

11.8.3.1. Pairing. — Let $\mathcal{S} : V \otimes \sigma^*(V) \to \mathbb{T}(0)$ be a symmetric perfect pairing. By the regularity, $\mathcal{S}_0$ can be extended to a pairing $\widetilde{V}_0 \otimes \sigma^* \widetilde{V}_\infty \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. Similarly, $\mathcal{S}_\infty$ can be extended to a pairing $\widetilde{V}_\infty \otimes \sigma^* \widetilde{V}_0 \to \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$, i.e., $(\widetilde{V}_0, \widetilde{V}_\infty)$ is a meromorphic prolongment of $(V, \mathbb{D}^\triangle, \mathcal{S})$. By using the uniqueness of the KMS-structure, we obtain an induced pairings:

- $\mathcal{S}_u : \mathcal{G}_u(V_0) \otimes \sigma^* \mathcal{G}_{u^\dagger}(V_\infty) \to \mathcal{O}_{\boldsymbol{C}_\lambda}$ satisfying $\mathcal{S}_u(\mathcal{N}_u \otimes \mathrm{id}) - \mathcal{S}_u(\mathrm{id} \otimes \sigma^* \mathcal{N}^\dagger_{u^\dagger}) = 0$. (See Lemma 6.1.5 for the signature.)
- $\mathcal{S}_u : \mathcal{G}_u \mathcal{H}(V_0) \otimes \sigma^* \mathcal{G}_{u^\dagger}\ \mathcal{H}(V_\infty) \to \mathcal{O}_{\boldsymbol{C}^*_\lambda}$.
- $\mathcal{S}_u : \mathcal{G}_u \mathcal{V}_0 \otimes \sigma^* \mathcal{G}_{u^\dagger} \mathcal{V}_\infty \to \mathcal{O}_{\boldsymbol{C}^*_\lambda \times X}$.

Then, we can show that the gluing of $\mathcal{G}_u(V_0)$ and $\mathcal{G}_{u^\dagger}(V_\infty)$ is compatible with the pairing. Thus, we obtain the symmetric pairing $\mathcal{S}_u : S_u(V) \otimes \sigma^* S_u(V) \to \mathbb{T}(0)$, which satisfies $\mathcal{S}_u\big(\mathcal{N}^\triangle_u \otimes \mathrm{id}\big) + \mathcal{S}_u\big(\mathrm{id} \otimes \sigma^*(\mathcal{N}^\triangle_u)\big) = 0$.

11.8.4. A characterization of purity and polarizability. — A result similar to the following lemma was shown in [**73**] with a different argument.

Lemma 11.8.6. — *If $\big(S_u(V), \mathcal{N}^\triangle_u, \mathcal{S}_u\big)$ is a polarized mixed twistor structure of weight 0 for each $u \in T$, then $(V, \mathbb{D}^\triangle, \mathcal{S})$ is a variation of polarized pure twistor structure of weight 0 after X is shrunk around D.*

Proof. — We have the polarized variation of pure twistor structure $\big(V^{(1)}_u, \mathbb{D}^{\triangle(1)}_u, \mathcal{S}^{(1)}_u\big)$ induced by the polarized mixed twistor structure $\big(S_u(V), \mathcal{N}^\triangle_u, \mathcal{S}_u\big)$. (See Sections 3.5.3, 3.6 and 3.7.5 of [**67**].) We have the corresponding tame harmonic bundle $\big(E^{(1)}_{0,u}, \overline{\partial}_{0,u}, \theta^{(1)}_{0,u}, h^{(1)}_{0,u}\big)$. We put

$$\big(E^{(1)}, \overline{\partial}_{E^{(1)}}, \theta^{(1)}, h^{(1)}\big) := \bigoplus_u \big(E^{(1)}_u, \overline{\partial}^{(1)}_u, \theta^{(1)}_u, h^{(1)}_u\big),$$
$$\big(E^{(1)}_u, \overline{\partial}^{(1)}_u, \theta^{(1)}_u, h^{(1)}_u\big) = \big(E^{(1)}_{0,u}, \overline{\partial}_{0,u}, \theta^{(1)}_{0,u}, h^{(1)}_{0,u}\big) \otimes L(u).$$

Let $\big(V^{(1)}, \mathbb{D}^{(1)\triangle}, \mathcal{S}^{(1)\triangle}\big)$ be the corresponding variation of polarized pure twistor structure. We have the regular meromorphic extension $(\widetilde{V}^{(1)}_0, \widetilde{V}^{(1)}_\infty)$. We recover $\bigoplus\big(S_u(V), \mathcal{N}^\triangle_u, \mathcal{S}^\triangle_u\big)$ by applying the construction in Sections 11.8.3–11.8.3.1 to $\big(V^{(1)}, \mathbb{D}^{(1)\triangle}, \mathcal{S}^{(1)\triangle}\big)$. Namely, we have the canonical isomorphisms $G_u : S_u(V^{(1)}) \simeq S_u(V)$ compatible with the nilpotent maps and the pairings.

Let us construct a C^∞-map $f : V^{(1)} \to V$. In the following, $U(\lambda_0)$ denotes a small neighbourhood of $\lambda_0 \in \boldsymbol{C}_\lambda$. Note we have the isomorphisms:

(252)
$$\mathrm{Gr}^{F^{(\lambda_0)}}_a(V^{(1)}_0) \simeq \bigoplus_{\mathfrak{p}(\lambda_0,u)=a} S_u(V^{(1)}_0)_{|U(\lambda_0)}, \quad \mathrm{Gr}^{F^{(\lambda_0)}}_a(V_0) \simeq \bigoplus_{\mathfrak{p}(\lambda_0,u)=a} S_u(V_0)_{|U(\lambda_0)}.$$

If λ_0 is generic, we have the unique *flat* isomorphism $f_{U(\lambda_0)} : \mathcal{P}^{(\lambda_0)}_*(V^{(1)}_0) \to \mathcal{P}^{(\lambda_0)}_*(V_0)$ with the following property:

- Under the isomorphisms (252), the induced map
$$\mathrm{Gr}_a^{F^{(\lambda_0)}}(f_{U(\lambda_0)}) : \mathrm{Gr}_a^{F^{(\lambda_0)}}(V_0^{(1)}) \simeq \mathrm{Gr}_a^{F^{(\lambda_0)}}(V_0)$$
is equal to the restriction of $\bigoplus_{\mathfrak{p}(\lambda_0,u)=a} G_u$.

Even if λ_0 is not generic, we can take a holomorphic (not necessarily flat) isomorphism $f_{U(\lambda_0)} : \mathcal{P}_*^{(\lambda_0)}(V_0^{(1)}) \to \mathcal{P}_*^{(\lambda_0)}(V_0)$ of the filtered bundles on $U(\lambda_0) \times (X, D)$ with the above property. If $f'_{U(\lambda_0)}$ is another isomorphism satisfying the above condition, we have

$$\bigl|\mathrm{id} - f_{U(\lambda_0)}^{-1} \circ f'_{U(\lambda_0)}\bigr|_{h^{(1)}} = O\bigl(|z|^{\varepsilon}\bigr). \tag{253}$$

Let $U(\mu_0)$ denote a neighbourhood of $\mu_0 \in \boldsymbol{C}_\mu$. We take isomorphisms $f^{\dagger}_{U(\mu_0)} : \mathcal{P}_*^{(\mu_0)}(V_\infty^{(1)}) \to \mathcal{P}_*^{(\mu_0)}(V_\infty)$ of filtered flat bundles on $U(\mu_0) \times (X^{\dagger}, D^{\dagger})$ with a similar property for each $\mu_0 \in \boldsymbol{C}_\mu$. Let us look at the morphism:

$$J := \mathcal{S}_0^{(1)} - \mathcal{S}_0 \circ \bigl(f_{U(\lambda_0)} \otimes \sigma^* f^{\dagger}_{U(\sigma(\lambda_0))}\bigr) : \mathcal{P}_*^{(\lambda_0)}(V_0) \otimes \sigma^* \mathcal{P}_*^{(\sigma(\lambda_0))}(V_\infty) \longrightarrow \mathcal{O}_{\mathcal{X}^{(\lambda_0)}}(*\mathcal{D}^{(\lambda_0)}).$$

Because of the conditions for $f_{U(\lambda_0)}$ and $f^{\dagger}_{U(\sigma(\lambda_0))}$, we have $|J| = O\bigl(|z|^{\varepsilon}\bigr)$ for some $\varepsilon > 0$ with respect to the metric $h^{(1)}$.

We take compact regions $W_1 \subset \boldsymbol{C}_\lambda$ and $W_2 \subset \boldsymbol{C}_\mu$ such that (i) $W_1 \cup W_2 = \mathbb{P}^1$, (ii) any $\lambda \in W_1 \cap W_2$ is generic. We can take a finite covering $W_1 \subset \bigcup U(\lambda_0)$. By gluing $f_{U(\lambda_0)}$ in C^∞, we obtain f_{W_1}. Similarly, we obtain $f^{\dagger}_{W_2}$. Note $f_{W_1|W_1\cap W_2} = f^{\dagger}_{W_2|W_1\cap W_2}$. By gluing them in C^∞, we obtain f.

Let $f_Q := f_{|\mathbb{P}^1\times Q}$. Let $V_Q := V_{|\mathbb{P}^1\times Q}$ and $V_Q^{(1)} := V^{(1)}_{|\mathbb{P}^1\times Q}$. Then, we have
$$f_Q^{-1} \circ \overline{\partial}_{V_Q} \circ f_Q - \overline{\partial}_{V_Q^{(1)}} = O\bigl(|z(Q)|^{\varepsilon}\bigr)$$
with respect to $h^{(1)}$ due to (253). Hence, $(V, \mathbb{D}^{\triangle})$ is a variation of pure twistor structure if Q is sufficiently close to O, due to Lemma 11.3.5. Because of the estimate of J above, there exists a constant $C > 0$ such that the following holds for any $u \in V^{(1)}_{|(\lambda,Q)}$ and $v \in V^{(1)}_{(\sigma(\lambda),Q)}$:
$$\bigl|\mathcal{S}^{(1)}(u, \sigma^* v) - \mathcal{S}\bigl(f_Q(u) \otimes \sigma^* f_Q(v)\bigr)\bigr| \leqslant C \cdot |z(Q)|^{\varepsilon} \cdot |u|_{h^{(1)}} \cdot |v|_{h^{(1)}}.$$
Hence, S gives the polarization due to Lemma 11.3.6. Thus, the proof of Lemma 11.8.6 is finished. □

Remark 11.8.7. — Lemma 11.8.6 can be generalized in the higher dimensional case. We omit the details here. □

Remark 11.8.8. — The converse of Lemma 11.8.6 was proved in [**67**]. Namely, if $(V, \mathbb{D}^{\triangle,\mathcal{S}})$ is a variation of polarized pure twistor structure of weight 0, then $\bigl(S_u(V), \mathcal{N}_u^{\triangle}, \mathcal{S}_u\bigr)$ is a polarized mixed twistor structure of weight 0 for each $u \in T$. □

CHAPTER 12

PROLONGATION AS $\mathcal{R}$-TRIPLE

In this chapter, we consider the $\mathcal{R}_X$-modules $\mathfrak{E}$ and the $\mathcal{R}$-triples $\mathfrak{T}(E)$ on X, associated to unramifiedly good wild harmonic bundles $(E, \overline{\partial}_E, \theta, h)$ (except in Section 12.8). In particular, we study their specialization along any monomial functions. These results are mainly preliminary for Theorem 19.1.3. (We refer to [**73**] for details on $\mathcal{R}$-module and $\mathcal{R}$-triple. We will give a review on them and their variants in Chapter 22.)

In Section 12.1, we construct $\mathcal{R}$-modules $\mathfrak{E}$ associated to unramifiedly good wild harmonic bundles $(E, \overline{\partial}_E, \theta, h)$. This construction is a natural generalization of that in the tame case which was studied in [**67**].

We investigate the basic properties of such $\mathcal{R}$-modules in Sections 12.2–12.4. It is our basic strategy to reduce the study to the tame case. Hence, we review in Section 12.2 our previous results on $\mathfrak{E}$ for the tame case [**67**]. Then, we study in Section 12.3 the $\mathcal{R}$-module associated to the tensor product of a tame harmonic bundle and a rank one wild harmonic bundle. Applying these preliminary results, we show the basic properties of $\mathfrak{E}$ in Section 12.4. In particular, we show in Section 12.4.3 the strict S-decomposability of $\mathfrak{E}$ along any monomial functions. And we show in Section 12.4.4 the strict S-decomposability of $P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{E})$ along any coordinate function. Sections 12.4.5–12.4.6 are preliminary for Section 12.7. We study the specialization and the reduction.

In Section 12.5, we construct a Hermitian sesqui-linear pairing $\mathfrak{C}$ of $\mathfrak{E}$, and thus we obtain an $\mathcal{R}$-triple $\mathfrak{T}(E) = (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$.

In Section 12.6, we give a characterization of $\mathfrak{T}(E)$ in the case $\dim X = 1$. This is one of the main differences between the tame case and the wild case. In the tame case, such a characterization is given in a much simpler way, because of the uniqueness of a meromorphic prolongment with regular singularity. However, we do not have such a nice uniqueness in the irregular case. So we need more considerations.

In Section 12.7, we study the specialization of $\mathcal{R}$-triples $\mathfrak{T}(E)$. In Proposition 12.7.1, we give a comparison between the specializations of the original $\mathcal{R}$-triple $\mathfrak{T}(E)$ and the reduced one. By using this proposition, we can conclude that the direct summands of $P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{T}(E))$ *generically* come from unramifiedly good wild harmonic bundles. Then, by Proposition 12.7.3, we show that they come from unramifiedly good wild harmonic bundles.

In Section 12.8, we shall address a similar issue in the ramified case. For simplicity, we restrict ourselves to the one dimensional case, which we will use in Section 17.2. The higher dimensional case will be argued in Lemma 19.2.2 with a slightly different method.

12.1. $\mathcal{R}$-module associated to unramifiedly good wild harmonic bundle

Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^n D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X \setminus D$. We assume that the coordinate system is admissible for the good set $\mathrm{Irr}(\theta)$. We use the notation $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{D} := \boldsymbol{C}_\lambda \times D$. We obtained the family of meromorphic λ-flat bundles $(\mathcal{QE}, \mathbb{D})$ on $(\mathcal{X}, \mathcal{D})$ in Section 11.1. We can naturally regard $\mathcal{QE}$ as a left $\mathcal{R}_X$-module. Let $U(\lambda_0)$ be a sufficiently small neighbourhood of λ_0. Let $\boldsymbol{\delta} = (1, \ldots, 1) \in \boldsymbol{R}^n$. For any $\boldsymbol{a} \in \boldsymbol{R}^n$, let $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{a}}\mathcal{E}$ be the union of $\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{b}}\mathcal{E}$ for $\boldsymbol{b} \in \boldsymbol{R}^n$ such that $b_i < a_i$ $(i = 1, \ldots, n)$. It is equal to $\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}-\varepsilon\boldsymbol{\delta}}\mathcal{E}$ for some $\varepsilon > 0$, and hence it is a locally free $\mathcal{O}_{U(\lambda_0)\times X}$-lattice of $\mathcal{Q}^{(\lambda_0)}\mathcal{E} := \mathcal{QE}_{|U(\lambda_0)\times X}$.

In particular, we have the lattice $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$. Let $\mathfrak{E}^{(\lambda_0)}$ denote the $\mathcal{R}_X$-submodule of $\mathcal{Q}^{(\lambda_0)}\mathcal{E}$ generated by $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$ over $\mathcal{R}_X$:

$$\mathfrak{E}^{(\lambda_0)} := \mathcal{R}_X \cdot \mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}.$$

Lemma 12.1.1. — *$\mathfrak{E}^{(\lambda_0)}$ is a coherent $\mathcal{R}_X$-module.*

Proof. — Since $\mathcal{QE}$ is a locally free $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-module, $\mathfrak{E}^{(\lambda_0)}$ is a pseudo-coherent $\mathcal{O}_{\mathcal{X}}$-module. Since $\mathfrak{E}^{(\lambda_0)}$ is finitely generated over $\mathcal{R}_X$, the claim of the lemma follows. See Proposition 22.1.4 below, for example. □

Let us take $\lambda_1 \in U(\lambda_1) \subset U(\lambda_0)$.

Lemma 12.1.2. — *We have $\mathfrak{E}^{(\lambda_0)}_{|U(\lambda_1)\times X} = \mathfrak{E}^{(\lambda_1)}$. As a result, we obtain the global $\mathcal{R}_X$-module $\mathfrak{E}$ on $\mathcal{X}$.*

Proof. — We have $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_1)\times X} \subset \mathcal{Q}^{(\lambda_1)}_{<\boldsymbol{\delta}}\mathcal{E}$. Hence, $\mathfrak{E}^{(\lambda_0)}_{|U(\lambda_1)\times X} \subset \mathfrak{E}^{(\lambda_1)}$. We would like to show the reverse implication. The composite of the following morphisms are denoted by π_i for $i = 1, \ldots, n$:

$$\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{\delta}}\mathcal{E} \longrightarrow \mathcal{Q}^{(\lambda_0)}_{\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_0)\times D_i} \longrightarrow {}^i\mathrm{Gr}_1^{F^{(\lambda_0)}}\big(\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_0)\times D_i}\big).$$

Here, ${}^{i}\mathrm{Gr}^{F^{(\lambda_0)}}$ is taken with respect to the naturally induced filtration ${}^{i}F^{(\lambda_0)}$ of $\mathcal{Q}_{\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}_{|U(\lambda_0)\times D_i}$. Let $\mathcal{K}(1,\lambda_0) := \bigl\{u \in \mathcal{KMS}(\mathcal{E}^0, i)\,\big|\,\mathfrak{p}(\lambda_0,u) = 1\bigr\}$. We have the generalized eigen-decomposition with respect to the induced endomorphism $\mathrm{Res}_i(\mathbb{D})$:

$$ {}^{i}\mathrm{Gr}_1^{F^{(\lambda_0)}}\bigl(\mathcal{Q}_{\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}_{|U(\lambda_0)\times D_i}\bigr) \stackrel{\cdot}{=} \bigoplus_{u\in\mathcal{K}(1,\lambda_0)} {}^{i}\mathbb{E}_u. $$

Here, $\mathrm{Res}_i(\mathbb{D}) - \mathfrak{e}(\lambda,u)$ is nilpotent on ${}^{i}\mathbb{E}_u$. We put

$$ \mathcal{K}_m := \bigcap_{i=1}^{m} \pi_i^{-1}\Bigl(\bigoplus_{u\neq(1,0)} {}^{i}\mathbb{E}_u\Bigr) \cap \bigcap_{i=m+1}^{n} \pi_i^{-1}(0). $$

We have $\mathcal{K}_0 = \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}$, and hence $\mathcal{K}_0 \subset \mathfrak{E}^{(\lambda_0)}$. It is easy to see $\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_1)}\mathcal{E} \subset \mathcal{K}_{n|U(\lambda_1)\times X}$. Hence, we only have to show $\mathcal{K}_m \subset \mathfrak{E}^{(\lambda_0)}$. We put $\widehat{\mathcal{K}}_m := \mathcal{K}_{m|\widehat{U(\lambda_0)\times O}}$ and $\widehat{\mathfrak{E}}^{(\lambda_0)} := \mathfrak{E}^{(\lambda_0)}_{|\widehat{U(\lambda_0)\times O}}$. We only have to show $\widehat{\mathcal{K}}_m \subset \widehat{\mathfrak{E}}^{(\lambda_0)}$. We use an induction on m.

Let us show $m-1 \Longrightarrow m$. Since $\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}$ is an unramifiedly good lattice, we have the irregular decomposition

$$ \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}} := \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}_{|\widehat{U(\lambda_0)\times O}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}. \tag{254} $$

Let $\mathbb{D}_0$ denote the family of logarithmic λ-connections of $\mathcal{Q}_{<\boldsymbol{\delta}}\widehat{\mathcal{E}}$ given as follows:

$$ \mathbb{D}_0 := \mathbb{D} - \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} d\mathfrak{a}\cdot \mathrm{id}_{\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}}. $$

Note $\mathrm{Res}_i(\mathbb{D}_0) = \mathrm{Res}_i(\mathbb{D})$. Let $\boldsymbol{v}$ be a frame of $\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}$, which is compatible with the above irregular decomposition (254), the parabolic filtrations ${}^{i}F^{(\lambda_0)}$, and the decomposition ${}^{i}\mathbb{E}^{(\lambda_0)}$. We also assume that the induced frame of ${}^{m}\mathrm{Gr}^{F^{(\lambda_0)}}(\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}})$ is compatible with the weight filtration of the nilpotent part $\mathcal{N}_m$ of the endomorphism on ${}^{m}\mathrm{Gr}^{F^{(\lambda_0)}}(\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}})$ induced by $\mathrm{Res}_m(\mathbb{D})$. (Note that the conjugacy classes of $\mathcal{N}_{m|(\lambda,Q)}$ are independent of $(\lambda,Q) \in U(\lambda_1)\times D_m$. The proof of this property can be reduced to the tame case, by using the map in Theorem 11.7.2, or the completion at (λ,O). The tame case was argued in [**67**]. See Lemma 12.47 of [**67**], for example.) Let $\mathfrak{a}_j$ be determined by $v_j \in \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}_j}$. Let $I(v_j)$ be the set of i such that ${}^{i}\deg^{\mathbb{E}^{(\lambda_0)}}(v_j) \neq 0$ and ${}^{i}\deg^{F^{(\lambda_0)}}(v_j) = 0$. For $\ell = m-1, m$, we put

$$ \widetilde{v}_j^{(\ell)} := \prod_{\substack{i\leqslant \ell\\ i\in I(v_j)}} z_i^{-1}\, v_j. $$

Then, $\widetilde{\boldsymbol{v}}^{(m)}$ is a frame of $\widehat{\mathcal{K}}_m$. We only have to show $\widetilde{v}_j^{(m)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$. If $m \notin I(v_j)$, we have $\widetilde{v}_j^{(m)} \in \widehat{\mathcal{K}}_{m-1}$, and thus there is nothing to prove. Let us consider the case $m \in I(v_j)$, i.e., $\widetilde{v}_j^{(m)} = z_m^{-1}\,\widetilde{v}_j^{(m-1)}$. Because $\mathbb{D}_0\widehat{\mathcal{K}}_{m-1} \subset \widehat{\mathcal{K}}_{m-1}\otimes\Omega^{1,0}(\log D)$, we have $z_1\eth_1\widetilde{v}_j^{(m-1)} - (z_1\partial_1\mathfrak{a}_j)\,\widetilde{v}_j^{(m-1)} \in \widehat{\mathcal{K}}_{m-1}$. Therefore, $\boldsymbol{z}^{\mathrm{ord}(\mathfrak{a}_j)}\,\widetilde{v}_j^{(m-1)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$. If the m-th component of $\operatorname{ord}(\mathfrak{a}_j)$ is not 0, we are done. Let us consider the case where the m-th component of $\operatorname{ord}(\mathfrak{a}_j)$ is 0. We have

$$\mathbb{D}_0(\partial_m)\widetilde{v}_j^{(m-1)} = \eth_m(\widetilde{v}_j^{(m-1)}) - (\partial_m \mathfrak{a}_j)\,\widetilde{v}_j^{(m-1)}.$$

Since we already know $\boldsymbol{z}^{\operatorname{ord}(\mathfrak{a}_j)}\,\widetilde{v}_j^{(m-1)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$, we have $(\partial_m \mathfrak{a}_j)\,\widetilde{v}_j^{(m-1)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$. We also have $\eth_m(\widetilde{v}_j^{(m-1)}) \in \widehat{\mathfrak{E}}^{(\lambda_0)}$, and thus $\mathbb{D}_0(\partial_m)\widetilde{v}_j^{(m-1)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$.

Let $[v_j]$ denote the induced section of ${}^m\mathrm{Gr}_0^{F^{(\lambda_0)}}\big(\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}\big)$. If $\mathcal{N}_m[v_j]=0$, we have

$$\Big(\mathbb{D}_0(\partial_m)\widetilde{v}_j^{(m-1)} - \mathfrak{e}(\lambda, u_m(v_j))\, z_m^{-1}\,\widetilde{v}_j^{(m-1)}\Big)_{|z_m=0} \in {}^mF_{<1}^{(\lambda_0)}\big(\mathcal{Q}_{\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}_{|z_m=0}\big).$$

Hence, $\mathbb{D}_0(\partial_m)\,\widetilde{v}_j^{(m-1)} - \mathfrak{e}(\lambda, u_m(v_j))\, z_m^{-1}\,\widetilde{v}_j^{(m-1)}$ is contained in $\widehat{\mathcal{K}}_{m-1} \subset \widehat{\mathfrak{E}}^{(\lambda_0)}$, and we obtain

$$\widetilde{v}_j^{(m)} = z_m^{-1}\,\widetilde{v}_j^{(m-1)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}.$$

In general, we have the number $p(v_j)$ determined by the following condition:

$$\big(\mathrm{Res}_m(\mathbb{D}) - \mathfrak{e}(\lambda, u_m(v_j))\big)^{p(v_j)}\big(\widetilde{v}_j^{(m-1)}\big) = 0,$$
$$\big(\mathrm{Res}_m(\mathbb{D}) - \mathfrak{e}(\lambda, u_m(v_j))\big)^{p(v_j)-1}\big(\widetilde{v}_j^{(m-1)}\big) \neq 0.$$

The induced section $\big[\mathbb{D}_0(\partial_m)\widetilde{v}_j^{(m-1)} - \mathfrak{e}(\lambda, u_m(v_j))\, z_m^{-1}\,\widetilde{v}_j^{(m-1)}\big]$ of ${}^m\mathrm{Gr}_1^{F^{(\lambda_0)}}\big(\mathcal{Q}_{\boldsymbol{\delta}}^{(\lambda_0)}\widehat{\mathcal{E}}\big)$ is contained in the subbundle generated by $[\widetilde{v}_q^{(m)}]$ such that $p(v_q) < p(v_j)$, by our choice of the frame $\boldsymbol{v}$. Then, we can show $\widetilde{v}_j^{(m)} \in \widehat{\mathfrak{E}}^{(\lambda_0)}$ by an induction on $p(v_j)$. Thus, the induction on m can proceed, and the proof of Lemma 12.1.2 is finished. □

Remark 12.1.3. — We have considered the $\mathcal{R}$-module associated to the good wild harmonic bundle on $X \setminus D$, where $D = \bigcup_{i=1}^n \{z_i = 0\}$. Let us consider the case where a harmonic bundle $(E', \overline{\partial}_{E'}, \theta', h')$ is given on $X \setminus D'$, where $D' = \bigcup_{i=1}^{\ell}\{z_i=0\}$. We put $(E, \overline{\partial}_E, \theta, h) := (E', \overline{\partial}_{E'}, \theta', h')_{|X\setminus D}$. We have the $\mathcal{R}$-module $\mathfrak{E}$ on $\mathcal{X}$ associated to $(E, \overline{\partial}_E, \theta, h)$.

We can construct an $\mathcal{R}$-module $\mathfrak{E}'$ on $\mathcal{X}$ from $(E', \overline{\partial}_{E'}, \theta', h')$ in the same way. Namely, let $\boldsymbol{\delta} = (1,\dots,1) \in \mathbb{Z}^{\ell}$ and let $\lambda_0 \in \boldsymbol{C}_{\lambda}$. We consider the $\mathcal{R}$-submodule $\mathfrak{E}'^{(\lambda_0)}$ of $\mathcal{QE}'_{|\mathcal{X}^{(\lambda_0)}}$ generated by $\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}'$. By varying λ_0 and gluing them, we obtain the $\mathcal{R}$-module $\mathfrak{E}'$.

Note that we have a natural isomorphism $\mathfrak{E}' \to \mathfrak{E}$. Indeed, we have natural isomorphisms $\mathcal{QE} = \mathcal{QE}'(*\mathcal{D})$ and $\mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E} = \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}'$, and hence $\mathfrak{E}^{(\lambda_0)} = \mathfrak{E}'^{(\lambda_0)}$, which induces the desired isomorphism. We will use it implicitly. □

12.2. Review of some results in the tame case

12.2.1. Some filtrations of $\mathfrak{E}$. — Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^n D_i$. We put $\underline{n} := \{1,\dots,n\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X \setminus D$. We have the associated $\mathcal{R}_X$-module $\mathfrak{E}$ on $\mathcal{X}$. For $\lambda_0 \in \boldsymbol{C}$, let $\mathcal{X}^{(\lambda_0)}$ denote a small neighbourhood of $\{\lambda_0\} \times X$ in $\mathcal{X}$. Note that ${}^{\underline{n}}V_{\boldsymbol{a}}^{(\lambda_0)}(^{\square}\mathcal{E})$ and $^{\square}\mathcal{E}$ in [**67**] are equal to $\mathcal{Q}_{\boldsymbol{a}+\boldsymbol{\delta}}^{(\lambda_0)}(\mathcal{E})$ and $\mathcal{Q}\mathcal{E}$ in this paper, respectively. Here $\boldsymbol{\delta} = (1, \dots, 1)$. We also have ${}^{\underline{n}}V_{<0}^{(\lambda_0)}(\mathfrak{E}) = {}^{\underline{n}}V_{<0}^{(\lambda_0)}(^{\square}\mathcal{E}) = \mathcal{Q}_{<\boldsymbol{\delta}}^{(\lambda_0)}(\mathcal{E})$. We will not distinguish them in the following argument.

Let I be any subset of $\underline{n}$. Let q_I denote the projection of $\boldsymbol{R}^n$ to $\boldsymbol{R}^I$. We have the filtration $\{{}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(^{\square}\mathcal{E}) \mid \boldsymbol{b} \in \boldsymbol{R}^I\}$ of $^{\square}\mathcal{E}_{|\mathcal{X}^{(\lambda_0)}}$, which is equal to the following in the notation of this paper:

$$
{}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(^{\square}\mathcal{E}) := \bigcup_{\substack{\boldsymbol{b}' \in \boldsymbol{R}^n \\ q_I(\boldsymbol{b}') = \boldsymbol{b}}} \mathcal{Q}_{\boldsymbol{b}'+\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}.
$$

It is also equal to $\mathcal{Q}_{\boldsymbol{b}'+\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E} \otimes \mathcal{O}\big(*\sum_{j \notin I} \mathcal{D}_j\big)$ for any $\boldsymbol{b}' \in \boldsymbol{R}^n$ such that $q_I(\boldsymbol{b}') = \boldsymbol{b}$. We also have the filtration $\big({}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}\mathfrak{E} \,\big|\, \boldsymbol{b} \in \boldsymbol{R}^I\big)$ of $\mathfrak{E}_{|\mathcal{X}^{(\lambda_0)}}$ given by

$$
{}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(\mathfrak{E}) := \mathfrak{E} \cap {}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(^{\square}\mathcal{E}).
$$

(See Section 15.2.4 and Corollary 15.63 of [**67**].) We recall the following lemma.

Lemma 12.2.1. — *Let $I \subset \underline{n}$ and $J := \underline{n} - I$. Let $\boldsymbol{b} \in \boldsymbol{R}^I_{<0}$ and $\boldsymbol{c} \in \boldsymbol{R}^J$. We set*

$$
{}^{I}T^{(\lambda_0)}(\boldsymbol{c}, \boldsymbol{b}) := \frac{{}^{\underline{n}}V_{\boldsymbol{b}+\boldsymbol{c}}^{(\lambda_0)}(\mathfrak{E})}{\sum_{\boldsymbol{d} \lneq \boldsymbol{c}} {}^{\underline{n}}V_{\boldsymbol{b}+\boldsymbol{d}}^{(\lambda_0)}(\mathfrak{E})}.
$$

Here, $\boldsymbol{d} \lneq \boldsymbol{c}$ means "$\boldsymbol{d} \leqslant \boldsymbol{c}$ and $\boldsymbol{d} \neq \boldsymbol{c}$". We put $c_i' := \max\{c_i - n < 0 \mid n \in \mathbb{Z}_{\geqslant 0}\}$ for any $i \in J$, $\boldsymbol{c}' := (c_i' \mid i \in J)$, and

$$
{}^{I}\widetilde{T}^{(\lambda_0)}(\boldsymbol{c}, \boldsymbol{b}) := \mathrm{Im}\Big(\prod_{i \in J} \eth_i^{c_i - c_i'} : {}^{J}\mathrm{Gr}_{\boldsymbol{c}'}^{V^{(\lambda_0)}}\, {}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(^{\square}\mathcal{E}) \longrightarrow {}^{J}\mathrm{Gr}_{\boldsymbol{c}}^{V^{(\lambda_0)}}\, {}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(^{\square}\mathcal{E})\Big).
$$

Then, the following holds:

- *The multiplication of $\eth_i$ induces the surjection ${}^{I}\widetilde{T}^{(\lambda_0)}(\boldsymbol{c} - \boldsymbol{\delta}_i, \boldsymbol{b}) \to {}^{I}\widetilde{T}^{(\lambda_0)}(\boldsymbol{c}, \boldsymbol{b})$ if $c_i \geqslant 0$, where $\boldsymbol{\delta}_i$ denotes the element of $\boldsymbol{R}^J$ whose j-th element is 0 $(j \neq i)$ or 1 $(j = i)$.*
- *We have the natural isomorphism ${}^{I}T^{(\lambda_0)}(\boldsymbol{c}, \boldsymbol{b}) \simeq {}^{I}\widetilde{T}^{(\lambda_0)}(\boldsymbol{c}, \boldsymbol{b})$.*

Proof. — The first claim is clear from the construction. The second claim is Lemma 15.46 of [**67**]. □

For a subset $I \subset \underline{n}$, let $\mathcal{R}_{X,I}$ denote the sheaf of subalgebras of $\mathcal{R}_X$ generated by $\mathcal{O}_{\mathcal{X}}$ and $\eth_j$ $(j \in I)$. We remark the following lemmas, which implicitly appeared in [**67**].

Lemma 12.2.2. — *Let $I \subset \underline{n}$ and $\boldsymbol{b} \in \boldsymbol{R}^I_{<0}$. Let ε be any sufficiently small positive number. Then, we have the following equality on $\mathcal{X}^{(\lambda_0)}$:*

$$
{}^{I}V_{\boldsymbol{b}}^{(\lambda_0)}(\mathfrak{E}) = \mathcal{R}_{X,\underline{n}-I} \cdot {}^{\underline{n}}V_{\boldsymbol{b}-\varepsilon\boldsymbol{\delta}_{\underline{n}-I}}^{(\lambda_0)}(\mathfrak{E}). \tag{255}
$$

Proof. — It is clear that the right-hand side of (255) is contained in the left-hand side. Hence, we only have to show that ${}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+\boldsymbol{c}}(\mathfrak{E})$ is contained in the right-hand side of (255) for any $\boldsymbol{c}\in\boldsymbol{R}^{\underline{n}-I}$, where we regard $\boldsymbol{b}+\boldsymbol{c}\in\boldsymbol{R}^I\times\boldsymbol{R}^{\underline{n}-I}=\boldsymbol{R}^n$.

Let $\boldsymbol{c}\in\boldsymbol{R}^{\underline{n}-I}$, and let f be a section of ${}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+\boldsymbol{c}}(\mathfrak{E})$. Let us show that f is contained in the right-hand side of (255). We put $\mathfrak{q}(\boldsymbol{c}):=\#\{j\in I\,|\,c_j\geqslant 0\}$. We use an induction on $\mathfrak{q}(\boldsymbol{c})$. If $\mathfrak{q}(\boldsymbol{c})=0$, there is nothing to prove. Assume $\mathfrak{q}(\boldsymbol{c})>0$, and let $c_i>0$. Due to Lemma 12.2.1, we have a section $g\in{}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{\delta}_i}(\mathfrak{E})$ such that

$$f-\eth_i g=\sum_{\boldsymbol{c}'\lneq\boldsymbol{c}}h_{\boldsymbol{c}'},\quad h_{\boldsymbol{c}'}\in{}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+\boldsymbol{c}'}(\mathfrak{E}).$$

We remark that the set of the parabolic weights is discrete. Hence, we can reduce the number $\mathfrak{q}$ after finite steps. Thus, we obtain (255). □

Lemma 12.2.3. — *Let $I\subset\underline{n}$ and $\boldsymbol{b}\in\boldsymbol{R}^I_{<0}$. Let $i\in\underline{n}-I$ and $c\in\boldsymbol{R}$. Then, we have*

$$ {}^{i}V^{(\lambda_0)}_{c}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E})=\sum_{(c',m)\in\mathcal{U}}\eth_i^m\big({}^{i}V^{(\lambda_0)}_{c'}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E})\big),$$
$$\mathcal{U}:=\big\{(c',m)\in\boldsymbol{R}_{<0}\times\mathbb{Z}_{\geqslant 0}\,\big|\,c'+m\leqslant c\big\}.$$

Proof. — If $c<0$, there is nothing to prove. Assume $c\geqslant 0$. We only have to show the following equality:

$$ {}^{i}V^{(\lambda_0)}_{c}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E})={}^{i}V^{(\lambda_0)}_{<c}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E})+\eth_i\big({}^{i}V^{(\lambda_0)}_{c-1}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E})\big). \tag{256}$$

Clearly, the right-hand side is contained in the left-hand side. We put $K:=\underline{n}-(\{i\}\cup I)$. Let $\boldsymbol{d}\in\boldsymbol{R}^K$, and let us show that ${}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+c\boldsymbol{\delta}_i+\boldsymbol{d}}(\mathfrak{E})$ is contained in the right-hand side. Let $\mathfrak{q}(\boldsymbol{d}):=\#\{j\,|\,d_j\geqslant 0\}$. We use an induction on $\mathfrak{q}(\boldsymbol{d})$. In the case $\mathfrak{q}(\boldsymbol{d})=0$, i.e., $\boldsymbol{d}\in\boldsymbol{R}^K_{<0}$, the claim easily follows from Lemma 12.2.1. Let f be a section of ${}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+c\boldsymbol{\delta}_i+\boldsymbol{d}}(\mathfrak{E})$. Due to Lemma 12.2.1, there exists a section g of ${}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+(c-1)\boldsymbol{\delta}_i+\boldsymbol{d}}(\mathfrak{E})$ such that the following holds:

$$f-\eth_i g=h_1+\sum_{\boldsymbol{d}'\lneq\boldsymbol{d}}h_{\boldsymbol{d}'},\qquad h_1\in{}^{i}V^{(\lambda_0)}_{<c}\,{}^{I}V^{(\lambda_0)}_{\boldsymbol{b}}(\mathfrak{E}),\qquad h_{\boldsymbol{d}'}\in{}^{\underline{n}}\!V^{(\lambda_0)}_{\boldsymbol{b}+c\boldsymbol{\delta}_i+\boldsymbol{d}'}(\mathfrak{E}).$$

Since the set of the parabolic weights is discrete, we can reduce the number $\mathfrak{q}$ after a finite number of steps. □

For any subset $I\subset\underline{n}:=\{1,\dots,n\}$, we put $D_I:=\bigcap_{i\in I}D_i$, and let $N^*_{D_I}X$ denote the conormal bundle of D_I in X.

Lemma 12.2.4. — *$\mathfrak{E}$ is holonomic, and the characteristic variety of $\mathfrak{E}$ is contained in $\mathcal{S}:=\bigcup_{I\subset\underline{n}}(\boldsymbol{C}_\lambda\times N^*_{D_I}X)$.*

Proof. — It is shown in Proposition 15.68 of [**67**] and its proof. We give a simplified proof. We put $F_0:=\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{\delta}}\mathcal{E}$. For $\boldsymbol{p}\in\mathbb{Z}^n_{\geqslant 0}$, we put $\eth^{\boldsymbol{p}}:=\prod\eth_i^{p_i}$. We set $F_m:=\sum_{|\boldsymbol{p}|\leqslant m}\eth^{\boldsymbol{p}}F_0$. Then, $\{F_m\}$ is a coherent filtration of $\mathfrak{E}$ on $\mathcal{X}^{(\lambda_0)}$. Let

$Q \in \mathcal{D}_K^{(\lambda_0)} \setminus \bigcup_{j \notin K} \mathcal{D}_j^{(\lambda_0)}$. For any $j \notin K$, we have $\eth_j F_0 \subset F_0$, and hence $\eth_j F_m \subset F_m$. Then, the action of $\eth_j$ on $\mathrm{Gr}^F(\mathfrak{E})$ around Q is trivial. It implies that the characteristic variety of $\mathfrak{E}$ is contained in $\boldsymbol{C}_\lambda \times N^*_{D_K} X$ around Q. □

12.2.2. Push-forward $i_{g\dagger}\mathfrak{E}$ and the V-filtration $U^{(\lambda_0)}$. — Let g be any function of the form $\boldsymbol{z}^{\boldsymbol{p}} = \prod_{j \in I} z_j^{p_j}$ for $p_j > 0$ ($j \in I$). Let $i_g : X \to X \times \boldsymbol{C}_t$ denote the graph. It is very important to investigate the push-forward $i_{g\dagger}\mathfrak{E}$ on $\mathcal{X} \times \boldsymbol{C}_t$. The support of $i_{g\dagger}\mathfrak{E}$ is the graph of g, which is naturally isomorphic to $\mathcal{X}$. And $i_{g\dagger}\mathfrak{E}$ is identified with $(i_{g*}\mathfrak{E})[\eth_t]$ on $\mathcal{X} \times \boldsymbol{C}_t$ (or simply denoted by $\mathfrak{E}[\eth_t]$), where $i_{g\dagger}$ (resp. i_{g*}) denotes the push-forward of $\mathcal{R}$-modules (resp. sheaves). The action of $\mathcal{R}_{X \times \boldsymbol{C}_t}$ is given by general formulas:

$$
\begin{aligned}
a \cdot (\eth_t^j \otimes u) &= \eth_t^j \otimes (a \cdot u) \qquad (a \in \mathcal{O}_X)\\
\eth_i \cdot (\eth_t^j \otimes u) &= \eth_t^j \otimes (\eth_i u) - \eth_t^{j+1} \otimes (\partial_i g) \cdot u\\
t \cdot (\eth_t^j \otimes u) &= \eth_t^j \otimes (g \cdot u) - j\lambda \eth_t^{j-1} \otimes u\\
\eth_t(\eth_t^j \otimes u) &= \eth_t^{j+1} \otimes u
\end{aligned}
\tag{257}
$$

We will implicitly use the following formula for $i \in I$:

$$
(p_i \eth_t t + \eth_i z_i)(\eth_t^j \otimes u) = -p_i \cdot j \cdot \lambda \eth_t^j \otimes u + \eth_t^j \otimes \eth_i(z_i u). \tag{258}
$$

Let π denote the projection $X \times \boldsymbol{C}_t \to X$. Let $V_0\mathcal{R}_{X \times \boldsymbol{C}_t}$ denote the sheaf of subalgebras of $\mathcal{R}_{X \times \boldsymbol{C}_t}$ generated by $\pi^*\mathcal{R}_X$ and $\eth_t t$. Recall that we have the V-filtration $U^{(\lambda_0)}$ of $i_{g\dagger}\mathfrak{E}$ on $\mathcal{X}^{(\lambda_0)}$ given as follows (Section 16.1 of [**67**]):

$$
\begin{aligned}
U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) &:= \pi^*\mathcal{R}_X \cdot \Big({}^I V_{b\boldsymbol{p}}^{(\lambda_0)}(\mathfrak{E}) \otimes 1\Big) \quad (b < 0),\\
U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) &= \sum_{\substack{c<0,\, n \in \mathbb{Z}_{\geqslant 0}\\ c+n \leqslant b}} \eth_t^n \cdot U_c^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \quad (b \geqslant 0).
\end{aligned}
$$

These modules are locally finitely generated $V_0\mathcal{R}_{X \times \boldsymbol{C}_t}$-modules, a property which immediately follows from Lemma 12.2.2. Since they are contained in a pseudo-coherent $\mathcal{O}_{\mathcal{X} \times \boldsymbol{C}_t}$-module $i_{g*}\mathcal{QE}[\eth_t]$, they are $\mathcal{R}_{X \times \boldsymbol{C}_t}$-coherent. (See Proposition 22.7.2.) The following properties can be checked by a direct calculation:

- $t \cdot U_b^{(\lambda_0)} \subset U_{b-1}^{(\lambda_0)}$ for any $b \in \boldsymbol{R}$, and $t \cdot U_b^{(\lambda_0)} = U_{b-1}^{(\lambda_0)}$ for $b < 0$.
- $\eth_t U_b^{(\lambda_0)} \subset U_{b+1}^{(\lambda_0)}$ for any $b \in \boldsymbol{R}$, and the induced morphisms $\eth_t : \mathrm{Gr}_{b-1}^{U^{(\lambda_0)}} \to \mathrm{Gr}_b^{U^{(\lambda_0)}}$ are surjective for any $b > 0$.

As in (16.3) of [**67**], we set

$$
\mathcal{KMS}(i_{g\dagger}\mathfrak{E}^0) := \bigcup_{i \in I} \big\{u \in \boldsymbol{R} \times \boldsymbol{C} \,\big|\, p_i \cdot u \in \mathcal{KMS}(\mathcal{E}^0, i)\big\}, \tag{259}
$$

$$
\mathcal{K}(i_{g\dagger}\mathfrak{E}, \lambda_0, b) := \big\{u \in \mathcal{KMS}(i_{g\dagger}\mathfrak{E}^0) \,\big|\, \mathfrak{p}(\lambda_0, u) = b\big\}. \tag{260}
$$

Here $\mathfrak{E}^0$ means the specialization of $\mathfrak{E}$ at $\lambda = 0$. The following lemma is proved in [**67**].

Lemma 12.2.5

- *The following endomorphism is nilpotent on* $\operatorname{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E})$*:*

$$\prod_{u\in\mathcal{K}(i_{g\dagger}\mathfrak{E},\lambda_0,b)}\bigl(-\eth_t t+\mathfrak{e}(\lambda,u)\bigr).$$

  (See Corollary 16.13 *of* [**67**].*)*
- *We also obtain that* $\operatorname{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E})$ *is strict, i.e., the multiplication of* $\lambda-\lambda_1$ *is injective for any* λ_1*. (See Proposition* 16.47 *of* [**67**].*)* □

More strongly, we have the following proposition.

***Proposition 12.2.6* (Proposition 16.49 of [67]).** — *For any* λ_0*,* $i_{g\dagger}\mathfrak{E}$ *is strictly* S*-decomposable along* t *at* λ_0 *with the above* V*-filtration* $U^{(\lambda_0)}$*. (See Section* 22.3 *for the notion of strict* S*-decomposability.)* □

Remark 12.2.7. — When we would like to know some property of $\widetilde{\psi}_{g,u}(\mathfrak{E})$, we only have to look at $U_b^{(\lambda_0)}$ for $b<0$. (See Section 22.3 for the functor $\widetilde{\psi}_{g,u}\cdot$.) □

12.2.3. Filtrations $\,{}^{i}V^{(\lambda_0)}$**.** — Let ${}^{J}V_0\mathcal{R}_X$ be as in Section 22.7.2. Let ${}^{J,t}V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$ denote the sheaf of subalgebras of $\mathcal{R}_{X\times\boldsymbol{C}_t}$ generated by $\pi^*({}^{J}V_0\mathcal{R}_X)$ and $\eth_t t$.

Let $b<0$. As in Section 16.1.4 of [**67**], for any element $\boldsymbol{c}\in\boldsymbol{R}^{I}_{\leqslant 0}\times\boldsymbol{R}^{\underline{n}-I}$, we put

$${}^{\underline{n}}V_{\boldsymbol{c}}^{(\lambda_0)}U_b^{(\lambda_0)}\bigl(i_{g\dagger}\mathfrak{E}\bigr):=\pi^*({}^{\underline{n}}V_0\mathcal{R}_X)\cdot\bigl({}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{c}+b\boldsymbol{p}}(\mathfrak{E})\otimes 1\bigr).$$

For any $\boldsymbol{c}\in\boldsymbol{R}^n$, we put

$${}^{\underline{n}}V_{\boldsymbol{c}}^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}):=\sum_{(\boldsymbol{n},\boldsymbol{a})\in S}\eth^{\boldsymbol{n}}\Bigl({}^{\underline{n}}V_{\boldsymbol{a}}^{(\lambda_0)}\bigl(U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\bigr)\Bigr),$$

where $S:=\bigl\{(\boldsymbol{n},\boldsymbol{a})\in\mathbb{Z}^{I}_{\geqslant 0}\times(\boldsymbol{R}^{I}_{\leqslant 0}\times\boldsymbol{R}^{\underline{n}-I})\bigm|\boldsymbol{n}+\boldsymbol{a}\leqslant\boldsymbol{c}\bigr\}$ and $\eth^{\boldsymbol{n}}:=\prod\eth_j^{n_j}$. Note that these modules are ${}^{\underline{n},t}V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$-modules, a property which can be checked by using (257) and (258). For $1\leqslant i\leqslant n$ and $c\in\boldsymbol{R}$, we put

$${}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}):=\sum_{\substack{\boldsymbol{c}\in\boldsymbol{R}^n\\ q_i(\boldsymbol{c})=c}}{}^{\underline{n}}V_{\boldsymbol{c}}^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}),$$

where q_i denotes the projection of $\boldsymbol{R}^n$ onto the i-th component. It is easy to check that these modules are finitely generated over ${}^{i,t}V\mathcal{R}_{X\times\boldsymbol{C}_t}$, and pseudo-coherent $\mathcal{O}_{\mathcal{X}\times\boldsymbol{C}_t}$-modules, and hence coherent ${}^{i,t}V\mathcal{R}_{X\times\boldsymbol{C}_t}$-modules, as remarked in Proposition 22.7.2 below. Thus, we obtain the filtration ${}^{i}V^{(\lambda_0)}$ on $U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ for $b<0$ by coherent ${}^{i,t}V\mathcal{R}_{X\times\boldsymbol{C}_t}$-modules.

We have the induced filtrations on $\operatorname{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E})$ which are also denoted by ${}^{i}V^{(\lambda_0)}$. Since the filtrations ${}^{i}V^{(\lambda_0)}$ are preserved by the action of $-\eth_t t$, they are compatible with the decomposition $\operatorname{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E})=\bigoplus_{\mathfrak{p}(\lambda_0,u)=b}\psi_{t,u}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. Hence, the filtrations of $\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ on $\mathcal{X}^{(\lambda_0)}$ are induced, which are also denoted by ${}^{i}V^{(\lambda_0)}$. Note that

${}^{i}V_a^{(\lambda_0)}\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ are coherent ${}^{i}V_0\mathcal{R}_X[\eth_t t]$-modules. Since $-\eth_t t+\mathfrak{e}(\lambda,u)$ is nilpotent on ${}^{i}V_a^{(\lambda_0)}\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$, they are also coherent ${}^{i}V_0\mathcal{R}_X$-modules.

Let N denote the nilpotent part of the action of $-\eth_t t$ on $\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$, and let $W(N)$ denote the weight filtration. We obtain the induced filtration ${}^{i}V^{(\lambda_0)}$ of $\mathrm{Gr}_h^{W(N)}\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ by coherent ${}^{i}V_0\mathcal{R}_X$-modules on $\mathcal{X}^{(\lambda_0)}$.

Proposition 12.2.8

- *The filtrations* ${}^{i}V^{(\lambda_0)}$ $(i=1,\dots,n)$ *are compatible with the primitive decomposition of* $\mathrm{Gr}_h^{W(N)}\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ *(Lemma 17.37 of* [**67**]*).*
- *The primitive part* $P\,\mathrm{Gr}_h^{W(N)}\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ *is strictly* S*-decomposable along* $z_i=0$ *with the filtration* ${}^{i}V^{(\lambda_0)}$ *for any* i *(Corollary 17.45 and Corollary 17.55 of* [**67**]*).* □

We give another description of the filtration ${}^{i}V^{(\lambda_0)}$ on $U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ $(b<0)$. Let $K_1:=I\cup\{i\}$ and $K_2:=\underline{n}-K_1$. Let $\boldsymbol{\delta}_j$ denote the element of $\boldsymbol{R}^n$ such that (i) the j-th component is 1, (ii) the other component is 0. For any subset $J\subset\underline{n}$, let $\boldsymbol{\delta}_J=\sum_{j\in J}\boldsymbol{\delta}_j$.

Lemma 12.2.9. — *Let $b<0$. Let $c\leqslant 0$ in the case $i\in I$, or $c<0$ in the case $i\notin I$. We have the following for a sufficiently small $\varepsilon>0$:*

$$\pi^*\big({}^{i}V_0\mathcal{R}_X\big)\cdot\big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}+c\boldsymbol{\delta}_i-\varepsilon\boldsymbol{\delta}_{K_2}}(\mathfrak{E})\otimes 1\big)={}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big).\tag{261}$$

Proof. — Let us consider the case $i\in I$. Let $\boldsymbol{c}_1\in\boldsymbol{R}^I_{\leqslant 0}=\boldsymbol{R}^{K_1}_{\leqslant 0}$. We have

$$\sum_{\boldsymbol{c}_2\in\boldsymbol{R}^{K_2}}{}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{c}_1+\boldsymbol{c}_2}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})=\sum_{\boldsymbol{c}_2\in\boldsymbol{R}^{K_2}}\pi^*\big({}^{\underline{n}}V_0\mathcal{R}_X\big)\cdot\big({}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{c}_1+\boldsymbol{c}_2+b\boldsymbol{p}}(\mathfrak{E})\otimes 1\big)\tag{262}$$
$$=\pi^*\big({}^{\underline{n}}V_0\mathcal{R}_X\big)\cdot\big({}^{K_1}V^{(\lambda_0)}_{\boldsymbol{c}_1+b\boldsymbol{p}}(\mathfrak{E})\otimes 1\big)=\pi^*\big({}^{\underline{n}}V_0\mathcal{R}_X\cdot\mathcal{R}_{X,K_2}\big)\cdot\big({}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{c}_1+b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{K_2}}(\mathfrak{E})\otimes 1\big).$$

Here, we have used Lemma 12.2.2. Even in the case $i\notin I$, we have the formally same equality as (262) for $\boldsymbol{c}_1\in\boldsymbol{R}^I_{\leqslant 0}\times\boldsymbol{R}_{<0}\subset\boldsymbol{R}^{K_1}$. Then, it is easy to derive (261). □

Lemma 12.2.10. — *Let $b<0$. In the case $c>0$ and $i\in I$, we have the following description:*

$$\begin{gathered}{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)=\sum_{(c',m)\in\mathcal{U}}\eth_i^m\big({}^{i}V_{c'}^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\big)\\ \mathcal{U}=\big\{(c',m)\in\boldsymbol{R}_{\leqslant 0}\times\mathbb{Z}_{\geqslant 0}\,\big|\,c'+m\leqslant c\big\}.\end{gathered}\tag{263}$$

In the case $c\geqslant 0$ and $i\notin I$, we have the following description:

$$\begin{gathered}{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)=\sum_{(c',m)\in\mathcal{U}}\eth_i^m\big({}^{i}V_{c'}^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\big)\\ \mathcal{U}=\big\{(c',m)\in\boldsymbol{R}_{<0}\times\mathbb{Z}_{\geqslant 0}\,\big|\,c'+m\leqslant c\big\}.\end{gathered}\tag{264}$$

Proof. — The case $i \in I$ is easy by the construction of ${}^{i}V^{(\lambda_0)}$ on $U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. Let us consider the case $i \notin I$. For $\boldsymbol{c}_2 \in \boldsymbol{R}^{K_2}$, we have

$$(265)\quad \sum_{\boldsymbol{c}_2} {}^{\underline{n}}V^{(\lambda_0)}_{c\boldsymbol{\delta}_i+\boldsymbol{c}_2} U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = \sum_{\boldsymbol{c}_2} {}^{\underline{n}}V_0\mathcal{R}_X \cdot \big({}^{\underline{n}}V^{(\lambda_0)}_{c\boldsymbol{\delta}_i+\boldsymbol{c}_2+b\boldsymbol{p}}(\mathfrak{E}) \otimes 1\big)$$
$$= {}^{\underline{n}}V_0\mathcal{R}_X \cdot \big({}^{K_1}V^{(\lambda_0)}_{c\boldsymbol{\delta}_i+b\boldsymbol{p}}(\mathfrak{E}) \otimes 1\big) = \sum_{(c',m)\in\mathcal{U}} \eth_i^m\Big({}^{\underline{n}}V_0\mathcal{R}_X \cdot \big({}^{K_1}V^{(\lambda_0)}_{c'\boldsymbol{\delta}_i+b\boldsymbol{p}}(\mathfrak{E}) \otimes 1\big)\Big)$$
$$= \sum_{(c',m)\in\mathcal{U}} \eth_i^m\Big({}^{\underline{n}}V_0\mathcal{R}_X \cdot \mathcal{R}_{X,K_2}\big({}^{\underline{n}}V^{(\lambda_0)}_{c'\boldsymbol{\delta}_i+b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{K_2}}(\mathfrak{E}) \otimes 1\big)\Big).$$

Here, we have used Lemma 12.2.3, and we omit to denote π^*. By using the description in Lemma 12.2.9, we obtain (264). □

12.3. The case where $\mathrm{Irr}(\theta)$ consists of only one element

12.3.1. The splitting of the associated $\mathcal{R}_X$-module into the tensor product

Let X, D, and $(E, \overline{\partial}_E, \theta, h)$ be as in Section 12.2.1. Let $\boldsymbol{m} \in \mathbb{Z}^n_{\leqslant 0} \setminus \{\boldsymbol{0}\}$. We put $\mathfrak{s}(\boldsymbol{m}) := \{j \mid m_j < 0\}$. Let $\mathfrak{a}$ be a meromorphic function of the form $\prod z_j^{m_j} \cdot \mathfrak{a}'$ where $\mathfrak{a}'$ is holomorphic and nowhere vanishing. Note $\mathrm{ord}(\mathfrak{a}) = \boldsymbol{m}$. Let $L(\mathfrak{a})$ be the unramifiedly good wild harmonic bundle on $X \setminus D$, which consists of the line bundle $\mathcal{O}_{X\setminus D}$ with the Higgs field $d\mathfrak{a}$ and the trivial Hermitian metric. We have the unramifiedly good wild harmonic bundle $(E', \overline{\partial}_{E'}, \theta', h') := (E, \overline{\partial}_E, \theta, h) \otimes L(\mathfrak{a})$. The associated coherent $\mathcal{R}_X$-module is denoted by $\mathfrak{E}'$.

Let $\mathcal{L}(\mathfrak{a})$ be the coherent $\mathcal{R}_X$-module $\mathcal{O}_{\mathcal{X}}\big(*\prod_{i\in\mathfrak{s}(\boldsymbol{m})} z_i\big) \cdot e$ with $\eth_j e = (\partial_j \mathfrak{a}) \cdot e$, which is the $\mathcal{R}_X$-module associated to the unramifiedly good wild harmonic bundle $L(\mathfrak{a})$. We obtain the $\mathcal{R}_X$-module $\mathfrak{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}(\mathfrak{a})$.

Lemma 12.3.1
- *On $\mathcal{X}^{(\lambda_0)}$, $\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})$ is generated by ${}^{\underline{n}}V^{(\lambda_0)}_{<0}(\mathfrak{E}) \otimes e$ over $\pi^*\mathcal{R}_X$.*
- *We have the natural isomorphism $\mathfrak{E}' \simeq \mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})$.*

Proof. — Since we have the natural isomorphisms $\mathcal{QE}' \simeq \mathcal{QE} \otimes \mathcal{L}(\mathfrak{a})$ and $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}' \simeq {}^{\underline{n}}V^{(\lambda_0)}_{<0}(\mathfrak{E}) \otimes e$, the second claim follows from the first claim and the definition of $\mathfrak{E}'$.

Let us consider the morphism

$$\Phi : \pi^*\mathcal{R}_X \otimes \big({}^{\underline{n}}V^{(\lambda_0)}_{<0}(\mathfrak{E}) \otimes e\big) \longrightarrow \mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})$$

induced by the inclusion ${}^{\underline{n}}V^{(\lambda_0)}_{<0}(\mathfrak{E}) \otimes e \subset \mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})$ and the $\mathcal{R}_X$-action. Let $F_m(\mathcal{R}_X)$ denote the submodule of $\mathcal{R}_X$ which consists of the differential operators of at most order m. On $\mathcal{X}^{(\lambda_0)}$, we define

$$F_m(\mathfrak{E}) := \mathrm{Im}\big(F_m(\mathcal{R}_X) \otimes {}^{\underline{n}}V^{(\lambda_0)}_{<0}\mathfrak{E} \longrightarrow \mathfrak{E}\big).$$

Since $\bigcup_m F_m(\mathfrak{E}) = \mathfrak{E}$ on $\mathcal{X}^{(\lambda_0)}$ by construction of $\mathfrak{E}$, we only have to show $F_m(\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a}) \subset \mathrm{Im}\,\Phi$ for any m. Let us consider the following claims:

(a_m) : $F_m(\mathfrak{E}) \otimes e \subset \operatorname{Im} \Phi$.

(b_m) : $F_m(\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a}) \subset \operatorname{Im} \Phi$.

The claim (a_0) clearly holds. Let us show $(a_m) \Longrightarrow (b_m)$. We only have to show $\boldsymbol{z}^{N\boldsymbol{m}}\big(F_m(\mathfrak{E}) \otimes e\big) \subset \operatorname{Im}(\Phi)$ for any N, which we show by an induction on N. In the case $N = 0$, the claim directly follows from (a_m). Assume we have already obtained $\boldsymbol{z}^{N\boldsymbol{m}}\big(F_m(\mathfrak{E}) \otimes e\big) \subset \operatorname{Im}(\Phi)$. Take $i \in \mathfrak{s}(\boldsymbol{m})$. Let $f \in F_m(\mathfrak{E})$. We have

$$\operatorname{Im}(\Phi) \ni z_i \eth_i\big(\boldsymbol{z}^{N\boldsymbol{m}} \cdot f \otimes e\big) = z_i \eth_i(\boldsymbol{z}^{N\boldsymbol{m}} \cdot f) \otimes e + (\boldsymbol{z}^{N\boldsymbol{m}} \cdot f) \otimes (z_i \partial_i \mathfrak{a}) \cdot e.$$

Because $z_i \eth_i(\boldsymbol{z}^{N\boldsymbol{m}} \cdot f) \in F_m(\mathfrak{E})$, we have $z_i \eth_i(\boldsymbol{z}^{N\boldsymbol{m}} \cdot f) \otimes e \in \operatorname{Im}(\Phi)$. Hence, we obtain $\boldsymbol{z}^{N\boldsymbol{m}} f \otimes (z_i \partial_i \mathfrak{a}) \cdot e \in \operatorname{Im}(\Phi)$, which implies $\boldsymbol{z}^{(N+1)\boldsymbol{m}} f \otimes e \in \operatorname{Im}(\Phi)$. Therefore, we obtain (b_m).

Let us show $(b_{m-1}) \Longrightarrow (a_m)$. For $f \in F_m(\mathfrak{E})$, we have $\eth_i(f \otimes e) = \eth_i f \otimes e + (\partial_i \mathfrak{a}) f \otimes e$. By the assumption, $\eth_i(f \otimes e)$ and $(\partial_i \mathfrak{a}) f \otimes e$ are contained in $\operatorname{Im}(\Phi)$. Hence, $\eth_i f \otimes e$ is also contained in $\operatorname{Im}(\Phi)$. Then, (a_m) follows. Thus, the proof of Lemma 12.3.1 is finished. □

For any subset $I \subset \underline{n} := \{1, \dots, n\}$, we put $D_I := \bigcap_{i \in I} D_i$, and let $N^*_{D_I} X$ denote the conormal bundle of D_I in X.

Corollary 12.3.2. — *$\mathfrak{E}'$ is holonomic. The characteristic variety of $\mathfrak{E}'$ is contained in $\mathcal{S} := \bigcup_{I \subset \underline{n}} (C_\lambda \times N^*_{D_I} X)$.*

Proof. — We use the notation in the proof of Lemma 12.2.4. We set $F_m(\mathcal{L}(\mathfrak{a})) := F_m(\mathcal{R}_X) \cdot e$. Let $F_m\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$ be the image of the following naturally defined map:

$$\bigoplus_{p+q \leqslant m} F_p(\mathfrak{E}) \otimes F_q\big(\mathcal{L}(\mathfrak{a})\big) \longrightarrow \mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}).$$

Let us show that $\bigoplus_m F_m\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$ is finitely generated over $\bigoplus_m F_m \mathcal{R}_X$. Let $\mathcal{H}$ denote the image of $F_1(\mathcal{R}_X) \cdot F_{m-1}\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big) \to F_m\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$. We only have to show $\mathcal{H} = F_m\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$. A section f of $F_m\big(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$ has an expression $\sum_{p+q \leqslant m} a_p \otimes b_q$, where $a_p \in F_p(\mathfrak{E})$ and $b_q \in F_q(\mathcal{L}(\mathfrak{a}))$. There exist sections $b'_m \in F_{m-1}(\mathcal{L}(\mathfrak{a}))$ and $v \in F_1(\mathcal{R}_X)$ such that $v \cdot b'_m = b_m$. Then, f is equivalent to

$$\sum_{q \leqslant m-2} a_p \otimes b_q + \big(a_1 \otimes b_{m-1} + (v a_0) \otimes b'_m\big)$$

modulo $\mathcal{H}$. By an easy descending induction, it is shown that f is equivalent to a section of $F_m(\mathfrak{E}) \otimes F_0\big(\mathcal{L}(-\mathfrak{a})\big)$ modulo $\mathcal{H}$.

Take $i \in \mathfrak{s}(\boldsymbol{m})$. Any section of $F_m(\mathfrak{E}) \otimes F_0\big(\mathcal{L}(-\mathfrak{a})\big)$ has an expression

$$a_m \otimes \big(\boldsymbol{z}^{-\boldsymbol{m}} \cdot z_i \partial_i \mathfrak{a} \cdot e\big).$$

It is equivalent to $-z_i \partial_i a_m \otimes (\boldsymbol{z}^{-\boldsymbol{m}} e) + a_m \otimes (m_i \boldsymbol{z}^{-\boldsymbol{m}} e)$ modulo $\mathcal{H}$. Hence, we only have to show $\boldsymbol{z}^{-\boldsymbol{m}} \cdot F_m\big(\mathfrak{E}\big) \otimes e$ is contained in $\mathcal{H}$. This can be shown by using Lemma 12.2.1 as in the proof of Lemma 12.2.2. Therefore, F_m gives a coherent filtration of $\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})$, due to Proposition 22.1.2.

Let us check that the support of $\mathrm{Gr}^F\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ as an $\mathcal{O}_{C_\lambda\times T^*X}$-module is contained in $\mathcal{S}$. By changing the coordinate system, we may assume that $\mathfrak{a}=\boldsymbol{z}^{\boldsymbol{m}}$. Let $I\subset\underline{n}$. Let $Q\in\mathcal{D}_I^{(\lambda_0)}\setminus\bigcup_{j\not\in I}\mathcal{D}_j^{(\lambda_0)}$. If $I\cap\mathfrak{s}(\boldsymbol{m})=\varnothing$, according to Lemma 12.2.4, the characteristic variety of $\mathfrak{E}'$ is contained in $\mathcal{S}$ around Q. Assume $I\cap\mathfrak{s}(\boldsymbol{m})\neq\varnothing$. Let $i\in I\cap\mathfrak{s}(\boldsymbol{m})$. For $j\in\mathfrak{s}(\boldsymbol{m})\setminus I$, we put $v_j:=m_j^{-1}(z_j\eth_j)-m_i^{-1}(z_i\eth_i)$. For $j\in\underline{n}\setminus(I\cup\mathfrak{s}(\boldsymbol{m}))$, we put $v_j:=\eth_j$. Because $v_j\,e=0$, we have $v_j\,F_0\subset F_0$ and hence $v_j\,F_m\subset F_m$ around Q. This implies that the characteristic variety of $\mathfrak{E}'$ is contained in $\boldsymbol{C}_\lambda\times N^*_{D_I}X$ around Q. □

12.3.2. Strictly S-decomposability of the associated $\mathcal{R}_X$-modules along a monomial function. — Let $1\leqslant\ell\leqslant n$. We put $\underline{l}:=\{1,\dots,\ell\}$. Let $g=\boldsymbol{z}^{\boldsymbol{p}}$ for some $\boldsymbol{p}\in\mathbb{Z}^\ell_{>0}$. We have a natural isomorphism $i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\simeq i_{g\dagger}\mathfrak{E}\otimes\pi^*\mathcal{L}(\mathfrak{a})$, where $\pi: X\times\boldsymbol{C}_t\to X$ be the natural projection. Take $\lambda_0\in\boldsymbol{C}_\lambda$. Let us consider the following for any $b\in\boldsymbol{R}$ on $\mathcal{X}^{(\lambda_0)}\times\boldsymbol{C}_t$:

$$U_b^{(\lambda_0)}\big(i_{g\dagger}(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a}))\big):=U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\pi^*\mathcal{L}(\mathfrak{a}).\tag{266}$$

Note that we have the isomorphisms:

$$\mathrm{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a}))\big)\simeq\mathrm{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\mathcal{L}(\mathfrak{a}).\tag{267}$$

Proposition 12.3.3

- *$i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ is strictly S-decomposable along t at any λ_0, and $U^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ is the V-filtration. As a result, $\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})$ is strictly S-decomposable along g.*
- *We have natural isomorphisms:*

$$\widetilde{\psi}_{t,u}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big)\simeq\widetilde{\psi}_{t,u}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\mathcal{L}(\mathfrak{a}).$$

Under these isomorphisms, the nilpotent parts N of $-\eth_t t$ are the same. In particular, we have natural isomorphisms for the primitive parts:

$$P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{t,u}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big)\simeq P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{t,u}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\mathcal{L}(\mathfrak{a}).$$

Proof. — The second claim follows from the first claim and the isomorphism (267). By construction of the filtration, we have

$$\begin{aligned}t\cdot U_a^{(\lambda_0)}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big)&=t\cdot U_a^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\pi^*\mathcal{L}(\mathfrak{a})\\&\subset U_{a-1}^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\pi^*\mathcal{L}(\mathfrak{a})=U_{a-1}^{(\lambda_0)}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big),\end{aligned}\tag{268}$$

$$\begin{aligned}\eth_t\cdot U_a^{(\lambda_0)}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big)&=\eth_t\cdot U_a^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\pi^*\mathcal{L}(\mathfrak{a})\\&\subset U_{a+1}^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes\pi^*\mathcal{L}(\mathfrak{a})=U_{a+1}^{(\lambda_0)}\big(i_{g\dagger}\big(\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big).\end{aligned}\tag{269}$$

Moreover, we have $t\cdot U_a^{(\lambda_0)}=U_{a-1}^{(\lambda_0)}$ $(a<0)$ and $\eth_t:\mathrm{Gr}_a^{U^{(\lambda_0)}}\simeq\mathrm{Gr}_{a+1}^{U^{(\lambda_0)}}$ $(a>-1)$.

From Lemma 12.2.5 and the isomorphism (267), we obtain that the following endomorphism is nilpotent on $\mathrm{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$:

$$\prod_{u\in\mathcal{K}(i_{g\dagger}\mathfrak{E},\lambda_0,b)}\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big).$$

We also obtain that $\mathrm{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ is strict.

Let us prepare to show that $U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ are $V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$-coherent. We only have to consider the case $b<0$. We have the natural inclusion ${}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})\otimes 1\subset U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. It induces the inclusion

$$\big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})\otimes 1\big)\otimes e\subset U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big).$$

Then, the $V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$-action induces the following morphism:

$$\Phi_b: V_0\mathcal{R}_{X\times\boldsymbol{C}_t}\otimes\Big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})\otimes 1\otimes e\Big)\longrightarrow U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big).$$

Lemma 12.3.4. — *Φ_b is surjective.*

Proof. — For $J=(j_1,\dots,j_n)$, we put $\eth^J=\prod_{i=1}^n\eth_i^{j_i}$ and $|J|=\sum j_i$. Let $F_m(V_0\mathcal{R}_{X\times\boldsymbol{C}_t})$ be as follows:

$$F_m(V_0\mathcal{R}_{X\times\boldsymbol{C}_t}):=\Big\{\sum_{|J|\leqslant m}a_J(\boldsymbol{z},t,\eth_tt)\cdot\eth^J\Big\}.$$

The inclusion ${}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})\otimes 1\subset U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ and the $\mathcal{R}_{X\times\boldsymbol{C}_t}$-action induce the following map:

$$F_m\big(V_0\mathcal{R}_{X\times\boldsymbol{C}_t}\big)\otimes\big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})\otimes 1\big)\longrightarrow U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}).$$

The image is denoted by $F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.

Lemma 12.3.5

- *We have $\eth_i\cdot F_mU^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\subset F_{m+1}U^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.*
- *We have $\eth_iz_i\cdot F_mU^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\subset F_mU^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.*

Proof. — The first claim is clear. For $f\in{}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\underline{l}}}(\mathfrak{E})$, we have

$$\eth_iz_i\cdot P(\boldsymbol{z},t,\eth_tt)\eth^J(f\otimes 1)-P(\boldsymbol{z},t,\eth_tt)\eth^J\cdot\eth_iz_i(f\otimes 1)\in F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}).$$

We also have $\eth_iz_i(f\otimes 1)=(\eth_iz_if)\otimes 1-p_i\eth_tt(f\otimes 1)$, where $p_i:=0$ if $i\notin\underline{l}$. Thus the second claim of Lemma 12.3.5 follows. □

Let us return to the proof of Lemma 12.3.4. According to Lemma 12.2.2, we have

$${}^{\underline{l}}V^{(\lambda_0)}_{b\boldsymbol{p}}(\mathfrak{E})\otimes 1\subset\bigcup_m F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}).$$

Since $U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ is generated by ${}^{\underline{l}}V^{(\lambda_0)}_{b\boldsymbol{p}}(\mathfrak{E})\otimes 1$ over $\mathcal{R}_X$, we obtain

$$\bigcup_m F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})=U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}).$$

Hence, to show the surjectivity of Φ_b, we only have to show $F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\otimes\pi^*\mathcal{L}(\mathfrak{a})\subset\mathrm{Im}(\Phi_b)$ for any m.

Let us consider the following claims:

(P_m) : $F_mU_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\otimes e\subset\mathrm{Im}(\Phi_b)$.

(Q_m) : $F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a}) \subset \mathrm{Im}(\Phi_b)$.

The claim (P_0) is clear. (Note the action of $\eth_t$ on $\mathcal{L}(\mathfrak{a})$ is trivial.) Let us show $(P_m) \Longrightarrow (Q_m)$. We show $\boldsymbol{z}^{N\boldsymbol{m}} \cdot F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \otimes e \subset \mathrm{Im}(\Phi_b)$ by an induction on N. The case $N = 0$ is (P_m). Let us show $N \Longrightarrow N+1$. Take $i \in \mathfrak{s}(\boldsymbol{m})$. Let $f \in F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. We have

$$\mathrm{Im}(\Phi_b) \ni z_i\eth_i(\boldsymbol{z}^{N\boldsymbol{m}} f \otimes e) = Nm_i\, \boldsymbol{z}^{N\boldsymbol{m}} f \otimes e + \boldsymbol{z}^{N\boldsymbol{m}}(z_i\eth_i f) \otimes e + \boldsymbol{z}^{N\boldsymbol{m}} f \otimes (z_i\partial_i\mathfrak{a})\, e.$$

We have $\boldsymbol{z}^{N\boldsymbol{m}} \cdot f \otimes e \in \mathrm{Im}(\Phi_b)$. By Lemma 12.3.5, we have $z_i\eth_i f \in F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$, and hence $\boldsymbol{z}^{N\boldsymbol{m}}(z_i\eth_i f) \otimes e \in \mathrm{Im}(\Phi_b)$. We obtain $\boldsymbol{z}^{N\boldsymbol{m}} f \otimes (z_i\partial_i\mathfrak{a}) \cdot e \in \mathrm{Im}(\Phi_b)$, which implies

$$\boldsymbol{z}^{(N+1)\boldsymbol{m}} f \otimes e \in \mathrm{Im}(\Phi_b).$$

Thus, we obtain (Q_m).

Let us show $(Q_m) \Longrightarrow (P_{m+1})$. Let $f \in F_mU_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. We have $\eth_i(f \otimes e) = (\eth_i f) \otimes e + f \otimes \eth_i e$. By the assumption, we have $\eth_i(f \otimes e) \in \mathrm{Im}(\Phi_b)$ and $f \otimes \eth_i e \in \mathrm{Im}(\Phi_b)$, and thus $(\eth_i f) \otimes e \in \mathrm{Im}(\Phi_b)$. Then, the claim (P_{m+1}) follows. Thus, the induction can proceed, and the proof of Lemma 12.3.4 is finished. □

Let us check that $U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$ is $V_0\mathcal{R}_{X\times \boldsymbol{C}_t}$-coherent. We have the inclusion

$$U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big) \subset i_{g\dagger}(\mathcal{Q}\mathcal{E}) \otimes \pi^*\mathcal{L}(\mathfrak{a}).$$

Hence, it is easy to check that $U_a^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)$ is a pseudo-coherent $\mathcal{O}_{\mathcal{X}\times \boldsymbol{C}_t}$-module. According to Lemma 12.3.4, $U_a^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big)$ is finitely generated over $V_0\mathcal{R}_{X\times \boldsymbol{C}_t}$. Thus, the desired coherence follows from Proposition 22.7.2. □

Remark 12.3.6. — Let $\mathcal{M}$ be an unramifiedly good meromorphic flat bundle with a unique irregular value. By applying the above argument, we obtain that the V-filtration of the D-module $\mathcal{M}$ along g is described as (266). □

12.3.3. Decomposition by the support. — Let $\underline{n} = \{1, \dots, n\}$. For each $J \subset \underline{n}$, we put $D_J := \bigcap_{j\in J} D_j$ and $D_J^\circ := D_J \setminus \bigcup_{i\notin J}(D_J \cap D_i)$. Recall the decomposition in Proposition 17.56 in [**67**]:

$$P\,\mathrm{Gr}_p^{W(N)}\, \widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E}) = \bigoplus_{J\subset \underline{n}} \mathcal{M}_{p,u,J}.$$

Here, $\mathcal{M}_{p,u,J}$ are the $\mathcal{R}_X$-modules which come from tame harmonic bundles $(E_{p,u,J}, \overline{\partial}_{p,u,J}, \theta_{p,u,J}, h_{p,u,J})$ on D_J°. The strict supports of $\mathcal{M}_{p,u,J}$ are $\mathcal{D}_J := \boldsymbol{C}_\lambda\times D_J$. According to Proposition 12.3.3, we obtain the following decomposition:

$$P\,\mathrm{Gr}_p^{W(N)}\, \widetilde{\psi}_{t,u}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big) = \bigoplus_{J\subset \underline{n}-\mathfrak{s}(\boldsymbol{m})} \mathcal{M}_{p,u,J} \otimes \mathcal{L}(\mathfrak{a}). \tag{270}$$

We use the notation in Section 12.2.3. For any $b < 0$ and $c \in \boldsymbol{R}$, we consider the ${}^{i,t}V_0\mathcal{R}_{X\times \boldsymbol{C}_t}$-module:

$${}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}(\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))\big) := {}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) \otimes \pi^*\mathcal{L}(\mathfrak{a}).$$

We would like to show that $\mathcal{M}_{p,u,J} \otimes \mathcal{L}(\mathfrak{a})$ are strictly S-decomposable along z_i at λ_0 with the induced filtration ${}^{i}V^{(\lambda_0)}$.

Lemma 12.3.7. — *The induced subsheaves ${}^{i}V_c^{(\lambda_0)}$ of $\mathrm{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))$ are ${}^{i}V_0\mathcal{R}_X$-coherent.*

Proof. — Let $K := \underline{n} - (\underline{l} \cup \{i\})$. Let $b < 0$. Let $c \leqslant 0$ in the case $i \leqslant \ell$, or $c < 0$ in the case $\ell \leqslant i \leqslant n$. We have the following naturally defined morphism:

$$\Phi : {}^{i,t}V_0\mathcal{R}_{X\times \boldsymbol{C}_t} \otimes \big({}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{b}\boldsymbol{p}+c\boldsymbol{\delta}_i-\varepsilon\boldsymbol{\delta}_K}(\mathfrak{E}) \otimes 1 \otimes e\big) \longrightarrow {}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \pi^*\mathcal{L}(\mathfrak{a})\big).$$

Lemma 12.3.8. — *The map Φ is surjective.*

Proof. — Since the argument is similar to that in the proof of Lemma 12.3.4, we give only an outline. Let $F_m{}^{i,t}V_0\mathcal{R}_{X\times \boldsymbol{C}_t}$ denote the intersection of $F_mV_0\mathcal{R}_{X\times\boldsymbol{C}_t}$ and ${}^{i,t}V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$. For b and c as above, we have the following naturally defined map:

$$F_m\big({}^{i,t}V_0\mathcal{R}_{X\times\boldsymbol{C}_t}\big) \otimes \big({}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{b}\boldsymbol{p}+c\boldsymbol{\delta}_i-\varepsilon\boldsymbol{\delta}_K}(\mathfrak{E}) \otimes 1\big) \longrightarrow {}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big).$$

The image is denoted by $F_m{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)$. As in Lemma 12.3.5, the following holds:

- For $j \neq i$, we have $\eth_j F_m{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \subset F_{m+1}{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.
- For any j, we have $\eth_j z_j F_m{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \subset F_m{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.

By the description in Lemma 12.2.9, ${}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ is $\bigcup F_m\big({}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\big)$. Hence, we only have to show $F_m\big({}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})\big) \otimes \mathcal{L}(\mathfrak{a}) \subset \mathrm{Im}(\Phi)$ for any m, which can be shown using an inductive argument as in the proof of Lemma 12.3.4. Thus, we obtain Lemma 12.3.8. □

Lemma 12.3.9. — *If $i \in \mathfrak{s}(\boldsymbol{m})$, ${}^{i}V_a^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})) = U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))$.*

Proof. — If $a < 0$, we have $z_i \cdot {}^{i}V_a^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = {}^{i}V_{a-1}^{(\lambda_0)}U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$. Hence, they are the same after tensoring $\mathcal{L}(\mathfrak{a})$. □

Lemma 12.3.10. — *In the case ($c > 0$ and $i \leqslant \ell$), we have the following description:*

$$\begin{aligned} &{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big) = \sum_{(c',m)\in\mathcal{U}} \eth_i^m\big({}^{i}V_{c'}^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big),\\ &\mathcal{U} = \big\{(c',m) \in \boldsymbol{R}_{\leqslant 0} \times \mathbb{Z}_{\geqslant 0} \,\big|\, c'+m \leqslant c\big\}. \end{aligned} \tag{271}$$

In the case ($c \geqslant 0$ and $\ell < i \leqslant n$), we have the following description:

$$\begin{aligned} &{}^{i}V_c^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})\big) = \sum_{(c',m)\in\mathcal{U}} \eth_i^m\big({}^{i}V_{c'}^{(\lambda_0)}U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{L}(\mathfrak{a})\big)\big),\\ &\mathcal{U} = \big\{(c',m) \in \boldsymbol{R}_{<0} \times \mathbb{Z}_{\geqslant 0} \,\big|\, c'+m \leqslant c\big\}. \end{aligned} \tag{272}$$

Proof. — By Lemma 12.3.9, we only have to consider the case $i \notin \mathfrak{s}(\boldsymbol{m})$. In the case ($c > 0$ and $i \leqslant \ell$), or ($c \geqslant 0$ and $\ell < i$), we obtain the following equality from Lemma 12.2.10:

$$ {}^{i}V_{c}^{(\lambda_0)} U_{b}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a}) = \eth_i\Big({}^{i}V_{c-1}^{(\lambda_0)} U_{b}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a})\Big) + {}^{i}V_{<c}^{(\lambda_0)} U_{b}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \otimes \mathcal{L}(\mathfrak{a}). $$

Then, we obtain (271) and (272) by an easy induction. □

Let us return to the proof of Lemma 12.3.7. According to Lemma 12.3.8 and Lemma 12.3.10, the induced subsheaves ${}^{i}V_{c}^{(\lambda_0)}$ of $\mathrm{Gr}_b^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))$ are finitely generated over ${}^{i}V_0\mathcal{R}_X$. It is easy to check the pseudo-coherence of ${}^{i}V^{(\lambda_0)}$ as $\mathcal{O}_{\mathcal{X}}$-modules. Hence, they are ${}^{i}V_0\mathcal{R}_X$-coherent by Proposition 22.7.2. Thus, the proof of Lemma 12.3.7 is finished. □

As a consequence, the induced subsheaves ${}^{i}V_{c}^{(\lambda_0)}$ of

$$ \widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})) \quad \text{and} \quad \mathrm{Gr}^{W(N)}\,\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a})) $$

are also ${}^{i}V_0\mathcal{R}_X$-coherent modules. The following claim immediately follows from Proposition 12.2.8 and Proposition 12.3.3.

Proposition 12.3.11

- *The filtrations ${}^{i}V^{(\lambda_0)}$ ($i = 1, \ldots, n$) are compatible with the primitive decomposition of $\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))$.*
- *For each i, the primitive part $P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E} \otimes \mathcal{L}(\mathfrak{a}))$ is strictly S-decomposable along $z_i = 0$ at any λ_0 with the induced filtration ${}^{i}V^{(\lambda_0)}$.*
- *In particular, $\mathcal{M}_{p,u,J} \otimes \mathcal{L}(\mathfrak{a})$ are strictly S-decomposable along z_i at any λ_0 with the induced filtration ${}^{i}V^{(\lambda_0)}$.* □

12.4. Specialization of the associated $\mathcal{R}_X$-module

12.4.1. Completion of $\mathfrak{E}$ along $\mathcal{D}_{\underline{n}}$. — We use the setting in Section 12.1. Let $\widehat{\mathcal{D}}_{\underline{n}}$ denote the completion of $\mathcal{X}$ along $\mathcal{D}_{\underline{n}}$. Let $\iota : \widehat{\mathcal{D}}_{\underline{n}} \to \mathcal{X}$ denote the canonical morphism. Recall that we have the irregular decomposition:

$$ (273) \qquad \iota^*\mathcal{Q}^{(\lambda_0)}\mathcal{E} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{Q}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}, \quad \iota^*\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\widehat{\mathcal{E}}_{\mathfrak{a}}. $$

It is easy to see that $\iota^*\mathfrak{E}^{(\lambda_0)}$ is generated by $\iota^*\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$ in $\iota^*\mathcal{Q}^{(\lambda_0)}\mathcal{E}$ over $\mathcal{R}_{\widehat{\mathcal{D}}_{\underline{n}}}$. Hence, we have the decomposition $\iota^*\mathfrak{E}^{(\lambda_0)} = \bigoplus_{\mathfrak{a}} \widehat{\mathfrak{E}}_{\mathfrak{a}}^{(\lambda_0)}$ corresponding to (273), where the $\mathcal{R}_{\widehat{\mathcal{D}}_{\underline{n}}}$-submodule $\widehat{\mathfrak{E}}_{\mathfrak{a}} \subset \mathcal{Q}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}$ is generated by $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\widehat{\mathcal{E}}_{\mathfrak{a}}$ on $U(\lambda_0) \times \widehat{D}_n$. By varying λ_0 and gluing them, we obtain

$$ (274) \qquad \iota^*\mathfrak{E} = \bigoplus_{\mathfrak{a}} \widehat{\mathfrak{E}}_{\mathfrak{a}}. $$

For each $\mathfrak{a} \in \mathrm{Irr}(\theta)$, we have the wild harmonic bundle $(E_{\mathfrak{a}}, \overline{\partial}_{\mathfrak{a}}, \theta_{\mathfrak{a}}, h_{\mathfrak{a}})$ such that $\mathrm{Irr}(\theta_{\mathfrak{a}}) = \{\mathfrak{a}\}$, which is obtained from $(E, \overline{\partial}_E, \theta, h)$ as the full reduction (Theorem 11.2.2).

Lemma 12.4.1. — *We have the natural isomorphism $\widehat{\mathfrak{E}}_{\mathfrak{a}} \simeq \iota^* \mathfrak{E}_{\mathfrak{a}}$.*

Proof. — By construction, we have natural isomorphisms $\mathcal{Q}\widehat{\mathcal{E}}_{\mathfrak{a}} \simeq \iota^* \mathcal{Q}\mathcal{E}_{\mathfrak{a}}$. The KMS-structure at λ_0 is unique, if it exists, a property which can be shown by using an argument similar to that in the proof of Lemma 2.8.3. Hence, we have the equality of the filtered bundles $\mathcal{Q}_*^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}} = \iota^* \mathcal{Q}_*^{(\lambda_0)}\mathcal{E}_{\mathfrak{a}}$ for each λ_0 under the above isomorphism. Then the claim of the lemma is clear by construction in Section 12.1. □

12.4.2. Holonomicity. — For any subset $I \subset \underline{n} := \{1, \ldots, n\}$, we put $D_I := \bigcap_{i \in I} D_i$, and let $N^*_{D_I} X$ denote the conormal bundle of D_I in X. Let $\mathcal{S}$ denote the union $\boldsymbol{C}_\lambda \times \bigcup_{I \subset \underline{n}} N^*_{D_I} X$.

Proposition 12.4.2. — *The characteristic variety of $\mathfrak{E}$ is contained in $\mathcal{S}$. In particular, $\mathfrak{E}$ is holonomic.*

Proof. — For $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n_{\geqslant 0}$, we put $\eth^J = \prod_{i=1}^n \eth_i^{j_i}$ and $|J| = \sum j_i$. Let $F_m \mathcal{R}_X = \big\{\sum_{|J| \leqslant m} a_J \cdot \eth^J\big\}$. We put $F_m(\mathfrak{E}) := F_m \mathcal{R}_X \cdot \mathcal{Q}^{(\lambda_0)}_{<\delta}(\mathcal{E})$ around $\{\lambda_0\} \times X$. Thus, we obtain a coherent filtration of $\mathfrak{E}$ around $\{\lambda_0\} \times X$.

Let $\mathrm{Gr}^F(\mathfrak{E})$ denote the associated $\mathcal{O}_{\boldsymbol{C}_\lambda \times T^*X}$-module. Let us show that the support of $\mathrm{Gr}^F(\mathfrak{E})$ is contained in $\mathcal{S}$, when we shrink X. We have the coherent filtrations of $\big\{F_m(\mathfrak{E}_{\mathfrak{a}}) \,\big|\, m = 1, 2, \ldots\big\}$ of $\mathfrak{E}_{\mathfrak{a}}$ constructed in the same way. Then, we have $\iota^* F_m(\mathfrak{E}) \simeq \bigoplus \iota^* F_m(\mathfrak{E}_{\mathfrak{a}})$ under the isomorphism (274) and Lemma 12.4.1. Let $\mathcal{Y}$ denote $\boldsymbol{C}_\lambda \times (T^*X \times_X \{O\})$. Let $\mathrm{Ch}(\mathfrak{E})$ and $\mathrm{Ch}(\mathfrak{E}_{\mathfrak{a}})$ denote the characteristic varieties of $\mathfrak{E}$ and $\mathfrak{E}_{\mathfrak{a}}$. Because $\iota^* \mathrm{Gr}^F(\mathfrak{E}) \simeq \bigoplus \iota^* \mathrm{Gr}^F(\mathfrak{E}_{\mathfrak{a}})$, we obtain that the completion of $\mathrm{Ch}(\mathfrak{E})$ along $\mathcal{Y}$ is the union of the completions of $\mathrm{Ch}(\mathfrak{E}_{\mathfrak{a}})$ along $\mathcal{Y}$. Hence, the above claim follows from Corollary 12.3.2. □

12.4.3. Strict S-decomposability along any monomial function. — Let us consider a monomial function $g = \boldsymbol{z}^{\boldsymbol{p}}$ for some $\boldsymbol{p} \in \mathbb{Z}^n_{\geqslant 0}$.

Proposition 12.4.3. — *$\mathfrak{E}$ is strictly S-decomposable along g. We have natural isomorphisms:*

$$\iota^* \widetilde{\psi}_{g,u}(\mathfrak{E}) \simeq \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \iota^* \widetilde{\psi}_{g,u}(\mathfrak{E}_{\mathfrak{a}}). \tag{275}$$

It is compatible with the nilpotent part N of $-\eth_t t$.

Proof. — We put $\mathfrak{s}(\boldsymbol{p}) := \{i \,|\, p_i \neq 0\}$. For any $b < 0$, we define

$$U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) := V_0 \mathcal{R}_{X \times \boldsymbol{C}_t} \cdot \big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p} - \varepsilon \boldsymbol{\delta}_{\underline{n} - \mathfrak{s}(\boldsymbol{p})}}(\mathcal{Q}\mathcal{E}) \otimes 1\big). \tag{276}$$

For any $b \geqslant 0$, we define

$$U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) := \sum_{\substack{c<0, m\in\mathbb{Z}_{\geqslant 0}\\ c+m\leqslant b}} \eth_t^m U_c^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}). \tag{277}$$

They are $V_0\mathcal{R}_{X\times \boldsymbol{C}_t}$-coherent modules. By construction, they satisfy the following conditions:

- $\bigcup_{a\in\boldsymbol{R}} U_a^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = i_{g\dagger}\mathfrak{E}$. For any $b \in \boldsymbol{R}$, there exists $\varepsilon > 0$ such that $U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = U_{b+\varepsilon}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$.
- $t \cdot U_a^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \subset U_{a-1}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ for any $a \in \boldsymbol{R}$, and $t \cdot U_a^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = U_{a-1}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ for any $a < 0$.
- $\eth_t : U_a^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) \subset U_{a+1}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E})$ for any $a \in \boldsymbol{R}$, and the induced morphisms $\eth_t : \mathrm{Gr}_a^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E}) \to \mathrm{Gr}_{a+1}^{U^{(\lambda_0)}}(i_{g\dagger}\mathfrak{E})$ are surjective for any $a > -1$.

The induced morphism $\mathcal{D}_{\underline{n}} \times \boldsymbol{C}_t \to X \times \boldsymbol{C}_t$ is also denoted by ι. Let us look at $\iota^*\big(i_{g\dagger}\mathfrak{E}\big)$ and the induced filtration $\iota^* U^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)$. We have the decomposition:

$$\iota^*\big(i_{g\dagger}\mathfrak{E}\big) = \bigoplus i_{g\dagger}\widehat{\mathfrak{E}}_{\mathfrak{a}}, \quad {}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{Q}\mathcal{E}) = \bigoplus_{\mathfrak{a}} {}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{Q}\widehat{\mathcal{E}}_{\mathfrak{a}}). \tag{278}$$

We have $U_b^{(\lambda_0)}(i_{g\dagger}\widehat{\mathfrak{E}}_{\mathfrak{a}})$ defined as in (276) and (277) for $\widehat{\mathfrak{E}}_{\mathfrak{a}}$ instead of $\mathfrak{E}$. Then, we have the decomposition for any b corresponding to (278):

$$\iota^* U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} U_b^{(\lambda_0)}(i_{g\dagger}\widehat{\mathfrak{E}}_{\mathfrak{a}}).$$

We have natural isomorphisms:

$$\widehat{\mathfrak{E}}_{\mathfrak{a}} \simeq \iota^*\mathfrak{E}_{\mathfrak{a}}, \qquad {}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{Q}\widehat{\mathcal{E}}_{\mathfrak{a}}) \simeq \iota^*\, {}^{\underline{n}}V^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{Q}\mathcal{E}_{\mathfrak{a}}).$$

Note the surjectivity of Φ_b in Lemma 12.3.4. Hence, we have the natural isomorphism:

$$U_b^{(\lambda_0)}(i_{g\dagger}\widehat{\mathfrak{E}}_{\mathfrak{a}}) \simeq \iota^* U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}_{\mathfrak{a}}).$$

Therefore, we have the following equality of the filtrations under the isomorphism $\iota^* i_{g\dagger}\mathfrak{E} \simeq \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \iota^* i_{g\dagger}\mathfrak{E}_{\mathfrak{a}}$:

$$\iota^* U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \iota^* U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}_{\mathfrak{a}}\big). \tag{279}$$

Recall that $\mathfrak{E}_{\mathfrak{a}}$ is strictly S-decomposable along g at any λ_0 with the V-filtration $U^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}_{\mathfrak{a}})$ (Proposition 12.3.3). Hence, we can conclude that $\mathfrak{E}$ is strictly S-decomposable along g at λ_0 with the V-filtration $U^{(\lambda_0)}(\mathfrak{E})$, due to Proposition 22.5.8 below. We obtain the isomorphism (275) from (279). □

We have the tame harmonic bundles $(E'_{\mathfrak{a}}, \overline{\partial}'_{\mathfrak{a}}, \theta'_{\mathfrak{a}}, h'_{\mathfrak{a}})$ such that $(E_{\mathfrak{a}}, \overline{\partial}_{\mathfrak{a}}, \theta_{\mathfrak{a}}, h_{\mathfrak{a}}) \simeq (E'_{\mathfrak{a}}, \overline{\partial}'_{\mathfrak{a}}, \theta'_{\mathfrak{a}}, h'_{\mathfrak{a}}) \otimes L(\mathfrak{a})$. The associated $\mathcal{R}$-modules are denoted by $\mathfrak{E}'_{\mathfrak{a}}$.

Corollary 12.4.4. — *We have isomorphisms:*

$$\iota^* \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E}) \simeq \bigoplus_{\mathfrak{a}} \iota^* \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E}'_{\mathfrak{a}}) \otimes \mathcal{L}(\mathfrak{a}), \tag{280}$$

$$\iota^* P \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E}) \simeq \bigoplus_{\mathfrak{a}} \iota^* P \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E}'_{\mathfrak{a}}) \otimes \mathcal{L}(\mathfrak{a}). \tag{281}$$

Proof. — It follows from Proposition 12.4.3. □

Recall the notion of $\mathcal{A}$-good wild harmonic bundle in Definition 7.1.3.

Corollary 12.4.5. — *If $(E, \overline{\partial}_E, \theta, h)$ is an unramifiedly $\mathcal{A}$-good wild harmonic bundle, then we have the vanishing $P \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E}) = 0$ unless $u \in \boldsymbol{R} \times \mathcal{A}$.*

Proof. — Under the assumption, $(E'_{\mathfrak{a}}, \overline{\partial}'_{\mathfrak{a}}, \theta'_{\mathfrak{a}}, h'_{\mathfrak{a}})$ are $\mathcal{A}$-tame harmonic bundles. Hence, we obtain the vanishing of $\iota^* P \operatorname{Gr}_p^{W(N)} \psi_u^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}'_{\mathfrak{a}}) = 0$ unless $u \in \mathcal{A}$ by the argument given in Section 19.5 of [**67**]. Although we considered only the case $\mathcal{A} = \sqrt{-1}\boldsymbol{R}$ in [**67**], the other cases can be shown with the same argument. □

12.4.4. Strict S-decomposability of $P \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}(\mathfrak{E})$ along a coordinate function. — Let $b < 0$. We use the notation in Section 12.3.3. For $1 \leqslant i \leqslant n$, we put $K := \underline{n} - (\mathfrak{s}(\boldsymbol{p}) \cup \{i\})$. In the case ($c \leqslant 0$ and $i \in \mathfrak{s}(\boldsymbol{p})$) or ($c < 0$ and $i \notin \mathfrak{s}(\boldsymbol{p})$), we define

$${}^{i}V_c^{(\lambda_0)} U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) := \operatorname{Im}\Big({}^{i,t}V_0\mathcal{R}_{X\times \boldsymbol{C}_t} \otimes \big({}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-c\boldsymbol{\delta}_i-\varepsilon\boldsymbol{\delta}_K}(\mathcal{Q}\mathcal{E}) \otimes 1\big) \longrightarrow U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)\Big).$$

(See Section 12.2.3 for ${}^{i,t}V_0\mathcal{R}_{X\times\boldsymbol{C}_t}$.) In the case ($c > 0$ and $i \in \mathfrak{s}(\boldsymbol{p})$), we put

$${}^{i}V_c^{(\lambda_0)} U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) := \sum_{(c',m)\in\mathcal{U}} \eth_i^m \cdot {}^{i}V_{c'}^{(\lambda_0)} U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big),$$

$$\mathcal{U} := \big\{(c',m) \in \boldsymbol{R}_{\leqslant 0} \times \mathbb{Z}_{\geqslant 0} \,\big|\, c' + m \leqslant c\big\}.$$

In the case ($c \geqslant 0$ and $i \notin \mathfrak{s}(\boldsymbol{p})$), we set

$${}^{i}V_c^{(\lambda_0)} U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) := \sum_{(c',m)\in\mathcal{U}} \eth_i^m \cdot {}^{i}V_{c'}^{(\lambda_0)} U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big),$$

$$\mathcal{U} := \big\{(c',m) \in \boldsymbol{R}_{<0} \times \mathbb{Z}_{\geqslant 0} \,\big|\, c' + m \leqslant c\big\}.$$

Thus, we obtain a filtration ${}^{i}V^{(\lambda_0)}$ of $U_b^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big)$ for each $b < 0$. (Note that we only have to consider the case $b < 0$ to investigate the property of $\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$.) The induced filtrations of $\operatorname{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}\big)$, $\widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ and $\operatorname{Gr}^{W(N)} \widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ are also denoted by ${}^{i}V^{(\lambda_0)}$. They are ${}^{i}V_0\mathcal{R}_X$-coherent.

Proposition 12.4.6

- *The filtrations ${}^{i}V^{(\lambda_0)}$ $(i = 1, \ldots, n)$ are compatible with the primitive decomposition of $\operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$.*
- *For each i, the primitive part $P \operatorname{Gr}_p^{W(N)} \widetilde{\psi}_{t,u}(i_{g\dagger}\mathfrak{E})$ is strictly S-decomposable along $z_i = 0$ at any λ_0 with the filtration ${}^{i}V^{(\lambda_0)}$.*

Proof. — We also have the induced filtration ${}^{i}V^{(\lambda_0)}$ of $\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\mathfrak{E}_{\mathfrak{a}})$. Under the isomorphism (280), we have the equality of the filtrations ${}^{i}V^{(\lambda_0)}$, by construction and Lemma 12.3.8. Then, the claim of Proposition 12.4.6 follows from Proposition 12.3.11 and Proposition 22.5.8 below. □

12.4.5. Comparison of the specializations. — According to Proposition 12.4.6, we have the decomposition:

$$P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\mathfrak{E})=\bigoplus_I \mathcal{M}_{u,p,I}. \tag{282}$$

Here, the support of $\mathcal{M}_{u,p,I}$ is $\mathcal{D}_I$, and the support of any non-trivial coherent $\mathcal{R}$-submodule of $\mathcal{M}_{u,p,I}$ is not contained in $\mathcal{D}_J$ $(J\supset I)$. Since $\mathcal{M}_{u,p,I}$ are strictly S-decomposable along any z_i, it is the push-forward of the $\mathcal{R}_{D_I}$-module $\mathcal{M}'_{u,p,I}$. Let $I^c:=\underline{n}-I$ and $z_{I^c}:=\prod_{j\in I^c}z_j$. The localization $\mathcal{M}'_{u,p,I}\otimes\mathcal{O}(*z_{I^c})$ is a family of meromorphic λ-flat bundles.

We take an auxiliary sequence $\boldsymbol{m}(0),\dots,\boldsymbol{m}(L)$ for $\operatorname{Irr}(\theta)$. By shrinking X, and by using Corollary 11.2.3 with twisting by good wild harmonic bundles of rank one, we obtain harmonic bundles $(E_{\mathfrak{a}}^{\boldsymbol{m}(i)},\overline{\partial}_{\mathfrak{a}}^{\boldsymbol{m}(i)},\theta_{\mathfrak{a}}^{\boldsymbol{m}(i)},h_{\mathfrak{a}}^{\boldsymbol{m}(i)})$ on $X\setminus D$ for $\mathfrak{a}\in\overline{\operatorname{Irr}}(\theta,\boldsymbol{m}(i))$ as the reductions at the level $\boldsymbol{m}(i)$, which induce $\mathcal{R}_X$-modules $\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}$. We have the decomposition:

$$P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)})=\bigoplus_I \mathcal{M}_{u,p,I}(E_{\mathfrak{a}}^{\boldsymbol{m}(i)}).$$

Lemma 12.4.7. — *Let k be determined by $\boldsymbol{m}(0)\in\mathbb{Z}_{<0}^k\times\mathbf{0}_{n-k}$. If $D_I\subset D(\leqslant k)$, we have the natural isomorphism:*

$$\Psi:\mathcal{M}_{u,p,I}(E_0^{\boldsymbol{m}(0)})\simeq\mathcal{M}_{u,p,I}. \tag{283}$$

Proof. — We may assume that $D_I\subset g^{-1}(0)$. Let $\iota_I:\widehat{D}_I\to X$ be the natural morphism. We have the following natural isomorphisms:

$$\begin{aligned}\iota_I^*P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\mathfrak{E})&\simeq P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{E})\\ &\simeq\bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\theta,\boldsymbol{m}(0))}P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(0)})\\ &\simeq P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{E}_0^{\boldsymbol{m}(0)})\simeq\iota_I^*P\operatorname{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}(\mathfrak{E}_0^{\boldsymbol{m}(0)}).\end{aligned} \tag{284}$$

Since the support of $\mathcal{M}_{u,p,I}$ and $\mathcal{M}_{u,p,I}(E_0^{\boldsymbol{m}(0)})$ are contained in $\mathcal{D}_I$, we obtain the desired isomorphism. □

Corollary 12.4.8. — *We have natural isomorphisms $\Psi:\mathcal{M}_{u,p,\underline{n}}\simeq\mathcal{M}_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(i)})$ for $i=0,1,\dots,L$.* □

12.4.6. C^∞-lift of a section. — This subsection is a preparation for Propositions 12.7.1 and 12.7.3. Let λ_0 be generic with respect to

$$\bigcup_{i=1}^{n} \{v \in \boldsymbol{R} \times \boldsymbol{C} \,|\, m_i \cdot v \in \mathcal{KMS}(\mathcal{PE}^0, i)\}.$$

Let $\mathcal{K}$ be a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$.

We take an auxiliary sequence $\boldsymbol{m}(0), \dots, \boldsymbol{m}(L)$ for $\mathrm{Irr}(\theta)$. Let f be a section of $\mathcal{M}_{u,p,I}(E_{\mathfrak{a}}^{\boldsymbol{m}(i)})$ on $\mathcal{K} \times X$. We would like to consider a C^∞-lift of f to $\mathcal{M}_{u,p,I}$. Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{n-k}$. For simplicity, we consider the case where $\boldsymbol{m}(i) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{n-k}$. We may assume $b := \mathfrak{p}(\lambda_0, u) < 0$. We will shrink X and $\mathcal{K}$ without mention.

We have a section $\widetilde{f} \in U_b^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)})$ on $\mathcal{K} \times X$ with the following property:

- The induced section $\widetilde{f}^{(1)}$ of $\mathrm{Gr}_b^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)$ is contained in $W_p(N)\widetilde{\psi}_{g,u}(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)})$. And the induced section $\widetilde{f}^{(2)}$ of $\mathrm{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}\big(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)$ is equal to f which is contained in $P\,\mathrm{Gr}_p^{W(N)}\widetilde{\psi}_{g,u}\big(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)$.

Recall $g = \boldsymbol{z}^{\boldsymbol{p}}$ for some $\boldsymbol{p} \in \mathbb{Z}^n_{\geqslant 0}$. We set $K := \{1 \leqslant i \leqslant n \,|\, p_i = 0\}$. By the definition of the filtration $U^{(\lambda_0)}$, we have the expression of f as a finite sum

$$f = \sum P_B \cdot \eth^B(a_B \otimes 1),$$

where $\eth^A = \prod_{i=1}^n \eth_i^{b_i}$ for $B = (b_1, \dots, b_n)$, $P_B \in \boldsymbol{C}[t\eth_t]$, and $a_B \in {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}(\mathcal{QE}_{\mathfrak{a}}^{\boldsymbol{m}(i)})$ for some $\varepsilon > 0$.

Let S be a small multi-sector in $\mathcal{K} \times (X \smallsetminus D(\leqslant k))$. Since λ_0 is assumed to be generic, we have a $\mathbb{D}$-flat splitting of the Stokes filtration at the level $\boldsymbol{m}(i)$:

$$ {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}\big(\mathcal{QE}\big)_{|\overline{S}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,\boldsymbol{m}(i))} {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}\big(\mathcal{QE}\big)_{\mathfrak{b},S}, \qquad \mathcal{QE}_{|\overline{S}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,\boldsymbol{m}(i))} \mathcal{QE}_{\mathfrak{b},S}.$$

They induce a decomposition of the $\mathcal{R}$-module $\mathfrak{E}_{|S} = \bigoplus \mathfrak{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},S}$, and each $\mathfrak{E}^{\boldsymbol{m}(i)}_{\mathfrak{b},S}$ is naturally isomorphic to $(\mathfrak{E}_{\mathfrak{b}}^{\boldsymbol{m}(i)})_{|S}$. Let $a_{B,S} \in {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}\big(\mathcal{QE}\big)_{\mathfrak{a},S}$ be the lift of a_B. We have the lift $\widetilde{f}_S$ of the restriction $\widetilde{f}_{|S}$ to $i_{g\dagger}\mathfrak{E}^{\boldsymbol{m}(i)}_{\mathfrak{a},S}$ on $S \times \boldsymbol{C}_t$:

$$\widetilde{f}_S = \sum P_B \cdot \eth^B \cdot \big(a_{B,S} \otimes 1\big).$$

If we are given another $\mathbb{D}$-flat splitting

$$ {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}(\mathcal{QE})_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,\boldsymbol{m}(i))} {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_K}(\mathcal{QE})'_{\mathfrak{b},S},$$

we obtain another lift:

$$\widetilde{f}'_S = \sum P_B \cdot \eth^B \cdot \big(a'_{B,S} \otimes 1\big).$$

The following lemma is obvious by construction, which is stated for reference in the subsequent argument.

Lemma 12.4.9. — *We take a frame $\boldsymbol{w}$ of ${}^{\boldsymbol{m}}\mathcal{V}^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_K}(\mathcal{QE})$, and let h_1 be the Hermitian metric for which $\boldsymbol{w}$ is orthonormal. Then, we have $a_{B,S} - a'_{B,S} = O\big(\exp(-\eta|\boldsymbol{z}^{\boldsymbol{m}(i)}|)\big)$ for some $\eta > 0$ with respect to h_1.* □

Shrinking X and $\mathcal{K}$ appropriately, we take a finite covering $\mathcal{K} \times (X \setminus D) = \bigcup S_i$ by small multi-sectors. We take a partition of unity $\{\chi_i\}$ subordinated to the covering $\{S_i\}$ such that χ_i depend only on $\arg(z_j)$ $(j = 1, \dots, k)$. For each S_i, we take a lift $\widetilde{f}_{S_i}$ as above. Let "$\otimes C^\infty$" denote the operation to take the tensor products with the sheaf of C^∞-functions. (Note that it is faithfully flat, according to [**54**].) Then, we obtain the following section:

$$\widetilde{f}_{C^\infty} = \sum \chi_i \cdot \widetilde{f}_{S_i} \in \big(i_{g\dagger}\mathfrak{E} \otimes C^\infty\big)_{|\mathcal{K}\times(X\setminus D(\leqslant k))\times \boldsymbol{C}_t}.$$

Lemma 12.4.10. — *$\widetilde{f}_{C^\infty}$ is a section of $U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}\big) \otimes C^\infty$ on $\mathcal{K} \times X \times \boldsymbol{C}_t$.*

Proof. — We have

$$\chi_i \cdot \widetilde{f}_{S_i} = \sum_B P_B \cdot \chi_i \cdot \eth^B(a_{B,S_i} \otimes 1) = \sum_B P_B \cdot \sum_L \eth^L(c_{B,L,S_i} \otimes 1).$$

Here, c_{B,L,S_i} are the product of a_{B,S_i} and $R_{B,L}(\chi_i)$, where $R_{B,L} \in \mathcal{R}_X$ are independent of S_i. By using $\sum \chi_i = 1$ and Lemma 12.4.9, we obtain that $\sum_i c_{A,L,S_i}$ is a C^∞-section of ${}^{\boldsymbol{m}}\mathcal{V}^{(\lambda_0)}_{\boldsymbol{bp}-\boldsymbol{\delta}_K}(\mathcal{Q})$. Then, the claim of the lemma follows. □

Because of Lemma 12.4.10, we obtain an induced section $\widetilde{f}^{(1)}_{C^\infty}$ of $\mathrm{Gr}^{U^{(\lambda_0)}}_b(i_{g\dagger}\mathfrak{E})\otimes C^\infty$ on $\mathcal{K} \times X$.

Lemma 12.4.11

- *$\widetilde{f}^{(1)}_{C^\infty}$ is contained in $W_p(N)\widetilde{\psi}_{g,u}(\mathfrak{E}) \otimes C^\infty$. Hence, it induces a section of $\widetilde{f}^{(2)}_{C^\infty}$ of $\mathrm{Gr}^{W(N)}_p\,\widetilde{\psi}_{g,u}(\mathfrak{E}) \otimes C^\infty$.*
- *$\widetilde{f}^{(2)}_{C^\infty}$ is contained in $P\,\mathrm{Gr}^{W(N)}_p\,\widetilde{\psi}_{g,u}(\mathfrak{E}) \otimes C^\infty$. Moreover, it is contained in $\mathcal{M}_{u,p,I} \otimes C^\infty$ for the decomposition $P\,\mathrm{Gr}^{W(N)}_p\,\widetilde{\psi}_{g,u}(\mathfrak{E}) \otimes C^\infty = \bigoplus \mathcal{M}_{u,p,J} \otimes C^\infty$.*
- *$(\widetilde{f}^{(2)}_{C^\infty})_{|\mathcal{K}\times\widehat{D}(\leqslant k)}$ is equal to $f_{|\mathcal{K}\times\widehat{D}(\leqslant k)}$ under the isomorphism*

$$\big(\mathcal{M}_{u,p,I} \otimes C^\infty\big)_{|\mathcal{K}\times\widehat{D}(\leqslant k)} \simeq \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,\boldsymbol{m}(i))} \big(\mathcal{M}_{u,p,I}(E^{\boldsymbol{m}(i)}_{\mathfrak{b}}) \otimes C^\infty\big)_{|\mathcal{K}\times\widehat{D}(\leqslant k)}.$$

- *In particular, if $D_I \subset D(\leqslant k)$, we have $\widetilde{f}^{(2)}_{C^\infty} = \Psi(f)$. (See Lemma 12.4.7 for Ψ.)*

Proof. — Let us show the first claim. Let $v \neq u$, and let $\widetilde{f}^{(1)}_{C^\infty,v}$ denote the $\widetilde{\psi}_{g,v}(\mathfrak{E}) \otimes C^\infty$-component of $\widetilde{f}^{(1)}_{C^\infty}$ for the decomposition $\mathrm{Gr}^{U^{(\lambda_0)}}_b(i_{g\dagger}\mathfrak{E}) \otimes C^\infty$. Assume that it is not 0. Then, there is an integer ℓ such that $\widetilde{f}^{(1)}_{C^\infty,v}$ is contained in $W_\ell(N)\widetilde{\psi}_{g,v}(\mathfrak{E}) \otimes C^\infty$ but not in $W_{\ell-1}(N)\widetilde{\psi}_{g,v}(\mathfrak{E}) \otimes C^\infty$. Let $[\widetilde{f}^{(1)}_{C^\infty,v}]$ denote the induced section of $\mathrm{Gr}_\ell^{W(N)} \widetilde{\psi}_{g,v}(\mathfrak{E}) \otimes C^\infty$. According to Proposition 12.4.6, we have the following decomposition as in (282):

$$\mathrm{Gr}_\ell^{W(N)} \widetilde{\psi}_{g,v}(\mathfrak{E}) = \bigoplus_J M_J.$$

Let $[\widetilde{f}^{(1)}_{C^\infty,v}] = \sum [\widetilde{f}^{(1)}_{C^\infty,v}]_J$ denote the corresponding decomposition. To show the first claim, we only have to show $[\widetilde{f}^{(1)}_{C^\infty,v}]_J = 0$ for any J. If $D_J \subset D(\leqslant k)$, we can calculate $[\widetilde{f}^{(1)}_{C^\infty,v}]_J$ after taking the completion along $D(\leqslant k)$. Hence, it is easy to check $[\widetilde{f}^{(1)}_{C^\infty,v}]_J = 0$ in this case.

Let us consider the case $D_J \not\subset D(\leqslant k)$. Let $\widetilde{f}^{(1)}_S$ denote the section of $\mathrm{Gr}_b^{U(\lambda_0)}(i_{g\dagger}\mathfrak{E}_{|S})$ induced by $\widetilde{f}_S$. By our choice of $\widetilde{f}_S$, the restriction of $\widetilde{f}^{(1)}_S$ to $S \cap \big(\mathcal{K} \times (X \setminus D(\leqslant k))\big)$ is contained in $W_{p-1}(N)\widetilde{\psi}_{g,u}(\mathfrak{E})_{|S}$. Hence, the restriction of $\widetilde{f}^{(1)}_{C^\infty}$ to $\mathcal{K} \times (X \setminus D(\leqslant k))$ is contained in $W_{p-1}(N)\widetilde{\psi}_{g,u}(\mathfrak{E}) \otimes C^\infty$. As a result, the restriction of $[\widetilde{f}^{(1)}_{C^\infty,v}]$ to $\mathcal{K} \times \big(X \setminus D(\leqslant k)\big)$ is 0.

We have the inclusion $M_J \to \widetilde{M}_J := M_J \otimes \mathcal{O}(*D(\leqslant k))$. It induces the inclusion $M_J \otimes C^\infty \to \widetilde{M} \otimes C^\infty$. Recall that $\widetilde{M}_J$ is the push-forward of a family of meromorphic λ-flat bundles on $\mathcal{D}_J$ to $\mathcal{X}$. Since the restriction of $[\widetilde{f}^{(1)}_{C^\infty,v}]_J \in \widetilde{M}_J \otimes C^\infty$ to $\mathcal{K} \times (X \setminus D(\leqslant k))$ is 0, we can conclude that $[\widetilde{f}^{(1)}_{C^\infty,v}]_J$ is 0.

The second and third claims can be shown by the same argument. The fourth claim follows from the third one. □

12.5. Construction of the $\mathcal{R}$-triple $(\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$

12.5.1. Statement. — We continue to use the setting in Section 12.1. Let $\sigma : \boldsymbol{C}^*_\lambda \to \boldsymbol{C}^*_\lambda$ be given by $\sigma(\lambda) = -\overline{\lambda}^{-1}$. We set $\boldsymbol{S} := \big\{\lambda \in \boldsymbol{C} \,\big|\, |\lambda| = 1\big\}$. Recall that we have the naturally induced Hermitian sesqui-linear pairing

$$C : \mathcal{E}_{|\boldsymbol{S}\times(X\setminus D)} \otimes \sigma^*\mathcal{E}_{|\boldsymbol{S}\times(X\setminus D)} \longrightarrow \mathcal{C}^\infty_{\boldsymbol{S}\times(X\setminus D)} \tag{285}$$

given by $C(f_1 \otimes \sigma^* f_2) := h(f_1, \sigma^* f_2)$, where the right-hand side means the sheaf of C^∞-functions on $\boldsymbol{S} \times (X \setminus D)$. We show the following proposition in Sections 12.5.2–12.5.5.

Proposition 12.5.1. — *We have a unique sesqui-linear pairing*

$$\mathfrak{C} : \mathfrak{E}_{|\boldsymbol{S}\times X} \otimes \sigma^*\mathfrak{E}_{|\boldsymbol{S}\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}},$$

whose restriction to $\boldsymbol{S} \times (X \setminus D)$ is equal to C.

As a result, we obtain an $\mathcal{R}_X$-triple $\mathfrak{T}(E) := (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$ associated to the unramifiedly good wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$.

12.5.2. The induced pairing of $\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$ and $\sigma^*\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$. — Let $\lambda_0 \in \boldsymbol{S}$. Note $\sigma(\lambda_0) = -\overline{\lambda}_0^{-1} = -\lambda_0$. Let $U(\lambda_0)$ be a sufficiently small neighbourhood of λ_0. Let $\boldsymbol{I}(\lambda_0) := U(\lambda_0) \cap \boldsymbol{S}$. We put $U(-\lambda_0) := \sigma(U(\lambda_0))$ and $\boldsymbol{I}(-\lambda_0) := \sigma(\boldsymbol{I}(\lambda_0))$.

Lemma 12.5.2. — *The restriction of* (285) *to* $\boldsymbol{I}(\lambda_0) \times (X \setminus D)$ *is extended to the following pairing:*

$$(286)\quad \mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X} \longrightarrow \Big\{F \in C^\infty\big(\boldsymbol{I}(\lambda_0) \times (X \setminus D)\big) \,\Big|\, |F| = O\Big(\prod_{i=1}^n |z_i|^{-2+\varepsilon}\Big) \text{ for some } \varepsilon > 0\Big\}.$$

Proof. — We use the notation in Section 9.1.1. As a preparation, we show that $g_{\mathrm{irr}}(-\lambda+\lambda_0)^\dagger \circ g_{\mathrm{irr}}(-\lambda+\lambda_0)^{-1}$ is bounded with respect to h on $\boldsymbol{I}(\lambda_0) \times (X \setminus D)$, if $U(\lambda_0)$ is sufficiently small. We put

$$g'_{\mathrm{irr}}(\lambda) := \exp\Big(\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \lambda\overline{\mathfrak{a}}\cdot\pi^\dagger_{\mathfrak{a}}\Big) = \prod_{i=0}^{L} \exp\Big(\sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(i))} \lambda\cdot\overline{\zeta_{\boldsymbol{m}(i)}(\mathfrak{b})}\cdot\pi_{\mathfrak{b}}^{\boldsymbol{m}(i)\dagger}\Big).$$

We have the boundedness of

$$g_{\mathrm{irr}}(-\lambda+\lambda_0)^\dagger \circ g'_{\mathrm{irr}}(-\lambda+\lambda_0)^{-1} = \exp\Big(\sum_{\mathfrak{a}} 2\sqrt{-1}\,\mathrm{Im}\big(\overline{(\lambda-\lambda_0)\mathfrak{a}}\big)\cdot\pi^\dagger_{\mathfrak{a}}\Big)$$

with respect to h. By the argument in the proof of Lemma 7.6.8, we can show the boundedness of $g'_{\mathrm{irr}}(-\lambda+\lambda_0)\circ g_{\mathrm{irr}}(-\lambda+\lambda_0)^{-1}$ with respect to h if $|\lambda-\lambda_0|$ is sufficiently small. Thus, we obtain the boundedness of $g_{\mathrm{irr}}(-\lambda+\lambda_0)^\dagger\circ g_{\mathrm{irr}}(-\lambda+\lambda_0)^{-1}$ with respect to h.

Then, we obtain the following estimate for $f_1 \in \mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X}$ and $f_2 \in \mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X}$:

$$\begin{aligned}(287)\quad \big|C(f_1,\sigma^*f_2)\big| &= \big|h\big(f_1(\lambda,\boldsymbol{z}), f_2(-\lambda,\boldsymbol{z})\big)\big| \\ &= \big|h\big(g_{\mathrm{irr}}(\lambda-\lambda_0)f_1(-\lambda,\boldsymbol{z}), g_{\mathrm{irr}}(-\lambda+\lambda_0)^\dagger f_2(-\lambda,\boldsymbol{z})\big)\big| \\ \leqslant \big|f_1\big|_{\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h}\cdot\big|f_2\big|_{\mathcal{P}^{(-\lambda_0)}_{\mathrm{irr}}h}\cdot&\big|g_{\mathrm{irr}}(-\lambda+\lambda_0)^\dagger\circ g_{\mathrm{irr}}(-\lambda+\lambda_0)^{-1}\big|_h \leqslant C\big|f_1\big|_{\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h}\cdot\big|f_2\big|_{\mathcal{P}^{(-\lambda_0)}_{\mathrm{irr}}h}.\end{aligned}$$

Thus, we obtain Lemma 12.5.2. □

12.5.3. Some estimates. — Recall that the coordinate system is assumed to be admissible for $\mathrm{Irr}(\theta)$. We take an auxiliary sequence $\boldsymbol{m}(0),\dots,\boldsymbol{m}(L)$ for $\mathrm{Irr}(\theta)$. Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0} \times \boldsymbol{0}_{n-k}$. Let $\mathbb{D}_1$ denote the restriction of $\mathbb{D}$ to the z_1-direction. Let S be a small multi-sector in $U(\lambda_0) \times \big(X \setminus D(\leqslant k)\big)$. Note $\sigma(S)$ is a small multi-sector in $U(-\lambda_0) \times \big(X \setminus D(\leqslant k)\big)$. We take $\mathbb{D}_1$-flat splittings of the full Stokes filtrations by the procedure in Section 3.7.5:

$$\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{\mathfrak{a},S}, \qquad \mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\sigma(\overline{S})} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{\mathfrak{a},S}.$$

Let $\boldsymbol{v}_{\mathfrak{a}}$ be frames of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}\big(\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}\big)$. We have the natural lift $\boldsymbol{v}_{\mathfrak{a},S}$ of $\boldsymbol{v}_{\mathfrak{a}|\overline{S}}$ to $\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{\mathfrak{a},S}$. Similarly, let $\boldsymbol{w}_{\mathfrak{a}}$ be frames of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_{\mathfrak{a}}\big(\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}\big)$, and let $\boldsymbol{w}_{\mathfrak{a},S}$ denote the lift to $\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{\mathfrak{a},S}$.

Lemma 12.5.3. — *Let* $S_{\boldsymbol{I}(\lambda_0)} := S\cap\big(\boldsymbol{I}(\lambda_0)\times X\big)$.

- *We have the vanishing* $h\big(v_{\mathfrak{a},S,p},\sigma^*(w_{\mathfrak{b},S,q})\big)=0$ *unless* $\mathfrak{a}-\mathfrak{b}\geqslant_S 0$.
- *In the case* $\mathfrak{a}-\mathfrak{b}>_S 0$, *we have the following estimate on* $S_{\boldsymbol{I}}$ *for some positive constants* C, N *and* ε:

$$h\big(v_{\mathfrak{a},S,p},\sigma^*w_{\mathfrak{b},S,q}\big)\exp\big(-(\lambda^{-1}+\overline{\lambda}_0)(\mathfrak{a}-\mathfrak{b})\big)=O\Big(|z_1|^{-N}\cdot\prod_{j=2}^{n}|z_j|^{-2+\varepsilon}\Big).$$

- *In the case* $\mathfrak{a}=\mathfrak{b}$, *we have* $h\big(v_{\mathfrak{a},S,p},\sigma^*w_{\mathfrak{a},S,q}\big)=O\Big(\prod_{j=1}^{n}|z_j|^{-2+\varepsilon}\Big)$ *for some* $\varepsilon>0$.

Proof. — For fixed $\mathfrak{a},\mathfrak{b}\in\mathrm{Irr}(\theta)$, we put $C_{p,q}:=h\big(v_{\mathfrak{a},S,p},\sigma^*w_{\mathfrak{b},S,q}\big)$, and then we obtain the matrix valued function $\boldsymbol{C}=(C_{p,q})$.

Lemma 12.5.4. — *We have the estimate* $|\boldsymbol{C}|=O\Big(\prod_{j=1}^{n}|z_j|^{-2+\varepsilon}\Big)$ *for some* $\varepsilon>0$ *on* $\boldsymbol{I}(\lambda_0)\times S$. *In particular, the third claim of Lemma* 12.5.3 *holds.*

Proof. — Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be frames of $\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_0)\times X}$ and $\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(-\lambda_0)\times X}$, respectively. Let $\boldsymbol{v}_S$ and $\boldsymbol{w}_S$ be the frames of $\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\overline{S}}$ and $\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\sigma(\overline{S})}$ given by $\boldsymbol{v}_{\mathfrak{a},S}$ and $\boldsymbol{w}_{\mathfrak{a},S}$ ($\mathfrak{a}\in\mathrm{Irr}(\theta)$). Let B_1 and B_2 be determined by $\boldsymbol{v}_S=\boldsymbol{v}\cdot B_1$ and $\boldsymbol{w}_S=\boldsymbol{w}\cdot B_2$. Then, B_i and B_i^{-1} ($i=1,2$) are bounded. Let $\boldsymbol{C}'$ be the matrix valued function whose (p,q)-entry is given by $h\big(v_p,\sigma^*w_q\big)$. By Lemma 12.5.2, $|\boldsymbol{C}'|=O\Big(\prod_{j=1}^{n}|z_j|^{-2+\varepsilon}\Big)$. Then, the claim of Lemma 12.5.4 follows. □

Let us return to the proof of Lemma 12.5.3. Let $A_{\mathfrak{a}}$ be the matrix-valued holomorphic function determined by

$$\mathbb{D}^f_1\boldsymbol{v}_{\mathfrak{a}}=\boldsymbol{v}_{\mathfrak{a}}\cdot\Big((\lambda^{-1}+\overline{\lambda}_0)\cdot\partial_1\mathfrak{a}\cdot dz_1+A_{\mathfrak{a}}\cdot dz_1/z_1\Big).\tag{288}$$

Similarly, let $B_{\mathfrak{b}}$ be the matrix-valued holomorphic function determined by

$$\mathbb{D}^f_1\boldsymbol{w}_{\mathfrak{b}}=\boldsymbol{w}_{\mathfrak{b}}\cdot\big((\lambda^{-1}-\overline{\lambda}_0)\cdot\partial_1\mathfrak{b}\cdot dz_1+B_{\mathfrak{b}}\cdot dz_1/z_1\big).\tag{289}$$

Put $r_1:=|z_1|$. From (288) and (289), we obtain the following relation on $S_{\boldsymbol{I}(\lambda_0)}$:

$$r_1\frac{d\boldsymbol{C}}{dr_1}=\Big((\lambda^{-1}+\overline{\lambda}_0)z_1\partial_1\mathfrak{a}-\overline{(\lambda^{-1}+\overline{\lambda}_0)z_1\partial_1\mathfrak{b}}\Big)\cdot\boldsymbol{C}+{}^tA_{\mathfrak{a}}\cdot\boldsymbol{C}+\boldsymbol{C}\,\sigma^*\overline{B}_{\mathfrak{b}}.$$

Then, the first and second claims of Lemma 12.5.3 follows from Lemma 12.5.4 and Lemma 20.3.2 below. □

Corollary 12.5.5. — *The pairing of* $\widetilde{\mathcal{F}}^S_{\mathfrak{a}}\big(\mathcal{P}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|S}\big)$ *and* $\sigma^*\widetilde{\mathcal{F}}^{\sigma(S)}_{\mathfrak{b}}\big(\mathcal{P}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\sigma(S)}\big)$ *is* 0 *unless* $\mathfrak{a}-\mathfrak{b}\geqslant_S 0$. □

12.5.4. The induced pairing of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$ and $\sigma^*\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}$

Lemma 12.5.6. — *The pairing C of $\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times(X\setminus D)}$ and $\sigma^*\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times(X\setminus D)}$ can be extended to*

$$(290)\quad \mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X}\otimes\sigma^*\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X}\longrightarrow \Big\{F\in C^\infty\big(\boldsymbol{I}(\lambda_0)\times(X\setminus D)\big)\,\Big|\,|F|=O\Big(\prod_{i=1}^n|z_i|^{-2+\varepsilon}\Big)\text{ for some }\varepsilon>0\Big\}.$$

Proof. — Let S, $\boldsymbol{v}_{\mathfrak{a},S}$ and $\boldsymbol{w}_{\mathfrak{a},S}$ be as in Section 12.5.3. We put

$$\widetilde{\boldsymbol{v}}_{\mathfrak{a},S}:=\boldsymbol{v}_{\mathfrak{a},S}\cdot\exp(-\overline{\lambda}_0\mathfrak{a}),\quad \widetilde{\boldsymbol{w}}_{\mathfrak{a},S}:=\boldsymbol{w}_{\mathfrak{a},S}\cdot\exp(\overline{\lambda}_0\mathfrak{a}).$$

They give frames $\widetilde{\boldsymbol{v}}_S$ and $\widetilde{\boldsymbol{w}}_S$ of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|S}$ and $\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\sigma(S)}$, respectively. If $U(\lambda_0)$ is sufficiently small, we have the following estimate for some $\varepsilon>0$ on $S_{\boldsymbol{I}(\lambda_0)}$, according to Corollary 12.5.3:

$$h(\widetilde{v}_{S,p},\sigma^*\widetilde{w}_{S,q})=O\Big(\prod_j^n|z_j|^{-2+\varepsilon}\Big).$$

Let $\widetilde{\boldsymbol{v}}$ and $\widetilde{\boldsymbol{w}}$ be frames of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_0)\times X}$ and $\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(-\lambda_0)\times X}$, respectively. Let B_1 and B_2 be determined by $\widetilde{\boldsymbol{v}}_S=\widetilde{\boldsymbol{v}}\cdot B_1$ and $\widetilde{\boldsymbol{w}}_S=\widetilde{\boldsymbol{w}}\cdot B_2$. Then, as remarked in Lemma 4.5.8, B_1 and B_1^{-1} are bounded on $S_{\boldsymbol{I}(\lambda_0)}$. We have a similar estimate for B_2. Hence, we obtain $h\big(\widetilde{v}_p,\sigma^*\widetilde{w}_q\big)=O\big(\prod_{j=1}^n|z_j|^{-2+\varepsilon}\big)$ for some $\varepsilon>0$ on $S_{\boldsymbol{I}(\lambda_0)}$. By varying S, we obtain Lemma 12.5.6. □

12.5.5. Construction of $\mathfrak{C}$. — We can regard (290) as the pairing

$$\mathfrak{C}:\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X}\otimes\sigma^*\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X}\longrightarrow\mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)}.$$

We would like to extend it to the $\mathcal{R}_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{R}_{|\boldsymbol{S}\times X}$-homomorphism

$$\mathfrak{C}:\mathfrak{E}^{(\lambda_0)}_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathfrak{E}^{(-\lambda_0)}_{|\boldsymbol{S}\times X}\longrightarrow\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}.$$

The argument is essentially the same as that in Section 18.1 of [**67**].

Lemma 12.5.7. — *Assume $\lambda\in\boldsymbol{I}(\lambda_0)$ is generic. If $\sum_{i=1}^m P_i\cdot u_i=0$ in $\mathcal{Q}\mathcal{E}^\lambda$ for $u_i\in\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^\lambda$ and $P_i\in\mathcal{R}_X$ $(i=1,\dots,m)$, then we have $\sum_{i=1}^m P_i\cdot C\big(u_i,\sigma^*v\big)=0$ in $\mathfrak{Db}_X$ for any $v\in\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^{-\lambda}$.*

Similarly, if $\sum_i^m Q_i\cdot u_i=0$ in $\mathcal{Q}\mathcal{E}^{-\lambda}$ for $u_i\in\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^{-\lambda}$ and $Q_i\in\mathcal{R}_X$ $(i=1,\dots,m)$, then we have $\sum_{i=1}^m\sigma^*(Q_i)\cdot C(v,\sigma^*u_i)=0$ in $\mathfrak{Db}_X$ for any $v\in\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^\lambda$.

Proof. — Let $\boldsymbol{v}$ be a frame of $\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^\lambda$. We have $\sum P_iC(u_i,\sigma^*v_j)=0$ on $X\setminus D$. We set $\boldsymbol{z}^{N\boldsymbol{\delta}}:=\prod_{j=1}^n z_j^N$. There exists a large number N such that

$$\sum P_i\cdot C(u_i,\sigma^*(\boldsymbol{z}^{N\boldsymbol{\delta}}v_j))=0$$

for any j. Hence, there exists a large N such that we have $\sum P_i\cdot C(u_i,\sigma^*v)=0$ for any $v\in\mathcal{Q}^{(\lambda_0)}_{-N\boldsymbol{\delta}}\mathcal{E}^\lambda$.

Since we have assumed λ is generic, any $v \in \mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}^{\lambda}$ have the expression $v = \sum \eth^J \cdot v_J$, where $v_J \in \mathcal{Q}^{(-\lambda_0)}_{-N\boldsymbol{\delta}}\mathcal{E}^{\lambda}$. Then, the claim of the lemma can be shown easily. □

Lemma 12.5.8. — *If $\sum_{i=1}^m P_i \cdot u_i = 0$ in $\mathfrak{E}_{|\boldsymbol{I}(\lambda_0)\times X}$ for $u_i \in \mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X}$ and $P_i \in \mathcal{R}_X$ $(i = 1, \dots, m)$, then we have $\sum_{i=1}^m P_i \cdot C\big(u_i, \sigma^* v\big) = 0$ in $\mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)}$ for any $v \in \mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X}$.*

Similarly, if $\sum_{i=1}^m Q_i \cdot u_i = 0$ in $\mathfrak{E}_{|\boldsymbol{I}(-\lambda_0)\times X}$ for $u_i \in \mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X}$ and $Q_i \in \mathcal{R}_X$ $(i = 1, \dots, m)$, then we have $\sum_{i=1}^m \sigma^*(Q_i)\cdot C(v, \sigma^* u_i) = 0$ in $\mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)}$ for any $v \in \mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X}$.

Proof. — It follows from Lemma 12.5.7 and the continuity. □

Let us finish the proof of Proposition 12.5.1. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be frames of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(\lambda_0)\times X}$ and $\mathcal{Q}^{(-\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}_{|U(-\lambda_0)\times X}$, respectively. For any $f \in \mathfrak{E}_{|\boldsymbol{I}(\lambda_0)\times X}$ and $g \in \mathfrak{E}_{|\boldsymbol{I}(-\lambda_0)\times X}$, we have the expressions $f = \sum P_i \cdot v_i$ and $g = \sum Q_j \cdot w_j$ for some $P_i, Q_j \in \mathcal{R}_X$. We put $\mathfrak{C}(f, \sigma^* g) := \sum P_i \cdot \sigma^* Q_j \cdot \mathfrak{C}(v_i, \sigma^* w_j)$. We can check the well-definedness by using Lemma 12.5.8. It is easy to check the morphism is $\mathcal{R}_X \otimes \sigma^*\mathcal{R}_X$-linear. Thus, we obtain a sesqui-linear pairing

$$\mathfrak{C} : \mathfrak{E}_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\mathfrak{E}_{|\boldsymbol{I}(-\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \tag{291}$$

whose restriction to $\boldsymbol{I}(\lambda_0) \times (X \setminus D)$ is equal to C.

Let $\mathfrak{C}'$ be another such $\mathcal{R}_X \otimes \sigma^*\mathcal{R}_X$-homomorphism. Then, we have $\mathfrak{C}' = \mathfrak{C}$ because of the strict S-decomposability of $\mathfrak{E}$ along the function $\prod_{i=1}^n z_i$ (Proposition 12.4.3). See [**73**], [**67**] or Proposition 22.10.7 below.

Hence, by varying λ_0 and gluing (291), we obtain the global sesqui-linear pairing

$$\mathfrak{C} : \mathfrak{E}_{|\boldsymbol{S}\times X} \otimes \sigma^*\mathfrak{E}_{|\boldsymbol{S}\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$$

with the desired property. Again, the uniqueness follows from the strict S-decomposability of $\mathfrak{E}$ along the function $\prod_{i=1}^n z_i$. Thus, we obtain Proposition 12.5.1. □

12.6. A characterization of the prolongment in the one dimensional case

12.6.1. Statement. — We give a remark on the uniqueness of $\mathcal{R}$-triples extending a variation of polarized pure twistor structure in the one dimensional case.

Let $X := \Delta$ and $D := \{O\}$. Let $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ be a strictly specializable $\mathcal{R}(*z)$-triple on X such that $\mathcal{S} = (\mathrm{id}, \mathrm{id}) : \mathcal{T} \to \mathcal{T}^*$ is a Hermitian sesqui-linear duality of weight 0. Recall that the underlying family of meromorphic λ-flat bundles $\mathcal{M}$ has the KMS-structure at each $\lambda_0 \in \boldsymbol{C}$ due to a lemma of Sabbah [**75**]. (It is reviewed in Lemma 12.6.7.) For simplicity, we assume that it is unramified. Let $\mathcal{I}$ denote the set of irregular values.

To simplify the claim, we assume that the restriction of $\mathcal{T}$ to $X \setminus D$ comes from a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$. By the assumption, we obtain that $(E, \overline{\partial}_E, \theta, h)$ is unramifiedly good wild. According to Proposition 12.5.1, we have the $\mathcal{R}(*z)$-triple $\mathfrak{T}(E)(*z) := \big(\mathcal{QE}, \mathcal{QE}, \mathfrak{C}(*z)\big)$ associated to $(E, \overline{\partial}_E, \theta, h)$. We would like to compare $\mathfrak{T}(E)(*z)$ and $\mathcal{T}$.

Theorem 12.6.1. — *Assume that $(P\,\mathrm{Gr}_\ell^W \tilde{\psi}_{z,u,\mathfrak{a}}(\mathcal{T}), \mathcal{S}_{\mathfrak{a},u,\ell})$ are polarized pure twistor structure of weight ℓ for any $\mathfrak{a} \in \mathcal{I}$, $u \in \boldsymbol{R} \times \boldsymbol{C}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. Then, we have a natural isomorphism $\mathfrak{T}(E)(*z) \simeq \mathcal{T}$.*

Remark 12.6.2. — Conversely, if $\mathcal{T} \simeq \mathfrak{T}(E)(*z)$, $(P\,\mathrm{Gr}_\ell^W \tilde{\psi}_{z,u,\mathfrak{a}}(\mathcal{T}), \mathcal{S}_{\mathfrak{a},u,\ell})$ are polarized pure twistor structure of weight ℓ for any $\mathfrak{a} \in \mathcal{I}$, $u \in \boldsymbol{R} \times \boldsymbol{C}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. See Corollary 12.7.2. □

Remark 12.6.3. — According to a result by Sabbah in [**75**], we do not have to assume that $(\mathcal{T}, \mathcal{S})_{|X \setminus D}$ comes from a harmonic bundle. Namely, if $(\mathcal{T}, \mathcal{S})_{|X \setminus D}$ comes from a variation of twistor structure with a pairing of weight 0, and if the assumption of the theorem is satisfied, the variation of twistor structure comes from a harmonic bundle. (It also follows from our argument below.) □

12.6.2. Reduction of $\mathcal{R}(*z)$-triples. — Let $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ be an $\mathcal{R}_X(*z)$-triple on X such that $\mathcal{S} = (\mathrm{id}, \mathrm{id})$ gives a Hermitian sesqui-linear duality of weight 0. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{D} := \boldsymbol{C}_\lambda \times D$. Assume the following:

- $\mathcal{M}$ is an unramifiedly good meromorphic prolongment of $\mathcal{M}_{|\mathcal{X} \setminus \mathcal{D}}$ in the sense of Definition 6.2.1. The set of irregular values is denoted by $\mathcal{I}$.
- $(\mathcal{T}, \mathcal{S})_{|X \setminus D}$ comes from a variation of twistor structure with a pairing of weight 0. Namely, $\mathcal{M}_{|\mathcal{X} \setminus \mathcal{D}}$ comes from a family of λ-flat bundles, and C is the restriction of a λ-holomorphic sesqui-linear pairing.

For any $\mathfrak{a} \in \mathcal{I}$, we have the full reduction $\mathcal{M}_\mathfrak{a} := \mathrm{Gr}_\mathfrak{a}^{\tilde{\mathcal{F}}}(\mathcal{M})$ on the product of $\boldsymbol{C}_\lambda$ and a neighbourhood X' of D. For simplicity of description, we replace X with X'.

12.6.2.1. Let $\lambda_0 \in \boldsymbol{S}$. Let $\mathcal{X}^{(\lambda_0)}$ denote a small neighbourhood of $\{\lambda_0\} \times X$ in $\mathcal{X}$. We use the symbol $\mathcal{D}^{(\lambda_0)}$ in a similar meaning. Let S be a small sector in $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$. Then, $\sigma(S)$ is a small sector in $\mathcal{X}^{(-\lambda_0)} \setminus \mathcal{D}^{(-\lambda_0)}$. Note that $\mathfrak{a} \leqslant_S \mathfrak{b}$ if and only if $\mathfrak{a} \geqslant_{\sigma(S)} \mathfrak{b}$ for any j and any $\mathfrak{a} \in \mathcal{I}$. By Lemma 12.6.10 below, we obtain an induced Hermitian sesqui-linear pairing

$$C_{0,\mathfrak{a}} : \mathcal{M}^\circ_{\mathfrak{a}|\boldsymbol{S} \times (X \setminus D)} \otimes \sigma^* \mathcal{M}^\circ_{\mathfrak{a}|\boldsymbol{S} \times (X \setminus D)} \longrightarrow \mathfrak{Db}_{\boldsymbol{S} \times (X \setminus D)/\boldsymbol{S}}.$$

In particular, we obtain an $\mathcal{R}$-triple $\mathrm{Gr}_\mathfrak{a}(\mathcal{T})^\circ := (\mathcal{M}^\circ_\mathfrak{a}, \mathcal{M}^\circ_\mathfrak{a}, C_{0,\mathfrak{a}})$ on $X \setminus D$ for $\mathfrak{a} \in \mathcal{I}$, which is equipped with a Hermitian sesqui-linear duality $\mathrm{Gr}_\mathfrak{a}(\mathcal{S})^\circ := (\mathrm{id}, \mathrm{id})$. We will prove the following lemma in Subsection 12.6.2.4.

Lemma 12.6.4. — *$C_{0,\mathfrak{a}}$ is naturally extended to a Hermitian sesqui-linear pairing:*

$$C_\mathfrak{a} : \mathcal{M}_{\mathfrak{a}|\boldsymbol{S} \times X} \otimes \sigma^* \mathcal{M}_{\mathfrak{a}|\boldsymbol{S} \times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{S} \times X/\boldsymbol{S}}(*z).$$

In particular, we obtain an $\mathcal{R}(*z)$-triple $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T}) := (\mathcal{M}_{\mathfrak{a}}, \mathcal{M}_{\mathfrak{a}}, C_{\mathfrak{a}})$ *on X for $\mathfrak{a} \in \mathcal{I}$ with a Hermitian sesqui-linear duality* $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{S}) = (\mathrm{id}, \mathrm{id})$.

12.6.2.2. We assume that $\mathcal{M}$ is strictly specializable and unramified. (See a remark in Subsection 12.6.1.) Since we have the natural isomorphism $\mathcal{M}_{|\widehat{\mathcal{D}}} \simeq \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{M}_{\mathfrak{a}|\widehat{\mathcal{D}}}$, the induced $\mathcal{R}$-triples $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T})$ are also strictly specializable along $z = 0$. Moreover, we have natural isomorphisms

$$\widetilde{\psi}_{z,\mathfrak{a},u}(\mathcal{M}) \simeq \widetilde{\psi}_{z,\mathfrak{a},u}(\mathrm{Gr}_{\mathfrak{a}} \mathcal{M}). \tag{292}$$

We will prove the following lemma in Subsection 12.6.2.5.

Lemma 12.6.5. — *Under* (292), *we have* $\widetilde{\psi}_{z,\mathfrak{a},u}C = \widetilde{\psi}_{z,\mathfrak{a},u}C_{\mathfrak{a}}$. *Namely, we have a natural isomorphism* $\widetilde{\psi}_{z,\mathfrak{a},u}\mathcal{T} \simeq \widetilde{\psi}_{z,\mathfrak{a},u}\big(\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T})\big)$ *for any* $\mathfrak{a} \in \mathcal{I}$.

12.6.2.3. Let $\mathcal{M}^{\vee}$ denote the dual of $\mathcal{M}$ as a family of meromorphic λ-flat bundles. Let $\Upsilon(\mathcal{T}^{\circ}, \mathcal{S}^{\circ})$ be the variation of twistor structure with the pairing of weight 0, obtained as the gluing of $\mathcal{M}_{|\mathcal{X}\setminus\mathcal{D}}$ and $\sigma^*\mathcal{M}^{\vee}_{|\mathcal{X}\setminus\mathcal{D}}$ by the isomorphism induced by C.

Lemma 12.6.6. — *The Stokes structures of $\mathcal{M}$ and $\sigma^*\mathcal{M}^{\vee}$ are the same in the sense of Definition* 6.2.2. *Namely,* $(\mathcal{M}, \sigma^*\mathcal{M}^{\vee})$ *is an unramifiedly good meromorphic prolongment of* $\Upsilon(\mathcal{T}^{\circ}, \mathcal{S}^{\circ})$.

Proof. — Let $\pi : \widetilde{X}(D) \to X$ be the real blow up. Due to the existence of C and Lemma 12.6.10 below, the Stokes filtrations of $\mathcal{M}$ and $\sigma^*\mathcal{M}$ are mutually dual on $\boldsymbol{S} \times \widetilde{X}(D)$. Hence their Stokes filtrations of $\mathcal{M}$ and $\sigma^*\mathcal{M}^{\vee}$ are the same on $\boldsymbol{S} \times \widetilde{X}(D)$. Let us compare them on $\boldsymbol{C}^*_{\lambda} \times \widetilde{X}(D)$. Let $m := \min\big\{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) \,\big|\, \mathfrak{a} \neq \mathfrak{b} \in \mathcal{I}\big\}$. Let us compare the partial Stokes filtrations $\mathcal{F}^{(m)\,P}$ ($P \in \boldsymbol{C}^*_{\lambda} \times \pi^{-1}(D)$) of $\mathcal{M}$ and $\sigma^*\mathcal{M}^{\vee}$. They are the same in the case $P \in \boldsymbol{S} \times \pi^{-1}(D)$. Note that the filtrations $\mathcal{F}^{(m)\,P}$ are constant, when P varies $\{\arg(\lambda) = m \arg(z)\}$. Hence, we obtain that the partial Stokes filtrations $\mathcal{F}^{(m)\,P}$ ($P \in \boldsymbol{C}^*_{\lambda} \times \pi^{-1}(D)$) of $\mathcal{M}$ and $\sigma^*\mathcal{M}^{\vee}$ are the same. Then, it is easy to obtain that the Stokes structure of $\mathcal{M}$ and $\sigma^*\mathcal{M}^{\vee}$ are the same on $\boldsymbol{C}^*_{\lambda} \times \pi^{-1}(D)$. □

We have a variation of twistor structure with an induced symmetric pairing $\mathrm{Gr}_{\mathfrak{a}}\big(\Upsilon(\mathcal{T}^{\circ}, \mathcal{S}^{\circ})\big)$ on $\mathbb{P}^1 \times (X \setminus D)$ as explained in Section 6.2. By construction, it is naturally isomorphic to $\Upsilon\big(\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T}^{\circ}, \mathcal{S}^{\circ})\big)$, and $(\mathcal{M}_{\mathfrak{a}}, \sigma^*\mathcal{M}^{\vee}_{\mathfrak{a}})$ gives a meromorphic prolongment of $\mathrm{Gr}_{\mathfrak{a}}\big(\Upsilon(\mathcal{T}^{\circ}, \mathcal{S}^{\circ})\big)$. Note that $\mathcal{M}_{\mathfrak{a}}$ are $\mathfrak{a}$-regular.

12.6.2.4. Proof of Lemma 12.6.4. — Let $\lambda_0 \in \boldsymbol{S}$. Let f be a section of $\mathcal{M}_{\mathfrak{a}|\mathcal{X}^{(\lambda_0)}}$, and g be a section of $\mathcal{M}_{\mathfrak{a}|\mathcal{X}^{(-\lambda_0)}}$. We shall show that $C_{0,\mathfrak{a}}\big(f, \sigma^*g\big)$ gives a section of $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}(*z)$ on $\boldsymbol{I}(\lambda_0) \times X$, where $\boldsymbol{I}(\lambda_0) := U(\lambda_0) \cap \boldsymbol{S}$.

Let $E^{(\lambda_0)}$ be a good lattice of $\mathcal{M}_{|\mathcal{X}^{(\lambda_0)}}$. We may assume that f is a section of $\mathrm{Gr}_{\mathfrak{a}}(E^{(\lambda_0)})$. Let us take a covering $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)} = \bigcup_{i=1}^{N} S_i$ by small sectors on which we have flat splittings $E^{(\lambda_0)}_{|\overline{S}_i} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} E^{(\lambda_0)}_{\mathfrak{a},S_i}$ of the full Stokes filtration $\mathcal{F}^{S_i}$ of

$E^{(\lambda_0)}_{|\overline{S}_i}$. We have the natural isomorphism $E^{(\lambda_0)}_{\mathfrak{a},S_i} \simeq \mathrm{Gr}_{\mathfrak{a}}(E^{(\lambda_0)})_{|\overline{S}_i}$. Hence, we obtain a lift of $f_{|\overline{S}_i}$ to $E^{(\lambda_0)}_{\mathfrak{a},S_i}$. By gluing them in C^∞ as in Section 3.6.8.2, we obtain a C^∞-section $\widetilde{f}$ of $E^{(\lambda_0)}$, which is called a C^∞-lift of f to $E^{(\lambda_0)}$. Similarly, we take a good lattice $E^{(-\lambda_0)}$ of $\mathcal{M}_{|\mathcal{X}^{(-\lambda_0)}}$ such that g is a section of $\mathrm{Gr}_{\mathfrak{a}}(E^{(-\lambda_0)})$, and we take a C^∞-lift $\widetilde{g}$ of g to $E^{(-\lambda_0)}$.

Let $E^{(\lambda_0)}_{C^\infty}$ and $E^{(-\lambda_0)}_{C^\infty}$ denote the sheaf of C^∞-sections of $E^{(\lambda_0)}$ and $E^{(-\lambda_0)}$, respectively. The pairing $C : E^{(\lambda_0)} \otimes \sigma^* E^{(-\lambda_0)} \to \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}(*z)$ can naturally be extended to $C : E^{(\lambda_0)}_{C^\infty} \otimes \sigma^* E^{(-\lambda_0)}_{C^\infty} \to \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}(*z)$. Hence, we have the section $C(\widetilde{f},\sigma^*\widetilde{g})$ of $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}(*z)$. Because $C(\widetilde{f},\sigma^*\widetilde{g}) = C_{\mathfrak{a}}(f,\sigma^* g)$ by construction of $\widetilde{f}$ and $\widetilde{g}$, we are done. □

12.6.2.5. Proof of Lemma 12.6.5. — By considering the tensor product with $\mathcal{L}(-\mathfrak{a})$, we may reduce the problem to the case $\mathfrak{a} = 0$. We obtain an isomorphism of the underlying $\mathcal{R}$-modules $\widetilde{\psi}_{z,u}(\mathcal{M}) \simeq \widetilde{\psi}_{z,u}(\mathcal{M}_0)$ from the isomorphism $\mathcal{M}_{|\widehat{\mathcal{D}}} \simeq \bigoplus_{\mathfrak{a}\in T} \mathcal{M}_{\mathfrak{a}|\widehat{\mathcal{D}}}$. Let us compare the specializations of sesqui-linear pairings C and C_0. Let $\lambda_0 \in \boldsymbol{S}$. Let $[f_1]$ be a section of $\widetilde{\psi}_{z,u}(\mathcal{M}_0)_{|\mathcal{D}^{(\lambda_0)}}$. Let $[f_2]$ be a section of $\widetilde{\psi}_{z,u}(\mathcal{M}_0)_{|\mathcal{D}^{(-\lambda_0)}}$. We would like to show $\widetilde{\psi}_{z,u}C([f_1],\sigma^*[f_2]) = \widetilde{\psi}_{z,u}C_0([f_1],\sigma^*[f_2])$.

We take $f_1 \in V^{(\lambda_0)}_{\mathfrak{p}(\lambda_0,u)}(\mathcal{M}_{0|\mathcal{X}^{(\lambda_0)}})$ which is a lift of $[f_1]$ in the sense that f_1 induces the element $[f_1] \in \widetilde{\psi}_{z,u}(\mathcal{M}_0) \subset \mathrm{Gr}^{V^{(\lambda_0)}}_{\mathfrak{p}(\lambda_0,u)}(\mathcal{M}_0)$. We also take $f_2 \in V^{(\lambda_0)}_{\mathfrak{p}(-\lambda_0,u)}(\mathcal{M}_{0|\mathcal{X}^{(-\lambda_0)}})$ which is a lift of $[f_2]$. Let χ be a test function on X which is constantly 1 around 0. We put $\omega_0 := \sqrt{-1}\,dz\,d\overline{z}/2\pi$. By definition, we have

$$\big\langle \widetilde{\psi}_{z,u}C_0([f_1],\sigma^*[f_2]),\, \rho\big\rangle = \operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle C_0(f_1,\sigma^* f_2),\ |z|^{2s}\chi(z)\,\rho\,\omega_0\Big\rangle.$$

We take a lattice $E^{(\lambda_0)}$ of $\mathcal{M}_{|\mathcal{X}^{(\lambda_0)}}$ such that (i) f_1 is a section of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_0(E^{(\lambda_0)})$, (ii) $E^{(\lambda_0)}$ is contained in $V^{(\lambda_0)}_{\mathfrak{p}(\lambda_0,u)}\mathcal{M}_{|\mathcal{X}^{(\lambda_0)}}$. We take a C^∞-lift $\widetilde{f}_1$ of f_1 to $E^{(\lambda_0)}$ as in the proof of Lemma 12.6.4. Similarly, we take a lattice $E^{(-\lambda_0)}$ of $\mathcal{M}_{|\mathcal{X}^{(-\lambda_0)}}$ such that (i) f_2 is a section of $\mathrm{Gr}^{\widetilde{\mathcal{F}}}_0(E^{(-\lambda_0)})$, (ii) $E^{(-\lambda_0)}$ is contained in $V^{(-\lambda_0)}_{\mathfrak{p}(-\lambda_0,u)}(\mathcal{M}_{|\mathcal{X}^{(-\lambda_0)}})$. We take a C^∞-lift $\widetilde{f}_2$ of f_2 to $E^{(-\lambda_0)}$. Then, we have $C_0(f_1,\sigma^* f_2) = C(\widetilde{f}_1,\sigma^*\widetilde{f}_2)$.

We have the section $[\widetilde{f}_1]$ of $\widetilde{\psi}_{z,u}(\mathcal{M})_{|\mathcal{D}^{(\lambda_0)}}$ induced by $[f_1]$ and the isomorphism $\widetilde{\psi}_{z,u}(\mathcal{M}) \simeq \widetilde{\psi}_{z,u}(\mathcal{M}_0)$. Similarly, $[f_2]$ naturally induces a section $[\widetilde{f}_2]$ of $\widetilde{\psi}_{z,u}(\mathcal{M})_{|\mathcal{D}^{(-\lambda_0)}}$. Let us show the following equality:

$$\operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \big\langle C(\widetilde{f}_1,\sigma^*\widetilde{f}_2),\ |z|^{2s}\chi\,\rho\,\omega_0\big\rangle = \big\langle \widetilde{\psi}_{z,u}C([\widetilde{f}_1],\sigma^*[\widetilde{f}_2]),\, \rho\big\rangle. \tag{293}$$

We can take a lift $\widetilde{f}_1'$ of $[\widetilde{f}_1]$ to $E^{(\lambda_0)}$ such that $\widetilde{f}_1' - \widetilde{f}_1 = O(|z|^N)$ for some sufficiently large N. Similarly, we can take a lift $\widetilde{f}_2'$ of $[\widetilde{f}_2]$ to $E^{(-\lambda_0)}$ such that $\widetilde{f}_2' - \widetilde{f}_2 = O(|z|^N)$ for some sufficiently large N. Then, it can be shown that both sides of (293) are equal to

$$\operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \big\langle C(\widetilde{f}_1', \sigma^*\widetilde{f}_2'),\, |z|^{2s}\chi\,\rho\,\omega_0\big\rangle.$$

Thus, we obtain $\widetilde{\psi}_{z,u}C = \widetilde{\psi}_{z,u}C_0$. □

12.6.3. Proof of Theorem 12.6.1. — Let us return to the setting in Subsection 12.6.1. According to Lemma 12.6.5, $P\operatorname{Gr}_j^{W(N)}\widetilde{\psi}_{z,\mathfrak{a},u}(\mathcal{T}_{\mathfrak{a}})$ and $P\operatorname{Gr}_j^{W(N)}\widetilde{\psi}_{z,\mathfrak{a},u}(\mathcal{T})$ are naturally isomorphic, compatible with the naturally induced Hermitian sesqui-linear dualities. Hence, we obtain that $\Upsilon(\operatorname{Gr}_{\mathfrak{a}}(\mathcal{T}^{\circ},\mathcal{S}^{\circ})) = \operatorname{Gr}_{\mathfrak{a}}(\Upsilon(\mathcal{T}^{\circ},\mathcal{S}^{\circ}))$ are variations of polarized pure twistor structure from the assumption in the theorem, according to a result by Sabbah [**73**], (It also follows from Lemma 11.8.6 and Corollary 22.12.5 below.) Then, the claim of Theorem 12.6.1 follows from Theorem 11.2.2. □

12.6.4. Strictly specializable holonomic $\mathcal{R}(*z)$-module on a disc (Appendix). — We recall a lemma due to Sabbah. It relates strict specializability of an $\mathcal{R}_X$-module with KMS-structure of the corresponding family of meromorphic λ-flat bundles. Let $X := \Delta$ and $D := \{0\}$. Let $\mathcal{K}$ be a neighbourhood of $\lambda_0 \in \boldsymbol{C}_\lambda$. We put $\mathcal{X} := \mathcal{K} \times X$ and $\mathcal{D} := \mathcal{K} \times D$.

Let $\mathcal{M}$ be a strict coherent holonomic $\mathcal{R}_X(*z)$-module on $\mathcal{X}$ whose characteristic variety is contained in $\mathcal{K} \times \big(N_X^*X \cup N_D^*X\big)$, where N_X^*X denotes the 0-section and N_D^*X denotes the conormal bundle of D in X. Then, $\mathcal{M}$ gives a family of meromorphic λ-flat bundles. Moreover, we assume that it is strictly specializable along z with ramified exponential twist. (See [**73**] or Sections 22.4.1–22.4.3 below.) We would like to show the following lemma. Although it was already shown by Sabbah in [**75**], we would like to give a partially different proof based on the arguments in [**58**], [**72**] and [**75**], just for our understanding.

Lemma 12.6.7. — *$\mathcal{M}$ has the KMS-structure at λ_0, i.e., we have the locally free $\mathcal{O}_{\mathcal{X}}$-submodules $\mathcal{P}_a^{(\lambda_0)}(\mathcal{M}) \subset \mathcal{M}$ ($a \in \boldsymbol{R}$) such that (i) $\bigcup_a \mathcal{P}_a^{(\lambda_0)}\mathcal{M} = \mathcal{M}$, (ii) $\mathcal{P}_*^{(\lambda_0)}\mathcal{M} = \big(\mathcal{P}_a^{(\lambda_0)}\mathcal{M} \,\big|\, a \in \boldsymbol{R}\big)$ is a good family of filtered λ-flat bundles with the KMS-structure at λ_0.*

Proof. — For any λ_1, let $\mathcal{M}^{\lambda_1}$ denote the restriction $\mathcal{M}/(\lambda-\lambda_1)\mathcal{M}$. For simplicity, we assume that $\mathcal{M}^{\lambda_0}$ is unramified. (The general case can be reduced to this case easily.) We have the irregular decomposition $\mathcal{M}^{\lambda_0}_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{\mathcal{M}}^{\lambda_0}_{\mathfrak{a}}$ with the good set of irregular values $\mathcal{I} \subset M(X,D)/H(X)$.

Lemma 12.6.8

- *$\mathcal{M}^{\lambda}$ is also unramified for each λ.*
- *The set of the irregular values of $\mathcal{M}^{\lambda}$ is given by $\mathcal{I}$.*
- *Let $\mathcal{M}^{\lambda}_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \widehat{\mathcal{M}}^{\lambda}_{\mathfrak{a}}$ be the irregular decomposition. Then,* $\operatorname{rank}(\widehat{\mathcal{M}}^{\lambda}_{\mathfrak{a}}) = \operatorname{rank}(\widehat{\mathcal{M}}^{\lambda_0}_{\mathfrak{a}})$

Proof. — We give only an outline. See [**75**] for more details. The following equality can be shown:

$$\operatorname{rank}(\widehat{\mathcal{M}}^{\lambda}_{\mathfrak{a}}) = \sum_{u \in \boldsymbol{R} \times \boldsymbol{C}/\mathbb{Z} \times \{0\}} \operatorname{rank} \widetilde{\psi}_{z,\mathfrak{a},u}(\mathcal{M})_{|\lambda}.$$

Then, the claim of the lemma follows. □

Let us return to the proof of Lemma 12.6.7. We put

$$V^{(\lambda_0)}_{\mathfrak{a},a}(\mathcal{M}) := V^{(\lambda_0)}_{a}\big(\mathcal{M} \otimes \mathcal{L}(-\mathfrak{a})\big) \otimes \mathcal{L}(\mathfrak{a}), \quad \mathcal{P}^{(\lambda_0)}_{a}(\mathcal{M}) := \bigcap_{\mathfrak{a} \in T} V^{(\lambda_0)}_{\mathfrak{a},a-1}(\mathcal{M}).$$

We shall show $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$ is a good lattice of $\mathcal{M}$. Using the arguments in [**58**] and [**72**], we can show the following general result:

- If we shrink $\mathcal{K}$ appropriately, we have a ramified covering $\phi_e : \mathcal{K}' \times X' \to \mathcal{K} \times X$ given by $\phi_e(\lambda, z) = \big(\lambda_0 + (\lambda - \lambda_0)^e, z^e\big)$ such that the restriction of $\phi_e^*\mathcal{M}$ to $\mathcal{K}'^* \times X'$ has an unramifiedly good lattice, where $\mathcal{K}'^* := \mathcal{K}' \setminus \{\lambda_0\}$.

We already know that each $\mathcal{M}^{\lambda}$ has a good lattice, and the set of the irregular values is T. Hence, it can be shown that we do not have to take a ramified covering, i.e., $\mathcal{M}_{|\mathcal{K}^* \times X}$ has an unramifiedly good lattice, where $\mathcal{K}^* := \mathcal{K} \setminus \{\lambda_0\}$. Then, it is easy to see that $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})_{|\mathcal{K}^* \times X}$ is a good lattice of $\mathcal{M}_{|\mathcal{K}^* \times X}$.

Take small numbers $0 < \varepsilon_1 < \varepsilon_2$ such that $\mathcal{K}_0 := \big\{\lambda \,\big|\, \varepsilon_1 \leqslant |\lambda - \lambda_0| \leqslant \varepsilon_2\big\}$ is contained in $\mathcal{K}$. We can take a locally free $\mathcal{O}_{\mathcal{X}}$-submodule $\mathcal{L} \subset \mathcal{M}$ such that $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})_{|\mathcal{K}_0 \times X} \subset \mathcal{L}_{|\mathcal{K}_0 \times X}$. Because of the $\mathcal{O}_{\mathcal{X}}$-local freeness of $\mathcal{L}$, we obtain $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M}) \subset \mathcal{L}$ by using Hartogs theorem. Because $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$ is the limit of $\mathcal{O}_{\mathcal{X}}$-coherent submodules of $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$, we can conclude that $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$ is $\mathcal{O}_{\mathcal{X}}$-coherent.

Let $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})^{\vee\vee}$ denote the $\mathcal{O}_{\mathcal{X}}$-submodule of $\mathcal{M}$ generated by $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$. It is $\mathcal{O}_{\mathcal{X}}$-locally free, and $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})^{\vee\vee}_{|\mathcal{X} \setminus \{(\lambda_0, O)\}} = \mathcal{P}^{(\lambda_0)}_a(\mathcal{M})_{|\mathcal{X} \setminus \{(\lambda_0, O)\}}$. We already know that $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})^{\vee\vee}_{|\mathcal{K}^* \times X}$ is a good lattice. Then, by using the non-degeneration of the irregular values and a well established argument in [**51**], it is easy to obtain that $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})^{\vee\vee}$ is a good lattice of $\mathcal{M}$. Once we know that $\mathcal{M}$ has good lattices, it is easy to show $\mathcal{P}^{(\lambda_0)}_a(\mathcal{M})^{\vee\vee} = \mathcal{P}^{(\lambda_0)}_a(\mathcal{M})$. By construction, the family $\mathcal{P}^{(\lambda_0)}_*(\mathcal{M})$ of good filtered λ-flat bundles has the KMS-structure at λ_0. Thus, we obtain Lemma 12.6.7. □

12.6.5. Preliminaries for the sesqui-linear pairing (Appendix)

12.6.5.1. Preliminaries for the compatibility with Stokes structure. — Let S be a small sector $\big\{r\, e^{\sqrt{-1}\theta} \,\big|\, \theta_1 < \theta < \theta_2,\ 0 < r < r_0\big\}$ in Δ^*. Let $\mathfrak{a}, \mathfrak{b} \in \boldsymbol{C}[z^{-1}]$ such that $\operatorname{Re}(\mathfrak{a} + \overline{\mathfrak{b}}) > 0$ on S.

Lemma 12.6.9. — *There does not exist a distribution F on Δ such that $F_{|S} = \exp\big(\mathfrak{a} + \overline{\mathfrak{b}} + \alpha \log z + \beta \log \overline{z}\big)$.*

Proof. — Let $\varepsilon > 0$ be a sufficiently small positive number. Let θ_1', θ_2' be $\theta_1 < \theta_1' < \theta_2' < \theta_2$. Let $\mathfrak{F} \subset C^\infty(\Delta)$ be the subspace generated by the functions of the following form:

$$\exp\Big(-\frac{1}{r^{2\varepsilon}}\Big) \cdot \rho(\theta) \cdot \frac{1}{r^A} Q(z, \overline{z}).$$

Here, (i) $\rho(\theta)$ is a C^∞-function whose support is contained in $[\theta_1', \theta_2']$, (ii) Q is a polynomial, (iii) $A \in \boldsymbol{R}$. The space $\mathfrak{F}$ is preserved by ∂_z and $\partial_{\overline{z}}$.

For $n \in \mathbb{Z}_{\leqslant 0}$, let L^2_n denote the space of distributions which are contained in the dual of the Sobolev space L^2_{-n} on Δ. Let F be a distribution on Δ such that $F_{|S} = \exp\big(\mathfrak{a} + \overline{\mathfrak{b}} + \alpha \log z + \beta \log \overline{z}\big)$. We have some $n \in \mathbb{Z}_{<0}$ such that $F \in L^2_n$. For any $g \in \mathfrak{F}$, we have $g \cdot F \in L^2_n$. We have

$$\partial_z (g \cdot F)_{|\Delta^*} = (\partial_z g) \cdot F_{|\Delta^*} + g \cdot (\partial_z F)_{|\Delta^*} = \partial g \cdot F_{|\Delta^*} + \big(g \cdot (\partial_z \mathfrak{a} + \alpha \cdot z^{-1})\big) \cdot F_{|\Delta^*}.$$

Note that $\partial_z g$ and $g \cdot \partial_z \mathfrak{a}$ are contained in $\mathfrak{F}$. By a similar calculation, there exists $\alpha \in \mathfrak{F}$ such that $\overline{\partial}\partial\big(g \cdot F\big)_{|\Delta^*} = \alpha \cdot F_{|\Delta^*}$ on Δ^*. Hence, there exist a large integer N and $\beta \in \mathfrak{F}$ such that $\overline{\partial}\partial\big(z^N \cdot g \cdot F\big) = \beta \cdot F$ on Δ. Using an inductive argument, we can show that there exist a large number $N(\ell)$ and $\beta^{(\ell)} \in \mathfrak{F}$ for each $\ell \in \mathbb{Z}_{>0}$ such that $(\overline{\partial}\partial)^\ell (z^{N(\ell)} g \cdot F) = \beta^{(\ell)} \cdot F \in L^2_n$. Then, we obtain that $z^{N(\ell)} g \cdot F$ is locally $L^2_{n+2\ell}$. Hence, if ℓ is sufficiently large, $z^{N(\ell)} g \cdot F$ has to be locally L^2. But, we can directly show that $z^N \cdot g \cdot F$ is not L^2 for any N if $g \neq 0$. Thus, we have arrived at a contradiction. □

The lemma can also be shown by a more direct argument by taking a sequence of test functions ρ_n satisfying (i) the supports of ρ_n are contained in S, (ii) $\rho_n \to 0$, (iii) $\langle F, \rho_n \rangle \to \infty$.

12.6.5.2. Compatibility of sesqui-linear pairing with Stokes filtrations. — Let $(\mathcal{V}_i, \nabla_i)$ $(i = 1, 2)$ be meromorphic flat bundles on (Δ, O), i.e., $\mathcal{V}_i$ are locally free $\mathcal{O}(*O)$-modules with connections $\nabla_i : \mathcal{V}_i \to \mathcal{V}_i \otimes \Omega^1_\Delta$. For simplicity, we assume that they are unramified. Let $\mathfrak{Db}_\Delta(*O)$ denote the sheaf of distributions on Δ^* with moderate growth at O (see [**72**] or a review in Section 22.8 below), and let $\mathcal{D}_\Delta(*O)$ denote the sheaf of meromorphic differential operators on Δ whose poles are contained in O. Let $\Delta^\dagger$ denote the conjugate of Δ. Let $C : \mathcal{V}_1 \otimes \overline{\mathcal{V}_2} \to \mathfrak{Db}_\Delta(*O)$ be a sesqui-linear pairing, i.e., $\big(\mathcal{D}_\Delta(*O) \otimes \mathcal{D}_{\Delta^\dagger}(*O)\big)$-homomorphism. Let S be a small sector in Δ^*, and let $\overline{S}$ denote the closure of S in the real blow up of Δ along O. We take a ∇_i-flat splitting of the full Stokes filtration $\mathcal{F}^S$:

$$\mathcal{V}_{i|\overline{S}} = \bigoplus_{\mathfrak{a} \in \operatorname{Irr}(\nabla_i)} \mathcal{V}_{i,\mathfrak{a},S}. \tag{294}$$

Then, we obtain an induced $\mathcal{D}_S \otimes \mathcal{D}_{S^\dagger}$-homomorphism $C_{\mathfrak{a},\mathfrak{b}} : \mathcal{V}_{1,\mathfrak{a},S} \otimes \overline{\mathcal{V}}_{2,\mathfrak{b},S} \to \mathfrak{Db}_S$ on S.

Lemma 12.6.10. — *We have* $C_{\mathfrak{a},\mathfrak{b}} = 0$ *unless* $\operatorname{Re}(\mathfrak{a} + \mathfrak{b}) \leqslant 0$ *on* S.

Proof. — Let $\widehat{O}$ denote the completion of Δ along O. We take a meromorphic frame of $\widehat{\boldsymbol{v}}^{(i)} = \big(\widehat{\boldsymbol{v}}^{(i)}_{\mathfrak{a}} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\nabla_i)\big)$ of $\mathcal{V}_{i|\widehat{O}}$ which is compatible with the irregular decomposition:

$$\nabla_i \widehat{\boldsymbol{v}}^{(i)} = \widehat{\boldsymbol{v}}^{(i)} \Big(\bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla_i)} \Big(d\mathfrak{a} + A^{(i)}_{\mathfrak{a}} \frac{dz}{z} \Big) \Big).$$

Here, $A^{(i)}_{\mathfrak{a}}$ are constant matrices with Jordan forms. For each small sector S' in Δ^*, let $\boldsymbol{v}^{(i)}_{S'}$ be the lift of $\widehat{\boldsymbol{v}}^{(i)}$ to $\mathcal{V}^{(i)}_{|S'}$, which is compatible with the decomposition (294). By varying S' and gluing them in C^∞ as in Section 3.6.8.2 above, we obtain a C^∞-frame $\boldsymbol{v}^{(i)}_{C^\infty} = \big(\boldsymbol{v}^{(i)}_{\mathfrak{a},C^\infty}\big)$. We may assume that $C\big(v^{(1)}_{\mathfrak{a},p,C^\infty}, v^{(2)}_{\mathfrak{b},q,C^\infty}\big)$ are also distributions on Δ. On the sector S, we may also assume that $v^{(i)}_{\mathfrak{a},p,C^\infty|S}$ are holomorphic and compatible with the decompositions (294). Let us consider the distribution $F_{p,q} := C\big(v^{(1)}_{\mathfrak{a},p,C^\infty}, v^{(2)}_{\mathfrak{b},q,C^\infty}\big)$ on Δ.

First, we consider the case where $v^{(1)}_{\mathfrak{a},p,C^\infty}$ and $v^{(2)}_{\mathfrak{b},q,C^\infty}$ are contained in the the bottom part of the weight filtration W associated to the nilpotent parts of $A^{(1)}_{\mathfrak{a}}$ and $A^{(2)}_{\mathfrak{b}}$, respectively. Then, we may assume to have the following equality on S:

$$\nabla_1 v^{(1)}_{\mathfrak{a},p,C^\infty|S} = \Big(d\mathfrak{a} + \alpha_p \frac{dz}{z}\Big) v^{(1)}_{\mathfrak{a},p,C^\infty|S}, \quad \nabla_2 v^{(2)}_{\mathfrak{a},q,C^\infty|S} = \Big(d\mathfrak{b} + \beta_q \frac{dz}{z}\Big) v^{(2)}_{\mathfrak{b},q,C^\infty|S}.$$

We have $F_{p,q|S} = B \cdot \exp\big(\mathfrak{a} + \alpha_p \log z + \overline{\mathfrak{b}} + \overline{\beta}_q \log \overline{z}\big)$ for some constant B. Due to Lemma 12.6.9, we have $B = 0$, i.e., $C_{\mathfrak{a},\mathfrak{b}}(v^{(1)}_{\mathfrak{a},p,C^\infty|S}, v^{(2)}_{\mathfrak{b},q,C^\infty|S}) = 0$.

We show $F_{p,q|S} = 0$ in general by using an induction on the degree with respect to the filtration W. We put $d^{(1)}(p) := \deg^W(v^{(1)}_{\mathfrak{a},p,C^\infty|S})$ and $d^{(2)}(q) := \deg^W(v^{(2)}_{\mathfrak{b},q,C^\infty|S})$. We will show $F_{p,q|S} = 0$ assuming $F_{p',q'|S} = 0$ for any $d^{(1)}(p') \leqslant d^{(1)}(p)$ and $d^{(2)}(q') < d^{(2)}(q)$. In that case, $F_{p,q|S}$ satisfies the differential equation:

$$dF_{p,q|S} = \Big(d\mathfrak{a} + d\overline{\mathfrak{b}} + \alpha_p \frac{dz}{z} + \beta_q \frac{d\overline{z}}{\overline{z}}\Big) \cdot F_{p,q|S}.$$

Hence, we obtain $F_{p,q|S} = B \cdot \exp\big(\mathfrak{a} + \overline{\mathfrak{b}} + \alpha_p \log z + \overline{\beta}_q \log \overline{z}\big)$ for some constant B. And by Lemma 12.6.9, we have $B = 0$. Similarly, we can show $F_{p,q|S} = 0$ assuming $F_{p',q'|S} = 0$ for any $d^{(1)}(p') < d^{(1)}(p)$ and $d^{(2)}(q') \leqslant d^{(2)}(q)$. Thus, we obtain $C_{\mathfrak{a},\mathfrak{b}} = 0$. □

12.7. Specialization of the associated $\mathcal{R}$-triples

12.7.1. Reduction and specialization. — Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D = \bigcup_{i=1}^n D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly $\mathcal{A}$-good wild harmonic bundle on $X \setminus D$. According to Proposition 12.5.1, we have the $\mathcal{R}_X$-triple $\mathfrak{T}(E) := (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$. Let us study its specialization along a monomial function $g = \boldsymbol{z}^{\boldsymbol{p}}$, where $\boldsymbol{p} = (p_i) \in \mathbb{Z}^n_{\geqslant 0}$. We use the notation in Section 12.4. According to Proposition 12.4.6, we have the decomposition:

$$P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{C})=\bigoplus_I \mathcal{M}_{u,p,I}. \tag{295}$$

Here, the support of $\mathcal{M}_{u,p,I}$ is $\mathcal{D}_I$, and the support of any non-trivial $\mathcal{R}$-submodule of $\mathcal{M}_{u,p,I}$ is not contained in $\mathcal{D}_J$ $(J\supset I)$. Since $\mathcal{M}_{u,p,I}$ is strictly S-decomposable along any z_i $(i\in I)$, it is the push-forward of the $\mathcal{R}_{D_I}$-modules $\mathcal{M}'_{u,p,I}$. We have the sesqui-linear pairing

$$C_{u,p,I}:\mathcal{M}_{u,p,I|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}_{u,p,I|\boldsymbol{S}\times X}\longrightarrow\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}},$$

induced by $\widetilde{\psi}_{g,u}(\mathfrak{C})\circ\big((\sqrt{-1}N)^p\otimes 1\big)$. We would like to know a more detailed property of the $\mathcal{R}$-triple

$$\mathcal{T}_{u,p,I}(E):=(\mathcal{M}_{u,p,I},\mathcal{M}_{u,p,I},C_{u,p,I}),$$

with the Hermitian sesqui-linear duality $\mathcal{S}_{u,p,I}=\big((-1)^p,1\big)$ of weight p.

For a subset $J\subset\underline{n}$, let $\mathrm{Irr}(\theta,J)$ denote the image of $\mathrm{Irr}(\theta)$ by the natural map

$$M(X,D)/H(X)\longrightarrow M(X,D)/M(X,D(\underline{n}-J)),$$

where $D(\underline{n}-J):=\bigcup_{i\in\underline{n}-J}D_i$. We take an auxiliary sequence $\boldsymbol{m}(0),\dots,\boldsymbol{m}(L)$ for $\mathrm{Irr}(\theta)$. For J, we have $\boldsymbol{m}(p)$ such that $m_i(p)<0$ for some $i\in J$ and $m_i(p+1)=0$ for any $i\in J$. We have a natural bijection $\mathrm{Irr}(\theta,\boldsymbol{m}(p))\simeq\mathrm{Irr}(\theta,J)$. We have the unramifiedly $\mathcal{A}$-good wild harmonic bundles $(E_{J,\mathfrak{a}},\overline{\partial}_{J,\mathfrak{a}},\theta_{J,\mathfrak{a}},h_{J,\mathfrak{a}})$ for $\mathfrak{a}\in\mathrm{Irr}(\theta,J)$ on $X\smallsetminus D$, which is obtained as the reduction of $(E,\overline{\partial}_E,\theta,h)$ at the level $\boldsymbol{m}(p)$. (We inductively use the one step reduction in Proposition 11.2.4 with twisting by good wild harmonic bundles of rank one.) We have the associated $\mathcal{R}_X$-triple $(\mathfrak{C}_{J,\mathfrak{a}},\mathfrak{C}_{J,\mathfrak{a}},\mathfrak{C}_{J,\mathfrak{a}})$ on X. We have the decomposition as in (295):

$$P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{C}_{J,\mathfrak{a}})=\bigoplus_I \mathcal{M}_{u,p,I}(E_{J,\mathfrak{a}}).$$

We have the induced sesqui-linear pairing

$$C_{u,p,I}(E_{J,\mathfrak{a}}):\mathcal{M}_{u,p,I}(E_{J,\mathfrak{a}})_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}_{u,p,I}(E_{J,\mathfrak{a}})_{|\boldsymbol{S}\times X}\longrightarrow\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}.$$

Thus, we obtain $\mathcal{R}_X$-triples $\mathcal{T}_{u,p,I}(E_{J,\mathfrak{a}})$.

Proposition 12.7.1. — *We have natural isomorphisms $\mathcal{T}_{u,p,I}(E)\simeq\mathcal{T}_{u,p,I}(E_{I,0})$.*

Proof. — We may assume $D_I\subset g^{-1}(0)$. Let $\iota_I:\widehat{D}_I\to X$ be the natural morphism. We have the following natural isomorphisms:

$$\begin{aligned}\iota_I^*P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{C})&\simeq P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{C})\\&\simeq\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,I)}P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{C}_{I,\mathfrak{a}})\\&\simeq P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\iota_I^*\mathfrak{C}_{I,0})\simeq\iota_I^*P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{C}_{I,0}).\end{aligned}\tag{296}$$

Since the support of $\mathcal{M}_{u,p,I}$ and $\mathcal{M}_{u,p,I}(E_{I,0})$ are contained in $\mathcal{D}_I$, we obtain the natural isomorphism $\mathcal{M}_{u,p,I}\simeq\mathcal{M}_{u,p,I}(E_{I,0})$. In the following, we may and will implicitly identify them. Let us compare the sesqui-linear pairings. For simplicity of description, we put $C'_{u,p,I} := C_{u,p,I}(E_{I,0})$. Because of the strict S-decomposability of $\mathcal{M}_{u,p,I}$ and $\mathcal{M}_{u,p,I}(E_{I,0})$ along any z_j (Proposition 12.4.6), we only have to compare $C_{u,p,I}$ and $C'_{u,p,I}$ on $X \setminus \bigcup_{j\not\in I} D_j$. The restrictions of $\mathcal{M}_{u,p,I}$ and $\mathcal{M}_{u,p,I}(E_{I,0})$ come from the family of λ-flat bundles $\mathcal{M}'_{u,p,I}$ and $\mathcal{M}'_{u,p,I}(E_{I,0})$ on $\boldsymbol{C}_\lambda \times D_I^\circ$, where $D_I^\circ := D_I \setminus \bigcup_{I\subsetneq J} D_J$. We only have to compare the restrictions of $C_{u,p,I}$ and $C'_{u,p,I}$ to $\mathcal{M}'_{u,p,I}$ and $\mathcal{M}'_{u,p,I}(E_{I,0})$. (See Proposition 22.10.7.) Let $\pi : X \to D_I$ be the natural projection. By considering the restriction to $\pi^{-1}(P)$ for $P \in D_I^\circ$, we may and will assume $I = \underline{n}$.

We take an auxiliary sequence $\boldsymbol{m}(0), \dots, \boldsymbol{m}(L)$ for $\mathrm{Irr}(\theta)$. We have the reductions $\big(E_{\mathfrak{a}}^{\boldsymbol{m}(i)}, \overline{\partial}_{\mathfrak{a}}^{\boldsymbol{m}(i)}, \theta_{\mathfrak{a}}^{\boldsymbol{m}(i)}, h_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)$ for $\mathfrak{a} \in \overline{\mathrm{Irr}}(\theta, \boldsymbol{m}(i))$. We have the associated $\mathcal{R}$-triples $\big(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}, \mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}, \mathfrak{C}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big)$ on X. We have the decomposition:

$$P\,\mathrm{Gr}_p^{W(N)} \widetilde{\psi}_{g,u}\big(\mathfrak{E}_{\mathfrak{a}}^{\boldsymbol{m}(i)}\big) = \bigoplus_J \mathcal{M}_{u,p,J}(E_{\mathfrak{a}}^{\boldsymbol{m}(i)}).$$

We have the induced sesqui-linear pairings $C_{u,p,J}(E_{\mathfrak{a}}^{\boldsymbol{m}(i)})$. We have the natural isomorphisms $\mathcal{M}_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(i)}) \simeq \mathcal{M}_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(i+1)})$, as remarked in Corollary 12.4.8. We only have to show $C_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(i)}) = C_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(i+1)})$ for $i = -1, \dots, L$ under the isomorphisms. (We put $E_0^{\boldsymbol{m}(-1)} := E$, formally.) By an easy inductive argument, we only have to compare $C_{u,p,\underline{n}}$ and $C_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(0)})$. To make the description simpler, we put $C_{u,p,\underline{n}}^{\boldsymbol{m}(0)} := C_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(0)})$.

Take a generic $\lambda_0 \in \boldsymbol{S}$. We take a small neighbourhood U_1 of λ_0, and we put $U_2 := \sigma(U_1)$ which is a neighbourhood of $-\lambda_0$. We put $U_i \cap \boldsymbol{S} = \boldsymbol{I}_i$ $(i = 1, 2)$. Take $f_i \in \mathcal{M}'_{u,p,\underline{n}|U_i\times D_{\underline{n}}} = \mathcal{M}'_{u,p,\underline{n}}(E_0^{\boldsymbol{m}(0)})_{|U_i\times D_{\underline{n}}}$. We would like to compare the distributions $C_{u,p,\underline{n}}(f_1, \sigma^* f_2)$ and $C_{u,p,\underline{n}}^{\boldsymbol{m}(0)}(f_1, \sigma^* f_2)$. We may assume $\mathfrak{p}(\lambda_0, u) < 0$ and $\mathfrak{p}(-\lambda_0, u) < 0$.

We take a lift $\widetilde{f}_1 \in \big(i_{g\dagger}\mathfrak{E}_0^{\boldsymbol{m}(0)}\big)_{|U_1\times(X\times\boldsymbol{C}_t)}$ of f_1 as in Section 12.4.6, i.e., (i) $\widetilde{f}_1 \in U_{\mathfrak{p}(\lambda_0,u)}^{(\lambda_0)}(i_{g\dagger}\mathfrak{E}_0^{\boldsymbol{m}(0)})$, (ii) the induced section $\widetilde{f}_1^{(1)}$ of $\mathrm{Gr}_{\mathfrak{p}(\lambda_0,u)}^{U^{(\lambda_0)}}\big(i_{g\dagger}\mathfrak{E}_0^{\boldsymbol{m}(0)}\big)$ is contained in $W_p(N)\psi_{g,u}^{(\lambda_0)}\mathfrak{E}_0^{\boldsymbol{m}(0)}$, (iii) the induced section $\widetilde{f}_1^{(2)}$ of $\mathrm{Gr}_p^{W(N)}\,\psi_{g,u}^{(\lambda_0)}\mathfrak{E}_0^{\boldsymbol{m}(0)}$ is equal to f_1. Similarly, we take a lift $\widetilde{f}_2 \in \big(i_{g\dagger}\mathfrak{E}_0^{\boldsymbol{m}(0)}\big)_{|U_2\times(X\times\boldsymbol{C}_t)}$ of f_2. Let χ be a test function on $\boldsymbol{C}_t$ which is constantly 1 around $t = 0$. Let $\omega_0 := (2\pi)^{-1}\sqrt{-1}dt\cdot d\bar{t}$. By definition, we have

$$(297)\quad \big\langle C_{u,p,\underline{n}}^{\boldsymbol{m}(0)}(f_1, \sigma^*(f_2)),\, \rho\big\rangle = \big\langle \widetilde{\psi}_{g,u}\mathfrak{C}_0^{\boldsymbol{m}(0)}\big((\sqrt{-1}N)^p \widetilde{f}_1^{(1)},\, \sigma^*\widetilde{f}_2^{(1)}\big), \rho\big\rangle$$
$$= (\sqrt{-1})^p \operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle \mathfrak{C}_0^{\boldsymbol{m}(0)}\big((-\eth_t t + \mathfrak{e}(\lambda,u))^p \widetilde{f}_1, \sigma^*\widetilde{f}_2\big),\, |t|^{2s}\cdot\chi\cdot\rho\cdot\omega_0\Big\rangle.$$

We take a lift $\widetilde{f}_{1,C^\infty}$ of $\widetilde{f}_1$ to $U^{(\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) \otimes C^\infty$, as in Section 12.4.6. Similarly, we take a lift $\widetilde{f}_{2,C^\infty}$ of $\widetilde{f}_2$ to $U^{(-\lambda_0)}\big(i_{g\dagger}\mathfrak{E}\big) \otimes C^\infty$. By construction, $\widetilde{f}_{b,C^\infty}$ $(b = 1, 2)$ are contained in $\mathcal{F}_0^{S\,\boldsymbol{m}(0)}$ for each small sector S. By using Corollary 12.5.5, we obtain the following equality on $\boldsymbol{I}_1 \times (X \setminus D)$:

$$(298)\qquad \mathfrak{C}\big((-\eth_t t + \mathfrak{e}(\lambda,u))^p \widetilde{f}_{1,C^\infty},\, \sigma^*\widetilde{f}_{2,C^\infty}\big) = \mathfrak{C}_0^{\boldsymbol{m}(0)}\big((-\eth_t t + \mathfrak{e}(\lambda,u))^p \widetilde{f}_1,\, \sigma^*\widetilde{f}_2\big).$$

Due to Lemma 12.4.11, the following holds:

$$(299)\quad (\sqrt{-1})^p \operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle \mathfrak{C}\big((-\eth_t t + \mathfrak{e}(\lambda,u))^p \widetilde{f}_{1,C^\infty},\, \sigma^*\widetilde{f}_{2,C^\infty}\big),\, |t|^{2s}\cdot\chi\cdot\rho\cdot\omega_0 \Big\rangle$$
$$= \big\langle \psi_{g,u}(\mathfrak{C})\big((\sqrt{-1}N)^p \widetilde{f}^{(1)}_{1,C^\infty},\, \sigma^*\widetilde{f}^{(1)}_{2,C^\infty}\big),\, \rho\big\rangle = C_{u,p,\underline{n}}(f_1, \sigma^* f_2).$$

From (297), (298) and (299), we obtain $C_{u,p,\underline{n}} = C^{\boldsymbol{m}(0)}_{u,p,\underline{n}}$ around generic λ_0. By continuity, we obtain $C_{u,p,\underline{n}} = C^{\boldsymbol{m}(0)}_{u,p,\underline{n}}$. Thus, the proof of Proposition 12.7.1 is finished. □

Corollary 12.7.2. — *The restriction of $\mathcal{T}_{u,p,I}$ to $X \setminus \bigcup_{j\notin I} D_j$ comes from an unramifiedly $\mathcal{A}$-good wild harmonic bundle $\big(E_{u,p,I}, \overline{\partial}_{u,p,I}, \theta_{u,p,I}, h_{u,p,I}\big)$ on D_I° with the twist by $\mathbb{T}^S(-p/2)$. The set of the irregular values of $\theta_{u,p,I}$ is contained in $\big\{\mathfrak{a} \in \operatorname{Irr}(\theta) \mid \mathfrak{a} = 0 \text{ in } \operatorname{Irr}(\theta, I)\big\}$.*

Proof. — Note that $(E_{I,0}, \overline{\partial}_{E,0}, \theta_{I,0}, h_{I,0})$ is tame with respect to the divisors D_ℓ ($\ell \in I$). Hence, $\mathcal{T}_{u,p,I}(E_{I,0})_{|D_I^\circ}$ comes from the harmonic bundle with the twist by $\mathbb{T}^S(-p/2)$. (See Section 18.4.9 of [**67**].) By considering the restriction to $\{0\} \times X$, we can easily check the other claims. □

12.7.2. The specializations come from wild harmonic bundles

Let $\mathfrak{T}(E_{u,p,I})$ denote the $\mathcal{R}_X$-triples $(\mathfrak{E}_{u,p,I}, \mathfrak{E}_{u,p,I}, \mathfrak{C}_{u,p,I})$ associated to the reductions $\big(E_{u,p,I}, \overline{\partial}_{u,p,I}, \theta_{u,p,I}, h_{u,p,I}\big)$ (Corollary 12.7.2) with the natural Hermitian sesqui-linear duality.

Proposition 12.7.3. — *There exists an isomorphism $\mathfrak{T}(E_{u,p,I}) \simeq \mathcal{T}_{u,p,I} \otimes \mathbb{T}^S(p/2)$.*

Proof. — Their restrictions to $X \setminus \bigcup_{j\notin I} D_j$ are the same. We only have to show $\mathfrak{E}_{u,p,I} \simeq \mathcal{M}_{u,p,I}$. (Note Proposition 22.10.7.) Since both of them are strictly S-decomposable along $G := \prod_{j\notin I} z_j$, we only have to compare their meromorphic structures, i.e., we only have to show $\mathfrak{E}_{u,p,I} \otimes \mathcal{O}(*G) = \mathcal{M}_{u,p,I} \otimes \mathcal{O}(*G)$. (See Lemma 22.4.10 below.) Note that they come from families of meromorphic λ-flat bundles on $\boldsymbol{C}_\lambda \times D_I$, which we have to compare. For this purpose, we only have to compare their restrictions to $\pi_j^{-1}(P)$ in D_I for $j \notin I$ and $P \in (D_j \cap D_I) \setminus \bigcup_{k\notin\{j\}\cup I}\big(D_j \cap D_k \cap D_I\big)$, where $\pi_j : D_I \to D_j \cap D_I$ denotes the projection. Therefore, we only have to consider the case $\dim D_I = 1$. Because of Proposition 12.7.1, we may assume $I = \{2, \dots, n\}$ and $\operatorname{Irr}(\theta) \subset \boldsymbol{C}[z_1^{-1}]$. We have the $\mathcal{R}$-triples $\operatorname{Gr}_{\mathfrak{a}}\big(\mathcal{T}_{u,p,I} \otimes \mathbb{T}^S(p/2)\big)^\circ$ for $\mathfrak{a} \in \operatorname{Irr}(\theta)$ on D_I° obtained from $\mathcal{T}_{u,p,I} \otimes \mathbb{T}^S(p/2)$ as the reductions (Section 12.6.2). Because of the uniqueness (Theorem 11.2.2), we only have to show that $\operatorname{Gr}_{\mathfrak{a}}\big(\mathcal{T}_{u,p,I} \otimes \mathbb{T}^S(p/2)\big)^\circ$ are variations of polarized pure twistor structures of weight 0. For this purpose, let us show that $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T}_{u,p,I})^{\circ}$ are isomorphic to the restriction of $\mathcal{T}_{u,p,I}(E_{1,\mathfrak{a}})$ to D_I°.

We have the isomorphism:

$$P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{E}(*\mathcal{D}_1))_{|\widehat{\mathcal{D}}_1} \simeq \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} P\,\mathrm{Gr}_p^{W(N)}\,\widetilde{\psi}_{g,u}(\mathfrak{E}_{1,\mathfrak{a}}(*\mathcal{D}_1))_{|\widehat{\mathcal{D}}_1}.$$

Thus, we have the isomorphism:

$$\mathcal{M}_{u,p,I}(*\mathcal{D}_1)_{|\widehat{\mathcal{D}}_1} \simeq \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \mathcal{M}_{u,p,I}(E_{1,\mathfrak{a}})(*\mathcal{D}_1)_{|\widehat{\mathcal{D}}_1}. \tag{300}$$

Note that $\mathcal{M}_{u,p,I}(*\mathcal{D}_1)$ and $\mathcal{M}_{u,p,I}(E_{1,\mathfrak{a}})(*\mathcal{D}_1)$ come from families of meromorphic λ-flat bundles on $\mathcal{D}_I$, which are denoted by $\mathcal{V}$ and $\mathcal{V}_{\mathfrak{a}}$. We regard them as $\mathcal{O}_{\mathcal{X}}(*\mathcal{D}_1)$-modules. By (300), we have the isomorphism

$$\widehat{\Psi} : \bigoplus \mathcal{V}_{\mathfrak{a}|\widehat{\mathcal{D}}_1} \simeq \bigoplus_{\mathfrak{a}} \mathrm{Gr}_{\mathfrak{a}}(\mathcal{V})_{|\widehat{\mathcal{D}}_1} = \mathcal{V}_{|\widehat{\mathcal{D}}_1}. \tag{301}$$

Lemma 12.7.4. — *The reduction* $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{V})$ *is naturally isomorphic to* $\mathcal{V}_{\mathfrak{a}}$ *for each* $\mathfrak{a} \in \mathrm{Irr}(\theta)$.

Proof. — By considering the tensor product with rank one harmonic bundle, we may assume that $\mathfrak{a} = 0$. We are given the isomorphism:

$$\widehat{\Phi} : \mathrm{Gr}_0(\mathcal{V})_{|\widehat{z_1=0}} \simeq \mathcal{V}_{0|\widehat{z_1=0}}.$$

Let $\lambda \neq 0$. The restriction of $\widehat{\Phi}$ to $\{\lambda = 0\}$ is denoted by $\widehat{\Phi}^{\lambda}$. Since the restrictions $\mathrm{Gr}_0\,\mathcal{V}_{|\{\lambda\}\times D_I}$ and $\mathcal{V}_{0|\{\lambda\}\times D_I}$ are λ-flat meromorphic bundles with regular singularity, $\widehat{\Phi}^{\lambda}$ is convergent, and it gives the holomorphic isomorphism $\mathrm{Gr}_0\,\mathcal{V}_{|\{\lambda\}\times D_I} \simeq \mathcal{V}_{0|\{\lambda\}\times D_I}$. Then, it is easy to see that $\widehat{\Phi}_{|\{\lambda\neq 0\}}$ comes from the flat isomorphism:

$$\mathrm{Gr}_0\,\mathcal{V}_{|\boldsymbol{C}_{\lambda}^*\times D_I} \longrightarrow \mathcal{V}_{0|\boldsymbol{C}_{\lambda}^*\times D_I}. \tag{302}$$

Let us show that the isomorphism (302) can be extended on $\boldsymbol{C}_{\lambda}\times D_I$. We take local frames $\boldsymbol{v}$ and $\boldsymbol{w}$ of $\mathrm{Gr}_0\,\mathcal{V}$ and $\mathcal{V}_0$ on some neighbourhood $\mathcal{U}$ around $(0,O)$ in $\mathcal{D}_I$. The morphism (302) is expressed by the matrix A with respect to $\boldsymbol{v}$ and $\boldsymbol{w}$. We may assume that the entries $a_{i,j}$ of A are holomorphic on $\mathcal{U}\setminus\{\lambda=0\}$. Moreover, $a_{i,j}$ has the formal expansion $\sum a_{i,j,k}(\lambda)\cdot z_1^k$, where $a_{i,j,k}(\lambda)$ are holomorphic with respect to λ. (Recall that we are given $\widehat{\Phi}$ on $\boldsymbol{C}_{\lambda}\times\{\widehat{z_1=0}\}$.) The power series are absolutely convergent on $\{|z_1|\leqslant R_1\}\times\{\delta_1\leqslant|\lambda|\leqslant\delta_2\}$. Hence, we have the following for any $\ell\geqslant 0$ and $|z_1|\leqslant R_1$:

$$\int_{|\lambda|=\delta_2} a_{i,j}\cdot\lambda^{\ell}\cdot d\lambda = \sum_k \int_{|\lambda|=\delta_2} a_{i,j,k}(\lambda)\cdot\lambda^{\ell}\cdot z^k\cdot d\lambda = 0.$$

Then, we can conclude that $a_{i,j}$ are holomorphic on $\{|z_1|\leqslant R_1\}\times\{|\lambda|\leqslant\delta_2\}$, which implies that the morphism (302) can be extended on $\boldsymbol{C}_{\lambda}\times D_I$. Since the completion along $\{z_1=0\}$ is an isomorphism, it is an isomorphism if we shrink X appropriately. □

Take a generic λ_0, and its small neighbourhood $U(\lambda_0)$. We have another direct construction of the flat isomorphism $\mathrm{Gr}_{\mathfrak{a}}\,\mathcal{V}_{|U(\lambda_0)\times X}\simeq\mathcal{V}_{\mathfrak{a}|U(\lambda_0)\times X}$. Let $b=\mathfrak{p}(\lambda_0,u)$. Let S be a small sector in $U(\lambda_0)\times(X\smallsetminus D_1)$. Because we have assumed that λ_0 is generic, we can take a $\mathbb{D}$-flat splitting of the full Stokes filtration (Proposition 3.7.18):

$$\tag{303} {}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\mathfrak{s}(\boldsymbol{p})}}(\mathcal{QE})_{|\overline{S}}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}{}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\mathfrak{s}(\boldsymbol{p})}}(\mathcal{QE})_{\mathfrak{a},\overline{S}}.$$

It induces the flat decompositions $\mathcal{QE}_{|S}=\bigoplus_{\mathfrak{a}}\mathcal{QE}_{\mathfrak{a},S}$ and

$$\tag{304} U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)_{|S\times\boldsymbol{C}_t}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{a},S}.$$

We have the corresponding decomposition of $\mathrm{Gr}^{U^{(\lambda_0)}}$. Since the decomposition is compatible with the action of $-\eth_t t$, we also obtain the decompositions:

$$\tag{305} \psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{|S}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{a},S},$$

$$\tag{306} P\,\mathrm{Gr}^{W(N)}_p\,\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{|S}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}P\,\mathrm{Gr}^{W(N)}_p\,\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{a},S},$$

$$\tag{307} \mathcal{V}_{|S}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\mathcal{V}_{\mathfrak{a},S}.$$

By the order $\leqslant_S$ on the set $\mathrm{Irr}(\theta)$ associated to the sector S, we obtain the filtrations $\widetilde{\mathcal{F}}^S$:

$$\begin{aligned}
\widetilde{\mathcal{F}}^S_{\mathfrak{a}}U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)_{|S}&=\bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}}U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{b},S},\\
\widetilde{\mathcal{F}}^S_{\mathfrak{a}}\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{|S}&=\bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}}\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{b},S},\\
\widetilde{\mathcal{F}}^S_{\mathfrak{a}}P\,\mathrm{Gr}^{W(N)}_p\,\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{|S}&=\bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}}P\,\mathrm{Gr}^{W(N)}_p\,\psi^{(\lambda_0)}_{g,u}\big(\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{b},S},\\
\widetilde{\mathcal{F}}^S_{\mathfrak{a}}\mathcal{V}_{|S}&=\bigoplus_{\mathfrak{b}\leqslant_S\mathfrak{a}}\mathcal{V}_{\mathfrak{b},S}.
\end{aligned}$$

Although the decompositions (304), (305), (306) and (307) depend on the choice of a splitting of $\mathcal{QE}$, the filtrations $\widetilde{\mathcal{F}}^S$ are well defined.

By construction, we have the natural isomorphism $F_{\mathfrak{a},S}:\mathcal{V}_{\mathfrak{a}|S}\simeq\mathcal{V}_{\mathfrak{a},S}$.

Lemma 12.7.5. — *If we shrink S, $F_S:=\bigoplus F_{\mathfrak{a},S}$ can be extended to the isomorphism $\bigoplus\mathcal{V}_{\mathfrak{a}|\overline{S}}\simeq\mathcal{V}_{|\overline{S}}$, and $F_{S|\widehat{Z}}=\pi^{-1}\circ\widehat{\Psi}$. The filtration $\widetilde{\mathcal{F}}^S(\mathcal{V}_{|S})$ is equal to the full Stokes filtration of $\mathcal{V}_{|S}$.*

Proof. — Let f be a section of $\mathcal{V}_{\mathfrak{a}}$. We take a lift $\widetilde{f}$ to $U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}_{1,\mathfrak{a}}\big)$. Namely, the induced section $\widetilde{f}^{(1)}$ of $\mathrm{Gr}^{U^{(\lambda_0)}}_b\big(i_{g\dagger}\mathfrak{E}_{1,\mathfrak{a}}\big)$ is contained in $W_p(N)\widetilde{\psi}_{u,g}(\mathfrak{E}_{1,\mathfrak{a}})$, and it induces $f\in\mathcal{V}_{\mathfrak{a}}\subset\mathrm{Gr}^{W(N)}_p\,\widetilde{\psi}_{u,g}(\mathfrak{E}_{1,\mathfrak{a}})$. Note that $U^{(\lambda_0)}_b(i_{g\dagger}\mathfrak{E}_{1,\mathfrak{a}})(*D_1)$ $(b<0)$ is the $V_0\mathcal{R}_{X\times\boldsymbol{C}_t}(*D_1)$-submodule of $i_{g\dagger}\mathcal{Q}^{(\lambda_0)}\mathcal{E}_{1,\mathfrak{a}}$ generated by ${}^{\underline{n}}V^{(\lambda_0)}_{b\boldsymbol{p}-\varepsilon\boldsymbol{\delta}_{\underline{n}-\mathfrak{s}(\boldsymbol{p})}}(\mathcal{QE}_{1,\mathfrak{a}})\otimes 1$ over

$\mathcal{R}_X(*\mathcal{D}_1)$. Hence, we have the expression

$$\widetilde{f}=\sum P_\ell\cdot(a_\ell\otimes 1),$$

where $P_\ell\in\mathcal{R}_X(*D_1)$ and $a_\ell\in {}^{\underline{n}}\mathcal{V}^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{QE}_{1,\mathfrak{a}})$.

Let $a_{\ell,S}$ be the lift of a_ℓ to ${}^{\underline{n}}\mathcal{V}^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{QE})_{\mathfrak{a},\overline{S}}$. We obtain the section $\widetilde{f}_S$ of $U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)_{\mathfrak{a},S}$ given as follows:

$$\widetilde{f}_S:=\sum P_\ell\cdot(a_{\ell,S}\otimes 1).$$

It induces a section f_S of $\mathcal{V}_{\mathfrak{a},S}$, which is equal to $F_{\mathfrak{a},S}(f_{|S})$. To show the first claim of the lemma, we only have to check that f_S induces the section of $\mathcal{V}_{|\overline{S}}$, and that $f_{S|\widehat{Z}}=\pi^{-1}(\widehat{\Psi}(f_{|\widehat{\mathcal{D}}_1}))$, after shrinking S.

We take a finite covering $U(\lambda_0)\times(X\setminus D_1)=\bigcup S_j$ by small multi-sectors such that $S_1=S$. We take a partition of unity (χ_i) subordinated to the covering (S_i) such that χ_i depend only on $\arg(z_1)$. On each S_j, let a_{ℓ,S_j} be the lift of a_ℓ to ${}^{\underline{n}}\mathcal{V}^{(\lambda_0)}_{\boldsymbol{bp}-\varepsilon\boldsymbol{\delta}_{\underline{n}-s(\boldsymbol{p})}}(\mathcal{QE})_{\mathfrak{a},\overline{S}_j}$. As in Section 12.4.6, we obtain a section $\widetilde{f}_{C^\infty}$ of $U^{(\lambda_0)}_b\big(i_{g\dagger}\mathfrak{E}(*\mathcal{D}_1)\big)\otimes C^\infty$ given as follows:

$$\widetilde{f}_{C^\infty}:=\sum\chi_i\cdot P_\ell\cdot(a_{\ell,S_i}\otimes 1).$$

As in Lemma 12.4.11, the induced section of $\mathrm{Gr}^{U(\lambda_0)}_b(i_{g\dagger}\mathfrak{E}\otimes C^\infty)$ is contained in $W_p(N)\widetilde{\psi}_{g,u}(\mathfrak{E})\otimes C^\infty$, and moreover it induces the section of $\mathcal{V}\otimes C^\infty\subset\mathrm{Gr}^{W(N)}_p\widetilde{\psi}_{g,u}(\mathfrak{E})\otimes C^\infty$ whose restriction to $\widehat{\mathcal{D}}_1$ is equal to $\widehat{\Psi}(f_{|\widehat{\mathcal{D}}_1})$. Note that we have $f_{C^\infty|S}=f_S$ after S is shrunk. Hence, we can conclude that f_S gives a section of $\mathcal{V}_{|\overline{S}}$, and $f_{S|\widehat{Z}}=\pi^{-1}(\widehat{\Psi}(f_{|\widehat{\mathcal{D}}_1}))$. □

Then, we obtain isomorphisms $\bigoplus\mathcal{V}_{\mathfrak{a}|\overline{S}}\simeq\mathrm{Gr}(\mathcal{V})_{|\overline{S}}$ for any such S. They are independent of the choice of a $\mathbb{D}$-flat spitting. By gluing them, we obtain the isomorphism $\bigoplus\mathcal{V}_{\mathfrak{a}|U(\lambda_0)\times X}\simeq\mathrm{Gr}(\mathcal{V})_{|U(\lambda_0)\times X}$. Since its restriction to $\widehat{\mathcal{D}}_1$ is equal to $\widehat{\Psi}$, it is equal to the isomorphism in Lemma 12.7.4.

Note that the underlying $\mathcal{R}_{D_I}$-modules of $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{T}_{u,p,I})$ and $\mathcal{T}_{u,p,I}(E_{1,\mathfrak{a}})$ are $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{V})_{|\boldsymbol{C}_\lambda\times D_I^\circ}$ and $\mathcal{V}_{\mathfrak{a}|\boldsymbol{C}_\lambda\times D_I^\circ}$, respectively. To show that $\mathrm{Gr}_{\mathfrak{a}}\big(\mathcal{T}_{u,p,I}\big)$ and $\mathcal{T}_{u,p,I}(E_{1,\mathfrak{a}})$ are isomorphic, we only have to compare the sesqui-linear pairings. By continuity, we only have to compare them on $\boldsymbol{I}(\lambda_0)\times S$, where $\boldsymbol{I}(\lambda_0)$ denotes the intersection of $\boldsymbol{S}$ and a small neighbourhood of some generic λ_0, and S denotes a small sector in $X\setminus D$. Such a comparison can be easily done by using the splittings (304), (305), (306) and (307). Thus, the proof of Proposition 12.7.3 is finished. □

12.8. Prolongation of ramified wild harmonic bundle on curve

In Section 12.5, we have constructed an $\mathcal{R}$-triple from an *unramifiedly* good wild harmonic bundle, which is strictly S-decomposable along any monomial functions.

We have several ways to argue a similar issue in the ramified case. In Lemma 19.2.2 below, we shall construct such a prolongation as the direct summand of the push-forward of the $\mathcal{R}$-triple associated to an unramifiedly good harmonic bundle. We can also obtain the prolongation directly by using the same method as that in the unramified case. We explain the latter only in the one dimensional case. We shall use it in Section 17.2.

Let $X := \Delta$ and $D := \{O\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle defined on $X \setminus D$, which is not necessarily unramified. We shall construct a family of meromorphic λ-flat bundles $\mathcal{QE}$, a strictly S-decomposable $\mathcal{R}$-submodule $\mathfrak{E} \subset \mathcal{QE}$, and an $\mathcal{R}$-triple $\mathfrak{T}(E) := (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$.

12.8.1. The family of meromorphic λ-flat bundles $\mathcal{QE}$. — We take an appropriate ramified covering $\varphi : (X', D') \to (X, D)$ such that $(E', \overline{\partial}_{E'}, \theta', h') := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is unramified. The ramification index is denoted by e. We have the associated family of meromorphic λ-flat bundles $(\mathcal{QE}', \mathbb{D}')$ on $\boldsymbol{C}_\lambda \times (X', D')$. Since it is $\mathrm{Gal}(X'/X)$-equivariant, we obtain the family of meromorphic λ-flat bundles $(\mathcal{QE}, \mathbb{D})$ on $\boldsymbol{C}_\lambda \times (X, D)$.

We have the irregular decomposition:

$$(\mathcal{QE}', \mathbb{D}')_{|U(\lambda_0) \times \widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (\mathcal{Q}\widehat{\mathcal{E}}'_{\mathfrak{a}}, \widehat{\mathbb{D}}'_{\mathfrak{a}}).$$

We put $\mathcal{Q}\widehat{\mathcal{E}}'_{\mathrm{irr}} := \bigoplus_{\mathfrak{a} \neq 0} \mathcal{Q}\widehat{\mathcal{E}}'_{\mathfrak{a}}$. Since $\mathcal{Q}\widehat{\mathcal{E}}'_{\mathrm{irr}}$ and $\mathcal{Q}\widehat{\mathcal{E}}'_0$ are $\mathrm{Gal}(X'/X)$-equivariant, we obtain the decomposition

$$\mathcal{Q}\widehat{\mathcal{E}} = \mathcal{Q}\widehat{\mathcal{E}}_0 \oplus \mathcal{Q}\widehat{\mathcal{E}}_{\mathrm{irr}}.$$

12.8.2. The good lattices. — For each λ_0, we have the good lattices $\mathcal{Q}^{(\lambda_0)}_a \mathcal{E}'$ of $\mathcal{QE}'$ on $U(\lambda_0) \times X'$. They are $\mathrm{Gal}(X'/X)$-equivariant. The descents are denoted by $\mathcal{Q}^{(\lambda_0)}_{a/e} \mathcal{E}'$. It has the KMS-structure at λ_0 with the index set $\mathcal{KMS}(\mathcal{QE}^0)$. We have the decomposition $\mathcal{Q}^{(\lambda_0)}_a \mathcal{E}'_{|U(\lambda_0) \times \widehat{D}'} = \bigoplus_{\mathfrak{a}} \mathcal{Q}^{(\lambda_0)}_a \widehat{\mathcal{E}}'_{\mathfrak{a}}$, which induces a decomposition

$$\mathcal{Q}^{(\lambda_0)}_c \mathcal{E}_{|U(\lambda_0) \times \widehat{D}} = \mathcal{Q}^{(\lambda_0)}_c \widehat{\mathcal{E}}_0 \oplus \mathcal{Q}^{(\lambda_0)}_c \widehat{\mathcal{E}}_{\mathrm{irr}}.$$

12.8.3. The $\mathcal{R}$-module $\mathfrak{E}$. — We naturally regard $\mathcal{QE}$ as an $\mathcal{R}_X$-module. Let $\mathfrak{E}^{(\lambda_0)}$ be the $\mathcal{R}_X$-submodule of $\mathcal{QE}$ on $U(\lambda_0) \times X$, generated by $\mathcal{Q}^{(\lambda_0)}_{<1} \mathcal{E}$. The following lemma is clear from the construction.

Lemma 12.8.1. — *We have the decomposition $\mathfrak{E}^{(\lambda_0)}_{|U(\lambda_0) \times \widehat{D}} = \mathcal{Q}\widehat{\mathcal{E}}_{\mathrm{irr}} \oplus \mathcal{Q}^{(\lambda_0)}_{\min} \widehat{\mathcal{E}}_0$, where $\mathcal{Q}^{(\lambda_0)}_{\min} \widehat{\mathcal{E}}_0$ denotes the $\mathcal{R}_{\widehat{D}}$-submodule of $\mathcal{Q}\widehat{\mathcal{E}}_0$ generated by $\mathcal{Q}^{(\lambda_0)}_{<1} \widehat{\mathcal{E}}_0$.* □

We have the wild harmonic bundles $(E'_{\mathfrak{a}}, \overline{\partial}_{E'_{\mathfrak{a}}}, \theta'_{\mathfrak{a}}, h'_{\mathfrak{a}})$ for $\mathfrak{a} \in \mathrm{Irr}(\theta')$ on $X' \setminus D'$, obtained as the full reduction of $(E', \overline{\partial}_{E'}, \theta', h')$. Note that $(E'_0, \overline{\partial}_{E'_0}, \theta'_0, h'_0)$ is equivariant, and we have the descent $(E_0, \overline{\partial}_{E_0}, \theta_0, h_0)$. We have the family of λ-flat bundles

$(\mathcal{P}\mathcal{E}_0, \mathbb{D}_0)$ and the $\mathcal{R}_X$-module $\mathfrak{E}_0$, associated to $(E_0, \overline{\partial}_{E_0}, \theta_0, h_0)$. Note we do not have to consider deformations because $(E_0, \overline{\partial}_{E_0}, \theta_0, h_0)$ is tame.

Lemma 12.8.2. — *We have the natural isomorphisms*

$$\mathcal{P}\mathcal{E}_{0|\widehat{D}} \simeq \mathcal{Q}\widehat{\mathcal{E}}_0, \quad \mathfrak{E}_{0|\widehat{D}} \simeq \mathcal{Q}^{(\lambda_0)}_{\min}\widehat{\mathcal{E}}_0.$$

Proof. — We have the natural isomorphisms $\mathcal{Q}^{(\lambda_0)}_c\widehat{\mathcal{E}}'_0 \simeq \mathcal{P}^{(\lambda_0)}_c\mathcal{E}'_{0|\widehat{D}'}$. Then, the first isomorphism is obtained. Since $\mathfrak{E}_{0|\widehat{D}}$ and $\mathcal{Q}^{(\lambda_0)}_{\min}\widehat{\mathcal{E}}_0$ are generated by $\mathcal{Q}^{(\lambda_0)}_{<1}\widehat{\mathcal{E}}_0$, the second isomorphism is obtained. □

Lemma 12.8.3. — *For $\lambda_1 \in U(\lambda_1) \subset U(\lambda_0)$, we have $\mathfrak{E}^{(\lambda_1)} = \mathfrak{E}^{(\lambda_0)}_{|U(\lambda_1)\times X}$ on $U(\lambda_1)\times X$.*

Proof. — We only have to compare the completion of them along $U(\lambda_1) \times D$. Both of them are isomorphic to the direct sum of $\mathcal{Q}_{\mathrm{irr}}\widehat{\mathcal{E}}$ and the completion of $\mathfrak{E}_0$. Hence, the claim of the lemma follows. □

Therefore, we obtain the global $\mathcal{R}_X$-module $\mathfrak{E}$.

Lemma 12.8.4. — *$\mathfrak{E}$ is strictly S-decomposable along the function z^n.*

Proof. — We only have to show that the completion $\mathfrak{E}_{|\widehat{D}}$ is strictly S-decomposable along the function z^n. It follows from the strict S-decomposability of $\mathfrak{E}_0$ ([**73**] and [**67**]). □

12.8.4. The sesqui-linear pairing $\mathfrak{C}$. — Recall that we have the pairing for each $\lambda_0 \in \boldsymbol{S}$:

$$C' : \mathcal{Q}^{(\lambda_0)}_{<1}\mathcal{E}'_{|\boldsymbol{I}(\lambda_0)\times X'} \otimes \sigma^*\mathcal{Q}^{(-\lambda_0)}_{<1}\mathcal{E}'_{|\boldsymbol{I}(-\lambda_0)\times X'} \longrightarrow \big\{f \in C^\infty\big(\boldsymbol{I}(\lambda_0) \times X'\big) \,\big|\, |f| = O\big(|z'|^{-2+\varepsilon}\big)\ \exists\varepsilon > 0\big\}.$$

It induces the pairing:

$$C : \mathcal{Q}^{(\lambda_0)}_{<1}\mathcal{E}_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\mathcal{Q}^{(-\lambda_0)}_{<1}\mathcal{E}_{|\boldsymbol{I}(-\lambda_0)\times X} \longrightarrow \big\{f \in C^\infty\big(\boldsymbol{I}(\lambda_0) \times X\big) \,\big|\, |f| = O\big(|z|^{-2+\varepsilon}\big)\ \exists\varepsilon > 0\big\}.$$

It induces $\mathfrak{C} : \mathfrak{E}_{|\boldsymbol{S}\times X} \otimes \sigma^*\mathfrak{E}_{|\boldsymbol{S}\times X} \to \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$, as in Section 12.5.5. Thus, we obtain the desired $\mathcal{R}$-triple $\mathfrak{T}(E) := (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$ even if $(E, \overline{\partial}_E, \theta, h)$ is ramified, in the case $\dim X = 1$.

Remark 12.8.5. — We only have to consider the unramified case to know the property of $\psi_g\mathfrak{T}(E)$. See Lemma 22.11.2. □

PART III

KOBAYASHI-HITCHIN CORRESPONDENCE

CHAPTER 13

PRELIMINARIES

This chapter is a collection of miscellaneous preliminaries for our study in Part III.

In Section 13.1, we recall some general properties of filtered flat sheaf in [**69**] related to μ_L-stability, with some minor generalization.

In Section 13.2, we study a Mehta-Ramanathan type theorem for filtered flat sheaves. This kind of result is always fundamental for the study of the μ_L-stability condition. Note that we do not have to assume that the filtered sheaf is good. This result will be useful in Chapter 16.

In Section 13.3, we construct Hermitian metrics for good filtered λ-flat bundles in a standard manner, which have some nice properties, although they are not pluri-harmonic. This is preliminary for Sections 13.4 and 13.5.

In Section 13.4, we collect some results related to wild harmonic bundles on projective curves. We review in Section 13.4.1 the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves, due to O. Biquard and P. Boalch [**10**]. Sections 13.4.2–13.4.4 are preliminary for the proof of Theorem 16.1.1. We show in Section 13.4.2 the convergence of the sequence of harmonic metrics for ε-perturbations. Then, we study in Section 13.4.3 the convergence of a sequence of Hermitian metrics, whose pseudo-curvatures converge to 0 in some sense. We also prove the continuity of harmonic metrics for a holomorphic family of stable good filtered flat bundles.

In Section 13.5, we give a sufficient condition for a harmonic bundle to be good wild. Proposition 13.5.1 can be regarded as a kind of "curve test".

We explain in Section 13.6 some basic properties of the good filtered flat bundles associated to good wild harmonic bundles. Since the arguments for the proof are essentially the same as those in [**66**] and [**69**], we give only outlines.

In Section 13.7, we explain a method of perturbation. The point is to kill the nilpotent part of the action of the residues on graded pieces. The content is almost the same as that in Section 2.1.6 of [**69**].

13.1. Preliminaries for μ_L-polystable filtered flat sheaves

13.1.1. Filtered flat sheaf and parabolic flat sheaf. — Let X be a complex manifold with a simple normal crossing divisor D with the decomposition into irreducible components $D = \bigcup_{i\in\Lambda} D_i$. A filtered flat sheaf on (X, D) is defined to be a pair of filtered sheaf $\boldsymbol{E}_* = \big({}_{\boldsymbol{a}}E \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^{\Lambda}\big)$ on (X, D) and a meromorphic flat connection ∇ of the $\mathcal{O}_X(*D)$-module $\boldsymbol{E} = \bigcup_{\boldsymbol{a}\in\boldsymbol{R}^{\Lambda}} {}_{\boldsymbol{a}}E$. (See Section 3.2 of [**66**] or Subsection 2.5.3 in this paper for filtered sheaf. We use the symbol ${}^{\diamond}E$ instead of ${}_{\boldsymbol{0}}E$.) If $\boldsymbol{E}_*$ is a filtered bundle, it is called a filtered flat bundle. Recall that $(\boldsymbol{E}_*, \nabla)$ is called regular, if $\nabla\big({}_{\boldsymbol{a}}E\big) \subset {}_{\boldsymbol{a}}E \otimes \Omega^1(\log D)$ is satisfied.

In this paper, a $\boldsymbol{c}$-parabolic sheaf with a meromorphic connection is called a $\boldsymbol{c}$-parabolic flat sheaf. By the operation of taking the $\boldsymbol{c}$-truncation, a filtered flat sheaf is equivalent to a $\boldsymbol{c}$-parabolic flat sheaf.

Remark 13.1.1. — Usually, the logarithmic property is contained in the definition of parabolic flat sheaf or parabolic Higgs sheaf. For example, in our previous paper [**66**], a parabolic Higgs sheaf means a pair of parabolic sheaf V_* with a Higgs field θ satisfying $\theta\big({}_{\boldsymbol{a}}V\big) \subset {}_{\boldsymbol{a}}V \otimes \Omega_X^{1,0}(\log D)$. It would be more appropriate that such an object is called regular parabolic Higgs sheaf or logarithmic parabolic Higgs sheaf. □

13.1.2. μ_L-Stability. — Let X be a smooth irreducible n-dimensional projective variety with a normal crossing hypersurface D and an ample line bundle L. The μ_L-(semi)stability condition for a filtered flat sheaf $(\boldsymbol{E}_*, \nabla)$ on (X, D) is defined in a standard manner. Namely, we say that $(\boldsymbol{E}_*, \nabla)$ is μ_L-stable (resp. μ_L-semistable) if we have $\mu_L(\boldsymbol{F}_*) < \mu_L(\boldsymbol{E}_*)$ (resp. $\mu_L(\boldsymbol{F}_*) \leqslant \mu_L(\boldsymbol{E}_*)$) for any sub-object $(\boldsymbol{F}_*, \nabla) \subset (\boldsymbol{E}_*, \nabla)$ such that $0 < \operatorname{rank} F < \operatorname{rank} E$, where $\mu_L(\boldsymbol{E}_*) := (\operatorname{rank} E)^{-1} \int_X c_1(L)^{n-1} \operatorname{par-c}_1(\boldsymbol{E}_*)$. We say that $(\boldsymbol{E}_*, \nabla)$ is μ_L-polystable, if it is a direct sum of μ_L-stable ones $\bigoplus(\boldsymbol{E}_{i*}, \nabla_i)$ such that $\mu_L(\boldsymbol{E}_*) = \mu_L(\boldsymbol{E}_{i*})$. By the correspondence of filtered sheaves and parabolic sheaves, we define the μ_L-stability, μ_L-semistability and μ_L-polystability conditions for parabolic flat sheaves.

As in [**66**], we say that $(\boldsymbol{E}_*, \nabla)$ is μ_L-polystable with trivial characteristic numbers, if it is μ_L-polystable and if each μ_L-stable component $(\boldsymbol{E}_{i*}, \nabla_i)$ satisfies $\mu_L(\boldsymbol{E}_{i\,*}) = \int_X \operatorname{par-c}_{2,L}(\boldsymbol{E}_{i\,*}) = 0$.

13.1.3. Canonical decomposition. — Let $\big(\mathcal{E}_*^{(i)}, \nabla^{(i)}\big)$ $(i = 1, 2)$ be μ_L-semistable $\boldsymbol{c}$-parabolic flat sheaves such that $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. Let $f : (\mathcal{E}_*^{(1)}, \nabla^{(1)}) \to (\mathcal{E}_*^{(2)}, \nabla^{(2)})$ be a non-trivial morphism. Let $(\mathcal{K}_*, \nabla_{\mathcal{K}})$ denote the kernel of f, which is equipped with the naturally induced parabolic structure and the flat connection. Let $\mathcal{I}$ denote the image of f, and $\widetilde{\mathcal{I}}$ denote the saturated subsheaf of $\mathcal{E}^{(2)}$ generated by $\mathcal{I}$, i.e., $\widetilde{\mathcal{I}}/\mathcal{I}$ is torsion, and $\mathcal{E}^{(2)}/\widetilde{\mathcal{I}}$ is torsion-free. The parabolic structures of $\mathcal{E}_*^{(1)}$ and $\mathcal{E}_*^{(2)}$ induce parabolic structures of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$, respectively. We denote the induced parabolic flat sheaves by $(\mathcal{I}_*, \nabla_{\mathcal{I}})$ and $(\widetilde{\mathcal{I}}_*, \nabla_{\widetilde{\mathcal{I}}})$. The following lemma can be shown using the argument in the proof of Lemma 3.9 of [**66**].

Lemma 13.1.2. — *$(\mathcal{K}_*, \nabla_{\mathcal{K}})$, $(\mathcal{I}_*, \nabla_{\mathcal{I}})$ and $(\widetilde{\mathcal{I}}_*, \nabla_{\widetilde{\mathcal{I}}})$ are also μ_L-semistable such that $\mu_L(\mathcal{K}_*) = \mu_L(\mathcal{I}_*) = \mu_L(\widetilde{\mathcal{I}}_*) = \mu_L(\mathcal{E}_*^{(i)})$. Moreover, $\mathcal{I}_*$ and $\widetilde{\mathcal{I}}_*$ are isomorphic in codimension one.* □

Lemma 13.1.3. — *Let $\left(\mathcal{E}_*^{(i)}, \nabla^{(i)}\right)$ $(i = 1, 2)$ be μ_L-semistable reflexive saturated parabolic flat sheaves such that $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. Assume either one of the following:*

1. *One of $(\mathcal{E}_*^{(i)}, \nabla^{(i)})$ is μ_L-stable, and $\operatorname{rank}(\mathcal{E}^{(1)}) = \operatorname{rank}(\mathcal{E}^{(2)})$ holds.*
2. *Both $(\mathcal{E}_*^{(i)}, \nabla^{(i)})$ $(i = 1, 2)$ are μ_L-stable.*

If there is a non-trivial map $f : (\mathcal{E}_*^{(1)}, \nabla^{(1)}) \to (\mathcal{E}_*^{(2)}, \nabla^{(2)})$, then f is an isomorphism.

Proof. — If $(\mathcal{E}_*^{(1)}, \nabla^{(1)})$ is μ_L-stable, the kernel of f is trivial due to Lemma 13.1.2. If $(\mathcal{E}_*^{(2)}, \nabla^{(2)})$ is μ_L-stable, the image of f and $\mathcal{E}^{(2)}$ are the same at the generic point of X. Thus, we obtain that f is generically an isomorphism in any case. Then, we obtain that f is an isomorphism in codimension one, due to Lemma 3.7 of [**66**]. Since both $\mathcal{E}_*^{(i)}$ $(i = 1, 2)$ are reflexive and saturated, we obtain that f is an isomorphism. □

Corollary 13.1.4. — *Let $(\mathcal{E}_*, \nabla)$ be a μ_L-polystable reflexive saturated parabolic flat sheaf. Then, we have a unique decomposition:*

$$(\mathcal{E}_*, \nabla) = \bigoplus_j (\mathcal{E}_*^{(j)}, \nabla^{(j)}) \otimes \boldsymbol{C}^{m(j)}.$$

Here, $(\mathcal{E}_*^{(j)}, \nabla^{(j)})$ are μ_L-stable with $\mu_L(\mathcal{E}_*^{(j)}) = \mu(\mathcal{E}_*)$, and they are mutually non-isomorphic. It is called the canonical decomposition. □

13.2. Mehta-Ramanathan type theorem

13.2.1. Statement. — Let X be an n-dimensional smooth irreducible projective variety with an ample line bundle L over a field k of characteristic 0, and let D be a simple normal crossing divisor of X.

Proposition 13.2.1. — *Let V_* be a parabolic sheaf on (X, D) with a meromorphic flat connection ∇. Then, it is μ_L-(semi)stable, if and only if the following holds:*

- *For any $m_1 > 0$, there exists $m > m_1$ such that $(V_*, \nabla)_{|Y}$ is μ_L-(semi)stable, where Y denotes a 1-dimensional complete intersection of generic hypersurfaces of L^m.*

The proof will be given in Sections 13.2.2–13.2.4. It is essentially the same as that in [**66**], where we closely follow the argument of V. Mehta, A. Ramanathan ([**61**], [**62**]) and Simpson ([**83**]). So, we will indicate only how to change in some points.

Before going into the proof, we give a remark on a characterization of semisimplicity of meromorphic flat connection. Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection on (X,D). We recall the following lemma, which we use implicitly in many places.

Lemma 13.2.2. — *$(\mathcal{E},\nabla)$ is simple (resp. semisimple) if and only if the associated Deligne-Malgrange filtered flat sheaf $(\boldsymbol{E}^{DM}_*,\nabla)$ is μ_L-stable (resp. μ_L-polystable).*

Proof. — We only have to show the "if" part. It was essentially observed by C. Sabbah [**73**]. Namely, we obtain $(\boldsymbol{F}^{DM}_*,\nabla)\subset(\boldsymbol{E}^{DM}_*,\nabla)$ for any flat subsheaf $\mathcal{F}\subset\mathcal{E}$. Because $\mu(\boldsymbol{F}^{DM}_*)=\mu(\boldsymbol{E}^{DM}_*)=0$, $\boldsymbol{F}^{DM}_*$ breaks the stability of $(\boldsymbol{E}^{DM}_*,\nabla)$ if $\mathcal{F}$ is non-trivial. □

Corollary 13.2.3. — *$(\mathcal{E},\nabla)$ is simple, if and only if $(\mathcal{E},\nabla)_{|Y}$ is simple, where Y denotes a complete intersection of generic hypersurfaces of L^m for arbitrary large m.*

Proof. — The "only if" part is trivial. The "if" part follows from Proposition 13.2.1 and Lemma 13.2.2. □

13.2.2. Preliminary. — The following lemma can be shown by using the argument in the proof of Proposition 3.2 of [**61**].

Lemma 13.2.4. — *Let $\mathcal{S}_0$ be any bounded family of torsion-free sheaves on X. There exists a large integer m_0 with the following property:*

- *Let $Y_1,\ldots,Y_r$ be generic hypersurfaces of L^{m_i} for $m_i\geqslant m_0$. Let $Y=Y_1\cap\cdots\cap Y_r$. Let $p\geqslant m_0$, and let $\mathcal{F}$ be any member of $\mathcal{S}_0$. Then, $H^0\bigl(Y,\mathcal{F}\otimes L^{-p}\bigr)=0$.* □

Lemma 13.2.5. — *Let $\mathcal{S}_1$ be a bounded family of torsion-free sheaves on X, and let $\mathcal{S}_2$ be a bounded family of line bundles on X. There exists a large integer m_0 with the following property:*

- *Let $Y_1,\ldots,Y_r$ be generic hypersurfaces of L^{m_i} for $m_i\geqslant m_0$. Let $Y=Y_1\cap\cdots\cap Y_r$. Let $\mathcal{F}\in\mathcal{S}_1$ and $\mathcal{L}\in\mathcal{S}_2$. Let ϕ be any morphism $\mathcal{L}\to\mathcal{F}\otimes\Omega^1_X$. Then, ϕ vanishes if and only if the induced morphism $\phi':\mathcal{L}_{|Y}\to\mathcal{F}_{|Y}\otimes\Omega^1_Y$ vanishes.*

Proof. — We put $Y^{(i)}:=Y_1\cap\cdots\cap Y_i$. If m_0 is sufficiently large, we have $H^0\bigl(Y^{(i)},\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_X\otimes L^{-p}\bigr)=0$ and $H^0\bigl(Y^{(i)},\mathcal{L}^{-1}\otimes\mathcal{F}\otimes L^{-p}\bigr)=0$ for any i and $p\geqslant m_0$, according to Lemma 13.2.4. We have the exact sequence:

$$0\longrightarrow L^{-m_{i+1}}\otimes\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{X|Y^{(i)}}\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{X|Y^{(i)}}\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{X|Y^{(i+1)}}\longrightarrow 0$$

Thus, the maps $H^0\bigl(Y^{(i)},\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_X\bigr)\to H^0\bigl(Y^{(i+1)},\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_X\bigr)$ are injective, if m_0 is sufficiently large. Hence, we only have to show that $\phi'=0$ implies that $\phi_{|Y}:\mathcal{L}_{|Y}\to\mathcal{F}\otimes\Omega^1_{X|Y}$ is trivial. We have the exact sequence:

$$0\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{F}\otimes L^{-m_j}_{|Y}\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{Y^{(j-1)}|Y}\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{Y^{(j)}|Y}\longrightarrow 0.$$

Hence, the maps $H^0\bigl(Y,\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{Y^{(j)}}\bigr)\to H^0\bigl(Y,\mathcal{L}^{-1}\otimes\mathcal{F}\otimes\Omega^1_{Y^{(j+1)}}\bigr)$ are injective, if m_0 is sufficiently large. Thus we are done. □

Let E be a torsion-free sheaf on X with a meromorphic flat connection $\nabla : E \longrightarrow E \otimes \Omega^1_X(*D)$. We remark the following obvious lemma as a reference for the subsequent argument.

Lemma 13.2.6. — *Let G be a subsheaf of E, and let $\widetilde{G}$ be the saturated subsheaf of E generated by G, i.e., $\widetilde{G}/G$ is torsion, and $E/\widetilde{G}$ is torsion-free. Then, $\nabla(G) \subset \widetilde{G} \otimes \Omega^1_X(*D)$ if and only if $\nabla(\widetilde{G}) \subset \widetilde{G} \otimes \Omega^1_X(*D)$. In this case, we say that $G \subset E$ generates a flat subsheaf.* □

Fix a constant C. Let $\mathcal{S}$ denote the family of the pairs $(\mathcal{L}, \phi)$ of line bundles $\mathcal{L}$ with $\deg(\mathcal{L}) \geqslant C$ and a non-trivial morphism $\phi : \mathcal{L} \longrightarrow E$.

Lemma 13.2.7. — *$\mathcal{S}$ is bounded. We also have the boundedness of the family of* $\mathrm{Cok}(\phi)_{tf}$, *where* $\mathrm{Cok}(\phi)_{tf}$ *denotes the quotient of* $\mathrm{Cok}(\phi)$ *by the torsion-part.*

Proof. — Let $\mathcal{S}'$ be the family of torsion-free quotient sheaves $\mathcal{G}$ of E which are of the form $\mathrm{Cok}(\phi)_{tf}$ for some $(\mathcal{L}, \phi) \in \mathcal{S}$. Note that $\deg(\mathrm{Cok}(\phi)_{tf})$ is bounded above. Due to a result of Grothendieck (Lemma 2.5 of [**32**]), the family $\mathcal{S}'$ is bounded. Hence, we have the boundedness of the family $\mathcal{S}''$ of saturated subsheaves $\mathcal{K}$ of E generated by $\phi(\mathcal{L})$ for some $(\mathcal{L}, \phi) \in \mathcal{S}$. By construction, for any $(\mathcal{L}, \phi) \in \mathcal{S}$, we have a member $\mathcal{K}$ of the bounded family $\mathcal{S}''$ such that $\phi(\mathcal{L}) \subset \mathcal{K}$. We have $0 \leqslant \deg(\mathcal{K}) - \deg(\mathcal{L}) \leqslant C'$ for some constant C'. Hence, we obtain that the family $\mathcal{S}$ is bounded. □

Lemma 13.2.8. — *There exists an integer m_0, depending only on (E, ∇) and the constant C, with the following property:*

- *Let $m_i \geqslant m_0$. Let $Y = \bigcap_{i=1}^{r} Y_i$, where Y_i denotes general hypersurfaces of L^{m_i}. Let $(\mathcal{L}, \phi) \in \mathcal{S}$. Then, $\phi(\mathcal{L})$ generates a flat subsheaf of $E \otimes \mathcal{O}(*D)$, if and only if $\phi(\mathcal{L}_{|Y})$ generates a flat subsheaf of $E \otimes \mathcal{O}(*D)_{|Y}$.*

Proof. — There exists a large integer N such that $\nabla(E) \subset E \otimes \Omega^1_X(N\,D)$. Then, $\nabla \circ \phi$ induces the $\mathcal{O}_X$-homomorphism $F_\phi : \mathcal{L} \longrightarrow \mathrm{Cok}(\phi)_{tf} \otimes \Omega^1_X(N\,D)$. According to Lemma 13.2.6, F_ϕ vanishes if and only if $\phi(\mathcal{L})$ generates a flat subsheaf of E. Due to Lemmas 13.2.5 and 13.2.7, if m_0 is sufficiently large, F_ϕ vanishes if and only if the induced map $F_{\phi,Y} : \mathcal{L}_{|Y} \longrightarrow \mathrm{Cok}(\phi)_{tf|Y} \otimes \Omega^1_Y(N\,D)$ vanishes. The latter condition is equivalent to that $\phi_{|Y}(\mathcal{L}_{|Y})$ generates a flat subsheaf of $E \otimes \mathcal{O}(*D)_{|Y}$. Thus, we are done. □

13.2.3. Family of degenerating curves.

— We recall the setting in [**61**], [**62**] and Section 3.4 of [**66**]. For simplicity, we assume $H^i(X, L^m) = 0$ for any $m \geqslant 1$ and $i > 0$. We put $S_m := H^0(X, L^m)$ for $m \in \mathbb{Z}_{\geqslant 1}$. For $\boldsymbol{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}_{\geqslant 1}$, we put $S_{\boldsymbol{m}} := \prod_{i=1}^{n-1} S_{m_i}$. Let $Z_{\boldsymbol{m}}$ denote the correspondence variety, i.e., $Z_{\boldsymbol{m}} = \{(x, s_1, \ldots, s_{n-1}) \in X \times S_{\boldsymbol{m}} \mid s_i(x) = 0,\ 1 \leqslant i \leqslant n-1\}$. The natural morphisms $Z_{\boldsymbol{m}} \longrightarrow S_{\boldsymbol{m}}$ and $Z_{\boldsymbol{m}} \longrightarrow X$ are denoted by $q_{\boldsymbol{m}}$ and $p_{\boldsymbol{m}}$, respectively. We put $Z^D_{\boldsymbol{m}} := Z_{\boldsymbol{m}} \times_X D$ and $Z^{D_j}_{\boldsymbol{m}} := Z_{\boldsymbol{m}} \times_X D_j$. Let $K_{\boldsymbol{m}}$ denote the function field of $S_{\boldsymbol{m}}$.

We put $Y_{\boldsymbol{m}} := Z_{\boldsymbol{m}} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$, $Y_{\boldsymbol{m}}^{D_j} := Z_{\boldsymbol{m}}^{D_j} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$ and $Y_{\boldsymbol{m}}^{D} := Z_{\boldsymbol{m}}^{D} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$. The decomposition into irreducible components of $Z_{\boldsymbol{m}}^{D} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$ is given by $\bigcup_j Z_{\boldsymbol{m}}^{D_j} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$.

We fix a sequence of integers $(\alpha_1, \dots, \alpha_{n-1})$ with $\alpha_i \geqslant 2$. We put $\alpha := \prod \alpha_i$. For a positive integer m, let (m) denote $(\alpha_1^m, \dots, \alpha_{n-1}^m)$. Let V_* be a coherent parabolic sheaf on (X, D). For each m, we can take an open subset $U_m \subset S_{(m)}$ such that, for all $s \in U_m$, (i) $q_{(m)}^{-1}(s)$ is smooth , (ii) $q_{(m)}^{-1}(s)$ intersects with the smooth part of D transversally, (iii) V_* is a parabolic bundle on an appropriate neighbourhood of $q_{(m)}^{-1}(s) \subset X$. In the following, we will shrink U_m, if necessary.

Mehta and Ramanathan constructed a family of degenerating curves. Take integers $\ell > m > 0$. Let A be a discrete valuation ring over k with the quotient field K. Then there exists a curve C over $\operatorname{Spec} A$ with a morphism $\varphi : C \to X \times \operatorname{Spec} A$ over $\operatorname{Spec} A$ with the properties: (i) C is smooth over k, (ii) the generic fiber C_K gives a sufficiently general K-valued point in U_ℓ, (iii) the special fiber C_k is reduced with smooth irreducible components C_k^i $(i = 1, \dots, \alpha^{\ell-m})$ which are sufficiently general k-valued points in U_m. We use the symbol D_C to denote $C \times_X D$.

Then, we obtain the parabolic bundle $\varphi^*(V_*)$ on (C, D_C), which is denoted by $V_{*|C}$. The restriction to C_K and C_k^i are denoted similarly. Let W_* be a parabolic subbundle of $V_{*\,|C_K}$. Recall that W can be extended to a subsheaf $\widetilde{W} \subset V_{|C}$, flat over $\operatorname{Spec} A$ with the properties: (i) $\widetilde{W}$ is a vector bundle over C, (ii) $\widetilde{W}_{|C_k^i} \to V_{|C_k^i}$ are injective.

We put $\Omega^1_{C/A} := \Omega^1_C(\log C_k)/\Omega^1_{\operatorname{Spec} A}(\log t)$, where t denotes the closed point of $\operatorname{Spec} A$. We have the induced meromorphic flat connection of V_* relative to A:

$$\nabla_C : V_* \longrightarrow V_* \otimes \Omega^1_{C/A}(*D_C).$$

The restriction to C_k^i is equal to the connection induced by the inclusion $C_k^i \subset X$. If W is a flat subbundle with respect to ∇_{C_K}, then, ∇_C preserves $\widetilde{W}$, and hence $\widetilde{W}_{|C_k^i}$ is also preserved by $\nabla_{C_k^i}$.

13.2.4. Proof of Proposition 13.2.1

Lemma 13.2.9. — *(V_*, ∇) is μ_L-semistable, if and only if there exists a positive integer m_0 such that $(V_*, \nabla)_{|Y_{(m)}}$ is μ_L-semistable for any $m \geqslant m_0$.*

Proof. — We only have to show the "only if" part. If $(V_*, \nabla)_{|Y_{(m)}}$ is μ_L-semistable for some m, then $(V_*, \nabla)_{|Y_{(\ell)}}$ is μ_L-semistable for any $\ell > m$, which we can show by an argument in [**61**]. (See also the first part of the proof of Lemma 3.31 of [**66**].)

We will show that V_* is not semistable if $V_{*|Y_{(m)}}$ are not semistable for any m. By shrinking U_m appropriately, we may have the subsheaf $W_{m\,*}$ of $p_{(m)}^* V_{*|q_{(m)}^{-1} U_m}$ such that (i) it is preserved by the induced relative connection of $p_{(m)}^* V_{*|q_{(m)}^{-1} U_m}$, (ii) $W_{m\,*|\,q_{(m)}^{-1}(s)}$ is the β-subobject of $(V_*, \nabla)_{|q_{(m)}^{-1}(s)}$ for any $s \in U_m$. By an argument in [**61**] (see also the proof of Lemma 3.31 of [**66**]), we can show the existence of subsheaf $\widetilde{W}$ of V such that $\widetilde{W}_{|q_{(m)}^{-1}(s)} = W_{m|q_{(m)}^{-1}(s)}$ for a sufficiently large m and for some $s \in S_{(m)}$. We can make m arbitrarily large. Then, $\widetilde{W}$ is preserved by ∇ according to Lemma 13.2.8. Thus, $\widetilde{W}$ contradicts the μ_L-semistability assumption on (V_*, ∇). □

Now, Proposition 13.2.1 follows from the next lemma.

Lemma 13.2.10. — *(V_*, ∇) is μ_L-stable, if and only if there exists a positive integer m_0 such that $(V_*, \nabla)_{|Y_{(m)}}$ is μ_L-stable for any $m \geqslant m_0$.*

Proof. — First, let us remark that $(V_*, \nabla)_{|q_{(m)}^{-1}(s)}$ has only obvious automorphisms for any sufficiently large m and general s, if (V_*, ∇) is μ_L-stable. To show this, we only have to consider the case where V_* is reflexive and saturated in the sense of Definition 3.17 of [**66**]. Then, the sheaf $\mathcal{H}om(V_*, V_*)$ is also reflexive. For any $\nabla_{|q_{(m)}^{-1}(s)}$-flat $f \in \mathcal{H}om(V_*, V_*)_{|q_{(m)}^{-1}(s)}$, we can take a lift $F \in \mathcal{H}om(V_*, V_*)$ such that $F_{|q_{(m)}^{-1}(s)} = f$ if m is sufficiently large. We would like to show $\nabla(F) = 0$, which is a section of $\mathcal{H}om(V_*, V_*) \otimes \Omega_X^1(N\,D)$. Because $\nabla(F)_{|q_{(m)}^{-1}(s)} = 0$, the claim follows from Lemma 13.2.5. Hence, F is an automorphism of (V_*, ∇), which is the multiplication by some constant. Thus, f is also a multiplication of some constant.

Assume that $(V_*, \nabla)_{|Y^{(m)}}$ is not stable for any m. Then, by an argument in [**62**] using the socle, we can show the existence of a subsheaf $0 \neq \widetilde{W}_{m*} \subset V_*$ for an arbitrarily large m, such that (i) $\widetilde{W}_{m|q_{(m)}^{-1}(s)}$ is preserved by $\nabla_{|q_{(m)}^{-1}(s)}$ for some general $s \in U_m$, (ii) $\mu_L(\widetilde{W}_{m*}) = \mu_L(V_*)$. (See also the proof of Lemma 3.32 of [**66**].) Due to Lemma 13.2.8, $\widetilde{W}$ is preserved by ∇. □

13.3. Auxiliary metrics

In this section, we fix a non-zero λ.

13.3.1. Regular case. — We recall a construction of a Hermitian metric for a regular filtered λ-flat bundle, which is not harmonic but satisfies some finiteness conditions. We put $X := \Delta$ and $D := \{O\}$. For any $\varepsilon > 0$, let g_ε denote the metric of $X \setminus D$ given by $|z|^{-2+2\varepsilon} dz\, d\bar{z}$. We put $\mu_m := \{\omega \in \boldsymbol{C} \,|\, \omega^m = 1\}$, which acts on X by the multiplication $(\omega, z) \mapsto \omega\, z$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a μ_m-equivariant regular filtered λ-flat bundle. The restriction to $X \setminus D$ is denoted by $(E, \mathbb{D}^\lambda)$. We have the parabolic filtration F of ${}^\diamond E_{|O}$. For each $a \in \mathcal{P}ar({}^\diamond E)$, we have the generalized eigen-decomposition $\mathrm{Gr}^F_a({}^\diamond E_{|O}) = \bigoplus_{\alpha \in \boldsymbol{C}} \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}({}^\diamond E_{|O})$. Recall $\mathcal{KMS}({}^\diamond E) := \big\{(a, \alpha) \,\big|\, \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}({}^\diamond E_{|O}) \neq 0\big\}$. For any $(a, \alpha) \in \mathcal{KMS}({}^\diamond E)$, we have the model bundle $(E_{0,a,\alpha}, \overline{\partial}_{0,a,\alpha}, \theta_{0,a,\alpha}, h_{0,a,\alpha})$ associated to $\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}({}^\diamond E_{|O})$ with the nilpotent part $N_{a,\alpha}$ of $\mathrm{Res}(\mathbb{D}^\lambda)$. (See Section 6.2 of [**67**], or Sections 7.4.1.2–7.4.1.3 in this paper.)

Namely, we set

$$E_{0,a,\alpha} := \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}({}^{\diamond}E_{|O}) \otimes \mathcal{O}_{X\setminus D}, \quad \theta_{0,a,\alpha} = N_{a,\alpha}\frac{dz}{z},$$

and then there exists a model metric $h_{0,a,\alpha}$ for $(E_{0,a,\alpha}, \theta_{0,a,\alpha})$ such that the parabolic structure is trivial.

We take $u(a,\alpha) = (b,\beta) \in \boldsymbol{R} \times \boldsymbol{C}$ such that $\mathfrak{k}(\lambda, u(a,\alpha)) = (a,\alpha)$. We have the model bundle $L\bigl(u(a,\alpha)\bigr)$. (See Section 6.1 of [**67**].) Namely, $L\bigl(u(a,\alpha)\bigr)$ is a line bundle $\mathcal{O}_{X\setminus D}\cdot e$ with the Higgs field $\beta\, dz/z$ and the metric $|e| = |z|^{-b}$.

Then, we obtain the following harmonic bundle:

$$(E_0, \overline{\partial}_0, \theta_0, h_0) := \bigoplus_{(a,\alpha)} (E_{0,a,\alpha}, \overline{\partial}_{0,a,\alpha}, \theta_{0,a,\alpha}, h_{0,a,\alpha}) \otimes L\bigl(u(a,\alpha)\bigr).$$

It is naturally equipped with a μ_m-action. Let $(\mathcal{P}_*\mathcal{E}_0^{\lambda}, \mathbb{D}_0^{\lambda})$ denote the associated filtered λ-flat bundle. By construction, we can take a μ_m-equivariant holomorphic isomorphism $\Phi : \mathcal{P}_0\mathcal{E}_0^{\lambda} \to {}^{\diamond}E$ with the following properties:

- Φ preserves the parabolic filtration.
- The induced map $\mathrm{Gr}^F(\Phi) : \mathrm{Gr}^F(\mathcal{P}_0\mathcal{E}^{\lambda}_{0|O}) \to \mathrm{Gr}^F({}^{\diamond}E_{|O})$ is compatible with the residues.

We obtain the induced μ_m-equivariant metric of E, which is also denoted by h_0.

Lemma 13.3.1

- *(E, h_0) is acceptable.*
- *The norm estimate holds for $(\boldsymbol{E}_*, \mathbb{D}^{\lambda}, h_0)$, i.e., let $\boldsymbol{v}$ be a frame of ${}^{\diamond}E$ such that (i) it is compatible with the parabolic filtration F, (ii) the induced frame of $\mathrm{Gr}^F({}^{\diamond}E_{|O})$ is compatible with the weight filtration W of the nilpotent part of $\mathrm{Res}(\mathbb{D}^{\lambda})$. We put $a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. Let h_1 be a Hermitian metric of E given by $h_1(v_i, v_j) := \delta_{i,j}\,|z|^{-2a(v_i)}\bigl(-\log|z|\bigr)^{k(v_i)}$. Then, h_0 and h_1 are mutually bounded.*
- *$\mathbb{D}^{\lambda} - \mathbb{D}_0^{\lambda}$ is bounded with respect to h_0 and g_{ε} for some $\varepsilon > 0$, under the identification of the bundles given by Φ.*
- *Let θ denote the section of $\mathrm{End}(E) \otimes \Omega^{1,0}$ associated to h_0 and $\mathbb{D}^{\lambda}$. (See [**82**], [**83**] or Section 2.2 of [**69**].) Then, $\theta - \theta_0$ is bounded with respect to h_0 and g_{ε} with respect to h_0, under the above identification of the bundles.*
- *$G(\mathbb{D}^{\lambda}, h_0)$ is bounded with respect to h_0 and g_{ε} for some $\varepsilon > 0$.*

Proof. — The first and second claims follow from the property of tame harmonic bundles [**67**]. The third claim is clear by construction. The fourth claim follows from the third one and the relation $\theta - \theta_0 = (1+|\lambda|^2)^{-1}(\mathbb{D}^{\lambda} - \mathbb{D}_0^{\lambda})$. (See Subsection 2.2.1 of [**69**].) We have $\theta_0 = O(1)\, dz/z$ with respect to h_0. Hence $\theta = O(1)\, dz/z$.

Let $R(h_0)$ denote the curvature of the holomorphic bundle E with the Hermitian metric h_0. Because $\dim X = 1$, we have the relation

$$G(\mathbb{D}^{\lambda}, h_0) = (1+|\lambda|^2)R(h) + (1+|\lambda|^2)^2\bigl[\theta, \theta^{\dagger}\bigr].$$

(See Subsection 2.2.3 of [**69**].) We have a similar relation for $G(\mathbb{D}_0^\lambda, h_0)$. Hence, we obtain the equality

$$G(\mathbb{D}^\lambda, h_0) = G(\mathbb{D}^\lambda, h_0) - G(\mathbb{D}_0^\lambda, h_0) = (1+|\lambda|^2)^2\big([\theta,\theta^\dagger] - [\theta_0,\theta_0^\dagger]\big).$$

Then, the last claim follows. □

Corollary 13.3.2. — *If there exists a tame harmonic metric h of $(E,\mathbb{D}^\lambda)$ adapted to $\boldsymbol{E}_*$, the metrics h and h_0 are mutually bounded.* □

We put $\widetilde{X} := \Delta^n$ and $\widetilde{D} := \{z_1 = 0\}$. We have the μ_m-action on $\widetilde{X}$ by the multiplication on z_1, which preserves $\widetilde{D}$. Let $(\widetilde{\boldsymbol{E}}_*, \widetilde{\mathbb{D}}^\lambda)$ be a μ_m-equivariant regular filtered λ-flat bundle on $(\widetilde{X},\widetilde{D})$. The restriction to $\widetilde{X}-\widetilde{D}$ is denoted by $(\widetilde{E},\widetilde{\mathbb{D}}^\lambda)$. For any $\varepsilon>0$, let $\widetilde{g}_\varepsilon$ denote the metric of $\widetilde{X}-\widetilde{D}$ given by $|z_1|^{-2+2\varepsilon}\,dz_1\,d\overline{z}_1 + \sum_{j=2}^n dz_j\,d\overline{z}_j$.

Lemma 13.3.3. — *There exists a μ_m-equivariant Hermitian metric $\widetilde{h}_0$ of $\widetilde{E}$ with the following properties:*

- *$(\widetilde{E},\widetilde{h}_0)$ is acceptable.*
- *The norm estimate holds for $(\widetilde{\boldsymbol{E}}_*, \widetilde{\mathbb{D}}^\lambda, \widetilde{h}_0)$, i.e., let $\boldsymbol{v}$ be a frame of ${}^\diamond E$ such that (i) it is compatible with the parabolic filtration F, (ii) the induced frame of $\mathrm{Gr}^F({}^\diamond E)$ is compatible with the weight filtration W of the nilpotent part of $\mathrm{Res}(\widetilde{\mathbb{D}}^\lambda)$. We put $a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. Let h_1 be a Hermitian metric of E given by $h_1(v_i,v_j) := \delta_{i,j}\,|z_1|^{-2a(v_i)}\big(-\log|z_1|\big)^{k(v_i)}$. Then, h_0 and h_1 are mutually bounded.*
- *$G(\widetilde{\mathbb{D}}^\lambda, \widetilde{h}_0)$ is bounded with respect to $\widetilde{h}_0$ and the metric $\widetilde{g}_\varepsilon$ for some $\varepsilon>0$.*
- *Let $\widetilde{\theta}$ denote the section of $\mathrm{End}(\widetilde{E})\otimes\Omega^{1,0}$ associated to $\widetilde{\mathbb{D}}^\lambda$ and $\widetilde{h}_0$. Then, $\widetilde{\theta} = O(1)\,dz_1/z_1$.*

Proof. — There exists a μ_m-equivariant regular filtered λ-flat bundle $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ on (X,D) such that $(\widetilde{\boldsymbol{E}}_*, \widetilde{\mathbb{D}}^\lambda)$ is the pull-back of $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ by the projection $(\widetilde{X},\widetilde{D}) \to (X,D)$. Hence, the claim follows from Lemma 13.3.1. □

13.3.2. Good filtered λ-flat bundle. — Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let g_ε denote the metric of $X\setminus D$ given by $|z_1|^{-2+2\varepsilon}\,dz_1\,d\overline{z}_1 + \sum_{j=2}^n dz_j\,d\overline{z}_j$. Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X,D). The restriction to $X\setminus D$ is denoted by E. We take a ramified covering $\varphi_e : (X',D') \to (X,D)$ given by $\varphi_e(z_1',z_2,\dots,z_n) = (z_1'^{\,e},z_2,\dots,z_n)$ such that $\varphi_e^*(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ is unramified. We have the irregular decomposition:

$$\varphi_e^*(\boldsymbol{E}_*,\mathbb{D}^\lambda)_{|\widehat{D}'} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D}^\lambda)} (\widehat{\boldsymbol{E}}'_{\mathfrak{a}*}, \widehat{\mathbb{D}}'^\lambda_{\mathfrak{a}}). \tag{308}$$

Here, the filtration of $\varphi_e^*\boldsymbol{E}_*$ is given as in Section 2.5.3.3. We have the natural $\mathrm{Gal}(X'/X)$-action on $\mathrm{Irr}(\mathbb{D}^\lambda)$. For $\mathfrak{a}\in\mathrm{Irr}(\mathbb{D}^\lambda)$, let $\mathrm{Stab}(\mathfrak{a})$ denote the stabilizer of the $\mathrm{Gal}(X'/X)$-action. Since $\widehat{\mathbb{D}}'^\lambda_{\mathfrak{a}} - d\mathfrak{a}$ is logarithmic, we have a $\mathrm{Stab}(\mathfrak{a})$-equivariant good filtered λ-flat bundle $(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda}_{\mathfrak{a}})$ on (X',D') such that $(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda}_{\mathfrak{a}})_{|\widehat{D}'}\simeq(\widehat{\boldsymbol{E}}'_{\mathfrak{a}*},\widehat{\mathbb{D}}'^{\lambda}_{\mathfrak{a}})$. The restrictions to $X'\setminus D'$ are denoted by $E_{\mathfrak{a}}$. We obtain an isomorphism:

$$\widehat{\psi}:\bigoplus_{\mathfrak{a}\in\operatorname{Irr}(\mathbb{D}^{\lambda})}(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda}_{\mathfrak{a}})_{|\widehat{D}'}\simeq\varphi_e^*(\boldsymbol{E}_*,\mathbb{D}^{\lambda})_{|\widehat{D}'}.$$

We may assume to have the $\operatorname{Gal}(X'/X)$-action on $\bigoplus(\boldsymbol{E}'_{\mathfrak{a}*},\,\mathbb{D}'^{\lambda}_{\mathfrak{a}})$ which is equal to the natural $\operatorname{Gal}(X'/X)$-action on $\varphi_e^*\boldsymbol{E}_*$ after the completion. Let N be any large number. We can take a $\operatorname{Gal}(X'/X)$-equivariant holomorphic (not necessarily flat) isomorphism $\psi_N:\bigoplus\boldsymbol{E}'_{\mathfrak{a}*}\simeq\varphi_e^*\boldsymbol{E}_*$ which is equal to $\widehat{\psi}$ on the N-th infinitesimal neighbourhood $\widehat{D}'^{(N)}$ of D'.

We have the induced action of $\operatorname{Stab}(\mathfrak{a})$ on $(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda}_{\mathfrak{a}})$. Let $\mathbb{D}'^{\lambda\,\mathrm{reg}}_{\mathfrak{a}}:=\mathbb{D}'^{\lambda}_{\mathfrak{a}}-d\mathfrak{a}$. Applying Lemma 13.3.3 to $(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda\,\mathrm{reg}}_{\mathfrak{a}})$, we can take a $\operatorname{Stab}(\mathfrak{a})$-equivariant Hermitian metric $h'_{\mathfrak{a}}$ of $E'_{\mathfrak{a}}$ with the property as in Lemma 13.3.3. We may assume $\omega^*h'_{\mathfrak{a}}=h'_{\omega^*\mathfrak{a}}$ on $E'_{\omega^*\mathfrak{a}}$. Hence, the metric $\bigoplus h'_{\mathfrak{a}}$ of $\bigoplus E'_{\mathfrak{a}}$ is $\operatorname{Gal}(X'/X)$-equivariant. We obtain the induced metrics $h'^{(0)}$ and $h^{(0)}$ of φ_e^*E and E, respectively. The metric $h^{(0)}$ is called an auxiliary metric.

Lemma 13.3.4. — *Let N be sufficiently large. By construction, we have the following properties:*

- *$(E,h^{(0)})$ is acceptable.*
- *The norm estimate holds for $(\boldsymbol{E}_*,\mathbb{D}^{\lambda},h^{(0)})$. (See Lemma 13.3.3.)*
- *$G(\mathbb{D}^{\lambda},h^{(0)})$ is bounded with respect to $h^{(0)}$ and g_{ε} for some $\varepsilon>0$.*
- *Let $\theta_{h^{(0)}}$ denote the section of $\operatorname{End}(E)\otimes\Omega^{1,0}$ induced by $h^{(0)}$ and $\mathbb{D}^{\lambda}$. Then, $|\theta_{h^{(0)}}|_{h^{(0)}}$ is of polynomial order with respect to z_1. More precisely,*

$$\varphi_e^*\theta_{h^{(0)}}-\bigoplus(1+|\lambda|^2)^{-1}d\mathfrak{a}\,\operatorname{id}_{\psi(E'_{\mathfrak{a}})}$$

is $O(1)\,dz'_1/z'_1+\sum_{j=2}^{n}O(1)\,dz_j$ with respect to $h'^{(0)}$.

Proof. — The claims hold for $\bigoplus(\boldsymbol{E}'_{\mathfrak{a}*},\mathbb{D}'^{\lambda}_{\mathfrak{a}})$ with the metric $\bigoplus h'_{\mathfrak{a}}$ by construction of $h'_{\mathfrak{a}}$. Under the identification of $\bigoplus\boldsymbol{E}'_{\mathfrak{a}*}$ and $\varphi_e^*\boldsymbol{E}_*$ via ψ, we have

$$\varphi_e^*\mathbb{D}^{\lambda}-\bigoplus_{\mathfrak{a}}\mathbb{D}'^{\lambda}_{\mathfrak{a}}=O\big(z_1^{N/2}\big).$$

Since N is sufficiently large, the claim of the lemmas are clear. □

13.3.3. Family version. — We put $X:=\Delta$ and $D:=\{O\}$. Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a good filtered λ-flat bundle on (X,D). The restriction to $X\setminus D$ is denoted by $(E,\mathbb{D}^{\lambda})$. We take a real number $c\notin\mathcal{P}ar(\boldsymbol{E}_*)$. We have the induced parabolic structure F of ${}_cE$. Take $\eta>0$ such that $10\eta<\operatorname{gap}({}_cE_*)$. (See Section 3.1 of [**66**] for gap.) For any $\varepsilon\geqslant 0$, let ω_{ε} denote the metric of $X\setminus D$ given as follows:

$$g_{\varepsilon}:=\big(\varepsilon^2|z|^{2\varepsilon}+\eta^2|z|^{2\eta}\big)\,\frac{dz\,d\overline{z}}{|z|^2}. \tag{309}$$

Let $F^{(\varepsilon)}$ denote ε-perturbation of F for any sufficiently small $\varepsilon > 0$, as explained in (II) of Section 13.7 below.

Lemma 13.3.5. — *If $\varepsilon_0 > 0$ is sufficiently small, there exists a family of Hermitian metrics $\{h_0^{(\varepsilon)} \mid 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ of E with the following properties:*

- *$G(\mathbb{D}^\lambda, h_0^{(\varepsilon)})$ are bounded with respect to $h_0^{(\varepsilon)}$ and g_ε. The estimate is uniform for ε.*
- *The norm estimate holds for $(E, F^{(\varepsilon)}, h_0^{(\varepsilon)})$.*
- *$\{h_0^{(\varepsilon)} \mid \varepsilon > 0\}$ converges to $h_0^{(0)}$ in the C^∞-sense locally on $X \setminus D$.*
- *Let t_ε be determined by $\det(h_0^{(\varepsilon)})/\det(h_0^{(0)})$. Then, t_ε and t_ε^{-1} are bounded, uniformly in ε.*

Proof. — In Sections 4.2–4.4 of [**69**], we give the construction of such a family of Hermitian metric for a regular λ-flat bundle. The result can be extended to the case where a meromorphic λ-flat bundle is not necessarily regular but good, with the same argument as in Section 13.3.2. □

13.4. Harmonic bundles on curves

13.4.1. Review of a result due to Biquard-Boalch. — Let C be a smooth connected complex projective curve, and D be a finite subset of C. Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (C, D). The restriction to $C \setminus D$ is denoted by (E, ∇). We recall the following result due to Biquard-Boalch [**10**]. (See also [**71**].)

Proposition 13.4.1. — *$(\boldsymbol{E}_*, \nabla)$ is polystable with* $\text{par-deg}(\boldsymbol{E}_*) = 0$, *if and only if there exists a wild harmonic metric h of (E, ∇) adapted to $\boldsymbol{E}_*$. Such a metric is unique up to obvious ambiguity. (See Theorem 16.1.1 for the higher dimensional version.)*

Proof. — We give an outline of a proof based on Simpson's method, for our later purpose. The claim is easy in the rank one case. Hence, we take a harmonic metric $h_{\det(E)}$ of $\bigl(\det(\boldsymbol{E}_*), \nabla\bigr)$. Let g_ε be a Kähler metric of $C \setminus D$, which is given by $|z|^{-2+\varepsilon}$ around $P \in D$ for a holomorphic coordinate z such that $z(P) = 0$.

Lemma 13.4.2. — *There exists a Hermitian metric $h^{(0)}$ of E with the following properties:*

- *$(E, h^{(0)})$ is acceptable.*
- *The norm estimate holds for $(E, \nabla, h^{(0)})$ at each $P \in D$. (See Lemma 13.3.1.)*
- *$G(\nabla, h^{(0)})$ is bounded with respect to $h^{(0)}$ and g_ε for some $\varepsilon > 0$.*
- *Let $\theta^{(0)}$ denote the section of $\mathrm{End}(E) \otimes \Omega^{1,0}$ induced by $h^{(0)}$ and ∇. Then, $|\theta^{(0)}|_{h^{(0)}}$ is of polynomial order with respect to $|z|^{-1}$ around each $P \in D$, where z denotes a holomorphic coordinate around P such that $z(P) = 0$.*
- *$\det(h^{(0)}) = h_{\det(E)}$.*

Proof. — Applying the construction in Section 13.3.2 around each $P \in D$, we obtain a metric of $E_{|\mathcal{U}\setminus D}$, where $\mathcal{U}$ denotes some neighbourhood of D. We extend it to a C^∞ Hermitian metric $h^{(0)\star}$ of E. Let s be determined by $h_{\det(E)} = \det(h^{(0)\star})\, s$. Then, s and s^{-1} are bounded on $C \setminus D$. Hence, we obtain the desired metric $h^{(0)}$ with an obvious modification. □

For any $\boldsymbol{F}_* \subset \boldsymbol{E}_*$, we have par-deg$(\boldsymbol{F}_*) = \deg(F, h^{(0)})$ due to a result of Simpson (Lemma 6.2 of [**82**]). Hence, $(\boldsymbol{E}_*, \nabla)$ is stable if and only if (E, ∇, h) is analytically stable. Then, due to a theorem of Simpson in [**81**] and [**82**], we obtain the harmonic metric $h = h^{(0)}\, s$ such that (i) s and s^{-1} are bounded with respect to $h^{(0)}$, (ii) ∇s is L^2 with respect to $h^{(0)}$, (iii) $\det(s) = 1$. (See also Proposition 2.49 of [**69**].) Let θ be the Higgs field associated to h and ∇. Let d'' denote the $(0,1)$-part of ∇, and let $\delta'_{h^{(0)}}$ denote the $(1,0)$-operator induced by d'' and $h^{(0)}$. Because of the L^2-property of $d''s$ and the self-adjointness of s, we obtain that $\delta'_{h^{(0)}} s$ is L^2 with respect to $h^{(0)}$.

Let us show that (E, ∇, h) is a *wild* harmonic bundle. For any $P \in D$, we take a coordinate neighbourhood (U_P, z) with $z(P) = 0$. Let $\psi : \mathbb{H} \to U_P \setminus \{P\}$ be given by $\psi(\zeta) = \exp(2\pi\sqrt{-1}\zeta)$. We put $K_n := \{\zeta \mid -1 < \operatorname{Re}\zeta < 1,\ n-1 < \operatorname{Im}\zeta < n+1\}$. Because $\theta = \theta^{(0)} + s^{-1}\delta'_{h_0} s/2$, there exists a constant C, which is independent of n, such that the following holds:

$$\int_{K_n} |\psi^*\theta|_h^2 \leqslant 2\int_{K_n} |\psi^*\theta^{(0)}|_{h^{(0)}}^2 + C.$$

Since $\int_{K_n} |\psi^*\theta|_h^2$ is the energy of the harmonic map on K_n corresponding to the harmonic bundle $\psi^*(E, \nabla, h)$ up to some positive constant multiplication, we obtain the following estimate (see [**27**]):

$$\sup_{K'_n} |\psi^*\theta|_h^2 \leqslant C_1 \sup_{K_n} |\psi^*\theta^{(0)}|_{h^{(0)}}^2 + C_2.$$

Here $K'_n := \{-2/3 \leqslant \operatorname{Re}(\zeta) < 2/3,\ n - 2/3 < \operatorname{Im}(\zeta) < n + 2/3\}$. Since the norm of $\theta^{(0)}$ is of polynomial order of $|z|^{-1}$ around P, the norm of θ is also of polynomial order of $|z|^{-1}$. In particular, the eigenvalues of θ are also of polynomial order. Thus, $(E, \overline{\partial}_E, \theta, h)$ is wild.

Conversely, we can show the "if" part by the same argument as that in Chapter 6 of [**82**].

We know the norm estimate for wild harmonic bundles on curves (Proposition 8.1.1). Hence, if h_i $(i = 1, 2)$ are wild harmonic metrics for (E, ∇) adapted to $\boldsymbol{E}_*$, they are mutually bounded. Then, by the same argument in the proof of Proposition 2.6 in [**66**], we can show the existence of a decomposition $(\boldsymbol{E}_*, \nabla) = \bigoplus(\boldsymbol{E}_{i\,*}, \nabla_i)$ such that (i) it is orthogonal with respect to both h_i $(i = 1, 2)$, (ii) $h_1 = a_i\, h_2$ on E_i for some $a_i > 0$. □

Remark 13.4.3. — We will also prove that if a harmonic metric h of (E, ∇) is adapted to $\boldsymbol{E}_*$, then (E, ∇, h) is a wild harmonic bundle (Proposition 13.5.3). □

13.4.2. Convergence of a sequence of harmonic bundles. — Let (C, D) be as in Section 13.4.1. Let $(E, \boldsymbol{F}, \nabla)$ be a stable good parabolic flat bundle on (C, D) such that par-deg$(E, \boldsymbol{F}) = 0$. Let $\boldsymbol{F}^{(\varepsilon)}$ denote an ε-perturbation of $\boldsymbol{F}$ as in the case (II) of Section 13.7 below. We have $\det(E, \boldsymbol{F}) = \det(E, \boldsymbol{F}^{(\varepsilon)})$. We take the harmonic metric $h_{\det E}$ of $\det(E, \boldsymbol{F}, \nabla)$. Let $h^{(\varepsilon)}$ be harmonic metrics of (E, ∇) adapted to $\boldsymbol{F}^{(\varepsilon)}$ such that $\det h^{(\varepsilon)} = h_{\det E}$. Let $\theta^{(\varepsilon)}$ denote the associated Higgs field. We have a straightforward generalization of Proposition 4.1 of [**69**].

Proposition 13.4.4. — *We have the convergences $h^{(\varepsilon)} \to h^{(0)}$ and $\theta^{(\varepsilon)} \to \theta^{(0)}$ in the C^∞-sense locally on $C \setminus D$.*

Proof. — Let η be a small positive number such that $\eta < \operatorname{gap}(E, \boldsymbol{F})/10$. Let ε_0 be a small positive number such that $10 \operatorname{rank}(E)\, \varepsilon_0 < \eta$. For any $0 \leqslant \varepsilon < \varepsilon_0$, let us take Kähler metrics g_ε of $C \setminus D$ with the following properties:

- Let $P \in D$. Let (U_P, z) be a holomorphic coordinate neighbourhood of P such that $z(P) = 0$. The restriction of g_ε is as in Section 13.3.3.
- $g_\varepsilon \to g_0$ in the C^∞-sense locally on $C \setminus D$.

Lemma 13.4.5. — *We can construct a family of Hermitian metrics $h_0^{(\varepsilon)}$ of $E_{|C \setminus D}$ with the following properties:*

- *$G(\mathbb{D}^\lambda, h_0^{(\varepsilon)})$ are uniformly bounded with respect to $h_0^{(\varepsilon)}$ and g_ε.*
- *The norm estimate holds for each $(E, \boldsymbol{F}^{(\varepsilon)}, h_0^{(\varepsilon)})$.*
- *$\{h_0^{(\varepsilon)} \mid \varepsilon > 0\}$ converges to $h_0^{(0)}$ in the C^∞-sense locally on $C \setminus D$.*
- *$\det(h_0^{(\varepsilon)}) = h_{\det(E)}$.*

Proof. — It can be shown by using the argument in Section 4.5.1 of [**69**] together with Lemma 13.3.5 (instead of Lemma 4.11 of [**69**]). □

Then, the claim of Proposition 13.4.4 can be shown by using the argument in Section 4.5 of [**69**]. □

13.4.3. Convergence of a sequence of Hermitian metrics. — Let (C, D) be as in Section 13.4.1. Let $(E, \boldsymbol{F}, \nabla)$ be a stable good parabolic flat bundle on (C, D) with par-deg$(E, \boldsymbol{F}) = 0$. For each $P \in D$, let (U_P, z) be a holomorphic coordinate around P such that $z(P) = 0$. Let $\boldsymbol{F}^{(\varepsilon)}$ be an ε-perturbation as in the case (II) of Section 13.7. We have $h_0^{(\varepsilon_i)}$ be a harmonic metric for each $(E, \boldsymbol{F}^{(\varepsilon_i)}, \nabla)$ for some sequence $\{\varepsilon_i\}$ such that $\varepsilon_i \to 0$. For simplicity of the description, we use ε instead of ε_i. We assume $\det h_0^{(\varepsilon)} = \det h_0^{(0)}$. Note that the sequence $h_0^{(\varepsilon)}$ $(\varepsilon > 0)$ converges to $h_0^{(0)}$ in the C^∞-sense locally on $C \setminus D$ (Proposition 13.4.4).

Let N be a large positive number, for example $N > 10$. We use Kähler metrics g_ε $(\varepsilon \geqslant 0)$ of $C \setminus D$ which are as follows on U_P for each $P \in D$:

$$\bigl(\varepsilon^{N+2}|z|^{2\varepsilon} + |z|^2\bigr)\frac{dz\, d\bar{z}}{|z|^2}.$$

We assume that $\{g_\varepsilon\}$ converges to g_0 for $\varepsilon \to 0$ in the C^∞-sense locally on $C \setminus D$. By using the argument in Section 5.1 of [**69**] with Proposition 13.4.4, we can show the following lemma, which is a straightforward generalization of Proposition 5.1 of [**69**].

Lemma 13.4.6. — *Let $h^{(\varepsilon)}$ $(\varepsilon > 0)$ be Hermitian metrics of $E_{|C\setminus D}$ with the following properties:*

1. *Let $s^{(\varepsilon)}$ be determined by $h^{(\varepsilon)} = h_0^{(\varepsilon)} s^{(\varepsilon)}$. Then, $s^{(\varepsilon)}$ is bounded with respect to $h_0^{(\varepsilon)}$, and we have $\det s^{(\varepsilon)} = 1$. We also have the finiteness of the L^2-norm $\|\nabla s^{(\varepsilon)}\|_{2,h_0^{(\varepsilon)},g_\varepsilon} < \infty$. (The estimates may depend on ε.)*
2. *$\|G(h^{(\varepsilon)})\|_{2,h^{(\varepsilon)},g_\varepsilon} < \infty$ and $\lim_{\varepsilon\to 0} \|G(h^{(\varepsilon)})\|_{2,h^{(\varepsilon)},g_\varepsilon} = 0$.*

Then the following claims hold.
- *The sequence $\{s^{(\varepsilon)}\}$ is weakly convergent to the identity of E in L_1^2 locally on $C \setminus D$.*
- *$|s^{(\varepsilon)}|_{h_0^{(\varepsilon)}}$ and $|(s^{(\varepsilon)})^{-1}|_{h_0^{(\varepsilon)}}$ are bounded on $C \setminus D$ uniformly in ε.* □

Corollary 13.4.7
- *The sequence $\{h^{(\varepsilon)}\}$ is convergent to $h_0^{(0)}$ weakly in L_1^2 locally on $C \setminus D$.*
- *The sequence $\{\nabla s^{(\varepsilon)}\}$ is convergent to 0 weakly in L^2 locally on $C \setminus D$.*
- *The sequence $\{\theta^{(\varepsilon)}\}$ is convergent to $\theta^{(0)}$ weakly in L^2 locally on $C \setminus D$.*
- *In particular, the sequences are convergent almost everywhere.* □

13.4.4. Continuity for a holomorphic family. — Let $\mathcal{C} \to \Delta$ be a holomorphic family of smooth projective curve, and $\mathcal{D} \to \Delta$ be a relative divisor. Let $(E, \boldsymbol{F}, \nabla)$ be a good filtered flat bundle on $(\mathcal{C}, \mathcal{D})$. Let t be any point of Δ. We denote the fibers by $\mathcal{C}_t$ and $\mathcal{D}_t$, and the restriction of $(E, \boldsymbol{F}, \nabla)$ to $(\mathcal{C}_t, \mathcal{D}_t)$ is denoted by $(E_t, \boldsymbol{F}_t, \nabla_t)$. We assume par-deg$(E_t, \boldsymbol{F}_t) = 0$ and that $(E_t, \boldsymbol{F}_t, \nabla_t)$ is stable for each t. For simplicity, we also assume that we are given a pluri-harmonic metric $h_{\det(E)}$ of $\det(E, \nabla)_{|\mathcal{C}\setminus\mathcal{D}}$ which is adapted to the induced parabolic structure.

Let $h_{H,t}$ be a harmonic metric of $(E_t, \boldsymbol{F}_t, \nabla_t)$ such that $\det(h_{H,t}) = h_{\det(E)\,|\,\mathcal{C}_t}$. They give the metric h_H of E. Let $\theta_{H,t}$ be the Higgs field obtained from $(E_t, \nabla, h^{(\varepsilon_t)})$, which is a section of $\mathrm{End}(E_t) \otimes \Omega^{1,0}_{\mathcal{C}_t}(\log \mathcal{D}_t)$. They give the section θ_H of $\mathrm{End}(E) \otimes \Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$, where $\Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$ denotes the sheaf of the logarithmic relative $(1,0)$-forms. The following lemma is a straightforward generalization of Proposition 4.2 of [**69**], and it can be proved by using an argument in the proof of Propositions 4.1 of [**69**] with Lemma 13.3.4.

Lemma 13.4.8. — *h_H and θ_H are continuous. Their derivatives of any degree along the fiber directions are also continuous.* □

13.5. Some characterizations of wildness of harmonic bundle

We fix a non-zero λ in this section.

13.5.1. Statements. — Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. Let $\pi_i : X \to D_i$ denote the natural projection. We put $D_i^\circ := D_i \setminus \bigcup_{j \neq i, 1 \leqslant j \leqslant \ell} D_j$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a good filtered flat bundle on (X, D). Let h be a pluri-harmonic metric of $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X \setminus D}$, and let $(E, \overline{\partial}_E, \theta)$ be the associated Higgs bundle. We will prove the following proposition in Section 13.5.4, after showing the special cases in Sections 13.5.2 and 13.5.3.

Proposition 13.5.1. — *Assume that there exist subsets $Z_i \subset D_i$ $(i = 1, \dots, \ell)$ with the following properties:*

- *The Lebesgue measure of Z_i are 0.*
- *$h_{|\pi_i^{-1}(P)}$ is adapted to $\boldsymbol{E}_{*|\pi_i^{-1}(P)}$ for any $P \in D_i^\circ \setminus Z_i$.*

Then, the following holds:

- *The harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is good and wild, and h is adapted to $\boldsymbol{E}_*$.*
- *If $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is unramified, $(E, \overline{\partial}_E, \theta, h)$ is also unramified, and the following holds for any $P \in D$:*

$$\operatorname{Irr}(\theta, P) = \big\{ (1 + |\lambda|^2)^{-1} \mathfrak{a} \,\big|\, \mathfrak{a} \in \operatorname{Irr}(\mathbb{D}^\lambda, P) \big\}. \tag{310}$$

We give a slightly different proposition. For simplicity, we assume that the determinant $\det(E, \overline{\partial}_E, \theta, h)$ is good and wild, and $\det(h)$ is adapted to $\det(\boldsymbol{E}_*)$. We prove the following proposition in Section 13.5.5.

Proposition 13.5.2. — *If h is adapted to $\boldsymbol{E}_*$, then the following holds:*

- *$h_{|\pi_i^{-1}(Q)}$ is adapted to $\boldsymbol{E}_{*\,|\,\pi_i^{-1}(Q)}$ for any $Q \in D_i^\circ$ and for any $i = 1, \dots, \ell$.*
- *$(E, \overline{\partial}_E, \theta, h)$ is good and wild.*
- *If $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is unramified, $(E, \overline{\partial}_E, \theta, h)$ is also unramified, and the following holds for any $P \in D$:*

$$\operatorname{Irr}(\theta, P) = \big\{ (1 + |\lambda|^2)^{-1} \mathfrak{a} \,\big|\, \mathfrak{a} \in \operatorname{Irr}(\mathbb{D}^\lambda, P) \big\}.$$

13.5.2. A characterization of wildness of harmonic bundles on a punctured disc. — We put $X := \big\{ z \in \boldsymbol{C} \,\big|\, |z| < 1 \big\}$ and $D := \{O\}$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X, D). The restriction to $X \setminus D$ is denoted by $(E, \mathbb{D}^\lambda)$.

Proposition 13.5.3. — *Let h be a harmonic metric of $(E, \mathbb{D}^\lambda)$ adapted to $\boldsymbol{E}_*$. Then, $(E, \mathbb{D}^\lambda, h)$ is a wild harmonic bundle. If $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is regular, then h is tame.*

Proof. — The second claim follows from the first claim and the comparison of the irregular values of λ-connection and the Higgs field (Theorem 7.4.5). We can take an auxiliary metric h_0 for $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ as in Section 13.3.2, which has the property as in Lemma 13.3.4. Let θ_{h_0} denote the section of $\operatorname{End}(E) \otimes \Omega^{1,0}$ associated to h_0 and $\mathbb{D}^\lambda$. Let d'' denote the $(0,1)$-part of $\mathbb{D}^\lambda$, and let δ'_{h_0} denote the $(1,0)$-operator induced by d'' and h_0. Let s be determined by $h = h_0\, s$, which is self-adjoint with respect to both the metrics h and h_0. Because h is adapted to $\boldsymbol{E}_*$, we have $|s|_{h_0} = O(|z|^{-\varepsilon})$ for any $\varepsilon > 0$.

Lemma 13.5.4. — *$|s|_{h_0}$ is bounded.*

Proof. — We have the inequality $\Delta \log \mathrm{tr}(s) \leqslant C\left|\Lambda G(h_0)\right|_{h_0}$ on $X \setminus D$. (See Section 2.2 of [**69**], for example.) Hence, we have $\Delta(\log \mathrm{tr}(s) + \varepsilon \log|z|) \leqslant C\left|\Lambda G(h_0)\right|_{h_0}$ on $X \setminus D$ for any $\varepsilon > 0$. Since $\log \mathrm{tr}(s) + \varepsilon \log|z|$ is bounded above, the inequality holds on X as distributions. Hence, the values of $\log \mathrm{tr}(s) + \varepsilon \log|z|$ is dominated by the values on the boundary. By taking $\varepsilon \to 0$, we obtain the boundedness of $\mathrm{tr}(s)$. Then, the claim of the lemma follows. □

Lemma 13.5.5. — *Let $(L, \mathbb{D}^\lambda)$ be a flat λ-bundle on $X \setminus D$ of rank one. Let h_1 be a harmonic metric of $(L, \mathbb{D}^\lambda)$. Let u be a holomorphic section of L on $X \setminus D$ with the following properties:*

- *Let f be determined by $\mathbb{D}^\lambda u = u\, f\, dz$. Then, f is meromorphic on X.*
- *$|u|_h \leqslant C\,|z|^a$ for some $a \in \boldsymbol{R}$.*

Then, there exist $b \in \boldsymbol{R}$ and $C_i > 0$ $(i = 1, 2)$ such that $C_1 \leqslant |u|_h\,|z|^b \leqslant C_2$.

Proof. — Taking the tensor product with an appropriate wild harmonic bundle on $X \setminus D$ of rank one, and replacing u with $e^g\, u$ for some holomorphic function g, we may assume $\mathbb{D}^\lambda u = 0$. Let d'' denote the $(0,1)$-part of $\mathbb{D}^\lambda$. We put $A := |u|_h^2$. Due to $\operatorname{rank} L = 1$, we have

(311) $$\overline{\partial}\partial \log A = R(d'', h) = (1 + |\lambda|^2)^{-1}\, G(\mathbb{D}^\lambda, h) = 0.$$

Hence, we have the expression $\log A = -b\, \log|z| + \operatorname{Re} F(z)$, where F is a holomorphic function on $X \setminus D$. By assumption, we have $\log A \leqslant -a\, \log r$ on $X \setminus D$, and hence F must be holomorphic on X. Then, the claim of the lemma follows. □

Let us return to the proof of Proposition 13.5.3. We have $\det(h) = \det(h_0)\, \det(s)$. By Lemma 13.5.5, we have $\det(s) \geqslant C_1\,|z|^{-N_1}$ for some positive constants C_1 and N_1. Hence, we obtain $C_2\,|z|^{N_2} \leqslant |s|_{h_0}$ and $|s^{-1}|_{h_0} \leqslant C_3|z|^{-N_3}$.

Recall the following formula (see [**81**] or Section 2.2 of [**69**]):

(312) $$(1 + |\lambda|^2)\Delta\, \mathrm{tr}(s) = -\,\mathrm{tr}\big(s\sqrt{-1}\Lambda G(h_0)\big) - \left|\mathbb{D}^\lambda(s)\cdot s^{-1/2}\right|_{h_0}^2.$$

Lemma 13.5.6. — *We have the finiteness $\int_{X\setminus D}\left|\mathbb{D}^\lambda s\cdot s^{-1/2}\right|_{h_0}^2 < \infty$.*

Proof. — Although it follows from a lemma in [**82**], we give a direct argument. Let ρ be an $\boldsymbol{R}_{\geqslant 0}$-valued C^∞-function on $\boldsymbol{R}$ such that $\rho(t) = 1$ for $t \leqslant 1$ and $\rho(t) = 0$ for $t \geqslant 2$. We can take such a function as $\rho'\, \rho^{-1/2}$ is bounded. We put $\chi_N(z) = \rho\big(-N^{-1}\log|z|\big)$. We have the following equality:

$$\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}(\chi_N\, s) = \chi_N\, \mathbb{D}^\lambda\mathbb{D}^{\lambda\star}s + \mathbb{D}^\lambda\chi_N\cdot\mathbb{D}^{\lambda\star}s + \mathbb{D}^\lambda s\cdot\mathbb{D}^{\lambda\star}\chi_N + s\,\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\chi_N$$

(See Section 2.1 of [**69**] for $\mathbb{D}^{\lambda\star}$.) By using (312), we obtain the following inequality for some constant A_1, which is independent of N:

$$\int \chi_N\left|\mathbb{D}^\lambda s\cdot s^{-1/2}\right|_{h_0}^2 \leqslant A_1 + 2\int\left|\mathbb{D}^\lambda s\right|_{h_0}\left|\mathbb{D}^{\lambda\star}\chi_N\right|.$$

The following inequality holds for some $A_2 > 0$:

(313)
$$\int \big|\mathbb{D}^\lambda s\big|_{h_0} \big|\mathbb{D}^{\lambda\star}\chi_N\big| \leqslant \Big(\int \chi_N^{-1}\big|\mathbb{D}^{\lambda\star}\chi_N\big|^2 |s^{1/2}|_{h_0}^2\Big)^{1/2}\Big(\int \chi_N |\mathbb{D}^\lambda s\cdot s^{-1/2}|_{h_0}^2\Big)^{1/2} \leqslant A_2\Big(\int \chi_N\,|\mathbb{D}^\lambda s\cdot s^{-1/2}|_{h_0}^2\Big)^{1/2}.$$

Hence, we obtain $\int \chi_N\,|\mathbb{D}^\lambda s\cdot s^{-1/2}|_{h_0}^2 \leqslant A_3$, independently of N. Thus, we obtain $\int \big|\mathbb{D}^\lambda(s)\cdot s^{-1/2}\big|_{h_0}^2 \leqslant A_3$. □

In particular, we obtain the finiteness $\int_{X\smallsetminus D}\big|d''s\big|_{h_0}^2 < \infty$ from Lemma 13.5.6. Due to the self-adjointness of s, we also obtain the finiteness $\int_{X\smallsetminus D}\big|\delta'_{h_0}s\big|_{h_0}^2 < \infty$.

We take a universal covering $\psi:\mathbb{H}\to\Delta^*$ given by $\psi(\zeta)=\exp(2\pi\sqrt{-1}\zeta)$. We put $K_n := \big\{\zeta\in\mathbb{H}\,\big|\,-1<\operatorname{Re}\zeta<1,\ n-1<\operatorname{Im}(\zeta)<n+1\big\}$. Let θ be the Higgs field corresponding to the harmonic bundle $(E,\mathbb{D}^\lambda,h)$. We have $\theta=\theta_{h_0}+s^{-1}\delta'_{h_0}s$. Due to the L^2-property of $\delta'_{h_0}s$ and the estimate $|s^{-1}|_{h_0}\leqslant C_3|z|^{-N_3}$, we obtain

$$\int_{K_n}\big|\psi^*\theta\big|_{h_0}\leqslant C_{12}\,e^{C_{13}n}.$$

Since h and h_0 are mutually bounded up to polynomial order of $|z|^{-1}$, we obtain

$$\int_{K_n}\big|\psi^*\theta\big|_h^2\leqslant C_{14}\,e^{C_{15}n}. \tag{314}$$

Recall that $|\theta|^2$ is the energy function for a harmonic map up to positive constant multiplication. By using a result in [**27**], we obtain $|\theta|_h^2\leqslant C_{16}|z|^{-C_{17}}$ from (314). In particular, θ is wild. □

Corollary 13.5.7. — *Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda,h)$ be as in Proposition* 13.5.3. *Then, the norm estimate holds for h.* □

13.5.3. Curve test in the smooth divisor case. — We put $X:=\Delta^n$ and $D_i:=\{z_i=0\}$. Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X,D_1). The restriction to $X\smallsetminus D_1$ is denoted by $(E,\mathbb{D}^\lambda)$. Let h be a pluri-harmonic metric of $(E,\mathbb{D}^\lambda)$ on $X\smallsetminus D_1$, and let $(E,\overline{\partial}_E,\theta)$ be the associated Higgs bundle. Let $\pi_i: X\to D_i$ denote the projection. For each $P\in D_1$, we put $\pi_1^{-1}(P)^*:=\pi_1^{-1}(P)\smallsetminus\{P\}$.

Proposition 13.5.8. — *Assume that there exists a subset $Z\subset D_1$ with the following properties:*
- *The Lebesgue measure of Z is* 0.
- *For any $P\in D_1\smallsetminus Z$, the restriction $h_{|\pi_1^{-1}(P)^*}$ is adapted to $\boldsymbol{E}_{*|\pi_1^{-1}(P)}$.*

Then, the following holds:
- *$(E,\overline{\partial}_E,\theta,h)$ is good wild harmonic bundle, and h is adapted to $\boldsymbol{E}_*$, i.e., $\boldsymbol{E}_*=\mathcal{P}_*\mathcal{E}^\lambda$.*

- *If* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *is unramified,* $(E, \overline{\partial}_E, \theta, h)$ *is also unramified, and*
$$\mathrm{Irr}(\theta) = \{(1+|\lambda|^2)^{-1}\mathfrak{a} \mid \mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)\}.$$

Proof. — In the following, we will shrink X without mention if it is necessary. By taking a ramified covering, we may assume that $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is unramified. We divide the proof into several steps.

13.5.3.1. Decomposition of (E, θ). — We have the expression $\theta = \sum f_i \, dz_i$. By Proposition 13.5.3 and the assumption, $(E, \overline{\partial}_E, \theta, h)_{|\pi_1^{-1}(Q)}$ are wild harmonic bundles for any $Q \in D_1 \setminus Z$. Hence, $P(t) = \det(t - z_1 f_1)$ is contained in $M(X, D_1)[t]$. We put $T := \{(1+|\lambda|^2)^{-1} z_1 \partial_1 \mathfrak{a} \mid \mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)\}$. According to Theorem 7.4.5, the following holds:

- Take any $Q \in D_1 \setminus Z$. Let α be a multi-valued meromorphic function on $\pi_1^{-1}(Q)$, which is a solution of $P(t)_{|\pi_1^{-1}(Q)} = 0$. Then, there exists $\mathfrak{b}(\alpha, Q) \in T$ such that the following holds:
$$\left|\mathfrak{b}(\alpha, Q)_{|\pi_1^{-1}(Q)} - \alpha\right| = O(1).$$

In other words, the assumption made in Section 13.5.6.1 (Appendix below) is satisfied by the good set of irregular values T. Applying Proposition 13.5.16 below, we obtain the decomposition $P(t) = \prod_{\mathfrak{b} \in T} P_{\mathfrak{b}}(t - \mathfrak{b})$ such that $P_{\mathfrak{b}}(t) \in H(X)[t]$. By using it, we obtain the decomposition

$$(E, f_1) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)} (E'_{\mathfrak{a}}, f_{1,\mathfrak{a}}), \tag{315}$$

such that $\beta - (1+|\lambda|^2)^{-1} z_1 \partial_z \mathfrak{a}$ is bounded for any solution β of $\det(t - z_1 f_{1,\mathfrak{a}}) = 0$. In particular, the restrictions $(E, \overline{\partial}_E, \theta, h)_{|\pi_1^{-1}(P)}$ are wild harmonic bundles for any $P \in D_1$. The decomposition (315) is preserved by f_i $(i = 2, \ldots, n)$ due to the commutativity $[f_i, f_1] = 0$. Note the decomposition (315) is holomorphic with respect to $\overline{\partial}_E$, but not necessarily holomorphic with respect to the $(0,1)$-part of $\mathbb{D}^\lambda$. Let $\pi_{\mathfrak{a}}$ denote the projection onto $E'_{\mathfrak{a}}$ with respect to (315).

Lemma 13.5.9. — *For each* $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)$, *we put* $f_{1,\mathfrak{a}}^{\mathrm{reg}} := f_{1,\mathfrak{a}} - (1+|\lambda|^2)^{-1} \partial_1 \mathfrak{a} \, \pi_{\mathfrak{a}}$. *Then,* $\det(t - z_1 f_{1,\mathfrak{a}}^{\mathrm{reg}})$ *are contained in* $H(X)[t]$, *and the coefficients of the restriction* $\det(t - z_1 f_{1,\mathfrak{a}}^{\mathrm{reg}})_{|D_1}$ *are constant.*

Proof. — The first claim follows from the construction. The second claim follows from Proposition 8.2.1 and the assumption $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is good. □

13.5.3.2. Norm estimate for $(\boldsymbol{E}_*, \mathbb{D}^\lambda, h)$. — Recall that Simpson's main estimate in Section 7.2 depends only on the behaviour of the eigenvalues of the Higgs field. Due to Corollary 7.2.10, the curvature $R(h_{|\pi_1^{-1}(P)})$ of $(E, \overline{\partial}_E, h)_{|\pi_1^{-1}(P)}$ is bounded with respect to $h_{|\pi_1^{-1}(P)}$ and the Poincaré metric $g_{\mathbf{P}}$ of $\pi_1^{-1}(P)^*$, and the estimate is uniform in $P \in D_1$.

As in Section 13.3.2, we take good filtered flat bundles $(\boldsymbol{E}_{\mathfrak{a}*}, \mathbb{D}^{\lambda}_{\mathfrak{a}})$ such that $\mathbb{D}^{\lambda}_{\mathfrak{a}}$ is $\mathfrak{a}$-regular for each $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$, and an isomorphism $\psi_N : \bigoplus \boldsymbol{E}_{\mathfrak{a}*} \simeq \boldsymbol{E}_*$ which is an approximation of the irregular decomposition in N-th order. Then, we take an auxiliary metric h_0 of $(E, \mathbb{D}^{\lambda})$, which has the property in Lemma 13.3.4. By taking a ramified covering, we may assume that $G(\mathbb{D}^{\lambda}, h_0)$ is bounded with respect to h_0 and the Euclidean metric of X. Let s be determined by $h = h_0\, s$. Let Δ^{λ}_i denote $-(1+|\lambda|^2)\partial_{z_i}\partial_{\overline{z}_i}$.

Lemma 13.5.10. — *s and s^{-1} are bounded with respect to h_0. Namely, h and h_0 are mutually bounded. In particular, h is adapted to $\boldsymbol{E}_*$, and more strongly, the norm estimate holds for $(\boldsymbol{E}_*, \mathbb{D}^{\lambda}, h)$.*

Proof. — Due to $G(h, \mathbb{D}^{\lambda}) = 0$, the boundedness of $G(h_0, \mathbb{D}^{\lambda})$ and an inequality in Subsection 2.2 of [**69**], we have a constant A such that the following holds on $\pi_1^{-1}(P)^*$, independently of $P \in D_1$:

$$\Delta^{\lambda}_1 \log \mathrm{tr}(s_{|\pi_1^{-1}(P)}) \leqslant A, \qquad \Delta^{\lambda}_1 \log \mathrm{tr}(s^{-1}_{|\pi_1^{-1}(P)}) \leqslant A.$$

Since we already know the boundedness of $s_{|\pi_1^{-1}(P)^*}$ and $s^{-1}_{|\pi_1^{-1}(P)^*}$ for any $P \in D_1$, the inequality holds on $\pi_1^{-1}(P)$ in the sense of distributions for such P. Hence $\big|s_{|\pi_1^{-1}(P)^*}\big|_{h_0}$ and $\big|s^{-1}_{|\pi_1^{-1}(P)^*}\big|_{h_0}$ are dominated by their values at $\partial\pi_1^{-1}(P)$ for any $P \in D_1$. Hence, we obtain the uniform boundedness of s and s^{-1}. □

13.5.3.3. Estimate for f_i $(i = 2, \dots, n)$. — We put $F_i' := \sum_{\mathfrak{a}}(1+|\lambda|^2)^{-1}\partial_i\mathfrak{a}\,\pi_{\mathfrak{a}}$ and $f_i^{\mathrm{reg}} := f_i - F_i'$ for $i > 1$. We would like to show that $\det(t - f_i^{\mathrm{reg}})$ is contained in $H(X)[t]$.

Lemma 13.5.11. — *Let $p_{\mathfrak{a},N}$ denote the projection onto $\psi_N(E_{\mathfrak{a}})$ with respect to the decomposition $E = \bigoplus \psi_N(E_{\mathfrak{a}})$. Then, we have the estimate $p_{\mathfrak{a},N} - \pi_{\mathfrak{a}} = O\big(|z_1|^{N/2}\big)$ with respect to h.*

Proof. — We already know $\big|\pi_{\mathfrak{a}} - p_{\mathfrak{a},N}\big|_{|\pi_1^{-1}(P)} = O(|z_1|^N)$ for each P (Proposition 7.5.1). Let $\overline{\partial}_{E,1}$ denote the restriction of $\overline{\partial}_E$ to the z_1-direction. Because the constants in Simpson's Main estimate (Section 7.2) depend only on the behaviour of the eigenvalues, we also have the following estimate on $X \setminus D_1$ with respect to h and the Poincaré metric of $X \setminus D_1$, for some $\varepsilon > 0$:

$$\big(\overline{\partial}_{E,1} + \lambda\, f_1^{\dagger}\, d\overline{z}_1\big)(\pi_{\mathfrak{a}} - p_{\mathfrak{a},N}) = \lambda\,\big[f_1^{\dagger}\, d\overline{z}_1,\ \pi_{\mathfrak{a}}\big] = O\big(\exp(-\varepsilon|z_1|^{-1})\big).$$

Then, the claim follows from Lemma 21.9.2 below and the uniform boundedness $\big|R(h_{|\pi_1^{-1}(P)})\big|_{h,g_{\mathrm{p}}} < C$. □

Corollary 13.5.12. — *We put $F_i := \sum_{\mathfrak{a}}(1+|\lambda|^2)^{-1}\partial_i\mathfrak{a}\, p_{\mathfrak{a},N}$ for $i > 1$. Then,*

$$F_i - F_i' = O(|z_1|^{N/3}).$$

Proof. — It follows from Lemma 13.5.11. □

Let θ_0 denote the section of $\mathrm{End}(E)\otimes\Omega^{1,0}$ induced by h_0 and $\mathbb{D}^\lambda$. We have the expression $\theta_0=\sum_{i=1}^n f_{0,i}\,dz_i$. Note the following estimate for $i>1$ with respect to h (Lemma 13.3.4):

$$f_{0,i}-F_i=O(1). \tag{316}$$

Lemma 13.5.13. — *There exists a subset $Z_1\subset D_1$ whose measure is 0, such that the following finiteness holds for any $P\in D_1-Z_1$:*

$$\int_{\pi_1^{-1}(P)}\bigl|f_i-f_{0,i}\bigr|_h^2\,\mathrm{dvol}_{g_{\mathbf{P}}}<\infty. \tag{317}$$

Here $g_{\mathbf{P}}$ denote the Poincaré metric of the punctured disc $\pi_1^{-1}(P)^*$.

Proof. — Let $\mathbb{D}_i^\lambda$ denote the restriction of $\mathbb{D}^\lambda$ to the z_i-direction. We have the equality

$$\Delta_i^\lambda\,\mathrm{tr}(s)=\bigl(s,A_i\bigr)_{h_0}-\bigl|\mathbb{D}_i^\lambda s\cdot s^{-1}\bigr|_{h_0}^2,$$

for some bounded section A_i of $\mathrm{End}(E)$. (See Section 2.2.5 of [**69**], for example.) Let χ be a cut function on a disc. For any $Q\in D_i\setminus D_1$, we have the following equality on $\pi_i^{-1}(Q)$:

$$\Delta_i^\lambda\,\mathrm{tr}\bigl(\chi\,s\bigr)=\chi\,(A_i,s)_h-\chi\,\bigl|\mathbb{D}_i^\lambda s\cdot s^{-1/2}\bigr|^2+2(1+|\lambda|^2)\,\mathrm{Re}\left(\frac{\partial\chi}{\partial z_i}\frac{\partial\,\mathrm{tr}(s)}{\partial\overline{z}_i}\right)+(\Delta_i^\lambda\chi)\,\mathrm{tr}(s).$$

Therefore, we obtain

$$\begin{aligned}0=\int_{\pi_i^{-1}(Q)}\chi\,(A_i,s)_h-\int_{\pi_i^{-1}(Q)}\chi\,|\mathbb{D}_i^\lambda s\cdot s^{-1/2}|^2\\+\int_{\pi_i^{-1}(Q)}2(1+|\lambda|^2)\,\mathrm{Re}\left(\frac{\partial\chi}{\partial z_i}\frac{\partial\,\mathrm{tr}\,s}{\partial\overline{z}_i}\right)+\int_{\pi_i^{-1}(Q)}(\Delta_i^\lambda\chi)\,\mathrm{tr}(s).\end{aligned}$$

Thus, we obtain the following inequality for some constants B_j $(j=1,2,3)$:

$$\begin{aligned}\int_{\pi_i^{-1}(Q)}\chi\,|\mathbb{D}_i^\lambda s|_h^2\leqslant B_1\int_{\pi_i^{-1}(Q)}\bigl(\chi+|\Delta_i^\lambda\chi|\bigr)\,|s|+B_2\int_{\pi_i^{-1}(Q)}|\partial_i\chi|\cdot|\mathbb{D}_i^\lambda s|_h\\ \leqslant B_3+B_2\int_{\pi_i^{-1}(Q)}|\partial_i\chi|\cdot|\mathbb{D}_i^\lambda s|_h.\end{aligned}$$

Let χ be such that (i) $\chi(r)=1$ for $r\leqslant r_0/3$, (ii) $\chi(r)=\exp\bigl(-(r-r_0)^{-1}\bigr)$ for $r_0-\eta<r<r_0$ $(\eta>0)$, (iii) $\chi(r)=0$ for $r\geqslant r_0$. Then, $(\partial\chi)\,\chi^{-1/2}$ is bounded.

Hence, we obtain the following inequality for some constants B_j $(j=4,5)$:

(318)
$$\int_{\pi_i^{-1}(Q)} \chi \left|\mathbb{D}_i^\lambda s\right|_h^2 \leqslant B_3 + B_4\Big(\int_{\pi_i^{-1}(Q)} |(\partial_i\chi)\,\chi^{-1/2}|^2\Big)^{1/2}\Big(\int_{\pi_i^{-1}(Q)} \chi\,|\mathbb{D}_i^\lambda s|_h^2\Big)^{1/2} \leqslant B_3 + B_5\Big(\int_{\pi_i^{-1}(Q)} \chi\,|\mathbb{D}_i^\lambda s|_h^2\Big)^{1/2}.$$

Hence, we have the uniform boundedness of the integrals $\int_{\pi_i^{-1}(Q)}\left|\mathbb{D}_i^\lambda s\right|_h^2$ for $i = 2,\dots,n$ and for $Q\in D_i\setminus D_1$, when we shrink X. Let d'' denote the $(0,1)$-part of $\mathbb{D}^\lambda$, and let δ'_{h_0} denote the $(1,0)$-operator induced by h_0 and d''. The restriction of δ'_{h_0} to the z_i-direction is denoted by $\delta'_{h_0,i}$. Since s is self-adjoint with respect to h_0, we obtain the uniform boundedness of the integrals $\int_{\pi_i^{-1}(Q)}|\delta'_{h_0,i}s|_h^2$ for $i=2,\dots,n$ and for $Q\in D_i\setminus D_1$.

Since we have the relation $\theta-\theta_0 = -(1+|\lambda|^2)^{-1}\,s^{-1}\delta'_{h_0}s$, we obtain the uniform boundedness of the integrals $\int_{\pi_i^{-1}(Q)}\left|f_i-f_{0,i}\right|^2|dz_i\,d\overline{z}_i|$ for $i=2,\dots,n$ and for $Q\in D_i\setminus D_1$. Then, the claim of Lemma 13.5.13 follows from Fubini's theorem. □

Let us complete the proof of Proposition 13.5.8. We obtain the following finiteness for any $P\in D_1\setminus Z_1$ and $i>1$, due to Corollary 13.5.12, the estimate (316) and Lemma 13.5.13:

(319)
$$\int_{\pi_1^{-1}(P)}\left|f_i^{\mathrm{reg}}\right|_h^2\,\mathrm{dvol}_{g_{\mathrm{P}}}<\infty.$$

Note that f_i^{reg} are holomorphic with respect to $\overline{\partial}_E$. Hence, we obtain the boundedness of $(f_i^{\mathrm{reg}})_{|\pi_1^{-1}(P)}$ $(i=2,\dots,n)$ for any $P\in D_1\setminus Z_1$ from (319), by using the norm estimate for wild harmonic bundles on curves.

By construction, we have $[\theta,f_i^{\mathrm{reg}}]=0$. Hence, we have the inequality $\Delta_1|f_i^{\mathrm{reg}}|_h^2\leqslant 0$ on $\pi_1^{-1}(P)^*$ for any $P\in D_1$ (Lemma 4.1 of [**82**]), where $\Delta_1=-\partial_{z_1}\partial_{\overline{z}_1}$. Since we already know the boundedness of $|f_i^{\mathrm{reg}}|_h^2$ on $\pi_1^{-1}(P)$ for any $P\in D_1\setminus Z_1$, we obtain $\Delta_1|f_i^{\mathrm{reg}}|_h^2\leqslant 0$ on $\pi_1^{-1}(P)$ as distributions, for any $P\in D_1\setminus Z_1$. Hence, the values of $|f_i^{\mathrm{reg}}|_h^2$ on $\pi_1^{-1}(P)^*$ are dominated by the values on $\partial\pi_1^{-1}(P)^*$ for $P\in D_1\setminus Z_1$. Then, we obtain the boundedness of $|f_i^{\mathrm{reg}}|_h^2$ on $X\setminus D$ by using the continuity. As a result, we obtain $\det(t-f_i^{\mathrm{reg}})\in H(X)[t]$. Together with Lemma 13.5.9, we can conclude that θ is good and wild. Thus, the proof of Proposition 13.5.8 is finished. □

13.5.4. Proof of Proposition 13.5.1. — We may and will assume that $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ is unramified. We will replace X with a small neighbourhood of the origin O without mention. We assume that the coordinate system is admissible for $\mathrm{Irr}(\mathbb{D}^\lambda) := \mathrm{Irr}(\mathbb{D}^\lambda,O)$. We have the expression $\theta=\sum_{i=1}^n f_i\,dz_i$. Due to Proposition 13.5.8, we already know that $\det(t-z_j\,f_j)\in M(X,D)[t]$ $(j=1,\dots,\ell)$ and $\det(t-f_j)\in M(X,D)[t]$ $(j=\ell+1,\dots,n)$.

Lemma 13.5.14. — *Let α be a multi-valued meromorphic function on X, which is a solution of* $\det(t - z_1 f_1) = 0$. *Then, there exists* $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)$ *such that*

$$\big|(1+|\lambda|^2)\alpha - z_1\partial_1\mathfrak{a}\big| = O(1).$$

Proof. — We put $T := \big\{(1+|\lambda|^2)^{-1} z_1\partial_1\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\mathbb{D}^\lambda)\big\}$. Due to Proposition 13.5.8, the polynomial $\det(t - z_1 f_1)$ satisfies the assumption made in Section 13.5.6.1 (Appendix below) with the good subset T. Hence, the claim of the lemma follows from Proposition 13.5.16 below. □

By Lemma 13.5.14, we obtain the decomposition $(E, f_1) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D}^\lambda)}(E_\mathfrak{a}, f_{1,\mathfrak{a}})$ such that the eigenvalues of

$$z_1 f_{1,\mathfrak{a}} - (1+|\lambda|^2)^{-1} z_1\partial_1\mathfrak{a}\ \mathrm{id}_{E_\mathfrak{a}}$$

are bounded. We have the corresponding decomposition $f_i = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathbb{D}^\lambda)} f_{i,\mathfrak{a}}$. We set

$$f^{\mathrm{reg}}_{i,\mathfrak{a}} := f_{i,\mathfrak{a}} - (1+|\lambda|^2)^{-1}\partial_i\mathfrak{a}\ \mathrm{id}_{E_\mathfrak{a}}\,.$$

Lemma 13.5.15. — $\det(t - z_i f^{\mathrm{reg}}_{i,\mathfrak{a}}) \in H(X)[t]$ *for* $i = 1, \ldots, \ell$, *and* $\det(t - f^{\mathrm{reg}}_{i,\mathfrak{a}}) \in H(X)[t]$ *for* $i = \ell+1, \ldots, n$.

Proof. — Let $1 \leqslant i \leqslant \ell$ and let $Q \in D_i^\circ$. Let X_Q be a small neighbourhood of Q such that $D_Q = X_Q \cap D$ is smooth. Let $\mathrm{Irr}(\mathbb{D}^\lambda, i)$ denote the image of $\mathrm{Irr}(\mathbb{D}^\lambda)$ by the map $M(X,D)/H(X) \to M(X,D)/M(X,D(\neq i))$, where $D(\neq i) := \bigcup_{1\leqslant j\leqslant \ell, j\neq i} D_i$. Due to Proposition 13.5.8, we have the decomposition $(E,\theta)_{|X_Q} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\mathbb{D}^\lambda,i)}(E_{\mathfrak{b},Q}, \theta_{\mathfrak{b},Q})$ with the following property:

- For the expression $\theta_{\mathfrak{b},Q} = \sum f_{\mathfrak{b},i,Q}\, dz_i$, we put $f^{\mathrm{reg}}_{\mathfrak{b},i,Q} := f_{\mathfrak{b},i,Q} - \partial_i\mathfrak{b}\ \mathrm{id}_{E_{\mathfrak{b},Q}}$. Then, the characteristic polynomials of $\det(t - z_i f^{\mathrm{reg}}_{\mathfrak{b},i,Q})$ and $\det(t - f^{\mathrm{reg}}_{\mathfrak{b},j,Q})$ $(j \neq i)$ are contained in $H(X_Q)[t]$.

We can observe $E_{\mathfrak{b},Q} = \bigoplus_{\mathfrak{a}\mapsto\mathfrak{b}} E_{\mathfrak{a}|X_Q\setminus D_Q}$ by the comparison of the eigenvalues of $z_1 f_1$. Hence, we obtain that $\det(t - z_i f_i^{\mathrm{reg}})$ $(i = 1, \ldots, \ell)$ and $\det(t - f_i^{\mathrm{reg}})$ $(i = \ell+1, \ldots, n)$ are contained in $H(X_Q)[t]$. Then, the claim of Lemma 13.5.15 follows from the Hartogs theorem. □

By using Proposition 8.2.1, we can show that the coefficients of $\det\big(t - z_i f^{\mathrm{reg}}_{i,\mathfrak{a}}\big)_{|D_i}$ are constants for $i = 1, \ldots, \ell$. Hence, $(E, \overline{\partial}_E, \theta, h)$ is unramifiedly good wild, and we have (310). We have the filtered λ-flat bundles $\mathcal{P}_*\mathcal{E}^\lambda$ obtained as in Section 7. By the assumption, we have $\mathcal{P}_*\mathcal{E}^\lambda_{|\pi_i^{-1}(P)} = \boldsymbol{E}_{*|\pi_i^{-1}(P)}$ for $i = 1, \ldots, \ell$ and $P \in D_i^\circ \setminus Z_i$. We can deduce $\mathcal{P}_*\mathcal{E}^\lambda = \boldsymbol{E}_*$. Thus, the proof of Proposition 13.5.1 is finished. □

13.5.5. Proof of Proposition 13.5.2. — We may assume $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is unramified. Due to Proposition 13.5.1, we only have to show the first claim. We put $(\boldsymbol{E}_{Q\,*}, \mathbb{D}^\lambda_Q) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|\pi_i^{-1}(Q)}$ and $E_Q := E_{|\pi_i^{-1}(Q)}$ for any $Q \in D_i^\circ$. We take an auxiliary metric h_Q for $(\boldsymbol{E}_{Q\,*}, \mathbb{D}^\lambda_Q)$ as in Section 13.3.2. We also assume $\det(h_Q) = \det(h_{|\pi_i^{-1}(Q)})$ by an obvious modification.

Let s_Q be determined by $h_{|\pi_i^{-1}(Q)} = h_Q\, s_Q$. Because h is adapted to $\boldsymbol{E}_*$, we have $|s_Q|_{h_Q} = O\big(|z_i|^{-\varepsilon}\big)$ for any $\varepsilon > 0$. Then, we can show $|s_Q|_{h_Q}$ is bounded by the same argument as that in the proof of Lemma 13.5.4. Because $\det(s_Q) = 1$, we also obtain that s_Q^{-1} is bounded with respect to h_Q. Then, we can conclude that $h_{|\pi_i^{-1}(Q)}$ is adapted to $\boldsymbol{E}_{Q,*}$, and the proof of Proposition 13.5.2 is finished. □

13.5.6. Decomposition of polynomials (Appendix)

13.5.6.1. Statement. — Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. We put $D_i^\circ := D_i \setminus \bigcup_{1\leqslant j\leqslant \ell, j\neq i} D_j$. Let π_i denote the projection of X to D_i. Let $P(\boldsymbol{z},t) = \sum_{j=0}^{r} a_j(\boldsymbol{z})\, t^j \in M(X,D)[t]$ be a monic polynomial in the variable t with $M(X,D)$-coefficients. Assume that we are given a good set of irregular values $T \subset M(X,D)$ (Definition 2.1.2) and subsets $Z_i \subset D_i$ ($i = 1,\dots,\ell$) such that the following holds for each i:

- The Lebesgue measure of Z_i is 0.
- Take any $Q \in D_i^\circ \setminus Z_i$. Let α be a multi-valued meromorphic function on $\pi_i^{-1}(Q)$, which is a solution of $P(t)_{|\pi_i^{-1}(Q)} = 0$. Then, there exists $\mathfrak{a}(\alpha,Q) \in T$ such that the following holds:
$$\big|\mathfrak{a}(\alpha,Q)_{|\pi_i^{-1}(Q)} - \alpha\big| = O(1).$$
In other brief words, α is decomposed into a multi-valued holomorphic function and the polar part $\mathfrak{a}(\alpha,Q)_{|\pi_i^{-1}(Q)}$.

We will prove the following proposition in Sections 13.5.6.2–13.5.6.3.

Proposition 13.5.16. — *We have a splitting $P(\boldsymbol{z},t) = \prod_{\mathfrak{a}\in T} P_{\mathfrak{a}}(\boldsymbol{z},t)$ in $M(X,D)[t]$ with the following properties:*

- *Each $P_{\mathfrak{a}}(\boldsymbol{z},t)$ is a monic in the variable t.*
- *Let α be a multi-valued meromorphic section on X which is a solution of $P_{\mathfrak{a}}(\boldsymbol{z},t) = 0$. Then, $\alpha - \mathfrak{a}$ is bounded. In other words, $P_{\mathfrak{a}}(\boldsymbol{z}, t-\mathfrak{a}) \in H(X)[t]$.*

13.5.6.2. Preliminaries. — Let $X := \Delta^n$ and $D := D_1$. Let $P \in M(X,D)[t]$ be a monic polynomial.

Lemma 13.5.17. — *Assume that there exists a subset $Z \subset D$ whose measure is 0, such that the following condition holds:*

- *Let $Q \in D \setminus Z$. For each root α of $P_{|\pi_1^{-1}(Q)}$, we have $|\alpha| = O(1)$.*

Then, the coefficients of P are holomorphic, and all roots of P are bounded on X.

Proof. — Let $P(\boldsymbol{z},t) = \sum a_j(\boldsymbol{z})\, t^j$. We obtain the boundedness of $a_{j|\pi_1^{-1}(Q)}$ for any $Q \in D \setminus Z$. Then, we obtain that a_j are holomorphic on X. □

Assume that we are given a good set of irregular values $S = \big\{\mathfrak{a} = \mathfrak{a}_m\, z_1^m\big\} \subset M(X,D)$ at the level m, and a subset $Z \subset D$ with the following properties:

- The Lebesgue measure of Z is 0.

- Let $Q \in D \setminus Z$. Let α be a root of $P_{|\pi_1^{-1}(Q)}$. Then, there exists $\mathfrak{a}(\alpha) \in S$ such that the following holds:

$$\alpha - \mathfrak{a}(\alpha)_{|\pi_1^{-1}(Q)} = O(|z_1|^{m+1}).$$

Lemma 13.5.18. — *We have a decomposition $P = \prod_{\mathfrak{a}\in S} P_{\mathfrak{a}}$ into monic polynomials with the following property:*

- $\bigl|\alpha - \mathfrak{a}\bigr| = O\bigl(|z_1|^{m+1}\bigr)$ *for each root α of $P_{\mathfrak{a}}$.*

Proof. — Let $r := \deg_t P$. We set $Q(\boldsymbol{z}, t) = z_1^{-rm} P(\boldsymbol{z}, z_1^m t)$. By Lemma 13.5.17, we obtain the expansion $Q(\boldsymbol{z}, t) = \sum_{j\geqslant 0} Q^j(\boldsymbol{z}, t)\, z_1^j$. We have the decomposition $Q^0 = \prod_{\mathfrak{a}\in S} Q^0_{\mathfrak{a}}$. It can be lifted to a decomposition $Q = \prod_{\mathfrak{a}\in S} Q_{\mathfrak{a}}$. By applying Lemma 13.5.17 to $Q_{\mathfrak{a}}\bigl(z_1^{-1}(t-\mathfrak{a})\bigr)$, we obtain that the coefficients of $Q_{\mathfrak{a}}\bigl(z_1^{-1}(t-\mathfrak{a})\bigr)$ are holomorphic on X. Hence, the induced decomposition $P = \prod_{\mathfrak{a}\in S} P_{\mathfrak{a}}$ has the desired property. □

13.5.6.3. Proof of Proposition 13.5.16. — We use an induction on $|T|$. Let $\boldsymbol{m}(0) := \min\bigl\{\mathrm{ord}(\mathfrak{a}) \bigm| \mathfrak{a} \in T\bigr\}$. In the case $\bigl|\eta_{\boldsymbol{m}(0)}(T)\bigr| = 1$, we pick an element $\mathfrak{a}_0 \in T$, and we put $\widetilde{P}(t) = P(t - \mathfrak{a}_0)$. Then, the polynomial $\widetilde{P}$ satisfies the assumption in Section 13.5.6.1 with the good set of irregular values $\widetilde{T} := \bigl\{\mathfrak{a} - \mathfrak{a}_0 \bigm| \mathfrak{a} \in T\bigr\}$ and we have $\min\bigl\{\mathrm{ord}(\mathfrak{b}) \bigm| \mathfrak{b} \in \widetilde{T}\bigr\} \gneq \boldsymbol{m}(0)$. Hence, we can reduce the problem to the case $\bigl|\eta_{\boldsymbol{m}}(T)\bigr| \geqslant 2$, which we will assume in the following argument.

We put $\mathcal{P} := \boldsymbol{z}^{-r\boldsymbol{m}(0)} P(\boldsymbol{z}^{\boldsymbol{m}(0)} t)$, which is monic in $M(X, D)[t]$. It can be shown that it is contained in $H(X)[t]$ by using Lemma 13.5.17 and Hartogs theorem. Let k be determined by $\boldsymbol{m}(0) \in \mathbb{Z}^k_{<0}$. Let $D_I := \bigcap_{i\in I} D_i$ for $I \subset \{1, \dots, k\}$. We set

$$T_I := \bigl\{(\boldsymbol{z}^{-\boldsymbol{m}(0)}\mathfrak{a})_{|D_I} \bigm| \mathfrak{a} \in T\bigr\}.$$

For $I \subset J \subset \{1, \dots, k\}$, we have the natural map $\phi_{I,J} : T_I \to T_J$ induced by the restriction to D_J.

Let Q be a point of $D_j^\circ := D_j \setminus \bigcup_{i\neq j, 1\leqslant i\leqslant k} D_i$. Let U_Q be a small neighbourhood of Q in X. By using Lemma 13.5.18, we obtain the decomposition $\mathcal{P}(t) = \prod_{\mathfrak{b}\in T_j} \mathcal{P}_{\mathfrak{b},Q}(t)$ on U_Q, such that $\mathcal{P}_{\mathfrak{b},Q}(t)_{|D_j}$ has the unique root $\mathfrak{b}$. For $\mathfrak{c} \in T_{\underline{\ell}}$, we put

$$\mathcal{P}_{\mathfrak{c},Q}(t) := \prod_{\phi_{j,\underline{\ell}}(\mathfrak{b})=\mathfrak{c}} \mathcal{P}_{\mathfrak{b},Q}(t).$$

By varying Q in D_j°, we obtain the decomposition $\mathcal{P} = \prod_{\mathfrak{c}\in T_{\underline{\ell}}} \mathcal{P}_{\mathfrak{c},j}(t)$ around D_j°. Because of $\mathcal{P} \in H(X)[t]$, we have $\mathcal{P}_{\mathfrak{c},j} \in H(X)[t]$, and the decomposition holds on X. Since $\mathcal{P}_{\mathfrak{c},j}(t)_{|D_{\underline{\ell}}}$ has the unique root $\mathfrak{c}$, we obtain that $\mathcal{P}_{\mathfrak{c},j}$ is independent of j, which we denote by $\mathcal{P}_{\mathfrak{c}}$.

We put $P_{\mathfrak{c}}^{\boldsymbol{m}(0)} := \boldsymbol{z}^{r\boldsymbol{m}(0)} \mathcal{P}_{\mathfrak{c}}\bigl(\boldsymbol{z}^{-\boldsymbol{m}(0)} t\bigr)$ and $T_{\mathfrak{c}} := \bigl\{\mathfrak{a} \in T \bigm| \eta_{\boldsymbol{m}}(\mathfrak{a}) = \mathfrak{c}\, \boldsymbol{z}^{\boldsymbol{m}(0)}\bigr\}$. Then, $P_{\mathfrak{c}}^{\boldsymbol{m}(0)} \in M(X, D)[t]$ has the following property:

- Take any $Q \in D_j^\circ$. Let α be a solution of $P_{\mathfrak{c}}^{\boldsymbol{m}(0)}(t)_{|\pi_j^{-1}(Q)} = 0$, which is a multi-valued meromorphic function on $\pi_j^{-1}(Q)$. Then, there exists $\mathfrak{a}(\alpha, Q) \in T_{\mathfrak{c}}$ such that the following holds:
$$\left|\mathfrak{a}(\alpha, Q)_{|\pi_j^{-1}(Q)} - \alpha\right| = O(1).$$

By applying the inductive assumption, we obtain the desired decompositions for $P_{\mathfrak{c}}^{\boldsymbol{m}(0)}(t)$, and thus for $P(t)$. Hence the induction can proceed, and the proof of Proposition 13.5.16 is finished. □

13.6. The filtered flat bundle associated to wild harmonic bundle

13.6.1. Polystability. — Let X be a connected smooth n-dimensional projective variety with an ample line bundle L. Let D be a simple normal crossing hypersurface. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a filtered λ-flat bundle on (X, D) which is generically good, i.e., we have a Zariski dense open subset D' of D such that $\boldsymbol{E}_*$ is good around every point $P \in D'$. We say that a pluri-harmonic metric h of E is generically adapted to $\boldsymbol{E}_*$, if it is adapted to $\boldsymbol{E}_*$ around P for each point $P \in D'$.

Proposition 13.6.1. — *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a filtered λ-flat sheaf which is saturated and generically good.*

- *Assume that we have a pluri-harmonic metric h of $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X \setminus D}$ which is generically adapted to $\boldsymbol{E}_*$. Then, $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is μ_L-polystable with* $\text{par-deg}_L(\boldsymbol{E}_*) = 0$. *The canonical decomposition (see Section* 13.1.3*) is orthogonal with respect to h. The restriction of h to any stable components of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is also pluri-harmonic.*
- *Let h' be another pluri-harmonic metric of $(E, \mathbb{D}^\lambda)$ generically adapted to $\boldsymbol{E}_*$. Then, we have a decomposition $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = \bigoplus (\boldsymbol{E}_{i\,*}, \mathbb{D}_i^\lambda)$ such that (i) it is orthogonal with respect to both of the metrics h and h', (ii) $h_i = a_i\, h'_i$ for some $a_i > 0$, where h_i and h'_i are the restrictions of h and h' to E_i, respectively. In particular, if $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is μ_L-stable, $h' = a\, h$ for some $a > 0$.*

Proof. — Let us consider the first claim. In the one dimensional case, it can be shown by Simpson's argument in Section 10 of [**81**] and Section 6 of [**82**]. We give only an outline. Note that we already know that $(E, \mathbb{D}^\lambda, h)$ is good wild according to Proposition 13.5.3. In particular, (E, h) is acceptable, which we will implicitly use. Let $(\boldsymbol{W}_*, \mathbb{D}^\lambda)$ be a filtered λ-flat subbundle of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$. We put $W := \boldsymbol{W}_{*|X \setminus D}$. Let h_W be the metric of W induced by h. Let $R(h_W)$ be the curvature of (W, h_W). The analytic degree $\deg^{an}(W)$ is defined to be $(\sqrt{-1}/2\pi) \int_X \operatorname{tr} R(h_W)$. According to Lemma 10.5 of [**81**] (see also Lemma 6.2 of [**82**]), it is equal to $\text{par-deg}(\boldsymbol{W}_*)$. Let π_W be the orthogonal projection of E onto W. By the Chern-Weil formula (see Lemma 2.34 of [**69**] for the Chern-Weil formula for λ-flat bundle), we have the following formula

$$\deg^{an}(W) = \frac{-1}{2\pi(1+|\lambda|^2)} \int_{X\setminus D} \left|\mathbb{D}^\lambda \pi_W\right|_h^2 \leqslant 0.$$

Hence, we obtain that $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is semistable. Moreover, if par-deg$(\boldsymbol{W}_*) = 0$, we have $\mathbb{D}^\lambda \pi_W = 0$. It implies $(E, \mathbb{D}^\lambda, h)$ is decomposed into $(W, \mathbb{D}^\lambda, h_W) \oplus (W', \mathbb{D}^\lambda, h_{W'})$ as harmonic bundles. Then, we obtain the decomposition $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = (\boldsymbol{W}_*, \mathbb{D}^\lambda) \oplus (\boldsymbol{W}'_*, \mathbb{D}^\lambda)$. Hence, $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is polystable.

We can reduce the higher dimensional case to the one dimensional case as follows. By considering the restriction of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ to sufficiently ample and generic curves, we obtain the μ_L-semistability and par-$\deg_L(\boldsymbol{E}_*) = 0$. Let $\boldsymbol{W}_* \subsetneq \boldsymbol{E}_*$ be a saturated filtered subsheaf such that par-$\deg_L(\boldsymbol{W}_*) =$ par-$\deg_L(\boldsymbol{E}_*)$. Let π_W denote the orthogonal projection to W which is defined outside of the subset with codimension two. By considering the restriction to the sufficiently ample general curves, we obtain $\mathbb{D}^\lambda \pi_W = 0$. In particular, π_W is holomorphic. By Hartogs theorem, π_W is defined on whole $X \setminus D$, and $\pi_W^2 = \pi_W$. Hence, we obtain the decomposition $E = W \oplus W'$, where $W' = \operatorname{Ker} \pi_W$. It is orthogonal and flat. Hence, we obtain the decomposition of harmonic bundles $(W, \mathbb{D}^\lambda_W, h_W) \oplus (W', \mathbb{D}^\lambda_{W'}, h_{W'})$.

Let P be a point of D around which $(E, \mathbb{D}^\lambda, h)$ is good. Then, $(W, \mathbb{D}^\lambda_W, h_W)$ and $(W', \mathbb{D}^\lambda_{W'}, h_{W'})$ are also good around P, and the decomposition is prolonged as $\boldsymbol{E}_* = \boldsymbol{W}_* \oplus \boldsymbol{W}'_*$ around P. By Hartogs property, we obtain the decomposition $\boldsymbol{E}_* = \boldsymbol{W}_* \oplus \boldsymbol{W}'_*$ on whole X. Hence, the μ_L-polystability follows. We also obtain that the restriction of h to any μ_L-stable components are pluri-harmonic. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = \bigoplus_{i=1}^{\ell} (\boldsymbol{E}_{i*}, \mathbb{D}^\lambda_i)$ be the canonical decomposition. We also have the decomposition $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = (\boldsymbol{E}_{1\,*}, \mathbb{D}^\lambda_1) \oplus (\boldsymbol{E}^\perp_{1\,*}, \mathbb{D}^{\lambda\perp}_1)$ whose restriction to $X \setminus D$ is orthogonal with respect to h. It is easy to derive $\boldsymbol{E}^\perp_{1\,*} = \bigoplus_{i=2}^{\ell} \boldsymbol{E}_{i*}$. Hence the orthogonality of the canonical decomposition is also obtained.

Let us show the second claim. In the case $\dim X = 1$, h and h' are mutually bounded due to the norm estimate. Therefore, the claim can be shown using the argument in the proof of Proposition 5.2 of [**66**]. Let us consider the higher dimensional case. For every point $P \in X \setminus D$, we have s_P such that $h'_{|P} = h_{|P}\, s_P$. By considering the curve case, we can show that s is flat with respect to $\mathbb{D}^\lambda$. Then the claim follows. □

Corollary 13.6.2. — *Let* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *and* h *be as in Proposition* 13.6.1. *Assume that* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *is the tensor product of* μ_L*-stable* $(\boldsymbol{E}_{0*}, \mathbb{D}^\lambda_0)$ *and a vector space* V*. Then,* h *is of the form* $h_0 \otimes g_V$*, where* h_0 *is a pluri-harmonic metric for* $(\boldsymbol{E}_{0*}, \mathbb{D}^\lambda_0)$ *as in Proposition* 13.6.1*, and* g_V *is a metric of* V*.*

Proof. — We take an inclusion $\boldsymbol{E}_{0*} \subset \boldsymbol{E}_*$. By restricting h, we obtain a pluri-harmonic metric h_0 for $(\boldsymbol{E}_{0*}, \mathbb{D}^\lambda_0)$ as in Proposition 13.6.1. By using the second claim of Proposition 13.6.1, we obtain that h is isomorphic to a direct sum of copies of h_0. □

Corollary 13.6.3. — *Let* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *and* h *be as in Proposition* 13.6.1. *Let* $\boldsymbol{E}'_*$ *be a filtered* λ*-flat subbundle of* $\boldsymbol{E}_*$ *such that* $\text{par-deg}_L(\boldsymbol{E}'_*) = 0$. *Let* E'' *be the orthogonal complement of* $E' := \boldsymbol{E}'_{*|X\setminus D}$ *in* E. *Then,* E'' *is naturally extended to a filtered* λ*-flat subbundle* $\boldsymbol{E}''_*$ *of* $\boldsymbol{E}_*$*, and we have* $\boldsymbol{E}_* = \boldsymbol{E}'_* \oplus \boldsymbol{E}''_*$.

Proof. — We can deduce this claim from the orthogonality of the canonical decomposition and Corollary 13.6.2. We can also deduce it directly from the proof of Proposition 13.6.1. □

13.6.2. Vanishing of the characteristic numbers. — Let $(E, \mathbb{D}^\lambda, h)$ be a good wild harmonic bundle on $X \setminus D$. We have the associated good filtered λ-flat bundle $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on (X, D).

Proposition 13.6.4. — *We have the vanishing of the characteristic numbers:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0, \quad \int_X \text{par-c}^2_{1,L}(\boldsymbol{E}_*) = 0.$$

Proof. — We can show it by the essentially same argument as the proof of Proposition 5.3 of [**66**]. We give only an outline with minor simplification. (i.e., we may simplify Lemma 5.4 of [**66**].)

Let $\pi : \widetilde{X} \to X$ be the blow up at crossing points of D. Let $\widetilde{D}$ denote the inverse image of D. Let $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \pi^{-1}(E, \overline{\partial}_E, \theta, h)$. As remarked in Section 11.7.4, $\mathcal{P}_*\widetilde{\mathcal{E}}^\lambda$ is obtained from $\mathcal{P}_*\mathcal{E}^\lambda$ by the procedure in Section 2.5.3.3. Hence, we may only have to consider the integrals over $\widetilde{X}$ by Lemma 14.3.5.

Let $\widetilde{h}_0$ be an ordinary metric for the parabolic bundle $\mathcal{P}_0\widetilde{\mathcal{E}}^\lambda$ as given in Chapter 4 of [**66**], where we considered ordinary metrics for parabolic Higgs bundles with possibly irrational parabolic weight. We apply it in the trivial Higgs field case. Then, we only have to show

$$\int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_0)^2\big) = 0, \quad \int_X \text{tr}\big(R(\widetilde{h}_0)\big)^2 = 0.$$

Let $\widetilde{h}_1$ be the Hermitian metric of $\widetilde{E}$ which is as in Section 11.7.4 around the crossing points of $\widetilde{D}$, and as in Section 4.2.6 of [**66**] around $\widetilde{D}_i$. Then, the following equality can be shown by using Lemmas 4.5 and 4.10 of [**66**] and Lemma 11.7.9:

$$\int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_0)^2\big) = \int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_1)^2\big), \quad \int_X \text{tr}\big(R(\widetilde{h}_0)\big)^2 = \int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_1)\big)^2.$$

We can show the following equalities by using the argument in the pages 62–63 of [**66**] and Lemma 5.2 of [**81**]:

$$\int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_1)^2\big) = \int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h})^2\big) = 0, \quad \int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h}_1)\big)^2 = \int_{\widetilde{X}} \text{tr}\big(R(\widetilde{h})\big)^2 = 0.$$

Thus, we are done. □

13.7. Perturbation

The construction of this section will be used in Sections 14.5 and 16.1. (We have already used such a perturbation for good filtered flat bundle on curves in Sections 13.4.2–13.4.3.) It is essentially the same as that given in Section 2.1.6 of [**69**].

Let X be a complex *surface* with a simple normal crossing divisor $D=\bigcup_{i\in\Lambda}D_i$. Let $(\boldsymbol{E}_*,\nabla)$ be a good filtered flat bundle on (X,D). Let $\boldsymbol{c}\in\boldsymbol{R}^{\Lambda}$ such that $c_i\notin\mathcal{P}ar(\boldsymbol{E}_*,i)$, and let ${}_{\boldsymbol{c}}E$ denote the $\boldsymbol{c}$-truncation. We have the induced filtration ${}^iF$ of ${}_{\boldsymbol{c}}E_{|D_i}$ for each $i\in\Lambda$. We have $\pi_a:{}^iF_a\big({}_{\boldsymbol{c}}E_{|D_i}\big)\longrightarrow{}^i\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$. We have the endomorphism $\mathrm{Res}_i(\nabla)$ on ${}^i\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$. Since the conjugacy classes of $\mathrm{Res}_i(\nabla)_{|P}$ are independent of the choice of $P\in D_i^{\circ}:=D_i\setminus\bigcup_{j\neq i}D_j$, the nilpotent part of $\mathrm{Res}_i(\nabla)$ induces the weight filtration W of ${}^i\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)_{|D_i^{\circ}}$. It can be extended to the filtration of $\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$ in the category of vector bundles. We put ${}^i\widetilde{F}_{(a,k)}:=\pi_a^{-1}(W_k)$, which is the filtration of ${}_{\boldsymbol{c}}E_{|D_i}$ indexed by $\mathcal{S}_i:=\big\{(a,k)\in\,]c_i-1,c_i]\times\mathbb{Z}\,\big|\,\mathrm{Gr}^W_k\big({}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\big)\neq 0\big\}$ with the lexicographic order. For $(\boldsymbol{a},\boldsymbol{k})=\big(a_i,k_i\,\big|\,i\in\Lambda\big)\in\prod\mathcal{S}_i$, we put

$$
{}_{(\boldsymbol{a},\boldsymbol{k})}\widetilde{E}=\mathrm{Ker}\Big({}_{\boldsymbol{c}}E\longrightarrow\bigoplus_{i\in\Lambda}{}_{\boldsymbol{c}}E_{|D_i}/{}^i\widetilde{F}_{(a_i,k_i)}\big({}_{\boldsymbol{c}}E_{|D_i}\big)\Big).
$$

Note that ${}_{(\boldsymbol{a},\boldsymbol{k})}\widetilde{E}$ is a good lattice. If $(\boldsymbol{E}_*,\nabla)$ is unramifiedly good, this claim is obvious. The general case can be reduced to the unramified case.

Let ε be a sufficiently small number. We take an increasing map $\varphi_i:\mathcal{S}_i\longrightarrow\boldsymbol{R}$ such that $|\varphi_i(a,k)-a|\leqslant C\,\varepsilon$ for some $C>0$. (Since we are interested in the family of the filtrations $\boldsymbol{F}^{(\varepsilon)}$ $(\varepsilon>0)$, this condition makes sense.) Then, ${}^i\widetilde{F}$ and φ_i give the $\boldsymbol{c}$-parabolic filtration $\boldsymbol{F}^{(\varepsilon)}=\big({}^iF^{(\varepsilon)}\,\big|\,i\in\Lambda\big)$. Thus, we obtain a good $\boldsymbol{c}$-parabolic flat bundle $\big({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\varepsilon)},\nabla\big)$ which is called a ε-perturbation of $({}_{\boldsymbol{c}}E,\boldsymbol{F},\nabla)$. By construction, we have the following convergence in the cohomology group $H^*(X,\boldsymbol{R})$:

$$
\lim_{\varepsilon\to 0}\mathrm{par\text{-}c}_1({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\varepsilon)})=\mathrm{par\text{-}c}_1({}_{\boldsymbol{c}}E,\boldsymbol{F}),\qquad\lim_{\varepsilon\to 0}\mathrm{par\text{-}ch}_2({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\varepsilon)})=\mathrm{par\text{-}ch}_2({}_{\boldsymbol{c}}E,\boldsymbol{F})
$$

The following proposition is standard. (See Proposition 3.28 of [**66**], for example.)

Proposition 13.7.1. — *Assume that $\big({}_{\boldsymbol{c}}E,\boldsymbol{F},\mathbb{D}^{\lambda}\big)$ is μ_L-stable. If ε is sufficiently small, then the ε-perturbation $\big({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\varepsilon)},\mathbb{D}^{\lambda}\big)$ is also μ_L-stable.* □

We will use two kinds of perturbations φ_i of parabolic weights.

(I) : The image of φ_i is contained in $\boldsymbol{Q}$ for each $i\in\Lambda$ (Section 14.3).

(II) : For simplicity, we assume $\varepsilon=m^{-1}$ and $0<10\,\mathrm{rank}\,E\,\varepsilon<\mathrm{gap}({}_{\boldsymbol{c}}E,\boldsymbol{F})$. (See Section 3.1 of [**66**] for gap.) Let $i\in\Lambda$. For each $a\in\mathcal{P}ar({}_{\boldsymbol{c}}E,\boldsymbol{F})$, we take $a'(\varepsilon,i)\in m^{-1}\mathbb{Z}$ such that $|a'(\varepsilon,i)-a|<m^{-1}$. Let $L(\varepsilon,i)\in\boldsymbol{R}$ be determined by

$$
L(\varepsilon,i)\ \mathrm{rank}(E):=\sum\big(a(\varepsilon,i)'-a\big)\ \mathrm{rank}\,{}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E).
$$

Then, we put $a(\varepsilon,i) := a'(\varepsilon,i) - L(\varepsilon,i)$ and $\varphi(a,k) := a(\varepsilon,i) + k\,\varepsilon$. By construction, we have the following equality:

$$\sum_{a,k} \varphi(a,k)\ \mathrm{rank}\big({}^i\mathrm{Gr}^{\widetilde{F}}_{a,k}({}_{\boldsymbol{c}}E)\big) = \sum_a a\ \mathrm{rank}\big({}^i\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}}E)\big).$$

Hence, we have $\text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\varepsilon)})$. The parabolic structure satisfies the SPW-condition in Definition 2.6 of [**69**]. Namely, for each i, we also have some $-1/m < \gamma_i \leqslant 0$ such that $\mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\varepsilon)}, i)$ is contained in $\big\{c_i + \gamma_i + p/m \,\big|\, p \in \mathbb{Z}_{\leqslant 0},\ -1 < \gamma_i + p/m \leqslant 0\big\}$.

Remark 13.7.2. — The construction given in this section is valid, when the base manifold X is a curve. □

CHAPTER 14

CONSTRUCTION OF AN INITIAL METRIC AND PRELIMINARY CORRESPONDENCE

In this chapter, we mainly study *graded semisimple* good filtered flat bundle. Almost all the results are minor generalization of those in Chapter 3 of [**69**] for $\lambda = 1$. We will often give only outlines for the proof.

In Sections 14.1–14.2, we explain local constructions of ordinary metrics. We give in Sections 14.1.6 and 14.2.5 the estimates which we will use in Sections 14.3 and 14.4. We also explain in Sections 14.1.7 and 14.2.6 the induced metrics on divisors, which will be used in Section 14.3.

In Section 14.3, we explain some formulas for the parabolic Chern character of good filtered λ-flat bundles. Our main purpose is to show the vanishing of characteristic numbers for good Deligne-Malgrange filtered bundle (Corollary 14.3.4), which will be significant in the proof of Theorems 16.2.1 and 16.2.4. Note that this vanishing holds even if the good Deligne-Malgrange filtered bundle is not graded semisimple.

In Section 14.4, we show Kobayashi-Hitchin correspondence for graded semisimple filtered flat bundles satisfying the SPW-condition in the surface case. This result will be used in the proof of Kobayashi-Hitchin correspondence for wild harmonic bundles (Theorem 16.1.1).

It also implies Bogomolov-Gieseker inequality for μ_L-stable good filtered flat bundles, which is explained in Section 14.5. Note that the inequality holds even if the μ_L-stable good filtered flat bundle satisfies neither graded semisimplicity nor the SPW-condition.

14.1. Around a crossing point

14.1.1. Taking a ramified covering. — Let $X = \Delta^2$, $D_j = \{z_j = 0\}$ and $D = D_1 \cup D_2$. In the following argument, we will replace X by a small neighbourhood of $O = (0,0)$ without mention. Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (X, D) such that $\mathrm{Res}_j(\nabla)$ is graded semisimple, i.e., the induced endomorphisms on ${}^j\mathrm{Gr}^F(^\diamond E)$

are semisimple. Let $\boldsymbol{c} = (c_1, c_2) \in \boldsymbol{R}^2$ such that $c_i \notin \mathcal{P}ar(\boldsymbol{E}_*, i)$. We assume the SPW-condition for $(\boldsymbol{E}_*, \nabla)$, i.e., there exist a positive integer m and $\gamma_i \in \boldsymbol{R}$ with $-1/m < \gamma_i \leqslant 0$, such that

$$\mathcal{P}ar({}_{\boldsymbol{c}}E, i) \subset \big\{c_i + \gamma_i + p/m \,\big|\, p \in \mathbb{Z},\ -1 < \gamma_i + p/m \leqslant 0\big\}.$$

Let d be an integer divisible by m rank$(E)!^3$. We put $X' := X$, $D'_j := D_j$, $D' := D$ and $O' = (0,0)$. Let $\varphi_d : (X', D') \to (X, D)$ be the ramified covering given by $\varphi_d(z_1, z_2) = (z_1^d, z_2^d)$. Let $\mathrm{Gal}(X'/X)$ denote the Galois group of X'/X. We have the filtered flat bundle $(\boldsymbol{E}'_*, \nabla')$ on (X', D') induced by $(\boldsymbol{E}_*, \nabla)$ and φ_d as in Section 2.5.3.3. We put $c'_i := d(c_i + \gamma_i)$ and $\boldsymbol{c}' := (c'_1, c'_2)$. By the assumption, $\mathcal{P}ar(\boldsymbol{E}'_*, i)$ is contained in $\big\{c'_i + n \,\big|\, n \in \mathbb{Z}\big\}$.

14.1.2. Taking an equivariant decomposition. — Since we have assumed that $(\boldsymbol{E}_*, \nabla)$ is good and that d is divisible by $(\mathrm{rank}\, E!)^3$, we have the good subset of irregular values $\mathrm{Irr}(\nabla', O') \in M(X', D')/H(X')$ and the irregular decomposition of the completion of $({}_{\boldsymbol{c}'}E', \nabla')$ at O'. For simplicity, we assume that the coordinate system (z_1, z_2) is admissible for $\mathrm{Irr}(\nabla', O')$. Let $\mathrm{Irr}(\nabla', D'_1) := \mathrm{Irr}(\nabla', O')$, and let $\mathrm{Irr}(\nabla', D'_2)$ denote the image of $\mathrm{Irr}(\nabla', O')$ via the naturally defined map $M(X', D')/H(X') \to M(X', D')/M(X', D'_1)$. We have the irregular decompositions for $j = 1, 2$:

$$({}_{\boldsymbol{c}'}E', \nabla)_{|\widehat{D}'_j} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla', D'_j)} \big({}_{\boldsymbol{c}'}E'_{\mathfrak{a}, \widehat{D}'_j}, \widehat{\nabla}_{\mathfrak{a}, \widehat{D}'_j}\big).$$

As remarked in Subsection 2.4.3, we have the decomposition on $\widehat{D}'$:

$$(320) \qquad {}_{\boldsymbol{c}'}E'_{|\widehat{D}'} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\nabla', D'_2)} {}_{\boldsymbol{c}'}E'_{\mathfrak{b}, \widehat{D}'}.$$

We have the naturally defined $\mathrm{Gal}(X'/X)$-action on $\mathrm{Irr}(\nabla', D'_i)$. For any $g \in \mathrm{Gal}(X'/X)$, we have $g \cdot {}_{\boldsymbol{c}'}E'_{\mathfrak{b}} = {}_{\boldsymbol{c}'}E'_{g \cdot \mathfrak{b}}$. Due to the graded semisimplicity assumption, the endomorphisms $\mathrm{Res}_j(\nabla')$ $(j = 1, 2)$ are semisimple. We have the eigen-decomposition:

$$ {}_{\boldsymbol{c}'}E'_{|D'_j} = \bigoplus_{\alpha \in \boldsymbol{C}} {}^j\mathbb{E}_\alpha.$$

Lemma 14.1.1. — *We can take a decomposition*

$$ {}_{\boldsymbol{c}'}E' = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla', O')} \bigoplus_{\boldsymbol{\alpha} \in \boldsymbol{C}^2} U_{\mathfrak{a}, \boldsymbol{\alpha}}$$

with the following properties:

- *For* $g \in \mathrm{Gal}(X'/X)$, *we have* $g \cdot U_{\mathfrak{a}, \boldsymbol{\alpha}} = U_{g \cdot \mathfrak{a}, \boldsymbol{\alpha}}$.
- *For* $\mathfrak{a} \in \mathrm{Irr}(\nabla', D'_1)$, *we put* $U^{(1)}_{\mathfrak{a}} := \bigoplus_{\boldsymbol{\alpha}} U_{\mathfrak{a}, \boldsymbol{\alpha}}$. *Then,* $U^{(1)}_{\mathfrak{a} | \widehat{D}'^{(N)}_1} = {}_{\boldsymbol{c}'}E'_{\mathfrak{a}, \widehat{D}'_1 | \widehat{D}'^{(N)}_1}$.
- *For* $\mathfrak{b} \in \mathrm{Irr}(\nabla', D'_2)$, *we put* $U^{(2)}_{\mathfrak{b}} := \bigoplus_{\mathfrak{a} \in T(\mathfrak{b})} U^{(1)}_{\mathfrak{a}}$, *where we have set* $T(\mathfrak{b}) := \big\{\mathfrak{a} \in \mathrm{Irr}(\nabla', O') \,\big|\, \mathfrak{a} \mapsto \mathfrak{b}\big\}$. *Then,* $U^{(2)}_{\mathfrak{b} | \widehat{D}'^{(N)}} = {}_{\boldsymbol{c}'}E'_{\mathfrak{b}, \widehat{D}' | \widehat{D}'^{(N)}}$.

- *For $j=1,2$, we have ${}^j\mathbb{E}_\alpha=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',O')}\bigoplus_{q_i(\boldsymbol{\alpha})=\alpha}U_{\mathfrak{a},\boldsymbol{\alpha}|D'_j}$, where $q_j:\boldsymbol{C}^2\to\boldsymbol{C}$ denotes the projection onto j-th component.*

Proof. — We have the decomposition $\mathrm{Irr}(\nabla',D'_2)=\coprod\mathfrak{o}_i$ by the orbits of the $\mathrm{Gal}(X'/X)$-action. We take representatives $\mathfrak{b}_i$ of $\mathfrak{o}_i$. For each $\mathfrak{b}_i$, let $\mathrm{Stab}(\mathfrak{b}_i)$ denote the stabilizer of $\mathfrak{b}_i$ with respect to the $\mathrm{Gal}(X'/X)$-action. We can take a $\mathrm{Stab}(\mathfrak{o}_i)$-invariant subbundle $U^{(2)}_{\mathfrak{b}_i}\subset{}_{\boldsymbol{c}'}E'$, such that $U^{(2)}_{\mathfrak{b}_i|\widehat{D}'(N)}={}_{\boldsymbol{c}'}E'_{\mathfrak{b}_i,\widehat{D}'|\widehat{D}'(N)}$. For $g\cdot\mathfrak{b}_i\in\mathfrak{o}_i$, we put $U^{(2)}_{g\cdot\mathfrak{b}_i}:=g\cdot U^{(2)}_{\mathfrak{b}_i}$. Thus, we obtain the decomposition ${}_{\boldsymbol{c}'}E'=\bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla',O')}U^{(2)}_{\mathfrak{b}}$. Let $\pi^{(2)}_{\mathfrak{b}}$ denote the projection onto $U^{(2)}_{\mathfrak{b}}$ with respect to the decomposition.

We have the decomposition $\mathrm{Irr}(\nabla',D'_1)=\coprod\mathfrak{p}_i$ by the orbits of the $\mathrm{Gal}(X'/X)$-action. We take representatives $\mathfrak{a}_i$ of $\mathfrak{p}_i$. For each $\mathfrak{a}_i$, let $\mathrm{Stab}(\mathfrak{p}_i)$ denote the stabilizer of $\mathfrak{a}_i$ with respect to the $\mathrm{Gal}(X'/X)$-action. By using the equivariant version of Lemma 3.6.30, we can take a $\mathrm{Stab}(\mathfrak{p}_i)$-invariant subbundle $U^{(1)}_{\mathfrak{a}_i}\subset{}_{\boldsymbol{c}'}E'$ with the following properties:

- $U^{(1)}_{\mathfrak{a}_i|\widehat{D}_1'^{(N)}}={}_{\boldsymbol{c}'}E'_{\mathfrak{a}_i,\widehat{D}'_1|\widehat{D}_1'^{(N)}}$.
- $U^{(1)}_{\mathfrak{a}_i|D'_2}$ is preserved by $\mathrm{Res}_2(\nabla')$ and $\pi^{(2)}_{\mathfrak{b}|D'_2}$ for any $\mathfrak{b}\in\mathrm{Irr}(\nabla',D'_2)$.

We put $U^{(1)}_{g\cdot\mathfrak{a}_i}:=g\cdot U^{(1)}_{\mathfrak{a}_i}$ for $g\cdot\mathfrak{a}_i\in\mathfrak{p}_i$, and then we obtain the decomposition ${}_{\boldsymbol{c}'}E'=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',O')}U^{(1)}_{\mathfrak{a}}$. By construction, the following holds:

(a) : $U^{(1)}_{\mathfrak{a}|\widehat{D}_1'^{(N)}}={}_{\boldsymbol{c}'}E'_{\mathfrak{a},\widehat{D}'_1|\widehat{D}_1'^{(N)}}$.

(b) : $U^{(1)}_{\mathfrak{a}|D'_2}$ are preserved by $\mathrm{Res}_2(\nabla')$.

(c) : The decomposition $\bigoplus_{\mathfrak{a}\in T(\mathfrak{b})}U^{(1)}_{\mathfrak{a}|D'_2}$ is preserved by $\pi^{(2)}_{\mathfrak{b}|D'_2}$.

Since we have $U^{(2)}_{\mathfrak{b}|O'}=\bigoplus_{\mathfrak{a}\in T(\mathfrak{b})}U^{(1)}_{\mathfrak{a}|O'}$, we obtain $U^{(2)}_{\mathfrak{b}|D'_2}=\bigoplus_{\mathfrak{a}\in T(\mathfrak{b})}U^{(1)}_{\mathfrak{a}|D'_2}$ from (c). By making the modification to $U^{(1)}_{\mathfrak{a}}$ which is trivial modulo $z_1^N\,z_2$, we obtain the decomposition $U^{(2)}_{\mathfrak{b}}=\bigoplus_{\mathfrak{a}\in T(\mathfrak{b})}U^{(1)}_{\mathfrak{a}}$. The conditions (a) and (b) are satisfied for the modified $U^{(1)}_{\mathfrak{a}}$. Let $\pi^{(1)}_{\mathfrak{a}}$ denote the projection onto $U^{(1)}_{\mathfrak{a}}$ with respect to the decomposition.

Since $U^{(1)}_{\mathfrak{a}|D'_j}$ $(j=1,2)$ are preserved by $\mathrm{Res}_j(\nabla')$, we have the eigen-decomposition $U^{(1)}_{\mathfrak{a}|D'_j}=\bigoplus{}^j\mathbb{E}_\alpha\big(U^{(1)}_{\mathfrak{a}|D'_j}\big)$, which are $\mathrm{Stab}(\mathfrak{p}_i)$-equivariant. Then, we can take a $\mathrm{Stab}(\mathfrak{p}_i)$-equivariant decomposition $U^{(1)}_{\mathfrak{a}_i}=\bigoplus_{\boldsymbol{\alpha}\in\boldsymbol{C}^2}U_{\mathfrak{a}_i,\boldsymbol{\alpha}}$ such that the following holds:

$$\bigoplus_{q_j(\boldsymbol{\alpha})=\alpha}U_{\mathfrak{a}_i,\boldsymbol{\alpha}|D_j}={}^j\mathbb{E}_\alpha\big(U_{\mathfrak{a}_i,\boldsymbol{\alpha}|D_j}\big).$$

We put $U_{g\cdot\mathfrak{a}_i,\boldsymbol{\alpha}}:=g\cdot U_{\mathfrak{a}_i,\boldsymbol{\alpha}}$. Thus, we obtain the decomposition ${}_{\boldsymbol{c}'}E'=\bigoplus_{\mathfrak{a}}\bigoplus_{\boldsymbol{\alpha}}U_{\mathfrak{a},\boldsymbol{\alpha}}$, which has the desired property, and the proof of Lemma 14.1.1 is finished. □

We consider $F = F^{\mathrm{irr}} + F^{\mathrm{reg}}$, where F^{irr} and F^{reg} are as follows:

$$F^{\mathrm{irr}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',O')} d\mathfrak{a}\,\mathrm{id}_{U^{(1)}_{\mathfrak{a}}},\quad F^{\mathrm{reg}} = \bigoplus_{\substack{\mathfrak{a}\in\mathrm{Irr}(\nabla',O'),\\ \boldsymbol{\alpha}\in\boldsymbol{C}^2}} \Big(\alpha_1\frac{dz_1}{z_1}+\alpha_2\frac{dz_2}{z_2}\Big)\,\mathrm{id}_{U_{\mathfrak{a},\boldsymbol{\alpha}}}.$$

We put $\nabla^{(0)} := \nabla - F$, which gives the holomorphic connection of ${}_{\boldsymbol{c}'}E'$, although it is not necessarily flat.

14.1.3. Frame and metric

Lemma 14.1.2. — *We can take frames* $\boldsymbol{v}_{\mathfrak{a},\boldsymbol{\alpha}} = \big(v_{\mathfrak{a},\boldsymbol{\alpha},j}\,\big|\,j=1,\dots,\operatorname{rank}U_{\mathfrak{a},\boldsymbol{\alpha}}\big)$ *of* $U_{\mathfrak{a},\boldsymbol{\alpha}}$ *such that* $g\cdot v_{\mathfrak{a},\boldsymbol{\alpha},j} = \omega(g,\mathfrak{a},\boldsymbol{\alpha},j)\,v_{g\cdot\mathfrak{a},\boldsymbol{\alpha},j}$ *for some* $\omega(g,\mathfrak{a},\boldsymbol{\alpha},j)\in\boldsymbol{C}$ *with* $|\omega(g,\mathfrak{a},\boldsymbol{\alpha},j)|=1$.

Proof. — For each $\mathfrak{a}_i$, we can take a frame $\boldsymbol{v}_{\mathfrak{a}_i,\boldsymbol{\alpha}} = (v_{\mathfrak{a}_i,\boldsymbol{\alpha},j})$ of $U_{\mathfrak{a},\boldsymbol{\alpha}}$ such that $g\cdot v_{\mathfrak{a}_i,\boldsymbol{\alpha},j} = \omega(g,\mathfrak{a}_i,\boldsymbol{\alpha},j)\,v_{\mathfrak{a}_i,\boldsymbol{\alpha},j}$ for any $g\in\mathrm{Stab}(\mathfrak{p}_i)$. For $\mathfrak{a}\in\mathfrak{p}_i$, we take $g(\mathfrak{a},\mathfrak{a}_i)\in G$ such that $g(\mathfrak{a},\mathfrak{a}_i)\cdot\mathfrak{a}_i=\mathfrak{a}$, and we put $v_{\mathfrak{a},\boldsymbol{\alpha},j} := g(\mathfrak{a},\mathfrak{a}_i)\cdot v_{\mathfrak{a}_i,\boldsymbol{\alpha},j}$. Thus, we obtain frames $\boldsymbol{v}_{\mathfrak{a},\boldsymbol{\alpha}}$ which have the desired property by construction. □

We obtain the frame $\boldsymbol{v} = \big(\boldsymbol{v}_{\mathfrak{a},\boldsymbol{\alpha}}\big)$ of ${}_{\boldsymbol{c}'}E'$. Let h'_0 denote the metric given by $h'_0(v_i,v_j) = |z_1|^{-2c'_1}|z_2|^{-2c'_2}\,\delta_{i,j}$. Since it is $\mathrm{Gal}(X'/X)$-equivariant, it induces a Hermitian metric h_0 of E, which is adapted to $\boldsymbol{E}_*$. We have the vanishing of the curvature $R(h_0)=0$.

14.1.4. Estimate of the connection 1-form with respect to the frame

Let A be the connection 1-form of $\nabla^{(0)}$ with respect to $\boldsymbol{v}$, i.e., $\nabla^{(0)}\boldsymbol{v} = \boldsymbol{v}\,A$. Corresponding to the decomposition ${}_{\boldsymbol{c}'}E' = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',D'_1)} U^{(1)}_{\mathfrak{a}}$, we have the decomposition $\boldsymbol{v} = (\boldsymbol{v}^{(1)}_{\mathfrak{a}})$. Correspondingly, we have the decomposition $A=\sum A_{\mathfrak{a},\mathfrak{b}}$. By our choice of the decomposition in Lemma 14.1.1, the following holds:

- We have $A_{\mathfrak{a},\mathfrak{b}}\equiv 0$ modulo $(z_1\,z_2)^N$ in the case $\mathrm{ord}(\mathfrak{a}-\mathfrak{b})<(0,0)$.
- If $\mathrm{ord}(\mathfrak{a}-\mathfrak{b})=(j,0)$ for some $j<0$, $A_{\mathfrak{a},\mathfrak{b}}$ are holomorphic and $A_{\mathfrak{a},\mathfrak{b}}\equiv 0$ modulo z_1^N.

In the case $\mathrm{ord}(\mathfrak{a}-\mathfrak{b})=(j,0)$, we have some refined estimate for the expression $A_{\mathfrak{a},\mathfrak{b}} = A_{\mathfrak{a},\mathfrak{b},1}\,dz_1 + A_{\mathfrak{a},\mathfrak{b},2}\,dz_2$:

- We may and will assume $A_{\mathfrak{a},\mathfrak{b},2} = O\big(z_1^N\,z_2\big)$ after taking some more ramified covering.
- For the decomposition $A_{\mathfrak{a},\mathfrak{b},1} = \sum A_{\mathfrak{a},\mathfrak{b},1,\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $A_{\mathfrak{a},\mathfrak{b},1,\boldsymbol{\alpha},\boldsymbol{\beta}} = O\big(z_1^N\,z_2\big)$ if $\alpha_2\neq\beta_2$, because the eigen-decomposition ${}_{\boldsymbol{c}'}E'_{\mathfrak{b},\widehat{D}'_2|D'_2} = \bigoplus\mathbb{E}_{\alpha}\big({}_{\boldsymbol{c}'}E'_{\mathfrak{b},\widehat{D}'_2|D'_2}\big)$ is flat with respect to the action of $\nabla^{(0)}(\partial_1)$.

Let us look at the term $A_{\mathfrak{a},\mathfrak{a}} = \sum A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta}}$. We have the following estimate in the case $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$ for the expression $A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta}} = A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta},1}\,dz_1 + A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta},2}\,dz_2$:

- We may assume $A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta},i} = O\big(z_i\big)$ after taking one more ramified covering.
- If $\alpha_1\neq\beta_1$, we have $A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta},2} = O\big(z_1\,z_2\big)$. If $\alpha_2\neq\beta_2$, we have $A_{\mathfrak{a},\mathfrak{a},\boldsymbol{\alpha},\boldsymbol{\beta},1} = O\big(z_1\,z_2\big)$.

14.1.5. Estimate of the associated $(1,0)$-forms. — Let $\theta_{\nabla'}$ be the $(1,0)$-form induced by ∇' and h'_0. (See [**83**] or Section 2.2 of [**69**].) We use the symbol $\theta_{\nabla^{(0)}}$ with a similar meaning. We put $\tau := 2^{-1}\big(c'_1\,dz_1/z_1 + c'_2\,dz_2/z_2\big)$. Then, $\theta_{\nabla^{(0)}} - \tau\,\mathrm{id}$ is C^∞ on X'. (See Section 2.2.5 of [**69**].) We have the decomposition $\theta_{\nabla^{(0)}} = \sum_{\mathfrak{b}\neq\mathfrak{b}'}\theta^{(2)}_{\nabla^{(0)},\mathfrak{b},\mathfrak{b}'} + \sum_{\mathfrak{b}}\theta^{(2)}_{\nabla^{(0)},\mathfrak{b}}$ corresponding to ${}_{\boldsymbol{c}'}E' = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla',D'_2)} U^{(2)}_{\mathfrak{b}}$. We also have the decomposition $\theta_{\nabla^{(0)}} = \sum_{\mathfrak{a}\neq\mathfrak{a}'}\theta^{(1)}_{\nabla^{(0)},\mathfrak{a},\mathfrak{a}'} + \sum_{\mathfrak{a}}\theta^{(1)}_{\nabla^{(0)},\mathfrak{a}}$ corresponding to ${}_{\boldsymbol{c}'}E' = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',D'_1)} U^{(1)}_{\mathfrak{a}}$. From the above estimate for A, we have the following estimates:

$$\theta_{\nabla^{(0)}} - \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla',D'_2)}\theta^{(2)}_{\nabla^{(0)},\mathfrak{b}} = O(|z_1|^N|z_2|^N),$$

$$\theta_{\nabla^{(0)}} - \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla',D'_1)}\theta^{(1)}_{\nabla^{(0)},\mathfrak{a}} = O\big(|z_1|^N\big).$$

(See Section 2.2.2 of [**69**] for the relation of A and $\theta_{\nabla^{(0)}}$, for example.) We have the decomposition $\theta_{\nabla^{(0)}} = \sum\big(\theta_{\nabla^{(0)}}\big)_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ corresponding to the decomposition ${}_{\boldsymbol{c}'}E' = \bigoplus U_{\boldsymbol{\alpha}}$, where $U_{\boldsymbol{\alpha}} = \bigoplus_{\mathfrak{a}} U_{\mathfrak{a},\boldsymbol{\alpha}}$. After taking some more ramified covering, we may and will assume to have the following:

- If $\alpha_1\neq\beta_1$, we have $(\theta_{\nabla^{(0)}})_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C^\infty\cdot z_1\,dz_1 + C^\infty\cdot z_1\,z_2\,dz_2$.
- If $\alpha_2\neq\beta_2$, we have $(\theta_{\nabla^{(0)}})_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C^\infty\cdot z_1\,z_2\,dz_1 + C^\infty\cdot z_2\,dz_2$.
- If $\alpha_i\neq\beta_i$ $(i=1,2)$, we have $(\theta_{\nabla^{(0)}})_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C^\infty\cdot z_1\,z_2\,dz_1 + C^\infty\cdot z_1\,z_2\,dz_2$.

We have $\theta_{\nabla'} = \theta_{\nabla^{(0)}} + F/2$.

Lemma 14.1.3. — *$\big[\theta_{\nabla'},\theta^\dagger_{\nabla'}\big]$ is bounded with respect to the metric h'_0 and the Euclidean metric of X'.*

Proof. — $\big[\theta_{\nabla'},\theta^\dagger_{\nabla'}\big] = \big[\theta_{\nabla^{(0)}},\theta^\dagger_{\nabla^{(0)}}\big] + \frac{1}{2}\big[F,\theta^\dagger_{\nabla^{(0)}}\big] + \frac{1}{2}\big[\theta_{\nabla^{(0)}},F^\dagger\big]$. The first term is C^∞ on X'. For the estimate of the other terms, let us take an auxiliary sequence $\mathcal{M} = (\boldsymbol{m})$ for the good set $\mathrm{Irr}(\nabla',O')$ as in Section 2.1. We use the symbols $\overline{\eta}_{\boldsymbol{m}}$ and $\zeta_{\boldsymbol{m}}$ in Section 2.1.3. Let $\overline{\mathrm{Irr}}(\nabla',\boldsymbol{m})$ denote the image of $\mathrm{Irr}(\nabla',O')$ via $\overline{\eta}_{\boldsymbol{m}}$. For $\mathfrak{c}\in\overline{\mathrm{Irr}}(\nabla',\boldsymbol{m})$, let $U^{\boldsymbol{m}}_{\mathfrak{c}}$ denote the sum of $U^{(1)}_{\mathfrak{a}}$ such that $\overline{\eta}_{\boldsymbol{m}}(\mathfrak{a}) = \mathfrak{c}$. Let $p^{\boldsymbol{m}}_{\mathfrak{c}}$ denote the projection onto $U^{\boldsymbol{m}}_{\mathfrak{c}}$ with respect to the decomposition ${}_{\boldsymbol{c}'}E' = \bigoplus_{\mathfrak{c}\in\overline{\mathrm{Irr}}(\nabla',\boldsymbol{m})} U^{\boldsymbol{m}}_{\mathfrak{c}}$. Then, we put as follows:

$$F^{\boldsymbol{m}} = \sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\nabla',\boldsymbol{m})} d\zeta_{\boldsymbol{m}}(\mathfrak{b})\,p^{\boldsymbol{m}}_{\mathfrak{b}}.$$

Note $F^{\mathrm{irr}} = \sum F^{\boldsymbol{m}}$. We have $\big[F^{\boldsymbol{m}},\theta^\dagger_{\nabla^{(0)}}\big] = O\big(|z_1|^{N/2}\,|z_2|^{N/2}\big)$ in the case $\boldsymbol{m}<\boldsymbol{0}$, and we have $\big[F^{\boldsymbol{m}},\theta^\dagger_{\nabla^{(0)}}\big] = O(|z_1|^{N/2})$ in the case $\boldsymbol{m} = (j,0)$ for some $j<0$. Hence, we obtain that $\big[F^{\mathrm{irr}},\theta^\dagger_{\nabla^{(0)}}\big]$ is bounded. We also have

$$\big[F^{\mathrm{reg}},\theta^\dagger_{\nabla^{(0)}}\big] \in C^\infty + C^\infty\cdot\frac{\overline{z}_1}{z_1} + C^\infty\cdot\frac{\overline{z}_2}{z_2}.$$

Thus the proof of Lemma 14.1.3 is finished. □

Lemma 14.1.4. — *$\big[\theta_{\nabla'},\theta_{\nabla'}\big]$ is bounded with respect to h_0' and the Euclidean metric of X'. After taking a refined ramified covering, we have $\big[\theta_{\nabla'},\theta_{\nabla'}\big]=O\big(|z_1\,z_2|\big)\,dz_1\,dz_2$.*

Proof. — The second claim immediately follows from the first one. We have $\big[\theta_{\nabla'},\theta_{\nabla'}\big]=\big[\theta_{\nabla^{(0)}},\theta_{\nabla^{(0)}}\big]+2\big[F,\theta_{\nabla^{(0)}}\big]$. The first term is C^∞.

Let us look at the second term $\big[F,\theta_{\nabla^{(0)}}\big]$. By construction, we have $\big[F^{\boldsymbol{m}},\theta_{\nabla^{(0)}}\big]=O(|z_1|^{N/2}\,|z_2|^{N/2})\,dz_1\,dz_2$ for $\boldsymbol{m}<\boldsymbol{0}$ and $\big[F^{\boldsymbol{m}},\theta_{\nabla^{(0)}}\big]=O\big(|z_1|^{N/2}\big)\,dz_1\,dz_2$ for $\boldsymbol{m}=(j,0)$. We also have

$$\left[F_1^{\mathrm{reg}}\frac{dz_1}{z_1},\theta_{\nabla^{(0)}}\right]\in C^\infty\cdot\frac{dz_1}{z_1}\,z_1\,z_2\,dz_2=C^\infty\cdot z_2\,dz_1\,dz_2.$$

Similarly, we have $[F_2^{\mathrm{reg}}dz_2/z_2,\theta_{\nabla^{(0)}}]\in C^\infty\cdot z_1\,dz_1\,dz_2$. Then, the claim of Lemma 14.1.4 follows. □

14.1.6. Estimates which will be used later. — Let $G(h_0')$ denote the pseudo curvature associated to ∇' and h_0'. (See [**82**], [**83**] or Section 2.2 of [**69**].)

Lemma 14.1.5. — *$G(h_0')$ is bounded with respect to h_0' and the Euclidean metric of X'.*

Proof. — We have $G(h_0')=2R(h_0')-4\big(\overline{\partial}^2+\theta_{\nabla'}^2-[\theta_{\nabla'}^\dagger,\theta_{\nabla'}]\big)$. Recall $R(h_0')=0$, and $\overline{\partial}^2$ is adjoint of $\theta^2_{\nabla'}$ with respect to h_0'. Hence, we obtain the boundedness of $G(h_0')$ from Lemmas 14.1.3 and 14.1.4. □

We have $\mathrm{tr}\big(R(h_0')^2\big)=4^{-1}\,\mathrm{tr}\big(G(h_0')^2\big)-4\overline{\partial}\,\mathrm{tr}(\theta^2_{\nabla'}\theta^\dagger_{\nabla'})$ (Lemma 2.31 of [**69**]). In Section 14.4, we would like to compare the integrals of $\mathrm{tr}\big(R(h_0')^2\big)=0$ and $\mathrm{tr}\big(G(h_0')^2\big)$. So, let us look at $\mathrm{tr}\big(\theta^2_{\nabla'}\theta^\dagger_{\nabla'}\big)$. We have the following.

Lemma 14.1.6. — *$\mathrm{tr}\big(\theta^2_{\nabla'}\,\theta^\dagger_{\nabla'}\big)$ is bounded with respect to h_0' and the Euclidean metric of X'.*

Proof. — The boundedness of $\mathrm{tr}\big(\theta^2_{\nabla'}\,\theta^\dagger_{\nabla^{(0)}}\big)$ follows from Lemma 14.1.4 and the smoothness of $\theta^\dagger_{\nabla^{(0)}}-\overline{\tau}\,\mathrm{id}$. Let us estimate $\mathrm{tr}\big(\theta^2_{\nabla'}\,F^\dagger\big)$. Let ${}_{\boldsymbol{c}'}E'=\bigoplus U^{\boldsymbol{m}}_{\mathfrak{b}}$ be the decomposition as in the proof of Lemma 14.1.3. We have the corresponding decomposition $\theta_{\nabla'}=\sum_{\mathfrak{b}\neq\mathfrak{b}'}\theta^{\boldsymbol{m}}_{\mathfrak{b},\mathfrak{b}'}+\sum_{\mathfrak{b}}\theta^{\boldsymbol{m}}_{\mathfrak{b}}$.

In the case $\boldsymbol{m}<\boldsymbol{0}$, we have $\theta^2_{\nabla'}-\sum_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\nabla',\boldsymbol{m})}\big(\theta^{\boldsymbol{m}}_{\mathfrak{b}}\big)^2=O(|z_1\,z_2|^{N/2})$. Because $\mathrm{tr}\big((\theta^{\boldsymbol{m}}_{\mathfrak{b}})^2\big)=0$, we have the following estimate:

$$\mathrm{tr}\big(\theta^2_{\nabla'}\,F^{\boldsymbol{m}\dagger}\big)=\sum\mathrm{tr}\big((\theta^{\boldsymbol{m}}_{\mathfrak{b}})^2\big)\,d\overline{\zeta_{\boldsymbol{m}}(\mathfrak{b})}+O\big(|z_1\,z_2|^{N/3}\big)=O\big(|z_1\,z_2|^{N/3}\big).$$

In particular, $\mathrm{tr}\big(\theta^2_{\nabla'}\,F^{\boldsymbol{m}\dagger}\big)$ is bounded in the case $\boldsymbol{m}<\boldsymbol{0}$.

In the case $\boldsymbol{m}=(j,0)$, we have $\big(\theta^2_{\nabla'}\big)-\sum_{\mathfrak{b}}(\theta^{\boldsymbol{m}}_{\mathfrak{b}})^2=O(|z_1|^{N/2})$, and hence we obtain

$$\mathrm{tr}\big(\theta^2_{\nabla'}\,F^{\boldsymbol{m}\dagger}\big)=\sum\mathrm{tr}\big((\theta^{\boldsymbol{m}}_{\mathfrak{b}})^2\big)\,d\overline{\zeta_{\boldsymbol{m}}(\mathfrak{b})}+O\big(|z_1|^{N/3}\big)=O\big(|z_1|^{N/3}\big).$$

In particular, $\mathrm{tr}\big(\theta^2_{\nabla'}\,F^{\boldsymbol{m}\dagger}\big)$ is bounded in the case $\boldsymbol{m}=(j,0)$.

Recall that we have assumed $\theta^2_{\nabla'} = O\big(|z_1\, z_2|\big)\, dz_1\, dz_2$ after taking some refined ramified covering, we obviously have the boundedness of $\mathrm{tr}\big(\theta^2_{\nabla'} F^{\mathrm{reg}\,\dagger}\big)$. Thus, we obtain Lemma 14.1.6. □

14.1.7. The induced metric of ${}^i\mathrm{Gr}^{F,\mathbb{E}}(^\diamond E)$. — For simplicity, we assume $c_i = \gamma_i = 0$ in this subsection. Let $(b,\beta) \in \mathcal{KMS}\big(^\diamond E_*, D_i\big)$. We consider the induced metric on ${}^i\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(^\diamond E)$. We restrict ourselves to the case $i = 1$ for simplicity of description. The other case can be argued similarly. We put $D_1^\circ := D_1 \setminus \{O\}$. We fix a positive C^∞-function ρ on X. Let $\chi := \rho\,|z_1|^2$. Then, h_0 and χ naturally induce the Hermitian metric $h_{b,\beta}$ of $\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(^\diamond E)_{|D_1^\circ}$ as follows. Let v_i $(i = 1, 2)$ be sections of ${}^1\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)$. We take lifts $\widetilde{v}_i$ of v_i to $^\diamond E$, i.e., $\widetilde{v}_{i|D_1}$ is contained in ${}^1F_b$, and mapped to v_i via the projection ${}^1F_b \to {}^1\mathrm{Gr}^F_b(E)$. Then, it can be shown that $\big(\chi^b\, h_0(\widetilde{v}_1, \widetilde{v}_2)\big)_{|D_1^\circ}$ is independent of the choice of lifts $\widetilde{v}_i$, which is denoted by $h_{b,\beta}(v_1, v_2)$.

Lemma 14.1.7

- *We have a holomorphic frame* $\boldsymbol{u}$ *of* ${}^1\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(^\diamond E)$ *on* D_1 *which is compatible with the induced parabolic structure at* O*, such that* $h_{b,\beta}(u_i, u_j) = \rho^b\, |z_1|^{-2\deg^F(u_i)}\, \delta_{i,j}$.
- *Let* $R(h_{b,\beta})$ *denote the curvature of* $\big({}^1\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(^\diamond E), h_{b,\beta}\big)$*. Then, the following holds on* D_1°*:*

$$\mathrm{tr}\big(R(h_{b,\beta})\big) - b\ \mathrm{rank}\, \mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(^\diamond E)\, \overline{\partial}\partial \log \rho = 0. \tag{321}$$

Proof. — The claims are clear from the construction. □

14.2. Around smooth point

14.2.1. Taking a ramified covering. — Let $X := \Delta^2$ and $D := D_1$. Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (X, D), which is graded semisimple. Let $c \in \boldsymbol{R}$ such that $c \notin \mathcal{P}ar(\boldsymbol{E}_*)$. We assume the SPW-condition, i.e., there exist a positive integer m and $\gamma \in \boldsymbol{R}$ with $-1/m < \gamma \leqslant 0$, such that

$$\mathcal{P}ar({}_cE) \subset \big\{c + \gamma + p/m \,\big|\, p \in \mathbb{Z},\ -1 < \gamma + p/m \leqslant 0\big\}.$$

Take an integer d which is divisible by $m\ \mathrm{rank}(E)!^3$. Let $\varphi_d : X' \to X$ be given by $\varphi_d(z_1, z_2) = (z_1^d, z_2)$. We have the filtered flat bundle $(\boldsymbol{E}'_*, \nabla')$ on (X', D') induced by $(\boldsymbol{E}_*, \nabla)$ and φ_d. We put $c' := d(c + \gamma)$. By the assumption, $\mathcal{P}ar(\boldsymbol{E}'_*)$ is contained in $\big\{c' + n \,\big|\, n \in \mathbb{Z}\big\}$. Since d is divisible by $\mathrm{rank}(E)!^3$, $(\boldsymbol{E}'_*, \nabla')$ is unramified, and we have the irregular decomposition:

$$_{c'}E'_{|\widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla')} E'_{\mathfrak{a}}.$$

We have the $\mathrm{Gal}(X'/X)$-action on $\mathrm{Irr}(\nabla')$, and $g \cdot E'_{\mathfrak{a}} = E'_{g\cdot\mathfrak{a}}$. From the graded semisimplicity assumption, $\mathrm{Res}(\nabla')$ is semisimple. We have the eigen-decomposition of $\mathrm{Res}(\nabla')$:

$$ {}_{c'}E'_{|D'} = \bigoplus_{\alpha \in \boldsymbol{C}} \mathbb{E}_\alpha. $$

14.2.2. Taking a decomposition. — Let N be sufficiently large. The following lemma can be shown using an argument similar to that employed in the proof of Lemma 14.1.1.

Lemma 14.2.1. — *We can take a decomposition*

$$ {}_{c'}E' = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla')} \bigoplus_{\alpha \in \boldsymbol{C}} U_{\mathfrak{a},\alpha} $$

with the following properties:

- *For* $g \in \mathrm{Gal}(X'/X)$, *we have* $g \cdot U_{\mathfrak{a},\alpha} = U_{g\cdot\mathfrak{a},\alpha}$.
- *We put* $U_{\mathfrak{a}} := \bigoplus_\alpha U_{\mathfrak{a},\alpha}$. *Then,* $U_{\mathfrak{a}|\widehat{D}'(N)} = {}_{c'}E'_{\mathfrak{a}|\widehat{D}'(N)}$.
- $\mathbb{E}_\alpha = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\nabla')} U_{\mathfrak{a},\alpha|D'}$. □

We consider

$$ F^{\mathrm{irr}} := \bigoplus_{\mathfrak{a}} d\mathfrak{a}\, \mathrm{id}_{U_{\mathfrak{a}}}, \quad F^{\mathrm{reg}} := \bigoplus_{\mathfrak{a},\alpha} \Big(\alpha \frac{dz_1}{z_1}\Big)\, \mathrm{id}_{U_{\mathfrak{a},\alpha}}, \quad F := F^{\mathrm{reg}} + F^{\mathrm{irr}}. \tag{322} $$

We put $\nabla^{(0)} := \nabla' - F$, which is a holomorphic connection of ${}_{c'}E'$, although it is not necessarily flat.

14.2.3. Metric. — Take a large number N. Let $\widetilde{h}'_0$ be a $\mathrm{Gal}(X'/X)$-invariant C^∞-metric of ${}_{c'}E'$ with the following properties:

- There exist C^∞-metrics $\widetilde{h}'_{0,\mathfrak{a}}$ of $U_{\mathfrak{a}}$ such that $\widetilde{h}'_0 \equiv \bigoplus \widetilde{h}'_{0,\mathfrak{a}}$ on the N-th infinitesimal neighbourhood of D.
- $\widetilde{h}'_{0|D}$ is compatible with the eigen-decomposition, i.e., ${}_{c'}E'_{|D'} = \bigoplus \mathbb{E}_\alpha$ is orthogonal with respect to $\widetilde{h}'_{0|D'}$

We remark that $R(h'_0) - \bigoplus_{\mathfrak{a}} R(h'_{0,\mathfrak{a}})$ is $O\big(|z_1|^N\big)$. Let ρ be a positive C^∞-function on X, and we put $\chi := \rho\, |z_1|^2$. Then, we set

$$ h'_0 := \varphi_d^*(\chi)^{-(c+\gamma)}\, \widetilde{h}'_0. $$

The induced metric of E is denoted by h_0.

14.2.4. Estimate of the associated $(1,0)$-form. — Let $\theta_{\nabla'}$ denote the $(1,0)$-form obtained from ∇' and h'_0. We use the symbol $\theta_{\nabla^{(0)}}$ with a similar meaning. Let $\tau := 2^{-1}(c+\gamma)\,\partial \log(\varphi_d^*\chi)$, and then $\theta_{\nabla^{(0)}} - \tau\, \mathrm{id}$ is C^∞ on X'. We have the decomposition $\theta_{\nabla^{(0)}} = \sum_{\mathfrak{b}} \theta_{\nabla^{(0)},\mathfrak{b}} + \sum_{\mathfrak{b}\neq\mathfrak{b}'} \theta_{\nabla^{(0)},\mathfrak{b},\mathfrak{b}'}$, corresponding to ${}_{c'}E' = \bigoplus_{\mathfrak{a}} U_{\mathfrak{a}}$. By construction, we have the following estimate:

$$ \theta_{\nabla^{(0)}} - \bigoplus \theta_{\nabla^{(0)},\mathfrak{b}} = O\big(|z_1|^N\big). \tag{323} $$

We also know that $\theta_{\nabla^{(0)}|D'}$ is compatible with the residue $\mathrm{Res}(\nabla')$. Let $\theta_{\nabla^{(0)}} = \sum_{\alpha,\beta}\theta_{\nabla^{(0)},\alpha,\beta}$ be the decomposition corresponding to ${}_{c'}E' = \bigoplus_\alpha\big(\bigoplus_{\mathfrak{a}} U_{\mathfrak{a},\alpha}\big)$. After taking a refined ramified covering, we may assume the following:

$$\theta_{\nabla^{(0)}} - \tau\,\mathrm{id} = O\big(|z_1|\big)\,dz_1 + O(1)\,dz_2,$$
$$\theta_{\nabla^{(0)},\alpha,\beta} = O\big(|z_1|\big)\,dz_1 + O\big(|z_1|\big)\,dz_2 \quad (\alpha\neq\beta). \tag{324}$$

Lemma 14.2.2. — *After taking a refined ramified covering, we have the boundedness of $\big[\theta_{\nabla'},\theta^\dagger_{\nabla'}\big]$ with respect to h'_0 and the Euclidean metric of X'.*

Proof. — We have $\theta_{\nabla'} = \theta_{\nabla^{(0)}} + F/2$, and hence

$$\big[\theta_{\nabla'},\theta^\dagger_{\nabla'}\big] = \big[\theta_{\nabla^{(0)}},\theta^\dagger_{\nabla^{(0)}}\big] + \frac{1}{2}\big[F,\theta^\dagger_{\nabla^{(0)}}\big] + \frac{1}{2}\big[\theta_{\nabla^{(0)}},F^\dagger\big] + \frac{1}{4}\big[F,F^\dagger\big].$$

The first term is C^∞ on X'. We have $\big[F^{\mathrm{irr}},\theta^\dagger_{\nabla^{(0)}}\big] = O\big(|z_1|^N\big)$ by (323). We also have the boundedness of $\big[F^{\mathrm{reg}},\theta^\dagger_{\nabla^{(0)}}\big]$ because of (324). We have $\big[F^{\mathrm{irr}},(F^{\mathrm{irr}})^\dagger\big] = O\big(|z_1|^N\big)$, $\big[F^{\mathrm{reg}},(F^{\mathrm{irr}})^\dagger\big] = O\big(|z_1|^N\big)$. We also have $\big[F^{\mathrm{reg}},(F^{\mathrm{reg}})^\dagger\big] = O(|z_1|^{-1})\,dz_1\,d\bar z_1$. Hence, we obtain the desired boundedness of $[\theta_{\nabla'},\theta^\dagger_{\nabla'}]$ after taking some refined ramified covering. □

Lemma 14.2.3. — *$\big[\theta_{\nabla'},\theta_{\nabla'}\big]$ is bounded with respect to h'_0 and the Euclidean metric of X'. We have $\big[\theta_{\nabla'},\theta_{\nabla'}\big] = O(|z_1|)\,dz_1\,dz_2$ after taking some refined ramified covering.*

Proof. — We have $\big[\theta_{\nabla'},\theta_{\nabla'}\big] = \big[\theta_{\nabla^{(0)}},\theta_{\nabla^{(0)}}\big] + \big[F,\theta_{\nabla^{(0)}}\big]$. The first term is C^∞, and we may assume that it is $O(|z_1|)\,dz_1\,dz_2$, after taking some more ramified covering. We have $\big[F^{\mathrm{irr}},\theta_{\nabla^{(0)}}\big] = O\big(|z_1|^N\big)\,dz_1\,dz_2$. From the compatibility of $\theta_{\nabla^{(0)}|D}$ and the residue, we obtain $\big[F^{\mathrm{reg}},\theta_{\nabla^{(0)}}\big] = O(1)\,dz_1\,dz_2$. After taking some more ramified covering, we may have $O(|z_1|)\,dz_1\,dz_2$. Thus, we are done. □

14.2.5. Estimate which will be used

Lemma 14.2.4. — *$G(h'_0)$ is bounded with respect to h'_0 and the Euclidean metric of X'.*

Proof. — It follows from Lemmas 14.2.2 and 14.2.3 with an argument similar to that employed in the proof of Lemma 14.1.5. □

We have $\mathrm{tr}\big(R(h'_0)^2\big) = 4^{-1}\,\mathrm{tr}\big(G(h'_0)^2\big) - 4\overline{\partial}\,\mathrm{tr}\big(\theta^2_{\nabla'}\theta^\dagger_{\nabla'}\big)$. Let us see the second term.

Lemma 14.2.5. — *$\mathrm{tr}\big(\theta^2_{\nabla'}\theta^\dagger_{\nabla'}\big)$ is bounded with respect to h'_0 and the Euclidean metric of X'.*

Proof. — The boundedness of $\mathrm{tr}\big(\theta^2_{\nabla'}\theta^\dagger_{\nabla^{(0)}}\big)$ follows from Lemma 14.2.3 and the smoothness of $\theta^\dagger_{\nabla^{(0)}} - \overline{\tau}\,\mathrm{id}$. Let η_j and ζ_j be the maps $M(X',D') \to M(X',D')$ given by $\eta_j\big(\sum a_k(z_2)z_1^k\big) = \sum_{k\leqslant j} a_k(z_2)z_1^k$ and $\zeta_j\big(\sum a_k(z_2)z_2^k\big) = a_j(z_2)z_1^j$. Let $\mathrm{Irr}(\nabla',j)$ denote the image of $\mathrm{Irr}(\nabla')$ by $\eta_j : M(X,D) \to M(X,D)$. For $\mathfrak{c} \in \mathrm{Irr}(\nabla',j)$, let $U^{(j)}_{\mathfrak{c}}$ denote the sum of $U_{\mathfrak{a}}$ such that $\eta_j(\mathfrak{a}) = \mathfrak{c}$. Let $p^{(j)}_{\mathfrak{c}}$ denote the projection onto $U^{(j)}_{\mathfrak{c}}$ with respect to the decomposition ${}_{c'}E' = \bigoplus_{\mathfrak{c}\in\mathrm{Irr}(\nabla',j)} U^{(j)}_{\mathfrak{c}}$. We have the corresponding decomposition $\theta_{\nabla'} = \sum \theta^{(j)}_{\mathfrak{c}}$. We set

$$F^{(j)} := \sum_{\mathfrak{b}\in\mathrm{Irr}(\nabla',j)} d\zeta_j(\mathfrak{b})\, p^{(j)}_{\mathfrak{b}}.$$

We have the decomposition $F^{\mathrm{irr}} = \sum F^{(j)}$. We obtain the following estimate as in the proof of Lemma 14.1.6:

$$\mathrm{tr}\big(\theta^2_{\nabla'}\, F^{(j)\dagger}\big) = \sum \mathrm{tr}\big((\theta^{(j)}_{\mathfrak{b}})^2\big)\, \overline{d\zeta_j(\mathfrak{b})} + O(|z_1|^{N/2}) = O(|z_1|^{N/2}).$$

Hence, $\mathrm{tr}\big(\theta^2_{\nabla'}\, F^{\mathrm{irr}\,\dagger}\big)$ is bounded. The boundedness of $\mathrm{tr}\big(\theta^2_{\nabla'}\, F^{\mathrm{reg}\,\dagger}\big)$ follows from $\theta^2_{\nabla'} = O(|z_1|)\, dz_1\, dz_2$. □

14.2.6. The induced metric of $\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)$. — For simplicity, we assume $c = \gamma = 0$ in this subsection. We remark that the eigenvalues of $\mathrm{Res}(\nabla')$ are given by $d\big(b+\beta\big)$, where $(b,\beta) \in \mathcal{KMS}({}^\diamond E_*)$. We have the decomposition $\mathbb{E}_\alpha = \bigoplus_{0\leqslant q\leqslant d-1} \mathbb{E}^q_\alpha$ such that $\mathrm{Gal}(X'/X) \ni \omega$ acts on $\mathbb{E}^q_\alpha$ by the multiplication of ω^q. The decomposition $\mathbb{E}_\alpha = \bigoplus \mathbb{E}^q_\alpha$ is orthogonal with respect to $\widetilde{h}'_0$.

Let $(b,\beta) \in \mathcal{KMS}({}^\diamond E_*)$. Then, h_0 and χ naturally induce a Hermitian metric $h_{b,\beta}$ of $\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)$ as follows. Let v_i $(i=1,2)$ be sections of $\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)$. We take lifts $\widetilde{v}_i$ of v_i to ${}^\diamond E$, i.e., $\widetilde{v}_{i|D}$ is contained in F_b, and mapped to v_i via the projection $F_b \to \mathrm{Gr}^F_b(E)$. Then, it can be shown that $\big(\chi^b\, h_0(\widetilde{v}_1, \widetilde{v}_2)\big)_{|D}$ is independent of the choice of $\widetilde{v}_i$, which is denoted by $h_{b,\beta}(v_1, v_2)$.

Let (q,α) and (b,β) be related by the relation $b = -q/d$ and $\beta = (\alpha+q)/d$. Let $h'_{b,\beta}$ denote the metric of $\mathbb{E}^q_\alpha$ induced by $\widetilde{h}'_0$.

Lemma 14.2.6. — *Let $R(h_{b,\beta})$ and $R(h'_{b,\beta})$ be the curvatures of $(\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E), h_{b,\beta})$ and $(\mathbb{E}^q_\alpha, h'_{b,\beta})$ respectively. Then, we have the following relation:*

$$\mathrm{tr}\big(R(h'_{b,\beta})\big) = \mathrm{tr}\big(R(h_{b,\beta})\big) - b\,\mathrm{rank}\,\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)\,\overline{\partial}\partial\log\rho. \tag{325}$$

Proof. — We take the isomorphism $\Phi : \mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E) \simeq \mathbb{E}^q_\alpha$ given as follows. Let v be a section of $\mathrm{Gr}^{F,\mathbb{E}}_{b,\beta}(E)$. We take a lift $\widetilde{v}$ of v to ${}^\diamond E$. Then, $\Phi(v) := (z_1^{-q}\varphi_d^*\widetilde{v})_{|D'}$ is contained in $\mathbb{E}^q_\alpha$, and it is independent of the choice of $\widetilde{v}$. Under the isomorphism Φ, we have $h'_{b,\beta} = h_{b,\beta}\,\rho^{-b}$. Then, (325) follows from a general formula. □

14.3. Some formulas for the parabolic Chern character

Let X be a smooth projective complex surface, and D be a simple normal crossing hypersurface with the decomposition into irreducible components $D = \bigcup_{i\in\Lambda} D_i$. Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (X,D). We have the bundles ${}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E)$ on D_i for each $\boldsymbol{c} \in \boldsymbol{R}^\Lambda$ and for each $(a,\alpha) \in \mathcal{KMS}\big({}_{\boldsymbol{c}}E_*, D_i\big)$. It is naturally equipped with the $\boldsymbol{c}_i$-parabolic structure at $D_i \cap \bigcup_{j\neq i} D_j$, where $\boldsymbol{c}_i := \big(c_j \,\big|\, j\in\Lambda, j\neq i\big)$.

Proposition 14.3.1. — *We have the following formula:*

$$\int_X 2\,\text{par-ch}_2(\boldsymbol{E}_*) \tag{326}$$
$$= \sum_{\substack{i\in\Lambda\\(a,\alpha)\in\mathcal{KMS}({}_{\boldsymbol{c}}E_*,D_i)}} \big(a+\operatorname{Re}(\alpha)\big)\Big(-\text{par-deg}_{D_i}\big({}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E)_*\big)+a\ \operatorname{rank}\big({}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E)\big)\,[D_i]^2\Big).$$

Here $[D_i]^2$ denotes the self intersection number of D_i.

Proof. — Note that the right-hand side is independent of the choice of $\boldsymbol{c}\in\boldsymbol{R}^{\Lambda}$. By using ε-perturbations explained in (I) of Section 13.7, we can reduce the problem to the case where the following conditions are satisfied:

- $\mathcal{P}ar(\boldsymbol{E}_*,i)\subset\boldsymbol{Q}$, $0\notin\mathcal{P}ar(\boldsymbol{E}_*,i)$, and $(\boldsymbol{E}_*,\nabla)$ is graded semisimple.

So, we will assume them in the following argument. We may assume $\boldsymbol{c}=(0,\dots,0)$.

We take d such that (i) $dw\in\mathbb{Z}$ for any $w\in\mathcal{P}ar(\boldsymbol{E}_*,D_i)$, (ii) d is divisible by $(\operatorname{rank}E!)^3$. We take a C^∞-metric h_0 of E on $X\setminus D$ which is as in Section 14.1 around crossing points $D_i\cap D_j$, and as in Section 14.2 around smooth points of D.

We take a Hermitian metrics g_i of $\mathcal{O}(D_i)$. The curvature of $(\mathcal{O}(D_i),g_i)$ is denoted by ω_i. Let $\sigma_i:\mathcal{O}\to\mathcal{O}(D_i)$ denote the canonical morphism. By using the functions $|\sigma_i|^2_{g_i}$, we obtain the C^∞-metric $h_{a,\alpha}$ of ${}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_{|D_i^\circ}$ for each $(a,\alpha)\in\mathcal{KMS}({}^\diamond E,D_i)$, as explained in Sections 14.1.7 and 14.2.6. It is compatible with the induced parabolic structure of $\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)$. Hence, we have

$$\frac{\sqrt{-1}}{2\pi}\int_{D_i}\operatorname{tr}\big(R(h_{a,\alpha})\big)=\text{par-deg}_{D_i}\big({}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*\big). \tag{327}$$

Let P be a point of the smooth part of D. We take a holomorphic coordinate (U,z_1,z_2) around P such that $D_U:=D\cap U=\{z_1=0\}$. Let $\varphi_P:U'\to U$ be a ramified covering as in Section 14.2. We have the irregular decomposition ${}^\diamond E'_{|\widehat{D}'_U}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla,P)}E'_{\mathfrak{a}}$. We put $\widehat{G}^{\mathrm{irr}}_P:=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla)}\mathfrak{a}\,\mathrm{id}_{E'_{\mathfrak{a}}}$. We can take a C^∞-equivariant section G^{irr}_P of $\mathrm{End}({}^\diamond E')$ such that $G^{\mathrm{irr}}_{P|\widehat{D}'^{(N)}_U}=\widehat{G}^{\mathrm{irr}}_{P|\widehat{D}'^{(N)}_U}$ for some large N. It induces an endomorphism of $E_{|U\setminus D}$. For any crossing point $P\in D$, we take a holomorphic coordinate (U,z_1,z_2) around P such that $U\cap D=\{z_1=0\}\cup\{z_2=0\}$. We take $\varphi_P:U'\to U$ and a decomposition ${}^\diamond E'=\bigoplus U_{\mathfrak{a}}$ as in Section 14.1. We put $G^{\mathrm{irr}}_P:=\bigoplus\mathfrak{a}\,\mathrm{id}_{U_{\mathfrak{a}}}$, which is $\mathrm{Gal}(U'/U)$-equivariant. Hence, it induces an endomorphism of $E_{|U\setminus D}$. By varying $P\in D$, gluing G^{irr}_P in C^∞, and extending it, we construct an endomorphism G^{irr} of E on $X\setminus D$.

Let ∇^u denote the unitary connection associated to h_0 and the $(0,1)$-part d'' of ∇. We obtain the C^∞-section $\mathcal{F}^{\mathrm{irr}}:=\nabla^uG^{\mathrm{irr}}$ of $\mathrm{End}(E)\otimes\Omega^{0,1}$ on $X\setminus D$. By construction, we have $\mathcal{F}^{\mathrm{irr}}_{|\widehat{D}'^{(N)}_U}=F^{\mathrm{irr}}_{|\widehat{D}'^{(N)}_U}$ around any smooth point P of D (Section 14.2). We also have $\mathcal{F}^{\mathrm{irr}}=F^{\mathrm{irr}}$ around any crossing point of D (Section 14.1).

Let $\sigma_i : \mathcal{O}_X \to \mathcal{O}_X(D_i)$ denote the canonical section. The norm of σ_i with respect to a chosen Hermitian metric is denoted by $|\sigma_i|$. Let $X_\delta := \bigcap\big\{|\sigma_i| \geqslant \delta\big\}$. We remark $R(h_0) = 0$ around the crossing points of D. Hence, we have

$$\int_{\partial X_\delta} \operatorname{tr}\big(\mathcal{F}^{\mathrm{irr}}\, R(h_0)\big) = \int_{\partial X_\delta} \operatorname{tr}\big(\nabla^u G^{\mathrm{irr}}\, R(h_0)\big) = \int_{\partial X_\delta} d\operatorname{tr}\big(G^{\mathrm{irr}}\, R(h_0)\big) = 0.$$

We put $\nabla^{(1)} := \nabla - \mathcal{F}^{\mathrm{irr}}$, and then $\theta_\nabla = \theta_{\nabla^{(1)}} + \mathcal{F}^{\mathrm{irr}}/2$. Recall the relation $R(h_0) = -2d''\theta_\nabla$. Hence we have

$$(328)\quad \Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X_\delta} \operatorname{tr}\big(R(h_0)^2\big) = -2\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X_\delta} \operatorname{tr}\big(d''\theta_\nabla\, R(h_0)\big)$$
$$= -2\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X_\delta} d\operatorname{tr}\big(\theta_\nabla\, R(h_0)\big) = -2\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{\partial X_\delta} \operatorname{tr}\big(\theta_{\nabla^{(1)}}\, R(h_0)\big).$$

We put $Y_{\delta,i} := \big\{|\sigma_i| = \delta\big\} \cap X_\delta$. Let us look at $\int_{Y_{\delta,i}} \operatorname{tr}\big(\theta_{\nabla^{(1)}}\, R(h_0)\big)$. The integrand is 0 around $D_i \cap D_j$ for any $j \neq i$. Let $P \in D_i \setminus \bigcup_{j\neq i} D_j$. Let (U, z_1, z_2), D_U and $\varphi_P : U' \to U$ be as above. We put $D'_U := \varphi_P^{-1}(D_U)$. We use the notation in Section 14.2 for this local argument. On U', we have $\nabla^{(1)} = \nabla^{(0)} + F^{\mathrm{reg}}$, where F^{reg} is as in (322). Since $\theta_{\nabla^{(0)}}$ is C^∞ on U', it does not contribute to the limit for $\delta \to 0$. (Note that $c = \gamma = 0$ in this case.) Thus we only have to look at the term $\operatorname{tr}\big(F^{\mathrm{reg}}\, R(h'_0)\big)/2$. Then, the limit of the contribution of $U \cap Y_{\delta,i}$ is as follows:

$$(329)\quad -\frac{1}{d}\int_{D'_U} \frac{\sqrt{-1}}{2\pi} \sum_{(a,\alpha)\in\mathcal{KMS}({}^\diamond E_*, D_i)} (d\alpha + da)\, \operatorname{tr} R(h'_{a,\alpha}) =$$
$$- \sum_{(a,\alpha)\in\mathcal{KMS}({}^\diamond E_*, D_i)} (\alpha + a)\frac{\sqrt{-1}}{2\pi}\int_{D_U}\Big(\operatorname{tr} R(h_{a,\alpha}) - a \operatorname{rank}{}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)\,\omega_i\Big).$$

Here, $h'_{a,\alpha}$ is as in Section 14.2.6, and we have used Lemma 14.2.6. We remark $R(h_0) = 0$ around every crossing point of D and the vanishing (321). Together with (327), we obtain

$$(330)\quad \lim_{\delta\to 0}\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X_\delta} \operatorname{tr}\big(R(h_0)^2\big)$$
$$= -\sum_i \sum_{(a,\alpha)\in\mathcal{KMS}({}^\diamond E_*, D_i)} (a+\alpha)\Big(\operatorname{par-deg}\big({}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)_*\big) - a \operatorname{rank}\big({}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}^\diamond E)\big)\,[D_i]^2\Big).$$

As shown in the proof of Proposition 4.18 of [**66**], we have the equality:

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X\setminus D} \operatorname{tr}\big(R(h_0)^2\big) = \int_X 2\operatorname{par-ch}_2(\boldsymbol{E}_*).$$

Taking the real part, we obtain the desired formula. □

Proposition 14.3.2. — *Let τ be a closed 2-form on X. Then, we have the following equality:*

$$\int_X \text{par-c}_1(\boldsymbol{E}_*)\,\tau = -\sum_i \sum_{(a,\alpha)\in\mathcal{KMS}({}_{\boldsymbol{c}}E,D_i)} \mathrm{Re}(a+\alpha)\ \mathrm{rank}\,{}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E)\int_{D_i}\tau.$$

Proof. — Recall that $(\sqrt{-1}/2\pi)\ \mathrm{tr}\,R(h_0)$ represents $\text{par-c}_1(\boldsymbol{E}_*)$. We also have the relation $\mathrm{tr}\,R(h_0) = -2\overline{\partial}\,\mathrm{tr}(\theta_\nabla)$. Then, it can be shown by the same argument as in the proof of Proposition 14.3.1. □

Remark 14.3.3. — See Section 3.5 of [**69**] for more formulas in the regular case. Similar formulas should hold even in the irregular case. □

Let L be an ample line bundle on X.

Corollary 14.3.4. — *Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection on (X,D) such that the associated Deligne-Malgrange filtered flat bundle $(\boldsymbol{E}^{DM}_*,\nabla)$ is good. (See Subsection* 2.7.1 *for Deligne-Malgrange filtered flat bundle.) Then, we have the vanishing of the characteristic numbers* $\text{par-deg}_L(\boldsymbol{E}^{DM}_*) = \int_X \text{par-ch}_2(\boldsymbol{E}^{DM}_*) = 0$.

Proof. — We only have to remark that any $(a,\alpha)\in\mathcal{KMS}(\boldsymbol{E}^{DM}_*,i)$ satisfy $a+\mathrm{Re}(\alpha)=0$ by the definition of $\boldsymbol{E}^{DM}_*$. □

14.3.1. Blow up and the parabolic characteristic numbers (Appendix)

Let X be a smooth projective surface with a simple normal crossing hypersurface D. Let $\pi:\widetilde{X}\to X$ be a blow up of X at a point P. Let $\boldsymbol{E}_*$ be a filtered bundle on (X,D). We have the induced filtered bundle $\widetilde{\boldsymbol{E}}_*$ on $(\widetilde{X},\widetilde{D})$.

Lemma 14.3.5. — *We have the following equality:*

$$\int_{\widetilde{X}} \text{par-ch}_2(\widetilde{\boldsymbol{E}}_*) = \int_X \text{par-ch}_2(\boldsymbol{E}_*), \tag{331}$$

$$\text{par-c}_1(\widetilde{\boldsymbol{E}}) = \pi^*\big(\text{par-c}_1(\boldsymbol{E}_*)\big). \tag{332}$$

See [**66**] *for* $\text{par-c}_1(\boldsymbol{E}_*)$ *and* $\text{par-ch}_2(\boldsymbol{E}_*)$, *and see* [**39**] *for a more systematic treatment.*

Proof. — The equality (332) can be reduced to the rank one case, which can be checked easily. Let us show (331). We give only an indication for a direct calculation. We use the formula (10) in [**66**] for $\text{par-ch}_2(\boldsymbol{E}_*)$. Let us consider the case $P\in D_1\cap D_2$. The other case can be shown similarly. Let $\widetilde{D}_i$ denote the proper transform of D_i, and let $\widetilde{D}_P$ denote the exceptional divisor.

Let ${}^iF$ $(i=1,2)$ denote the parabolic filtration of ${}^\diamond E_{|D_i}$, and let ${}^i\mathrm{Gr}^F_a({}^\diamond E) := {}^iF_a({}^\diamond E_{|D_i})/{}^iF_{<a}({}^\diamond E_{|D_i})$. For any $\boldsymbol{a}=(a_1,a_2)\in\boldsymbol{R}^2$, we put

$${}^{\underline{2}}F_{\boldsymbol{a}}({}^\diamond E_{|P}) := {}^1F_{a_1}({}^\diamond E_{|P})\cap{}^2F_{a_2}({}^\diamond E_{|P}),\quad {}^{\underline{2}}\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E) := \frac{{}^{\underline{2}}F_{\boldsymbol{a}}({}^\diamond E_{|P})}{\sum_{\boldsymbol{b}\lneq\boldsymbol{a}}{}^{\underline{2}}F_{\boldsymbol{b}}({}^\diamond E_{|P})}.$$

We put $\mathcal{P}ar(i) := \{a \,|\, {}^i\mathrm{Gr}^F_a({}^\diamond E) \neq 0\}$ and $\mathcal{P}ar(P) := \{\boldsymbol{a} \in \boldsymbol{R}^2 \,|\, {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E) \neq 0\}$.

We take a splitting ${}^\diamond E_{|P} = \bigoplus U_{\boldsymbol{a}}$ of the filtrations ${}^iF$ $(i = 1,2)$ indexed by $\mathcal{P}ar(P)$, i.e., it is taken as ${}^2F_{\boldsymbol{a}} = \bigoplus_{\boldsymbol{b}\leqslant \boldsymbol{a}} U_{\boldsymbol{b}}$ holds for each $\boldsymbol{a} = (a_1, a_2)$. We put ${}^PF_b({}^\diamond E_{|P}) := \bigoplus_{a_1+a_2\leqslant b} U_{\boldsymbol{a}}$, which is independent of the choice of a splitting. We have $\operatorname{rank} {}^PF_{-1} = \sum_{a_1+a_2\leqslant -1} \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E)$. We have the exact sequence:

$$0 \longrightarrow \pi^*({}^\diamond E) \longrightarrow {}^\diamond \widetilde{E} \longrightarrow {}^PF_{-1} \otimes \mathcal{O}_{\widetilde{D}_P}(\widetilde{D}_P) \longrightarrow 0.$$

Hence, we obtain the following equality, by using Lemma 3.23 of [**66**]:

$$\int_{\widetilde{X}} \text{par-ch}_2({}^\diamond \widetilde{E}) - \int_X \text{par-ch}_2({}^\diamond E) = -\frac{1}{2} \sum_{a_1+a_2\leqslant -1} \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E). \tag{333}$$

Let ${}^P\widetilde{F}$ denote the parabolic filtration of ${}^\diamond \widetilde{E}_{|\widetilde{D}_P}$, and we set

$$\mathrm{Gr}^{{}^P\widetilde{F}}_a({}^\diamond \widetilde{E}) := {}^P\widetilde{F}_a({}^\diamond \widetilde{E}_{|\widetilde{D}_P}) / {}^P\widetilde{F}_{<a}({}^\diamond \widetilde{E}_{|\widetilde{D}_P}).$$

Let $\mathcal{P}ar(\widetilde{D}_P) := \{a \,|\, \mathrm{Gr}^{{}^P\widetilde{F}}_a({}^\diamond \widetilde{E}) \neq 0\}$. We have the following equality:

$$- \sum_{a\in\mathcal{P}ar(\widetilde{D}_P)} a \deg_{\widetilde{D}_P} \mathrm{Gr}^{{}^P\widetilde{F}}_a({}^\diamond \widetilde{E}) = \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant -1}} (a_1+a_2+1) \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E). \tag{334}$$

Let ${}^i\widetilde{F}({}^\diamond \widetilde{E}_{|\widetilde{D}_i})$ denote the parabolic filtration of ${}^\diamond \widetilde{E}_{|\widetilde{D}_i}$, and we set

$${}^i\mathrm{Gr}^{\widetilde{F}}_a({}^\diamond \widetilde{E}) := {}^i\widetilde{F}_a({}^\diamond \widetilde{E}_{|\widetilde{D}_i}) / {}^i\widetilde{F}_{<a}({}^\diamond \widetilde{E}_{|\widetilde{D}_i}).$$

We have the following equality:

$$\begin{aligned} - \sum_{i=1,2} \sum_{a\in\mathcal{P}ar(i)} a \deg_{\widetilde{D}_i}\big({}^i\mathrm{Gr}^{\widetilde{F}}_a({}^\diamond \widetilde{E})\big) + \sum_{i=1,2} \sum_{a\in\mathcal{P}ar(i)} a \deg_{D_i}\big({}^i\mathrm{Gr}^F_a({}^\diamond E)\big) \\ = - \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant -1}} (a_1+a_2) \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E). \end{aligned} \tag{335}$$

We have the following equality:

$$\begin{aligned} \frac{1}{2} \sum_{a\in\mathcal{P}ar(\widetilde{D}_P)} a^2 \operatorname{rank}\big(\mathrm{Gr}^{{}^P F}_a({}^\diamond \widetilde{E})\big)\, [\widetilde{D}_P]^2 = \\ -\frac{1}{2} \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant -1}} (a_1+a_2+1)^2 \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E) - \frac{1}{2} \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2>-1}} (a_1+a_2)^2 \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E). \end{aligned} \tag{336}$$

We have

$$\begin{aligned} \frac{1}{2} \sum_{i=1,2} \sum_{a\in\mathcal{P}ar(i)} a^2 \operatorname{rank} {}^i\mathrm{Gr}^{\widetilde{F}}_a({}^\diamond \widetilde{E})\, [\widetilde{D}_i]^2 - \frac{1}{2} \sum_{i=1,2} \sum_{a\in\mathcal{P}ar(i)} a^2 \operatorname{rank} {}^i\mathrm{Gr}^F_a({}^\diamond E)\, [D_i]^2 \\ = -\frac{1}{2} \sum_{i=1,2} \sum_{a\in\mathcal{P}ar(i)} a^2 \operatorname{rank} {}^i\mathrm{Gr}^{\widetilde{F}}_a({}^\diamond \widetilde{E}) = -\frac{1}{2} \sum_{\boldsymbol{a}\in\mathcal{P}ar(P)} (a_1^2+a_2^2) \operatorname{rank} {}^2\mathrm{Gr}^F_{\boldsymbol{a}}({}^\diamond E). \end{aligned} \tag{337}$$

We have

$$(338)\quad \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2>-1}}(a_1+a_2)\,a_2\,r_{\boldsymbol{a}}+\sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant-1}}(a_1+a_2+1)\,a_2\,r_{\boldsymbol{a}}$$
$$+\sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2>-1}}a_1\,(a_1+a_2)\,r_{\boldsymbol{a}}+\sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant-1}}a_1\,(a_1+a_2+1)\,r_{\boldsymbol{a}}-\sum_{\boldsymbol{a}\in\mathcal{P}ar(P)}a_1\,a_2\,r_{\boldsymbol{a}}$$
$$=\sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2>-1}}(a_1^2+a_2^2+a_1a_2)\,r_{\boldsymbol{a}}+\sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar(P)\\ a_1+a_2\leqslant-1}}(a_1^2+a_2^2+a_1a_2+a_1+a_2)\,r_{\boldsymbol{a}},$$

where $r_{\boldsymbol{a}}:=\operatorname{rank}{}^2\mathrm{Gr}_{\boldsymbol{a}}({}^\diamond E)$. Taking summation of (333)–(338) and using the formula (10) in [**66**], we obtain (331). □

14.4. Preliminary correspondence

Let X be a smooth irreducible projective complex surface, and D be a simple normal crossing hypersurface with the decomposition into irreducible components $D=\bigcup_{i\in\Lambda}D_i$. We also assume that D is ample. Let L be an ample line bundle on X, and ω be a Kähler form which represents $c_1(L)$. We will not distinguish ω and the associated Kähler metric. Let $(\boldsymbol{E}_*,\nabla)$ be a μ_L-stable good filtered flat bundle on (X,D), which is graded semisimple. Let $\boldsymbol{c}\in\boldsymbol{R}^\Lambda$ such that $c_i\notin\mathcal{P}ar(\boldsymbol{E}_*,i)$. We assume the SPW-condition for each $i\in\Lambda$, i.e., there exist a positive integer m and $\gamma_i\in\boldsymbol{R}$ with $-1/m<\gamma_i\leqslant 0$, such that

$$\mathcal{P}ar(\boldsymbol{E}_*)\subset\big\{c_i+\gamma_i+p/m\,\big|\,p\in\mathbb{Z},\ -1<\gamma_i+p/m<0\big\}.$$

We take a large integer d which is divisible by $m\operatorname{rank}(E)!^3$. For $\varepsilon=1/d$, we take a Kähler metric ω_ε of $X\setminus D$ as in Section 4.3.1 of [**66**].

Proposition 14.4.1. — *There exists a Hermitian metric h_{HE} of E on $X\setminus D$ satisfying the following conditions:*

- *Hermitian-Einstein condition $2^{-1}\Lambda_{\omega_\varepsilon}G(h_{HE})=a\,\mathrm{id}_E$ for some constant a determined by the following equation:*

$$(339)\qquad a\,\frac{\sqrt{-1}}{2\pi}\frac{\operatorname{rank}E}{2}\int_{X\setminus D}\omega_\varepsilon^2=a\,\frac{\sqrt{-1}}{2\pi}\frac{\operatorname{rank}(E)}{2}\int_X\omega^2=\text{par-deg}_\omega(\boldsymbol{E}_*).$$

- *h_{HE} is adapted to $\boldsymbol{E}_*$.*
- $\deg_{\omega_\varepsilon}(E,h_{HE})=\text{par-deg}_\omega(\boldsymbol{E}_*)$.
- *We have the following equalities:*

$$\int_X 8\,\text{par-ch}_2(\boldsymbol{E}_*)=\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\Big(G(h_{HE})^2\Big),$$
$$4\int_X\text{par-c}_1^2(\boldsymbol{E}_*)=\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\big(G(h_{HE})\big)^2.$$

Proof. — We take a C^∞-metric h_0 of $E_{|X\setminus D}$ which is as in Section 14.1 around every crossing point of D, and as in Section 14.1 around every smooth point of D. It satisfies the following properties:

Lemma 14.4.2

- *$G(h_0)$ is bounded with respect to h_0 and ω_ε.*
- *The following equalities hold:*

$$\frac{1}{4}\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}(G(h_0)^2)=\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}(R(h_0)^2)=\int_X 2\,\text{par-ch}_2(\boldsymbol{E}_*),\tag{340}$$

$$\frac{1}{4}\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\big(G(h_0)\big)^2=\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\big(R(h_0)\big)^2=\int_X \text{par-c}_1(\boldsymbol{E}_*)^2.\tag{341}$$

Proof. — The first claim follows from Lemmas 14.1.5 and 14.2.4. The equality (340) follows from Lemma 14.1.6, Lemma 14.2.5 and the Chern-Weil formula to express the Chern character by the curvature (See Proposition 4.18 of [**66**] in the trivial Higgs case, for example.) The equality (341) follows from the general formula $\operatorname{tr}\big(G(h_0)\big)=2\operatorname{tr}R(h_0)$ and the Chern-Weil formula. (See Lemma 4.16 of [**66**], for example.) □

Make a modification $h_{in}=h_0\exp(-g')$ as in the proof of Proposition 6.1 of [**66**], such that $\Delta_{\omega_\varepsilon}\det(h_{in})$ is constant. Then, h_{in} satisfies the following conditions:

- h_{in} is adapted to the parabolic structure of $\boldsymbol{E}_*$.
- $G(h_{in})$ is bounded with respect to h_{in} and ω_ε.
- Let V be any saturated coherent subsheaf of E, and let π_V denote the orthogonal projection of E onto V. Then $\overline{\partial}\pi_V$ is L^2 with respect to h_{in} and ω_ε, if and only if there exists a saturated coherent subsheaf ${}_{\boldsymbol{c}}V$ of ${}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X\setminus D}=V$ ([**52**] and [**89**]). Moreover we have $\text{par-deg}_\omega({}_{\boldsymbol{c}}V_*)=\deg_{\omega_\varepsilon}(V,h_{in,V})$, where $h_{in,V}$ denotes the metric of V induced by h_{in}.
- $\operatorname{tr}\Lambda_{\omega_\varepsilon}G(h_{in})=2\operatorname{rank}(E)\,a$ for the constant a determined by the equation (339).
- The following equalities hold:

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\Big(G(h_{in})^2\Big)=\int_X 8\,\text{par-ch}_2(\boldsymbol{E}_*),$$
$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X\setminus D}\operatorname{tr}\Big(G(h_{in})\Big)^2=\int_X 4\,\text{par-c}_1^2(\boldsymbol{E}_*).$$

Then, the proposition follows from Simpson's theorem in [**81**] and [**82**]. (See also Proposition 2.49 of [**69**].) □

14.5. Bogomolov-Gieseker inequality

Let Y be an n-dimensional smooth connected projective variety over $\boldsymbol{C}$. Let L be an ample line bundle on Y, and let D be a simple normal crossing hypersurface of Y.

Corollary 14.5.1. — *Let $(\boldsymbol{E}_*,\nabla)$ be a μ_L-stable good filtered flat bundle on (Y,D). Then, Bogomolov-Gieseker inequality holds for $\boldsymbol{E}_*$. Namely, we have the following inequality:*

$$\int_Y \text{par-ch}_{2,L}(\boldsymbol{E}_*) \leqslant \frac{\int_Y \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2\operatorname{rank} E}.$$

Proof. — According to Mehta-Ramanathan type theorem (Proposition 13.2.1), the problem can be reduced to the case $\dim Y = 2$. By using ε-perturbations as in the case (II) of Section 13.7, we can reduce the problem to the case where $(\boldsymbol{E}_*,\nabla)$ is graded semisimple, and satisfy the SPW-condition. Then, the claim follows from Proposition 14.4.1 and the inequality for the curvature of Hermitian-Einstein metric (see [**81**]). □

Corollary 14.5.2. — *Let $(\boldsymbol{E}_*,\nabla)$ be a μ_L-stable good filtered flat bundle on (Y,D) with the trivial characteristic numbers* $\text{par-deg}_L(\boldsymbol{E}_*) = \int_Y \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. *Then, we have* $\text{par-c}_{1,L}(\boldsymbol{E}_*) = 0$.

Proof. — Due to Proposition 13.2.1, the problem can be reduced to the case $\dim Y = 2$. We have

$$\int_Y \text{par-c}_1(\boldsymbol{E}_*)\, c_1(L) = \text{par-deg}_L(\boldsymbol{E}_*) = 0,$$

$$0 = 2\operatorname{rank}(E)\int_Y \text{par-ch}_2(\boldsymbol{E}_*) \leqslant \int_Y \text{par-c}_1^2(\boldsymbol{E}_*).$$

Then, the claim follows from the Hodge index theorem. □

CHAPTER 15

PRELIMINARIES FOR THE RESOLUTION OF TURNING POINTS

We give a preparation for the resolution of "turning points" for Higgs fields underlying harmonic bundles. In Section 15.1, we explain a procedure to construct a projective morphism via which given ideals are transformed to principal ideals.

When we are given an algebraic equation on the rational function field of an irreducible variety, we sometimes hope that the polar part of a solution may ramify only along the poles. We show in Section 15.2 that it can be achieved if we take the pull-back via an appropriate morphism. It will be used in Section 16.3.3.

It is significant to ask, for a given Higgs field θ, whether there exists a projective birational map φ such that $\varphi^*\theta$ is good in the sense of Definition 7.1.4. We show in Section 15.3 the existence of such a map under some condition. This result will be useful for the resolution of turning points (Sections 16.2–16.3) and the correspondence between wild harmonic bundles and polarized wild pure twistor D-modules (Section 19.3).

15.1. Resolution for a tuple of ideals

Let G be a finite group. Let Y be a normal complex analytic space provided with a G-action. Let $I_1,\ldots,I_r$ be ideal sheaves of $\mathcal{O}_Y$. Assume that the G-action induces the permutation of $\{I_1,\ldots,I_r\}$. We would like to give a canonical construction to obtain a normal complex analytic space Y' equipped with a G-action and a G-equivariant birational projective morphism $\psi : Y' \to Y$ such that the following holds:

- Let I_j' denote the ideal sheaf of $\mathcal{O}_{Y'}$ generated by $\psi^{-1}(I_j)$. Then, I_j' is invertible. Moreover, for each point P of Y', the ideals I_j' are totally ordered with respect to inclusion relation around P. (We may permit that $I_j' = I_k'$ around P.)

We put $Y^{(0)} := Y$. Using an inductive argument, we will construct $Y^{(k)}$ with the following properties:

- We have G-actions on $Y^{(k)}$ and G-equivariant maps $\psi_k : Y^{(k)} \to Y$.
- We put $I_j^{(k)} := \psi_k^{-1}(I_j) \cdot \mathcal{O}_{Y^{(k)}}$. For each closed point $P \in Y^{(k)}$, we have an ordered subset
$$S(P) := \{i_1(P), \dots, i_k(P)\} \subset \{1, \dots, r\}$$
such that (i) $I_{i_h(P)}$ $(h = 1, \dots, k)$ are invertible, (ii) $I^{(k)}_{i_1(P)} \supset I^{(k)}_{i_2(P)} \supset \cdots \supset I^{(k)}_{i_k(P)}$, (iii) $I_j^{(k)}$ $(j \notin S(P))$ are contained in $I^{(k)}_{i_k(P)}$, around P.

Assume that $Y^{(k)}$ has already been constructed. We have the following G-equivariant ideal sheaf on $Y^{(k)}$:
$$I_Y^{(k)} := \sum_{|J|=k+1} \bigcap_{i \in J} I_i^{(k)}.$$
For each closed point $P \in Y^{(k)}$ and for each $h \notin S(P)$, we have $\bigcap_{j=1}^k I^{(k)}_{i_j(P)} \cap I_h^{(k)} = I_h^{(k)}$. Hence, the following holds around P:
$$I_Y^{(k)} = \sum_{h \notin S(P)} I_h^{(k)}.$$
Let $\pi_k : Y^{(k+1)} \to Y^{(k)}$ be the blow up of $Y^{(k)}$ along $I_Y^{(k)}$. Let $\psi_{k+1} := \psi_k \circ \pi_k$. We put $I_j^{(k+1)} := \psi_{k+1}^{-1}(I_j) \cdot \mathcal{O}_{Y^{(k+1)}} = \pi_k^{-1}(I_j^{(k)}) \cdot \mathcal{O}_{Y^{(k+1)}}$. Let P' be any closed point of $Y^{(k+1)}$, and let $P := \pi_k(P')$. Around P', the ideals $I_j^{(k+1)}$ $(j \in S(P))$ are invertible. We have $\sum_{j \notin S(P)} I_j^{(k+1)} = \pi_k^{-1}\big(I_Y^{(k)}\big) \cdot \mathcal{O}_{Y^{(k+1)}}$, which is invertible by definition of blow up. Hence, one of $I_j^{(k+1)}$ $(j \notin S(P))$ is invertible, and we have $I_h^{(k+1)} \subset I_j^{(k+1)}$ for any $h \notin S(P)$. Thus, the inductive construction can proceed. Let Y' be the normalization of $Y^{(r+1)}$, and let $\psi : Y' \to Y$ be the naturally induced morphism. By construction, (Y', ψ) has the desired property.

Let $\widetilde{G}$ be a finite group, and let $\widetilde{Y}$ be a complex analytic space with a $\widetilde{G}$-action. Assume that we are given a surjective homomorphism $\widetilde{G} \to G$, and an equivariant non-ramified covering $F : \widetilde{Y} \to Y$ which induces $\widetilde{Y}/\widetilde{G} \simeq Y/G$. We have the induced ideals $\widetilde{I}_j$ $(j = 1, \dots, r)$ of $\widetilde{Y}$, which are the pull-back of I_j. The $\widetilde{G}$-action induces the permutation of $\{\widetilde{I}_1, \dots, \widetilde{I}_r\}$. Applying the above procedure, we obtain the $\widetilde{G}$-equivariant map $\widetilde{\psi} : \widetilde{Y}' \to \widetilde{Y}$.

Lemma 15.1.1. — *We have the natural isomorphism $\widetilde{Y}' \simeq \widetilde{Y} \times_Y Y'$. The induced map $F' : \widetilde{Y}' \to Y'$ is the non-ramified Galois covering, whose Galois group is the kernel K of $\widetilde{G} \to G$. In particular, we have the natural isomorphism $\widetilde{Y}'/\widetilde{G} \simeq Y'/G$.*

Proof. — Inductively, we can check the following:
- We have the natural isomorphism $\widetilde{Y}^{(k)} \simeq Y^{(k)} \times_Y \widetilde{Y}$, and the induced map $F^{(k)} : \widetilde{Y}^{(k)} \to Y^{(k)}$ is unramified K-covering.
- $\widetilde{I}_j^{(k)} = (F^{(k)})^{-1}(I_j^{(k)}) \cdot \mathcal{O}_{\widetilde{Y}^{(k)}}$ and $\widetilde{I}_Y^{(k)} = (F^{(k)})^{-1}(I_Y^{(k)}) \cdot \mathcal{O}_{\widetilde{Y}^{(k)}}$.

Then, the claim of the lemma follows. □

Remark 15.1.2. — The construction can also work in the category of complex algebraic geometry. □

15.2. Separation of the ramification and the polar part of a Higgs field

15.2.1. Statement. — Let X be a smooth projective variety over $\boldsymbol{C}$. Let D be a simple normal crossing hypersurface of X. Let $U = \operatorname{Spec} R$ be an affine Zariski open subset of X. For simplicity, we assume that the ideal sheaf $\mathcal{O}_U(-D \cap U)$ is principal, and let $g \in R$ be a generator of $\mathcal{O}_U(-D \cap U)$. Let R_g denote the localization of R with respect to g. Let K denote the quotient field of R, which is the same as the rational function field of X. In the following, an algebraic closure $\overline{K}$ of K is fixed. Let $\mathcal{P}(T) \in R_g[T]$ be a monic of degree r. We will prove the following proposition in Sections 15.2.2–15.2.3.

Proposition 15.2.1. — *There exists a birational projective morphism $F : X_0 \to X$ with the following properties:*

- *$D_0 := F^{-1}(D)$ is a simple normal crossing hypersurface, and the restriction $X_0 \setminus D_0 \to X \setminus D$ is an isomorphism.*
- *Let $U_0 := F^{-1}(U)$. For any $P \in U_0 \cap D_0$, we take a holomorphic coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around P such that $D_P := D_0 \cap X_P = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We take a ramified covering $\psi_P : \widetilde{X}_P \to X_P$ given by $\psi_P(\zeta_1, \ldots, \zeta_n) = (\zeta_1^M, \ldots, \zeta_\ell^M, \zeta_{\ell+1}, \ldots, \zeta_n)$, where M is divisible by $r!$. We put $\widetilde{D}_P := \psi_P^{-1}(D_P)$. Then, there exist a finite subset $S \subset M(\widetilde{X}_P, \widetilde{D}_P)$ and monic polynomials $\mathcal{Q}_{\mathfrak{a}} \in H(\widetilde{X}_P)[T]$ for $\mathfrak{a} \in S$ such that the following holds:*

$$(F \circ \psi_P)^* \mathcal{P}(T) = \prod_{\mathfrak{a} \in S} \mathcal{Q}_{\mathfrak{a}}(T - \mathfrak{a}).$$

  Here, $M(\widetilde{X}_P, \widetilde{D}_P)$ denote the ring of meromorphic functions on $\widetilde{X}_P$ which admit poles along $\widetilde{D}_P$, and $H(\widetilde{X}_P)$ denote the ring of holomorphic functions on $\widetilde{X}_P$.

Remark 15.2.2. — Let α be a (possibly) multi-valued root of the polynomial $(F \circ \psi_P)^* \mathcal{P}$. The claim of Proposition 15.2.1 can be reworded in the way that the polar parts of α can ramify only along D_0. □

15.2.2. Construction of the space. — Let K'/K be the Galois extension associated to $\mathcal{P}(T)$. Let G denote the Galois group of K' over K. Let $\pi : X' \to X$ be the normalization of X in K'. We put $D' := \pi^{-1}(D)$. The pull-back $U' := \pi^{-1}(U)$ is the affine scheme $\operatorname{Spec} R'$, where R' is the normalization of R in K'. We have the roots α_i $(i = 1, \ldots, r)$ of $\mathcal{P}(T)$ in R'_g.

We take a large number N such that $g^N \cdot \alpha_i \in R'$ for every i. Let $I_{i,j}$ denote the ideal of R' generated by g^N and $g^N \cdot (\alpha_i - \alpha_j)$. They naturally induce the ideal sheaves $\mathcal{I}_{i,j}$ of $\mathcal{O}_{X'}$. Note that the closed subset associated to $\mathcal{I}_{i,j}$ is contained in D'.

We apply the construction in Section 15.1 to X' with the ideal sheaves $\mathcal{I}_{i,j}$ and the G-action. Then, we obtain the normal variety X'_1 with the G-action and the G-equivariant morphism $F'_1 : X'_1 \to X'$. We put $U'_1 := (F'_1)^{-1}(U')$, $D'_1 := (F'_1)^{-1}(D')$ and $\alpha_{1,i} := (F'_1)^{-1}(\alpha_i)$. The pull-back of g is denoted by g_1. Note that $X'_1 \setminus D'_1 \to X' \setminus D'$ is an isomorphism. Since the ideals sheaves $(F'_1)^{-1}\mathcal{I}_{i,j} \cdot \mathcal{O}_{X'_1}$ are invertible, either one of the following holds, for each $Q \in U'_1 \cap D'_1$ and for each pair i, j:

$(A1)$ **:** g_1^N generates $(F'_1)^{-1}\mathcal{I}_{i,j} \cdot \mathcal{O}_{X'_1}$ around Q. In this case, $\alpha_{1,i} - \alpha_{1,j} \in \mathcal{O}_{X'_1,Q}$, where $\mathcal{O}_{X'_1,Q}$ denote the local ring at Q.

$(A2)$ **:** $g_1^N \cdot (\alpha_{1,i} - \alpha_{1,j})$ generates $(F'_1)^{-1}\mathcal{I}_{i,j} \cdot \mathcal{O}_{X'_1}$ around Q. In this case, there exists $\chi_{i,j} \in \mathcal{O}_{X'_1,Q}$ such that $1 = (\alpha_{1,i} - \alpha_{1,j}) \cdot \chi_{i,j}$ in $\mathcal{O}_{X'_1,Q}$.

Let $X_1 := X'_1/G$, which is also a normal variety. We have the induced morphism $F_1 : X_1 \to X$. We put $D_1 := F_1^{-1}(D)$, which is the same as D'_1/G. Note that the restriction $X_1 \setminus D_1 \to X \setminus D$ is an isomorphism, and in particular, $X_1 \setminus D_1$ is smooth.

We take a smooth projective variety X_0 with the birational projective morphism $F_{1,0} : X_0 \to X_1$ such that (i) $D_0 := F_{1,0}^{-1}(D_1)$ is a simple normal crossing divisor, (ii) $X_0 \setminus D_0 \simeq X_1 \setminus D_1$. Let $\pi_0 : X'_0 \to X_0$ denote the normalization of X_0 in K', and we put $D'_0 := \pi_0^{-1}(D_0)$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccc} X'_0 & \xrightarrow{F'_{1,0}} & X'_1 & \xrightarrow{F'_1} & X' \\ {\scriptstyle \pi_0}\downarrow & & {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ X_0 & \xrightarrow{F_{1,0}} & X_1 & \xrightarrow{F_1} & X \end{array}$$

We put $F := F_1 \circ F_{1,0}$, $F' := F'_1 \circ F'_{1,0}$, $U_0 := F^{-1}(U)$ and $U'_0 := \pi_0^{-1}(U_0)$. We also put $\alpha_{0,i} := (F')^{-1}\alpha_i$, which are the algebraic sections of $\mathcal{O}_{X'_0}(*D'_0)$ on U'_0. The pull-back of g is denoted by g_0. By construction, either one of the following holds for each $Q \in U'_0 \cap D'_0$ and each i, j:

$(A_0 1)$ **:** $\alpha_{0,i} - \alpha_{0,j} \in \mathcal{O}_{U'_0,Q}$.

$(A_0 2)$ **:** There exists $\chi_{i,j} \in \mathcal{O}_{U'_0,Q}$ such that $1 = (\alpha_{0,i} - \alpha_{0,j}) \cdot \chi_{i,j}$.

15.2.3. Proof of Proposition 15.2.1. — Let us show that (X_0, F_0) has the desired property. Let P be any point of D_0. We take an affine Zariski open neighbourhood $\mathcal{U}_P$ of P in X_0 with an etale morphism $\varphi_P : \mathcal{U}_P \to \boldsymbol{C}^n$ such that $D_P := D_0 \cap \mathcal{U}_P = \varphi_P^{-1}\big(\bigcup_{i=1}^{\ell}\{w_i = 0\}\big)$. Let $g_i := \varphi_P^*(w_i)$ $(i = 1, \ldots, \ell)$. Let $\widetilde{K}_P$ denote the Galois extension over K associated to the polynomials $T^M - g_i$ $(i = 1, \ldots, \ell)$, where M is as in the claim of Proposition 15.2.1. Let $\widetilde{K}'_P$ denote the minimal field which contains $\widetilde{K}_P$ and K'. Due to a general theory ([**29**], for example), $\widetilde{K}'_P$ is the Galois extension over K, and the Galois group $\widetilde{G}_P$ of $\widetilde{K}'_P/\widetilde{K}_P$ is the same as the Galois group of $K'/K' \cap \widetilde{K}_P$.

Let $\mathcal{U}'_P$, $\widetilde{\mathcal{U}}_P$ and $\widetilde{\mathcal{U}}'_P$ denote the normalization of $\mathcal{U}_P$ in K', $\widetilde{K}_P$ and $\widetilde{K}'_P$, respectively. We have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{U}}'_P & \xrightarrow{\lambda'_P} & \mathcal{U}'_P \\ {\scriptstyle\widetilde{\pi}_P}\downarrow & & \downarrow{\scriptstyle\pi_P} \\ \widetilde{\mathcal{U}}_P & \xrightarrow{\lambda_P} & \mathcal{U}_P \end{array}$$

We have the natural isomorphism $\widetilde{\mathcal{U}}_P \simeq \widetilde{\mathcal{U}}'_P/\widetilde{G}_P$. Let $\widetilde{D}_P := \lambda_P^{-1}(D_P)$. We remark that $\widetilde{\pi}_P$ is unramified at the generic points of $\widetilde{D}_P$, because M is divisible by $r!$.

In the following argument, we will work in the category of complex analytic spaces. Let $\widetilde{P} = \lambda_P^{-1}(P)$. We take a point $Q \in \widetilde{\pi}_P^{-1}(\widetilde{P})$. Let $\widetilde{G}_Q := \{\sigma \in \widetilde{G}_P \,|\, \sigma(Q) = Q\}$. We take a $\widetilde{G}_Q$-invariant small open connected neighbourhood V_Q of Q in $\widetilde{\mathcal{U}}'_P$. By shrinking V_Q, we may assume that $V_{\widetilde{P}} := V_Q/\widetilde{G}_Q$ naturally gives a small neighbourhood of $\widetilde{P}$ in $\widetilde{\mathcal{U}}_P$. Let $\pi_Q : V_Q \to V_{\widetilde{P}}$ denote the induced projection.

We have the minimal reduced closed analytic subspace $R_{\widetilde{P}} \subset V_{\widetilde{P}}$ such that π_Q is non-ramified covering on the non-empty open set $V_{\widetilde{P}} - R_{\widetilde{P}}$. We fix a base point $P_0 \in V_{\widetilde{P}} - R_{\widetilde{P}}$, and let $\pi_1(V_{\widetilde{P}} - R_{\widetilde{P}}, P_0)$ denote the fundamental group. Since V_Q is normal, it is smooth in codimension one. Since $V_Q - \pi^{-1}(R_{\widetilde{P}})$ is connected, we have the naturally induced surjection $\rho_Q : \pi_1(V_{\widetilde{P}} - R_{\widetilde{P}}, P_0) \to \widetilde{G}_Q$. For any smooth point A of $\widetilde{R}_P$, let γ_A denote the element of $\pi_1(V_{\widetilde{P}} - R_{\widetilde{P}}, P_0)$ which is induced by a path connecting A and P_0, and a small loop around $R_{\widetilde{P}}$ near A. Let $\sigma_A := \rho_Q(\gamma_A)$.

Let $\alpha_{Q,j}$ denote the pull-back of $\alpha_{0,j}$ by $V_Q \to \mathcal{U}'_P$, and let $\widetilde{D}_Q := \pi_Q^{-1}(\widetilde{D}_P)$.

Lemma 15.2.3. — *$\alpha_{Q,j}$ are $\widetilde{G}_Q$-invariant in $\mathcal{O}_{V_Q,Q}(*\widetilde{D}_Q)/\mathcal{O}_{V_Q,Q}$.*

Proof. — Let A be as above. For each j, either one of the following holds:

(A_Q1) : $\sigma_A^*(\alpha_{Q,j}) - \alpha_{Q,j} \in \mathcal{O}_{V_Q,Q}$.

(A_Q2) : There exists $\chi_j \in \mathcal{O}_{V_Q,Q}$ such that $1 = \chi_j \cdot (\sigma_A^*(\alpha_{Q,j}) - \alpha_{Q,j})$.

We have the closed analytic subset $Z_A \subset V_Q$ of codimension 1 such that (i) $Q \in Z_A$ and $A \in \pi_Q(Z_A)$, (ii) $Z_A \not\subset \pi_Q^{-1}(\widetilde{D}_P)$, (iii) $(\sigma_A^*(\alpha_{Q,j}) - \alpha_{Q,j})_{Q_1}$ is contained in the maximal ideal at Q_1 for any $Q_1 \in Z_A$. Hence, we can conclude that (A_Q2) cannot happen.

Since the elements γ_A generate $\pi_1(V_{\widetilde{P}} - R_{\widetilde{P}}, P_0)$, the claim of Lemma 15.2.3 follows. □

Because the $\widetilde{G}_Q$-invariant part of $\mathcal{O}_{V_Q,Q}(*\widetilde{D}_Q)/\mathcal{O}_{V_Q,Q}$ is naturally isomorphic to $\mathcal{O}_{V_P,P}(*\widetilde{D}_P)/\mathcal{O}_{V_P,P}$, we have $\beta_{P,j} \in \mathcal{O}_{V_P,P}(*\widetilde{D}_P)/\mathcal{O}_{V_P,P}$ induced by $\alpha_{Q,j}$, and hence the finite subset $S = \{\beta_{P,j}\}$. We take a lift of S to $\mathcal{O}_{V_P,P}(*\widetilde{D}_P)$, which is also denoted by S. We have $\mathfrak{a}(j) \in S$ for each j such that $\alpha_{Q,j} - \pi_Q^*(\mathfrak{a}(j)) \in \mathcal{O}_{V_Q}$. Then, we can conclude that $(X_0, D_0) \to (X, D)$ has the desired property, and the proof of Proposition 15.2.1 is finished. □

15.3. Resolution for generically good Higgs field

15.3.1. Statement. — Let X be a complex manifold, and let D be a normal crossing divisor. In the following, Zariski open subset means the complement of a closed analytic subset. Let $\mathcal{A}$ be a $\boldsymbol{Q}$-vector subspace of $\boldsymbol{C}$. Let (E,θ) be a holomorphic Higgs bundle on $X \setminus D$ of rank r. We state some conditions on (E,θ).

(Generically $\mathcal{A}$-good) : We have a Zariski open subset D' of D such that θ is $\mathcal{A}$-good around every point of D'. (See Definition 7.1.5.) Note that the condition implies that the eigenvalues of θ are multi-valued meromorphic 1-forms on (X,D), which may ramify along some hypersurface of X.

If (E,θ) is generically $\mathcal{A}$-good, the following condition makes sense.

(RD) : The ramification of the eigenvalues of θ may happen only along D.

We have another slightly more complicated condition. For any $P \in D$, we take a holomorphic coordinate neighbourhood $(X_P, z_1, \dots, z_n)$ around P such that $D_P := D \cap X_P = \bigcup_{i=1}^{\ell(P)} \{z_i = 0\}$. We take a ramified covering $\psi_P : \widetilde{X}_P \to X_P$ given by $\psi_P(\zeta_1, \dots, \zeta_n) = (\zeta_1^M, \dots, \zeta_{\ell(P)}^M, \zeta_{\ell(P)+1}, \dots, \zeta_n)$, where M is divisible by $r!$. We put $\widetilde{D}_P := \psi_P^{-1}(D_P)$. We have the expression:

$$\psi_P^*(\theta) = \sum_{i=1}^{\ell(P)} f_i \frac{d\zeta_i}{\zeta_i} + \sum_{i=\ell(P)+1}^{n} f_i \, d\zeta_i.$$

Let $M(\widetilde{X}_P, \widetilde{D}_P)$ denote the ring of meromorphic functions on $\widetilde{X}_P$ which admit poles along $\widetilde{D}_P$, and let $H(\widetilde{X}_P)$ denote the ring of holomorphic functions on $\widetilde{X}_P$. If (E,θ) is generically $\mathcal{A}$-good, the characteristic polynomials $\det(T - f_i)$ are contained in $M(\widetilde{X}_P, \widetilde{D}_P)[T]$. Then, the following condition also makes sense.

(SRP) : For each $P \in D$ and $i = 1, \dots, n$, we have a finite subset $S_i(P) \subset M(\widetilde{X}_P, \widetilde{D}_P)$ and monic polynomials $\mathcal{Q}_{P,i,\mathfrak{b}} \in H(\widetilde{X}_P)[T]$ ($\mathfrak{b} \in S_i(P)$), such that the following holds:

$$\det(T - f_i) = \prod_{\mathfrak{b} \in S_i(P)} \mathcal{Q}_{P,i,\mathfrak{b}}(T - \mathfrak{b}).$$

We will prove the following proposition in Section 15.3.4 after the preparation in Sections 15.3.2–15.3.3.

Proposition 15.3.1. — *Assume that (E,θ) is generically $\mathcal{A}$-good, and moreover it satisfies either (RD) or (SRP). Then, there exists a complex manifold X' with a birational projective morphism $\varphi : X' \to X$ such that (i) $D' := \varphi^{-1}(D)$ is normal crossing, (ii) $X' \setminus D' \simeq X \setminus D$, (iii) $\varphi^*(\theta)$ is $\mathcal{A}$-good (Definition 7.1.6).*

Corollary 15.3.2. — *Let (E_i, θ_i, h_i) $(i = 1,2)$ be good wild harmonic bundles on (X,D). There exists a projective birational morphism $\varphi : (X',D') \to (X,D)$ such that $\varphi^*\big((E_1,\theta_1,h_1) \otimes (E_2,\theta_2,h_2)\big)$ is good.*

Proof. — It is easy to see that $\mathrm{id}_{E_1} \otimes \theta_2 + \theta_1 \otimes \mathrm{id}_{E_2}$ satisfies the condition (RD). □

Remark 15.3.3. — If (RD) holds, (SRP) also holds. Hence, we only have to consider the case (SRP) in Proposition 15.3.1. We consider the case (RD) too, partially because it is easy to state and argue, and partially because it suffices for some of our purposes. □

Proposition 15.3.1 is a preparation for the resolution of turning points (Theorem 16.2.1) and the correspondence between wild harmonic bundles and polarized wild pure twistor D-modules (Theorem 19.1.3).

15.3.2. The irregular values and the logarithmic 1-forms. — Let $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let (E, θ) be a generically unramifiedly $\mathcal{A}$-good Higgs bundle on $X \setminus D$. We have the expression

$$\theta = \sum_{i=1}^{\ell} f_i \frac{dz_i}{z_i} + \sum_{i=\ell+1}^{n} f_i \, dz_i. \tag{342}$$

We shall observe that a (not necessarily good) set of irregular values is associated to θ in a functorial way, under some assumptions.

Let $M(X, D)$ denote the ring of meromorphic functions on X which admit poles along D, and let $H(X)$ denote the ring of holomorphic functions on X. Let $CL(X, D, \mathcal{A})$ denote the space of logarithmic 1-forms of the form $\sum a_i \, dz_i/z_i$ ($a_i \in \mathcal{A}$). In the case $\mathcal{A} = \boldsymbol{C}$, it is simply denoted by $CL(X, D)$. Recall the following lemma. (See Proposition 3.13 in Chapter II of [**24**].)

Lemma 15.3.4. — *Let ω be a meromorphic 1-form on X which admits poles along D.*

- *Assume that $d\omega$ is logarithmic on (X, D), then there exists $\mathfrak{a} \in M(X, D)$ such that $\omega - d\mathfrak{a}$ is logarithmic on (X, D), by shrinking X around the origin $(0, \ldots, 0)$ appropriately. Such an $\mathfrak{a}$ is unique modulo $H(X)$.*
- *If moreover $d\omega$ is holomorphic on X, there exists a unique $\kappa \in CL(X, D)$ such that $\omega - d\mathfrak{a} - \kappa$ is holomorphic on X.* □

15.3.2.1. The case (RD). — In this subsection, we assume the following:

(UR) : The eigenvalues of f_i are single-valued on X.

Remark 15.3.5. — If (RD) is satisfied, the above condition is satisfied on some ramified covering of (X, D). □

We have the homogeneous generalized eigen-decomposition $E = \bigoplus E^{(p)}$ on a Zariski open subset of $X \setminus D$, and f_i have the single-valued eigenvalues $\alpha_i^{(p)}$ on $E^{(p)}$. We put $\omega^{(p)} := \sum_{i=1}^{\ell} \alpha_i^{(p)} \, dz_i/z_i + \sum_{i=\ell+1}^{n} \alpha_i^{(p)} \, dz_i$.

Lemma 15.3.6

- *We have $\kappa^{(p)} \in CL(X, D, \mathcal{A})$ and meromorphic functions $\mathfrak{a}^{(p)} \in M(X, D)$ such that $\omega^{(p)} - \big(d\mathfrak{a}^{(p)} + \kappa^{(p)}\big)$ are holomorphic on X.*
- *Such a $\kappa^{(p)}$ is unique, and such an $\mathfrak{a}^{(p)}$ is unique modulo $H(X)$.*

Proof. — Since θ is generically $\mathcal{A}$-good, $d\omega^{(p)}$ are holomorphic around any general point $Q \in D$. Hence, $d\omega^{(p)}$ are holomorphic 2-forms on X. Due to Lemma 15.3.4, we can take $\mathfrak{a}^{(p)} \in M(X,D)$ and $\kappa^{(p)} \in CL(X,D)$ such that $\omega^{(p)} - d\mathfrak{a}^{(p)} - \kappa^{(p)}$ is holomorphic. Since θ is generically $\mathcal{A}$-good, we obtain $\kappa^{(p)} \in CL(X,D,\mathcal{A})$. □

Let $\boldsymbol{\kappa}(\theta)$ denote the tuple $\bigl(\kappa^{(p)}\bigr)$ of logarithmic 1-forms on (X,D), and let $\mathrm{Irr}(\theta)$ denote the subset $\bigl\{\mathfrak{a}^{(p)}\bigr\}$ in $M(X,D)/H(X)$. They are uniquely determined by θ.

Let $\widetilde{X} := \Delta_\zeta^n$ and $\widetilde{D} := \bigcup_{j=1}^k \{\zeta_j = 0\}$. Let $\psi : \widetilde{X} \to X$ be a morphism such that $\psi^{-1}(D) = \widetilde{D}$. We have the induced tuple $\psi^*\boldsymbol{\kappa}(\theta)$ of logarithmic 1-forms on $(\widetilde{X},\widetilde{D})$, and the induced subset $\psi^*\mathrm{Irr}(\theta) \subset M(\widetilde{X},\widetilde{D})/H(\widetilde{X})$.

Lemma 15.3.7. — *We have the functoriality* $\mathrm{Irr}(\psi^*\theta) = \psi^*\mathrm{Irr}(\theta)$. *We also have* $\boldsymbol{\kappa}(\psi^*\theta) \equiv \psi^*\boldsymbol{\kappa}(\theta)$ *modulo the holomorphic exact* 1*-forms.*

Proof. — Since $\psi^*\omega^{(p)} - d\psi^*(\mathfrak{a}^{(p)}) - \psi^*\kappa^{(p)}$ are holomorphic, the claim of the lemma is clear. □

15.3.2.2. The case (SRP). — In this subsection, we assume the following:

(UR-SRP) : We have the finite subsets $S_i \subset M(X,D)$ and polynomials $\mathcal{P}_{\mathfrak{b},i} \in H(X)[T]$ ($\mathfrak{b} \in S_i$) such that

$$\det(T - f_i) = \prod_{\mathfrak{b}\in S_i} \mathcal{P}_{\mathfrak{b},i}(T - \mathfrak{b}).$$

We also assume that the roots of $\mathcal{P}_{\mathfrak{b},i}(T)$ are unramified along a Zariski dense open subset of D.

Remark 15.3.8. — If (SRP) is satisfied, the above condition is satisfied on some ramified covering of (X,D). □

We have a normal complex space X' with a finite morphism $\psi : X' \to X$ such that the roots of $\psi^*\det(T - f_i)$ are single-valued for any i. We have the homogeneous generalized eigen-decomposition $\psi^*E = \bigoplus E^{(p)}$ on a Zariski open subset of X', and $\psi^* f_i$ have single-valued eigenvalues $\alpha_i^{(p)}$ on $E^{(p)}$. We put $\omega^{(p)} := \sum_{i=1}^{\ell} \alpha_i^{(p)}\, \psi^* dz_i/z_i + \sum_{i=\ell+1}^{n} \alpha_i^{(p)}\, \psi^* dz_i$. We regard them as multi-valued meromorphic 1-forms on (X,D) which may ramify outside of $X \setminus D$. For each $\alpha_i^{(p)}$, we have the corresponding $\mathfrak{b}_i^{(p)} \in S_i$. We put $\omega_0^{(p)} := \sum_{i=1}^{\ell} \mathfrak{b}_i^{(p)}\, dz_i/z_i + \sum_{i=\ell+1}^{n} \mathfrak{b}_i^{(p)}\, dz_i$, which are single-valued meromorphic 1-forms on (X,D).

Lemma 15.3.9

- *There exists* $\mathfrak{a}^{(p)} \in M(X,D)$ *such that* $\omega_0^{(p)} - d\mathfrak{a}^{(p)}$ *is a logarithmic* 1*-form. Such an* $\mathfrak{a}^{(p)}$ *is unique modulo* $H(X)$*.*
- *There exists a unique* $\kappa^{(p)} \in CL(X,D,\mathcal{A})$ *such that* $\omega^{(p)} - d\mathfrak{a}^{(p)} - \kappa^{(p)}$ *is a multi-valued holomorphic* 1*-form on* X*, i.e., it is of the form* $\sum \beta_i^{(p)}\, \psi^* dz_i$ *where* $\beta_i^{(p)}$ *are holomorphic on* X'*.*

Proof. — Let Q be any point of D such that (i) (E,θ) is unramifiedly $\mathcal{A}$-good around Q, (ii) any roots of $\mathcal{P}_{\mathfrak{b},i}$ ($i=1,\dots,\ell$, $\mathfrak{b}\in S_i$) are unramified around Q. On a small neighbourhood X_Q of Q, $\omega^{(p)}$ are single-valued meromorphic 1-forms, and $d\omega^{(p)}$ are holomorphic 2-forms. Because $\omega^{(p)}-\omega_0^{(p)}$ are logarithmic, $d\omega_0^{(p)}$ are logarithmic on X_Q. By varying Q in a Zariski dense subset in D, we obtain that $d\omega_0^{(p)}$ are logarithmic. Due to Lemma 15.3.4, we can take $\mathfrak{a}^{(p)}\in M(X,D)$ such that $\omega_0^{(p)}-d\mathfrak{a}^{(p)}$ are logarithmic. We obtain that $\omega^{(p)}-d\mathfrak{a}^{(p)}$ are logarithmic on X_Q. Since θ is generically $\mathcal{A}$-good, we can find $\kappa_Q^{(p)}\in CL(X,D,\mathcal{A})$ such that $\tau^{(p)}:=\omega^{(p)}-\big(\kappa^{(p)}+d\mathfrak{a}^{(p)}\big)$ are holomorphic on X_Q. Hence, $\tau^{(p)}$ are holomorphic on X'. □

Let $\boldsymbol{\kappa}(\theta)$ denote the tuple of the logarithmic 1-forms $\big(\kappa^{(p)}\big)$, and let $\mathrm{Irr}(\theta)$ denote the subset $\{\mathfrak{a}^{(p)}\}$ of $M(X,D)/H(X)$. They are uniquely determined by θ.

Let $\widetilde{X}:=\Delta^n$ and $\widetilde{D}:=\bigcup_{j=1}^m\{\zeta_j=0\}$. Let $\psi:\widetilde{X}\to X$ be a morphism such that $\psi^{-1}(D)=\widetilde{D}$. We have the induced tuple of logarithmic 1-forms $\psi^*\boldsymbol{\kappa}(\theta)$, and the induced subset $\psi^*\mathrm{Irr}(\theta)\subset M(\widetilde{X},\widetilde{D})/H(\widetilde{X})$.

Lemma 15.3.10. — *$\psi^*(E,\theta)$ also satisfies the condition* **(UR-SRP)**. *We have the functoriality $\mathrm{Irr}(\psi^*\theta)=\psi^*\mathrm{Irr}(\theta)$. We also have $\boldsymbol{\kappa}(\psi^*\theta)\equiv\psi^*\boldsymbol{\kappa}(\theta)$ modulo the holomorphic exact forms.*

Proof. — Since $\psi^*\omega^{(p)}-d\psi^*(\mathfrak{a}^{(p)})-\psi^*\kappa^{(p)}$ are multi-valued holomorphic 1-forms on $\widetilde{X}$, the claim of the lemma follows. □

15.3.3. The ideal sheaves associated to a set of irregular values. — Let $X:=\Delta^n$ and $D:=\bigcup_{i=1}^{\ell}\{z_i=0\}$. Let $\mathcal{I}$ be a finite subset of $M(X,D)/H(X)$. We take a lift of $\mathcal{I}$ to $M(X,D)$, which is given by $\big\{\mathfrak{a}^{(p)}\,\big|\,p=1,\dots,s\big\}$. For $j=1,\dots,\ell$, we put $m_j(\mathfrak{a}^{(p)}):=\mathrm{ord}_{z_j}(\mathfrak{a}^{(p)})$ in the case where $\{z_j=0\}$ is contained in the poles of $\mathfrak{a}^{(p)}$, and $m_j(\mathfrak{a}^{(p)}):=0$ otherwise. We set $m_j(\mathcal{I}):=\min_p m_j(\mathfrak{a}^{(p)})$, which is well defined for $\mathcal{I}$. We put $\xi(\mathcal{I}):=\prod_{j=1}^{\ell}z_j^{-m_j(\mathcal{I})}$, which is uniquely determined by $\mathcal{I}$ and independent of the coordinate system up to the multiplication by an invertible function. Let $I(\mathfrak{a}^{(p)},\mathcal{I})$ ($p=1,\dots,s$) denote the ideal generated by $\xi(\mathcal{I})\cdot\mathfrak{a}^{(p)}$ and $\xi(\mathcal{I})$. The tuple of the ideal $\boldsymbol{I}(\mathcal{I})$ is uniquely determined by $\mathcal{I}$.

Remark 15.3.11. — The construction is independent of the choice of a coordinate system, and it can be globalized. Namely, let X be a complex manifold, and let D be a normal crossing divisor of X. If we are given a finite subset $\mathcal{I}\subset M(X,D)/H(X)$, we obtain the tuple of the ideals $\boldsymbol{I}(\mathcal{I})$, whose restriction to holomorphic coordinate neighbourhoods are given as above.

We also have the purely algebraic construction of the ideal $\boldsymbol{I}(\mathcal{I})$, when X is algebraic. □

Let $\widetilde{X}:=\Delta^n$ and $\widetilde{D}:=\bigcup_{j=1}^{\ell}\{\zeta_j=0\}$. Let $\psi:(\widetilde{X},\widetilde{D})\to(X,D)$ be a ramified covering. We have the induced subset $\psi^{-1}\mathcal{I}\subset M(\widetilde{X},\widetilde{D})/H(\widetilde{X})$ and the induced ideals

$\psi^{-1}\boldsymbol{I}(\mathcal{I}) := \big(\psi^{-1}I(\mathfrak{a}^{(p)},\mathcal{I})\cdot\mathcal{O}_{\widetilde{X}}\,\big|\,p=1,\dots,s\big)$. It is easy to see $\boldsymbol{I}(\psi^{-1}\mathcal{I}) = \psi^{-1}\boldsymbol{I}(\mathcal{I})$. If ψ is a ramified covering, we can reconstruct $\boldsymbol{I}(\mathcal{I})$ as the Galois descent of $\boldsymbol{I}(\psi^{-1}\mathcal{I})$.

15.3.4. Construction of a resolution. — Let us return to the setting in Section 15.3.1. Let P be any point of D. We take a coordinate neighbourhood U around P. We take a ramified covering $\varphi:(U^{\circ},D^{\circ})\to(U,D)$ such that either (UR) or (UR-SRP) is satisfied for $\varphi^*\theta$. Applying the construction in Sections 15.3.2.1–15.3.2.2, we obtain the subset $\mathcal{I}_P := \mathrm{Irr}(\varphi^*\theta)\subset M(U^{\circ},D^{\circ})/H(U^{\circ})$, and the set of logarithmic 1-forms $\boldsymbol{\kappa}(\varphi^*\theta)$. Applying the procedure of Section 15.3.3 to $\mathcal{I}_P$, we obtain the tuple of ideals $\boldsymbol{I}(\mathcal{I}_P)$ of $\mathcal{O}_{U^{\circ}}$. Applying the procedure in Section 15.1 to them, we obtain the complex analytic space U' with a birational projective morphism $U'\to U$. According to Lemma 15.1.1 and the remark in the last part of Section 15.3.3, we can globalize the construction, and thus we obtain the complex analytic space X' with the birational projective morphism $X'\to X$. Taking a resolution of singularities, we obtain a complex manifold X_1 with a birational projective morphism $F_1: X_1\to X$ such that (i) $D_1 := F_1^{-1}(D)$ is normal crossing, (ii) $X_1\setminus D_1\simeq X\setminus D$. We put $(E_1,\theta_1) := F_1^*(E,\theta)$.

Let P_1 be any point of D_1. We take a small neighbourhood U of $P=F_1(P_1)$. Let U_1 be a coordinate neighbourhood around P_1 such that $F_1(U_1)\subset U$. We take a ramified covering (U°,D°) of U, as above. We have $\mathcal{I}_P$ and $\boldsymbol{\kappa}(\varphi^*\theta)$ on (U°,D°) as above. We take a lift of $\mathcal{I}_P$ to $M(U^{\circ},D^{\circ})$ which is given by $\{\mathfrak{a}^{(p)}\}$. Let $\xi(\mathcal{I}_P)$ be as in Section 15.3.3. Let $\psi_1:(U_1^{\circ},D_1^{\circ})\to(U_1,D_1)$ be a ramified covering such that the composite $U_1^{\circ}\to U$ factors through U°, and so we have $F_1^{\circ}:(U_1^{\circ},D_1^{\circ})\to\big(U^{\circ},D^{\circ}\big)$. Then, either (UR) or (UR-SRP) holds for $\psi_1^*\theta_1$. We put $\mathfrak{a}_1^{(p)} := (F_1^{\circ})^*\mathfrak{a}^{(p)}$, and then $\mathrm{Irr}(\psi_1^*\theta_1)\subset M(U_1^{\circ},D_1^{\circ})/H(U_1^{\circ})$ is the same as $(F_1^{\circ})^*\mathcal{I}_P = \{\mathfrak{a}_1^{(p)}\}$. We also have $\boldsymbol{\kappa}(\psi_1^*\theta_1)\equiv(F_1^{\circ})^*\boldsymbol{\kappa}(\varphi^*\theta)$ modulo the exact holomorphic 1-forms. By construction, either one of the following holds, for each p:

(Case 1) : $I(\mathfrak{a}^{(p)},\mathcal{I}_P)\cdot\mathcal{O}_{U_1^{\circ}}$ is generated by $(F_1^{\circ})^*\xi(\mathcal{I}_P)$.

(Case 2) : $I(\mathfrak{a}^{(p)},\mathcal{I}_P)\cdot\mathcal{O}_{U_1^{\circ}}$ is generated by $(F_1^{\circ})^*\big(\xi(\mathcal{I}_P)\cdot\mathfrak{a}^{(p)}\big)$.

In the case 1, since $(F_1^{\circ})^*\xi(\mathcal{I}_P)\cdot\mathfrak{a}_1^{(p)}$ is the multiplication of $(F_1^{\circ})^*\xi(\mathcal{I}_P)$ with a holomorphic function, $\mathfrak{a}_1^{(p)}$ is holomorphic on U_1°. Let us consider the case 2. Let $(z_{1,1},\dots,z_{1,n})$ be a coordinate system of U_1° such that $D_1^{\circ}=\bigcup_{j=1}^{\ell_1}\{z_{1,j}=0\}$. Then, $(F_1^{\circ})^*\xi(\mathcal{I}_P)$ is the product of an invertible function with a monomial $\prod_{j=1}^{\ell_1}z_{1,j}^{q_j}$ for some $q_j\geqslant 0$. Since $(F_1^{\circ})^*\xi(\mathcal{I}_P)\cdot\mathfrak{a}_1^{(p)}$ divides $(F_1^{\circ})^*\xi(\mathcal{I}_P)$, we obtain that $\mathfrak{a}_1^{(p)}$ is the product of an invertible function with $\prod_{j=1}^{\ell_1}z_{1,j}^{r_j(p)}$ for some $r_j(p)\leqslant 0$.

We put $\mathfrak{a}_1^{(p,q)} := \mathfrak{a}_1^{(p)}-\mathfrak{a}_1^{(q)}$. Let $\widetilde{\mathcal{I}}_{P_1}\subset M(U_1^{\circ},D_1^{\circ})/H(U_1^{\circ})$ be given by $\{\mathfrak{a}_1^{(p,q)}\}$. Applying the procedure in Section 15.3.3 to $\widetilde{\mathcal{I}}_{P_1}$, we obtain the tuple of ideals $I(\widetilde{\mathcal{I}}_{P_1})$.

Applying the procedure in Section 15.1, making the globalization, and taking a resolution of singularities, we obtain a complex manifold X_2 with a birational projective morphism $F_{2,1} : X_2 \to X_1$ such that (i) $D_2 := F_{2,1}^{-1}(D_1)$ is normal crossing, (ii) $X_2 \setminus D_2 \simeq X_1 \setminus D_1$. We set $(E_2, \theta_2) := F_{2,1}^*(E_1, \theta_1)$ and $F_2 := F_1 \circ F_{2,1}$.

Let P_2 be any point of D_2. Let $P_1 := F_{2,1}(P_2)$ and $P := F_2(P_2)$. We take neighbourhoods U_1 and U of P_1 and P as above, respectively. We also take ramified coverings $(U_1^\circ, D_1^\circ) \to (U_1, D_1)$ and $(U^\circ, D^\circ) \to (U, D)$ as above. Let U_2 be a coordinate neighbourhood around P_2 such that $F_{2,1}(U_2) \subset U_1$. Let $\psi_2 : (U_2^\circ, D_2^\circ) \to (U_2, D_2)$ be a ramified covering such that the composite $U_2^\circ \to U_1$ factors through U_1°. So we have $F_{2,1}^\circ : (U_2^\circ, D_2^\circ) \to (U_1^\circ, D_1^\circ)$. Then, either (UR) or (UR-SRP) holds for $\psi_2^*\theta_2$. We put $\mathfrak{a}_2^{(p)} := (F_{2,1}^\circ)^*\mathfrak{a}_1^{(p)}$, and then $\mathrm{Irr}(\psi_2^*\theta_2)$ is the same as $\{\mathfrak{a}_2^{(p)}\}$. We also have $\boldsymbol{\kappa}(\psi_2^*\theta_2) \equiv (F_{2,1}^\circ)^*\boldsymbol{\kappa}(\psi_1^*\theta_1)$ modulo the holomorphic exact 1-forms. Let $(w_1, \ldots, w_n)$ be a coordinate system of (U_2°, D_2°) such that $D_2^\circ = \bigcup_{k=1}^{\ell_2}\{w_k = 0\}$. By construction of X_2, either one of the following holds for each $\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)}$:

(Case 1) : $\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)}$ is holomorphic on U_2°.

(Case 2) : If $\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)}$ admits a pole, $\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)}$ is the product of an invertible function with $\prod_{k=1}^{\ell_2} w_k^{s_k(p,q)}$ for some $s_k(p,q) \leqslant 0$. In this sense, $\mathrm{ord}(\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)})$ exists in $\mathbb{Z}_{\leqslant 0}^{\ell_2}$.

Moreover, the ideals $(F_{2,1}^\circ)^{-1}I\big(\mathfrak{a}_1^{(p,q)}, \widetilde{\mathcal{I}}_{P_1}\big)\cdot\mathcal{O}_{U_2^\circ}$ are totally ordered with respect to the inclusion relation. Hence, the set of $\mathrm{ord}(\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)})$ for non-holomorphic $\mathfrak{a}_2^{(p)} - \mathfrak{a}_2^{(q)}$ is totally ordered with respect to $\leqslant_{\mathbb{Z}^{\ell_2}}$. By construction of X_1, either one of the following holds for each p: (i) $\mathfrak{a}_2^{(p)}$ is holomorphic, or (ii) $\mathrm{ord}(\mathfrak{a}_2^{(p)})$ exists in $\mathbb{Z}_{\leqslant 0}^{\ell_2}$. Hence, $\mathrm{Irr}(\psi_2^*\theta_2)$ is a good set of irregular values.

By construction, each eigenvalue of $\psi_2^*\theta_2$ is of the form $d\mathfrak{a} + \kappa + \tau$, where (i) $\mathfrak{a}$ belongs to the good set of irregular values $\mathrm{Irr}(\psi_2^*\theta_2)$, (ii) $\kappa = \sum_{k=1}^{\ell_2} a_k \cdot dw_k/w_k$ ($a_k \in \mathcal{A}$), (iii) τ is a multi-valued holomorphic 1-form on U_2°. Hence θ_2 is $\mathcal{A}$-good, and thus the proof of Proposition 15.3.1 is finished. ∎

CHAPTER 16

KOBAYASHI-HITCHIN CORRESPONDENCE AND SOME APPLICATIONS

In this chapter, we study existence problem of pluri-harmonic metrics for good filtered flat bundles. As an application, we show the existence of a resolution of turning points for a meromorphic flat bundle. (See *Introduction.*)

In Section 16.1, we establish Kobayashi-Hitchin correspondence between good wild harmonic bundles and μ_L-polystable good filtered flat bundles with trivial characteristic numbers (Theorem 16.1.1). This is a rather straightforward generalization of the correspondence in the tame case.

We study in Sections 16.2–16.3 a characterization of semisimplicity of meromorphic flat bundles by the existence of $\sqrt{-1}\boldsymbol{R}$-good wild pluri-harmonic metric (Theorem 16.2.4). We also show the resolution of turning points for a meromorphic flat bundle on a projective variety (Theorem 16.2.1). These two results are intimately related. (They will be slightly refined in Section 16.4.)

In contrast to the tame case, we cannot directly apply Theorem 16.1.1 to show Theorem 16.2.4, because the Deligne-Malgrange filtered sheaf of a meromorphic flat bundle is not necessarily *good* even in the surface case. We need the existence of a resolution of turning points as in Theorem 16.2.1. Conversely, once we know Theorem 16.2.4, it is rather easy to show Theorem 16.2.1.

Our argument will proceed as follows. We have already established Theorem 16.2.1 for the surface case in [**68**] and Proposition 2.7.10. Applying Theorem 16.1.1, we obtain Theorem 16.2.4 for the surface case. Then, we will show a variant of Theorem 16.2.4 in the higher dimensional case, in which we do not assume the existence of a good Deligne-Malgrange lattice. By using it, we show Theorem 16.2.1 in the higher dimensional case. After that, it is easy to show Theorem 16.2.4. This is a nice interaction between the theories of wild harmonic bundles and meromorphic flat bundles.

In Section 16.4, we refine the results in Section 16.2, i.e., we consider the case where X is not necessarily projective but proper algebraic.

16.1. Kobayashi-Hitchin correspondence for good filtered flat bundles

Let X be an n-dimensional connected smooth projective variety with an ample line bundle L, and let D be a simple normal crossing hypersurface of X with the decomposition into irreducible components $D = \bigcup_{i\in\Lambda} D_i$. Let $(\boldsymbol{E}_*, \nabla)$ be a μ_L-stable good filtered flat bundle on (X, D) with trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, and we put $(E, \nabla) := (\boldsymbol{E}_*, \nabla)_{|X\setminus D}$. Recall $\text{par-c}_1(\boldsymbol{E}_*) = 0$ (Corollary 14.5.2). For each $\boldsymbol{c} \in \boldsymbol{R}^{\Lambda}$, we have the determinant line bundle $\det\bigl({}_{\boldsymbol{c}}E\bigr)$ of the bundle ${}_{\boldsymbol{c}}E$, on which we have the induced parabolic structure and the induced flat connection. Thus we obtain the canonically determined filtered flat bundle $\bigl(\det \boldsymbol{E}_*, \nabla\bigr)$ on (X, D) of rank one. We also have $\text{par-c}_1\bigl(\det \boldsymbol{E}_*\bigr) = \text{par-c}_1\bigl(\boldsymbol{E}_*\bigr) = 0$. Therefore, we can take a pluri-harmonic metric $h_{\det E}$ of $(\det(E), \nabla)$ which is adapted to the parabolic structure of $\det \boldsymbol{E}_*$. It is determined up to positive constant multiplication.

Theorem 16.1.1. — *There exists a unique pluri-harmonic metric h of (E, ∇) with the following properties:*
- $\det(h) = h_{\det E}$.
- *(E, ∇, h) is a good wild harmonic bundle on $X \setminus D$.*
- *h is adapted to the parabolic structure of $\boldsymbol{E}_*$.*

Proof. — It can be proved by the argument in the tame case [**69**], after the preparation in Chapters 13–14. Hence, we give only an indication.

Let us begin with a remark. If we have a pluri-harmonic metric h of (E, ∇) adapted to $\boldsymbol{E}_*$, the corresponding harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is good and wild, according to Proposition 13.5.2. Let us show the uniqueness of such a metric. Let h_i $(i = 1, 2)$ be pluri-harmonic metrics of (E, ∇) which are adapted to $\boldsymbol{E}_*$ such that $\det(h_i) = h_{\det(E)}$. Let $C \subset X$ be any sufficiently ample and generic complete intersection curve, which intersects with D transversally. The restriction of $(\boldsymbol{E}_*, \nabla)_{|C}$ is stable, according to Proposition 13.2.1. Because of the weak norm estimate of good wild harmonic bundles (Theorem 7.4.3), the metrics $h_{i|C\setminus D}$ are adapted to $\boldsymbol{E}_{*|C}$. Then, we obtain $h_{1|C\setminus D} = h_{2|C\setminus D}$ due to Proposition 13.4.1. Since such curves C cover a Zariski open subset of $X \setminus D$, we obtain $h_1 = h_2$ on $X \setminus D$. Thus, we obtain the uniqueness.

Let us show the existence. In the curve case, it was shown by Biquard and Boalch (Proposition 13.4.1). Let us consider the surface case.

Lemma 16.1.2. — *We have the desired pluri-harmonic metric h in the case* $\dim X = 2$.

Proof. — The argument is essentially the same as that in the proof of Theorem 5.4 in [**69**]. So, we give only an indication. Let $({}_{\boldsymbol{c}}E, \boldsymbol{F})$ be the $\boldsymbol{c}$-truncation. For a large integer d, we put $\varepsilon = 1/d$, and we take an ε-perturbation $\boldsymbol{F}^{(\varepsilon)}$ as explained in (II) of Section 13.7. Then, we have the Hermitian metric $h^{(\varepsilon)}_{HE}$ of (E, ∇) such that (i) it is adapted to $\boldsymbol{F}^{(\varepsilon)}$, (ii) $\det(h^{(\varepsilon)}_{HE}) = h_{\det}$, (iii) it satisfies the conditions in Proposition 14.4.1.

We would like to take the limit of the sequence $\{h^{(\varepsilon)}_{HE}\}$ in $\varepsilon \to 0$. Let m be large. By using Lemma 13.4.6, Lemma 13.4.8 and the argument in Section 5.2 of [**69**], we obtain the pluri-harmonic metric h of (E, ∇) satisfying $\det(h) = h_{\det(E)}$ and the following property:

- $h_{|s^{-1}(0)}$ is adapted to $\boldsymbol{E}_{*|s^{-1}(0)}$ for any sufficiently general $s \in H^0(X, L^m)$.

According to Proposition 13.5.1, (E, ∇, h) is a good wild harmonic bundle, and h is adapted to $\boldsymbol{E}_*$. Thus, the proof of Lemma 16.1.2 is finished. □

Now, we can show the existence in the general case by using the same argument as that in the proof of Theorem 5.16 of [**69**], and thus the proof of Theorem 16.1.1 is finished. □

We say that a μ_L-polystable good filtered flat bundle has trivial characteristic numbers, if each μ_L-stable component has trivial characteristic numbers.

Corollary 16.1.3. — *Let $(\boldsymbol{E}_*, \nabla)$ be a good filtered flat bundle on (X, D). We put $E := \boldsymbol{E}_{*|X \setminus D}$. It is μ_L-polystable with trivial characteristic numbers, if and only if there exists a pluri-harmonic metric h of (E, ∇) adapted to the prolongment $\boldsymbol{E}_*$.*

Proof. — The "only if" part follows from Theorem 16.1.1. Let us see the "if" part. Let (E, ∇, h) be a good wild harmonic bundle, and let $(\boldsymbol{E}_*, \nabla)$ be the associated filtered flat bundle. According to Proposition 13.6.1, $(\boldsymbol{E}_*, \nabla)$ is μ_L-polystable with $\text{par-deg}_L(\boldsymbol{E}_*) = 0$. Moreover, if $(\boldsymbol{E}_{0*}, \nabla_0)$ is a stable component of $(\boldsymbol{E}_*, \nabla)$, and then the restriction of h to $E_0 := \boldsymbol{E}_{0*|X \setminus D}$ gives the pluri-harmonic metric of (E_0, ∇_0) adapted to the prolongment $\boldsymbol{E}_{0*}$. Then, we obtain $\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_{0*}) = 0$ from Proposition 13.6.4. □

We also obtain the following corollary on the uniqueness of pluri-harmonic metrics, from Theorem 16.1.1 and Proposition 13.6.1.

Corollary 16.1.4. — *Let $(\boldsymbol{E}_*, \nabla)$ be a μ_L-polystable good filtered flat bundle on (X, D) with the trivial characteristic numbers. Let $(\boldsymbol{E}_*, \nabla) = \bigoplus_{i \in \Gamma} (\boldsymbol{E}_{i*}, \nabla_i) \otimes \boldsymbol{C}^{m(i)}$ be the canonical decomposition, where $(\boldsymbol{E}_{i*}, \nabla_i)$ are μ_L-stable. We take a pluri-harmonic metric h_i of (E_i, ∇_i) adapted to the prolongment $\boldsymbol{E}_{i*}$ for each $i \in \Gamma$. Then, any pluri-harmonic metric h adapted to $\boldsymbol{E}_*$ is of the form $h = \bigoplus h_i \otimes g_i$, where g_i are Hermitian metrics of $\boldsymbol{C}^{m(i)}$.* □

Remark 16.1.5. — Recall that, for a given good filtered ∇-flat bundle $(\boldsymbol{E}_*, \nabla)$, we have a deformation $(\boldsymbol{E}^{(2)}_*, \nabla)$ as explained in Section 4.5.2. Although we considered a pluri-harmonic metric h adapted to $\boldsymbol{E}_*$, it might be more natural to consider a pluri-harmonic metric adapted to $\boldsymbol{E}^{(2)}_*$. See the proof of Theorem 19.4.1. □

16.2. Applications to algebraic meromorphic flat bundles

16.2.1. Existence of a good model. — Let X be an n-dimensional smooth connected projective variety with a simple normal crossing hypersurface D. We will prove the following theorem in Section 16.3.

Theorem 16.2.1. — *Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection on (X,D). Then, there exists a smooth projective variety X' with a birational morphism $\varphi : X' \to X$ such that (i) $D' := \varphi^{-1}(D)$ is simply normal crossing, (ii) $X' \setminus D' \simeq X \setminus D$, (iii) the Deligne-Malgrange lattice of $\varphi^*(\mathcal{E},\nabla)$ is good.*

Before going into the proof, we give a direct corollary. Let K be a finite extension of $\boldsymbol{C}(x_1,\ldots,x_n)$. Let M be a finite dimensional K-vector space with $\mathrm{Der}(K/\boldsymbol{C})$-action. We have the following corollary.

Corollary 16.2.2. — *There exist a smooth projective variety X with $K(X) \simeq K$, and a meromorphic connection $(\mathcal{E},\nabla)$ on X with $(\mathcal{E},\nabla)\otimes K \simeq M$, such that the Deligne-Malgrange lattice of $(\mathcal{E},\nabla)$ is good.* □

Remark 16.2.3. — Very recently, Kedlaya established the existence of resolution of turning points in a more general situation with a completely different method. See [**46**] and [**47**]. □

16.2.2. Characterization of semisimplicity. — Let X be an n-dimensional smooth connected projective variety, and let D be a simple normal crossing hypersurface of X. Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection on (X,D). We will prove the following theorem in Section 16.3.

Theorem 16.2.4. — *$(\mathcal{E},\nabla)$ is semisimple, if and only if the following holds:*
- *Let $\varphi : \widetilde{X} \to X$ be a birational projective morphism such that the Deligne-Malgrange filtered flat bundle $(\widetilde{\boldsymbol{E}}_*^{DM},\nabla)$ associated to $\varphi^*(\mathcal{E},\nabla)$ is good, as in Theorem 16.2.1. Then, there exists a pluri-harmonic metric h of $(\widetilde{\mathcal{E}},\widetilde{\nabla}) := \varphi^*(\mathcal{E},\nabla)_{|\widetilde{X}-\varphi^{-1}(D)}$ which is adapted to $\widetilde{\boldsymbol{E}}_*^{DM}$. Note that $(\widetilde{\mathcal{E}},\widetilde{\nabla},h)$ is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle.*

We give a complement on the uniqueness. Such a metric h is unique up to obvious ambiguity as in Corollary 16.1.4, which follows from Proposition 13.6.1.

Proposition 16.2.5. — *Let $(\mathcal{E},\nabla)$ be semisimple, and let h be the metric as in Theorem 16.2.4. Then, (i) if $(\mathcal{E},\nabla)$ is simple, h is uniquely determined up to constant multiplication, (ii) the canonical decomposition of $(\mathcal{E},\nabla)$ is orthogonal with respect to h, (iii) if $(\mathcal{E},\nabla)$ is a tensor product of a simple meromorphic flat bundle $(\mathcal{E}',\nabla')$ and a vector space V, then h is the tensor product of h' for $(\mathcal{E}',\nabla')$ and the metric of V.* □

16.2.3. Good Deligne-Malgrange lattice. — It seems better to give an explicit statement for the filtered flat bundle associated to a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle, which is implicitly used in the statement of Theorem 16.2.4. Let X be a complex manifold, and let D be a simple normal crossing hypersurface of X.

Proposition 16.2.6. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on (X, D). It is $\sqrt{-1}\boldsymbol{R}$-good wild, if and only if $(\mathcal{P}_*\mathcal{E}^1, \mathbb{D}^1)$ is the Deligne-Malgrange filtered bundle associated to the meromorphic flat bundle $(\mathcal{P}\mathcal{E}^1, \mathbb{D}^1)$. In particular, $\mathcal{P}_0\mathcal{E}^1$ is the good Deligne-Malgrange lattice of $(\mathcal{P}\mathcal{E}^1, \mathbb{D}^1)$ in this case.*

Proof. — In the case $\dim X = 1$, the claim follows from the comparison of the KMS-spectra in Proposition 8.2.1. The general case can be reduced to the above case. □

16.2.4. Pull-back. — We give an application of Theorem 16.2.4. Let X and Y be irreducible smooth projective varieties. Let $F : Y \to X$ be a rational morphism. Let $(\mathcal{V}, \nabla)$ be a semisimple meromorphic flat connection on X. Assume that $F(Y)$ is not contained in the pole of $(\mathcal{V}, \nabla)$. We have the meromorphic flat bundle $F^*(\mathcal{V}, \nabla)$ on Y.

Theorem 16.2.7. — *$F^*(\mathcal{V}, \nabla)$ is also semisimple.*

Proof. — By replacing X birationally, we may assume the following:

- We have a simple normal crossing hypersurface D_X such that $(\mathcal{V}, \nabla)$ is a semisimple meromorphic flat bundle on (X, D_X).
- The Deligne-Malgrange filtered flat bundle $\boldsymbol{V}^{DM}_{X\,*}$ associated to $(\mathcal{V}, \nabla)$ is good.

We take a good wild pluri-harmonic metric h_X of $(\mathcal{V}, \nabla)_{|X \setminus D_X}$, adapted to $\boldsymbol{V}^{DM}_{X\,*}$. Let $(E_X, \overline{\partial}_{E_X}, \theta_X, h_X)$ denote the corresponding good wild harmonic bundle on $X \setminus D_X$. The associated meromorphic flat bundle $(\mathcal{P}\mathcal{E}^1_X, \mathbb{D}^1_X)$ is naturally isomorphic to $(\mathcal{V}, \nabla)$. By construction, θ is good and wild. Moreover, the eigenvalues of the endomorphism $\mathrm{Res}_{D_{X,i}}(\theta)$ on ${}^i\mathrm{Gr}^F({}^\diamond E_X)$ on $D_{X,i}$ are purely imaginary.

By replacing Y birationally, we may assume to have a regular morphism $F : Y \to X$ such that $D_Y := F^{-1}(D_X)$ is a simple normal crossing hypersurface.

Lemma 16.2.8. — *$(E_Y, \overline{\partial}_{E_Y}, \theta_Y, h_Y) := F^{-1}(E_X, \overline{\partial}_{E_X}, \theta_X, h_X)$ is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on $Y \setminus D_Y$.*

Proof. — Note that $F^*({}^\diamond E_X)$ gives a prolongment of E_Y to a locally free $\mathcal{O}_Y$-module on Y. Let P be any point of D_Y. We take a small holomorphic coordinate neighbourhood V around P. We also take a small holomorphic coordinate neighbourhood U around $F(P)$. We put $D_U := D_X \cap U$ and $D_V := D_Y \cap V$. We can take a ramified covering $\varphi_U : (\widetilde{U}, \widetilde{D}_U) \to (U, D_U)$ such that we have the decomposition:

$$\varphi_U^*({}^\diamond E_X, \theta_X) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta_X, F(P))} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}). \tag{343}$$

Here, $\theta_{\mathfrak{a}} - d\mathfrak{a} \cdot \mathrm{id}_{E_{\mathfrak{a}}}$ are logarithmic with respect to $\varphi_U^*({}^\diamond E_X)$ for any $\mathfrak{a} \in \mathrm{Irr}(\theta_X, F(P))$. We can take a ramified covering $\varphi_V : (\widetilde{V}, \widetilde{D}_V) \to (V, D_V)$ such that the composite

$F \circ \varphi_V : \widetilde{V} \to U$ factors through $\widetilde{U}$, i.e., there exists $\widetilde{F} : \widetilde{V} \to \widetilde{U}$ such that $F \circ \varphi_V = \varphi_U \circ \widetilde{F}$. From (343), we obtain the decomposition around P:

$$\varphi_V^*(F^*({}^\diamond E_X), \theta_Y) = \bigoplus_{\mathfrak{a}} \big(\widetilde{F}^*(E_{\mathfrak{a}}), \theta_{Y,\mathfrak{a}}\big).$$

And $\theta^{\rm reg}_{\widetilde{V}} := \bigoplus \big(\theta_{Y,\mathfrak{a}} - d\widetilde{F}^*\mathfrak{a} \cdot \mathrm{id}_{\widetilde{F}^*(E_{\mathfrak{a}})}\big)$ is logarithmic.

Let us show that the eigenvalues of the residue of θ_Y are constant on each component of D_Y, and that they are purely imaginary. Let P be a smooth point of D_Y. Let V and U be as above. Let $(w_1, \dots, w_m)$ and $(z_1, \dots, z_n)$ be holomorphic coordinate on $\widetilde{V}$ and $\widetilde{U}$ respectively, such that $\widetilde{D}_V = \{w_1 = 0\}$ and $\widetilde{D}_U = \bigcup_{j=1}^{\ell}\{z_j = 0\}$. Let d_i $(i = 1, \dots, \ell)$ be the order of $\widetilde{F}^*(z_i)$ with respect to w_1. Let Q be any point of $\widetilde{D}_V$. Then, $\mathrm{Res}_{w_1}(\varphi_V^*\theta_Y)_{|Q}$ on $\varphi_V^* F^*({}^\diamond E_X)_{|Q}$ is given by $\sum_{i=1}^{\ell} d_i \cdot \mathrm{Res}_{z_i}(\varphi_U^*\theta_X)_{|F(Q)}$. Hence, we obtain that the eigenvalues of $\mathrm{Res}_{w_1}(\varphi_V^*\theta_Y)$ are constant on $\widetilde{D}_V$, and they are purely imaginary.

We have the expression $\theta^{\rm reg}_{\widetilde{V}} = f_1 \cdot dw_1/w_1 + \sum_{j=2}^{m} f_j \cdot dw_j$. Then, we can conclude that the coefficients of $P(t) := \det(t - f_1)_{|\widetilde{D}_V}$ are constant, and that the solutions of $P(t) = 0$ are purely imaginary. Thus, the proof of Lemma 16.2.8 is finished. □

We have the meromorphic flat bundle $(\mathcal{P}\mathcal{E}_Y^1, \mathbb{D}_Y^1)$ associated to $(E_Y, \overline{\partial}_{E_Y}, \theta_Y, h_Y)$. We also have the good lattices $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_Y^1$ of $\mathcal{P}\mathcal{E}_Y^1$. We have the naturally defined morphism $F^*(\mathcal{P}_0\mathcal{E}_X^1, \mathbb{D}_X^1) \to (\mathcal{P}_0\mathcal{E}_Y^1, \mathbb{D}_Y^1)$, which induces the morphism $\iota : F^*(\mathcal{P}\mathcal{E}_X^1) \to \mathcal{P}\mathcal{E}_Y^1$ of locally free $\mathcal{O}_Y(*D_Y)$-modules. Since the restriction of ι to $Y \setminus D_Y$ is an isomorphism, ι is an isomorphism on Y. Hence, $F^*(\mathcal{V}, \nabla)$ has the good lattices $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_Y^1$. By using Proposition 8.2.1, we can show that $\mathcal{P}_*\mathcal{E}_Y^1 = \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_Y^1 \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^{\ell}\big)$ is the Deligne-Malgrange filtered flat bundle associated to $F^*(\mathcal{V}, \nabla)$. Then, we can conclude that $F^*(\mathcal{V}, \nabla)$ is semisimple due to Theorem 16.2.4. □

16.3. Proof of Theorems 16.2.1 and 16.2.4

16.3.1. Existence of a pluri-harmonic metric. — Let X be an n-dimensional smooth connected projective variety, and let D be a simple normal crossing hypersurface of X. Let $(\mathcal{E}, \nabla)$ be a flat meromorphic connection on (X, D).

Proposition 16.3.1. — *If $(\mathcal{E}, \nabla)$ is simple, there exists a pluri-harmonic metric h of $(E, \nabla) := (\mathcal{E}, \nabla)_{|X \setminus D}$ with the following property, which is unique up to positive constant multiplication:*

- *Let P be any smooth point of D around which $(\mathcal{E}, \nabla)$ is good. Then, (E, ∇, h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle around P, and h is adapted to the Deligne-Malgrange filtered bundle $\boldsymbol{E}_*^{DM}$ around P.*

In particular, if the Deligne-Malgrange filtered bundle $\boldsymbol{E}_*^{DM}$ associated to $(\mathcal{E}, \nabla)$ is good, then h is adapted to $\boldsymbol{E}_*^{DM}$, and (E, ∇, h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on (X, D).

Proof. — First, let us consider the case $\dim X = 2$. We can take an appropriate birational projective morphism $\varphi : \widetilde{X} \to X$ such that $\varphi^*(\mathcal{E}, \nabla)$ has a good formal structure. (See [**68**].) Note that $\varphi^*(\mathcal{E}, \nabla)$ is also simple. So, we may assume that $(\mathcal{E}, \nabla)$ has a good formal structure from the beginning. Then, the associated Deligne-Malgrange filtered bundle is good (Proposition 2.7.10). Because $\varphi^*(\mathcal{E}, \nabla)$ is simple, the associated Deligne-Malgrange filtered bundle is μ_L-stable. We also have the vanishing of the parabolic characteristic numbers, according to Corollary 14.3.4. Hence, we obtain the desired pluri-harmonic metric due to Theorem 16.1.1 (or Lemma 16.1.2).

The higher dimensional case can be reduced to the surface case by using Corollary 13.2.3 and an argument in the proof of Theorem 5.16 of [**69**]. □

16.3.2. Pull-back via the birational projective morphism. — Let $\varphi : X' \to X$ be any birational projective morphism. If $(\mathcal{E}, \nabla)$ is simple, then $\varphi^*(\mathcal{E}, \nabla)$ is also simple. Therefore, we have pluri-harmonic metrics $h_{\mathcal{E}}$ and $h_{\varphi^*\mathcal{E}}$ for $(\mathcal{E}, \nabla)$ and $\varphi^*(\mathcal{E}, \nabla)$ as in Proposition 16.3.1.

Lemma 16.3.2. — *$\varphi^* h_{\mathcal{E}} = a \cdot h_{\varphi^*\mathcal{E}}$ for some $a > 0$.*

Proof. — Let C be a sufficiently generic and ample curve in X. Then, $h_{\mathcal{E}|C}$ and $h_{\varphi^*\mathcal{E}|C}$ are the pluri-harmonic metrics for the simple meromorphic connection $(\mathcal{E}, \nabla)_{|C}$. Thus, we have $\varphi^* h_{\mathcal{E}|C} = a_C \cdot h_{\varphi^*\mathcal{E}|C}$ because of the uniqueness. Since such curves C cover Zariski open subsets of X and X', the claim of the lemma follows. □

16.3.3. Proof of Theorem 16.2.1 in the case that $(\mathcal{E}, \nabla)$ is simple. — Assume that $(\mathcal{E}, \nabla)$ is simple. We can take a pluri-harmonic metric $h_{\mathcal{E}}$ as in Proposition 16.3.1. Let $(E, \overline{\partial}_E, \theta, h_{\mathcal{E}})$ be the corresponding harmonic bundle on $X \setminus D$.

Lemma 16.3.3. — *There exists a birational projective morphism $F : X_1 \to X$ with the following properties:*

- *$D_1 := F^{-1}(D)$ is a simple normal crossing hypersurface, and $X_1 \setminus D_1 \simeq X \setminus D$.*
- *We put $(E_1, \overline{\partial}_{E_1}, h_1, \theta_1) := F^*(E, \overline{\partial}_E, h, \theta)$. Then, θ_1 is generically $\sqrt{-1}\boldsymbol{R}$-good, and (SRP) holds. (See Section* 15.3.1*.)*

Proof. — We take a covering $X = \bigcup \mathcal{U}^{(p)}$ by affine Zariski open subsets $\mathcal{U}^{(p)} = \operatorname{Spec} R^{(p)}$ with etale morphisms $\varphi_p : \mathcal{U}^{(p)} \to \boldsymbol{C}^n$ such that $\varphi_p^{-1}\big(\bigcup_{i=1}^{\ell(p)}\{z_i = 0\}\big) = \mathcal{U}_p \cap D$. Let $g^{(p)} = \prod_{i=1}^{\ell(p)} \varphi_p^{-1}(z_i)$ be the defining equation of $D \cap \mathcal{U}^{(p)}$. Let $R^{(p)}_{g^{(p)}}$ denote the localization of $R^{(p)}$ with respect to $g^{(p)}$. Let $\tau_i^{(p)} := \varphi_p^* dz_i / z_i$ for $i = 1, \dots, \ell(p)$, and $\tau_i^{(p)} := \varphi_p^* dz_i$ for $i = \ell(p)+1, \dots, n$. They give a frame of $\Omega^1(\log D)$ on $\mathcal{U}^{(p)}$. We have the expression $\theta = \sum f_i^{(p)} \tau_i^{(p)}$. We obtain the characteristic polynomials $\mathcal{P}_i^{(p)}(T) := \det\big(T - f_i^{(p)}\big)$. Let P be any point of D around which $(E, \overline{\partial}_E, \theta, h)$ is good wild. Then, the coefficients of $\mathcal{P}_i^{(p)}(T)$ are meromorphic functions around P whose poles are contained in D. Hence, we obtain $\mathcal{P}_i^{(p)}(T) \in R^{(p)}_{g^{(p)}}[T]$.

Applying Proposition 15.2.1 to each $\mathcal{P}_i^{(p)}$, we obtain a smooth projective variety $X_{1,i}^{(p)}$ with a regular projective birational morphism $F_{1,i}^{(p)} : X_{1,i}^{(p)} \to X$ such that $(F_{1,i}^{(p)})^*\mathcal{P}_i^{(p)}$ satisfies the condition in Proposition 15.2.1. Taking the fiber products of $X_{1,i}^{(p)}$ over X, the normalization, and the resolution of the singularities, we obtain a birational projective morphism $F_1 : X_1 \to X$ such that each $F_1^*\mathcal{P}_i^{(p)}$ satisfies the condition in Proposition 15.2.1. The morphism factors through each $X_{1,i}^{(p)}$.

Due to Lemma 16.3.2, θ_1 is generically $\sqrt{-1}\boldsymbol{R}$-good. Let us show that (SRP) holds. We put $\mathcal{U}_1^{(p)} := F_1^{-1}(\mathcal{U}^{(p)})$. For any $P \in D_1 \cap \mathcal{U}_1^{(p)}$, we take a holomorphic coordinate $(U, z_1, \dots, z_n)$ around P such that $U \cap D_1 = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We take a ramified covering $\psi_P : \widetilde{U} \to U$ given by $\psi_P(\zeta_1, \dots, \zeta_n) = (\zeta_1^M, \dots, \zeta_\ell^M, \zeta_{\ell+1}, \dots, \zeta_n)$, where M is divisible by $r!$. We put $\widetilde{D} := \psi_P^{-1}D_1$. Let $\widetilde{\psi}_P := F_1 \circ \psi_P$. By construction, the eigenvalues of $\widetilde{\psi}_P^* f_i^{(p)}$ are of the form $\beta+\gamma$, where $\beta \in \mathcal{O}_{\widetilde{U}}(*\widetilde{D})$ and γ are multivalued holomorphic functions on $\widetilde{U}$. For the expression $\psi_P^*\theta = \sum_{i=1}^{\ell} f_i \cdot d\zeta_i/\zeta_i + \sum_{i=\ell+1}^{n} f_i \cdot d\zeta_i$, we have $f_i = \sum_j a_{i,j} \cdot \widetilde{\psi}_p^* f_j^{(p)}$, where $a_{i,j} \in H(\widetilde{U}_P)$. Note the commutativity of $f_j^{(p)}$. Then, it is easy to observe that (SRP) holds for θ_1. Thus we obtain Lemma 16.3.3. □

Hence, we may and will assume that (SRP) holds from the beginning. Applying Proposition 15.3.1 to θ, we can take a birational projective morphism $\varphi : X' \to X$ such that $\varphi^*\theta$ is $\sqrt{-1}\boldsymbol{R}$-good, i.e., $(\widetilde{E}, \widetilde{\nabla}, \widetilde{h}) := \varphi^*(E, \nabla, h)$ is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle. Then, we have the associated prolongment $(\widetilde{\boldsymbol{E}}_*, \widetilde{\nabla})$, which is good. Due to Lemma 16.3.2 and Proposition 16.3.1, it is the same as the Deligne-Malgrange filtered flat sheaf of $\varphi^*\mathcal{E}$ around a general point P of the smooth part of D. Hence, we can conclude that $\widetilde{\boldsymbol{E}}_*$ is the same as the Deligne-Malgrange filtered sheaf associated to $(\varphi^*\mathcal{E}, \widetilde{\nabla})$, and it is good. Thus, we obtain the desired resolution in the case where $(\mathcal{E}, \nabla)$ is simple.

16.3.4. End of proof of Theorems 16.2.1 and 16.2.4. — Let us consider the general case. A meromorphic flat connection $(\mathcal{E}, \nabla)$ is the extension of simple meromorphic flat connections $(\mathcal{E}_i, \nabla_i)$ $(i = 1, \dots, N)$. We can take a birational projective morphism $\varphi : X' \to X$ such that (i) $D' := \varphi^{-1}(D)$ is simple normal crossing, (ii) $X' \setminus D' \simeq X \setminus D$, (iii) any $\varphi^*(\mathcal{E}_i, \nabla_i)$ have the good Deligne-Malgrange lattices. We may also assume that the union of the set of the irregular values of $\varphi^*(\mathcal{E}_i, \nabla_i)$ are also good around each point of φ^*D. (For example, we only have to apply Proposition 15.3.1, to the meromorphic Higgs bundle corresponding to $\bigoplus(\mathcal{E}_i, \nabla_i)$.) Then, the Deligne-Malgrange lattice of $\varphi^*(\mathcal{E}, \nabla)$ is good, according to Corollary 2.7.11. Thus, the proof of Theorem 16.2.1 is finished.

Let us show Theorem 16.2.4. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection on (X, D). We take a birational projective morphism $\varphi : \widetilde{X} \to X$ such that the Deligne-Malgrange filtered bundle $\widetilde{\boldsymbol{E}}_*^{DM}$ associated to $(\widetilde{\mathcal{E}}, \widetilde{\nabla}) := \varphi^*(\mathcal{E}, \nabla)$ is good (Theorem 16.2.1). Note that $(\mathcal{E}, \nabla)$ is semisimple, if and only if $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ is semisimple. If $(\mathcal{E}, \nabla)$

is semisimple, we can take a pluri-harmonic metric h with the desired property due to Proposition 16.3.1. Conversely, if we have a pluri-harmonic metric h of $(\widetilde{\mathcal{E}},\widetilde{\nabla})$ adapted to $\widetilde{\boldsymbol{E}}_*^{DM}$, then $(\widetilde{\boldsymbol{E}}_*^{DM},\widetilde{\nabla})$ is μ_L-polystable due to Proposition 13.6.1. By Lemma 13.2.2, we obtain that $(\widetilde{\mathcal{E}},\widetilde{\nabla})$ is semisimple, and thus $(\mathcal{E},\nabla)$ is semisimple. □

16.4. Minor refinement of the result in Section 16.2

Let X be a smooth proper algebraic variety over $\boldsymbol{C}$. Let D be a simply normal crossing hypersurface of X. Let $(\mathcal{E},\nabla)$ be a meromorphic flat bundle. We set $(E,\nabla):=(\mathcal{E},\nabla)_{|X\setminus D}$.

Proposition 16.4.1. — *$(\mathcal{E},\nabla)$ is semisimple, if and only if there exists a pluri-harmonic metric h of (E,∇) such that the following holds:*

- *Let P be any smooth point of D around which $(\mathcal{E},\nabla)$ is good. Then, (E,∇,h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle around P, and h is adapted to the Deligne-Malgrange filtered bundle $\boldsymbol{E}_*^{DM}$ around P.*

Such a metric is unique up to an obvious ambiguity, in the following sense:

- *If $(\mathcal{E},\nabla)$ is simple, it is unique up to positive constant multiplication.*
- *Let $(\mathcal{E},\nabla)=\bigoplus_i(\mathcal{E}_i,\nabla_i)\otimes V_i$ be the canonical decomposition, i.e., $(\mathcal{E}_i,\nabla_i)$ are simple, V_i are vector spaces, and $(\mathcal{E}_i,\nabla_i)\not\simeq(\mathcal{E}_j,\nabla_j)$ for $i\neq j$. Then, the metric h is of the form $\bigoplus h_i\otimes g_i$, where h_i is a pluri-harmonic metric for $(\mathcal{E}_i,\nabla_i)$ as above, and g_i are metrics of V_i.*

Proof. — Assume that $(\mathcal{E},\nabla)$ is semisimple. By using Chow's lemma and Theorem 16.2.1, we can take a projective birational morphism $\varphi:X'\to X$ such that (i) $D':=\varphi^{-1}(D)$ is simply normal crossing, (ii) the Deligne-Malgrange lattice of $(\mathcal{E}',\nabla'):=\varphi^*(\mathcal{E},\nabla)$ is a locally free $\mathcal{O}_X$-module and good. We put $(E',\nabla'):=(\mathcal{E}',\nabla')_{|X'\setminus D'}$. By using Theorem 16.1.1 or Theorem 16.2.4, we can take a pluri-harmonic metric h' of (E',∇') such that $\mathcal{P}_*E'$ is the same as the Deligne-Malgrange filtered bundle of $(\mathcal{E}',\nabla')$. There is a Zariski closed subset $W\subset X$ such that φ induces $X'\setminus\varphi^{-1}(W)\simeq X\setminus W$.

Lemma 16.4.2. — *There exists a pluri-harmonic metric h of (E,∇) such that $h_{|X\setminus W}=h'_{|X'\setminus\varphi^{-1}(W)}$ under the natural identification.*

Proof. — According to Theorem 6.1.3 of [**73**], (E',∇',h') naturally induces a pure twistor D-module $\mathcal{T}(E')$ of weight 0 with the polarization $(\mathrm{id},\mathrm{id})$ on $X'\setminus D'$. By the Hard Lefschetz theorem for regular polarized pure twistor D-module (Theorem 6.1.1 of [**73**]), we obtain the pure twistor D-module $\varphi_\dagger^0\mathcal{T}(E')$ of weight 0 on $X\setminus D$. Let $\mathcal{T}_E$ be the direct summand of $\varphi_\dagger^0\mathcal{T}(E')$ whose strict support is $X\setminus D$. It is easy to observe that the underlying D-module is naturally isomorphic to (E,∇), because of the regularity. Moreover, by the Hard Lefschetz theorem, $\mathcal{T}_E$ is equipped with the polarization whose restriction to $X \setminus W$ is the same as the polarization of $\mathcal{T}(E')$ under the natural identification. Thus, we obtain a pluri-harmonic metric h of (E, ∇). □

If $P \in D \setminus W$, $(\mathcal{E}, \nabla)$ is good around P, and h is adapted to $\boldsymbol{E}_*^{DM}$ by construction. Let P be any point of D around which $(\mathcal{E}, \nabla)$ is good. By applying Proposition 13.5.1, we obtain that h is adapted to $\boldsymbol{E}_*^{DM}$. Hence, h has the desired property.

Conversely, assume that $(\mathcal{E}, \nabla)$ is equipped with a metric h as above. Let $\mathcal{E}_1 \subset \mathcal{E}$ be an $\mathcal{O}_X(*D)$-submodule such that $\nabla \mathcal{E}_1 \subset \mathcal{E}_1 \otimes \Omega^1_X(*D)$. We put $E_1 := \mathcal{E}_{1|X \setminus D}$, which is a flat subbundle of $E = \mathcal{E}_{|X \setminus D}$. Let E_2 denote the orthogonal complement of E_1 in E with respect to h.

Lemma 16.4.3. — *E_2 is a flat subbundle of E.*

Proof. — Let us consider a morphism F from a smooth projective curve C to X such that (i) $F(C)$ is not contained in D, (ii) $(\mathcal{E}, \nabla)$ is good around each point of $F(C) \cap D$. We have $F^*(\mathcal{E}_1, \nabla) \subset F^*(\mathcal{E}, \nabla)$, and F^*h. Note that semisimplicity is related with the polystability as in Lemma 13.2.2. Applying Corollary 16.1.4, we obtain that F^*E_2 is flat with respect to $F^*\nabla$. Because a Zariski open set of X is covered by such curves, we obtain that E_2 is also flat. □

Let $\boldsymbol{E}_*^{DM}$ and $\boldsymbol{E}_{1*}^{DM}$ be the Deligne-Malgrange filtered sheaf associated to $(\mathcal{E}, \nabla)$ and $(\mathcal{E}_1, \nabla)$, respectively. Let π be the orthogonal projection of E onto E_1. Let us observe that it can be extended to a projection $\boldsymbol{E}_* \to \boldsymbol{E}_{1*}$. Let P be a smooth point of D around which $(\mathcal{E}, \nabla)$ is good. Because $\boldsymbol{E}_*^{DM} \simeq \mathcal{P}_*E_1 \oplus \mathcal{P}_*E_2$ around P, π has the desired extension around P. By using the Hartogs property of reflexive sheaves, we obtain the desired extension on X. It implies that E_2 can be extended to a meromorphic flat subbundle $\mathcal{E}_2$ of $\mathcal{E}$, and we have the flat decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Moreover, $\mathcal{E}_i$ are also equipped with the metrics h_i satisfying the condition in the statement of Proposition 16.4.1. Hence, by an easy inductive argument, we can conclude that $\mathcal{E}$ is semisimple. We can also deduce that the canonical decomposition of $(\mathcal{E}, \nabla)$ is orthogonal with respect to h.

Let h and h' be metrics for $(\mathcal{E}, \nabla)$ as in Proposition 16.4.1. Let s be the endomorphism of E determined by $h = h' \cdot s$, which is self-adjoint with respect to both h and h'. Let C and F be as in the proof of Lemma 16.4.3. By applying Corollary 16.1.4, we obtain that F^*s is ∇-flat. Hence, we obtain that s is ∇-flat. It induces the ∇-flat decomposition $E = \bigoplus E_i$ such that (i) it is orthogonal with respect to both h and h', (ii) $h_{E_i} = a_i\, h'_{E_i}$ for some $a_i > 0$. By using the argument in the previous paragraph, we can show that the decomposition can be extended to that of meromorphic flat bundles $\mathcal{E} = \bigoplus \mathcal{E}_i$. In particular, we obtain the uniqueness in the case where $(\mathcal{E}, \nabla)$ is simple. By using the argument in the proof of Corollary 13.6.2, we obtain that the restriction of h to a direct summand of the canonical decomposition has the form as in the statement of Proposition 16.4.1. Thus, the proof of Proposition 16.4.1 is finished. □

Corollary 16.4.4. — *There exists a birational projective morphism $\varphi : X' \to X$ such that (i) $D' := \varphi^{-1}(D)$ is simply normal crossing, (ii) $X' \setminus D' \simeq X \setminus D$, (iii) the Deligne-Malgrange lattice of $\varphi^*(\mathcal{E}, \nabla)$ is a locally free $\mathcal{O}_X$-module and good. Namely, Theorem* 16.2.1 *holds without the assumption that X is projective.*

Proof. — We only have to apply the argument in Subsections 16.3.2–16.3.4, by replacing Proposition 16.3.1 with Proposition 16.4.1. □

PART IV

APPLICATION TO WILD PURE TWISTOR D-MODULES

CHAPTER 17

WILD PURE TWISTOR D-MODULES

In Section 17.1, we recall the notion of wild pure twistor D-modules and their basic property due to Sabbah ([**73**] and [**75**]). Note that we will give a review on $\mathcal{R}$-modules, $\mathcal{R}$-triples and variants in Chapter 22.

We show in Section 17.2 the correspondence between wild pure twistor D-modules and wild harmonic bundles on curves. Although, we will study a more general correspondence in Chapter 19, we need the correspondence in the curve case for the proof of the Hard Lefschetz theorem in Chapter 18.

In Section 17.3, we study the Gysin map for wild pure twistor D-modules, which is a preparation for Section 18.4.

17.1. Review of wild pure twistor D-modules due to Sabbah

17.1.1. Wild pure twistor D-modules and some property. — We recall the notions of pure twistor D-module and wild pure twistor D-modules defined in [**73**] and [**75**]. See those papers for more details and precision. (See also [**77**] and [**78**] for the original work due to Saito on pure Hodge modules.) First, we recall the definition of pure twistor D-module in [**73**], which is given in an inductive way on the dimension of the support. Let X be a complex manifold.

Definition 17.1.1. — Let w be an integer. The category $\mathrm{MT}_{\leqslant d}(X, w)$ is defined to be the full subcategory of the category of $\mathcal{R}_X$-triples whose objects $\mathcal{T}$ satisfy the following conditions:

- (HSD) : The underlying $\mathcal{R}_X$-modules of $\mathcal{T}$ are holonomic and strictly S-decomposable. The dimensions of their strict supports are less than d.
- (MT_0) : By (HSD), we have the decomposition $\mathcal{T} = \bigoplus_Z \mathcal{T}_Z$ by the strict supports, where Z runs through the irreducible closed subsets of X. In the case $\dim Z = 0$, $\mathcal{T}_Z$ is a push-forward of a pure twistor structure in dimension 0 via the inclusion $Z \to X$.
- $(MT_{>0})$: Let U be any open subset of X, and f be any holomorphic function on U. Let u be any element of $\boldsymbol{R} \times \boldsymbol{C}$, and ℓ be any integer. Then the induced $\mathcal{R}$-triples $\mathrm{Gr}_\ell^W \widetilde{\psi}_{f,u} \mathcal{T}_{|U}$ are objects of $\mathrm{MT}_{\leqslant d-1}(U, w+\ell)$. Here W denotes the weight filtration of the induced nilpotent map on $\widetilde{\psi}_{f,u} \mathcal{T}_{|U}$.

In the case $d \geqslant \dim X$, we denote the category by $\mathrm{MT}(X, w)$. The objects of $\mathrm{MT}(X, w)$ are called pure twistor D-modules of weight w on X. □

See [**73**] for the fundamental properties of pure twistor D-modules. Note that $\mathcal{T}_Z$ with $\dim Z = 0$ can be obtained from $\mathcal{T}$ by a successive use of the vanishing cycle functor.

Let $\mathcal{A}$ be a $\boldsymbol{Q}$-vector subspace of $\boldsymbol{C}$.

Definition 17.1.2. — Let $\mathcal{T}$ be a pure twistor D-module on X. We have the decomposition $\mathcal{T} = \bigoplus \mathcal{T}_Z$ by the strict supports. For the definition of $\mathcal{A}$-wild pure twistor D-modules, we impose the following condition inductively on the dimension of the supports of Z:

($MT_{>0}(\mathrm{rami}, \exp, \mathcal{A})$) : Let U be any open subset of X, and g be a holomorphic function on U. For any $n \in \mathbb{Z}_{>0}$ and $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$, $\mathcal{T}_Z$ is strictly specializable along g with ramified exponential twist by $\mathfrak{a}$. And the induced $\mathcal{R}_X$-triples $\mathrm{Gr}_\ell^{W(N)} \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_Z)$ are wild pure twistor D-modules of weight $w + \ell$ for any $u \in \boldsymbol{R} \times \boldsymbol{C}$ and $\ell \in \mathbb{Z}$. Moreover, $\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_Z) = 0$ unless $u \in \boldsymbol{R} \times \mathcal{A}$. □

Let $\mathrm{MT}^{\mathrm{wild}}(X, w, \mathcal{A})$ denote the category of $\mathcal{A}$-wild pure twistor D-modules on X of weight w. Let $\mathrm{MT}^{\mathrm{wild}}_{\leqslant d}(X, w, \mathcal{A})$ denote the category of $\mathcal{A}$-wild pure twistor D-modules on X of weight w such that the dimensions of the strict supports are less than d. For a subvariety $Y \subset X$, let $\mathrm{MT}^{\mathrm{wild}}_Y(X, w, \mathcal{A})$ denote the category of $\mathcal{A}$-wild pure twistor D-modules on X of weight w whose strict supports are contained in Y.

Remark 17.1.3. — In the following argument, we often omit to distinguish $\mathcal{A}$. In [**67**], a $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module satisfying the regularity condition is called a pure imaginary pure twistor D-module. □

We recall some properties of wild pure twistor D-modules. We refer to [**73**] and [**75**] for more details.

Lemma 17.1.4. — *Kashiwara's equivalence holds. Namely,* $\mathrm{MT}^{\mathrm{wild}}_Y(X, w, \mathcal{A}) = \mathrm{MT}^{\mathrm{wild}}(Y, w, \mathcal{A})$ *for a complex submanifold* $Y \subset X$.

Proof. — According to [**73**], $\mathrm{MT}_Y(X, w) = \mathrm{MT}(Y, w)$. It is easy to check that the condition ($MT_{>0}(\mathrm{rami}, \exp, \mathcal{A})$) is preserved. □

The following lemma is clear from the definition.

Lemma 17.1.5. — *Let* $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ *be an* $\mathcal{R}_X$*-triple.*

- *If* $\mathcal{T}$ *is a wild pure twistor* D*-module on* X*, the restriction* $\mathcal{T}_{|U}$ *is also a wild pure twistor* D*-module on* U *for any open subset* U *of* X.
- *Assume (i) there exists an open covering* $X = \bigcup U_i$ *such that* $\mathcal{T}_{|U_i}$ *are wild pure twistor* D*-modules on* U_i*, (ii)* $\mathcal{M}'$ *and* $\mathcal{M}''$ *are holonomic (see Subsection* 22.2.3*). Then,* $\mathcal{T}$ *is also a wild pure twistor* D*-module on* X. □

Lemma 17.1.6. — *Let $\mathcal{T} \in \mathrm{MT}^{\mathrm{wild}}(X, w, \mathcal{A})$ and $\mathcal{T}' \in \mathrm{MT}^{\mathrm{wild}}(X, w', \mathcal{A})$ for $w > w'$. Any morphism $\mathcal{T} \to \mathcal{T}'$ of $\mathcal{R}$-triples is trivial.*

Proof. — It can be reduced to the claim for pure twistor D-modules in the sense of [**73**]. (See Proposition 4.1.8 of [**73**].) □

Let $\mathrm{MTW}^{\mathrm{wild}}_{\leqslant d}(X, \mathcal{A})$ be the category of $\mathcal{R}$-triples $\mathcal{T}$ with increasing filtrations W indexed by $\mathbb{Z}$ such that $\mathrm{Gr}^W_\ell \mathcal{T} \in \mathrm{MT}^{\mathrm{wild}}_{\leqslant d}(X, \ell, \mathcal{A})$. Let $\mathrm{MTN}^{\mathrm{wild}}(X, w, \mathcal{A})$ be the category of $\mathcal{R}$-triples $\mathcal{T}$ with morphisms $\mathcal{N} : \mathcal{T} \to \mathcal{T} \otimes \mathbb{T}(-1)$ such that (i) $\mathcal{N}^m = 0$ for any sufficiently large m, (ii) $\mathrm{Gr}^{W(\mathcal{N})}_\ell(\mathcal{T}) \in \mathrm{MT}^{\mathrm{wild}}_{\leqslant d}(X, w + \ell, \mathcal{A})$ for the monodromy weight filtration $W(\mathcal{N})$. A morphism $\varphi : (\mathcal{T}_1, \mathcal{N}_1) \to (\mathcal{T}_2, \mathcal{N}_2)$ is defined to be a morphism of $\mathcal{R}$-triples such that $\varphi \circ \mathcal{N}_1 = \mathcal{N}_2 \circ \varphi$.

Lemma 17.1.7

(a_d) *The category $MT^{\mathrm{wild}}_{\leqslant d}(X, w, \mathcal{A})$ is abelian. All morphisms are strict and strictly specializable.*

(b_d) *The category $\mathrm{MTW}^{\mathrm{wild}}_{\leqslant d}(X, \mathcal{A})$ is abelian. Any morphism $\varphi : (\mathcal{T}_1, W_1) \to (\mathcal{T}_2, W_2)$ is strict with respect to the filtrations W_i. All morphisms are strict.*

(c_d) *The category $\mathrm{MTN}^{\mathrm{wild}}_{\leqslant d}(X, w, \mathcal{A})$ is abelian. Any morphism $\varphi : (\mathcal{T}_1, \mathcal{N}_1) \to (\mathcal{T}_2, \mathcal{N}_2)$ is strict with respect to $W(\mathcal{N}_1)$ and $W(\mathcal{N}_2)$. The filtrations on* $\mathrm{Im}(\varphi)$, $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Cok}(\varphi)$ *induced by $W(\mathcal{N}_i)$ $(i = 1, 2)$ are equal to the monodromy weight filtrations of the induced morphisms. All morphisms are strict.*

Proof. — The proof of the lemma can be carried out with the argument in [**73**] (based on [**77**] for pure Hodge modules), and many claims can be reduced to the case of pure twistor D-modules. We give only an outline, and we refer to [**73**] and [**75**] for more details. Let us see $(a_d) \Longrightarrow (c_d)$. Let $\varphi : (\mathcal{T}_1, \mathcal{N}_1) \to (\mathcal{T}_2, \mathcal{N}_2)$. We have the induced morphisms $\mathrm{Gr}^{W(\mathcal{N})}_\ell(\varphi) : \mathrm{Gr}^{W(\mathcal{N})}_\ell(\mathcal{T}_1) \to \mathrm{Gr}^{W(\mathcal{N})}_\ell(\mathcal{T}_2)$ of pure twistor D-modules. By using Lemma 17.1.6, it is easy to derive that φ is strict with respect to $W(\mathcal{N}_1)$ and $W(\mathcal{N}_2)$. Therefore, $\mathrm{Ker}\,\varphi$, $\mathrm{Im}(\varphi)$ and $\mathrm{Cok}(\varphi)$ are equipped with the naturally induced filtration W, and we have the natural isomorphisms $\mathrm{Gr}^W_\ell(\mathrm{Ker}(\varphi)) \simeq \mathrm{Ker}(\mathrm{Gr}^W_\ell(\varphi))$, $\mathrm{Gr}^W_\ell(\mathrm{Im}(\varphi)) \simeq \mathrm{Im}(\mathrm{Gr}^W_\ell(\varphi))$ and $\mathrm{Gr}^W_\ell(\mathrm{Cok}(\varphi)) \simeq \mathrm{Cok}(\mathrm{Gr}^W_\ell(\varphi))$. Since we have the isomorphisms $\mathcal{N}_1^\ell : \mathrm{Gr}^W_\ell(\mathrm{Ker}(\varphi)) \simeq \mathrm{Gr}^W_{-\ell}(\mathrm{Ker}(\varphi))$, the induced filtration W is the same as the weight filtration of the induced nilpotent map $\mathcal{N}$ on $\mathrm{Ker}(\varphi)$. Similar claims hold for $\mathrm{Im}(\varphi)$ and $\mathrm{Cok}(\mathrm{Im}\,\varphi)$. We can show $(a_d) \Longrightarrow (b_d)$ in a similar way.

Let us see $(c_{d-1}) \Longrightarrow (a_d)$. Let $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism of wild pure twistor D-modules of weight w. We may assume that $\mathcal{T}_i$ have the same strict support Z. Since, it is a morphism of pure twistor D-modules, we already know that (i) φ is strict and strictly specializable along any function g, (ii) $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$ and $\mathrm{Cok}(\varphi)$ are pure twistor D-modules whose strict supports are Z. Let g be any function. We have the induced morphism $\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi) : (\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_1), \mathcal{N}_1) \to (\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_2), \mathcal{N}_2)$ to which we can apply (c_{d-1}). We obtain that $\mathrm{Gr}^{W(N)}_\ell \mathrm{Ker}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi))$, $\mathrm{Gr}^{W(N)}_\ell \mathrm{Im}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi))$

and $\mathrm{Gr}_\ell^{W(N)} \mathrm{Cok}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi))$ are wild pure twistor D-modules of weight $w+\ell$. In particular, we also obtain that $\mathrm{Cok}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)$ is strict. Due to Lemma 22.11.1, we obtain the natural isomorphisms $\mathrm{Ker}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)) \simeq \widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Ker}\,\varphi)$, $\mathrm{Im}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)) \simeq \widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Im}\,\varphi)$ and $\mathrm{Cok}(\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)) \simeq \widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Cok}\,\varphi)$. Hence, we obtain that $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$ and $\mathrm{Cok}(\varphi)$ are also wild pure twistor D-modules of weight w. □

In the proof, we have obtained the following.

Corollary 17.1.8. — *Let $\mathcal{T}_i$ $(i=1,2)$ be wild pure twistor D-modules of weight w. Let $\varphi:\mathcal{T}_1\longrightarrow\mathcal{T}_2$ be a morphism of wild pure twistor D-modules. For any holomorphic function g, $\mathfrak{a}\in \boldsymbol{C}[t_n^{-1}]$ and $u\in\boldsymbol{R}\times\boldsymbol{C}$, we have the natural isomorphisms $\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Ker}\,\varphi)\simeq \mathrm{Ker}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)$, $\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Im}\,\varphi)\simeq\mathrm{Im}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)$, and $\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Cok}\,\varphi)\simeq\mathrm{Cok}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi)$. We also have*

$$P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Ker}\,\varphi)\simeq \mathrm{Ker}\,P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi),$$
$$P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Im}\,\varphi)\simeq \mathrm{Im}\,P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi),$$
$$P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\mathrm{Cok}\,\varphi)\simeq \mathrm{Cok}\,P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\varphi).$$

In particular, let $\mathcal{C}$ be a complex in $\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})$. By taking the functors $\widetilde{\psi}_{g,\mathfrak{a},u}$ and $P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}$, we have the complex $\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{C})$ in $\mathrm{MTN}^{\mathrm{wild}}(X,w,\mathcal{A})$, and the complex $P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{C})$ in $\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})$. Then, we have natural isomorphisms:

$$H^i\big(\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{C})\big)\simeq\widetilde{\psi}_{g,\mathfrak{a},u}H^i(\mathcal{C}),\quad H^i\big(P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{C})\big)\simeq P\,\mathrm{Gr}_\ell^{W(N)}\,\widetilde{\psi}_{g,\mathfrak{a},u}H^i(\mathcal{C}).$$

□

17.1.2. Polarizable wild pure twistor D-module.

— We recall the definition of polarization of wild pure twistor D-modules. ([**73**] and [**75**]. See also Saito's original work [**77**].)

Definition 17.1.9. — Let $\mathcal{T}\in\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})$. Let $\mathcal{S}:\mathcal{T}\longrightarrow\mathcal{T}^*(-w)$ be a Hermitian sesqui-linear duality of weight w. It is called a polarization, if the following inductive conditions are satisfied:

- (MTP_0) : We have the decomposition $\mathcal{S}=\bigoplus\mathcal{S}_Z$, corresponding to $\mathcal{T}=\bigoplus\mathcal{T}_Z$. If $\dim Z=0$, $(\mathcal{T}_Z,\mathcal{S}_Z)$ is the push-forward of a polarized pure twistor structure of weight w.
- $(MTP_{>0}(\mathrm{rami},\exp))$: Let U be an open subset of X, and let f be a holomorphic function on U. For each $\mathfrak{a}\in\boldsymbol{C}[t_n^{-1}]$, $u\in\boldsymbol{R}\times\mathcal{A}$ and $\ell\in\mathbb{Z}_{\geqslant 0}$, the induced Hermitian sesqui-linear duality
$$\mathcal{S}_{f,\mathfrak{a},u,\ell}:P\,\mathrm{Gr}_\ell^W\,\widetilde{\psi}_{f,\mathfrak{a},u}(\mathcal{T})\longrightarrow P\,\mathrm{Gr}_\ell^W\,\widetilde{\psi}_{f,\mathfrak{a},u}(\mathcal{T})^*(-w-\ell)$$
is a polarization of $P\,\mathrm{Gr}_\ell^W\,\widetilde{\psi}_{f,\mathfrak{a},u}(\mathcal{T})$ as an $\mathcal{A}$-wild pure twistor D-module of weight $w+\ell$.

An $\mathcal{A}$-wild pure twistor D-module $\mathcal{T}$ is called polarizable, if it has a polarization. □

Let $\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})^{(p)}$ denote the full subcategory of $\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})$ which consists of the polarizable $\mathcal{A}$-wild pure twistor D-modules of weight w. When we are given a subvariety $Y\subset X$, let $\mathrm{MT}^{\mathrm{wild}}_{Y}(X,w,\mathcal{A})^{(p)}$ (resp. $\mathrm{MT}^{\mathrm{wild}}_{Y,ss}(X,w,\mathcal{A})^{(p)}$) denote the full subcategory of $\mathcal{A}$-wild pure twistor D-modules on X of weight w whose supports are contained in Y (resp. whose strict supports are exactly Y). We say that $\mathcal{T}\in\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})$ is simple, if it does not contain any non-trivial subobjects. We say that $(\mathcal{T},\mathcal{S})$ of weight w is simple, if $\mathcal{T}$ is simple.

Lemma 17.1.10. — *Let $(\mathcal{T},\mathcal{S})$ be a polarized wild pure twistor D-module of weight w, whose strict support is Z. We have a Zariski closed subset $Z_0\subset Z$ such that $(\mathcal{T},\mathcal{S})_{|X\setminus Z_0}$ is a push-forward of a variation of polarized pure twistor structure on $Z\setminus Z_0$.*

Proof. — It can be reduced to the claim for pure twistor D-modules in the sense of [**73**]. Then, it follows from Proposition 4.1.9 of [**73**]. □

Proposition 17.1.11. — *Let $(\mathcal{T},\mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight w. Let $\mathcal{T}'\subset\mathcal{T}$ be wild pure twistor D-submodule of weight w. Then, the composition $\mathcal{S}'$ of the following morphisms*

$$\mathcal{T}'\longrightarrow\mathcal{T}\xrightarrow{\ \mathcal{S}\ }\mathcal{T}^*(-w)\longrightarrow\mathcal{T}'^*(-w)$$

gives a polarization of $\mathcal{T}'$, and we have the decomposition $(\mathcal{T},\mathcal{S})=(\mathcal{T}',\mathcal{S}')\oplus(\mathcal{T}'',\mathcal{S}'')$.

Proof. — We have the decomposition of polarized pure twistor D-modules as above, which is shown in Proposition 4.2.5 of [**73**]. It gives the decomposition of polarized $\mathcal{A}$-wild pure twistor D-modules. □

Corollary 17.1.12. — *Let $(\mathcal{T},\mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight w. Then, we have a decomposition $(\mathcal{T},\mathcal{S})=\bigoplus(\mathcal{T}_i,\mathcal{S}_i)$, where $\mathcal{T}_i$ are simple $\mathcal{A}$-wild pure twistor D-modules of weight w. In particular, the abelian category $\mathrm{MT}^{\mathrm{wild}}(X,w,\mathcal{A})^{(p)}$ is semisimple.* □

Proposition 17.1.13. — *Let $(\mathcal{T},\mathcal{S})$ be a simple polarized $\mathcal{A}$-wild pure twistor D-module of weight w.*

- *Let (V,S) be a polarized pure twistor structure of weight 0 in dimension 0. Then, the naturally defined Hermitian sesqui-linear duality $\mathcal{T}\otimes V\to\mathcal{T}^*\otimes V^*(-w)$ is a polarization.*
- *Let V be a pure twistor structure of weight 0 in dimension 0. For any polarization $\widetilde{\mathcal{S}}$ of $\mathcal{T}\otimes V$, there exists a polarization S of V such that $\widetilde{\mathcal{S}}=\mathcal{S}\otimes S$.*

Proof. — The first claim is clear. Because of Corollary 17.1.12, we have the decomposition $V=\bigoplus V_i$ into rank one objects such that $(\mathcal{T},\widetilde{\mathcal{S}})=\bigoplus(\mathcal{T}\otimes V_i,\widetilde{\mathcal{S}}_i)$. Because $\mathcal{T}$ is simple, there exists a positive number $a_i>0$ such that $\widetilde{\mathcal{S}}_i=a_i\cdot\mathcal{S}$, under some identification $V_i\simeq\mathbb{T}^S(0)$. Then, the second claim follows. □

Proposition 17.1.14. — *Let $\mathcal{T}$ be a polarizable $\mathcal{A}$-wild pure twistor D-module of weight w.*

- *We have the canonical decomposition $\mathcal{T} = \bigoplus \mathcal{T}_i$, where $\mathcal{T}_i$ are the direct sum of some simple objects $\mathcal{T}_i'$ such that $\mathcal{T}_i' \not\simeq \mathcal{T}_j'$ $(i \neq j)$.*
- *Any polarization $\mathcal{S}$ of $\mathcal{T}$ is a direct sum of the polarizations $\mathcal{S}_i$ of $\mathcal{T}_i$.*

Proof. — The first claim follows from Corollary 17.1.12. The second claim is obvious. □

Corollary 17.1.15. — *Let $\mathcal{T}$ be an $\mathcal{A}$-wild pure twistor D-module of weight w. Let $\mathcal{S}_i$ $(i = 1, 2)$ be polarizations of $\mathcal{T}$. Then, there exists an automorphism φ of $\mathcal{T}$ such that $\varphi : (\mathcal{T}, \mathcal{S}_1) \to (\mathcal{T}, \mathcal{S}_2)$ is an isomorphism. In other words, a polarization of $\mathcal{T}$ is unique up to obvious ambiguity.*

Proof. — It follows from Proposition 17.1.13 and Proposition 17.1.14. □

17.1.3. Polarized graded wild Lefschetz twistor D-module. — We recall the definition and some results for polarized graded wild Lefschetz twistor D-module due to Sabbah ([**73**] and [**75**]). We consider the graded objects with the lower index. It is easy to give a modification for the upper index case.

Let ε be an integer. A graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight w of type ε is a pair $(\mathcal{T}, \mathcal{L})$ as follows:

- $\mathcal{T} = \bigoplus_{j \in \mathbb{Z}} \mathcal{T}_j$ is a graded $\mathcal{R}$-triple. Each $\mathcal{T}_j$ is an $\mathcal{A}$-wild pure twistor D-module of weight $w - \varepsilon \cdot j$.
- $\mathcal{L}$ is a graded morphism $\mathcal{T} \to \mathcal{T}(\varepsilon)[2]$ such that $\mathcal{L}^j : \mathcal{T}_j \to \mathcal{T}_{-j}(\varepsilon \cdot j)$ are isomorphisms for any $j \geqslant 0$.

We put $P\mathcal{T}_j := \operatorname{Ker} \mathcal{L}^{j+1} \cap \mathcal{T}_j$ for each $j \geqslant 0$, and $P\mathcal{T}_j = 0$ for $j < 0$. Then, we obtain the naturally defined primitive decomposition $\mathcal{T}_j = \bigoplus \mathcal{L}^k\big(P\mathcal{T}_{j+2k}(-\varepsilon k)\big)$ for any j, where $\mathcal{L}^k\big(P\mathcal{T}_{j+2k}(-\varepsilon k)\big)$ denote the image of $\mathcal{L}^k : P\mathcal{T}_{j+2k}(-\varepsilon k) \to \mathcal{T}_j$.

Recall that the Hermitian adjoint $\mathcal{T}^*$ has the natural grading given by $(\mathcal{T}^*)_j := (\mathcal{T}_{-j})^*$. It is naturally equipped with an induced morphism $\mathcal{L}^* : \mathcal{T}^* \to \mathcal{T}^*(\varepsilon)[2]$, and $(\mathcal{T}^*, \mathcal{L}^*)$ is a graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight w of type ε. We have the natural isomorphism $(P\mathcal{T}_j)^* \simeq (\mathcal{L}^*)^j\big(P\mathcal{T}_j^*(-\varepsilon j)\big)$.

Let $\mathcal{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ be a graded Hermitian sesqui-linear duality satisfying $\mathcal{L}^* \circ \mathcal{S} + \mathcal{S} \circ \mathcal{L} = 0$. We have the composite:

$$\mathcal{S}_{-j} \circ \mathcal{L}^j : P\mathcal{T}_j \longrightarrow \mathcal{L}^j\big(P\mathcal{T}_j(-\varepsilon j)\big)(\varepsilon j) \longrightarrow (P\mathcal{T}_j)^*(-w + \varepsilon j)$$

Then, $\mathcal{S}$ is called a polarization of $(\mathcal{T}, \mathcal{L})$, if the composite $\mathcal{S}_{-j} \circ \mathcal{L}^j$ gives a polarization of $P\mathcal{T}_j$ for each $j \geqslant 0$.

The following lemma, due to Sabbah, is one of the most important properties of graded $\mathcal{A}$-wild Lefschetz twistor D-module. Saito originally proved the corresponding property for his Hodge modules.

Lemma 17.1.16. — *Assume ε is 1 or -1. Let $(\mathcal{T},\mathcal{L},\mathcal{S})$ (resp. $(\mathcal{T}',\mathcal{L}',\mathcal{S}')$) be a polarized graded $\mathcal{A}$-wild Lefschetz twistor D-module of type ε of weight w (resp. $w-\varepsilon$). Let $c:\mathcal{T}\to\mathcal{T}'[1]$ and $v:\mathcal{T}'\to\mathcal{T}[1](\varepsilon)$ be graded morphisms such that $v\circ c=\mathcal{L}$ and $c\circ v=\mathcal{L}'$. Assume that c and v are adjoint with respect to $\mathcal{S}$ and $\mathcal{S}'$, i.e., $\mathcal{S}'\circ c=v^*\circ\mathcal{S}$. Then we have the decomposition $\mathcal{T}'=\operatorname{Im}c\oplus\operatorname{Ker}v$.*

Proof. — It can be shown by using the argument in the proof of Proposition 4.2.10 of [**73**]. We give only an outline. We may assume that both $\mathcal{T}$ and $\mathcal{T}'$ have the same strict support Z. By the result for smooth polarized graded Lefschetz twistor structures (Lemma 5.2.15 of [**77**] or Proposition 2.19 of [**73**]), we have such a decomposition on the generic part of Z. Because of the strict S-decomposability, we can easily obtain the decomposition on Z. □

17.1.4. Bi-graded wild Lefschetz twistor D-module. — We recall the notion of bi-graded wild Lefschetz twistor D-module due to Sabbah ([**73**] and [**75**]). We consider the bi-graded objects with lower indices. It is easy to give a modification in the other cases.

Let ε_a $(a=1,2)$ be integers. A bi-graded $\mathcal{A}$-wild Lefschetz twistor D-modules of type $(\varepsilon_1,\varepsilon_2)$ of weight w is defined naturally, i.e., it is a tuple $(\mathcal{T},\mathcal{L}_1,\mathcal{L}_2)$ as follows:

- $\mathcal{T}$ is a bi-graded $\mathcal{R}$-triple $\bigoplus_{i,j\in\mathbb{Z}}\mathcal{T}_{i,j}$. Each $\mathcal{T}_{i,j}$ is a $\mathcal{A}$-wild pure twistor D-module of weight $w-i\varepsilon_1-j\varepsilon_2$.
- $\mathcal{L}_1$ is a tuple of morphisms $\mathcal{T}_{i,j}\to\mathcal{T}_{i-2,j}(\varepsilon_1)$ $(i,j\in\mathbb{Z})$ such that
$$\mathcal{L}_1^i:\mathcal{T}_{i,j}\longrightarrow\mathcal{T}_{-i,j}(i\varepsilon_1)$$
is an isomorphism for each $i\geqslant 0$.
- $\mathcal{L}_2$ is a tuple of morphisms $\mathcal{T}_{i,j}\to\mathcal{T}_{i,j-2}(\varepsilon_2)$ $(i,j\in\mathbb{Z})$ such that
$$\mathcal{L}_2^j:\mathcal{T}_{i,j}\longrightarrow\mathcal{T}_{i,-j}(j\varepsilon_2)$$
is an isomorphism for each $j\geqslant 0$.
- $\mathcal{L}_1$ and $\mathcal{L}_2$ commute.

We put $P\mathcal{T}_{i,j}=\operatorname{Ker}\mathcal{L}_1^{i+1}\cap\operatorname{Ker}\mathcal{L}_2^{j+1}\cap\mathcal{T}_{i,j}$. They obviously induce the primitive decomposition as in the case of graded $\mathcal{A}$-wild Lefschetz twistor D-modules. The Hermitian adjoint $\mathcal{T}^*$ naturally has the grading and the morphisms $\mathcal{L}_i^*$ $(i=1,2)$, for which $(\mathcal{T}^*,\mathcal{L}^*)$ is a bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module.

A bi-graded morphism $\mathcal{S}:\mathcal{T}\to\mathcal{T}^*(-w)$ such that $\mathcal{L}_a^*\circ\mathcal{S}+\mathcal{S}\circ\mathcal{L}_a=0$ $(a=1,2)$ induces the morphism:
$$\mathcal{S}\circ\mathcal{L}_1^i\circ\mathcal{L}_2^j:P\mathcal{T}_{i,j}\longrightarrow(P\mathcal{T}_{i,j})^*(-w+i\varepsilon_1+j\varepsilon_2).$$
If $\mathcal{S}\circ\mathcal{L}_1^i\circ\mathcal{L}_2^j$ gives a polarization of $P\mathcal{T}_{i,j}$, $\mathcal{S}$ is called a polarization of $(\mathcal{T},\mathcal{L}_1,\mathcal{L}_2)$.

Let $d:\mathcal{T}\to\mathcal{T}(\varepsilon_1+\varepsilon_2)$ be a morphism satisfying the following:

- $d:\mathcal{T}_{i,j}\to\mathcal{T}_{i-2,j-2}(\varepsilon_1+\varepsilon_2)$ and $d\circ d=0$.
- d is anti-commutative with $\mathcal{L}_a$ $(a=1,2)$, i.e., $d\circ\mathcal{L}_a+\mathcal{L}_a\circ d=0$.
- d is self-adjoint with respect to $\mathcal{S}$, i.e., $d^*\circ\mathcal{S}=\mathcal{S}\circ d$.

We have the naturally defined bi-graded structure on $\mathcal{T}^{(1)} := \operatorname{Ker} d/\operatorname{Im} d$. We also have the naturally defined maps $\mathcal{L}_a^{(1)}$ $(a=1,2)$ and the Hermitian sesqui-linear duality $\mathcal{S}^{(1)}$.

The following lemma, due to Sabbah, is one of the most important properties of bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module. Saito originally proved the corresponding property for his Hodge modules.

Lemma 17.1.17. — *$(\mathcal{T}^{(1)}, \mathcal{L}_1^{(1)}, \mathcal{L}_2^{(1)}, \mathcal{S}^{(1)})$ is a polarized bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight w of type $(\varepsilon_1, \varepsilon_2)$.*

Proof. — It can be shown using the same argument as that in the proof of Proposition 4.2.10 of [**73**]. We give only an outline. We may assume that $\mathcal{T}$ has the strict support Z which is irreducible. We use an induction on the dimension of Z. By the result for smooth polarized bi-graded Lefschetz twistor structures (Proposition 4.22 of [**77**], Theorem 4.5 of [**33**], Lemma 2.1.20 of [**73**]), the claim holds on the generic part of Z. By the strict S-decomposability, it follows that the morphisms $(\mathcal{L}_1^{(1)})^{j_1} : \mathcal{T}^{(1)}_{j_1,j_2} \simeq \mathcal{T}^{(1)}_{-j_1,j_2}(j_1\cdot\varepsilon_1)$ are isomorphisms for $j_1 \geqslant 0$, and that the morphisms $(\mathcal{L}_2^{(1)})^{j_2} : \mathcal{T}^{(1)}_{j_1,j_2} \simeq \mathcal{T}^{(1)}_{j_1,-j_2}(j_2\cdot\varepsilon_2)$ are isomorphisms for $j_2 \geqslant 0$. We only have to show that the composite $\mathcal{S}^{(1)} \circ (\mathcal{L}_1^{(1)})^{j_1} \circ (\mathcal{L}_2^{(1)})^{j_2}$ gives a polarization of $P\mathcal{T}^{(1)}_{j_1,j_2}$ for each $(j_1,j_2) \in \mathbb{Z}^2_{\geqslant 0}$.

Let g be any holomorphic function. For any $\ell \in \mathbb{Z}_{\geqslant 0}$, $u \in \boldsymbol{R}\times\boldsymbol{C}$ and $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$, we consider

$$\begin{aligned}(\mathcal{T}_{g,\mathfrak{a},u,\ell})_{j_1,j_2} &:= P\operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{j_1,j_2})\\ &= \operatorname{Ker}\Big(\mathcal{N}^{\ell+1} : \operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{j_1,j_2}) \longrightarrow \operatorname{Gr}^W_{-\ell-2} \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{j_1,j_2})(-\ell-1)\Big).\end{aligned}$$

Then, $\mathcal{T}_{g,\mathfrak{a},u,\ell} := \bigoplus_{j_1,j_2}\big(\mathcal{T}_{g,\mathfrak{a},u,\ell}\big)_{j_1,j_2}$ with the induced morphisms $\mathcal{L}_a$ $(a=1,2)$ is a bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight $w+\ell$. We have the following morphisms induced by $\mathcal{N}^\ell$ and $\mathcal{S}$:

$$\operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{j_1,j_2}) \longrightarrow \operatorname{Gr}^W_{-\ell} \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{j_1,j_2})(-\ell) \longrightarrow \operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}_{-j_1,-j_2})^*(-w-\ell)$$

They induce a morphism $(\mathcal{T}_{g,\mathfrak{a},u,\ell})_{j_1,j_2} \to \big((\mathcal{T}_{g,\mathfrak{a},u,\ell})_{-j_1,-j_2}\big)^*(-w-\ell)$, which gives a Hermitian sesqui-linear duality $\mathcal{S}_{g,\mathfrak{a},u,\ell}$ of the bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module $\mathcal{T}_{g,\mathfrak{a},u,\ell}$ of weight $w+\ell$.

Lemma 17.1.18. — *$\mathcal{S}_{g,\mathfrak{a},u,\ell}$ is a polarization of $\mathcal{T}_{g,\mathfrak{a},u,\ell}$.*

Proof. — We have the natural isomorphism $P\big(\mathcal{T}_{g,\mathfrak{a},u,\ell}\big)_{j_1,j_2} \simeq P\operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(P\mathcal{T}_{j_1,j_2})$, via which $\mathcal{S}_{g,\mathfrak{a},u,\ell}$ is the same as the polarization of $P\operatorname{Gr}^W_\ell \widetilde{\psi}_{g,\mathfrak{a},u}(P\mathcal{T}_{j_1,j_2})$. Thus, we obtain Lemma 17.1.18. □

We have the induced morphisms $d : (\mathcal{T}_{g,\mathfrak{a},u,\ell})_{j_1,j_2} \to (\mathcal{T}_{g,\mathfrak{a},u,\ell})_{j_1-1,j_2-1}$. By the inductive assumption, the induced tuple $\big(\mathcal{T}^{(1)}_{g,\mathfrak{a},u,\ell}, \mathcal{L}^{(1)}_1, \mathcal{L}^{(1)}_2, \mathcal{S}^{(1)}_{g,\mathfrak{a},u,\ell}\big)$ is a polarized bi-graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight $w+\ell$. In particular, $\mathcal{S}^{(1)}_{g,\mathfrak{a},u,\ell} \circ (\mathcal{L}^{(1)}_1)^{j_1} \circ (\mathcal{L}^{(1)}_2)^{j_2}$ gives a polarization of $P\big(\mathcal{T}^{(1)}_{g,\mathfrak{a},u,\ell}\big)_{j_1,j_2}$.

By Corollary 17.1.8, we have a natural isomorphism

$$(\mathcal{T}^{(1)}_{g,\mathfrak{a},u,\ell})_{j_1,j_2} \simeq P\,\mathrm{Gr}^W_\ell\,\widetilde{\psi}_{g,\mathfrak{a},u}\big(\mathcal{T}^{(1)}_{j_1,j_2}\big),$$

compatible with the induced Lefschetz morphisms. We obtain the isomorphism $P\big(\mathcal{T}^{(1)}_{g,\mathfrak{a},u,\ell}\big)_{j_1,j_2} \simeq P\,\mathrm{Gr}^W_\ell\,\widetilde{\psi}_{g,\mathfrak{a},u}\big(P\mathcal{T}^{(1)}_{j_1,j_2}\big)$, and the induced Hermitian sesqui-linear duality of $P\,\mathrm{Gr}^W_\ell\,\widetilde{\psi}_{g,\mathfrak{a},u}\big(P\mathcal{T}^{(1)}_{j_1,j_2}\big)$ is equal to the polarization $\mathcal{S}^{(1)}_{g,\mathfrak{a},u,\ell} \circ (\mathcal{L}^{(1)}_1)^{j_1} \circ (\mathcal{L}^{(1)}_2)^{j_2}$. Therefore we can conclude that $\mathcal{S}^{(1)} \circ (\mathcal{L}^{(1)}_1)^{j_1} \circ (\mathcal{L}^{(1)}_2)^{j_2}$ is a polarization of $P\mathcal{T}^{(1)}_{j_1,j_2}$, and we obtain Lemma 17.1.17. □

17.2. Wild harmonic bundles and wild pure twistor D-modules on curves

17.2.1. Statement. — Let X be a smooth complex curve. Let $\mathcal{A}$ be a vector subspace of $\boldsymbol{C}$ over $\boldsymbol{Q}$. Let D be a discrete subset of X. Let $(V, \mathbb{D}^\triangle)$ be a variation of pure twistor structure of weight w on $X \setminus D$ with a pairing. We say that $(V, \mathbb{D}^\triangle, S)$ is $\mathcal{A}$-wild on (X, D), if the corresponding harmonic bundle is wild on (X, D).

Let $\mathrm{VPT}^{\mathrm{wild}}(X, D, w, \mathcal{A})$ denote the category of $\mathcal{A}$-wild variation of polarized pure twistor structure of weight w on (X, D). Let $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(X, D, w, \mathcal{A})$ denote the category of polarized $\mathcal{A}$-wild pure twistor D-modules of weight w, such that (i) their strict supports are X, (ii) their restriction to $X \setminus D$ comes from a polarized variation of pure twistor structure. In this subsection, for both categories, morphisms are defined to be isomorphisms. For a given $(\mathcal{T}, \mathcal{S}) \in \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(X, D, w, \mathcal{A})$, its restriction to $X \setminus D$ comes from a harmonic bundle after a suitable Tate twist. It is easy to see that the harmonic bundle is wild on (X, D). Hence, we have a naturally defined functor

$$\Phi : \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(X, D, w, \mathcal{A}) \longrightarrow \mathrm{VPT}^{\mathrm{wild}}(X, D, w, \mathcal{A}).$$

Proposition 17.2.1. — *Φ is an equivalence.*

We only have to consider the case $w = 0$. In the following argument, we omit to distinguish $\mathcal{A}$.

Remark 17.2.2. — We will show the higher dimensional version (Theorem 19.1.3) in Section 19.1.2. Because we need Proposition 17.2.1 in the proof of the Hard Lefschetz theorem (Theorem 18.1.1), we include it here. See also Corollary 19.1.4 for a variant of the statement. □

17.2.2. From wild harmonic bundle to wild pure twistor D-module

Let D be a discrete subset of X. Let $(E,\overline{\partial}_E,\theta,h)$ be a wild harmonic bundle defined on $X\setminus D$. By applying the construction in Section 12.8, we obtain the $\mathcal{R}_X$-triple $\mathfrak{T}(E):=(\mathfrak{E},\mathfrak{E},\mathfrak{C})$ with the Hermitian sesqui-linear duality $\mathfrak{S}:\mathfrak{T}(E)\to\mathfrak{T}(E)$. Let us show that $(\mathfrak{T}(E),\mathfrak{S})$ is a polarized wild pure twistor D-module. We may and will assume that $X=\Delta$ and $D=\{O\}$. We already know that it is strictly S-decomposable (Lemma 12.8.4). It is easy to check that $\mathfrak{E}$ is holonomic.

Let $g(z):=z^n$ and $\mathfrak{a}\in \boldsymbol{C}[t_m^{-1}]$. Let us show that $P\operatorname{Gr}_\ell^{W(N)}\widetilde{\psi}_{g,\mathfrak{a},u}\mathfrak{T}(E)$ with the induced Hermitian sesqui-linear duality is a polarized pure twistor structure of weight ℓ. Let $\varphi_m:\boldsymbol{C}_{t_m}\to\boldsymbol{C}_t$ given by $\varphi_m(t_m)=t_m^m$. The induced map $X\times\boldsymbol{C}_{t_m}\to X\times\boldsymbol{C}_t$ is also denoted by φ_m. Let $\pi:\widetilde{X}\to X$ be given by $\pi(\zeta)=\zeta^{me}$ such that $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}):=\pi^{-1}(E,\overline{\partial}_E,\theta,h)$ is unramifiedly good. The induced morphism $\widetilde{X}\times\boldsymbol{C}_{t_m}\to X\times\boldsymbol{C}_{t_m}$ is also denoted by π. We put $\widetilde{\pi}:=\varphi_m\circ\pi$. Let $i_g:X\to X\times\boldsymbol{C}_t$ denote the graph of g, and let Γ denote the image. Let ω_m be a primitive m-th root of 1. We have

$$\widetilde{\pi}^{-1}\Gamma=\bigcup_{p=0}^{m-1}\big\{(\zeta,t_m)\,\big|\,t_m-\omega_m^p\cdot\zeta^{ne}=0\big\}.$$

Let $j_p:\widetilde{X}\to\widetilde{X}\times\boldsymbol{C}_{t_m}$ be the graph of the functions $\gamma_p:=\omega_m^p\cdot\zeta^{ne}$. We put $\mathfrak{a}_p:=\mathfrak{a}\big(\omega_m^p\cdot\zeta^{ne}\big)$.

We have the $\mathcal{R}_{\widetilde{X}}$-triple $\widetilde{\mathfrak{T}}=(\widetilde{\mathfrak{E}},\widetilde{\mathfrak{E}},\widetilde{\mathfrak{C}})$ with the Hermitian sesqui-linear duality $\mathfrak{S}:=(\mathrm{id},\mathrm{id})$ associated to $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h})$. We have the unramifiedly good wild harmonic bundles $(\widetilde{E}_p,\overline{\partial}_{\widetilde{E}_p},\theta_p,\widetilde{h}_p):=\big(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}\big)\otimes L(-\mathfrak{a}_p)$. We have the associated $\mathcal{R}_{\widetilde{X}}$-triples $\mathfrak{T}_p:=\big(\widetilde{\mathfrak{E}}_p,\widetilde{\mathfrak{E}}_p,\widetilde{\mathfrak{C}}_p\big)$ with the Hermitian sesqui-linear duality $\mathfrak{S}_p:=(\mathrm{id},\mathrm{id})$. We obtain the $\mathcal{R}_{\widetilde{X}\times\boldsymbol{C}_{t_m}}$-triple $j_{p\dagger}\mathfrak{T}_p$ and the localization $j_{p\dagger}\mathfrak{T}_p(*t_m)$.

Lemma 17.2.3. — *$P\operatorname{Gr}_\ell^{W(N)}\widetilde{\psi}_{t_m,u}(j_{p\dagger}\mathfrak{T}_p)$ with the induced Hermitian sesqui-linear dualities are polarized pure twistor D-modules of weight ℓ with 0-dimensional supports.*

Proof. — It follows from Corollary 12.7.2. □

Lemma 17.2.4. — *$\pi_\dagger j_{p\dagger}\mathfrak{T}_p(*t_m)$ are strictly specializable along t_m, and*

$$P\operatorname{Gr}_\ell^{W(N)}\widetilde{\psi}_{t_m,u}(\pi_\dagger j_{p\dagger}\mathfrak{T}_p)$$

with the induced Hermitian sesqui-linear duality are polarized pure twistor structures of weight ℓ.

Proof. — It can be shown by using the argument in [**77**], [**73**] or Sections 14.6 of [**67**] (see Lemma 18.3.6 and Proposition 18.3.7 below in this paper) with Lemma 17.2.3. Note Assumption 18.3.4 below is satisfied in this case. □

Lemma 17.2.5. — *$\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{T}(E)(*t)\big)\otimes\mathfrak{L}(-\mathfrak{a})$ is a direct summand of*

$$\bigoplus_{p=0}^{m-1}\pi_{\dagger}\big(j_{p\dagger}\mathfrak{T}_p(*t_m)\big).$$

Proof. — Let $i_{g\dagger}\mathfrak{E}(*t)$ denote $i_{g\dagger}\mathfrak{E}\otimes_{\mathcal{O}_{\mathcal{X}\times\boldsymbol{C}_t}}\mathcal{O}_{\mathcal{X}\times\boldsymbol{C}_t}(*t)$. Let $j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)$ denote $j_{p\dagger}\widetilde{\mathfrak{E}}_p\otimes_{\mathcal{O}_{\widetilde{\mathcal{X}}\times\boldsymbol{C}_{t_m}}}\mathcal{O}_{\widetilde{\mathcal{X}}\times\boldsymbol{C}_{t_m}}(*t_m)$. We have the following natural isomorphism:

$$\pi^{\dagger}\big(\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{E}(*t)\big)\big)\simeq\widetilde{\pi}^{\dagger}\big(i_{g\dagger}\mathfrak{E}(*t)\big)\simeq\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}(*t_m).$$

Therefore, we have

$$\bigoplus_{p=0}^{m-1}j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)\simeq\pi^{\dagger}\big(\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{E}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})\big).\tag{344}$$

Note that the multiplication of $\zeta\cdot t_m$ on $j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)$ is invertible. Hence, we obtain the following isomorphism from (344) by using Lemma 22.7.1:

$$\bigoplus_{p=0}^{m-1}\pi_{\dagger}\big(j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)\big)\simeq\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})\otimes\pi_*\mathcal{O}_{\widetilde{\mathcal{X}}\times\boldsymbol{C}_{t_m}}(*t_m).\tag{345}$$

We have the natural μ_{em}-action on $\pi_{\dagger}\big(\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)\big)$, which is the same as the one induced by the μ_{em}-action on $\pi_*\mathcal{O}_{\widetilde{\mathcal{X}}\times\boldsymbol{C}_{t_m}}(*t_m)$ in the right-hand side of (345). We have a natural morphism from $\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{E}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})$ to the μ_{em}-invariant part. Its restriction outside of $\{t_m=0\}$ is an isomorphism, and the multiplication by t_m is invertible on both sides of (345). Hence, $\varphi_m^{\dagger}\big(i_{g\dagger}\mathfrak{E}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})$ is identified with the μ_{em}-invariant part of $\pi_{\dagger}\big(\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}_p(*t_m)\big)$.

To show that the isomorphism is compatible with the sesqui-linear pairings, we only have to compare them on $(X\times\boldsymbol{C}_m)\smallsetminus(X\times\{0\})$, where the claim is clear. □

Lemma 17.2.6. — *$P\,\mathrm{Gr}_{\ell}^{W(N)}\widetilde{\psi}_{t,\mathfrak{a},u}\big(\mathfrak{T}(E)\big)$ with the naturally induced Hermitian sesqui-linear duality is a polarized pure twistor structure of weight ℓ.*

Proof. — Since $P\,\mathrm{Gr}_{\ell}^{W(N)}\widetilde{\psi}_{t,\mathfrak{a},u}\big(\mathfrak{T}(E)\big)$ is a direct summand of

$$\bigoplus_p P\,\mathrm{Gr}_{\ell}^{W(N)}\widetilde{\psi}_{t_m,u}\big(\pi_{\dagger}j_{p\dagger}\mathfrak{T}_p\big),$$

Lemma 17.2.6 follows from Lemma 17.2.4. □

Corollary 17.2.7. — *The above $\mathcal{R}_X$-triple $\mathfrak{T}(E)$ with the Hermitian duality a polarized wild pure twistor D-module of weight* 0. □

17.2.3. End of the proof of Proposition 17.2.1. — By Corollary 17.2.7, we obtain that Φ is essentially surjective. Let us show that Φ is fully faithful. It is easy to deduce that Φ is faithful from Lemma 22.4.10. Let us show that Φ is full.

Let $(\mathcal{T},\mathcal{S})\in\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(X,D,0)$, where $\mathcal{T}=(\mathcal{M},\mathcal{M},C)$ and $\mathcal{S}=(\mathrm{id},\mathrm{id})$, and let $(E,\overline{\partial}_E,\theta,h)$ be the corresponding wild harmonic bundle on $X\smallsetminus D$. We have the associated wild pure twistor D-module $\mathfrak{T}(E)=(\mathfrak{E},\mathfrak{E},\mathfrak{C})$ of weight 0 with the polarization $\mathfrak{S} = (\mathrm{id}, \mathrm{id})$. We only have to show that the isomorphism $\mathfrak{T}(E)_{|X \setminus D} \simeq \mathcal{T}_{|X \setminus D}$ can be extended to $\mathfrak{T}(E) \simeq \mathcal{T}$. Since the property is local, we may and will assume $X = \Delta$ and $D = \{O\}$. We have the natural isomorphism $\mathcal{M}_{|(X \setminus D) \times \boldsymbol{C}_\lambda} \simeq \mathfrak{E}_{|(X \setminus D) \times \boldsymbol{C}_\lambda}$. We only have to show that it can be extended to an isomorphism $\mathcal{M} \simeq \mathfrak{E}$. Since both modules are strictly S-decomposable, we only have to show $\mathcal{M}(*z) \simeq \mathfrak{E}(*z)$. (See Lemma 22.4.10.)

Let $\varphi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a ramified covering such that $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is unramified. We obtain the $\mathcal{R}_{\widetilde{X}}(*\widetilde{D})$-triples $\varphi^\dagger \mathcal{T}(*z)$ and $\varphi^\dagger \mathfrak{T}(E)(*z)$. They are the same on $\widetilde{X} - \widetilde{D}$. By the assumption that $(\mathcal{T}, \mathcal{S})$ is a polarized wild pure twistor D-module, $P\operatorname{Gr}_\ell^{W(N)} \psi_{\widetilde{z},\mathfrak{a},u}(\varphi^\dagger \mathcal{T}(*z))$ with the naturally induced Hermitian sesqui-linear duality are polarized pure twistor structures of weight ℓ. Due to Theorem 12.6.1, we obtain that $\varphi^* \mathcal{M}(*z)$ is naturally isomorphic to $\varphi^* \mathfrak{E}(*z)$, and hence $\mathcal{M}(*z) \simeq \mathfrak{E}(*z)$. Thus, we are done. □

17.3. Gysin map for wild pure twistor D-modules

We consider the Gysin maps for wild pure twistor D-modules, following M. Saito's argument in the Hodge case (see [**77**]).

17.3.1. $\mathcal{R}$-module $j_* j^* \mathcal{M}$. — Let X be a complex manifold, and let $i_Y : Y \subset X$ be a smooth hypersurface of X. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{Y} := \boldsymbol{C}_\lambda \times Y$. We have the following $\mathcal{R}_X$-submodule of $\mathcal{O}_{\mathcal{X}}(*\mathcal{Y})$:

$$j_* j^* \mathcal{O}_{\mathcal{X}} := \mathcal{R}_X \cdot \mathcal{O}_{\mathcal{X}}(\mathcal{Y}) \subset \mathcal{O}_{\mathcal{X}}(*\mathcal{Y}).$$

If $Y = \{t = 0\}$, it is equal to $\sum_m \mathcal{O}_{\mathcal{X}} \cdot (\lambda \cdot t^{-1})^m \cdot t^{-1}$. Thus, it is holonomic and strict. For an $\mathcal{R}_X$-module $\mathcal{M}$, we define

$$j_* j^* \mathcal{M} := \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{X}}} j_* j^* \mathcal{O}_{\mathcal{X}} \subset \mathcal{M}(*\mathcal{Y}).$$

In other words, $j_* j^* \mathcal{M}$ is the submodule of $\mathcal{M}(*\mathcal{Y})$ generated by $\mathcal{M}(\mathcal{Y}) := \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Y})$ over $\mathcal{R}_X$. If the support of any submodule of $\mathcal{M}$ is not contained in $\mathcal{Y}$, we have the injection $\mathcal{M} \to j_* j^* \mathcal{M}$.

In the following, let $\mathcal{M}$ be a strict, holonomic $\mathcal{R}_X$-module, and we assume that Y is *strictly non-characteristic* with respect to $\mathcal{M}$. (See Section 3.7 of [**73**].)

Lemma 17.3.1. — *Assume that X has a global coordinate system $(t, z_1, \dots, z_{n-1})$ with $Y = \{t = 0\}$. Then, the $\mathcal{R}_X$-modules $\mathcal{M}$ and $j_* j^* \mathcal{M}$ are strictly specializable along t.*

- *$\mathcal{KMS}(\mathcal{M}, t)$ is contained in $\mathbb{Z}_{\leqslant -1} \times \{0\} \subset \boldsymbol{R} \times \boldsymbol{C}$, and the V-filtration for $\mathcal{M}$ is given by $V_n(\mathcal{M}) = t^{-1-n}\mathcal{M}$ for $n \in \mathbb{Z}_{\leqslant -1}$ and $V_n(\mathcal{M}) = \mathcal{M}$ for $n \in \mathbb{Z}_{\geqslant 0}$.*
- *$\mathcal{KMS}(j_* j^* \mathcal{M}, t)$ is contained in $\mathbb{Z} \times \{0\} \subset \boldsymbol{R} \times \boldsymbol{C}$. The V-filtration for $j_* j^* \mathcal{M}$ is given by*

$$V_n\big(j_* j^* \mathcal{M}\big) = \begin{cases} t^{-n} \mathcal{M} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{Y}) & (n \in \mathbb{Z}_{\leqslant 0}), \\ \sum_{p+q \leqslant n,\ q \leqslant 0} \eth_t^p V_q & (n \in \mathbb{Z}_{\geqslant 1}). \end{cases}$$

The induced morphism $t : \psi_{t,0}(j_*j^*\mathcal{M}) \to \psi_{t,-\boldsymbol{\delta}_0}(j_*j^*\mathcal{M})$ *is an isomorphism, and* $\eth_t : \psi_{t,-\boldsymbol{\delta}_0}(j_*j^*\mathcal{M}) \to \psi_{t,0}(j_*j^*\mathcal{M})$ *is trivial. Note* $V_n(j_*j^*\mathcal{M}) = V_n(\mathcal{M})$ *for* $n \in \mathbb{Z}_{\leqslant -1}$, *and* $\boldsymbol{\delta}_0 = (1,0)$.
Remark that the filtration V *is independent of the choice of a coordinate system.*

Proof. — The first claim follows from Lemma 3.7.4 of [**73**]. Let us consider the second claim. It is easy to check Condition 22.3.1. By construction, $V_n(j_*j^*\mathcal{M}) = V_n(\mathcal{M})$ for $n \leqslant -1$. Hence, $\mathrm{Gr}_n^V(j_*j^*\mathcal{M})$ are strict for $n \leqslant -1$. We can easily check (i) $t : \mathrm{Gr}_0^V \to \mathrm{Gr}_{-1}^V$ is an isomorphism, (ii) $\eth_t : \mathrm{Gr}_{-1}^V \to \mathrm{Gr}_0^V$ is 0. In particular, it follows that $\mathrm{Gr}_0^V(j_*j^*\mathcal{M})$ is strict. We can also easily check the vanishing of the action of $-\eth_t t - n\lambda$ on $\mathrm{Gr}_n^V(j_*j^*\mathcal{M})$. It follows that $t : \mathrm{Gr}_n^V \to \mathrm{Gr}_{n-1}^V$ are injective for $n \geqslant 1$, and hence we obtain that $\mathrm{Gr}_n^V(j_*j^*\mathcal{M})$ are strict for $n \geqslant 1$. Thus, we are done. □

Let $\mathcal{N}_{Y/X}$ denote the sheaf of sections of the normal bundle of Y in X. We put $\mathcal{N}_{\mathcal{Y}/\mathcal{X}} := \lambda \cdot p_\lambda^*\mathcal{N}_{Y/X}$, where p_λ denotes the projection $\mathcal{X} \to X$.

Lemma 17.3.2. — *We have a naturally induced isomorphism:*

$$\Lambda' : j_*j^*(\mathcal{M})/\mathcal{M} \simeq \lambda^{-1} \cdot i_{Y\dagger}i_Y^\dagger\mathcal{M}.$$

Proof. — We have naturally induced isomorphisms

$$j_*j^*(\mathcal{M})/\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_\mathcal{X}} j_*j^*(\mathcal{O}_\mathcal{X})/\mathcal{O}_\mathcal{X}, \quad i_{Y\dagger}i_Y^\dagger\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_\mathcal{X}} i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X}.$$

Hence, we only have to consider the case $\mathcal{M} = \mathcal{O}_\mathcal{X}$. Recall that $i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X}$ is isomorphic to $i_{Y*}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}} \otimes \mathrm{Sym}^\bullet \mathcal{N}_{\mathcal{Y}/\mathcal{X}})$. It is equipped with the filtration given by $V_m = i_{Y*}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}} \otimes \mathrm{Sym}^{\bullet \leqslant m} \mathcal{N}_{\mathcal{Y}/\mathcal{X}})$ for $m \in \mathbb{Z}_{\geqslant 0}$. Note that $V_0 = i_{Y*}\mathcal{N}_{\mathcal{Y}/\mathcal{X}} \subset i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X}$ generates $i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X}$ over $\mathcal{R}_X$.

We have the subsheaf $\mathcal{O}_\mathcal{X}(\mathcal{Y}) \subset j_*j^*\mathcal{O}_\mathcal{X}$ and a naturally defined surjection

$$\mathcal{O}_\mathcal{X}(\mathcal{Y}) \longrightarrow \mathcal{O}_\mathcal{X}(\mathcal{Y})_{|\mathcal{Y}} \simeq p_\lambda^*\mathcal{N}_{Y/X} \simeq \lambda^{-1} \cdot V_0.$$

We claim (i) it is uniquely extended to a morphism $j_*j^*\mathcal{O}_\mathcal{X} \to \lambda^{-1} \cdot i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X}$ as $\mathcal{R}_X$-modules, (ii) the kernel is $\mathcal{O}_\mathcal{X}$. Because of the uniqueness, we only have to check these properties locally. Then, the claims can be checked by a direct calculation. □

We have the isomorphism $\Lambda : j_*j^*\mathcal{M}/\mathcal{M} \simeq i_{Y\dagger}i_Y^\dagger\mathcal{M}$ given by $\Lambda = -\sqrt{-1}\lambda \cdot \Lambda'$. In the case $X = \{(t, z_1, \dots, z_{n-1})\}$, $Y = \{t = 0\}$ and $\mathcal{M} = \mathcal{O}_\mathcal{X}$, we have $\Lambda(\sqrt{-1}t^{-1}) = [\eth_t]$, where the latter denotes the naturally defined section of $\mathcal{N}_{\mathcal{Y}/\mathcal{X}}$.

17.3.2. $\mathcal{R}$-module $j_!j^*\mathcal{M}$. — Let $\mathcal{M}$ and Y be as above. Let us construct an $\mathcal{R}_X$-module $j_!j^*\mathcal{M}$. First, let us consider the case $\mathcal{M} = \mathcal{O}_\mathcal{X}$. As an $\mathcal{O}_\mathcal{X}$-module, we define

$$j_!j^*\mathcal{O}_\mathcal{X} := \mathcal{O}_\mathcal{X} \oplus i_{Y\dagger}i_Y^\dagger\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{X} \oplus i_{Y*}\big(\mathrm{Sym}^\bullet \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \otimes \mathcal{N}_{\mathcal{Y}/\mathcal{X}}\big).$$

The action of $\Theta_\mathcal{X}$ on $j_!j^*\mathcal{O}_\mathcal{X}$ is given as follows:

$$v \cdot (s_1, s_2) = \big(v \cdot s_1,\ v \cdot s_2 + \pi(v_{|Y}) \cdot s_{1|Y}\big).$$

Here, "$|Y$" denotes the restriction to Y, π denotes a projection of $\Theta_{\mathcal{X}|\mathcal{Y}} \to \mathcal{N}_{\mathcal{Y}/\mathcal{X}}$, and $v \cdot s_2$ is given by the natural $\mathcal{R}_X$-module structure on $i_{Y\dagger}i_Y^{\dagger}\mathcal{O}_{\mathcal{X}}$. It is uniquely extended to an action of $\mathcal{R}_X$ on $j_!j^*\mathcal{O}_{\mathcal{X}}$. We only have to check it on a coordinate neighbourhood such that $Y = \{t = 0\}$ by a direct calculation.

In the general case, we put $j_!j^*\mathcal{M} := \mathcal{M} \otimes j_!j^*\mathcal{O}_{\mathcal{X}}$. (Note that Y is assumed to be strictly non-characteristic with respect to $\mathcal{M}$.) By construction, we have the following exact sequence of $\mathcal{R}_X$-modules:

$$0 \longrightarrow i_\dagger i^\dagger \mathcal{M} \longrightarrow j_!j^*\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow 0 \tag{346}$$

Lemma 17.3.3. — *Assume that X has a global coordinate system $(t, z_1, \dots, z_{n-1})$ with $Y = \{t = 0\}$. The $\mathcal{R}_X$-module $j_!j^*\mathcal{M}$ is strictly specializable along t. The set $\mathcal{KMS}(j_!j^*\mathcal{M})$ is contained in $\mathbb{Z} \times \{0\}$. The V-filtration is given by $V_n(j_!j^*\mathcal{M}) = V_n(\mathcal{M}) \oplus V_n(i_\dagger i^\dagger\mathcal{M})$. More concretely,*

$$V_n(j_!j^*\mathcal{M}) = \begin{cases} t^{-n-1}\mathcal{M} & (n \in \mathbb{Z}_{\leqslant -1}), \\ \mathcal{M} \oplus \sum_{p \leqslant n} i_{Y*}\big(\mathrm{Sym}^p \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \otimes \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \otimes \mathcal{M}\big) & (n \in \mathbb{Z}_{\geqslant 0}). \end{cases}$$

The induced map $t : \psi_{t,0}(j_!j^*\mathcal{M}) \to \psi_{t,-\boldsymbol{\delta}_0}(j_!j^*\mathcal{M})$ is 0, and $\eth_t : \psi_{t,-\boldsymbol{\delta}_0}(j_!j^*\mathcal{M}) \to \psi_{t,0}(j_!j^*\mathcal{M})$ is an isomorphism.

Note that the filtration V is independent of the choice of a coordinate system. □

17.3.3. Sesqui-linear pairings $j_!j^*C$ and j_*j^*C. — Let $(\mathcal{M}', \mathcal{M}'', C)$ be an $\mathcal{R}_X$-triple such that $\mathcal{M}'$ and $\mathcal{M}''$ are strict and holonomic. Let Y be strictly non-characteristic with respect to $\mathcal{M}'$ and $\mathcal{M}''$. We would like to construct a sesqui-linear pairing j_*j^*C (resp. $j_!j^*C$) of $j_!j^*\mathcal{M}'$ (resp. $j_*j^*\mathcal{M}'$) and $j_*j^*\mathcal{M}''$ (resp. $j_!j^*\mathcal{M}''$).

We explain the construction of j_*j^*C. Let us consider the case where X has a global coordinate system with $Y = \{t = 0\}$. Let $\lambda_0 \in \boldsymbol{S}$, and let $U(\lambda_0)$ denote a small neighbourhood of λ_0 in $\boldsymbol{C}^*_\lambda$. We put $\boldsymbol{I}(\lambda_0) := U(\lambda_0) \cap \boldsymbol{S}$. Let f (resp. g) be a local section of $\mathcal{M}'$ (resp. $\mathcal{M}''$) on $U(\lambda_0) \times X$ (resp. $\sigma(U(\lambda_0)) \times X$). Let ϕ be a C^∞-top form on X with compact support. Then, $\big\langle C(f, \sigma^*g),\ |t|^{2s} \cdot \bar{t}^{-1}\phi\big\rangle$ gives a continuous function on $\boldsymbol{I}(\lambda_0) \times \big\{s \in \boldsymbol{C} \mid \mathrm{Re}(s) \gg 0\big\}$, which is holomorphic with respect to s.

Lemma 17.3.4. — *$\big\langle C(f, \sigma^*g), |t|^{2s} \cdot \bar{t}^{-1}\phi\big\rangle$ gives a continuous function on the set $\boldsymbol{I}(\lambda_0) \times \big\{s \in \boldsymbol{C} \mid \mathrm{Re}(s) > -1\big\}$ which is holomorphic with respect to s.*

Proof. — Since it can be shown by a standard argument, we give only an outline. Let $\omega := dt \cdot d\bar{t} \cdot \prod_{j=1}^{n-1} dz_j \cdot d\bar{z}_j$. We take a sufficiently large N. A test function ϕ on X can be decomposed as

$$\phi = \sum_{i+j \leqslant N} \phi_{i,j} \cdot t^i \cdot \bar{t}^j \cdot \omega + \varphi \cdot \omega,$$

where $\phi_{i,j}$ are test functions which are constant with respect to t around $t = 0$, and φ is a test function such that $\varphi = O\big(|t|^N\big)$ around $t = 0$. If N is sufficiently large, the contribution of φ to $\langle C(f,\sigma^*g),|t|^{2s}\cdot\overline{t}^{-1}\phi\rangle$ is holomorphic around $s=0$. Let us consider

$$F_{i,j}(s):=\langle C(f,\sigma^*g),\ |t|^{2s}\cdot t^i\cdot\overline{t}^{j-1}\cdot\phi_{i,j}\cdot\omega\rangle=\langle C\big(t^{i+1}f,\ \sigma^*(t^jg)\big),\ |t|^{2(s-1)}\phi_{i,j}\cdot\omega\rangle$$

Note $t^{i+1}f\in V_{-i-2}(\mathcal{M}')$ and $t^jg\in V_{-j-1}(\mathcal{M}'')$. We put

$$b_{M,L}:=\prod_{m=0}^{L}\big(-\partial_t t+i+2+m\big)^M.$$

Using a standard argument, we obtain the following equality, modulo continuous functions on $\boldsymbol{I}(\lambda_0)\times\boldsymbol{C}_s$ which are holomorphic with respect to s:

$$\text{(347)}\quad \langle C\big(b_{M,L}(t^{i+1}f),\sigma^*(t^jg)\big),\ |t|^{2(s-1)}\cdot\phi_{i,j}\cdot\omega\rangle\equiv \prod_{m=0}^{L}(s+i+1+m)^M\langle C\big(t^{i+1}f,\sigma^*(t^jg)\big),\ |t|^{2(s-1)}\cdot\phi_{i,j}\cdot\omega\rangle.$$

If M is sufficiently large, there exists a local section $P\in V_0\mathcal{R}_X$ such that $b_{M,L}(t^{i+1}f)=P\cdot(t^{L+1+i+1}f)$. Hence, the left-hand side of (347) is

$$\text{(348)}\qquad \langle C\big(t^{i+1}f,\ \sigma^*(t^jg)\big),\ t^{L+1}\cdot|t|^{2(s-1)}\cdot Q\phi_{i,j}\cdot\omega\rangle$$

for some $Q\in V_0\mathcal{R}_X$. If L is sufficiently large, (348) is continuous and holomorphic with respect to s on $\boldsymbol{I}(\lambda_0)\times\{\mathrm{Re}(s)>-1\}$. Thus, the claim of the lemma follows. □

Let $\mathcal{M}''(\mathcal{Y}):=\mathcal{M}''\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}(\mathcal{Y})$, which is a $V_0\mathcal{R}_X$-submodule of $j_*j^*\mathcal{M}''$. We obtain the following pairing:

$$\text{(349)}\qquad \widetilde{C}:\mathcal{M}'_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}''(\mathcal{Y})_{|\boldsymbol{S}\times X}\longrightarrow\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}},$$

$$\text{(350)}\qquad \langle\widetilde{C}(f,\sigma^*(t^{-1}g)),\ \phi\rangle=\langle C(f,\sigma^*g),\ |t|^{2s}\cdot\overline{t}^{-1}\cdot\phi\rangle_{|s=0}.$$

Due to Lemma 22.10.9, the restriction of $\widetilde{C}$ to $\mathcal{M}'_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}''_{|\boldsymbol{S}\times X}$ is equal to C.

Lemma 17.3.5. — *The pairing $\widetilde{C}$ is a $V_0\mathcal{R}_X\otimes\sigma^*V_0\mathcal{R}_X$-homomorphism. It is independent of the choice of a coordinate system.*

Proof. — Let us consider $\langle\widetilde{C}\big(Pf,\sigma^*(Q(t^{-1}g))\big),\phi\rangle$. We have $Q'\in V_0\mathcal{R}_X$ such that $Qt^{-1}=t^{-1}Q'$ in $\mathcal{R}_X$. By definition, we have

$$\text{(351)}\quad \Big\langle\widetilde{C}\big(Pf,\sigma^*(Q(t^{-1}g))\big),\ \phi\Big\rangle=\Big\langle\widetilde{C}\big(Pf,\sigma^*(t^{-1}Q'g)\big),\ \phi\Big\rangle$$
$$=\Big\langle C\big(Pf,\sigma^*(Q'g)\big),\ |t|^{2s}\overline{t}^{-1}\cdot\phi\Big\rangle_{|s=0}=\Big\langle C(f,\sigma^*(g)),\ {}^tP\cdot\overline{\sigma^*({}^tQ't^{-1})}\big(|t|^{2s}\cdot\phi\big)\Big\rangle_{|s=0}.$$

Note ${}^tQ't^{-1}=t^{-1}\,{}^tQ$. There is a C^∞-top form φ with compact support, such that the following holds:

$${}^tP\cdot\overline{\sigma^*({}^tQ)}\big(|t|^{2s}\cdot\phi\big)=|t|^{2s}\cdot\big({}^tP\big)\cdot\overline{\sigma^*({}^tQ)}\phi+s\cdot|t|^{2s}\varphi.$$

Thus, (351) can be rewritten as follows:

$$\Big\langle C(f,\sigma^*(g)),\ \overline{t}^{-1}|t|^{2s}\cdot({}^tP)\cdot\overline{\sigma^*({}^tQ)}\cdot\phi\Big\rangle_{|s=0}=\Big\langle\widetilde{C}(f,\sigma^*(t^{-1}g)),\ {}^tP\cdot\overline{\sigma^*({}^tQ)}\phi\Big\rangle.$$

Thus, we obtain the first claim. The second claim is clear by construction. □

Note that $j_!j^*\mathcal{M}'$ and $j_*j^*\mathcal{M}''$ are generated by $\mathcal{M}'$ and $\mathcal{M}''(\mathcal{Y})$ over $\mathcal{R}_X$, respectively. Let $\overline{\eth}_t:=-\lambda^{-1}\cdot\overline{\partial}_t$, which is identified with $\sigma^*(\eth_t)$. We would like to extend $\widetilde{C}$ to the pairing

$$j_*j^*C: j_!j^*\mathcal{M}'\otimes\sigma^*\big(j_*j^*\mathcal{M}''\big)\longrightarrow \mathfrak{Db}_{S\times X/S} \tag{352}$$

by the formula

$$j_*j^*C\Big(\sum\eth_t^k a_k,\ \sigma^*\Big(\sum\eth_t^\ell b_\ell\Big)\Big)=\sum\eth_t^k\overline{\eth}_t^\ell\widetilde{C}(a_k,\sigma^*b_\ell) \tag{353}$$

for $a_k\in\mathcal{M}'$ and $b_\ell\in\mathcal{M}''(\mathcal{Y})$. We have to check the well-definedness.

Lemma 17.3.6. — *If $\sum_\ell\eth_t^\ell b_\ell=0$ in $j_*j^*\mathcal{M}'$, we have $\sum_\ell\overline{\eth}_t^\ell\widetilde{C}(a_k,\sigma^*b_\ell)=0$.*

Proof. — Since it can be shown by a standard argument (see [**73**] or [**67**], for example), we give only an outline. We use the notation in the proof of Lemma 17.3.4. We put $\Phi:=\sum_\ell\overline{\eth}_t^\ell\widetilde{C}(a_k,\sigma^*b_\ell)$. Since the support of Φ is contained in $t=0$, Φ is of the form $\sum_{p,q}A_{p,q}\cdot\partial_t^p\overline{\partial}_t^q\delta_Y$, where $A_{p,q}$ denote distributions on Y, and δ_Y denotes the δ-function at Y. We put $b_{M,L}:=\prod_{m=0}^L(-\partial_t\cdot t+m)^M$. If M is sufficiently large, there exists $Q\in V_0\mathcal{R}_X$ such that $b_{M,L}a_k=Q\cdot t^{L+1}\cdot a_k$. Hence $b_{M,L}\Phi=0$ for any large L. We also have the relation $(-\partial_t t+m)\partial^p\delta_Y=(p+m)\partial^p\delta_Y$. Then, it is easy to derive $\Phi=0$. □

Lemma 17.3.7. — *If $\sum_{k=0}^N\eth_t^k a_k=0$ in $j_!j^*\mathcal{M}'$, we have $\sum_{k=0}^N\eth_t^k\widetilde{C}(a_k,\sigma^*b_\ell)=0$.*

Proof. — In the case $N=0$, the claim is clear. Let us consider the case $N=1$. Assume $a_0+\eth_t a_1=0$ in $j_!j^*\mathcal{M}'=0$. Then, we have $a_{1|Y}=0$ and $a_0+\eth_t a_1=0$ in $\mathcal{M}'$. In particular, we have $a_1\in V_{-2}\mathcal{M}'$, and hence $a_1=t\cdot a_1'$ for some $a_1'\in\mathcal{M}'$. We have

$$\begin{aligned}\eth_t\widetilde{C}(a_1,\sigma^*b_\ell)=\eth_t\widetilde{C}(ta_1',\sigma^*b_\ell)&=\eth_t t\widetilde{C}(a_1',\sigma^*b_\ell)\\ &=\widetilde{C}(\eth_t(ta_1'),\ \sigma^*b_\ell)=\widetilde{C}(\eth_t a_1,\sigma^*b_\ell).\end{aligned} \tag{354}$$

In particular, we obtain $\widetilde{C}(a_0,\sigma^*b_\ell)+\eth_t\widetilde{C}(a_1,\sigma^*b_\ell)=\widetilde{C}(a_0+\eth_t a_1,\sigma^*b_\ell)=0$.

Let us show the claim for general N, assuming the claim for $N-1$. If $\sum_{j=0}^N\eth_t^j a_j=0$ in $j_!j^*\mathcal{M}'$, we have $a_N\in V_{-2}$ and $a_N=t\cdot a_N'$ for some $a_N'\in\mathcal{M}'$. We have the following vanishing in $j_!j^*\mathcal{M}$:

$$\sum_{j=0}^{N-2}\eth_t^j a_j+\eth_t^{N-1}\big(a_{N-1}+\eth_t ta_N'\big)=0.$$

By the inductive assumption on N, the following holds:

$$\sum_{j=0}^{N-2} \eth_t^j \widetilde{C}\big(a_j,\ \sigma^* b_\ell\big) + \eth_t^{N-1} \widetilde{C}\big(a_{N-1} + \eth_t(ta'_N),\ \sigma^* b_\ell\big) = 0.$$

We have $\widetilde{C}(a_{N-1} + \eth_t(ta'_N), \sigma^* b_\ell) = \widetilde{C}(a_{N-1}, \sigma^* b_\ell) + \eth_t \widetilde{C}(t \cdot a'_N, \sigma^* b_\ell)$ as in (354). Then, the induction can proceed. □

Thus, we obtain the pairing $j_* j^* C$ given by (353) in the case where X is equipped with a coordinate system such that $Y = \{t = 0\}$.

Lemma 17.3.8. — *$j_* j^* C$ is an $\mathcal{R}_{X|\mathbf{S}\times X} \otimes \sigma^* \mathcal{R}_{X|\mathbf{S}\times X}$-homomorphism. It is independent of the choice of a coordinate system. As a result, we obtain the globally well defined pairing $j_* j^* C$.*

Proof. — The first claim is clear by construction. The restriction of $j_* j^* C$ to $\mathcal{M}' \otimes \mathcal{M}''(\mathcal{Y})$ is equal to $\widetilde{C}$ given in (349), which is is independent of the choice of a coordinate system. Since $j_! j^* \mathcal{M}'$ and $j_* j^* \mathcal{M}''$ are generated by $\mathcal{M}'$ and $\mathcal{M}''(\mathcal{Y})$ over $\mathcal{R}_X$, the extension of $\widetilde{C}$ to an $\mathcal{R}_X \otimes \sigma^* \mathcal{R}_X$-homomorphism of $j_! j^* \mathcal{M}'$ and $j_* j^* \mathcal{M}''$ is unique. Hence, $j_* j^* C$ is independent of the choice of a coordinate system. □

Similarly, we have the globally defined sesqui-linear pairing $j_! j^* C$ of $j_* j^* \mathcal{M}'$ and $j_! j^* \mathcal{M}''$ given by the local formula

$$j_! j^* C\Big(\sum \eth_t^\ell b_\ell,\ \sigma^*\Big(\sum \eth_t^k a_k\Big)\Big) = \sum \eth_t^\ell \overline{\eth}_t^k \widetilde{C}(b_\ell, \sigma^* a_k) \tag{355}$$

for $b_\ell \in \mathcal{M}'(\mathcal{Y})$ and $a_k \in \mathcal{M}''$.

17.3.4. $\mathcal{R}$-triples $j_* j^* \mathcal{T}$ and $j_! j^* \mathcal{T}$. — We obtain the following $\mathcal{R}$-triples:

$$j_* j^* \mathcal{T} := \big(j_! j^* \mathcal{M}', j_* j^* \mathcal{M}'', j_* j^* C\big), \quad j_! j^* \mathcal{T} := \big(j_* j^* \mathcal{M}', j_! j^* \mathcal{M}'', j_! j^* C\big).$$

Lemma 17.3.9. — *We have the natural identification $(j_* j^* \mathcal{T})^* = j_! j^* (\mathcal{T}^*)$ and $j_* j^* (\mathcal{T}^*) = j_! j^* (\mathcal{T})^*$.*

Proof. — We only have to show one of them. We only have to consider the case where X has a coordinate system with $Y = \{t = 0\}$. We have $\big(j_* j^* C\big)^* (t^{-1} g, \sigma^*(f)) := \overline{\sigma^* \big(j_* j^* C(f, \sigma^*(t^{-1} g))\big)}$ for local sections f and g of $j_! j^* \mathcal{M}'$ and $j_* j^* \mathcal{M}''$, respectively. Hence, we obtain the following equality for local sections $f \in \mathcal{M}'$ and $g \in \mathcal{M}''(\mathcal{Y})$:

$$\begin{aligned} \Big\langle \overline{\sigma^* \big(j_* j^* C(f, \sigma^*(t^{-1} g))\big)},\ \phi \Big\rangle &= \sigma^* \overline{\big\langle j_* j^* C(f, \sigma^*(t^{-1} g)),\ \overline{\sigma^* \phi} \big\rangle} \\ &= \sigma^* \overline{\big\langle C(f, \sigma^*(g)),\ |t|^{2s} \overline{t}^{-1} \overline{\sigma^* \phi} \big\rangle_{|s=0}} = \big\langle C^*(g, \sigma^* f),\ |t|^{2s} t^{-1} \phi \big\rangle_{|s=0} \\ &= \big\langle j_! j^* C^*(t^{-1} g, \sigma^* f),\ \phi \big\rangle. \end{aligned} \tag{356}$$

Then, the claim of the lemma follows. □

It is easy to observe that j_*j^* and $j_!j^*$ are functorial, i.e., a morphism of $\mathcal{R}$-triples $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ naturally induces

$$j_*j^*\varphi : j_*j^*\mathcal{T}_1 \longrightarrow j_*j^*\mathcal{T}_2, \quad j_!j^*\varphi : j_!j^*\mathcal{T}_1 \longrightarrow j_!j^*\mathcal{T}_2.$$

17.3.5. Cok($\mathcal{T}$) and Ker($\mathcal{T}$). — We have the natural morphism $\mathcal{T} \to j_*j^*\mathcal{T}$. The cokernel is denoted by $\mathrm{Cok}(\mathcal{T})$. We also have the natural morphism $j_!j^*\mathcal{T} \to \mathcal{T}$. The kernel is denoted by $\mathrm{Ker}(\mathcal{T})$. It is easy to observe that Ker and Cok are functorial as in the case of j_*j^* and $j_!j^*$. The underlying sesqui-linear pairings of $\mathrm{Cok}(\mathcal{T})$ and $\mathrm{Ker}(\mathcal{T})$ are denoted by $[j_*j^*C]$ and $[j_!j^*C]$, respectively. We also use the symbols j_*j^*C and $j_!j^*C$, if there is no risk of confusion.

Lemma 17.3.10. — *We have the natural identification* $\mathrm{Ker}(\mathcal{T})^* = \mathrm{Cok}(\mathcal{T}^*)$. *Together with Lemma* 17.3.9, *we obtain the following identification of the exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}^* & \longrightarrow & j_*j^*(\mathcal{T}^*) & \longrightarrow & \mathrm{Cok}(\mathcal{T}^*) & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & \mathcal{T}^* & \longrightarrow & j_!j^*(\mathcal{T})^* & \longrightarrow & \mathrm{Ker}(\mathcal{T})^* & \longrightarrow & 0 \end{array}$$

Proof. — It can be checked directly from the definition. □

17.3.6. Pull-back of polarized pure twistor structure in the strictly non-characteristic case. — Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be a wild pure twistor D-module on X of weight 0 with a polarization $\mathcal{S} = (\varphi', \varphi'')$. Let $i_Y : Y \subset X$ be a smooth hypersurface which is strictly non-characteristic with respect to $\mathcal{T}$. Recall that we have in this case the polarized wild pure twistor D-module $i_Y^\dagger\mathcal{T} = (i_Y^\dagger\mathcal{M}', i_Y^\dagger\mathcal{M}'', C_Y)$ on Y of weight 0 with the induced polarization $i_Y^\dagger\mathcal{S} = (i_Y^\dagger\varphi', i_Y^\dagger\varphi'')$. (See [**73**] for more details.) It can be seen as follows. We only have to consider the case where $\mathcal{M}' = \mathcal{M}'' =: \mathcal{M}$ and $\varphi' = \varphi'' = \mathrm{id}$, to which the general case can be reduced.

Locally Y is defined by a coordinate function t. Note that $i_Y^\dagger\mathcal{M}$ is equal to $\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M})$ in this case, as shown in Lemma 3.7.4 of [**73**]. Moreover, the nilpotent part of $-\eth_t t$ on $\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M})$ is trivial. We also have the induced sesqui-linear pairing $C_Y := \psi_{t,-\boldsymbol{\delta}_0}C$. Hence, $i_Y^\dagger\mathcal{T}$ is given by $\psi_{t,-\boldsymbol{\delta}_0}\mathcal{T}$ locally. It is a wild pure twistor D-module with a polarization $(\mathrm{id}, \mathrm{id})$.

We obtain the strict S-decomposability of $i_Y^\dagger\mathcal{M}$ from the local expression as $\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M})$ and the twistor property of $\mathcal{T}$. We can glue the locally defined sesqui-linear pairings by using the uniqueness of the extension of the sesqui-linear pairings for strictly S-decomposable $\mathcal{R}$-modules. (See [**73**], [**67**] or Proposition 22.10.7 below.) Thus, we obtain a globally defined $\mathcal{R}$-triple. It is a wild pure twistor D-module on Y of weight 0 with a polarization $(\mathrm{id}, \mathrm{id})$.

17.3.7. Some isomorphisms and a polarization of $\mathrm{Ker}(\mathcal{T})$**.** — Let $(\mathcal{T},\mathcal{S})$ and Y be as in Subsection 17.3.6. Let Λ be the isomorphism given in Section 17.3.1.

Lemma 17.3.11. — *The pair of the morphisms* $\varphi_1 := \big(\Lambda, \mathrm{id}\big)$ *gives an isomorphism*

$$i_{Y\dagger} i_Y^{\dagger}\mathcal{T} \simeq \mathrm{Ker}(\mathcal{T}) \otimes \mathbb{T}^S(-1/2).$$

Proof. — We only have to consider the case $\mathcal{M}' = \mathcal{M}'' =: \mathcal{M}$. We only have to check the compatibility of φ_1 and the sesqui-linear pairings. Since $i_{Y\dagger}i_Y^{\dagger}\mathcal{M}$ is strictly S-decomposable, we only have to check the compatibility on the generic part. Hence, we only have to consider the case where $\mathcal{T}$ comes from a harmonic bundle on X.

Let $\mathbb{T}^S(0) = (\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}, C_0)$. We have the isomorphisms $i_{Y\dagger}i_Y^{\dagger}\mathcal{M} \simeq \mathcal{M}\otimes i_{Y\dagger}i_Y^{\dagger}\mathcal{O}_{\mathcal{X}}$ and $j_*j^*\mathcal{M}/\mathcal{M} \simeq \mathcal{M}\otimes j_*j^*\mathcal{O}_{\mathcal{X}}/\mathcal{O}_{\mathcal{X}}$. For local sections $f\otimes a \in \mathcal{M}\otimes j_*j^*\mathcal{O}_{\mathcal{X}}/\mathcal{O}_{\mathcal{X}}$ and $g\otimes b \in \mathcal{M}\otimes i_{Y\dagger}i_Y^{\dagger}\mathcal{O}_{\mathcal{X}}$, we have

$$[j_!j^*C]\big(f\otimes a,\, \sigma^*(g\otimes b)\big) = C(f,\sigma^*g)\cdot [j_!j^*C_0](a,\sigma^*b).$$

For local sections $f\otimes a$ and $g\otimes b$ of $\mathcal{M}\otimes i_{Y\dagger}i_Y^{\dagger}\mathcal{O}_{\mathcal{X}}$, we have

$$i_{Y\dagger}i_Y^{\dagger}C\big(f\otimes a,\ \sigma^*(g\otimes b)\big) = C(f,\sigma^*g)\cdot i_{Y\dagger}i_Y^{\dagger}C_0(a,\sigma^*b).$$

Hence, we only have to consider the case $\mathcal{T} = \mathbb{T}^S(0)$, $X=\Delta$ and $Y=\{0\}$.

Let $\omega = \chi\cdot\sqrt{-1}dt\cdot d\overline{t}/2\pi$, where χ is a test function on U which is constantly 1 around 0. We can check $j_!j^*C_0\big(t^{-1},\sigma^*1\big) = t^{-1}$ directly from the definition. Then, we have

$$\big\langle j_!j^*C_0\big(t^{-1},\sigma^*(\eth_t\cdot 1)\big),\ \omega\big\rangle = \big\langle -\lambda^{-1}\overline{\partial}_t(t^{-1}),\,\omega\big\rangle = -\lambda^{-1}.$$

Hence, we obtain

$$\Big\langle \sqrt{-1}\lambda[j_!j^*C_0]\big(\sqrt{-1}t^{-1},\sigma^*(\eth_t\cdot 1)\big),\ \omega\Big\rangle = 1.$$

We also have $\big\langle i_{Y\dagger}i_Y^{\dagger}C_0([\eth_t],\sigma^*[\eth_t]),\,\omega\big\rangle = 1$. (See Section 1.6.d of [**73**].) Now, we can check the compatibility of φ_1 and the sesqui-linear pairings easily. □

Lemma 17.3.12. — *The pair of morphisms* $\varphi_2 := \big(-\mathrm{id},\Lambda\big)$ *gives an isomorphism*

$$\mathrm{Cok}(\mathcal{T})\otimes\mathbb{T}^S(1/2) \longrightarrow i_{Y\dagger}i_Y^{\dagger}\mathcal{T}.$$

Proof. — Using the same argument as in the proof of Lemma 17.3.11, we can reduce the issue to the case $\mathcal{T} = \mathbb{T}^S(0)$, $X=\Delta$ and $Y=\{0\}$. We have $j_*j^*C_0\big(\eth_t\cdot 1,\,\sigma^*(t^{-1})\big) = \lambda\partial_t(\overline{t}^{-1})$ for the sections $\eth_t\cdot 1 \in j_!j^*\mathcal{O}_{\mathcal{X}}$ and $t^{-1}\in j_*j^*\mathcal{O}_{\mathcal{X}}$. Let ω be as in the proof of Lemma 17.3.11. Then, we have

$$\big\langle j_*j^*C_0\big(\eth_t\cdot 1,\,\sigma^*(t^{-1})\big),\omega\big\rangle = \big\langle\lambda\partial_t(\overline{t}^{-1}),\,\omega\big\rangle = \lambda.$$

Hence, we obtain

$$\Big\langle(\sqrt{-1}\lambda)^{-1}[j_*j^*C_0]\big(-\eth_t\cdot 1,\sigma^*(\sqrt{-1}t^{-1})\big),\,\omega\Big\rangle = 1.$$

Then, we can check the desired compatibility easily. □

We obtain an isomorphism $\varphi_3 = (-\Lambda, \Lambda) : \mathrm{Cok}(\mathcal{T})\otimes\mathbb{T}^S(1/2) \simeq \mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2)$. We obtain the following isomorphisms:

$$(357)\quad \mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2) \xrightarrow{\varphi_3^{-1}} \mathrm{Cok}(\mathcal{T})\otimes\mathbb{T}^S(1/2) \xrightarrow{\mathrm{Cok}(\mathcal{S})} \mathrm{Cok}(\mathcal{T}^*)\otimes\mathbb{T}^S(1/2) \longrightarrow \mathrm{Ker}(\mathcal{T})^*\otimes\mathbb{T}^S(1/2) \longrightarrow \big(\mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2)\big)^*.$$

The third map is given in Lemma 17.3.10. The composite is denoted by φ_4.

Lemma 17.3.13. — *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2) & \xrightarrow{\varphi_4} & \big(\mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2)\big)^* \\ \varphi_1\uparrow & & \downarrow\varphi_1^* \\ i_{Y\dagger}i_Y^\dagger\mathcal{T} & \xrightarrow{i_{Y\dagger}i_Y^\dagger\mathcal{S}} & i_{Y\dagger}i_Y^\dagger\mathcal{T}^* \end{array}$$

Hence, φ_4 gives a polarization of $\mathrm{Ker}(\mathcal{T})\otimes\mathbb{T}^S(-1/2)$.

Proof. — The first component of φ_4 is the composite of the following:

$$j_*j^*\mathcal{M}'/\mathcal{M}' \xleftarrow{-\Lambda^{-1}} i_{Y\dagger}i_Y^\dagger\mathcal{M}' \xleftarrow{i_{Y\dagger}i_Y^\dagger\varphi'} i_{Y\dagger}i_Y^\dagger\mathcal{M}'' \xleftarrow{\mathrm{id}} i_{Y\dagger}i_Y^\dagger\mathcal{M}'' \xleftarrow{-1} i_{Y\dagger}i_Y^\dagger\mathcal{M}''.$$

Hence, the first component of $\varphi_1^*\circ\varphi_4\circ\varphi_1$ is

$$\Lambda\circ\big((-\Lambda)^{-1}\circ i_{Y\dagger}i_Y^\dagger\varphi'\circ(-1)\big)\circ\mathrm{id} = i_{Y\dagger}i_Y^\dagger\varphi'.$$

It is equal to the first component of $i_{Y\dagger}i_Y^\dagger\mathcal{S}$. The equality for the second components can be checked in a similar way. □

Let $\mathcal{S}_{\mathrm{Ker}(\mathcal{T})} : \mathrm{Ker}(\mathcal{T}) \to \mathrm{Ker}(\mathcal{T})^*\otimes\mathbb{T}^S(1)$ denote the induced polarization of $\mathrm{Ker}(\mathcal{T})$.

17.3.8. Relation with the Lefschetz morphism. — Let $(\mathcal{T},\mathcal{S})$ and Y be as above. Let a_X denote the obvious map from X to a point. Let c be the cohomology class $2\pi[Y]$. When we take a C^∞-form ω representing c, we have the induced map $\mathcal{L}_c = (-L_\omega, L_\omega) : a^i_{X\dagger}\mathcal{T} \to a^{i+2}_{X\dagger}\mathcal{T}\otimes\mathbb{T}(1)$, i.e.,

$$a^{-i}_{X\dagger}\mathcal{M}' \xleftarrow{-L_\omega} a^{-i-2}_{X\dagger}\mathcal{M}', \qquad a^i_{X\dagger}\mathcal{M}'' \xrightarrow{L_\omega} a^{i+2}_{X\dagger}\mathcal{M}''.$$

Here, L_ω is given by the multiplication of $\lambda^{-1}\omega$. The map $\mathcal{L}_c$ is called the Lefschetz morphism. (See [**73**]. See also Subsection 22.2.4.)

From the exact sequence $0 \to \mathrm{Ker}(\mathcal{T}) \to j_!j^*\mathcal{T} \to \mathcal{T} \to 0$ of $\mathcal{R}_X$-triples, we obtain the following morphism:

$$i^*_{Y,\mathrm{Gys}} : a^i_{X\dagger}(\mathcal{T}) \longrightarrow a^{i+1}_{X\dagger}\big(\mathrm{Ker}(\mathcal{T})\big).$$

From the exact sequence $0 \to \mathcal{T} \to j_*j^*\mathcal{T} \to \mathrm{Cok}(\mathcal{T}) \to 0$, we obtain the following morphism:

$$i^{\mathrm{Gys}}_{Y*} : a^i_{X\dagger}\,\mathrm{Cok}(\mathcal{T}) \longrightarrow a^{i+1}_{X\dagger}(\mathcal{T}).$$

Recall that we have obtained the isomorphism

$$\varphi_3^{-1} : \mathrm{Ker}(\mathcal{T}) \otimes \mathbb{T}^S(-1/2) \longrightarrow \mathrm{Cok}(\mathcal{T}) \otimes \mathbb{T}^S(1/2).$$

Lemma 17.3.14. — *The following diagram is commutative:*

$$\begin{array}{ccc} a^i_{X\dagger}(\mathcal{T}) & \xrightarrow{\mathcal{L}_c} & a^{i+2}_{X\dagger}(\mathcal{T}) \otimes \mathbb{T}^S(1) \\ {\scriptstyle i^*_{Y,\mathrm{Gys}}}\downarrow & & \uparrow{\scriptstyle i^{\mathrm{Gys}}_{Y*}} \\ a^{i+1}_{X\dagger}(\mathrm{Ker}(\mathcal{T})) & \xrightarrow{a^{i+1}_X(\varphi_3^{-1})} & a^{i+1}_{X\dagger}(\mathrm{Cok}(\mathcal{T})) \otimes \mathbb{T}^S(1) \end{array}$$

Proof. — Let $n := \dim X$. Note that we recall the construction of $a_{X\dagger}\mathcal{M}'$ and $a_{X\dagger}\mathcal{M}''$ in Subsection 22.2.4. We use the notation there. For a local defining function t of Y, the section $\lambda^{-1}[\eth_t]dt$ of $i_{Y\dagger}i^\dagger_Y\mathcal{O}_{\mathcal{X}} \otimes \Omega^\bullet_{\mathcal{X}}$ is independent of the choice of t. Hence, it defines a global section of $i_{Y\dagger}i^\dagger_Y\mathcal{O}_{\mathcal{X}} \otimes \Omega^\bullet_{\mathcal{X}}$. Similarly, we have the well-defined global section $\lambda^{-1}dt/t$ of $j_*j^*\mathcal{O}_{\mathcal{X}}/\mathcal{O}_{\mathcal{X}} \otimes \Omega^\bullet_{\mathcal{X}}$.

Let s be a C^∞-section of $\mathcal{M}'' \otimes \Omega^\bullet_{\mathcal{X}}$ such that $ds = 0$. The induced map $a^i_{X\dagger}\mathcal{M}'' \to a^{i+1}_{X\dagger}(i_\dagger i^\dagger\mathcal{M}'')$ in $i^*_{Y,\mathrm{Gys}}$ is given by the following correspondence:

$$\begin{array}{ccccc} & & j_!j^*\mathcal{M}'' \otimes \Omega^\bullet_{\mathcal{X}} & \longrightarrow & \mathcal{M}'' \otimes \Omega^\bullet_{\mathcal{X}} \\ & & \downarrow & & \\ i_\dagger i^\dagger\mathcal{M}'' \otimes \Omega^{\bullet+1}_{\mathcal{X}} & \longrightarrow & j_!j^*\mathcal{M}'' \otimes \Omega^{\bullet+1}_{\mathcal{X}} & & \\ & & (s,0) & \longmapsto & s \\ & & \downarrow & & \\ (-1)^n\lambda^{-1}[\eth_t]\cdot dt\cdot s_{|Y} & \longmapsto & \big(0,(-1)^n\lambda^{-1}[\eth_t]\cdot dt\cdot s_{|Y}\big) & & \end{array}$$

Here, $(-1)^n$ appears because of the shift of the degree. By the isomorphism $\Lambda : i_\dagger i^\dagger\mathcal{M}'' \simeq j_*j^*\mathcal{M}''/\mathcal{M}''$ in $\varphi_3^{-1} : \mathrm{Ker}(\mathcal{T}) \simeq \mathrm{Cok}(\mathcal{T}) \otimes \mathbb{T}(1)$, we have the correspondence:

$$\lambda^{-1}[\eth_t]\cdot dt\cdot s_{|Y} \longleftrightarrow \sqrt{-1}\lambda^{-1}dt/t\cdot s_{|Y}.$$

We take a Hermitian metric g on the line bundle $\mathcal{O}(Y)$. Let $\sigma : \mathcal{O} \to \mathcal{O}(Y)$ be the canonical section. We obtain the function $\log|\sigma|^2_g$. Recall that the C^∞-form $\sqrt{-1}\overline{\partial}\partial\log|\sigma|^2_g$ represents the cohomology class $2\pi[Y]$. Then, we obtain the image of $(-1)^n\sqrt{-1}\lambda^{-1}dt/t\cdot s_{|Y}$ via the map $a^{i+1}_{X\dagger}(j_*j^*\mathcal{M}''/\mathcal{M}'') \to a^{i+2}_{X\dagger}(\mathcal{M}'')$ in i^{Gys}_{Y*}, by the following correspondence:

$$\begin{array}{ccc} & j_*j^*\mathcal{M}''\otimes\Omega_{\mathcal{X}}^{\bullet+1} \longrightarrow (j_*j^*\mathcal{M}''/\mathcal{M}'')\otimes\Omega_{\mathcal{X}}^{\bullet+1} \\ & \downarrow \\ \mathcal{M}''\otimes\Omega_{\mathcal{X}}^{\bullet+2} \longrightarrow & j_*j^*\mathcal{M}''\otimes\Omega_{\mathcal{X}}^{\bullet+2} \end{array}$$

$$(-1)^n\sqrt{-1}\lambda^{-1}\partial\log|\sigma|_g^2\cdot s \longmapsto (-1)^n\sqrt{-1}\lambda^{-1}dt/t\cdot s_{|Y}$$
$$\downarrow$$
$$\sqrt{-1}\lambda^{-1}\overline{\partial}\partial\log|\sigma|_g^2\cdot s \longmapsto \sqrt{-1}\lambda^{-1}\overline{\partial}\partial\log|\sigma|_g^2\cdot s$$

Thus, we can conclude that L_ω is equal to the $\big(a^i_{X\dagger}\mathcal{M}''\to a^{i+2}_{X\dagger}\mathcal{M}''\big)$-component of $i^{\mathrm{Gys}}_{Y*}\circ a^{i+1}_X(\varphi_3^{-1})\circ i^*_{Y,\mathrm{Gys}}$. Similarly we can check that $L_{-\omega}$ is equal to the $\big(a^{-i-2}_{X\dagger}\mathcal{M}'\to a^{-i}_{X\dagger}\mathcal{M}'\big)$-component of $i^{\mathrm{Gys}}_{Y*}\circ a^{i+1}_X(\varphi_3^{-1})\circ i^*_{Y,\mathrm{Gys}}$. □

Corollary 17.3.15. — *We have the following commutative diagram:*

(358)
$$\begin{array}{ccccc} a^{-i}_{X\dagger}(\mathcal{T}) & \xrightarrow{\mathcal{L}_c^{j-1}} & a^{-i+2j-2}_{X\dagger}(\mathcal{T})\otimes\mathbb{T}^S(j-1) & \xrightarrow{\mathcal{L}_c} & a^{-i+2j}_{X\dagger}(\mathcal{T})\otimes\mathbb{T}^S(j) \\ {\scriptstyle i^*_{Y,\mathrm{Gys}}}\downarrow & & {\scriptstyle i^*_{Y,\mathrm{Gys}}}\downarrow & & {\scriptstyle i^{\mathrm{Gys}}_{Y*}}\uparrow \\ a^{-i+1}_{X\dagger}(\mathrm{Ker}(\mathcal{T})) & \xrightarrow{\mathcal{L}_c^{j-1}} & a^{-i+2j-1}_{X\dagger}(\mathrm{Ker}(\mathcal{T}))\otimes\mathbb{T}^S(j-1) & \xrightarrow{\varphi_3^{-1}} & a^{-i+2j-1}_{X\dagger}(\mathrm{Cok}(\mathcal{T}))\otimes\mathbb{T}^S(j) \end{array}$$

Here, $a^{-i+2j-1}_{X\dagger}(\varphi_3^{-1})$ is denoted just by φ_3^{-1}. □

17.3.9. A lemma. — Let us consider the case where X is a projective variety and Y is a smooth ample hypersurface. We recall the following lemma.

Lemma 17.3.16. — *Let $\mathcal{M}$ be an $\mathcal{R}_X$-module on $\boldsymbol{C}_\lambda\times X$, which is strict, coherent and holonomic. Then, we have the vanishing $a^i_{X\dagger}\big(j_*j^*\mathcal{M}\big)=0$ for $i>0$.*

Moreover, the following holds:

- *If $i\geqslant 1$, $a^i_{X\dagger}\big(j_*j^*\mathcal{M}/\mathcal{M}\big)\simeq a^{i+1}_{X\dagger}\mathcal{M}$.*
- *If $i=0$, $a^0_{X\dagger}\big(j_*j^*\mathcal{M}/\mathcal{M}\big)\to a^1_{X\dagger}\mathcal{M}$ is surjective.*

Proof. — Let us show the first claim. We shall argue such a vanishing on a small compact neighbourhood $\mathcal{K}$ of $\lambda_0\in\boldsymbol{C}^*_\lambda$. We put $\mathcal{X}^{(\lambda_0)}:=\mathcal{K}\times X$. For an $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-coherent sheaf $\mathcal{F}$, we have $a^i_{X*}\big(\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(*\mathcal{Y})\big)=0$ for $i>0$, because $\mathcal{O}_{\mathcal{X}}(\mathcal{Y})$ is relatively ample. If $\mathcal{K}$ is sufficiently small, $\mathcal{M}_{\mathcal{X}^{(\lambda_0)}}$ has a finite filtration by $\mathcal{R}$-modules such that the associated graded $\mathcal{R}$-module is the limit of $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-coherent submodules. Hence, we obtain the vanishing $a^i_{X*}\big(j_*j^*\mathcal{M}\otimes p_\lambda^*\Omega^{r,0}_X\big)=0$ for $i>0$ and any $r\geqslant 0$, where $\Omega^{r,0}$ denote the sheaf of holomorphic r-forms on X. We can deduce the first claim from the vanishing. The other claim follows from the exact sequence $0\to\mathcal{M}\to j_*j^*\mathcal{M}\to j_*j^*\mathcal{M}/\mathcal{M}\to 0$. □

CHAPTER 18

THE HARD LEFSCHETZ THEOREM

We show the Hard Lefschetz theorem for polarized wild pure twistor D-modules, which is essentially due to M. Saito and C. Sabbah. This chapter is included for rather expository purpose.

18.1. Statement

Let X and Y be complex manifolds, and let $f : X \to Y$ be a projective morphism. Let $\mathcal{A}$ be a $\boldsymbol{Q}$-vector subspace of $\boldsymbol{C}$. Let $(\mathcal{T}, \mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight w on X. We have the graded $\mathcal{R}$-triple $\bigoplus f_\dagger^i \mathcal{T}$ on Y. We have the Lefschetz morphism $\mathcal{L}_c : f_\dagger^i \mathcal{T} \to f_\dagger^{i+2} \mathcal{T} \otimes \mathbb{T}^S(1)$ associated to the Chern class c of a relatively ample line bundle. We also have the induced Hermitian sesqui-linear duality $\bigoplus f_\dagger^i \mathcal{S}$ of $\bigoplus f_\dagger^i \mathcal{T}$ of weight w. (See [**73**].) Let us show the following theorem in this chapter.

Theorem 18.1.1. — *$\left(\bigoplus f_\dagger^i \mathcal{T}, \mathcal{L}_c, \bigoplus f_\dagger^i \mathcal{S}\right)$ is a polarized $\mathcal{A}$-wild Lefschetz twistor D-module of weight w on Y.*

Let $\mathcal{M}$ be the underlying $\mathcal{R}$-module of $\mathcal{T}$. We obtain the D-module by taking the specialization at $\mathcal{X}^\lambda$ for $\lambda \neq 0$, which is denoted by $\mathcal{M}^\lambda$.

Corollary 18.1.2. — *Let $(\mathcal{T}, \mathcal{S})$ be a polarized wild pure twistor D-module on X. The Hard Lefschetz Theorem holds for the D-module $\mathcal{M}^\lambda$ $(\lambda \neq 0)$, i.e., $L_c^i : f_\dagger^{-i}(\mathcal{M}^\lambda) \to f_\dagger^i(\mathcal{M}^\lambda)$ is an isomorphism. As a result, we have an isomorphism $Rf_\dagger(\mathcal{M}^\lambda) \simeq \bigoplus R^i f_\dagger(\mathcal{M}^\lambda)$ in the derived category of cohomologically holonomic complexes on Y. (See* [**23**].*)* □

The Hard Lefschetz Theorem was proved by Beilinson-Bernstein-Deligne-Gabber [**7**] for regular holonomic D-modules of geometric origin by the method of mod p-reductions. Then, it was vastly generalized in the original work due to M. Saito for polarized pure Hodge modules [**77**]. Using the argument of M. Saito, C. Sabbah proved it for regular polarized pure twistor D-modules [**73**]. We closely follow the argument of Saito and Sabbah. (But, we follow Saito's argument more closely in some part.)

18.1.1. Plan of the proof. — We use the induction on the dimensions of the strict supports $\mathrm{Supp}(\mathcal{T})$ and $f\big(\mathrm{Supp}(\mathcal{T})\big)$. Let us consider the following claim for any $n \geqslant m$:

$P(n,m)$ **:** The claim of Theorem 18.1.1 holds, in the case where $\dim \mathrm{Supp}(\mathcal{T}) \leqslant n$ and $\dim f(\mathrm{Supp}(\mathcal{T})) \leqslant m$ hold.

The claim $P(0,0)$ is obvious. The proof is divided into the following three steps.

Step 1 : Prove the Hard Lefschetz Theorem for a polarized wild pure twistor D-module in the case where X is a smooth projective curve and Y is a point (Proposition 18.2.1).

Step 2 : Give the argument for $P(n-1,m-1) \Longrightarrow P(n,m)$.

Step 3 : Give the argument for $P(n-1,0) \Longrightarrow P(n,0)$.

We will show Step 1 in Section 18.2. A rather detailed argument for Step 2 is explained in Chapter 14.6 of [**67**] in the regular case. Hence, we will indicate how to modify the statements in Section 18.3. We will give a rather detailed argument for Step 3 in Section 18.4 by following M. Saito, although it is just a translation of his argument to the case of wild pure twistor D-modules. We also rely on some results of Sabbah in [**73**]. In the following argument, we omit to distinguish $\mathcal{A}$.

18.2. Step 1

18.2.1. Statement. — Let X be a smooth projective curve, and let D be a finite subset of X. Let a_X denote the obvious morphism of X to a point. Let ω be a Kähler form of X.

Proposition 18.2.1. — *Let $(\mathcal{T},\mathcal{S})$ be a polarized wild pure twistor D-module on X of weight w. Then, the push-forward $\big(\bigoplus a^i_{X\dagger}\mathcal{T},\mathcal{L}_\omega,a_{X\dagger}\mathcal{S}\big)$ is a polarized graded Lefschetz twistor structure of weight w.*

We only have to consider the case where the strict support of $\mathcal{T}$ is X.

18.2.2. The L^2-cohomology for $(\mathcal{P}_*\mathcal{E}^\lambda,\mathbb{D}^\lambda,h)$. — Let X be a smooth projective curve. Let D be a finite subset of X. We take a Kähler metric $g_{X\setminus D}$ of $X\setminus D$ which is Poincaré-like around each point of D. Let ω_1 denote the associated Kähler form. We assume $\int_{X\setminus D}\omega_1=\int_X\omega$.

Let $(E,\overline{\partial}_E,\theta,h)$ be a wild harmonic bundle on $X\setminus D$. We have the associated filtered λ-flat bundle $(\mathcal{P}_*\mathcal{E}^\lambda,\mathbb{D})$ for $\lambda\in\boldsymbol{C}$. Recall that the norm estimate holds for

$(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda, h)$ (Proposition 8.1.1). Because of the results in Section 5.1, we have the following quasi-isomorphisms of the complexes of sheaves:

$$\mathcal{S}(\mathcal{P}_*\mathcal{E}^\lambda \otimes \Omega_X^{\bullet,0}) \longrightarrow \mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda) \longrightarrow \mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda). \tag{359}$$

In particular, we obtain the following corollary.

Corollary 18.2.2. — *The cohomology groups* $H^\bullet\big(\Gamma\big(\mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$ *associated to* $\mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ *are finite dimensional, where* Γ *denotes the functor taking global sections.*

Proof. — The hyper-cohomology groups associated to $\mathcal{S}(\mathcal{P}_*\mathcal{E}^\lambda \otimes \Omega_X^{\bullet,0})$ are finite dimensional. Hence, we obtain Corollary 18.2.2 by the quasi-isomorphisms in (359). □

Let Harm^i denote the space introduced in Section 8.4.1.

Proposition 18.2.3. — *The induced map* $\mathrm{Harm}^i \to H^i\big(\Gamma\big(\mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$ *is an isomorphism.*

Proof. — We only have to apply Lemma 8.4.13 with Corollary 18.2.2. □

We have the factorization $\mathrm{Harm}^i \subset \Gamma\big(\mathcal{L}^i_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big) \subset \Gamma\big(\mathcal{L}^i(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)$. Hence, we obtain the following factorization:

$$\mathrm{Harm}^i \simeq H^i\Big(\Gamma\big(\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\Big) \simeq H^i\Big(\Gamma\big(\mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\Big).$$

In the case $\lambda \neq 0$, we also have the factorization $\mathrm{Harm}^\bullet \subset \Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big) \subset \Gamma\big(\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)$, according to Proposition 8.4.5 and Corollary 7.5.4. It induces the following factorization:

$$\mathrm{Harm}^i \simeq H^i\Big(\Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\Big) \simeq H^i\Big(\Gamma\big(\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\Big). \tag{360}$$

We have the multiplication L_{ω_1} of ω_1 on $\mathrm{Harm}^\bullet$ and $H^i\big(\Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$, which is compatible with (360). We also have the multiplication L_ω of ω on $H^i\big(\Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$.

Lemma 18.2.4. — *The isomorphism* (360) *is compatible with* L_{ω_1} *and* L_ω.

Proof. — There exists τ such that (i) $\omega - \omega_1 = d\tau$, (ii) τ is bounded with respect to $g_{X\setminus D}$. Hence, it is easy to show $L_\omega = L_{\omega_1}$ on $H^i\big(\Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$. Then, the claim of the lemma follows. □

Remark 18.2.5. — Recall that a λ-connection is equivalent to an ordinary connection in the case $\lambda \neq 0$. Let $\mathcal{P}_{\min}\mathcal{E}^\lambda$ denote the D_X-submodule of $\mathcal{P}\mathcal{E}^\lambda$ generated by $\mathcal{P}_{<1}\mathcal{E}^\lambda$ over D_X. If λ is generic, it is the same as the standard minimal extension, and we obtain an isomorphism between the cohomology group of the D-module $\mathcal{P}_{\min}\mathcal{E}^\lambda$ and the L^2-cohomology group $H^\bullet\big(\Gamma\big(\mathcal{L}^\bullet(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big)\big)$ from the quasi-isomorphism (359). For this isomorphism, we do not need the harmonicity of h but only the norm estimates. □

18.2.3. Deformation caused by variation of the irregular values. — Recall $\mathcal{Q}_*\mathcal{E}^0 := \mathcal{P}_*\mathcal{E}^0$, and hence we have the isomorphism $\overline{\Phi}^0 : \mathrm{Harm}^i \to \mathbb{H}^i\big(\mathcal{S}(\mathcal{Q}_*\mathcal{E}^0 \otimes \Omega_X^{\bullet,0})\big)$. In the case $\lambda \neq 0$, let $\mathcal{Q}h$ be a Hermitian metric of $\mathcal{E}^\lambda$ on $X \setminus D$ whose restriction to a neighbourhood of D is as in Section 5.1.1 for the filtered λ-flat bundle $(\mathcal{Q}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$. We can construct the metric $\mathcal{Q}h^{(1+|\lambda|^2)}$ as in Section 5.1.3 with $T = 1 + |\lambda|^2$. Then, $\mathcal{Q}h^{(1+|\lambda|^2)}$ and h are mutually bounded (Lemma 5.1.6). Hence, we obtain the following quasi-isomorphisms because of the results in Section 5.1.3:

$$\begin{aligned}(361)\quad \overline{\mathcal{L}}_{\mathrm{poly}}(\mathcal{P}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda) \xleftarrow{\simeq} \overline{\mathcal{L}}_{\mathrm{poly}}(\mathcal{E}^\lambda, \mathcal{Q}h^{(1+|\lambda|^2)}, h) \xrightarrow{\simeq} \overline{\mathcal{L}}_{\mathrm{poly}}(\mathcal{Q}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\\ \xrightarrow{\simeq} \mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{Q}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda) \xleftarrow{\simeq} \mathcal{S}(\mathcal{Q}_*\mathcal{E}^\lambda \otimes \Omega_X^{\bullet,0}).\end{aligned}$$

They induce the isomorphisms of the associated cohomology groups. Therefore, we obtain a natural isomorphism:

$$(362)\qquad \overline{\Phi}^\lambda : \mathrm{Harm}^i \longrightarrow \mathbb{H}^i\big(\mathcal{S}(\mathcal{Q}_*\mathcal{E}^\lambda \otimes \Omega_X^{\bullet,0})\big).$$

Lemma 18.2.6. — *We have the compatibility* $\overline{\Phi}^\lambda \circ L_{\omega_1} = L_\omega \circ \overline{\Phi}^\lambda$.

Proof. — The multiplication L_ω is compatible with the quasi-isomorphisms in (361). Then, Lemma 18.2.6 follows from Lemma 18.2.4. □

If $|\lambda|$ is sufficiently small, due to Proposition 8.4.5 and Proposition 10.2.3, we have the following inclusions which are quasi-isomorphisms:

(363)

$$\mathrm{Harm}^\bullet \subset \Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{E}^\lambda, \mathcal{Q}h^{(1+|\lambda|^2)}, h)\big) \subset \Gamma\big(\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{Q}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big) \subset \Gamma\big(\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{Q}_*\mathcal{E}^\lambda, \mathbb{D}^\lambda)\big).$$

It is easy to see that the composite of (363) is equal to the isomorphism $\overline{\Phi}^\lambda$, if $|\lambda|$ is sufficiently small.

18.2.4. Quasi-isomorphisms for local families. — Let us consider the family case. Let $U(\lambda_0)$ denote a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$. We have the family of flat λ-connections $(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D})$, for which we take a Hermitian metric h_1 as in Subsection 5.3.1. We have the following quasi-isomorphisms of associated complexes of sheaves (the left-hand side is defined only in the case $\lambda_0 \neq 0$):

$$(364)\qquad \overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}) \longrightarrow \mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}) \longleftarrow \mathcal{S}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E} \otimes \Omega_X^{\bullet,0}).$$

Let p_X denote the projection $U(\lambda_0) \times X \to U(\lambda_0)$. By considering the push-forward by p_X, we obtain isomorphisms:

$$R^i p_{X*}\overline{\mathcal{L}}^\bullet_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}) \xrightarrow{\simeq} R^i p_{X*}\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}) \xleftarrow{\simeq} R^i p_{X*}\mathcal{S}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E} \otimes \Omega_X^{\bullet,0}).$$

These sheaves are $\mathcal{O}_{U(\lambda_0)}$-coherent.

Around $\lambda_0 = 0$, we have the inclusion $\mathrm{Harm}^\bullet \otimes \mathcal{O}_{U(0)} \to p_{X*}\big(\mathcal{L}^\bullet_{\mathrm{poly}}(\mathcal{Q}^{(0)}_*\mathcal{E}, \mathbb{D})\big)$ due to Proposition 8.4.5 and Proposition 10.4.2. Therefore, we obtain a morphism of the coherent $\mathcal{O}_{U(0)}$-modules:

$$\overline{\Phi}^{(0)}:\bigoplus_i \mathrm{Harm}^i\otimes\mathcal{O}_{U(0)}\longrightarrow\bigoplus_i R^ip_{X*}\big(\mathcal{S}\big(\mathcal{Q}_*^{(0)}\mathcal{E}\otimes\Omega_X^{\bullet,0}\big)\big). \tag{365}$$

Lemma 18.2.7

- *$\overline{\Phi}^{(0)}$ is an isomorphism.*
- *The specialization of $R^ip_{X*}\big(\mathcal{S}\big(\mathcal{Q}_*^{(0)}\mathcal{E}\otimes\Omega_X^{\bullet,0}\big)\big)$ at $\lambda\in U(0)$ is naturally isomorphic to $\mathbb{H}^i\big(\mathcal{S}\big(\mathcal{Q}_*\mathcal{E}^\lambda\otimes\Omega_X^\bullet\big)\big)$, and the specialization of $\overline{\Phi}^{(0)}$ is the same as $\overline{\Phi}^\lambda$ under the isomorphism.*

Proof. — For simplicity of the description, we put $\mathcal{S}^{(0)}:=\mathcal{S}(\mathcal{Q}_*^{(0)}\mathcal{E}\otimes\Omega^{\bullet,0})$. For any $\lambda_1\in U(0)$, let $\mathcal{S}^{(0)}_{|\lambda_1}$ denote the specialization of $\mathcal{S}^{(0)}$ at $\{\lambda_1\}\times X$. We remark that we have the natural quasi-isomorphism $\mathcal{S}^{(0)}_{|\lambda_1}\to\mathcal{S}\big(\mathcal{Q}_*\mathcal{E}^{\lambda_1}\otimes\Omega^{\bullet,0}\big)$. By using the remark in the last paragraph of Subsection 18.2.3, we obtain the following commutative diagram of the $\mathcal{O}_{U(0)}$-coherent sheaves:

$$\begin{array}{ccccccc}
\mathrm{Harm}^i\otimes\mathcal{O}_{U(0)} & \xrightarrow{\lambda-\lambda_1} & \mathrm{Harm}^i\otimes\mathcal{O}_{U(0)} & \longrightarrow & \mathrm{Harm}^i & \longrightarrow & 0\\
{\scriptstyle\overline{\Phi}^{(0)}}\big\downarrow & & {\scriptstyle\overline{\Phi}^{(0)}}\big\downarrow & & {\scriptstyle\overline{\Phi}^{\lambda_1}}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow\\
R^ip_{X*}\mathcal{S}^{(0)} & \xrightarrow{\lambda-\lambda_1} & R^ip_{X*}\mathcal{S}^{(0)} & \longrightarrow & R^ip_{X*}\mathcal{S}^{(0)}_{|\lambda_1} & \longrightarrow & R^{i+1}p_{X*}\mathcal{S}^{(0)}
\end{array}$$

Here, $\lambda-\lambda_1$ means the multiplication of $\lambda-\lambda_1$. Then, it is easy to show the claims of the lemma. □

Let us see around $\lambda_0\neq 0$. Let $T_1(\lambda):=1+|\lambda|^2$. We have the metric $h_1^{(T_1(\lambda))}$ as constructed in Section 5.3.4.

Lemma 18.2.8. — *If we shrink $U(\lambda_0)$ appropriately, the metrics $h_1^{(T_1(\lambda))}$ and h are mutually bounded.*

Proof. — Let h_0 be a Hermitian metric for the family of filtered λ-flat bundles $(\mathcal{P}_*^{(\lambda_0)}\mathcal{E},\mathbb{D})$ as in Subsection 5.3.1. Let S be a small sector in $U(\lambda_0)\times(X\setminus D)$. We can take a $\mathbb{D}$-flat splitting $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{\overline{S}}=\bigoplus_{\mathfrak{a},S}\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$ of the full Stokes filtration. Let $p^{(j)}_{\mathfrak{a},S}$ denote the projection onto $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$. Let $F_S(w)$ be given as in (198):

$$F_S(w):=\exp\big(w\,\mathcal{B}_S\big),\quad \mathcal{B}_S:=\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\mathfrak{a}\,p_{\mathfrak{a},S}.$$

According to Corollary 10.4.3, the metrics $F_S(-\overline{\lambda}+\overline{\lambda}_0)^*h_0$ and h are mutually bounded on S. By construction of $\mathcal{Q}^{(\lambda_0)}\mathcal{E}$ (see Subsection 4.5.3), the metrics $F_S(\overline{\lambda}_0)^*h_0$ and h_1 are mutually bounded on S. Hence, h and $F_S(-\overline{\lambda})^*h_1$ are mutually bounded. It implies the claim of Lemma 18.2.8. □

Then, we obtain the following inclusion, due to Proposition 8.4.5 and Proposition 10.2.3:

$$\bigoplus_i \mathrm{Harm}^i \otimes \mathcal{O}_{U(\lambda_0)} \longrightarrow p_{X*}\Big(\overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, h_1^{(T_1(\lambda))}\big)\Big).$$

We also have the following quasi-isomorphisms of the complexes of sheaves due to the results in Section 5.3:

$$(366)\quad \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, h_1^{(T_1(\lambda))}\big) \xleftarrow{\simeq} \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(\mathcal{Q}^{(\lambda_0)}\mathcal{E}, h_1, h_1^{(T_1(\lambda))}\big) \xrightarrow{\simeq} \overline{\mathcal{L}}^{\bullet}_{\mathrm{poly}}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}\big) \xrightarrow{\simeq} \mathcal{L}^{\bullet}_{\mathrm{poly}}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}\big) \xleftarrow{\simeq} \mathcal{S}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E} \otimes \Omega^{\bullet,0}_X\big).$$

Thus, we obtain a morphism of coherent $\mathcal{O}_{U(\lambda_0)}$-sheaves:

$$\overline{\Phi}^{(\lambda_0)} : \bigoplus_i \mathrm{Harm}^i \otimes \mathcal{O}_{U(\lambda_0)} \longrightarrow \bigoplus_i R^i p_{X*}\big(\mathcal{S}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E} \otimes \Omega^{\bullet,0}_X\big)\big).$$

Lemma 18.2.9

- $\overline{\Phi}^{(\lambda_0)}$ *is an isomorphism.*
- *The specialization of* $R^i p_{X*}\big(\mathcal{S}\big(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}\otimes\Omega^{\bullet,0}_X\big)\big)$ *at* $\lambda \in U(\lambda_0)$ *is naturally isomorphic to* $\mathbb{H}^i\big(\mathcal{S}\big(\mathcal{Q}_*\mathcal{E}^{\lambda} \otimes \Omega^{\bullet,0}\big)\big)$, *and the specialization of* $\overline{\Phi}^{(\lambda_0)}$ *at* λ *is the same as* $\overline{\Phi}^{\lambda}$ *under the isomorphism.*

Proof. — The claims can be shown by the arguments in the proof of Lemma 18.2.7. For simplicity of the description, we put $\mathcal{S}^{(\lambda_0)} := \mathcal{S}(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E} \otimes \Omega^{\bullet,0})$. For any $\lambda_1 \in U(\lambda_0)$, let $\mathcal{S}^{(\lambda_0)}_{|\lambda_1}$ denote the specialization of $\mathcal{S}^{(\lambda_0)}$ at $\{\lambda_1\} \times X$. We remark that we have the natural quasi-isomorphism $\mathcal{S}^{(\lambda_0)}_{|\lambda_1} \to \mathcal{S}\big(\mathcal{Q}_*\mathcal{E}^{\lambda_1} \otimes \Omega^{\bullet,0}\big)$. By using the commutative diagrams (85) and (86), we obtain the following commutative diagram of the $\mathcal{O}_{U(\lambda_0)}$-coherent sheaves:

$$\begin{array}{ccccccc}
\mathrm{Harm}^i \otimes \mathcal{O}_{U(\lambda_0)} & \xrightarrow{\lambda-\lambda_1} & \mathrm{Harm}^i \otimes \mathcal{O}_{U(\lambda_0)} & \longrightarrow & \mathrm{Harm}^i & \longrightarrow & 0 \\
{\scriptstyle \overline{\Phi}^{(\lambda_0)}}\downarrow & & {\scriptstyle \overline{\Phi}^{(\lambda_0)}}\downarrow & & {\scriptstyle \overline{\Phi}^{\lambda_1}}\downarrow\simeq & & \downarrow \\
R^i p_{X*}\mathcal{S}^{(\lambda_0)} & \xrightarrow{\lambda-\lambda_1} & R^i p_{X*}\mathcal{S}^{(\lambda_0)} & \longrightarrow & R^i p_{X*}\mathcal{S}^{(\lambda_0)}_{|\lambda_1} & \longrightarrow & R^{i+1} p_{X*}\mathcal{S}^{(\lambda_0)}
\end{array}$$

Here, $\lambda - \lambda_1$ means the multiplication of $\lambda - \lambda_1$. Then, it is easy to show the both claims of the lemma. □

18.2.5. Global isomorphism. — Let $\Omega^{\bullet,0}_X$ denote the holomorphic de Rham complex on X. We have the complex $\mathfrak{E} \otimes \Omega^{\bullet,0}_X$ induced by $\mathbb{D}$ and $\lambda\partial_X$. We would like to compare the hyper-cohomology of $\mathfrak{E} \otimes \Omega^{\bullet,0}_X$ and the harmonic forms. Let $U(\lambda_0)$ denote a small neighbourhood of λ_0 in $\boldsymbol{C}_{\lambda}$. Let $\mathfrak{E}^{(\lambda_0)}$ denote the restriction of $\mathfrak{E}$ to $U(\lambda_0) \times X$.

Lemma 18.2.10. — *The naturally defined morphism $\mathcal{S}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}\otimes\Omega_X^{\bullet,0})\to\mathfrak{E}^{(\lambda_0)}\otimes\Omega_X^{\bullet,0}$ is a quasi-isomorphism. Therefore, we obtain an isomorphism:*

$$\Phi^{(\lambda_0)}:\bigoplus_i \operatorname{Harm}^i\otimes\mathcal{O}_{U(\lambda_0)}\longrightarrow\bigoplus_i R^ip_{X*}\big(\mathfrak{E}^{(\lambda_0)}\otimes\Omega_X^{\bullet,0}\big).$$

Proof. — We only have to show the acyclicity of the quotient complex of $\mathcal{S}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}\otimes\Omega_X^{\bullet,0})\to\mathfrak{E}^{(\lambda_0)}\otimes\Omega_X^{\bullet,0}$. For that purpose, we may replace them with their completions at $U(\lambda_0)\times D$. For the regular part which comes from a tame harmonic bundle, the issue was studied in [**73**] and [**67**]. For the irregular parts, both of them are acyclic. □

For $U(\lambda_1)\subset U(\lambda_0)$, we have $\mathfrak{E}^{(\lambda_1)}=\mathfrak{E}^{(\lambda_0)}_{|U(\lambda_1)\times X}$. According to Lemmas 18.2.7 and 18.2.9, we obtain $\Phi^{(\lambda_0)}_{|U(\lambda_1)}=\Phi^{(\lambda_1)}$. Hence, we can glue $\{\Phi^{(\lambda_0)}\mid\lambda_0\in\boldsymbol{C}\}$, and we obtain the global isomorphism:

$$\Phi:\bigoplus_i\operatorname{Harm}^i\otimes\mathcal{O}_{\boldsymbol{C}_\lambda}\longrightarrow\bigoplus R^ip_{X*}\big(\mathfrak{E}\otimes\Omega_X^{\bullet,0}\big).\tag{367}$$

Lemma 18.2.11. — *We have the compatibility $\Phi\circ L_{\omega_1}=L_\omega\circ\Phi$.*

Proof. — Since the specialization of $R^ip_{X*}\big(\mathfrak{E}\otimes\Omega_X^{\bullet,0}\big)$ at λ is naturally identified with $\mathbb{H}^i\big(\mathcal{S}(\mathcal{Q}_*\mathcal{E}^\lambda\otimes\Omega^{\bullet,0})\big)$, the claim follows from Lemma 18.2.6. □

18.2.6. Twist. — To consider the push-forward of $\mathcal{R}$-triples, we twist the de Rham complex as in [**73**] which we refer to for more details and precision. Let $\Omega_{\mathcal{X}}^{1,0}:=\lambda^{-1}p_\lambda^*\Omega_X^{1,0}\subset p_\lambda^*\Omega_X^{1,0}\big(*(X\times 0)\big)$, and $\Omega_{\mathcal{X}}^{p,0}:=\bigwedge^p\Omega_{\mathcal{X}}^{1,0}$. The derivation ∂_X for $\Omega_X^{\bullet,0}$ induces the derivation of $\Omega_{\mathcal{X}}^{\bullet,0}$. We have the natural isomorphism $p_\lambda^*\Omega_X^{\bullet,0}\simeq\Omega_{\mathcal{X}}^{\bullet,0}$, via which ∂_X is identified with $\lambda\,\partial_X$ on $p_\lambda^*\Omega_X^\bullet$.

We have the derivation on $\mathfrak{E}\otimes\Omega_{\mathcal{X}}^{\bullet,0}$ induced by $\mathbb{D}^f$ and ∂_X, which is also denoted by $\mathbb{D}^f$. We have the natural isomorphism $(\mathfrak{E}\otimes\Omega_{\mathcal{X}}^{\bullet,0},\mathbb{D}^f)\simeq(\mathfrak{E}\otimes\Omega_X^{\bullet,0},\mathbb{D})$ induced by the natural isomorphism $\Omega_{\mathcal{X}}^{1,0}\simeq p_\lambda^*\Omega_X^{1,0}$ given by the multiplication by λ. Shifting the degree, we consider the complex $(\mathfrak{E}\otimes\Omega_{\mathcal{X}}^{1+\bullet,0},\mathbb{D}^f)$. We have a similar twist $\mathcal{S}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}\otimes\Omega_{\mathcal{X}}^{1+\bullet,0})$.

We put $\Omega_{\mathcal{X}}^{p,q}:=\Omega_{\mathcal{X}}^{p,0}\otimes p_\lambda^{-1}\Omega_X^{0,q}$. The derivation $-\partial_X-\overline{\partial}_X$ induces the derivation on $\Omega_{\mathcal{X}}^{1+\bullet,\bullet}$. By the natural isomorphism $\Omega_{\mathcal{X}}^{1+\bullet,\bullet}\simeq p_\lambda^{-1}\Omega_X^{\bullet,\bullet}$, $-\partial_X-\overline{\partial}_X$ corresponds to $-\lambda\partial_X-\overline{\partial}_X$. By the natural isomorphism, we obtain the complex of sheaves $\mathcal{K}^\bullet_{\mathrm{poly}}\big(\mathcal{Q}_*^{(\lambda_0)}\mathcal{E},\mathbb{D}\big)$ corresponding to $\mathcal{L}^\bullet_{\mathrm{poly}}\big(\mathcal{Q}_*^{(\lambda_0)}\mathcal{E},\mathbb{D}\big)$. Then, we have the following quasi-isomorphisms:

$$\mathcal{K}^\bullet_{\mathrm{poly}}\big(\mathcal{Q}_*^{(\lambda_0)}\mathcal{E},\mathbb{D}\big)\longleftarrow\mathcal{S}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}\otimes\Omega_{\mathcal{X}}^{1+\bullet,0})\longrightarrow\mathfrak{E}^{(\lambda_0)}\otimes\Omega_{\mathcal{X}}^{1+\bullet,0}.\tag{368}$$

If $\lambda_0=0$, we have the induced inclusion:

$$\operatorname{Harm}^p\otimes\mathcal{O}_{U(0)}\longrightarrow\Gamma\Big(\mathcal{L}^p_{\mathrm{poly}}\big(\mathcal{Q}_*^{(0)}\mathcal{E},\mathbb{D}\big)\Big)\simeq\Gamma\Big(\mathcal{K}^{p-1}_{\mathrm{poly}}\big(\mathcal{Q}_*^{(0)}\mathcal{E},\mathbb{D}\big)\Big).$$

Thus, we obtain the isomorphism $\widetilde{\Phi}^{(0)}:\operatorname{Harm}^{1+\bullet}\simeq Rp_{X*}\big(\mathfrak{E}\otimes\Omega_{\mathcal{X}}^{1+\bullet,0}\big)$.

If $\lambda_0 \neq 0$, we have the corresponding twist for the diagram (366):

$$\overline{\mathcal{K}}^{\bullet}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, h_1^{(T_1(\lambda))}) \xleftarrow{\simeq} \overline{\mathcal{K}}^{\bullet}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}, h_1, h_1^{(T_1(\lambda))}) \xrightarrow{\simeq} \overline{\mathcal{K}}^{\bullet}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, \mathbb{D}) \xrightarrow{\simeq} \mathcal{K}^{\bullet}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, \mathbb{D}) \xleftarrow{\simeq} \mathcal{S}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}}). \tag{369}$$

We also have the induced inclusion:

$$\mathrm{Harm}^{p} \otimes \mathcal{O}_{U(\lambda_0)} \longrightarrow \Gamma\Big(\overline{\mathcal{K}}^{p-1}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, h_1^{(T_1(\lambda))})\Big).$$

We obtain induced isomorphisms $\widetilde{\Phi}^{(\lambda_0)} : \mathrm{Harm}^{1+\bullet} \otimes \mathcal{O}_{U(\lambda_0)} \to Rp_{X*}\big(\mathfrak{E}^{(\lambda_0)} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}}\big)$. By gluing them, we obtain $\widetilde{\Phi} : \mathrm{Harm}^{1+\bullet} \otimes \mathcal{O}_{\boldsymbol{C}_\lambda} \to Rp_{X*}\big(\mathfrak{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}}\big)$.

We put $\mathcal{H}^p := \mathrm{Harm}^{p+1} \otimes \mathcal{O}_{\boldsymbol{C}_\lambda}$. We obtain an induced isomorphism

$$\widetilde{\Phi} : \mathcal{H}^{\bullet} \longrightarrow a^{\bullet}_{X\dagger}\mathfrak{E}.$$

18.2.7. The graded Lefschetz twistor structure on the space of harmonic forms. — Let $C^0(\boldsymbol{S}, \boldsymbol{C})$ denote the sheaf of continuous functions on $\boldsymbol{S} = \big\{|\lambda| = 1\big\}$. Recall that the natural multiplication and the integration induces the following sesqui-linear pairing:

$$C^p_{\mathfrak{H}} : \mathcal{H}^{-p}_{|\boldsymbol{S}} \otimes \sigma^*(\mathcal{H}^p)_{|\boldsymbol{S}} \longrightarrow C^0(\boldsymbol{S}, \boldsymbol{C}).$$

Thus, we obtain the $\mathcal{R}$-triples $\mathfrak{H}^p := (\mathcal{H}^{-p}, \mathcal{H}^p, C^p_{\mathfrak{H}})$ $(p = -1, 0, 1)$. We put $\mathfrak{H}^{\bullet} := \bigoplus \mathfrak{H}^p$. We have the Lefschetz morphism $\mathcal{L}_{\omega_1} = (-L_{\omega_1}, L_{\omega_1}) : \mathfrak{H}^{-1} \to \mathfrak{H}^1 \otimes \mathbb{T}^S(1)$. We regard it as the morphism $\mathfrak{H}^{\bullet} \to \mathfrak{H}^{\bullet} \otimes \mathbb{T}^S(1)$. We also have the Hermitian sesqui-linear duality $\mathcal{S}_{\mathfrak{H}} : \mathfrak{H}^{\bullet} \to (\mathfrak{H}^{\bullet})^*$. The following lemma can be shown using the argument for Hodge-Simpson theorem 2.2.4 in [**73**].

Lemma 18.2.12. — *$(\mathfrak{H}^{\bullet}, \mathcal{L}_{\omega_1}, \mathcal{S}_{\mathfrak{H}})$ is a polarized graded Lefschetz twistor structure of weight* 0. □

18.2.8. Compatibility with the sesqui-linear pairing. — Let C^p_1 denote the induced sesqui-linear pairing:

$$C^p_1 : R^{-p}p_{X*}(\mathfrak{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}})_{|\boldsymbol{S}} \otimes \sigma^*\big(R^p p_{X*}(\mathfrak{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}})\big)_{|\boldsymbol{S}} \longrightarrow C^0(\boldsymbol{S}, \boldsymbol{C}).$$

We have $a^p_{X\dagger}(\mathcal{T}) = \big(a^{-p}_{X\dagger}(\mathfrak{E}), a^p_{X\dagger}(\mathfrak{E}), C^p_1\big)$.

Lemma 18.2.13. — *C^p_1 and $C^p_{\mathfrak{H}}$ are compatible with the isomorphism $\widetilde{\Phi}$. Namely, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Harm}^{1-p} \otimes \mathcal{O}_{\boldsymbol{C}_\lambda|\boldsymbol{S}} \otimes \sigma^*\big(\mathrm{Harm}^{1+p} \otimes \mathcal{O}_{\boldsymbol{C}_\lambda}\big)_{|\boldsymbol{S}} & \xrightarrow{C^p_{\mathfrak{H}}} & C^0(\boldsymbol{S}, \boldsymbol{C}) \\ {\scriptstyle \widetilde{\Phi} \otimes \sigma^* \widetilde{\Phi}} \big\downarrow \simeq & & = \big\downarrow \\ R^{-p}p_{X*}(\mathfrak{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}})_{|\boldsymbol{S}} \otimes \sigma^*\big(R^p p_{X*}(\mathfrak{E} \otimes \Omega^{1+\bullet,0}_{\mathcal{X}})\big)_{|\boldsymbol{S}} & \xrightarrow{C^p_1} & C^0(\boldsymbol{S}, \boldsymbol{C}). \end{array} \tag{370}$$

Proof. — Let $\lambda_0 \in \boldsymbol{C}$ such that $|\lambda_0| = 1$. We have the following pairings:

$$\overline{\mathcal{K}}^{-p}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, h_1^{(T_1)})_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\bigl(\overline{\mathcal{K}}^{p}_{\mathrm{poly}}(\mathcal{Q}^{(-\lambda_0)}_{*}\mathcal{E}, h_1^{(T_1)})\bigr)_{|\boldsymbol{I}(\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \otimes \Omega^{1,1}_{\mathcal{X}}$$

$$\overline{\mathcal{K}}^{-p}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, h_1, h_1^{(T_1)})_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\bigl(\overline{\mathcal{K}}^{p}_{\mathrm{poly}}(\mathcal{Q}^{(-\lambda_0)}_{*}\mathcal{E}, h_1, h_1^{(T_1)})\bigr)_{|\boldsymbol{I}(\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \otimes \Omega^{1,1}_{\mathcal{X}}$$

$$\overline{\mathcal{K}}^{-p}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, \mathbb{D})_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\bigl(\overline{\mathcal{K}}^{p}_{\mathrm{poly}}(\mathcal{Q}^{(-\lambda_0)}_{*}\mathcal{E}, \mathbb{D})\bigr)_{|\boldsymbol{I}(\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \otimes \Omega^{1,1}_{\mathcal{X}}$$

$$\mathcal{K}^{-p}_{\mathrm{poly}}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E}, \mathbb{D})_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\bigl(\mathcal{K}^{p}_{\mathrm{poly}}(\mathcal{Q}^{(-\lambda_0)}_{*}\mathcal{E}, \mathbb{D})\bigr)_{|\boldsymbol{I}(\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \otimes \Omega^{1,1}_{\mathcal{X}}$$

$$\mathcal{S}(\mathcal{Q}^{(\lambda_0)}_{*}\mathcal{E} \otimes \Omega^{1-p,0}_{\mathcal{X}})_{|\boldsymbol{I}(\lambda_0)\times X} \otimes \sigma^*\bigl(\mathcal{S}\bigl(\mathcal{Q}^{(-\lambda_0)}_{*}\mathcal{E} \otimes \Omega^{1+p,0}_{\mathcal{X}}\bigr)\bigr)_{|\boldsymbol{I}(\lambda_0)\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{I}(\lambda_0)\times X/\boldsymbol{I}(\lambda_0)} \otimes \Omega^{1,1}_{\mathcal{X}}$$

(The continuity of distributions follows from the remark in Subsection 5.3.1.2.) They are compatible with the quasi-isomorphisms in (366) and (368). Hence, we obtain the commutativity of the diagram (370). □

Then, we obtain the isomorphism of $\mathcal{R}$-triples $\mathfrak{H}^\bullet \simeq a^\bullet_{X\dagger}(\mathfrak{T})$. It is clearly compatible with the induced Hermitian sesqui-linear dualities. We also have the compatibility with the Lefschetz morphisms by Lemma 18.2.11. Hence, Proposition 18.2.1 follows from Lemma 18.2.12. □

18.3. Step 2

In this section, most of the proof is referred to our previous paper [**67**]. However, we should emphasize that it is essentially due to Saito and Sabbah, as remarked in [**67**].

18.3.1. Preliminary I. — Let $f : X \to Y$ be a projective morphism of complex manifolds. Let Z be a closed subvariety of X. Let c be the first Chern class of a relatively ample line bundle on X. In this subsection, the following assumption is imposed.

Assumption 18.3.1. — *The claim of Theorem* 18.1.1 *holds for the morphism f and polarized wild pure twistor D-modules whose strict supports are contained in Z.* □

Let us consider a tuple $(\mathcal{T}, M, N, \mathcal{S})$ on X as follows:

Condition 18.3.2
- An $\mathcal{R}_X$-triple $\mathcal{T}$ with any increasing filtration M such that $\mathrm{Gr}_i^M \mathcal{T}$ are wild pure twistor D-modules of weight $w+i$. The support of $\mathcal{T}$ is contained in Z.
- A nilpotent map $N : \mathcal{T} \to \mathcal{T}(-1)$ whose weight filtration is equal to M.
- A Hermitian sesqui-linear duality $\mathcal{S} : \mathcal{T} \to \mathcal{T}^*(-w)$. We assume that N is skew adjoint with respect to $\mathcal{S}$, i.e., $N^* \circ \mathcal{S} + \mathcal{S} \circ N = 0$.
- $\mathcal{S} \circ N^j$ gives a polarization of $P\,\mathrm{Gr}_j^M \mathcal{T}$ for each $j \geqslant 0$. □

We put $\widehat{\mathcal{T}}^i := \mathrm{Gr}^W_{-i} \mathcal{T}$, and then we obtain the graded $\mathcal{R}$-triple $\widehat{\mathcal{T}} = \bigoplus_j \widehat{\mathcal{T}}^j$. We have the naturally induced nilpotent map $\widehat{N} : \widehat{\mathcal{T}}^j \to \widehat{\mathcal{T}}^{j+2}(-1)$ and the Hermitian sesqui-linear duality $\widehat{\mathcal{S}} : \widehat{\mathcal{T}} \to \widehat{\mathcal{T}}^*(-w)$. By the condition, the tuple $\big(\widehat{\mathcal{T}}, \widehat{N}, \widehat{\mathcal{S}}\big)$ is a polarized graded wild Lefschetz twistor D-module of weight w and type -1.

We obtain the complex $f_\dagger \mathcal{T}$ of $\mathcal{R}_Y$-triples with the nilpotent map $f_\dagger N$ and the Hermitian sesqui-linear duality $f_\dagger \mathcal{S}$. (See [**73**].) We have the induced filtration M on $f_\dagger \mathcal{T}$, which is the weight filtration of $f_\dagger N$. We also have the Lefschetz map $\mathcal{L}_c : f_\dagger \mathcal{T} \to f_\dagger \mathcal{T}[2](1)$.

By taking the cohomology, we obtain $\bigoplus f_\dagger^j \mathcal{T}$ with $\bigoplus f_\dagger^j N$, $\bigoplus f_\dagger^j \mathcal{S}$, and the Lefschetz map $\mathcal{L}_c$. We also have the induced filtration M on each $f_\dagger^j \mathcal{T}$. We put $\widetilde{\mathcal{T}}^{i,j} := \mathrm{Gr}^M_{-i} f_\dagger^j \mathcal{T}$. The induced nilpotent map and sesqui-linear duality are denoted by $\widetilde{N}$ and $\widetilde{\mathcal{S}}$. The Lefschetz map is also denoted by $\widetilde{\mathcal{L}}_c$.

Proposition 18.3.3
- *Let $(\mathcal{T}, M, N, \mathcal{S})$ be as in Condition* 18.3.2. *Under Assumption* 18.3.1, *the tuple $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{S}}, \widetilde{N}, \widetilde{\mathcal{L}}_c)$ is a polarized wild bi-graded Lefschetz twistor D-module of type $(-1,1)$ of weight w.*
- *The induced filtration M on $f_\dagger^j \mathcal{T}$ is equal to the weight filtration of $f_\dagger^j N$.*

Proof. — It can be shown by using Lemma 17.1.6, Lemma 17.1.17 and the argument in the proof of Proposition 14.133 in [**67**]. □

18.3.2. Preliminary II. — Let $f : X \to Y$ be a projective morphism of a complex manifolds. Let g be a holomorphic function on Y. Let c be the first Chern class of a relatively ample line bundle on X. We put $\widetilde{g} = g \circ f$. Let Z be a closed subvariety of X such that $\widetilde{g}$ is not constantly 0 on Z. We impose the following assumption in this subsection.

Assumption 18.3.4. — *The claim of Theorem* 18.1.1 *holds for the morphism f and polarized wild pure twistor D-modules whose strict supports are contained in $\widetilde{g}^{-1}(0) \cap Z$.* □

Let us consider a tuple $(\mathcal{T}, \mathcal{S}, \mathfrak{a})$ on X as follows:

Condition 18.3.5

- $\mathcal{T}$ is a holonomic $\mathcal{R}_X$-triple, $\mathcal{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ is a Hermitian sesqui-linear duality, and $\mathfrak{a} \in \boldsymbol{C}[t_m^{-1}]$.
- $\mathcal{T}$ is strictly S-decomposable along $\widetilde{g}$. The support of $\mathcal{T}$ is contained in Z. Moreover, it is strictly specializable along $\widetilde{g}$ with ramified exponential twist by $\mathfrak{a}$.
- For each $u \in \boldsymbol{R} \times \boldsymbol{C}$, we put $\mathcal{T}_{\mathfrak{a},u} := \psi_{\widetilde{g},\mathfrak{a},u}\mathcal{T}$. The induced nilpotent map and Hermitian sesqui-linear duality are denoted by $N_{\mathfrak{a},u}$ and $\mathcal{S}_{\mathfrak{a},u}$. Let M denote the monodromy weight filtration of $N_{\mathfrak{a},u}$. Then, $(\mathcal{T}_{\mathfrak{a},u}, M, N_{\mathfrak{a},u}, \mathcal{S}_{\mathfrak{a},u})$ satisfies Condition 18.3.2. □

We put $\widehat{\mathcal{T}}_{\mathfrak{a},u} := \mathrm{Gr}^M \mathcal{T}_{\mathfrak{a},u}$. The induced nilpotent map and the Hermitian sesqui-linear duality are denoted by $\widehat{N}_{\mathfrak{a},u}$ and $\widehat{\mathcal{S}}_{\mathfrak{a},u}$. The tuple $\big(\widehat{\mathcal{T}}_{\mathfrak{a},u}, \widehat{N}_{\mathfrak{a},u}, \widehat{\mathcal{S}}_{\mathfrak{a},u}\big)$ is a polarized graded wild Lefschetz twistor D-module of weight w and type -1.

Lemma 18.3.6. — *Let $(\mathcal{T}, \mathcal{S}, \mathfrak{a})$ be as in Condition* 18.3.5. *Then, $f_\dagger^j \mathcal{T}$ is strictly specializable along g with ramified exponential twist by the $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$, and we have the natural isomorphism $\widetilde{\psi}_{g,\mathfrak{a},u} f_\dagger^j \mathcal{T} \simeq f_\dagger^j \widetilde{\psi}_{\widetilde{g},\mathfrak{a},u}\mathcal{T}$.*

Proof. — Applying Proposition 18.3.3 to any $\big(\mathcal{T}_{\mathfrak{a},u}, M, N_{\mathfrak{a},u}, \mathcal{S}_{\mathfrak{a},u}\big)$, we obtain $f_\dagger^j \mathcal{T}_{\mathfrak{a},u}$ are strict. Hence, the claim follows from Proposition 22.11.3 and Proposition 22.11.5. □

For any $u \in \boldsymbol{R} \times \boldsymbol{C}$, we put

$$\widetilde{\mathcal{T}}^{i,j}_{\mathfrak{a},u} := \mathrm{Gr}^M_{-i} \widetilde{\psi}_{g,\mathfrak{a},u} f_\dagger^j \mathcal{T}.$$

Here, M denotes the weight filtration for the induced nilpotent maps for $\widetilde{\psi}_{g,\mathfrak{a},u}$. Then, we obtain the bi-graded $\mathcal{R}$-triples $\widetilde{\mathcal{T}}_{\mathfrak{a},u} = \bigoplus \widetilde{\mathcal{T}}^{i,j}_{\mathfrak{a},u}$. The induced nilpotent map and the sesqui-linear duality are denoted by $\widetilde{N}_{\mathfrak{a},u}$ and $\widetilde{\mathcal{S}}_{\mathfrak{a},u}$. The Lefschetz map $\widetilde{\mathcal{L}}_{\mathfrak{a},u}$ is also induced.

Proposition 18.3.7. — *Let $(\mathcal{T}, \mathcal{S}, \mathfrak{a})$ be as in Condition* 18.3.5. *Under Assumption* 18.3.4, *$\big(\widetilde{\mathcal{T}}_{\mathfrak{a},u}, \widetilde{\mathcal{S}}_{\mathfrak{a},u}, \widetilde{N}_{\mathfrak{a},u}, \widetilde{\mathcal{L}}_{\mathfrak{a},u}\big)$ is a polarized bi-graded wild Lefschetz twistor D-module of type $(-1, 1)$ of weight w.*

Proof. — It can be shown by using Proposition 18.3.3 and the same argument as that in the proof of Proposition 14.139 of [**67**]. □

Let $(\mathcal{T}, \mathcal{S})$ be as in Condition 18.3.5 with $\mathfrak{a} = 0$. We put $\mathcal{T}_0 := \phi_{\widetilde{g},0}(\mathcal{T})(-1/2)$, and $\widehat{\mathcal{T}}_0^i := \mathrm{Gr}^M_{-i} \mathcal{T}_0$. Then, we obtain the graded $\mathcal{R}_X$-triple $\widehat{\mathcal{T}}_0$. We have the induced nilpotent maps $\widehat{N}_0$ and the sesqui-linear duality $\widehat{\mathcal{S}}_0$. The tuple $\big(\widehat{\mathcal{T}}_0, \widehat{\mathcal{S}}_0, \widehat{N}_0\big)$ is a polarized graded wild Lefschetz twistor D-module of weight $w+1$. We have the maps $\mathrm{Can} : \widehat{\mathcal{T}}_{-\boldsymbol{\delta}_0} \to \widehat{\mathcal{T}}_0$ and $\mathrm{Var} : \widehat{\mathcal{T}}_0 \to \widehat{\mathcal{T}}_{-\boldsymbol{\delta}_0}(-1)$, such that $\widehat{N}_{-\boldsymbol{\delta}_0} = \mathrm{Var} \circ \mathrm{Can}$ and $\widehat{N}_0 = \mathrm{Can} \circ \mathrm{Var}$. The maps Can and Var are adjoint with respect to the sesqui-linear dualities. The nilpotent map N_0 induces the nilpotent map $\widetilde{N}_0$ on $f^j_{\dagger}\mathcal{T}_0$. Let M denote the weight filtration. We put $\widetilde{\mathcal{T}}_0^{i,j} := \operatorname{Gr}^M_{-i} f^j_{\dagger}\mathcal{T}_0$. The induced sesqui-linear duality is denoted by $\widetilde{\mathcal{S}}_0$. We also have the induced Lefschetz map $\widetilde{\mathcal{L}}_0$.

Proposition 18.3.8

- *The underling $\mathcal{R}_Y$-modules of $f^i_{\dagger}\mathcal{T}$ are strictly S-decomposable along g.*
- *The tuple $(\widetilde{\mathcal{T}}_0, \widetilde{\mathcal{S}}_0, \widetilde{N}_0, \widetilde{\mathcal{L}}_0)$ is a bi-graded wild Lefschetz twistor D-module of weight $w+1$ and type $(-1,1)$.*

Proof. — It can be shown by using Proposition 18.3.3, Lemma 17.1.16 and the same argument as that in the proof of Proposition 14.140 of [**67**]. □

Let $\mathcal{M}'_i$ and $\mathcal{M}''_i$ are the underlying $\mathcal{R}_Y$-modules of $f^i_{\dagger}\mathcal{T}$. Due to the S-decomposability of $\mathcal{M}'_i$ and $\mathcal{M}''_i$, we obtain the sesqui-linear pairing of $\psi_{g,0}(\mathcal{M}'_i)$ and $\psi_{g,0}(\mathcal{M}''_i)$ by the specialization. As a result, we obtain the $\mathcal{R}$-triple $\phi_{g,0}f^i_{\dagger}\mathcal{T}$.

Lemma 18.3.9. — *$\phi_{g,0}f^i_{\dagger}\mathcal{T}(-1/2)$ and $f^i_{\dagger}\mathcal{T}_0$ are isomorphic.*

Proof. — It follows from Lemma 22.10.6 and Proposition 18.3.8. □

18.3.3. The explanation for Step 2. — The following assumption for the induction is imposed in this subsection.

Assumption 18.3.10. — *Let n and m be non-negative integers. Let $f : X \to Y$ be a projective morphism. Let $(\mathcal{T}, \mathcal{S})$ be a polarized wild pure twistor D-module of weight w on X whose strict support* $\operatorname{Supp}(\mathcal{T})$ *is irreducible. We put $n(\mathcal{T}) = \dim \operatorname{Supp}(\mathcal{T})$ and $m(\mathcal{T}) = \dim f(\operatorname{Supp}(\mathcal{T}))$. The claim of Theorem* 18.1.1 *holds for $\mathcal{T}$ if $n(\mathcal{T}) \leqslant n$ and $m(\mathcal{T}) \leqslant m$ are satisfied.* □

Under the assumption, we show that the claim of Theorem 18.1.1 holds for $\mathcal{T}$ such that $n(\mathcal{T}) \leqslant n+1$ and $m(\mathcal{T}) \leqslant m+1$ are satisfied.

Since the claim is local on Y, we may replace Y with any open subset of Y. Let g be any holomorphic function on Y such that g is not constantly 0 on $f\big(\operatorname{Supp}(\mathcal{T})\big)$. Due to Proposition 18.3.8, the underlying $\mathcal{R}_Y$-modules of $\bigoplus f^i_{\dagger}\mathcal{T}$ are strictly S-decomposable along g. Then, we have the decomposition $f^j_{\dagger}\mathcal{T} = \bigoplus_Z (f^j_{\dagger}\mathcal{T})_Z$ by the strict supports. The decomposition is compatible with the Lefschetz map $\mathcal{L}_c$ and the induced Hermitian sesqui-linear duality $\bigoplus f^j_{\dagger}\mathcal{S}$. By using Proposition 18.3.8 and Lemma 18.3.9, we can check that $\bigoplus_j (f^j_{\dagger}\mathcal{T})_Z$ with $(\mathcal{L}_c)_Z$ and $\bigoplus_j (f^j_{\dagger}\mathcal{S})_Z$ is a polarized graded Lefschetz twistor structure if $\dim Z = 0$. Then, by using Proposition 18.3.7, we can check that $\big(\bigoplus f^j_{\dagger}\mathcal{T}, \mathcal{L}_c, \bigoplus_j f^j_{\dagger}\mathcal{S}\big)$ is a polarized graded wild Lefschetz twistor D-module of weight w. Thus, we finish Step 2 in the proof of Theorem 18.1.1.

18.4. Step 3

18.4.1. Statement. — Let X be a smooth projective variety with an ample line bundle $\mathcal{O}_X(1)$. Let a_X denote the obvious morphism of X to a point. Let $(\mathcal{T},\mathcal{S})$ be a polarized wild pure twistor D-module on X of weight w, whose strict support is an irreducible closed subset Z of X with $\dim Z = n$. We will show the following proposition in this section.

Proposition 18.4.1. — *Assume $P(n-1,0)$ holds. Then, $\left(\bigoplus_i a^i_{X\dagger}\mathcal{T},\mathcal{L}_c,\bigoplus_i a^i_{X\dagger}\mathcal{S}\right)$ is a polarized graded Lefschetz twistor structure of weight w, where $\mathcal{L}_c$ denotes the Lefschetz morphism associated to $\mathcal{O}_X(1)$.*

We follow an argument in [**77**] very closely. We also use some results in [**73**]. We may assume $w=0$, $\mathcal{T}=(\mathcal{M},\mathcal{M},C)$, and $\mathcal{S}=(\mathrm{id},\mathrm{id})$.

18.4.2. Preliminary. — We fix an embedding $X\subset\mathbb{P}^N$. Let X_0 be the intersection $X\cap H_1\cap H_2$, where H_i denote general hyperplanes such that X_0 is strictly non-characteristic with respect to $\mathcal{T}$. Let $\widetilde{X}$ be the blow up of X along X_0. We obtain the following diagram:

$$X \xleftarrow{\ \pi\ } \widetilde{X} \xrightarrow{\ p\ } \mathbb{P}^1.$$

Here, p is the Lefschetz pencil for H_1 and H_2. We have the $\mathcal{R}$-triple $(\widetilde{\mathcal{T}},\widetilde{\mathcal{S}})$ on $\widetilde{X}$, as in [**73**]. Let $\widetilde{\mathcal{M}}$ denote the underlying $\mathcal{R}_{\widetilde{X}}$-module of $\widetilde{\mathcal{T}}$. We have the naturally induced Hermitian sesqui-linear duality $\widetilde{\mathcal{S}}$ of $\widetilde{\mathcal{T}}$.

Lemma 18.4.2. — *$\left(\bigoplus_i p^i_\dagger\widetilde{\mathcal{T}},\mathcal{L}_c,\bigoplus_i p^i\widetilde{\mathcal{S}}\right)$ is a polarized graded wild Lefschetz twistor D-module on $\mathbb{P}^1$, where $\mathcal{L}_c$ denotes the Lefschetz map associated to $\pi^*\mathcal{O}_X(1)$.*

Proof. — See (1) and (2) in Chapter 6.4 of [**73**]. □

Lemma 18.4.3. — *We have the vanishing $\pi^j_\dagger\widetilde{\mathcal{T}}=0$ for $j\neq 0$, and the canonical decomposition $(\pi_\dagger\widetilde{\mathcal{T}},\pi_\dagger\widetilde{\mathcal{S}})=(\mathcal{T},\mathcal{S})\oplus(\mathcal{T}_1,\mathcal{S}_1)$ of $\mathcal{R}_X$-triples with Hermitian sesqui-linear duality.*

Proof. — See (3) in Section 6.4 of [**73**]. We just give a remark on the vanishing $\pi^j_\dagger\widetilde{\mathcal{T}}=0$ for $j\neq0$. According to (3) in Section 6.4 of [**73**], it can be shown that the underlying $\mathcal{R}_X$-modules of $\pi^j_\dagger\widetilde{\mathcal{T}}$ are strict. Hence, we only have to show the vanishing of the specialization $\pi^j_\dagger\pi^\dagger\mathcal{M}^1$ at $\lambda=1$, where $\mathcal{M}^1$ denotes the specialization of $\mathcal{M}$ along $\lambda=1$. Hence, we can reduce the issue to the case of D_X-modules. By the projection formula, we have the isomorphism $\pi_\dagger\pi^\dagger\mathcal{M}^1\simeq\mathcal{M}^1\otimes^L\pi_\dagger\mathcal{O}_{\widetilde{X}}$ in the derived category of D_X-modules. We have the vanishing $\pi^j_\dagger\mathcal{O}_{\widetilde{X}}=0$ for $j\neq0$, and $\pi^0_\dagger\mathcal{O}_X$ is isomorphic to the direct sum $\mathcal{O}_X\oplus M_2$, where M_2 is locally isomorphic to the push-forward of $\mathcal{O}_{X_0}$. (See Section 5.3.9 of [**77**], for example.) Since X_0 is non-characteristic with respect to $\mathcal{M}^1$, we obtain that $\mathcal{M}^1\otimes^L M_2\simeq\mathcal{M}^1\otimes M_2$. □

By Lemma 18.4.2 and Deligne's result [**23**], the spectral sequence $a^i_{\mathbb{P}^1\dagger}p^j_\dagger\widetilde{\mathcal{T}} \Longrightarrow a^{i+j}_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}$ degenerates at the E^2-level. Hence, due to Proposition 18.3.3, $a^i_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}$ is a pure twistor structure of weight i. According to Lemma 18.4.3, $a^i_{X\dagger}\mathcal{T}$ is a direct summand of $a^i_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}$. Hence, $a^i_{X\dagger}\mathcal{T}$ is a pure twistor structure of weight i.

We have the Leray filtration:

$$L^1a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{M}} \subset L^0a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{M}} \subset L^{-1}a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}.$$

Because of the degeneration of the spectral sequence, we have the isomorphisms $\mathrm{Gr}^i_L\big(a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}\big) \simeq a^i_{\mathbb{P}^1\dagger}p^{\ell-i}_\dagger\widetilde{\mathcal{M}}$. We also have the Leray filtration for pure twistor structures of weight ℓ:

$$L^1a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \subset L^0a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \subset L^{-1}a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}.$$

We have the isomorphisms $\mathrm{Gr}^i_L\big(a^\ell_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}\big) \simeq a^i_{\mathbb{P}^1\dagger}p^{\ell-i}_\dagger\widetilde{\mathcal{T}}$.

We will use the following general lemma in linear algebra.

Lemma 18.4.4. — *Let V be a pure twistor structure of weight w with a polarization S. Let V_1 be a pure twistor structure of weight w with a monomorphism $\varphi : V_1 \to V$. We have the following maps:*

$$V_1 \xrightarrow{\ \varphi\ } V \xrightarrow{\ S\ } V^* \otimes \mathbb{T}(-w) \xrightarrow{\ \varphi^*\ } V_1^* \otimes \mathbb{T}(-w).$$

Then, the composite S_1 gives a polarization of V_1.

Proof. — We can reduce the problem to the case $w = 0$. Then, the claim is the same as the following:

- Let V be a $\boldsymbol{C}$-vector space, and let V_1 be a subspace of V. Let h be a Hermitian metric of V. Let h_1 denote the restriction of h to V_1. We have the induced anti-linear maps $\varphi_h : V \to V^\vee$ and $\varphi_{h_1} : V_1 \to V_1^\vee$. Let i denote the inclusion $V_1 \to V$. Then, the following diagram is commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\ \varphi_{h_1}\ } & V_1^\vee \\ {\scriptstyle i}\big\downarrow & & \big\uparrow{\scriptstyle i^\vee} \\ V & \xrightarrow{\ \varphi_h\ } & V^\vee \end{array}$$

The claim can be checked directly. □

18.4.3. Gysin maps. — Let Y denote a general fiber of p. For $Y \subset X$, we apply the construction in Section 17.3. We have obtained the polarized wild pure twistor D-module $(\mathrm{Ker}(\mathcal{T}), \mathcal{S}_{\mathrm{Ker}(\mathcal{T})})$ of weight -1. (See Section 17.3.7.) Since the support of $\mathrm{Ker}(\mathcal{T})$ is contained in Y, the inductive assumption can be applied.

Recall we have obtained the morphisms $i^*_{Y,\mathrm{Gys}}$ and $i^{\mathrm{Gys}}_{Y,*}$ in Section 17.3.8.

Lemma 18.4.5. — *The following holds:*

- $i^*_{Y,\mathrm{Gys}} : a^i_{X\dagger}\mathcal{T} \to a^{i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})$ *is an isomorphism for* $i < -1$ *and a monomorphism for* $i = -1$.
- *The morphism* $(i^*_{Y,\mathrm{Gys}})^* : \big(a^{i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})\big)^* \to a^i_{X\dagger}(\mathcal{T})^*$ *is an isomorphism if* $i<-1$, *and an epimorphism if* $i = -1$. *In other words,* $i^{\mathrm{Gys}}_{Y*} : a^i_{X\dagger}\operatorname{Cok}(\mathcal{T}) \to a^{i+1}_{X\dagger}\mathcal{T}$ *is an isomorphism for* $i > 0$ *and an epimorphism for* $i = 0$.

Proof. — Since we already know that $a^i_{X\dagger}\operatorname{Ker}(\mathcal{T})$, $a^i_{X\dagger}\operatorname{Cok}(\mathcal{T})$ and $a^i_{X\dagger}\mathcal{T}$ are pure twistor structures with appropriate weights (Section 18.4.2), the claim follows from Lemma 17.3.16. □

Lemma 18.4.6. — *The morphism* $L^i_c : a^{-i}_{X\dagger}\mathcal{M} \to a^i_{X\dagger}\mathcal{M}$ *is an isomorphism for* $i \geqslant 2$.

Proof. — We identify $j_*j^*\mathcal{M}/\mathcal{M}$ and $i_{Y\dagger}i^\dagger_Y\mathcal{M}$ by Λ. (See Section 17.3.1 for Λ.) According to Corollary 17.3.15, we have the factorization of L^i_c up to signature, as follows:

$$a^{-i}_{X\dagger}\mathcal{M} \xrightarrow{b_1} a^{-i+1}_{X\dagger}i_{Y\dagger}i^\dagger_Y\mathcal{M} \xrightarrow{L^{i-1}_c} a^{i-1}_{X\dagger}i_{Y\dagger}i^\dagger_Y\mathcal{M} \xrightarrow{b_2} a^i_{X\dagger}\mathcal{M}.$$

Due to Lemma 18.4.5, b_j $(j = 1, 2)$ are isomorphisms. Since we can apply the inductive assumption to $\operatorname{Ker}(\mathcal{T})$, the middle arrow is also an isomorphism. Thus, we obtain Lemma 18.4.6. □

Lemma 18.4.7. — *The following diagram is commutative:*

$$\begin{array}{ccc} a^{-i}_{X\dagger}\mathcal{T} & \xrightarrow{\mathcal{S}\circ\mathcal{L}^i_c} & \big(a^{-i}_{X\dagger}\mathcal{T}\big)^*\otimes\mathbb{T}^S(i) \\ {\scriptstyle i^*_{Y,\mathrm{Gys}}}\downarrow & & \uparrow{\scriptstyle (i^*_{Y,\mathrm{Gys}})^*} \\ a^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T}) & \xrightarrow{\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}\circ\mathcal{L}^{i-1}_c} & \big(a^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})\big)^*\otimes\mathbb{T}^S(i) \end{array} \tag{371}$$

Here $(i^*_{Y,\mathrm{Gys}})^*$ *denotes the Hermitian adjoint of* $i^*_{Y,\mathrm{Gys}}$, *and* $\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}$ *denotes the polarization of* $\operatorname{Ker}(\mathcal{T})$ *given in Section* 17.3.7.

Proof. — We have the commutativity of the diagram (358) in Corollary 17.3.15. Note that $\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}$ can be factorized as follows:

$$\operatorname{Ker}(\mathcal{T}) \xrightarrow{\varphi_3^{-1}} \operatorname{Cok}(\mathcal{T})\otimes\mathbb{T}^S(1) \xrightarrow{\varphi_5} \operatorname{Cok}(\mathcal{T}^*)\otimes\mathbb{T}^S(1) \longrightarrow \operatorname{Ker}(\mathcal{T})^*\otimes\mathbb{T}^S(1).$$

Here, φ_3 is given in Section 17.3.7, φ_5 is induced by the polarization $\mathcal{S}$ of $\mathcal{T}$, and the right arrow is the natural identification given in Lemma 17.3.10.

We have the following commutative diagram:

(372)

$$\begin{array}{ccccccc} a^i_{X\dagger}\mathcal{T} & \xrightarrow{\mathcal{S}} & a^i_{X\dagger}(\mathcal{T}^*) & \longrightarrow & a^i_{X\dagger}(\mathcal{T}^*) & \longrightarrow & \big(a^{-i}_{X\dagger}\mathcal{T}\big)^* \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a^{i-1}_{X\dagger}\operatorname{Cok}(\mathcal{T}) & \xrightarrow{\varphi_5} & a^{i-1}_{X\dagger}\operatorname{Cok}(\mathcal{T}^*) & \longrightarrow & a^{i-1}_{X\dagger}\big(\operatorname{Ker}(\mathcal{T})^*\big) & \longrightarrow & \big(a^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})\big)^* \end{array}$$

The horizontal arrows are the natural identifications. Composing (358) and (372), we obtain the commutativity of the diagram (371). □

18.4.4. The case $i \geqslant 2$. — Let us consider the case $i \geqslant 2$. Let $Pa^{-i}_{X\dagger}\mathcal{T}$ denote the kernel of $\mathcal{L}_c^{i+1} : a^{-i}_{X\dagger}\mathcal{T} \to a^{i+2}_{X\dagger}\mathcal{T} \otimes \mathbb{T}^S(i+1)$.

Lemma 18.4.8. — *We have the following commutative diagram:*

$$(373)\qquad \begin{array}{ccc} Pa^{-i}_{X\dagger}\mathcal{T} & \xrightarrow{\mathcal{S}\circ\mathcal{L}^i} & (Pa^{-i}_{X\dagger}\mathcal{T})^* \otimes \mathbb{T}^S(i) \\ \simeq\downarrow & & \simeq\uparrow \\ Pa^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T}) & \xrightarrow{\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}\circ\mathcal{L}_c^{i-1}} & \big(Pa^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})\big)^* \otimes \mathbb{T}^S(i) \end{array}$$

Proof. — By Lemma 18.4.6, we have the decompositions $a^{-i}_{X\dagger}(\mathcal{M}) = Pa^{-i}_{X\dagger}(\mathcal{M}) \oplus \mathcal{N}_1$ and $a^{i}_{X\dagger}\mathcal{M} = Pa^{i}_{X\dagger}\mathcal{M} \oplus \mathcal{N}_2$:

$$Pa^{-i}_{X\dagger}(\mathcal{M}) := \operatorname{Ker}\big(L_c^{i+1} : a^{-i}_{X\dagger}\mathcal{M} \longrightarrow a^{i+2}_{X\dagger}\mathcal{M}\big), \quad \mathcal{N}_1 := \operatorname{Im}\big(L_c : a^{-i-2}_{X\dagger}\mathcal{M} \longrightarrow a^{-i}_{X\dagger}\mathcal{M}\big),$$
$$Pa^{i}_{X\dagger}(\mathcal{M}) := \operatorname{Im}\big(L_c^{i} : Pa^{-i}_{X\dagger}\mathcal{M} \longrightarrow a^{i}_{X\dagger}\mathcal{M}\big), \qquad \mathcal{N}_2 := \operatorname{Im}\big(L_c^{i} : \mathcal{N}_1 \longrightarrow a^{i}_{X\dagger}\mathcal{M}\big).$$

Let $\mathcal{T}_1$ denote the image of $\mathcal{L}_c : a^{-i+2}_{X\dagger}\mathcal{T} \otimes \mathbb{T}^S(-1) \to a^{-i}_{X\dagger}\mathcal{T}$. Then, the underlying $\mathcal{R}$-modules of $Pa^{-i}_{X\dagger}\mathcal{T}$ (resp. $\mathcal{T}_1$) are given by $Pa^{i}_{X\dagger}\mathcal{M}$ and $Pa^{-i}_{X\dagger}\mathcal{M}$ (resp. $\mathcal{N}_2$ and $\mathcal{N}_1$). We also have a similar decomposition for $a^{-i+1}_{X\dagger}(\operatorname{Ker}(\mathcal{T}))$. Then, we can directly check that the morphisms in the diagram (371) preserves the decompositions. Hence, we obtain (373) from (371). □

Since the lower horizontal arrow gives a polarization of $Pa^{-i+1}_{X\dagger}\operatorname{Ker}(\mathcal{T})$ by the inductive assumption, the Hermitian sesqui-linear duality $\mathcal{S}\circ\mathcal{L}_c^i$ gives a polarization of $Pa^{-i}_{X\dagger}\mathcal{T}$.

18.4.5. The case $i = 1$. — In the case $i = 1$, we obtain the following commutative diagram from (371):

$$(374)\qquad \begin{array}{ccc} a^{-1}_{X\dagger}\mathcal{T} & \xrightarrow{\mathcal{S}\circ\mathcal{L}_c} & (a^{-1}_{X\dagger}\mathcal{T})^* \otimes \mathbb{T}^S(1) \\ i^*_{Y,\mathrm{Gys}}\downarrow & & (i^*_{Y,\mathrm{Gys}})^*\uparrow \\ a^{0}_{X\dagger}\operatorname{Ker}(\mathcal{T}) & \xrightarrow{\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}} & \big(a^{0}_{X\dagger}\operatorname{Ker}(\mathcal{T})\big)^* \otimes \mathbb{T}^S(1) \end{array}$$

Since we already know that $L_c^3 : a^{-3}_{X\dagger}\mathcal{M} \to a^{3}_{X\dagger}\mathcal{M}$ is an isomorphism, we have the following decompositions:

$$a^{-1}_{X\dagger}\mathcal{M} = L_c \cdot a^{-3}_{X\dagger}\mathcal{M} \oplus \operatorname{Ker}\Big(L_c^2 : a^{-1}_{X\dagger}\mathcal{M} \longrightarrow a^{3}_{X\dagger}\mathcal{M}\Big),$$
$$a^{1}_{X\dagger}\mathcal{M} = L_c^2 \cdot a^{-3}_{X\dagger}\mathcal{M} \oplus \operatorname{Ker}\Big(L_c : a^{1}_{X\dagger}\mathcal{M} \longrightarrow a^{3}_{X\dagger}\mathcal{M}\Big).$$

This decomposition is compatible with the sesqui-linear pairing. We set

$$Pa^{-1}_{X\dagger}\mathcal{T} := \big(\operatorname{Ker} L_c, \operatorname{Ker} L_c^2, C_1\big), \quad \mathcal{T}_1 := \big(\operatorname{Im} L_c^2, \operatorname{Im} L_c, C_2\big).$$

Here, C_i denote the naturally induced sesqui-linear pairings. We have $a_{X\dagger}^{-1}\mathcal{T} = Pa_{X\dagger}^{-1}\mathcal{T} \oplus \mathcal{T}_1$. We already know that $a_{X\dagger}^{-1}\mathcal{T}$ is a pure twistor structure of weight -1. (Section 18.4.2). Hence, $Pa_{X\dagger}^{-1}\mathcal{T}$ and $\mathcal{T}_1$ are also pure twistor structures of weight -1. The morphism $\mathcal{S}\circ\mathcal{L}_c : a_{X\dagger}^{-1}\mathcal{T} \to (a_{X\dagger}^{-1}\mathcal{T})^* \otimes \mathbb{T}^S(1)$ preserves the decomposition. It is easy to show that the induced map $\mathcal{T}_1 \to \mathcal{T}_1^* \otimes \mathbb{T}^S(1)$ is an isomorphism, by using the fact that $L_c^3 : a_{X\dagger}^{-3}\mathcal{M} \to a_{X\dagger}^{3}\mathcal{M}$ is an isomorphism.

We have the primitive decomposition $a_{X\dagger}^{0}\big(\mathrm{Ker}(\mathcal{T})\big) = Pa_{X\dagger}^{0}\big(\mathrm{Ker}(\mathcal{T})\big) \oplus \mathcal{T}_2$. Here, the underlying $\mathcal{R}_X$-modules of $Pa_{X\dagger}^{0}\,\mathrm{Ker}(\mathcal{T})$ are given by

$$\mathrm{Ker}\Big(L_c : a_{X\dagger}^{0}(j_*j^*\mathcal{M}/\mathcal{M}) \longrightarrow a_{X\dagger}^{2}(j_*j^*\mathcal{M}/\mathcal{M})\Big),$$
$$\mathrm{Ker}\Big(L_c : a_{X\dagger}^{0}(i_{Y\dagger}i_Y^{\dagger}\mathcal{M}) \longrightarrow a_{X\dagger}^{2}(i_{Y\dagger}i_Y^{\dagger}\mathcal{M})\Big).$$

The underlying $\mathcal{R}$-modules of $\mathcal{T}_2$ are given by

$$\mathrm{Im}\Big(L_c : a_{X\dagger}^{-2}(j_*j^*\mathcal{M}/\mathcal{M}) \longrightarrow a_{X\dagger}^{0}(j_*j^*\mathcal{M}/\mathcal{M})\Big),$$
$$\mathrm{Im}\Big(L_c : a_{X\dagger}^{-2}(i_{Y\dagger}i_Y^{\dagger}\mathcal{M}) \longrightarrow a_{X\dagger}^{0}(i_{Y\dagger}i_Y^{\dagger}\mathcal{M})\Big).$$

Lemma 18.4.9. — *The morphism $i^*_{Y,\mathrm{Gys}} : a_{X\dagger}^{-1}\mathcal{T} \to a_{X\dagger}^{0}\,\mathrm{Ker}(\mathcal{T})$ induces the morphisms:*

$$Pa_{X\dagger}^{-1}\mathcal{T} \longrightarrow Pa_{X\dagger}^{0}\,\mathrm{Ker}(\mathcal{T}), \qquad \mathcal{T}_1 \longrightarrow \mathcal{T}_2.$$

Proof. — We put $\mathcal{M}_0 := i_{Y\dagger}i_Y^{\dagger}\mathcal{M}$. We will identify $\mathcal{M}_0$ and $j_*j^*\mathcal{M}/\mathcal{M}$ via Λ. We have the following commutative diagram:

$$\begin{array}{ccccc}
a_{X\dagger}^{-3}\mathcal{M} & \xrightarrow{L_c} & a_{X\dagger}^{-1}\mathcal{M} & \xrightarrow{L_c^2} & a_{X\dagger}^{3}\mathcal{M} \\
\downarrow & & f\downarrow & & \uparrow\simeq \\
a_{X\dagger}^{-2}\mathcal{M}_0 & \xrightarrow{L_c} & a_{X\dagger}^{0}\mathcal{M}_0 & \xrightarrow{L_c} & a_{X\dagger}^{2}\mathcal{M}_0
\end{array}$$

Hence, we obtain $f(\mathrm{Im}\,L_c) \subset \mathrm{Im}(L_c)$ and $f(\mathrm{Ker}\,L_c^2) \subset \mathrm{Ker}\,L_c$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
a_{X\dagger}^{-2}\mathcal{M}_0 & \xrightarrow{L_c} & a_{X\dagger}^{0}\mathcal{M}_0 & \xrightarrow{L_c} & a_{X\dagger}^{2}\mathcal{M}_0 \\
\simeq\uparrow & & g\downarrow & & \downarrow \\
a_{X\dagger}^{-3}\mathcal{M} & \xrightarrow{L_c^2} & a_{X\dagger}^{1}\mathcal{M} & \xrightarrow{L_c} & a_{X\dagger}^{3}\mathcal{M}
\end{array}$$

Hence, we obtain $g\big(\mathrm{Ker}(L_c)\big) \subset \mathrm{Ker}(L_c)$ and $g\big(\mathrm{Im}(L_c)\big) \subset \mathrm{Im}(L_c^2)$. Thus, Lemma 18.4.9 is proved. □

We obtain the following commutative diagram from (374) and Lemma 18.4.9:

$$\begin{array}{ccc} Pa^{-1}_{X\dagger}\mathcal{T} & \xrightarrow{\mathcal{S}\circ\mathcal{L}_c} & (Pa^{-1}_{X\dagger}\mathcal{T})^*\otimes\mathbb{T}^S(1) \\ {\scriptstyle b_1}\big\downarrow & & {\scriptstyle b_1^*}\big\uparrow \\ Pa^0_{X\dagger}\operatorname{Ker}(\mathcal{T}) & \xrightarrow{\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}} & Pa^0_{X\dagger}\operatorname{Ker}(\mathcal{T})^*\otimes\mathbb{T}^S(1) \end{array}$$

We know that b_1 is monomorphic, and b_1^* is epimorphic (Lemma 18.4.5). By the inductive assumption, $\mathcal{S}_{\operatorname{Ker}(\mathcal{T})}$ gives a polarization of $Pa^0_{X\dagger}\operatorname{Ker}(\mathcal{T})$. Hence, we can conclude that $\mathcal{L}_c : Pa^{-1}_{X\dagger}\mathcal{T}\to Pa^1_{X\dagger}\mathcal{T}\otimes\mathbb{T}^S(1)$ is an isomorphism, and $(Pa^{-1}_{X\dagger}\mathcal{T},\mathcal{S}')$ is a polarized pure twistor structure of weight -1 due to Lemma 18.4.4, where $\mathcal{S}':=\mathcal{S}\circ\mathcal{L}_c$. Since the induced map $\mathcal{T}_1\to\mathcal{T}_1^*\otimes\mathbb{T}^S(1)$ is an isomorphism, we also obtain that $\mathcal{L}_c : a^{-1}_{X\dagger}\mathcal{T}\to a^1_{X\dagger}\mathcal{T}\otimes\mathbb{T}^S(1)$ is an isomorphism.

18.4.6. The case $i=0$. — We already know $\mathcal{L}_c^2 : a^{-2}_{X\dagger}\mathcal{T}\to a^2_{X\dagger}\mathcal{T}\otimes\mathbb{T}(2)$ is an isomorphism. In particular, we obtain the decompositions of $\mathcal{R}$-modules $a^0_{X\dagger}\mathcal{M} = \operatorname{Im} L_c\oplus Pa^0_{X\dagger}\mathcal{M}$ and the $\mathcal{R}$-triples $a^0_{X\dagger}\mathcal{T}=\operatorname{Im}\mathcal{L}_c\oplus Pa^0_{X\dagger}\mathcal{T}$, where $Pa^0_{X\dagger}\mathcal{M}:=\operatorname{Ker} L_c$ and $Pa^0_{X\dagger}\mathcal{T}:=\operatorname{Ker}\mathcal{L}_c$. We already know that $\operatorname{Im}\mathcal{L}_c$ and $Pa^0_{X\dagger}\mathcal{T}$ are pure twistor structures of weight 0. The decomposition is compatible with the Hermitian sesqui-linear duality. Let $\mathcal{S}_0$ denote the induced Hermitian sesqui-linear duality of $Pa^0_{X\dagger}\mathcal{T}$. We would like to show that $\mathcal{S}_0$ gives a polarization of $Pa^0_{X\dagger}\mathcal{T}$.

We have the following pure twistor structures:

$$L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}=\Big(\frac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}}{L^1a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}},\ L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}},\ C_1\Big),\quad \frac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}{L^1a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}=(L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}})^*.$$

Here, C_1 denotes the naturally induced sesqui-linear pairing.

Proposition 18.4.10. — *We have the following factorizations:*

$$Pa^0_{X\dagger}\mathcal{T}\longrightarrow L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}\longrightarrow a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}},\qquad a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*\longrightarrow\frac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}{L^1a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}\longrightarrow Pa^0_{X\dagger}\mathcal{T}^*.$$

Proof. — We only have to show the first factorization. We already know the twistor property of $Pa^0_{X\dagger}\mathcal{T}$, $L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}$ and $a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}$. Hence, we only have to show such a factorization for the specialization at $\lambda=1$. Namely, we only have to show that $Pa^0_{X\dagger}\mathcal{M}^1$ is contained in $L^0a^0_{\widetilde{X}}\widetilde{\mathcal{M}}^1$. We put $\widetilde{Y}:=Y$ and let $i_{\widetilde{Y}}:\widetilde{Y}\to\widetilde{X}$ denote the natural inclusion.

Lemma 18.4.11. — *We have the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^1 & \longrightarrow & j_*j^*\mathcal{M}^1 & \longrightarrow & j_*j^*\mathcal{M}^1/\mathcal{M}^1 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\big\uparrow & & {\scriptstyle f_2}\big\uparrow & & {\scriptstyle =}\big\uparrow & & \\ 0 & \longrightarrow & \pi_\dagger\widetilde{\mathcal{M}}^1 & \longrightarrow & \pi_\dagger(\tilde{j}_*\tilde{j}^*\widetilde{\mathcal{M}}^1) & \longrightarrow & \pi_\dagger(\tilde{j}_*\tilde{j}^*\widetilde{\mathcal{M}}^1/\widetilde{\mathcal{M}}^1) & \longrightarrow & 0 \end{array} \tag{375}$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_{Y\dagger}i_Y^{\dagger}\mathcal{M}^1 & \longrightarrow & j_!j^*\mathcal{M}^1 & \longrightarrow & \mathcal{M}^1 & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & \\
0 & \longrightarrow & \pi_\dagger\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big) & \longrightarrow & \pi_\dagger\big(\widetilde{j}_!\widetilde{j}^*\widetilde{\mathcal{M}}^1\big) & \longrightarrow & \pi_\dagger\big(\widetilde{\mathcal{M}}^1\big) & \longrightarrow & 0
\end{array}
\tag{376}
$$

Here f_1 and g_1 are the natural morphisms.

Proof. — We will use commutative diagrams in Section 22.6.5 below. Because of the non-characteristic condition, we obtain (375) and (376) from (429) and (430) respectively, by taking the tensor product with $\mathcal{M}^1$. □

Let us return to the proof of Proposition 18.4.10. We obtain the following commutative diagrams:

(377)

$$
\begin{array}{ccc}
a^i_{X\dagger}\big(j_*j^*\mathcal{M}^1/\mathcal{M}^1\big) & \longrightarrow & a^{i+1}_{X}(\mathcal{M}^1) \\
\big\uparrow & & \big\uparrow \\
a^i_{\widetilde{X}\dagger}\big(\widetilde{j}_*\widetilde{j}^*\widetilde{\mathcal{M}}^1/\widetilde{\mathcal{M}}^1\big) & \longrightarrow & a^{i+1}_{\widetilde{X}\dagger}(\widetilde{\mathcal{M}}^1)
\end{array}
\qquad
\begin{array}{ccc}
a^i_{X\dagger}(\mathcal{M}^1) & \longrightarrow & a^{i+1}_{X\dagger}\big(i_{Y\dagger}i_Y^{\dagger}\mathcal{M}^1\big) \\
\big\downarrow & & \big\downarrow \\
a^i_{\widetilde{X}\dagger}(\widetilde{\mathcal{M}}^1) & \longrightarrow & a^{i+1}_{\widetilde{X}\dagger}\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big)
\end{array}
$$

We also have the following commutative diagram:

$$
\begin{array}{ccccc}
a^0_{X\dagger}\mathcal{M}^1 & \longrightarrow & a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 & \longrightarrow & a^1_{\widetilde{X}\dagger}\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big) \\
{\scriptstyle L_c}\big\downarrow & & {\scriptstyle L_c}\big\downarrow & & {\scriptstyle \simeq}\big\downarrow \\
a^2_{X\dagger}\mathcal{M}^1 & \longrightarrow & a^2_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 & \longleftarrow & a^1_{\widetilde{X}\dagger}\big(\widetilde{j}_*\widetilde{j}^*\widetilde{\mathcal{M}}^1/\widetilde{\mathcal{M}}^1\big)
\end{array}
\tag{378}
$$

The right arrow is the natural isomorphism. From (377) and (378), the morphism $L_c : a^0_{X\dagger}\mathcal{M}^1 \to a^2_{X\dagger}\mathcal{M}^1$ is factorized as follows:

$$
a^0_{X\dagger}\mathcal{M}^1 \longrightarrow a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 \longrightarrow a^1_{\widetilde{X}\dagger}\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big) = a^1_{X\dagger}\big(i_{Y\dagger}i_Y^{\dagger}\mathcal{M}^1\big) \xrightarrow{\ \simeq\ } a^2_{X\dagger}\mathcal{M}^1
$$

Hence, we obtain

$$
Pa^0_{X\dagger}\mathcal{M}^1 \subset \operatorname{Ker}\big(\psi : a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 \longrightarrow a^1_{\widetilde{X}\dagger}\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big)\big).
$$

Note we have the decomposition $p^1_\dagger\widetilde{\mathcal{M}}^1 = N_1 \oplus N_2$, where the strict support of N_1 is $\mathbb{P}^1$ and the support of N_2 is 0-dimensional. We have $a^j_{\mathbb{P}^1\dagger}(N_2) = 0$ for $j \neq 0$, and $f \in a^{-1}_{\mathbb{P}^1\dagger}(N_1)$ is 0 if and only if the restriction of f to some general point is 0. Hence, we obtain the injectivity of the natural morphism $\operatorname{Gr}^{-1}_L a^0_{\widetilde{X}\dagger}(\widetilde{\mathcal{M}}^1) \simeq a^{-1}_{\mathbb{P}^1\dagger}p^1_\dagger\widetilde{\mathcal{M}}^1 \to a^1\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\widetilde{\mathcal{M}}^1\big)$. Hence, we obtain $Pa^0_{X\dagger}\mathcal{M}^1 \subset L^0a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1$. Then, the claim of Proposition 18.4.10 follows. □

Because $\mathcal{L}_c : L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}} \to L^1 a^2_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}$ is an isomorphism, the following induced morphism is injective:

$$Pa^0_{X\dagger}\mathcal{M} \longrightarrow \operatorname{Gr}^0_L a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}} = \frac{L^0 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}}{L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}} \subset \frac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}}{L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}}.$$

Due to Proposition 18.4.10, we obtain the following morphisms:

$$Pa^0_{X\dagger}\mathcal{T} \longrightarrow L^0 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \longrightarrow \operatorname{Gr}^0_L a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \simeq a^0_{\mathbb{P}^1\dagger} p^0_\dagger \widetilde{\mathcal{T}} \tag{379}$$

$$a^0_{\mathbb{P}^1\dagger} p^0_\dagger \widetilde{\mathcal{T}}^* \simeq \operatorname{Gr}^0_L a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^* \longrightarrow \frac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}{L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*} \longrightarrow Pa^0_{X\dagger}\mathcal{T}^* \tag{380}$$

The composite of (379) is denoted by F. The composite of (380) is the adjoint F^*. They are morphisms of pure twistor structures of weight 0. Since $Pa^0_X\mathcal{M} \to a^0_{\mathbb{P}^1} p^0 \mathcal{M}$ is injective, F is a monomorphism.

We have the primitive decomposition:

$$a^0_{\mathbb{P}^1\dagger} p^0_\dagger \widetilde{\mathcal{T}} = \bigoplus_{i\geqslant 0} a^0_{\mathbb{P}^1\dagger}\Big(\mathcal{L}^i_c \cdot P p^{-2i}_\dagger \widetilde{\mathcal{T}}\Big).$$

Lemma 18.4.12. — *We have the following factorization of F:*

$$Pa^0_{X\dagger}\mathcal{T} \longrightarrow a^0_{\mathbb{P}^1\dagger} P p^0_\dagger \widetilde{\mathcal{T}} \longrightarrow a^0_{\mathbb{P}^1\dagger} p^0_\dagger \widetilde{\mathcal{T}}.$$

Proof. — Because of the twistor property, we only have to show that the image of $Pa^0_{X\dagger}\mathcal{M}^1 \to a^0_{\mathbb{P}^1\dagger} p^0_\dagger \widetilde{\mathcal{M}}^1$ is contained in $a^0_{\mathbb{P}^1\dagger} P p^0_\dagger \widetilde{\mathcal{M}}^1$. The composite of the following morphisms is 0:

$$Pa^0_{X\dagger}\mathcal{M}^1 \longrightarrow L^0 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 \xrightarrow{L_c} L^0 a^2_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1.$$

Hence, the composite of $Pa^0_{X\dagger}\mathcal{M}^1 \to \operatorname{Gr}^0_L a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1 \to \operatorname{Gr}^0_L a^2_{\widetilde{X}\dagger}\widetilde{\mathcal{M}}^1$ is 0. This implies the above claim. □

Lemma 18.4.13. — *The following diagram is commutative:*

$$\begin{array}{ccc} Pa^0_{X\dagger}\mathcal{T} & \xrightarrow{F} & a^0_{\mathbb{P}^1\dagger} P p^0_\dagger \widetilde{\mathcal{T}} \\ \mathcal{S}_0 \big\downarrow & & \mathcal{S}_1 \big\downarrow \\ Pa^0_{X\dagger}\mathcal{T}^* & \xleftarrow{F^*} & a^0_{\mathbb{P}^1\dagger} P p^0_\dagger \widetilde{\mathcal{T}}^* \end{array} \tag{381}$$

Here $\mathcal{S}_1$ denote the naturally induced polarization of $a^0_{\mathbb{P}^1\dagger} P p^0_\dagger \widetilde{\mathcal{T}}$.

Proof. — We have started from the following commutative diagram:

$$\begin{array}{ccccc} Pa^0_{X\dagger}\mathcal{T} & \longrightarrow & a^0_{X\dagger}\mathcal{T} & \longrightarrow & a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Pa^0_{X\dagger}\mathcal{T}^* & \longleftarrow & a^0_{X\dagger}\mathcal{T}^* & \longleftarrow & a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^* \end{array}$$

We obtain the commutativity of the following diagram, because it is obtained as the factorization (Proposition 18.4.10):

$$\begin{array}{ccccc} Pa^0_{X\dagger}\mathcal{T} & \longrightarrow & L^0 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} & \longrightarrow & a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Pa^0_{X\dagger}\mathcal{T}^* & \longleftarrow & \dfrac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}{L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*} & \longleftarrow & a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^* \end{array}$$

Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc} Pa^0_{X\dagger}\mathcal{T} & \longrightarrow & L^0 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} & \longrightarrow & \mathrm{Gr}^0_L\, a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Pa^0_{X\dagger}\mathcal{T}^* & \longleftarrow & \dfrac{a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*}{L^1 a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^*} & \longleftarrow & \mathrm{Gr}^0_L\, a^0_{\widetilde{X}\dagger}\widetilde{\mathcal{T}}^* \end{array} \tag{382}$$

Since (381) is obtained as the factorization of (382) in Lemma 18.4.12, we obtain the desired commutativity. □

Since F is monomorphic, $(Pa^0_{X\dagger}\mathcal{T}, \mathcal{S}_0)$ is polarized, and hence Proposition 18.4.1 is proved. □

CHAPTER 19

CORRESPONDENCES

In this chapter, we study some correspondences and their application to a conjecture of Kashiwara. In Sections 19.1–19.3, we establish the correspondence between wild harmonic bundles and polarized wild pure twistor D-modules on complex manifolds (Theorem 19.1.3). The argument is essentially the same as that in the tame case in [**67**], although we have some additional difficulties caused by Stokes structures and ramification. In Section 19.4, we show the correspondence between semisimple algebraic holonomic D-modules and polarizable wild pure twistor D-modules on projective varieties (Theorem 19.4.1). As an easy consequence, we obtain Kashiwara's conjecture (Theorem 19.4.2).

19.1. Wild harmonic bundles and wild pure twistor D-modules

19.1.1. Preliminary. — Let X be a complex manifold, and Z be any closed irreducible subvariety of X. Let $\mathcal{A}$ be a $\boldsymbol{Q}$-vector subspace of $\boldsymbol{C}$. In the following, "Zariski open" means "the complement of some closed analytic subset". Let U be a smooth Zariski open subset of Z. Recall that a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on U is called $\mathcal{A}$-wild on (Z, U), if we have a complex manifold $\widetilde{Z}$ and a birational projective morphism $\varphi : \widetilde{Z} \to Z$ satisfying the following properties:

- $\widetilde{D} := \widetilde{Z} \setminus \varphi^{-1}(U)$ is a normal crossing divisor.
- $\varphi^*(E, \overline{\partial}_E, \theta, h)_{|U}$ is an $\mathcal{A}$-wild harmonic bundle on $(\widetilde{Z}, \widetilde{D})$.

Definition 19.1.1. — Let $(V, \mathbb{D}^{\triangle}, S)$ be a variation of polarized pure twistor structure of weight w defined on U. We say that $(V, \mathbb{D}^{\triangle}, S)$ is an $\mathcal{A}$-wild variation of polarized pure twistor structure of weight w on (Z, U), if the underlying harmonic bundle is $\mathcal{A}$-wild on (Z, U). □

Let $(\mathcal{T}, \mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight w whose strict support is Z. As remarked in Lemma 17.1.10, there exists a Zariski open subset U of Z such that $(\mathcal{T}, \mathcal{S})_{|X \setminus Y}$ comes from a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on U after a suitable Tate twist, where $Y := Z \setminus U$. We will prove the following lemma in Section 19.3.

Lemma 19.1.2. — *$(E, \overline{\partial}_E, \theta, h)$ is $\mathcal{A}$-wild on (Z, U).*

19.1.2. Statement. — Let Z be an irreducible closed subset of X, and let U be a Zariski open subset of Z, and $Y := Z \setminus U$. Let $\mathrm{VPT}^{\mathrm{wild}}(Z, U, w, \mathcal{A})$ denote the category of $\mathcal{A}$-wild variations of polarized pure twistor structure of weight w on (Z, U). Let $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})$ denote the category of polarized $\mathcal{A}$-wild pure twistor D-modules such that (i) their strict supports are Z, i.e., the support of any non-zero direct summand is Z, (ii) their restriction to $X \setminus Y$ come from variations of pure polarized twistor structure of weight w on U. In the both categories, the morphisms are the isomorphisms. By Lemma 19.1.2, we obtain a functor

$$\Phi : \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A}) \longrightarrow \mathrm{VPT}^{\mathrm{wild}}(Z, U, w, \mathcal{A}).$$

The following theorem is one of the main results in this monograph.

Theorem 19.1.3. — *Φ is an equivalence of categories.*

Note that this theorem implies the following:

- Let $(V, \mathbb{D}^{\triangle}, S)$ be an $\mathcal{A}$-wild variation of polarized pure twistor structure of weight w on (Z, U). Then, there exists $(\mathcal{T}, \mathcal{S}) \in \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})$ such that $(\mathcal{T}, \mathcal{S})_{|X \setminus Y}$ comes from $(V, \mathbb{D}^{\triangle}, S)$. In other words, $(V, \mathbb{D}^{\triangle}, S)$ can be extended to a polarized wild pure twistor D-module on X. It is called a minimal extension of $(V, \mathbb{D}^{\triangle}, S)$.
- If $(\mathcal{T}', \mathcal{S}')$ is another minimal extension of $(V, \mathbb{D}^{\triangle}, S)$, we have $(\mathcal{T}, \mathcal{S}) \simeq (\mathcal{T}', \mathcal{S}')$, where the restriction of the isomorphism to $X \setminus Y$ is the identity of $(V, \mathbb{D}^{\triangle}, S)$. Namely, a minimal extension is unique.
- In particular, if $(V, \mathbb{D}^{\triangle}, S) = \bigoplus (V_i, \mathbb{D}_i^{\triangle}, S_i)$, we have the corresponding decomposition $(\mathcal{T}, \mathcal{S}) = \bigoplus (\mathcal{T}_i, \mathcal{S}_i)$.

We will show the essential surjectivity of Φ in Section 19.2, and the full faithfulness in Section 19.3.

19.1.2.1. Variant. — Let Z, U, Y be as above. Let $(V, \mathbb{D}^{\triangle})$ be a variation of pure twistor structure of weight w defined on a smooth Zariski open subset U of Z. We say that $(V, \mathbb{D}^{\triangle})$ is a *polarizable* $\mathcal{A}$-wild variation of pure twistor structure of weight w on (Z, U), if there exists a polarization S of $(V, \mathbb{D}^{\triangle})$ such that $(V, \mathbb{D}^{\triangle}, S)$ is an $\mathcal{A}$-wild variation of *polarized* pure twistor structure of weight w on (Z, U). Let $\mathrm{VPT}^{\mathrm{wild}}(Z, U, w, \mathcal{A})'$ denote the category of the *polarizable* $\mathcal{A}$-wild variations of pure twistor structure of weight w on (Z, U). Morphisms in this category are defined to be morphisms for variations of twistor structure on U.

Let $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})'$ denote the category of *polarizable* $\mathcal{A}$-wild pure twistor D-modules of weight w such that (i) their strict supports are Z, (ii) their restriction to

$X \setminus Y$ come from variations of twistor structure of weight w. (We consider the full subcategory of the category of wild pure twistor D-modules of weight w on X.) By Lemma 19.1.2, we obtain the following functor:

$$\Phi : \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})' \longrightarrow \mathrm{VPT}^{\mathrm{wild}}(Z, U, w, \mathcal{A})'.$$

Corollary 19.1.4. — *Φ is an equivalence of categories.*

Proof. — The essential surjectivity follows from Theorem 19.1.3. Let us consider the full faithfulness. Note that both the categories $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})'$ and $\mathrm{VPT}^{\mathrm{wild}}(Z, U, w, \mathcal{A})$ are semisimple. Hence, we only have to show the following:

- $\Phi(\mathcal{T})$ is simple if $\mathcal{T} \in \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})$ is simple.
- Let $\mathcal{T}_i \in \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, w, \mathcal{A})$ $(i = 1, 2)$ be simple. If $\Phi(\mathcal{T}_1) \simeq \Phi(\mathcal{T}_2)$, then $\mathcal{T}_1 \simeq \mathcal{T}_2$.

Both claims follow from Theorem 19.1.3. □

19.2. Prolongation to polarized wild pure twistor D-modules

We shall prove the essential surjectivity of Φ in Theorem 19.1.3. The argument is essentially the same as that in the tame case [**67**].

In Subsection 19.2.1, we will reduce the issue to the local and unramified case, in which we have already constructed an $\mathcal{R}$-triple in Chapter 12. We only have to show that it is a wild polarized pure twistor D-module (Proposition 19.2.1). In Subsection 19.2.2, we give a preparation for the functoriality of the family of filtered λ-flat bundles associated to an unramifiedly good wild harmonic bundle. In Subsection 19.2.3, we shall show that the $\mathcal{R}$-triple in Chapter 12 is strictly S-decomposable along any function. In Subsection 19.2.4, we study the specialization along a monomial function with an exponential twist. Then, in Subsection 19.2.5, we argue the specialization along any function with exponential twist.

Recall that we have already studied the specialization of the $\mathcal{R}$-triple along monomial functions without exponential twist, in Sections 12.4 and 12.7. Namely, we already know that it is strictly S-decomposable along any monomial function (Proposition 12.4.3), and we also know that we obtain a polarized graded wild Lefschetz twistor D-module as the specialization along a monomial function (Proposition 12.7.3. More precisely, we use the inductive assumption on the dimension of the strict support.) The specialization along any function can be reduced to that along monomial functions by Hironaka's resolution and the propositions in Subsection 18.3.2.

19.2.1. Reduction to the local and unramified case. — Let $X = \Delta_z^n$ and $D = \bigcup_{i=1}^{\ell} D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly $\mathcal{A}$-good wild harmonic bundle on $X \setminus D$. We have constructed the $\mathcal{R}_X$-module $\mathfrak{E}$ with the sesqui-linear pairing $\mathfrak{C} : \mathfrak{E}_{|\boldsymbol{S}\times X} \otimes \sigma^* \mathfrak{E}_{|\boldsymbol{S}\times X} \to \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$. (See Sections 12.1 and 12.5.) We will prove the following proposition in Sections 19.2.2–19.2.5.

Proposition 19.2.1. — *$\mathfrak{T}(E) := (\mathfrak{E},\mathfrak{E},\mathfrak{C})$ is an $\mathcal{A}$-wild pure twistor D-module of weight* 0*. The natural Hermitian sesqui-linear duality $\mathcal{S} = (\mathrm{id},\mathrm{id})$ gives a polarization.*

Let X and D be as above. Let $(E,\overline{\partial}_E,\theta,h)$ be an $\mathcal{A}$-good wild harmonic bundle, which is not necessarily unramified. We show the next lemma by assuming Proposition 19.2.1.

Lemma 19.2.2. — *There exists an $\mathcal{A}$-wild pure twistor D-module $(\mathfrak{E}_1,\mathfrak{E}_1,\mathfrak{C}_1)$ of weight* 0 *with the natural polarization* (id, id) *satisfying the following:*

- *The restriction to $X \setminus D$ is isomorphic to the polarized pure twistor D-module associated to $(E,\overline{\partial}_E,\theta,h)$.*
- *Let $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{D} := \boldsymbol{C}_\lambda \times D$. Then, $\mathfrak{E}_1 \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$ is naturally isomorphic to $\mathcal{Q}\mathcal{E}$. In particular, we naturally have $\mathfrak{E}_1 \subset \mathcal{Q}\mathcal{E}$.*

Proof. — Take a ramified covering $\varphi : (\widetilde{X},\widetilde{D}) \to (X,D)$ such that $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}) := \varphi^{-1}(E,\overline{\partial}_E,\theta,h)$ is unramified. We obtain the associated polarized $\mathcal{A}$-wild pure twistor D-module $\mathfrak{T}(\widetilde{E}) = (\widetilde{\mathfrak{E}},\widetilde{\mathfrak{E}},\widetilde{\mathfrak{C}})$ with $\widetilde{\mathfrak{S}} = (\mathrm{id},\mathrm{id})$ as in Proposition 19.2.1, which is $\mathrm{Gal}(\widetilde{X}/X)$-equivariant. Hence, we obtain a polarized $\mathcal{A}$-wild pure twistor D-module $P\varphi^0_{\dagger}\mathfrak{T}(\widetilde{E})$ on X with the $\mathrm{Gal}(\widetilde{X}/X)$-action. The invariant part of $P\varphi^0_{\dagger}\mathfrak{T}(\widetilde{E})_{|X\setminus D}$ is isomorphic to the polarized pure twistor D-module associated to $(E,\overline{\partial}_E,\theta,h)$. Note that a wild pure twistor D-module has the decomposition by strict supports, because it is assumed to be strict S-decomposable in definition. We have the direct summand $\mathfrak{T}_1 = (\mathfrak{E}_1,\mathfrak{E}_1,\mathfrak{C}_1)$ of the invariant part $P\varphi_{\dagger}\mathfrak{T}(\widetilde{E})^{\mathrm{Gal}(\widetilde{X}/X)}$ such that the strict support of $\mathfrak{T}_1$ is X, with the naturally induced polarization $\mathfrak{S}_1 = (\mathrm{id},\mathrm{id})$. Then, $(\mathfrak{T}_1,\mathfrak{S}_1)$ gives a prolongment of $(E,\overline{\partial}_E,\theta,h)$.

We have $\varphi^*\mathcal{Q}\mathcal{E} = \mathcal{Q}\widetilde{\mathcal{E}}$, and hence $\mathcal{Q}\mathcal{E}$ is naturally isomorphic to the $\mathrm{Gal}(\widetilde{X}/X)$-invariant part of $\varphi_{\dagger}\mathcal{Q}\widetilde{\mathcal{E}}$. We also have $\widetilde{\mathfrak{E}} \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(*\widetilde{\mathcal{D}}) = \mathcal{Q}\widetilde{\mathcal{E}}$. Hence, we obtain a natural morphism $\mathfrak{E}_1 \to \mathcal{Q}\mathcal{E}$ satisfying $\mathfrak{E}_1 \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D}) \simeq \mathcal{Q}\mathcal{E}$. □

Let X be a complex manifold with a normal crossing hypersurface D. Let $(E,\overline{\partial}_E,\theta,h)$ be an $\mathcal{A}$-good wild harmonic bundle on (X,D). We show the next lemma by assuming Proposition 19.2.1 and hence Lemma 19.2.2.

Lemma 19.2.3. — *There exists an $\mathcal{A}$-wild pure twistor D-module $(\mathfrak{E}_1,\mathfrak{E}_1,\mathfrak{C}_1)$ of weight* 0 *with the polarization* (id, id)*, whose restriction to $X \setminus D$ is isomorphic to the polarized pure twistor D-module associated to $(E,\overline{\partial}_E,\theta,h)$.*

Proof. — Let U be an open subset of X with a holomorphic coordinate system $(z_1,\ldots,z_n)$ such that $D \cap U = \bigcup_{j=1}^{\ell}\{z_j = 0\}$. By applying Lemma 19.2.2, we obtain the wild pure twistor D-module $\mathfrak{T}_U := (\mathfrak{E}_U,\mathfrak{E}_U,\mathfrak{C}_U)$ with the polarization (id, id). Let U_i $(i = 1,2)$ be such two open sets. Then, the restriction of $\mathfrak{E}_{U_i} \otimes \mathcal{O}(*\mathcal{D})$ to $\boldsymbol{C}_\lambda \times (U_1 \cap U_2)$ is naturally isomorphic to $\mathcal{Q}\mathcal{E}_{|\boldsymbol{C}_\lambda \times (U_1\cap U_2)}$. Let g be a holomorphic function on $U_1 \cap U_2$ such that $U_1 \cap U_2 \cap D = \{g = 0\}$. Since both $\mathfrak{E}_{U_i|U_1\cap U_2}$ are strictly S-decomposable along g, the isomorphism on $(U_1 \cap U_2) \setminus D$ can be extended to an isomorphism on $U_1 \cap U_2$. (See Lemma 22.4.10.) Hence, by varying U and gluing $\mathfrak{T}_U$, we obtain the globally defined $\mathcal{R}_X$-triple $\mathfrak{T}(E) := (\mathfrak{E}_1, \mathfrak{E}_1, \mathfrak{C}_1)$ with the Hermitian sesqui-linear duality $\mathfrak{S}(E) := (\mathrm{id}, \mathrm{id})$. Let us check that $\big(\mathfrak{T}(E), \mathfrak{S}(E)\big)$ is a polarized wild pure twistor D-module of weight 0. According to Lemma 17.1.5, we only have to show that $\mathfrak{E}_1$ is a good $\mathcal{R}_X$-module. (See Subsection 22.2.3.)

Note that $\mathfrak{E}_1$ is naturally contained in $\mathcal{Q}\mathcal{E}$. Let K be any compact subset of X. For any $\lambda_0 \in \boldsymbol{C}_\lambda$, there exist a large number N and a neighbourhood $\mathcal{U}$ of $\{\lambda_0\} \times K$ in $\mathcal{X}$, such that an $\mathcal{O}_{\mathcal{U}}$-module $\mathfrak{E}_1 \cap \mathcal{Q}_{N\boldsymbol{\delta}}^{(\lambda_0)}\mathcal{E}$ generates $\mathfrak{E}_1$. Hence, $\mathfrak{E}_1$ is good. Thus, the proof of Lemma 19.2.3 is completed. □

Let us show the essential surjectivity of Φ assuming Proposition 19.2.1, and hence Lemma 19.2.3. Let $\big((E, \overline{\partial}_E, \theta, h)\big)$ be an $\mathcal{A}$-wild harmonic bundle on (Z, U). We take a complex manifold $\widetilde{Z}$ and a birational projective morphism $\varphi : \widetilde{Z} \to Z$ such that (i) the complement of $\varphi^{-1}(U)$ is normal crossing, (ii) $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is a good $\mathcal{A}$-wild harmonic bundle. By applying Lemma 19.2.3, we take a polarized $\mathcal{A}$-wild pure twistor D-module $(\mathfrak{T}_1, \mathfrak{S}_1)$ on $\widetilde{Z}$ whose restriction to $\varphi^{-1}(U)$ is isomorphic to the polarized pure twistor D-module associated to $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)$. According to Theorem 18.1.1, $\big(\bigoplus_i \varphi_\dagger^i \mathfrak{T}_1, \mathcal{L}_c, \bigoplus_i \varphi_\dagger^i \mathfrak{S}_1\big)$ is a polarized graded $\mathcal{A}$-wild Lefschetz twistor D-module on X, where $\mathcal{L}_c$ denotes a Lefschetz map associated to a line bundle relatively ample with respect to φ. Let $P\varphi_\dagger^0\mathfrak{T}_1$ be the primitive part of $\varphi_\dagger^0\mathfrak{T}_1$. We have the decomposition $P\varphi_\dagger^0\mathfrak{T}_1 = \mathfrak{T}_2 \oplus \mathfrak{T}_2'$, where the strict support of $\mathfrak{T}_2$ is Z, and the support of $\mathfrak{T}_2'$ is strictly smaller than Z. We have the naturally induced polarization $\mathfrak{S}_2$ on $\mathfrak{T}_2$. Then, $(\mathfrak{T}_2, \mathfrak{S}_2)$ gives the desired prolongment of $(E, \overline{\partial}_E, \theta, h)$. Thus, the proof of the essential surjectivity of Φ is reduced to that of Proposition 19.2.1.

In the rest of this section, we will prove Proposition 19.2.1. We use an induction on $\dim X$.

19.2.2. Pull-back of the associated family of meromorphic λ-flat bundles

Let $X := \Delta_z^n$ and $D := \bigcup_{i=1}^{\ell} D_i$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly $\mathcal{A}$-good wild harmonic bundle on $X \setminus D$. Let $\varphi : X_1 \to X$ be a birational projective morphism such that $D_1 := \varphi^{-1}(D)$ is a normal crossing divisor.

Lemma 19.2.4. — *$(E_1, \overline{\partial}_{E_1}, \theta_1, h_1) := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is an unramifiedly $\mathcal{A}$-good wild harmonic bundle on $X_1 \setminus D_1$.*

Proof. — Because the claim is local on X, we may assume to have the decomposition $(E, \overline{\partial}_E, \theta) = \bigoplus (E_{\mathfrak{a}}, \overline{\partial}_{E_{\mathfrak{a}}}, \theta_{\mathfrak{a}}) \otimes L(\mathfrak{a})$. Then, the claim is clear. □

Let $\mathcal{P}_{<0}\mathcal{E}^\lambda$ be the union of $\mathcal{P}_{\boldsymbol{b}}\mathcal{E}^\lambda$ for $\boldsymbol{b} \in \boldsymbol{R}^\ell$ such that $b_i < 0$ $(i = 1, \dots, \ell)$.

Lemma 19.2.5. — *We have $\varphi^*\mathcal{P}\mathcal{E}^\lambda = \mathcal{P}\mathcal{E}_1^\lambda$ and $\varphi^*\mathcal{P}_{<0}\mathcal{E}^\lambda \subset \mathcal{P}_{<0}\mathcal{E}_1^\lambda$.*

Proof. — Because $\varphi^*\mathcal{P}_{<0}\mathcal{E}^\lambda = \mathcal{O}_{X_1}\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}\mathcal{P}_{<0}\mathcal{E}^\lambda$, we only have to show $\varphi^{-1}\mathcal{P}_{<0}\mathcal{E}^\lambda\subset\mathcal{P}_{<0}\mathcal{E}_1^\lambda$, which follows from the definition. (See Subsection 7.4.1.) □

Let $\mathcal{Q}_{<0}\mathcal{E}^\lambda$ be the union of $\mathcal{Q}_{\boldsymbol{b}}\mathcal{E}^\lambda$ for $\boldsymbol{b}\in\boldsymbol{R}^\ell$ such that $b_i<0$ $(i=1,\dots,\ell)$. We obtain the following lemma from Lemma 4.5.7.

Lemma 19.2.6. — *We have $\varphi^*\mathcal{Q}_{<0}\mathcal{E}^\lambda\subset\mathcal{Q}_{<0}\mathcal{E}_1^\lambda$ and $\varphi^*\mathcal{Q}\mathcal{E}^\lambda=\mathcal{Q}\mathcal{E}_1^\lambda$.* □

It is easy to derive the following lemma from Lemma 19.2.6 by using the local freeness of $\mathcal{Q}^{(\lambda_0)}\mathcal{E}_1$ and $\mathcal{Q}^{(\lambda_0)}_{<0}\mathcal{E}_1$.

Lemma 19.2.7. — *For any λ_0, we have $\varphi^*\mathcal{Q}^{(\lambda_0)}_{<0}\mathcal{E}\subset\mathcal{Q}^{(\lambda_0)}_{<0}\mathcal{E}_1$ and $\varphi^*\mathcal{Q}^{(\lambda_0)}\mathcal{E}=\mathcal{Q}^{(\lambda_0)}\mathcal{E}_1$.* □

19.2.3. Strict S-decomposability. — Let X, D and $(E,\overline{\partial}_E,\theta,h)$ be as above. Let g be any holomorphic function on X. Let us show that $\mathfrak{E}$ is strictly S-decomposable along g. Let $\widetilde{X}$ be a complex manifold with a projective birational morphism $\varphi:\widetilde{X}\to X$ such that (i) $\widetilde{X}-\varphi^{-1}\big(g^{-1}(0)\cup D\big)\simeq X-\big(g^{-1}(0)\cup D\big)$, (ii) $\varphi^{-1}\big(g^{-1}(0)\cup D\big)$ is a normal crossing divisor. We obtain the $\mathcal{A}$-good wild harmonic bundle $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}):=\varphi^{-1}(E,\overline{\partial}_E,\theta,h)$ and the associated $\mathcal{R}_{\widetilde{X}}$-triple $\mathfrak{T}(\widetilde{E}):=(\widetilde{\mathfrak{E}},\widetilde{\mathfrak{E}},\widetilde{\mathfrak{C}})$. Let $g_1:=g^a\cdot\prod_{j=1}^{\ell}z_j^{n_j}$ for some $a,n_j\in\mathbb{Z}_{\geqslant 0}$. We put $\widetilde{g}_1:=g_1\circ\varphi$.

- By Proposition 12.4.3, $\mathfrak{T}(\widetilde{E})$ is strictly S-decomposable along $\widetilde{g}_1$.
- For any $u\in\boldsymbol{R}\times\boldsymbol{C}$, according to Proposition 12.7.3 and the inductive assumption on $\dim X$, $\bigoplus_\ell\operatorname{Gr}^W_\ell\widetilde{\psi}_{\widetilde{g}_1,u}(\mathfrak{T}(\widetilde{E}))$ with the induced morphism $\mathcal{N}$ and Hermitian sesqui-linear duality is a polarized graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight 0.

Namely, $\mathfrak{T}(\widetilde{E})$ and the Hermitian sesqui-linear duality (id, id) satisfy Condition 18.3.5 with $\mathfrak{a}=0$. According to Proposition 18.3.8, $\varphi^i_\dagger\mathfrak{T}(\widetilde{E})$ is strictly S-decomposable along g_1. Hence, we have the decomposition $\varphi^0_\dagger\widetilde{\mathfrak{E}}=\mathcal{M}_1\oplus\mathcal{M}_2$ such that (i) $\mathcal{M}_1$ contains no non-trivial $\mathcal{R}$-submodules whose supports are contained in $D\cup g^{-1}(0)$, (ii) the support of $\mathcal{M}_2$ is contained in $g^{-1}(0)\cup D$. By construction, we have the natural identification of the restrictions of $\mathfrak{E}$ and $\mathcal{M}_1$ to $\boldsymbol{C}_\lambda\times\big(X-\big(D\cup\{g=0\}\big)\big)$.

Lemma 19.2.8. — *The above identification can be naturally extended to a morphism $\kappa:\mathfrak{E}\to\mathcal{M}_1$ on $\boldsymbol{C}_\lambda\times X$.*

Proof. — It can be shown using the same argument as in Section 19.3.1 of [**67**]. We give only an outline. We consider the corresponding right $\mathcal{R}_X$-modules, $\mathfrak{E}\otimes\omega_{\mathcal{X}}$ and $\widetilde{\mathfrak{E}}\otimes\omega_{\widetilde{\mathcal{X}}}$. By Lemma 19.2.7, any section of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\mathcal{E}\otimes\omega_{\mathcal{X}}$ naturally induces a section of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}\widetilde{\mathcal{E}}\otimes\omega_{\widetilde{\mathcal{X}}}\subset\widetilde{\mathfrak{E}}\otimes\omega_{\widetilde{\mathcal{X}}}$. Hence, it naturally induces a section of $\varphi^0_\dagger\big(\widetilde{\mathfrak{E}}\otimes\omega_{\widetilde{\mathcal{X}}}\big)$. (See Lemma 14.11 of [**67**].) Hence, we obtain $F:\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}(\mathcal{E})\otimes\omega_{\mathcal{X}}\to\mathcal{M}_1\otimes\omega_{\mathcal{X}}$.

Let f_i $(i=1,2)$ be local sections of $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}}(\mathcal{E})\otimes\omega_{\mathcal{X}}$, and let P_i $(i=1,2)$ be local sections of $\mathcal{R}_X$. If $f_1\cdot P_1=f_2\cdot P_2$ in $\mathfrak{E}\otimes\omega_{\mathcal{X}}$, the restrictions of $F(f_i)\cdot P_i$ to

$\boldsymbol{C}_\lambda \times \big(X - \big(D \cup g^{-1}(0)\big)\big)$ are the same. By using the strict S-decomposability of $\mathcal{M}_1$ along $g \cdot \prod_{j=1}^{\ell} z_j$, we obtain $F(f_1) \cdot P_1 = F(f_2) \cdot P_2$ on X.

Since $\mathfrak{E} \otimes \omega_{\mathcal{X}}$ is generated by $\mathcal{Q}^{(\lambda_0)}_{<\boldsymbol{\delta}} \mathcal{E} \otimes \omega_{\mathcal{X}}$ over $\mathcal{R}_X$, we obtain the desired map. □

Let us show that κ is an isomorphism. Since $\mathfrak{E}_{|\boldsymbol{C}_\lambda \times (X \setminus D)}$ comes from a harmonic bundle, we know that $\mathfrak{E}_{|\boldsymbol{C}_\lambda \times (X \setminus D)}$ is strictly S-decomposable along $g_{|X \setminus D}$. (See [**73**] or [**67**].) Hence, $\kappa_{|X \setminus D}$ is an isomorphism, due to Lemma 22.4.10. We already know that $\mathfrak{E}$ is strictly S-decomposable $\prod_{j=1}^{\ell} z_j$. Hence, κ is an isomorphism, due to Lemma 22.4.10 again. In particular, we obtain that $\mathfrak{E}$ is strictly S-decomposable along g.

19.2.4. Specialization along monomial functions with exponential twist

We study the specialization along monomial functions before considering the general case.

Proposition 19.2.9. — *Let $g := \prod_{i=1}^{k} z_i^{m_i}$.*

- *$\mathfrak{T}(E)$ is strictly specializable along g with ramified exponential twist by any $\mathfrak{a} \in \boldsymbol{C}[t_m^{-1}]$*
- *$P\,\mathrm{Gr}_\ell^{W(N)}\, \widetilde{\psi}_{g,\mathfrak{a},u} \mathfrak{T}(E)$ are $\mathcal{A}$-wild pure twistor D-modules of weight ℓ with the naturally induced polarization for any $u \in \boldsymbol{R} \times \boldsymbol{C}$ and $\mathfrak{a} \in \boldsymbol{C}[t_m^{-1}]$.*
- *Moreover, $P\,\mathrm{Gr}_\ell^{W(N)}\, \widetilde{\psi}_{g,\mathfrak{a},u} \mathfrak{T}(E) = 0$ unless $u \in \boldsymbol{R} \times \mathcal{A}$.*

Proof. — We first consider the unramified case, i.e., $\mathfrak{a} \in \boldsymbol{C}[t^{-1}]$, and then we argue the general case.

19.2.4.1. Unramified case. — Note that $(E, \overline{\partial}_E, \theta, h) \otimes L(-g^*\mathfrak{a})$ is not necessarily good. According to Proposition 15.3.1, we can take a birational projective morphism $(\widetilde{X}, \widetilde{D}) \to (X, D)$ such that (i) $\widetilde{X} - (\varphi^* g)^{-1}(0) \simeq X - g^{-1}(0)$, (ii) $\varphi^{-1}\big((E, \overline{\partial}_E, \theta, h) \otimes L(-g^*\mathfrak{a})\big)$ is an unramifiedly $\mathcal{A}$-good wild harmonic bundle on $(\widetilde{X}, \widetilde{D})$. We set $\widetilde{g} := \varphi^* g$. We set

$$(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \varphi^*(E, \overline{\partial}_E, \theta, h), \quad (\widetilde{E}', \overline{\partial}_{\widetilde{E}'}, \widetilde{\theta}', \widetilde{h}') := (\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) \otimes L\big(-\widetilde{g}^*\mathfrak{a}\big).$$

We have the associated $\mathcal{R}_{\widetilde{X}}$-triples $\mathcal{T}(\widetilde{E}) := (\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{E}}, \widetilde{\mathfrak{C}})$ and $\mathcal{T}(\widetilde{E}') := (\widetilde{\mathfrak{E}}', \widetilde{\mathfrak{E}}', \widetilde{\mathfrak{C}}')$ with the Hermitian sesqui-linear dualities $(\mathrm{id}, \mathrm{id})$. Let i_g (resp. $i_{\widetilde{g}}$) denote the graph $X \to X \times \boldsymbol{C}_t$ (resp. $\widetilde{X} \to \widetilde{X} \times \boldsymbol{C}_t$) for the function g (resp. $\widetilde{g}$). Since $(\widetilde{E}', \overline{\partial}_{\widetilde{E}'}, \widetilde{\theta}', \widetilde{h}')$ is also unramifiedly $\mathcal{A}$-good wild, the following holds:

- The $\mathcal{R}_{\widetilde{X} \times \boldsymbol{C}_t}$-triple $i_{\widetilde{g}\dagger} \mathfrak{T}(\widetilde{E}')$ is strictly S-decomposable along t.
- According to Proposition 12.7.3 and the inductive assumption on $\dim X$, $\bigoplus_\ell \mathrm{Gr}_\ell^W \widetilde{\psi}_{\widetilde{g},u}\big(\mathfrak{T}(\widetilde{E}')\big)$ with the induced morphism $\mathcal{N}$ and Hermitian sesqui-linear duality is a polarized graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight 0 for each $u \in \boldsymbol{R} \times \boldsymbol{C}$. Moreover, $\widetilde{\psi}_{\widetilde{g},u}\big(\mathfrak{T}(\widetilde{E}')\big) = 0$ unless $u \in \boldsymbol{R} \times \mathcal{A}$, by Corollary 12.4.5.

Because $i_{\widetilde{g}\dagger} \mathfrak{T}(\widetilde{E}')(*t) \simeq i_{\widetilde{g}\dagger} \mathfrak{T}(\widetilde{E})(*t) \otimes \mathfrak{L}(-\widetilde{g}^*\mathfrak{a})$, we obtain the following:

- The $\mathcal{R}_{\widetilde{X}\times \boldsymbol{C}_t}(*t)$-triple $i_{\widetilde{g}\dagger}\mathfrak{T}(\widetilde{E})(*t)\otimes \mathfrak{L}(-\widetilde{g}^*\mathfrak{a})$ is strictly specializable along t.
- $\bigoplus_\ell \mathrm{Gr}_\ell^W \widetilde{\psi}_{\widetilde{g},u,\mathfrak{a}}(\mathfrak{T}(\widetilde{E}))$ with the induced morphism $\mathcal{N}$ and Hermitian sesqui-linear duality is a polarized graded $\mathcal{A}$-wild Lefschetz twistor D-module of weight 0 for each $u\in \boldsymbol{R}\times\boldsymbol{C}$. Moreover, $\widetilde{\psi}_{\widetilde{g},u,\mathfrak{a}}(\mathfrak{T}(\widetilde{E}))=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$.

Namely, $\mathfrak{T}(\widetilde{E})$ and the Hermitian sesqui-linear duality (id, id) satisfy Condition 18.3.5 with $\mathfrak{a}$. We also have $i_{g\dagger}\varphi_\dagger^i\mathfrak{T}(\widetilde{E})(*t)\otimes\mathfrak{L}(-g^*\mathfrak{a})=0$ unless $i=0$. According to Lemma 18.3.6 and Proposition 18.3.7, the following holds:

- The $\mathcal{R}_{X\times\boldsymbol{C}_t}(*t)$-triple $i_{g\dagger}\varphi_\dagger^0\mathfrak{T}(\widetilde{E})(*t)\otimes\mathfrak{L}(-g^*\mathfrak{a})$ is strictly specializable along t.
- $\bigoplus_\ell \mathrm{Gr}_\ell^W \widetilde{\psi}_{g,u,\mathfrak{a}}\varphi_\dagger^0(\mathfrak{T}(\widetilde{E}))$ with the induced morphism $\mathcal{N}$ and Hermitian sesqui-linear duality is a polarized graded wild Lefschetz twistor D-module of weight 0 for each $u\in\boldsymbol{R}\times\boldsymbol{C}$. Moreover, $\widetilde{\psi}_{g,u,\mathfrak{a}}\varphi_\dagger^0(\mathfrak{T}(\widetilde{E}))=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$.

Let us show that $i_{g\dagger}\varphi_\dagger^0\mathfrak{T}(\widetilde{E})(*t)$ and $i_{g\dagger}\mathfrak{T}(E)(*t)$ are isomorphic. We only have to show that $\varphi_\dagger^0\mathfrak{T}(\widetilde{E})(*g)$ and $\mathfrak{T}(E)(*g)$ are isomorphic. By construction, their restrictions to $X\setminus\{g=0\}$ are naturally isomorphic. As in Section 19.2.3, the isomorphism of the underlying $\mathcal{R}$-modules can be extended to $\mathfrak{E}(*g)\simeq\varphi_\dagger^0\widetilde{\mathfrak{E}}(*g)$, and hence $\varphi_\dagger^0\mathfrak{T}(\widetilde{E})(*g)\simeq\mathfrak{T}(E)(*g)$. Hence, we are done in the unramified case.

19.2.4.2. General case. — Let $\varphi_m : X\times\boldsymbol{C}_{t_m}\longrightarrow X\times\boldsymbol{C}_t$ be induced by $\varphi_m(t_m)=t_m^m$. Let $\pi_m:\widetilde{X}\to X$ be the ramified covering given by $\pi_m(\zeta_1,\dots,\zeta_n)=(\zeta_1^m,\dots,\zeta_k^m,\zeta_{k+1},\dots,\zeta_n)$. The induced morphism $\widetilde{X}\times\boldsymbol{C}_{t_m}\longrightarrow X\times\boldsymbol{C}_{t_m}$ is also denoted by π_m. We put $\widetilde{\pi}_m:=\varphi_m\circ\pi_m$. Let $\Gamma_g:=\{t-g=0\}\subset X\times\boldsymbol{C}_t$. Let ω_m be a primitive m-th root of 1. We have the following decomposition:

$$\widetilde{\pi}_m^{-1}\Gamma_g=\bigcup_{p=0}^{m-1}\bigl\{t_m-\omega_m^p\cdot g(\zeta_1,\dots,\zeta_n)=0\bigr\}.$$

Let $j_p:\widetilde{X}\longrightarrow\widetilde{X}\times\boldsymbol{C}_{t_m}$ be the graph of $\omega_m^p\cdot g(\zeta_1,\dots,\zeta_n)$.

We set $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}):=\pi_m^{-1}(E,\overline{\partial}_E,\theta,h)$, which is unramifiedly $\mathcal{A}$-good wild. We have the associated $\mathcal{R}$-module $\widetilde{\mathfrak{E}}$ and the associated $\mathcal{R}$-triple $\mathfrak{T}(\widetilde{E})$. The direct sum $\bigoplus_p j_{p\dagger}\mathfrak{T}(\widetilde{E})(*t_m)\otimes\mathfrak{L}(-\mathfrak{a})$ is $\mathrm{Gal}(\widetilde{X}/X)$-equivariant.

Lemma 19.2.10. — *$\varphi_m^*i_{g\dagger}\mathfrak{T}(E)(*t)\otimes\mathfrak{L}(-\mathfrak{a})$ is identified with the* $\mathrm{Gal}(\widetilde{X}/X)$*-invariant part of*

$$\pi_{m\dagger}\Bigl(\bigoplus_p j_{p\dagger}\mathfrak{T}(\widetilde{E})(*t_m)\otimes\mathfrak{L}(-\mathfrak{a})\Bigr).$$

Proof. — We have the following natural isomorphisms:

$$\pi_m^*\bigl(\varphi_m^*(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t))\bigr)\simeq\widetilde{\pi}_m^*(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t))\simeq\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}\otimes\mathcal{O}(*t_m).$$

Therefore, we have

$$\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}\otimes\mathcal{O}(*t_m)\otimes\mathcal{L}(-\mathfrak{a})\simeq\pi_m^*\Bigl(\varphi_m^*\bigl(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\bigr)\otimes\mathcal{L}(-\mathfrak{a})\Bigr).$$

By using Lemma 22.7.1, we obtain

$$\pi_{m\dagger}\Big(\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}\otimes\mathcal{O}(*t_m)\otimes\mathcal{L}(-\mathfrak{a})\Big)\simeq\varphi_m^*\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})\otimes\pi_{m\dagger}\pi_m^*\mathcal{O}(*t_m).$$

Hence, $\varphi_m^*\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})$ is identified with the $\mathrm{Gal}(\widetilde{X}/X)$-invariant part of $\pi_{m\dagger}\Big(\bigoplus j_{p\dagger}\widetilde{\mathfrak{E}}\otimes\mathcal{L}(-\mathfrak{a})\Big)$.

We have the sesqui-linear pairing of $\varphi_m^*\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})$ induced by φ_m^*C. The sesqui-linear pairing of $\bigoplus j_{p\dagger}\mathfrak{E}\otimes\mathcal{L}(-\mathfrak{a})$ also induces a sesqui-linear pairing of $\varphi_m^*\big(i_{g\dagger}\mathfrak{E}\otimes\mathcal{O}(*t)\big)\otimes\mathcal{L}(-\mathfrak{a})$. Since the restriction of them to $X\times\boldsymbol{C}_{t_m}\setminus\{t_m=0\}$ are the same, they are the same on $X\times\boldsymbol{C}_{t_m}$. (Note the sesqui-linear pairing of $\mathcal{R}(*t_m)$-triples has the values in the moderate distributions.) Thus, we obtain Lemma 19.2.10. $\square$

Let us return to the proof of Proposition 19.2.9. By the previous result in the unramified case, $j_{p\dagger}\mathfrak{T}(\widetilde{E})(*t_m)\otimes\mathfrak{L}(-\mathfrak{a})$ are strictly specializable along t_m, and $\Big(\mathrm{Gr}^W\widetilde{\psi}_{t_m,u,\mathfrak{a}}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\big),\mathcal{N}_p,\mathcal{S}_p\Big)$ are polarized graded $\mathcal{A}$-wild Lefschetz twistor D-modules of weight 0, where $\mathcal{N}_p$ and $\mathcal{S}_p$ are the naturally induced nilpotent maps and Hermitian sesqui-linear duality. Moreover, $\widetilde{\psi}_{t_m,u,\mathfrak{a}}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\big)=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$. By Proposition 18.3.7 and Lemma 18.3.6, the following holds:

- $\pi_{m\dagger}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\otimes\mathcal{L}(-\mathfrak{a})\big)$ are strictly specializable along t_m.
- $\big(\mathrm{Gr}^W\widetilde{\psi}_{t_m,u,\mathfrak{a}}\pi_{m\dagger}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\big),\mathcal{N}'_p,\mathcal{S}'_p\big)$ are polarized graded $\mathcal{A}$-wild Lefschetz twistor D-modules of weight 0, where $\mathcal{N}'_p$ and $\mathcal{S}'_p$ denote the induced nilpotent maps and Hermitian sesqui-linear duality. Moreover, $\widetilde{\psi}_{t_m,u,\mathfrak{a}}\pi_{m\dagger}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\big)=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$.

Then, the first claim of Proposition 19.2.9 follows from Lemma 19.2.10. Since $P\,\mathrm{Gr}_\ell^{W(N)}\widetilde{\psi}_{g,\mathfrak{a},u}\mathfrak{T}(E)$ is a direct summand of $\bigoplus P\,\mathrm{Gr}_\ell^W\widetilde{\psi}_{t_m,u,\mathfrak{a}}\pi_{m\dagger}\big(j_{p\dagger}\mathfrak{T}(\widetilde{E})\big)$, the second and third claims of Proposition 19.2.9 follows. $\square$

19.2.5. End of Proof of Proposition 19.2.1. — Let g be any function on X. Let $\mathfrak{a}\in\boldsymbol{C}[t_m^{-1}]$. We take a complex manifold X_1 and a birational projective morphism $\varphi: X_1\to X$ such that (i) $X_1-\varphi^{-1}(g^{-1}(0))\simeq X-g^{-1}(0)$, (ii) $\varphi^{-1}(g^{-1}(0)\cup D)$ is normal crossing. We have the unramifiedly $\mathcal{A}$-good wild harmonic bundle $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}):=\varphi^{-1}(E,\overline{\partial}_E,\theta,h)$. We have the associated $\mathcal{R}$-triple $\mathfrak{T}(\widetilde{E})=(\widetilde{\mathfrak{E}},\widetilde{\mathfrak{E}},\widetilde{\mathfrak{C}})$. Let $\widetilde{g}:=g\circ\varphi$. Due to Proposition 19.2.9, the following holds:

- $\mathfrak{T}(\widetilde{E})$ is strictly specializable along $\widetilde{g}$ with ramified exponential twist by $\mathfrak{a}$.
- $P\,\mathrm{Gr}_\ell^{W(N)}\widetilde{\psi}_{\widetilde{g},u,\mathfrak{a}}\mathfrak{T}(\widetilde{E})$ is an $\mathcal{A}$-wild pure twistor D-module of weight ℓ, and the naturally induced Hermitian sesqui-linear duality gives a polarization. Moreover, $\widetilde{\psi}_{\widetilde{g},u,\mathfrak{a}}\mathfrak{T}(\widetilde{E})=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$.

By construction, we have $\varphi_\dagger^i\mathfrak{T}(\widetilde{E})(*g)=0$ unless $i=0$. According to Proposition 18.3.7, the following holds:

- $\varphi_\dagger^0\mathfrak{T}(\widetilde{E})$ is strictly specializable along g with ramified exponential twist by $\mathfrak{a}$.

- $P\,\mathrm{Gr}_{\ell}^{W(N)}\,\widetilde{\psi}_{g,u,\mathfrak{a}}\varphi_{\dagger}^{0}\mathfrak{T}(\widetilde{E})$ is an $\mathcal{A}$-wild pure twistor D-module of weight ℓ, and the naturally induced Hermitian sesqui-linear duality gives a polarization. Moreover, $\widetilde{\psi}_{g,u,\mathfrak{a}}\varphi_{\dagger}^{0}\mathfrak{T}(\widetilde{E})=0$ unless $u\in\boldsymbol{R}\times\mathcal{A}$.

As in Section 19.2.3, we have the natural isomorphism $\mathfrak{T}(E)(*g)\simeq\varphi_{\dagger}^{0}\mathfrak{T}(\widetilde{E})(*g)$. Thus, the proof of Proposition 19.2.1 is finished. □

19.3. Wildness and uniqueness

We shall show Lemma 19.1.2 and the full faithfulness of Φ in Theorem 19.1.3. The faithfulness is clear by strict S-decomposability of wild pure twistor D-modules. Indeed, let $f:\mathcal{M}_1\longrightarrow\mathcal{M}_2$ be a morphism of strictly S-decomposable $\mathcal{R}$-modules whose strict supports are Z. If its restriction to a Zariski open subset of Z is 0, it is 0 on Z. Hence, we only have to show Lemma 19.1.2 and that the induced functor Φ is full. Let us rewrite the claims. Let Z be an irreducible closed subset of X. Let $(\mathcal{T},\mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight w whose strict support is Z. There exists a Zariski open subset U of Z such that $(\mathcal{T},\mathcal{S})_{|X\setminus Y}$ comes from a harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on U after a suitable Tate twist, where $Y:=Z\setminus U$.

Proposition 19.3.1

- $(E,\overline{\partial}_E,\theta,h)$ *is $\mathcal{A}$-wild on (Z,U). In particular, we obtain a functor*
$$\Phi:\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z,U,w,\mathcal{A})\longrightarrow\mathrm{VPT}^{\mathrm{wild}}(Z,U,w,\mathcal{A}).$$
- *Recall that we have constructed the polarized $\mathcal{A}$-wild pure twistor D-module $(\mathfrak{T},\mathfrak{S})$ of weight 0 associated to $(E,\overline{\partial}_E,\theta,h)$ which is $\mathcal{A}$-wild on (Z,U). Then, we have a natural isomorphism $(\mathcal{T},\mathcal{S})\simeq(\mathfrak{T},\mathfrak{S})\otimes\mathbb{T}^S(-w/2)$.*

The first claim is Lemma 19.1.2, and the second claim implies that Φ is full.

In Subsection 19.3.1, we consider the case $\dim Z=1$. To argue the higher dimensional case, we reduce the issue to the local case in Subsection 19.3.2. We give a preparation in Subsection 19.3.3. We study the restriction to curves in Subsection 19.3.4, and the behaviour around a good point in Subsection 19.3.5. Then, we finish the proof in Subsection 19.3.6.

Although much part of the argument is essentially the same as that in the tame case, we have some additional difficulties. In the tame case, it is rather easy to show that the corresponding harmonic bundle is tame by using a convenient curve test, i.e., for a given harmonic bundle, it is tame if its restrictions to curves are tame. Although we have a curve test in the wild case (Proposition 13.5.1), we need preliminaries to apply it, and an additional argument to use Proposition 15.3.1.

Another difficulty is caused for the second claim by Stokes structures. Recall that we have the uniqueness of an extension of a flat bundle to regular singular meromorphic flat bundle. Hence, if we are given regular polarized pure twistor D-modules $\mathcal{T}_i$ $(i=1,2)$ whose restrictions to a Zariski open subset are isomorphic, it is rather easy to obtain that the underlying $\mathcal{R}$-modules of $\mathcal{T}_i$ are isomorphic. Because we do not have such uniqueness in the irregular case, we need some arguments to obtain the desired isomorphisms. Hence, we will study the comparison of the associated meromorphic objects.

Recall that we have already established the correspondence in the case where Z is smooth and one dimensional (Section 17.2), in which the wildness is easy to show, but the uniqueness is not so easy. Note that we have used the uniqueness result (Theorem 12.6.1) in an essential way.

19.3.1. The case $\dim Z = 1$. — Assume $\dim Z = 1$. Let $P \in Z \setminus U$. We take a small neighbourhood X_P of P. We have the decomposition $\mathcal{T}_{|X_P} = \bigoplus_i \mathcal{T}_i$ by strict supports, corresponding to the decomposition into irreducible components $Z \cap X_P = \bigcup_{i\in\Lambda} Z_i$. We only have to show the claims of Proposition 19.3.1 for each $\mathcal{T}_i$. Hence, we may assume (i) $X = \Delta^n$, (ii) Z is not contained in $\{z_n = 0\}$. Let $(\mathcal{T}, \mathcal{S})$ be a polarized $\mathcal{A}$-wild pure twistor D-module of weight 0, whose strict support is Z. We assume that the restriction of $(\mathcal{T}, \mathcal{S})$ to $X \setminus \{z_n = 0\}$ comes from a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $U = Z \setminus \{z_n = 0\}$. We may assume $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ and $\mathcal{S} = (\mathrm{id}, \mathrm{id})$. Let $\pi : X \to \Delta$ be the projection onto the n-th component. Let $\widetilde{Z} := \Delta_z$, and let $\varphi : \widetilde{Z} \to Z$ be a normalization. The composite $\pi \circ \varphi$ is denoted by $\widetilde{\pi}$. We may assume $\widetilde{\pi}(z) = z^m$ for some $m > 0$.

Let $\mathcal{T}' = (\mathcal{M}', \mathcal{M}', C')$ denote the direct summand of $P\pi_\dagger^0\mathcal{T}$ whose strict support is Δ. It is an $\mathcal{A}$-wild pure twistor D-module of weight 0 with the naturally induced polarization $\mathcal{S}' = (\mathrm{id}, \mathrm{id})$. We also have the corresponding $\mathcal{A}$-wild harmonic bundle $(E', \overline{\partial}_{E'}, \theta', h') := \pi_*(E, \overline{\partial}_E, \theta, h)$ on Δ^*. According to Proposition 17.2.1, $(\mathcal{T}', \mathcal{S}')$ is naturally isomorphic to the $\mathcal{A}$-wild pure twistor D-module associated to $(E', \overline{\partial}_{E'}, \theta', h')$ by the construction in Section 17.2.2 or Section 19.2.

Lemma 19.3.2. — *$(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \varphi^*(E, \overline{\partial}_E, \theta, h)$ is an $\mathcal{A}$-wild harmonic bundle. Namely, the first claim of Proposition* 19.3.1 *holds in the case* $\dim Z = 1$.

Proof. — Since $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$ is a direct summand of $\widetilde{\pi}^*(E', \overline{\partial}_{E'}, \theta', h')$, the claim is clear. □

Let us show the second claim of Proposition 19.3.1 in the case $\dim Z = 1$. We have the wild pure twistor D-module $\mathfrak{T}(\widetilde{E}) = (\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{E}}, \widetilde{\mathfrak{C}})$ with the polarization $\widetilde{\mathfrak{S}} = (\mathrm{id}, \mathrm{id})$ on $\widetilde{Z}$, associated to $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$. Let $\mathcal{L}_c$ be the Lefschetz map of $\varphi_\dagger^\bullet\mathfrak{T}(\widetilde{E})$ for a relatively ample line bundle with respect to φ. Let $P\varphi_\dagger^0\mathfrak{T}(\widetilde{E})$ be the kernel of $\mathcal{L}_c : \varphi_\dagger^0\mathfrak{T}(\widetilde{E}) \to \varphi_\dagger^2\mathfrak{T}(\widetilde{E})$. Let $\mathcal{T}_1 = (\mathcal{M}_1, \mathcal{M}_1, C_1)$ be the direct summand of $P\varphi_\dagger^0\mathfrak{T}(\widetilde{E})$ whose strict support is Z, which is equipped with the naturally induced polarization $\mathcal{S}_1$. Let $\mathcal{M}_1$ denote the underlying $\mathcal{R}_X$-module of $\mathcal{T}_1$. We would like to show $(\mathcal{T}_1, \mathcal{S}_1) \simeq (\mathcal{T}, \mathcal{S})$. For that purpose, we only have to show that the natural isomorphism $\mathcal{M}_{1|X\setminus\{z_n=0\}} \simeq \mathcal{M}_{|X\setminus\{z_n=0\}}$ can be extended to an isomorphism on X.

Lemma 19.3.3. — *We have a natural isomorphism* $\pi_\dagger\mathcal{M}(*z_n)\simeq\mathcal{M}'(*z_n)$.

Proof. — By construction of $\mathcal{M}'(*z_n)$, we have a natural morphism $\iota:\mathcal{M}'\to\pi_\dagger\mathcal{M}$, and the support of $\mathrm{Ker}(\iota)$ and $\mathrm{Cok}(\iota)$ are contained in $\{z_n=0\}$. Because of the coherence, we obtain the vanishing of $\mathrm{Cok}(\iota)$ and $\mathrm{Ker}(\iota)$ after the localization. □

In general, for a given $\mathcal{R}_X$-module $\mathcal{N}$ on $\boldsymbol{C}_\lambda\times X$, let $\mathcal{N}^\lambda$ denote the specialization of $\mathcal{N}$ to $\{\lambda\}\times X$. Let $\lambda\neq 0$. We set $\mathcal{V}^\lambda:=L^{-n+1}\varphi^\dagger\mathcal{M}^\lambda(*z_n)$. Note $L^j\varphi^\dagger\mathcal{M}^\lambda(*z_n)=0$ for any $j\neq -n+1$. As remarked in Lemma 22.6.3, it gives a meromorphic λ-flat bundle on $(\widetilde{Z},\widetilde{D})$.

Lemma 19.3.4. — *We have a natural isomorphism* $\mathrm{tr}:\varphi^0_\dagger\mathcal{V}^\lambda\to\mathcal{M}^\lambda(*z_n)$.

Proof. — As remarked in Section 22.6.1, we have the trace map

$$\mathrm{tr}:\varphi_\dagger L\varphi^*\mathcal{M}^\lambda[1-n]\longrightarrow\mathcal{M}^\lambda.$$

If we take the localization with respect to z_n, we have $\varphi_\dagger L\varphi^*\mathcal{M}^\lambda[1-n](*z_n)\simeq\varphi^0_\dagger\mathcal{V}^\lambda$. Thus, we obtain the desired morphism. To check that it is an isomorphism, we only have to see that the restriction is an isomorphism on $X\setminus\{z_n=0\}$, and it is clear. □

Lemma 19.3.5. — *We have a natural isomorphism* $\mathcal{V}^\lambda\simeq\mathcal{Q}\widetilde{\mathcal{E}}^\lambda=\widetilde{\mathfrak{E}}^\lambda(*z)$.

Proof. — Because $\mathcal{M}'^\lambda(*z_n)=\mathcal{Q}\mathcal{E}'^\lambda$ as in Proposition 17.2.1, we have the natural inclusion $\mathcal{Q}\widetilde{\mathcal{E}}^\lambda\subset\widetilde{\pi}^*\mathcal{M}'^\lambda(*z_n)$. Note $\mathcal{Q}\widetilde{\mathcal{E}}^\lambda$, $\mathcal{V}^\lambda$ and $\widetilde{\pi}^*\mathcal{M}'^\lambda(*z_n)$ are locally free $\mathcal{O}_{\widetilde{Z}}(*\widetilde{D})$-modules. Because the restriction of $\mathcal{V}^\lambda$ and $\mathcal{Q}\widetilde{\mathcal{E}}^\lambda$ to $\varphi^{-1}(U)$ are the same in $\widetilde{\pi}^*\mathcal{M}'^\lambda(*z_n)_{|\varphi^{-1}(U)}$, we only have to show that $\mathcal{V}^\lambda$ is also contained in $\widetilde{\pi}^*\mathcal{M}'^\lambda(*z_n)$. We obtain the following natural isomorphisms from Lemma 19.3.3 and Lemma 19.3.4:

$$\widetilde{\pi}^*\mathcal{M}'^\lambda(*z_n)\simeq\widetilde{\pi}^*\big(\pi^0_\dagger\mathcal{M}^\lambda(*z_n)\big)\simeq\widetilde{\pi}^*\pi^0_\dagger\varphi^0_\dagger\big(\mathcal{V}^\lambda\big)\simeq\widetilde{\pi}^*\widetilde{\pi}_\dagger\big(\mathcal{V}^\lambda\big).$$

Since $\widetilde{\pi}$ is a ramified covering, $\widetilde{\pi}_\dagger$ is the same as the push-forward for $\mathcal{O}$-modules, as explained in Lemma 22.7.1. Hence, $\mathcal{V}^\lambda$ is naturally identified with a direct summand of $\widetilde{\pi}^*\widetilde{\pi}_\dagger(\mathcal{V}^\lambda)$. Thus, we are done. □

Lemma 19.3.6. — *We have a natural isomorphism* $\mathcal{M}^\lambda(*z_n)\simeq\mathcal{M}^\lambda_1(*z_n)$ *for* $\lambda\neq 0$.

Proof. — It follows from Lemma 19.3.4 and Lemma 19.3.5. □

The rest is essentially the same as the argument given in Section 19.4.2 in [**67**]. We give an outline with minor simplification. Let $\lambda_0\in\boldsymbol{C}_\lambda$. Let $U(\lambda_0)$ denote a small neighbourhood of λ_0. Let $\boldsymbol{v}=(v_i)$ be a frame of $\mathcal{Q}^{(\lambda_0)}_{<0}\widetilde{\mathcal{E}}$. Each v_i naturally induces a section $\varphi_\dagger(v_i)$ of $\mathcal{M}_1(*z_n)$. The tuple of $\varphi_\dagger(v_i)$ is denoted by $\varphi_\dagger(\boldsymbol{v})$. Let $\varphi_\dagger(\boldsymbol{v})\cdot\mathcal{R}_X(*z_n)$ denote the submodule of $\mathcal{M}_1(*z_n)$ generated by $\varphi_\dagger(\boldsymbol{v})$ over $\mathcal{R}_X(*z_n)$.

Lemma 19.3.7. — *We have* $\varphi_\dagger(\boldsymbol{v})\cdot\mathcal{R}_X(*z_n)=\mathcal{M}_1(*z_n)$. *In other words,* $\varphi_\dagger(\boldsymbol{v})$ *generates* $\mathcal{M}_1(*z_n)$ *over* $\mathcal{R}_X(*z_n)$.

In particular, $\mathcal{M}^\lambda_1(*z_n)$ *is generated by the restriction of* $\varphi_\dagger(\boldsymbol{v})$ *to* $\{\lambda\}\times X$.

Proof. — It is easy to show the coincidence of the restriction of them to $X \setminus \{z_n = 0\}$. Then, the claim follows from Lemma 22.4.9, for example. □

Let $\mathcal{E}$ denote the $\mathcal{R}_U$-module on $\boldsymbol{C}_\lambda \times U$ associated to $(E, \overline{\partial}_E, \theta, h)$. Let $\iota_U : U \to X$. Then, we obtain the $\mathcal{R}_X$-module $\iota_{U\dagger}\mathcal{E}$. As remarked in Lemma 19.25 of [**67**], any section f of $\iota_{U\dagger}\mathcal{E}$ (resp. $\iota_{U\dagger}\mathcal{E}^\lambda$) on $U(\lambda_0) \times X$ (resp. X) is uniquely expressed as follows:

$$f = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^{n-1}} \varphi_\dagger(v_i) \cdot f_{\boldsymbol{n},i} \cdot \prod_{j=1}^{n-1} \eth_j^{n_j}. \tag{383}$$

Here, $f_{\boldsymbol{n},i}$ are holomorphic functions on $U(\lambda_0) \times U$ (resp. U).

Lemma 19.3.8. — *Let f be a section of $\mathcal{M}_1^\lambda(*z_n)$ on X. Then, each $f_{\boldsymbol{n},i}$ is meromorphic on $\widetilde{Z}$.*

Proof. — Let z be a holomorphic coordinate of $\widetilde{Z}$. Then, $\varphi^{-1}(\partial/\partial z_n)$ is expressed as a linear combination of $\partial/\partial z$ and $\varphi^{-1}(\partial/\partial z_i)$ $(i = 1, \ldots, n-1)$ with meromorphic coefficients. Then, the claim of Lemma 19.3.8 follows from Lemma 19.3.7. □

Lemma 19.3.9. — *We have a natural isomorphism $\mathcal{M}(*z_n) \simeq \mathcal{M}_1(*z_n)$.*

Proof. — We only have to show $\mathcal{M}(*z_n) \subset \mathcal{M}_1(*z_n)$ in $\iota_{U\dagger}\mathcal{E}$. Let f be a section of $\mathcal{M}(*z_n)$ on $U(\lambda_0) \times X$. We have the expression of f as in (383). The restrictions $f_{\boldsymbol{n},i|\{\lambda\} \times X}$ are meromorphic on X for each $\lambda \neq 0$ contained in $U(\lambda_0)$. Hence, as remarked in Lemma 19.23 of [**67**], we obtain that $f_{\boldsymbol{n},i}$ is meromorphic on $U(\lambda_0) \times X$. Then, it is easy to see that f is contained in $\mathcal{M}_1(*z_n)$. Thus, we obtain Lemma 19.3.9. □

Then, we obtain $\mathcal{M} = \mathcal{M}_1$ due to Lemma 22.4.10. Thus, the second claim of Proposition 19.3.1 is finished in the case $\dim Z = 1$.

19.3.2. Preliminaries for the general case. — Before going to the case $\dim Z > 1$, we give some lemmas to show that we can shrink X and U arbitrarily in the proof. Let Z be a closed analytic subset of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle defined on a Zariski open subset U of Z.

Lemma 19.3.10. — *Assume the following:*

(Locally wild) : *For any point P of Z, there exists a neighbourhood $\mathcal{U}_P$ of P in X such that $(E, \overline{\partial}_E, \theta, h)_{|\mathcal{U}_P \cap U}$ is wild on $(Z \cap \mathcal{U}_P, U \cap \mathcal{U}_P)$.*

Then, $(E, \overline{\partial}_E, \theta, h)$ is wild on (Z, U).

Proof. — We take a birational projective morphism $\varphi : Z_1 \to Z$ such that $D_1 := \varphi^{-1}(Z \setminus U)$ is a normal crossing hypersurface of Z_1. By the assumption, the eigenvalues of $\varphi^{-1}(\theta)$ are (possibly multi-valued) meromorphic 1-forms. By replacing φ and U appropriately, we can assume that the ramification of the 1-forms may happen only along D_1. For each point of $P_1 \in Z_1$, the local wildness condition is satisfied. Hence, $\varphi^{-1}(\theta)$ is generically good. Due to Proposition 15.3.1, we can take an appropriate refinement of φ such that $\varphi^*\theta$ is good. Then, $\varphi^*(E, \overline{\partial}_E, \theta, h)$ is good wild. □

Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on (Z, U). We have the polarized wild pure twistor D-module $(\mathfrak{T}, \mathfrak{S})$ of weight 0, associated to $(E, \overline{\partial}_E, \theta, h)$ as in Section 19.2. Assume that we are given another polarized wild pure twistor D-module $(\mathcal{T}, \mathcal{S})$ of weight 0 whose strict support is Z, such that the restriction to $X \setminus Y$ comes from $(E, \overline{\partial}_E, \theta, h)$, where $Y := Z \setminus U$. Then, we have the natural isomorphism $F : (\mathfrak{T}, \mathfrak{S})_{|X\setminus Y} \simeq (\mathcal{T}, \mathcal{S})_{|X\setminus Y}$

Lemma 19.3.11. — *Assume the following:*

(Locally extendable) : *For any point $P \in Z$, there exists a neighbourhood $\mathcal{U}_P$ of P in X such that the isomorphism $F_{|\mathcal{U}_P\setminus Y}$ can be extended to an isomorphism $F_{\mathcal{U}_P} : (\mathfrak{T}, \mathfrak{S})_{|\mathcal{U}_P} \to (\mathcal{T}, \mathcal{S})_{|\mathcal{U}_P}$.*

Then, F can be extended to an isomorphism $\widetilde{F} : (\mathfrak{T}, \mathfrak{S}) \to (\mathcal{T}, \mathcal{S})$.

Proof. — By the strict S-decomposability of the underlying $\mathcal{R}$-modules of pure twistor D-modules, we have the following general and easy fact:

- Let $\mathcal{T}$ be a pure twistor D-module whose strict support is Z_1. Let φ be an automorphism of $\mathcal{T}$. If the restriction of φ to some non-empty Zariski open subset of Z_1 is equal to the identity, then φ is equal to the identity.

Then, we obtain $F_{\mathcal{U}_P|\mathcal{U}_P\cap\mathcal{U}_Q} = F_{\mathcal{U}_Q|\mathcal{U}_P\cap\mathcal{U}_Q}$, and thus we obtain the global isomorphism. □

By Lemmas 19.3.10 and 19.3.11, we may shrink X arbitrarily to show Proposition 19.3.1 in the following argument without mention.

Let us discuss shrinking U. Let $U' \subset U$ be a Zariski open subset. We put $Y := Z \setminus U$ and $Y' := Z \setminus U'$.

Lemma 19.3.12. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on U. Then, $(E, \overline{\partial}_E, \theta, h)$ is wild on (Z, U) if and only if $(E, \overline{\partial}_E, \theta, h)_{|U'}$ is wild on (Z, U').*

Proof. — We argue only the only if part. The other case can be discussed similarly. We can take a projective birational morphism $\varphi : \widetilde{Z} \to Z$ such that (i) $\widetilde{Z}$ is smooth, (ii) $\varphi^{-1}(Y)$ and $\varphi^{-1}(Y')$ are normal crossing, (iii) $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)_{U'}$ is good wild on $(\widetilde{Z}, \varphi^{-1}(Y'))$. Then, it is easy to check that $\varphi^{-1}(E, \overline{\partial}_E, \theta, h)_U$ is good wild on $(\widetilde{Z}, \varphi^{-1}(Y))$. □

Let $(E, \overline{\partial}_E, \theta, h)$ be wild on (Z, U). We have constructed polarized wild pure twistor D-modules $(\mathfrak{T}, \mathfrak{S})$ and $(\mathfrak{T}', \mathfrak{S}')$ associated to $(E, \overline{\partial}_E, \theta, h)$ on (Z, U), and $(E, \overline{\partial}_E, \theta, h)_{|U'}$ on (Z, U') respectively. By construction, we have a natural isomorphism $(\mathfrak{T}, \mathfrak{S}) \simeq (\mathfrak{T}', \mathfrak{S}')$ whose restriction to U' is the identity. Hence, we may and will shrink U arbitrarily for the proof of Proposition 19.3.1 in the following argument.

19.3.3. Meromorphic flat connections. — In the following argument, we will not distinguish meromorphic λ-flat bundles and meromorphic flat bundles in the case $\lambda \neq 0$. Let us consider the case $\dim Z = k+1 > 1$. By shrinking U, we may assume $Z \setminus U = g^{-1}(0) \cap Z$ for some function g. Let $\varphi' : \widetilde{Z}' \to Z$ be a birational projective morphism such that $\widetilde{Z}' - (\varphi')^{-1}(U)$ is a normal crossing hypersurface. Note that the Higgs bundle $(\varphi')^{-1}(E, \overline{\partial}_E, \theta)$ can be extended to a meromorphic Higgs sheaf on $\widetilde{Z}'$ given by $(\varphi')^* \mathcal{M}^0$. By shrinking U, if necessary, we can take a birational projective morphism $\varphi : \widetilde{Z} \to Z$, which factors through $\widetilde{Z}'$, such that the following holds:

- $\widetilde{Z} - \varphi^{-1}(U)$ is a normal crossing hypersurface.
- The ramification of the eigenvalues of $\varphi^{-1}(\theta)$ may happen only along $\widetilde{D} := \widetilde{Z} - \varphi^{-1}(U)$.

We put $\mathcal{V}^\lambda := L^{-n+k+1}\varphi^* \mathcal{M}^\lambda \otimes \mathcal{O}(*\widetilde{D})$ for $\lambda \neq 0$. It is a holonomic D-module whose characteristic variety is contained in $\widetilde{Z} \cup \pi^{-1}(\widetilde{D})$, where $\pi : T^*\widetilde{Z} \to \widetilde{Z}$ denotes the natural projection of the cotangent bundle. Hence $\mathcal{V}^\lambda$ is a meromorphic λ-flat connection. It is an $\mathcal{O}_{\widetilde{Z}}(*\widetilde{D})$-reflexive module [**58**].

Lemma 19.3.13. — *$\varphi^0_\dagger \mathcal{V}^\lambda$ and $\mathcal{M}^\lambda(*g)$ are naturally isomorphic.*

Proof. — We have the trace map $\mathrm{tr} : \varphi_\dagger L\varphi^* \mathcal{M}^\lambda[k+1-n] \to \mathcal{M}^\lambda$, as remarked in Section 22.6.1. After localization with respect to g, we have $\varphi_\dagger L\varphi^*\mathcal{M}^\lambda(*g)[k+1-n] \simeq \varphi^0_\dagger \mathcal{V}^\lambda$. Then, we obtain a naturally defined morphism $\mathrm{tr} : \varphi^0_\dagger \mathcal{V}^\lambda \to \mathcal{M}^\lambda(*g)$. Since the restriction $\mathrm{tr}_{|U}$ is an isomorphism, the support of $\mathrm{Ker}(\mathrm{tr})$ and $\mathrm{Cok}(\mathrm{tr})$ are contained in $g^{-1}(0)$. Since the action of g on $\varphi^0_\dagger \mathcal{V}^\lambda$ and $\mathcal{M}^\lambda(*g)$ is invertible, we obtain that tr induces an isomorphism on Z. Thus, we obtain Lemma 19.3.13. □

19.3.4. Restriction to a curve. — Let $\widetilde{C}$ be a smooth curve in $\widetilde{Z}$ transversal to the smooth part of $\widetilde{D}$. We have the wild harmonic bundles $(E_{\widetilde{C}}, \overline{\partial}_{E_{\widetilde{C}}}, \theta_{\widetilde{C}}, h_{\widetilde{C}}) := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)_{|\widetilde{C} \setminus \widetilde{D}}$. We have the associated meromorphic λ-flat bundle $(\mathcal{Q}\mathcal{E}^\lambda_{\widetilde{C}}, \mathbb{D}^\lambda_{\widetilde{C}})$ on $\widetilde{C}$. We put $C := \varphi(\widetilde{C})$.

Proposition 19.3.14. — *We have a natural isomorphism $\mathcal{Q}\mathcal{E}^\lambda_{\widetilde{C}} \simeq \mathcal{V}^\lambda_{|\widetilde{C}}$. The harmonic bundle $(E_{\widetilde{C}}, \overline{\partial}_{E_{\widetilde{C}}}, \theta_{\widetilde{C}}, h_{\widetilde{C}})$ is $\mathcal{A}$-wild.*

Proof. — In the proof of the proposition, we may and will shrink X and Z without mention.

Lemma 19.3.15. — *We can take a function f on X with the following property:*

- *$C \cap U$ is contained in the smooth part of $f^{-1}(0) \cap Z$.*

Proof. — Let I_C denote the ideal sheaf of $\mathcal{O}_X$ corresponding to C. Let $P \in C \cap \{g=0\}$. We have some functions $f_1, \ldots, f_N$ which generates the ideal sheaf I_C around P. If $Q \in C \cap U$ is sufficiently close to P, $f_1, \ldots, f_N$ generate I_C at Q. In particular, one of $df_{i|Q}$ is not 0 on $T_Q Z$, which gives a desired function. □

Lemma 19.3.16. — *We can take functions $f_1, \dots, f_k$ with the following properties:*
- $C \subset Z \cap \bigcap_{i=1}^{k} f_i^{-1}(0)$.
- $C \cap U$ *is contained in the smooth part of* $Z \cap \bigcap_{i=1}^{j} f_i^{-1}(0)$.
- $f_{j+1}^{-1}(0)$ *and* $Z \cap \bigcap_{i=1}^{j} f_i^{-1}(0)$ *are transversal at* $C \cap U$.

Proof. — We only have to apply Lemma 19.3.15 inductively. □

We put $Z^{(0)} := Z$, $\widetilde{Z}^{(0)} := \widetilde{Z}$, $\widetilde{C}^{(0)} := \widetilde{C}$ and $g^{(0)} := g$. Let $Z^{(j)}$ denote the irreducible component of $Z \cap \bigcap_{i \leqslant j} f_i^{-1}(0)$, which contains C. We take functions $g^{(j)} = a^{(j)} g^{(j-1)}$ such that the singular part of $Z \cap \bigcap_{i \leqslant j} f_i^{-1}(0)$ is contained in $(g^{(j)})^{-1}(0)$. We take $\widetilde{Z}^{(j)}$ inductively as follows: Assume that we are given $\widetilde{Z}^{(j)}$ with a birational projective morphism $\varphi^{(j)} : \widetilde{Z}^{(j)} \to Z$ and the curve $\widetilde{C}^{(j)} \subset \widetilde{Z}^{(j)}$ such that $\varphi^{(j)}(\widetilde{C}^{(j)}) = C$. We take a birational projective morphism $\mu^{(j)} : \widetilde{Z}_1^{(j)} \to \widetilde{Z}^{(j)}$ with the following properties:
- It is bi-holomorphic on $\widetilde{Z}^{(j)} \setminus (g^{(j)})^{-1}(0)$.
- We put $F_j := (\varphi^{(j)} \circ \mu^{(j)})^*(f_{j+1})$ and $G_j := (\varphi^{(j)} \circ \mu^{(j)})^*(g^{(j+1)})$. Then, $F_j^{-1}(0) \cup G_j^{-1}(0)$ is normal crossing.

Let $\widetilde{C}^{(j+1)}$ be the proper transform of $\widetilde{C}^{(j)}$. Let $\widetilde{Z}^{(j+1)}$ denote the irreducible component of $F_j^{-1}(0)$ which contains $\widetilde{C}^{(j+1)}$. We have the induced map $\varphi^{(j+1)} : \widetilde{Z}^{(j+1)} \to Z^{(j+1)}$. Note that the intersection of $\widetilde{Z}^{(j+1)}$ and another irreducible component of $F_j^{-1}(0)$ is contained in $G_j^{-1}(0)$. Let $\mathcal{V}^{\lambda(j)}$ be the pull-back of $\mathcal{V}^{\lambda}$ via the induced morphism $\widetilde{Z}^{(j)} \to \widetilde{Z}$.

We obtain the polarized $\mathcal{A}$-wild pure twistor D-modules $\mathcal{T}^{(j)}$ of weight 0 whose strict supports are $Z^{(j)}$, inductively as follows:
- We put $\mathcal{T}^{(0)} := \mathcal{T}$. Assume that we are given $\mathcal{T}^{(j)}$ on $Z^{(j)}$. We have the polarized $\mathcal{A}$-wild pure twistor D-module $P\,\mathrm{Gr}_0^W \psi_{f_{j+1},-\boldsymbol{\delta}_0} \mathcal{T}^{(j)}$, which is decomposed by the strict supports. Let $\mathcal{T}^{(j+1)}$ denote the $Z^{(j+1)}$-component.

Let $\mathcal{M}^{(j)}$ denote the underlying $\mathcal{R}$-module of $\mathcal{T}^{(j)}$.

Lemma 19.3.17. — *We have a natural isomorphism* $\varphi_{\dagger}^{(j)} \mathcal{V}^{\lambda(j)} \simeq \mathcal{M}^{(j)\lambda}(*g^{(j)})$.

Proof. — We use an induction. The case $j = 0$ is the claim of Lemma 19.3.13. Assume that the claim holds for j, and we will derive the isomorphism for $j+1$. We put $\mathcal{V}_1^{\lambda(j)} := \mu^{(j)*} \mathcal{V}^{\lambda(j)} \otimes \mathcal{O}_{\widetilde{Z}_1^{(j)}}(*G_j)$. We put $\kappa^{(j)} := \varphi^{(j)} \circ \mu^{(j)}$. Then, we have the following natural isomorphisms:

$$\kappa_{\dagger}^{(j)} \mathcal{V}_1^{\lambda(j)} \simeq \varphi_{\dagger}^{(j)} \mu_{\dagger}^{(j)} \mu^{(j)*} \mathcal{V}^{\lambda(j)}(*g^{(j+1)}) \simeq \varphi_{\dagger}^{(j)} \mathcal{V}^{\lambda(j)}(*g^{(j+1)}) \simeq \mathcal{M}^{(j)\lambda}(*g^{(j+1)}).$$

Here, we have used the isomorphism $\mu_{\dagger}^{(j)} \mu^{(j)*} \mathcal{V}^{\lambda(j)}(*g^{(j+1)}) \simeq \mathcal{V}^{\lambda(j)}(*g^{(j+1)})$ induced by the trace map. Then, we obtain the following isomorphisms:

$$\kappa_{\dagger}^{(j)}\big(\psi_{F_j,-1} \mathcal{V}_1^{\lambda(j)}\big) \simeq \psi_{f_{j+1},-1}\big(\kappa_{\dagger}^{(j)} \mathcal{V}_1^{\lambda(j)}\big) \simeq \psi_{f_{j+1},-1} \mathcal{M}^{(j)\lambda}(*g^{(j+1)}). \tag{384}$$

Let $\widetilde{Z}_i^{(j+1)}$ denote the irreducible components of $F_j^{-1}(0)$ which are not contained in $G_j^{-1}(0)$. (We put $\widetilde{Z}_1^{(j+1)} := \widetilde{Z}^{(j+1)}$.) We have the decomposition by the supports:

$$\psi_{F_j,-1}\mathcal{V}_1^{\lambda(j)} = \bigoplus_i \mathcal{N}_{\widetilde{Z}_i^{(j+1)}}.$$

And $\mathcal{N}_{\widetilde{Z}_1^{(j+1)}} \simeq \iota_\dagger \mathcal{V}^{\lambda(j+1)}$, where ι denotes the inclusion $\widetilde{Z}^{(j+1)} \to \widetilde{Z}_1^{(j)}$.

We have the following natural isomorphisms:

$$\mathcal{M}^{(j+1)}(*g^{(j+1)}) \simeq P\,\mathrm{Gr}_0^W\,\psi^{(\lambda_0)}_{f_{j+1},-\boldsymbol{\delta}_0}\mathcal{M}^{(j)}(*g^{(j+1)}) \simeq \psi^{(\lambda_0)}_{f_{j+1},-\boldsymbol{\delta}_0}\mathcal{M}^{(j)}(*g^{(j+1)}).$$

We also have the following isomorphism:

$$\Big(\psi^{(\lambda_0)}_{f_{j+1},-\boldsymbol{\delta}_0}\mathcal{M}^{(j)}(*g^{(j+1)})\Big)^{\lambda} \simeq \psi_{f_{j+1},-1}\mathcal{M}^{(j)\lambda}(*g^{(j+1)}).$$

Hence, the restriction of (384) to the $Z^{(j+1)}$-component gives the desired isomorphism. Thus, we obtain Lemma 19.3.17. □

Let us return to the proof of Proposition 19.3.14. By construction, we have $\widetilde{Z}^{(k)} = \widetilde{C}^{(k)} \simeq \widetilde{C}$. Under the isomorphism, we have $\mathcal{V}^{\lambda(k)} \simeq \mathcal{V}^{\lambda}_{|\widetilde{C}}$ and $\varphi^{(k)} = \varphi_{|\widetilde{C}}$. We also have $C = Z^{(k)}$, and $\mathcal{T}^{(k)}$ gives a polarized $\mathcal{A}$-wild pure twistor D-module of weight 0 whose strict supports is C. We have obtained the following isomorphism:

$$\varphi^{(k)}_\dagger \mathcal{V}^{\lambda(k)} \simeq \mathcal{M}^{(k)\lambda}(*g^{(k)}). \tag{385}$$

The rest of the argument is essentially the same as the proof of Lemma 19.3.5. We give only an outline. We may assume to have a coordinate w on $\widetilde{C}$ such that $\widetilde{C} \cap \widetilde{D} = \{w=0\}$. We take a projection $\pi : X \to \Delta$ such that the composite $\Psi := \pi \circ \varphi^{(k)} : \widetilde{Z}^{(k)} \to \Delta$ is given by $\Psi(w) = w^\ell$ for some $\ell > 0$. Let $\mathcal{T}' = (\mathcal{M}', \mathcal{M}', \mathcal{C}')$ denote the direct summand of the polarized $\mathcal{A}$-wild pure twistor D-module $\pi^0_\dagger \mathcal{T}^{(k)}$ whose strict support is Δ. We have the corresponding $\mathcal{A}$-wild harmonic bundle $(E', \overline{\partial}_{E'}, \theta', h')$ on Δ^*, which is obtained as the push-forward of $(E, \overline{\partial}_E, \theta, h)_{|C}$. As in Lemma 19.3.3, we have the natural isomorphisms:

$$\Psi_\dagger \mathcal{V}^{\lambda(k)} \simeq \pi_\dagger \varphi^{(k)}_\dagger \mathcal{V}^{\lambda(k)} \simeq \pi_\dagger \mathcal{M}^{(k)\lambda}(*g^{(k)}) \simeq \mathcal{M}'^{\lambda}(*w).$$

As in the proof of Lemma 19.3.5, we have natural inclusions $\mathcal{V}^{\lambda(k)} \to \Psi^*\mathcal{M}'^{\lambda}(*w)$ and $\mathcal{QE}^{\lambda}_{\widetilde{C}} \to \Psi^*\mathcal{M}'^{\lambda}(*w)$. Since the restriction of $\mathcal{V}^{\lambda(k)}$ and $\mathcal{QE}^{\lambda}_{\widetilde{C}}$ to $\widetilde{Z}^{(k)} \setminus \{G^{(k)} = 0\}$ are the same, we obtain $\mathcal{V}^{\lambda(k)} \simeq \mathcal{QE}^{\lambda}_{\widetilde{C}}$, and thus the first claim of Proposition 19.3.14 is proved. Since $(E_{\widetilde{C}}, \overline{\partial}_{E_{\widetilde{C}}}, \theta_{\widetilde{C}}, h_{\widetilde{C}})$ is a direct summand of $(\varphi^{(k)})^{-1}(E', \overline{\partial}_{E'}, \theta', h')$, the harmonic bundle $(E_{\widetilde{C}}, \overline{\partial}_{E_{\widetilde{C}}}, \theta_{\widetilde{C}}, h_{\widetilde{C}})$ is also $\mathcal{A}$-wild. Thus, we obtain the second claim of Proposition 19.3.14. □

19.3.5. Around a good point. — For distinction, we set $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$. Let us fix $\lambda_1 \neq 0$. We continue to use the notation in Subsection 19.3.4. Let P be a smooth point of $\widetilde{D}$ around which $\mathcal{V}^{\lambda_1}$ is good. (Recall that the set of good points is non-empty and Zariski open. See [**58**].)

Lemma 19.3.18. — *On an appropriate neighbourhood $\mathcal{U}$ of P, $\big(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}\big)_{|\mathcal{U}\setminus\widetilde{D}}$ is $\mathcal{A}$-good wild on $(\mathcal{U},\widetilde{D}\cap\mathcal{U})$, and $\mathcal{QE}^{\lambda}_{|\mathcal{U}}$ is naturally isomorphic to $\mathcal{V}^{\lambda}_{|\mathcal{U}}$ for any $\lambda\neq 0$.*

Proof. — Take a ramified covering $\eta:(\mathcal{U}',D')\longrightarrow(\mathcal{U},\widetilde{D}\cap\mathcal{U})$ such that $\eta^*(\mathcal{V}^{\lambda_1},\mathbb{D}^{\lambda_1})$ is unramified. Let $\mathcal{I}$ denote the set of the irregular values of $\eta^*(\mathcal{V}^{\lambda_1},\mathbb{D}^{\lambda_1})$. We may assume that $\mathcal{U}$ is equipped with a coordinate system $(z_1,\dots,z_n)$ such that $\widetilde{D}\cap\mathcal{U}=\{z_1=0\}$. Let $\pi:\mathcal{U}\longrightarrow\widetilde{D}\cap\mathcal{U}$ denote the projection given by the coordinate.

We apply Proposition 19.3.14 to the harmonic bundle

$$\big(E_{\pi^{-1}(Q)},\overline{\partial}_{E_{\pi^{-1}(Q)}},\theta_{\pi^{-1}(Q)},h_{\pi^{-1}(Q)}\big)=\big(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h}\big)_{|\pi^{-1}(Q)}.$$

Since the meromorphic λ_1-flat bundle $\eta^*\mathcal{QE}^{\lambda_1}_{\pi^{-1}(Q)}$ is unramified, the meromorphic λ-flat bundle $\eta^*\mathcal{QE}^{\lambda}_{\pi^{-1}(Q)}$ is also unramified for every λ, according to Theorem 7.4.5. Moreover, the set of irregular values are given as follows:

$$\operatorname{Irr}\big(\eta^*\mathcal{QE}^{\lambda}_{\pi^{-1}(Q)},\mathbb{D}^{\lambda}\big)=\big\{\mathfrak{a}_{|\eta^{-1}\pi^{-1}(Q)}\,\big|\,\mathfrak{a}\in\mathcal{I}\big\}.$$

According to Lemma 2.7.14, $\eta^*\mathcal{V}^{\lambda}$ also has an unramifiedly good lattice V^{λ}.

We have the generalized eigen-decomposition $V^{\lambda}_{|Q}=\bigoplus_{\alpha}\mathbb{E}_{\alpha}\big(V^{\lambda}_{|Q}\big)$ with respect to $\operatorname{Res}(\mathbb{D}^{\lambda})$. Let us consider the following set:

$$\mathcal{S}p(Q,\lambda):=\Big\{\alpha+n\cdot\lambda\,\Big|\,\mathbb{E}_{\alpha}\big(V^{\lambda}_{|Q}\big)\neq 0,\ n\in\mathbb{Z}\Big\}.$$

It is well known and easy to see that the set $\mathcal{S}p(Q,\lambda)$ is independent of the choice of lattices V^{λ}. It is also independent of Q.

We have the set $\mathcal{KMS}\big(\eta^{-1}\widetilde{E}_{|\pi^{-1}(Q)}\big)$ of the KMS-spectra of the unramifiedly good wild harmonic bundle $\eta^{-1}(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta},\widetilde{h})_{|\pi^{-1}(Q)}$ at $\lambda=0$ for each Q. We have a discrete subset $A(Q)\subset\boldsymbol{C}^*$ such that the map $\mathfrak{e}(\lambda)$ gives the bijection between $\mathcal{KMS}\big(\eta^{-1}\widetilde{E}_{|\pi^{-1}(Q)}\big)$ and $\mathcal{S}p(Q,\lambda)$ for any $\lambda\in\boldsymbol{C}^*-A(Q)$. (See Proposition 8.2.1.) Then, we can conclude that the sets $\mathcal{KMS}\big(\eta^{-1}\widetilde{E}_{|\pi^{-1}(Q)}\big)$ are independent of the choice of Q, and they are denoted by $\mathcal{KMS}\big(\eta^{-1}\widetilde{E},D'\big)$.

Let λ be generic with respect to the set $\mathcal{KMS}\big(\eta^{-1}\widetilde{E},D'\big)$, namely, the map $\mathfrak{e}(\lambda)$: $\mathcal{KMS}\big(\eta^{-1}\widetilde{E},D'\big)\longrightarrow\boldsymbol{C}$ is injective. For each $c\in\boldsymbol{R}$, we can take a good lattice ${}_cV'$ of $\eta^*\mathcal{V}^{\lambda}$ with the following property:

- The set of the residue $\operatorname{Res}_{D'}(\mathbb{D}^{\lambda})\in\operatorname{End}({}_cV'_{|D'})$ is given by the following set:

$$\Big\{\mathfrak{e}(\lambda,u)\,\Big|\,u\in\mathcal{KMS}(\eta^{-1}\widetilde{E},D'),\ c-1<\mathfrak{p}(\lambda,u)\leqslant c\Big\}.$$

Thus, we obtain an unramifiedly good filtered λ-flat bundle $(V'_*,\mathbb{D}^{\lambda})$ on $\big(\mathcal{U}',\,D'\big)$. By taking the descent, we obtain a good filtered λ-flat bundles $(V_*,\mathbb{D}^{\lambda})$ on $\big(\mathcal{U},\widetilde{D}\cap\mathcal{U}\big)$. By construction, we have $\mathcal{V}^{\lambda}=\bigcup{}_cV$.

Let $T:=(1+|\lambda|^2)$, and let us consider the deformations $V'^{(T)}_*$ and $V^{(T)}_*$ (Section 4.5.2). We have the natural isomorphism $(\eta^*\mathcal{V}^{\lambda})^{(T)}_{|\pi^{-1}(Q)}\simeq\eta^*\mathcal{PE}_{\pi^{-1}(Q)}$ (Section 12.1) and hence $(\mathcal{V}^{\lambda})^{(T)}_{|\pi^{-1}(Q)}\simeq\mathcal{PE}_{\pi^{-1}(Q)}$. By using the characterization of the lattices

(Proposition 8.3.1), we obtain $(V_*^{(T)})_{|\pi^{-1}(Q)} \simeq \mathcal{P}_*\mathcal{E}^{\lambda}_{\pi^{-1}(Q)}$. Hence, we obtain that $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})_{|\mathcal{U}\setminus\widetilde{D}}$ is wild and good, due to Proposition 13.5.1. Since the eigenvalues of $\mathrm{Res}(\mathbb{D}^0)$ on $\mathcal{Q}\widetilde{\mathcal{E}}^0$ are the same as those of the residue on $\mathcal{Q}\widetilde{\mathcal{E}}^0_{|\widetilde{C}}$, the harmonic bundle is $\mathcal{A}$-good wild. For any $\lambda \neq 0$, we have the isomorphism $\mathcal{Q}\mathcal{E}^{\lambda}_{|\pi^{-1}(Q)} \simeq \mathcal{V}^{\lambda}_{|\pi^{-1}(Q)}$, and hence we obtain the isomorphism $\mathcal{Q}\mathcal{E}^{\lambda} \simeq \mathcal{V}^{\lambda}$. Thus, we obtain Lemma 19.3.18. □

19.3.6. End of Proof of Proposition 19.3.1. — Let us show the first claim of Proposition 19.3.1. Due to Lemma 19.3.18, $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$ satisfies the assumption made in Section 15.3.1. Hence, we can take a birational projective morphism $\varphi' : \widetilde{Z}' \to \widetilde{Z}$ such that $\varphi'^*\varphi^*\theta$ is good due to Proposition 15.3.1, and so we can assume $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$ is a good wild harmonic bundle on $(\widetilde{Z}, \widetilde{D})$ from the beginning.

Lemma 19.3.19

- *For any $\lambda \neq 0$, we have a natural isomorphism $\mathcal{Q}\mathcal{E}^{\lambda} \simeq \mathcal{V}^{\lambda}$.*
- *$(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h})$ is an $\mathcal{A}$-good wild harmonic bundle.*

Proof. — There exists a closed analytic subset $W \subset D$ such that $\mathcal{V}^{\lambda}$ is good on $\widetilde{Z}\setminus W$. Then, we have the isomorphism $\mathcal{Q}\mathcal{E}^{\lambda}_{|\widetilde{Z}\setminus W} \simeq \mathcal{V}^{\lambda}_{|\widetilde{Z}\setminus W}$ because of Lemma 19.3.18. Then, it can be extended to an isomorphism on $\widetilde{Z}$ by the reflexivity of $\mathcal{Q}\mathcal{E}^{\lambda}$ and $\mathcal{V}^{\lambda}$. Thus, we obtain the first claim of Lemma 19.3.19. The second claim also follows from Lemma 19.3.18. □

Hence, we can conclude that $(E, \overline{\partial}_E, \theta, h)$ $\mathcal{A}$-wild harmonic bundle on (Z, U). Thus, the first claim of Proposition 19.3.1 is proved.

In the above argument, we obtain the $\mathcal{A}$-good wild harmonic bundle $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) = \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ on $(\widetilde{Z}, \widetilde{D})$. We have the associated polarized $\mathcal{A}$-wild pure twistor D-module $(\mathfrak{T}(\widetilde{E}), \widetilde{\mathfrak{S}})$ on $\widetilde{Z}$ whose underlying $\mathcal{R}_{\widetilde{Z}}$-module is denoted by $\widetilde{\mathfrak{E}}$. Let $\mathcal{T}_1$ denote the direct summand of $\varphi^0_{\dagger}\mathfrak{T}(\widetilde{E})$ whose strict support is Z. We have the naturally induced polarization $\mathcal{S}_1$ of $\mathcal{T}_1$. We would like to show $(\mathcal{T}_1, \mathcal{S}_1) \simeq (\mathcal{T}, \mathcal{S})$, which is the second claim of Proposition 19.3.1. Let $\mathcal{M}_1$ denote the underlying $\mathcal{R}$-module of $\mathcal{T}_1$. We only have to show $\mathcal{M}_1 \simeq \mathcal{M}$.

The rest is essentially the same as the argument in Section 19.4.3 of [**67**]. We give only an outline. We put $r := n - k - 1$.

Lemma 19.3.20. — *We have a natural isomorphism $\varphi_{\dagger}\widetilde{\mathfrak{E}}(*g) \simeq \mathcal{M}(*g)$.*

Proof. — By shrinking U, we may assume to have holomorphic functions $a_1, \ldots, a_r$ with the following properties:

- Z is one of the irreducible components of $\bigcap_{j=1}^{r} a_j^{-1}(0)$.
- We put $w_j := \sum_{i=1}^{n} \partial_{z_i} a_j \cdot \partial_{z_i}$. Then, $w_1, \ldots, w_r$ form a frame of the normal bundle of U in X.

We have the $\mathcal{R}_U$-module $\mathcal{E}$ on $\boldsymbol{C}_\lambda \times U$, associated to $(E, \overline{\partial}_E, \theta, h)$. Let ι denote the immersion $U \to X$. We only have to show that $\mathcal{M}(*g)$ and $\varphi_\dagger \widetilde{\mathfrak{E}}(*g)$ are the same in $\iota_\dagger \mathcal{E}$.

Let s be any section of $\mathcal{M}(*g)$ on $U(\lambda_0) \times X$. When we regard s as the section of $\iota_\dagger \mathcal{E}$, it is expressed as a finite sum of the following form:

$$s = \sum_{\boldsymbol{n} \in \mathbb{Z}^r_{\geqslant 0}} u_{\boldsymbol{n}}(s) \cdot \prod_{j=1}^{r} w_j^{n_j}.$$

Here $u_{\boldsymbol{n}}(s)$ are holomorphic sections of $\mathcal{E}$ on $U(\lambda_0) \times U$. We can naturally regard them as sections of $\widetilde{\mathcal{E}}$ on $U(\lambda_0) \times (\widetilde{Z} - \widetilde{D})$.

Lemma 19.3.21. — *$u_{\boldsymbol{n}}(s)$ are meromorphic sections of $\mathcal{Q}\widetilde{\mathcal{E}}$ on $U(\lambda_0) \times \widetilde{Z}$.*

Proof. — Let $\widetilde{C}$ be a curve in $\widetilde{Z}$ as in Section 19.3.4, and let $C = \varphi(\widetilde{C})$. We also use the other objects and the notation in Section 19.3.4. Let us see that s induces sections $s^{(j)}$ of $\mathcal{M}^{(j)}(*g^{(j)})$ inductively. We may assume that f_j are coordinate functions on X, by replacing X with $X \times \boldsymbol{C}^k$. Let us consider the specialization along f_1. Since $s_{|X \setminus \{g=0\}}$ is contained in $V^{(\lambda_0)}_{-1}\big(\mathcal{M}(*g)\big)_{|X \setminus \{g=0\}}$, where $V^{(\lambda_0)}$ is the V-filtration along f_1, we obtain $s \in V^{(\lambda_0)}_{-1}\big(\mathcal{M}(*g)\big)$. Hence, it induces the section $s^{(1)}$ of $\mathrm{Gr}^{V^{(\lambda_0)}}_{-1}\big(\mathcal{M}(*g)\big)$. Since the support of $\widetilde{\psi}_{f_1,u}(\mathcal{M})$ is contained in $\{g = 0\}$ unless $u \in \mathbb{Z}_{<0} \times \{0\}$, we have $\mathrm{Gr}^{V^{(\lambda_0)}}_{-1}\big(\mathcal{M}(*g)\big) = \psi^{(\lambda_0)}_{f_1,-\boldsymbol{\delta}_0}\big(\mathcal{M}(*g)\big)$. For a similar reason, we obtain the natural isomorphism $\psi^{(\lambda_0)}_{f_1,-\boldsymbol{\delta}_0}\mathcal{M}(*g) \simeq \mathcal{M}^{(1)}(*g)$. Thus, $s^{(1)}$ is induced. By the same procedure, we obtain $s^{(j+1)}$ from $s^{(j)}$. By applying Lemma 19.3.8, we obtain that the restrictions of $u_{\boldsymbol{n}}(s)$ to $\widetilde{C}$ are meromorphic sections of $\mathcal{Q}\widetilde{\mathcal{E}}_{|U(\lambda_0) \times \widetilde{C}}$. Then, we can conclude that $u_{\boldsymbol{n}}(s)$ are meromorphic sections of $\mathcal{Q}\widetilde{\mathcal{E}}$ on $U(\lambda_0) \times \widetilde{Z}$. Thus, we obtain Lemma 19.3.21. □

Let us return to the proof of Lemma 19.3.20. Let $\Gamma_\varphi : \widetilde{Z} \to \widetilde{Z} \times X$ denote the graph. Each $u_{\boldsymbol{n}}(s) \cdot \prod_{j=1}^r w_j^{n_j}$ gives a section of $\Gamma_{\varphi\dagger}\mathcal{Q}\widetilde{\mathcal{E}}$, and hence it induces a section of $\varphi_\dagger \mathcal{Q}\widetilde{\mathcal{E}}$ (See Lemma 14.11 of [**67**], for example.) Therefore, s is naturally contained in $\varphi_\dagger \widetilde{\mathfrak{E}}(*g) \simeq \varphi_\dagger \mathcal{Q}\widetilde{\mathcal{E}}$, and we obtain $\mathcal{M}(*g) \subset \varphi_\dagger \widetilde{\mathfrak{E}}(*g)$. By using the coherence property of $\mathcal{M}$ and $\widetilde{\mathfrak{E}}$ and Lemma 22.4.9, we obtain $\mathcal{M}(*g) = \varphi_\dagger \widetilde{\mathfrak{E}}(*g)$. Thus, we obtain Lemma 19.3.20. □

We have the natural isomorphism $\mathcal{M}_1(*g) \simeq \varphi^0_\dagger \widetilde{\mathfrak{E}}(*g)$, and hence $\mathcal{M}_1(*g) \simeq \mathcal{M}(*g)$. Since both $\mathcal{M}_1(*g)$ and $\mathcal{M}$ are strictly S-decomposable along g, we obtain that $\mathcal{M}_1$ and $\mathcal{M}$ are isomorphic due to Lemma 22.4.10. Thus, the proof of Proposition 19.3.1 is accomplished. □

19.4. Application to algebraic semisimple holonomic D-modules

19.4.1. Main theorems. — Let X be a smooth proper complex algebraic variety. Let Z be an irreducible closed subvariety of X. Let $\mathrm{Hol}_Z(X)$ denote the category of holonomic D-modules whose supports are contained in Z. (We consider the full subcategory of D-modules, i.e., morphisms are not necessarily isomorphisms.) Let $\mathrm{Hol}_{Z,ss}(X)$ denote the category of semisimple holonomic D-modules whose strict supports are exactly Z, and let $\mathrm{Hol}_{Z,s}(X)$ denote the category of simple holonomic D-modules whose strict supports are Z.

For an integer w, let $\mathrm{MT}^{\mathrm{wild}}_{Z,ss}(X,w,\sqrt{-1}\boldsymbol{R})^{(p)}$ denote the category of polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules of weight w, whose strict supports are exactly Z, i.e., the support of any non-zero direct summand is Z. (We consider the full subcategory of $\mathrm{MT}^{\mathrm{wild}}(X,w)$, i.e., morphisms are not necessarily isomorphisms.) Let $\mathrm{MT}^{\mathrm{wild}}_{Z,s}(X,w,\sqrt{-1}\boldsymbol{R})^{(p)}$ denote the category of simple polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules whose strict supports are Z.

We have the naturally defined functor $\Xi_{DR}:\mathrm{MT}^{\mathrm{wild}}_{Z,ss}(X,w,\sqrt{-1}\boldsymbol{R})^{(p)}\to\mathrm{Hol}_Z(X)$ by taking the specialization of the underlying $\mathcal{R}$-modules at $\lambda=1$. We will prove the following theorem in Sections 19.4.2–19.4.3.

Theorem 19.4.1. — *Fix an integer w. The functor Ξ_{DR} naturally induces an equivalence of the categories $\mathrm{MT}^{\mathrm{wild}}_{Z,ss}(X,w,\sqrt{-1}\boldsymbol{R})^{(p)}$ and $\mathrm{Hol}_{Z,ss}(X)$. It also induces an equivalence of the categories $\mathrm{MT}^{\mathrm{wild}}_{Z,s}(X,w,\sqrt{-1}\boldsymbol{R})^{(p)}$ and $\mathrm{Hol}_{Z,s}(X)$.*

Before going to the proof, we give the following consequence, which is one of the main motivations for this study.

***Theorem 19.4.2* (Kashiwara's conjecture).** — *Let X be a smooth complex algebraic variety, and let $\mathcal{F}$ be an algebraic semisimple holonomic D-module.*

- *Let $F:X\to Y$ be a projective morphism of smooth algebraic varieties. Let L denote the Lefschetz morphism for some line bundle which is relatively ample with respect to F. Then, $F^j_\dagger(\mathcal{F})$ are also semisimple for any j, and the induced morphisms $L^j:F^{-j}_\dagger\mathcal{F}\to F^j_\dagger\mathcal{F}$ $(j\geqslant 0)$ are isomorphic for $j\geqslant 0$. In particular, $F_\dagger(\mathcal{F})$ is isomorphic to $\bigoplus F^i_\dagger(\mathcal{F})[-i]$ in the derived category of cohomologically holonomic D_Y-modules.*
- *Let g be an algebraic function on X, and let $\mathfrak{a}\in\boldsymbol{C}[t_n^{-1}]$. Then, $\mathrm{Gr}^W\psi_{g,\mathfrak{a}}(\mathcal{F})$ and $\mathrm{Gr}^W\phi_g(\mathcal{F})$ are also semisimple, where (i) $\psi_{g,\mathfrak{a}}$ denotes the nearby cycle functor with ramified exponential twist by $\mathfrak{a}$ (see Section 22.6.3), (ii) ϕ_g denotes the vanishing cycle functor, (iii) Gr^W is taken with respect to the weight filtration of the naturally induced nilpotent maps.*

Proof. — We take a smooth proper variety $\overline{X}$ which contains X as a Zariski open subset such that $\overline{X}-X$ is a normal crossing hypersurface, by using Nagata's embedding and Hironaka's resolution. We take the minimal extension $\overline{\mathcal{F}}$ of $\mathcal{F}$ on $\overline{X}$.

By Theorem 19.4.1, we can take a polarizable $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module $\overline{\mathcal{T}}$ on $\overline{X}$ such that $\Xi_{DR}(\overline{\mathcal{T}}) \simeq \overline{\mathcal{F}}$. Let $\mathcal{T}$ denote the restriction of $\overline{\mathcal{T}}$ to X. We have $\Xi_{DR}(\mathcal{T}) = \mathcal{F}$. Because $F^j_\dagger(\mathcal{F}) \simeq \Xi_{DR}(F^j_\dagger(\mathcal{T}))$, the first claim follows from Theorem 18.1.1.

We also have

$$\operatorname{Gr}^W \psi_{g,\mathfrak{a}}(\mathcal{F}) \simeq \bigoplus_{\substack{u=(a,\alpha)\\ -1\leqslant a<0}} \Xi_{DR}\big(\operatorname{Gr}^W \widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T})\big), \quad \operatorname{Gr}^W \phi_g(\mathcal{F}) \simeq \Xi_{DR}\big(\operatorname{Gr}^W \phi_g(\mathcal{T})\big).$$

Hence, the second claim is also clear. □

19.4.1.1. For the proof of Theorem 19.4.1, we only have to show the following two claims:

(A) : $\Xi_{DR}(\mathcal{T})$ is simple for any $\mathcal{T} \in \operatorname{MT}^{\mathrm{wild}}_{Z,s}\big(X, w, \sqrt{-1}\boldsymbol{R}\big)^{(p)}$.

(B) : Ξ_{DR} induces an equivalence between the categories $\operatorname{MT}^{\mathrm{wild}}_{Z,s}\big(X, w, \sqrt{-1}\boldsymbol{R}\big)^{(p)}$ and $\operatorname{Hol}_{Z,s}(X)$.

19.4.2. Construction of wild pure twistor D-module. — Let M be a simple holonomic D-module whose strict support is Z. There exists a smooth Zariski open subset $U \subset Z$ such that $M_{|X-(Z\setminus U)}$ comes from a flat bundle on U. We can take a smooth projective variety $\widetilde{Z}$ and a birational projective morphism $\pi : \widetilde{Z} \to Z$ such that $\widetilde{M} := L^{-\dim X+\dim Z}\pi^* M \otimes \mathcal{O}(*\widetilde{D})$ is a meromorphic flat connection on $(\widetilde{Z}, \widetilde{D})$, where $\widetilde{D} := \widetilde{Z} - \pi^{-1}(U)$. (See Proposition 22.6.2 and Lemma 22.6.3.) According to Theorem 16.2.1, we may and will assume that it is a meromorphic flat bundle, and that the Deligne-Malgrange lattice associated to $\widetilde{M}$ is good.

We have the meromorphic flat bundle $V = \widetilde{M}^{(2)}$, obtained as the deformation of $\widetilde{M}$ by the procedure explained in Section 4.5.2 with $T = 2$. (Recall that we used the deformation in the construction of $\mathcal{QE}$.)

Lemma 19.4.3. — *V is also simple.*

Proof. — We only have to consider the case $\dim Z = 1$ due to Mehta-Ramanathan type theorem (Proposition 13.2.1). Let $V' \subset V$ be a flat subbundle such that $0 < \operatorname{rank}(V') < \operatorname{rank}(V)$. Let P be a point of $\widetilde{Z}$, where the connection of V has a pole. Let U_P be a small neighbourhood of P, and let $\varphi_P : \widetilde{U}_P \to U_P$ be a ramified covering such that both $\varphi_P^* V$ and $\varphi_P^* V'$ have an unramifiedly good Deligne-Malgrange lattice. Then, it can be easily checked that the irregular decompositions are compatible. Hence, we have the naturally defined morphism $V'^{(1/2)} \to V^{(1/2)} = \widetilde{M}$ with $0 < \operatorname{rank} V'^{(1/2)} < \operatorname{rank} \widetilde{M}$, which contradicts with the simplicity of $\widetilde{M}$. □

Due to Theorem 16.1.1, we can take a pluri-harmonic metric h of $V_{|\widetilde{Z}-\widetilde{D}}$, which is adapted to the Deligne-Malgrange filtered bundle $\boldsymbol{V}^{DM}_*$ associated to V. Let $(E, \overline{\partial}_E, h, \theta)$ be the corresponding $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on $\widetilde{Z} - \widetilde{D}$. We have $\mathcal{P}\mathcal{E}^1 = V$ and $\mathcal{Q}\mathcal{E}^1 = \widetilde{M}$. Let $(\mathfrak{T}(E), \mathfrak{S})$ be the associated $\sqrt{-1}\boldsymbol{R}$-wild polarized pure twistor D-module of weight 0 on $\widetilde{Z}$. Let $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ be the $\sqrt{-1}\boldsymbol{R}$-wild polarized pure twistor D-module of weight 0, which is a direct summand of $P\pi_\dagger^0\mathfrak{T}(E)$ whose strict support is Z. We have the induced polarization $\mathcal{S}$ of $\mathcal{T}$.

We would like to show $\mathcal{M}_{|\lambda=1} \simeq M$. We can take a divisor D of X such that $(Z \setminus U) \subset D$. We may assume $U = Z \setminus D$ from the beginning. We have the trace map $\mathrm{tr} : \pi_\dagger^0\widetilde{M} \to M(*D)$. The restriction to $X \setminus D$ is an isomorphism. Hence, the supports of $\mathrm{Ker}(\mathrm{tr})$ and $\mathrm{Cok}(\mathrm{tr})$ are contained in D. Both $\pi_\dagger^0\widetilde{M}$ and $M(*D)$ are algebraically localized on $X \setminus D$, and hence we can conclude that they are isomorphic on X. Thus, we obtain the isomorphism $\mathcal{M}_{|\lambda=1} \otimes \mathcal{O}(*D) \simeq M \otimes \mathcal{O}(*D)$. Since M is simple, it is strictly S-decomposable along D. We know that $\mathcal{M}_{|\lambda=1}$ is also strictly S-decomposable along D. Hence, we can conclude $\mathcal{M}_{|\lambda=1} \simeq M$ by the same argument as in the proof of Lemma 22.4.10.

19.4.3. End of the proof of Theorem 19.4.1. — Let $(\mathcal{T}, \mathcal{S})$ be a simple polarized $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module of weight 0 whose strict support is Z. Let $M := \Xi_{DR}(\mathcal{T})$. We have a Zariski open subset $U \subset Z$ such that $(\mathcal{T}, \mathcal{S}) \in \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, 0, \sqrt{-1}\boldsymbol{R})$. (See Subsection 19.1.2 for $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}$.) Let $(E, \overline{\partial}_E, \theta, h)$ be the corresponding harmonic bundle on U, which is $\sqrt{-1}\boldsymbol{R}$-wild on (Z, U). By shrinking U, we may assume to have a divisor D of X such that $Z \setminus U = Z \cap D$. We can take a smooth projective variety $\widetilde{Z}$ with a projective birational morphism $\varphi : \widetilde{Z} \to Z$ such that (i) $\widetilde{D} := \varphi^{-1}(Z \setminus U)$ is simply normal crossing, (ii) $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) := \varphi^{-1}(E, \overline{\partial}_E, \theta, h)$ is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on $(\widetilde{Z}, \widetilde{D})$. We have the associated polarized $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-module $(\mathfrak{T}(\widetilde{E}), \widetilde{\mathfrak{S}})$ on $\widetilde{Z}$. We know that $\mathcal{T}$ is isomorphic to the direct summand of $P\varphi_\dagger^0\mathfrak{T}(\widetilde{E})$ by the correspondence in Theorem 19.1.3.

Due to Theorem 16.2.4, $(\mathcal{P}\widetilde{\mathcal{E}}^1, \mathbb{D}^1)$ is semisimple. By using Proposition 13.6.1, we can take an orthogonal decomposition $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}, \widetilde{h}) = \bigoplus(\widetilde{E}_i, \overline{\partial}_{\widetilde{E}_i}, \widetilde{\theta}_i, \widetilde{h}_i)$ such that $(\mathcal{P}\mathcal{E}_i, \mathbb{D}^1_i)$ associated to $(\widetilde{E}_i, \overline{\partial}_{\widetilde{E}_i}, \widetilde{\theta}_i, \widetilde{h}_i)$ are simple. We have the decomposition $\mathfrak{T}(\widetilde{E}) = \bigoplus_i \mathfrak{T}(\widetilde{E}_i)$, and $P\varphi_\dagger^0\mathfrak{T}(\widetilde{E}) = \bigoplus P\varphi_\dagger^0\mathfrak{T}(\widetilde{E}_i)$, which induces a decomposition of $(\mathcal{T}, \mathcal{S})$. Since we have assumed $(\mathcal{T}, \mathcal{S})$ is simple, we obtain that $(\mathcal{P}\widetilde{\mathcal{E}}^1, \mathbb{D}^1)$ is also simple. (Note that the restriction of φ is an isomorphism on a generic part of Z.) Using the argument in the proof of Lemma 19.4.3, we obtain that $\mathcal{Q}\widetilde{\mathcal{E}}^1$ is simple. By the construction of M, we have the isomorphism $M \otimes \mathcal{O}(*D) \simeq \varphi_\dagger^0\mathcal{Q}\widetilde{\mathcal{E}}^1 \otimes \mathcal{O}(*D)$ induced by the trace morphism. Hence, the support of any non-trivial submodule of M is contained in D. Because M is strictly S-decomposable along any function g on X such that $Z \not\subset g^{-1}(0)$, we obtain that M is also simple. Thus, the claim (A) is proved.

By the construction in Section 19.4.2, we obtain the essential surjectivity in the claim of (B). Let us show the full faithfulness. Let $(\mathcal{T}_i, \mathcal{S}_i)$ $(i = 1, 2)$ be simple polarized $\sqrt{-1}\boldsymbol{R}$-wild pure twistor D-modules of weight 0. We put $M_i := \Xi_{DR}(\mathcal{T}_i)$.

Let $(E_i, \overline{\partial}_{E_i}, \theta_i, h_i)$ be $\sqrt{-1}\boldsymbol{R}$-wild harmonic bundles on (Z, U) underlying $(\mathcal{T}_i, \mathcal{S}_i)$. Let $\varphi : \widetilde{Z} \to Z$ be a birational projective morphism such that $(\widetilde{E}_i, \overline{\partial}_{\widetilde{E}_i}, \widetilde{\theta}_i, \widetilde{h}_i) := \varphi^*(E_i, \overline{\partial}_{E_i}, \theta_i, h_i)$ are $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on $(\widetilde{Z}, \widetilde{D})$, where $\widetilde{D}$ denotes some normal crossing divisor of $\widetilde{Z}$. Let $(\mathcal{Q}\widetilde{\mathcal{E}}^1_i, \widetilde{\mathbb{D}}^1_i)$ denote the meromorphic flat bundles associated to $(\widetilde{E}_i, \overline{\partial}_{\widetilde{E}_i}, \widetilde{\theta}_i, \widetilde{h}_i)$.

Let $f : M_1 \simeq M_2$ be a morphism. By Lemma 19.3.19, we obtain the induced isomorphism $(\mathcal{Q}\widetilde{\mathcal{E}}^1_1, \widetilde{\mathbb{D}}^1_1) \simeq (\mathcal{Q}\widetilde{\mathcal{E}}^1_2, \widetilde{\mathbb{D}}^1_2)$ as meromorphic flat bundles. It induces an isomorphism of the associated Deligne-Malgrange filtered flat bundles $\mathcal{Q}_*\widetilde{\mathcal{E}}^1_1 \simeq \mathcal{Q}_*\widetilde{\mathcal{E}}^1_2$. It induces $\mathcal{P}_*\widetilde{\mathcal{E}}^1_1 \simeq \mathcal{P}_*\widetilde{\mathcal{E}}^1_2$. Hence, we obtain $\widetilde{f} : (\widetilde{E}_1, \overline{\partial}_{\widetilde{E}_1}, \widetilde{\theta}_1, \widetilde{h}_1) \simeq (\widetilde{E}_2, \overline{\partial}_{\widetilde{E}_2}, \widetilde{\theta}_2, \widetilde{h}_2)$. (See Corollary 16.1.4 or Proposition 13.6.1.) Because of the correspondence in Theorem 19.1.3, we obtain an isomorphism $F : (\mathcal{T}_1, \mathcal{S}_1) \simeq (\mathcal{T}_2, \mathcal{S}_2)$. By construction, we have the coincidence of the restrictions of $\Xi_{DR}(F)$ and f to a Zariski open subset of Z. Because M_i are simple, we obtain $\Xi_{DR}(F) = f$, i.e., Ξ_{DR} is full.

Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism such that $\Xi_{DR}(F) = 0$. We obtain the vanishing of the induced morphism of variations of pure twistor structure on U. Then, we obtain $F = 0$. Hence, Ξ_{DR} is fully faithful. Thus, the proof of Theorem 19.4.1 is finished. □

PART V

APPENDIX

CHAPTER 20

PRELIMINARIES FROM ANALYSIS ON MULTI-SECTORS

In Section 20.1, we give a generalization of Hukuhara-Malmquist type theorem. Namely, we study a lifting of a formal solution of some non-linear differential equation to a solution on small open multi-sectors. It is one of the fundamental tools in the study of Stokes structures in Chapters 3–4. For the proof, we use a classical argument explained in [**94**].

In Section 20.2, we give some estimates of sections on a sector. Lemma 20.2.1 has been used for comparison of irregular decompositions (Sections 7.6 and 10.2). Lemmas 20.2.2 and 20.2.3 have been used in the study of L^2-cohomology associated to wild harmonic bundles on curves (Chapter 5).

In Section 20.3, we give some estimates of the growth order of solutions of some differential equations on multi-sectors. In Section 20.3.1, we give an estimate of the growth order of a section whose derivative rapidly decays. In Section 20.3.2, we give an estimate of the growth order of a flat section. We reformulate it in Sections 20.3.3–20.3.4. These results have been used implicitly in many places. For example, they have been used to give a characterization of Stokes filtrations.

20.1. Hukuhara-Malmquist type theorem

20.1.1. Statement. — We study a lifting of a formal solution of some non-linear differential equation to a solution on small open multi-sectors. See [**53**] and [**94**] for the history and the classical arguments in the one dimensional case. For the higher dimensional case, it was studied in [**53**] more generally but in a different manner. We give some statements and the outline of a proof in a way conveneint for our purposes. Although we have used it for analysis on a multi-sector around 0 (Chapters 3–4), it is more conventional and easier to consider it on a multi-sector around ∞. The translation can be done in a straightforward way.

We set $S^1 := \{w \in \boldsymbol{C} \,\big|\, |w| = 1\}$ for which we use a polar coordinate. Let $(\theta^{(0)}, \theta_1^{(0)}, \dots, \theta_n^{(0)})$ be a point of $(S^1)^{n+1}$. Let W be a compact region in $\boldsymbol{C}^M$, where M denotes some non-negative integer. Given

$$R \in \boldsymbol{R}_{>0}, \quad \boldsymbol{R}_y = (R_1, \dots, R_n) \in \boldsymbol{R}_{>0}^n, \quad \varepsilon \in \,]0, 2\pi[, \quad \boldsymbol{\varepsilon}_y = (\varepsilon_1, \dots, \varepsilon_n) \in \,]0, 2\pi[^n,$$

let $S(R, \varepsilon, \boldsymbol{R}_y, \boldsymbol{\varepsilon}_y)$ denote the multi-sector in $\boldsymbol{C}^* \times (\boldsymbol{C}^*)^n \times W$:

$$S(R, \varepsilon, \boldsymbol{R}_y, \boldsymbol{\varepsilon}_y) := \big\{x \in \boldsymbol{C} \,\big|\, |x| > R, |\arg(x) - \theta^{(0)}| < \varepsilon\big\} \times \prod_{i=1}^{n} \big\{y \in \boldsymbol{C} \,\big|\, |y_i| > R_i, |\arg(y_i) - \theta_i^{(0)}| < \varepsilon_i\big\} \times W.$$

We fix a multi-sector $S\big(R^{(0)}, \varepsilon^{(0)}, \boldsymbol{R}_y^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}\big)$ for some given $R^{(0)}$, $\varepsilon^{(0)}$, $\boldsymbol{R}^{(0)}$ and $\boldsymbol{\varepsilon}_y^{(0)}$. Let a be a non-negative integer such that $2\varepsilon^{(0)}(a+1) < \pi$, and let $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{>0}^n$. Let $\lambda_i(w)$ $(i = 1, \dots, d)$ be holomorphic functions on W satisfying the following condition:

- Let us consider the region
$$H(\varepsilon^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) := \big\{(\theta, \theta_1, \dots, \theta_n) \in (S^1)^{n+1} \,\big|\, |\theta - \theta^{(0)}| \leqslant \varepsilon^{(0)}, |\theta_i - \theta_i^{(0)}| \leqslant \varepsilon_i^{(0)}\big\}$$
and the functions
$$F_j(\theta, \theta_1, \dots, \theta_n, w) := \mathrm{Re}\Big(\lambda_j(w) \exp\big(\sqrt{-1}\big(\textstyle\sum b_i\theta_i + (a+1)\theta\big)\big)\Big)$$
for $j = 1, \dots, d$. Then, the following holds for each j:
  - If $\{F_j = 0\} \cap \big(H(\varepsilon^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) \times W\big)$ is not empty, it is connected and contained in $H(\overline{\varepsilon}_j^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) \times W$ for some $\overline{\varepsilon}_j^{(0)} < \varepsilon^{(0)}$.

Let us consider the following differential equation for $\boldsymbol{C}^d$-valued functions u on $S(R^{(0)}, \varepsilon^{(0)}, \boldsymbol{R}_y^{(0)}, \boldsymbol{\varepsilon}_y^{(0)})$:

$$x^{-a}\frac{\partial u}{\partial x} = \Lambda \cdot u + p\big(x, \boldsymbol{y}, w, u(x, \boldsymbol{y}, w)\big). \tag{386}$$

- Λ is a diagonal d-th square matrix whose (j,j)-th entries are of the form $\lambda_j(\boldsymbol{y}, w)\boldsymbol{y}^{\boldsymbol{b}}$.
  - $\boldsymbol{y}^{\boldsymbol{b}} := \prod_{i=1}^n y_i^{b_i}$.
  - $\lambda_j(\boldsymbol{y}, w)$ are holomorphic functions independent of the variable x, and satisfy
$$\big|\lambda_j(\boldsymbol{y}, w) - \lambda_j(w)\big| = O\Big(\sum_{i=1}^n |y_i|^{-1}\Big).$$
- $p(x, \boldsymbol{y}, w, u) = \big(p_j(x, \boldsymbol{y}, w, u) \,\big|\, j = 1, \dots, d\big)$ is a $\boldsymbol{C}^d$-valued holomorphic function on $S(R^{(0)}, \varepsilon^{(0)}, \boldsymbol{R}_y^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) \times \boldsymbol{C}^d$, and it is decomposed into
$$p^{(0)}(x, \boldsymbol{y}, w) + p^{(1)}(x, \boldsymbol{y}, w) \cdot u + p^{(2)}(x, \boldsymbol{y}, w, u)$$
with the following properties:
  - $p^{(0)}(x, \boldsymbol{y}, w) = O\big(|x|^{-m}|\boldsymbol{y}|^{-m}\big)$ for any m. Here, $|\boldsymbol{y}|$ denotes $\prod_{i=1}^n |y_i|$.

- $p^{(1)}(x, \boldsymbol{y}, w) \cdot u$ is linear with respect to $u = (u_1, \dots, u_d)$. We have $|p^{(1)}(x, \boldsymbol{y}, w)| \leqslant g(x)\,|\boldsymbol{y}^{\boldsymbol{b}}|$, where $g(x) \to 0$ in $|x| \to \infty$.
- $p^{(2)}(x, \boldsymbol{y}, w, u)$ is polynomial with respect to u, and the coefficients are $O\big(|x|^{-m}\,|\boldsymbol{y}|^{-m}\big)$ for any m. We also assume that it does not contain the constant and linear terms with respect to u.

Take a positive number $\varepsilon' < \varepsilon^{(0)}$. We assume $\varepsilon' > \overline{\varepsilon}_j^{(0)}$, if $\{F_j = 0\} \cap \big(H(\varepsilon^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) \times W\big)$ is not empty. We will show the following proposition in Sections 20.1.2–20.1.9.

Proposition 20.1.1. — *If R and $\boldsymbol{R}_y^{(0)}$ are sufficiently large, we have a solution u of (386) on $S(R, \varepsilon', \boldsymbol{R}_y^{(0)}, \boldsymbol{\varepsilon}_y^{(0)})$, such that $|u| = O\big(|x|^{-m}|\boldsymbol{y}|^{-m}\big)$ for any $m > 0$.*

We follow the argument in Chapter 14 of the standard text book [**94**]. We recommend the reader to read it to understand the idea. In the following argument, we may and will assume $\theta^{(0)} = \theta_i^{(0)} = 0$. We will also consider only the case $a = 0$, because the general case can easily be reduced to the case $a = 0$ by the change of the variables $\xi = x^{a+1}$. (See [**94**].)

20.1.1.1. Remark for simplification of the proof. — Assume the following:

$(*)$: $\{F_j = 0\} \cap \big(H(\varepsilon^{(0)}, \boldsymbol{\varepsilon}_y^{(0)}) \times W\big)$ is not empty for each $j = 1, \dots, d$

Then, the proof below can be simplified, i.e., the arguments in Sections 20.1.6–20.1.8 are not necessary: Since we only have to consider the case **(B)** in Section 20.1.2 under the assumption $(*)$, the integral equation $\mathcal{P}_{x_1}(u) = u$ is independent of the choice of x_1, which implies that $U(m_0')$ in Section 20.1.5 is independent of m_0. Hence, we obtain $U(m_0) = O\big(|x|^{-m}\,|\boldsymbol{y}|^{-N}\big)$ for any m in the end of Section 20.1.5. Actually, we have the uniqueness in Proposition 20.1.1 if the condition $(*)$ is satisfied.

We have applied Proposition 20.1.1 for lifting of a formal irregular decomposition to a decomposition on multi-sectors, for example, in Sections 3.5.1, and 3.6.1. For this application, we only have to consider the case where the condition $(*)$ is satisfied. Although we keep the proof for generality, the reader can skip Sections 20.1.6–20.1.8.

20.1.2. Integral transforms. — We put $T := \big\{z \,\big|\, |\arg(z)| \leqslant \varepsilon^{(0)}\big\}$. We take a positive number x_1. Let $x_1 + T$ denote the subset $\big\{x_1 + z \,\big|\, z \in T\big\}$ of $\boldsymbol{C}$. Note that the set $\big\{x \,\big|\, |\arg(x)| < \varepsilon',\ |x| > R\big\}$ is contained in $x_1 + T$, if R is sufficiently large. For each $x \in x_1 + T$, we take a tuple of paths $\Gamma(x, x_1) = \big(\gamma_1(x, x_1), \dots, \gamma_d(x, x_1)\big)$ contained in $x_1 + T$ as in Chapter 14.3 of [**94**]:

(A) : If $F_j < 0$ on $H(\varepsilon^{(0)}, \boldsymbol{\varepsilon}_y^{(0)})$, let $\gamma_j(x, x_1)$ be the segment connecting x_1 to x.

(B) : Otherwise, we can take some $\theta^{(1)}$ such that $F_j(\theta^{(1)}, \theta_1, \dots, \theta_n, w) > 0$ for any $|\theta_i| < \varepsilon_i^{(0)}$ and $w \in W$. Let $\gamma_j(x, x_1)$ denote the path connecting ∞ and x on the line $\big\{x + s\, \exp(\sqrt{-1}\theta^{(1)}) \,\big|\, s > 0\big\}$. In this case, $\gamma_j(x, x_1)$ is independent of x_1.

We put $S(x_1, \boldsymbol{R}_y) := (x_1 + T) \times \prod_{i=1}^{n} \{ y \in \boldsymbol{C} \,|\, |y_i| > R_i, |\arg(y_i)| < \varepsilon_i^{(0)} \} \times W$. For a $\boldsymbol{C}^d$-valued holomorphic function $v = (v_1, \dots, v_d)$ on $S(x_1, \boldsymbol{R}_y^{(0)})$, we consider the following integral, if it is convergent:

$$\int_{\Gamma(x,x_1)} \exp\big((x-t)\Lambda\big)\, v(t, \boldsymbol{y}, w)\, dt := \left(\int_{\gamma_j(x,x_1)} \exp\big((x-t)\, \lambda_j(\boldsymbol{y}, w)\, \boldsymbol{y}^{\boldsymbol{b}}\big)\, v(t, \boldsymbol{y}, w)\, dt \;\Big|\; j = 1, \dots, d \right).$$

Then, we consider the following integral transform:

$$\mathcal{P}_{x_1}(v)(x, \boldsymbol{y}, w) := \int_{\Gamma(x,x_1)} \exp\big((x-t)\Lambda\big)\, p\big(t, \boldsymbol{y}, w, v(t, \boldsymbol{y}, w)\big)\, dt.$$

We will construct the solution of the integral equation $\mathcal{P}_{x_1}(u) = u$ in Sections 20.1.3–20.1.4.

20.1.3. Preliminary estimate. — We put $\lambda_0 := \min_{j,\boldsymbol{w}} |\lambda_j(\boldsymbol{w})|$. If $R_i^{(0)}$ are sufficiently large, there is a positive constant μ_0 such that the following holds for any $(t, \boldsymbol{y}, w) \in \gamma_j(x, x_1) \times \prod_{i=1}^{n} \{ y \in \boldsymbol{C} \,|\, |y_i| > R_i^{(0)}, |\arg(y_i)| < \varepsilon_i^{(0)} \} \times W$. (See Lemma 14.1 in [**94**]):

$$\operatorname{Re}\Big((x-t)\, \lambda_j(\boldsymbol{w})\, \boldsymbol{y}^{\boldsymbol{b}} \Big) \leqslant -|x-t|\, \lambda_0\, \mu_0\, |\boldsymbol{y}^{\boldsymbol{b}}|. \tag{387}$$

Then, we can show the following lemma by using the argument in the proof of Lemma 14.2 in [**94**].

Lemma 20.1.2. — *There exists a constant $K_m > 0$, which is taken independently from large $\boldsymbol{R}_y$ and x_1, with the following property:*

- *Let $\chi(x, \boldsymbol{y}, w)$ be a $\boldsymbol{C}^d$-valued holomorphic function on $S(x_1, \boldsymbol{R}_y)$ such that*

$$\big|\chi(x, \boldsymbol{y}, w)\big| \leqslant c\, |x|^{-m} |\boldsymbol{y}|^{-N}$$

for some $c > 0$, $m > 0$ and $N > 0$. We put

$$\psi(x, \boldsymbol{y}, w) := \int_{\Gamma(x,x_1)} \exp\big((x-t)\, \Lambda\big)\, \chi(t, \boldsymbol{y}, w)\, dt.$$

Then, the following inequality holds:

$$\big|\psi(x, \boldsymbol{y}, w)\big| \leqslant \left(\frac{K_m}{|\boldsymbol{y}^{\boldsymbol{b}}|} \right) c\, |x|^{-m} |\boldsymbol{y}|^{-N}.$$

We may assume $K_m \leqslant K_{m+1}$ holds. For each m, we take a small $\gamma_m > 0$ such that $\gamma_m K_m < 1/2$ and $\gamma_m \geqslant \gamma_{m+1}$. We can take $x_1^{(1)}(m)$ such that the following holds for any $(x, \boldsymbol{y}, w) \in S\big(x_1^{(1)}(m), \boldsymbol{R}_y^{(0)}\big)$:

$$\big|p_1(x, \boldsymbol{y}, w)\big| \leqslant \frac{1}{2} \gamma_m\, |\boldsymbol{y}^{\boldsymbol{b}}|.$$

Take a constant C_0 which is independent of m and N. We can take $x_1^{(2)}(m) > x_1^{(1)}(m)$ such that the following holds for any $|u^{(i)}| \leqslant C_0$ $(i=1,2)$ and $(x,\boldsymbol{y},w) \in S\big(x_1^{(2)}(m), \boldsymbol{R}_y^{(0)}\big)$:

$$\big|p_2(x,\boldsymbol{y},w,u^{(1)}) - p_2(x,\boldsymbol{y},w,u^{(2)})\big| \leqslant \frac{1}{2}\gamma_m \big|u^{(2)} - u^{(1)}\big|.$$

Thus, we obtain the following for any $|u^{(i)}| \leqslant C_0$ $(i=1,2)$ and $(x,\boldsymbol{y},w) \in S(x_1^{(2)}(m), \boldsymbol{R}_y^{(0)})$:

$$\big|p(x,\boldsymbol{y},w,u^{(1)}) - p(x,\boldsymbol{y},w,u^{(2)})\big| \leqslant \gamma_m \big|u^{(2)} - u^{(1)}\big|.$$

20.1.4. Construction of limit for each (m,N). — For each $m>0$ and $N>0$, we can take $C_{m,N} > 0$ such that $\big|p_0(x,\boldsymbol{y},w)\big| < C_{m,N}\,|x|^{-m}\,|\boldsymbol{y}|^{-N}$. We can take $x_1^{(3)}(m,N) > x_1^{(2)}(m)$ such that the following inequality holds on $S\big(x_1^{(3)}(m,N), \boldsymbol{R}_y^{(0)}\big)$:

$$\frac{1}{1-\gamma_m K_m}\frac{K_m}{|\boldsymbol{y}^{\boldsymbol{b}}|}\,C_{m,N}\,|x|^{-m}\,|\boldsymbol{y}|^{-N} \leqslant C_0, \qquad C_{m,N}\,|x|^{-m}\,|\boldsymbol{y}|^{-N} \leqslant C_0. \tag{388}$$

We put $v_0 := 0$ and $v_i := \mathcal{P}_{x_1^{(3)}(m,N)}(v_{i-1})$ for $i \geqslant 1$, inductively.

Lemma 20.1.3. — *The limit $v_\infty(m,N) := \lim_{i\to\infty} v_i$ exists on $S\big(x^{(3)}(m,N), \boldsymbol{R}_y^{(0)}\big)$, and satisfies the following estimate on $S\big(x^{(3)}(m,N), \boldsymbol{R}_y^{(0)}\big)$:*

$$\big|v_\infty(m,N)\big| \leqslant \frac{1}{1-\gamma_m K_m} C_{m,N}\frac{K_m}{|\boldsymbol{y}^{\boldsymbol{b}}|}\,|x|^{-m}\,|\boldsymbol{y}|^{-N}.$$

Proof. — Using an inductive argument, we can show

$$|v_{i+1} - v_i| \leqslant \big(\gamma_m K_m\big)^i\, C_{m,N}\,\frac{K_m}{|\boldsymbol{y}^{\boldsymbol{b}}|}\,|x|^{-m}\,|\boldsymbol{y}|^{-N},$$

$$|v_{i+1}| \leqslant \frac{1}{1-\gamma_m K_m}\, C_{m,N}\,\frac{K_m}{|\boldsymbol{y}^{\boldsymbol{b}}|}\,|x|^{-m}\,|\boldsymbol{y}|^{-N}.$$

Then, the claim follows. □

20.1.5. Estimate for any N and some fixed m_0. — We fix m_0 and N_0, and we put $U(m_0) := v_\infty(m_0, N_0)$ which is given on $S\big(x^{(3)}(m_0,N_0), \boldsymbol{R}_y^{(0)}\big)$.

Lemma 20.1.4. — $\big|U(m_0)\big| \leqslant C_N'\,|x|^{-m_0}\,|\boldsymbol{y}|^{-N}$ *on* $S\big(x^{(3)}(m_0,N_0), \boldsymbol{R}_y^{(0)}\big)$ *for any* N.

Proof. — Let $C_{m_0,N}$ be a constant as in Section 20.1.4. For each N, we take a large $R_i^{(3)}(m_0,N)$ $(i=1,\dots,n)$ such that the following holds on the region $S(m_0,N) := \bigcup_{i=1}^n S\big(x_1^{(3)}(m_0,N_0), \boldsymbol{R}_{y,i}^{(3)}(m_0,N)\big)$:

$$\frac{1}{1-\gamma_{m_0}K_{m_0}} C_{m_0,N}\,\frac{K_{m_0}}{|\boldsymbol{y}^{\boldsymbol{b}}|}\,|x|^{-m_0}\,|\boldsymbol{y}|^{-N} \leqslant C_0, \quad C_{m_0,N}\,|x|^{-m_0}\,|\boldsymbol{y}|^{-N} \leqslant C_0.$$

Here, $\boldsymbol{R}_{y,i}^{(3)}(m_0,N)$ denotes the tuple whose j-th entry is $R_j^{(0)}$ for $(j \neq i)$ and whose i-th entry is $R_i^{(3)}(m_0,N)$.

We put $v_0 = 0$ and $v_i := \mathcal{P}_{x_1^{(3)}(m_0,N_0)}(v_{i-1})$ for $i \geqslant 1$, inductively. Using the same argument as that in the proof of Lemma 20.1.3, we can show the existence of the limit $U(m_0)^{(N)} := \lim_{i\to\infty} v_i$ which satisfies

$$\left|U(m_0)^{(N)}\right| \leqslant \frac{1}{1-\gamma_{m_0}K_{m_0}} C_{m_0,N} \frac{K_{m_0}}{|\boldsymbol{y}^{\boldsymbol{b}}|} |x|^{-m_0} |\boldsymbol{y}|^{-N}.$$

By construction, the restriction of $U(m_0)$ to $S\big(x_1^{(3)}(m_0,N_0), \boldsymbol{R}_{y,i}^{(3)}(m_0,N)\big)$ is equal to $U(m_0)^{(N)}$. Hence, we obtain $\left|U(m_0)\right| \leqslant C''_{m_0,N}|x|^{-m_0}|\boldsymbol{y}|^{-N}$ on $S(m_0,N)$. Thus, we are done. □

To obtain the estimate $\left|U(m_0)\right| \leqslant C_{1,m,N}|x|^{-m}|\boldsymbol{y}|^{-N}$ for any m and N, we would like to compare the two solutions $U(m_0)$ and $U(m)$ of the differential equation (386) in the case $a = 0$. Note that we do not have the uniqueness of the solutions of (386) in general, and that the integral equation $\mathcal{P}_{x_1}(u) = u$ depends on the choice of x_1. (As remarked in the end of Subsection 20.1.1, if the condition (∗) is satisfied, the proof is completed in this stage.) We give some preparations in Sections 20.1.6–20.1.8 to estimate the ambiguity of the solutions of (386), and we will finish the proof of Proposition 20.1.1 in Section 20.1.9.

In the following argument, we may and will assume $x_1^{(i)}(m) \leqslant x_1^{(i)}(m+1)$ for $i = 1, 2, 3$.

20.1.6. Estimate of a solution of the linear differential equation. — Take any $x_2(m) > x_1^{(1)}(m)$. Let us consider the following linear differential equation for $\boldsymbol{C}^d$-valued holomorphic functions u on $S(x_2(m), \boldsymbol{R}_y^{(0)})$:

$$\frac{\partial u}{\partial x} = \Lambda \cdot u + p_1(x, \boldsymbol{y}, w)\cdot u. \tag{389}$$

We also consider the following integral transform:

$$\mathcal{R}_{x_2(m)}u = \int_{\Gamma(x,x_2(m))} \exp\Big((x-t)\Lambda\Big)\, p_1(t,\boldsymbol{y},w)\cdot u(t,\boldsymbol{y},w)\,dt.$$

The following lemma is easy to show.

Lemma 20.1.5. — *Let $m_1 \leqslant m$. If $|u| \leqslant C\,|x|^{-m_1}|\boldsymbol{y}|^{-N}$, we have*

$$\left|\mathcal{R}_{x_2(m)}(u)\right| \leqslant \frac{1}{2}(K_{m_1}\gamma_{m_1})\,C\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}.$$

Hence, if u satisfies $|u| \leqslant C\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}$ and $\mathcal{R}_{x_2(m)}(u) = u$, we obtain $u = 0$.

Proof. — Recall that we have assumed $x_1^{(1)}(m_1) \leqslant x_1^{(1)}(m)$. The first claim immediately follows from our choice of K_{m_1} and γ_{m_1}. If u satisfies $|u| \leqslant C\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}$ and $\mathcal{R}_{x_2(m)}(u) = u$, we obtain

$$|u| \leqslant C(K_{m_1}\gamma_{m_1}/2)^j\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}$$

for any j using an inductive argument. Hence, we obtain $u = 0$. □

Let u be a solution of the equation (389) on $S\big(x_2(m), \boldsymbol{R}_y^{(0)}\big)$ satisfying $|u| \leqslant C_N\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}$ for some m_1 and N.

Lemma 20.1.6. — *We have the estimate* $|u| \leqslant C_N'\,|x|^{-m}\,|\boldsymbol{y}|^{-N}$.

Proof. — We have nothing to prove in the case $m_1 \geqslant m$, and so we assume $m_1 < m$. We have the estimate $\big|\mathcal{R}_{x_2(m)}u\big| \leqslant \gamma_{m_1}\,K_{m_1}\,C_N\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}$. It is easy to show the following equality:

$$\frac{\partial}{\partial x}\big(\mathcal{R}_{x_2(m)}(u)-u\big)=\Lambda\big(\mathcal{R}_{x_2(m)}(u)-u\big).$$

Then, we have the estimate $\mathcal{R}_{x_2(m)}u-u=O\big(\exp(-\eta\,|x|\,|\boldsymbol{y}^{\boldsymbol{b}}|)\big)$ for some $\eta>0$. In particular, we have

$$\big|\mathcal{R}_{x_2(m)}u-u\big| \leqslant C_{10}\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}.$$

By an easy induction, we can show

$$\big|\mathcal{R}^{i+1}_{x_2(m)}(u)-\mathcal{R}^{i}_{x_2(m)}(u)\big| \leqslant (\gamma_{m_1}\,K_{m_1})^i\,C_{10}\,|x|^{-m_1}\,|\boldsymbol{y}|^{-N}.$$

Hence, we have the limit $\mathcal{R}^{\infty}_{x_2(m)}(u):=\lim_{i\to\infty}\mathcal{R}^{i}_{x_2(m)}(u)$ which satisfies

$$\mathcal{R}_{x_2(m)}\big(\mathcal{R}^{\infty}_{x_2(m)}(u)\big)=\mathcal{R}^{\infty}_{x_2(m)}(u),$$

$$\big|\mathcal{R}^{\infty}_{x_2(m)}(u)\big| \leqslant \Big(C_N+\frac{C_{10}}{1-\gamma_{m_1}K_{m_1}}\Big)|x|^{-m_1}\,|\boldsymbol{y}|^{-N}.$$

Hence, we can conclude $\mathcal{R}^{\infty}_{x_2(m)}(u)=0$ because of Lemma 20.1.5.

Because $\mathcal{R}_{x_2(m)}u-u=O\big(\exp(-\eta\,|x|\,|\boldsymbol{y}^{\boldsymbol{b}}|)\big)$, we also have

$$\big|\mathcal{R}_{x_2(m)}u-u\big| \leqslant C_{11}\,|x|^{-m}\,|\boldsymbol{y}|^{-N}.$$

We can show $\big|\mathcal{R}^{i+1}_{x_2(m)}(u)-\mathcal{R}^{i}_{x_2(m)}(u)\big| \leqslant (\gamma_m\,K_m)^i\,C_{11}\,|x|^{-m}\,|\boldsymbol{y}|^{-N}$ by an easy induction. Hence, we have

$$|u|=\big|u-\mathcal{R}^{\infty}_{x_2(m)}(u)\big| \leqslant \frac{C_{11}}{1-\gamma_m K_m}|x|^{-m}|\boldsymbol{y}|^{-N}.$$

Thus, the proof of Lemma 20.1.6 is finished. □

20.1.7. Existence of a small solution of some equation. — Let us consider the following differential equation on $S\big(x_2(m), \boldsymbol{R}_y^{(0)}\big)$:

$$\frac{\partial u}{\partial x}=\Lambda\cdot u+p_1(x,\boldsymbol{y},w)\cdot u+q(x,\boldsymbol{y},w). \tag{390}$$

Here, $q(x,\boldsymbol{y},w)$ satisfy $\big|q(x,\boldsymbol{y},w)\big| \leqslant C\,|x|^{-m}\,|\boldsymbol{y}|^{-N}$.

Lemma 20.1.7. — *We have a solution v of the equation* (390) *satisfying* $|v| \leqslant C'|x|^{-m}\,|\boldsymbol{y}|^{-N}$.

Proof. — We consider the following integral transform:

$$\mathcal{Q}_{x_2(m)}(u) = \int_{\Gamma(x,x_2(m))} \exp\big((x-t)\,\Lambda\big)\Big(p_1(t,\boldsymbol{y},w)\cdot u(t,\boldsymbol{y},w) + q(t,\boldsymbol{y},w)\Big)\,dt.$$

If $|u_i| \leqslant C\,|x|^{-m}|\boldsymbol{y}|^{-N}$ $(i=1,2)$, we have

$$\big|\mathcal{Q}_{x_2(m)}(u_1) - \mathcal{Q}_{x_2(m)}(u_2)\big| = \big|\mathcal{R}_{x_2(m)}(u_1-u_2)\big| \leqslant \gamma_m\,K_m\,C\,|x|^{-m}\,|\boldsymbol{y}|^{-N}.$$

We put $v_0 := 0$ and $v_i = \mathcal{Q}_{x_2(m)}(v_{i-1})$ for $i \geqslant 1$ inductively. Then, we have $|v_{i+1} - v_i| \leqslant (\gamma_m K_m)^i\,C\,|x|^{-m}|\boldsymbol{y}|^{-N}$. Hence, we have the limit $v_\infty := \lim_{i\to\infty} v_i$ which satisfies

$$\mathcal{Q}_{x_2(m)}v_\infty = v_\infty, \quad |v_\infty| \leqslant \frac{1}{1-\gamma_m K_m} C\,|x|^{-m}\,|\boldsymbol{y}|^{-N}.$$

Thus, the proof of Lemma 20.1.7 is finished. □

20.1.8. Estimate of the difference of two solutions of the equation (386)

Let us consider the equation (386) in the case $a = 0$.

Lemma 20.1.8. — *Let u_i $(i=1,2)$ be solutions of (386) on $S\big(x_2(m), \boldsymbol{R}_y^{(0)}\big)$, which satisfy $|u_i| \leqslant C_N\,|x|^{-m_0}\,|\boldsymbol{y}|^{-N}$ for any N and some $m_0 > 0$. Then, we have $|u_1 - u_2| \leqslant C'_N\,|x|^{-m}|\boldsymbol{y}|^{-N}$ for any N.*

Proof. — We may and will assume $m_0 < m$. We put $V := u_1 - u_2$. We have the following equality:

$$\frac{\partial V}{\partial x} = \Lambda\cdot V + p_1(x,\boldsymbol{y},w)\cdot V + q(x,\boldsymbol{y},w).$$

Here, $q(x,\boldsymbol{y},w) := p_2\big(x,\boldsymbol{y},w,u_1(x,\boldsymbol{y},w)\big) - p_2\big(x,\boldsymbol{y},w,u_2(x,\boldsymbol{y},w)\big)$. By assumption, we have $\big|q(x,\boldsymbol{y},w)\big| \leqslant B_N\,|x|^{-m}\,|\boldsymbol{y}|^{-N}$ for any N. Due to Lemma 20.1.7, we can take $v(m)$ on $S(x_2(m), \boldsymbol{R}_y^{(0)})$ satisfying

$$\frac{\partial v(m)}{\partial x} = \Lambda\cdot v(m) + p_1(x,\boldsymbol{y},w)\cdot v(m) + q(x,\boldsymbol{y},w), \quad \big|v(m)\big| \leqslant C_1\,|x|^{-m}|\boldsymbol{y}|^{-N}.$$

We have $\big|V - v(m)\big| \leqslant C_{1,N}\,|x|^{-m_0}|\boldsymbol{y}|^{-N}$ and the following equality:

$$\frac{\partial(V - v(m))}{\partial x} = \Lambda\big(V - v(m)\big) + p_1(x,\boldsymbol{y},w)\big(V - v(m)\big).$$

Due to Lemma 20.1.6, we obtain $\big|V - v(m)\big| \leqslant C_{2,N}|x|^{-m}\,|\boldsymbol{y}|^{-N}$. Hence, we obtain $\big|V\big| \leqslant C_{3,N}|x|^{-m}\,|\boldsymbol{y}|^{-N}$. Thus, we are done. □

20.1.9. End of the proof of Proposition 20.1.1. — Let us show $|U(m_0)| \leqslant C_{m,N}|x|^{-m}|\boldsymbol{y}|^{-N}$ for any m and N, where $U(m_0)$ was constructed in Section 20.1.5. Take $m > m_0$, and let us compare $U(m_0)$ and $U(m)$ on $S\big(x_1^{(3)}(m, N_0), \boldsymbol{R}_y^{(0)}\big)$. Both of them satisfy the differential equation (386), and both of them are dominated by $C_N\,|x|^{-m_0}\,|\boldsymbol{y}|^{-N}$ for any N. Because of Lemma 20.1.8, we obtain the estimate $\big|U(m_0) - U(m)\big| \leqslant A_{m,N}|x|^{-m}\,|\boldsymbol{y}|^{-N}$ for any N. Thus, we obtain the desired estimate $|U(m_0)| \leqslant A'_{m,N}|x|^{-m}\,|\boldsymbol{y}|^{-N}$, and the proof of Proposition 20.1.1 is finished. □

20.2. Estimates of some integrals on a sector

20.2.1. Exponential decay

Let $S_\infty(r_0, \theta_0, \theta_1)$ denote a sector $\big\{x = r\,e^{\sqrt{-1}\theta}\,\big|\,\theta_0 < \theta < \theta_1, r_0 < r\big\}$ around ∞ for $\theta_0, \theta_1 \in \boldsymbol{R}$ and $r_0 > 0$. Fix any $\theta^{(0)}$ such that $\theta_0 < \theta^{(0)} < \theta_1$. In the following argument, r_0' and θ_i' $(i = 0, 1)$ denote the numbers such that (i) $r_0' > r_0$ is appropriately large, (ii) $\theta_0 < \theta_0' < \theta^{(0)} < \theta_1' < \theta_1$, (iii) $|\theta_i' - \theta^{(0)}|$ are sufficiently small. Let Y be a complex manifold. Let P be any point of Y. In the following argument, U_P denotes a small compact neighbourhood of P.

Let (V, ∇) be a flat bundle on $\mathcal{Y} := S_\infty(r_0, \theta_0, \theta_1) \times Y$ relative to Y (i.e., we consider derivations only along $S_\infty(r_0, \theta_0, \theta_1)$-direction). Assume we are given a frame $\boldsymbol{v}$ such that $\nabla \boldsymbol{v} = \boldsymbol{v}\,(d\mathfrak{a} + A\,dx/x)$, where A is a constant matrix, and $\mathfrak{a}$ is a polynomial $\sum_{j=1}^k \mathfrak{a}_j\,x^j$ whose coefficients $\mathfrak{a}_j$ are holomorphic functions on Y. We assume $\mathfrak{a}_k$ is nowhere vanishing. Let $\omega = \sum \omega_i\,v_i$ be a ∇-closed section of $V \otimes \Omega^1_{\mathcal{Y}/Y}$ on $\mathcal{Y}$ such that $\big|\omega_i\big| = O\big(\exp(-\varepsilon|x|^L)\big)$ for some $\varepsilon > 0$, where we use the Euclidean metric $dr\,dr + r^2\,d\theta\,d\theta$ on $S_\infty(r_0, \theta_0, \theta_1)$.

Lemma 20.2.1. — *On some small $S_\infty(r_0', \theta_0', \theta_1') \times U_P$, there exists a section $\tau = \sum \tau_i\,v_i$ of V satisfying $\nabla\tau = \omega$ and $O(|\tau_i|) = O\big(\exp(-2^{-1}\varepsilon|x|^L)\big)$.*

Proof. — We may replace $\boldsymbol{v}$ with $\boldsymbol{v}\exp(-A\log x)$, and so we may assume $\operatorname{rank}(V) = 1$ and $A = 0$. Formally, we put

$$\tau(r, \theta, Q) := \exp(\mathfrak{a}) \int_{\gamma(r,\theta)} \exp(-\mathfrak{a})\,\omega \tag{391}$$

for some path $\gamma(r, \theta)$ connecting (r, θ, Q) and (r_2, θ_2, Q), where (r_2, θ_2) is a base point of $S_\infty(r_0, \theta_0, \theta_1) \cup \{\infty\}$. Then, it satisfies $\nabla\tau = \omega$, if the integral (391) converges. The problem is how to choose the path $\gamma(r, \theta)$ so that the desired estimate holds.

In the case $L > k$, we choose the path $\gamma(r, \theta)$ connecting (r, θ) and ∞ on the ray $\big\{(s, \theta)\,\big|\, s \geqslant r\big\}$. Then, τ satisfies the desired estimate. So we only have to consider the case $L \leqslant k$.

We have $\mathrm{Re}(\mathfrak{a}) = r^k\,\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\big)+O\big(r^{k-1}\big)$. We also have the following equality:

$$\frac{\partial}{\partial r}\Big(r^{-k+1}\exp\big(-\mathrm{Re}(\mathfrak{a})-\varepsilon\,r^L\big)\Big)=(-k+1)\,r^{-k}\exp\big(-\mathrm{Re}(\mathfrak{a})-\varepsilon\,r^L\big) - \Big(r^{-k+1}\frac{\partial\,\mathrm{Re}(\mathfrak{a})}{\partial r}+\varepsilon\,L\,r^{-k+L}\Big)\exp\big(-\mathrm{Re}(\mathfrak{a})-\varepsilon\,r^L\big). \tag{392}$$

Note we have $r^{-k+1}\partial_r\,\mathrm{Re}(\mathfrak{a}) = k\,\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\big)+O(r^{-1})$. In the case $k=L$, we will have to be concerned with the zero set of the function $r^{-k+1}\partial_r\,\mathrm{Re}(\mathfrak{a})+\varepsilon\,k = k\big(\mathrm{Re}(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta})+\varepsilon\big)+O(r^{-1})$ to use (392) for the estimate of the integral of $\exp\big(-\mathrm{Re}(\mathfrak{a})-\varepsilon\,r^k\big)$.

In the following, $S_\infty(r_0',\theta_0',\theta_1')$ will be denoted by S, for simplicity of the description. If $S\times U_P$ is sufficiently small, we may assume that one of the following holds:

(A) : $\partial_\theta\,\mathrm{Re}(\mathfrak{a})\neq 0$ on $S\times U_P$.

(B) : $\partial_r\,\mathrm{Re}(\mathfrak{a})\neq 0$ on $S\times U_P$.

Let us consider the case (A). We may assume that $-\mathrm{Re}(\mathfrak{a})$ is increasing with respect to θ. We may also assume that one of the following holds for sufficiently small U_P, if we choose θ_0' appropriately:

(A1) : $L=k$ and $-\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta_0'}\big)-\varepsilon<0$ on U_P.

(A2) : $L=k$ and $-\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta_0'}\big)-\varepsilon>0$ on U_P.

(A3) : $L<k$ and $-\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta_0'}\big)<0$ on U_P.

(A4) : $L<k$ and $-\mathrm{Re}\big(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta_0'}\big)>0$ on U_P.

In the case (A1), let γ_1 be the path connecting ∞ and (r,θ_0') on the ray $\big\{(s,\theta_0')\,\big|\,s\geqslant r\big\}$, and let γ_2 be the path connecting (r,θ_0') and (r,θ) on the arc $\big\{re^{\sqrt{-1}\varphi}\,\big|\,\theta_0'\leqslant\varphi\leqslant\theta\big\}$. The contribution of γ_2 to the integral (391) is dominated by $\exp(-\varepsilon\,r^L)$. If $S\times U_P$ is sufficiently small, the contribution of γ_1 is dominated by the following for some $C>0$:

$$\exp\big(\mathrm{Re}(\mathfrak{a})_{\theta_0',r,Q}\big)\int_r^\infty \exp\big(-\mathrm{Re}(\mathfrak{a})_{\theta_0',s,Q}-\varepsilon\,s^k\big)\,ds\leqslant C\,\exp\big(-\varepsilon\,r^k\big).$$

Here, we have used (392). Thus we are done in the case (A1).

In the case (A2), let γ_1 be the path connecting (r_0',θ_0') and (r_0',θ) on the arc $\{r_0'\,e^{\sqrt{-1}\varphi}\,|\,\theta_0'\leqslant\varphi\leqslant\theta\}$, and let γ_2 be the path connecting (r_0',θ) and (r,θ) on the ray $\big\{s\,e^{\sqrt{-1}\theta}\,\big|\,r_0\leqslant s\leqslant r\big\}$. We have $\mathrm{Re}(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta})<-\varepsilon$ on $S\times U_P$. Hence, if $S\times U_P$ is sufficiently small, we may have $\mathrm{Re}(\mathfrak{a})<-(2/3)\,\varepsilon\,r^k$ on $S\times U_P$. Then, the contribution of γ_1 is dominated by $\exp\big(-(2/3)\,\varepsilon\,r^k\big)$. If r_0' is sufficiently large, the contribution of γ_2 can be estimated as follows for some $C_1>0$, by using (392):

$$\exp\big(\mathrm{Re}(\mathfrak{a})_{\theta,r,Q}\big)\int_{r_0'}^r\exp\big(-\mathrm{Re}(\mathfrak{a})_{\theta,s,Q}-\varepsilon\,s^k\big)\,ds\leqslant C_1\Big(\exp\big(\mathrm{Re}(\mathfrak{a})_{\theta,r,Q}\big)+\exp\big(-\varepsilon\,r^k\big)\Big).$$

Thus we are done in the case (A2).

In the case (A3), we take the paths γ_j $(j=1,2)$ as in the case (A1). The contribution of γ_2 is dominated by $\exp\bigl(-\varepsilon\,|x|^L\bigr)$. We have the following estimate, by using (392):

$$\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta_0',r,Q}\bigr)\int_r^\infty \exp\bigl(-\operatorname{Re}(\mathfrak{a})_{\theta_0',s,Q}-\varepsilon\,s^L\bigr)\,ds\sim r^{-k+1}\exp(-\varepsilon\,r^L).$$

Thus, we are done in the case (A3).

In the case (A4), we take the paths γ_j $(j=1,2)$ as in the case (A2). The contribution of γ_1 is dominated by $\exp(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q})$. If r_0' is sufficiently large, we have $\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\leqslant\operatorname{Re}(\mathfrak{a})_{\theta_0',r,Q}<2^{-1}\operatorname{Re}(\mathfrak{a}_k(Q)e^{\sqrt{-1}\theta_0'})\,r^k\leqslant-\varepsilon\,r^L$ for any $r>r_0'$. Hence, the contribution of γ_1 is dominated as desired. By using (392), we obtain the following estimate:

$$\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)\int_{r_0'}^r \exp\bigl(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}-\varepsilon\,s^L\bigr)\,ds \\ \leqslant C_1\,r^{-k+1}\exp(-\varepsilon\,r^L)+C_2\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr).$$

Then, we can obtain the desired estimate for the contribution of γ_2.

Let us consider the case (B). If $S\times U_P$ is sufficiently small, we may assume that one of the following holds:

(B1) : $L=k$ and $-\operatorname{Re}\bigl(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\bigr)-\varepsilon<-\varepsilon/10$ on $S\times U_P$.

(B2) : $L=k$ and $-\operatorname{Re}\bigl(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\bigr)-2\varepsilon/3>\varepsilon/10$ on $S\times U_P$.

(B3) : $L<k$ and $-\operatorname{Re}\bigl(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\bigr)<0$ on $S\times U_P$.

(B4) : $L<k$ and $-\operatorname{Re}\bigl(\mathfrak{a}_k\,e^{\sqrt{-1}k\theta}\bigr)>0$ on $S\times U_P$.

In the cases (B1) and (B3), let γ be the path connecting ∞ and (r,θ) on the ray $\bigl\{s\,e^{\sqrt{-1}\theta}\bigm| s\geqslant r\bigr\}$. In the case (B1), we obtain the following estimate, by using (392):

$$\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)\int_r^\infty \exp\bigl(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}-\varepsilon\,s^k\bigr)\leqslant C_1\exp(-\varepsilon\,r^k).$$

The case (B3) can be estimated similarly and easily.

In the cases (B2) and (B4), let us take the paths γ_j $(j=1,2)$ as in the case (A2). In the case (B2), the contribution of γ_1 is dominated by $\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)$. If r_0' is sufficiently large, $\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)$ is dominated by $\exp\bigl(-2^{-1}\varepsilon\,r^k\bigr)$. The contribution of γ_2 can be dominated as follows, by using (392):

$$\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)\int_{r_0'}^r \exp\bigl(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}-\varepsilon\,s^k\bigr)\,ds \\ \leqslant \exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)\int_{r_0'}^r \exp\bigl(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}-(2/3)\,\varepsilon\,s^k\bigr)\,ds \\ \leqslant C_1\Bigl(\exp\bigl(-(2/3)\,\varepsilon\,r^k\bigr)+\exp\bigl(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\bigr)\Bigr). \tag{393}$$

Note that the integral $\int_{r_0'}^{r} \exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q} - (2/3)\,\varepsilon\, s^k\big)\, ds$ is estimated more easily than $\int_{r_0'}^{r} \exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q} - \varepsilon\, s^k\big)\, ds$, because $r^{-k+1}\partial_r \operatorname{Re}(\mathfrak{a}) + 2\varepsilon\, k/3 < -\eta < 0$ for some $\eta > 0$ on $S \times U_P$. The case (B4) can be estimated as in the case (A4). Thus the proof of Lemma 20.2.1 is finished. □

20.2.2. Polynomial order. — We continue to use the setting in Section 20.2.1. Let $\omega = \sum \omega_i\, v_i$ be a ∇-closed C^∞-section of $V \otimes \Omega^1$ on $\mathcal{Y}$ such that $|\omega_i| = O\big(|x|^N\big)$ for some N.

Lemma 20.2.2. — *There exists a C^∞-section $\tau = \sum \tau_i\, v_i$ of V on $S = S_\infty(r_0', \theta_0', \theta_1')$ such that (i) $\nabla\tau = \omega$, (ii) $|\tau_i| = O\big(|x|^{N+\ell}\big)$, where $\ell \geqslant 0$ is independent of ω.*

Proof. — As in the proof of Lemma 20.2.1 we may assume $\operatorname{rank}(V) = 1$ and $A = 0$ by making a meromorphic transform. We remark that an additional growth order ℓ may appear, but it is independent of ω. We may assume that one of (A) or (B) holds as in the proof of Lemma 20.2.1, if $S \times U_P$ is sufficiently small.

Let us consider the case (A). We may assume that $-\operatorname{Re}(\mathfrak{a})$ is increasing with respect to θ. If U_P is sufficiently small, and if we choose θ_0' appropriately, one of the following holds:

(A1') : $-\operatorname{Re}(\mathfrak{a}_k\, e^{\sqrt{-1}k\theta_0'}) < 0$ on U_P.

(A2') : $-\operatorname{Re}(\mathfrak{a}_k\, e^{\sqrt{-1}k\theta_0'}) > 0$ on U_P.

Let us consider the case (A1'). We take the paths γ_1 and γ_2 as in the case (A1) in the proof of Lemma 20.2.1. Then, $|\tau(r,\theta,Q)|$ is dominated by

$$\exp\big(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\big)\int_{\gamma_1} \exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta_0',s,Q}\big)\, s^N\, ds \\ + \exp\big(\operatorname{Re}(\mathfrak{a})_{\theta,r,Q}\big)\int_{\gamma_2} \exp\big(-\operatorname{Re}(\mathfrak{a})_{\varphi,r,Q}\big)\, r^N\, d\varphi.$$

The contribution of γ_2 is dominated by r^N. We have the following equality:

$$(394)\quad \frac{\partial}{\partial s}\Big(s^{-k+1+N}\exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}\big)\Big) \\ = (N-k+1)\, s^{-k+N}\exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}\big) - s^{-k+1+N}\,\frac{\partial \operatorname{Re}(\mathfrak{a})_{\theta,s,Q}}{\partial s}\exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta,s,Q}\big).$$

Note $s^{-k+1}\,\partial_s \operatorname{Re}(\mathfrak{a})_{\theta,s,Q} = k\ \operatorname{Re}(\mathfrak{a}_k)_{\theta,s,Q} + O(s^{-1})$. Hence, we obtain

$$\int_{\gamma_1} \exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta_0',s,Q}\big)\, s^N\, ds \leqslant C_1\, r^{N-k+1}\exp\big(-\operatorname{Re}(\mathfrak{a})_{\theta_0',r,Q}\big).$$

Therefore, the contribution of γ_1 is also dominated by r^N.

For the case (A2'), we take the paths γ_i $(i = 1, 2)$ as in the case (A2) in the proof of Lemma 20.2.1. The contribution of γ_1 is dominated by $\exp(\operatorname{Re}(\mathfrak{a}))$. It can be shown as in the proof of Lemma 20.2.1 by using (394), that the contribution of γ_2 is dominated by r^N.

Let us consider the case (B). If $S \times U_P$ is sufficiently small, one of the following holds for some $\delta > 0$:

(B1') : $-\operatorname{Re}(\mathfrak{a}) < -\delta$ on $S \times U_P$.

(B2') : $-\operatorname{Re}(\mathfrak{a}) > \delta$ on $S \times U_P$.

We take the paths γ as in the cases (B1) or (B2) in the proof of Lemma 20.2.1, respectively. We can obtain the desired estimate as in the cases (A1') and (A2') above, by using (394). □

Let $\omega = \sum \omega_i\, v_i$ be a C^∞-section of $V \otimes \Omega^2$ on $\mathcal{Y}$ such that $|\omega_i| = O(|x|^N)$.

Lemma 20.2.3. — *There exists a C^∞-section $\tau = \sum \tau_i\, v_i$ of $V \otimes \Omega^1_{\mathcal{Y}/Y}$ on $S_\infty(r_0', \theta_0', \theta_1') \times U_P$ such that (i) $\nabla\tau = \omega$, (ii) $\tau_i = O(|x|^{N+\ell})$, where $\ell \geqslant 0$ is independent of ω.*

Proof. — As in the proof of Lemma 20.2.2, we may assume $\operatorname{rank}(V) = 1$ and $A = 0$, and we may assume that one of (A), (B1') or (B2') holds. We have the expression $\omega = f(r, \theta, Q)\, dr\, d\theta$. In the case (A), we put

$$\tau(r, \theta, Q) := -\Big(\exp\big(\mathfrak{a}_{\theta,r,Q}\big) \int_{\theta_0'}^{\theta} \exp\big(-\mathfrak{a}_{\varphi,r,Q}\big) f(r, \varphi, Q)\, d\varphi\Big)\, dr.$$

In the case (B1') or (B2'), we take the paths γ as in the proof of Lemma 20.2.2, and we put

$$\tau(r, \theta, Q) := \Big(\exp(\mathfrak{a}_{r,\theta,Q}) \int_\gamma \exp(-\mathfrak{a}_{s,\theta,Q}) f(s, \theta, Q)\, ds\Big)\, d\theta.$$

(We note that the contribution of γ_1 is 0 in the case (B2').) Then, τ has the desired property. □

20.3. Some Estimates on a multi-sector

20.3.1. Estimate of growth order of some integrals on multi-sectors

Let S_x and S_w be small multi-sectors around ∞:

$$S_x = \big\{(x_1, \dots, x_\ell) \in \boldsymbol{C}^\ell \,\big|\, R_{x,i} < |x_i|,\ |\arg(x_i) - \theta_{x,i}| < \delta_{x,i},\ (i = 1, \dots, \ell)\big\},$$
$$S_w = \big\{(w_1, \dots, w_p) \in \boldsymbol{C}^p \,\big|\, R_{w,i} < |w_i|,\ |\arg(w_i) - \theta_{w,i}| < \delta_{w,i},\ (i = 1, \dots, p)\big\}.$$

Let Y be a compact region in $\boldsymbol{C}^m$. Let λ be a holomorphic function on Y. Let $\boldsymbol{m} \in \mathbb{Z}^\ell_{>0}$ and $\boldsymbol{n} \in \mathbb{Z}^p_{>0}$. Let $\boldsymbol{f}$ be a $\boldsymbol{C}^d$-valued holomorphic function on $S_x \times S_w \times Y$ satisfying the following differential equation:

$$d_x \boldsymbol{f} = \Big(d_x\big(\lambda(y)\, \boldsymbol{x}^{\boldsymbol{m}}\, \boldsymbol{w}^{\boldsymbol{n}}\big) + \sum_{i=1}^{\ell} A_i(\boldsymbol{x}, \boldsymbol{w}, y) \frac{dx_i}{x_i}\Big) \boldsymbol{f} + \boldsymbol{\omega}.$$

Here, A_i denotes $M_d(\boldsymbol{C})$-valued holomorphic functions, and we consider the derivations only with respect to the variables $x_1, \dots, x_\ell$. Assume the following for some $\varepsilon > 0$:

- $\boldsymbol{\omega} = O\big(|\boldsymbol{x}|^{-N}|\boldsymbol{w}|^{-N}\big)$ holds for any $N > 0$, where we put $|\boldsymbol{x}| := \prod_{i=1}^{\ell}|x_i|$ and $|\boldsymbol{w}| := \prod_{j=1}^{p}|w_j|$.
- $\mathrm{Re}\big(\lambda(y)\,\boldsymbol{x}^{\boldsymbol{m}}\,\boldsymbol{w}^{\boldsymbol{n}}\big) < -\varepsilon\,\big|\boldsymbol{x}^{\boldsymbol{m}}\,\boldsymbol{w}^{\boldsymbol{n}}\big|$ on S.
- $|A_i|$ and $|\boldsymbol{\omega}|$ are sufficiently smaller than $\varepsilon\,\big|\boldsymbol{x}^{\boldsymbol{m}}\,\boldsymbol{w}^{\boldsymbol{n}}\big|$.
- Take some point $\boldsymbol{x}_0 \in S_x$. Then, $\big|\boldsymbol{f}_{|\{\boldsymbol{x}_0\}\times S_w\times Y}\big| = O\big(|\boldsymbol{w}|^{-N}\big)$ holds for any $N > 0$.

Let $T_x := \big\{(x_1, \dots, x_\ell) \in \boldsymbol{C}^\ell \,\big|\, \big|\arg(x_i) - \theta_{x,i}\big| < \delta_{x,i}\big\}$.

Lemma 20.3.1. — *Under these conditions, we have $\boldsymbol{f} = O\big(|\boldsymbol{x}|^{-N}|\boldsymbol{w}|^{-N}\big)$ for any N and for any $\boldsymbol{x} \in (\boldsymbol{x}_0 + T_x) \times S_w \times Y$.*

Proof. — The following is minor generalization of the argument in Lemma 14.2 of [**94**]. Let $\xi_i := x_i^{m_i}$. Let $\boldsymbol{\xi}_0 = \big(\xi_1^{(0)}, \dots, \xi_\ell^{(0)}\big)$ correspond to $\boldsymbol{x}_0$. Let $T_\xi := \big\{(\xi_1, \dots, \xi_\ell) \,\big|\, |\arg(\xi_i) - m_i\,\theta_{x,i}| < m_i\,\delta_{x,i},\ (i = 1, \dots, \ell)\big\}$. By the above correspondence, $\boldsymbol{x}_0 + T_x$ is contained in $\boldsymbol{\xi}_0 + T_\xi$. Hence, we only have to obtain the estimate on $(\boldsymbol{\xi}_0 + T_\xi) \times S_w \times Y$.

We put $\boldsymbol{\xi}^{(i)} := \big(\xi_1, \dots, \xi_i, \xi_{i+1}^{(0)}, \dots, \xi_\ell^{(0)}; \boldsymbol{w}, y\big)$ for $i = 0, \dots, \ell$. We have $\boldsymbol{\xi}^{(0)} = (\boldsymbol{\xi}_0, \boldsymbol{w}, y)$ and $\boldsymbol{\xi}^{(\ell)} = \big(\xi_1, \dots, \xi_\ell; \boldsymbol{w}, y\big)$. Let $\tau_i(s)$ be the segment connecting $\boldsymbol{\xi}^{(i-1)}$ and $\boldsymbol{\xi}^{(i)}$:

$$\tau_i(s) = \big(\xi_1, \dots, \xi_{i-1}, \xi_i^{(0)} + s\,(\xi_i - \xi_i^{(0)}),\, \xi_{i+1}^{(0)}, \dots, \xi_\ell^{(0)}; \boldsymbol{w}, y\big), \quad (0 \leqslant s \leqslant 1).$$

On the path $\tau_i(s)$, we have the following inequalities for some constants C_i:

$$\begin{aligned}(395)\quad \frac{d|\boldsymbol{f}|^2}{ds} &\leqslant 2\,\mathrm{Re}\Big(\lambda\prod_{j<i}\xi_j\,(\xi_i - \xi_i^{(0)})\prod_{j>i}\xi_j^{(0)}\,\boldsymbol{w}^{\boldsymbol{n}}\Big)\,|\boldsymbol{f}|^2 \\ &\quad + C_0\Big(|\xi_i|^{-1}|\xi_i - \xi_i^{(0)}|\,|A_i|\,|\boldsymbol{f}|^2 + |\xi_i - \xi_i^{(0)}|\,\big|\iota_{\partial_{\xi_i}}\boldsymbol{\omega}\big|\,|\boldsymbol{f}|\Big) \\ &\leqslant -C_1\prod_{j<i}|\xi_j|\,|\xi_i - \xi_i^{(0)}|\prod_{j>i}|\xi_j^{(0)}|\,|\boldsymbol{w}^{\boldsymbol{n}}|\,|\boldsymbol{f}|^2 + C_2\,|\xi_i - \xi_i^{(0)}|\,\big|\iota_{\partial_{\xi_i}}\boldsymbol{\omega}\big|.\end{aligned}$$

Here, we have used $|\boldsymbol{f}| \leqslant 2(1 + |\boldsymbol{f}|^2)$.

Let us show the following estimate for some constant $B_N^{(i)}$ independent of $\boldsymbol{\xi}$, by an induction on i:

$$\big|\boldsymbol{f}(\boldsymbol{\xi}^{(i)})\big|^2 \leqslant B_N^{(i)}\prod_{j\leqslant i}|\xi_j|^{-N}\prod_{j>i}|\xi_j^{(0)}|^{-N}\,|\boldsymbol{w}|^{-N}.$$

From (395), we obtain

$$\begin{aligned}(396)\quad \big|\boldsymbol{f}(\boldsymbol{\xi}^{(i)})\big|^2 &\leqslant \exp\Big(-C_1\prod_{j<i}|\xi_j|\,|\xi_i - \xi_i^{(0)}|\prod_{j>i}|\xi_j^{(0)}|\,|\boldsymbol{w}^{\boldsymbol{n}}|\Big)\,\big|\boldsymbol{f}(\boldsymbol{\xi}^{(i-1)})\big|^2 \\ &\quad + \int_0^1 \exp\Big(-C_1(1-s)\prod_{j<i}|\xi_j|\,|\xi_i - \xi_i^{(0)}|\prod_{j>i}|\xi_j^{(0)}|\,|\boldsymbol{w}^{\boldsymbol{n}}|\Big)\,C_2\,|\xi_i - \xi_i^{(0)}|\,\big|\iota_{\partial_{\xi_i}}\boldsymbol{\omega}\big|\,ds.\end{aligned}$$

The first term can be dominated by

$$B_N^{(i-1)}\exp\Bigl(-C_1\prod_{j<i}|\xi_j|\,|\xi_i-\xi_i^{(0)}|\prod_{j>i}|\xi_j^{(0)}|\,|\boldsymbol{w}^{\boldsymbol{n}}|\Bigr)\prod_{j<i}|\xi_j|^{-N}\prod_{j\geqslant i}|\xi_j^{(0)}|^{-N}\,|\boldsymbol{w}|^{-N}.$$

In the case $|\xi_i^{(0)}|\leqslant|\xi_i|/2$, we have $|\xi_i-\xi_i^{(0)}|\geqslant|\xi_i|/2$, and hence the exponential term is very small. In the case $|\xi_i^{(0)}|>|\xi_i|/2$, we have $|\xi_i^{(0)}|^{-N}\leqslant C_4\,|\xi_i|^{-N}$, and hence the first term can be estimated appropriately.

Let us look at the second term of (396). For $s>1/2$, we have $|\xi_i^{(0)}+s(\xi_i-\xi_i^{(0)})|\geqslant|\xi_i|/2$. Hence, the integrand can be dominated by

$$C_{4,N}\,|\xi_i-\xi_i^{(0)}|\prod_{j<i}|\xi_i|^{-N}\,|\xi_i^{(0)}+s(\xi_i-\xi_i^{(0)})|^{-N}\prod_{j>i}|\xi_j^{(0)}|^{-N}\,|\boldsymbol{w}|^{-N}\\ \leqslant C_{5,N}\prod_{j\leqslant i}|\xi_j|^{-N+1}\prod_{j>i}|\xi_j^{(0)}|^{-N}\,|\boldsymbol{w}|^{-N}. \tag{397}$$

Thus, $\int_{1/2}^{1}$ can be estimated appropriately. For $s<1/2$, the exponential term in the integrand is dominated by

$$\exp\Bigl(\frac{-C_1}{2}\prod_{j<i}|\xi_j|\,|\xi_i-\xi_i^{(0)}|\prod_{j>i}|\xi_j^{(0)}|\,|\boldsymbol{w}^{\boldsymbol{n}}|\Bigr).$$

The term $|\iota_{\partial_{\xi_i}}\boldsymbol{\omega}|$ is dominated by $C_{6,N}\prod_{j<i}|\xi_j|^{-N}\prod_{j\geqslant i}|\xi_j^{(0)}|^{-N}|\boldsymbol{w}|^{-N}$, and $|\xi_i-\xi_i^{(0)}|$ is dominated by $|\xi_i|$. Hence, using the same argument for the estimate of the first term, we can estimate $\int_0^{1/2}$. Thus, the induction can proceed. The estimate for $|\boldsymbol{f}(\boldsymbol{\xi}^{(\ell)})|=|\boldsymbol{f}(\boldsymbol{\xi})|$ gives the claim of the lemma. □

20.3.2. Estimate of flat sections. — Let S_x, S_w, T_x and Y be as in Section 20.3.1. Let $1\leqslant k\leqslant\ell$, $\boldsymbol{m}\in\mathbb{Z}_{>0}^k$ and $\boldsymbol{n}\in\mathbb{Z}_{>0}^p$. Let R be a C^∞-section of $M_d(\boldsymbol{C})\otimes\Omega_{S_x}^1$ on $S_x\times S_w\times Y$:

$$R=\sum_{i=1}^{\ell}R_i^{(1,0)}\frac{dx_i}{x_i}+\sum_{i=1}^{\ell}R_i^{(0,1)}\frac{d\overline{x}_i}{\overline{x}_i}.$$

We assume the estimates $|R_i^{(p,q)}|\leqslant B_0|\boldsymbol{x}^{\boldsymbol{m}}\boldsymbol{w}^{\boldsymbol{n}}|$ for $(p,q)=(1,0),(0,1)$ and $i=1,\dots,\ell$. Let $\boldsymbol{f}$ be a $\boldsymbol{C}^d$-valued C^∞-function on $S_x\times S_w\times Y$ satisfying the following differential equation:

$$d_x\boldsymbol{f}=R\,\boldsymbol{f}.$$

We assume $\boldsymbol{f}(\boldsymbol{x}_0,\boldsymbol{w},y)\neq0$ for some $\boldsymbol{x}_0\in S_x$.

Lemma 20.3.2. — *We have the following estimate on $(\boldsymbol{x}_0+T_x)\times S_w\times Y$ for some $C_0>0$:*

$$\left|\log\Bigl(\frac{|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{w},y)|^2}{|\boldsymbol{f}(\boldsymbol{x}_0,\boldsymbol{w},y)|^2}\Bigr)\right|\leqslant B_0\,C_0\,|\boldsymbol{x}^{\boldsymbol{m}}\boldsymbol{w}^{\boldsymbol{n}}|\,\log\Bigl(\prod_{i=k+1}^{\ell}|x_i|\Bigr).$$

The constant C_0 is independent of $\boldsymbol{f}$.

Proof. — Let $\xi_i = x_i^{m_i}$ for $i = 1, \dots, k$, and $\xi_i = x_i$ for $i = k+1, \dots, \ell$. We use the notation in the proof of Lemma 20.3.1. We only have to obtain the estimate on $(\boldsymbol{\xi}_0 + T_\xi) \times S_w \times Y$. On the paths $\tau_i(s)$ $(i = 1, \dots, k)$, we have

$$\Big|\frac{d}{ds}|\boldsymbol{f}|^2\Big| \leqslant C_1\, B_0 \prod_{j=1}^{i-1} |\xi_j|\, |\xi_i - \xi_i^{(0)}| \prod_{j=i+1}^{k} |\xi_j^{(0)}|\, |\boldsymbol{w}^{\boldsymbol{n}}|\, |\boldsymbol{f}|^2.$$

Hence, we obtain

$$\Big|\log\Big(\frac{|\boldsymbol{f}(\boldsymbol{\xi}^{(i)})|^2}{|\boldsymbol{f}(\boldsymbol{\xi}^{(i-1)})|^2}\Big)\Big| \leqslant C_1\, B_0 \prod_{j=1}^{i-1} |\xi_j|\, |\xi_i - \xi_i^{(0)}| \prod_{j=i+1}^{k} |\xi_j^{(0)}|\, |\boldsymbol{w}^{\boldsymbol{n}}|.$$

Note we have $\prod_{j=1}^{i-1} |\xi_j|\, |\xi_i - \xi_i^{(0)}| \prod_{j=i+1}^{k} |\xi_j^{(0)}| \leqslant C_2 \prod_{j=1}^{k} |\xi_j|$. Hence, we obtain

$$\Big|\log\Big(\frac{|\boldsymbol{f}(\boldsymbol{\xi}^{(k)})|^2}{|\boldsymbol{f}(\boldsymbol{\xi}^{(0)})|^2}\Big)\Big| \leqslant C_3\, B_0 \prod_{j=1}^{k} |\xi_j|\, |\boldsymbol{w}^{\boldsymbol{n}}|. \tag{398}$$

On the paths $\tau_i(s)$ $(i = k+1, \dots, \ell)$, we have

$$\Big|\frac{d}{ds}|\boldsymbol{f}|^2\Big| \leqslant B_0\, \frac{|\xi_i - \xi_i^{(0)}|}{|\xi_i^{(0)} + s\,(\xi_i - \xi_i^{(0)})|} \prod_{j=1}^{k} |\xi_j|\, |\boldsymbol{w}^{\boldsymbol{n}}|\, |\boldsymbol{f}|^2.$$

By an elementary geometric argument, we can show the following inequality on $\tau_i(s)$ for some $0 < C_4 < 1$:

$$\frac{|\xi_i - \xi_i^{(0)}|}{|\xi_i^{(0)} + s\,(\xi_i - \xi_i^{(0)})|} \leqslant \frac{C_4\, |\xi_i - \xi_i^{(0)}|}{|\xi_i^{(0)}| + s\,|\xi_i - \xi_i^{(0)}|}. \tag{399}$$

Hence, we obtain

$$\Big|\log\Big(\frac{|\boldsymbol{f}(\boldsymbol{\xi}^{(i)})|}{|\boldsymbol{f}(\boldsymbol{\xi}^{(i-1)})|}\Big)\Big| \leqslant C_5\, B_0\, \log\Big(1 + |\xi_i^{(0)}|^{-1}\, |\xi_i - \xi_i^{(0)}|\Big) \prod_{j=1}^{k} |\xi_j|\, |\boldsymbol{w}^{\boldsymbol{n}}|.$$

Note $1 + |\xi_i^{(0)}|^{-1}\, |\xi_i - \xi_i^{(0)}| \leqslant C_6 |\xi_i|$. Thus we are done. □

Lemma 20.3.3. — *If moreover $|R_i^{(p,q)}| \leqslant B_0\, |x_i^{-1} \boldsymbol{x}^{\boldsymbol{m}} \boldsymbol{w}^{\boldsymbol{n}}| + B_0$ are satisfied for $(p,q) = (1,0), (0,1)$ and $i = k+1, \dots, \ell$, the following holds for some C_0':*

$$\Big|\log\Big(\frac{|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}, y)|^2}{|\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{w}, y)|^2}\Big)\Big| \leqslant B_0\, C_0' \Big(|\boldsymbol{x}^{\boldsymbol{m}}\, \boldsymbol{w}^{\boldsymbol{n}}| + \log\Big(\prod_{i=k+1}^{\ell} |x_i|\Big)\Big).$$

Proof. — We obtain (398) without any change. On $\tau_i(s)$ $(i = k+1, \dots, \ell)$, we have

$$\frac{d}{ds} \log |\boldsymbol{f}|^2 \leqslant B_0\, \frac{|\xi_i - \xi_i^{(0)}|}{|\xi_i^{(0)} + s\,(\xi_i - \xi_i^{(0)})|} + B_0\, \frac{|\xi_i - \xi_i^{(0)}|}{|\xi_i^{(0)} + s\,(\xi_i - \xi_i^{(0)})|^2} \prod_{j=1}^{k} |\xi_j|\, |\boldsymbol{w}^{\boldsymbol{n}}|.$$

The contribution of the first term can be dominated by $\log\big(1+|\xi_i^{(0)}|^{-1}|\xi_i-\xi_i^{(0)}|\big)$. By using (399), it is easy to show that the contribution of the second term can be dominated by $\prod_{i=1}^{k}|\xi_i|\,|\boldsymbol{w}^{\boldsymbol{n}}|$. □

Remark 20.3.4. — We have a variant of Lemma 20.3.5 corresponding to Lemma 20.3.3. Namely, if $\operatorname{ord}(R)\geqslant \boldsymbol{m}(1)$, then we have an estimate

$$\left|\log\Big(\frac{\big|g\,\exp(\lambda^{-1}\mathfrak{a})\big|(\lambda,\boldsymbol{z},\boldsymbol{y})}{\big|g\,\exp(\lambda^{-1}\mathfrak{a})\big|(\lambda,\boldsymbol{z}_0,\boldsymbol{y})}\Big)\right|\leqslant C|\boldsymbol{z}^{\boldsymbol{m}(1)}|+C\log\Big(|z_{i(0)}|^{-1}\prod_{i=k+1}^{\ell}|z_i|^{-1}\Big).$$

If $m_{i(0)}<-1$, the term $|z_{i(0)}|^{-1}$ in the logarithm is not necessary. See Subsection 2.6.1 for the order $\operatorname{ord}R$. □

20.3.3. Estimate for a flat section of a family of λ-flat bundles for $\lambda\neq 0$

We give a special version of Lemma 20.3.2, which will often be used in our subsequent arguments. Let X be a product of Δ_z^ℓ and a complex manifold Y. Let $\mathcal{K}$ denote a point or a compact region of $\boldsymbol{C}_\lambda^*$. Let $\mathcal{X}:=X\times\mathcal{K}$ and $\mathcal{D}:=D\times\mathcal{K}$. Let $\pi:\widetilde{\mathcal{X}}(\mathcal{D})\to\mathcal{X}$ denote the real blow up of $\mathcal{X}$ along $\mathcal{D}$. Let $S_{\boldsymbol{z}}$ be a multi-sector of $(\Delta^*)^\ell$. Let S be a product of $S_{\boldsymbol{z}}$ and a compact region U of $Y\times\mathcal{K}$. Let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(\mathcal{D})$.

Let $\boldsymbol{m}\in\mathbb{Z}_{<0}^k$ for some $k\in[1,\ell]$, and let $\boldsymbol{m}(1)=\boldsymbol{m}+\boldsymbol{\delta}_{i(0)}$ for some $i(0)\in[1,k]$. Let $\mathfrak{a}$ be the product of $\boldsymbol{z}^{\boldsymbol{m}}$ and a nowhere vanishing holomorphic function $\mathfrak{a}_{\boldsymbol{m}}$ on $\mathcal{X}$. Let R be a holomorphic section of $M_d(\boldsymbol{C})\otimes p_\lambda^*\big(\boldsymbol{z}^{\boldsymbol{m}(1)}\Omega_X^{1,0}(\log D)\big)$ on $\overline{S}$.

Lemma 20.3.5. — *Let g be a $\boldsymbol{C}^d$-valued holomorphic function on S such that $\lambda\,d_{\boldsymbol{z}}g=(d\mathfrak{a}+R)\,g$. Let $\boldsymbol{z}_0$ be any point of $S_{\boldsymbol{z}}$. Then, we have the following estimate for some $C>0$:*

$$\left|\log\Big(\frac{\big|g\,\exp(\lambda^{-1}\mathfrak{a})\big|(\lambda,\boldsymbol{z},\boldsymbol{y})}{\big|g\,\exp(\lambda^{-1}\mathfrak{a})\big|(\lambda,\boldsymbol{z}_0,\boldsymbol{y})}\Big)\right|\leqslant C|\boldsymbol{z}^{\boldsymbol{m}(1)}|\,\log\Big(|z_{i(0)}|^{-1}\prod_{i=k+1}^{\ell}|z_i|^{-1}\Big).\tag{400}$$

If $m_{i(0)}<-1$, the term $|z_{i(0)}|^{-1}$ in the logarithm is not necessary. (If $g_{|\boldsymbol{z}_0\times U}=0$, we use the convention $\log(0/0)=0$.)

Proof. — We put $f=g\,\exp\big(\lambda^{-1}(\mathfrak{a}-\mathfrak{a}_{|\boldsymbol{z}_0\times U})\big)$. By changing the variables $x_i=z_i^{-1}$, we apply Lemma 20.3.2 to f. Then, we obtain the desired estimate. (Note we may assume that R and g are defined on S' which is slightly larger than S.) □

Corollary 20.3.6. — *Let g be as in Lemma* 20.3.5. *Assume the following:*

- *$k=\ell$ and $\operatorname{Re}(\lambda^{-1}\mathfrak{a})<-\delta|\boldsymbol{z}^{\boldsymbol{m}}|$ for some $\delta>0$.*

If we shrink S appropriately in the radius direction, we have $|g|=O\big(\exp(-\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|)\big)$ for some $\varepsilon>0$. □

Corollary 20.3.7. — *Let g be as in Lemma* 20.3.5. *Assume $g_{|\boldsymbol{z}_0\times U}$ is nowhere vanishing. We also assume the following:*

- $k=\ell$ *and* $\mathrm{Re}(\lambda^{-1}\mathfrak{a})>\delta|\boldsymbol{z}^{\boldsymbol{m}}|$ *for some* $\delta>0$.

If we shrink S *appropriately in the radius direction, we have* $|g|\geqslant C_1\exp\big(\varepsilon|\boldsymbol{z}^{\boldsymbol{m}}|\big)$ *for some* $C_1>0$ *and* $\varepsilon>0$. □

20.3.4. Estimate for a flat section of a family of λ-flat bundles around $\lambda=0$

We give another special version of Lemma 20.3.2. Let X be a product of Δ_z^ℓ and a complex manifold Y. Let $\mathcal{K}$ denote a neighbourhood of 0_λ in $\boldsymbol{C}_\lambda$. Let $\mathcal{X}:=X\times\mathcal{K}$ and $\mathcal{D}:=D\times\mathcal{K}$. Let $\mathcal{X}^0:=X\times\{0_\lambda\}$. We put $W:=\mathcal{X}^0\cup\mathcal{D}$. Let $\widetilde{\mathcal{X}}(W)$ denote the real blow up of $\mathcal{X}$ along W. Let S_λ and $S_{\boldsymbol{z}}$ be multi-sectors of $\mathcal{K}\setminus\{0_\lambda\}$ and $(\Delta^*)^\ell$, respectively. Let S be a product of S_λ, $S_{\boldsymbol{z}}$ and a compact region U of Y, and let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(W)$.

Let $\boldsymbol{m}\in\mathbb{Z}^k_{<0}$ for some $k\in[1,\ell]$, and let $\boldsymbol{m}(1)=\boldsymbol{m}+\boldsymbol{\delta}_{i(0)}$ for some $i(0)\in[1,k]$. Let $\mathfrak{a}$ be the product of $\boldsymbol{z}^{\boldsymbol{m}}$ and a nowhere vanishing holomorphic function $\mathfrak{a}_{\boldsymbol{m}}$ on $\mathcal{X}$. Let R be a holomorphic section of $M_d(\boldsymbol{C})\otimes p_\lambda^*\big(\boldsymbol{z}^{\boldsymbol{m}(1)}\Omega_X(\log D)\big)$ on $\overline{S}$.

Lemma 20.3.8. — *Let* g *be a* $\boldsymbol{C}^d$*-valued holomorphic function on* S *such that*

$$\lambda\, d_{\boldsymbol{z}}g=(d\mathfrak{a}+R)\,g.$$

Let $\boldsymbol{z}_0$ *be any point of* $S_{\boldsymbol{z}}$. *Then, we have the following estimate for some* $C>0$*:*

$$\left|\log\Big(\frac{\big|g\,\exp\big(\lambda^{-1}\mathfrak{a}\big)\big|(\lambda,\boldsymbol{z},\boldsymbol{y})}{\big|g\,\exp\big(\lambda^{-1}\mathfrak{a}\big)\big|(\lambda,\boldsymbol{z}_0,\boldsymbol{y})}\Big)\right|\leqslant C|\boldsymbol{z}^{\boldsymbol{m}(1)}\,\lambda^{-1}|\,\log\Big(|z_{i(0)}|^{-1}\prod_{i=k+1}^{\ell}|z_i|^{-1}\Big).\tag{401}$$

In the case $m_{i(0)}<-1$, *the term* $|z_{i(0)}|^{-1}$ *in the logarithm is not necessary. (We use the convention* $\log(0/0)=0$ *in the case* $g_{|S_\lambda\times\boldsymbol{z}_0\times U}=0$.*)*

Proof. — We put $f=g\,\exp\big(\lambda^{-1}(\mathfrak{a}-\mathfrak{a}_{|S_\lambda\times\boldsymbol{z}_0\times U})\big)$. By changing the variables $w=\lambda^{-1}$ and $x_i=z_i^{-1}$, we apply Lemma 20.3.2 to f. Then, we obtain the desired estimate. □

Corollary 20.3.9. — *Let* g *be as in Lemma* 20.3.8. *Assume the following:*

- $|g|$ *is bounded on* $S_\lambda\times\boldsymbol{z}_0\times U$.
- $k=\ell$ *and* $\mathrm{Re}(\lambda^{-1}\mathfrak{a})<-\delta|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}}|$ *for some* $\delta>0$.

If we shrink S *appropriately in the radius direction, we have* $|g|=O\big(\exp\big(-\varepsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}}|\big)\big)$ *for some* $\varepsilon>0$. □

Remark 20.3.10. — We have a variant of Lemma 20.3.5 corresponding to Lemma 20.3.3. See Remark 20.3.4. □

CHAPTER 21

ACCEPTABLE BUNDLES

We studied acceptable bundles in [**65**], [**66**] and [**67**] after [**21**], [**81**] and [**82**]. In this chapter, we give a review and some complements. The most important one is local freeness (Theorem 21.3.1) in Section 21.3. Although there is a considerable overlap with [**65**] and [**67**], the author thinks it appropriate to give the details in view of significance of the theory of acceptable bundles for us.

In Section 21.1, we recall basic facts on holomorphic vector bundles on Kähler manifolds. We review in Section 21.2 the fundamental property of the twisted metrics for acceptable bundles, with minor refinement. Roughly speaking, if N is sufficiently negative, we have the vanishing of higher L^2-cohomology, due to which we can extend holomorphic sections on a hyperplane to those on the whole space. And if N is sufficiently positive, the norms of holomorphic sections are pluri-subharmonic, due to which we do not have to care the distinctions between L^2-estimates and growth estimates, or curve-wise estimates and global estimates.

In Section 21.3, we give the statement of the main theorem (Theorem 21.3.1) of this chapter. It briefly means the local freeness of the sheaves of holomorphic sections satisfying some growth estimate. It was shown in [**67**] for acceptable bundles underlying tame harmonic bundles. The main body of the proof is almost the same. We need only some easy changes for the general case. However, it is used to obtain a filtered λ-flat bundle from an unramifiedly good wild harmonic bundle, which is one of the most fundamental results for us. Hence, we give a rather detailed outline of the proof in Sections 21.4–21.7.

The theory of acceptable bundles on curves was well established by Simpson in [**81**] and [**82**]. We explain some complements in Section 21.4 for our use, which are more or less well known. In Section 21.4.2, we explain that the parabolic filtration has the splitting by the weight decomposition with respect to the Galois group after taking an appropriate ramified covering. In Section 21.4.3, we review that the parabolic weights are essentially the logarithm of the limit of the monodromy. Then, we show in Section

21.4.4 that the parabolic weights are well defined for the irreducible components of the divisors in the higher dimensional case. (We have used the property of tame harmonic bundles for the control of the parabolic weights in [**67**].)

To show an $\mathcal{O}$-module is locally free, it is always essential to show that holomorphic sections on a hypersurface can be extended to those on the whole space, in an appropriate sense. We study such problems in Sections 21.5–21.6 by the method in [**65**] and [**67**]. We change the way of construction of a cocycle (Section 21.5.2). Then, we show Theorem 21.3.1 in Section 21.7 by using the argument in [**65**] and [**67**].

In Section 21.8, we show that a small deformation of an acceptable bundle is also naturally extended to a filtered bundle, which is useful to show Theorem 9.1.2.

In Section 21.9 we give some complements. In Section 21.9.1, we study estimates of sup norms for L^2-solutions of the $\overline{\partial}$-equation. Lemma 21.9.1 is used in Sections 21.9.2, 7.4.2 and 8.2.3. We also give a variant of such an estimate in Section 21.9.2, which is used in Sections 9.4.1 and 13.5.3. We give in Section 21.9.3 an estimate of a unitary connection form with respect to a holomorphic frame compatible with the parabolic structure. We show that it is bounded up to log order. We give in Section 7.5.2 a refinement if the acceptable bundle comes from a good wild harmonic bundle. (See also Section 10.3.2.) Such an estimate is used to obtain an estimate of harmonic forms on a punctured disc (Section 8.4.2).

21.1. Some general results on vector bundles on Kähler manifolds

21.1.1. Vanishing of L^2-cohomology and some estimates. — We recall some results of Andreotti-Vesentini in [**3**]. Let (Y, g) be a complete Kähler manifold, not necessarily compact. The volume form of Y is denoted by dvol. Let $(E, \overline{\partial}_E, h)$ be a Hermitian holomorphic bundle over Y. Let $(\cdot,\cdot)_{h,g}$ denote the induced fiber-wise Hermitian metric of $E \otimes \Omega_Y^{p,q}$. The space of C^∞ (p,q)-forms with compact support is denoted by $A_c^{p,q}(E)$. For any $\eta_i \in A_c^{p,q}(E)$ $(i = 1, 2)$, we define

$$\langle \eta_1, \eta_2 \rangle_h := \int (\eta_1, \eta_2)_{h,g} \cdot \mathrm{dvol}, \qquad \|\eta\|_h^2 := \langle \eta, \eta \rangle_h.$$

The completion of $A_c^{p,q}(E)$ with respect to the norm $\|\cdot\|_h$ is denoted by $A_h^{p,q}(E)$.

Let $\overline{\partial}_E^* : A_c^{p,q}(E) \to A_c^{p,q-1}(E)$ denote the formal adjoint of $\overline{\partial}_E : A_c^{p,q}(E) \to A_c^{p,q+1}(E)$. We set $\Delta'' = \overline{\partial}_E^*\overline{\partial}_E + \overline{\partial}_E\overline{\partial}_E^*$. We have the maximal closed extensions $\overline{\partial}_E : A_h^{p,q}(E) \to A_h^{p,q+1}(E)$ and $\overline{\partial}_E^* : A_h^{p,q}(E) \to A_h^{p,q-1}(E)$. We denote the domains of $\overline{\partial}_E$ and $\overline{\partial}_E^*$ by $\mathrm{Dom}(\overline{\partial}_E)$ and $\mathrm{Dom}(\overline{\partial}_E^*)$ respectively.

Proposition 21.1.1 **(Proposition 5 of [3]).** — *$A_c^{p,q}(E)$ is dense in $W^{p,q} := \mathrm{Dom}(\overline{\partial}_E) \cap \mathrm{Dom}(\overline{\partial}_E^*)$ with respect to the the graph norm $\|\eta\|_h^2 + \|\overline{\partial}_E\eta\|_h^2 + \|\overline{\partial}_E^*\eta\|_h^2$. (See also [**21**]).* □

***Proposition 21.1.2* (Theorem 21 in [3]).** — *Assume that there exists a positive number $c>0$ satisfying the following:*

For any $\eta \in W^{p,q}$, we have $\|\overline{\partial}_E\eta\|_h^2 + \|\overline{\partial}_E^*\eta\|_h^2 \geqslant c\cdot\|\eta\|_h^2$.

Then, for any C^∞-element $\eta \in A_h^{p,q}(E)$ such that $\overline{\partial}_E(\eta)=0$, there exists a C^∞-solution $\rho\in A_h^{p,q-1}(E)$ satisfying the equation $\overline{\partial}_E(\rho)=\eta$. □

21.1.2. Kodaira inequality. — We recall some inequality due to K. Kodaira [**49**] and M. Cornalba–P. Griffiths [**21**]. (See also [**65**] and [**67**].) For a Kähler manifold Y, we have the operator $\Lambda:\Omega_Y^{p,q}\to\Omega_Y^{p-1,q-1}$ which is the adjoint of the multiplication by the Kähler form (see Page 62 of [**48**]). Let E be a holomorphic vector bundle with a Hermitian metric h over Y. We have the metric connection of E induced by the holomorphic structure $\overline{\partial}_E$ and the Hermitian metric h. We denote the curvature by $R(h)$. Let $R(\Omega_Y^{0,q})$ denote the curvature of the Levi-Civita connection on $\Omega_Y^{0,q}$. We have the following inequality:

(402)
$$\big\|\overline{\partial}_E\eta\big\|_h^2+\big\|\overline{\partial}_E^*\eta\big\|_h^2 \geqslant \sqrt{-1}\big\langle\Lambda(R(\Omega_Y^{0,q}))\cdot\eta,\,\eta\big\rangle_h-\sqrt{-1}\big\langle\Lambda\big(R(h)\eta\big)-\Lambda R(h)\cdot\eta,\,\eta\big\rangle_h.$$

(See Section 2.8.2 of [**67**].)

We give a more specific formula for $(0,1)$-forms. We denote the Ricci curvature of the Kähler metric g by $\mathrm{Ric}(g)$. We can naturally regard $\mathrm{Ric}(g)$ as a section of $\mathrm{End}(E)\otimes\Omega_Y^{1,1}$. Let f be a local section of $\mathrm{End}(E)\otimes\Omega_Y^{1,1}$, and η be a local element of $A_c^{0,1}(E)$. We put

$$\langle\langle f,\eta\rangle\rangle_h := -\sqrt{-1}\Big(\Lambda\big(f\cdot\eta\big)-\Lambda(f)\cdot\eta,\ \eta\Big)_h. \tag{403}$$

Let $\varphi_1,\dots,\varphi_d$ be a local orthonormal frame of $\Omega^{1,0}$, and let $e_1,\dots,e_r$ be a local orthonormal frame of E. For local expressions $f=\sum f_{\mu,\nu,i,\bar{j}}\cdot e_\mu^\vee\otimes e_\nu\otimes(\varphi_i\wedge\overline{\varphi}_j)$, $\eta=\sum\eta_{\mu,i}\cdot e_\mu\otimes\overline{\varphi}_i$, we can rewrite (403) as

$$\sum f_{\mu,\nu,i,\bar{j}}\cdot\eta_{\mu,i}\cdot\overline{\eta}_{\nu,j}, \tag{404}$$

where $e_1^\vee,\dots,e_r^\vee$ denote the dual frame.

For any $\eta\in\mathrm{Dom}(\overline{\partial}_E)\cap\mathrm{Dom}(\overline{\partial}_E^*)\subset A_h^{0,1}(E)$, we have the following inequality:

$$\|\overline{\partial}_E(\eta)\|_h^2+\|\overline{\partial}_E^*(\eta)\|_h^2\geqslant\int\langle\langle R(h)+\mathrm{Ric}(g),\eta\rangle\rangle_h\,\mathrm{dvol}.$$

(See [**49**] and [**21**]. See also Proposition 4.5 of [**65**].)

21.2. Twist of the metric of an acceptable bundle

Let X be a complex manifold, and D be a normal crossing divisor of X. Let $g_{\mathbf{p}}$ be a Poincaré-like metric of $X\setminus D$, i.e., for any point $P\in D$, we can take a neighbourhood U of P in X such that $U\setminus D$ is bi-holomorphic to the product of some discs and punctured discs, and then $g_{\mathbf{p}}$ and the Poincaré metric of $U\setminus D$ are mutually bounded around P. Recall that $(E, \overline{\partial}_E, h)$ is called acceptable, if the curvature $R(h)$ is bounded with respect to h and $g_{\mathbf{P}}$. (See also Definition 2.44 of [**67**].)

Let us consider the case $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell} D_i$, where we put $D_i := \{z_i = 0\}$. Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle over $X \setminus D$. For $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$ and $N \in \boldsymbol{R}$, we set

$$\tau(\boldsymbol{a}, N) := -\sum_{i=1}^{\ell} a_i \cdot \log |z_i|^2 + N \cdot \Big(\sum_{i=1}^{\ell} \log(-\log |z_i|^2) + \sum_{i=\ell+1}^{n} \log(1 - |z_i|^2)\Big). \tag{405}$$

We consider the following metric for any $\boldsymbol{a}$ and N:

$$h_{\boldsymbol{a},N} := h \cdot e^{-\tau(\boldsymbol{a},N)} = h \cdot \prod_{i=1}^{\ell} |z_i|^{2a_i} \cdot \big(-\log |z_i|^2\big)^{-N} \cdot \prod_{i=\ell+1}^{n} \big(1 - |z_i|^2\big)^{-N}. \tag{406}$$

We use the symbols $|\cdot|_{\boldsymbol{a},N}$, $\|\cdot\|_{\boldsymbol{a},N}$, $(\cdot,\cdot)_{\boldsymbol{a},N}$ and $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{a},N}$ instead of $|\cdot|_{h_{\boldsymbol{a},N}}$, $\|\cdot\|_{h_{\boldsymbol{a},N}}$, $(\cdot,\cdot)_{h_{\boldsymbol{a},N}}$ and $\langle\cdot,\cdot\rangle_{h_{\boldsymbol{a},N}}$. We also use the symbols $A^{p,q}_{\boldsymbol{a},N}(E)$ instead of $A^{p,q}_{h_{\boldsymbol{a},N}}(E)$.

21.2.1. The case where N is sufficiently negative. — We have the following equality for any section $\eta \in A^{0,q}_c(E)$ (See the proof of Lemma 2.45 of [**67**]): (1)

$$\sqrt{-1}\Big(\Lambda\big(\overline{\partial}\partial\tau(\boldsymbol{a}, N)\eta\big) - \Lambda\big(\overline{\partial}\partial\tau(\boldsymbol{a}, N)\big) \cdot \eta\Big) = -N \cdot q \cdot \eta. \tag{407}$$

We obtain the following inequality from (402) and (407):

$$\begin{aligned}\big\|\overline{\partial}_E\eta\big\|^2_{\boldsymbol{a},N} + \big\|\overline{\partial}^*_E\eta\big\|^2_{\boldsymbol{a},N} \geqslant{}& \sqrt{-1}\langle\Lambda(\Omega^{0,q}_{X\setminus D}) \cdot \eta,\ \eta\rangle_{\boldsymbol{a},N} \\ &- \sqrt{-1}\Big\langle\Lambda\big(R(h)\cdot\eta\big) - \Lambda R(h)\cdot\eta,\ \eta\Big\rangle_{\boldsymbol{a},N} - N \cdot q \cdot \|\eta\|^2_{\boldsymbol{a},N}.\end{aligned} \tag{408}$$

There exists a positive constant $C > 0$, depending only on the bound of the curvature $R(h)$, such that the following holds on $X \setminus D$ for any $q = 1, \ldots, n$ and for any $\eta \in A^{0,q}_c(E)$:

$$\Big|\Big\langle\Lambda(\Omega^{0,q}_{X\setminus D}) \cdot \eta - \Lambda\big(R(h)\cdot\eta\big) + \Lambda\big(R(h)\big)\cdot\eta,\ \eta\Big\rangle_h\Big| \leqslant C \cdot |\eta|^2_h.$$

If we take a negative integer N such that $N < -C - 1$ for the above constant C, we obtain the following inequalities for any $q \geqslant 1$ and for any $\eta \in A^{0,q}_c(E)$, due to the inequality (408):

$$\big\|\overline{\partial}_E\eta\big\|^2_{\boldsymbol{a},N} + \big\|\overline{\partial}^*_E\eta\big\|^2_{\boldsymbol{a},N} \geqslant \|\eta\|^2_{\boldsymbol{a},N}. \tag{409}$$

Lemma 21.2.1. — *Let C be a positive constant as above. If $N < -C - 1$, we have the vanishing of any higher cohomology group $H^i\big(A^{0,\cdot}_{\boldsymbol{a},N}(E), \overline{\partial}_E\big)$ $(i > 0)$.*

Proof. — It follows from Proposition 21.1.1 and Proposition 21.1.2 and (409). □

1. The signature in Lemma 2.45 of [**67**] is opposite.

21.2.1.1. Refinement. — Let $Z^{0,q}_{\boldsymbol{a},N}(E)$ denote the kernel of the natural morphism $\overline{\partial}_E : A^{0,q}_{\boldsymbol{a},N}(E) \to A^{0,q+1}_{\boldsymbol{a},N}(E)$. Let $A^{*,0,q}_{\boldsymbol{a},N}(E)$ be the space of the sections τ of $E \otimes \Omega^{0,q}$ such that τ and $\overline{\partial}^*_E\tau$ are L^2 with respect to $h_{\boldsymbol{a},N}$ and $g_{\mathbf{p}}$. It is also obtained as the completion of $A^{0,q}_c(E)$ with respect to the norm $\|\tau\|_{\boldsymbol{a},N} + \|\overline{\partial}^*_E\tau\|_{\boldsymbol{a},N}$. The kernel of the morphism $\overline{\partial}^*_E : A^{*,0,q}_{\boldsymbol{a},N}(E) \to A^{*,0,q-1}_{\boldsymbol{a},N}(E)$ is denoted by $Z^{*,0,q}_{\boldsymbol{a},N}(E)$.

Let $\widetilde{A}^{0,q}_{\boldsymbol{a},N}(E)$ denote the space of the sections of $E \otimes \Omega^{0,q}$ such that τ, $\overline{\partial}_E\tau$ and $\overline{\partial}^*_E\tau$ are L^2. It is also obtained as the completion of $A^{0,q}_c(E)$ with respect to the norm $\|\tau\|'_{\widetilde{A},\boldsymbol{a},N} := \|\tau\|_{\boldsymbol{a},N} + \|\overline{\partial}_E\tau\|_{\boldsymbol{a},N} + \|\overline{\partial}^*_E\tau\|_{\boldsymbol{a},N}$. We set

$$\widetilde{Z}^{0,q}_{\boldsymbol{a},N}(E) := \operatorname{Ker}\Big(\overline{\partial}_E : \widetilde{A}^{0,q}_{\boldsymbol{a},N}(E) \longrightarrow A^{0,q+1}_{\boldsymbol{a},N}(E)\Big),$$
$$\widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E) := \operatorname{Ker}\Big(\overline{\partial}^*_E : \widetilde{A}^{0,q}_{\boldsymbol{a},N}(E) \longrightarrow A^{*,0,q-1}_{\boldsymbol{a},N}(E)\Big).$$

If N is as in Lemma 21.2.1, for any $q \geqslant 1$, $\|\tau\|_{\widetilde{A},\boldsymbol{a},N} := \|\overline{\partial}_E\tau\|_{\boldsymbol{a},N} + \|\overline{\partial}^*_E\tau\|_{\boldsymbol{a},N}$ determines a norm equivalent to $\|\tau\|'_{\widetilde{A},\boldsymbol{a},N}$ due to (409). The Hermitian pairing $\langle\tau_1,\tau_2\rangle_{\widetilde{A},\boldsymbol{a},N} := (\overline{\partial}_E\tau_1, \overline{\partial}_E\tau_2)_{\boldsymbol{a},N} + (\overline{\partial}^*_E\tau_1, \overline{\partial}^*_E\tau_2)_{\boldsymbol{a},N}$ induces the norm $\|\cdot\|_{\widetilde{A},\boldsymbol{a},N}$.

Lemma 21.2.2. — *If N is as in Lemma* 21.2.1, *the following holds for any $q \geqslant 1$ and any $\boldsymbol{a} \in \boldsymbol{R}^\ell$.*

- *We have the decomposition $\widetilde{A}^{0,q}_{\boldsymbol{a},N}(E) = \widetilde{Z}^{0,q}_{\boldsymbol{a},N}(E) \oplus \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$.*
- *$\overline{\partial}_E : \big(\widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E), \|\cdot\|_{\widetilde{A},\boldsymbol{a},N}\big) \to \big(Z^{0,q+1}_{\boldsymbol{a},N}(E), \|\cdot\|_{\boldsymbol{a},N}\big)$ is an isomorphism.*
- *For any $\rho \in \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$, there exists $\mu \in \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E)$ such that $\rho = \overline{\partial}^*_E\mu$.*

Proof. — We use a descending induction on q. The claim is trivial for any sufficiently large q. Assume that we have already obtained the decomposition $\widetilde{A}^{0,q+1}_{\boldsymbol{a},N}(E) = \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E) \oplus \widetilde{Z}^{*,0,q+1}_{\boldsymbol{a},N}(E)$, and $\widetilde{Z}^{*,0,q+1}_{\boldsymbol{a},N}(E) \subset \overline{\partial}^*_E\big(\widetilde{Z}^{0,q+2}_{\boldsymbol{a},N}(E)\big)$. For any $\kappa \in Z^{0,q+1}_{\boldsymbol{a},N}(E)$, let us consider the linear map $F_\kappa : \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E) \to \boldsymbol{C}$ given by $F_\kappa(\tau) := (\tau,\kappa)_{\boldsymbol{a},N}$. It is continuous with respect to $\|\cdot\|_{\widetilde{A},\boldsymbol{a},N}$. By the Riesz representation theorem, there exists $\nu \in \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E)$ such that $(\tau,\kappa)_{\boldsymbol{a},N} = \langle\tau,\nu\rangle_{\widetilde{A},\boldsymbol{a},N} = (\overline{\partial}^*_E\tau, \overline{\partial}^*_E\nu)_{\boldsymbol{a},N}$ for any $\tau \in \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E)$. For any $\tau \in \widetilde{Z}^{*,0,q+1}_{\boldsymbol{a},N}(E)$, we obviously have $(\overline{\partial}^*_E\tau, \overline{\partial}^*_E\nu)_{\boldsymbol{a},N} = 0$, and we also have $(\tau,\kappa)_{\boldsymbol{a},N} = 0$ because $\tau = \overline{\partial}^*_E\tau'$ for some τ'. Hence, $(\tau,\kappa)_{\boldsymbol{a},N} = (\overline{\partial}^*_E\tau, \overline{\partial}^*_E\nu)_{\boldsymbol{a},N}$ holds for any $\tau \in \widetilde{A}^{0,q+1}_{\boldsymbol{a},N}(E)$, which implies $\kappa = \overline{\partial}_E\big(\overline{\partial}^*_E\nu\big)$. In particular, κ is contained in the image of $\overline{\partial}_E : \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E) \to Z^{0,q+1}_{\boldsymbol{a},N}(E)$. Then, $\overline{\partial}_E : \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E) \to Z^{0,q+1}_{\boldsymbol{a},N}(E)$ is an isomorphism, by definition of $\|\cdot\|_{\widetilde{A},\boldsymbol{a},N}$ and $\|\cdot\|_{\boldsymbol{a},N}$.

Due to (409), we have $\widetilde{Z}^{0,q}_{\boldsymbol{a},N}(E) \cap \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E) = 0$. Let $\omega \in \widetilde{A}^{0,q}_{\boldsymbol{a},N}(E)$. Applying the above result to $\overline{\partial}_E\omega$, we can take $\omega_1 \in \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$ such that $\overline{\partial}_E\omega_1 = \overline{\partial}_E\omega$. Then, $\omega - \omega_1 \in \widetilde{Z}^{0,q}_{\boldsymbol{a},N}(E)$. Thus, we obtain the decomposition. $\widetilde{A}^{0,q}_{\boldsymbol{a},N}(E) = \widetilde{Z}^{0,q}_{\boldsymbol{a},N}(E) \oplus \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$.

Let $\rho \in \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$. Applying the above to $\overline{\partial}_E\rho \in Z^{0,q+1}_{\boldsymbol{a},N}(E)$, we can find $\nu \in \widetilde{Z}^{0,q+1}_{\boldsymbol{a},N}(E)$ such that $\overline{\partial}_E\overline{\partial}^*_E\nu = \overline{\partial}_E\rho$. Then, $\overline{\partial}^*_E\nu \in \widetilde{Z}^{*,0,q}_{\boldsymbol{a},N}(E)$ and we have $\overline{\partial}_E(\overline{\partial}^*_E\nu - \rho) = 0$. Then, we obtain $\rho = \overline{\partial}^*_E\nu$. Thus, the induction can proceed. □

Lemma 21.2.3. — *Let N be as in Lemma* 21.2.1. *For any $\omega \in Z^{0,1}_{\boldsymbol{a},N}(E)$, we have a section f of E such that $\overline{\partial} f = \omega$ and $\|f\|_{\boldsymbol{a},N} \leqslant \|\omega\|_{\boldsymbol{a},N}$.*

Proof. — Let us consider the linear map $F_\omega : \widetilde{Z}^{0,1}_{\boldsymbol{a},N}(E) \to \boldsymbol{C}$ as in the proof of Lemma 21.2.2, which is continuous with respect to $\|\cdot\|_{\widetilde{A},\boldsymbol{a},N}$. By the Riesz representation theorem, there exists $\rho \in \widetilde{Z}^{0,1}_{\boldsymbol{a},N}(E)$ with $\|\rho\|_{\widetilde{A},\boldsymbol{a},N} \leqslant \|\omega\|_{\boldsymbol{a},N}$, such that $(\tau,\omega)_{\boldsymbol{a},N} = (\overline{\partial}^*_E\tau, \overline{\partial}^*_E\rho)_{\boldsymbol{a},N}$. As in the proof of Lemma 21.2.2, we have $\omega = \overline{\partial}_E(\overline{\partial}^*_E\rho)$. Since we have $\|\overline{\partial}^*_E\rho\|_{\boldsymbol{a},N} = \|\rho\|_{\widetilde{A},\boldsymbol{a},N} \leqslant \|\omega\|_{\boldsymbol{a},N}$, we are done. □

21.2.2. The case where N is sufficiently positive. — Let $\pi_i : X \setminus D \to D_i$ denote the natural projection for $i = 1, \ldots, n$. We put $D_i^\circ := D_i \setminus \bigcup_{j\neq i, j\leqslant \ell} D_j$. Let P be a point of D_i°, then we obtain the curve $\pi_i^{-1}(P)$ which is isomorphic to Δ^* $(i \leqslant \ell)$ or Δ $(\ell < i \leqslant n)$. Let Δ'' denote the Laplacian on $\pi_i^{-1}(P)$ with respect to the Euclidean metric. The restriction of the metric $h_{\boldsymbol{a},N}$ to $E_{|\pi_i^{-1}(P)}$ is also denoted by $h_{\boldsymbol{a},N}$. Similarly, the restriction of $|\cdot|_{\boldsymbol{a},N}$ to $\pi_i^{-1}(P)$ is denoted by the same symbol. We can easily show the following lemma. (See Lemma 2.49 and Corollary 2.50 of [**67**].)

Lemma 21.2.4. — *There exists a positive constant N_0, depending only on the estimate of the curvature $R(h)$, such that the following holds:*

- *Let P be a point of D_i°, and U be an open subset of the curve $\pi_i^{-1}(P)$. Let F be a holomorphic section of $E_{|U}$. Then the inequalities $\Delta''|F|^2_{\boldsymbol{a},N} \leqslant 0$ and $\Delta''\log|F|_{\boldsymbol{a},N} \leqslant 0$ hold on U for any $N \geqslant N_0$. In particular, $|F|^2_{\boldsymbol{a},N}$ and $\log|F|_{\boldsymbol{a},N}$ are subharmonic on U.*

As a result, for any holomorphic section F of E, the functions $\log|F|_{\boldsymbol{a},N}$ and $|F|_{\boldsymbol{a},N}$ are pluri-subharmonic, if $N \geqslant N_0$. □

We explain various consequences of this pluri-subharmonicity. Since we will be interested in the behaviour of holomorphic sections around the origin O, we put $X(C) := \big\{(z_1,\ldots,z_n) \in X \,\big|\, |z_i| \leqslant C\big\}$, $D_i(C) := D_i \cap X(C)$ and $D(C) = D \cap X(C)$ for any $0 < C < 1$, and we will consider the restrictions of holomorphic sections to $X(C)^* := X(C) \setminus D(C)$.

21.2.2.1. L^2-estimates on curves and on $X \setminus D$. — The following corollary allows us to derive the L^2-property on curves from the L^2-property on $X \setminus D$, which easily follows from the pluri-subharmonicity in Lemma 21.2.4. (See Corollary 2.51 of [**67**].)

Corollary 21.2.5. — *Let F be a holomorphic section of E on $X \setminus D$ such that $F \in A^{0,0}_{\boldsymbol{a},N}(E)$. Let N_0 be as in Lemma 21.2.4, and let $M > \max\{N_0, N\}$. For any*

$1 \leqslant j \leqslant \ell$ and $P \in D_j^\circ$, we have the following finiteness:

$$\int_{\pi_j^{-1}(P)\cap X^*(C)} \big|F_{|\pi_j^{-1}(P)\cap X^*(C)}\big|^2_{\boldsymbol{a},M} \,\mathrm{dvol}_{\pi_j^{-1}(P)} < \infty.$$

Here $\mathrm{dvol}_{\pi_j^{-1}(P)}$ denotes the volume form of $\pi_j^{-1}(P)$ with respect to the restriction $g_{\mathbf{p}\,|\,\pi_j^{-1}(P)}$. □

21.2.2.2. From L^2-estimate to growth estimate on a curve. — Let us consider the case $X = \Delta$ and $D = \{O\}$. We can show the following lemma by the argument in Lemma 7.12 of [**66**], although it is stated for acceptable bundles induced by tame harmonic bundles.

Lemma 21.2.6. — *Let f be a holomorphic section of E on $X\smallsetminus D$ such that $\|f\|_{b,N} < \infty$ for some $b, N \in \boldsymbol{R}$. Let N_0 be as in Lemma 21.2.4, and $M > \max\{N_0, |N|+2\}$. Then, the following holds on $X(1/4)$:*

$$|f(z)|_h^2 \leqslant B\|f\|_{b,N}^2 |z|^{-2b}(-\log|z|)^M.$$

Here, $B > 0$ is independent of f.

Proof. — Let dvol (resp. $\mathrm{dvol}_{g_{\mathbf{p}}}$) denote the volume form associated to Euclidean metric (resp. Poincaré metric). From the subharmonicity, we obtain the following inequalities for any $0 < |z| \leqslant 1/4$:

$$\begin{aligned}(410)\quad \log|f(z)|^2_{b,M} &\leqslant \frac{4}{\pi|z|^2}\int_{|w-z|\leqslant|z|/2} \log|f(w)|^2_{b,M}\,\mathrm{dvol}\\ &\leqslant \log\Big(\frac{4}{\pi|z|^2}\int_{|w-z|\leqslant|z|/2}|f(w)|^2_{b,M}\,\mathrm{dvol}\Big) \leqslant \log\Big(\frac{20}{\pi}\int_{|w-z|\leqslant|z|/2}|f(w)|^2_{b,N}\,\mathrm{dvol}_{g_{\mathbf{p}}}\Big)\\ &\leqslant \log\Big(\frac{20}{\pi}\|f\|_{b,N}\Big).\end{aligned}$$

Thus, we are done. □

21.2.2.3. Refinement of growth estimates on a curve. — Let us consider the case $X = \Delta$ and $D = \{O\}$. We can show the following lemma by the argument in Lemma 7.13 of [**66**], although it is stated for acceptable bundles induced by tame harmonic bundles.

Lemma 21.2.7. — *Let f be a holomorphic section such that $|f|_h = O\big(|z|^{-a-\varepsilon}\big)$ for any $\varepsilon > 0$ on $X(C)$. Let N_0 be as in Lemma 21.2.4, and let $M > N_0$. We set $H(z) := |f(z)|_h^2 \cdot |z|^{2a} \cdot (-\log|z|)^{-M}$. Then, $H(z)$ is bounded. More strongly, $\max_{|z|\leqslant C'}|H(z)| = \max_{|z|=C'} H(z)$ for any $0 < C' < C$.*

Proof. — We put $H_\varepsilon := H(z)\cdot|z|^{2\varepsilon}$ for any $\varepsilon > 0$. We have the subharmonicity of $\log H_\varepsilon$ on $X^*(C)$. By the assumption, we have $\lim_{|z|\to 0}\log H_\varepsilon(z) = -\infty$. Hence, $\max_{|z|\leqslant C'}|H_\varepsilon(z)| = \max_{|z|=C'} H_\varepsilon(z)$ for any $0 < C' < C$. By taking the limit for $\varepsilon \to 0$, we obtain the desired estimate. □

21.2.2.4. Growth estimates on curves and on $X \setminus D$. — Let us return to the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell} D_i$. We would like to remark that the growth estimates on curves imply the growth estimate on $X \setminus D$. The following lemma is a refinement of Corollary 2.53 of [**67**].

Proposition 21.2.8. — *Let F be a holomorphic section of E on $X \setminus D$. Assume that we are given numbers $a_i \in \boldsymbol{R}$ $(i = 1, \dots, \ell)$ such that the following holds:*

- $\big|F_{|\pi_i^{-1}(P)}\big|_h = O\big(|z_i|^{-a_i-\varepsilon}\big)$ *for any $\varepsilon > 0$ and for any $i = 1, \dots, \ell$ and $P \in D_i^{\circ}$.*

Let N_0 be as in Lemma 21.2.4, and let $M > N_0$. Take $0 < C < 1$. Then, there exists a constant B, which is independent of F, such that the following holds on $X^*(C)$:

$$\big|F\big|_h^2 \leqslant B \cdot \prod_{j=1}^{\ell} |z_j|^{-2a_j} \cdot \big(-\log|z_j|\big)^M \cdot \max_{\substack{|z_j|=C\\ 1\leqslant j\leqslant \ell}} \big|F\big|_h^2.$$

Proof. — We only have to use Lemma 21.2.7 inductively. □

Corollary 21.2.9. — *Let F be a holomorphic section of E over $X \setminus D$ such that $\|F\|_{\boldsymbol{a},N} < \infty$. Let N_0 be as in Lemma 21.2.4, and let $M > N_0$. Then, the following holds on $X^*(C)$ for some $0 < C < 1$:*

$$\big|F\big|_h^2 \leqslant B \cdot \prod_{j=1}^{\ell} |z_j|^{-2a_j} \cdot \big(-\log|z_j|\big)^M \cdot \max_{\substack{|z_j|=C\\ 1\leqslant j\leqslant \ell}} \big|F\big|_h^2.$$

In particular, $F \in {}_{\boldsymbol{a}}E$. (See Section 21.3 for ${}_{\boldsymbol{a}}E$.)

Proof. — We obtain the L^2-estimates for the restrictions of F to curves transversal to the smooth part of D, due to Corollary 21.2.5. This implies the growth estimates of the restriction to curves due to Lemma 21.2.6. Then, we obtain the growth estimate of $|F|_h$ on $X^*(C)$ by Proposition 21.2.8. □

21.3. Prolongation to filtered bundle

Let $X := \Delta^n$, and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle on $X \setminus D$ of rank r. We naturally identify E with the sheaf of its holomorphic sections. Let $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$. The i-th components of $\boldsymbol{a}$ are denoted by a_i. For any open subset $U \subset X$, we define

$${}_{\boldsymbol{a}}E(U) := \Big\{ f \in E(U \setminus D) \,\Big|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z|^{-a_i-\varepsilon}\Big)\ \forall \varepsilon > 0 \Big\}.$$

By taking sheafification, we obtain an $\mathcal{O}_X$-module ${}_{\boldsymbol{a}}E$. In the case $\boldsymbol{a} = (0, \dots, 0)$, we prefer the symbol ${}^{\diamond}E$.

Theorem 21.3.1. — *Assume that* $\det(E, \overline{\partial}_E, h)$ *is flat, for simplicity. Then,* ${}_{\boldsymbol{a}}E$ *is locally free for any* $\boldsymbol{a} \in \boldsymbol{R}^\ell$*. Moreover, the family* $\{{}_{\boldsymbol{a}}E \,|\, \boldsymbol{a} \in \boldsymbol{R}^\ell\}$ *forms a filtered bundle.*

This theorem implies that (i) the images ${}^iF_b({}_{\boldsymbol{a}}E_{|D_i})$ of ${}_{\boldsymbol{a}(i,b)}E_{|D_i} \to {}_{\boldsymbol{a}}E_{|D_i}$ are subbundles for any $b \in]a_i - 1, a_i]$, where the j-the components of $\boldsymbol{a}(i,b)$ and $\boldsymbol{a}$ are equal for $j \neq i$, and the i-th component of $\boldsymbol{a}(i,b)$ is b, (ii) the induced filtrations ${}^iF$ $(i = 1, \dots, \ell)$ of ${}_{\boldsymbol{a}}E_{|D_i}$ are compatible in the sense of Definition 4.37 of [**67**]. (See Section 2.5.3.)

The tuple of filtrations $({}^iF \,|\, i = 1, \dots, \ell)$ is often denoted by $\boldsymbol{F}$ in this situation. Let $\boldsymbol{v} = (v_1, \dots, v_r)$ be a frame of ${}_{\boldsymbol{a}}E$ compatible with $\boldsymbol{F}$. Namely, the numbers $a_i(v_k)$ are attached to any v_k and $i = 1, \dots, \ell$, such that $\{v_{k|D_i} \,|\, a_i(v_k) \leqslant b\}$ gives a frame of ${}^iF_b({}_{\boldsymbol{a}}E_{|D_i})$ for each pair of $i = 1, \dots, \ell$ and $b \in]a_i - 1, a_i]$. The numbers $a_j(v_i)$ are often denoted by ${}^j\deg^F(v_i)$ in this situation, and called the degree with respect to ${}^jF$. We set

$$v'_k := v_k \cdot \prod_{i=1}^{\ell} |z_i|^{a_i(v_k)}. \tag{411}$$

Let $H(h, \boldsymbol{v}')$ denote the Hermitian-matrix valued function whose (p,q)-entries are given by $h(v'_p, v'_q)$.

Theorem 21.3.2. — *There exist positive constants* C *and* N *such that the following holds around* O*:*

$$C^{-1} \cdot \Big(-\sum_{i=1}^{\ell} \log|z_i|\Big)^{-N} \leqslant H(h, \boldsymbol{v}') \leqslant C \cdot \Big(-\sum_{i=1}^{\ell} \log|z_i|\Big)^{N}. \tag{412}$$

In other words, $\boldsymbol{v}'$ *is adapted to* h *up to log order.*

The holomorphic bundle $\mathrm{End}(E)$ with the induced Hermitian metric h is also acceptable. We can show the following proposition by using the weak norm estimate (Theorem 21.3.2).

Proposition 21.3.3. — ${}^\diamond\mathrm{End}(E)$ *is naturally isomorphic to the sheaf of the endomorphisms* f *of* ${}_{\boldsymbol{a}}E$ *for any* $\boldsymbol{a} \in \boldsymbol{R}^\ell$*, such that* $f_{|D_i}$ *preserves the filtration* ${}^iF$ *for each* $i = 1, \dots, \ell$*.* □

We will prove Theorems 21.3.1 and 21.3.2 in Sections 21.4–21.7.

21.4. One dimensional case

21.4.1. Weak norm estimate. — In the one dimensional case, Theorem 21.3.1 was proved by Simpson in [**81**]. (See also [**82**]). For any $a \in \boldsymbol{R}$, we have the associated vector bundle ${}_aE$ with the parabolic filtration F. We set

$$\mathcal{P}ar({}_aE) := \big\{b \in]a-1, a] \,\big|\, \mathrm{Gr}^F_b({}_aE) \neq 0\big\}.$$

Moreover, he showed the functoriality with respect to dual, tensor product and direct sum. In particular, if $\boldsymbol{v}$ is a frame of ${}_aE$ compatible with the parabolic filtration, the dual frame $\boldsymbol{v}^\vee$ of $E^\vee$ naturally gives a frame of ${}_{-a-1+\varepsilon}E^\vee$ for some sufficiently small $\varepsilon > 0$. Let us prove Theorem 21.3.2 in the one dimensional case, by using this fact. Let $\boldsymbol{v}' = (v_i')$ be as in (411). By using Lemma 21.2.7, we obtain $H(h, \boldsymbol{v}') \leqslant C_1 \cdot \bigl(-\log|z|\bigr)^{M_1}$ for some $C_1, M_1 > 0$. We also obtain $H(h^\vee, \boldsymbol{v}^{\vee\prime}) \leqslant C_2 \cdot \bigl(-\log|z|\bigr)^{M_2}$, where $h^\vee$ denotes the induced metric of $E^\vee$. Then, we obtain that $\boldsymbol{v}'$ is adapted.

21.4.2. Pull-back and descent. — For any positive integer c, let $\psi_c : X \to X$ be given by $\psi_c(z) = z^c$. We put $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{h}) := \psi_c^{-1}(E, \overline{\partial}_E, h)$, which is also acceptable. For any $a, b \in \boldsymbol{R}$, let $\nu(a,b) \in \mathbb{Z}$ be determined by $b - 1 < \nu(a,b) + a \leqslant b$. For any $b \in \boldsymbol{R}$, it is easy to check

$$ {}_b\widetilde{E} = \sum_a z^{-\nu(ca,b)} \cdot \psi_c^{-1}({}_aE). $$

More concretely, let $\boldsymbol{v}$ be a frame of ${}^\diamond E$ compatible with the parabolic filtration F. Let $a_i := \deg^F(v_i)$. We set

$$ \widetilde{v}_i := z^{-\nu(ca_i,b)} \cdot \psi_c^{-1}(v_i). $$

It is easy to check that $\widetilde{\boldsymbol{v}} = (\widetilde{v}_i)$ is a frame of ${}_b\widetilde{E}$ compatible with the parabolic filtration, by using the weak norm estimate. Hence, we have

$$ \mathcal{P}ar({}_b\widetilde{E}) = \bigl\{ ca + \nu(ca,b) \bigm| a \in \mathcal{P}ar({}^\diamond E) \bigr\}. $$

Let $\mu_c := \bigl\{ w \in \boldsymbol{C} \bigm| w^c = 1 \bigr\}$. Let ω be a generator. We have the natural μ_c-action on X given by the multiplication, which is lifted to an action on ${}_b\widetilde{E}$. Note $(\omega^{-1})^*(\widetilde{v}_i) = \omega^{\nu(ca_i,b)}\widetilde{v}_i$.

Assume that $0 \leqslant b < 1/2$. If c is a sufficiently large integer, we have (i) $0 \leqslant \nu(ca,b) \leqslant c-1$, (ii) $\mathcal{P}ar({}^\diamond E) \to \mathbb{Z}$ given by $\nu(ca,b)$ is injective. We have the decomposition

$$ {}_b\widetilde{E}_{|O} = \bigoplus_{0 \leqslant p \leqslant c-1} V_p, \quad V_p = \bigl\langle \widetilde{v}_{j|O} \bigm| \nu(ca_j, b) = p \bigr\rangle, \tag{413} $$

where ω acts on V_p as the multiplication by ω^p.

For each $p \in \bigl\{0 \leqslant p \leqslant c-1 \bigm| V_p \neq 0\bigr\}$, there exists $\chi(p) \in \mathcal{P}ar({}^\diamond E)$ such that $p = \nu\bigl(\chi(p)a, b\bigr)$. Thus, we obtain the map

$$ \chi : \bigl\{0 \leqslant p \leqslant c-1 \bigm| V_p \neq 0\bigr\} \longrightarrow \mathcal{P}ar({}^\diamond E). $$

By setting $\varphi(p) := \nu\bigl(\chi(p)a, b\bigr) + \chi(p)a \in \mathcal{P}ar({}_bE)$, we obtain the map

$$ \varphi : \bigl\{0 \leqslant p \leqslant c-1 \bigm| V_p \neq 0\bigr\} \longrightarrow \mathcal{P}ar({}_b\widetilde{E}). $$

The decomposition (413) gives a splitting of the parabolic filtration F of ${}_b\widetilde{E}$ in the following sense:

$$F_d({}_b\widetilde{E}_{|O}) = \bigoplus_{\varphi(p)\leqslant d} V_p.$$

Conversely, let $\widetilde{\boldsymbol{u}}$ be a frame of ${}_b\widetilde{E}$. Assume that it is μ_c-equivariant in the sense that $\omega^*\widetilde{u}_j = \omega^{-p_j}\cdot\widetilde{u}_j$ for some $0\leqslant p_j\leqslant c-1$. Note that $\widetilde{u}_{j|O}\in V_{p_j}$, and hence the frame $\widetilde{\boldsymbol{u}}$ is compatible with the filtration F. We set $u_j := z^{p_j}\cdot\widetilde{u}$. Since u_j is μ_c-invariant, it induces a section of E, which is also denoted by u_j. By using the weak norm estimate, it is easy to check that $\boldsymbol{u}=(u_j)$ is a frame of ${}^\diamond E$ compatible with the parabolic filtration.

21.4.3. Parabolic structure and monodromy. — Let $X:=\Delta$ and $D:=\{O\}$. Let $(E,\overline{\partial}_E,h)$ be an acceptable bundle of rank R on $X\setminus D$.

Lemma 21.4.1. — *There exists a C^1-orthonormal frame $\boldsymbol{e}$ of E with the following properties.*
- *There exists a diagonal matrix Γ whose (i,i)-entries α_i satisfy $0\leqslant\alpha_i<1$.*
- *Let A be determined by*

$$\overline{\partial}_E\boldsymbol{e} = \boldsymbol{e}\cdot\Big(-\frac{\Gamma}{2}+A\Big)\cdot\frac{d\overline{z}}{\overline{z}}.$$

Then $A = O\big((-\log|z|)^{-1}\big)$.

Proof. — See Lemma 7.10 of [**66**] with the simplified proof due to the referee of the paper. Although the lemma is stated for tame harmonic bundles, it holds for any acceptable bundles. □

We set $S(\Gamma) := \big\{-\alpha_1,\dots,-\alpha_R\big\}$.[(2)] We recall the following lemma. Although it is also stated for tame harmonic bundles, it holds for any acceptable bundles.

Lemma 21.4.2 **(Lemma 7.17, [66]).** — *$\mathcal{P}ar({}^\diamond E)=S(\Gamma)$ holds. The multiplicity of an eigenvalue α_i is equal to* $\dim\mathrm{Gr}^F_{-\alpha_i}(E)$. □

For $0<r<1$, let $P(r)$ denote the characteristic polynomial of the monodromy along the loop $r\cdot e^{\sqrt{-1}\theta}$ $(0\leqslant\theta\leqslant 2\pi)$ of the unitary connection $\partial_E+\overline{\partial}_E$. We have the limit $\lim_{r\to 0}P(r)$ whose roots are $\alpha_1,\dots,\alpha_R$. By Lemma 21.4.2, $\lim_{r\to0}P(r)$ determines $\mathcal{P}ar({}^\diamond E)$.

2. $S(\Gamma)$ in Page 75 of [**66**] should be corrected.

21.4.4. Control of the parabolic weights in the higher dimensional case

Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let $\pi : X \to D$ be the projection. Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle on $X \setminus D$. For any point $Q \in D$, we have the acceptable bundle $(E_Q, \overline{\partial}_Q, h_Q) := (E, \overline{\partial}_E, h)_{|\pi^{-1}(Q)\setminus D}$.

Lemma 21.4.3. — *The following holds for any $Q_i \in D$ $(i = 1, 2)$:*

$$\mathcal{P}ar({}^\diamond E_{Q_1}) := \mathcal{P}ar({}^\diamond E_{Q_2}), \quad \dim \mathrm{Gr}^F_a({}^\diamond E_{Q_1}) = \dim \mathrm{Gr}^F_a({}^\diamond E_{Q_2}).$$

Proof. — Let $\nabla := \overline{\partial}_E + \partial_E$ denote the unitary connection associated to $(E, \overline{\partial}_E, h)$. We use the natural identification $X = \Delta_{z_1} \times D$. For $0 < r < 1$, let $P(r, Q_i)$ denote the characteristic polynomials of the monodromy along the loops $\big(r \cdot e^{\sqrt{-1}\theta}, Q_i\big)$ $(0 \leqslant \theta \leqslant 2\pi)$. We only have to show $\lim_{r\to 0} P(r, Q_1) = \lim_{r\to 0} P(r, Q_2)$.

Let $\gamma(t) : [0, 1] \to D$ be a C^∞-map such that $\gamma(0) = Q_1$ and $\gamma(1) = Q_2$. For any $0 < r < 1$, we have the map

$$\Phi_r : [0, 1] \times [0, 2\pi] \longrightarrow X \setminus D, \quad \Phi_r(t, \theta) = \big(r \cdot e^{\sqrt{-1}\theta}, \gamma(t)\big).$$

Take any $v \in E_{|(r,Q)}$ such that $|v| = 1$. We have the section $V(r)$ of $\Phi_r^* E$ determined by the following conditions:

$$V(r)_{|(r,Q)} = v, \quad \nabla(\partial_\theta)V(r) = 0, \quad \big(\nabla(\partial_t)V(r)\big)_{|\theta=0} = 0.$$

Let R denote the curvature of ∇. Then, we have

$$\nabla(\partial_\theta)\nabla(\partial_t)V(r) = \Phi_r^* R(\partial_t, \partial_\theta)V(r) = O\big((-\log r)^{-1}\big).$$

Hence, we obtain the following estimate with respect to h:

$$\nabla(\partial_t)V(r)_{|\theta=2\pi} = O\big((-\log r)^{-1}\big).$$

Then, we obtain $\lim_{r\to 0} P(r, Q_1) = \lim_{r\to 0} P(r, Q_2)$. □

21.5. Extension of holomorphic sections I

21.5.1. Statement. — We set $X_0 := \Delta_z^{n-1} = \big\{(z_1, \dots, z_{n-1}) \,\big|\, |z_i| < 1\big\}$, and $D_0 := \bigcup_{i=1}^{\ell}\{z_i = 0\}$ for some $\ell \leqslant n - 1$. Let $\Delta_w := \big\{w \in \boldsymbol{C} \,\big|\, |w| < 1\big\}$. We put $X := X_0 \times \Delta_w$ and $D := D_0 \times \Delta_w$. We also set $X^{(0)} := X_0 \times \{0\}$ and $D^{(0)} := D_0 \times \{0\}$.

Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle on $X \setminus D$. Let f be a holomorphic section of ${}^\diamond(E_{|X^{(0)}\setminus D^{(0)}})$ on $X^{(0)} \setminus D^{(0)}$. Take $0 < R < 1$, and we set $X(R) := \big\{|z_i| < R, |w| < R\big\}$. The following lemma is the counterpart of Lemma 8.51 of [**67**].

Lemma 21.5.1. — *There exists a holomorphic section F of ${}^\diamond E$ on $X(R)$ such that $F_{|X(R)\cap(X^{(0)}\setminus D^{(0)})} = f$.*

21.5.2. Construction of a cocycle. — The argument is essentially the same as that in [**67**]. We have to modify the construction of an appropriate cocycle (Section 8.4.2 of [**67**]). We can argue as follows.

We use the polar coordinate $w = r \cdot e^{\sqrt{-1}\theta}$ for Δ_w. Let ∂_r denote the vector field $\partial/\partial r$. Let f be a holomorphic section of ${}^\diamond(E_{|X^{(0)}\setminus D^{(0)}})$. By using Proposition 21.2.8, we obtain that f is bounded up to log order, i.e., the following holds for some $C, N > 0$:

$$|f|_h \leqslant C \cdot \Big(-\sum_{i=1}^{\ell} \log|z_i|\Big)^N.$$

By using the unitary connection $\nabla := \overline{\partial}_E + \partial_E$, we extend f to a continuous section F of E on $X \setminus D$ by the condition $\nabla_{\partial_r} F = 0$ and $F_{|X_0} = f$. We have $|F(z,w)|_h \leqslant |f(z)|_h$ by the construction.

Lemma 21.5.2. — *We have the following estimate on $X \setminus \big(D \cup (X^{(0)})\big)$ for some $C > 0$ with respect to h and $g_{\mathbf{p}}$:*

$$\big|\nabla F(z,w)\big| \leqslant C \cdot \big|f(z)\big| \cdot |w|.$$

Proof. — In the following, $C_i > 0$ denote positive constants. We put $\overline{v}_i := (-\log|z_i|) \cdot \overline{z}_i \partial/\partial \overline{z}_i$ $(i = 1, \dots, \ell)$ and $\overline{v}_i := \partial/\partial \overline{z}_i$ $(i = \ell+1, \dots, n-1)$ for simplicity of description. Because $\nabla_{\partial_r} F = 0$ and $[\partial_r, \overline{v}_i] = 0$, we have $\nabla_{\partial_r} \nabla_{\overline{v}_i} F = R(h)(\partial_r, \overline{v}_i) F$. Hence, we have

$$\frac{\partial}{\partial r} \big|\nabla_{\overline{v}_i} F\big|_h^2(z,w) = 2\operatorname{Re} h\big(\nabla_{\partial_r}\nabla_{\overline{v}_i} F, \nabla_{\overline{v}_i} F\big) = 2\operatorname{Re} h\Big(R(h)(\partial_r, \overline{v}_i)F, \nabla_{\overline{v}_i} F\Big) \leqslant C_1 \cdot \big|f(z)\big|_h \cdot \big|\nabla_{\overline{v}_i} F\big|_h(z,w).$$

Note $\nabla_{\overline{v}_i} F(z,0) = 0$. Hence, we obtain $\big|\nabla_{\overline{v}_i} F\big|_h(z,w) \leqslant C_2 \cdot \big|f(z)\big|_h \cdot r$. We also have

$$\frac{\partial}{\partial r} \big|\nabla_{\partial_\theta} F\big|_h^2(z,w) = 2\operatorname{Re} h\Big(R(h)(\partial_r, \partial_\theta)F, \nabla_{\partial_\theta} F\Big) \leqslant C_3 \cdot r \cdot \big|f(z)\big|_h \cdot \big|\nabla_{\partial_\theta} F\big|_h(z,w).$$

Note $\nabla_{\partial_\theta} F(z,0) = 0$ on the real blow up of $X \setminus D$ along $X^{(0)} \setminus D^{(0)}$. Hence, we obtain $\big|\nabla_{r^{-1}\partial_\theta} F\big|_h \leqslant C_4 \cdot r \cdot |f(z)|_h$. Thus we are done. □

We take a C^∞-metric $h_{\mathcal{O}(-X^{(0)})}$ of the line bundle $\mathcal{O}\big(-X^{(0)}\big)$ on X. We naturally regard $\overline{\partial} F(z,w)$ as a section of $E \otimes \mathcal{O}\big(-X^{(0)}\big) \otimes \Omega^{0,1}_{X\setminus D}$ on $X \setminus D$. Then, the norm of $\overline{\partial} F(z,w)$ is bounded with respect to $h' = h \otimes h_{\mathcal{O}(-X^{(0)})}$ and $g_{\mathbf{p}}$ up to polynomial orders in $-\log|z_i|$ $(i = 1, \dots, \ell)$.

Let χ be a non-negative C^∞-function on Δ_w such that $\chi(w) = 1$ for $|w| \leqslant 1/2$ and $\chi(w) = 0$ for $|w| \geqslant 2/3$. We obtain the following C^1-section of E on $X \setminus D$:

$$\rho := \chi \cdot F(z,w).$$

Then, ρ satisfies the following properties in Proposition 8.40 in [**67**].

- ρ is bounded up to polynomial order in $-\log|z_i|$ $(i = 1, \dots, \ell)$ with respect to h.

- We regard $\overline{\partial}\rho$ as a section of $E \otimes \mathcal{O}_X(-X^{(0)}) \otimes \Omega^{0,1}$ on $X \setminus D$. Then, it is bounded with respect to $h \otimes h_{\mathcal{O}(-X^{(0)})}$ and $g_{\mathbf{p}}$ up to polynomial order in $-\log|z_i|$ $(i = 1, \dots, \ell)$.

21.5.3. Proof of Lemma 21.5.1. — By using ρ as above, Lemma 21.4.3, and the argument in Sections 8.4.3–8.4.4 of [**67**] we can show the extension property of holomorphic sections as in Proposition 8.46 and Lemma 8.51 of [**67**]. To explain how to use Lemma 21.4.3 and the above cocycle, we give an outline.

Let $g_{\mathbf{p}}$ be the Poincaré metric of $X \setminus D$. Let π_i denote the projection of $X \setminus D$ to $\{z_i = 0\} \times \Delta_w$. Let D_i° be the image. For any point $Q \in D_i^\circ$, let $(E_Q, \overline{\partial}_Q, h_Q)$ denote the restriction of $(E, \overline{\partial}_E, h)$ to $\pi_i^{-1}(Q) \setminus D$. By Lemma 21.4.3, the set $\mathcal{P}ar(^\diamond E, i) := \mathcal{P}ar(^\diamond E_Q)$ is independent of the choice of $Q \in D_i^\circ$.

For $\boldsymbol{a} \in \boldsymbol{R}^\ell$ and $N \in \boldsymbol{R}$, we consider the following metrics:

$$h_{\boldsymbol{a},N} = h \cdot \prod_{i=1}^{\ell} |z_i|^{2a_i} \big(-\log|z_i|^2\big)^{-N} \cdot \prod_{i=\ell+1}^{n-1} (1 - |z_i|^2)^{-N} \cdot (1 - |w|^2)^{-N}$$

$$\widetilde{h}_{\boldsymbol{a},N} := h_{\boldsymbol{a},N} \otimes h_{\mathcal{O}(-X^{(0)})}$$

Let $\|\cdot\|_{\widetilde{h}_{\boldsymbol{a},N}}$ denote the L^2-norm with respect to $\widetilde{h}_{\boldsymbol{a},N}$ and $g_{\mathbf{p}}$. As in Section 21.2.1, due to the argument in Section 2.8.6 of [**67**], there exists a negative number N_0 such that the following holds for any $N < N_0$ and for any C^∞-section η of $E \otimes \mathcal{O}(-X^{(0)}) \otimes \Omega^{0,1}$ with compact support:

$$\|\overline{\partial}_E \eta\|^2_{\widetilde{h}_{\boldsymbol{a},N}} + \|\overline{\partial}^*_E \eta\|^2_{\widetilde{h}_{\boldsymbol{a},N}} \geqslant \|\eta\|^2_{\widetilde{h}_{\boldsymbol{a},N}}.$$

According to Andreotti-Vesentini (Propositions 21.1.1 and 21.1.2), it implies the following:

- Let $N < N_0$. Let $\boldsymbol{a}$ be any element of $\boldsymbol{R}^\ell$. Let τ be a $\overline{\partial}_E$-closed section of $E \otimes \mathcal{O}(-X^{(0)}) \otimes \Omega^{0,1}$ which is L^2 with respect to $\widetilde{h}_{\boldsymbol{a},N}$ and $g_{\mathbf{p}}$. Then, there exists a section ω of $E \otimes \mathcal{O}(-X^{(0)})$ such that (i) it is L^2 with respect to $\widetilde{h}_{\boldsymbol{a},N}$ and $g_{\mathbf{p}}$, (ii) $\overline{\partial}_E \omega = \tau$.

We take $N < N_0$. We take a small number $\varepsilon > 0$ such that $]-1, -1+\varepsilon] \cap \mathcal{P}ar(^\diamond E, i) = \varnothing$ for any $i = 1, \dots, \ell$. Let $\boldsymbol{\delta} = (1, \dots, 1) \in \boldsymbol{R}^\ell$. We regard $\overline{\partial}_E \rho$ as an L^2-section of $E \otimes \mathcal{O}(-X^{(0)}) \otimes \Omega^{0,1}$ with respect to $\widetilde{h}_{\varepsilon\boldsymbol{\delta},N}$. Then, we can find a section ω of $E \otimes \mathcal{O}(-X^{(0)})$ such that (i) it is L^2 with respect to $\widetilde{h}_{\varepsilon\boldsymbol{\delta},N}$ and $g_{\mathbf{p}}$, (ii) $\overline{\partial}_E \omega = \overline{\partial}_E \rho$.

We obtain a section $F := \rho - \omega$ of E on $X \setminus D$, which is L^2 with respect to $h_{\varepsilon\boldsymbol{\delta},N}$ and $g_{\mathbf{p}}$. By our construction, $\overline{\partial}_E F = 0$. By using the L^2-property of ω as a section of $E \otimes \mathcal{O}(-X_0)$, we obtain that $\omega_{|X^{(0)} \setminus D^{(0)}} = 0$. Therefore, $F_{|X^{(0)} \setminus D^{(0)}} = f$.

By using Corollary 2.51 of [**67**], we obtain that the restrictions $F_{|\pi_i^{-1}(Q) \setminus D}$ are L^2 with respect to $h_{\varepsilon\boldsymbol{\delta},N}$ for any $Q \in D_i^\circ$. By Lemma 21.2.6, we obtain

$$F_{|\pi_i^{-1}(Q)} \in \varepsilon(E_{|\pi_i^{-1}(Q)}).$$

Because $]-1,-1+\varepsilon]\cap \mathcal{P}ar({}^\diamond E,i)=\varnothing$ for any $i=1,\dots,\ell$, we obtain

$$F_{|\pi_i^{-1}(Q)}\in {}^\diamond(E_{|\pi_i^{-1}(Q)}).$$

Hence, F is a section of ${}^\diamond E$ due to Proposition 21.2.8. □

21.6. Extension of holomorphic sections II

21.6.1. Statement. — We put $X:=\Delta^n$ and $D:=\bigcup_{i=1}^{\ell}D_i$. We also set

$$X^{(1)}:=\big\{(z_1,\dots,z_n)\in X\,\big|\,z_1=z_2\big\},\quad D^{(1)}:=X^{(1)}\cap D,$$
$$X_0:=\big\{(z_1,\dots,z_n)\in X\,\big|\,z_1=z_2=0\big\}.$$

Condition 21.6.1. — Let $(E,\overline{\partial}_E,\theta)$ be an acceptable bundle over $X\setminus D$. Let ε_1 and ε_2 be positive numbers such that $\varepsilon_1+\varepsilon_2<1$. Assume that $\mathcal{P}ar\big({}^\diamond E,i\big)$ is contained in $]-\varepsilon_i,0]$ for $i=1,2$. □

We remark $\mathcal{P}ar\big(E,i\big)\cap\,]0,1-\varepsilon_i]=\varnothing$ under the condition. The following proposition is the counterpart of Proposition 8.46 of [**67**].

Proposition 21.6.2. — *Let $(E,\overline{\partial}_E,h)$ be an acceptable bundle satisfying Condition* 21.6.1. *Let f be a holomorphic section of ${}^\diamond(E_{|X^{(1)}\setminus D^{(1)}})$ on $X^{(1)}$. Then there exists a neighbourhood U of X_0 in X, and there exists a holomorphic section $\widetilde{f}\in\Gamma(U,{}^\diamond E)$, such that $\widetilde{f}_{|X^{(1)}\cap U}=f_{|X^{(1)}\cap U}$.*

21.6.2. Preliminary from geometry on the blow up

21.6.2.1. Metrics and the curvatures of $\mathcal{O}_{\mathbb{P}^1}(i)$. — We recall the contents in Section 4.7.3 in [**65**] with minor corrections. Let $\mathbb{P}^1$ denote the one dimensional complex projective space. We use the homogeneous coordinate $[t_0:t_1]$. The points $[0:1]$ and $[1:0]$ are denoted by 0 and ∞ respectively. We use the coordinates $t=t_0/t_1$ and $s=t_1/t_0$. We have the line bundle $\mathcal{O}_{\mathbb{P}^1}(i)$ over $\mathbb{P}^1$. It is the gluing of $\boldsymbol{C}^2=\big\{(t,\zeta_1)\big\}=\mathcal{O}_{\mathbb{P}^1}(i)_{|\mathbb{P}^1\setminus\{\infty\}}$ and $\boldsymbol{C}^2=\big\{(s,\zeta_2)\big\}=\mathcal{O}_{\mathbb{P}^1}(i)_{|\mathbb{P}^1\setminus\{0\}}$. The relations are given by $s=t^{-1}$ and $t^{-i}\cdot\zeta_1=\zeta_2$.

For $\xi=(t,\zeta_1)=(s,\zeta_2)\in\mathcal{O}_{\mathbb{P}^1}(i)$, we define

$$h_i(\xi,\xi):=|\zeta_1|^2\big(1+|t|^2\big)^{-i}=|\zeta_2|^2\big(1+|s|^2\big)^{-i}.$$

Then, h_i is a smooth Hermitian metric of the line bundle $\mathcal{O}_{\mathbb{P}^1}(i)$. For any $a,b\in\boldsymbol{R}$, we have the possibly singular metrics $h_{i,(a,b)}$ of $\mathcal{O}_{\mathbb{P}^1}(i)$ given by

$$h_{i,(a,b)}(\xi,\xi):=h_i(\xi,\xi)\big(1+|t|^{-2}\big)^{-a}\big(1+|t|^2\big)^{-b}=h_i(\xi,\xi)\big(1+|s|^2\big)^{-a}\big(1+|s|^{-2}\big)^{-b},$$

for $\xi=(t,\zeta_1)=(s,\zeta_2)\in\mathcal{O}_{\mathbb{P}^1}(i)$. Around $t=0$ (resp. $s=0$), $|t|^{-2a}h_{i,(a,b)}$ (resp. $|s|^{-2b}h_{i,(a,b)}$) is a C^∞-metric. The curvature $R(h_{i,(a,b)})$ is as follows:

$$R(h_{i,(a,b)})=(a+b+i)\frac{dt\cdot d\bar{t}}{\big(1+|t|^2\big)^2}. \tag{414}$$

21.6.2.2. An open subset of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ with a complete Kähler metric

We are mainly interested in the case $i=-1$. We regard $\mathcal{O}_{\mathbb{P}^1}(-1)$ as a complex manifold. We set

$$Y := \big\{\xi\in\mathcal{O}_{\mathbb{P}^1}(-1)\,\big|\,h_{-1,(0,0)}(\xi,\xi)<1\big\}. \tag{415}$$

Let π denote the projection of Y onto $\mathbb{P}^1$. The image of the 0-section $\mathbb{P}^1\to Y$ is also denoted by $\mathbb{P}^1$. We have the normal crossing divisor $D'=\mathbb{P}^1\cup\pi^{-1}(0)\cup\pi^{-1}(\infty)$ of Y. The manifold $Y\setminus D'$ is isomorphic to $\big\{(t,x)\in \boldsymbol{C}^{*\,2}\,\big|\,|x|^2(1+|t|^2)<1\big\}$.

We have a complete Poincaré-like Kähler metric of $Y\setminus D'$. For example, we can construct it as follows. As a contribution of the 0-section $\mathbb{P}^1$, we put $\tau_1 := -\log\Big[\big(1+|t|^2\big)\cdot|x|^2\Big]$ and

$$g_1 := \frac{1}{\tau_1^2}\left(\frac{\bar{t}\cdot dt}{1+|t|^2}+\frac{dx}{x}\right)\cdot\left(\frac{t\cdot d\bar{t}}{1+|t|^2}+\frac{d\bar{x}}{\bar{x}}\right)+\frac{1}{\tau_1}\frac{dt\cdot d\bar{t}}{\big(1+|t|^2\big)^2}.$$

As a contribution of $\pi^{-1}(\infty)$, we put $\tau_2:=\log\big(1+|t|^2\big)$ and

$$g_2 := \frac{1}{\tau_2}\Big(-1+\frac{|t|^2}{\tau_2}\Big)\cdot\frac{dt\cdot d\bar{t}}{\big(1+|t|^2\big)^2}.$$

As the contribution of the divisor $\pi^{-1}(0)$, we put $\tau_3:=\log(1+|t|^2)-\log|t|^2=\log(1+|s|^2)$, where we use $s=t^{-1}$ and

$$g_3 := \frac{1}{\tau_3}\cdot\Big(-1+\frac{|s|^2}{\tau_3}\Big)\frac{ds\cdot d\bar{s}}{\big(1+|s|^2\big)^2}.$$

Then, we set $g:=g_1+g_2+g_3$. It is easy to check that g is a complete Kähler metric. Note the following formulas:

$$\begin{aligned}
\overline{\partial}\partial\log\tau_1 &= \frac{1}{\tau_1^2}\Big(\frac{\bar{t}\cdot dt}{1+|t|^2}+\frac{dx}{x}\Big)\wedge\Big(\frac{t\cdot d\bar{t}}{1+|t|^2}+\frac{d\bar{x}}{\bar{x}}\Big)+\frac{1}{\tau_1}\frac{dt\wedge d\bar{t}}{\big(1+|t|^2\big)^2}=:\omega_1\\
\overline{\partial}\partial\log\tau_2 &= \frac{1}{\tau_2}\Big(-1+\frac{|t|^2}{\tau_2}\Big)\cdot\frac{dt\wedge d\bar{t}}{\big(1+|t|^2\big)^2}=:\omega_2\\
\overline{\partial}\partial\log\tau_3 &= \frac{1}{\tau_3}\Big(-1+\frac{|s|^2}{\tau_3}\Big)\cdot\frac{ds\wedge d\bar{s}}{\big(1+|s|^2\big)^2}=:\omega_3.
\end{aligned}$$

We put $\omega:=\omega_1+\omega_2+\omega_3$. Then, $\sqrt{-1}\omega$ is the Kähler form corresponding to g. We set

$$H_0 := \frac{1}{\tau_1}+\frac{1}{\tau_2}\Big(-1+\frac{|t|^2}{\tau_2}\Big)+\frac{1}{\tau_3}\Big(-1+\frac{|s|^2}{\tau_3}\Big)>0.$$

Then, the following holds:

$$\omega^2=\det(g)dt\wedge d\bar{t}\wedge dx\wedge d\bar{x}=\Big(\frac{2}{\tau_1^2|x|^2\big(1+|t|^2\big)^2}\times H_0\Big)\cdot dt\wedge d\bar{t}\wedge dx\wedge d\bar{x}.$$

We put $H_1:=H_0\cdot\big(1+|t|^2\big)^{-1}\cdot\big(1+|s|^2\big)^{-1}$. Recall $\mathrm{Ric}(g)=\overline{\partial}\partial(\det(g))$.

Lemma 21.6.3

- *Let C be a number such that $0 < C < 1$. We have $H_1 \sim \big(\big|\log|t|\big| + 1\big)^{-2}$ on the domain $\big\{\xi \in \mathcal{O}_{\mathbb{P}^1}(-1) \,\big|\, h_{-1,(0,0)}(\xi,\xi) \leqslant C\big\}$.*
- *We have the equality $\operatorname{Ric}(g) - \overline{\partial}\partial \log(H_1) = -\overline{\partial}\partial \log \tau_1^2$.* □

21.6.2.3. Inequality and vanishing. — We put $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. We also put $\Delta_z^2 := \{(z_1, z_2) \,|\, |z_i| < 1\}$ and $D_i' := \{z_i = 0\} \subset \Delta_z^2$ $(i = 1, 2)$. Let $\varphi : \widetilde{\Delta_z^2} \to \Delta_z^2$ denote the blow up of Δ_z^2 at the origin $O = (0, 0)$. We have the exceptional divisor $\varphi^{-1}(O)$ and the proper transforms $\widetilde{D'}_i$ of D_i' $(i = 1, 2)$.

We put $\widetilde{X} := \widetilde{\Delta_z^2} \times \Delta_w^{n-2}$. Then we have the composite ψ of the natural morphisms:

$$\widetilde{X} \xrightarrow{\varphi \times \mathrm{id}} \Delta_z^2 \times \Delta_w^{n-2} \longrightarrow \Delta_z^n.$$

Here the latter morphism is the natural isomorphism given by $w_i = z_{i+2}$ $(i = 1, \dots, n-2)$. We put $\widetilde{D} := \psi^{-1}(D)$, which is the same as

$$\Big[\big(\varphi^{-1}(O) \cup \widetilde{D_1'} \cup \widetilde{D_2'}\big) \times \Delta_w^{n-2}\Big] \cup \Big[\widetilde{\Delta}_z^2 \times \Big(\bigcup_{i=1}^{\ell-2}\{w_i = 0\}\Big)\Big].$$

The restriction of ψ to $\widetilde{X} - \widetilde{D}$ gives an isomorphism $\widetilde{X} - \widetilde{D} \simeq X \smallsetminus D$.

We can take a holomorphic embedding ι of Y in (415) to $\widetilde{\Delta}^2$ satisfying the following:

- The image of the 0-section $\mathbb{P}^1$ is the exceptional divisor $\varphi^{-1}(O)$.
- We have $\iota^{-1}(\widetilde{D}_1') = \pi^{-1}(\infty)$ and $\iota^{-1}(\widetilde{D}_2') = \pi^{-1}(0)$.

We put $\overline{X} := Y \times \Delta_w^{n-2}$. Then we have the naturally induced morphism $\overline{X} \to \widetilde{X}$, which is also denoted by ι. We have the point $[1:1] \in \mathbb{P}^1$, and we put

$$\overline{D} := \iota^{-1}(\widetilde{D}), \quad \overline{X}^{(1)} := \pi^{-1}([1:1]) \times \Delta_w^{n-2}, \quad \overline{D}^{(1)} := \overline{X}^{(1)} \cap \overline{D}.$$

The composite $\psi \circ \iota$ is denoted by ψ_1. The metric $g_{\overline{X}-\overline{D}}$ is induced from the metric g of $Y \smallsetminus D'$ and the Poincaré metric of $\Delta_w^{n-2} \smallsetminus \bigcup_{i=1}^{\ell-2}\{w_i = 0\}$.

Let $(E, \overline{\partial}_E)$ be a holomorphic bundle with a Hermitian metric h over $X \smallsetminus D$. We assume that $(E, \overline{\partial}_E, h)$ is acceptable. We denote the curvature of $\psi_1^{-1}(E, \overline{\partial}_E, h)$ by $\psi_1^{-1}R(h)$.

Let ε_i $(i = 1, 2)$ be as in Condition 21.6.1. We can pick positive numbers ε, a and b satisfying

$$a + b = 1, \quad 0 < a + 2\varepsilon < 1 - \varepsilon_1, \quad 0 < b + 2\varepsilon < 1 - \varepsilon_2. \tag{416}$$

We set $\boldsymbol{\delta} := (1, \dots, 1) \in \boldsymbol{R}^{\ell}$. Let $h_{\varepsilon\boldsymbol{\delta},N}$ be as in (406). Let $\widetilde{h}_{N,\varepsilon,a,b}$ denote the metric of the bundle $\psi_1^{-1}(E)(-\overline{X}^{(1)}) := \psi_1^{-1}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ on $\overline{X} - \overline{D}$ given as follows:

$$\widetilde{h}_{N,\varepsilon,a,b} := \psi_1^{-1}h_{\varepsilon\boldsymbol{\delta},N} \cdot h_{-1,a,b} \cdot H_1^{-1} \cdot \tau_1^{2+\varepsilon}\big(\tau_2 \cdot \tau_3\big)^{\varepsilon}. \tag{417}$$

For simplicity, we use the symbol $\widetilde{h}$ instead of $\widetilde{h}_{N,\varepsilon,a,b}$.

Lemma 21.6.4. — *If N is sufficiently negative, the following inequality holds for any $\eta \in A_c^{0,1}(\psi_1^{-1}E(-\overline{X}^{(1)}))$:*

$$\langle\langle R(\tilde{h}) + \mathrm{Ric}(g), \eta\rangle\rangle_{\tilde{h}} \geqslant \varepsilon \|\eta\|_{\tilde{h}}^2.$$

(See (403) *and* (404) *for the definition of* $\langle\langle\cdot,\cdot\rangle\rangle_{\tilde{h}}$*.)*

Proof. — Recall that $\overline{X} \setminus \overline{D}$ is isomorphic to the product of $Y \setminus D'$ and $\Delta_w^{n-2} \setminus \bigcup_{i=1}^{\ell-2}\{w_i = 0\}$, which is compatible with the metrics. We can apply the inequality (409) to the $\left(\Delta_w^{n-2} \setminus \bigcup_{i=1}^{\ell-2}\{w_i = 0\}\right)$-direction. Hence we only have to consider $(Y \setminus D')$-direction, i.e., we may and will assume $n = 2$. We have the following equality:

$$\begin{aligned}(418)\quad R(\tilde{h}) + \mathrm{Ric}(g) &= R(\psi_1^{-1}h_{\varepsilon\boldsymbol{\delta},N}) + R(h_{-1,a,b}) - \overline{\partial}\partial \log H_1 \\ &\quad + (2+\varepsilon)\cdot \overline{\partial}\partial \log \tau_1 + \varepsilon \cdot \overline{\partial}\partial(\log\tau_2 + \log\tau_3) + \mathrm{Ric}(g) \\ &= R(\psi_1^{-1}h_{\varepsilon\boldsymbol{\delta},N}) + \varepsilon\cdot(\omega_1+\omega_2+\omega_3).\end{aligned}$$

Here we have used $R(h_{-1,a,b}) = 0$ due to (414) and our choice of a and b. By taking sufficiently negative N, we can assume the following inequality for any $\eta \in A_c^{0,1}(E)$ on $X \setminus D$:

$$(419)\qquad \langle\langle R(h_{\varepsilon\boldsymbol{\delta},N}), \eta\rangle\rangle_{\varepsilon\boldsymbol{\delta},N} \geqslant 0.$$

Then, by a fiber-wise linear algebraic argument using the expression (404), it is easy to see that the following inequality holds for any $\eta \in A_c^{0,1}(\psi_1^{-1}(E))$:

$$\langle\langle \psi^{-1}R(h_{\varepsilon\boldsymbol{\delta},N}), \eta\rangle\rangle_{\tilde{h}} \geqslant 0.$$

We also obtain $\varepsilon\big\langle\langle\omega_1+\omega_2+\omega_3,\eta\rangle\big\rangle_{\tilde{h}} \geqslant \varepsilon\cdot\|\eta\|_{\tilde{h}}$, by using (404). Thus we are done. □

Corollary 21.6.5. — *If N is sufficiently negative, the first cohomology group $H^1\big(A_{\tilde{h}}^{0,\cdot}(\psi_1^{-1}E(-\overline{X}^{(1)}))\big)$ vanishes.*

Proof. — It immediately follows from Lemma 21.6.4, Proposition 21.1.1 and Proposition 21.1.2. □

The contribution of $h_{-1,a,b}\cdot H_1^{-1}\cdot \tau_1^{2+\varepsilon}\cdot\big(\tau_2\cdot\tau_3\big)^{\varepsilon}$ to the metric $\widetilde{h}$ is equivalent to the following, on a curve transversal with $\{t=0\}$:

$$|t|^{2a}\cdot\big(-\log|t|\big)^2\cdot|t|^{2\varepsilon}\big(-\log|t|\big)^{2\varepsilon} = |t|^{2(a+\varepsilon)}\cdot\big(-\log|t|\big)^{2+2\varepsilon}.$$

We have a similar estimate on a curve transversal with $\{s=0\}$. The contribution is equivalent to $(-\log|x|)^{2+\varepsilon}$ on a curve transversal with $\{x=0\}$.

21.6.3. Proof of Proposition 21.6.2. — We impose the additional condition $\psi_1\big(\overline{X}^{(1)}\big) = X^{(1)}$. We remark that $\psi_1(\overline{X})$ is a neighbourhood of X_0. We take numbers ε, a and b as in (416). Note

$$\mathcal{P}ar\big(E,1\big) \cap]0, a+2\varepsilon] \subset \mathcal{P}ar\big(E,1\big) \cap]0, 1-\varepsilon_1] = \varnothing.$$

Similarly, we have $\mathcal{P}ar\big(E,2\big) \cap]0, b+2\varepsilon] = \varnothing$. Moreover, we may assume that $\mathcal{P}ar\big(E,i\big) \cap]0,\varepsilon] = \varnothing$ for $i = 3,\dots,\ell$, if ε is sufficiently small.

Let us take a sufficiently negative number N such that (409) holds. We take the metric $\widetilde{h} = \widetilde{h}_{N,\varepsilon,a,b}$ of the bundle $\psi_1^{-1}E(-\overline{X}^{(1)})$ as in (417). We also put $\widetilde{h}_0 := \psi_1^{-1}h_{\mathbf{0},N} \cdot h_{-1,a,b} \cdot H_1^{-1} \cdot \tau_1^{2+\varepsilon}\big(\tau_2 \cdot \tau_3\big)^{\varepsilon}$. We remark that we can use Corollary 21.6.5 in this setting. We take the metrics $\widehat{h}$ and $\widehat{h}_0$ of $\psi_1^{-1}E$:

$$\widehat{h} := \psi_1^{-1}h_{\varepsilon\boldsymbol{\delta},N} \cdot h_{0,a,b} \cdot H_1^{-1} \cdot \tau_1^{2+\varepsilon}\big(\tau_2 \cdot \tau_3\big)^{\varepsilon},$$
$$\widehat{h}_0 := \psi_1^{-1}h_{\mathbf{0},N} \cdot h_{0,a,b} \cdot H_1^{-1} \cdot \tau_1^{2+\varepsilon}\big(\tau_2 \cdot \tau_3\big)^{\varepsilon}.$$

Take an embedding $\kappa : \Delta_\zeta \to \mathbb{P}^1 \setminus \{0,\infty\}$ such that $\kappa(0) = [1:1] \in \mathbb{P}^1$. We take a holomorphic function η on $\pi^{-1}(\kappa(\Delta_\zeta))$ such that $\eta^{-1}(0)$ is the intersection of the exceptional divisor $\varphi^{-1}(O)$ and $\pi^{-1}(\kappa(\Delta_\zeta))$.

The section f induces a holomorphic section of $\big(\psi_1^{-1}E\big)_{|\overline{X}^{(1)}}$ over $\overline{X}^{(1)} \setminus \overline{D}^{(1)}$, which is denoted by f. By using the argument in Subsection 21.5.2, we can take a C^1-section ρ of $\psi_1^{-1}E$ over $\overline{X} - \overline{D}$ satisfying the following:

- The support of ρ is contained in $\pi^{-1}(\kappa(\Delta_\zeta)) \times \Delta_w^{n-2}$.
- ρ is bounded as a section of $\psi_1^{-1}E$ up to polynomial order of $-\log|z_i|$ ($i = 3,\dots,\ell$) and $-\log|\eta|$, with respect to $\widehat{h}_0$. In particular, ρ is an L^2-section of $\psi_1^{-1}E$ with respect to $\widehat{h}$.
- $\overline{\partial}\rho$ is bounded as a section of $\psi_1^{-1}E(-\overline{X}^{(1)}) \otimes \Omega^{1,0}_{\overline{X}-\overline{D}}$ up to polynomial order of $-\log|z_i|$ ($i = 3,\dots,\ell$) and $-\log|\eta|$, with respect to the metric $\widetilde{h}_0$ of $\psi_1^{-1}E(-\overline{X}^{(1)}) \otimes \Omega^{1,0}$ and the metric $g_{\overline{X}-\overline{D}}$ of $\Omega^{0,1}_{\overline{X}-\overline{D}}$. The restriction of $\overline{\partial}\rho$ to $\overline{X} - (\overline{D} \cup \overline{X}^{(1)})$ are C^∞. In particular, $\overline{\partial}\rho$ is contained in $A^{0,1}_{\widetilde{h}}\big(\psi_1^{-1}E(-\overline{X}^{(1)})\big)$.
- We have $\rho_{|\overline{X}^{(1)}-\overline{D}^{(1)}} = f$.

Due to Corollary 21.6.5, we can pick $G \in A^{0,0}_{\widetilde{h}}\big(\psi_1^{-1}E(-\overline{X}^{(1)})\big)$ such that $\overline{\partial}G = \overline{\partial}\rho$. We put $\widetilde{f} := \rho - G$, which satisfies $\overline{\partial}\widetilde{f} = 0$ and $\widetilde{f} \in A^{0,0}_{\widehat{h}}(\psi_1^{-1}E)$. By using the L^2-property of G as a section of $\psi_1^{-1}E \otimes \mathcal{O}(-\overline{X}^{(1)})$, we obtain that $G_{|\overline{X}^{(1)}-\overline{D}^{(1)}} = 0$. Therefore, $\widetilde{f}_{|\overline{X}^{(1)}-\overline{D}^{(1)}} = f$.

We have an open subset V of X such that $\psi_1\big(\overline{X}-\overline{D}\big) = V \setminus D$. By the identification, we can regard $\widetilde{f}$ as a holomorphic section of $E_{|V\setminus D}$. We would like to show that $\widetilde{f}$ gives a section of ${}^\diamond E$ on V. We put $D_i^\circ := D_i \setminus \bigcup_{j\leqslant \ell, j\neq i} D_j$.

Lemma 21.6.6. — *Let P be any point of $V \cap D_1^\circ$. We have $\big|\widetilde{f}_{|\pi_1^{-1}(P)\cap V}\big| = O\big(|z_1|^{-\kappa}\big)$ for any $\kappa > 0$.*

Proof. — We put $C_P := \pi_1^{-1}(P)\cap V$. We may assume that the closure of C_P in $\boldsymbol{C}^n$ is contained in X, by shrinking V. Let us take a small neighbourhood U of P in D_1°. We may assume $C_P \times U \subset V$. The metrics $\widehat{h}_{|C_P\times U}$ and $h\cdot|z_1|^{2(a+\varepsilon)}$ are mutually bounded up to polynomial order of $-\log|z_1|$. Therefore $\widetilde{f}_{|C_P\times U}$ is L^2, with respect to the metric $h\cdot|z_1|^{2(a+\varepsilon)}\cdot\big(-\log|z_1|\big)^{-M}$ for $M>0$ and the metric $g_{X\setminus D\,|\,C_P\times U}$. Due to Corollary 21.2.5, the restriction $\widetilde{f}_{|C_P}$ is L^2 with respect to $h\cdot|z_1|^{2(a+\varepsilon)}\cdot\big(-\log|z_1|\big)^{-M}$ and the metric $g_{X\setminus D\,|\,C_P}$. We also remark Lemma 21.2.6. Because $\mathcal{P}ar\big({}^\diamond E,1\big)\cap]0,a+2\varepsilon] = \varnothing$, we obtain the desired estimate for $\widetilde{f}_{|C_P}$. □

Similarly, we can show the following lemma.

Lemma 21.6.7. — *We have $\big|\widetilde{f}_{|\pi_i^{-1}(P)\cap V}\big|_h = O\big(|z_i|^{-\kappa}\big)$ for any $\kappa>0$, any $P\in D_i^\circ\cap V$, and $i=1,2,\dots,\ell$.* □

Lemma 21.6.8. — *$\widetilde{f}$ is a section of ${}^\diamond E$ over V.*

Proof. — It follows from Lemma 21.6.7 and Proposition 21.2.8. □

As a result, we obtain a holomorphic section $\widetilde{f}$ of ${}^\diamond E$ over V such that $\widetilde{f}_{|X^{(1)}} = f$. Thus the proof of Proposition 21.6.2 is accomplished. □

21.7. Proof of Theorems 21.3.1 and 21.3.2

We use an induction on the dimension of X. As we have already remarked, the one dimensional case of Theorem 21.3.1 was proved by Simpson in [**81**] and [**82**]. We remark that the claim for $\det(E,\overline{\partial}_E,h)$ is easy because it is assumed to be flat.

21.7.1. Preliminary prolongation. — Let $(E,\overline{\partial}_E,h)$ be an acceptable bundle on $X\setminus D$. We impose the following assumptions in this section.

- Let ε_i $(i=1,2)$ be small positive numbers such that $\operatorname{rank}(E)\cdot(\varepsilon_1+\varepsilon_2) < 1/2$. Then, $\mathcal{P}ar({}^\diamond E,i)$ is contained in $]-\varepsilon_i,0[$ for $i=1,2$.
- Pick a point $({}^0z_3,\dots,{}^0z_n)\in(\Delta^*)^{n-2}$, and set $C_0 := \{(z,z,{}^0z_3,\dots,{}^0z_n)\in X\}$. Then, $\mathcal{P}ar(E_{|C_0})$ is contained in $]-\varepsilon_1-\varepsilon_2,0[$. Note that this condition is independent of the choice of $({}^0z_3,\dots,{}^0z_n)$.

We set $\widetilde{b}_i := \sum_{b\in\mathcal{P}ar({}^\diamond E,i)} b\cdot\operatorname{rank}{}^i\operatorname{Gr}_b^F({}^\diamond E)$, and $\widetilde{\boldsymbol{b}} := (\widetilde{b}_i)$.

Lemma 21.7.1. — *${}_{\widetilde{\boldsymbol{b}}}\det(E)_{|C_0}$ is isomorphic to $\det\big({}^\diamond(E_{|C_0})\big)$.*

Proof. — Take any $P_i\in D_i^\circ$ $(i=1,2)$. We have

$$\sum_{b\in\mathcal{P}ar({}^\diamond E_{|C_0})} b\cdot\dim\operatorname{Gr}_b^F({}^\diamond E_{|C_0}) - \sum_{i=1,2}\sum_{b\in\mathcal{P}ar({}^\diamond E,i)} b\cdot\dim{}^i\operatorname{Gr}_b^F\big({}^\diamond E_{|\pi_i^{-1}(P_i)}\big)\in\mathbb{Z}.$$

By the assumption, it has to be 0. Then, the claim follows. □

Lemma 21.7.2. — *Under the assumption, the $\mathcal{O}_X$-sheaf ${}^\diamond E$ is locally free.*

Proof. — We use the notation in Subsection 21.6. Note that we are also using the induction on $\dim X$ to show the theorems. Due to the inductive assumption on $\dim X$, we have the local freeness of ${}^\diamond(E_{|X^{(1)}\setminus D^{(1)}})$. Pick a frame $\boldsymbol{v}=(v_1,\ldots,v_r)$ of ${}^\diamond(E_{|X^{(1)}\setminus D^{(1)}})$ over $X^{(1)}$. For each v_i, we pick a section $\widetilde{v}_i$ of ${}^\diamond E$ over a neighbourhood U of X_0 such that $\widetilde{v}_{i\,|\,U\cap X^{(1)}}=v_{i\,|\,U\cap X^{(1)}}$, by using Proposition 21.6.2. We may assume that v_i are defined on X by shrinking X.

Let us show that $\widetilde{\boldsymbol{v}}$ is a frame of ${}^\diamond E$ around X_0. We set $\Omega(\widetilde{\boldsymbol{v}}):=\widetilde{v}_1\wedge\cdots\wedge\widetilde{v}_r$. The restriction $\widetilde{\boldsymbol{v}}_{|\pi_i^{-1}(P)}$ gives a tuple of holomorphic sections of ${}^\diamond(E_{|\pi_i^{-1}(P)})$ for any $P\in D_i^\circ$. Hence $\Omega(\widetilde{\boldsymbol{v}})_{|\pi_i^{-1}(P)}$ is a holomorphic section of $\det\big({}^\diamond(E_{|\pi_i^{-1}(P)})\big)={}_{\widetilde{b}_i}\big(\det(E)_{|\pi_i^{-1}(P)}\big)$. This implies $\Omega(\widetilde{\boldsymbol{v}})$ is a holomorphic section of ${}_{\widetilde{\boldsymbol{b}}}\det(E)$.

We have the natural isomorphism $({}_{\widetilde{\boldsymbol{b}}}\det(E))_{|X^{(1)}}\simeq\det\big({}^\diamond(E_{|X^{(1)}})\big)$ due to Lemma 21.7.1. Since $\Omega(\widetilde{\boldsymbol{v}})_{|X^{(1)}}$ is a frame of $\det\big({}^\diamond(E_{|X^{(1)}})\big)$, we obtain $\Omega(\widetilde{\boldsymbol{v}})_{|O}\neq 0$. By shrinking X, we may assume that $\Omega(\widetilde{\boldsymbol{v}})$ is a frame of ${}_{\boldsymbol{b}}\det E$.

Let f be any section of ${}^\diamond E$. We have an expression $f=\sum f_i\cdot\widetilde{v}_i$, where f_i are holomorphic on $X\setminus D$. It is easy to observe that $f\wedge\widetilde{v}_2\wedge\cdots\wedge\widetilde{v}_r=f_1\cdot\Omega(\widetilde{\boldsymbol{v}})$ is a section of ${}_{\widetilde{\boldsymbol{b}}}\det(E)$. Since $\Omega(\widetilde{\boldsymbol{v}})$ is a frame, f_1 is holomorphic on X. Similarly, we obtain that f_i $(i=2,\ldots,r)$ are also holomorphic on X. Hence, $\widetilde{\boldsymbol{v}}$ is a frame of ${}^\diamond E$ on X. □

21.7.2. Proof of Theorem 21.3.1

21.7.2.1. Step 1. — For a real number a, let $\kappa_{1/2}(a)\in\,]-1/2,1/2]$ and $\nu_{1/2}(a)\in\mathbb{Z}$ be determined by $a=\kappa_{1/2}(a)+\nu_{1/2}(a)$. We set $\eta:=(10\operatorname{rank}E)^{-1}$. We can take a positive integer c such that the following conditions are satisfied:

1. The maps $\mathcal{P}ar({}^\diamond E,i)\to\mathbb{Z}$ given by $a\mapsto\nu_{1/2}(c\cdot a)$ are injective for $i=1,\ldots,\ell$.
2. $\{\kappa_{1/2}(c\cdot a_i)\,|\,a_i\in\mathcal{P}ar(E,i)\}\subset\,]-\eta,\eta[$ for $i=1,2,\ldots,\ell$.
3. $\{\kappa_{1/2}(c\cdot a)\,|\,a\in\mathcal{P}ar(E_{|C_0})\}\subset\,]-\eta,\eta[$, where C_0 is a curve as in Section 21.7.1.

We set $\boldsymbol{\eta}:=(\eta,\ldots,\eta)\in\boldsymbol{R}^\ell$. For a positive integer c, let $\psi_c:X\setminus D\to X\setminus D$ be given by

$$\psi_c(z_1,\ldots,z_n)=(z_1^c,\ldots,z_\ell^c,z_{\ell+1},\ldots,z_n).$$

We put $(E_1,\overline{\partial}_{E_1},h_1):=\psi_c^{-1}(E,\overline{\partial}_E,h)\otimes L(-\boldsymbol{\eta})$. We have the natural isomorphism ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)\simeq{}^\diamond E_1$. We also have the following, due to the result in the one dimensional case (Section 21.4.2):

$$\mathcal{P}ar({}^\diamond E_1,i)=\big\{-\eta+\kappa_{1/2}(c\cdot a_i)\,|\,a_i\in\mathcal{P}ar(E,i)\big\}\subset\,]-(5\operatorname{rank}E)^{-1},0[,$$
$$\mathcal{P}ar\big({}^\diamond E_{1|C_0}\big)=\big\{-2\eta+\kappa_{1/2}(c\cdot a)\,\big|\,a\in\mathcal{P}ar(E_{|C_0})\big\}\subset\,]-2\cdot(5\operatorname{rank}E)^{-1},0[.$$

By Lemma 21.7.2, ${}^{\diamond}E_1$ is locally free. Hence, the sheaf ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)$ is a locally free $\mathcal{O}_X$-module.

Let $\mu_c := \{z \in \boldsymbol{C} \,|\, z^c = 1\}$. We fix a generator ω. We have the natural μ_c^{ℓ}-action on $X \smallsetminus D$ given by

$$(\omega_1, \dots, \omega_\ell) \cdot (z_1, \dots, z_n) = \big(\omega_1 \cdot z_1, \dots, \omega_\ell \cdot z_\ell,\, z_{\ell+1}, \dots, z_n\big).$$

It is lifted to the action on ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)$. The i-th component of μ_c^{ℓ} is denoted by $\mu_c^{(i)}$, which acts on ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)_{|D_i}$. We have the decomposition:

$${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)_{|\,D_i} = \bigoplus_{0\leqslant p\leqslant c-1} {}^{i}V_p.$$

Here the generator ω of $\mu_c^{(i)}$ acts as the multiplication of ω^p on ${}^{i}V_p$. We have the following morphism given as in Section 21.4.2:

$$\varphi_i : \big\{p \,\big|\, 0 \leqslant p \leqslant c-1,\ {}^{i}V_p \neq 0\big\} \longrightarrow \mathcal{P}ar\big({}_{\boldsymbol{\eta}}(\psi_c^{-1}E), i\big).$$

We consider the filtration ${}^{i}F'$ of ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)_{|\,D_i}$ in the category of vector bundles on D_i, given as follows for any $\eta - 1 < b < \eta$:

$$(420)\qquad {}^{i}F'_b := \bigoplus_{\varphi_i(p)\leqslant b} {}^{i}V_p.$$

By the splittings (420), it is easy to see that the tuple of filtrations $({}^{i}F' \,|\, i = 1, \dots, \ell)$ are compatible.

We set $\boldsymbol{\delta}_i := (\overbrace{0, \dots, 0}^{i-1}, 1, 0, \dots, 0) \in \boldsymbol{R}^{\ell}$. For any $-1 < b < 0$, we consider the subsheaf ${}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big)'$ of ${}_{\boldsymbol{\eta}}\psi_c^{-1}(E)$ given as follows:

$${}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big)' = \operatorname{Ker}\Big(\pi : {}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big) \longrightarrow \frac{{}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)_{|D_i}}{{}^{i}F'_{\eta+b}}\Big).$$

Here π denotes the naturally defined morphism of $\mathcal{O}_X$-modules.

Lemma 21.7.3. — *We have the following, for any $-1 < b < 0$:*

$${}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big)' = {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big).$$

In other words, the parabolic filtration ${}^{i}F$ is equal to ${}^{i}F'$.

Proof. — Let f be a holomorphic section of ${}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big)$. It can be also regarded as a section of ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)$. Let P be a point of D_i°. We have the element $f(P)$ of ${}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)_{|P} = {}_{\eta}\big(\psi_c^{-1}E_{|\pi_i^{-1}(P)}\big)_{|P}$. By using the result in the curve case, we obtain that $f(P)$ is contained in ${}^{i}F'_{b+\eta\,|\,P}$. (See Section 21.4.2.) Let $\overline{f}$ denote the image of f via the projection π. Then $\overline{f}(P) = 0$ for any $P \in D_i^{\circ}$. It implies $\overline{f} = 0$ on D_i. Hence, we obtain $f \in {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}\big(\psi_c^{-1}E\big)'$.

Conversely, pick a section $f \in {}_{\eta+b\boldsymbol{\delta}_i}\big(\psi_{\boldsymbol{c}}^{-1}E\big)'$. By using the result in the curve case (Section 21.4.2), we obtain the following inequality for any $P \in D_i^\circ$:

$$\big|f_{|\pi_i^{-1}(P)}\big|_h = O\big(|z_i|^{-b-\eta-\varepsilon}\big) \quad (\forall\, \varepsilon > 0).$$

Then we obtain $f \in {}_{\eta+b\boldsymbol{\delta}_i}\big(\psi_{\boldsymbol{c}}^{-1}E\big)$ due to Proposition 21.2.8. In all, we obtain ${}_{\eta+b\boldsymbol{\delta}_i}\big(\psi_{\boldsymbol{c}}^{-1}E\big) = {}_{\eta+b\boldsymbol{\delta}_i}\big(\psi_{\boldsymbol{c}}^{-1}E\big)'$. □

21.7.2.2. Step 2. — By shrinking X, let us take a μ_c^ℓ-equivariant frame $\boldsymbol{v} = (v_i)$ of ${}_{\eta}\psi_{\boldsymbol{c}}^{-1}E$ in the sense

$$(\omega_1, \dots, \omega_\ell)^* v_i = \prod_{j=1}^{\ell} \omega_j^{-p_j(v_i)} \cdot v_i$$

for some $0 \leqslant p_j(v_i) \leqslant c-1$. Note that $\boldsymbol{v}$ is compatible with the parabolic filtrations ${}^{j}F$ $(j = 1, \dots, \ell)$. We set

$$\overline{v}_i := \prod z_j^{p_j(v_i)} \cdot v_i.$$

Since they are μ_c^ℓ-invariant, we obtain a tuple $\overline{\boldsymbol{v}} = (\overline{v}_1, \dots, \overline{v}_r)$ of sections of E. By using the result in the curve case, we obtain that they are sections of ${}^\diamond E$. (See Section 21.4.2). Moreover, the restrictions $\overline{\boldsymbol{v}}_{|\pi_i^{-1}(P)}$ for $P \in D_i^\circ$ are frames of ${}^\diamond\big(E_{|\pi_i^{-1}(P)}\big)$. Hence, we can conclude that $\overline{\boldsymbol{v}}$ is a frame of ${}^\diamond E$ on X. Therefore, we obtain that ${}^\diamond E$ is a locally free $\mathcal{O}_X$-module. Then, Theorem 21.3.1 follows.

21.7.2.3. Complement. — Let us show directly that the induced filtrations are compatible. We have the map $\chi_i : \big\{p \,\big|\, 0 \leqslant p \leqslant c-1,\ {}^{i}V_p \neq 0\big\} \to \mathcal{P}ar\big({}^\diamond E, i\big)$ as in Section 21.4.2. We set $a_i(v_j) := \chi_i\big(p_i(v_j)\big)$. We consider the filtration ${}^{i}F'_b$ of ${}^\diamond E_{|D_i}$ by vector subbundles over D_i, given as follows:

$${}^{i}F'_b := \big\langle \overline{v}_{j\,|\,D_i} \,\big|\, a_i(v_j) \leqslant b \big\rangle.$$

For any $-1 < b \leqslant 0$, we consider the subsheaf ${}_{b\cdot\boldsymbol{\delta}_i}(E)'$ of ${}^\diamond E$ given as follows:

$${}_{b\cdot\boldsymbol{\delta}_i}\big(E\big)' := \operatorname{Ker}\Big(\pi : {}^\diamond E \longrightarrow \frac{{}^\diamond E_{|D_i}}{{}^{i}F'_b}\Big).$$

Here π denotes the naturally defined morphism. Then ${}_{b\cdot\boldsymbol{\delta}_i}\big(E\big)'$ is locally free.

Lemma 21.7.4. — *${}_{b\boldsymbol{\delta}_i}E = {}_{b\boldsymbol{\delta}_i}\big(E\big)'$ and ${}^{i}F'_b = {}^{i}F_b$.*

Proof. — Let f be a holomorphic section of ${}_{b\boldsymbol{\delta}_i}E$. We can also regard it as a section of ${}^\diamond E$. By applying the result in the curve case (Section 21.4.2) to $f_{|\pi_i^{-1}(P)} \in {}^\diamond\big(E_{|\pi_i^{-1}(P)}\big)$, we obtain that $f(P) \in {}^{i}F'_{|P}$ for any $P \in D_i^\circ$. Then it is easy to derive that f is contained in ${}_{b\boldsymbol{\delta}_i}\big(E\big)'$.

Conversely, let f be a holomorphic section of ${}_{b\boldsymbol{\delta}_i}\big(E\big)'$. Applying the result in the curve case (Section 21.4.2) to $f_{|\pi_i^{-1}(P)}$, we obtain $f_{|\pi_i^{-1}(P)} \in {}_{b}(E_{|\pi_i^{-1}(P)})$ for any $P \in D_i^\circ$. Then we obtain $f \in {}_{b\boldsymbol{\delta}_i}E$ by Proposition 21.2.8. Therefore we obtain ${}_{b\boldsymbol{\delta}_i}E = {}_{b\boldsymbol{\delta}_i}\big(E\big)'$, and thus ${}^{i}F_b = {}^{i}F'_b$. □

By our construction, ${}^{i}F'$ is the filtration in the category of the vector bundles over D_i, and the tuple $({}^{i}F' \mid i = 1, \dots, \ell)$ is compatible. Hence, the filtration ${}^{i}F$ is a filtration in the category of vector bundles over D_i, and the tuple $({}^{i}F \mid i = 1, \dots, \ell)$ is compatible.

21.7.3. Proof of Theorem 21.3.2. — Let $\boldsymbol{v}$ be a frame of ${}_{\boldsymbol{b}}E$ compatible with the parabolic filtrations $({}^{i}F \mid i = 1, \dots, \ell)$. We obtain the numbers ${}^{i}\!\mathfrak{b}(v_j) := {}^{i}\deg^F(v_j)$. We put

$$v_j' := v_j \cdot \prod_{i=1}^{\ell} |z_i|^{{}^{i}\mathfrak{b}(v_j)}, \qquad \boldsymbol{v}' = (v_j').$$

Let us show that $\boldsymbol{v}'$ is adapted up to log order. By our construction of $\boldsymbol{v}'$ and Proposition 21.2.8, there exists $C_1 > 0$ such that the following holds:

$$H(h, \boldsymbol{v}') \leqslant C_1 \cdot \Big(-\sum \log |z_i|\Big)^M.$$

Let $\boldsymbol{v}^\vee$ denote the dual frame of $\boldsymbol{v}$. Let P be a point of D_i°. According to the functoriality in the curve case (Section 21.4.1), $\boldsymbol{v}^\vee_{|\pi_i^{-1}(P)}$ is a frame of ${}_{-b_i+(1-\varepsilon)}E^\vee{}_{|\pi_i^{-1}(P)}$, which is compatible with the parabolic filtration. Hence, $\boldsymbol{v}^\vee$ is a frame of ${}_{-\boldsymbol{b}+(1-\varepsilon)\boldsymbol{\delta}}E^\vee$ for some $\varepsilon > 0$. We have ${}^{i}\deg^F(v_j^\vee) = \deg^F\big(v^\vee_{j\,|\pi_i^{-1}(P)}\big) = -{}^{i}\mathfrak{b}(v_j)$ for any point $P \in D_i^\circ$. We put

$$\boldsymbol{v}^{\vee\prime} = (v_j^{\vee\prime}), \qquad v_j^{\vee\prime} := v_j^\vee \cdot \prod_{i=1}^{\ell} |z_i|^{-{}^{i}\mathfrak{b}(v_j)}.$$

Due to Proposition 21.2.8, we obtain

$$H(h^\vee, \boldsymbol{v}^{\vee\prime}) \leqslant C_2 \cdot \Big(-\sum_{i=1}^{\ell} \log |z_i|\Big)^M.$$

Here, $h^\vee$ denote the induced metric of $E^\vee$. This implies

$$C_3 \cdot \Big(-\sum_{i=1}^{\ell} \log |z_i|\Big)^{-M} \leqslant H(h, \boldsymbol{v}').$$

Thus, we obtain Theorem 21.3.2. □

21.8. Small deformation

21.8.1. Statement. — Let $X := \Delta^\ell$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$. Let $g_{\mathbf{p}}$ denote the Poincaré metric of $X \setminus D$. Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle on $X \setminus D$ of rank r. For any $0 < C < 1$, we set $X(C) := \big\{(z_1, \dots, z_n) \in X \,\big|\, |z_i| < C\big\}$, $D(C) := D \cap X(C)$ and $X^*(C) := X(C) \setminus D(C)$.

Let Y be an open subset of $\boldsymbol{C}_\lambda$ with a base point $\boldsymbol{y}_0$, and let p denote the projection $Y \times (X \setminus D) \to X \setminus D$. Let $\mathcal{A}$ be a section of $p^*\big(\mathrm{End}(E) \otimes \Omega^{0,1}_{X \setminus D}\big)$ with the following properties:

- $\big(\overline{\partial}_E+\overline{\partial}_\lambda+\mathcal{A}\big)^2=0$. The holomorphic bundle $\big(p^{-1}E,\overline{\partial}_E+\overline{\partial}_\lambda+\mathcal{A}\big)$ is denoted by $(\mathcal{E},\overline{\partial}_{\mathcal{E}})$.
- $|\mathcal{A}|_{h,g_{\mathbf{p}}}$ is bounded, and $|\mathcal{A}_{\boldsymbol{y}}-\mathcal{A}_{\boldsymbol{y}_0}|_{h,g_{\mathbf{p}}}\leqslant B\cdot|\boldsymbol{y}-\boldsymbol{y}_0|$ for some $B>0$, where $\mathcal{A}_{\boldsymbol{y}}:=\mathcal{A}_{|\boldsymbol{y}\times(X\smallsetminus D)}$.
- $\mathcal{A}_{\boldsymbol{y}_0}=0$ and $\operatorname{tr}\mathcal{A}=0$.

We have the associated filtered sheaf $\mathcal{E}_*$ as in Subsection 21.3.

Theorem 21.8.1. — *There exist a positive number $C>0$ and a neighbourhood $\mathcal{U}_0$ of $\boldsymbol{y}_0$ such that the following holds:*

- *The restriction of $\mathcal{E}_*$ to $\mathcal{U}_0\times X(C)$ is a filtered bundle.*
- *Let $\boldsymbol{v}$ be a frame of ${}_{\boldsymbol{a}}\mathcal{E}$ compatible with the parabolic filtrations $\big({}^iF\,\big|\,i=1,\dots,\ell\big)$. We set $a_i(v_j):={}^i\!\deg^F(v_j)$ and*
$$v_j':=v_j\cdot\prod_{i=1}^{\ell}|z_i|^{a_i(v_j)}.$$
Then, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that the following holds:
$$C_\varepsilon^{-1}\cdot\prod_{i=1}^{\ell}|z_i|^{\varepsilon}\leqslant H(h,\boldsymbol{v}')\leqslant C_\varepsilon\cdot\prod_{i=1}^{\ell}|z_i|^{-\varepsilon}.$$

In the following, $\mathcal{U}_i$ will denote a neighbourhood of $\boldsymbol{y}_0$, and C_i will denote a constant such that $0<C_i<1$.

21.8.2. Extension of holomorphic sections. — Let us argue the extension problem of holomorphic sections on $\{\boldsymbol{y}_0\}\times X$ to those on $\mathcal{U}\times X$. First, we remark that an L^2-estimate implies a growth estimate.

Lemma 21.8.2. — *Let F be a holomorphic section of $\mathcal{E}$ on $\mathcal{U}\times(X\smallsetminus D)$ such that $\|F\|_{\boldsymbol{a},N}<\infty$. Let $\mathcal{U}'$ be relatively compact in $\mathcal{U}$. Then, there exist $M>0$ and $0<C<1$ such that the following holds on $\mathcal{U}'\times X^*(C)$:*
$$|F|_h=O\Big(\prod_{i=1}^{\ell}|z_i|^{-a_i}\big(-\log|z_i|\big)^M\Big).$$

Proof. — Let $\mathbb{H}$ denote the upper half plane. We use the coordinate $\zeta=\xi+\sqrt{-1}\eta$. Let $\varphi_C:\mathbb{H}^\ell\times\Delta^{n-\ell}\to(\Delta^*)^\ell\times\Delta^{n-\ell}=X\smallsetminus D$ be the map given by
$$\varphi_C(\zeta_1,\dots,\zeta_\ell,z_{\ell+1},\dots,z_n)=\big(e^{2\pi\sqrt{-1}\zeta_1},\dots,e^{2\pi\sqrt{-1}\zeta_\ell},Cz_{\ell+1},\dots,Cz_n\big).$$
Let g be the Euclidean metric of $\mathbb{H}^\ell\times\Delta^{n-\ell}$. For any $\kappa>0$, there exists $C(\kappa)>0$ such that $|\varphi_{C(\kappa)}^*R(h)|_{\varphi_{C(\kappa)}^*h,g}\leqslant\kappa$ on $\{\eta_j\geqslant C(\kappa),\,j=1,\dots,\ell\}\subset\mathbb{H}^\ell\times\Delta^{n-\ell}$.

For $\boldsymbol{n}=(n_1,\dots,n_\ell)$, we put
$$K_{\boldsymbol{n}}:=\prod_{i=1}^{\ell}\big\{(\xi_i,\eta_i)\,\big|\,-1<\xi_i<1,n_i-1<\eta_i<n_i+1\big\}\times\Delta^{n-\ell}.$$

Let κ be sufficiently small. If $n_i > C(\kappa)$, due to a theorem of Uhlenbeck, we can take an orthonormal frame $\boldsymbol{e_n}$ of $\varphi_{C(\kappa)}^*(E, \overline{\partial}_E, h)$ on $K_{\boldsymbol{n}}$ such that the connection form $A_{\boldsymbol{n}}$ is sufficiently small with respect to h and g, where $A_{\boldsymbol{n}}$ is determined by $\overline{\partial}_E \boldsymbol{e_n} = \boldsymbol{e_n} \cdot A_{\boldsymbol{n}}$.

The induced map $\mathcal{U} \times \mathbb{H}^\ell \times \Delta^{n-\ell} \to \mathcal{U} \times (X \setminus D)$ is also denoted by $\varphi_{C(\kappa)}$. We have the orthonormal frame of $(\varphi_{C(\kappa)}^* \mathcal{E})_{|\mathcal{U} \times K_{\boldsymbol{n}}}$ induced by $\boldsymbol{e_n}$, which is denoted by $\widetilde{\boldsymbol{e}}_{\boldsymbol{n}}$. Let $\widetilde{A}_{\boldsymbol{n}}$ be given by $\overline{\partial}_{\mathcal{E}} \widetilde{\boldsymbol{e}}_{\boldsymbol{n}} = \widetilde{\boldsymbol{e}}_{\boldsymbol{n}} \cdot \widetilde{A}_{\boldsymbol{n}}$. By the assumption, $\widetilde{A}_{\boldsymbol{n}}$ are uniformly bounded with respect to g. We have the expression $F_{|K_{\boldsymbol{n}}} = \sum F_{\boldsymbol{n},i} \cdot \widetilde{\boldsymbol{e}}_{\boldsymbol{n},i}$ on each $\mathcal{U} \times K_{\boldsymbol{n}}$. We put $\boldsymbol{F_n} = \big(F_{\boldsymbol{n},i}\big)$ which satisfies $\overline{\partial} \boldsymbol{F_n} + \widetilde{A}_{\boldsymbol{n}} \boldsymbol{F_n} = 0$. We also have

$$\int_{K_{\boldsymbol{n}}} \big|\boldsymbol{F_n}\big|^2 \cdot \mathrm{dvol}_g \leqslant B \cdot \big\|F\big\|_{\boldsymbol{a},N}^2 \cdot \prod_{i=1}^{\ell} e^{2a_i \cdot n_i} \cdot n_i^{|N|+2}.$$

Here, B is a positive constant depending on $C(\kappa)$. If we take a relatively compact subset $K'_{\boldsymbol{n}} \subset K_{\boldsymbol{n}}$, we can obtain the estimate of the sup norm of $\boldsymbol{F_n}$ on $\mathcal{U}' \times K'_{\boldsymbol{n}}$ by a standard bootstrapping argument. Thus, the claim of Lemma 21.8.2 follows. □

We set $\boldsymbol{\delta} := (1, \ldots, 1) \in \boldsymbol{R}^\ell$.

Lemma 21.8.3. — *There exist $\mathcal{U}_1$ and $C_1 > 0$ satisfying the following:*
- *Let f be any holomorphic section of ${}_{\boldsymbol{a}}E$ on $X \setminus D$. For any $\varepsilon > 0$, there exists a section $F^{(\varepsilon)}$ of ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{U}_1 \times X(C_1)$ such that $F^{(\varepsilon)}_{|\boldsymbol{y}_0 \times X(C)} = f_{|X(C)}$.*

Proof. — Let $N < 0$ be as in Lemma 21.2.2. For any $\kappa > 0$, there exists a small neighbourhood $\mathcal{U}_2$ such that $\big\|\mathcal{A}_{\boldsymbol{y}}\big\|_h \leqslant \kappa$ for any $\boldsymbol{y} \in \mathcal{U}_2$. Note that the norm of the morphism

$$\mathcal{A}_{\boldsymbol{y}} : A^{0,q}_{\boldsymbol{b},N}(E) \longrightarrow A^{0,q+1}_{\boldsymbol{b},N}(E)$$

is dominated by $\big|\mathcal{A}_{\boldsymbol{y}}\big|_h$ for any $\boldsymbol{b} \in \boldsymbol{R}^\ell$. We can regard f as a section of $A^{0,0}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta},N}(E)$. Then, by using a standard argument in Section 2.9.1 of [**67**], we can find a holomorphic section $F^{(\varepsilon)}$ of $\mathcal{E}$ on $\mathcal{U}_2 \times (X \setminus D)$ such that (i) $F^{(\varepsilon)}_{|\boldsymbol{y}_0 \times (X \setminus D)} = f$, (ii) $\big\|F^{(\varepsilon)}\big\|_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta},N} < \infty$. By using Lemma 21.8.2, we obtain $F^{(\varepsilon)} \in {}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{U}_1 \times X(C_1)$ for some appropriate $\mathcal{U}_1 \subset \mathcal{U}_2$ and $0 < C_1 < 1$. □

21.8.3. Construction of local frames. — By shrinking X, we take a frame $\boldsymbol{u} = (u_i)$ of ${}_{\boldsymbol{a}}E$ on $X \setminus D$ compatible with the parabolic structure. Let $a_j(u_i)$ denote the parabolic degree of u_i with respect to the filtration ${}^jF$, and we put $\boldsymbol{a}(u_i) = \big(a_j(u_i) \,|\, j = 1, \ldots, \ell\big)$.

By the assumption $\mathrm{tr}\,\mathcal{A} = 0$, we have $\det \mathcal{E} = p^* \det(E)$. We put

$$\widetilde{a}_j := \sum_{b \in \mathcal{P}ar({}_{\boldsymbol{a}}E, j)} b \cdot \mathrm{rank}\, {}^j\mathrm{Gr}^F_b\big({}_{\boldsymbol{a}}E\big) = \sum_{i=1}^{r} a_j(u_i).$$

The tuple $(\widetilde{a}_j \,|\, 1 \leqslant j \leqslant \ell)$ is denoted by $\widetilde{\boldsymbol{a}}$. Note $\det({}_{\boldsymbol{a}}E) \simeq {}_{\widetilde{\boldsymbol{a}}}\det E$.

Take $\varepsilon > 0$ such that $10r^2 \cdot \varepsilon < |b - b'|$ for any distinct $b, b' \in \mathcal{P}ar(E, i)$ and any $i = 1, \dots, \ell$. We extend u_i to the section $u_i^{(\varepsilon)}$ of ${}_{\boldsymbol{a}(u_i)+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{U}_1 \times X(C_1)$. Since ε is sufficiently small, $\Omega(\boldsymbol{u}^{(\varepsilon)}) := u_1^{(\varepsilon)} \wedge \cdots \wedge u_r^{(\varepsilon)}$ gives a holomorphic section of ${}_{\tilde{\boldsymbol{a}}}\det \mathcal{E}$. Let $Z^{(\varepsilon)}$ denote the 0-set of $\Omega(\boldsymbol{u}^{(\varepsilon)})$. Since $\Omega(\boldsymbol{u}^{(\varepsilon)})_{|(\boldsymbol{y}_0, O)} \neq 0$ in ${}_{\tilde{\boldsymbol{a}}}\det \mathcal{E}_{|(\lambda_0, O)}$, there exist $\mathcal{U}_3(\varepsilon)$ and $0 < C_3(\varepsilon) < C_1$ such that $Z^{(\varepsilon)} \cap \big(\mathcal{U}_3(\varepsilon) \times X(C_3(\varepsilon))\big) = \varnothing$.

Lemma 21.8.4. — *$\boldsymbol{u}^{(\varepsilon)}$ gives a frame of ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $(\mathcal{U}_1 \times X(C)) \smallsetminus Z^{(\varepsilon)}$, in particular on $\mathcal{U}_3(\varepsilon) \times X(C_3(\varepsilon))$.*

Proof. — Let P be any point of $\mathcal{U}_1 \times D(C) \smallsetminus Z^{(\varepsilon)}$. Let $\mathcal{V}$ be a small neighbourhood of P. The restriction of $\boldsymbol{u}^{(\varepsilon)}$ to $\mathcal{V} \smallsetminus (\mathcal{U}_1 \times D)$ is a frame. Let f be a section of ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{V}$. We have the expression $f = \sum f_i \cdot u_i^{(\varepsilon)}$, where f_i are holomorphic on $\mathcal{V} \smallsetminus (\mathcal{U}_1 \times D)$. Since ε is sufficiently small, it is easy to observe that $f \wedge u_2^{(\varepsilon)} \wedge \cdots \wedge u_r^{(\varepsilon)}$ is a holomorphic section of ${}_{\tilde{\boldsymbol{a}}}\det \mathcal{E}$ on $\mathcal{V}$. It is equal to $f_1 \cdot \Omega(\boldsymbol{u}^{(\varepsilon)})$, and hence f_1 is holomorphic on $\mathcal{V}$. We obtain that the other f_i are also holomorphic on $\mathcal{V}$ in the same way. Hence, $\boldsymbol{u}^{(\varepsilon)}$ gives a frame of ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{V}$. □

We set $u_j^{(\varepsilon)\prime} := \prod_{i=1}^{\ell} |z_i|^{a_i(u_j)} \cdot u_j^{(\varepsilon)}$.

Lemma 21.8.5. — *Let P be any point of $(\mathcal{U}_1 \times D(C_1)) \smallsetminus \mathcal{Z}^{(\varepsilon)}$, and let $\mathcal{V}_P$ be any neighbourhood of P. Then, there exists a constant $B_P > 0$ such that the following holds on $\mathcal{V}_P$:*

$$B_P^{-1} \prod_{i=1}^{\ell} |z_i|^{2r^2\varepsilon} \leqslant H(h, \boldsymbol{u}^{(\varepsilon)\prime}) \leqslant B_P \prod_{i=1}^{\ell} |z_i|^{-2r^2\varepsilon}.$$

Proof. — The right inequality is clear. Let $\boldsymbol{u}^{(\varepsilon)\vee}$ be the dual frame of $\boldsymbol{u}^{(\varepsilon)}$ on $(\mathcal{U}_1 \times X(C_1)) \smallsetminus Z^{(\varepsilon)}$. There exists a constant $B_{P\,i} > 0$ $(i = 1, 2)$ such that the following holds:

$$\big|u_j^{(\varepsilon)\vee}\big|_h \leqslant B_{P\,1} \frac{\big|u_1^{(\varepsilon)} \wedge \cdots \wedge \check{u}_j^{(\varepsilon)} \cdots \wedge u_r^{(\varepsilon)}\big|}{|\Omega(\boldsymbol{u}^{(\varepsilon)})|_h} \leqslant B_{P\,2} \prod_{i=1}^{\ell} \big|z_i\big|^{a(u_i) - r\varepsilon}.$$

Hence, we obtain the following for some $B_{3\,i} > 0$:

$$H(h^\vee, \boldsymbol{u}^{(\varepsilon)\prime\vee}) \leqslant B_{P\,3} \prod_{i=1}^{\ell} |z_i|^{-2r^2\varepsilon}.$$

Thus we obtain Lemma 21.8.5. □

21.8.4. Proof of Theorem 21.8.1. — Let us fix small $\varepsilon_0 > 0$ as above, and we take a frame $\boldsymbol{u}^{(\varepsilon_0)}$ of ${}_{\boldsymbol{a}+\varepsilon_0\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{U}_3(\varepsilon_0) \times X\big(C_3(\varepsilon_0)\big)$ as above. Let us show that $u_j^{(\varepsilon_0)}$ are actually sections of ${}_{\boldsymbol{a}}\mathcal{E}$ on $\mathcal{U}_3(\varepsilon_0) \times X\big(C_3(\varepsilon_0)\big)$, and then the first claim of Theorem 21.8.1 follows.

Let $0 < \varepsilon \leqslant \varepsilon_0$. Let P be a point of $(\mathcal{U}_3(\varepsilon_0) \times D) \setminus Z^{(\varepsilon)}$, and let $\mathcal{V}$ be a small neighbourhood of P such that $\boldsymbol{u}^{(\varepsilon)}$ gives a frame of ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{V}$. We have the expression $u_i^{(\varepsilon_0)} = \sum b_{j,i} \cdot u_j^{(\varepsilon)}$. By using Lemma 21.8.5, we obtain that $b_{j,i}$ are holomorphic on $\mathcal{V}$, and thus ${}_{\boldsymbol{a}+\varepsilon_0\boldsymbol{\delta}}\mathcal{E} = {}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ on $\mathcal{V}$.

For each $I \subset \underline{\ell}$, we put $I^c := \underline{\ell} - I$, and

$$D_I^\circ := \bigcap_{i \in I} D_i \setminus \Big(\bigcup_{j \in I^c} D_j \Big).$$

We have the stratification $Z^{(\varepsilon)} = \coprod_I \coprod_j Z_{I,j}^{(\varepsilon)}$ satisfying the following:

- $Z_{I,j}^{(\varepsilon)} \subset \mathcal{U}_1 \times D_I^\circ$ and $\dim Z_{I,j} = j$.
- $Z_{I,j}^{(\varepsilon)}$ are smooth and locally closed in $\mathcal{U}_1 \times X$. Moreover, the induced morphism $Z_{I,j}^{(\varepsilon)} \to \mathcal{U}_1 \times X \to X$ is an immersion.

Let P be any point of $Z_{I,j}$. We show that ${}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E} = {}_{\boldsymbol{a}+\varepsilon_0\boldsymbol{\delta}}\mathcal{E}$ around P, by using the ascending induction on $|I|$ and the the descending induction on j.

Recall $\dim Y = 1$. We take a small neighbourhood T_P of P in $Z_{I,j}$. We take a small tubular neighbourhood $T_P^{(1)}$ of T_P in $\mathcal{U}_1 \times D_I^\circ$. We have identifications $T_P^{(1)} \simeq T_P \times N_P$ and $T_P \simeq T_P \times \{(0,\dots,0)\}$, where N_P denotes a $(1+n-|I|-j)$-dimensional multi-disc. We take a small tubular neighbourhood $T_P^{(2)}$ of $T_P^{(1)}$ in $\mathcal{U}_1 \times X(C_1)$. We have identifications $T_P^{(2)} \simeq T_P^{(1)} \times \Delta_w^{|I|}$ and $T_P^{(1)} \simeq T_P^{(1)} \times \{(0,\dots,0)\}$, where $\Delta_w^{|I|}$ is an $|I|$-dimensional multi-disc. Moreover, $T_P^{(2)} \cap (\mathcal{U}_1 \times D) = T_P^{(1)} \times \bigcup_{i \in I} \{z_i = 0\}$. By the inductive assumption, we have the following estimate on $T_P \times \partial N_P \times (\Delta^*)^I$ for any $\varepsilon' > \varepsilon$:

$$\big| u_i^{(\varepsilon_0)} \big|_h = O\Big(\prod_{k \in I} |z_k|^{-a_k - \varepsilon'} \Big). \tag{421}$$

Since $\big| u_i^{(\varepsilon_0)} \big|_h$ is pluri-subharmonic in the Y-direction, (421) holds on $T_P \times N_P \times (\Delta^*)^I$ for any $\varepsilon' > \varepsilon$. It means $u_i^{(\varepsilon_0)} \in {}_{\boldsymbol{a}+\varepsilon\boldsymbol{\delta}}\mathcal{E}$ around P. Since $\varepsilon > 0$ is arbitrary, we obtain that $u_i^{(\varepsilon_0)} \in {}_{\boldsymbol{a}}\mathcal{E}$. Thus, the proof of the first claim of Theorem 21.8.1 is completed.

Let us show the second claim of the theorem. Because we have also obtained ${}_{\boldsymbol{a}(u_i)+\varepsilon_0\boldsymbol{\delta}}\mathcal{E} = {}_{\boldsymbol{a}(u_i)}\mathcal{E}$, the following holds for any $\varepsilon > 0$:

$$\big| u_i^{(\varepsilon_0)} \big|_h = O\Big(\prod_{j=1}^{\ell} |z_j|^{-a_j(u_i) - \varepsilon} \Big).$$

Hence, there exists $C_\varepsilon > 0$ such that the following holds:

$$H(h, \boldsymbol{u}^{(\varepsilon_0)\prime}) \leqslant C_\varepsilon \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}.$$

By the argument in the proof of Lemma 21.8.5, we also obtain that there exists C'_ε such that

$$H(h^\vee, \boldsymbol{u}^{(\varepsilon_0)\prime\vee}) \leqslant C'_\varepsilon \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}.$$

Thus, the second claim of Theorem 21.8.1 is proved. □

21.9. Complement

21.9.1. Preliminary estimate for sup norms. — Let $X := \Delta^n$, $D_i := \{z_i = 0\}$ and $D := \bigcup_{i=1}^k D_i$ for some $k \leqslant n$. We use the Poincaré metric $g_{\mathbf{p}}$ of $X \setminus D$. We put $X(C) := \{(z_1, \dots, z_n) \in X \mid |z_i| \leqslant C\}$, $D_i(C) := D_i \cap X(C)$ and $D(C) := D \cap X(C)$ for $0 < C < 1$. We set

$$(422) \qquad Z(C) := \{(z_1, \dots, z_n) \in X(C) \setminus D(C) \mid |z_i| = C \ (i = 1, \dots, k)\}.$$

Let $(E, \overline{\partial}_E, h)$ be an acceptable bundle on $X \setminus D$. We use the notation in Section 21.2. We assume $|R(h)|_{h,g_{\mathbf{p}}} \leqslant C_0$ for some given constant C_0. Let ω be a C^∞-section of $E \otimes \Omega^{0,1}$ on $X \setminus D$. Assume $\sup_{X \setminus D} |\omega|_{\boldsymbol{a},N} < \infty$ for simplicity. Let f be a C^∞-section of $E \otimes \Omega^{0,1}$ on $X \setminus D$ such that $\overline{\partial} f = \omega$ and $\|f\|_{\boldsymbol{a},N} \leqslant \|\omega\|_{\boldsymbol{a},N}$. Note $\|\omega\|_{\boldsymbol{a},N} \leqslant \eta \cdot \sup |\omega|_{\boldsymbol{a},N}$, where $\eta > 0$ is independent of ω.

Lemma 21.9.1. — *There exists a constant C_1, depending on C_0, C and N, such that the following holds:*

$$\sup_{X(C) \setminus D(C)} |f|_{\boldsymbol{a},N+k} \leqslant C_1 \cdot \sup_{X \setminus D} |\omega|_{\boldsymbol{a},N}.$$

Proof. — We give only an outline of the proof. Let $\mathbb{H}$ denote the upper half plane $\{\zeta \in \boldsymbol{C} \mid \mathrm{Im}(\zeta) > 0\}$. We have the universal covering map $\varphi : \mathbb{H} \to \Delta^*$ given by $\varphi(\zeta) = \exp(\sqrt{-1}\zeta)$, which induces $\varphi : \mathbb{H}^k \times \Delta^{n-k} \to X \setminus D$. We put $\mathcal{U}(C) := \varphi^{-1}(X(C) \setminus D(C))$. Let g and dvol_g be the Euclidean metric of $\mathbb{H}^k \times \Delta^{n-k}$ and the associated volume form. Note that the fiber-wise norm of $\varphi^*\omega$ with respect to $\varphi^* h$ and g is dominated by $\varphi^*|\omega|_{\boldsymbol{a},N}$. Let B_ε be an ε-multi-ball contained in $\mathcal{U}(C)$. We have $\int_{B_\varepsilon} |f|^2_{\boldsymbol{a},N+k} \mathrm{dvol}_g \leqslant \|f\|^2_{\boldsymbol{a},N} \leqslant \|\omega\|^2_{\boldsymbol{a},N}$. We take the natural diffeomorphism $\psi : B_1 \simeq B_\varepsilon \subset \mathbb{H}^k \times \Delta^{n-k}$, where $B_1 := \{(w_1, \dots, w_n) \mid |w_i| < 1\}$. Let g_1 and dvol_{g_1} denote the Euclidean metric and the associated volume form of B_1. We fix a sufficiently small ε depending only on C, C_0 and N, such that the curvature of $\psi^*\varphi^*(E, \overline{\partial}_E, h_{\boldsymbol{a},N+k})$ is sufficiently small with respect to $\psi^*\varphi^* h_{\boldsymbol{a},N+k}$ and g_1 to which a theorem of Uhlenbeck can be applied. Then, we can take an orthogonal frame $\boldsymbol{e}$ of $\psi^*\varphi^*(E, h_{\boldsymbol{a},N+k})$ for which the $(0,1)$-form A is sufficiently small with respect to g_1, where A is given by $\overline{\partial}_E \boldsymbol{e} = \boldsymbol{e} \cdot A$. We have the expressions $\psi^*\varphi^* f = \sum F_i \cdot e_i$ and $\psi^*\varphi^*\omega = \sum G_i \cdot e_i$. The L^2-norm of $\boldsymbol{F} = (F_i)$ with respect to dvol_{g_1} is dominated by $C'_1(C_0, C, N) \cdot \|\omega\|_{\boldsymbol{a},N}$, where $C'_1(C_0, C, N)$ denotes a constant depending only on C_0, C and N. The norm of $\boldsymbol{G} = (G_i)$ with respect to g_1 is dominated by $|\omega|_{\boldsymbol{a},N}$. We have the relation $\overline{\partial}\boldsymbol{F} + [A, \boldsymbol{F}] = \boldsymbol{G}$. By using the standard bootstrapping argument, we obtain the desired estimate for the sup-norm of $\boldsymbol{F}$ in terms of $\sup_{X\setminus D}|\omega|_{\boldsymbol{a},N}$. □

21.9.2. A priori refinement of estimate for the sup norm. — Let ρ be a C^∞-section of E on $X \setminus D$ such that the following holds for some large M:

$$|\rho|_h = O\Big(\prod_{i=1}^k |z_i|^M\Big), \quad |\overline{\partial}_E\rho|_{h,g_{\mathrm{P}}} = O\Big(\prod_{i=1}^k |z_i|^M\Big).$$

Let $Z(C)$ be as in (422). We define

$$|\rho|_{\infty,Z(C)} := \sup_{Z(C)}|\rho|_h, \quad \|\overline{\partial}_E\rho\|_\infty := \sup_{X\setminus D}\Big|\overline{\partial}_E\rho\cdot\prod_{i=1}^k|z_i|^{-M}\Big|_{h,g_{\mathrm{P}}}.$$

Lemma 21.9.2. — *Let $M' < M$. There exists a constant C_2, depending only on C_0, C and M', such that the following holds:*

$$\sup_{X(C)\setminus D(C)}\Big(|\rho|_h\cdot\prod_{i=1}^k|z_i|^{-M'}\Big) \leqslant C_2\Big(|\rho|_{\infty,Z(C)} + \|\overline{\partial}_E\rho\|_\infty\Big).$$

Proof. — In the following, $C_i > 0$ denote constants depending only on C_0, C and M'. Take $M' < M_1 < M_2 < M$. According to Lemma 21.2.3, we can take a C^∞-section g of E on $X \setminus D$ satisfying the equality $\overline{\partial}_E g = \overline{\partial}_E\rho$ and the following estimate:

$$\int_{X\setminus D}|g|_h^2\cdot\prod_{i=1}^k|z_i|^{-2M_2}\Big(-\sum_{i=1}^k\log|z_i|\Big)^{C_{11}}\mathrm{dvol} \leqslant C_{10}\cdot\|\overline{\partial}_E\rho\|_\infty^2.$$

According to Lemma 21.9.1, we obtain $|g|_h \leqslant C_{15}\|\overline{\partial}_E\rho\|_\infty\cdot\prod_{i=1}^k|z_i|^{M_1}$ on $X(C)\setminus D(C)$. Then, we apply Proposition 21.2.8 to $\rho - g$. □

21.9.3. Connection form of an acceptable bundle. — Let X, D and $(E, \overline{\partial}_E, h)$ be as in Section 21.3. As remarked in Theorem 21.3.1, we obtain the prolongment ${}_{\boldsymbol{a}}E$ for each $\boldsymbol{a} \in \boldsymbol{R}^\ell$ from $(E, \overline{\partial}_E, h)$. Let $\boldsymbol{v}$ be a holomorphic frame of ${}_{\boldsymbol{a}}E$ compatible with the parabolic structures. Then, the C^∞-section F of $\mathrm{End}(E)\otimes\Omega^{1,0}$ is determined by $F(\boldsymbol{v}) = \partial_E\boldsymbol{v}$.

Lemma 21.9.3. — *F is bounded up to log order, with respect to h and the Poincaré metric g_{P} of $X \setminus D$.*

Proof. — Let $\boldsymbol{v}'$ be as in Section 21.3. Let h_0 be the Hermitian metric of E determined by $h_0(v_i', v_j') = \delta_{i,j}$. We have $R(h_0) = 0$. Let s be the endomorphism of E determined by $h = h_0\cdot s$, which is self-adjoint with respect to both h and h_0. We have $\partial_h - \partial_{h_0} = s^{-1}\partial_{h_0}s$ and $R(h) = \overline{\partial}_E(s^{-1}\partial_{h_0}s)$.

Let π_1 be the natural projection of $X \setminus D$ to D_1. We only have to show the following inequality for some positive constants C and N which are independent of $P \in D_1 \setminus \bigcup_{1<j\leqslant \ell} D_j$:

$$\big|s^{-1}\partial_{h_0} s_{|\pi_1^{-1}(P)}\big|_{h_0,g_{\mathbf{P}}} \leqslant C\cdot\Big(-\sum_{j=1}^{\ell}\log|z_j|\Big)^N.$$

Let κ_1 be an $\boldsymbol{R}_{\geqslant 0}$-valued C^∞-function on $\boldsymbol{R}$ such that $\kappa_1(t)=1$ for $t\leqslant 1/2$ and $\kappa_1(t)=0$ for $t\geqslant 2/3$. Let κ_2 be an $\boldsymbol{R}_{\geqslant 0}$-valued C^∞-function on $\boldsymbol{R}$ such that $\kappa_2(t)=0$ for $t\leqslant 1/3$ and $\kappa_2(t)=1$ for $t\geqslant 1$. We put $\chi_L(z):=\kappa_1\big(-L^{-1}\log|z|\big)\cdot\kappa_2(-\log|z|)$ for any positive number L. Let ρ be any C^∞-function on $\pi_1^{-1}(P)$. In the following, we will consider integrals over $\pi_1^{-1}(P)$. We often use $\int$ instead of $\int_{\pi_1^{-1}(P)}$ to save space. We have

$$(423)\quad \int_{\pi_1^{-1}(P)}\big(s^{-1}\partial_{h_0}(\chi_L s),\,\partial_{h_0}(\chi_L s)\big)_{h_0}\cdot\rho \\ =\int\big(\overline{\partial}(s^{-1}\partial_{h_0}(\chi_L s)),\,\chi_L s\big)_{h_0}\cdot\rho-\int\big(s^{-1}\partial_{h_0}(\chi_L s),\,\chi_L s\big)_{h_0}\cdot\overline{\partial}\rho.$$

We have the following equalities:

$$(424)\quad \int_{\pi_1^{-1}(P)}\big(\overline{\partial}(s^{-1}\partial_{h_0}(\chi_L\cdot s)),\,\chi_L s\big)_{h_0}\rho \\ =\int\big(R(h)\chi_L,\chi_L s\big)_{h_0}\rho+\int\operatorname{tr}\big(\overline{\partial}\chi_L\cdot\partial_{h_0}s\cdot\chi_L\big)\rho+\int\big(\overline{\partial}\partial\chi_L,\,\chi_L s\big)_{h_0}\rho \\ =\int\big(R(h)\chi_L,\,\chi_L s\big)_{h_0}\rho-\int\operatorname{tr}\big(\overline{\partial}\chi_L\cdot s\cdot\partial\chi_L\big)\rho-\int\operatorname{tr}\big(\chi_L\cdot\overline{\partial}\chi_L\cdot s\big)\partial\rho$$

$$(425)\quad \int_{\pi_1^{-1}(P)}\big(s^{-1}\partial_{h_0}(\chi_L s),\,\chi_L s\big)_{h_0}\overline{\partial}\rho=\int\operatorname{tr}\big(\partial_{h_0}(\chi_L s)\cdot\chi_L\cdot\overline{\partial}\rho\big) \\ =\int\partial\big(\operatorname{tr}(\chi_L^2 s)\cdot\overline{\partial}\rho\big)-\int\operatorname{tr}\big(\chi_L s\cdot\partial\chi_L\big)\overline{\partial}\rho-\int\operatorname{tr}\big(\chi_L^2 s\big)\cdot\partial\overline{\partial}\rho.$$

Let $C_2(P):=-\sum_{2\leqslant j\leqslant\ell}\log|z_j(P)|$. We have the following estimates, for some sufficiently large N:

$$(426)\quad \begin{aligned}\big|s_{|\pi_1^{-1}(P)}\big|_{h_0}&\leqslant\big(-\log|z_1|+C_2(P)\big)^N,\\ \big|R(h)_{|\pi_1^{-1}(P)}\big|_{h_0,g_{\mathbf{P}}}&\leqslant\big(-\log|z_1|+C_2(P)\big)^N.\end{aligned}$$

We put $\rho_M:=(-\log|z_1|)^{-M}$ for some sufficiently large M. By using (423), (424), (425) and (426), we obtain the following for some constants C_3 and M':

$$\int_{\pi_1^{-1}(P)}\big(s^{-1}\partial_{h_0}(\chi_L s),\,\partial_{h_0}(\chi_L s)\big)\cdot\rho_M\leqslant C_3\cdot C_2(P)^{M'}.$$

By making $L \to \infty$, we obtain

$$\int_{\substack{\pi_1^{-1}(P)\\ |z_1|\leqslant e^{-1}}} \big(s^{-1}\partial_{h_0}(s),\, \partial_{h_0}(s)\big)\cdot \rho_M \leqslant C_3\cdot C_2(P)^{M'}.$$

By making M larger, we obtain the following estimate for some constants C_4 and M'':

$$\int_{\substack{\pi_1^{-1}(P)\\ |z_1|\leqslant e^{-1}}} \big(s^{-1}\partial_{h_0}(s), s^{-1}\partial_{h_0}(s)\big)\cdot \rho_M \leqslant C_4\cdot C_2(P)^{M''}.$$

We can take $G_P\cdot dz_1/z_1$ for each P with the following property for some N_1:

$$\overline{\partial}_{z_1}\left(G_P\frac{dz_1}{z_1}\right) = R(h)_{|\pi_1^{-1}(P)},\quad \big|G_P\big|_{h_0}\leqslant C_5\cdot\big(-\log|z_1|+C_2(P)\big)^{N_1}.$$

(We can use Lemma 5.2.3, for example.) We put $\psi_P := s^{-1}\partial_{h_0}s_{|\pi_1^{-1}(P)} - G_P\cdot dz_1/z_1$. Then, we obtain $\overline{\partial}_{z_1}\psi_P = 0$ and the following inequality for some C, M_1 and N_3, which are independent of P:

$$\int_{\pi_1^{-1}(P)} \big|\psi_P\big|^2_{h_0,g_{\mathbf{p}}}\big(-\log|z_1|+C_2(P)\big)^{-M_1}\,\mathrm{dvol}_{g_{\mathbf{p}}} < C\cdot C_2(P)^{N_3}. \tag{427}$$

By the holomorphic property, we obtain $\big|\psi_P\big|_{h_0,g_{\mathbf{p}}}\leqslant C_{10}\cdot\big(-\log|z_1|+C_2(P)\big)^{N_4}$ for some sufficiently large N_4 and C_{10}, which are independent of the choice of P, by the following standard argument. Let $\boldsymbol{u}=(u_j)$ be a frame of ${}^{\diamond}\mathrm{End}(E)$ compatible with the parabolic structure. We have the expression $\psi_P=\sum\psi_{P,j}\cdot u_{j|\pi_1^{-1}(P)}$. We obtain the estimate for the integrals of $|\psi_{P,j}|^2$ with appropriate weight from (427). Since $\psi_{P,j}$ are holomorphic, we obtain the estimate for the sup norms of $|\psi_{P,j}|$.

Thus, we obtain the desired estimate for $s^{-1}\partial_{h_0}s$, and the proof of Lemma 21.9.3 is finished. □

CHAPTER 22

REVIEW ON $\mathcal{R}$-MODULES, $\mathcal{R}$-TRIPLES AND VARIANTS

In Section 22.1, we recall the basic facts on finiteness of filtered rings and filtered modules from [**43**] for reader's convenience.

In Sections 22.2–22.3, we recall some basic property of $\mathcal{R}_X$-modules and their specializations. In Section 22.4, we recall the notion of $\mathcal{R}_X(*t)$-module. We refer to [**73**] for more details and precision. ($\mathcal{R}(*t)$-modules are called $\widetilde{\mathcal{R}}$-modules there.) See [**77**] for the original work due to M. Saito on filtered D-modules. See also [**67**].

We would like to reduce the study of wild objects to the study of tame objects. One of the important tools is formal completion, for which we give a preparation in Section 22.5. In Subsection 22.5.1, we review formal complex spaces. We refer to [**4**], [**8**] and [**50**] for more details and precision. Then, we recall basic facts on formal D-modules and formal $\mathcal{R}$-modules in Subsections 22.5.2 and 22.5.3. We can obtain the claims in these subsections by directly applying a general theory explained in the appendix of [**43**]. We explain in Subsection 22.5.4 how to use completions in showing the strict S-decomposability which is the main motivation for us to consider formal $\mathcal{R}$-modules.

In Section 22.6, we give some miscellaneous preliminaries on D-modules for references in our argument. We just indicate where they are used in this monograph. The contents in Subsections 22.6.1 and 22.6.2 are implicitly used for the proof of theorems in Chapter 19. We prepare the nearby cycle functor with ramified exponential twist in Subsection 22.6.3 to state Theorem 19.4.2. Subsections 22.6.4 and 22.6.5 are preliminary for Section 18.4.

Section 22.7 is also complements. We give in Subsection 22.7.1 a remark on push-forward of $\mathcal{R}$-modules via a ramified covering, which has been implicitly used in Part IV. Subsection 22.7.2 is a preparation for the argument in Sections 12.2–12.4.

We review in Section 22.8 some basic property of the sheaves of distributions with moderate growth satisfying some continuity, introduced by Sabbah in [**73**], which is one of the basic ingredients for the definition of wild pure twistor D-modules. It is used in Chapter 12 and Part IV.

In Sections 22.9–22.10, we recall the notion of $\mathcal{R}$-triples and their specialization. In Section 22.11, we recall the notion of $\mathcal{R}(*t)$-triples. We refer to [**73**] for more details and precision on their properties. ($\mathcal{R}(*t)$-triples are called $\widetilde{\mathcal{R}}$-triples there.) In Section 22.12, we compare pure twistor structures and $\mathcal{R}$-triples in dimension 0. We also give, for a regular singular $\mathcal{R}$-triple on a disc, a comparison of the specializations as a $\mathcal{R}$-triple and as a variation of twistor structure.

22.1. Filtered rings

For the reader's convenience, we recall some general results on filtered rings and their modules taken from the Appendix of the nice textbook [**43**] due to Kashiwara, which we refer to for more details and precision.

22.1.1. Coherence. — Let $\mathcal{A}$ be a sheaf of rings on a topological space X. We have the standard notions of *locally finitely generated $\mathcal{A}$-modules* and *locally finitely presented $\mathcal{A}$-modules.*

***Proposition 22.1.1* (Proposition A.4 of [43]).** — *Let $\mathcal{F}$ be a locally finitely generated $\mathcal{A}$-module. Let $x \in X$, and suppose that the stalk $\mathcal{F}_x$ vanishes. Then, there exists a neighbourhood U of x such that $\mathcal{F}_{|U} = 0$.* □

An $\mathcal{A}$-module $\mathcal{F}$ is called *pseudo-coherent* if the following holds:

- Let U be any open subset of X. Let $\mathcal{G}$ be a locally finitely generated $\mathcal{A}_{|U}$-submodule of $\mathcal{F}_{|U}$. Then, $\mathcal{G}$ is locally finitely presented.

Any $\mathcal{A}$-submodule of a pseudo-coherent $\mathcal{A}$-module is also pseudo-coherent.

An $\mathcal{A}$-module $\mathcal{F}$ is called *coherent*, if it is pseudo-coherent and locally finitely generated. The category of coherent $\mathcal{A}$-modules is abelian. It is stable under extensions. (See Proposition A.6 of [**43**] for more details.) The sheaf of rings $\mathcal{A}$ is called *coherent* if it is coherent as an $\mathcal{A}$-module. If $\mathcal{A}$ is coherent, an $\mathcal{A}$-module is coherent if and only if it is locally finitely presented.

An $\mathcal{A}$-module $\mathcal{F}$ is called *Noetherian*, if the following holds:

- $\mathcal{F}$ is a coherent $\mathcal{A}$-module.
- For any $x \in X$, the stalk $\mathcal{F}_x$ is a Noetherian $\mathcal{A}_x$-module.
- Let U be any open subset of X. Let $\{\mathcal{G}_i\}$ be any family of coherent $\mathcal{A}_{|U}$-submodules of $\mathcal{F}_{|U}$. Then, $\sum \mathcal{G}_i$ is a coherent $\mathcal{A}_{|U}$-module.

A sheaf of rings $\mathcal{A}$ is called Noetherian, if it is Noetherian as an $\mathcal{A}$-module.

22.1.2. Filtered rings and filtered modules. — A sheaf of rings $\mathcal{A}$ is called filtered, if it is equipped with an increasing sequence of submodules $\{F_n(\mathcal{A}) \,|\, n = 0, 1, 2, \ldots\}$ such that (i) $\mathcal{A} = \bigcup F_n(\mathcal{A})$, (ii) $1 \in F_0(\mathcal{A})$, (iii) $F_n(\mathcal{A}) \cdot F_m(\mathcal{A}) \subset$

$F_{n+m}(\mathcal{A})$. If an $\mathcal{A}$-module $\mathcal{M}$ is equipped with an increasing sequence of submodules $\{F_n(\mathcal{M}) \mid n = 0, 1, 2, \ldots\}$ such that $F_n(\mathcal{A})F_m(\mathcal{M}) \subset F_{n+m}(\mathcal{M})$, the pair $(\mathcal{M}, F)$ is called a filtered $\mathcal{A}$-module. We say that $(\mathcal{M}, F)$ is a *coherent* $\mathcal{A}$-module, if $\bigoplus F_n(\mathcal{M})$ is a coherent $\bigoplus F_n(\mathcal{A})$-module. In that case, F is called a *coherent filtration.* Similarly, F is called a *locally finitely generated*, if $\bigoplus F_n(\mathcal{M})$ is locally finitely generated over $\bigoplus F_n(\mathcal{A})$.

The following proposition is useful to check the coherence of a filtration.

***Proposition 22.1.2* (Lemma A.27 of [43]).** — *Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module, and $\mathcal{N}$ be an $\mathcal{A}$-submodule. A filtration $F(\mathcal{N})$ of $\mathcal{N}$ is coherent, if and only if it is locally finitely generated.* □

Recall the following general and fundamental theorem.

***Proposition 22.1.3* (Theorem A.20 of [43]).** — *Let $(\mathcal{A}, F)$ be a filtered ring. Assume the following:*

- *$F_0(\mathcal{A})$ is Noetherian.*
- *$\mathrm{Gr}_k(\mathcal{A})$ are coherent $F_0(\mathcal{A})$-module.*
- *Let $m \in \mathbb{Z}_{>0}$, and let U be an open subset of X. Let $\mathcal{N}$ be an $\mathcal{A}$-submodule of $\mathcal{A}_{|U}^{\oplus m}$ such that $\mathcal{N} \cap F_k(\mathcal{A})^{\oplus m}$ are coherent $F_0(\mathcal{A})_{|U}$-modules for all k. Then, $\mathcal{N}$ is locally finitely generated over $\mathcal{A}$.*

Then, $\mathcal{A}$ is Noetherian. □

The following proposition is useful to check coherence of an $\mathcal{A}$-module.

***Proposition 22.1.4* (Lemma A.22 of [43]).** — *Let $(\mathcal{A}, F)$ be a filtered ring satisfying the conditions in Proposition* 22.1.3. *Let $\mathcal{M}$ be an $\mathcal{A}$-module such that (i) locally finitely generated over $\mathcal{A}$, (ii) pseudo-coherent as an $F_0(\mathcal{A})$-module. Then, $\mathcal{M}$ is a coherent $\mathcal{A}$-module.* □

The following proposition is useful to show a Noetherian property of a sheaf of rings $\mathcal{A}$.

***Proposition 22.1.5* (Theorem A.29, Theorem A.31 of [43]).** — *Let $(\mathcal{A}, F)$ be a filtered ring on X. Assume that $F_0(\mathcal{A})$ and $\mathrm{Gr}^F(\mathcal{A})$ are Noetherian rings, and $\mathrm{Gr}^F_k(\mathcal{A})$ are locally finitely generated $F_0(\mathcal{A})$-modules for any k. Then, $\mathcal{A}$ is a Noetherian ring, and it satisfies the conditions in Proposition* 22.1.3. *Moreover, the Rees ring $\bigoplus F_n(\mathcal{A})$ is also Noetherian.* □

22.2. $\mathcal{R}$-modules

22.2.1. The sheaf of algebras $\mathcal{R}_X$. — Let us recall the notion of $\mathcal{R}_X$-modules introduced in [73]. Let X be a complex manifold. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$, where $\boldsymbol{C}_\lambda$ denotes the complex line with the coordinate λ. Let $p : \mathcal{X} \to X$ denote the natural projection. Let Θ_X denote the tangent sheaf of X. We have the subsheaf $\lambda \cdot p^*\Theta_X$ of $p^*\Theta_X$. We denote $\lambda \cdot p^*\Theta_X$ by $\Theta_{\mathcal{X}}$. Since we do not consider the tangent sheaf of $\mathcal{X}$, there is no risk of confusion. Let $\mathcal{D}_X$ denote the sheaf of differential operators on X. We have the sheaf $p^*\mathcal{D}_X = \mathcal{O}_{\mathcal{X}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{D}_X$, which is the sheaf of relative differential operators of $\mathcal{X}$ over $\boldsymbol{C}_\lambda$. Let $\mathcal{R}_X$ denote the sheaf of subalgebras of $p^*\mathcal{D}_X$ generated by $\mathcal{O}_{\mathcal{X}}$ and $\Theta_{\mathcal{X}}$. The notions of left $\mathcal{R}$-modules and right $\mathcal{R}$-modules are naturally defined as in the case of $\mathcal{D}_X$-modules. *In this paper, we usually consider left $\mathcal{R}_X$-modules.*

The sheaf $\mathcal{R}_X$ is equipped with the increasing filtration F by the order of differential operators, via which it is a sheaf of filtered algebras. The associated graded sheaf is isomorphic to $p^* \operatorname{Sym}^\bullet \Theta_X$. According to Propositions 22.1.4 and 22.1.5, we obtain the following.

Proposition 22.2.1

- *The sheaf of algebras $\mathcal{R}_X$ is Noetherian, and it satisfies the conditions in Proposition* 22.1.3. *The Rees ring is also Noetherian.*
- *Let M be an $\mathcal{R}_X$-module such that (i) pseudo-coherent as an $\mathcal{O}_X$-module, (ii) locally finitely generated as an $\mathcal{R}_X$-module. Then, M is a coherent $\mathcal{R}_X$-module.* □

Let λ_0 be any point of $\boldsymbol{C}$. We say that an $\mathcal{R}_X$-module $\mathcal{M}$ is *strict* at λ_0, if the multiplication of $\lambda - \lambda_0$ on $\mathcal{M}$ is injective. If $\mathcal{M}$ is strict at any $\lambda_0 \in \boldsymbol{C}$, then it is called *strict*.

22.2.2. Left and right $\mathcal{R}$-module. — Let ω_X denote the canonical line bundle of X, i.e., it is the sheaf of holomorphic $(n, 0)$-forms, where $n = \dim X$. Then, $p^*\omega_X$ is a right $\mathcal{R}_X$-module, on which the right $\mathcal{R}_X$-action is given as the restriction of the natural right $p^*\mathcal{D}_X$-action. Let $\omega_{\mathcal{X}} := \lambda^{-n} \cdot p^*\omega_X$ as the subsheaf of $p^*\omega_X \otimes \mathcal{O}_{\mathcal{X}}\big(*(\{0\} \times X)\big)$. It is naturally a right $\mathcal{R}_X$-module.

As in the case of $\mathcal{D}_X$-modules, for any left $\mathcal{R}_X$-module $\mathcal{M}$, we have the natural right $\mathcal{R}_X$-module structure on $\mathcal{M}^r := \omega_{\mathcal{X}} \otimes \mathcal{M}$. By this functor, the categories of left $\mathcal{R}_X$-modules and right $\mathcal{R}_X$-modules are naturally equivalent.

22.2.3. Holonomic $\mathcal{R}$-module. — Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module on $U \subset \mathcal{X}$. As in Section 22.1, we have the notion of coherent filtration for $\mathcal{M}$. If $\mathcal{M}$ has a coherent filtration F, we obtain the coherent $p^* \operatorname{Sym} \Theta_X$-module $\operatorname{Gr}^F \mathcal{M}$. It naturally induces a coherent sheaf on $\boldsymbol{C}_\lambda \times T^*X$. Its support $\operatorname{Ch}(\mathcal{M})$ is called the characteristic variety. We can show in a standard way that $\operatorname{Ch}(\mathcal{M})$ is independent of the choice of a coherent filtration.

Let $\mathcal{M}$ be an $\mathcal{R}_X$-module on $U \subset \mathcal{X}$. We say that $\mathcal{M}$ is *good*, if the following condition is satisfied:

- Let K be any compact subset of X such that $\{\lambda_0\} \times K \subset U$. Then, there exist a neighbourhood U' of $\{\lambda_0\} \times K$ in U, a finite filtration F of $\mathcal{M}_{|U'}$ such that $\mathrm{Gr}^F(\mathcal{M}_{|U'})$ has a coherent filtration.

If $\mathcal{M}$ is good, we have the globally well-defined characteristic variety $\mathrm{Ch}(\mathcal{M}) \subset \boldsymbol{C}_\lambda \times T^*X$. If there is a Lagrangian subvariety L of T^*X such that $\mathrm{Ch}(\mathcal{M}) \subset \boldsymbol{C}_\lambda \times L$, then $\mathcal{M}$ is called holonomic. As a similar but weaker condition, we say that $\mathcal{M}$ is locally good, if there exists a covering $U = \bigcup U_j$ such that $\mathcal{M}_{|U_j}$ are good. The characteristic variety makes sense for any locally good $\mathcal{R}_X$-module. Goodness is required for good behaviour with respect to the push-forward via a proper morphism.

22.2.4. Push-forward and pull-back. — The push-forward and pull-back of $\mathcal{R}$-modules are defined as in the case of D-modules. Their properties are also similar. Let $f : X \to Y$ be a holomorphic map of complex manifolds. The induced morphism $\mathcal{X} \to \mathcal{Y}$ is also denoted by F. The left and right $(\mathcal{R}_X, F^{-1}\mathcal{R}_Y)$-module $\mathcal{R}_{X\to Y}$ is given as follows:

$$\mathcal{R}_{X\to Y} := \mathcal{O}_{\mathcal{X}} \otimes_{F^{-1}\mathcal{O}_{\mathcal{Y}}} F^{-1}\mathcal{R}_Y.$$

The pull-back of a *left* $\mathcal{R}_Y$-module $\mathcal{N}$ is given as follows:

$$f^\dagger \mathcal{N} := \mathcal{R}_{X\to Y} \otimes^L_{F^{-1}\mathcal{R}_Y} F^{-1}\mathcal{N} \in D^b(\mathcal{R}_X).$$

The push-forward of a *right* $\mathcal{R}_X$-module $\mathcal{M}$ is given as follows:

$$f_\dagger \mathcal{M} := RF_!\big(\mathcal{M} \otimes^L_{\mathcal{R}_X} \mathcal{R}_{X\to Y}\big) \in D^b(\mathcal{R}_Y).$$

Here $\otimes^L$ denotes the derived tensor product, and $RF_!$ denotes the ordinary push-forward with compact support. (See [**73**] for more precision.) The push-forward of a left module is defined by using the equivalence of the categories of left and right $\mathcal{R}$-modules as mentioned above. Namely, we set $\mathcal{R}_{Y\leftarrow X} := \omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} F^{-1}\big(\mathcal{R}_Y \otimes_{\mathcal{O}_{\mathcal{Y}}} \omega_{\mathcal{Y}}^{-1}\big)$, and define

$$f_\dagger(\mathcal{M}) := RF_!\big(\mathcal{R}_{Y\leftarrow X} \otimes^L_{\mathcal{R}_X} \mathcal{M}\big)$$

for a left $\mathcal{R}_X$-module $\mathcal{M}$. The i-th cohomology sheaves of $f_\dagger\mathcal{M}$ are denoted by $f^i_\dagger\mathcal{M}$. We refer to Section 1.4 of [**73**] for more details and precision. We mention one result: if $\mathcal{M}$ is good (holonomic), then $f^i_\dagger\mathcal{M}$ are also good (holonomic), which can be shown in a standard way as in [**43**].

We recall the construction of the push-forward for a left $\mathcal{R}_X$-module in the case where Y is a point for explanation in Subsection 17.3.8. Let $\Omega^{r,s}_X$ denote the C^∞-vector bundle of (r,s)-forms on X. Let $\mathcal{X}^0 := \{0\} \times X$. We obtain the $\boldsymbol{C}_\lambda$-holomorphic bundle $\Omega^{r,s}_{\mathcal{X}} := p^*\Omega^{r,s}_X \otimes \mathcal{O}_{\mathcal{X}}(r\mathcal{X}^0)$, and the double complex $\big(\Omega^{r,s}_{\mathcal{X}}, \partial_X, \overline{\partial}_X\big)$. The associated total complex is denoted by $(\Omega^\bullet_{\mathcal{X}}, d)$. The left $\mathcal{R}_X$-module structure on $\mathcal{M}$ and the exterior derivative d naturally induce the complex $(\mathcal{M} \otimes \Omega^\bullet_{\mathcal{X}}, d)$. Then, $f_\dagger\mathcal{M}$ is given as follows in this case:

$$f_\dagger\mathcal{M} = F_!\big(\Omega^\bullet_{\mathcal{X}}[\dim X] \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}\big).$$

A closed q-form ω on X induces a morphism of complexes $L_\omega : \mathcal{M}\otimes\Omega^\bullet_{\mathcal{X}} \to \mathcal{M}\otimes\Omega^{\bullet+q}_{\mathcal{X}}$, which induces $L_\omega : f^i_\dagger\mathcal{M} \to f^{i+q}_\dagger\mathcal{M}$. More generally, if $X = Y\times Z$, we naturally have $f_\dagger(\mathcal{M}) \simeq F_!\big(p_Z^{-1}\Omega^\bullet_{\mathcal{Z}}[\dim Z]\otimes_{p_Z^{-1}\mathcal{O}_{\mathcal{Z}}}\mathcal{M}\big)$

22.3. Specialization of $\mathcal{R}$-modules

Sabbah introduced the notions of V-filtration, nearby cycle functor and vanishing cycle functor for $\mathcal{R}$-modules in [**73**], which are natural generalization of those for filtered D-modules in the original work by M. Saito [**77**]. (See also [**67**].)

22.3.1. Strict specializability. — Let $\boldsymbol{C}_t$ be a complex line with a coordinate t. Let X_0 be a complex manifold. We put $X := X_0\times\boldsymbol{C}_t$. We identify X_0 and $X_0\times\{0\}$. Recall $\mathcal{X} := \boldsymbol{C}_\lambda\times X$ and $\mathcal{X}_0 := \boldsymbol{C}_\lambda\times X_0$. For any point $\lambda_0\in\boldsymbol{C}_\lambda$, let $\mathcal{X}^{(\lambda_0)}$ denote a small neighbourhood of $\{\lambda_0\}\times X$. We use the symbol $\mathcal{X}_0^{(\lambda_0)}$ in a similar meaning. Let $p_{X_0} : X\to X_0$ denote the projection. Let $V_0\mathcal{R}_X$ denote the sheaf of subalgebras of $\mathcal{R}_X$ generated by $p^*_{X_0}\mathcal{R}_{X_0}$ and $\eth_t t$.

Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module on $\mathcal{X}^{(\lambda_0)}$. Let $V^{(\lambda_0)}$ be an increasing filtration of coherent $V_0\mathcal{R}_X$-submodules indexed by $\boldsymbol{R}$. We recall some conditions for $V^{(\lambda_0)}$.

Condition 22.3.1
- For any $a\in\boldsymbol{R}$, and for any $P\in\mathcal{X}_0^{(\lambda_0)}$, there exists $\varepsilon>0$ such that $V_a^{(\lambda_0)}(\mathcal{M}) = V_{a+\varepsilon}^{(\lambda_0)}(\mathcal{M})$ around P. Moreover, $\bigcup_{a\in\boldsymbol{R}}V_a^{(\lambda_0)}(\mathcal{M}) = \mathcal{M}$.
- $t\cdot V_a^{(\lambda_0)}(\mathcal{M})\subset V_{a-1}^{(\lambda_0)}(\mathcal{M})$ for any $a\in\boldsymbol{R}$, and $t\cdot V_a^{(\lambda_0)}(\mathcal{M}) = V_{a-1}^{(\lambda_0)}(\mathcal{M})$ for any $a<0$.
- $\eth_t\cdot V_a^{(\lambda_0)}(\mathcal{M})\subset V_{a+1}^{(\lambda_0)}(\mathcal{M})$ for any $a\in\boldsymbol{R}$, and the induced morphisms $\eth_t$: $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})\to\mathrm{Gr}_{a+1}^{V^{(\lambda_0)}}(\mathcal{M})$ are surjective for any $a>-1$. □

Condition 22.3.2. — Let $(\mathcal{M},V^{(\lambda_0)})$ be as in Condition 22.3.1. Recall that $\mathcal{M}$ is called strictly specializable along t at λ_0 with respect to $V^{(\lambda_0)}$ if the following holds:
- $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$ is strict for each $a\in\boldsymbol{R}$, i.e., the multiplication of $\lambda-\lambda_1$ on $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$ is injective for any $\lambda_1\in\boldsymbol{C}$.
- For any $P\in\mathcal{X}_0^{(\lambda_0)}$, there exists a finite subset $\mathcal{K}(a,\lambda_0,P)\subset\boldsymbol{R}\times\boldsymbol{C}$ such that the action of $\prod_{u\in\mathcal{K}(a,\lambda_0)}\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)$ on $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$ is nilpotent around P.

The union $\bigcup_{a\in\boldsymbol{R}}\mathcal{K}(a,\lambda_0,P)$ is denoted by $\mathcal{KMS}(\mathcal{M},t,P)$. We also put $\mathcal{K}(a,\lambda_0) := \bigcup_{P\in\mathcal{X}_0}\mathcal{K}(a,\lambda_0,P)$ and $\mathcal{KMS}(\mathcal{M},t) := \bigcup_{P\in\mathcal{X}_0}\mathcal{KMS}(\mathcal{M},t,P)$. □

If $(\mathcal{M},V^{(\lambda_0)})$ is strictly specializable along t, for $u\in\mathcal{KMS}(\mathcal{M},t)$, we put

$$\psi_{t,u}^{(\lambda_0)}(\mathcal{M}) := \bigcup_N \mathrm{Ker}\Big(\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)^N\Big).$$

Thus, we obtain an $\mathcal{R}_{X_0}$-module $\psi_{t,u}^{(\lambda_0)}(\mathcal{M})$, and the decomposition

$$\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}) = \bigoplus_{u\in\mathcal{K}(a,\lambda_0)} \psi_{t,u}^{(\lambda_0)}(\mathcal{M}). \tag{428}$$

Remark 22.3.3. — If we consider local issues, we may assume that the set $\mathcal{K}(a,\lambda_0)$ is finite. We will often assume it implicitly. □

Lemma 22.3.4. — *Such a filtration $V^{(\lambda_0)}$ is unique, if it exists. The index set $\mathcal{KMS}(\mathcal{M},t)$ is also uniquely determined if $\psi_{t,u}^{(\lambda_0)}(\mathcal{M})\neq 0$ for any $u\in\mathcal{KMS}(\mathcal{M},t)$.*

Proof. — Although it is proved in [**73**] and [**67**], we give a sketch of the proof. We only have to argue the issue locally around any point of $\mathcal{X}_0^{(\lambda_0)}$. Let $V^{(\lambda_0)}$ and $\widetilde{V}^{(\lambda_0)}$ be filtrations satisfying the above conditions, with the index sets $\mathcal{K}(a,\lambda_0)$ and $\widetilde{\mathcal{K}}(a,\lambda_0)$. Let $a<0$. We have $V_a^{(\lambda_0)}\subset \widetilde{V}_b^{(\lambda_0)}$ for some b. Note $V_{a-1}^{(\lambda_0)}\subset \widetilde{V}_{b-1}^{(\lambda_0)}$. Hence, we obtain the induced map $V_a^{(\lambda_0)}/V_{a-1}^{(\lambda_0)}\to \widetilde{V}_b^{(\lambda_0)}/\widetilde{V}_{b-1}^{(\lambda_0)}$. For any $a-1<e\leqslant a$, the action of $\prod_{a-1<c\leqslant e}\prod_{u\in\mathcal{K}(c,\lambda_0)}\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)$ is locally nilpotent on $V_e^{(\lambda_0)}/V_{a-1}^{(\lambda_0)}$. For any $b-1<e\leqslant b$, the action of $\prod_{b-1<c\leqslant e}\prod_{u\in\widetilde{\mathcal{K}}(c,\lambda_0)}\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)$ is locally nilpotent on $\widetilde{V}_e^{(\lambda_0)}/\widetilde{V}_{b-1}^{(\lambda_0)}$. For any $a-1<c\leqslant a$ and $b-1<e\leqslant b$, we have the induced map

$$\Phi_{c,e}: V_c^{(\lambda_0)}/V_{a-1}^{(\lambda_0)} \longrightarrow \widetilde{V}_b^{(\lambda_0)}/\widetilde{V}_e^{(\lambda_0)}.$$

We obtain the vanishing of the restriction of $\Phi_{c,e}$ to $\mathcal{X}_0^{(\lambda_0)}-(\{\lambda_0\}\times X_0)$ if $c<e$. By using the strictness of $\mathrm{Gr}^{V^{(\lambda_0)}}$ and $\mathrm{Gr}^{\widetilde{V}^{(\lambda_0)}}$, we obtain the vanishing of $\Phi_{c,e}$ if $c<e$. Hence, we obtain that $V_a^{(\lambda_0)}\subset\widetilde{V}_a^{(\lambda_0)}$. Moreover, we obtain that $\mathcal{K}(a,\lambda_0)=\widetilde{\mathcal{K}}(a,\lambda_0)$ if $-\eth_t t+\mathfrak{e}(\lambda_0,u)$ has non-trivial kernel for each $u\in\mathcal{K}(a,\lambda_0)\cup\widetilde{\mathcal{K}}(a,\lambda_0)$. □

Lemma 22.3.5. — *Assume that $\mathcal{M}$ on $\mathcal{X}^{(\lambda_0)}$ is strictly specializable along t at λ_0 with the index set $\mathcal{KMS}(\mathcal{M},t)$. For simplicity, $\mathcal{X}^{(\lambda_0)}$ is assumed to be the product of X and a neighbourhood of λ_0, and $\mathcal{KMS}(\mathcal{M})$ is assumed to be finite.*

Assume that $|\lambda_1-\lambda_0|$ is sufficiently small. Then $\mathcal{M}$ is strictly specializable along t at λ_1 with the index set $\mathcal{KMS}(\mathcal{M},t)$. Moreover, on a neighbourhood $\mathcal{X}^{(\lambda_1)}\subset\mathcal{X}^{(\lambda_0)}$ of $\{\lambda_1\}\times X$, we have natural isomorphisms

$$\psi_{t,u}^{(\lambda_0)}(\mathcal{M})_{|\mathcal{X}^{(\lambda_1)}}\simeq\psi_{t,u}^{(\lambda_1)}(\mathcal{M}).$$

Proof. — We will construct the filtration $V^{(\lambda_1)}$ of $\mathcal{M}$ on $\mathcal{X}^{(\lambda_1)}$. In the following argument, we restrict $\mathcal{M}$ and the associated modules to $\mathcal{X}^{(\lambda_1)}$. For any real number $d\in\boldsymbol{R}$, we consider $S(d):=\big\{c\in\boldsymbol{R}\,\big|\,\exists\,u\in\mathcal{K}(c,\lambda_0),\ \mathfrak{p}(\lambda_1,u)=d\big\}$. Since $|\lambda_1-\lambda_0|$ is small, we have $\big|S(d)\big|\leqslant 1$. First, let us consider the case $S(d)=\{c\}$. Let $\pi_c: V_c^{(\lambda_0)}(\mathcal{M})\to\mathrm{Gr}_c^{V^{(\lambda_0)}}(\mathcal{M})$ be the projection, and we define

$$V_d^{(\lambda_1)}(\mathcal{M}):=\pi_c^{-1}\Big(\bigoplus_{\substack{u\in\mathcal{K}(c,\lambda_0)\\ \mathfrak{p}(\lambda_1,u)\leqslant d}}\psi_{t,u}^{(\lambda_0)}(\mathcal{M})\Big).$$

Let us consider the case $S(d) = \varnothing$. In that case, we put $d_0 := \max\{d' \leqslant d \mid S(d') \neq \varnothing\}$, and we define $V_d^{(\lambda_1)}(\mathcal{M}) := V_{d_0}^{(\lambda_1)}(\mathcal{M})$. Then it is easy to check that $V^{(\lambda_1)}$ is the desired filtration. □

Lemma 22.3.6. — *Assume that $\mathcal{M}$ is strictly specializable along t at λ_0 with a filtration $V^{(\lambda_0)}$.*

- *The multiplication $t : V_{<0}^{(\lambda_0)}(\mathcal{M}) \to V_{<0}^{(\lambda_0)}(\mathcal{M})$ is injective.*
- *The induced maps $\eth_t : \operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}) \to \operatorname{Gr}_{a+1}^{V^{(\lambda_0)}}(\mathcal{M})$ are isomorphisms for any $a > -1$.*

Proof. — Let us show the first claim. Let $f \in V_{<0}^{(\lambda_0)}$ such that $t \cdot f = 0$. Assume $f \neq 0$, and we will deduce a contradiction. Note that $\eth_t^N f \neq 0$ for any N, and that $V_{<0}^{(\lambda_0)} \cap \mathcal{R}_X \cdot f$ is $V_0\mathcal{R}_X$-coherent. Hence, there exists an N such that $\eth_t^N f \notin V_{<0}^{(\lambda_0)}$. It implies the existence of $a \in \boldsymbol{R}_{<0}$ such that $f \in V_a^{(\lambda_0)} \setminus V_{<a}^{(\lambda_0)}$. Hence, we obtain a non-zero element $[f] \in \operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$. Because $t \cdot f = 0$, we have $\eth_t t[f] = 0$. Hence, $[f] \notin \psi_{t,u}^{(\lambda_0)}(\mathcal{M})$ for any $u \in \mathcal{K}(a, \lambda_0)$, which is a contradiction. Therefore, we can conclude that $f = 0$.

As for the second claim, it is easy to check that the endomorphisms $t \cdot \eth_t$ of $\operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$ are injective for any $a > -1$. Thus, we are done. □

22.3.2. Compatibility of a morphism and V-filtrations. — Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{R}_X$-modules on $\mathcal{X}^{(\lambda_0)}$, which are strictly specializable along t at λ_0.

Proposition 22.3.7. — *Let ϕ be a morphism $\mathcal{M} \to \mathcal{N}$.*

1. *$\phi\big(V_a^{(\lambda_0)}(\mathcal{M})\big)$ is contained in $V_a^{(\lambda_0)}(\mathcal{N})$ for each $a \in \boldsymbol{R}$.*
2. *Assume that $\operatorname{Gr}^{V^{(\lambda_0)}}(\phi) : \operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{M}) \to \operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{N})$ is strict, namely, $\operatorname{Cok}(\operatorname{Gr}_a^{V^{(\lambda_0)}}(\phi))$ is strict for each a. Then ϕ is strictly compatible with the filtrations $V^{(\lambda_0)}(\mathcal{M})$ and $V^{(\lambda_0)}(\mathcal{N})$, i.e., $V_a^{(\lambda_0)}(\mathcal{M}) \cap \operatorname{Im}(\phi) = \phi\big(V_a^{(\lambda_0)}(\mathcal{N})\big)$ for each a.*
3. *Under the above assumption,* $\operatorname{Ker}(\phi)$, $\operatorname{Cok}(\phi)$ *and* $\operatorname{Im}(\phi)$ *are strictly specializable along t at λ_0 with the induced filtrations, and we have the natural isomorphisms for any a:*

$$\operatorname{Ker}(\operatorname{Gr}_a^{V^{(\lambda_0)}}(\phi)) \simeq \operatorname{Gr}_a^{V^{(\lambda_0)}}(\operatorname{Ker}(\phi)), \quad \operatorname{Cok}(\operatorname{Gr}_a^{V^{(\lambda_0)}}(\phi)) \simeq \operatorname{Gr}_a^{V^{(\lambda_0)}}(\operatorname{Cok}(\phi))$$
$$\operatorname{Im}(\operatorname{Gr}_a^{V^{(\lambda_0)}}(\phi)) \simeq \operatorname{Gr}_a^{V^{(\lambda_0)}}(\operatorname{Im}(\phi)).$$

Proof. — The first claim can be shown by the argument in Lemma 22.3.4. To show the second claim, let us consider the induced morphism $\phi' : V_d^{(\lambda_0)}/V_{<c}^{(\lambda_0)}(\mathcal{M}) \to V_d^{(\lambda_0)}/V_{<c}^{(\lambda_0)}(\mathcal{N})$.

Lemma 22.3.8. — *ϕ' is strict with respect to the induced filtration $V^{(\lambda_0)}$, i.e., the following holds for any $c \leqslant d' \leqslant d$:*

$$\operatorname{Im}(\phi') \cap \Big(V^{(\lambda_0)}_{d'}/V^{(\lambda_0)}_{<c}(\mathcal{N})\Big) = \phi'\Big(V^{(\lambda_0)}_{d'}/V^{(\lambda_0)}_{<c}(\mathcal{M})\Big).$$

Proof. — We only have to consider the case where $\mathcal{X}^{(\lambda_0)}$ is the product of X and a neighbourhood $U(\lambda_0)$ of λ_0 in $\boldsymbol{C}_\lambda$. There exists a dense subset $U^*(\lambda_0) \subset U(\lambda_0)$ such that $\mathfrak{e}(\lambda) : \mathcal{KMS}(\mathcal{M},t) \to \boldsymbol{C}$ is injective for each $\lambda \in U^*(\lambda_0)$. We set $\mathcal{X}_0^{(\lambda_0)*} := U^*(\lambda_0) \times X_0$. We have the decomposition:

$$V^{(\lambda_0)}_{d}/V^{(\lambda_0)}_{<c}(\mathcal{M})_{|\mathcal{X}_0^{(\lambda_0)*}} = \bigoplus_{\substack{u \in \mathcal{KMS}(\mathcal{M},t)\\ c \leqslant \mathfrak{p}(\lambda_0,u) \leqslant d}} \psi^{(\lambda_0)}_u(\mathcal{M})_{|\mathcal{X}_0^{(\lambda_0)*}}.$$

It gives a splitting of the induced filtration $V^{(\lambda_0)}$ on $V^{(\lambda_0)}_{d}/V^{(\lambda_0)}_{<c}(\mathcal{M})_{|\mathcal{X}_0^{(\lambda_0)*}}$. We have similar decomposition for $V^{(\lambda_0)}_{d}/V^{(\lambda_0)}_{<c}(\mathcal{N})_{|\mathcal{X}_0^{(\lambda_0)*}}$. Since $\phi'_{|\mathcal{X}_0^{(\lambda_0)*}}$ preserves the decompositions, it is strict with respect to $V^{(\lambda_0)}$. By using the strictness of $\operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{M})$, $\operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{N})$ and $\operatorname{Cok}\operatorname{Gr}^{V^{(\lambda_0)}}(\phi)$, we obtain that ϕ' is strict with respect to the filtrations $V^{(\lambda_0)}$. □

Lemma 22.3.9. — *Let $(\lambda_0, P) \in \mathcal{X}^{(\lambda_0)}$. For any c, there exists $e < 0$ such that $V^{(\lambda_0)}_e(\mathcal{N}) \cap \operatorname{Im}\phi \subset \phi\big(V^{(\lambda_0)}_c(\mathcal{M})\big)$ around (λ_0, P).*

Proof. — Let $\mathcal{Q}_N$ denote the $V_0\mathcal{R}_X$-submodule of $V^{(\lambda_0)}_{-1}(\mathcal{N})$ which consists of the sections a such that $t^N a \in \operatorname{Im}(\phi) \cap V^{(\lambda_0)}_{-N-1}(\mathcal{N})$. There exists a large N_1 such that $\mathcal{Q}_N = \mathcal{Q}_{N+1}$ for any $N \geqslant N_1$ around (λ_0, P). We set $\mathcal{Q} := \mathcal{Q}_{N_1}$. We have $t^N\mathcal{Q} = \operatorname{Im}\phi \cap V^{(\lambda_0)}_{-N-1}(\mathcal{N})$ for any $N \geqslant N_1$ by our choice. There exists a large number f such that $t^{N_1}\mathcal{Q} \subset \phi\big(V^{(\lambda_0)}_f(\mathcal{M})\big)$. Take N_2 such that $f - N_2 \leqslant c$ and $N_2 \geqslant N_1$. Then, we obtain $V^{(\lambda_0)}_{-1-N_1-N_2}(\mathcal{N}) \cap \operatorname{Im}\phi \subset \phi\big(V^{(\lambda_0)}_c(\mathcal{M})\big)$. □

Let h be an element of $V^{(\lambda_0)}_d(\mathcal{M})$ such that $\phi(h) \in V^{(\lambda_0)}_c(\mathcal{N})$ for some $c < d$. Let e be as in Lemma 22.3.9 for the c. By using Lemma 22.3.8 inductively, there exists an element $h_1 \in V^{(\lambda_0)}_c(\mathcal{M})$ such that $\phi(h - h_1) \in V^{(\lambda_0)}_e(\mathcal{N})$. By the choice of e, there exists $h_2 \in V^{(\lambda_0)}_c(\mathcal{M})$ such that $\phi(h_2) = \phi(h - h_1)$, i.e., $\phi(h) = \phi(h_1 + h_2)$, which means the strictness of ϕ with respect to the filtrations $V^{(\lambda_0)}$. Thus, we obtain the second claim of Proposition 22.3.7.

The third claim easily follows from the second claim. Note we use Lemma 22.3.6 to show $V_{a-1}(\operatorname{Ker}(\phi)) = t \cdot V_a(\operatorname{Ker}(\phi))$ for $a < 0$. □

Definition 22.3.10. — A morphism $\phi : \mathcal{M} \to \mathcal{N}$ is called strictly specializable along t, if $\operatorname{Gr}^{V^{(\lambda_0)}}(\phi) : \operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{M}) \to \operatorname{Gr}^{V^{(\lambda_0)}}(\mathcal{N})$ is strict. □

According to Proposition 22.3.7, if ϕ is strictly specializable, it is strictly compatible with the V-filtration.

22.3.3. The functor $\psi_{t,u}$. — Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module on $\mathcal{X}$. We say that $\mathcal{M}$ is strictly specializable along t if it is strictly specializable along t at any λ_0. If $\mathcal{M}$ is strictly specializable along t, according to Lemma 22.3.5, $\{\psi_{t,u}^{(\lambda_0)}\mathcal{M} \mid \lambda_0 \in \boldsymbol{C}\}$ determines the globally defined $\mathcal{R}_{X_0}$-module $\psi_{t,u}(\mathcal{M})$ for each $u \in \mathcal{KMS}(\mathcal{M}, t)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be coherent $\mathcal{R}_X$-modules on $\mathcal{X}$, which are strictly specializable along t. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{R}_X$-modules. According to Proposition 22.3.7, the morphism ϕ preserves the V-filtrations $V^{(\lambda_0)}$ at any $\lambda_0 \in \boldsymbol{C}_\lambda$. We obtain induced morphisms $\mathrm{Gr}^{V^{(\lambda_0)}}(\phi) : \mathrm{Gr}^{V^{(\lambda_0)}}(\mathcal{M}) \to \mathrm{Gr}^{V^{(\lambda_0)}}(\mathcal{N})$. Since the decomposition (428) is obtained as a generalized eigen-decomposition, we obtain induced morphisms $\psi_{t,u}^{(\lambda_0)}(\phi) : \psi_{t,u}^{(\lambda_0)}(\mathcal{M}) \to \psi_{t,u}^{(\lambda_0)}(\mathcal{N})$. We can glue them to obtain a morphism $\psi_{t,u}(\phi) : \psi_{t,u}(\mathcal{M}) \to \psi_{t,u}(\mathcal{N})$ of $\mathcal{R}_{X_0}$-modules on $\mathcal{X}_0$, which is clear from the construction of $V^{(\lambda_1)}$ from $V^{(\lambda_0)}$ given in Lemma 22.3.5.

Let $\boldsymbol{\delta}_0$ denote the element $(1,0) \in \boldsymbol{R} \times \boldsymbol{C}$. If an $\mathcal{R}_X$-module $\mathcal{M}$ is strictly specializable along t, we have the naturally induced morphisms of $\mathcal{R}_{X_0}$-modules

$$t : \psi_{t,u}\mathcal{M} \longrightarrow \psi_{t,u-\boldsymbol{\delta}_0}\mathcal{M}, \qquad \eth_t : \psi_{t,u}\mathcal{M} \longrightarrow \psi_{t,u+\boldsymbol{\delta}_0}\mathcal{M}.$$

In particular, we put

$$\mathrm{can} = -\eth_t : \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M} \longrightarrow \psi_{t,0}\mathcal{M}, \qquad \mathrm{var} = t : \psi_{t,0}\mathcal{M} \longrightarrow \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M}.$$

Remark 22.3.11. — If we consider right $\mathcal{R}$-modules, can is given by $\eth_t$. □

Lemma 22.3.12. — *Assume that $\mathcal{M}$ is strictly specializable along t.*

- *If we have a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, then $\mathcal{M}_i$ $(i = 1, 2)$ are strictly specializable along t.*
- *Assume that $\mathcal{M}$ is supported in $\mathcal{X}_0$. Then we have $V_{<0}^{(\lambda_0)}\mathcal{M} = 0$ for any $\lambda_0 \in \boldsymbol{C}_\lambda$. We also have $\psi_{t,u}\mathcal{M} = 0$, if u does not contained in $\mathbb{Z}_{\geqslant 0} \times \{0\}$.*

Proof. — Let us show the first claim. Since $\mathrm{id} \oplus 0$ preserves the filtration $V^{(\lambda_0)}$ on $\mathcal{X}^{(\lambda_0)}$, the decomposition and $V^{(\lambda_0)}$ are compatible. Then, the first claim is clear. See [**73**] or Proposition 14.42 of [**67**] for the second claim. □

Let $\mathcal{S}^2(X,t)$ denote the full subcategory of strictly specializable $\mathcal{R}_X$-modules along t. Let $\mathcal{S}^2_{X_0}(X,t) \subset \mathcal{S}^2(X,t)$ denote the full subcategory of objects whose supports are contained in $\mathcal{X}_0$.

Corollary 22.3.13. — *We have the equivalence* $\mathcal{S}^2_{X_0}(X,t) \simeq \big(\text{strict } \mathcal{R}_{X_0}\text{-modules}\big)$. □

22.3.4. Strict S-decomposability. — The following condition essentially appeared in [**77**].

Condition 22.3.14. — Let $\mathcal{M}$ on $\mathcal{X}^{(\lambda_0)}$ be strictly specializable along t at λ_0 with the filtration $V^{(\lambda_0)}$. Recall that $\mathcal{M}$ is called strictly S-decomposable along t at λ_0, if moreover $\psi_{t,0}^{(\lambda_0)}(\mathcal{M}) = \mathrm{Im}(\mathrm{can}) \oplus \mathrm{Ker}(\mathrm{var})$ holds.

We say that $\mathcal{M}$ on $\mathcal{X}$ is strictly S-decomposable along t, if it is strictly S-decomposable along t at any λ_0. □

Proposition 22.3.15. — *Let $\mathcal{M}$ be strictly specializable along t.*

1. *The following conditions are equivalent:*
   - $\mathrm{var}: \psi_{t,0}\mathcal{M} \to \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M}$ *is injective.*
   - *Let $\mathcal{M}'$ be a submodule of $\mathcal{M}$ such that the support of $\mathcal{M}'$ is contained in $\mathcal{X}_0$. Then $\mathcal{M}' = 0$.*
   - *Let $\mathcal{M}'$ be a submodule of $\mathcal{M}$ such that the support of $\mathcal{M}'$ is contained in $\mathcal{X}_0$. Assume that $\mathcal{M}' \in \mathcal{S}^2(X,t)$. Then $\mathcal{M}' = 0$.*
2. *Assume* $\mathrm{can}: \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M} \to \psi_{t,0}\mathcal{M}$ *is surjective. Let $\mathcal{M}'' \in \mathcal{S}^2(X,t)$ be a quotient of $\mathcal{M}$ such that the support of $\mathcal{M}''$ is contained in $\mathcal{X}_0$. Then $\mathcal{M}'' = 0$.*
3. *$\mathcal{M}$ is strictly S-decomposable along t if and only if the following holds:*
   - *We have the decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$. Here the support of $\mathcal{M}''$ is contained in X_0, and $\mathcal{M}'$ has neither submodules or quotients contained in $\mathcal{S}^2(X,t)$ whose support is contained in $\mathcal{X}_0$.*

Proof. — See [**73**]. (See also [**77**].) □

22.3.5. The functors $\widetilde{\psi}^{(\lambda_0)}_{t,u}$ and $\widetilde{\psi}_{t,u}$. — Let $\mathcal{M}$ be an $\mathcal{R}_X$-module, which is strictly specializable along t. For any $u = (a,\alpha) \in \mathcal{KMS}(\mathcal{M},t)$ such that $u \notin \mathbb{Z}_{\geqslant 0} \times \{0\}$, the $\mathcal{R}_{X_0}$-module $\widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})$ is defined in [**67**] as the inductive limit of $t: \psi^{(\lambda_0)}_{t,u-N\cdot\boldsymbol{\delta}_0}(\mathcal{M}) \to \psi^{(\lambda_0)}_{t,u-(N+1)\cdot\boldsymbol{\delta}_0}(\mathcal{M})$. It is equal to $\psi^{(\lambda_0)}_{t,u}(\mathcal{M}(*t))$. (See [**73**] or Section 22.4.1.) It is easy to observe that $\widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})_{|\mathcal{X}^{(\lambda_1)}} \simeq \widetilde{\psi}^{(\lambda_1)}_{t,u}(\mathcal{M})$ in the situation of Lemma 22.3.5. Hence, we obtain the $\mathcal{R}_{X_0}$-module $\widetilde{\psi}_{t,u}(\mathcal{M})$ on $\mathcal{X}_0$ as the gluing of $\widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})$. We have a natural isomorphism $\widetilde{\psi}_{t,u}(\mathcal{M}) \simeq \lim_{N\to\infty} \psi_{t,u-N\cdot\boldsymbol{\delta}_0}(\mathcal{M})$. We have the canonical inclusions $\psi^{(\lambda_0)}_{t,u}(\mathcal{M}) \to \widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})$ and $\psi_{t,u}(\mathcal{M}) \to \widetilde{\psi}_{t,u}(\mathcal{M})$ (Lemma 14.52 of [**67**]).

Remark 22.3.16. — If u is contained in $\boldsymbol{R}_{<0} \times \{0\} \subset \boldsymbol{R} \times \boldsymbol{C}$, we have the canonical morphism $\widetilde{\psi}_{t,u}(\mathcal{M}) \simeq \psi_{t,u}(\mathcal{M})$. We will not distinguish them. □

Remark 22.3.17. — In the case $u = (0,0) \in \boldsymbol{R} \times \boldsymbol{C}$ (it is also denoted by 0), we also use the notation $\phi_{t,0}$ instead of $\psi_{t,0}$. It is called the vanishing cycle functor. □

Remark 22.3.18. — Let u be an element of $\mathcal{KMS}(\mathcal{M},t)$. Let us consider the following set:

$$S(u) := \big\{\lambda \in \boldsymbol{C}^* \,\big|\, \exists b \in \mathbb{Z}_{\geqslant 0},\ \text{s.t. } \mathfrak{e}\big(\lambda, u - b\cdot\boldsymbol{\delta}_0\big) = 0,\ \mathfrak{p}\big(\lambda_0, u - b\cdot\boldsymbol{\delta}_0\big) \geqslant 0\big\}.$$

Then, $S(u)$ is discrete in $\boldsymbol{C}_\lambda$. For any $\lambda_0 \in \boldsymbol{C}^* - S(u)$, we have the canonical isomorphism $\psi^{(\lambda_0)}_{t,u}(\mathcal{M}) \simeq \widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})$. □

Remark 22.3.19. — Let $u \in (\boldsymbol{R}\times\boldsymbol{C})\setminus(\mathbb{Z}_{\geqslant 0}\times\{0\})$. We have a natural isomorphism $\psi_u^{(\lambda_0)}(\mathcal{M}) \simeq \widetilde{\psi}^{(\lambda_0)}(\mathcal{M})$, depending on λ_0. If $\mathfrak{p}(\lambda_0,u)\leqslant 0$, the natural morphism $\psi^{(\lambda_0)}(\mathcal{M}_i)\to\widetilde{\psi}^{(\lambda_0)}(\mathcal{M}_i)$ is an isomorphism. In the case $\mathfrak{p}(\lambda_0,u)>0$, we take $n_0\in\mathbb{Z}_{\geqslant 0}$ such that $-1<\mathfrak{p}(\lambda_0,u)-n_0\leqslant 0$, and then we have a natural isomorphism $\eth_t^{n_0}:\psi_{u-n\boldsymbol{\delta}}^{(\lambda_0)}(\mathcal{M}_i)\to\psi_u^{(\lambda_0)}(\mathcal{M}_i)$, from which we obtain $\psi_u^{(\lambda_0)}(\mathcal{M})\simeq\widetilde{\psi}^{(\lambda_0)}(\mathcal{M})$. □

Remark 22.3.20. — Let $\mathcal{M}_i$ $(i=1,2)$ be strict coherent $\mathcal{R}_X$-modules, which are strictly specializable along t. Let $\phi:\mathcal{M}_1\to\mathcal{M}_2$ be a morphism. Let $u\in(\boldsymbol{R}\times\boldsymbol{C})\setminus(\mathbb{Z}_{\geqslant 0}\times\{0\})$. If the cokernel of $\widetilde{\psi}_u(\phi):\widetilde{\psi}_u(\mathcal{M}_1)\to\widetilde{\psi}_u(\mathcal{M}_2)$ is strict, then the cokernel of $\psi_u(\phi)$ is also strict. Indeed, we can deduce it by using Remark 22.3.19. □

22.3.6. The general case. — Let Y be a complex manifold, and let $\mathcal{M}$ be an $\mathcal{R}_Y$-module on $\boldsymbol{C}_\lambda\times Y$. Let U be an open subset of Y, and let f be a holomorphic function on U. Then we obtain the $\mathcal{R}_{U\times\boldsymbol{C}_t}$-module $\iota_\dagger\mathcal{M}_{|\boldsymbol{C}_\lambda\times U}$ on $\boldsymbol{C}_\lambda\times U\times\boldsymbol{C}_t$, where $\iota:U\to U\times\boldsymbol{C}_t$ denotes the graph embedding.

Definition 22.3.21

- Let f and U be as above. A coherent $\mathcal{R}_Y$-module $\mathcal{M}$ is called strictly specializable (resp. S-decomposable) along f, if $i_{f\,\dagger}\mathcal{M}_{|U\times\boldsymbol{C}_t}$ is strictly specializable (resp. S-decomposable) along t.
- A coherent $\mathcal{R}_Y$-module $\mathcal{M}$ is called strictly specializable (resp. S-decomposable), if it is strictly specializable (resp. S-decomposable) along any holomorphic function defined on any open subset of Y. □

If $\mathcal{M}$ is strictly specializable along a holomorphic function f on Y, we define

$$\psi_{f,u}(\mathcal{M}):=\psi_{t,u}(\iota_\dagger\mathcal{M}),\qquad \widetilde{\psi}_{f,u}(\mathcal{M}):=\widetilde{\psi}_{t,u}(\iota_\dagger\mathcal{M}).$$

Note that if $\mathcal{M}$ is strictly specializable, $\mathcal{M}$ is strict. Indeed, let us deduce a contradiction, by assuming that $\mathcal{M}$ is not strict. Let s be a local section such that $(\lambda-\lambda_0)\,s=0$ for some λ_0. If $\mathrm{Supp}(s)=\mathrm{Supp}(\mathcal{M})$, by shrinking X, we may assume that there exists a holomorphic function f such that (i) $\mathrm{Supp}\,\mathcal{M}\not\subset f^{-1}(0)$, (ii) $\mathrm{Supp}(s)\subset f^{-1}(0)$. Then we obtain that $\psi_{f,0}(\mathcal{M})$ is not strict. If $\mathrm{Supp}(s)=\mathrm{Supp}(\mathcal{M})$, by shrinking X, we may assume that the characteristic variety of $\mathcal{M}$ is smooth. Then, we obtain that $\psi_{f,-\boldsymbol{\delta}}(\mathcal{M})$ is not strict. Hence, we have arrived at a contradiction.

Recall the compatibility of push-forward and specialization due to Sabbah (Theorem 3.3.15 in [**73**]). See also Saito's work [**77**].

Proposition 22.3.22. — *Let $F:Y\to Z$ be a proper morphism. Let g be a holomorphic function on Z, and we set $\widetilde{g}:=F^*g$. Let $\mathcal{M}$ be a coherent $\mathcal{R}_Y$-module which is strictly specializable along $\widetilde{g}$, and good with respect to F. If $F^i_{0\,\dagger}\widetilde{\psi}_{\widetilde{g},u}(\mathcal{M})$ are strict for any $u\in\boldsymbol{R}\times\boldsymbol{C}$ and any i, then $F^i_\dagger(\mathcal{M})$ are also strictly specializable along g, and $\widetilde{\psi}_{g,u}F^i_\dagger(\mathcal{M})\simeq F^i_{0\dagger}\widetilde{\psi}_{\widetilde{g},u}(\mathcal{M})$.* □

We also remark the following in [**77**] and [**73**].

Lemma 22.3.23. — *Let Y, Z, F and g be as in Proposition 22.3.22. Assume that F is a closed immersion. Let $\mathcal{M}$ be a coherent $\mathcal{R}_Y$-module. If $F_\dagger\mathcal{M}$ is strictly specializable along g, then $\mathcal{M}$ is also strictly specializable.*

Proof. — We give a sketch of a proof. We only have to consider the case $Z = Y \times \Delta_t$. We have the decomposition $F_\dagger\mathcal{M} = \bigoplus_{n=0}^{\infty} F_*\mathcal{M}\,\eth_t^n$. We only have to show that the V-filtrations $V^{(\lambda_0)}$ are compatible with the decomposition. Let $v \in V_a^{(\lambda_0)}$. We have the decomposition $v = \sum_{i=0}^{N} v_n$. It is easy to show that $\lambda^N v_N \in V_a^{(\lambda_0)}$. We have $b \geqslant a$ such that $v_N \in V_b^{(\lambda_0)}$ and $v_N \notin V_{<b}^{(\lambda_0)}$. By using the strictness of $\mathrm{Gr}_b^{V^{(\lambda_0)}}$ for any b, we obtain $b = a$. Namely, $v_N \in V_a^{(\lambda_0)}$. Then, by an easy induction, we obtain that $v_m \in V_a^{(\lambda_0)}$ for any m. □

22.3.7. Strict support. — Let Z be an irreducible subvariety of X. Let $\mathcal{M}$ be a strictly S-decomposable $\mathcal{R}_X$-module, whose support is contained in Z. If there is non-zero $\mathcal{R}_X$-submodule whose support is strictly smaller than Z, then Z is called the strict support of $\mathcal{M}$.

We obtain the following lemma by using the decomposition given in Proposition 22.3.15 and an easy inductive argument. (See [**77**] and [**73**].)

Lemma 22.3.24. — *Let $\mathcal{M}$ be a holonomic strictly S-decomposable $\mathcal{R}_X$-module. Then we have a unique decomposition $\mathcal{M} = \bigoplus_Z \mathcal{M}_Z$, where Z runs through the irreducible closed subsets of X, satisfying the following:*
- *$\mathcal{M}_Z$ are strictly S-decomposable $\mathcal{R}_X$-modules.*
- *The strict support of $\mathcal{M}_Z$ is Z.*

It is called the decomposition by the strict supports. □

22.4. $\mathcal{R}_X(*t)$-modules

22.4.1. Strict specializable $\mathcal{R}_X(*t)$-modules. — Let $X = X_0 \times \boldsymbol{C}_t$. We identify X_0 with the $\{t = 0\}$ in X. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{X}_0 := \boldsymbol{C}_\lambda \times X_0$. Let $\mathcal{X}^{(\lambda_0)}$ denote a small neighbourhood of $\{\lambda_0\} \times X$ in $\boldsymbol{C}_\lambda \times X$. We use the symbol $\mathcal{X}_0^{(\lambda_0)}$ with a similar meaning. For simplicity, we assume that $\mathcal{X}^{(\lambda_0)}$ is the product of a neighbourhood of λ_0 and X. We use the symbols $\mathcal{X}_0$ and $\mathcal{X}_0^{(\lambda_0)}$ with similar meanings.

Let $\mathcal{R}_X(*t)$ denote $\mathcal{R}_X \otimes \mathcal{O}_\mathcal{X}(*t)$, which is naturally a sheaf of algebras. We refer to [**73**] and [**75**] for general and foundational properties of $\mathcal{R}_X(*t)$-modules. We have the standard correspondence between the left $\mathcal{R}_X(*t)$-modules and the right $\mathcal{R}_X(*t)$-modules given by $\mathcal{M}$ and $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{X}(*t)} \omega_\mathcal{X}(*t)$. In this paper, we consider *left* $\mathcal{R}_X(*t)$-modules. Coherence, holonomicity and strictness for $\mathcal{R}_X(*t)$-modules are defined as in the case of $\mathcal{R}_X$-modules.

We recall the notion of strict specializability in this situation.

Definition 22.4.1. — Let $\mathcal{M}$ be a left coherent $\mathcal{R}_X(*t)$-module. It is called strictly specializable along t at λ_0, if we have an increasing filtration $V^{(\lambda_0)}$ of $\mathcal{M}_{|\mathcal{X}^{(\lambda_0)}}$ by coherent $V_0\mathcal{R}_X$-modules indexed by $\boldsymbol{R}$, such that the following holds:

- For any $a \in \boldsymbol{R}$ and $P \in \mathcal{X}_0$, there exists $\varepsilon > 0$ such that $V_a^{(\lambda_0)}(\mathcal{M}) = V_{a+\varepsilon}^{(\lambda_0)}(\mathcal{M})$ around P. Moreover, $\bigcup_a V_a^{(\lambda_0)}(\mathcal{M}) = \mathcal{M}$.
- $\mathrm{Gr}^{V^{(\lambda_0)}}(\mathcal{M})$ is a strict $\mathcal{R}_{X_0}$-module, i.e., the multiplication of $\lambda - \lambda_1$ is injective for any $\lambda_1 \in \boldsymbol{C}$.
- $t \cdot V_a^{(\lambda_0)}(\mathcal{M}) = V_{a-1}^{(\lambda_0)}(\mathcal{M})$ and $\eth_t \cdot V_a^{(\lambda_0)} \subset V_{a+1}^{(\lambda_0)}(\mathcal{M})$ for any $a \in \boldsymbol{R}$.
- For each $a \in \boldsymbol{R}$ and $P \in \mathcal{X}_0$, there is a finite subset $\mathcal{K}(a, \lambda_0, P) \subset \boldsymbol{R} \times \boldsymbol{C}$ such that the action of $\prod_{u \in \mathcal{K}(a,\lambda_0)} \big(-\eth_t t + \mathfrak{e}(\lambda, u)\big)$ on $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})$ is nilpotent on a neighbourhood of P.

We say that $\mathcal{M}$ is strictly specializable along t, if it is strictly specializable along t at any λ_0. □

The union $\bigcup_a \mathcal{K}(a, \lambda_0, P)$ is denoted by $\mathcal{KMS}(\mathcal{M}, t, P)$. We also put $\mathcal{K}(a, \lambda_0) := \bigcup_{P \in \mathcal{X}_0^{(\lambda_0)}} \mathcal{K}(a, \lambda_0, P)$ and $\mathcal{KMS}(\mathcal{M}, t) := \bigcup_{P \in \mathcal{X}_0^{(\lambda_0)}} \mathcal{KMS}(\mathcal{M}, t, P)$.

Remark 22.4.2. — If we consider local issues, we implicitly assume that the set $\mathcal{K}(a, \lambda_0)$ is finite. □

Let $\mathcal{M}$ and $\mathcal{K}(a, \lambda_0)$ be as in Definition 22.4.1. For $u \in \mathcal{K}(a, \lambda_0)$, we put on $\mathcal{X}_0^{(\lambda_0)}$

$$\widetilde{\psi}_{t,u}^{(\lambda_0)}(\mathcal{M}) := \bigcup_N \mathrm{Ker}\Big(\big(-\eth_t t + \mathfrak{e}(\lambda, u)\big)^N : \mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}) \longrightarrow \mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M})\Big).$$

By the strictness, we have the decomposition:

$$\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}) = \bigoplus_{u \in \mathcal{K}(a,\lambda_0)} \widetilde{\psi}_{t,u}^{(\lambda_0)}(\mathcal{M}).$$

We implicitly assume $\widetilde{\psi}_u^{(\lambda_0)}(\mathcal{M}) \neq 0$ for each $u \in \mathcal{K}(a, \lambda_0)$. Because the multiplication by t induces an isomorphism $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}) \simeq \mathrm{Gr}_{a-1}^{V^{(\lambda_0)}}(\mathcal{M})$, we have the bijection:

$$\mathcal{K}(a, \lambda_0) \simeq \mathcal{K}(a-1, \lambda_0), \quad u \longleftrightarrow u - \boldsymbol{\delta}_0.$$

Here $\boldsymbol{\delta}_0 = (1, 0) \in \boldsymbol{R} \times \boldsymbol{C}$.

Lemma 22.4.3

- *The filtration $V^{(\lambda_0)}$ as in Definition* 22.4.1 *is unique, if it exists.*
- *Assume that $\mathcal{M}$ is strictly specializable along t at λ_0. For simplicity, $\mathcal{X}^{(\lambda_0)}$ is assumed to be the product of X and a neighbourhood of λ_0, and $\mathcal{KMS}(\mathcal{M}, t)$ is assumed to be finite. Then, $\mathcal{M}$ is also strictly specializable along t at any λ_1 such that $|\lambda_1 - \lambda_0|$ is sufficiently small. The filtration $V^{(\lambda_1)}$ of $\mathcal{M}_{|\mathcal{X}^{(\lambda_1)}}$ can be constructed from $V^{(\lambda_0)}$, and we have $\widetilde{\psi}^{(\lambda_0)}_{t,u|\mathcal{X}_0^{(\lambda_1)}} = \widetilde{\psi}_{t,u}^{(\lambda_1)}$ on $\mathcal{X}_0^{(\lambda_1)} \subset \mathcal{X}_0^{(\lambda_0)}$.*

Proof. — See [**73**] or the arguments in Lemma 22.3.4 and Lemma 22.3.5. □

As a result, if $\mathcal{M}$ on $\mathcal{X}$ is strictly specializable along t, the set $\mathcal{KMS}(\mathcal{M},t)$ is independent of the choice of λ_0. By gluing $\widetilde{\psi}^{(\lambda_0)}_{t,u}(\mathcal{M})$, we obtain the $\mathcal{R}_{\mathcal{X}_0}$-module $\widetilde{\psi}_{t,u}(\mathcal{M})$ on $\mathcal{X}_0$ for each $u \in \mathcal{KMS}(\mathcal{M},t)$. Formally, we put $\widetilde{\psi}_{t,u}(\mathcal{M}) := 0$ for $u \notin \mathcal{KMS}(\mathcal{M},t)$. The multiplication by t induces the isomorphism $\widetilde{\psi}_{t,u}(\mathcal{M}) \simeq \widetilde{\psi}_{t,u-\boldsymbol{\delta}_0}(\mathcal{M})$. We put $N := -\eth_t t + \mathfrak{e}(\lambda,u)$ on $\widetilde{\psi}_{t,u}(\mathcal{M})$, which is the nilpotent part of $-\eth_t t$.

Let $\mathcal{M}_i$ $(i=1,2)$ be strictly specializable along t. The following lemma can be shown by the same argument as in the proof of Lemma 22.3.4.

Lemma 22.4.4. — *Any morphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ preserves the filtrations $V^{(\lambda_0)}$ for any λ_0. In particular, we have the induced morphism $\widetilde{\psi}_{t,u}(\varphi) : \widetilde{\psi}_{t,u}(\mathcal{M}_1) \to \widetilde{\psi}_{t,u}(\mathcal{M}_2)$ for each $u \in \boldsymbol{R}\times\boldsymbol{C}$.* □

A morphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is called strictly specializable, if the cokernel of $\mathrm{Gr}^{V^{(\lambda_0)}}_a(\varphi)$ is strict for each a. The following lemma can be shown by using the same argument as of the proof of Proposition 22.3.7.

Lemma 22.4.5. — *If $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is strictly specializable along t, it is strictly compatible with the filtrations $V^{(\lambda_0)}$, i.e., $V^{(\lambda_0)}_a(\mathcal{M}_2)\cap \mathrm{Im}(\varphi) = \varphi\big(V^{(\lambda_0)}_a(\mathcal{M}_1)\big)$ for each a.*

As a result, $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$ and $\mathrm{Cok}(\varphi)$ are also strictly specializable along t, and we have the natural isomorphisms $\widetilde{\psi}_{t,u}\,\mathrm{Ker}(\varphi) \simeq \mathrm{Ker}\,\widetilde{\psi}_{t,u}(\varphi)$, $\widetilde{\psi}_{t,u}\,\mathrm{Im}(\varphi) \simeq \mathrm{Im}\,\widetilde{\psi}_{t,u}(\varphi)$ and $\widetilde{\psi}_{t,u}\,\mathrm{Cok}(\varphi) \simeq \mathrm{Cok}\,\widetilde{\psi}_{t,u}(\varphi)$. □

We refer to Section 3.4 of [**73**] for the following lemma.

Lemma 22.4.6. — *Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module which is strictly specializable along t at λ_0 with the filtration $V^{(\lambda_0)}$.*

- *$\mathcal{M}(*t) := \mathcal{R}_X(*t)\otimes_{\mathcal{R}_X}\mathcal{M}$ is a coherent $\mathcal{R}_X(*t)$-module which is strictly specializable along t at λ_0.*
- *The filtration $V^{(\lambda_0)}(\mathcal{M}(*t))$ is given by $V^{(\lambda_0)}_a(\mathcal{M}(*t)) = V^{(\lambda_0)}_a(\mathcal{M})$ for $a<0$, and $V^{(\lambda_0)}_a(\mathcal{M}(*t)) = t^{-n}V^{(\lambda_0)}_{a-n}(\mathcal{M})$ for $a \geqslant 0$, where n is chosen as $a-n<0$.*
- *We have the natural isomorphism $\widetilde{\psi}_{t,u}(\mathcal{M}) \simeq \widetilde{\psi}_{t,u}(\mathcal{M}(*t))$ for any $u\in\boldsymbol{R}\times\boldsymbol{C}$.*
- □

22.4.2. Pull-back via a n-th ramified covering. — We put $X^{(n)} := X_0\times \boldsymbol{C}_{t_n}$. Let $\varphi_n : X^{(n)} \to X$ be the morphism induced by $\varphi_n^*(t) = t_n^n$. Let $\mathcal{M}$ be an $\mathcal{R}_X(*t)$-module. Since $\mathcal{R}_{X^{(n)}}(*t_n) = \mathcal{O}_{\mathcal{X}^{(n)}}\otimes_{\varphi_n^{-1}(\mathcal{O}_{\mathcal{X}})}\varphi_n^{-1}\mathcal{R}_X(t)$ is flat over $\varphi_n^{-1}\mathcal{R}_X(*t)$, we have $\varphi_n^{\dagger}\mathcal{M} \simeq \varphi_n^*\mathcal{M}$. If $\mathcal{M}$ is $\mathcal{R}_X(*t)$-coherent, $\varphi_n^{\dagger}\mathcal{M}$ is $\mathcal{R}_{X^{(n)}}(*t_n)$-coherent. If $\mathcal{M}$ is a holonomic $\mathcal{R}_X(*t)$-module, $\varphi_n^{\dagger}\mathcal{M}$ is a holonomic $\mathcal{R}_{X^{(n)}}(*t_n)$-module. The following lemma can be checked directly.

Lemma 22.4.7. — *If $\mathcal{M}$ is strictly specializable along t, then so is $\varphi_n^\dagger\mathcal{M}$. The V-filtration $V^{(\lambda_0)}$ of $\varphi_n^\dagger\mathcal{M}$ is given as follows:*

$$V_a^{(\lambda_0)}(\varphi_n^\dagger\mathcal{M}) = \sum_{nb-c\leqslant a} \varphi_n^{-1}(V_b^{(\lambda_0)}\mathcal{M})\cdot t_n^c.$$

In particular, we have the following natural isomorphism of $\mathcal{R}_{X_0}$-modules:

$$\widetilde{\psi}_{t_n,u}(\varphi_n^\dagger\mathcal{M}) \simeq \bigoplus_{(u',c)\in\mathcal{S}} \widetilde{\psi}_{t,u'}(\mathcal{M}),$$

$$\mathcal{S} = \big\{(u',c)\,\big|\, u'\in\mathcal{KMS}(\mathcal{M},t),\ c\in\mathbb{Z},\ 0\leqslant c\leqslant n-1,\ n\cdot u' - c\cdot\boldsymbol{\delta}_0 = u\big\}$$

Here, $\boldsymbol{\delta}_0 := (1,0)\in\boldsymbol{R}\times\boldsymbol{C}$. □

22.4.3. Exponential twist. — Let $\mathfrak{a}\in\boldsymbol{C}[t_n^{-1}]$. We have the $\mathcal{R}_{\boldsymbol{C}_{t_n}}(*t_n)$-module $\mathcal{L}(\mathfrak{a})$ given as follows:

$$\mathcal{L}(\mathfrak{a}) = \mathcal{O}_{\boldsymbol{C}_{t_n}}(*t_n)\cdot e,\qquad \eth_{t_n}e = \partial_{t_n}\mathfrak{a}\cdot e.$$

The pull-back via the projection $X^{(n)}\to\boldsymbol{C}_{t_n}$ is also denoted by $\mathcal{L}(\mathfrak{a})$.

For an $\mathcal{R}_X(*t)$-module $\mathcal{M}$, we have the $\mathcal{R}_{X^{(n)}}(*t_n)$-module $\varphi_n^\dagger\mathcal{M}\otimes_{\mathcal{O}_{X^{(n)}}}\mathcal{L}(-\mathfrak{a})$. If it is strictly specializable along t_n, we say that $\mathcal{M}$ is strictly specializable along t with ramified exponential twist by $\mathfrak{a}$, and we define for any $u\in\boldsymbol{R}\times\boldsymbol{C}$

$$\widetilde{\psi}_{t,\mathfrak{a},u}(\mathcal{M}) := \widetilde{\psi}_{t_n,u}\big(\varphi_n^\dagger\mathcal{M}\otimes\mathcal{L}(-\mathfrak{a})\big).$$

For an $\mathcal{R}_X$-module $\mathcal{M}$, we have the induced $\mathcal{R}_X(*t)$-module $\mathcal{M}(*t)$. We define

$$\widetilde{\psi}_{t,\mathfrak{a},u}(\mathcal{M}) := \widetilde{\psi}_{t,\mathfrak{a},u}\big(\mathcal{M}(*t)\big),$$

if the right-hand side can be defined.

22.4.4. Comparison of strictly S-decomposable $\mathcal{R}$-modules. — Let $X = \boldsymbol{C}_t\times X_0$ for some complex manifold X_0. We identify X_0 with $\{t=0\}$.

Lemma 22.4.8. — *Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module whose support is contained in X_0. Then, $\mathcal{M}(*t)=0$.*

A similar claim holds for coherent D-modules.

Proof. — We only have to show the claim locally. Let $\pi: T^*X\to X$ denote the projection of the cotangent bundle. We take a coherent filtration F of $\mathcal{M}$. The associated graded module $\mathrm{Gr}^F(\mathcal{M})$ induces a coherent $\mathcal{O}_{T^*X}$-module, and the support is contained in $\pi^{-1}(X_0)$. Hence, the action of t on $\mathcal{M}$ is locally nilpotent, and we obtain $\mathcal{M}(*t)=0$. □

Let $\mathcal{M}_i$ $(i=1,2)$ be coherent $\mathcal{R}_X$-modules, and let $f:\mathcal{M}_1\to\mathcal{M}_2$ be a morphism.

Lemma 22.4.9. — *If the restriction $f_{|X\smallsetminus\{t=0\}}$ is an isomorphism, the induced morphism $\mathcal{M}_1(*t)\to\mathcal{M}_2(*t)$ is an isomorphism. A similar claim holds for D-modules.*

Proof. — We obtain $\mathrm{Ker}(f)(*t) = \mathrm{Cok}(f)(*t) = 0$ due to Lemma 22.4.8. □

Let X be a general complex manifold, and let g be a holomorphic function on X. Let $\mathcal{M}_i$ $(i=1,2)$ be coherent $\mathcal{R}_X$-modules such that (i) they are strictly S-decomposable along g, (ii) they do not have any $\mathcal{R}_X$-submodules whose supports are contained in $\{g=0\}$. By the condition (ii), we have the inclusions $\mathcal{M}_i \subset \mathcal{M}_i(*g)$.

Lemma 22.4.10. — *If we have an isomorphism $\mathcal{M}_1(*g) \simeq \mathcal{M}_2(*g)$, we obtain an isomorphism $\mathcal{M}_1 \simeq \mathcal{M}_2$ as the restriction. A similar claim holds for D-modules.*

Proof. — Let $\mathcal{M}_i^{(\lambda_0)}$ denote the restriction of $\mathcal{M}_i$ to a small neighbourhood of $\{\lambda_0\} \times X$. We may assume that $X = X_0 \times \boldsymbol{C}_t$ and $g=t$. The V-filtrations of $\mathcal{M}_i^{(\lambda_0)}$ $(i=1,2)$ induce the V-filtrations of $\mathcal{M}_i^{(\lambda_0)}(*t)$ via which $\mathcal{M}_i^{(\lambda_0)}(*t)$ are strictly specializable along t at λ_0. By the uniqueness of such V-filtrations, they are the same under the identification $\mathcal{V}^{(\lambda_0)} := \mathcal{M}_1^{(\lambda_0)}(*t) = \mathcal{M}_2^{(\lambda_0)}(*t)$. In particular, we have $A^{(\lambda_0)} := V_{<0}^{(\lambda_0)}(\mathcal{M}_1^{(\lambda_0)}) = V_{<0}^{(\lambda_0)}(\mathcal{M}_2^{(\lambda_0)})$ in $\mathcal{V}^{(\lambda_0)}$. Since both $\mathcal{M}_i^{(\lambda_0)}$ $(i=1,2)$ are generated by $A^{(\lambda_0)}$ in $\mathcal{V}^{(\lambda_0)}$, we obtain $\mathcal{M}_1^{(\lambda_0)} = \mathcal{M}_2^{(\lambda_0)}$ for any λ_0, and thus $\mathcal{M}_1 = \mathcal{M}_2$. □

22.5. Formal $\mathcal{R}$-modules

22.5.1. Formal complex spaces. — We recall some basic facts on formal complex spaces. We refer to [**4**], [**8**] and [**50**] for more details and precision. Let X be a complex space. Let T be an analytic subvariety of X, which consists of the underlying topological space $|T|$ and the structure sheaf $\mathcal{O}_T$. Let $\mathcal{I}_T$ denote the sheaf of ideals of $\mathcal{O}_X$, corresponding to T. We have the analytic subspace $T^{(n)}$ corresponding to $\mathcal{I}_T^n$, i.e., $T^{(n)} = \big(|T|, \mathcal{O}_X/\mathcal{I}_T^n\big)$. As the limit, we obtain the ringed space $\widehat{T} := \big(|T|, \varprojlim \mathcal{O}_X/\mathcal{I}_T^n\big)$, which is called the completion of X along T.

Proposition 22.5.1

1. *$\widehat{T}$ is a formal complex space in the sense of* [**8**], *and the morphism $\iota_T : \widehat{T} \to X$ is flat.*
2. *The sheaf of algebras $\mathcal{O}_{\widehat{T}}$ is coherent and Noetherian.*
3. *Coherent sheaves $\mathcal{F}$ on $\widehat{T}$ are equivalent to systems of coherent sheaves $\mathcal{F}^{(n)}$ $(n=1,2,\dots,)$ on $T^{(n)}$ such that $\mathcal{F}^{(n)} \otimes \mathcal{O}_{T^{(m)}} = \mathcal{F}^{(m)}$ for $n \geqslant m$.*

Proof. — The first claim is Lemma 1.6 of [**8**]. The second claim is Lemma 1.1 and Corollary 1.5 of [**8**]. (See Section 22.1 for Noetherian property in this situation.) The third claim is Lemma 1.2 of [**8**]. □

Lemma 22.5.2. — *The extension $\iota_T^{-1}\mathcal{O}_X \to \mathcal{O}_{\widehat{T}}$ is faithfully flat. We have* $\operatorname{Supp}(\mathcal{F}) \cap |T| = \varnothing$ *for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\iota_T^*\mathcal{F} = 0$.*

Proof. — Let P be any point of $|T|$. We have the completion $\widehat{P}$. Let $\iota_P : \widehat{P} \to X$ an $\iota_{P,T} : \widehat{P} \to \widehat{T}$ denote the canonical morphisms. Since $\mathcal{O}_{\widehat{P}}$ is the completion of the local ring $\mathcal{O}_{X,P}$ at P, the morphism ι_P is faithfully flat. In the case $\iota_T^*\mathcal{F} = 0$, we have $\iota_P^*\mathcal{F} = 0$, and hence $\iota_P^{-1}\mathcal{F} = 0$. Since P can be any point of $|T|$, we obtain $\iota_T^{-1}\mathcal{F} = 0$. The second claim follows from the first claim and Proposition 22.1.1. □

22.5.2. Formal D-modules. — Let X be a complex manifold. Let Z be an analytic subvariety of X. Let $\widehat{Z}$ be the completion of X along Z. Let $\iota : \widehat{Z} \to X$ denote the canonical morphism. Let Θ_X denote the tangent sheaf of X. We put $\Theta_{\widehat{Z}} := \iota^*\Theta_X$, which acts on $\mathcal{O}_{\widehat{Z}}$ as differential operators. Let $\mathcal{D}_{\widehat{Z}}$ denote the sheaf of differential operators of $\mathcal{O}_{\widehat{Z}}$, which is generated by $\Theta_{\widehat{Z}}$ and $\mathcal{O}_{\widehat{Z}}$ with the standard relations. If we are given a coordinate system $(z_1, \dots, z_n)$ of X, $\Theta_{\widehat{Z}}$ is the free $\mathcal{O}_{\widehat{Z}}$-module with the base $\partial_i := \partial/\partial z_i$, and $\mathcal{D}_{\widehat{Z}}$ is the sheaf of algebras generated by $\mathcal{O}_{\widehat{Z}}$ and $\Theta_{\widehat{Z}}$ with the relation $[\partial_i, z_j] = \delta_{i,j}$. By applying the argument in Appendix A.1 of [**43**], we obtain the following standard proposition.

Proposition 22.5.3

- *The sheaf of algebras $\mathcal{D}_{\widehat{Z}}$ is Noetherian, and it has the property in Proposition* 22.1.3. *The Rees ring is also Noetherian.*
- *Let M be a $\mathcal{D}_{\widehat{Z}}$-module such that (i) pseudo-coherent as an $\mathcal{O}_{\widehat{Z}}$-module, (ii) locally finitely generated as a $\mathcal{D}_{\widehat{Z}}$-module. Then, M is a coherent $\mathcal{D}_{\widehat{Z}}$-module.*

Proof. — We give only an outline. We have the standard filtration of $\mathcal{D}_{\widehat{Z}}$ by the order of the differential operators, i.e., $F_m\mathcal{D}_{\widehat{Z}} = \left\{\sum_{|J|\leqslant m} a_J \cdot \partial^J\right\}$ on a coordinate neighbourhood, where $\partial^J = \prod \partial_i^{j_i}$ and $|J| = \sum j_i$ for $J = (j_1, \dots, j_n)$. Then, $F_0 D_{\widehat{Z}} = \mathcal{O}_{\widehat{Z}}$ is Noetherian (Proposition 22.5.1). Since $\mathrm{Gr}^F D_{\widehat{Z}}$ is locally a polynomial algebra over $F_0 D_{\widehat{Z}}$, it is also Noetherian. Then, the first claim of the proposition follows from Proposition 22.1.5. The second claim follows from Proposition 22.1.4. □

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. By the standard formalism, $\iota^*\mathcal{M}$ has a $\mathcal{D}_{\widehat{Z}}$-module structure. Thus we obtain the functor ι^* from the category of $\mathcal{D}_X$-modules to the category of $\mathcal{D}_{\widehat{Z}}$-modules.

Lemma 22.5.4. — *The functor ι^* is exact. If $\iota^*\mathcal{M} = 0$ for a coherent $\mathcal{D}_{\widehat{Z}}$-module, then* $\mathrm{Supp}(\mathcal{M}) \cap |Z| = \varnothing$.

Proof. — The first claim follows from Proposition 22.5.1. If $\iota^*\mathcal{M} = 0$, we obtain $\iota^{-1}\mathcal{M} = 0$ because of Lemma 22.5.2. Then, the second claim follows from Proposition 22.1.1. □

Lemma 22.5.5. — *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then $\iota^*\mathcal{M}$ is a coherent $\mathcal{D}_{\widehat{Z}}$-module.*

Proof. — Because of the Noetherian property of $\mathcal{D}_X$ and $\mathcal{D}_{\widehat{Z}}$, the coherence is equivalent to the locally finitely presentedness. Then, the claim is clear. □

22.5.3. Formal $\mathcal{R}$-modules. — Let X be a complex manifold, and let Z be an analytic subvariety of X. Let $\boldsymbol{C}_\lambda$ denote the complex line with the coordinate λ. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{Z} := \boldsymbol{C}_\lambda \times Z$. The morphisms induced by $Z \to X$ are denoted by ι. The completion of $\mathcal{X}$ along $\mathcal{Z}$ is denoted by $\widehat{\mathcal{Z}}$. Let p_λ denote the projection of $\mathcal{X}$ or $\widehat{\mathcal{Z}}$ onto X or $\widehat{Z}$. Let $\mathcal{R}_{\widehat{\mathcal{Z}}}$ be the sheaf of subalgebras of $p_\lambda^*\mathcal{D}_{\widehat{Z}}$ generated by $\mathcal{O}_{\widehat{\mathcal{Z}}}$ and $\lambda \cdot p_\lambda^*\Theta_{\widehat{Z}}$. If a holomorphic coordinate system $(z_1, \dots, z_n)$ of X is given, let $\eth_i$ denote $\lambda \cdot \partial_i$, as usual. The following lemma can be shown using the argument for Proposition 22.5.3.

Proposition 22.5.6

- *The sheaf of algebras $\mathcal{R}_{\widehat{\mathcal{Z}}}$ is Noetherian, and it has the property in Proposition* 22.1.3. *The Rees algebra is also Noetherian.*
- *Let $\mathcal{M}$ be an $\mathcal{R}_{\widehat{\mathcal{Z}}}$-module such that (i) pseudo-coherent as an $\mathcal{O}_{\widehat{\mathcal{Z}}}$-module, (ii) locally finitely generated as an $\mathcal{R}_{\widehat{\mathcal{Z}}}$-module. Then, $\mathcal{M}$ is a coherent $\mathcal{R}_{\widehat{\mathcal{Z}}}$-module.* □

Let $\mathcal{M}$ be an $\mathcal{R}_X$-module on $\mathcal{X}$. By the standard formalism, $\iota^*\mathcal{M}$ has the $\mathcal{R}_{\widehat{\mathcal{Z}}}$-module structure on $\widehat{\mathcal{Z}}$. Thus we obtain the functor ι^* of the category of $\mathcal{R}_X$-modules on $\mathcal{X}$ to the category of $\mathcal{R}_{\widehat{\mathcal{Z}}}$-modules on $\widehat{\mathcal{Z}}$. The following claims can be shown by the arguments in Section 22.5.2.

Lemma 22.5.7

- *The functor ι^* is exact. If $\iota^*\mathcal{M} = 0$ for a coherent $\mathcal{R}_X$-module, then* $\operatorname{Supp}(\mathcal{M}) \cap |\mathcal{Z}| = \varnothing$.
- *If $\mathcal{M}$ is a coherent $\mathcal{R}_X$-module, then $\iota^*\mathcal{M}$ is a coherent $\mathcal{R}_{\widehat{\mathcal{Z}}}$-module.* □

22.5.4. A criterion for strict S-decomposability. — Let X_0 be a complex manifold, and let Z_0 be a closed submanifold of X_0. Let $X := X_0 \times \boldsymbol{C}_t$ and $Z := Z_0 \times \boldsymbol{C}_t$. Let $\iota : \widehat{Z}_0 \to X_0$ denote the natural morphism. The induced morphisms are also denoted by ι. Let $\mathcal{X}^{(\lambda_0)}$ denote $U(\lambda_0) \times X$, where $U(\lambda_0)$ is some neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$. We use the symbols like $\mathcal{Z}^{(\lambda_0)}$, $\mathcal{X}_0^{(\lambda_0)}$ and $\widehat{\mathcal{Z}}_0^{(\lambda_0)}$ with similar meanings.

Let $\mathcal{M}_i$ $(i = 1, 2)$ be coherent $\mathcal{R}_X$-modules equipped with filtrations $V^{(\lambda_0)}(\mathcal{M}_i)$ satisfying Condition 22.3.1. We obtain the $\mathcal{R}_{\widehat{\mathcal{Z}}}$-modules $\iota^*\mathcal{M}_i$ $(i = 1, 2)$ on $\widehat{\mathcal{Z}}^{(\lambda_0)}$ with the induced filtrations $\iota^*V^{(\lambda_0)}$ indexed by $\boldsymbol{R}$. Assume that we are given an isomorphism $\varphi : \iota^*\mathcal{M}_1 \to \iota^*\mathcal{M}_2$ of $\mathcal{R}_{\widehat{\mathcal{Z}}}$-modules strictly compatible with the filtrations.

Proposition 22.5.8. — *If $\mathcal{M}_1$ is strictly specializable (S-decomposable) along t at λ_0 with respect to $V^{(\lambda_0)}(\mathcal{M}_1)$, then $\mathcal{M}_2$ is also strictly specializable (S-decomposable) along t at λ_0 with respect to $V^{(\lambda_0)}(\mathcal{M}_2)$, after shrinking X_0 around Z_0. (See Condition* 22.3.2 *and Condition* 22.3.14 *for strict specializability and strict S-decomposability.)*

Proof. — We have an induced isomorphism $\iota^* \operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_1) \simeq \iota^* \operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_2)$ as in Proposition 22.3.7. Let N_2 be the kernel of $\lambda - \lambda_0$ on $\operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_2)$. Since $\operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_1)$ is strict, $\iota^* \operatorname{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_2)$ is also strict due to Lemma 22.5.7. Hence, we obtain $\iota^* N_2 = 0$. Then, we obtain $N_2 = 0$ by shrinking X_0, due to Lemma 22.5.7. By using coherence, we obtain that $\mathrm{Gr}_a^{V^{(\lambda_0)}}(\mathcal{M}_2)$ is strict, after shrinking $U(\lambda_0)$ and X_0 appropriately. We can check the other conditions by using Lemma 22.5.7. □

22.6. Preliminaries for $\mathcal{D}$-modules

We recall some basic facts for $\mathcal{D}$-modules and $\mathcal{R}$-modules. (See [**11**], [**43**], [**38**], [**73**], [**67**], for example.)

22.6.1. Pull-back of $\mathcal{D}$-modules. — Let $f : X \to Y$ be a morphism of complex manifolds, and let $\mathcal{F}$ be a $\mathcal{D}_Y$-module. We put $\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$, which is equipped with the left $\mathcal{D}_X$-action and the right $f^{-1}\mathcal{D}_Y$-action. Recall that the pull-back $f^\dagger\mathcal{F}$ is defined to be $f^\dagger\mathcal{F} := \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{F}$. Let $L^i f^\dagger$ denote the i-th derived functor of $f^\dagger$. Let *for* denote the natural functor from the category of $\mathcal{D}$-modules to the category of $\mathcal{O}$-modules. Let $L^i f^*$ denote the i-th derived functor of the pull-back f^* for $\mathcal{O}$-modules. According to the following lemma, we do not have to distinguish the sheaves $L^i f^\dagger \mathcal{F}$ and $L^i f^* \mathcal{F}$.

Lemma 22.6.1. — *$for(L^i f^\dagger \mathcal{M}) \simeq L^i f^*(for(\mathcal{M}))$*

Proof. — See Proposition 2.3.8 of [**11**], for example. □

Recall the following well known results.

Proposition 22.6.2

- *If $\mathcal{F}$ is a holonomic $\mathcal{D}_Y$-module, then $L^i f^\dagger \mathcal{F}$ are also holonomic.*
- *Assume that f is a closed embedding of complex manifolds, and $\mathcal{F} = f_\dagger \mathcal{F}_0$ for some $\mathcal{D}_X$-module $\mathcal{F}_0$. Let $k := \dim X - \dim Y$. Then, we have $L^i f^* \mathcal{F} = 0$ for $i \neq k$ and $L^k f^* \mathcal{F} \simeq \mathcal{F}_0$.*

Proof. — See Theorem 3.2.13 of [**11**] for the first claim, for example. See Proposition 4.32 of [**43**] for the second claim, for example. □

Recall that we have the trace map whose construction is explained in Chapter 4.9 of [**43**], for example:

$$\mathrm{tr}_f : f_\dagger\big(Lf^\dagger\mathcal{F}\big)[\dim X] \longrightarrow \mathcal{F}[\dim Y].$$

22.6.2. Holonomic $\mathcal{D}$-modules and meromorphic connections. — We recall the following general lemma. Let X be a complex manifold, and let D be a divisor of X. Let $\pi : T^*X \to X$ denote the natural projection of the cotangent bundle.

Lemma 22.6.3. — *Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module such that (i) the characteristic variety of $\mathcal{F}$ is contained in $\pi^{-1}(D)\cup T^*_X X$, (ii) $\mathcal{F}$ is also an $\mathcal{O}_X(*D)$-module. Then, locally on X, there exists an $\mathcal{O}_X$-coherent subsheaf $\mathcal{F}_0\subset\mathcal{F}$ such that $\mathcal{F}=\mathcal{O}_X(*D)\cdot\mathcal{F}_0$. In particular, $\mathcal{F}$ gives a meromorphic flat connection on (X,D).*

Proof. — Since the claim is local, we may assume that D is given as $g^{-1}(0)$. We may also take a coherent filtration F of $\mathcal{F}$. Then, $\mathrm{Gr}^F(\mathcal{F})$ is a coherent $\mathcal{O}_{T^*X}$-module, and the support of $\mathrm{Gr}^F(\mathcal{F})$ is contained in $T^*_X X\cup\pi^{-1}(D)$. Hence, there exists a large number i_0 such that the support of $\mathrm{Gr}^F_i(\mathcal{F})$ is contained in $\pi^{-1}(D)$ for any $i\geqslant i_0$. For such an i, there exists a large number N such that $g^N\cdot F_i(\mathcal{F})\subset F_{i_0}(\mathcal{F})$. Hence, $F_{i_0}(\mathcal{F})$ generates $\mathcal{F}$ over $\mathcal{O}_X(*D)$. □

22.6.3. Nearby cycle functor with ramified exponential twist. — Recall the notion of nearby cycle functor with ramified exponential twist, by following Deligne and Sabbah ([**25**], [**75**]). Let X_0 be a complex manifold, and let $X:=X_0\times\boldsymbol{C}_t$. We set $X^{(n)}:=X_0\times\boldsymbol{C}_{t_n}$. Let $\varphi_n:X^{(n)}\to X$ be a morphism induced by $\varphi_n(t_n)=t_n^n$. For a given $\mathfrak{a}\in\boldsymbol{C}[t_n^{-1}]$, we set $L(\mathfrak{a}):=\mathcal{O}_{X^{(n)}}(*t_n)\cdot e$ with the meromorphic flat connection $\nabla e=e\cdot d\mathfrak{a}$. It is naturally a holonomic $\mathcal{D}_{X^{(n)}}$-module.

Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. By taking the pull-back and the tensor product with $L(-\mathfrak{a})$, we obtain a holonomic $\mathcal{D}_{X^{(n)}}$-module $\varphi_n^\dagger\mathcal{F}\otimes L(-\mathfrak{a})$. Applying the nearby cycle functor, we obtain a holonomic $\mathcal{D}_{X_0}$-module $\psi_{t_n}\big(\varphi_n^\dagger\mathcal{F}\otimes L(-\mathfrak{a})\big)$.

Let Y be a complex manifold, g be a holomorphic function on Y, and $\mathcal{F}$ be a holonomic $\mathcal{D}_Y$-module. Then, for any $\mathfrak{a}\in\boldsymbol{C}[t_n^{-1}]$, we define

$$\psi_{g,\mathfrak{a}}(\mathcal{F}):=\psi_{t_n}\Big(\varphi_n^*\big(i_{g\dagger}\mathcal{F}\big)\otimes\mathcal{L}(-\mathfrak{a})\Big).$$

The functor $\psi_{g,\mathfrak{a}}$ is called the nearby cycle functor with ramified exponential twist by $\mathfrak{a}$.

22.6.4. The functors $j_!j^*$ and j_*j^*. — Let X be a complex manifold, and let $i_Y:Y\subset X$ be a smooth hypersurface of X. Let Θ_X denote the tangent sheaf of X. Let $\mathcal{N}_{Y/X}$ denote the sheaf of the sections of the normal bundle of Y in X. We put $j_*j^*\mathcal{O}_X:=\mathcal{O}_X(*Y)$.

We also have the following $\mathcal{D}_X$-module $j_!j^*\mathcal{O}_X$. As an $\mathcal{O}_X$-module, we set

$$j_!j^*\mathcal{O}_X:=\mathcal{O}_{\mathcal{X}}\oplus i_{Y\dagger}i_Y^\dagger\mathcal{O}_X=\mathcal{O}_X\oplus i_{Y*}\big(\mathrm{Sym}^\cdot\mathcal{N}_{Y/X}\otimes\mathcal{N}_{Y/X}\big).$$

The action of Θ_X is given by $v\cdot(s,t)=\big(v\cdot s,\ v\cdot t+\pi(v_{|Y})\cdot s_{|Y}\big)$, where "$|Y$" denotes the restriction to Y, π denotes the projection of $\Theta_{X|Y}\to\mathcal{N}_{Y/X}$, and $v\cdot t$ is given by the natural $\mathcal{D}_X$-module structure on $i_{Y\dagger}i_Y^\dagger\mathcal{O}_X$. It can be uniquely extended to an action of $\mathcal{D}_X$.

Let Ω_X denote the canonical line bundle of X. For a $\mathcal{D}_X$-module M, we have the derived dual module $\boldsymbol{D}_X(M):=R\mathcal{H}om_{\mathcal{D}_X}(M,\mathcal{D}_X)\otimes\Omega_X^{-1}[\dim X]$ in the derived category of $\mathcal{D}_X$-modules.

Lemma 22.6.4. — *We have a natural isomorphism* $\boldsymbol{D}_X\big(j_*j^*\mathcal{O}_X\big) \simeq j_!j^*\mathcal{O}_X$

Proof. — Note that we have the canonical isomorphisms $(j_!j^*\mathcal{O}_X)_{|X\smallsetminus Y} \simeq \mathcal{O}_{X|X\smallsetminus Y} \simeq \boldsymbol{D}_X\big(j_*j^*\mathcal{O}_X\big)_{|X\smallsetminus Y}$. We have the $\mathcal{O}_X$-submodule $\mathcal{O}_X$ of $j_!j^*\mathcal{O}_X$, which generates $j_!j^*\mathcal{O}_X$ over $\mathcal{D}_X$. Hence, an automorphism φ of $j_!j^*\mathcal{O}_X$ is the identity, if the restriction $\varphi_{|X\smallsetminus D}$ is the identity. Therefore, we only have to show that the canonical isomorphism can be extended locally.

Then, we only have to consider the case $X = \Delta^n$ and $Y = \{z_1 = 0\}$. Moreover, it can be reduced to the case $n = 1$. We consider the following exact sequence of left $\mathcal{D}_\Delta$-modules on a disc Δ:

$$0 \longrightarrow \mathcal{D}_\Delta \xrightarrow{\varphi} \mathcal{D}_\Delta \xrightarrow{\psi} \mathcal{O}_\Delta(*O) \longrightarrow 0, \quad \varphi(P_1) = P_1\cdot\partial_t\cdot t, \quad \psi(P_2) = P_2\cdot t^{-1}.$$

Hence, as a right $\mathcal{D}_\Delta$-module, $R\mathcal{H}om_{\mathcal{D}_\Delta}(\mathcal{O}_\Delta(*O),\mathcal{D}_\Delta)[1]$ is the cokernel of the following morphism:

$$\mathcal{D}_\Delta \longrightarrow \mathcal{D}_\Delta, \quad Q \longmapsto \partial_t\cdot t\cdot Q.$$

We have the following exact sequence of right $\mathcal{D}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \partial_t\mathcal{D}_\Delta/\partial_t t\mathcal{D}_\Delta & \longrightarrow & \mathcal{D}_\Delta/\partial_t t\mathcal{D}_\Delta & \longrightarrow & \mathcal{D}_\Delta/\partial_t\mathcal{D}_\Delta & \longrightarrow & 0\\
 & & \simeq\big\downarrow & & \simeq\big\downarrow & & \simeq\big\downarrow & & \\
0 & \longrightarrow & \boldsymbol{C}[\partial_t] & \longrightarrow & \mathcal{D}_\Delta/\partial_t t\mathcal{D}_\Delta & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & 0
\end{array}$$

Then, it is easy to check $\mathcal{D}_\Delta/\partial_t t\mathcal{D}_\Delta \simeq j_!j^*\mathcal{O}_\Delta$. □

Let M be a $\mathcal{D}_X$-module. We put $j_*j^*M := M\otimes^L\mathcal{O}_X(*Y)$ and $j_!j^*M := M\otimes^L j_!j^*\mathcal{O}_X$, where "$\otimes^L$" denotes the derived tensor product for $\mathcal{D}$-modules.

Lemma 22.6.5. — *We have a natural isomorphism* $j_!j^*M \simeq \boldsymbol{D}_X\big(j_*j^*\boldsymbol{D}_X(M)\big)$.

Proof. — Because $j_*j^*\boldsymbol{D}_XM \simeq \boldsymbol{D}_X(M)\otimes^L\mathcal{O}_X(*Y)$, we have $\boldsymbol{D}_X\big(j_*j^*\boldsymbol{D}_X(M)\big) \simeq M\otimes^L\boldsymbol{D}_X\big(\mathcal{O}_X(*Y)\big)$. We also have the natural isomorphism $j_!j^*M \simeq M\otimes^L j_!j^*\mathcal{O}_X$. Then, Lemma 22.6.5 follows from Lemma 22.6.4. □

If Y is non-characteristic with respect to M, we have $j_!j^*M = M\otimes j_!j^*\mathcal{O}_X$. In that case, we have the following exact sequences:

$$0 \longrightarrow M \longrightarrow j_*j^*M \longrightarrow j_*j^*M/M \longrightarrow 0,$$
$$0 \longrightarrow i_\dagger i^\dagger M \longrightarrow j_!j^*M \longrightarrow M \longrightarrow 0.$$

22.6.5. Some commutative diagrams. — Let X be a projective variety with an embedding $X\subset\mathbb{P}^N$. Let X_0 be the intersection $X\cap H_1\cap H_2$, where H_i denote general hyperplanes. Let $\pi:\widetilde{X}\to X$ be the blow up of X along X_0. We put $Y := H_1\cap X$. We have the natural embedding $Y\subset\widetilde{X}$ whose image is denoted by $\widetilde{Y}$.

We have the exact sequences for $\mathcal{D}_{\widetilde{X}}$-modules as in Section 22.6.4:

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}} \longrightarrow 0,$$
$$0 \longrightarrow i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{j}_!\widetilde{j}^*\mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow 0.$$

Because $\pi_{\dagger}^{p}\mathcal{O}_{\widetilde{X}} = 0$ and $\pi_{\dagger}^{p}\big(i_{\widetilde{Y}}\mathcal{O}_{\widetilde{Y}}\big) = 0$ for $p \neq 0$, we obtain $\pi_{\dagger}^{p}\big(\widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}\big) = 0$ and $\pi_{\dagger}^{p}\big(\widetilde{j}_!\widetilde{j}^*\mathcal{O}_{\widetilde{X}}\big) = 0$ for $p \neq 0$. We also recall the following lemma.

Lemma 22.6.6. — *We have the following commutative diagram of $\mathcal{D}_X$-modules:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & j_*j^*\mathcal{O}_X & \longrightarrow & j_*j^*\mathcal{O}_X/\mathcal{O}_X & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\big\uparrow & & {\scriptstyle f_2}\big\uparrow & & {\scriptstyle =}\big\uparrow & & \\ 0 & \longrightarrow & \pi_{\dagger}(\mathcal{O}_{\widetilde{X}}) & \longrightarrow & \pi_{\dagger}\big(\widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & \pi_{\dagger}\big(\widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & 0 \end{array} \tag{429}$$

Here, f_1 is the natural morphism.

By taking the dual, we also have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i_{Y\dagger}i_Y^{\dagger}\mathcal{O}_X & \longrightarrow & j_!j^*\mathcal{O}_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & {\scriptstyle =}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & \\ 0 & \longrightarrow & \pi_{\dagger}\big(i_{\widetilde{Y}\dagger}i_{\widetilde{Y}}^{\dagger}\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & \pi_{\dagger}\big(\widetilde{j}_!\widetilde{j}^*\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & \pi_{\dagger}\big(\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & 0 \end{array} \tag{430}$$

Here, g_1 is the natural morphism.

Proof. — We have the following commutative diagram:

$$\begin{array}{ccccc} \pi_{\dagger}\mathcal{O}_{\widetilde{X}} & \longrightarrow & \pi_{\dagger}\mathcal{O}_{\widetilde{X}} & \longrightarrow & \mathcal{O}_X \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \pi_{\dagger}\big(\widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}\big) & \longrightarrow & \pi_{\dagger}\pi^{\dagger}(j_*j^*\mathcal{O}_X) & \longrightarrow & j_*j^*\mathcal{O}_X \end{array}$$

Let f_2 be the composite of the lower horizontal arrows. Then, we only have to show that the induced map $f_3 : \pi_{\dagger}\big(\widetilde{j}_*\widetilde{j}^*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}\big) \to j_*j^*\mathcal{O}_X/\mathcal{O}_X$ is the identity. We only have to compare the restrictions of f_3 with the identity on $X - X_0$. Then, the claim is clear. □

22.7. Complement

22.7.1. Push-forward of $\mathcal{R}$-modules by ramified covering. — Let $X := \Delta_z^n$, $\widetilde{X} := \Delta_\zeta^n$, $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$, and $\widetilde{D} := \bigcup_{i=1}^{\ell}\{\zeta_i = 0\}$. Let $f : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a ramified covering given by $f(\zeta_1, \dots, \zeta_n) = (\zeta_1^{m_1}, \dots, \zeta_\ell^{m_\ell}, \zeta_{\ell+1}, \dots, \zeta_n)$. Let $\mathcal{M}$ be an $\mathcal{R}_{\widetilde{X}}$-module. We obtain the $\mathcal{R}_{\widetilde{X}}$-module $\mathcal{M}' := \mathcal{M} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(*\widetilde{\mathcal{D}})$, on which ζ_i $(i = 1, \dots, \ell)$ are invertible. Let us consider the push-forward via f.

Lemma 22.7.1. — *The push-forward of $\mathcal{M}'$ as an $\mathcal{R}$-module is isomorphic to the push-forward as an $\mathcal{O}$-module. A similar claim holds also for $\mathcal{D}$-modules.*

Proof. — We only have to consider the case of right $\mathcal{R}$-modules. Let $\mathcal{C}$ be an $\mathcal{R}_{\widetilde{X}}$-free resolution of $\mathcal{O}_{\widetilde{X}} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{R}_X$. The push-forward of $\mathcal{M}'$ as an $\mathcal{R}$-module is given by

$$f_*(\mathcal{M}' \otimes_{\mathcal{R}_{\widetilde{X}}} \mathcal{C}). \tag{431}$$

Here f_* denotes the push-forward of $\mathcal{O}$-modules. Let $\mathcal{R}_{\widetilde{X}}(*\widetilde{D}) := \mathcal{R}_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(*\widetilde{\mathcal{D}})$ and $\mathcal{R}_X(*D) := \mathcal{R}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*\mathcal{D})$. Because $\mathcal{M}' := \mathcal{M} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(*\widetilde{\mathcal{D}})$, (431) is equal to

$$f_*\big(\mathcal{M}' \otimes_{\mathcal{R}_{\widetilde{X}}(*\widetilde{D})} (\mathcal{R}_{\widetilde{X}}(*\widetilde{D}) \otimes_{\mathcal{R}_{\widetilde{X}}} \mathcal{C})\big). \tag{432}$$

Note $\mathcal{R}_{\widetilde{X}}(*\widetilde{D}) \otimes_{\mathcal{R}_{\widetilde{X}}} \mathcal{C}$ is quasi-isomorphic to $\mathcal{R}_{\widetilde{X}}(*\widetilde{D})$. Hence, (432) is naturally isomorphic to $f_*(\mathcal{M}')$. □

22.7.2. Some sheaves of algebras. — Let $X := \Delta^n$. For any subset $J \subset \underline{n}$, let ${}^J V_0\mathcal{R}_X$ denote the sheaf of subalgebras of $\mathcal{R}_X$ generated by $\mathcal{O}_X$, $\eth_j z_j$ $(j \in J)$ and $\eth_j$ $(j \in \underline{n} - J)$. It is equipped with the filtration by the order of differential operators. The following lemma can be shown by the argument in the proof of Proposition 22.5.3.

Proposition 22.7.2

- *The sheaf of algebras ${}^I V_0\mathcal{R}_X$ is Noetherian, and it has the property in Proposition* 22.1.3. *The Rees ring is also Noetherian.*
- *Let $\mathcal{M}$ be an ${}^I V_0\mathcal{R}_X$-module which is (i) pseudo-coherent as an $\mathcal{O}_X$-module, (ii) locally finitely generated as a ${}^I V_0\mathcal{R}_X$-module. Then, it is a coherent ${}^I V_0\mathcal{R}_X$-module.* □

22.8. The sheaves $\mathfrak{Db}_{X\times T/T}$ and $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}$

We recall some sheaves, following Sabbah in [**72**] and [**75**], to which we refer for more details and precision. For simplicity, we consider the case where X is an open subset of $\boldsymbol{C}^n$. Let $(x_1, \dots, x_{2n})$ be a real coordinate system, and denote $\partial_i = \partial/\partial x_i$. For $J = (j_1, \dots, j_n)$, we put $\partial^J := \prod_{i=1}^n \partial_i^{j_i}$ and $|J| := \sum_{i=1}^n j_i$. Let V be an open subset of $X \times T$. Let $\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$ denote the space of C^∞-sections of $\Omega^{n,n}_{X\times T/T}$ on V with compact supports. For any compact subset $K \subset V$ and $m \in \mathbb{Z}_{>0}$, we have the semi-norm $\|f\|_{m,K} = \sup_{|J|\leqslant m}\sup_K |\partial_J f|$. For any closed subset $Z \subset X$, let $\mathcal{E}^{<Z,(n,n)}_{X\times T/T,c}(V)$ denote the subspace of $\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$, which consists of the sections f such that $(\partial^J f)_{|Z} = 0$ for any J. We have the induced semi-norms $\|\cdot\|_{m,K}$ on the space $\mathcal{E}^{<Z,(n,n)}_{X\times T/T,c}(V)$. By the semi-norms, the spaces $\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$ and $\mathcal{E}^{<Z,(n,n)}_{X\times T/T,c}(V)$ are locally convex topological spaces. Let $C^0_c(T)$ denote the space of continuous functions on T with compact supports. It is a normed vector space with the sup norms.

Let $\mathfrak{Db}_{X\times T/T}(V)$ denote the space of continuous $C^\infty(T)$-linear maps from $\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$ to $C^0_c(T)$. In the case $X = X_0 \times \boldsymbol{C}_t$, let $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$ denote the space of continuous $C^\infty(T)$-linear maps from $\mathcal{E}^{<X_0,(n,n)}_{X\times T/T,c}(V)$ to $C^0_c(T)$. Any elements of $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$ are called distributions with moderate growth. Let $\mathfrak{Db}_{X\times T/T,X_0}(V)$ be the space of continuous $C^\infty(T)$-linear maps $\Phi : \mathcal{E}^{(n,n)}_{X\times T/T,c}(V) \to C^0_c(T)$ whose supports are contained in $X_0\times T$, i.e., $\Phi(f)=0$ if $f=0$ on some neighbourhood of $X_0\times T$. They give the sheaves $\mathfrak{Db}_{X\times T/T}$, $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}$ and $\mathfrak{Db}_{X\times T/T,X_0}$ on $X\times T$. We have the following lemma as in [**54**].

Lemma 22.8.1. — *The natural sequence*

$$0\longrightarrow \mathfrak{Db}_{X\times T/T,X_0}(V)\xrightarrow{\varphi}\mathfrak{Db}_{X\times T/T}(V)\xrightarrow{\psi}\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)\longrightarrow 0$$

is exact.

Proof. — Let us show the surjectivity of ψ. We may assume that V is a product of T with an open subset X_1 of X. Let $\Phi\in\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$. It naturally induces a continuous linear map $F_\Phi : \mathcal{E}^{<X_0,(n,n)}_{X,c}(X_1)\to C^0(T)$. According to the Hahn-Banach theorem (see [**70**]), there exists a continuous linear map $\mathcal{E}^{(n,n)}_{X,c}(X_1)\to C^0(T)$, whose restriction to $\mathcal{E}^{<X_0,(n,n)}_{X,c}(X_1)$ is equal to F_Φ. Then, we obtain a continuous $C^\infty(T)$-linear map $\mathcal{E}^{(n,n)}_{X,c}(X_1)\otimes C^\infty_c(T)\to C^0_c(T)$. By using the continuity, we obtain a $C^\infty(T)$-linear map $\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)\to C^0_c(T)$, whose restriction to $\mathcal{E}^{<X_0,(n,n)}_{X\times T/T,c}(V)$ is equal to the given Φ. Thus, we obtain the surjectivity of ψ. The injectivity of φ is clear. Let H be the space of functions $f\in\mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$ such that $f=0$ on some neighbourhood of $X_0\times T$. We can show that H is dense in $\mathcal{E}^{<X_0,(n,n)}_{X\times T/T,c}(V)$ using the argument of Lemma 4.3 in [**54**]. Then, $\mathrm{Im}(\varphi)=\mathrm{Ker}(\psi)$ follows. □

Note that $\mathfrak{Db}_{X\times T/T}(V)$ are naturally $\boldsymbol{C}[t]$-modules. Let $\mathfrak{Db}_{X\times T/T}(*t)$ denote the sheafification of presheaves $V\longmapsto\mathfrak{Db}_{X\times T/T}(V)\otimes_{\boldsymbol{C}[t]}\boldsymbol{C}[t,t^{-1}]$.

Lemma 22.8.2. — *$\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$ is naturally isomorphic to $\mathfrak{Db}_{X\times T/T}(*t)(V)$. It is also isomorphic to the image of $\mathfrak{Db}_{X\times T/T}(V)\to\mathfrak{Db}_{X\times T/T}(V\setminus X_0)$.*

Proof. — Due to Lemma 22.8.1, we have the natural injection $\iota:\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)\to\mathfrak{Db}_{X\times T/T}(*t)(V)$. It is easy to observe that any element of $\mathfrak{Db}_{X\times T/T}(V)(*t)(V)$ naturally determines a continuous $C^\infty(T)$-linear function $\mathcal{E}^{<X_0(n,n)}_{X\times T/T,c}(V)\to C^0_c(T)$. Hence, ι is surjective. The second claim immediately follows from Lemma 22.8.1. □

The following lemma can be shown by using a standard argument.

Lemma 22.8.3. — *Let Φ be an element of $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$. Let K be any compact region. We have some $m\in\mathbb{Z}_{>0}$ and $C>0$ such that $\sup_T\bigl|\Phi(f)\bigr|\leqslant C\cdot\|f\|_{m,K}$ for any $f\in\mathcal{E}^{<X_0\,(n,n)}_{X\times T/T,c}(V)$ with* $\mathrm{Supp}(f)\subset K$.

Proof. — Assume that the claim does not hold. We have a sequence $C_m \to \infty$ and $f_m \in \mathcal{E}^{<X_0\,(n,n)}_{X\times T/T,c}(V)$ with $\mathrm{Supp}(f_m) \subset K$ such that $\sup_T\big|\Phi(f_m)\big| > C_m \cdot \|f_m\|_{m,K}$. For $g_m := f_m \cdot \|f_m\|^{-1} \cdot C_m^{-1}$, we have $\sup_T\big|\Phi(g_m)\big| > 1$ and $\|g_m\|_{m,K} \to 0$. Thus, we have arrived at the contradiction. □

Corollary 22.8.4. — *Let* $\Phi \in \mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$. *We have the well defined pairing* $\Phi\big(|t|^s \cdot f\big) \in C^0_c(T)$ *for* $f \in \mathcal{E}^{(n,n)}_{X\times T/T,c}(V)$ *with* $\mathrm{Supp}(f) \subset K$, *if* $\mathrm{Re}(s)$ *is sufficiently large.* □

22.8.1. Pull-back via a ramified covering. — Let $\varphi_n : \boldsymbol{C}_{t_n} \to \boldsymbol{C}_t$ be given by $\varphi_n^*(t) = t_n^n$. Let $X^{(n)} := X_0 \times \boldsymbol{C}_{t_n}$, and let $\varphi_n : X^{(n)} \times T \to X \times T$ be the induced map. Let $V^{(n)} := \varphi_n^{-1}(V)$. We have the well defined continuous map

$$\varphi_{n\,*} : \mathcal{E}^{<X_0\,(n,n)}_{X^{(n)}\times T/T,c}(V^{(n)}) \longrightarrow \mathcal{E}^{<X_0\,(n,n)}_{X\times T/T,c}(V).$$

Hence, we have the induced map

$$\varphi_n^* : \mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V) \longrightarrow \mathfrak{Db}^{\mathrm{mod}}_{X^{(n)}\times T/T}(V^{(n)}).$$

We have the following general lemma.

Lemma 22.8.5. — *Let* ρ *be a test function on* X_0, *and let* $\tau \in \mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$. *If* $m \not\equiv 0$ *modulo* n, *we have the vanishing* $\big\langle \varphi_n^*\tau,\, |t_n|^{2s} t_n^m \varphi_n^*(\chi) \cdot \rho\big\rangle = 0$ *for any* s *such that* $\mathrm{Re}(s)$ *is sufficiently large.*

Proof. — We only have to consider $t_n \mapsto t_n \cdot a$ for a primitive n-th root a. □

22.8.2. Exponential twist. — Sabbah observed the following in [**75**].

Lemma 22.8.6. — *Let* $\Phi \in \mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$ *and* $\mathfrak{a} \in \boldsymbol{C}[t^{-1}]$. *Then,*

$$\exp\big(2\sqrt{-1}\,\mathrm{Im}(\lambda\mathfrak{a})\big)\Phi \in \mathfrak{Db}_{X\times T/T}(V \setminus X_0)$$

is contained in $\mathfrak{Db}^{\mathrm{mod}}_{X\times T/T}(V)$.

Proof. — For any $f \in \mathcal{E}^{<X_0,(n,n)}_{X\times T/T,c}(V)$, we have $\exp\big(2\sqrt{-1}\,\mathrm{Im}(\lambda\overline{\mathfrak{a}})\big) f \in \mathcal{E}^{<X_0,(n,n)}_{X\times T/T,c}(V)$. Hence, the claim of the lemma follows. □

22.9. $\mathcal{R}$-triples

22.9.1. Hermitian sesqui-linear pairing. — We recall the notion of $\mathcal{R}$-triples in [**73**] for the convenience of readers. We refer to it for more details and precision. Let X be a complex manifold. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$. We set $\boldsymbol{S} := \big\{\lambda \in \boldsymbol{C} \,\big|\, |\lambda| = 1\big\}$. Let $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$ be as in Section 22.8. Let $\sigma : \boldsymbol{C}^*_\lambda \to \boldsymbol{C}^*_\lambda$ given by $\sigma(\lambda) = -\overline{\lambda}^{-1}$. The induced map $\boldsymbol{S} \times X \to \boldsymbol{S} \times X$ is also denoted by σ. We have the natural $(\mathcal{R}_X)_{|\boldsymbol{S}\times X}$-action on $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$. We also have the $(\sigma^*\mathcal{R}_X)_{|\boldsymbol{S}}$-action on $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$ locally given by $\sigma^*(\eth_i) = -\lambda^{-1}\overline{\partial}_i =: \overline{\eth}_i$. Thus, $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$ is a left $(\mathcal{R}_X \otimes \sigma^*\mathcal{R}_X)_{|\boldsymbol{S}\times X}$-module.

Let $\mathcal{M}'$ and $\mathcal{M}''$ be left $\mathcal{R}$-modules on $\mathcal{X}$. Recall that a Hermitian sesqui-linear pairing of $\mathcal{M}'$ and $\mathcal{M}''$ is defined to be a morphism of left $(\mathcal{R}_X \otimes \sigma^* \mathcal{R}_X)_{|\mathbf{S}\times X}$-modules:

$$C : \mathcal{M}'_{|\mathbf{S}\times X} \otimes \sigma^* \mathcal{M}''_{|\mathbf{S}\times X} \longrightarrow \mathfrak{Db}_{\mathbf{S}\times X/\mathbf{S}}.$$

It is also called a sesqui-linear pairing. We often denote the pairing $C(x, \sigma^* y)$ by $C(x, \overline{y})$ as in [**73**]. See [**73**] for a sesqui-linear pairing of right $\mathcal{R}$-modules.

Let C be a sesqui-linear pairing of $\mathcal{M}'$ and $\mathcal{M}''$. Then a sesqui-linear pairing C^* of $\mathcal{M}''$ and $\mathcal{M}'$ is given by the following formula:

$$C^*(x, \sigma^* y) = \overline{\sigma^* C(y, \sigma^* x)}. \tag{433}$$

Here x and y denote local sections of $\mathcal{M}''_{|\mathbf{S}\times X}$ and $\mathcal{M}'_{|\mathbf{S}\times X}$ respectively.

The following lemma is proved in [**73**].

Lemma 22.9.1. — *Let $\mathcal{M}_i$ $(i = 1, 2)$ be strictly S-decomposable $\mathcal{R}_X$-modules whose strict supports are irreducible closed subsets Z_i. If $Z_1 \neq Z_2$, then there does not exist any non-trivial sesqui-linear pairing of $\mathcal{M}_1$ and $\mathcal{M}_2$.* □

22.9.2. $\mathcal{R}_X$-triple. — A left $\mathcal{R}_X$-triple is a tuple $(\mathcal{M}', \mathcal{M}'', C)$ of $\mathcal{R}_X$-modules $\mathcal{M}'$, $\mathcal{M}''$ and a sesqui-linear pairing C of $\mathcal{M}'$ and $\mathcal{M}''$. Let $\mathcal{T}_i = (\mathcal{M}'_i, \mathcal{M}''_i, C_i)$ $(i = 1, 2)$ be $\mathcal{R}_X$-triples. A morphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ is a tuple of morphisms $\varphi' : \mathcal{M}'_2 \to \mathcal{M}'_1$ and $\varphi'' : \mathcal{M}''_1 \to \mathcal{M}''_2$ satisfying

$$C_1\big(\varphi'(x), \sigma^*(y)\big) = C_2\big(x, \sigma^*(\varphi''(y))\big).$$

Here x and y denote local sections of $\mathcal{M}'_{2|\mathbf{S}\times X}$ and $\mathcal{M}''_{1|\mathbf{S}\times X}$. The category $\mathcal{R}_X$-Triples of $\mathcal{R}_X$-triples is abelian.

22.9.3. Tate twist and Hermitian adjoint. — For any half integer $k \in \frac{1}{2}\mathbb{Z}$, the k-th Tate object $\mathbb{T}^S(k)$ is defined to be the tuple $\big(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}, C_k\big)$, where C_k is given as follows:

$$C_k(f, \sigma^* g) = \big(\sqrt{-1}\lambda\big)^{-2k} \cdot f \cdot \overline{\sigma^* g}.$$

The k-th Tate twist $\mathcal{T}(k)$ of $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ is the tuple

$$\big(\mathcal{M}', \mathcal{M}'', (\sqrt{-1}\lambda)^{-2k} C\big).$$

In other words, it is the tensor product $\mathcal{T} \otimes \mathbb{T}^S(k)$. We will not distinguish $\mathcal{T}(k)$ and $\mathcal{T} \otimes \mathbb{T}^S(k)$. A morphism $\varphi = (\varphi', \varphi'') : \mathcal{T} \to \mathcal{T}'$ naturally induces the morphism $\varphi = (\varphi', \varphi'') : \mathcal{T}(k) \to \mathcal{T}'(k)$ for any $k \in \frac{1}{2}\mathbb{Z}$.

For a left $\mathcal{R}$-triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$, the Hermitian adjoint $\mathcal{T}^*$ is defined to be the tuple $(\mathcal{M}'', \mathcal{M}', C^*)$, where C^* is given as in (433). For any morphism $\varphi = (\varphi', \varphi'') : \mathcal{T}_1 \to \mathcal{T}_2$, we obtain the morphism $\varphi^* = (\varphi'', \varphi') : \mathcal{T}_2 \to \mathcal{T}_1$.

The Hermitian adjoint of the Tate object $\mathbb{T}^S(-k)$ is $\big(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}, (-1)^{-2k} C_k\big)$. We fix the isomorphism

$$\big((-1)^{2k}\,\mathrm{id}, \mathrm{id}\big) : \mathbb{T}^S(k) \simeq \mathbb{T}^S(-k)^*.$$

It induces the isomorphism $\mathcal{T}(k) \simeq \big(\mathcal{T}^*(-k)\big)^*$.

22.9.4. Hermitian sesqui-linear duality of $\mathcal{R}$-triples. — Let $\mathcal{T}$ be an $\mathcal{R}_X$-triple, and let w be an integer. Recall that a morphism $\mathcal{S}:\mathcal{T}\to\mathcal{T}^*(-w)$ is called a Hermitian sesqui-linear duality of weight w, if the equality $\mathcal{S}=(-1)^w\mathcal{S}^*$ holds. In the case $\mathcal{S}=(S',S'')$, the condition is equivalent to $S'=(-1)^wS''$.

Let $\mathcal{S}=(S',S'')$ be a morphism $\mathcal{T}\to\mathcal{T}^*(-w)$. Via the canonical isomorphism $\mathbb{T}^S(k)\simeq\mathbb{T}^S(-k)^*$, we obtain the map $\mathcal{T}(k)\to\mathcal{T}(k)^*(-w+2k)$, which is given by $\mathcal{S}(k):=((-1)^{2k}S',S'')$. Then, $\mathcal{S}$ is a Hermitian sesqui-linear duality of weight w, if and only if $\mathcal{S}(k)$ is a Hermitian sesqui-linear duality of weight $w+k$.

Let $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$ be an $\mathcal{R}_X$-triple with a Hermitian sesqui-linear duality $\mathcal{S}=(S',S'')$ of weight w. Later, we will be mainly interested in the case where $\mathcal{S}$ is an isomorphism, i.e., S' and S'' are isomorphisms. Then we have the $\mathcal{R}_X$-triple $\widetilde{\mathcal{T}}=(\mathcal{M}'',\mathcal{M}'',\widetilde{C})$ and the canonical isomorphism $(S',\mathrm{id}):\mathcal{T}\to\widetilde{\mathcal{T}}$, where $\widetilde{C}$ is given by $\widetilde{C}(x,\sigma^*y)=C(S'x,\sigma^*y)$ for local sections x and y of $\mathcal{M}''$.

For an $\mathcal{R}_X$-triple $\mathcal{T}$ of the form $(\mathcal{M}',\mathcal{M}',C)$, an isomorphism $(\mathrm{id},\mathrm{id}):\mathcal{T}\to\mathcal{T}^*$ is a Hermitian sesqui-linear duality of weight 0, if and only if the following equality holds for local sections x and y on appropriate open subsets:

$$C(x,\sigma^*y)=C^*(x,\sigma^*y):=\sigma^*\overline{C(y,\sigma^*x)}.$$

22.9.5. Push-forward. — Let $f:X\to Y$ be a morphism of complex manifolds. For an $\mathcal{R}_X$-triple $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$, the push-forward $f_\dagger\mathcal{T}\in D\big(\mathcal{R}_Y\text{-Triples}\big)$ is defined. The i-th cohomology is denoted by $f_\dagger^i\mathcal{T}$. We refer to [**73**] for details. Here, we just recall the description of $f_\dagger^i\mathcal{T}$ in the case $X=Y\times Z$. We use the notation in Subsection 22.2.4. We put $n=\dim Z$, and $\varepsilon(m)=(-1)^{m(m-1)/2}$ for $m\in\mathbb{Z}$. Then, the pairing $f_\dagger^i(C)$ of $f_\dagger^{-i}(\mathcal{M}')$ and $f_\dagger^i(\mathcal{M})$ is denoted as follows:

$$f_\dagger^i(C)\Big(\eta^{n-i}\cdot m',\sigma^*(\eta^{n+i}\cdot m'')\Big)=\frac{\varepsilon(n+i)}{(2\pi\sqrt{-1})^n}\int\eta^{n-i}\cdot\overline{\sigma^*(\eta^{n+i})}\cdot C(m',\sigma^*m'').$$

Here, $\eta^{n-i}\cdot m'$ and $\eta^{n+i}\cdot m''$ are sections of $\Omega_{\mathcal{Z}}^{n-i}\otimes\mathcal{M}'$ and $\Omega_{\mathcal{Z}}^{n+i}\otimes\mathcal{M}''$, respectively.

22.10. Specialization of $\mathcal{R}$-triples

We recall the specialization of sesqui-linear pairings introduced by Sabbah [**73**] (see also [**5**], [**6**]) which we refer to for more details and precision.

22.10.1. Specialization along a coordinate function

22.10.1.1. Preliminary I. — Let $\boldsymbol{C}_t$ be a complex line with a coordinate t. Let X_0 be an $(n-1)$-dimensional complex manifold. We put $X:=X_0\times\boldsymbol{C}_t$. We identify X_0 and $X_0\times\{0\}$. Let $\mathcal{M}'$ and $\mathcal{M}''$ be coherent $\mathcal{R}_X$-modules which are strictly specializable along t. Let C be a sesqui-linear pairing of $\mathcal{M}'$ and $\mathcal{M}''$. Let u be any element of

$\boldsymbol{R}\times\boldsymbol{C}$. We recall the construction due to Sabbah to obtain the specialization along t:

$$\widetilde{\psi}_{t,u}C : \big(\widetilde{\psi}_{t,u}\mathcal{M}'\otimes\sigma^*\widetilde{\psi}_{t,u}\mathcal{M}''\big)_{|\boldsymbol{S}\times X_0} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

For $\lambda_0\in\boldsymbol{S}$, let U be a small neighbourhood of λ_0, and $\boldsymbol{I}:=U\cap\boldsymbol{S}$. Let W_0 be an open subset of X_0. We put $W:=W_0\times\boldsymbol{C}_t$. Let m be a section of $\mathcal{M}'$ on $U\times W$, and μ be a section of $\mathcal{M}''$ on $\sigma(U)\times W$. Let us take any C^∞ $(n-1,n-1)$-form ϕ on W_0 whose support is compact. We also take a C^∞-function χ on $\boldsymbol{C}_t$ with the compact support such that $\chi=1$ around the origin $O\in\boldsymbol{C}_t$. By considering the push-forward for $\boldsymbol{S}\times X\to\boldsymbol{S}$, we obtain the following continuous function on $\boldsymbol{I}\times\big\{s\in\boldsymbol{C}\,\big|\,2\operatorname{Re}(s)+|k|>R_0\big\}$, which is holomorphic with respect to s:

$$\mathcal{I}^{(k)}_{C(m,\overline{\mu}),\phi}(s):=\begin{cases}\big\langle C(m,\overline{\mu}),\ |t|^{2s}t^k\cdot\chi(t)\cdot\phi\wedge\frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t}\big\rangle & (k\geqslant 0)\\[2ex] \big\langle C(m,\overline{\mu}),\ |t|^{2s}\bar{t}^{-k}\cdot\chi(t)\cdot\phi\wedge\frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t}\big\rangle & (k\leqslant 0)\end{cases}\tag{434}$$

Here, R_0 denotes some real number depending only on m and μ. (See Corollary 22.8.4.) The following lemma can be shown by the argument in [**73**] or the proof of Lemma 14.72 and Lemma 14.73 of [**67**].

Lemma 22.10.1. — *Let $k\geqslant 0$. We have the following formula:*

$$\lambda\cdot\big(s+k+\lambda^{-1}\mathfrak{e}(\lambda,u)\big)\cdot\mathcal{I}^{(k)}_{C(m,\overline{\mu}),\phi}(s)=\mathcal{I}^{(k)}_{C(m',\overline{\mu}),\phi}(s)+F.\tag{435}$$

Here we put $m':=(-\eth_t t+\mathfrak{e}(\lambda,u))\cdot m$, and F is a continuous function on $\boldsymbol{I}\times\boldsymbol{C}$ which is holomorphic with respect to the variable s.

We also have the following equality:

$$-\lambda^{-1}\cdot\Big(s+k+\frac{\mathfrak{e}(\lambda,u)}{\lambda}\Big)\cdot\mathcal{I}^{(-k)}_{C(m,\overline{\mu}),\phi}(s)=\mathcal{I}^{(-k)}_{C(m,\overline{\mu}'),\phi}+F.\tag{436}$$

Here we put $\mu'=\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)\cdot\mu$, and F is as above. □

22.10.1.2. Preliminary II. — Let m be a section of $V^{(\lambda_0)}_c\mathcal{M}'$ such that $0\neq\pi_c(m)\in\psi^{(\lambda_0)}_{t,u_0}\mathcal{M}'$ via the projection $\pi_c:V^{(\lambda_0)}_c\mathcal{M}'\to\operatorname{Gr}^{V^{(\lambda_0)}}_c\mathcal{M}'$ for some $u_0\in\boldsymbol{R}\times\boldsymbol{C}$ such that $\mathfrak{p}(\lambda_0,u_0)=c$. Let b_m be a polynomial such that (i) $b_m(-\eth_t t)m\in V_{-1}\mathcal{R}\cdot m$, (ii) it is of the following form:

$$b_m(x)=\big(x+\mathfrak{e}(\lambda,u_0)\big)^{\nu(u_0)}\cdot\prod_{u\in S_0}\big(x+\mathfrak{e}(\lambda,u)\big)^{\nu(u)}.$$

Here S_0 denotes a finite subset of $\boldsymbol{R}\times\boldsymbol{C}$ such that $\mathfrak{p}(\lambda_0,u)<c$ for any $u\in S_0$. For any positive integer M, we put

$$B^{(M)}_m(x):=\prod_{\nu=0}^{M-1}b_m(x+\nu\lambda).$$

By construction, there exists a finite subset $S_1(M)\subset\boldsymbol{R}\times\boldsymbol{C}$ such that the following holds:

- $B^{(M)}_m(x)=\prod_{u\in S_1(M)}\big(x+\mathfrak{e}(\lambda,u)\big)^{\nu'(u)}$.

• For any $u \in S_1(M)$, we have $\mathfrak{p}(\lambda_0, u) \leqslant c$. If $\mathfrak{p}(\lambda_0, u) = c$, then $u = u_0$. Moreover, ν' is bounded on $\bigcup_M S_1(M)$.

Lemma 22.10.2. — *We have the following equality:*

$$\Big(\prod_{u \in S_1(M)} \lambda^{\nu'(u)}\big(s + k + \lambda^{-1}\mathfrak{e}(\lambda, u)\big)^{\nu'(u)}\Big) \cdot \mathcal{I}^{(k)}_{C(m,\overline{\mu}),\phi}(s) = \mathcal{I}^{(k)}_{C(m',\overline{\mu}),\phi} + F. \tag{437}$$

Here $m' = B_m^{(M)}(-\eth_t t) \cdot m$, and F has the property as in Lemma 22.10.1. *The first term in the right-hand side is holomorphic with respect to the variable s on the half plane $\{s \in \boldsymbol{C} \,|\, 2\operatorname{Re}(s) + k > R_0 - M\}$.*

Proof. — We obtain the equality (437) by using Lemma 22.10.1 inductively. By the construction of $B_m^{(M)}$, we have $B_m^{(M)}(-\eth_t t) \cdot m = t^M P \cdot m$ for some $P \in V_0\mathcal{R}_X$. Hence, $\mathcal{I}^{(k)}_{C(m',\overline{\mu}),\phi}$ in (437) is given on $\boldsymbol{I} \times \{s \in \boldsymbol{C} \,|\, 2\operatorname{Re}(s) + k > R_0 - M\}$. □

Then, we obtain the following lemma.

Lemma 22.10.3. — *There exist a discrete subset S_2 of $\boldsymbol{R} \times \boldsymbol{C}$ and a number ν such that the following holds:*

- $\mathcal{I}^{(k)}_{C(m,\overline{\mu}),\phi} \cdot \prod_{u \in S_2}\big(s + k + \lambda^{-1} \cdot \mathfrak{e}(\lambda, u)\big)^{\nu}$ *is continuous on $\boldsymbol{I} \times \boldsymbol{C}$, and it is holomorphic with respect to s.*
- *For any element $u \in S_2$, we have $\mathfrak{p}(\lambda_0, u) \leqslant c$. If $\mathfrak{p}(\lambda_0, u) = c$, then $u = u_0$.* □

Note $\sigma(\lambda_0) = -\lambda_0$ for $\lambda_0 \in \boldsymbol{S}$. Let μ be an element of $V_d^{(-\lambda_0)}\mathcal{M}''$ such that $0 \neq \pi_d(\mu) \in \psi^{(-\lambda_0)}_{t,u_1}\mathcal{M}''$ via the projection $V_d^{(-\lambda_0)}\mathcal{M}'' \to \operatorname{Gr}_d^{V^{(-\lambda_0)}}\mathcal{M}''$, where $d = \mathfrak{p}(-\lambda_0, u_1)$. We obtain the following lemma in a similar way.

Lemma 22.10.4. — *There exist a discrete subset $S_3 \subset \boldsymbol{R} \times \boldsymbol{C}$ and a number ν with the following properties:*

- $\mathcal{I}^{(-k)}_{C(m,\overline{\mu}),\phi} \cdot \prod_{u \in S_3}\big(s + k + \lambda^{-1}\mathfrak{e}(\lambda, u)\big)^{\nu}$ *is given on $\boldsymbol{I} \times \boldsymbol{C}$, and it is holomorphic with respect to s.*
- *For any $u \in S_3$, we have $\mathfrak{p}(-\lambda_0, u) \leqslant d$. If $\mathfrak{p}(-\lambda_0, u) = d$, then $u = u_0$.* □

22.10.1.3. Construction of the specialization $\psi_{t,u}C$. — Let $[m]$ be a section of $\psi^{(\lambda_0)}_{t,u_0}\mathcal{M}'$ on $U \times W_0$, and m be a section of $\mathcal{M}'$ on $U \times W_0 \times \boldsymbol{C}_t$ such that $\pi_c(m) = [m]$, where we put $c := \mathfrak{p}(\lambda_0, u_0)$, and π_c denotes the projection $V_c^{(\lambda_0)}\mathcal{M}' \to \operatorname{Gr}_c^{V^{(\lambda_0)}}\mathcal{M}'$. Let $[\mu]$ be a section of $\psi^{(-\lambda_0)}_{t,u_0}\mathcal{M}''$ on $\sigma(U) \times W_0$, and μ be a section of $\mathcal{M}''$ on $\sigma(U) \times W_0 \times \boldsymbol{C}_t$ such that $\pi_d(\mu) = [\mu]$, where we put $d = \mathfrak{p}\big(-\lambda_0, u_0\big)$, and π_d denotes the projection $V_d^{(-\lambda_0)}\mathcal{M}'' \to \operatorname{Gr}_d^{V^{(-\lambda_0)}}\mathcal{M}''$.

We put $\mathcal{I}_{C(m,\overline{\mu}),\phi}(s) := \mathcal{I}^{(0)}_{C(m,\overline{\mu}),\phi}(s)$. Recall that we have a discrete subset S of $\boldsymbol{R} \times \boldsymbol{C}$ such that $\mathcal{I}_{C(m,\overline{\mu}),\phi}(s)\prod_{u \in S}\big(s + \lambda^{-1}\,\mathfrak{e}(\lambda, u)\big)^{\nu}$ is given on $\boldsymbol{I} \times \boldsymbol{C}$. For any $u \in S$, we have $\mathfrak{p}(\lambda_0, u) \leqslant \mathfrak{p}(\lambda_0, u_0)$ and $\mathfrak{p}(-\lambda_0, u) \leqslant \mathfrak{p}(-\lambda_0, u_0)$. If one of the equalities holds,

$u = u_0$. Hence, we have $\mathfrak{e}(\lambda_0, u) \neq \mathfrak{e}(\lambda_0, u_0)$ for any $u \in S \setminus \{u_0\}$. (See Lemma 14.80 in [**67**], for example.)

We would like to define

$$\big\langle \psi_{t,u_0}^{(\lambda_0)} C\big([m], [\overline{\mu}]\big), \phi \big\rangle := \operatorname*{Res}_{s+\lambda^{-1}\mathfrak{e}(\lambda,u_0)} \big(\mathcal{I}_{C(m,\overline{\mu}),\phi}(s)\big). \tag{438}$$

The right-hand side of (438) gives a continuous function on $\boldsymbol{I}$. The well definedness of (438) can be checked as in [**73**] or Lemma 14.82 of [**67**]. Thus, we obtain the local section $\psi_{t,u_0}^{(\lambda_0)} C\big([m], [\overline{\mu}]\big)$ of $\mathfrak{Db}_{\boldsymbol{I}\times X_0/\boldsymbol{I}}$. By varying $\lambda_0 \in \boldsymbol{S}$ and gluing them, we obtain

$$\psi_{t,u_0} C : \psi_{t,u_0}\mathcal{M}' \otimes \sigma^*\psi_{t,u_0}\mathcal{M}'' \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

We can easily check that $\psi_{t,u_0} C$ gives a Hermitian sesqui-linear pairing of $\psi_{t,u_0}\mathcal{M}'$ and $\psi_{t,u_0}\mathcal{M}''$.

We put $N := -\eth_t t + \mathfrak{e}(\lambda, u_0)$, which induces the nilpotent map on $\psi_{t,u_0}\mathcal{M}'$ and $\psi_{t,u_0}\mathcal{M}''$. As in [**73**] or Lemma 14.84 of [**67**], we have the following equality:

$$\psi_{t,u_0} C\big(N[m], [\overline{\mu}]\big) = (\sqrt{-1}\lambda)^2 \cdot \psi_{t,u_0} C\big([m], [\overline{N\mu}]\big).$$

22.10.1.4. The induced pairing $\widetilde{\psi}_{t,u} C$. — We construct the induced pairing:

$$\widetilde{\psi}_{t,u} C : \widetilde{\psi}_{t,u}\mathcal{M}' \otimes \sigma^*\widetilde{\psi}_{t,u}\mathcal{M}'' \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

For local sections m and μ of $\psi_{t,u}^{(\lambda_0)}(\mathcal{M}')$ and $\psi_{t,u}^{(-\lambda_0)}(\mathcal{M}'')$, we have the following equality as in Lemma 14.85:

$$\psi_{t,u-\boldsymbol{\delta}_0}^{(\lambda_0)} C\big([t\cdot m], \overline{[t\cdot\mu]}\big) = \psi_{t,u}^{(\lambda_0)} C\big([m], \overline{[\mu]}\big). \tag{439}$$

Here, we put $\boldsymbol{\delta}_0 = (1, 0) \in \boldsymbol{R}\times\boldsymbol{C}$.

Let m be a section of $\widetilde{\psi}_{t,u}^{(\lambda_0)}(\mathcal{M}')$, and μ be a section of $\widetilde{\psi}_{t,u}^{(-\lambda_0)}(\mathcal{M}'')$. We pick a sufficiently large integer N, $m_1 \in \psi_{t,u-N\boldsymbol{\delta}_0}^{(\lambda_0)}(\mathcal{M}')$ and $\mu_1 \in \psi_{t,u-N\boldsymbol{\delta}_0}^{(-\lambda_0)}(\mathcal{M}'')$ corresponding to m and μ respectively. Then, the pairing $\widetilde{\psi}_{t,u}^{(\lambda_0)} C(m, \overline{\mu})$ is defined to be $\psi_{t,u-N\boldsymbol{\delta}_0}^{(\lambda_0)} C(m_1, \overline{\mu_1})$. By varying λ_0 and gluing $\widetilde{\psi}_{t,u}^{(\lambda_0)} C$, we obtain $\widetilde{\psi}_{t,u} C$.

22.10.1.5. The nearby cycle functor of a strictly specializable $\mathcal{R}$-triple. — Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be a coherent $\mathcal{R}_X$-triple. We say that $\mathcal{T}$ is strictly specializable along t, if $\mathcal{M}'$ and $\mathcal{M}''$ are strictly specializable along t.

Let $\mathcal{T}$ be an $\mathcal{R}_X$-triple which is strictly specializable along t. For any $u \in \boldsymbol{R}\times\boldsymbol{C}$, we obtain the following induced sesqui-linear pairings:

$$\psi_{t,u} C : \big(\psi_{t,u}(\mathcal{M}') \otimes \sigma^*\psi_{t,u}(\mathcal{M}'')\big)_{|\boldsymbol{S}\times X_0} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

Thus, we obtain the $\mathcal{R}_{X_0}$-triple $\psi_{t,u}(\mathcal{T}) = \big(\psi_{t,u}\mathcal{M}', \psi_{t,u}\mathcal{M}'', \psi_{t,u} C\big)$. We also have the following modified pairing:

$$\widetilde{\psi}_{t,u} C : \big(\widetilde{\psi}_{t,u}(\mathcal{M}') \otimes \sigma^*\widetilde{\psi}_{t,u}(\mathcal{M}'')\big)_{|\boldsymbol{S}\times X_0} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

Thus, we obtain the $\mathcal{R}_{X_0}$-triple $\widetilde{\psi}_{t,u}(\mathcal{T}) := \big(\widetilde{\psi}_{t,u}\mathcal{M}', \widetilde{\psi}_{t,u}\mathcal{M}'', \widetilde{\psi}_{t,u} C\big)$.

We have the following relations:

$$\widetilde{\psi}_{t,u}C\big(N[m],[\overline{\mu}]\big) = (\sqrt{-1}\lambda)^2 \cdot \widetilde{\psi}_{t,u}C\big([m],\overline{N}[\overline{\mu}]\big),$$
$$\psi_{t,u}C\big(N[m],[\overline{\mu}]\big) = (\sqrt{-1}\lambda)^2 \cdot \psi_{t,u}C\big([m],\overline{N}[\overline{\mu}]\big).$$

Here $N := -\eth_t t + \mathfrak{e}(\lambda,u)$. Hence, we have the following morphism

$$\mathcal{N} := (-\sqrt{-1}N, \sqrt{-1}N) : \widetilde{\psi}_{t,u}\mathcal{T} \longrightarrow \widetilde{\psi}_{t,u}\mathcal{T} \otimes \mathbb{T}^S(-1).$$

It induces the weight filtrations W on $\widetilde{\psi}_{t,u}\mathcal{T}$ or $\psi_{t,u}\mathcal{T}$. The primitive part of Gr is denoted by $P\operatorname{Gr}_h^W \widetilde{\psi}_{t,u}\mathcal{T}$.

22.10.1.6. Vanishing cycle functor. — The specialization $\psi_{t,0}C = \widetilde{\psi}_{t,0}C$ is not appropriate for our use. We recall the construction of the Hermitian sesqui-linear pairing $\phi_{t,0}C$ of $\psi_{t,0}\mathcal{M}'$ and $\psi_{t,0}\mathcal{M}''$ only in the simple case where $\mathcal{M}'$ and $\mathcal{M}''$ are strictly S-decomposable along t, by following the older version of [**73**]. See [**73**] for a more general case.

Since we have assumed that $\mathcal{M}'$ is strictly S-decomposable along t, we have the decomposition $\mathcal{M}' = \mathcal{M}'_1 \oplus \mathcal{M}'_2$ as in the claim (3) of Proposition 22.3.15. Similarly we have the decomposition $\mathcal{M}'' = \mathcal{M}''_1 \oplus \mathcal{M}''_2$. We also have the decomposition of the Hermitian sesqui-linear pairing $C = C_1 \oplus C_2$, where C_i are pairings of $\mathcal{M}'_i$ and $\mathcal{M}''_i$. (See Lemma 22.9.1.) Since the supports of $\mathcal{M}'_2$ and $\mathcal{M}''_2$ are contained in X_0, we have $\psi_{t,0}\mathcal{M}'_2 = \mathcal{M}'_2$ and $\psi_{t,0}\mathcal{M}''_2 = \mathcal{M}''_2$. Therefore, we put $\phi_{t,0}C_2 := C_2$. We only have to define $\phi_{t,0}C_1$. Hence, we may assume $\psi_{t,0} = \operatorname{Im}\operatorname{can}$ for $\mathcal{M}'$ and $\mathcal{M}''$ from the beginning. Recall that can and var are induced by the left action of $-\eth_t$ and t. For $x \in \psi_{t,0}\mathcal{M}'$ and $y \in \psi_{t,0}\mathcal{M}''$, we define

$$\phi_{t,0}C(x,y) := (\sqrt{-1}\lambda)^{-1}\psi_{t,-\boldsymbol{\delta}_0}C(\operatorname{var} x, y_{-1}) = (\sqrt{-1}\lambda)\cdot\psi_{t,-\boldsymbol{\delta}_0}C(x_{-1}, \operatorname{var} y),$$

where $x_{-1} \in \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M}'$ and $y_{-1} \in \psi_{t,-\boldsymbol{\delta}_0}\mathcal{M}''$ are sections such that $x = -\sqrt{-1}\operatorname{can} x_{-1}$ and $y = \sqrt{-1}\operatorname{can} y_{-1}$.

Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be an $\mathcal{R}_X$-triple. We say that it is strictly S-decomposable along t, if $\mathcal{M}'$ and $\mathcal{M}''$ are strictly S-decomposable along t. If it is strictly S-decomposable, the vanishing cycle $\phi_{t,0}\mathcal{T}$ is defined to be $(\psi_{t,0}\mathcal{M}', \psi_{t,0}\mathcal{M}'', \phi_{t,0}C)$. By construction, we have the following morphisms of $\mathcal{R}_{X_0}$-triples:

$$\operatorname{Can} := (\operatorname{var}, \sqrt{-1}\operatorname{can}) : \psi_{t,-\boldsymbol{\delta}_0}\mathcal{T} \longrightarrow \phi_{t,0}\mathcal{T}(-1/2),$$
$$\operatorname{Var} := (-\sqrt{-1}\operatorname{can}, \operatorname{var}) : \phi_{t,0}\mathcal{T}(1/2) \longrightarrow \psi_{t,-\boldsymbol{\delta}_0}\mathcal{T}.$$

22.10.2. The general case. — Let Y be a complex manifold, and let $\mathcal{T}$ be a coherent $\mathcal{R}_Y$-triple. Let W be an open subset of Y, and let f be a holomorphic function on W. Let $\iota : W \to W \times \boldsymbol{C}_t$ denote the graph embedding. We obtain an $\mathcal{R}_{W\times \boldsymbol{C}_t}$-triple $\iota_\dagger\big(\mathcal{T}_{|\boldsymbol{C}_\lambda\times W}\big)$ on $\boldsymbol{C}_\lambda \times (W \times \boldsymbol{C}_t)$. We say that $\mathcal{T}$ is strictly specializable (S-decomposable) along f, if $\iota_\dagger\big(\mathcal{T}_{|\boldsymbol{C}_\lambda\times U}\big)$ is strictly specializable (S-decomposable) along t. If $\mathcal{T}$ is strictly specializable, $\widetilde{\psi}_{f,u}\mathcal{T}$ (resp. $\psi_{f,u}\mathcal{T}$) are defined to be $\widetilde{\psi}_{t,u}\big(\iota_\dagger\mathcal{T}\big)$

(resp. $\psi_{t,u}\iota_{\dagger}\mathcal{T}$) for any $u \in \boldsymbol{R} \times \boldsymbol{C}$. If $\mathcal{T}$ is strictly S-decomposable along f, $\phi_{f,0}(\mathcal{T})$ is defined to be $\phi_{t,0}(\iota_{\dagger}\mathcal{T})$.

We can show the following lemma by using the argument due to Sabbah. (See Theorem 3.3.15 and Corollary 3.6.35 in [**73**]. See also [**77**] for the original work due to Saito.)

Lemma 22.10.5. — *Let $F : X \to Y$ be a proper morphism of complex manifolds. Let g be any holomorphic function on Y. We put $\widetilde{g} := g \circ F$. Let $\mathcal{T}$ be a coherent $\mathcal{R}_X$-triple such that (i) $\mathcal{T}$ is strictly specializable along $\widetilde{g}$ and good with respect to F, (ii) $F_{\dagger}^{j}\psi_{\widetilde{g},u}(\mathcal{T})$ are strict. Then, $F_{\dagger}^{j}\mathcal{T}$ are also strictly specializable along g. Moreover, we have natural isomorphisms*

$$\widetilde{\psi}_{g,u}F_{\dagger}^{j}\mathcal{T} \simeq F_{\dagger}^{j}\widetilde{\psi}_{\widetilde{g},u}\mathcal{T}, \quad \psi_{g,u}F_{\dagger}^{j}\mathcal{T} \simeq F_{\dagger}^{j}\psi_{\widetilde{g},u}\mathcal{T}$$

for any $u \in \boldsymbol{R} \times \boldsymbol{C}$. □

Lemma 22.10.6. — *Let X, Y, F, g, and $\mathcal{T}$ be as in Lemma 22.10.5. Assume moreover that $\mathcal{T}$ and $F_{\dagger}^{j}\mathcal{T}$ ($j \in \mathbb{Z}$) are strictly S-decomposable along $\widetilde{g}$ and g, respectively. Then, we have a natural isomorphism $F_{\dagger}^{j}\phi_{\widetilde{g},0}\mathcal{T} \simeq \phi_{g,0}F_{\dagger}^{j}\mathcal{T}$ for any j.* □

The lemma was proved by Sabbah without the additional assumption of strict S-decomposability. (Theorem 3.3.15 and Corollary 3.6.35 of [**73**].) We will later give a direct argument in this restricted case (Section 22.10.5) just for our understanding.

22.10.3. Uniqueness. — We recall a rather general remark on the uniqueness of the prolongation of Hermitian sesqui-linear pairings (Proposition 14.97 of [**67**]). Let X be a complex manifold, and let f be a holomorphic function on X. We put $X' := X - f^{-1}(0)$. Let $\mathcal{M}'$ and $\mathcal{M}''$ be $\mathcal{R}_X$-modules such that (i) they are strictly specializable along f, (ii) the morphism can : $\psi_{f,-\boldsymbol{\delta}_0}(\mathcal{M}') \to \psi_{f,0}(\mathcal{M}')$ is surjective. Let C_a ($a = 1, 2$) be sesqui-linear pairings of $\mathcal{M}'$ and $\mathcal{M}''$.

Proposition 22.10.7. — *If $C_1 = C_2$ on $\boldsymbol{S} \times X'$, we have $C_1 = C_2$ on $\boldsymbol{S} \times X$.*

Proof. — Let $i_f : X \to X \times \boldsymbol{C}$ denote the graph embedding. We only have to show $i_{f\dagger}C_1 = i_{f\dagger}C_2$. Thus, we may assume that X is of the form $X_0 \times \boldsymbol{C}$, and that $f = t$ is the coordinate of $\boldsymbol{C}$ from the beginning.

For $\lambda_0 \in \boldsymbol{S}$, let U be a small neighbourhood of λ_0 in $\boldsymbol{C}_{\lambda}$, and $\boldsymbol{I} := \boldsymbol{S} \cap U$. Let W_0 be an open subset of X_0, and W be the product of W_0 and an open subset W_1 of $\boldsymbol{C}_t$. Let m' and m'' be sections of $\mathcal{M}'$ and $\mathcal{M}''$ respectively on $U \times W$ and $\sigma(U) \times W$ respectively. We put $A(m', \overline{m''}) := C_1(m', \overline{m''}) - C_2(m', \overline{m''})$.

As a preparation, let us consider the case where λ_0 is generic.

Lemma 22.10.8. — *Assume that λ_0 is generic with respect to the set $\mathcal{KMS}(\mathcal{M}', t) \cup \mathcal{KMS}(\mathcal{M}'', t) \cup \{(0,0)\}$. Then we have $A(m', \overline{m''}) = 0$.*

Proof. — We use an argument due to Sabbah (Proposition 3.7.6 in [**73**]). Since the support of $A(m',\overline{m''})$ is contained in $\boldsymbol{I}\times W_0$, we have the following expression:

$$A(m',\overline{m''})=\sum_{a+b\leqslant p}\eta_{a,b}\cdot\partial_t^a\cdot\overline{\partial}_t^b\cdot\delta_{\boldsymbol{I}\times W_0}.$$

Here $\eta_{a,b}$ are sections of $\mathfrak{Db}_{\boldsymbol{I}\times W_0/\boldsymbol{I}}$, and $\delta_{\boldsymbol{I}\times W_0}$ denote the delta distribution for $\boldsymbol{I}\times W_0$ in $\boldsymbol{I}\times W$. We only have to show $\eta_{a,b}=0$ for any a and b.

Let us consider the case where $m'\in V^{(\lambda_0)}_{<0}(\mathcal{M}')$. For any $p\in\mathbb{Z}_{\geqslant 0}$, we have a finite subset $S(p)\subset\mathcal{KMS}(\mathcal{M}',t)$ such that the following holds:

- $\mathfrak{p}(\lambda_0,u)<0$ for any element $u\in S(p)$.
- We put $B_p(x):=\prod_{u\in S(p)}\big(x+\mathfrak{e}(\lambda,u)\big)$. Then there is a section P_p of $V_0\mathcal{R}_X$ such that $B_p(-\eth_t t)\cdot m'=P_p\cdot t^{p+1}\cdot m'$.

If p is sufficiently large, we have the vanishing

$$B_p(-\eth_t t)A(m',m'')=P_pt^{p+1}A(m',m'')=0.$$

Note the following equality:

$$\big(-\eth_t t+\mathfrak{e}(\lambda,u)\big)\cdot\eth_t^a\cdot\delta_{W_0}=\big(a\lambda+\mathfrak{e}(\lambda,u)\big)\cdot\eth_t^a\cdot\delta_{W_0}=\mathfrak{e}(\lambda,u-a\cdot\boldsymbol{\delta}_0)\cdot\eth_t^a\cdot\delta_{W_0}.$$

Note $\mathfrak{p}(\lambda_0,u-a\cdot\boldsymbol{\delta}_0)<0$. Since λ_0 is assumed to be generic, we have $a\lambda_0+\mathfrak{e}(\lambda_0,u)\neq 0$. Thus we obtain $\eta_{a,b}=0$ in the case $m'\in V^{(\lambda_0)}_{<0}(\mathcal{M}')$. Since $\mathcal{M}'$ is generated by $V^{(\lambda_0)}_{<0}\mathcal{M}'$ around λ_0, the general case can be reduced to the case $m'\in V^{(\lambda_0)}_{<0}(\mathcal{M}')$. □

Let us return to the proof of Proposition 22.10.7. Let ϕ be any test function on W. By taking the push-forward via the projection $p:\boldsymbol{I}\times W\to\boldsymbol{I}$, we obtain the distribution $F:=p_*\big(\phi\cdot A(m',\overline{m''})\big)$, which gives a continuous function on $\boldsymbol{I}$. Due to Lemma 22.10.8, F vanishes on neighbourhoods of any generic $\lambda\in\boldsymbol{I}$. Therefore we obtain the vanishing of F on $\boldsymbol{I}$. It means $C_1=C_2$ on $\boldsymbol{I}\times W$. □

22.10.4. The evaluation of distributions and the value of holomorphic functions. — Let X and X_0 be as in Subsection 22.10.1. For any $M\in\boldsymbol{R}$, we set $\mathcal{U}(M):=\big\{s\in\boldsymbol{C}\,\big|\,\mathrm{Re}(s)>M\big\}$. Let $\mathcal{M}'$ and $\mathcal{M}''$ be strictly specializable $\mathcal{R}_X$-modules. Let C be a Hermitian sesqui-linear pairing of $\mathcal{M}'$ and $\mathcal{M}''$. Let $\lambda_0\in\boldsymbol{S}$, and let $U(\lambda_0)$ be a small neighbourhood of λ_0 in $\boldsymbol{C}$. We put $\boldsymbol{I}(\lambda_0):=\boldsymbol{S}\cap U(\lambda_0)$. Let $m\in V^{(\lambda_0)}_{<0}(\mathcal{M}')\otimes C^{\infty}$ and $\mu\in V^{(-\lambda_0)}_{0}(\mathcal{M}'')\otimes C^{\infty}$ be C^{∞}-sections on $U(\lambda_0)\times X$ and $\sigma(U(\lambda_0))\times X$, respectively. There exists a large M, depending on m and μ such that (i) the following pairing makes sense for any $s\in\mathcal{U}(M-k)$, $k\in\mathbb{Z}_{\geqslant 0}$ and a C^{∞} (n,n)-form φ on X with compact support

$$\mathcal{I}^{(k)}_{C(m,\sigma^*\mu),\varphi}(s):=\Big\langle C(m,\sigma^*\mu),\,|t|^{2s}\cdot t^k\cdot\varphi\Big\rangle,$$

(ii) it gives a continuous function on $\boldsymbol{I}(\lambda_0)\times\mathcal{U}(M-k)$, which is holomorphic with respect to s. We put $\mathcal{I}_{C(m,\sigma^*\mu),\varphi}:=\mathcal{I}^{(0)}_{C(m,\sigma^*\mu),\varphi}$.

Lemma 22.10.9. — *$\mathcal{I}_{C(m,\sigma^*\mu),\varphi}(s)$ is naturally extended to a continuous function on $\boldsymbol{I}(\lambda_0) \times \mathcal{U}(-\delta)$ for some $\delta > 0$, and is holomorphic with respect to s. And we have $\mathcal{I}_{C(m,\sigma^*\mu),\varphi}(0) = \Big\langle C\big(m, \sigma^*\mu\big),\, \varphi\Big\rangle$.*

Proof. — We may and will assume that m and μ are holomorphic sections. Fix $R > 0$. Using an argument in [**73**] (or the proof of Lemma 14.76 and Lemma 14.77 in [**67**]), we can show that there exist a large number $N > 0$ and a finite subset $S \subset \boldsymbol{R} \times \boldsymbol{C}$ with the following properties:

- $\prod_{u \in S}\big(s + \lambda^{-1}\mathfrak{e}(\lambda, u)\big)^N \cdot \mathcal{I}_{C(m,\sigma^*\mu),\varphi}(s)$ is naturally extended to a continuous function on $\boldsymbol{I}(\lambda_0) \times \mathcal{U}(-R)$, which is holomorphic with respect to s.
- $\mathfrak{p}(\lambda_0, u) < 0$ and $\mathfrak{p}(-\lambda_0, u) \leqslant 0$ for any $u \in S$.

Note the general formula $\mathfrak{p}(\lambda, u) + \mathfrak{p}(\sigma(\lambda), u) = -2\operatorname{Re}\big(\mathfrak{e}(\lambda, u)/\lambda\big)$. Then, the first claim follows.

Let us show the second claim. By the above argument, we know that $\mathcal{I}^{(k)}_{C(m,\sigma^*\mu),\varphi}(s)$ is a continuous function on $\boldsymbol{I}(\lambda_0) \times \mathcal{U}(-\delta)$, which is holomorphic with respect to s. We prepare the following lemma.

Lemma 22.10.10. — *There exists a $k_0 \in \mathbb{Z}_{>0}$ such that the following holds for any $m \in V^{(\lambda_0)}_{<0}$, $\mu \in V^{(-\lambda_0)}_0$, any C^∞ (n,n)-form φ with compact support, and any $k \geqslant k_0$:*

$$\mathcal{I}^{(k)}_{C(m,\sigma^*\mu),\varphi}(0) = \big\langle C(m, \sigma^*\mu),\, \varphi t^k\big\rangle.$$

Proof. — Let m_i and μ_j be finite generators of $V^{(\lambda_0)}_{<0}\mathcal{M}'$ and $V^{(-\lambda_0)}_0\mathcal{M}''$ over $V_0\mathcal{R}$. We can take some k_0 such that the following holds for any m_i, μ_j, φ, and $k \geqslant k_0$:

$$\mathcal{I}^{(k)}_{C(m_i,\sigma^*\mu_j),\varphi}(0) = \big\langle C(m_i, \sigma^*\mu_j),\, \varphi t^k\big\rangle.$$

For any $m = \sum a_i \cdot m_i$ and $\mu = \sum b_j \cdot \mu_j$, we have

$$\big\langle C\big(m, \sigma^*\mu\big),\, \varphi|t|^{2s}t^k\big\rangle = \sum\Big\langle C\big(m_i, \sigma^*\mu_j\big),\, a_i^*\overline{\sigma^*(b_j^*)}\big(\varphi|t|^{2s}t^k\big)\Big\rangle.$$

Here, $a^* = \sum(-1)^{|J|}\eth_J \cdot a_J$ for $a = \sum a_J \cdot \eth^J$. Note $a_i^*\overline{\sigma^*(b_j^*)}(\varphi|t|^{2s}t^k)$ is of the form $|t|^{2s}t^kG_{i,j}$, where $G_{i,j}$ are C^∞ (n,n)-forms with compact supports. Then, the claim of Lemma 22.10.10 follows. □

By a descending induction on $k \geqslant 0$, let us show the following equality for any $m \in V^{(\lambda_0)}_{<0}$, $\mu \in V^{(-\lambda_0)}_0$ and any C^∞ (n,n)-form φ with compact supports:

$$\mathcal{I}^{(k)}_{C(m,\sigma^*\mu),\varphi}(0) = \big\langle C(m, \sigma^*\mu),\, \varphi t^k\big\rangle.$$

Assume that it holds for $k+1$. Let $m' := \big(-\eth_t t + \mathfrak{e}(\lambda, u)\big)m$. Then, we have

$$\begin{aligned}(440)\quad \big\langle C(m', \sigma^*\mu),\, \varphi|t|^{2s}t^k\big\rangle &= \big\langle C(m, \sigma^*\mu),\, (t\bar{\eth}_t + \mathfrak{e}(\lambda, u))(\varphi|t|^{2s}t^k)\big\rangle \\ &= \big(\lambda(s+k) + \mathfrak{e}(\lambda, u)\big)\cdot\big\langle C(m, \sigma^*\mu),\, \varphi|t|^{2s}t^k\big\rangle + \big\langle C(m, \sigma^*\mu),\, \bar{\eth}_t\varphi \cdot |t|^{2s}t^{k+1}\big\rangle.\end{aligned}$$

By a similar calculation, we obtain

$$(441)\quad \big\langle C(m',\sigma^*\mu),\varphi t^k\big\rangle = \big(\lambda k+\mathfrak{e}(\lambda,u)\big)\big\langle C(m,\sigma^*\mu),\varphi t^k\big\rangle + \big\langle C(m,\sigma^*\mu),\eth_t\varphi\cdot t^{k+1}\big\rangle.$$

We take a polynomial b_m such that $b_m(-\eth_t t)m = t\cdot P(-\eth_t t)m \in V_{-1}\mathcal{R}\cdot m$. It is of the following form:

$$b_m(x) = \prod_{i=1}^{M}\big(x+\mathfrak{e}(\lambda,u_i)\big).$$

Here we have $\mathfrak{p}(\lambda_0,u_i) < 0$ for each u_i. We put $e_i := \mathfrak{e}(\lambda,u_i)$, for simplicity of description. By using (440) inductively, we obtain the following formula:

(442)

$$\Big\langle C\big(P(-\eth_t t)m,\sigma^*\mu\big),\ \varphi|t|^{2s}t^{k+1}\Big\rangle = \prod_{i=1}^{M}\big(\lambda(s+k)+e_i\big)\cdot\Big\langle C(m,\sigma^*\mu),\ \varphi|t|^{2s}t^k\Big\rangle$$
$$+\sum_{j=0}^{M-1}\prod_{i=j+1}^{M}\big(\lambda(s+k)+e_i\big)\cdot\Big\langle C\Big(\prod_{i=1}^{j}(-\eth_t t+e_i)m,\sigma^*\mu\Big),\ \eth_t\varphi\cdot|t|^{2s}t^{k+1}\Big\rangle.$$

We also have the following formula from (441):

$$(443)\quad \Big\langle C\big(P(-\eth_t t)m,\sigma^*\mu\big),\ \varphi\cdot t^{k+1}\Big\rangle = \Big\langle C(m,\sigma^*\mu),\varphi t^k\Big\rangle\cdot\prod_{i=1}^{M}(\lambda k+e_i)$$
$$+\sum_{j=0}^{M-1}\prod_{i=j+1}^{M}(\lambda k+e_i)\cdot\Big\langle C\Big(\prod_{i=1}^{j}(-\eth_t t+e_i)m,\sigma^*\mu\Big),\ \eth_t\varphi\cdot t^{k+1}\Big\rangle.$$

Then, we obtain $\mathcal{I}^{(k)}_{C(m,\sigma^*\mu),\varphi}(0) = \big\langle C(m,\sigma^*\mu),\ \varphi t^k\big\rangle$ from (442) and (443). Thus, the induction can proceed, and the proof of Lemma 22.10.9 is finished. □

22.10.5. Proof of Lemma 22.10.6. — Under the isomorphism $\psi_{t,0}F_\dagger^j\mathcal{M}_i \simeq F_\dagger^j\psi_{t,0}(\mathcal{M}_i)$ for any j, we would like to show the equality $\phi_{t,0}(F_\dagger^jC) = F_\dagger^j\phi_{t,0}(C)$. We have the morphisms can : $\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M}')\to\psi_{t,0}(\mathcal{M}')$ and var : $\psi_{t,0}(\mathcal{M}')\to\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M}')$. We also have similar morphisms for $\mathcal{M}''$, which are denoted by the same symbols. They induce the morphisms can : $F_{0\dagger}^{-i}\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M}')\to F_{0\dagger}^{-i}\psi_{t,0}(\mathcal{M}')$ and var : $F_{0\dagger}^{-i}\psi_{t,0}(\mathcal{M}')\to F_{0\dagger}^{-i}\psi_{t,-\boldsymbol{\delta}_0}(\mathcal{M}')$. We have similar morphisms for $\mathcal{M}''$. Recall the following relations:

$$(444)\qquad \phi_{t,0}\big(F_\dagger^iC\big)\big(-\sqrt{-1}\operatorname{can}(a),\ \sigma^*b\big) = (\sqrt{-1}\lambda)\psi_{t,-\boldsymbol{\delta}_0}\big(F_\dagger^iC\big)\big(a,\ \sigma^*\operatorname{var}(b)\big),$$

$$(445)\qquad \phi_{t,0}\big(F_\dagger^iC\big)\big(a,\ \sigma^*\sqrt{-1}\operatorname{can}(b)\big) = (\sqrt{-1}\lambda)\psi_{t,-\boldsymbol{\delta}_0}\big(F_\dagger^iC\big)\big(\operatorname{var}(a),\ \sigma^*b\big).$$

The pairing $F_\dagger^i\phi_{t,0}(C)$ satisfies a similar relation with $\psi_{t,-\boldsymbol{\delta}_0}F_\dagger^iC = F_\dagger^i\psi_{t,-\boldsymbol{\delta}_0}C$.

By the assumption, we have a decomposition $F_\dagger^{-i}(\mathcal{M}') = M_1'\oplus M_2'$ such that (i) M_1' has no non-zero $\mathcal{R}_Y$-submodule whose support is contained in Y_0, (ii) the support of M_2' is contained in Y_0. We have a similar decomposition $F_\dagger^i\mathcal{M}'' = M_1''\oplus M_2''$. By using Proposition 22.11.5 and the relations between the pairings $\phi_{t,0}(F_{\dagger}^{i}C)$, $F_{\dagger}^{i}\phi_{t,0}(C)$ and $\psi_{t,-\boldsymbol{\delta}_0}(F_{\dagger}^{i}C)$ like (444) and (445), we can show the following:

- $\psi_{t,0}(M_1')$ and $\psi_{t,0}(M_2'')$ are orthogonal with respect to both the pairings $\phi_{t,0}(F_{\dagger}^{i}C)$ and $F_{\dagger}^{i}\phi_{t,0}(C)$. Similarly, $\psi_{t,0}(M_1'')$ and $\psi_{t,0}(M_2')$ are orthogonal with respect to both pairings.
- $\phi_{t,0}(F_{\dagger}^{i}C)$ and $F_{\dagger}^{i}\phi_{t,0}(C)$ are equal on $\psi_{t,0}(M_1')_{|\boldsymbol{S}\times Y}\otimes\sigma^{*}\psi_{t,0}(M_1'')_{|\boldsymbol{S}\times Y}$.

Hence, we only have to compare the induced sesqui-linear pairings of $\psi_{t,0}(M_2')$ and $\sigma^{*}\psi_{t,0}(M_2'')$.

We may assume that F_0 is the projection, i.e., $X_0=Y_0\times Z$. Let $n=\dim Z$. Then, $F_{\dagger}\mathcal{M}'$ can be expressed by $F_{*}(\mathcal{M}'\otimes\Omega_{\mathcal{Z}}^{n+\bullet})$. (See Subsection 22.2.4 for $\Omega_{\mathcal{Z}}^{\bullet}$.) Similarly, we have $F_{\dagger}\mathcal{M}''=F_{*}(\mathcal{M}''\otimes\Omega_{\mathcal{Z}}^{n+\bullet})$. Let $\lambda_0\in\boldsymbol{S}$. Let u and v be local sections of $V_0^{(\lambda_0)}M_2'$ and $V_0^{(-\lambda_0)}M_2''$ respectively. We can regard them as sections of $F_{\dagger}^{-i}\mathcal{M}'$ and $F_{\dagger}^{i}(\mathcal{M}'')$ respectively, and we have $\phi_{t,0}F_{\dagger}^{i}(C)(u,v)=F_{\dagger}^{i}C(u,v)$. We take lifts $\widetilde{u}$ and $\widetilde{v}$ of u and v to $F_{*}(\mathcal{M}'\otimes\Omega_{\mathcal{Z}}^{n-i})$ and $F_{*}(\mathcal{M}''\otimes\Omega_{\mathcal{Z}}^{n+i})$, respectively. Let φ be a test function on Y_0. Let ρ be a test function on $\boldsymbol{C}_t$ which is constantly 1 around $t=0$, and let $\omega:=\rho\cdot\sqrt{-1}dt\cdot d\overline{t}/2\pi$. We put $B_{n,i}:=(-1)^{(n+i)(n+i-1)/2}(2\pi\sqrt{-1})^{-n}$. Then, we have the following equalities:

$$(446)\quad \langle\phi_{t,0}(F_{\dagger}^{i}C)(u,\sigma^{*}v),\,\varphi\rangle=\Big\langle F_{\dagger}^{i}C(u,\sigma^{*}v),\,\varphi\cdot\omega\Big\rangle=B_{n,i}\Big\langle C(\widetilde{u},\sigma^{*}\widetilde{v}),\,F_0^{*}\varphi\cdot\omega\Big\rangle.$$

Let $[\widetilde{u}]$ and $[\widetilde{v}]$ denote the lift of u and v to $F_{0*}\big(\mathcal{M}'\otimes\Omega_{\mathcal{Z}}^{n-i}\big)$ and $F_{0*}\big(\mathcal{M}''\otimes\Omega_{\mathcal{Z}}^{n+i}\big)$ respectively, which are induced by $\widetilde{u}$ and $\widetilde{v}$. We have the following equality:

$$(447)\qquad \langle F_{0\dagger}^{i}\phi_{t,0}(C)(u,\sigma^{*}v),\,\varphi\rangle=B_{n,i}\langle\phi_{t,0}(C)([\widetilde{u}],\sigma^{*}[\widetilde{v}]),\,F_0^{*}\varphi\rangle.$$

We take an appropriate locally finite open covering $Z=\bigcup_{j\in S}U_j$ and the partition of unity $\{\chi_j\mid j\in S\}$ which is subordinate to the covering. We may assume to have $u_j^{(1)}\in V_{-1}^{(\lambda_0)}(\mathcal{M}')\otimes\Omega_{\mathcal{Z}}^{n-i}$ and $u_j^{(2)}\in V_{<0}^{(\lambda_0)}(\mathcal{M}')\otimes\Omega_{\mathcal{Z}}^{n-i}$ on U_j such that $\widetilde{u}_{|U_j}=\eth_t u_j^{(1)}+u_j^{(2)}$. Then, we have the following equality:

$$(448)\quad \Big\langle\phi_{t,0}(C)\big([\widetilde{u}],\sigma^{*}[\widetilde{v}]\big),\,F_0^{*}\varphi\Big\rangle=\sum_{j\in S}\Big\langle\phi_{t,0}(C)\big([\widetilde{u}],\sigma^{*}[\widetilde{v}]\big),\,\chi_jF_0^{*}\varphi\Big\rangle$$
$$=\sum_{j\in S}\Big\langle\phi_{t,0}(C)\big([\eth_t u_j^{(1)}],\sigma^{*}[\widetilde{v}]\big),\,\chi_jF_0^{*}\varphi\Big\rangle=\sum_{j\in S}\lambda\Big\langle\psi_{t,-\boldsymbol{\delta}_0}(C)\big([u_j^{(1)}],\,\sigma^{*}[t\widetilde{v}]\big),\;\chi_jF_0^{*}\varphi\Big\rangle$$
$$=\sum_{j\in S}\operatorname*{Res}_{s=-1}\lambda\Big\langle C\big(u_j^{(1)},\sigma^{*}\widetilde{v}\big),\;\chi_jF_0^{*}\varphi\cdot|t|^{2s}\overline{t}\cdot\omega\Big\rangle$$
$$=\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^{*}\widetilde{v}\Big),\;F_0^{*}\varphi\cdot\eth_t\big(|t|^{2(s+1)}\big)\cdot\omega\Big\rangle.$$

This can be rewritten as follows:

$$\begin{aligned}
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\eth_t\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)}\Big),\,\sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\omega\Big\rangle\\
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^*(\overline{\eth}_t\widetilde{v})\Big),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\omega\Big\rangle\\
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\eth_t\omega\Big\rangle.
\end{aligned}\tag{449}$$

It is equal to

$$\begin{aligned}
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C(\widetilde{u},\sigma^*\widetilde{v}),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\omega\Big\rangle\\
&+\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(2)},\,\sigma^*(\widetilde{v})\Big),\ F_0^*\varphi\cdot|t|^{2s+2}\cdot\omega\Big\rangle\\
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^*(\overline{\eth}_t\widetilde{v})\Big),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\omega\Big\rangle\\
&-\operatorname*{Res}_{s=-1}\frac{1}{s+1}\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^*(\widetilde{v})\Big),\ F_0^*\varphi\cdot|t|^{2(s+1)}\cdot\eth_t\omega\Big\rangle.
\end{aligned}\tag{450}$$

By the assumption, the support of $F_{0*}C(\widetilde{u},\sigma^*\widetilde{v})$ is contained in $t=0$. Hence, the first term of (450) is 0. Let us look at the second term. We have $u_j^{(2)}\in V_{<0}^{(\lambda_0)}$ and $\widetilde{v}\in V_0^{(-\lambda_0)}$. Due to Lemma 22.10.9,

$$G(s):=\Big\langle C\big(\sum\chi_j\cdot u_j^{(2)},\,\sigma^*(\widetilde{v})\big),\ F_0^*\varphi\cdot|t|^{2s+2}\cdot\omega\Big\rangle.$$

is holomorphic on the region $\{\operatorname{Re}(s)>-1-\delta\}$ for some $\delta>0$, and the following holds:

$$G(-1)=\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(2)},\,\sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot\omega\Big\rangle.\tag{451}$$

Hence, the second term in (450) is given by (451). For the same reason, the third term is as follows:

$$-\Big\langle C\Big(\sum_{j\in S}\chi_j\cdot u_j^{(1)},\,\sigma^*(\overline{\eth}_t\widetilde{v})\Big),\ F_0^*\varphi\cdot\omega\Big\rangle.\tag{452}$$

We have a similar equality for the fourth term. Hence, the sum of the third and the forth terms in (450) can be rewritten as follows:

$$\begin{aligned}
&-\Big\langle C\Big(\sum\chi_j\cdot u_j^{(1)},\,\sigma^*(\overline{\eth}_t\widetilde{v})\Big),\ F_0^*\varphi\cdot\omega\Big\rangle-\Big\langle C\Big(\sum\chi_j\cdot u_j^{(1)},\,\sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot\eth_t\omega\Big\rangle\\
&\qquad=\Big\langle C\Big(\sum\chi_j\cdot\eth_t u_j^{(1)},\,\sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot\omega\Big\rangle.
\end{aligned}\tag{453}$$

Hence (450) can be written as follows:

$$(454)\quad \Big\langle C\Big(\sum \chi_j\cdot u_j^{(2)},\, \sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot\omega\Big\rangle + \Big\langle C\Big(\sum \chi_j\cdot \eth_t u_j^{(1)},\, \sigma^*\widetilde{v}\Big),\ F_0^*\varphi\cdot\omega\Big\rangle = \big\langle C\big(\widetilde{u},\sigma^*\widetilde{v}\big),\ F_0^*\varphi\cdot\omega\big\rangle.$$

From (447), (448) (450) and (454), we obtain

$$\big\langle F^i_{0\dagger}\phi_{t,0}(C)(u,\sigma^*v),\ \varphi\big\rangle = \big\langle C\big(\widetilde{u},\sigma^*\widetilde{v}\big),\ F_0^*\varphi\cdot\omega\big\rangle.$$

Together with (446), we obtain $\phi_{t,0}(F^i_{\dagger}C)(u,\sigma^*v) = F^i_{0\dagger}\phi_{t,0}(C)(u,\sigma^*v)$. Thus, Lemma 22.10.6 is proved. □

22.11. $\mathcal{R}(*t)$-triples

We recall the notion of $\mathcal{R}_X(*t)$-triple due to Sabbah in [**73**], where it is called $\widetilde{\mathcal{R}}_X$-triples. We use the notation in Section 22.10.1. Let $\mathcal{M}'$ and $\mathcal{M}''$ be $\mathcal{R}_X(*t)$-modules. An $\mathcal{R}_X(*t)\otimes\sigma^*\mathcal{R}_X(*t)$-homomorphism $C : \mathcal{M}'_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}''_{|\boldsymbol{S}\times X}\to\mathfrak{Db}^{\mathrm{mod}}_{\boldsymbol{S}\times X/\boldsymbol{S}}$ is called a Hermitian sesqui-linear pairing of $\mathcal{M}'$ and $\mathcal{M}''$. It is also called a sesqui-linear pairing. Such $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$ is called an $\mathcal{R}_X(*t)$-triple.

Let $\mathcal{T}_i=(\mathcal{M}'_i,\mathcal{M}''_i,C_i)$ $(i=1,2)$ be $\mathcal{R}_X(*t)$-triples. A morphism of $\mathcal{R}_X(*t)$-triples $\varphi:\mathcal{T}_1\to\mathcal{T}_2$ is defined to be a pair of $\varphi':\mathcal{M}'_2\to\mathcal{M}'_1$ and $\varphi'':\mathcal{M}''_1\to\mathcal{M}''_2$ satisfying $C_1\big(\varphi'(x),\sigma^*(y)\big)=C_2\big(x,\sigma^*(\varphi''(y))\big)$. The category of $\mathcal{R}_X(*t)$-triples is abelian.

For an $\mathcal{R}_X(*t)$-triple $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$, we have the induced sesqui-linear pairing

$$C^*:\mathcal{M}''_{|\boldsymbol{S}\times X}\otimes\sigma^*\mathcal{M}'_{|\boldsymbol{S}\times X}\longrightarrow\mathfrak{Db}^{\mathrm{mod}}_{\boldsymbol{S}\times X/\boldsymbol{S}}.$$

given by $C^*(x,\sigma^*y)=\overline{\sigma^*C(y,\sigma^*x)}$. The tuple $\mathcal{T}^*=(\mathcal{M}'',\mathcal{M}',C^*)$ is called the Hermitian adjoint of $\mathcal{T}$. A morphism $\varphi=(\varphi',\varphi''):\mathcal{T}_1\to\mathcal{T}_2$ naturally induces $\varphi^*=(\varphi'',\varphi'):\mathcal{T}_2^*\to\mathcal{T}_1^*$.

Let k be any half integer. For an $\mathcal{R}_X(*t)$-triple $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$, we put $\mathcal{T}(k):=\big(\mathcal{M}',\mathcal{M}'',(\sqrt{-1}\lambda)^{-2k}C\big)$, which is called the k-th Tate twist of $\mathcal{T}$. It can be regarded as $\mathcal{T}\otimes\mathbb{T}^S(k)$. We have the isomorphism $\mathcal{T}(k)\simeq\big(\mathcal{T}^*(-k)\big)^*$ as in the case of $\mathcal{R}_X$-triples.

For an $\mathcal{R}_X(*t)$-triple $\mathcal{T}$, a morphism $\mathcal{S}:\mathcal{T}\to\mathcal{T}^*(-w)$ is called a Hermitian sesqui-linear duality of weight w, if the equality $\mathcal{S}=(-1)^w\mathcal{S}^*$ holds.

22.11.1. Specialization. — Let $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',C)$ be a coherent $\mathcal{R}_X(*t)$-triple. We say that $\mathcal{T}$ is strictly specializable, if $\mathcal{M}'$ and $\mathcal{M}''$ are strictly specializable along $t=0$. In that case, we have the induced pairing for any $u\in\boldsymbol{R}\times\boldsymbol{C}$ ([**73**]) as in the case of $\mathcal{R}_X$-triples:

$$\widetilde{\psi}_{t,u}C:\widetilde{\psi}_{t,u}(\mathcal{M}')_{|\boldsymbol{S}\times X_0}\otimes\sigma^*\widetilde{\psi}_{t,u}(\mathcal{M}'')_{|\boldsymbol{S}\times X_0}\longrightarrow\mathfrak{Db}_{\boldsymbol{S}\times X_0/\boldsymbol{S}}.$$

Thus, we obtain the $\mathcal{R}_{X_0}$-triple $\widetilde{\psi}_{t,u}(\mathcal{T})=\big(\widetilde{\psi}_{t,u}(\mathcal{M}'),\widetilde{\psi}_{t,u}(\mathcal{M}''),\widetilde{\psi}_{t,u}(C)\big)$.

Lemma 22.11.1. — *Let $\mathcal{T}_i$ $(i=1,2)$ be strictly specializable $\mathcal{R}_X(*t)$-triples. Let $\varphi = (\varphi', \varphi'') : \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism such that $\psi_{t,u}(\varphi')$ and $\psi_{t,u}(\varphi'')$ are strict for any $u \in \boldsymbol{R}\times\boldsymbol{C}$. Then,* $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$ *and* $\mathrm{Cok}(\varphi)$ *are also strictly specializable along t, and we have the natural isomorphisms $\widetilde{\psi}_{t,u}\,\mathrm{Ker}(\varphi) \simeq \mathrm{Ker}\,\widetilde{\psi}_{t,u}(\varphi)$, $\widetilde{\psi}_{t,u}\,\mathrm{Im}(\varphi) \simeq \mathrm{Im}\,\widetilde{\psi}_{t,u}(\varphi)$ and $\widetilde{\psi}_{t,u}\,\mathrm{Cok}(\varphi) \simeq \mathrm{Cok}\,\widetilde{\psi}_{t,u}(\varphi)$ for any $u \in \boldsymbol{R}\times\boldsymbol{C}$.*

Proof. — Because Lemma 22.4.5, we only have to compare the induced sesqui-linear pairings, which can be checked easily. □

22.11.2. Pull-back via ramified covering. — Let φ_n be as in Section 22.4.2. Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be an $\mathcal{R}_X(*t)$-triple. We have the induced pairing as explained in [**75**] (see also Section 22.8):

$$\varphi_n^{\dagger}C : \varphi_n^{\dagger}\mathcal{M}'_{|\boldsymbol{S}\times X^{(n)}} \otimes \sigma^*\varphi_n^{\dagger}\mathcal{M}''_{|\boldsymbol{S}\times X^{(n)}} \longrightarrow \mathfrak{Db}^{\mathrm{mod}}_{\boldsymbol{S}\times X^{(n)}/\boldsymbol{S}}.$$

Thus, we obtain an $\mathcal{R}_{X^{(n)}}(*t_n)$-triple $\varphi_n^{\dagger}(\mathcal{T}) = \big(\varphi_n^{\dagger}\mathcal{M}', \varphi_n^{\dagger}\mathcal{M}'', \varphi_n^{\dagger}C\big)$.

Let $\boldsymbol{I}(\lambda_0)$ denote a small neighbourhood of λ_0 in $\boldsymbol{S}$, and let $\boldsymbol{I}(-\lambda_0) = \sigma(\boldsymbol{I}(\lambda_0))$.

Lemma 22.11.2. — *Assume that $\mathcal{T}$ is strictly specializable along t. Let $[\tau_1] \in \widetilde{\psi}_{t,(u+m_1\boldsymbol{\delta})/n}(\mathcal{M}')_{|\boldsymbol{I}(\lambda_0)\times X_0}$ and $[\tau_2] \in \widetilde{\psi}_{t,(u+m_2\boldsymbol{\delta})/n}(\mathcal{M}'')_{|\boldsymbol{I}(-\lambda_0)\times X_0}$. We have the following relation:*

$$\widetilde{\psi}_{t_n,u}(\varphi_n^{\dagger}C)\Big([t_n^{m_1}\varphi_n^{\dagger}\tau_1], \sigma^*\big([t_n^{m_2}\varphi_n^{\dagger}\tau_2]\big)\Big) = \begin{cases} \widetilde{\psi}_{t,(u+m\boldsymbol{\delta}_0)/n}(C)\big([\tau_1], \sigma^*[\tau_2]\big) & m_1 = m_2 = m \\ 0 & m_1 \neq m_2 \end{cases}$$

As a result, we have the following decomposition of $\mathcal{R}_{X_0}$-triples:

$$\widetilde{\psi}_{t_n,u}(\varphi_n^{\dagger}\mathcal{T}) \simeq \bigoplus_{\substack{nu' - c\boldsymbol{\delta}_0 = u \\ 0\leqslant c\leqslant n-1}} \widetilde{\psi}_{t,u'}(\mathcal{T}).$$

Proof. — Let ρ be a C^{∞}-top form on X_0 with compact support. Let $\chi_1 = \chi_1(|t_n|)$ be a test function on $\boldsymbol{C}_{t_n}$ which are constantly 1 around 0, and let $\chi := \chi_1\cdot\sqrt{-1}dt\cdot d\overline{t}/2\pi$. We have

$$(455)\quad \Big\langle \widetilde{\psi}_{t_n,u}(\varphi_n^{\dagger}C)\big([t_n^{m_1}\varphi_n^{\dagger}\tau_1],\, \sigma^*[t_n^{m_2}\varphi_n^{\dagger}\tau_2]\big),\ \rho\Big\rangle$$
$$= \operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle \varphi_n^{\dagger}C\big(t_n^{m_1}\varphi_n^{\dagger}\tau_1,\, \sigma^*(t_n^{m_2}\varphi_n^{\dagger}\tau_2)\big),\ |t_n|^{2s}\chi\cdot\rho\Big\rangle$$
$$= \operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle \varphi_n^{\dagger}C\big(\tau_1, \sigma^*\tau_2\big),\ |t_n|^{2s}t_n^{m_1}\overline{t}_n^{m_2}\cdot\chi\cdot\rho\Big\rangle.$$

If $m_1 \neq m_2$, (455) is 0 by Lemma 22.8.5. In the case $m_1 = m_2 = m$, (455) is equal to

(456)

$$\operatorname*{Res}_{s+\mathfrak{e}(\lambda,u)} \Big\langle C(\tau_1,\sigma^*\tau_2),\ |t|^{2(s+m)/n}\chi\cdot\rho\Big\rangle\cdot n = \operatorname*{Res}_{n\overline{s}-m+\mathfrak{e}(\lambda,u)} \Big\langle C(\tau_1,\sigma^*\tau_2),\ |t|^{2\overline{s}}\chi\cdot\rho\Big\rangle\cdot n$$
$$= \operatorname*{Res}_{\overline{s}+\mathfrak{e}(\lambda,(u+m\boldsymbol{\delta}_0)/n)} \Big\langle C(\tau_1,\sigma^*\tau_2),\ |t|^{2\overline{s}}\chi\cdot\rho\Big\rangle = \Big\langle \widetilde{\psi}_{t,(u+m\boldsymbol{\delta}_0)/n} C\big([\tau_1],\sigma^*[\tau_2]\big),\ \rho\Big\rangle.$$

Thus we are done. □

22.11.3. Exponential twist. — Let $\mathcal{L}(\mathfrak{a})$ be as in Section 22.4.3. For any $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$, we have the naturally defined sesqui-linear pairing

$$C_{\mathfrak{a}} : \mathcal{L}(\mathfrak{a})_{|\mathcal{S}\times X^{(n)}} \otimes \sigma^*\mathcal{L}(\mathfrak{a})_{|\mathcal{S}\times X^{(n)}} \longrightarrow \mathfrak{Db}_{\mathcal{S}\times X^{(n)}/\mathcal{S}},$$

given by $C_{\mathfrak{a}}(e,\sigma^*e) = \exp\big(-2\sqrt{-1}\operatorname{Im}(\lambda\overline{\mathfrak{a}})\big)$. We put $\mathfrak{L}(\mathfrak{a}) := \big(\mathcal{L}(\mathfrak{a}),\mathcal{L}(\mathfrak{a}),C_{\mathfrak{a}}\big)$.

Let $\mathcal{T}$ be an $\mathcal{R}_X(*t)$-triple. Sabbah observed that $\varphi_n^\dagger(\mathcal{T})\otimes\mathfrak{L}(-\mathfrak{a})$ is well defined as the $\mathcal{R}_{X^{(n)}}(*t_n)$-triple (see Lemma 22.8.6):

$$\varphi_n^\dagger(\mathcal{T})\otimes\mathfrak{L}(-\mathfrak{a}) := \Big(\varphi_n^\dagger\mathcal{M}'\otimes\mathcal{L}(-\mathfrak{a}),\ \varphi_n^\dagger\mathcal{M}''\otimes\mathcal{L}(-\mathfrak{a}),\ \varphi_n^\dagger C\cdot C_{-\mathfrak{a}}\Big).$$

It is naturally identified with $\Big(\varphi_n^\dagger\mathcal{M}',\ \varphi_n^\dagger\mathcal{M}'',\ \exp\big(2\sqrt{-1}\operatorname{Im}(\lambda\overline{\mathfrak{a}})\big)\varphi_n^\dagger C\Big)$. If the underlying $\mathcal{R}_{X^{(n)}}(*t_n)$-modules of $\varphi_n^\dagger(\mathcal{T})\otimes\mathfrak{L}(-\mathfrak{a})$ are strictly specializable, we obtain the following $\mathcal{R}_{X_0}$-triples for any $u\in\boldsymbol{R}\times\boldsymbol{C}$:

$$\widetilde{\psi}_{t,\mathfrak{a},u}(\mathcal{T}) := \widetilde{\psi}_{t_n,u}\big(\varphi_n^\dagger(\mathcal{T})\otimes\mathfrak{L}(-\mathfrak{a})\big).$$

We have the well defined map:

$$\mathcal{N} = \big(-\sqrt{-1}N,\sqrt{-1}N\big) : \widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T}) \longrightarrow \widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T})\otimes\mathbb{T}^S(-1).$$

We have the monodromy weight filtration $W(\mathcal{N})$ of $\mathcal{N}$ in the category of $\mathcal{R}_{X_0}$-triples.

Note that we have the natural isomorphism $(\mathrm{id},\mathrm{id}) : \mathfrak{L}(\mathfrak{a})^* \simeq \mathfrak{L}(\mathfrak{a})$. A Hermitian sesqui-linear duality $\mathcal{S} : \mathcal{T}\to\mathcal{T}^*(-w)$ of weight w naturally induces $\varphi_n^\dagger\mathcal{T}\otimes\mathfrak{L}(-\mathfrak{a}) \to \big(\varphi_n^\dagger\mathcal{T}\otimes\mathfrak{L}(-\mathfrak{a})\big)^*(-w)$, which is also a Hermitian sesqui-linear duality of weight w. If $\varphi_n^\dagger(\mathcal{T})\otimes\mathfrak{L}(-\mathfrak{a})$ is strictly specializable along t_n, we have the induced morphisms:

$$\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{S}) : \widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T}) \longrightarrow \widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T})^*(-w),$$
$$\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{S}) : \operatorname{Gr}_\ell^{W(\mathcal{N})}\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T}) \longrightarrow \operatorname{Gr}_{-\ell}^{W(\mathcal{N})}\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T})^*(-w).$$

Therefore, we have the induced Hermitian sesqui-linear duality:

$$\mathcal{S}_{t,u,\mathfrak{a},\ell} := \widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{S})\circ\mathcal{N}^\ell : P\operatorname{Gr}_\ell^{W(\mathcal{N})}\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T}) \longrightarrow P\operatorname{Gr}_\ell^{W(\mathcal{N})}\widetilde{\psi}_{t,u,\mathfrak{a}}(\mathcal{T})^*(-w-\ell).$$

22.11.4. Push-forward. — We put $Y := Y_0 \times \boldsymbol{C}_t$. Let $F_0 : X_0 \to Y_0$ be a morphism, and let $F : X \to Y$ denote the induced morphism. Let $\mathcal{M}$ be an $\mathcal{R}_X(*t)$-module. We assume that F is proper on the support of $\mathcal{M}$. We have the well defined push-forward $F_\dagger\mathcal{M}$ in the derived category of $\mathcal{R}_Y(*t)$-modules. The i-th cohomology sheaves are denoted by $F_\dagger^i\mathcal{M}$. Recall the following compatibility in [**73**] and [**75**]. (See also [**77**].)

Proposition 22.11.3. — *Assume that $\mathcal{M}$ is strictly specializable along t, and good with respect to F. If $F_{0\dagger}^i\widetilde{\psi}_{t,u}(\mathcal{M})$ are strict for any $u \in \boldsymbol{R}\times\boldsymbol{C}$ and any i, then $F_\dagger^i(\mathcal{M})$ are also strictly specializable along $t=0$, and $\widetilde{\psi}_{t,u}F_\dagger^i(\mathcal{M}) \simeq F_{0\dagger}^i\widetilde{\psi}_{t,u}(\mathcal{M})$.* □

Let $Y^{(n)} := Y_0 \times \boldsymbol{C}_{t_n}$. The induced morphism $Y^{(n)} \to Y$ is also denoted by φ_n. Recall the following compatibility in [**75**], which can be shown directly.

Proposition 22.11.4. — *For any $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$ and for any holonomic $\mathcal{R}_X(*t)$-module $\mathcal{M}$, we have $F_\dagger\big(\varphi_n^\dagger(\mathcal{M}) \otimes \mathcal{L}(-\mathfrak{a})\big) \simeq \varphi_n^\dagger F_\dagger(\mathcal{M}) \otimes \mathcal{L}(-\mathfrak{a})$.* □

Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be an $\mathcal{R}_X$-triple (resp. $\mathcal{R}_X(*t)$-triple). We assume that the support of $\mathcal{T}$ is proper with respect to F. We have the well defined push-forward $F_\dagger(\mathcal{T}) = \big(F_\dagger(\mathcal{M}')^{\mathrm{op}}, F_\dagger(\mathcal{M}''), F_\dagger C\big)$ in the derived category of $\mathcal{R}_Y$-triples (resp. $\mathcal{R}_Y(*t)$-triples). The i-th cohomology triples are denoted by $F_\dagger^i(\mathcal{T}) = \big(F_\dagger^{-i}\mathcal{M}', F_\dagger^i\mathcal{M}'', F_\dagger^i C\big)$.

Recall the compatibility of the specialization and the push-forward for the sesqui-linear pairing in [**73**].

Proposition 22.11.5. — *Assume that both $\mathcal{M}'$ and $\mathcal{M}''$ satisfy the assumption in Proposition* 22.11.3. *Then, we have the compatibility $\widetilde{\psi}_{t,u}(F_\dagger^i C) = F_{0\dagger}^i(\widetilde{\psi}_{t,u}C)$. Namely, $\widetilde{\psi}_{t,u}F_\dagger^i\mathcal{T} \simeq F_{0\dagger}^i(\widetilde{\psi}_{t,u}\mathcal{T})$.*

We also have $F_\dagger^i\big(\varphi_n^\dagger\mathcal{T} \otimes \mathfrak{L}(-\mathfrak{a})\big) \simeq \varphi_n^\dagger F_\dagger^i(\mathcal{M}) \otimes \mathfrak{L}(-\mathfrak{a})$. □

22.11.5. Specialization of $\mathcal{R}_X$-triples along a function g with ramified exponential twist. — Let g be a non-trivial function on X. Let $i_g : X \to X \times \boldsymbol{C}$ denote the graph. Let $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ be a coherent $\mathcal{R}_X$-triple. We have the $\mathcal{R}_{X\times\boldsymbol{C}}$-triple $i_{g\dagger}\mathcal{T}$, and the induced $\mathcal{R}_{X\times\boldsymbol{C}_t}(*t)$-triple $(i_{g\dagger}\mathcal{T})(*t)$. Let $\mathfrak{a} \in \boldsymbol{C}[t_n^{-1}]$. We say that $\mathcal{T}$ is strictly specializable along g with ramified exponential twist by $\mathfrak{a}$, if $\varphi_n^\dagger\big(i_{g\dagger}\mathcal{T}(*t)\big) \otimes \mathcal{L}(-\mathfrak{a})$ is strictly specializable along t_n. In that case, we define

$$\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T}) := \widetilde{\psi}_{t,\mathfrak{a},u}\big(i_{g\dagger}\mathcal{T}(*t)\big)$$

for any $u \in \boldsymbol{R}\times\boldsymbol{C}$. If a Hermitian sesqui-linear duality $\mathcal{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ is given, we have the induced Hermitian sesqui-linear duality

$$\mathcal{S}_{g,\mathfrak{a},u,\ell} : P\,\mathrm{Gr}_\ell^{W(\mathcal{N})}\big(\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T})\big) \longrightarrow P\,\mathrm{Gr}_\ell^{W(\mathcal{N})}\big(\widetilde{\psi}_{g,\mathfrak{a},u}(\mathcal{T})\big)^*(-w-\ell).$$

22.12. Comparison

22.12.1. Twistor structure and $\mathcal{R}$-module in dimension 0. — Recall that an $\mathcal{R}$-triple in dimension 0 is a tuple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ of coherent $\mathcal{O}_{\boldsymbol{C}_\lambda}$-modules $\mathcal{M}'$ and $\mathcal{M}''$ with an $\mathcal{O}_{\boldsymbol{C}_\lambda^*}$-homomorphism

$$C : \mathcal{M}'_{|\boldsymbol{C}_\lambda^*} \otimes \sigma^* \mathcal{M}''_{|\boldsymbol{C}_\lambda^*} \longrightarrow \mathcal{O}_{\boldsymbol{C}_\lambda^*}.$$

If $\mathcal{M}'$ and $\mathcal{M}''$ are locally free, $\mathcal{T}$ is called strict. If the pairing is perfect, $\mathcal{T}$ is called perfect.

Let V be a twistor structure on $\mathbb{P}^1$. We have the associated $\mathcal{R}$-triple

$$\Theta(V) = \big(V_0^\vee, \sigma^*(V_\infty), C_V\big),$$

where $V_0 := V_{|\boldsymbol{C}_\lambda}$, $V_\infty := V_{|\boldsymbol{C}_\mu}$ and C_V is the naturally defined Hermitian sesqui-linear pairing. It is easy to observe that Θ gives an equivalence of the categories of algebraic vector bundles on $\mathbb{P}^1$ and strict perfect $\mathcal{R}$-triples in dimension 0, which preserves tensor products and direct sums. We also have $\Theta(\sigma^*(V)^\vee) = \Theta(V)^*$. A perfect strict $\mathcal{R}$-triple is called pure twistor structure of weight n, if it is isomorphic to $\Theta(V)$ for some pure twistor structure V of weight n.

Recall $\mathbb{T}^S(k) := \big(\mathcal{O}_{\boldsymbol{C}_\lambda} e_0^{(2k)}, \mathcal{O}_{\boldsymbol{C}_\lambda} e_\infty^{(2k)}, (\sqrt{-1}\lambda)^{-2k}\big)$ for a half-integer k. The Tate twist in the category of $\mathcal{R}$-triples in dimension 0 is given by the tensor product with $\mathbb{T}^S(k)$. The canonical isomorphism $\mathbb{T}^S(k) \to \mathbb{T}^S(-k)^*$ is given by $\big((-1)^{2k}, 1\big)$. We take a complex number a such that $a^2 = -\sqrt{-1}$. We fix the isomorphism

$$\Phi^{(n)} = \big(\Phi^{(n)\prime}, \Phi^{(n)\prime\prime}\big) : \Theta(\mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow \mathbb{T}^S(-n/2),$$

given by $\Phi^{(n)\prime}(e_0^{(-n)}) = a^n f_0^{(-n)}$ and $\Phi^{(n)\prime\prime}\big(\sigma^*(f_\infty^{(n)})\big) = a^{-n} e_\infty^{(-n)}$. Note the commutativity of the following induced diagram (Lemma 3.149 of [**67**]):

$$\begin{array}{ccc} \Theta\big(\mathcal{O}_{\mathbb{P}^1}(n)\big)^* & \longrightarrow & \Theta\big(\mathcal{O}_{\mathbb{P}^1}(-n)\big) \\ \uparrow & & \downarrow \\ \mathbb{T}^S(-n/2)^* & \longrightarrow & \mathbb{T}^S(n/2) \end{array} \tag{457}$$

The functor Θ preserves the Tate twists in the both categories, and it also preserves the compatibility between the Tate twist and the adjunction because of the commutativity of (457).

For an integer n, a morphism $\mathcal{S} : \mathcal{T} \to \mathcal{T}^* \otimes \mathbb{T}^S(-n)$ is called a Hermitian sesqui-linear duality of weight n, if $(-1)^n \mathcal{S}^* = \mathcal{S}$ is satisfied. Note that a morphism $S : V \otimes \sigma^*(V) \to \mathbb{T}(-n)$ induces $\Psi_S : V \to \sigma^*(V)^\vee \otimes \mathbb{T}(-n)$, and hence $\Theta(\Psi_S) : \Theta(V) \to \Theta(V)^* \otimes \mathbb{T}^S(-n)$.

Lemma 22.12.1. — *S is a pairing of weight n, if and only if $\Theta(\Psi_S)$ is a Hermitian sesqui-linear duality of weight n.*

Proof. — S is a pairing of weight n, if and only if $(1 \otimes \iota) \circ \sigma^* \Psi_S^\vee = (-1)^n \Psi_S$. By the functor Θ, it is transferred to the condition that $\Theta(\Psi_S)$ is a Hermitian sesqui-linear duality of weight n. □

Remark 22.12.2. — Let $(V^\triangle, \mathbb{D}^\triangle)$ be a variation of twistor structure on $\mathbb{P}^1 \times X$, obtained as the gluing of $(V_0, \mathbb{D}_0)$ and $(V_\infty, \mathbb{D}_\infty^\dagger)$. Then, we obtain an $\mathcal{R}_X$-triple $(V_0^\vee, \sigma^* V_\infty, C_V)$. □

Let $\mathcal{T}$ be a pure twistor structure of weight n. A Hermitian-sesqui-linear duality $\mathcal{S} : \mathcal{T} \to \mathcal{T}^* \otimes \mathbb{T}^S(-n)$ of weight n is a polarization, if $(\mathcal{T}, \mathcal{S})$ is isomorphic to $\big(\Theta(V), \Theta(\Psi_S)\big)$ for some polarized pure twistor structure (V, S) of weight n (Corollary 3.151 of [**67**]).

22.12.2. Comparison of specializations. — Let $X = \Delta^n$ and $D = \{O\}$. We put $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{D} := \boldsymbol{C}_\lambda \times D$. Let $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ be a regular coherent strictly specializable $\mathcal{R}_X(*z)$-triple on X such that $\mathcal{S} = (\mathrm{id}, \mathrm{id})$ is a Hermitian sesqui-linear duality of weight 0. Note that $\mathcal{M}_i$ has the KMS-structure at each $\lambda_0 \in \boldsymbol{C}_\lambda$. Assume that the restriction $(\mathcal{T}, \mathcal{S})_{|X \setminus D} = \Theta(V^\vee, \mathbb{D}^\triangle, \mathcal{S}_1)$ for a variation of twistor structure $(V, \mathbb{D}^\triangle, \mathcal{S})$ with a pairing of weight 0 on $\mathbb{P}^1 \times (X \setminus D)$. We use the identifications $V_0 = \mathcal{M}_1$ and $\sigma^* V_\infty^\vee = \mathcal{M}_2$.

As explained in Section 11.8.3, for any $u \in \boldsymbol{R} \times \boldsymbol{C}$, we obtain the vector bundle $S_u(V)$ with the nilpotent map and the symmetric pairing

$$\mathcal{N}_u^\triangle : S_u(V) \longrightarrow S_u(V) \otimes \mathbb{T}(-1), \quad \mathcal{S}_u : S_u(V) \otimes \sigma^* S_u(V) \longrightarrow \mathbb{T}(0).$$

We have the specialization $\widetilde{\psi}_{z,u}(\mathcal{T}) = \big(\widetilde{\psi}_{z,u}\mathcal{M}, \widetilde{\psi}_{z,u}\mathcal{M}, \widetilde{\psi}_{z,u}C\big)$ for any $u \in T$ with the induced nilpotent map

$$\mathcal{N}_u = \big(-\sqrt{-1}N, \sqrt{-1}N\big) : \widetilde{\psi}_{z,u}(\mathcal{T}) \longrightarrow \widetilde{\psi}_{z,u}(\mathcal{T}) \otimes \mathbb{T}^S(-1),$$

where $N = -\eth_t t + \mathfrak{e}(\lambda, u)$ is the nilpotent part of $-\eth_t t$. We also have the induced Hermitian sesqui-linear duality of weight 0

$$\widetilde{\psi}_{z,u}(\mathcal{S}) : \widetilde{\psi}_{z,u}(\mathcal{T}) \longrightarrow \widetilde{\psi}_{z,u}(\mathcal{T})^*.$$

Proposition 22.12.3. — *We have a natural isomorphism*

$$\Theta\big(S_u(V), \mathcal{N}_u^\triangle, \mathcal{S}_u\big) \simeq \big(\widetilde{\psi}_{z,u}(\mathcal{T}), \mathcal{N}_u, \widetilde{\psi}_{z,u}(\mathcal{S})\big).$$

We give some consequences. By taking the primitive part with respect to the weight filtration of $\mathcal{N}_u$, we obtain the $\mathcal{R}$-triples $P\,\mathrm{Gr}_\ell^W \widetilde{\psi}_{z,u}(\mathcal{T})$ for any $u \in T$ and $\ell \in \mathbb{Z}$ with the Hermitian sesqui-linear duality

$$\mathcal{S}_{u,\ell} = \widetilde{\psi}_{z,u}(\mathcal{S}) \circ \mathcal{N}_u^\ell : P\,\mathrm{Gr}_\ell^W \widetilde{\psi}_{z,u}(\mathcal{T}) \longrightarrow P\,\mathrm{Gr}_\ell^W \widetilde{\psi}_{z,u}(\mathcal{T})^* \otimes \mathbb{T}^S(-\ell).$$

The following fact implicitly appeared in [**67**].

Proposition 22.12.4. — *$\big(S_u(V), \mathcal{N}_u^{\triangle}, \mathcal{S}_u\big)$ is a polarized mixed twistor structure of weight 0, if and only if $\big(P\,\mathrm{Gr}_\ell^W\,\widetilde{\psi}_{z,u}(\mathcal{T}), \mathcal{S}_{u,\ell}\big)$ are polarized pure twistor structure of weight ℓ for any $\ell \in \mathbb{Z}_{\geqslant 0}$.*

Proof. — Recall that $\big(S_u(V), \mathcal{N}_u^{\triangle}, \mathcal{S}_u\big)$ is a polarized mixed twistor structure, if (i) $\mathrm{Gr}_\ell^W\, S_u(V)$ are pure twistor structure of weight ℓ for any $\ell \in \mathbb{Z}$, (ii) the induced pairings $\mathcal{S}_u\big(\mathcal{N}_u^\ell \otimes \mathrm{id}\big)$ on $P\,\mathrm{Gr}_\ell^W\, S_u(V)$ $(\ell \geqslant 0)$ give polarizations. Here, W denote the weight filtration associated to $\mathcal{N}$. Then, Proposition 22.12.4 follows from Proposition 22.12.3. □

Corollary 22.12.5. — *$(V, \mathbb{D}^{\triangle}, \mathcal{S})$ is a variation of polarized pure twistor structure, if $\big(P\,\mathrm{Gr}_\ell^W\,\widetilde{\psi}_{z,u}(\mathcal{T}), \mathcal{S}_{u,\ell}\big)$ are pure twistor structures for any $u \in \boldsymbol{R} \times \boldsymbol{C}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. It comes from a tame harmonic bundle.*

Proof. — The first claim follows from Proposition 22.12.4 and Lemma 11.8.6. We obtain the tameness from the regularity of the meromorphic Higgs field. □

22.12.2.1. Proof of Proposition 22.12.3. — Let $\widehat{u} = u + (1,0) \in \boldsymbol{R} \times \boldsymbol{C}$. We have the natural identification $\mathcal{G}_{\widehat{u}}(V_0) \simeq \widetilde{\psi}_{z,u}(\mathcal{M}_1)$ by construction. We also have the natural identification $\mathcal{G}_{\widehat{u}}(V_0) \simeq \mathcal{G}_u(V_0)$ given by the multiplication of z. We will not distinguish them in the following argument. We also have natural identification $\widetilde{\psi}_{z,u}(\mathcal{M}_2) \simeq \mathcal{G}_u\big(\sigma^* V_\infty^\vee\big) \simeq \sigma^* \mathcal{G}_{-u}\big(V_\infty^\vee\big) \simeq \sigma^* \mathcal{G}_u(V_\infty)$. Note Lemma 6.1.5 for the signature of the KMS-spectrum.

By construction, $\Theta(S_u(V))$ is a tuple of $\mathcal{O}_{\boldsymbol{C}_\lambda}$-modules $\mathcal{G}_u(V_0)$ and $\sigma^*\mathcal{G}_u(V_\infty)^\vee \simeq \mathcal{G}_u(\sigma^*(V_\infty^\vee))$, and an induced pairing $C_{0,u}$. We shall show that $C_{0,u} = \psi_u C$ under the above identification, which implies $\Theta(S_u(V)) \simeq \widetilde{\psi}_{z,u}(\mathcal{T})$.

We only have to compare them around generic $\lambda_0 \in \boldsymbol{C}_\lambda^*$. In the following, $U(\lambda_0)$ denotes a neighbourhood of λ_0, and $\mathcal{X}^{(\lambda_0)} := U(\lambda_0) \times X$ and $\mathcal{D}^{(\lambda_0)} := U(\lambda_0) \times D$. We put $\mathcal{X}^{(\sigma(\lambda_0))} := \sigma(U(\lambda_0)) \times X$ and $\mathcal{D}^{(\sigma(\lambda_0))} := \sigma(U(\lambda_0)) \times D$. We have the generalized eigen-decomposition of the monodromy along the loop with the counter-clockwise direction:

$$\mathcal{P}_a^{(\lambda_0)}(V_0) = \bigoplus_{a-1<\mathfrak{p}(\lambda_0,u)\leqslant a} \mathcal{P}_a^{(\lambda_0)}(V_0)_{\mathfrak{e}^f(\lambda,u)}.$$

Here, eigenvalues of the monodromy on $\mathcal{P}_a^{(\lambda_0)}(V_0)_{\mathfrak{e}^f(\lambda,u)}$ are $\mathfrak{e}^f(\lambda,u)$. Note that $\mathcal{P}_{\mathfrak{p}(\lambda,u)}^{(\lambda_0)}(V_0)_{\mathfrak{e}^f(\lambda,u)}$ is naturally isomorphic to $\mathcal{G}_u\mathcal{V}_{0|\mathcal{X}^{(\lambda_0)}}$ in Section 11.8.1.4. We will use the identification in the following. We also have the generalized eigen-decomposition of $\mathcal{P}_a^{(\sigma(\lambda_0))}V_0$ with respect to the monodromy along the loop. It is easy to observe that the sesqui-linear pairing is compatible with the decompositions. Hence, we obtain the sesqui-linear pairing C_u of $\mathcal{G}_u\mathcal{V}_{0|\mathcal{X}^{(\lambda_0)}\setminus\mathcal{D}^{(\lambda_0)}}$ and $\sigma^*\mathcal{G}_u\mathcal{V}_{0|\mathcal{X}^{(\sigma(\lambda_0))}\setminus\mathcal{D}^{(\sigma(\lambda_0))}}$ to the sheaf of λ-holomorphic C^∞-functions on $\mathcal{X}^{(\lambda_0)} \setminus \mathcal{D}^{(\lambda_0)}$.

Let $f \in \mathcal{G}_{\widehat{u}}(V_0)$ on $U(\lambda_0)$ and $g \in \sigma^*\mathcal{G}_{\widehat{u}}(V_\infty)^\vee$ on $\sigma(U(\lambda_0))$. Let F_0 be the section of $\mathcal{G}_{\widehat{u}}\mathcal{V}_0$ on $\mathcal{X}^{(\lambda_0)}$ such that $F_{0|\mathcal{D}^{(\lambda_0)}} = f$ and $(-\eth_z z + \mathfrak{e}(\lambda,u))^L F_0 = 0$ for some large L. Such F_0 is called the lift of f to $\mathcal{G}_{\widehat{u}}\mathcal{V}_0$. Similarly, we take the lift G_0 of g to $\sigma^*\mathcal{G}_{\widehat{u}}\mathcal{V}_\infty^\vee$

on $\mathcal{X}^{(\sigma(\lambda_0))}$. Let $\omega_0 = \sqrt{-1}(2\pi)^{-1}\, dz\, d\bar{z}$. Let χ be a test function on X which is constantly 1 around 0. Hence, we obtain

$$\widetilde{\psi}_{z,u} C_u\big(f, \sigma^* g\big) = \operatorname*{Res}_{s+\lambda^{-1}\mathfrak{e}(\lambda,u)} \Big\langle C_u\big(F_0, \sigma^* G_0\big),\, |z|^{2s}\, \chi\, \omega_0 \Big\rangle. \tag{458}$$

By the elementary argument in the proof of Lemma 18.11 of [**67**], we can show that the function $C_u\big(F_0, \sigma^* G_0\big)$ is of the following form:

$$|z|^{2\lambda^{-1}\mathfrak{e}(\lambda,u)-2} \sum_{k=0}^{M} a_k(\lambda)\, (\log|z|^2)^k. \tag{459}$$

Here, a_k are holomorphic functions of λ, and M denotes some positive integer. And we can show the following equality by a direct calculation (see Lemma 2.34 of [**67**] for more general case):

$$\operatorname*{Res}_{s+\lambda^{-1}\mathfrak{e}(\lambda,u)} \Big\langle C_u\big(F_0, \sigma^* G_0\big),\, |z|^{2s}\, \chi\, \omega_0 \Big\rangle = a_0(\lambda).$$

Let F be the section of $\mathcal{G}_{\widehat{u}}\mathcal{H}(V_0)$ on $U(\lambda_0)$ such that $\Phi^{\mathrm{can}}(F) = f$. If we regard F as a multi-valued flat section of $\mathcal{G}_{\widehat{u}}\mathcal{V}_0$, we have an expression

$$F = \exp\big(-\lambda^{-1}\mathfrak{e}(\lambda, \widehat{u})\, \log z\big)\, \Big(F_0 + \sum_{j=1}^{N} F_j (\log z)^j\Big)$$

such that $(-\eth_z z + \mathfrak{e}(\lambda, u))^L F_j = 0$ for some large L. Similarly, let G be the section of $\sigma^* \mathcal{G}_{\widehat{u}}\mathcal{H}(V_\infty)^\vee$ on $\sigma(U(\lambda_0))$ such that $\Phi^{\mathrm{can}}(G) = f$. We have similar expression for G, i.e.,

$$G = \exp\big(-\lambda^{-1}\mathfrak{e}(\lambda, \widehat{u})\log z\big)\Big(G_0 + \sum_{j=1}^{N} G_j\, (\log z)^j\Big)$$

such that $(-\eth_z z + \mathfrak{e}(\lambda, u))^L G_j = 0$ for some large L. Because $C_u\big(F_i, \sigma^* G_j\big)$ has an expression as in (459), $C_u\big(F, \sigma^* G\big)$ is described as a polynomial in $(\log|z|^2)$ whose coefficients are holomorphic functions in λ. Because $C_u\big(F, \sigma^* G\big)$ is constant in the X-direction, we obtain that

$$C_{0,u}(f, \sigma^* g) = C_u\big(F, \sigma^* G\big) = a_0(\lambda).$$

Hence, we obtain $\widetilde{\psi}_{z,u} C = C_{0,u}$.

It is easy to compare the Hermitian sesqui-linear dualities under the identification. Let us compare the nilpotent maps. Let $\mathrm{Res}^{\mathrm{nil}}$ denote the nilpotent part of the residue. We have

$$\Theta(\mathcal{N}_u^{\triangle}) = \big(\mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_0^\vee)^\vee \otimes t_0^{(-1)},\, \sigma^* \mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_{V_\infty^\vee}) \otimes \sigma^* t_\infty^{(-1)}\big).$$

We have $\mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_0^\vee)^\vee \otimes t_0^{(-1)} = -\mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_0) \otimes t_0^{(-1)}$. Under the identification of $\Theta^S(n)$ and $\Theta(n)$ in Subsection 22.12.1, it is identified with $-\sqrt{-1}N \otimes e_0^{(2)}$, where N is the nilpotent part of $-\eth_t t$. We have $\sigma^* \mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_{V_\infty^\vee}) \otimes \sigma^* t_\infty^{(-1)} = -\mathrm{Res}^{\mathrm{nil}}(\mathbb{D}_{\sigma^* V_\infty^\vee}) \otimes \sigma^* t_\infty^{(-1)}$, which is identified with $\sqrt{-1}N \otimes e_0^{(-2)}$. Hence, we have $\mathcal{N}_u^{\triangle} = \mathcal{N}_u$. Thus, the proof of Proposition 22.12.3 is finished. □

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