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LINEAR STABILITY OF BLACK HOLES
[d'après M. Dafermos et I. Rodnianski]

by **Sergiu KLAINERMAN**

> *The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unravelled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate.*
>
> S. Chandrasekhar, *The mathematical theory of black holes.*

1. INTRODUCTION

While the splendorous remains, a layer of complexity has now been unravelled. I report on the recent, remarkable, ongoing progress made on the linear stability of black holes, more precisely on the boundedness and decay properties of solutions to linear equations in a Kerr spacetime. The Kerr spacetimes $\mathcal{K}(m, a)$ are explicit solutions of the Einstein vacuum equations (discovered by R. Kerr in 1963) depending on two parameters $0 \leq a \leq m$, corresponding physically to black holes of mass m and angular momentum am. The case $a = m = 0$ corresponds to the Minkowski space while $a = 0, m > 0$, corresponds to the much older Schwarzschild solution (K. Schwarzschild 1915).

The problem of linear stability of the Kerr family is an old problem which has received a lot of attention in the Physics literature immediately after the discovery of these fascinating solutions of the Einstein equations in vacuum, which, embedded in the larger 3-parameter family of the so-called Kerr-Newman spacetimes, form the basis of our understanding of black holes. The obvious question raised by the discovery of any interesting, explicit solution of a complex, non-linear system, such as the Einstein equations, is that of their stability under small perturbations. Roughly the problem here is to show that all spacetime developments of initial data sets, sufficiently close to the initial data set of a Kerr spacetime, behaves in the large like (possible another) Kerr solution. This is not only a deep mathematical question but one with serious astrophysical implications. Indeed if the Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts. The Einstein equations are, of course, nonlinear and hyperbolic, thus the issue of stability is an extremely difficult and a dicey one. Given the geometric, covariant structure of the equations, with no universal notions of space and time variables, it is not even a-priori clear what that means. Linear stability, though still tricky, is somewhat easier to define. It is clear, for example, that any first order approximation of the equations, at the level of the space-time metric, in any reasonable coordinate system, will generate some system of wave equations in the Kerr background we want to perturb. Thus it is natural to ask, and this must certainly be relevant to the full nonlinear problem, whether solutions to linear wave equations in a fixed Kerr background are well behaved. If it turns out that solutions of these linear equations are amplified, due to the non-trivial features of the background geometry, then there is a reasonable chance that the background itself might be unstable.

It is not enough, however, to establish that solutions are not amplified; to have a chance to prove non-linear stability we also need to show that solutions decay at a sufficient rate. There is a lot of confusion in this regard among some physicists who seem to believe that somehow the lack of linear instability is a strong indication of nonlinear stability. This, of course, is not true even near solutions of minimal energy of simple nonlinear PDE's, as the case of the Burger equation $\partial_t u + u\partial_x u = 0$ easily demonstrates. The solution $u = 0$ is a global minimum for the energy integral $E(t) = \int |u(t,x)|^2 dx$, yet any compactly supported, smooth, small perturbation of the zero initial leads to blow up in finite time.

To be useful, a result on linear stability has to establish, *quantitatively*, not just a lack of amplification but also a realistic decay. In fact all known stability results, for strongly nonlinear wave equations (Einstein equations are quasilinear), depend on precise decay information for the linearized solutions.

The methods by which one establishes these decay estimates are also a very important issue. Thus, in the Minkowski space $\mathbb{R}^{1+3}$, it is easy to derive decay estimates for solution to the standard wave equation $\square\phi = 0$ using explicit representation formulas in the physical or Fourier variables. These formulas, however, depend heavily on the specific features of the Minkowski space and do not survive under small perturbations of the Minkowski metric. In other words, such methods are intrinsically not *robust.* A far more useful method for deriving decay estimates for the wave equation, and more generally for linear field equations, is that of *invariant vector fields*, see **[28]**, **[29]**. That method, first introduced to prove stability results for quasilinear wave equations, plays a fundamental role in all known proofs of the stability of the Minkowski space, see **[12]**, **[30]**, **[32]**, **[5]**.

In the case of the Kerr metric, or rather the more accessible case of the Schwarzschild metric, one can use the specific symmetries of the space to separate variables and then concentrate on the pointwise properties of the corresponding eigenvalue problem. This method is not only not robust but, to our knowledge, was not even satisfactory to derive unconditional decay results for general solutions of the wave equation. In the physics literature, where the problem of linear stability of Schwarzschild and Kerr spacetimes has received a tremendous amount of attention (see e.g. **[40]**, **[45]**, **[39]**, **[38]**, **[41]**, the monograph **[9]** and the references therein), this method of mode decomposition led to nothing more, in the words of Press and Teukolsky (see **[38]**), than *"an unsuccessful search for instabilities"*. On the other hand mathematical rigorous efforts based on this approach can only lead to statements of decay without a rate or precise rates of decay of specific modes, both of which, in principle, compatible with the scenario in which a general solution of the corresponding linear problem is not even uniformly bounded. For the results in this direction, see **[34]**, **[31]**, **[21]** in Schwarzschild and an attempt **[22]** in Kerr. Moreover, even if ultimately successful, such methods would leave us with a heavy machinery to prove some form of linear stability without any clue on how to approach to the non-linear problem.

A simple version of the vector field method was first used by Kay and Wald, see **[27]**, to prove the boundedness of solutions of the wave equation in a Schwarzschild spacetime. The first attempt to use the vector field method, to prove integrated local energy decay in Schwarzschild is due to Blue and Soffer **[6]**. Their work however had serious flaws. The first complete results on pointwise decay for solutions of the wave equation on the Schwarzschild background have been obtained, independently, by Blue-Sterbenz **[7]**, and Dafermos-Rodnianski **[16]**. In **[16]** Dafermos and Rodnianski also introduced the crucial red shift vector field, which led to stronger decay rates along the event horizon in Schwarzschild and, more importantly, played a central role in extending the boundedness and decay results to Kerr space-times, see **[20]**, **[17]**. Other important contributions were made by S. Alinhac in **[3]**, Dafermos-Rodnianski in **[19]**, Marzuola-Metcalfe-Tataru-Tohaneanu in **[35]** and Luk in **[33]** for the problem in Schwarzschild, and by Tataru-Tohaneanu in **[43]** and Andersson-Blue in **[4]**, for Kerr spacetimes.

I will review these results following, mainly the works of Dafermos-Rodnianski, in particular their general exposition in **[17]** and the recent paper **[18]**.

2. INITIAL VALUE PROBLEM

We recall that an initial data set consists of a 3-dimensional manifold Σ, a complete Riemannian metric $g_{(0)}$, a symmetric 2-tensor $k_{(0)}$, and a well specified set of initial conditions corresponding to the matterfields under consideration. These have to be restricted to a well known set of constraint equations. We restrict the discussion to *asymptotically flat* initial data sets, i.e. outside a sufficiently large compact set K, $\Sigma_{(0)} \setminus K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^3$ and admits a system of coordinates in which $g_{(0)}$ is asymptotically euclidean and $k_{(0)}$ vanishes, at appropriate order. A *Cauchy development* of an initial data set is a globally hyperbolic spacetime $(\mathcal{M}, \mathbf{g})$, verifying the Einstein field equations, in the presence of a matterfield with energy momentum $\mathbf{Q}$,

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}, \tag{2.1}$$

and an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i_*(g_{(0)}), i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$.

In what follows I will restrict the discussion to the Einstein vacuum equations, i.e. the case when the energy momentum tensor vanishes identically and the equations take the purely geometric form

$$\mathbf{R}_{\alpha\beta} = 0. \tag{2.2}$$

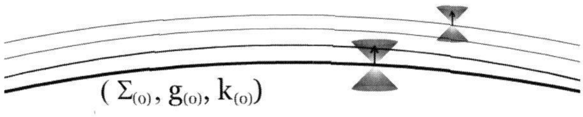


FIGURE 1.

The most primitive question asked about the initial value problem, solved in a satisfactory way, for very large classes of evolution equations, is that of local existence and uniqueness of solutions. For the Einstein equations this type of result was first established by Y. F.-Bruhat **[23]** with the help of wave coordinates(1). According to this result any smooth initial data set admits a unique, smooth, local (up to an isometry) *globally hyperbolic*(2) Cauchy development. In the case of nonlinear systems of differential equations the local existence and uniqueness result leads, through a straightforward extension argument, to a global result concerning the maximal time interval of existence. If this interval is bounded the solution must become infinite at its upper boundary. The formulation of the same type of result for the Einstein equations is a little more subtle; something similar was achieved in **[10]**.

THEOREM 1 (Bruhat-Geroch). — *For each smooth initial data set there exists a unique, smooth, maximal, future, globally hyperbolic development (MFGHD).*

(1) These allow one to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations.
(2) Any past directed, in-extendable causal curve of the development intersects Σ_0.

Thus any construction, obtained by an evolutionary approach from a specific initial data set, must be necessarily contained in its maximal development MFGHD. This may be said to solve the problem of global[(3)] existence and uniqueness in General Relativity; all further questions, one could say, concern the qualitative properties of these maximal developments. The central issue becomes that of existence and character of singularities

2.1. Special solutions

We recall that EVE admits a remarkable family of explicit, stationary solutions given by the two parameter family of Kerr solutions among which one distinguishes the Schwarzschild family of solutions, of mass $m > 0$,

$$g_S = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \tag{2.3}$$

Though the metric seems singular at $r = 2m$ it turns out that one can glue together two regions $r > 2m$ and two regions $r < 2m$ of the Schwarzschild metric to obtain a metric which is smooth along $\mathscr{H} = \{r = 2m\}$, see **[24]**, called the Schwarzschild horizon. The portion of $r < 2m$ to the future of the hypersurface $t = 0$ is a *black hole* whose future boundary $r = 0$ is singular. The similar region to the past of $t = 0$ is called a *white hole*. The region $r > 2m$, called the domain of outer communication, is free of singularities.

To see how to explicitly extend the metric, introduce the *tortoise* coordinate $r_* = r + 2m \ln(r/2m - 1)$ and the Kruskal null coordinates, $U = e^{-(t-r_*)/4m}$, $V = e^{(t+r_*)/4m}$, relative to which the metric takes the form $ds^2 = -\frac{32m^3 e^{-r/2M}}{r} dU dV + r^2 d\sigma^2$. Observe now that $r = 2M$ corresponds precisely to $U \cdot V = 0$. Indeed r is an implicit function of $U \cdot V$ through the relation $(\frac{r}{2m} - 1)e^{\frac{r}{2m}} = -UV$. In the new coordinates, after a simple conformal compactification, the completed space-time has the form given in Figure 3A.

Here the boundaries $\mathscr{I}^+$ and $\mathscr{I}^-$, called future and past null infinities, are idealized boundaries of the space-time corresponding to end points, of future directed, respectively past directed, null geodesics. The points $\mathbf{i}^+$ and $\mathbf{i}^-$ correspond to end points of future and past time-like geodesics while $\mathbf{i}^0$ corresponds to space-like infinity.

The Schwarzschild family is included in a larger two parameter family of solutions $\mathscr{K}(a, m)$ discovered by Kerr. A given Kerr space-time, with $0 \leq a < m$, has a well

(3) A proper definition of global solutions in GR requires a special discussion concerning the proper time of causal geodesics.

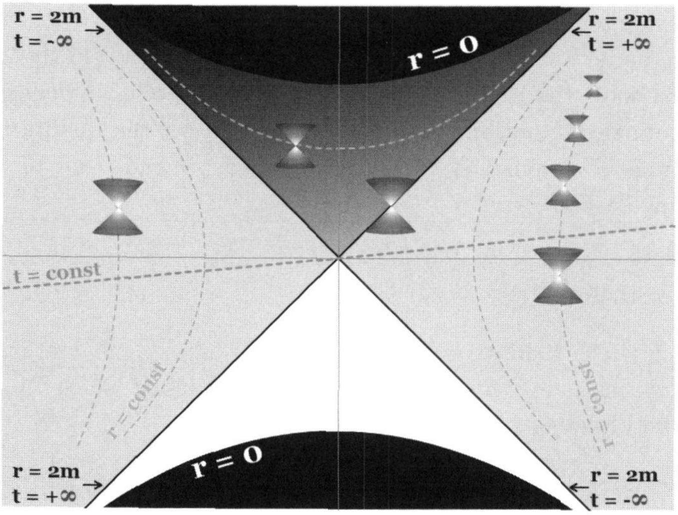


FIGURE 2. Kruskal's maximally extended Schwarzschild space-time. Note the two disconnected external regions, $r > 2m$, the black and white holes and the curvature singularity at $r = 0$. Note the behavior of light cones at the event horizon, $r = 2m$.

defined domain of outer communication $r > r_+ := m + (m^2 - a^2)^{1/2}$. In Boyer-Lindquist coordinates, well adapted to $r > r_+$, the Kerr metric has the form

$$\begin{aligned} g_K = & - \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\ & + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned}$$

with $\Sigma = r^2 + a^2 \cos^2 \theta, \Delta = r^2 + a^2 - 2mr$. As in the Schwarzschild case, the exterior Kerr metric extends smoothly across the Kerr event horizon, $\mathscr{H} = \{r = r_+\}$. It can be shown that the future and past sets of any point in the domain of outer communication intersect any time-like curve, passing through points of arbitrary large values of r, in finite time as measured relative to proper time along the curve. This fact is violated by points in the region $r \le r_+$, which consists of the union between a *black hole* region, extended towards the future, and a *white hole* region to the past. Thus physical signals (i.e. future time-like or null geodesics) which initiate at points in $r \le r_+$ cannot be registered by far away observers[(4)]. The extended Kerr is singular only at $r = 0$. Thus the singularities in Kerr cannot have any effect on the domain of outer communication which is, in fact, entirely smooth even analytic. The boundary

[(4)] They must end in the singularity at $r = 0$, in Schwarzschild space-time. Their behavior in Kerr is more complicated.

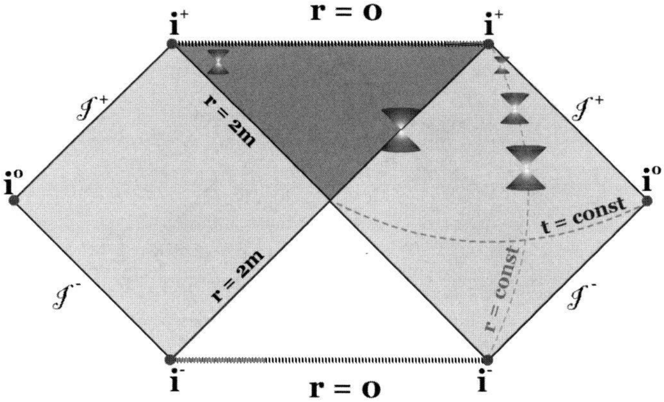


FIGURE 3A. Complete Penrose diagram of Schwarzschild. Note the black hole and white hall regions, singularity at $r = 0$, event horizon $r = 2m$ and the boundaries at infinity.

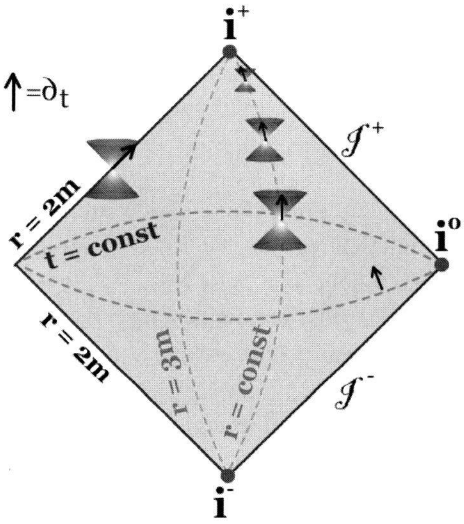


FIGURE 3B. The right disconnected exterior region of Schwarzschild. Note that $\mathbb{T} = \partial_t$ becomes null along the horizon $r = 2m$ and vanishes on the bifurcate sphere where the two branches of the horizon meet.

of the domain of outer communication $\{r = r_+\}$ is called the *event horizon*. In the non-degenerate case, $a < m$, the event horizon consists of two null hypersurfaces intersecting transversally on a compact 2 sphere.

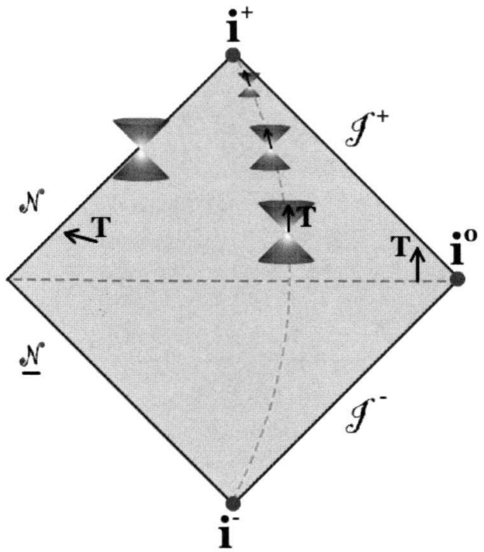


FIGURE 4A. Exterior domain of Kerr. Note that the stationary vector field $\mathbb{T}$, which is time-like in the far away (asymptotic) region of space-time, becomes space-like inside the ergo-region, near the horizon $\mathscr{H} = \mathscr{N} \cup \underline{\mathscr{N}}$.

The exterior Kerr metrics are *stationary*, which means, roughly, that the coefficients of the metric are independent of the time variable t. One can reformulate this by saying that the vector field $\mathbb{T} = \partial_t$ is Killing[5] (everywhere in the domain of outer communication) and time-like at points with r large, i.e. the so-called *asymptotic region* (where the space-time is close to flat). One can also easily check that $\mathbb{T}$ is tangent to the horizon $\mathscr{H} = \mathscr{N} \cup \underline{\mathscr{N}}$, which is itself a null hypersurface, i.e. the restriction of the metric to the tangent space to $\mathscr{H}$ is degenerate (see Figure 4A). In addition to being stationary the coefficients of the Kerr metric are independent of the circular variable ϕ. Thus Kerr is stationary and *axially symmetric*. The Schwarzschild metrics corresponding to $a = 0$ are not just axially symmetric but spherically symmetric, which means that the metric is left invariant by the whole rotation group of the standard sphere $\mathbb{S}^2$. A well known theorem of Birkhoff shows that they are the only such solutions of the vacuum Einstein equations. Another peculiarity of a Schwarzschild metric, not true in the case of Kerr, is that the stationary Killing vector field $\mathbb{T} = \partial_t$ is orthogonal to the hypersurface $t = 0$. A stationary space-time which has this property is called *static*. This is also equivalent to the fact that the Schwrazschild metric is invariant with respect to the reflection $t \to -t$. Moreover $\mathbb{T}$

[5] A vector field X is said to be Killing if its associated 1 parameter flow consists of isometries of g, i.e. the Lie derivative of the metric g with respect to X vanishes, $\mathscr{L}_X g = 0$.

is time-like for all $r > 2m$ and null along the Schwarzschild horizon $\mathscr{H} = \{r = 2m\}$. This is not the case for Kerr solutions in which case $\mathbb{T} = \partial_t$ is only time-like for $r > m + (m^2 - a^2\cos^2\theta)^{1/2}$, null for $r = m + (m^2 - a^2\cos^2\theta)^{1/2}$ and space-like in the region between r_+ and $r = m + (m^2 - a^2\cos^2\theta)^{1/2}$, called the *ergosphere*. Finally we remark that the Kerr family has unacceptable features for $a > m$.

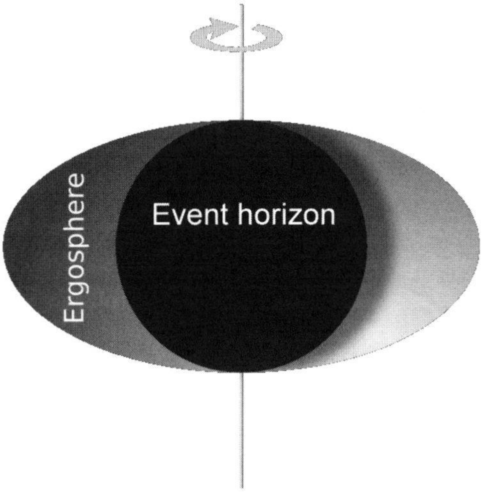


FIGURE 4B. Kerr solution, on a fixed slice, as a rotating black hole. Note the axis of symmetry and the presence of the ergosphere outside the event horizon.

To summarize:

1. The Kerr family $\mathscr{K}(a, m)$, $0 \le a \le m$ provides a two parameter family of asymptotically flat solutions of the Einstein vacuum equations exhibiting a smooth domain of outer communication and its complement, separated by the event horizon $\{r = r_+\}$. For $a < m$ the event horizon consists of two null hypersurfaces intersecting transversally on a compact 2 sphere.
2. All solutions are stationary, i.e they admit a Killing vector field $\mathbb{T}$ which is time-like in the *asymptotic region*. The Schwarzschild space-time (i.e. $a = 0$) is also static. Moreover the Kerr family is axially symmetric, i.e. it admits another, circular, Killing vector field $\mathbb{Z}$ which vanishes on the axis of symmetry. The Schwarzschild space-time is spherically symmetric.
3. The stationary vector field $\mathbb{T}$ is tangent along the horizon and space-like for all $a > 0$. It remains space-like in a small region of DOC called ergo-region. In the case $a = 0$, $\mathbb{T}$ is null along the horizon and time-like everywhere in DOC.
4. In all cases $0 \le a < m$, DOC contains trapped null geodesics, i.e. null geodesics which are entirely contained in a region of *DOC* with a bounded value of r.

In the case $a = 0$, all trapped null geodesics are either tangent to the time-like surface $\{r = 3m\}$ or asymptotic to it.

2.2. Stationary space-times

We formalize below the notion of an asymptotically flat stationary, vacuum, space-time. Assume that $(\mathcal{M}, \mathbf{g})$ is a smooth vacuum Einstein space-time of dimension $3+1$ and $\mathbb{T}$ is a smooth Killing vector field on $\mathcal{M}$. Assume given a space-like hypersurface $\Sigma_0 \subseteq \mathcal{M}$ such that outside a sufficiently large compact set K of Σ, every orbit of $\mathbb{T}$ intersects Σ_0 at only one point. Moreover we assume the existence of a coordinate system (x^0, x^1, x^2, x^3) in $\mathcal{M}^{(\mathrm{end})} = \mathbb{T}(\Sigma_0 \setminus K)$ (i.e. the union of orbits of $\mathbb{T}$ which intersect $\Sigma_0 \setminus K$) such that $\mathbb{T} = \partial_t$ and, with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, the components of the space-time metric verify[6], for some $k \geq 0$, $\mathbf{g}_{00} = -1 + \frac{2m}{r} + O_k(r^{-2})$, $\mathbf{g}_{ij} = \delta_{ij} + O_k(r^{-1})$, $\mathbf{g}_{0i} = -\epsilon_{ijk}\frac{2J^j x^k}{r^3} + O_k(r^{-3})$, for some $m > 0$, $J = (J^1, J^2, J^3) \in \mathbb{R}$ such that, $|J|^2 < m^2$. We can then define the exterior region, or domain of outer communication, by

$$\mathcal{E} = \mathcal{I}^-(\mathcal{M}^{(\mathrm{end})}) \cap \mathcal{I}^+(\mathcal{M}^{(\mathrm{end})}),$$

where $\mathcal{I}^-(\mathcal{M}^{(\mathrm{end})})$, $\mathcal{I}^+(\mathcal{M}^{(\mathrm{end})})$ denote, respectively, the past and future sets of $\mathcal{M}^{(\mathrm{end})}$. One further assumes that $\mathcal{E}$ is globally hyperbolic, i.e. any inextensible time-like or null curve in $\mathcal{E}$ must intersect Σ_0. Finally we define $\mathcal{B} \cup \mathcal{W}$, the union of the black hole and white hole regions, as the complement of $\mathcal{E}$ in $\mathcal{M}$ and the event horizon $\mathcal{H}$ as the boundary of $\mathcal{E}$. One can show that $\mathcal{H}$ is achronal (i.e. no points in $\mathcal{H}$ can be connected by time-like curves) and that $\mathbb{T}$ must be tangent to $\mathcal{H}$. One can also show, using the theorem of Hawking, that $\mathcal{H}$ is non-expanding (see appendix).

One can easily check that the event horizon $\mathcal{H}$ of any of the Kerr family $\mathcal{K}(a, m)$, $0 \leq a < m$, verifies the following properties:

1. $\mathcal{H}$ is spanned by two smooth null hypersurfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ which intersect transversally along a 2 sphere S. Moreover $\mathcal{N}$(resp. $\underline{\mathcal{N}}$) is spanned by the union of the future (past), in-extendible, complete, null geodesics orthogonal to S.
2. Both $\mathcal{N}$ and $\underline{\mathcal{N}}$ have vanishing null second fundamental forms (see appendix).

The second condition is in fact an easy consequence of the non-expanding nature of $\mathcal{H}$ and the Einstein vacuum equations. A fundamental conjecture in General Relativity is to prove that the converse is true, i.e. any, regular, stationary solution of the Einstein vacuum equations verifying the above properties must be isometric to $\mathcal{K}(a, m)$, $0 \leq a < m$. The simple motivation behind this conjecture is that one expects, due to gravitational radiation, that general, dynamic, solutions of the Einstein

(6) We denote by $O_k(r^a)$ any smooth function in $\mathcal{M}^{(\mathrm{end})}$ which verifies $|\partial^i f| = O(r^{a-i})$ for any $0 \leq i \leq k$.

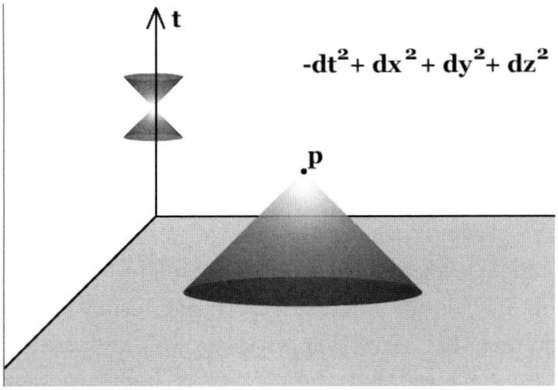


FIGURE 5A. Minkowski space in standard coordinates.

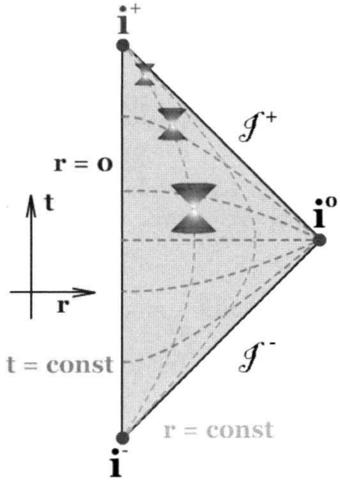


FIGURE 5B. Penrose diagram of the Minkowski space. Note that both the past of $\mathscr{I}^+$ and future of $\mathscr{I}^-$ exhaust the entire space.

field equations settle down, asymptotically, into a stationary regime. Thus the conjecture, if true, would give a description of all the asymptotic states of the Einstein vacuum equations. The conjecture is, essentially, solved in the analytic case (see **[14]** for an up to date account) and only partially solved in the category of smooth space-times, see **[26]** and **[1]**.

In the next section we attempt to give a somewhat precise formulation of the problem of stability of Kerr.

3. STABILITY OF KERR

3.1. Stability of the Minkowski space

The Minkowski space $\mathbb{R}^{3+1}$ is, of course, the simplest solution of the Einstein vacuum equations. Is it stable? Among all Kerr solutions, the Minkowski space is also the only one free of singularities, or geodesically complete. Roughly speaking this means that any freely moving observer in $\mathcal{M}$ can be extended indefinitely, as measured relative to its proper time. Such a space-time is said to have a regular MFGHD. Does this property persist under small perturbations?

The result stated below is a rough version of the global stability of Minkowski, the complete result also provides very precise information about the decay of the curvature tensor along null and time-like directions as well as many other geometric information concerning the causal structure of the corresponding space-time, see **[12]**, **[30]**, **[32]** and **[5]**. Of particular interest are *peeling properties* i.e. the precise decay rates of various components of the curvature tensor along future null geodesics.

THEOREM 2 (Global Stability of Minkowski). — *Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular MFGHD.*

3.2. Cosmic censorship

In general, however, we expect maximal developments to be incomplete, with singular boundaries. An important result in this direction is the recent formation of trapped surfaces result of D. Christodoulou **[11]**. Together with the well known singularity theorem of R. Penrose, his result shows that there exists a large class of regular initial data whose MFGHD is incomplete. The unavoidable presence of singularities, for sufficiently large initial data sets, as well as the analysis of explicit examples (such as Schwarzschild and Kerr) have led Penrose to formulate two fundamental conjectures, concerning the character of general solutions to the Einstein equations. Here I restrict my discussion only to the so called weak cosmic censorship conjecture (WCC), which is the only one relevant to the issue of stability of Kerr. To understand the statement of (WCC), consider the different behavior of null rays in Schwarzschild and Minkowski space-times. In Minkowski space light originating at any point $p = (t_0, x_0)$ propagates, towards future, along the null rays of the null cone $t - t_0 = |x - x_0|$. Any free observer in $\mathbb{R}^{1+3}$, following a straight time-like line, will necessarily meet this light cone in finite time, thus experiencing the event p. On the other hand, any point p in the trapped region $r < 2m$ of the Schwarzschild space, is such that all null rays initiating at p remain trapped in the region $r < 2m$. In particular events causally connected to the singularity at $r = 0$ cannot influence events in the domain of outer communication $r > 2m$, which is thus entirely free of singularities. The same holds true in any Kerr solution with $0 \leq a < m$.

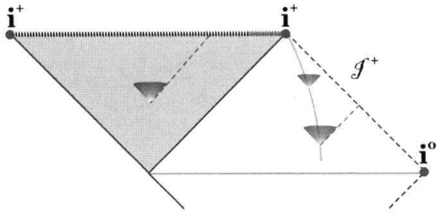


FIGURE 6. Behavior of null geodesics in the domain of outer communication by contrast to those in a black hole.

WCC is an optimistic extension of this fact to the future developments of general, asymptotically flat initial data. The desired conclusion of the conjecture is that any such development, with the possible exception of a non-generic set of initial conditions, has the property that any *sufficiently distant observer* will never encounter singularities or any other effects propagating from them. To make this more precise one needs define what a sufficiently distant observer means. This is typically done by introducing the notion of future null infinity $\mathscr{I}^+$ which, roughly speaking, provides end points for the null geodesics which propagate to asymptotically large distances. The future null infinity is constructed by conformally embedding the physical space-time $(\mathcal{M}, \mathbf{g})$ under consideration to a larger space-time[7] $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$, $\bar{\mathbf{g}} = \Omega^2 \mathbf{g}$ in $\mathcal{M}$, with a null boundary $\mathscr{I}^+$ (where $\Omega = 0, d\Omega \neq 0$).

DEFINITION 1. — *The future null infinity $\mathscr{I}^+$ is said to be complete if any future null geodesic along it can be indefinitely extended relative to an affine parameter.*

Weak Cosmic Censorship

Generic asymptotically flat initial data have maximal future developments possessing a complete future null infinity.

Using the language introduced above, we are finally ready to state the following.

CONJECTURE 3 (Global stability of Kerr). — *Any small perturbation of the initial data set of a Kerr space-time has a global future development with a complete future null infinity which, within its domain of outer communication*[8]*, behaves asymptotically like a (another) Kerr solution.*

4. STABILITY OF MINKOWSKI SPACE

To understand what would be needed in a proof of stability of Kerr it pays to review some of the main ideas in the proof of stability of the Minkowski space. For lack of space and time I will be very schematic. Also, for brevity, I will be discussing only the proof in [**12**] and [**30**]. I will just note that the proof in [**32**] is based also on a variation of the vector field method discussed below, even though the geometric set-up is different.

[7] Note however that the boundary of this extended space-time is not smooth, generically.
[8] That means, roughly, outside the black hole region.

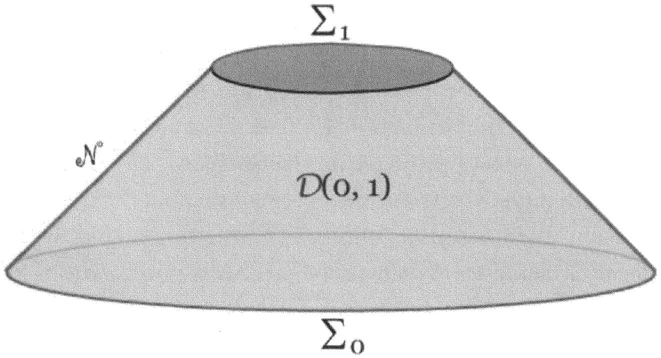


FIGURE 7. Domain of integration for Equation (4.2) with null boundary $\mathcal{N}$ and two space-like pieces, Σ_0, Σ_1.

4.1. Vector field method

The centerpiece, keystone, of the entire proof is a geometric method to derive decay estimates for components of the curvature tensor based on a generalization of the energy method for wave equations. The method has two distinct parts, a geometric version of the multiplier method and the method of commuting vector fields.

1. *Multiplier method.* — One starts with the Bianchi identities which, due to the vanishing of their trace, take the form of a Maxwell type system[9]:

$$\mathbf{D}_{[\epsilon}\mathbf{R}_{\alpha\beta]\gamma\delta} = 0, \qquad \mathbf{D}^{\delta}\mathbf{R}_{\alpha\beta\gamma\delta} = 0. \tag{4.1}$$

A remarkable feature of this system is the existence of a fully symmetric, traceless, covariant four tensor $\mathbf{Q}_{\alpha\beta\gamma\delta}$, depending quadratically on $\mathbf{R}$, which verifies the divergence condition

$$\mathbf{D}^{\delta}\mathbf{Q}_{\alpha\beta\gamma\delta} = 0,$$

and such that $\mathbf{Q}(X_1, X_2, X_3, X_4)$ is positive for any future, causal vector fields X_1, X_2, X_3, X_4. Thus, for any three such vector fields X, Y, Z we find with $\mathbf{Q}_{\mu}(X, Y, Z)$ the one form obtained by contraction with X, Y, Z,

$$\mathbf{D}^{\mu}\mathbf{Q}_{\mu}(X, Y, Z) = \mathrm{Err}(X, Y, Z) \tag{4.2}$$

$$\mathrm{Err}(X, Y, Z) = \frac{1}{2}\big(\mathbf{Q}({}^{(X)}\pi, Y, Z) + \mathbf{Q}(X, {}^{(Y)}\pi, Z) + \mathbf{Q}(X, Y, {}^{(Z)}\pi)\big)$$

where ${}^{(X)}\pi = \mathcal{L}_X\mathbf{g}$ is the deformation tensor of X. We integrate the above identity on past domains of dependence[10] $\mathcal{D}(0, 1)$, sandwiched between two space-like hypersurfaces Σ_0 and Σ_1 with future unit normals denoted by T.

[9] The two equations in (4.1) are in fact equivalent.
[10] $\mathcal{D}$ is such that the causal past set of any point in $\mathcal{D}$, in the slab between Σ_0, Σ_1, is included in $\mathcal{D}$.

Let $\mathcal{N}$ denote the null boundary of $\mathcal{D}(0,1)$ and L the geodesic null generator (i.e. $\mathbf{D}_L L = 0$ and $\mathbf{g}(L,L) = 0$), of $\mathcal{N}$, normalized by the condition $\mathbf{g}(L,T) = -1$ on $\Sigma_0 \cap \mathcal{N}$. Then, with $\mathbf{Q} = \mathbf{Q}[\mathbf{R}]$, $\mathrm{Err} = \mathrm{Err}[\mathbf{R}]$ as above,

(4.3)

$$\int_{\mathcal{N}} \mathbf{Q}(X,Y,Z,L) + \int_{\Sigma_1} \mathbf{Q}(X,Y,Z,T) = \int_{\Sigma_0} \mathbf{Q}(X,Y,Z,T) - \iint_{\mathcal{D}(0,1)} \mathrm{Err}(X,Y,Z).$$

Clearly, if X,Y,Z are Killing we deduce, $\mathrm{Err}[\mathbf{R}](X,Y,Z) = 0$, and thus derive a conservation law. In the particular case when the vector fields X,Y,Z are also causal we derive a very useful coercive estimate for the left-hand side of (4.3) in terms of the integral on Σ_0, which may be interpreted as initial condition. In view of the fact that the *energy-momentum* $\mathbf{Q}$ is traceless with respect to any pair of indices, the same remains true if we replace Killing vector fields by conformal Killing ones, i.e. such that ${}^{(X)}\pi$ is proportional to the metric $\mathbf{g}$ or, in other words, the traceless part ${}^{(X)}\hat{\pi}$ vanishes identically.

2. *Commuting vector field method.* — In addition to the procedure outlined above, the generalized energy method allows us to make use of commutation with selected vector fields. In fact, for any vector field X one can show that a suitable modified Lie derivative of $\mathbf{R}$, denoted by $\widehat{\mathcal{L}_X}\mathbf{R}$, verifies the following version of (4.1)

$$\mathbf{D}^\delta(\widehat{\mathcal{L}_X}\mathbf{R})_{\alpha\beta\gamma\delta} = J_{\alpha\beta\gamma}({}^{(X)}\hat{\pi}, \mathbf{R}). \tag{4.4}$$

We can thus replace $\mathbf{Q} = \mathbf{Q}[\mathbf{R}]$ with $\mathbf{Q}[\widehat{\mathcal{L}_X}\mathbf{R}]$ and repeat the procedure above to derive integral inequalities for suitable directional derivatives of $\mathbf{R}$.

The procedure outlined above, based on Killing and conformal Killing vector fields, seems to require a space-time with a lot of symmetries, such as the Minkowski space. It pays at this point to consider how the method works in that case.

4.2. Minkowski space $\mathbb{R}^{n+1}$

The Minkowski space $\mathbb{R}^{n+1}$ comes equipped with two important geometric structures:

I. *Family of Killing and conformal Killing vector fields*

- Generators of translations in the x^μ directions: $\mathbb{T}_\mu = \frac{\partial}{\partial x^\mu}$.
- Generators of rotations in the (μ,ν) plane: $\mathbb{L}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$.
- Generator of scaling: $\mathbb{S} = x^\mu \partial_\mu$.
- Generators of inverted translations[11]: $\mathbb{K}_\mu = 2x_\mu x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho)\frac{\partial}{\partial x^\mu}$.

(11) Observe that the vector fields $\mathbb{K}_\mu$ can be obtained applying the standard inversion to the vector fields $\mathbb{T}_\mu$.

Of particular importance for us are the causal vector fields $\mathbb{T}_0 = \partial_t$ and $\mathbb{K}_0 = (t^2 + r^2)\partial_t + 2tx^i\partial_i$, which can be used to derive coercive energy identities. Here $r^2 = |x|^2 = (x^1)^2 + \cdots (x^n)^2$.

II. *Canonical double null foliation.* — This is given by the level surfaces of two optical functions $u = t - r$ and $\underline{u} = t + r$, i.e. solutions of the Eikonal equation $\mathbf{m}^{\alpha\beta}\partial_\alpha u \partial_\beta u = \mathbf{m}^{\alpha\beta}\partial_\alpha \underline{u} \partial_\beta \underline{u} = 0$. With respect to $u, \underline{u}$ the vector fields $\mathbb{T}_0$, $\mathbb{S}$ and $\mathbb{K}_0$ take the form

$$\mathbb{T}_0 = \frac{1}{2}(L + \underline{L}), \quad \mathbb{S} = \frac{1}{2}(\underline{u}L + u\underline{L}), \quad \mathbb{K}_0 = \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L}) \tag{4.5}$$

where $L = -\mathbf{m}^{\alpha\beta}\partial_\beta u \partial_\alpha = \partial_t + \partial_r$ and $\underline{L} = -\mathbf{m}^{\alpha\beta}\partial_\beta \underline{u} \partial_\alpha = \partial_t - \partial_r$ are the null generators of the corresponding null hypersurfaces. Observe also that the rotation vector fields $\mathbb{L}_{ij} = x_i\partial_j - x_j\partial_i$ (denoted also by $\mathbb{O}_{ij}$) are tangent to the leaves of both foliations.

To see how these vector fields can be used consider solutions of the standard wave equation $\square\phi = 0$, with compactly supported data. Let $\mathbf{Q} = \mathbf{Q}[\phi]$ be the associated energy momentum tensor (see (6.2) in Section 6), i.e. $\mathbf{D}^\beta Q_{\alpha\beta} = 0$. The standard energy identity, associated to the time translation $\mathbb{T}_0 = \partial_t$ allows us to derive the standard energy conservation identity

$$\int_{\Sigma_t} |\partial\phi|^2 = \int_{\Sigma_0} |\partial\phi|^2 \le I_0$$

with I_0 a constant depending only on the initial data of ϕ and $|\partial\phi|^2 = \sum_{\alpha=0}^n |\partial_\alpha\phi|^2$. Using the causal conformal Killing vector field $\mathbb{K}_0$ (see details in Section 6 for dimensions $n \ge 3$), we can also estimate

$$\int_{\Sigma_t} |\phi|^2 \lesssim \int_{\Sigma_0} (1 + r^2)|\partial\phi|^2 \le I_0.$$

The Killing vector fields $\mathbb{T}_\mu$ and $\mathbb{L}_{\mu\nu}$ commute with $\square$, while $\mathbb{S}$ preserves the space of solutions to $\square\phi = 0$ (since $[\square, \mathbb{S}] = 2\square$). This leads us to introduce the *generalized Sobolev* norms

$$\mathcal{Q}_k[\phi](t) = \sum_{j=0}^{k} \sum_{X_{i_1}, .., X_{i_j}} \|\mathcal{L}_{X_{i_1}} \mathcal{L}_{X_{i_2}} ... \mathcal{L}_{X_{i_j}} \phi\|_{L^2(\Sigma_t)} \tag{4.6}$$

with the sum taken over all Killing vector fields $\mathbb{T}, \mathbb{L}$ and scaling vector field $\mathbb{S}$.

The crucial point of this method is that these generalized energy type norms are bounded by initial data, i.e.,

$$\mathcal{Q}_k[\phi](y) \lesssim \mathcal{Q}_k[\phi](0) \lesssim I_0.$$

PROPOSITION 1 (Global Sobolev inequalities). — *Let ϕ be an arbitrary function in R^{n+1} such that $\mathcal{Q}_k[\phi]$ is finite for some $k > \frac{n}{2}$. Then for $t > 0$, we have with $u = t - |x|$ and $\underline{u} = t + |x|$*

$$|\phi(t,x)| \le c\frac{1}{(1+\underline{u})^{\frac{n-1}{2}}(1+|u|)^{\frac{1}{2}}}\mathcal{Q}_k[\phi]. \tag{4.7}$$

Since $\mathcal{Q}_k[\phi]$ is bounded, for solutions of $\square\phi = 0$, depending only on initial data at $t = 0$, we deduce a strong, realistic, uniform decay estimate.

A similar analysis can be done for solutions of the Maxwell equations or the linearized Bianchi equations in Minkowski space. It is also important to realize that one can be more economical with the vector fields we use. Thus, for example, one can derive the same information using only the vector fields $\mathbb{T}_0$, $\mathbb{S}$, $\mathbb{K}_0$ and rotations $\mathbb{O}_{ij} = \mathbb{L}_{ij}$, $i, j = 1, \dots, n$. The upshot of the vector field method is that it allows us to derive realistic decay estimates by a flexible procedure which can be easily generalized to perturbations of the Minkowski space.

4.3. Deformation method

Since a general perturbation of Minkowski space cannot preserve any symmetries the best we can hope for is to substitute them by approximate symmetries. We are thus looking to replace some of the conformal Killing vector fields of Minkowski with *almost conformal Killing*, i.e. vector fields whose deformation tensors are sufficiently small so that we can still derive useful estimates for the curvature tensor. The idea is to define these vector fields starting from two special functions whose role is to replace the optical functions $u, \underline{u}$ of the Minkowski space. In the original proof of **[12]** this is done by choosing a suitable defined optical function u and a suitable time function t. The function $\underline{u}$ is then defined to be $\underline{u} = t - 2u$. In **[30]** one picks instead two exact optical functions u and $\underline{u}$. One can then define vector fields $\mathbb{T}_0, \mathbb{S}, \mathbb{K}_0$ by mimicking the formulas (4.5) (with $L = -\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta$, $\underline{L} = 2\mathbb{T}_0 - L$ and $\mathbb{T}_0$ the unit future normal to the maximal foliation Σ_t) and rotation vector fields by a geometric method tied u, t or $u, \underline{u}$. To make the method work we need to make sure that the errors generated in the energy inequalities derived above are sufficiently small. To see, very roughly, what this entails consider (in the case of the (t, u) foliations of **[12]**) a quantity of the form:

$$\mathcal{Q}(t) \quad = \quad \int_{\Sigma_t} \mathbf{Q}[\widehat{\mathcal{L}_{\mathbb{O}}}\mathbf{R}](\mathbb{K}_0, \mathbb{K}_0, \mathbb{T}_0, \mathbb{T}_0).$$

Based on the vector field method outlined above one can show that the time dependent quantity $\mathcal{Q}(t)$ verifies, schematically, an identity of the form

$$\mathcal{Q}(t) \quad = \quad \mathcal{Q}(0) + \mathcal{E}(t), \qquad \mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

with

$$\mathcal{E}_1(t) = \int_0^t ds \int_{\Sigma_s} Q[\widehat{\mathcal{L}_{\mathbb{O}}}\mathbf{R}](\mathbb{K}_0, \mathbb{K}_0, \mathbb{T}_0)$$

and $E_2(t)$ the additional error term generated by the right-hand side of (4.4). Here $Q(\widehat{\mathcal{L}_{\mathbb{O}}}\mathbf{R}, \mathbb{K}_0, \mathbb{K}_0, \mathbb{T}_0)$ is an expression quadratic in $\widehat{\mathcal{L}_{\mathbb{O}}}\mathbf{R}$ and linear in the deformation tensors of $\mathbb{K}_0$ and $\mathbb{T}_0$. To make this work, i.e. obtain a global bound for $\mathcal{Q}(t)$, by a Gronwall inequality, we see that we need appropriate (and compatible!) decay estimates for both $\mathbf{R}$ and the traceless parts of the deformation tensors of $\mathbb{K}_0, \mathbb{T}_0$ and $\mathbb{O}$.

We summarize the above considerations as follows:

1. The proof of stability of Minkowski space in **[12]** and **[30]** requires precise decay information for the curvature tensor $\mathbf{R}$.
2. In a first approximation one may assume that $\mathbf{R}$ verifies a linear field equation(12) in Minkowski space (linearized Bianchi). The vector field method allows one to derive realistic decay estimates for components of $\mathbf{R}$.
3. One can derive, essentially, the same decay estimates for the true curvature tensor of a perturbed solution of the Einstein equations, by a deformation method in which one deforms part of the geometric structure of the Minkowski space ((u, t) or $(u, \underline{u})$) and an appropriate number of conformal Killing vector fields (i.e. $\mathbb{T}_0, \mathbb{K}_0, \mathbb{S}_0$ and rotations $\mathbb{O}$). The key here is to derive, simultaneously, suitable decay estimates for $\mathbf{R}$ and the traceless parts of the deformation tensors of these vector fields. These estimates have to be strong enough to be able to control the error terms generated in the energy estimates.

4.4. Non-linear stability of Kerr

In view of the above discussion a proof of the non-linear stability of the Kerr family requires:

1. A robust method to derive decay estimates for linear field equations in a fixed Kerr background. Such a method has to take into account the geometric features of the Kerr metric, such as the event horizon, ergo-region and trapped null geodesics. It cannot rely only on the continuous symmetries of the Kerr metric, i.e. its Killing vector fields, which are both too limited and have serious degeneracies.

(12) Note also that the result of Lindblad-Rodnianski **[32]** is based on a linearization at the level of the metric, which brings in the standard wave equation.

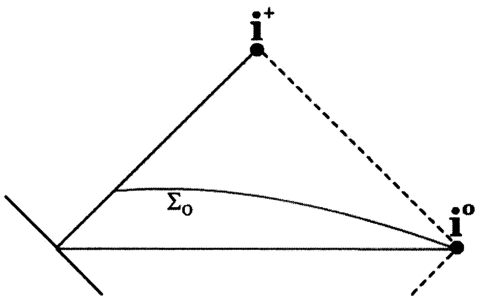
2. Find an effective linearization procedure, such as the linearized Bianchi equations[13] in the stability of Minkowski space, to which the methods sketched above apply.
3. Find a way to deform the geometry of the Kerr solution, taking into account that any small perturbation of a Kerr metric may lead, asymptotically, to a different Kerr metric.

5. LINEAR STABILITY OF THE KERR FAMILY

As discussed above a first, essential step, in the proof of stability of the Kerr solution is to establish its linear stability, which amounts to prove appropriate decay estimates for solutions to the specific linear field equations in a fixed Kerr background which arise by a suitable linearization. In a somewhat simplified version of linear stability, one would like to show, by robust methods, that all solutions of the covariant wave equation

$$\square_{\mathbf{g}}\phi = 0, \quad 0 \leq a < m, \tag{5.1}$$

in $\mathcal{K}(a,m)$, $0 \leq a < m$ (or more generally a fixed stationary, axially symmetric space-time with a non-degenerate horizon) with reasonable initial data on a space-like hypersurface Σ_0, as in the figure below, are *well behaved*[14] in the future of Σ_0 (see figure below).



A more elementary task, and yet very difficult in the rotating case[15], $a > 0$, is to show that solutions remain bounded in the entire exterior region of the space-time.

[13] Note however that the exact analog of the Bianchi equations in a Kerr background are ill posed.
[14] Decay at rates comparable to those in the flat case.
[15] The much simpler non-rotating case $a = 0$, corresponding to the Schwarzschild space-time, was solved previously in work by Kay and Wald.

5.1. Difficulties

The following are the main difficulties one has to overcome to prove linear stability, in the sense discussed above.

- $\mathcal{K}(a, m)$ has only two linearly independent Killing vector fields, the stationary one $\mathbb{T}$ and the axially symmetric one $\mathbb{Z}$. In the Schwarzschild case we have, of course, an action of the full rotation group $SO(3)$ and thus two linearly independent rotation vector fields.
- The stationary Killing field $\mathbb{T}$ degenerates in the ergo-region of $\mathcal{K}(a, m)$, i.e. it becomes space-like. The presence of an ergo-region is connected, physically, with the so called Penrose process according to which energy can be extracted from a rotating black hole and thus contribute to linear instability. This phenomenon is also known in the Physics literature as super-radiance. Even in Schwarzschild, $\mathbb{T}$ loses its time-like character on the horizon. Thus the basic energy identity provided by $\mathbb{T}$ loses information near the horizon, for $a = 0$, and loses coercivity, thus seemingly useless, for $a > 0$.
- $\mathcal{K}(a, m)$ possesses a family of trapped null geodesics, i.e. future null geodesics which neither go to $\mathcal{I}^+$ nor penetrate the black hole region. Though, fortunately, these are unstable they provide however very serious technical difficulties to derive decay information. In the case $a = 0$ the situation is somewhat simpler as one can show that all trapped geodesics are restricted, or asymptotic, to the surface $r = 3m$.

5.2. Main new ideas

I try to summarize below some of the main new ideas which have crystallized in the wake of the pioneering works of Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski, Tataru-Tohaneanu, Andersson-Blue, mentioned in the introduction.

- The introduction (by Dafermos-Rodnianski) of a new vector field defined in a neighborhood of the horizon (called the *red shift* vector field), which I will denote by $\mathbb{H}$, with coercive properties in a small neighborhood of the horizon, which compensates for the degeneracy of the stationary vector field $\mathbb{T}$.
- A robust mechanism, due to Dafermos-Rodnianski, for proving boundedness of solutions for Kerr space-times with $a \ll m$, despite the notorious problem of super-radiance. This is based on a decomposition, invariant relative to the actions of $\mathbb{T}$ and $\mathbb{Z}$, into super-radiant and sub-radiant modes and the properties of the red shift vector field $\mathbb{H}$.
- Discovery on an effective treatment of the trapped region, based on the fact that all trapped null geodesics are unstable. In Schwarzschild this can be achieved by a suitable modification of the so called Morawetz vector field, which I will denote

by $\mathbb{M}$. In $\mathcal{K}(a,m)$, for a small enough, there are three competing methods **[17]**, **[43]**, **[4]** which deal effectively with the trapped region. They all depend, in one form or another, on the integrability properties of the geodesic flow, remarkable fact due to Carter **[8]**.

- Decay in both Schwarzschild and Kerr is due to a third vector field, which is a suitable modification of $\mathbb{K}_0$ from Minkowski space(16). Recently, in **[18]**, Dafermos and Rodnianski gave a new, more flexible, treatment of how to generate decay from null infinity without using $\mathbb{K}_0$.
- Traditionally energy estimates require integration, using appropriate vector fields, on large causal domains. Thus one was restricted to look for vector fields which are coercive in such regions and, unfortunately, there are not enough of those. The new methods, especially those of Dafermos-Rodnianski, point the way to a more flexible use of vector fields by concentrating on specific geometric regions where degeneracies occur (such as the event horizon) and finding new *non-causal* vector fields (such as the red shift $\mathbb{H}$), which provides an effective cure for the missing information. The lack of causality of $\mathbb{H}$ can then be compensated by patching it with other vector fields, such as $\mathbb{T}$ or $\mathbb{M}$. A similar patching procedure can be implemented in a neighborhood of null infinity, see **[18]**.

5.3. Main results

The first result, on boundedness of solutions to the wave equation (5.1), applies to the exterior region of a fixed stationary, axially symmetric space-time $\mathcal{M}$, sufficiently close to Schwarzschild, see **[20]**.

THEOREM 3 (Boundedness). — *Any solution* (5.1) *with reasonable initial data on a space-like hypersurface Σ_0, is globally bounded*(17) *in the future of Σ_0. The result applies in particular to Kerr space-times $\mathcal{K}(a,m)$ with $a \ll m$. The same method can also be applied to derive boundedness of axially symmetric solutions of* (5.1) *for the whole range $0 \le a < m$.*

The next result concerns decay of solutions in the Schwarzschild case $a = 0$. The result is expressed relative to the pair of optical functions $u = t - r^*$ and $\underline{u} = t + r^*$ where $r^* = r + 2m\ln(r - 2m)$. Observe that along the horizon, to the future of Σ_0, we have $u = -\infty$ but, for the region we are interested in, we have $\underline{u}$ finite.

(16) Such a vector field is also used in the stability of the Minkowski space.
(17) It also has bounded, non-degenerate, total energy.

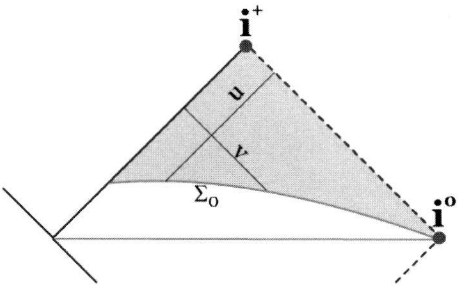


FIGURE 7A. Decay in Schwarzschild can be measured with respect to the double null foliation given by the level hypersurfaces of $u = t - r^*$ and $v = \underline{u} = t + r^*$.

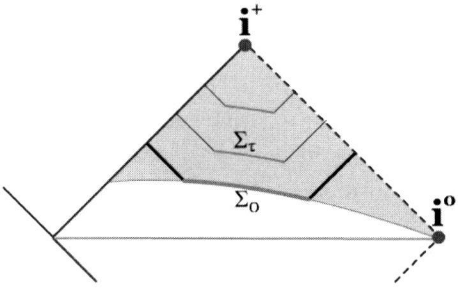


FIGURE 7B. Decay in Kerr can be measured with respect to a foliation Σ_τ obtained from Σ_0, by using the $\mathbb{T}$- flow. Note that Σ_0 consists of two null portions and a space-like one in the middle.

THEOREM 4. — *Let Σ_0 as in Figure 8A above, in the exterior of the Schwarzschild space-time $\mathcal{K}(0, m)$. Any solution to the wave equation (5.1), with reasonable initial data on Σ_0, admits the following estimates:*

1. *There exists a constant C such that, uniformly*[18] *on all points to the future of Σ_0*
$$|\phi| \le \frac{C}{\underline{u}}.$$
2. *For any $R > 2m$, we have, with a constant C_R, for all $r \ge R$,*
$$|r\phi| \le \frac{C_R}{|u|^{1/2}}.$$

A similar theorem can be stated and proved for $\mathcal{K}(a, m)$ with $a > 0$ sufficiently small. In this case however the functions $u = t - r^*$, $\underline{u} = t + r^*$ where $r_* = r + r_+ \ln(r - r_+)$, $r_+ = m + \sqrt{m^2 - a^2}$ are not optical functions. To avoid this problem one can measure decay in a different way. The idea is to start with hypersurface Σ_0, as in Figure 8B, and translate it using the flow ϕ_τ associated to the stationary Killing vector field $\mathbb{T} = \partial_t$. This defines a foliation $\Sigma_\tau = \phi_\tau(\Sigma_0)$.

THEOREM 5. — *Let Σ_0 and foliation Σ_τ defined as above (see Figure 8B), in the exterior of the Kerr space-time $\mathcal{K}(a, m)$, with a sufficiently small. Any solution to the wave equation (5.1), with reasonable initial data on Σ_0, admits the following estimates:*

[18] The result has been recently improved by J. Luk, see **[33]** using geometric methods. A similar result was also announced by Tataru in **[42]** using Fourier methods.

1. *There exists a constant C such that, uniformly*[19],
$$|r^{1/2}\phi| \leq C\tau^{-1+\delta}.$$
2. *Also, uniformly,*
$$|r\phi| \leq C\tau^{-\frac{1-\delta}{2}}.$$

6. VECTOR FIELD METHOD FOR THE WAVE EQUATION

We discuss here modifications of the vector field method for the wave equation in a globally hyperbolic Lorentzian space-time $(\mathcal{M}, \mathbf{g})$,

$$\square_{\mathbf{g}}\phi = 0. \tag{6.1}$$

Multiplier method. We start with the energy momentum tensor,

$$\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}[\phi] = \mathbf{D}_\alpha\phi\mathbf{D}_\beta\phi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\big(\mathbf{g}^{\mu\nu}\mathbf{D}_\mu\phi\mathbf{D}_\nu\phi\big). \tag{6.2}$$

One can easily check that $\mathbf{Q}_{\mu\nu}$ is symmetric and verifies the local conservation laws $\mathbf{D}^\nu\mathbf{Q}_{\mu\nu} = 0$ as well as the positive energy conditions $\mathbf{Q}(X,Y) \geq 0$, for all causal, future oriented vector fields X, Y. Unlike the Bel-Robinson tensor encountered above, the energy-momentum tensor of the wave equations is not traceless, in the interesting physical dimension $n = 3$. Indeed $\mathbf{g}^{\mu\nu}\mathbf{Q}_{\mu\nu} = -\frac{n-1}{2}\mathbf{g}^{\mu\nu}\mathbf{D}_\mu\phi\mathbf{D}_\nu\phi$.

Given a vector field X with deformation tensor ${}^{(X)}\pi = \mathcal{L}_X\mathbf{g}$, i.e., ${}^{(X)}\pi_{\alpha\beta} = \mathbf{D}_a X_\beta + \mathbf{D}_b X_\alpha$, we have

$$\mathbf{D}^\mu(\mathbf{Q}_{\mu\nu}X^\nu) = \frac{1}{2}\mathbf{Q}^{\mu\nu\,(X)}\pi_{\mu\nu}. \tag{6.3}$$

We integrate (6.3) on a past domain of dependence[20] sandwiched between two space-like hypersurfaces Σ_0 and Σ_1 with future unit normal T.

Let $\mathcal{N}$ denote the null boundary of the future set of $\mathcal{D}(0,1)$ and L the geodesic null generator (i.e. $\mathbf{D}_L L = 0$ and $\mathbf{g}(L,L) = 0$), of $\mathcal{N}$, normalized by the condition $\mathbf{g}(L,T) = -1$ on $\Sigma_0 \cap \mathcal{N}$.

Integrating (6.3) in $\mathcal{D}(0,1)$ we derive the formula

$$\int_{\mathcal{N}} \mathbf{Q}(X,L) + \int_{\Sigma_1} \mathbf{Q}(X,T) = \int_{\Sigma_0} \mathbf{Q}(X,T) - \iint_{\mathcal{D}(0,1)} \frac{1}{2}\mathbf{Q}\cdot{}^{(X)}\pi.$$

This formula is particularly useful if X is Killing and time-like in which case ${}^{(X)}\pi = 0$ and the two boundary integrands on the left are positive. In the particular case when $\mathbf{g}$ is the Minkowski metric and $X = \mathbb{T}_0 = \partial_t$ is the time derivative

[19] Note that the loss of δ was recently removed.

[20] $\mathcal{D}(0,1)$ is such that the causal past set of any point in $\mathcal{D}$, in the slab between Σ_0, Σ_1, is included in $\mathcal{D}$.

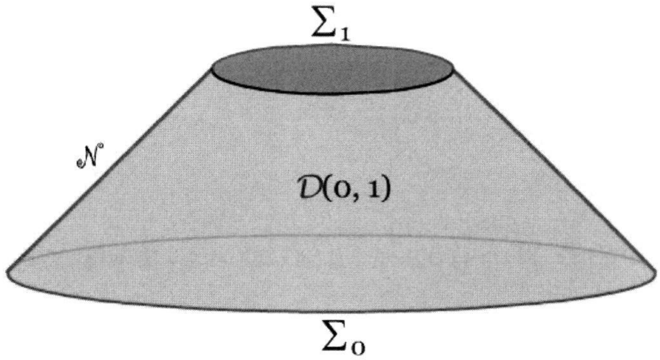


FIGURE 8C. Domain of integration for Equation (6.3) with null boundary $\mathcal{N}$ and two space-like pieces, Σ_0, Σ_1.

with respect to the standard coordinate t, we derive the standard law of conservation of energy. The method turns out to be useful, even if X is not Killing, by adding a lower order correction to the pointwise identity (6.3).

More precisely we modify the energy momentum $\mathbf{Q}$ as follows,

$$\mathbf{Q}_{(w)}(X,Y) \;=\; \mathbf{Q}(X,Y) + \frac{1}{2} w\,\phi \cdot Y(\phi) - \frac{1}{4} Y(w)\,\phi^2,$$

with w a scalar function to be chosen appropriately.

PROPOSITION 2. — *The following integral identity holds true in a past domain of dependence as above,*

$$\int_{\mathcal{N}} \mathbf{Q}_{(w)}(X,L) + \int_{\Sigma_1} \mathbf{Q}_{(w)}(X,T) \;=\; \int_{\Sigma_1} \mathbf{Q}_{(w)}(X,T) - \int_{\mathcal{D}(0,1)} \mathrm{Err}(\phi; w, X)$$

with integrand $\mathrm{Err} = \mathrm{Err}(\phi; w, X)$ *given by*

$$\mathrm{Err} = \frac{1}{2}\big(\mathbf{Q}\cdot {}^{(X)}\pi + w\cdot \mathbf{g}(d\phi, d\phi)\big) - \frac{1}{4}\Box(w)\phi^2. \tag{6.4}$$

Proof. — Consider $P_\mu = \mathbf{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\phi\mathbf{D}_\mu\phi - \frac{1}{4}\mathbf{D}_\mu w\phi^2$ and calculate its divergence, $\mathbf{D}^\mu P_\mu = \frac{1}{2}\pi^{\mu\nu}\mathbf{Q}_{\mu\nu} + \frac{1}{2}w\mathbf{D}^\mu\phi\mathbf{D}_\mu\phi - \frac{1}{4}(\Box w)\phi^2$ which we then integrate on our causal domain $\mathcal{D}$. □

REMARK. — Typically we want to choose $w = \frac{n-1}{2}\mathrm{tr}\,{}^{(X)}\pi$ to cancel the lagrangian term in $\pi^{\mu\nu}\mathbf{Q}_{\mu\nu} = \hat{\pi}^{\mu\nu}\mathbf{Q}_{\mu\nu} - \frac{n-1}{2}\mathrm{tr}\pi\mathbf{D}^\mu\phi\mathbf{D}_\mu\phi$. In some situations, as in Examples 2, 3 below, it pays to choose instead $w = \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi$.

Below are two important examples (both due originally to C. Morawetz) in Minkowski space, in a domain $\mathcal{D} = \{t,x)/t_0 \le t \le t_1, |x| \le t - r_0\} \subset \mathbb{R}^{n+1}$ sandwiched between $\Sigma_0 = \{t = t_0\}$ and $\Sigma_1 = \{t = t_1\}$. Thus, $L = \partial_t + \partial_r$ and $T = \mathbb{T} = \partial_t$.

Example 1. — Let X be the conformal Killing vector field $\mathbb{K}_0 = (t^2 + |x|^2)\partial_t + 2tx^i\partial_i$ with deformation tensor ${}^{(\mathbb{K}_0)}\pi = -4t\mathbf{m}$. Thus $\mathrm{tr}({}^{(\mathbb{K}_0)}\pi) = -4(n+1)t$. Since $\mathrm{tr}(\mathbf{Q}) = -\frac{n-1}{2}\mathbf{g}(d\phi, d\phi)$, we choose $w = \frac{n-1}{2}(\mathrm{tr}\,{}^{(\mathbb{K}_0)}\pi)$ to make the term $\mathbf{Q}\cdot{}^{(X)}\pi + w\mathbf{g}(d\phi, d\phi)$ vanish identically. We derive the conservation law, with $w = \frac{n-1}{2}(\mathrm{tr}\,{}^{(\mathbb{K}_0)}\pi)$:

$$\int_{\mathcal{N}} \mathbf{Q}_{(w)}(\mathbb{K}_0, L) + \int_{\Sigma_1} \mathbf{Q}_{(w)}(\mathbb{K}_0, T) = \int_{\Sigma_0} \mathbf{Q}_{(w)}(\mathbb{K}_0, T). \tag{6.5}$$

We can easily check that both $\int_{\Sigma_1} \mathbf{Q}_{(w)}(\mathbb{K}_0, T)$ and $\int_{\mathcal{N}} \mathbf{Q}_{(w)}(\mathbb{K}_0, L)$ are positive. In fact one can show, for $n \geq 3$ (see **[29]**), for a small constant $c > 0$, with $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$

$$\mathbf{Q}_{(w)}(\mathbb{K}_0, T) \geq c\big((t+r)^2|L\phi|^2 + (t-r)^2|\underline{L}\phi|^2 + r^2|\nabla\phi|^2 + |\phi|^2\big).$$

Example 2. — Start with $X = \partial_r$. We have

$${}^{(X)}\pi_{00} = {}^{(X)}\pi_{0i} = 0,\ {}^{(X)}\pi_{ij} = \frac{2}{r}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right), \quad i,j = 1,\dots,n.$$

Hence, $\mathrm{tr}\,{}^{(X)}\pi = \frac{2(n-1)}{r}$. Thus, choosing $w = \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi$, we have, with $\nabla\!\!\!/$ denoting the induced covariant differentiation on the spheres $S(t,r)$ of constant t and r

$$\mathbf{Q}\cdot{}^{(X)}\pi + w\,\mathbf{g}(d\phi, d\phi) = \mathbf{D}_\alpha\mathbf{D}_\beta\,{}^{(X)}\pi^{\alpha\beta} = \frac{2}{r}|\nabla\!\!\!/\phi|^2$$
$$\frac{1}{4}\Box_{\mathbf{g}}(w)\phi^2 = \frac{n-1}{4}\Delta\left(\frac{1}{r}\right)\phi^2.$$

In the particular case when $n = 3$, since $\Delta(\frac{1}{r}) = -4\pi\delta_0$, we deduce

$$\mathrm{Err}(\phi; w = \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi, X = \partial_r) = \frac{1}{r}|\nabla\!\!\!/\phi|^2 + 2\pi\delta_0\phi^2.$$

Therefore, with $w = \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi = \frac{2}{r}$

$$2\pi\int_{t_0}^{t_1}|\phi(t,0)|^2dt + \int_{\mathcal{D}}\frac{1}{r}|\nabla\!\!\!/\phi|^2 = \int_{\Sigma_0}\mathbf{Q}_{(w)}(\partial_r, \partial_t) - \int_{\Sigma_1}\mathbf{Q}_{(w)}(\partial_r, \partial_t) - \int_{\mathcal{N}}\mathbf{Q}_{(w)}(\partial_r, L).$$

One can easily bound the surface integrals on the right-hand side by energy estimates (using vector field $X = \mathbb{T} = \partial_t$) and thus derive a very useful space-time inequality for the left-hand side.

Example 3. — Take as vector field $X' = f(r)X = f(r)\partial_r$. We have, in general, ${}^{(fX)}\pi_{\alpha\beta} = f\,{}^{(X)}\pi_{\alpha\beta} + \mathbf{D}_\alpha f X_\beta + \mathbf{D}_\beta X_\alpha$, $\mathrm{tr}({}^{(fX)}\pi) = f\mathrm{tr}\,{}^{(X)}\pi + 2X(f)$. Hence, for $X = \partial_r$, we deduce

$$\begin{aligned}\mathbf{Q}\cdot{}^{(fX)}\pi &= f\mathbf{Q}\cdot{}^{(X)}\pi + 2f'(r)|\partial_r\phi|^2 - f'(r)\mathbf{g}(d\phi, d\phi)\\ &= f\,{}^{(X)}\pi(d\phi, d\phi) + 2f'|\partial_r\phi|^2 - \big(\frac{1}{2}f\mathrm{tr}\,{}^{(X)}\pi + f'\big)\mathbf{g}(d\phi, d\phi).\end{aligned}$$

Therefore, for $w = \frac{1}{2}\mathrm{tr}\,^{(fX)}$, since $^{(X)}\pi(d\phi, d\phi) = |\nabla\!\!\!/\phi|^2$ and $\mathrm{tr}\,^{(X)}\pi = \frac{2(n-1)}{r}$,

$$\begin{aligned}\mathbf{Q}\cdot\,^{(fX)}\pi + w\,\mathbf{g}(d\phi, d\phi) &= f(r)|\nabla\!\!\!/\phi|^2 + 2f'(r)|\partial_r\phi|^2\\ \frac{1}{4}\square w &= \frac{1}{4}\Delta\Big(\frac{(n-1)f(r)}{r} + f'(r)\Big).\end{aligned}$$

For $n = 3$, with F a primitive of f, i.e. $f(r) = F'(r)$, $\frac{1}{4}\square\frac{1}{2}\mathrm{tr}(^{(fX)}\pi) = \frac{1}{4}\Delta^2 F$ and, hence, with $w = w(fX) = \frac{1}{2}\mathrm{tr}(^{(fX)}\pi)$,

$$\mathrm{Err}(\phi; w, fX) = f(r)\,^{(X)}\pi(d\phi, d\phi) + 2f'(r)|\partial_r\phi|^2 - \frac{1}{4}\phi^2\,\Delta^2 F(r).$$

To obtain a coercive estimate we thus need $f, f' \geq 0$ and $\Delta^2 F \leq 0$. One can easily check that $f(r) = \frac{r^\lambda}{1+r^\lambda}$, $0 \leq \lambda \leq 1$ verifies these requirements. In the particular case $\lambda = 1$ we derive.

PROPOSITION 3. — *The following estimate holds true for arbitrary solutions of* $\square\phi = 0$ *in* $\mathbb{R}^{3+1}$, *for an arbitrary* $R > 0$,

$$\int_0^\infty \int_{|x|\leq R} (|\mathbf{D}\phi|^2 + |\phi|^2) \lesssim \int_{\Sigma_0} |\mathbf{D}\phi|^2.$$

To summarize: The multiplier method consists in finding vector fields X and scalars $w = w(X)$ such that at least one of the following statements holds true in a past causal domain $\mathcal{D}$:

- The vector field X is coercive, i.e. we have both $\mathrm{Err}(\phi; w(X), X) \geq 0$ and $\mathbf{Q}_{(w)}(X, L)$, $\mathbf{Q}_{(w)}(X, T)$ are positive at the future boundary of $\mathcal{D}$.
- The vector field X is positive, i.e. $\mathrm{Err}(\phi; w(X), X) \geq 0$, and we have a way to estimate the boundary terms along $\mathcal{N}$ and Σ_1.

In practice it is very hard to find good vector fields X which achieve either of the two conditions. As we have seen, in the stability of the Minkowski space, one defines vector fields X, analogous to $\mathbb{T}_0, \mathbb{K}_0$ on Minkowski space, such that the integrand Err is sufficiently small so that the corresponding space-time integral can be controlled. Also, as we shall see in the next section, it is very difficult to find globally defined vector fields X and scalars $w = w(X)$ for which $\mathrm{Err}(\phi; w(X), X)$ has definite sign. The new idea, pursued by Dafermos-Rodnianski, is to concentrate in regions of space-times, not necessarily causal domains (such as a small neighborhood of the event horizon in Schwarzschild or the entire ergo-region in Kerr), where the natural Killing vector fields of the space-time are degenerate and look for new vector fields for which $\mathrm{Err}(\phi; X.\lambda)$ has a sign in the restricted region. Once we control these degenerate regions we can hope to get a global coercive vector field by a patching procedure.

6.1. Commuting vector field method

As in the stability of the Minkowski space it is not enough to derive estimates by the multiplier method. One needs in addition to commute the equation with enough suitable vector fields. In the case of the wave equation this is provided by the following.

LEMMA 1. — *For an arbitrary vector field X we have,*

$$\square_{\mathbf{g}}(X\phi) = X(\square_{\mathbf{g}}\phi) - {}^{(X)}\pi^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi - \big(2\mathbf{D}^{\beta\,(X)}\pi_{\alpha\beta} - \mathbf{D}_\alpha(tr^{(X)}\pi)\big)\mathbf{D}^\alpha\phi.$$

In particular, if X is Killing and $\square_{\mathbf{g}}\phi = 0$ we infer that $\square_{\mathbf{g}}(X\phi) = 0$ and therefore we can apply to $X(\phi)$ the same multiplies method estimates as for ϕ. There are cases, however, where the error terms obtained by commutation are not small but contain instead terms which lead, by integration, to positive bulk integrals. This, as we shall see, is the case of the red shift vector field discussed below.

7. RED SHIFT

In **[17]** Dafermos and Rodnianski prove a general result concerning the existence of a red shift vector field in a neighborhood of a non-degenerate Killing horizon. This is a null hypersurface $\mathcal{N}$ with a null generator L (see appendix for definitions) which is the restriction to $\mathcal{N}$ of a Killing vector field $\mathbb{N}$, with complete orbits and flow $(\phi_\tau)_{\tau\geq 0}$, and such that $\omega = \mathbf{g}(\mathbf{D}_L L, \underline{L}) < 0$, for an adapted null companion[21] $\underline{L}$. It is easily seen that the future horizon of any $\mathcal{K}(a,m)$ with $0 \leq a < m$ verifies these assumptions. The result below, however, is a lot more general.

PROPOSITION 4 (Dafermos-Rodnianski). — *Given such a null hypersurface, there exist a neighborhood $\mathcal{U}$ of $\mathcal{N}$ and a strictly time-like, smooth vector field $\mathbb{H}$ on $\mathcal{U}$, both invariant*[22]*, with respect to the $\mathbb{N}$-flow ϕ_τ, $\tau \geq 0$, such that in $\mathcal{U}$, for a constant $c > 0$*

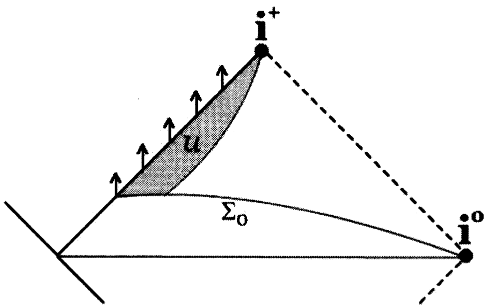
$$ {}^{(\mathbb{H})}\pi \cdot \mathbf{Q} \geq c\,\mathbf{Q}(\mathbb{H},\mathbb{H}). \tag{7.1}$$

Moreover, given any $\Lambda > 0$, we can choose $\mathbb{H}$ such that, all along $\mathcal{N}$

$$ {}^{(\mathbb{H})}\pi \cdot \mathbf{Q} \geq c\, e_3(\phi)^2 + \Lambda\big((e_4(\phi)^2 + |\nabla\!\!\!/\phi|^2\big).$$

The proof of the proposition is based on the following lemma.

(21) In fact ω can be made constant, related to the surface gravity of the Killing horizon.
(22) Or $\mathbb{T}$-invariant in a stationary metric such as Kerr.



LEMMA 2. — *Assume given a small portion of null hypersurface $\mathcal{N}$, in a neighborhood of a compact cross section S, with an adapted null pair (e_3, e_4) (see appendix) such that $\omega = \mathbf{g}(\mathbf{D}_4 e_4, e_3) < 0$. Extend e_3 in a small, space-time, neighborhood of S by solving the differential equation*

$$\mathbf{D}_X X \quad = \quad -A(X + N), \quad X|_{\mathcal{N}} = e_3,$$

where N is an arbitrary smooth extension of e_4 and A a sufficiently large positive constant, whose size depends on $\Lambda > 0$ below. Then, in a full neighborhood of S, along $\mathcal{N}$, we have

$$\frac{1}{2}\mathbf{Q} \cdot {}^{(X)}\pi \geq c e_3(\phi)^2 + \Lambda\big((e_4\phi)^2 + |\nabla\phi|^2\big). \tag{7.2}$$

Proof. — See appendix. □

The proof of the proposition follows easily by applying the lemma to the case when L is the restriction of the Killing vector field $\mathbb{N}$ (recall that $\mathcal{N}$ is a Killing horizon) with complete orbits. In that case it suffices to construct X in a small neighborhood of S (restricted, say, to a space-like hypersurface Σ passing through S) and then extend it by using the flow $(\phi_\tau)_{\tau\geq 0}$ of $\mathbb{N}$ (or $\mathbb{T}$ in a stationary space-time such as Kerr), in a whole neighborhood of the horizon of the form $\mathcal{U} = \cup_{\tau\geq 0}\phi_\tau(U)$ where U is a neighborhood of S in Σ. Since $\mathcal{L}_{\mathbb{N}}\, {}^{(X)}\pi = \mathcal{L}_{\mathbb{N}}\mathcal{L}_X \mathbf{g} = \mathcal{L}_X\mathcal{L}_{\mathbb{N}}\mathbf{g} = 0$ the positivity property of $\mathbf{Q} \cdot {}^{(X)}\pi$ on the neighborhood of S in Σ is preserved all through the neighborhood $\mathcal{U}$ of $\mathcal{N}$. Moreover, the same is true for the deformation tensor of the vector field $\mathbb{H} = \mathbb{N} + X$.

It remains, however, to check whether it is realistic to expect that $L = \mathbb{N}$ is both Killing and verifies the condition $-\omega > 0$. This property defines in fact *non-degenerate* Killing horizons. In the particular case of the Schwarzschild space-time one can check directly that the stationary Killing field $\mathbb{T}$ verifies both properties along the event horizon. The same is true for all Kerr solutions with $a < m$, but in that case the vector field $\mathbb{N}$ differs from $\mathbb{T}$, which is space-like on the horizon.

In fact the following general result holds true, see **[2]**.

PROPOSITION 5. — *Any non-expanding, bifurcate, null hypersurface* $(\mathcal{N}, \underline{\mathcal{N}}, S)$ *admits a future directed Killing vector field* $\mathbb{N}$*, defined in a neighborhood of* S*, which is tangent to the null generators of the horizon. Moreover, given an arbitrary null geodesic vector field* L *on* $\mathcal{N}$ *with affine parameter* $\underline{u}$*,* $\mathbb{N}$ *must be of the form* $\mathbb{N} = \kappa\, \underline{u}\, L$ *for some constant* $\kappa > 0$.

7.1. The red shift vector field as commutator

The red shift vector field provides useful estimates near horizon even when used as commutator. In view of Lemma 1 we have, with π the deformation tensor of vector field $\mathbb{H}$, $\square_{\mathbf{g}}(\mathbb{H}\phi) = -\pi^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi + \cdots$, where we ignore the terms linear in the first derivatives of ϕ, which may be assumed as having been already estimated. One can easily check that $\pi_{3a} = 0$, see Appendix 10.2. Thus, $\pi^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi$ does not contain the derivatives $\mathbf{D}_4\mathbf{D}_a\phi$. Hence (see Appendix 10.2), since $\pi_{44} = -2\omega, \pi_{34} = \underline{\omega}$,

$$-\pi^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi \;=\; 2\omega\mathbf{D}_3^2\phi - 2\underline{\omega}\mathbf{D}_3\mathbf{D}_4\phi + \pi_{4a}\mathbf{D}_3\mathbf{D}_a + \pi_{ab}\mathbf{D}_a\mathbf{D}_b + \cdots .$$

One can also eliminate the term $\mathbf{D}_3\mathbf{D}_4\phi$ using the equation $\square\phi = 0$ since the principal terms of $\square$, expressed relative to our null frame, are of the form $-2\underline{\omega}\mathbf{D}_3\mathbf{D}_4 + \delta^{ab}\mathbf{D}_a\mathbf{D}_b$. We deduce that

$$\square_{\mathbf{g}}(\mathbb{H}\phi) = 2\omega\mathbf{D}_3^2\phi + A_a\mathbf{D}_3\mathbf{D}_a\phi + B_{ab}\mathbf{D}_a\mathbf{D}_b\phi + \cdots \tag{7.3}$$

with bounded A, B. Now, when applying the multiplier method to (7.3), i.e. replacing ϕ with $\mathbb{H}(\phi)$ in the previous step, we can take advantage of the negative sign of $2\omega\mathbf{D}_3^2\phi$ and absorb all other second derivatives choosing the constant $\Lambda > 0$ in (7.2) sufficiently large.

7.2. Modified Morawetz vector field in Schwarzschild

To take care of the trapped region $r = 3m$ in Schwarzschild one needs to construct a vector field of the form $h(r)\partial_r$ similar to the one of Morawetz in Example 3 above. In fact it is better to work with the modified Regge-Wheeler coordinate $r^* = r + 2m\log(r - 2m) - 3m - 2m\log m$, such that $r* = 0$ for $r = 3m$. In these coordinates the Schwarzschild metric takes the form $\mu\big(-dt^2 + (dr^*)^2\big) + r^2 d\sigma^2$ with $\mu = (1 - \frac{2m}{r})$. Observe that $\frac{dr^*}{dr} = \mu^{-1}$ and $\partial_{r^*}\mu = \frac{2m}{r^2}\mu$. Hence $\Gamma^{r^*}_{r^*r^*} = \frac{m}{r^2} = \Gamma^{r^*}_{tt}$ and $\Gamma^t_{r^*r^*} = \Gamma^{r^*}_{r^*t} = \Gamma^{r^*}_{tt} = \Gamma^t_{tt} = 0$. Also, for an arbitrary orthonormal frame e_1, e_2 on the spheres of constants r and t, $\mathbf{D}_a e_b = \nabla_a e_b - \frac{\delta_{ab}}{r}\partial_{r^*}$.

We look for a vector field $X = f\partial_{r^*}$ and scalar $w = w(f)$ such that $\mathrm{Err}(\phi; w, X) \geq 0$ for an open neighborhood, in r^* of $r^* = 0$. To motivate the calculations consider first $X = \partial_{r^*}$ for which we can easily calculate the only non-xero components, with respect to the frame $\partial_t, \partial_{r^*}, e_1, e_2$ of its deformation tensor, i.e. ${}^{(X)}\pi_{tt} = -2\frac{m}{r^2}\mu$,

${}^{(X)}\pi_{r^*r^*} = 2\frac{m}{r^2}\mu$, $\pi_{ab} = \frac{2\mu}{r}\delta_{ab}$. Thus, $\mathrm{tr}\,{}^{(X)}\pi = \frac{4m}{r^2} + \frac{4\mu}{r}$. Also, since, $\mathbf{g}(d\phi, d\phi) = -\mu^{-1}(\partial_t\phi)^2 + \mu^{-1}(\partial_{r^*}\phi)^2 + |\nabla\!\!\!/\phi|^2$ we derive

$$\begin{aligned}\mathbf{Q}^{\alpha\beta}\,{}^{(X)}\pi_{\alpha\beta} &= {}^{(X)}\pi(d\phi, d\phi) - \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi\mathbf{g}(d\phi, d\phi)\\ &= \left(\frac{2m}{r^2} - \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi\right)\mathbf{g}(d\phi, d\phi) + 2\frac{r-3m}{r^2}|\nabla\phi|^2\\ &= 2\frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 - \frac{2\mu}{r}\mathbf{g}(d\phi, d\phi).\end{aligned}$$

Thus,

$$\frac{1}{2}\mathbf{Q}^{\alpha\beta}\,{}^{(X)}\pi_{\alpha\beta} = \frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 - \frac{\mu}{r}\mathbf{g}(d\phi, d\phi).$$

To eliminate the lagrangian term we are led to choose $w = \frac{\mu}{r}$ for which

$$\frac{1}{2}\mathbf{Q}_{(w)} \cdot {}^{(X)}\pi = \frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2$$

which, unlike the case of Minkowski space, does not have a definite sign.

We look for a modification of X of the form $fX = f(r^*)\partial_{r^*}$. As in Example 3 above we find

$$\begin{aligned}\mathbf{Q}\cdot{}^{(fX)}\pi &= f\mathbf{Q}\cdot{}^{(X)}\pi + 2\mathbf{Q}(df, X) = f\mathbf{Q}\cdot{}^{(X)}\pi + 2f'\mu^{-1}\mathbf{Q}(X,X)\\ &= 2f\frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 - \frac{2f\mu}{r}\mathbf{g}(d\phi, d\phi) + 2f'\mu^{-1}\mathbf{Q}_{r^*r^*}\\ &= 2f\frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 - \frac{2f\mu}{r}\mathbf{g}(d\phi, d\phi) + 2f'\mu^{-1}\left((\partial_{r^*}\phi)^2 - \frac{1}{2}\mu\mathbf{g}(d\phi, d\phi)\right)\\ &= 2f\frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 + 2f'\mu^{-1}(\partial_{r^*}\phi)^2 - \left(f' + \frac{2f\mu}{r}\right)\mathbf{g}(d\phi, d\phi). \qquad (7.4)\end{aligned}$$

Recalling Formula (6.4), with $w = f' + \frac{2\mu}{r}$, and setting $W = -\frac{1}{4}\Delta(w)$, we derive

$$\mathrm{Err}(\phi, w, f\partial_{r^*}) = f\frac{r-3m}{r^2}|\nabla\!\!\!/\phi|^2 + f'\mu^{-1}(\partial_{r^*}\phi)^2 + W\phi^2.$$

To obtain a coercive estimate we need to choose a function $f = f(r^*)$ such that $f' \geq 0$, $f\frac{r-3m}{r^2} \geq 0$ and $W \geq 0$. This cannot be done, but one can find an f which verifies the first two properties and such that $W > 0$ in a small neighborhood of $r = 3m$.

Therefore, if the function ϕ is given by its decomposition into spherical harmonics

$$\phi = \sum_{\ell \geq 0} \phi_\ell,$$

then for the part $\phi_L = \sum_{\ell \geq L} \phi_\ell$, with L sufficiently large, for which

$$\int_{\mathbb{S}^2} |\nabla\!\!\!/\phi_L|^2 \geq \frac{L(L+1)}{r^2}\int_{\mathbb{S}^2} |\phi_L|^2,$$

we can find an appropriate function f_L, bounded, increasing and vanishing at $r = 3m$, and a scalar w_L such that $\mathrm{Err}_L = \mathrm{Err}(\phi_L; w_L, f_L(r_*)\partial_{r^*})$ has the lower bound

$$\int_{\mathbb{S}^2} \mathrm{Err}_L \geq c \int_{\mathbb{S}^2} \left(f_L \frac{r-3m}{r^2} (|\nabla\!\!\!/ \phi_L|^2 + |\partial_t \phi_L|^2) + 2 f_L' \mu^{-1} (\partial_{r^*} \phi_L)^2 + F|\phi_L|^2 \right)$$

for some positive function F. For the remaining first L harmonics, one can find functions f_ℓ and scalars w_ℓ such that, for $\mathrm{Err}_l = \mathrm{Err}(\phi_\ell; w_\ell, f_\ell(r_*)\partial_{r^*})$

$$\int_{\mathbb{S}^2} \mathrm{Err}_l \geq c \int_{\mathbb{S}^2} \left(F(|\nabla\!\!\!/ \phi_\ell|^2 + |\partial_t \phi_\ell|^2) + 2 f' \mu^{-1} (\partial_{r^*} \phi_\ell)^2 + F|\phi_\ell|^2 \right).$$

Combining we obtain

$$\int_{\mathbb{S}^2} \mathrm{Err}(\phi) \geq c \int_{\mathbb{S}^2} \left(f \frac{r-3m}{r^2} (|\nabla\!\!\!/ \phi|^2 + |\partial_t \phi|^2) + 2 f' \mu^{-1} (\partial_{r^*} \phi)^2 + F|\phi|^2 \right).$$

Two alternative approaches for obtaining a positive definite quantity, without a decomposition into spherical harmonics, have been advanced. One relies on combining (7.4) with an appropriate choice of a scalar w and the red shift vector field, see **[35]**. The other, **[18]**, exploits a combination given by the expression

$$\mathbf{Q}_{w_1}[\phi] \cdot {}^{(f_1 X)}\pi + \mathbf{Q}_{w_2}[\mathbb{O}(\phi)] \cdot {}^{(f_2 X)}\pi$$

with angular momentum vector fields $\mathbb{O}$. In all of these approaches the generated expression degenerates, relative to the principle terms, at the photon-sphere $r = 3m$, thus necessitating a loss of regularity to obtain a non-degenerate estimate.

The corresponding construction in Kerr with small angular momentum is much more subtle, as the trapped set is no longer confined to a co-dimension one manifold $r = 3m$ in physical space. Its structure has to be now captured in the cotangent space, where it is governed by the geodesic flow. In Kerr, the geodesic flow is integrable, which equivalently can be expressed in terms of the separability of the wave equation—respecting the decomposition

$$\phi(t, r, \varphi, \theta) = \sum_{m \geq 0} \int e^{i\omega t} e^{im\varphi} \sum_{\lambda} S_{\lambda,m}(a\omega, \theta) u^{\omega}_{\lambda,m}(r),$$

where $S_{\lambda,m}(a\omega, \theta)$ are the oblate spheroidal harmonics and λ is the Carter constant—an additional, to ω and m, integral of motion—or existence of a Killing (Carter) tensor. In the Kerr case with $a \ll m$, the (degenerate) analog of the Morawetz estimate can be derived with the help of three different approaches. In the first, one replaces a vector field $f(r^*)\partial_{r^*}$ by an appropriately constructed pseudo-differential operator, **[43]**. The second approach, **[17]**, combines different X estimates with appropriately defined functions $f^{\omega}_{m,\lambda}$ and scalars $w^{\omega}_{m,\lambda}$ dependent on the geometric frequencies ω, m, λ. In the third approach, **[4]**, one explores a combination of X type identities for ϕ and for the quantity obtained by fusing the Carter tensor and ϕ. All three approaches rely on the integrability of the geodesic flow and in particular imply the estimate

$$\int_{\Sigma_t} \mathrm{Err}(\phi) \geq c \int_{\Sigma_t} \left(f_1(|\nabla\!\!\!/\phi|^2 + |\partial_t\phi|^2) + f_2(\partial_{r^*}\phi)^2 + F|\phi|^2\right)$$

for some nonnegative function f_1, vanishing in a neighborhood of $r = 3m$, and positive functions f_2 and F.

8. BOUNDEDNESS RESULTS

8.1. Simplest case

Consider first a static space-time $(\mathcal{M}, \mathbf{g})$ which is the MFGHD of an initial data set Σ_0 and such that the Killing vector field $\mathbb{T}$ is everywhere time-like and orthogonal to Σ_0 (23). Let t be the time function associated to t, i.e $\mathbb{T}(t) = 1$ and $t = 0$ on Σ_0. Starting with a local system of coordinates $x = (x^1, \dots, x^n)$ on Σ_0 and parametrizing points along the orbits γ of $\mathbb{T}$ by the parameter t and the x coordinates on $\gamma \cap \Sigma_0$ we easily see that $\mathcal{M} = \Sigma_0 \times \mathbb{R}$ and the space-time $\mathbf{g}$ metric takes the form

$$\mathbf{g} = -n^2(x)dt^2 + g_{ij}(x)dx^i dx^j, \tag{8.1}$$

with $x = (x^1, \dots, x^n)$ an arbitrary coordinate system on Σ_0 and g a Riemannian metric. Our assumptions imply, for a sufficiently small constant λ_0, uniformly in $\mathcal{M}$

$$\lambda_0 \leq n \leq \lambda_0^{-1}, \qquad \lambda_0|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq \lambda_0^{-1}|\xi|^2.$$

Also, relative to our system of coordinates, $\mathbb{T} = \partial_t$. We normalize $\mathbb{T}$ by introducing the vector field $e_{(0)} = n^{-1}\mathbb{T} = n^{-1}\partial_t$, unit future normal to the space-like foliation Σ_t defined by the level surfaces of t. We decompose a space-time vector field X relative to the unit time-like $e_{(0)}$,

$$X = X^0 e_{(0)} + \underline{X}, \qquad \mathbf{g}(e_{(0)}, \underline{X}) = 0, \tag{8.2}$$

and define the positive definite Riemannian metric,

$$h(X, Y) = X^0 \cdot Y^0 + g(\underline{X}, \underline{Y}). \tag{8.3}$$

Given an arbitrary tensor-field π we denote by $|\pi|$ its norm with respect to the metric h.

(23) This implies, in particular, that all orbits of $\mathbb{T}$ are complete, see **[13]**, and must intersect Σ_0 (orthogonally), see **[15]**.

PROPOSITION 6. — *Any solution ϕ of the wave equation*

$$\Box_{\mathbf{g}}\phi = 0, \qquad \phi|_{t=0} = \phi_{(0)}, \quad \partial_t\phi|_{t=0} = \phi_{(1)} \tag{8.4}$$

with smooth, compactly supported, initial data $\phi_{(0)}, \phi_{(1)}$ on Σ_0 is globally bounded.

Proof. — According to our general procedure we have, with $\mathbf{Q}$ the energy-momentum tensor

$$\int_{\Sigma_t} Q(\mathbb{T}, e_{(0)}) = \int_{\Sigma_0} Q(\mathbb{T}, e_{(0)}) \leq C. \tag{8.5}$$

Hence, since $Q(\mathbf{T}, e_{(0)}) = \frac{1}{2}n|\mathbf{D}\phi|^2$, and $\lambda_0 \leq n \leq \lambda_0^{-1}$ we deduce

$$\int_{\Sigma_t} |\mathbf{D}\phi|^2 \leq \lambda_0^{-2} \int_{\Sigma_0} |\mathbf{D}\phi|^2 \lesssim C \tag{8.6}$$

with a constant C depending only on λ_0 and the initial data. In view of our definition above, we have $|\mathbf{D}\phi|^2 = (e_{(0)}\phi)^2 + |\nabla\phi|^2$, where ∇ denotes the induced covariant derivative on Σ_t. We plan to bound the L^∞ norm of ϕ in terms of the L^2 norms of its higher derivative, according to Sobolev inequality, $\|\phi(t)\|_{L^\infty} \lesssim \sum_{i=1}^{s} \|\nabla^i\phi(t)\|_{L^2}$ for $s > \frac{3}{2}$. To get the higher derivative we commute $\Box$ with $\mathbf{T}$. Since $\mathbb{T}$ is Killing we must have $\Box\mathbb{T}(\phi) = 0$ and therefore, repeating the first step

$$\int_{\Sigma_t} |\mathbf{D}(\mathbb{T}\phi)|^2 \lesssim C$$

from which, in particular, $\int_{\Sigma_t} |\partial_t^2\phi|^2 \lesssim C$. Now we can write[24] $\Box = -n^{-2}\partial_t^2 + \Delta_{\mathbf{g}}$ from which we infer that $\|\Delta_{\mathbf{g}}\phi\|_{L^2(\Sigma_t)}$ is uniformly bounded. Using the Bochner identity for Δ_g, the boundedness of the curvature tensor of g (and of derivatives of n) and the first derivative estimates already established we then deduce that $\|\nabla^2\phi\|_{L^2(\Sigma_t)}$ is uniformly bounded in t. Using the vanishing of ϕ at infinity (on each Σ_t) and elliptic estimates, we can also derive a bound for $\|\phi\|_{L^2(\Sigma_t)}$. We can repeat the procedure, by commuting $\Box_{\mathbf{g}}$ once more with $\mathbb{T}$, to establish bounds for all higher derivatives $\|\nabla^k\phi\|_{L^2(\Sigma_t)}$, $k \geq 0$. Thus, by Sobolev, ϕ is uniformly bounded. □

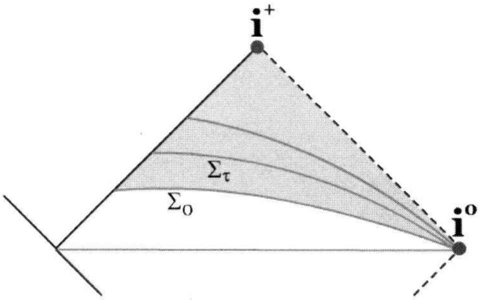
8.2. First degenerate case

We assume next the more realistic hypothesis that $\mathbb{T}$ is not time-like everywhere but degenerates in fact along a horizon, i.e. a null hypersurface $\mathcal{N}$ along which $\mathbb{T}$ is tangent to its generators. This, of course, is the situation in Schwarzschild. Since we have to work with space-like hypersurfaces transversal to the horizon we will not make use[25] of the condition that $\mathbb{T}$ is hypersurface orthogonal. We choose an original

(24) Note that $\Delta_{\mathbf{g}}$ differs from Δ_g by first order terms in $\nabla\phi$.

(25) It can be shown, however, that a stationary space-time with $\mathbb{T}$ tangent to the generators of the horizon must be in fact static.

space-like hypersurface Σ_0 and translate it using the flow of $\mathbb{T}$ to obtain a space-like foliation Σ_τ, as in the picture below.



It is easy to show that away from the horizon we still have $\mathbf{Q}(\mathbb{T}, e_0) \geq C|\mathbf{D}\phi|^2$. The constant C however degenerates as we approach the horizon. Yet some control remains. Thus, precisely on the horizon, we have using an adapted null frame, as in appendix, normalized such that $e_0 = \frac{1}{2}(e_3 + e_4)$, and such that $\mathbb{T} = -\omega e_4$ with $\omega = \mathbf{g}(\mathbf{D}_4 e_4, e_3) < 0$. Therefore, the energy density $\mathbf{Q}(\mathbb{T}, e_0) = -\frac{1}{2}\omega\big(\mathbf{Q}(e_4, e_4) + \mathbf{Q}(e_3, e_4) = -\frac{1}{2}\omega\big((e_4\phi)^2 + |\nabla\!\!\!/\phi|^2\big)$. In other words we are only missing the transversal derivative $e_3(\phi)$. Similarly, the flux density $\mathbf{Q}(\mathbb{T}, e_4) = -\omega\mathbf{Q}(e_4, e_4) = -\omega|e_4(\phi)|^2$, i.e we are missing the angular derivatives $\nabla\!\!\!/\phi$. Through a clever argument Kay and Wald, see **[27]**, were able to overcome these difficulties and still derive a boundedness result without using any new vector field.

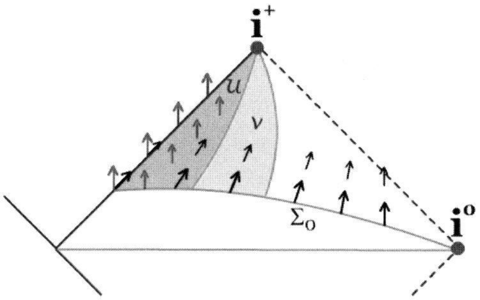
PROPOSITION 7. — *Any solution ϕ of the wave equation* (8.7)

$$\square_{\mathbf{g}}\phi = 0, \qquad \phi|_{t=0} = \phi_{(0)}, \quad \partial_t\phi|_{t=0} = \phi_{(1)} \tag{8.7}$$

in Schwarzschild space-time with smooth initial data $\phi_{(0)}, \phi_{(1)}$ on Σ_0, decaying sufficiently fast at infinity, is globally bounded in the domain of outer communication $\mathbf{E}$.

Proof. — The red shift vector field of Dafermos-Rodnianski provides a far more powerful and compelling proof, which holds in fact for any stationary space-time in which $\mathbb{T}$ is everywhere time-like in the complement of the event horizon. The idea is that, precisely in a neighborhood of the horizon $\mathscr{N}$, where the energy identity due to $\mathbb{T}$ becomes degenerate, we gain the missing information from the red shift vector field $\mathbb{H}$. Indeed, along the horizon $\mathbb{H}$ is future time-like. Hence the energy density and flux density associated to the red shift vector field $\mathbb{H}$ provide precisely the information we would get from the Killing field $\mathbb{T}$ if there was no degeneracy at $\mathscr{N}$.

So far this information is purely local. To obtain a useful estimate we need to also make use of the fact that ${}^{\mathbb{H}}\pi \cdot \mathbf{Q} \geq c\!\!\!,\,\mathbf{Q}(\mathbb{H}, \mathbb{H})$ in a space-time neighborhood $\mathscr{U}$ of $\mathscr{N}$, as in Proposition 4. We first extend $\mathbb{H}$ to our entire domain $\mathscr{D} = \mathscr{I}^+(\Sigma_0) \cap \mathscr{E}$ by making sure that it coincides with $\mathbb{T}$ away from a slightly larger, $\mathbb{T}$-invariant neighborhood $\mathcal{V}$. We can also arrange that the extended $\mathbb{H}$ is also $\mathbb{T}$ invariant and that, in $\mathcal{V} \setminus \mathcal{U}$, we have $|\, \mathbf{Q} \cdot {}^{\mathbb{H}}\pi \,| \lesssim \mathbf{Q}(\mathbb{T}, \mathbb{T})$. Indeed this can be first arranged on Σ_0, by an extension of the form $f\mathbb{H} + (1-f)\mathbb{T}$ (with a smooth f such that $f = 1$ on $\mathcal{U} \cap \Sigma_0$ and $f = 0$ in the complement of $\mathcal{V} \cap \Sigma$) and then extended to the entire domain $\mathcal{D}$ by using the pushforward with ϕ_τ.



We then apply Proposition 2 for vector field $X = \mathbb{H}$ and $w = 0$, in the domain $\mathcal{D}(0,\tau)$, the region of $\mathcal{D}$ between Σ_0 and Σ_τ. Since ${}^{\mathbb{H}}\pi = 0$ in the complement of $\mathcal{V}$ we have

$$\int_{\mathcal{N}(0,\tau)} \mathbf{Q}(\mathbb{H}, e_4) + \int_{\Sigma_\tau} \mathbf{Q}(\mathbb{H}, e_0) = \int_{\Sigma_\tau} \mathbf{Q}(\mathbb{H}, e_0) - \iint_{\mathcal{U}(0,\tau)} {}^{\mathbb{H}}\pi \cdot \mathbf{Q} - \iint_{\mathcal{V}(0,\tau)\setminus\mathcal{U}(0,\tau)} {}^{\mathbb{H}}\pi \cdot Q.$$

Since ${}^{\mathbb{H}}\pi \cdot \mathbf{Q} \geq c|\mathbf{D}\phi|^2$ in $\mathcal{U}$, $|\, {}^{\mathbb{H}}\pi \cdot \mathbf{Q}| \lesssim |\mathbf{D}\phi|^2$ in $\mathcal{V} \setminus \mathcal{U}$ and $\int_{\mathcal{N}(0,\tau)} \mathbf{Q}(\mathbb{H}, e_4) \geq 0$, $\mathbf{Q}(\mathbb{H}, e_0) \geq |\mathbf{D}\phi|^2$, we deduce[26]

$$\begin{aligned} F(\tau) := \int_{\Sigma_\tau} |\mathbf{D}\phi|^2 &\lesssim \int_{\Sigma_0} |\mathbf{D}\phi|^2 - \iint_{\mathcal{U}(0,\tau)} |\mathbf{D}\phi|^2 + \int_{\mathcal{D}(0,\tau)\setminus\mathcal{U}(0,\tau)} |\mathbf{D}\phi|^2 \\ &\lesssim F(0) - \int_0^\tau F(\tau')d\tau' + \int_0^\tau \int_{\Sigma_{\tau'}\setminus\mathcal{U}} |\mathbf{D}\phi|^2 \\ &\lesssim F(0) - \int_0^\tau F(\tau')d\tau' + C\tau. \end{aligned}$$

Thus, by Gronwall we derive a global bound for $F(\tau)$, i.e. a bound for the L^2 norm of all first derivatives of ϕ.

To estimate the higher derivative we commute the wave equation not only with $\mathbb{T}$ but also with the red shift vector field $\mathbb{H}$. Indeed, commutation with $\mathbb{T}$ provides estimates for $\sup_{\tau\geq 0} \|\mathbf{D}\mathbb{T}(\phi)\|_{L^2(\Sigma_\tau)}$, estimate which degenerates only near the horizon $\mathcal{N}$. This degeneracy is more than compensated by commuting the wave operator

[26] In the last line of the inequality below we make use of the fact that, away from the neighborhood $\mathcal{U}$ of the horizon, the energy identity provided by $\mathbf{T}$ gives us a bound for $\int_{\Sigma_{\tau'}\setminus\mathcal{U}} |\mathbf{D}\phi|^2$ in terms of initial conditions.

with $\mathbb{H}$. Thus, repeated commutations with $\mathbb{T}$ and $\mathbb{H}$ and elliptic theory, as in the simpler case explained below, provide bounds for all higher derivatives of ϕ. $\square$

8.3. The super-radiant regime

The method of proof described above can be extended to the case when the vector field $\mathbb{T}$ becomes space-like in a neighborhood of the horizon, as is the case in Kerr. The major difficulty in this case is that the global energy density associated $\mathbb{T}$ is not positive definite in the ergo-region and therefore ceases to provide any useful information, at least in a first approximation. The effect of super-radiance is well described in the physics literature, starting with the pioneering work of Penrose [**36**] and Zel′dovich [**44**], and provides an amplification mechanism for linear waves.

Nevertheless, Dafermos-Rodnianski were able to extend their methods to cover the case of axially symmetric stationary space-times which are sufficiently close to Schwarzschild. Thus, in addition to $\mathbb{T}$ the space-time has a second Killing vector field $\mathbb{Z}$, with circular orbits, tangent to the horizon $\mathcal{N}$. One can show, in this case, for a constant $\gamma > 0$, and a suitably defined null generator $L = e_4$, $\mathbb{T} = L - \gamma\mathbb{Z}$ along the horizon $\mathcal{N}$. In other words $\mathcal{N}$ is also a Killing horizon for a Kerr space-time.

Thus, the flux density associated to $\mathbb{T}$ is $\mathbf{Q}(\mathbb{T}, L) = \mathbf{Q}(L - \gamma\mathbb{Z}, L) = |\mathbf{L}\phi|^2 - \gamma(\mathbb{Z}\phi)(\mathbf{L}\phi) = (\mathbb{T}\phi)^2 + \gamma(\mathbb{Z}\phi)(\mathbb{T}\phi)$. Therefore, if $|\mathbb{T}\phi| > \gamma|\mathbb{Z}\phi|$, we must have $\mathbf{Q}(\mathbb{T}, L) > 0$. This suggests a decomposition of $\phi = \phi_\sharp + \phi_\flat$ such that $\mathbf{Q}[\phi_\sharp](\mathbb{T}, L) \geq 0$. It can be made precise by decomposing ϕ with respect to Fourier frequencies $\omega \in \mathbb{R}$ relative to $\mathbb{T}$, and discrete frequencies m, relative to $\mathbb{Z}$. Thus, by a simple cut-off, $\phi_\sharp$ will be restricted to the frequency range $|\omega| > |\gamma|m$, called *sub-radiant regime*, while $\phi_\flat$, the super-radiant part of ϕ, has frequencies in the range $\omega \leq \gamma m$. We expect that the arguments used in the previous subsection would work to treat the non super-radiant part $\phi_\sharp$, for which $\mathbb{T}$ continues to provide a coercive energy identity. The real new issue is $\phi_\flat$. One can show, and this is the main new insight of Dafermos-Rodnianski [**20**], that in stationary axisymmetric space-times near Schwarzschild, in particular in $\mathcal{K}(a, m)$ with $a \ll m$, the super-radiant frequencies of $\square_g\phi = 0$ are not trapped. The quantitative manifestation of this fact is reflected in the existence of a "simple" vector field $X = f(r^*)\partial_{r^*}$ and a scalar $w = w(X)$ with the property that

$$\mathbf{Q}_{(w)}[\phi_\flat] \cdot {}^{(X)}\pi \geq C_{r_1,r_2}\chi_{r_1,r_2}(|\mathbf{D}\phi_\flat|^2 + |\phi_\flat|^2)$$

with a characteristic function χ_{r_1,r_2} equal to one in the region $2m < r_1 \leq r \leq r_2 < \infty$. The relative ease of the choice of X hinges on the fact that for $\phi_\flat$ the lagrangian term $\mathbf{g}(d\phi_\flat, d\phi_\flat)$ is positive in a neighborhood of $r = 3m$—the trapped set in Schwarzschild. This inequality leads to a non-degenerate version of the Morawetz estimate and together with the red shift estimate allows one to control $\phi_\flat$. I should note that the actual analysis is complicated by coupling between $\phi_\sharp$ and $\phi_\flat$ introduced by cut-offs in the physical space which are, unfortunately, required to justify the time Fourier frequencies ω.

9. DECAY MECHANISM

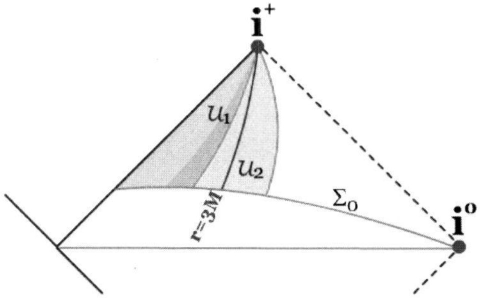
The proof of decay in both Theorems 4 and 5 hinges on two basic steps plus a final iteration procedure based on the pigeon hole principle. We consider below the simpler case of decay in Schwarzschild. We consider $\mathbb{T}$-invariant regions obtained by intersecting space-time domains of the form $2m < R_1 < R < R_2$ or $2m \le r < R$ with the future of Σ_0, in the exterior domain $\mathcal{E}$. Also, in what follows, $\mathcal{N}$ is the portion of the horizon $r = 2m$ to the future of Σ_0.

Step I. — The goal of the first step is to derive an estimate of the form

$$\iint_{\mathcal{V}} |\mathbf{D}\phi|^2 \le C_{\mathcal{V}} I_0, \tag{9.1}$$

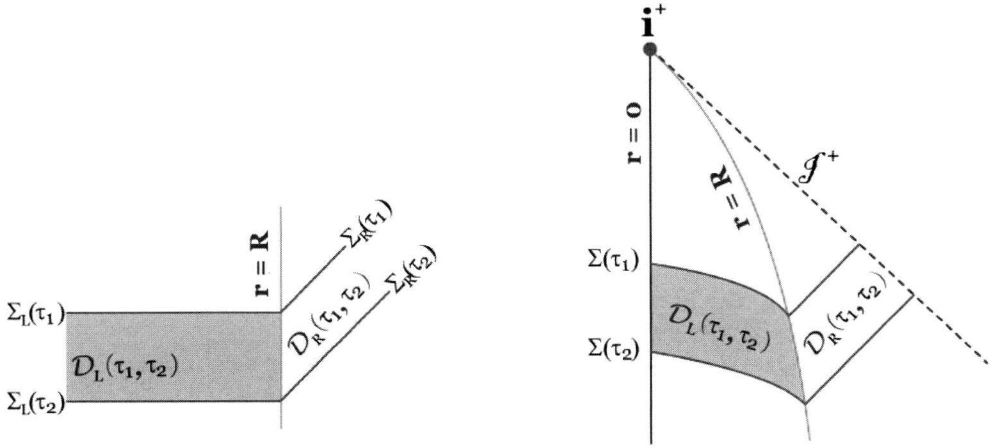
where $\mathcal{V}$ is an arbitrary large neighborhood of $\mathcal{N}$, containing the trapped region and I_0 a constant depending only on the initial data of ϕ (it depends in fact on the $L^2(\Sigma \cap \mathcal{V})$ of the first two derivatives of ϕ. The proof of such an estimate requires the following substeps:

1. Using the red shift vector field $\mathbb{H}$ one can control the non-degenerate energy in a small $\mathbb{T}$-invariant neighborhood $\mathcal{U}_1$ of the horizon (such as $2m \le r < r_1 \ll 3m$), at the expense of having to control the space-time integral of $|\mathbf{D}\phi|^2$ in the complement of $\mathcal{U}_1$ in a somewhat larger neighborhood (such as $r \le r_1 + \epsilon$).
2. Using the modified Morawetz vector field $\mathbb{M}$ one can control the space-time integral of $f(r)\big(|\mathbf{D}\phi|^2 + |\phi|^2\big)$, with $f(r)$ vanishing of order 2 at $r = 3m$, in a sufficiently large $\mathbb{T}$-invariant neighborhood $\mathcal{U}_2$ of the trapped region $r = 3m$ which intersects $\mathcal{U}_1$ (such as $r_1 - \epsilon < r < R$, for an arbitrary $R > 3m$).
3. Commuting the equation with $\mathbb{T}$ and using elliptic theory (or, alternatively, commuting also with the angular momentum vector fields) we derive a similar estimate for $f(r)|\mathbf{D}^2\phi|^2 + |\mathbf{D}\phi|^2$. Thus, by losing one derivative, we control the space-time integral of $|\mathbf{D}\phi|^2$ in $\mathcal{U}_2$.
4. Combining this last estimate with the previous estimate in $\mathcal{U}_1$ we derive a space-time estimate for $|\mathbf{D}\phi|^2$ in $\mathcal{V} = \mathcal{U}_1 \cup \mathcal{U}_2$. We also derive a non-degenerate estimate along the horizon.



Step II. — The goal now is to derive a decay estimate by using the previous step together with asymptotic information from future null infinity. Originally this was done by using a natural adaptation of the vector field $\mathbb{K}_0$ of Minkowski space. Here I will sketch instead the new procedure of [**18**]. To simplify matters I will first present their argument in Minkowski space. It will be quite transparent from the proof how to adapt it to the Schwarzschild case. In fact, once the first step above has been accomplished (which is a lot more delicate in a Kerr background because of the extended trapped region) the same proof also applies to Kerr.

The idea is to foliate Minkowski space by hypersurfaces $\Sigma(\tau) = \Sigma_L\tau \cup \Sigma_R(\tau)$ divided by $r = R$, for a fixed value R. The left piece is a space-like hyperplane $\Sigma_L(\tau) = \{(t = \tau, x)/|x| \leq R\}$ while the right piece is the null hypersurface $\Sigma_R(\tau) = \{(t, x)/t - |x| = \tau - R, |x| \geq R\}$. Let $\mathscr{D}_L(\tau_1, \tau_2)$ and $\mathscr{D}_R(\tau_1, \tau_2)$ be the regions to the left and right with for $\tau_1 \leq \tau_2$ as in the figure below (the figure on the right is the same as that on the left, but viewed in the compactified Penrose diagram of the Minkowski space).



We start with the following estimate:

$$\iint_{\mathcal{D}_L(\tau,\infty)} (|\mathbf{D}\phi|^2 + |\phi|^2) \lesssim C_R E(\tau) \tag{9.2}$$

where $E(\tau)$ is the non-degenerate energy of the slice $\Sigma(\tau)$, i.e.

$$E(\tau) := \int_{\Sigma(\tau)} \mathbf{Q}[\phi](\mathbb{T}, N) \tag{9.3}$$

with N normal to $\Sigma(\tau)$, i.e. $N = \partial_t$ on Σ_L and $N = L = \partial_t + \partial_r$ on Σ_R. Thus,

$$E(\tau) = \int_{\Sigma_L(\tau)} |\mathbf{D}\phi|^2 + \int_{\Sigma_R(\tau)} \big((L\phi)^2 + |\nabla\!\!\!/\phi|^2\big).$$

This is, essentially, the estimate obtained at the first step. It also follows by using a variation of the Morawetz vector field discussed before. Observe also that $E(\tau)$ is monotonically decreasing[27], i.e. $E(\tau_2) \le E(\tau_1)$, in Minkowski space.

In the region $\mathcal{D}_R$ we apply the energy estimate[28] of Proposition 2 with $X = r^p(\partial_t + \partial_r) = r^p L$, $0 \le p \le 2$ and appropriate choice of w. We derive the identity

$$\begin{aligned}\int_{\Sigma_R(\tau_2)} r^p(\hat{L}\phi)^2 + \iint_{\mathcal{D}_R(\tau_1,\tau_2)} r^{p-1}\big((\hat{L}\phi)^2 + (2-p)|\nabla\!\!\!/\phi|^2\big) \\ + \int_{\mathcal{I}^+(\tau_1,\tau_2)} r^p(\nabla\!\!\!/\phi)^2 = \int_{\Sigma_R(\tau_1)} r^p(\hat{L}\phi)^2 + \int_{\mathcal{D}_L\cap\mathcal{D}_R} r^p\big(|\nabla\!\!\!/\phi|^2 - |\hat{L}\phi|^2\big)\end{aligned} \tag{9.4}$$

where $\hat{L}\phi = \frac{1}{2}(\partial_t + \partial_r)\phi + \frac{1}{2r}\phi$. Ignoring the boundary term at future null infinity we derive, for $p = 2$

$$\int_{\Sigma_R(\tau_2)} r^2(\hat{L}\phi)^2 + \iint_{\mathcal{D}_R(\tau_1,\tau_2)} r(\hat{L}\phi)^2 \lesssim \int_{\Sigma_R(\tau_1)} r^2(\hat{L}\phi)^2 + I_R(\tau_1,\tau_2)$$

with

$$I_R(\tau_1,\tau_2) = \int_{\mathcal{D}_L\cap\mathcal{D}_R(\tau_1,\tau_2)} r^2\big(|\nabla\!\!\!/\phi|^2 - |\hat{L}\phi|^2\big) \lesssim R^2 \int_{\mathcal{D}_L\cap\mathcal{D}_R(\tau_1,\tau_2)} |\mathbf{D}\phi|^2.$$

Averaging with respect to R (in a small interval near a fixed value) and using (9.2), we derive

$$\iint_{\mathcal{D}_R(\tau_1,\tau_2)} r(\hat{L}\phi)^2 \lesssim \int_{\Sigma_R(\tau_1)} r^2(\hat{L}\phi)^2 + C_R R^2 E(\tau_1). \tag{9.5}$$

Hence, in fact

$$\int_{\Sigma_R(\tau_2)} r^2(\hat{L}\phi)^2 + \iint_{\mathcal{D}_R(\tau_1,\tau_2)} r(\hat{L}\phi)^2 \lesssim \int_{\Sigma(\tau_1)} r^2(\hat{L}\phi)^2 + C_R E(\tau_1).$$

(27) In Schwarzschild or Kerr we expect some bounded amplification.

(28) Alternatively one can proceed exactly as in **[18]** by multiplying directly the wave equation, in null coordinates.

Using the pigeonhole principle applied to (9.5) we infer that there exists a dyadic sequence $\sigma_n \to \infty$ such that

$$\int_{\Sigma_R(\sigma_{n+1})} r(\hat{L}\phi)^2 \lesssim \sigma_n^{-1}\left(\int_{\Sigma_R(\tau_1)} r^2(\hat{L}\phi)^2 + C_R E(\tau_1)\right). \tag{9.6}$$

We now consider (9.4) with $p = 1$. After averaging in R exactly as before and applying once more (9.2) we deduce

$$\int_{\Sigma_R(\sigma_n)} r(\hat{L}\phi)^2 + \iint_{\mathcal{D}_R(\sigma_n,\sigma_{n-1})} (|\hat{L}\phi|^2 + |\nabla\!\!\!/\phi|^2) \lesssim \int_{\Sigma_R(\sigma_{n-1})} r(\hat{L}\phi)^2 + C_R\, E(\sigma_{n-1}).$$

Using (9.6) we derive

$$\iint_{\mathcal{D}_R(\sigma_n,\sigma_{n-1})} (|\hat{L}\phi|^2 + |\nabla\!\!\!/\phi|^2) \lesssim \sigma_n^{-1}\left(\int_{\Sigma(\tau_1)} r^2(\hat{L}\phi)^2 + C_R E(\tau_1)\right) + C_R E(\sigma_{n-1}).$$

Observe that $(\hat{L}\phi)^2 = (L\phi)^2 + \phi^2 + \frac{1}{r}\partial_r(\phi^2)$. Thus, after an integration by parts

$$\iint_{\mathcal{D}_R(\sigma_n,\sigma_{n-1})} (\hat{L}\phi)^2 = \iint_{\mathcal{D}_R(\sigma_n,\sigma_{n-1})} (L\phi)^2 - \int_{\mathcal{D}_L\cap\mathcal{D}_R(\sigma_n,\sigma_{n-1})} \phi^2.$$

Hence,

$$\iint_{\mathcal{D}_R(\sigma_n,\sigma_{n-1})} (|L\phi|^2 + |\nabla\!\!\!/\phi|^2) \lesssim \sigma_n^{-1}\left(\int_{\Sigma(\tau_1)} r^2(L\phi)^2 + C_R E(\tau_1)\right) + C_R E(\sigma_{n-1}).$$

On the other hand, in view of (9.2)

$$\iint_{\mathcal{D}_L(\sigma_n,\sigma_{n-1})} (|\mathbf{D}\phi|^2 + \phi^2) \lesssim C_R E(\sigma_{n-1}).$$

Adding the last two inequalities together, we derive

$$\int_{\sigma_{n-1}}^{\sigma_n} E(\tau)d\tau \lesssim C\sigma_n^{-1}\left(\int_{\Sigma(\tau_1)} r^2(L\phi)^2 + C_R E(\tau_1)\right). + C_R E(\sigma_{n-1}).$$

Thus, with $I_1 = \int_{\Sigma(\tau_1)} r^2(L\phi)^2 + C_R E(\tau_1)$ depending only on the initial norm on $\Sigma(\tau_1)$ and R

$$\int_{\sigma_{n-1}}^{\sigma_n} E(\tau)d\tau \lesssim I_1\sigma_n^{-1} + C_R E(\sigma_{n-1}).$$

Finally we deduce, by another simple application of the pigeonhole principle and the monotonicity of $E(\tau)$, that for all $\tau \geq \tau_1$,

$$E(\tau) \lesssim \tau^{-2} I_1 \tag{9.7}$$

as desired.

10. APPENDIX

10.1. Null hypersurfaces

Consider a null hypersurface $\mathcal{N}$ embedded in $\mathcal{M}$, with unit normal L (which itself is tangent to $\mathcal{N}$). Clearly $\mathcal{N}$ is generated by all null geodesics tangent to L orthogonal to a 2-surface S. In what follows we assume that S has the topology of a 2-sphere.

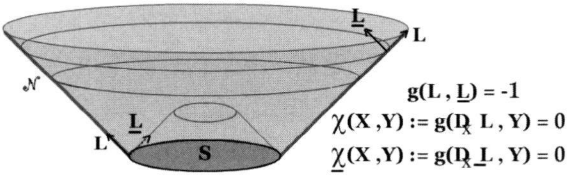
DEFINITION. — *The null second fundamental form of a null hypersurface $\mathcal{N}$ is defined by*

$$\chi(X,Y) \quad = \quad \mathbf{g}(\mathbf{D}_X L, Y), \tag{10.1}$$

where L is a fixed null vector field tangent to the null generators of $\mathcal{N}$ and X, Y arbitrary vector fields tangent to $\mathcal{N}$.

Observe that the definition depends tensorially on the choice of L, i.e. if $L' = aL$ we have $\chi' = a\chi$. The trace $\operatorname{tr}\chi$ can be defined, relative to an arbitrary frame L, e_1, e_2, with $\mathbf{g}(e_a, e_b) = \delta_{ab}$, by $\operatorname{tr}\chi = \chi_{11} + \chi_{22}$. One can easily check that the definition is independent of the frame or the choice of null normal L. The hypersurface $\mathcal{N}$ is said to be non-expanding if the trace of χ vanishes identically.

We can foliate $\mathcal{N}$ by the level surfaces of an affine parameter s of L, i.e. $L(s) = 1$, $s = 0$ on S. We can then define the null companion $\underline{L}$ of L, at any point p of $\mathcal{N}$, to be the unique null normal orthogonal to the level surface passing through p such that $\mathbf{g}(L, \underline{L}) = -1$.



10.2. Red shift vector field

Consider first an arbitrary null hypersurface $\mathcal{N}$ and a null pair $(e_4 = L, e_3 = \underline{L})$, $\mathbf{g}(e_3, e_4) = -1$ with L null, tangent to $\mathcal{N}$ and $\underline{L}$ hypersurface orthogonal, i.e. orthogonal to a foliation of $\mathcal{N}$ by 2-surfaces (see appendix). We complete the null pair to a null frame (e_1, e_2, e_3, e_4) with e_1, e_2 an orthonormal frame tangent to the foliation. We easily check the following

$$\begin{aligned} \mathbf{D}_4 e_4 &= -\omega e_4, \\ \mathbf{D}_4 e_3 &= \omega e_3 + \underline{\eta}_1 e_2 + \underline{\eta}_2 e_2 \\ \mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3 \end{aligned} \tag{10.2}$$

where $\omega = \mathbf{g}(D_4 e_4, e_3)$, $\underline{\eta}_a = \mathbf{g}(\mathbf{D}_4 e_3, e_a)$, $\zeta_a = \mathbf{g}(\mathbf{D}_a e_4, e_3)$, $\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b)$ depend only on the original choice of the null pair (e_3, e_4) along $\mathcal{N}$.

We extend e_3 in a small neighborhood of $\mathcal{N}$ by solving the equation

$$\mathbf{D}_3 e_3 = -\underline{\omega} e_3 \tag{10.3}$$

with $\underline{\omega}$ an arbitrary function on $\mathcal{N}$ which we hope to choose later. The deformation tensor of $X = e_3$ can be easily calculated along $\mathcal{N}$

$$\pi_{44} = -2\omega,\ \pi_{34} = \underline{\omega},\ \pi_{33} = 0,\ \pi_{3a} = 0,\ \pi_{4a} = \underline{\eta}_a - \zeta_a,\ \pi_{ab} = \underline{\chi}_{ab}.$$

Therefore,

$$\begin{aligned}\mathbf{Q}\cdot\pi &= \mathbf{Q}_{33}\pi_{44} + 2\mathbf{Q}_{34}\pi_{34} - 2\mathbf{Q}_{3a}\pi_{4a} + \mathbf{Q}_{ab}\pi_{ab}\\ &= -\omega(e_3\phi)^2 + 2\underline{\omega}\,|\nabla\!\!\!/\phi|^2 - 2\nabla_3\phi\,\nabla\!\!\!/\phi\cdot(\underline{\eta}-\zeta) + \underline{\chi}_{ab}\nabla_a\phi\nabla_b\phi\\ &\quad - \frac{1}{2}\mathrm{tr}\underline{\chi}\big(-2e_3(\phi)\cdot e_4(\phi) + |\nabla\!\!\!/\phi|^2\big)\\ &= -\omega(e_3\phi)^2 + 2\underline{\omega}\,|\nabla\!\!\!/\phi|^2 + \hat{\underline{\chi}}_{ab}\nabla_a\phi\nabla_b\phi - 2e_3\phi\,\nabla\!\!\!/\phi\cdot(\underline{\eta}-\zeta) + \mathrm{tr}\underline{\chi}(e_3\phi)(e_4\phi).\end{aligned}$$

By assuming $-\omega \geq \kappa > 0$ and $\underline{\omega}$ sufficiently large positive, we deduce, for some positive constant c, $\frac{1}{2}\mathbf{Q}\cdot\pi + \mathrm{tr}\underline{\chi}(e_3\phi)(e_4\phi) \geq c\big((e_3\phi)^2 + |\nabla\!\!\!/\phi|^2\big)$. To get rid of the term $\mathrm{tr}\underline{\chi}(e_3\phi)(e_4\phi)$ we need to make a modification of Equation (10.3). We use instead

$$\mathbf{D}_3 e_3 \;=\; -\underline{\omega} e_3 - A e_4. \tag{10.4}$$

With this modification all components of π remain the same, except $\pi_{33} = A$. Thus,

$$\begin{aligned}\mathbf{Q}\cdot\pi = &- \omega(e_3\phi)^2 + 2\underline{\omega}\,|\nabla\!\!\!/\phi|^2 + A(e_4\phi)^2 + \hat{\underline{\chi}}_{ab}\nabla_a\phi\nabla_b\phi\\ &- 2e_3\phi\,\nabla\!\!\!/\phi\cdot(\underline{\eta}-\zeta) + \mathrm{tr}\underline{\chi}(e_3\phi)(e_4\phi).\end{aligned}$$

Thus, choosing $-\omega \geq \kappa > 0$ and constants $\underline{\omega}, A$ sufficiently large, we deduce, for some positive $c > 0$

$$\frac{1}{2}\mathbf{Q}\cdot\pi \geq c(e_3\phi)^2 + \Lambda\big(|e_4\phi|^2 + |\nabla\!\!\!/\phi|^2\big)$$

with $\Lambda > 0$ arbitrarily large, provided that $\underline{\omega}$ and A are sufficiently large.

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