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NEUMANN AND DIRICHLET HEAT KERNELS IN INNER UNIFORM DOMAINS

Pavel Gyrya, Laurent Saloff-Coste

Abstract. — This monograph focuses on the heat equation with either the Neumann or the Dirichlet boundary condition in unbounded domains in Euclidean space, Riemannian manifolds, and in the more general context of certain regular local Dirichlet spaces. In works by A. Grigor'yan, L. Saloff-Coste and K-T. Sturm, the equivalence between

— the parabolic Harnack inequality,
— the two-sided Gaussian heat kernel estimate,
— the Poincaré inequality and the volume doubling property,

is established in a very general context. We use this result to provide precise two-sided heat kernel estimates in a large class of domains described in terms of their inner intrinsic metric and called inner (or intrinsically) uniform domains. Perhaps surprisingly, we treat both the Neumann boundary condition and the Dirichlet boundary condition using essentially the same approach albeit with the additional help of a Doob's h-transform in the case of Dirichlet boundary condition.

The main results are new even when applied to Euclidean domains with smooth boundary where they capture the global effect of the condition of inner uniformity as, for instance, in the case of domains that are the complement of a convex set in Euclidean space.

***Résumé* (Le noyau de la chaleur avec condition de Neumann ou de Dirichlet dans les domaines intérieurement uniformes).** — Ce texte traite de l'étude du noyau de la chaleur avec condition de Neumann ou condition de Dirichlet au bord dans les domaines euclidiens non-bornés, mais aussi les domaines non-bornés dans les variétés riemanniennes et, plus généralement, les domaines non-bornés de certain espaces de Dirichlet réguliers locaux.

Les travaux de A. Grigor'yan, L. Saloff-Coste et K-T. Sturm, ont montré l'équivalence, dans un large contexte, des propriétés suivantes:

— l'inégalité de Harnack parabolique,
— les estimations gaussiennes du noyau de la chaleur,

— l'inégalité de Poincaré et la propriété de doublement du volume.

Nous utilisons ce résultat pour obtenir des estimations précises du noyau de la chaleur pour une large classe de domaines définis en termes de leur distance intrinsèque et appelés domaines intérieurement (ou intrinséquement) uniformes. De façon peut être surprenante, nous traitons le problème avec la condition de Neumann au bord et celui avec la condition de Dirichlet au bord par la même approche, mais avec l'aide supplémentaire d'une transformation de Doob dans le cas de la condition de Dirichlet.

Les résultats principaux que nous obtenons sont nouveaux même dans le cas des domaines euclidiens a bord régulier où ils capturent l'effet de la condition d'uniformité intérieure comme, par exemple, dans le cas des domaines qui sont le complement d'un convexe fermé de $\mathbb{R}^n$.

CONTENTS

CHAPTER 1

INTRODUCTION

This monograph is based on joint work of the authors that served as the basis for Pavel Gyrya's 2007 Ph.D. Thesis at Cornell. As Pavel left academia after his Ph.D. and since there was no pressing need for publication, the authors decided to publish the results in the form of a monograph, thus allowing for a more detailled and more unified presentation of the work. The second author had the pleasure to lecture on this material at Université Paris XI (Orsay, France) at the invitation of Alano Ancona. We thank Janna Lierl for her careful reading of the manuscript. Many thanks to Alano Ancona, Hiroaki Aikawa, Alexander Bendikov, Zhen-Qing Chen, Piotr Hajłaz and Theo Sturm for answering various queries in detailed and useful ways.

1.1. Goals: informal description

The aim of this monograph is to prove precise two-sided global estimates for the heat kernel with Dirichlet boundary condition in good domains in a good ambient space. Roughly speaking, a good ambient space is one where a precise two-sided global Gaussian heat kernel estimate holds.

Thanks to the work of Grigor'yan, Saloff-Coste and Sturm, we know that any local regular Dirichlet space whose intrinsic distance yields a complete metric structure in which the volume doubling condition and Poincaré inequality are satisfied is a good ambient space as defined above. The most important and simplest example is the Euclidean space $\mathbb{R}^n$ but there are many further interesting examples: Riemannian manifolds with non-negative Ricci curvature, nilpotent Lie groups equipped with a sub-Riemannian invariant structure, etc.

In the classical study of harmonic functions with Dirichlet boundary condition, attention has often been restricted to important and interesting but somewhat unnatural Euclidean domains such as the region above the graph of a Lipschitz function. See, e.g., [**3**] and the references therein. A more recent picture of what is known and further references are in [**1, 4**].

One problem with such Euclidean domains is that they do not have an obvious generalization in the setting of manifolds or, more generally, in the setting of Dirichlet spaces, especially if one thinks in global terms and considers unbounded domains. What is needed for an easy generalization to a wider setting is a metric definition of what constitutes a good domain in purely metric terms. Such a definition has emerged over a rather long period of time and it will be used extensively in this monograph. Ultimately, a good domain for us is an *inner uniform domain.* A useful survey concerning this notion is in [**93**]. The precise definition will be given later in this introduction but what is important to note here is that it only involves the natural inner metric of the domain and nothing else.

Our original hope, at the start of this project, was to provide a simple proof of a precise two-sided Gaussian estimate for the heat kernel with Dirichlet boundary condition in any Euclidean domain above the graph of a Lipschitz function. Such estimates are derived in [**99**] and, in a more restrictive case, in [**87**]. As an alternative test case not covered by these references, we were also interested in treating the complement of an arbitrary convex set.

We were thrilled to discover that, thanks to recent works of Aikawa and his collaborators [**1, 2**] (and also Ancona [**4**]) concerning the boundary Harnack principle for harmonic functions with Dirichlet boundary condition, we were able to handle the much larger and more satisfying class of all inner uniform Euclidean domains. Indeed, this meant that the same ideas would also apply in a much larger and very natural context and provide great many further examples. Some of these examples are treated in details in the last chapter of this monograph. In the next few sections, we give a more detailed description of our techniques and results.

From the viewpoint developed in this monograph, two-sided heat kernel estimates go hand in hand with the validity of a certain form of the parabolic Harnack inequality. This connection is clearly established in the work of Grigor'yan, Saloff-Coste and Sturm [**52, 81, 82, 83, 89, 92**] in situations when there is no boundary (i.e., complete Riemannian manifolds) or when the Neumann boundary condition is assumed at the boundary (e.g., for Euclidean domains with smooth boundary). In the case of domains with the Dirichlet boundary condition, the relation between two-sided heat kernel estimates and some version of the parabolic Harnack inequality is not so clear. It is one of the contributions of this monograph to establish a clear straightforward connection. Various versions of the parabolic boundary Harnack inequality have been discussed in previous work. See [**40, 41, 42, 60, 61, 62, 80**]. These versions are often stronger than (or different from) the one considered in this monograph.

The work presented in this monograph calls for further developments in several directions and we comment here only on the most obvious ones. We have chosen to emphasize the case when heat kernel bounds of a completely global nature can be established. By this we mean that we only discuss situations when the heat kernel $h(t,x,y)$ of interest can be estimated for all (t,x,y). As our estimates come in the form of almost matching upper and lower bounds, this puts very strong global requirement on the underlying space and the domains we treat. Indeed, there are many good reasons to view the small scales and large scales behaviors of the heat kernel as two separate issues. It is obvious that the methods we use can (and should) be developed from a localized viewpoint without difficulties. Of particular interest is the study of the Dirichlet heat kernel on compact (inner uniform) domains and its relation to the notion of intrinsic ultracontractivity. This will be discussed elsewhere.

1.2. Smooth unbounded Euclidean domains

Neumann boundary condition. — Let $U \subset \mathbb{R}^n$ be an unbounded Euclidean domain with smooth boundary ∂U. Let λ be the Lebesgue measure. We use the notation $d\lambda(x) = dx$ when no confusions can possibly arise. Without any kind of difficulty, we can consider the spaces $\mathscr{C}_c^\infty(\overline{U})$ (smooth functions with compact support in $\overline{U}$) and $\mathscr{C}_c^\infty(U)$ (smooth functions with compact support in U) as well as the Sobolev spaces $W^1(U)$ and $W_0^1(U)$. The space $W^1(U)$ is the space of all functions in $L^2(U)$ whose distributional first derivatives can be represented by functions in $L^2(U)$, equipped with the norm

$$\|f\|_{W^1(U)} = \left(\|f\|_2^2 + \|\nabla f\|_2^2\right)^{1/2}$$

where

$$\nabla f = \left(\frac{\partial}{\partial x_i} f\right)_1^n, \quad |\nabla f| = \left(\sum_1^n \left|\frac{\partial}{\partial x_i} f\right|^2\right)^{1/2}.$$

The smoothness of the boundary implies that $\mathscr{C}_c^\infty(\overline{U})$ is dense in $W^1(U)$. The space $W_0^1(U)$ is the closure of $\mathscr{C}_c^\infty(U)$ in $W^1(U)$.

Let $\Delta = \sum_1^n \left(\frac{\partial}{\partial x_i}\right)^2$ be the Laplace operator. Let $\vec{\nu}$ be the outward unit normal vector to ∂U. A classical solution of the heat equation in U with Neumann boundary condition is a function $u \in \mathscr{C}^\infty((0,\infty) \times \overline{U})$ such that

$$\text{(1.1)} \qquad \begin{cases} \partial_t u = \Delta u & \text{in } (0,\infty) \times U \\ \frac{\partial}{\partial \vec{\nu}} u = 0 & \text{on } (0,\infty) \times \partial U. \end{cases}$$

The Neumann heat kernel in U is the fundamental solution of this equation, i.e., the solution $p_U^N(t,x,y) : (0,\infty) \times \overline{U}$ of (1.1) (in t, y for fixed $x \in \overline{U}$) such that

$$\lim_{t \to 0} p_U^N(t,x,y) = \delta_x(y).$$

For any function $f \in L^2(U)$,

$$(t,x) \mapsto v(t,x) = \int_U p_U^N(t,x,y) f(y) dy$$

is the unique solution of (1.1) with $\lim_{t\to 0} v(t,x) = f(x)$ (in $L^2(U)$). In other words, p_U^N is the kernel of the heat semigroup $P_{U,t}^N = e^{t\Delta_U^N}$ associated with the local regular Dirichlet form (considered over the set $\overline{U}$)

$$\mathscr{E}_U^N(f,g) = \int_U \nabla f \cdot \nabla g \, d\lambda, \quad f, g \in W^1(U).$$

Note that Δ_U^N is the closed extension of $(\Delta, \mathscr{C}_c^\infty(\overline{U}) \cap \{\frac{\partial}{\partial \nu} f = 0\})$ associated with $(\mathscr{E}_U^N, W^1(U))$.

This monograph provides global two-sided Gaussian estimates for the Neumann heat kernel p_U^N (under specific assumptions on U). The simplest instance of this problem is when U is the upper-half space $\mathbb{R}_+^n = \{x = (x_i)_1^n : x_n > 0\}$. In this case the Neumann heat kernel is given explicitly by

$$p_{\mathbb{R}_+^n}^N(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t} + \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y'\|^2/4t}$$

where $y' = (y_1, \dots, y_{n-1}, -y_n)$ if $y = (y_1, \dots, y_n)$. This formula plainly indicates that the heat kernel $p_{\mathbb{R}_+^n}^N(t,x,y)$ and the Gauss kernel $(4\pi t)^{-n/2} e^{-\|x-y\|^2/4t}$ are very similar.

Less trivial examples are obtained by considering the interior and exterior of a paraboloid of revolution:

$$IP = \left\{ x = (x_i)_1^n : x_n > \sum_1^{n-1} x_i^2 \right\}, \quad EP = \left\{ x = (x_i)_1^n : x_n < \sum_1^{n-1} x_i^2 \right\}.$$

The domain IP is an unbounded convex set and it is well-known that, for any convex domain U, $p_U^N(t,x,y)$ admits upper and lower Gaussian bounds of the type

$$\frac{C}{V_U(x,\sqrt{t})} e^{-c\|x-y\|^2/t}, \quad x,y \in \overline{U},\ t > 0,$$

where $V_U(x,r)$ is the Lebesgue volume of the trace in U of the Euclidean ball of center x and radius r. The exterior EP of the paraboloid is perhaps more interesting. We will show that the heat kernel $p_{EP}^N(t,x,y)$ admits upper and lower Gaussian bounds of the type

$$\frac{C}{t^{n/2}} e^{-c\rho(x,y)^2/t}, \quad \rho(x,y) = \rho_{EP}(x,y), \quad x,y \in \overline{U},\ t > 0,$$

where, for any domain U, $\rho_U(x,y)$ denotes the intrinsic geodesic distance in U, i.e., the infimum of the lengths of curves joining x and y in U. In fact, we will show that such heat kernel bounds hold true for any unbounded domain U that is inner uniform. This covers the case of EP because, as we shall prove, the complement of a convex set is always inner uniform.

Dirichlet boundary condition. — Next, we consider classical solutions of the heat equation in a domain U (with smooth boundary) with Dirichlet boundary condition. A classical solution of the heat equation in U with Dirichlet boundary condition is a function $u \in \mathscr{C}^\infty((0,\infty) \times \overline{U})$ such that

$$(1.2) \qquad \begin{cases} \partial_t u = \Delta u & \text{in } (0,\infty) \times U \\ u = 0 & \text{on } (0,\infty) \times \partial U \end{cases}$$

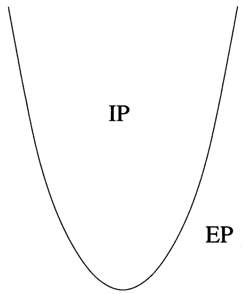


FIGURE 1. The interior and exterior domains of the parabola

The Dirichlet heat kernel in U is the fundamental solution of this equation, i.e., the solution $p_U^D(t,x,y)$ of (1.2) (in $(t,y) \in (0,\infty) \times \overline{U}$, for fixed $x \in U$) such that

$$\lim_{t\to 0} p_U^D(t,x,y) = \delta_x(y).$$

For any function $f \in L^2(U)$,

$$(t,x) \mapsto v(t,x) = \int_U p_U^D(t,x,y) f(y)dy$$

is the unique solution of (1.2) with $\lim_{t\to 0} v(t,x) = f(x)$ (in $L^2(U)$). In other words, p_U^D is the kernel of the heat semigroup $P_{U,t}^D = e^{t\Delta_U^D}$ associated with the local regular Dirichlet form

$$\mathscr{E}_U^D(f,g) = \int_U \nabla f \cdot \nabla g \, d\lambda, \quad f,g, \in W_0^1(U).$$

Note that Δ_U^D is the closed extension of $(\Delta, \mathscr{C}_c^\infty(U))$ associated with $(\mathscr{E}_U^D, W_0^1(U))$.

One of the aims of this monograph is to provide a precise two-sided Gaussian estimate for the Dirichlet heat kernel when U is an unbounded inner uniform domain (see the next section).

As in the Neumann case, the simplest instance of this problem is when $U = \mathbb{R}_+^n = \{x = (x_i)_1^n : x_n > 0\}$ is the upper-half space, in which case $p_{\mathbb{R}_+^n}^D$ is given by the formula

$$p_{\mathbb{R}_+^n}^D(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t} - \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y'\|^2/4t}$$

with $y' = (y_1, \ldots, y_{n-1}, -y_n)$ as before. This formula is harder to interpret than the formula for the corresponding Neumann case because, when x and/or y are close to the boundary and/or t is large, there are significant cancellations occurring between the two terms in the formula. In fact, $p_{\mathbb{R}_+^n}^D(t,x,y)$ admits upper and lower bounds of the type

$$\frac{Cx_n y_n}{t^{n/2}(x_n + \sqrt{t})(y_n + \sqrt{t})} e^{-c\|x-y\|^2/t}$$

which capture both the (linear) vanishing at the boundary and the additional long time decay $p_{\mathbb{R}_+^n}^D(t,x,y) \sim c_n t^{-1-n/2}$ due to the Dirichlet boundary condition.

Let us now consider what happens in the cases of the interior (IP) and exterior (EP) of a paraboloid. Interestingly, the situation for the Dirichlet heat kernel is rather different from that for the Neumann heat kernel. On the one hand, for the interior of a paraboloid, the Dirichlet condition has a huge effect on the behavior of the heat kernel and obtaining two-sided heat kernel estimates (i.e., upper and lower estimates that matches closely, in some sense) becomes a very delicate matter. This case will not be discussed in this monograph (the interior of a paraboloid is not an inner uniform domain). Some results are contained in the work of Bañuelos et al [**9**]. On the other hand, for the exterior of a paraboloid, the boundary effect is much tamer and we will give a sharp two-sided estimate of the Dirichlet heat kernel p^D_{EP}. This two-sided estimate involves in crucial ways a function h that we call a *harmonic profile* (or simply a profile, for short) of the domain. More precisely, given a domain U, call *harmonic profile* of U any continuous positive function h in U which is harmonic in U and vanishes continuously at the boundary (for the readers familiar with the theory of the Martin boundary, h is a function associated with "the point at infinity" of the Martin boundary). For many domains, such a profile is unique (up to multiplication by a positive constant). For instance, the profile of $\mathbb{R}^n_+ = \{x = (x_i)_1^n : x_n > 0\}$ is $h(x) = x_n$. The profile of $U = \mathbb{R}^n \setminus \overline{B(0,1)} = \{x = (x_i)_1^n : \|x\| > 1\}$ is $h(x) = \log \|x\|$ if $n = 2$ and $h(x) = 1 - \|x\|^{-n+2}$ if $n > 2$. In studying the Dirichlet heat kernel, we will make extensive use of the celebrated technique known as Doob's transform (or h-transform, or Doob's h-transform)) for which, in the present case, the most basic ingredient is the harmonic profile h of U.

For any unbounded intrinsically uniform domain with profile h, we will prove upper and lower bounds for the Dirichlet heat kernel $p^D_U(t,x,y)$ of the type

$$\frac{Ch(x)h(y)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} e^{-c\rho_U(x,y)^2/t}, \quad x,y \in \overline{U},\ t > 0.$$

To be very concrete, in the case of the exterior $V = EP$ of the parabola $y = x^2$ in the plane, that is, $V = \{x = (x_1,x_2) = x_2 < x_1^2\}$, the profile of V is

$$h(x) = \sqrt{2\left(\sqrt{x_1^2 + (1/4 - x_2)^2} + 1/4 - x_2\right)} - 1.$$

The heat kernel estimates above indicate in particular that, for any fixed $x, y \in V$, $p^D_V(t,x,y)$ is bounded above and below in a time neighborhood of $+\infty$ by $t^{-3/2}$ (compare with the upper-half plane for which the corresponding behavior given earlier is in t^{-2}).

Unfortunately, this type of explicit expression for h is mostly restricted to dimension 2 where conformal techniques are available.

1.3. Inner uniform domains

As explained in the previous two sections, this work is devoted to the study of the Neumann and Dirichlet heat kernels in unbounded inner uniform domain (bounded domains will be treated elsewhere). Inner uniformity is a simple condition that involves the intrinsic geodesic distance (inner distance) of the domain. Any domain U in $\mathbb{R}^n$ can be equipped with its inner Euclidean metric ρ_U where $\rho_U(x,y)$ is defined as the infimum of the Euclidean lengths of all rectifiable curves γ joining x and y in U. A domain $U \subset \mathbb{R}^n$ is said to be an *inner uniform domain* if there are constants $c_0, C_0 \in (0,\infty)$ such that for any two points x,y in U there exists a rectifiable curve $\gamma : [0,1] \to U$ of length at most $C_0 \rho_U(x,y)$ joining x and y and satisfying

$$\forall\, z \in \gamma([0,1]),\ \ \operatorname{dist}(z,\partial U) \geq c_0 \frac{\rho_U(x,z)\rho_U(y,z)}{\rho_U(x,y)}.$$

This last condition is the same as requiring

$$\forall\, z \in \gamma([0,1]),\ \ \operatorname{dist}(z,\partial U) \geq c_0' \min\{\rho_U(x,z), \rho_U(y,z)\}.$$

The reason we write $\operatorname{dist}(z,\partial U)$ on the left-hand side of these inequalities is that the distance to the boundary can be computed using either the usual Euclidean distance or the inner distance since both give the same result.

Let us recall for comparison that a *uniform domain* is a domain U for which there are constants $c_0, C_0 \in (0,\infty)$ such that for any two points x,y in U there exists a rectifiable curve $\gamma : [0,1] \to U$ of length at most $C_0\|x-y\|$ joining x to y and satisfying

$$\forall\, z \in \gamma([0,1]),\ \ \operatorname{dist}(z,\partial U) \geq c_0 \frac{\|x-z\|\|y-z\|}{\|x-y\|}.$$

It is easy to see that a uniform domain is nothing else than an inner uniform domain whose inner distance is comparable to the Euclidean distance, i.e., such that $\forall\, x,y \in U,\ \ \rho_U(x,y) \leq C_1\|x-y\|$. The slitted plane $\mathbb{R}^2 \setminus \{(x,y) : x=0,\ y \in (-\infty, 0]\}$ is a simple example of an inner uniform domain that is not uniform.

In the literature, uniformity and inner uniformity are often defined in terms of length of curves instead of the purely metric definitions discussed above. That our definition above agrees with these definitions in Euclidean space is a non-trivial but well-known fact (at least, for the specialists of the subject). We will review this fact in Chapter 3 in the more general context of interest to us in this monograph.

Inner uniform domains form a very large class of domains and can have a rather wild boundary. In fact, this is already the case for uniform domains. It is worth pointing out that inner uniformity is not very easy to check for the simple reason that it is not easy to compute the inner distance.

We show in Chapter 6 that the complement of any convex set in $\mathbb{R}^n$ is inner uniform (many such domains are not uniform). This is less obvious than it first appears and it produces an interesting simple class of inner uniform domains.

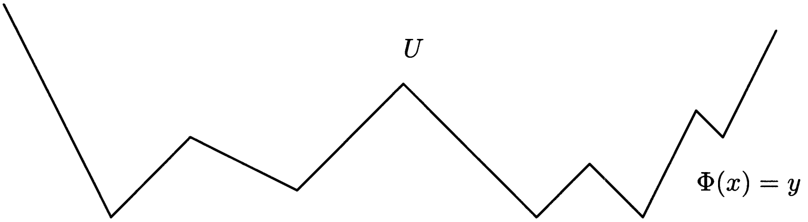
If $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, the region above its graph

$$U(\Phi) = \{x = (x_i)_1^n : x_n > \Phi(x_1,\dots,x_{n-1})\}$$

is an important example of uniform domain.

As a toy example of some interest, let us mention the following collection of vertically slitted upper-half planes. Given a (finite or) countable family $\mathbf{f} = \{(x_i, y_i))\} \subset$

FIGURE 2. The domain above the graph of the Lipschitz function Φ

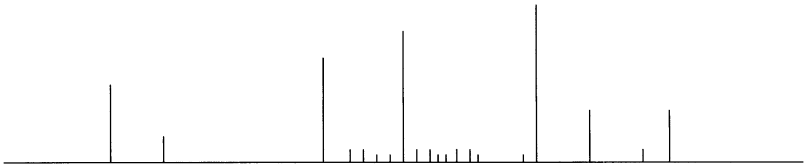


$\mathbb{R}^2_+$ of points in the upper-half plane, let $\mathbb{R}^2_{+\mathbf{f}}$ be the upper-half plane with the vertical segments

$$s_i = \{z = (x_i, y) : 0 < y \leq y_i\}$$

deleted. It is easy to check that the domain $\mathbb{R}^2_{+\mathbf{f}}$ is inner uniform if and only if there is a constant $c > 0$ such that for any pair (i, j), $|x_i - x_j| \geq c \min\{y_i, y_j\}$. Such domains are never uniform if there is at least one non trivial slit.

FIGURE 3. The slitted upper-half plane $\mathbb{R}^2_{+\mathbf{f}}$



We can now describe the main results proved in this monograph. Let U be an unbounded domain in $\mathbb{R}^n$. Assume that U is inner uniform and, for now, has smooth boundary. In this case, we will show that the Neumann heat kernel $p_U^N(t, x, y)$ in U is bounded above and below by functions of the type

$$\frac{C}{t^{n/2}} e^{-c\rho_U(x,y)^2/t},$$

uniformly for all $t > 0$ and $x, y \in U$.

Further, any such domain admits a harmonic profile h (unique up to multiplication by a constant) and, if we set

$$V_{h^2}(x, r) = \int_{B_U(x,r)} h^2 d\lambda, \quad B_U(x, r) = \{z \in U : \rho_U(x, z) < r\},$$

then the Dirichlet heat kernel $p_U^D(t, x, y)$ in U is bounded above and below by functions of the type

$$\frac{Ch(x)h(y)}{\sqrt{V_{h^2}(x, \sqrt{t}) V_{h^2}(y, \sqrt{t})}} e^{-c\rho_U(x,y)^2/t}, \quad x, y \in U, \ t > 0.$$

For the Neumann heat kernel estimate, the crucial point is that the volume $\lambda(B_U(x,r))$ of the inner metric ball in U is comparable to r^n, uniformly over all $x\in U$ and $r>0$ (a simple consequence of the hypothesis that U is unbounded and inner uniform) and that the Poincaré inequality

$$\forall\, f\in W^1(B),\quad \min_{\xi\in\mathbb{R}}\int_B |f-\xi|^2 d\lambda \le P_0 r^2\int_B |\nabla f|^2 d\lambda \tag{1.3}$$

is satisfied for every inner metric ball $B=B_U(x,r)$. The fact that the Poincaré inequality (1.3) holds is one of the main novel technical result obtained in this work and is the key to all further developments presented here.

For the Dirichlet heat kernel, the crucial point of our approach is that the modified volume function $V_{h^2}(x,r)$ satisfies the doubling property

$$\forall\, x\in U,\ \forall r>0,\quad V_{h^2}(x,2r)\le D_0 V_{h^2}(x,r)$$

and that the modified Poincaré inequality

$$f\in W^1(B,h^2d\lambda),\quad \min_{\xi\in\mathbb{R}}\int_B |f-\xi|^2 h^2 d\lambda \le P_0 r^2\int_B |\nabla f|^2 h^2 d\lambda \tag{1.4}$$

is satisfied for every ball $B=B_U(x,r)$. These properties are obtained using the work of Aikawa and his collaborators [**1, 2**] which, for any unbounded inner uniform domain, suffices to provide both the existence of a (unique) profile h and its fundamental properties (through the elliptic boundary Harnack principle). In particular, once the basic properties of h are established, the proofs of (1.3) and (1.4) are similar.

In addition to the heat kernel estimates stated above, our approach naturally yields parabolic Harnack inequalities and Hölder continuity estimates, up to the boundary. To state these results, let $Q(x,r)=(0,r^2)\times B_U(x,r)$ be a space time cylinder. Let $Q_+=(3r^2/4,r^2)\times B_U(x,r/2)$, $Q_-=(r^2/4,r^2/2)\times B_U(x,r/2)$, $Q'=(r^2/4,3r^2/4)\times B_U(x,r/2)$ be sub-cylinders in Q. In what follows, all boundaries are computed in $\mathbb{R}^n$.

First, consider the Neumann heat equation in an (unbounded) inner uniform domain U (for now, with smooth boundary). We will show that there exists a constant $H_0\in(0,\infty)$ such that if u is a non-negative solution of the heat equation in $Q(x,r)$ with vanishing normal derivative along the part of ∂U that intersects $\partial B_U(x,r)$ (the boundary is taken in $\mathbb{R}^n$) then

$$\sup_{Q_-}\{u\}\le H_0\inf_{Q_+}\{u\}. \tag{1.5}$$

Further, there are constants $H_1,\alpha\in(0,\infty)$ such that if u is a solution of the heat equation in $Q(x,r)$ with vanishing normal derivative along the part of ∂U that intersects $\partial B_U(x,r)$ then

$$\sup_{Q'}\left\{\frac{|u(y,s)-u(z,t)|}{(|t-s|^{1/2}+\rho_U(y,z))^\alpha}\right\}\le\frac{H_1}{r^\alpha}\sup_Q\{|u|\}. \tag{1.6}$$

Second, consider the Dirichlet heat equation in an unbounded inner uniform domain U (for now, with smooth boundary). Let h be the harmonic profile of U. We will show that there exists a constant $H_0\in(0,\infty)$ such that if u is a non-negative solution of the heat equation in $Q(x,r)$ which vanishes continuously along the part of ∂U that intersects $\partial B_U(x,r)$ then

$$\sup_{Q_-}\{u/h\} \le H_0 \inf_{Q_+}\{u/h\}. \tag{1.7}$$

Further, there are constants $H_1, \alpha \in (0,\infty)$ such that if u is a solution of the heat equation in $Q(x,r)$ which vanishes continuously along the part of ∂U that intersects $\partial B_U(x,r)$ then

$$\sup_{Q'}\left\{\frac{|u(y,s)/h(y) - u(z,t)/h(z)|}{(|t-s|^{1/2} + \rho_U(y,z))^\alpha}\right\} \le \frac{H_1}{r^\alpha}\sup_Q\{|u/h|\}. \tag{1.8}$$

It turns out that the smoothness assumption made on the boundary of U in the discussion above is absolutely irrelevant. However, working without this assumption introduces technical difficulties in the very definition of the objects involved. First of all, in the Neumann case, one needs to make precise what is meant by the Neumann heat kernel and, more generally, by local solutions of the heat equation with Neumann condition on the boundary. In the Dirichlet case, there might be irregular points on the boundary and the notion of a solution vanishing along ∂U must be understood in an appropriate sense. Nevertheless, as we shall see, these technical difficulties can be overcome. In particular, the constants appearing in the various estimates discussed above depend only on the dimension and the inner uniformity constants of the domain and not otherwise on the domain U itself.

We close this section with remarks concerning the sharpness of the inner uniformity condition with respect to the estimates described above. In the case of the Neumann boundary condition, the main advantage of the inner uniformity condition is that it prevents the domain from having bottlenecks (see the Poincaré inequality). However, there are many domains that are not inner uniform but satisfy similar Neumann heat kernel estimates. The simplest example is the class of unbounded convex domains, many of which are not inner uniform (i.e., IP, the inside of a paraboloid discussed above). In any convex domain U, the Neumann heat kernel satisfies the two-sided estimate

$$\frac{c}{V(x,\sqrt{t})}e^{-C\|x-y\|^2/t} \le p_U^N(t,x,y) \le \frac{C}{V(x,\sqrt{t})}e^{-c\|x-y\|^2/t}$$

where $V(x,r)$ is the volume of the trace of the Euclidean ball centered at x and of radius r in U. This is well known and can be proved using the techniques that will be used in this monograph (the volume $V(x,r)$ is doubling and Poincaré inequality holds). So, in a sense, inner uniformity is very far from being a necessary condition for the type of Neumann heat kernel estimates we derive.

The situation for the Dirichlet heat kernel is very different. Crucial to our analysis and to the type of results we obtain in the Dirichlet case is the doubling condition for the modified volume function V_{h^2} based on the measure $h^2 d\lambda$ where h is the harmonic profile of U. The inner uniformity condition is reasonably close to be optimal in this respect. Indeed, roughly speaking, the doubling condition for V_{h^2} is associated to the power function behavior of h (at infinity and at the boundary). Such power function behavior of h fails for unbounded domains whose "opening at infinity" grows much slower than a (possibly twisted) linear cone. A typical example of this is IP, the inside of a paraboloid, in which case the profile h grows at infinity in the vertical direction much faster than a power function. Similarly, at the local level, the doubling condition would fails around the point 0 in the planar domain above the graph of $x \mapsto |x|^\epsilon$, for any $\epsilon \in (0,1)$ (at 0, the profile h of any of these domains vanishes faster than any power function).

1.4. Remarks on the Doob h-transform technique

Central to our treatment of the heat equation on U with Dirichlet boundary condition is the celebrated technique of Doob's h-transform. See, e.g., [**30**, Chapter 5] for an introduction. It is perhaps remarkable that, after being used so many times by so many authors in so many different contexts, this technique still yields interesting new results. In order to take full advantage of the Doob h-transform technique, we have to solve two serious technical obstacles.

The first is of a fairly general nature and concerns the description in useful terms of the domain $h^{-1}W_0^1(U)$ of the Dirichlet form obtained after h-transform. This problem is discussed at some length in [**35**] where it is solved in the case of operators of the type $\Delta + V$ in Euclidean space but left partially open for Euclidean domains with the Dirichlet boundary condition (the basic setup of [**35**] is somewhat different from ours). In Chapter 5, we obtain some general results in this direction, describing in concrete terms the Dirichlet form obtained by h-transform when the function h is a harmonic profile of the domain U. See Proposition 5.7. Similar results are also obtained using different techniques and in greater generality, in [**29, 45**].

The second obstacle is much more concrete and technically much more difficult. In order to obtain good two-sided heat kernel estimates for the Dirichlet heat kernel in U with profile h via the Doob h-transform technique, we need to prove a weighted Poincaré inequality and a weighted doubling property in the domain U equipped with the weight h^2. Such properties are certainly not true without very strong assumptions on the domain U. Here, we base our results on the hypothesis that U is inner uniform and on consequences of this hypothesis on the behavior of the profile h which are drawn from the work of Aikawa, Ancona, and others. As explained at the end of the previous section, the condition of inner uniformity is not too far from being optimal for such a strong conclusion to hold.

1.5. Harnack-type Dirichlet spaces

In the previous two sections, we described the simplest applications of our results in the context of unbounded Euclidean domains. Even in this classical case, the tools we will use are best described in the general context of (local, regular) Dirichlet forms. One reason for this is that, given an unbounded Euclidean inner uniform domain, we will introduce and work with the (abstract) completion $\widetilde{U}$ of the domain U equipped with its inner metric. In general, $\widetilde{U}$ is different from $\overline{U}$ (the closure of U in $\mathbb{R}^n$) and is not a subset of $\mathbb{R}^n$. Working in $\widetilde{U}$ is critical for an easy application of our techniques. An early use of this simple idea is found in the work of M. Brelot. See [**21**].

Another important motivation for working in the context of regular local Dirichlet spaces is that it opens up the possibility of applying the results to a wide array of different settings including uniformly elliptic operators in $\mathbb{R}^n$, Riemannian manifolds and manifolds equipped with a sub-Riemannian structure (e.g., nilpotent Lie groups). Further, the Dirichlet space framework allows applications to spaces that are much less smooth than manifolds, e.g., polytopal complexes and other such structures.

For us, a Dirichlet space is a locally compact separable metrizable space X equipped with a non-negative Borel measure μ that is finite on compact sets and positive on non-empty open sets. In addition, (X,μ) is equipped with a Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, $\mathcal{D}(\mathcal{E})\subset L^2(X,\mu)$. See [**47**]. We will always assume that this Dirichlet form is strictly local and regular. The hypothesis that $\mathcal{E}$ is strictly local means that, in some sense, the associated infinitesimal generator is a differential type operator. In terms of Markov processes, it means that the associated process has continuous sample paths. The hypothesis that $\mathcal{E}$ is regular involves a relation with the topology of X. Let $\mathcal{C}_c(X)$ be the space of compactly supported continuous functions on X. The Dirichlet space $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ is regular if $\mathcal{D}(\mathcal{E})\cap\mathcal{C}_c(X)$ is dense in $\mathcal{D}(\mathcal{E})$ for the norm $(\|f\|_2^2+\mathcal{E}(f,f))^{1/2}$ and dense in $\mathcal{C}_c(X)$ for the sup norm.

The Dirichlet form $\mathcal{E}(f,f)$ can be written in the form $\int_X d\Gamma(f,f)$ where Γ is a measure valued symmetric bilinear form (in the classical case, $d\Gamma(f,f)=|\nabla f|^2 d\lambda$). This leads to the introduction of the intrinsic distance (in general, this "distance" may well take the value ∞)

$$\rho(x,y)=\sup\{f(x)-f(y): f\in\mathcal{D}(\mathcal{E})\cap\mathcal{C}_c(X): d\Gamma(f,f)\le d\mu\}.$$

See [**16, 90, 91**] and the references therein. Informally, $\rho(x,y)$ is the supremum of the differences $f(x)-f(y)$ over all functions f whose gradient has length at most 1, μ almost everywhere. This notion of intrinsic distance (and some of its variants) plays a very important role in a large part of the modern theory of Dirichlet spaces.

Now, we make some basic but fundamental hypotheses by assuming that ρ is continuous, defines the underlying topology of X, and turns (X,ρ) into a complete metric space. To understand these hypotheses, let us consider two examples. In our first example, we let $X=\{x=(x_1^n): x_n\ge 0\}$ be the closed upper-half space equipped with the Lebesgue measure and the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ where $\mathcal{E}(f,f)=\int|\nabla f|^2 d\lambda$, $\mathcal{D}(\mathcal{E})=W^1(X)$. This is a local regular Dirichlet form. The associated intrinsic distance is the Euclidean distance on X and (X,ρ) is complete. In our second example, we let $X=\{x=(x_1^n): x_n>0\}$ be the (open) upper-half space equipped with the Lebesgue measure and the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ where $\mathcal{E}(f,f)=\int|\nabla f|^2 d\lambda$, $\mathcal{D}(\mathcal{E})=W_0^1(X)$. This is again a local regular Dirichlet form. The intrinsic distance as defined above is not the Euclidean distance because $\rho(x,y)\le\min\{1,\max\{x_n,y_n\}\}\|x-y\|$. The intrinsic distance still defines the usual topology on X but (X, ρ) is not complete. In this case, there is incompatibility between being regular, and being complete in the sense that, if one completes (X, ρ), one looses regularity.

The qualitative assumptions on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ described above are very important but we will need to make more quantitative hypotheses. Let

$$B(x, r) = \{y \in X : \rho(x, y) < r\}$$

be the ρ-ball of radius r around x. Let

$$V(x, r) = \mu(B(x, r))$$

be its measure. We will make the following two crucial hypotheses:

1. *The doubling property.* There exists $D_0 \in (0, \infty)$ such that, for all $x \in X$ and $r > 0$, $V(x, 2r) \leq D_0 V(x, r)$.
2. *Poincaré inequality.* There exists $P_0 \in (0, \infty)$ such that, for all $x \in X$, $r > 0$ and $f \in \mathcal{D}(\mathcal{E})$,

$$\min_{\xi \in \mathbb{R}} \int_{B(x,r)} |f - \xi|^2 d\mu \leq P_0 r^2 \int_{B(x,r)} d\Gamma(f, f).$$

We call a Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfying all of these properties a *Harnack-type Dirichlet space*. The reason is that properties (1)-(2) above are equivalent to a parabolic Harnack inequality. This is the content of Sturm's generalization [**92**] of the results of Grigor'yan [**52**] and Saloff-Coste [**81, 82, 83**]. See also the works by Biroli and Mosco [**16, 17, 18**]. If one wants to be more precise, the spaces defined above should be called "globally Harnack-type Dirichlet spaces" but we will not use this terminology here.

1.6. Inner uniform domains in Harnack-type Dirichlet spaces

We can now describe the main results obtained in this monograph in their full generality. Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack Dirichlet space that admits a carré du champ (this natural hypothesis will be explained later). Recall (see, e.g., [**90, 91**]), that the intrinsic metric ρ of a Harnack-type Dirichlet space is a length metric (i.e., can be computed as the minimal length of continuous curves). It follows that, in the space (X, ρ), one can naturally define both the notion of inner metric ρ_U of a domain $U \subset X$ and the notion of inner uniform domain. More precisely, the distance $\rho_U(x, y)$ between two points in U is the infimum of the lengths of continuous paths in U joining x to y.

For a domain $U \subset X$, one can define the Neumann-type Dirichlet form

$$(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$$

and the Dirichlet type Dirichlet form

$$(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$$

associated with the original local regular form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on X. If U is inner uniform in X, using the hypothesis that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet form with a carré du champ, we will show that the Neumann-type Dirichlet form associated to U is regular on the set $\widetilde{U}$, the completion of U with respect its inner metric ρ_U. Moreover, we will show that the space $\widetilde{U}$ equipped with the (restriction of the) measure μ and the Neumann-type Dirichlet form is, itself, a Harnack-type Dirichlet space. It then follows that the associated Neumann-type heat kernel in U admits upper and lower bounds of the type

$$\frac{c_1}{V(x,\sqrt{t})} e^{-c_2 \rho_U(x,y)^2/t}$$

for all $t > 0$ and $x, y \in U$ (or $\widetilde{U}$). Of course, the parabolic Harnack inequality that gives their name to Harnack-type Dirichlet spaces also holds for local weak solutions of the heat equation in $\widetilde{U}$ and these solutions are Hölder continuous.

Considering now the Dirichlet-type Dirichlet form in the unbounded intrinsic domain $U \subset X$, by a direct generalization of Aikawa arguments in [**1**], we will first show that the elliptic boundary Harnack inequality holds in U. This first crucial step leads to the construction and main properties of the harmonic profile h of U (a positive harmonic function in U which, locally in $\widetilde{U}$, belongs to the domain $\mathcal{D}(\mathcal{E}_U^D)$, i.e., vanishes on the boundary of U, in the appropriate sense). The second crucial step is to prove that the modified form

$$\mathcal{E}_h(f,f) = \mathcal{E}_U^D(hf,hf), \quad f \in h^{-1}\mathcal{D}(\mathcal{E}_U^D)$$

is a strictly local regular Harnack-type Dirichlet form on $(\widetilde{U}, h^2 d\mu)$. This is a very non-trivial statement. The modified form $\mathcal{E}_h$ can be viewed as the main technical object in the well known Doob's transform (or h-transform) technique. Once it is proved that this modified form turns $\widetilde{U}$ into a Harnack-type Dirichlet space, all the desired conclusions follow immediately. In particular, the Dirichlet heat kernel in U admits upper and lower bounds of the type

$$\frac{c_1 h(x)h(y)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} e^{-c_2\rho_U(x,y)^2/t}, \quad x, y \in U, \ t > 0,$$

where $V_{h^2}(x,r) = \int_{B(x,r)} h^2 d\mu$.

1.7. Mixed boundary conditions

So far, we have considered the two type of boundary problems, Neumann and Dirichlet, separately. One of the advantages of working in the general setting of Dirichlet spaces is that it actually makes the treatment of some mixed boundary problems very easy. Consider again, for simplicity, a smooth unbounded Euclidean domain U. Let V be a closed subset of ∂U. Consider the heat equation in U with mixed boundary conditions (Dirichlet along V, Neumann along $\partial U \setminus V$)

$$\begin{cases} \partial_t u = \Delta u & \text{in } (0,\infty) \times U \\ u = 0 & \text{on } (0,\infty) \times V \\ \frac{\partial}{\partial \vec{\nu}} u = 0 & \text{on } (0,\infty) \times (\partial U \setminus V) \end{cases} \tag{1.9}$$

and its fundamental solution, the mixed heat kernel $p^M_{U,V}(t,x,y)$. Now, assume that U is inner uniform. In the simple case considered here where the boundary of U is smooth, the completion $\widetilde{U}$ of U with respect to the inner distance can be identified with the closure $\overline{U}$ of U in $\mathbb{R}^n$. Then, according to the results described earlier, the metric space $(\overline{U}, \rho_U)$ equipped with the Dirichlet form $(\mathcal{E}^N_U, W^1(U))$ is a Harnack-type Dirichlet space. Now, for any closed subset V of $\partial U = \overline{U} \setminus U$, the set $\Omega = \overline{U} \setminus V$ is an open connected set in $\overline{U}$ and it is essentially obvious that this open set Ω is inner uniform in $\overline{U}$ with inner distance ρ_Ω equal to the original inner distance ρ_U in U. This means that we can apply the results described earlier concerning the heat equation with Dirichlet boundary condition in an inner uniform domain (here, Ω) of a Harnack-type Dirichlet space (here, the space $(\overline{U}, \mathcal{E}^N_U, W^1(U))$). These results translate immediately and straightforwardly into results concerning our original mixed problem in the Euclidean domain U.

To illustrate this by one of the simplest possible examples, consider the open upper-half plane $U = \mathbb{R}^2_+ = \{(x,y) : x > 0\}$ and let $V = \{(x,y) : y = 0, x \leq 0\}$ be the non-positive half of the real axis. The profile h for the mixed problem corresponding to this data is given explicitly in polar coordinates by (abusing notation) $h(z) = \Re(z^{1/2}) = r^{1/2}\cos(\theta/2)$. Indeed, this function is harmonic in U, has vanishing normal derivative along the positive semi-axis, and vanishes continuously along the negative semi-axis. Importantly for our purpose, a direct computation shows that $h \in W^1_{\mathrm{loc}}(\overline{U})$ (it is well known and easy to check that $z \mapsto \Re(z^{1/2})$ is locally in W^1 of the slitted plane. The analysis of the mixed problem above is essentially equivalent to that of the Dirichlet problem in the slitted plane). The conclusion is that the heat kernel $p^M_{U,V}(t,x,y)$ for the mixed problem specified above is bounded above and below by an expression of the type

$$\frac{c_1 h(x)h(y)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} e^{-c_2\rho_U(x,y)^2/t}, \quad x,y \in U,\ t > 0,$$

where $V_{h^2}(x,r) = \int_{B(x,r)} h^2 d\mu$ and $h(z) = r^{1/2}\cos(\theta/2)$ as above.

CHAPTER 2

HARNACK-TYPE DIRICHLET SPACES

This chapter is devoted to the description and main properties of the spaces that will serve as candidates for the underlying ambient space in our study. The first section describes some of the models we have in mind. The remaining sections introduce the general setup (strictly local regular Dirichlet spaces of Harnack-type) that will be used throughout the monograph. This general setup covers all the models described earlier as special cases

2.1. Model spaces

2.1.1. The n-dimensional Euclidean space. — The most classical and most important model for us is the Euclidean n-space $\mathbb{R}^n$. The Euclidean structure provides us with a variety of different objects: the Lebesgue measure, the Euclidean metric, the Laplace operator

$$\Delta = \sum_1^n \left(\frac{\partial}{\partial x_i}\right)^2$$

(and the associated potential theory), the length of the gradient

$$|\nabla f| = \left(\sum_1^n |\partial f/\partial x_i|^2\right)^{1/2},$$

and more. These objects are interrelated and it is not entirely obvious to see which ones are the most fundamental for a given type of problem (see, e.g., the development of abstract potential theory [**31**] or the more recent theory of Sobolev space on metric spaces [**57, 59, 85**]).

For us here, the crucial structure is captured by the Dirichlet form

$$\mathcal{E}(f,f) = \int |\nabla f|^2 d\lambda, \quad f \in W^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, d\lambda).$$

The space $W^1(\mathbb{R}^n)$ is the Sobolev space of all L^2-functions whose first order partial derivatives in the sense of distributions can be represented by L^2 functions. The same definition applies to any open set $U \subset \mathbb{R}^n$ and yields the Sobolev space $W^1(U)$.

From this Dirichlet form we can recover the Laplace operator (by integration by parts through Green's formula) and the Euclidean metric. If we let $B(x,r)$ be the Euclidean open ball of radius r around x, we have

$$\lambda(B(x,r)) = V(x,r) = \Omega_n r^n$$

where Ω_n is the volume of the unit ball. The standard Poincaré inequality states that there is a constant P_0 such that, for any ball $B(x,r)$,

$$\inf_{\xi\in\mathbb{R}} \int_{B(x,r)} |f-\xi|^2 d\lambda \le P_0 r^2 \int_{B(x,r)} |\nabla f|^2 d\lambda, \;\; f \in W^1(B(x,r)).$$

This is equivalent to saying that the lowest non-zero Neumann eigenvalue in the ball $B(x,r)$ is bounded below by $(P_0 r^2)^{-1}$. Note that there is a similar inequality (also often called Poincaré inequality) for functions in $W_0^1(B(x,r))$, namely,

$$\int_{B(x,r)} |f|^2 d\lambda \le P_0' r^2 \int_{B(x,r)} |\nabla f|^2 d\lambda, \;\; f \in W_0^1(B(x,r)),$$

which is equivalent to say that the lowest Dirichlet eigenvalue is bounded below by $(P_0' r^2)^{-1}$. Despite their great similarity, these two inequalities capture rather different properties and should be clearly distinguished. For the purpose of this monograph, it is the Neumann Poincaré inequality that plays a crucial role.

The fundamental solution of the heat equation $(\partial_t - \Delta)u = 0$ is the heat kernel

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t},$$

which is also the kernel of the heat semigroup $e^{t\Delta}$.

The elliptic Harnack inequality goes back to Carl Gustav Axel von Harnack (1851-1888), see, e.g., [**67**]. It states that there exists a constant H_0 (depending only on the dimension n) such that, for any ball $B = B(x,r)$ and any non-negative harmonic function u in B,

$$\sup_{B(x,r/2)} \{u\} \le H_0 \inf_{B(x,r/2)} \{u\}.$$

One of the best known consequences of this inequality is the (strong) Liouville property: any function that is bounded below and harmonic in $\mathbb{R}^n$ must be constant.

The parabolic version of the Harnack inequality is much less known and did not attract much attention before the work of Jürgen Moser [**75**] concerning uniformly elliptic second order differential operators in divergence form. See [**74, 75, 76**]. To state the parabolic Harnack inequality, consider a time space cylinder

$$Q = (s, s+r^2) \times B(x,r).$$

In this cylinder, consider the upper cylinder

$$Q_+ = (s + 3r^2/4, s + r^2) \times B(x, r/2)$$

and the middle cylinder

$$Q_- = (s + r^2/4, s + r^2/2) \times B(x, r/2).$$

Then there exists a constant H_0 (depending only on the dimension n) such that any non-negative solution u of the heat equation in Q satisfies

$$\sup_{Q_-}\{u\} \le H_0 \inf_{Q_+}\{u\}.$$

All these properties of the n-dimensional Euclidean space will serve as models in what follows.

2.1.2. Uniformly elliptic divergence form operators. — In $\mathbb{R}^n$, consider a measurable matrix-valued function $a : x \mapsto a(x)$ with the property that a is symmetric and satisfies

$$\forall \xi, \zeta \in \mathbb{R}^n, \forall x \in \mathbb{R}^n, \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \ge \epsilon\|\xi\|^2, \quad \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\zeta_j \le \epsilon^{-1}\xi\cdot\zeta.$$

Call this property uniform ellipticity. The associated second order divergence form differential operator

$$L_a = \sum_i \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right)$$

is then said to be uniformly elliptic. Note that this operator is actually not properly defined as a differential operator unless a is differentiable. In general, what is well defined is the Dirichlet form

$$\mathcal{E}(f,f) = \int \sum_{i,j} a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}d\lambda, \quad f \in W^1(\mathbb{R}^n).$$

The Riemannian type distance associated with the inverse of the matrix a is, in some sense, the natural distance associated with this Dirichlet form. Because of the uniform ellipticity hypothesis, it is uniformly comparable to the Euclidean distance.

Through the work of De Giorgi [**36**], Nash [**77**], Moser [**74, 75, 76**], Aronson [**5, 6, 7**], Aronson and Serrin [**8**] and others, it became clear that these operators share many of the properties of the Laplace operator. Particularly relevant to us here is the fact that the kernel of the semigroup e^{tL_a} satisfies upper and lower bounds of the type

$$\frac{c_1}{t^{n/2}}e^{-c_2\|x-y\|^2/t}.$$

This is a celebrated result of Aronson [**5, 6**] which is also implicitly at the center of Nash's paper [**77**]. See also [**43**]. The Hölder regularity of the associated solutions was obtained in [**36**] in the elliptic case and in [**77**] in the parabolic case. The elliptic and parabolic Harnack inequalities were derived by Moser [**74, 75, 76**]. See also [**8**].

What makes this class of examples important to us is that its study ultimately depends on extracting from the classical case of the Laplacian the truly fundamental properties that are needed. This is particularly clear in the work of Nash and Moser.

It is interesting to mention here that the symmetry of the matrix $(a_{i,j})$ is not really essential. The global Harnack inequality and the global two-sided heat kernel estimate do not require this symmetry (see, e.g., [**8**]). The results obtained in this monograph also extend to the case when $(a_{i,j})$ is not symmetric but this requires taking a more general point of view and it is not covered in this monograph. It will appear in the Cornell Ph.D. thesis of Janna Lierl.

2.1.3. Riemannian manifolds. — Riemannian manifolds offer a very rich generalization of Euclidean space. Each comes equipped with a metric g (the Riemannian tensor), a measure μ (the Riemannian volume element), a distance ρ, geodesic balls $B(x,r)$ and a volume function $V(x,r) = \mu(B(x,r))$, $x \in M, r > 0$, a Laplace-Beltrami operator $\Delta =$ div grad and, most importantly for us, a natural Dirichlet form

$$\int |\nabla f|^2 d\mu, \quad f \in W^1(M),$$

where $|\nabla f| = g(\nabla f, \nabla f)^{1/2}$ is the length of the gradient $\nabla f =$ grad f of the function f. The space $W^1(M)$ can be defined more or less as in $\mathbb{R}^n$. It is the space of those functions in $L^2(M,\mu)$ which, locally, have distributional first order derivatives that can all be represented by locally integrable functions and such that $|\nabla f|$ is in $L^2(M,\mu)$. A Riemannian manifold (M,g) is complete if the associated metric space (M,ρ) is complete.

Complete Riemannian manifolds form an interesting class for us because they may or may not satisfy the various properties we want to impose which include the doubling volume property, the Poincaré inequality, the parabolic Harnack inequality, the two-sided Aronson type heat kernel estimate, etc. The following theorem due independently to Grigor'yan [**52**] and Saloff-Coste [**81**] shows that all these properties are very strongly interrelated. See [**83**, Section 5.5] for details.

Theorem 2.1. — *For any complete Riemannian manifold, the following properties are equivalent:*

1. *The volume doubling property together with the Poincaré inequality on geodesic balls;*
2. *The parabolic Harnack inequality;*
3. *The two-sided heat kernel bound*

$$\frac{c_1}{V(x,\sqrt{t})} e^{-c_2\rho(x,y)^2/t} \leq p(t,x,y) \leq \frac{c_3}{V(x,\sqrt{t})} e^{-c_4\rho(x,y)^2/t}.$$

Interesting classes of manifolds satisfying the equivalent properties listed above are: complete manifolds with non-negative Ricci curvature; connected Lie groups of polynomial volume growth (this includes all connected nilpotent Lie groups); covers of compact manifolds with fundamental groups of polynomial volume growth. See [**81, 83, 84**] and the references therein. On hyperbolic n-space, these properties fail at large scale (e.g., the volume growth is exponential).

2.1.4. Sub-Riemannian geometry. — In this short section, we discuss the important framework associated with sub-Riemannian geometries. Since our aim is simply to illustrate how this framework fits into our general picture, we only consider the simplest and perhaps most basic sub-Riemannian geometries. See, e.g., [**72, 81, 82, 100**] for further details.

There are at least two very natural and different ways to generalize Euclidean space. One leads to the notion of Riemannian manifold which was discussed above. The other leads to Lie groups equipped with invariant sub-Riemannian geometries. Let G be a unimodular connected Lie group with Haar measure $d\mu$ (unimodular means that the Haar measure is both left and right invariant). Let $\mathfrak{g}$ be its Lie algebra viewed as the space of all left-invariant vector fields. Let $\mathcal{X} = \{X_1, \dots, X_k\} \subset \mathfrak{g}$ be a family of left-invariant vector fields which generates $\mathfrak{g}$, algebraically. This means that the span of the X_i's and all their Lie brackets of all orders is $\mathfrak{g}$ itself. This is known as the Hörmander condition because, by a celebrated (and more general) theorem of Hörmander [**64**], this condition implies that the operator $L = \sum_1^k X_i^2$ is hypoelliptic.

Given this structure, we obtain a Dirichlet form by setting

$$\mathcal{E}_{\mathcal{X}}(f,f) = \int \sum_1^k |X_i f|^2 d\mu, \quad f \in W^1_{\mathcal{X}}(G),$$

where $W^1_{\mathcal{X}}(G)$ is the set of all functions f in $L^2(G)$ such that the distributions $X_i f$ can be represented by functions in $L^2(G)$, $i = 1, \dots, k$. Let $p_{\mathcal{X}}(t,x,y)$ be the associated heat kernel (see, e.g., [**100**]).

Associated with $\mathcal{X}$ is a natural left-invariant distance function $\rho_{\mathcal{X}}$ on G which is given by the infimum of the lengths of absolutely continuous paths that stay tangent to the linear span of $\mathcal{X}$ (such paths are often called horizontal paths). Because of the Hörmander condition, this distance defines the natural topology of G and $(G, \rho_{\mathcal{X}})$ is a complete metric space. Denote by $B_{\mathcal{X}}(x,r)$ the corresponding balls and set $V_{\mathcal{X}}(r) = \mu(B_{\mathcal{X}}(x,r))$ (by invariance, $\mu(B_{\mathcal{X}}(x,r))$ is independent of x). Any such geometry behaves very nicely locally. But for the volume doubling property to hold globally, we need to restrict our attention to the class of Lie groups with polynomial volume growth (see, e.g., [**100**]). These include all nilpotent Lie groups and, in particular, the Heisenberg group which is $\mathbb{R}^3$ equipped with the product

$$g_1 \cdot g_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \tfrac{1}{2}(x_1 y_2 - x_2 y_1))$$

where $g_i = (x_i, y_i, z_i)$, $i = 1, 2$. Let us recall here that a locally compact, compactly generated group has polynomial volume growth if, for one (equivalently, any) compact generating neighborhood V of the identity element, there are constants A, C such that, for all n, $\mu(V^n) \le C n^A$.

In the present setting, the following theorem holds. See, e.g., [**81, 82, 83, 84, 100**].

Theorem 2.2. — *For any unimodular connected Lie group G equipped with a Hörmander family $\mathcal{X}$ of left-invariant vector fields, the following properties are equivalent:*

1. *Polynomial volume growth;*
2. *The volume doubling property together with the Poincaré inequality on geodesic balls;*

3. The parabolic Harnack inequality;
4. The two-sided heat kernel bound of the type

$$\frac{c_1}{V_{\mathcal{X}}(\sqrt{t})}e^{-c_2\rho_{\mathcal{X}}(x,y)^2/t} \leq p_{\mathcal{X}}(t,x,y) \leq \frac{c_3}{V_{\mathcal{X}}(\sqrt{t})}e^{-c_4\rho_{\mathcal{X}}(x,y)^2/t}.$$

2.1.5. Polytopal complexes. — The aim of this short section is again to present the simplest cases of a large class of illustrative examples. This time, we leave the category of smooth spaces for that of polytopal complexes. For an excellent treatment of this type of spaces from a point of view closely related to ours, see [**39**]. To obtain the simplest examples, start with the square lattice $\mathbb{Z}^n$ in Euclidean n-space. Now, fix $k \in \{1, 2, , \dots, n\}$ and flesh out this lattice by adding all the faces of dimension less or equal to k of each of the unit cubes with integer coordinates. For $k = 1$, we obtain a one dimensional complex (the natural Cayley graph of the square lattice). For $k = n$, we recover $\mathbb{R}^n$. Call E^n_k the corresponding space. It can be turned into a metric space by using the length of curves in E^n_k to define the distance between points. There is a natural measure on E^n_k given by the k-dimensional Lebesgue measure on each k-dimensional face.

Given an open unit cube Q in dimension k, the Sobolev space $W^1(Q)$ is the space of functions in $L^2(Q)$ whose first partial derivatives in Q in the sense of distributions can be represented by L^2 functions. The natural norm on $W^1(Q)$ is $\left(\|f\|^2_{2,Q} + \|\nabla f\|^2_{2,Q}\right)^{1/2}$. To define the domain $W^1(E^n_k)$ of the natural Dirichlet form on E^n_k, we need to use the following trace theorem: For any $(k-1)$-dimensional face F of Q, there exists a continuous operator T^Q_F from $W^1(Q)$ to $L^2(F)$ which, restricted to smooth functions in $\overline{Q}$, is simply the trace operator $T^Q_F f = f\,|_F$. Write $E^n_k = \bigcup_{\mathbf{l}} \overline{Q_{\mathbf{l}}}$ where $\overline{Q_{\mathbf{l}}}$ is the k-dimensional unit cube with $\mathbf{l} = (l_1, \dots, l_k)$ as its integer point with smallest coordinates. Let $W^1(E^n_k)$ be the space of all functions in $L^2(E^n_k)$ such that, for each $\mathbf{l}$, $f \in W^1(Q_{\mathbf{l}})$, for each $\mathbf{l}, \mathbf{m}$ such that $\overline{Q_{\mathbf{l}}}, \overline{Q_{\mathbf{m}}}$ share a $k-1$ dimensional face F, $T^{Q_{\mathbf{l}}}_F f = T^{Q_{\mathbf{m}}}_F f$, and

$$\mathcal{E}_{E^n_k}(f,f) = \sum_{\mathbf{l}\in\mathbb{Z}^k} \int_{Q_{\mathbf{l}}} |\nabla f|^2 d\mu < \infty.$$

Now, $(\mathcal{E}_{E^n_k}, W^1(E^n_k))$ is the natural Dirichlet form on E^n_k and it induces a heat semigroup, its infinitesimal generator and its heat kernel. It is not hard to check that the doubling volume property and the Poincaré inequality hold. It follows from the general theory to be presented in next sections that the parabolic Harnack inequality holds true, local solutions of the heat equation are Hölder continuous, and the heat kernel $p(t, x, y)$ is continuous and bounded by

$$\frac{c_1}{V(x,\sqrt{t})}e^{-c_2\rho(x,y)^2/t} \leq p(t,x,y) \leq \frac{c_3}{V(x,\sqrt{t})}e^{-c_4\rho(x,y)^2/t}$$

for all $x, y \in E^n_k, t > 0$. Moreover, $V(x,r) \simeq r^k$ for $r \in [0,1]$ whereas $V(x,r) \simeq r^n$ for $r \in (1,\infty)$.

2.2. Local regular Dirichlet spaces

In this important section, we introduce the basic framework that will be used throughout this monograph. We refer the reader to [**47**, Ch. 1] for details.

Let (X, μ) be a locally compact separable metric space equipped with a Borel measure μ which is finite on compact sets and strictly positive on any non-empty open set. This X is our basic underlying space. We let $\mathscr{C}_c(X)$ be the space of continuous functions with compact support in X and $\mathscr{C}_0(X)$ its closure for the sup norm (continuous functions vanishing at infinity).

We let $\|f\|_2$ be the norm of f in the Hilbert space $L^2(X, \mu)$ and set

$$\langle f, g\rangle = \int_X f\bar{g}d\mu.$$

In most of this monograph, functions actually take real values. In particular, one can take $L^2(X, \mu)$ to be the L^2 space of real valued functions on X. Because Markov semigroups preserve real functions and positivity, no difficulties arise when it is necessary to use complex valued functions (e.g., when using interpolation techniques or analytic extensions).

2.2.1. Regular Dirichlet forms. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a densely defined non-negative definite closed symmetric form on $L^2(X, \mu)$. A function v is called a normal contraction of the function u if for all $x, y \in X$,

$$|v(x) - v(y)| \leq |u(x) - u(y)| \quad \text{and} \quad |v(x)| \leq |u(x)|.$$

We assume that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is Markovian, that is, has the property that if $u \in \mathcal{D}(\mathcal{E})$ and v is a normal contraction of u then $v \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$. Such a form is called a Dirichlet form. For any Dirichlet form, the set $\mathcal{D}(\mathcal{E}) \cap L^\infty(X)$ is an algebra ([**47**, Theorem 1.4.2]).

A core for $(X, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a subset $\mathscr{C}$ of both $\mathscr{C}_c(X)$ and $\mathcal{D}(\mathcal{E})$ that is dense in $\mathscr{C}_c(X)$ in uniform norm and dense in $\mathcal{D}(\mathcal{E})$ in the norm

$$\left(\|f\|_2^2 + \mathcal{E}(f, f)\right)^{1/2}.$$

When referring to the Hilbert space $\mathcal{D}(\mathcal{E})$, we always mean $\mathcal{D}(\mathcal{E})$ equipped with the norm above. Note that the notion of core depends on the precise choice of the topological space X as a well as on the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

A Dirichlet form that admits a core is called regular. Any regular Dirichlet form admits a core $\mathscr{C}$ with the following properties: (1) $\mathscr{C}$ is an algebra and (2) for any compact set K and relatively compact open set V containing K, there exists a function u in $\mathscr{C}$ such that $u \equiv 1$ on K and $u \equiv 0$ on $X \setminus V$ ([**47**, Problem 1.4.1]). Again, note that the notion of regularity involves the choice of the topological space X over which the form is defined.

2.2.2. Energy and carré du champ. — A Dirichlet form is called strictly local if for any two functions $u, v \in \mathcal{D}(\mathcal{E})$ with compact supports such that v is constant in a neighborhood of the support of u, we have $\mathcal{E}(u, v) = 0$. Unless explicitly stated, we assume throughout this monograph that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular strictly local Dirichlet form. All the examples in Section 2.1 are of this type.

Any strictly local regular Dirichlet form can be written in terms of an energy measure $\Gamma(u, v)$ so that

$$\mathcal{E}(u, v) = \int d\Gamma(u, v)$$

where for $u, v \in \mathcal{D}(\mathcal{E})$, $\Gamma(u, v)$ is a signed Radon measure on X. More precisely, the measure-valued quadratic form $\Gamma(u, u)$ is defined for $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X)$ by

$$\forall \phi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(X), \int \phi d\Gamma(u, u) = \mathcal{E}(u, \phi u) - \frac{1}{2}\mathcal{E}(u^2, \phi).$$

It is extended to $\mathcal{D}(\mathcal{E})$ by

$$d\Gamma(u, u) = \sup\{d\Gamma(v_n, v_n) : v_n = \min\{n, \max\{u, -n\}\},\ n = 1, 2 \ldots.\}.$$

The measure-valued symmetric bilinear form $\Gamma(u, v)$, $u, v \in \mathcal{D}(\mathcal{E})$, is obtained by polarization. The measure-valued symmetric form Γ satisfies the Leibniz rule and the chain rule. See, e.g., [**47**, Section 3.2] and also [**73**]. It is local in the sense that for any open set V and functions $u_1, u_2, v \in \mathcal{D}(\mathcal{E})$ such that $u_1 = u_2$ a.e. in V.

$$\Gamma(u_1, v)|_V = \Gamma(u_2, v)|_V.$$

Definition 2.3. — For any open set $V \subset X$, set

$$\mathcal{F}_{\text{loc}}(V) = \{u \in L^2_{\text{loc}}(V) : \forall \text{ compact } K \subset V, \exists\, u^\# \in \mathcal{D}(\mathcal{E}), u = u^\#\big|_K \text{ a.e.}\}.$$

For $u, v \in \mathcal{F}_{\text{loc}}(V)$ and for any relatively compact set $\Omega \subset V$, set

$$\Gamma_{V,\Omega}(u, v) = \Gamma(u^\#, v^\#)\big|_\Omega$$

where $u^\#, v^\#$ are any elements of $\mathcal{D}(\mathcal{E})$ such that $u = u^\#|_\Omega$, $v = v^\#|_\Omega$, a.e. in Ω. Since $\Gamma_{V,\Omega}(u, v)|_{\Omega'} = \Gamma_{V,\Omega'}(u, v)$ a.e. on Ω' if $\Omega' \subset \Omega$, we can define $\Gamma_V(u, v)$ on all of V by setting

$$\Gamma_V(u, v)|_\Omega = \Gamma_{V,\Omega}(u, v)$$

for all relatively compact set $\Omega \subset V$.

We will make repeated use of the following subspaces of $\mathcal{F}_{\text{loc}}(V)$.

Definition 2.4. — For any open set $V \subset X$, set

$$\mathcal{F}(V) = \{u \in \mathcal{F}_{\text{loc}}(V) : \int_V |u|^2 d\mu + \int_V d\Gamma_V(u, u) < \infty\}$$

and

$$\mathcal{F}_c(V) = \{u \in \mathcal{F}(V) : \text{ the essential support of } u \text{ is compact in } V\}.$$

The next lemma clarifies the nature of the space $\mathcal{F}_c(V)$. The proof uses straightforward cutoff function arguments and is omitted. For the last statement, see [**47**, Lemma 3.2.5]. For the definition of the notion of quasi-continuity (associated with a the given regular Dirichlet form), see [**47**, Chapter 2, pp. 67] and [**47**, Theorem 2.1.3, 2.1.7].

Lemma 2.5. — *Let $V \subset X$ be an open set. The space $\mathcal{F}_c(V)$ is a subspace of $\mathcal{D}(\mathcal{E})$ and*

$$\mathcal{F}_c(V) = \{f \in \mathcal{D}(\mathcal{E}) : \text{ the essential support of } f \text{ is a compact subset of } V\}.$$

The space $\mathcal{F}_{\mathrm{loc}}(V) \cap L^\infty_{\mathrm{loc}}(V)$ is an algebra. Moreover, for $f, g \in \mathcal{F}_{\mathrm{loc}}(V) \cap L^\infty_{\mathrm{loc}}(V)$ and any compact $K \subset V$

$$\Gamma(fg, fg)(K) \le 2\left(\int_K g^2 d\Gamma(f,f) + \int_K f^2 d\Gamma(g,g)\right).$$

In this last formula, quasi-continuous versions of f, g on K must be used on the right-hand side.

In the best case scenario, the energy measure $\Gamma(u,v)$ is actually absolutely continuous with respect to the measure μ for any $u, v \in \mathcal{D}(\mathcal{E})$. The following definition introduces this property.

Definition 2.6. — The strictly local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is said to admit a carré du champ if, for any $u, v \in \mathcal{D}(\mathcal{E})$, the measure $\Gamma(u,v)$ is absolutely continuous with respect to μ. When that is the case, the Radon-Nikodym derivative

$$\Upsilon(u,v) = \frac{d\Gamma(u,v)}{d\mu} \in L^1(X,\mu), \quad u, v \in \mathcal{D}(\mathcal{E}),$$

is called the carré du champ and, for any open set V and functions $u, v \in \mathcal{F}_{\mathrm{loc}}(V)$, we let

$$\Upsilon_V(u,v) = \frac{d\Gamma_V(u,v)}{d\mu} \in L^1_{\mathrm{loc}}(V,\mu).$$

Lemma 2.7. — *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strictly local regular Dirichlet form which admits a carré du champ. Then, for any open set V and $f, g \in \mathcal{F}_{\mathrm{loc}}(V) \cap L^\infty_{\mathrm{loc}}(V)$, we have*

$$\Upsilon_V(fg, fg) \le 2\left(f^2 \Upsilon_V(g,g) + g^2 \Upsilon_V(f,f)\right).$$

Remark 2.8. — In most cases, we will drop the reference to the open set V when referring to the energy measure $\Gamma_V(f,g)$ or the carré du champ $\Upsilon_V(f,g)$, $f, g \in \mathcal{F}_{\mathrm{loc}}(V)$ and write Γ and Υ instead.

2.2.3. The intrinsic distance associated with a Dirichlet form. — The fact that there is a simple way to attach a "distance like function" ρ to any strictly local regular Dirichlet form $\mathcal{E}$ is an important observation that is captured in the following definition. Note that, without further hypotheses, this definition should be manipulated with great care because slight modification of the definition may lead to different quantities. An excellent discussion is in [**90, 91**]. In each of the model examples discussed earlier in Section 2.1, the natural distance of the model can be obtained from the natural Dirichlet form of the model using this definition.

Definition 2.9. — Given a strictly local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$, for any $x, y \in X$, set

$$\rho(x,y) = \rho_{\mathcal{E}}(x,y) = \sup\{u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(X),\ \ d\Gamma(u,u) \leq d\mu\}.$$

In this definition, the condition $d\Gamma(u,u) \leq d\mu$ is understood to mean that $\Gamma(u,u)$ is absolutely continuous w.r.t. μ with Radon-Nikodym derivative bounded by 1 μ a.e. on X. The function ρ depends on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, X, and the topology of X. It is always lower semicontinuous, symmetric, and satisfies the triangle inequality $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. It is only a pseudo-distance because it is well possible that $\rho(x,y) = +\infty$ or $\rho(x,y) = 0$. Even if ρ is finite for all x, y, it may not define the original topology of X (for instance, it is possible that $\rho \equiv 0$!). It is possible that ρ is finite on a dense set but $+\infty$ a.e. on $X \times X$ (see, e.g., [**14**]).

Among the possible variants of this definition, let us consider

$$\rho^*(x,y) = \sup\{u(x) - u(y) : u \in \mathcal{F}_{\text{loc}}(X) \cap \mathcal{C}(X),\ \ d\Gamma(u,u) \leq d\mu\}. \tag{2.1}$$

In full generality, it is possible that $\rho^* \neq \rho$. However, we will work throughout this monograph under the following qualitative hypotheses:

(A1) The pseudo-distance ρ is finite everywhere, continuous, and defines the original topology of X.
(A2) The metric space (X, ρ) is complete.

Under these hypotheses, the metric ρ has many nice properties. In particular, assuming (A1)-(A2) for ρ is equivalent to assuming (A1)-(A2) for ρ^* and, if (A1)-(A2) hold then $\rho = \rho^*$. See [**88, 90, 91**].

Definition 2.10. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$. Assume that (A1)-(A2) hold true. For $x \in X$ and $r > 0$, let

$$B(x,r) = \{y \in X : \rho(x,y) < r\}$$

be the open ball of radius r around x in the metric space (X, ρ).

In any metric space, define the length of a continuous curve $\gamma : I = [a,b] \mapsto X$ by

$$L(\gamma) = \sup\left\{\sum_1^n \rho(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, a \leq t_0 < \cdots < t_n \leq b\right\}.$$

In general, $L(\gamma) \geq \rho(\gamma(a), \gamma(b))$. A metric space is a length space if $\rho(x, y)$ is equal to the infimum of the lengths of continuous curves joining x and y. A length space is a geodesic length space if, for any pair x, y there exists a continuous curve $\gamma : I = [0, 1] \mapsto X$ with $\gamma(0) = x, \gamma(1) = y$ and

$$\forall\, s, t \in I, \;\; \rho(\gamma(s), \gamma(t)) = |t - s| \rho(x, y).$$

Such a curve is called a minimal geodesic. The crucial property for us is that (X, ρ) is a length space. The fact that minimal geodesic exist is related to the completeness of the space (more precisely, to the fact that balls are relatively compact). Note that length spaces are locally path connected (every neighborhood of a point contains another neighborhood that is path connected). The next theorem gathers properties discussed in [**88, 89, 90, 91**].

Theorem 2.11. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$. Assume that* (A1)-(A2) *hold true. The following properties are satisfied:*

1. *For any subset $V \subset X$, the distance function*

$$f_V : X \mapsto [0, \infty), \; x \mapsto \rho(x, V)$$

   is in $\mathcal{F}_{\mathrm{loc}}(X) \cap \mathcal{C}(X)$ and $d\Gamma(f_V, f_V) \leq d\mu$.
2. *For each $x \in X$ and $r > 0$, the ball $B(x, r)$ is connected. Moreover, $\overline{B(x, r)} = \{y : \rho(x, y) \leq r\}$ and is compact.*
3. *The metric space (X, ρ) is a geodesic length space.*
4. *Let $\gamma : I = [a, b] \mapsto X$ be a continuous path without self-intersections joining x to y. Let U be an open set containing $\gamma(I)$ and $\mathcal{N}(U, \gamma)$ be the set of all open subsets of U that contain $\gamma(I)$. The length of γ can be computed as follows:*

$$L(\gamma) = \sup \{u(x) - u(y) : V \in \mathcal{N}(U, \gamma), u \in \mathcal{C}(V) \cap \mathcal{F}_{\mathrm{loc}}(V), \; d\Gamma(u, u) \leq d\mu\}$$

For an excellent introduction to length spaces including a discussion of the abstract version of the Hopf-Rinow-Cohn-Vossen theorem, see [**22**].

Finally, we introduce the intrinsic distance ρ_U associated with an open set $U \subset X$.

Definition 2.12. — Let U be an open set in a length metric space (X, ρ). Set

$$\rho_U(x, y) = \inf \{L(\gamma) : \gamma : [0, 1] \mapsto U \text{ continuous}, \gamma(0) = x, \gamma(1) = y\}.$$

If U is not connected, there are points x, y such that $\rho_U(x, y) = \infty$. However, if U is connected then $\rho_U(x, y)$ is finite for all $x, y \in U$ (see, e.g., [**22**, Ex. 2.4.15]).

The fourth statement in Theorem 2.11 is very relevant to the last definition and gives the following.

Proposition 2.13. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$. Assume that* (A1)-(A2) *hold true. Then*

$$\rho_U(x, y) = \sup \{u(x) - u(y) : u \in \mathcal{C}(U) \cap \mathcal{F}_{\mathrm{loc}}(U), \; d\Gamma(u, u) \leq d\mu\}.$$

As we shall see later in Remark 2.35, the distance function ρ_U can be viewed as the distance $\rho^*_{\mathcal{E}^D_U}$ defined at (2.1) with respect to the regular Dirichlet form $(\mathcal{E}^D_U, (\mathcal{E}^D_U))$ which is, by definition, the closure of $(\mathcal{E}^D_U, \mathcal{F}_c(U))$. This form is discussed in detail in Section 2.4.1 below. We call it the Dirichlet-type Dirichlet form on U, as it corresponds to imposing the Dirichlet boundary condition along ∂U. This observation and [**88**, Lemma 1'] yield the fact that, for any fixed $x \in U$, the function $f_x : y \mapsto \rho_U(x,y)$ is in $\mathcal{C}(U) \cap \mathcal{F}_{\mathrm{loc}}(U)$ and satisfies $d\Gamma(f_x, f_x) \le 1$, a.e..

The following definition will be crucial for our purpose.

Definition 2.14. — Let U be an open set in a length metric space (X,ρ). Let $\widetilde{U}$ be the completion of the metric space (U, ρ_U), equipped with the natural extension of ρ_U to $\widetilde{U} \times \widetilde{U}$.

Of course, this means that $\widetilde{U}$ is the quotient of the set of all Cauchy sequences in U by the usual equivalence relation.

Definition 2.15. — The open metric balls in (U, ρ_U) and $(\widetilde{U}, \rho_U)$ are denoted respectively by

$$B_U(x,r) = \{y \in U : \rho(x,y) < r\}, \quad B_{\widetilde{U}}(x,r) = \{y \in \widetilde{U} : \rho_U(x,y) < r\}$$

where, in the first case, $x \in U$, and in the second case, $x \in \widetilde{U}$.

In the case of Euclidean domains, M. Brelot considered the Dirichlet problem in terms of $\widetilde{U}$ in [**21**].

Remark 2.16. — Observe that $(\widetilde{U}, \rho_U)$ is a length metric space. Observe also that it is possible that $\widetilde{U}$ is not locally compact. For instance, in the Euclidean plane, let

$$U = (0,1) \times (0,1) \cup \left(\cup_{k=1}^{\infty} [(2^{-k-1/2}, 2^{-k}) \times (0,\infty)]\right).$$

Then $\widetilde{U}$ is given by

$$\widetilde{U} = [0,1] \times [0,1] \cup \left(\cup_{k=1}^{\infty} [[2^{-k-1/2}, 2^{-k}] \times (0,\infty)]\right)$$

and, if we let x_0 be the point $x_0 = (0,1)$, none of the balls $B_{\widetilde{U}}(x_0, r)$, $r > 0$, is compact in $\widetilde{U}$ because any such ball contains infinitely many disjoint balls of any fixed radius less than $r/8$.

2.2.4. The doubling property. — Having at our disposal the intrinsic distance of the previous section, we can now introduce the notion of volume doubling property.

Definition 2.17. — We say that a measure metric space (X, ρ, μ) has the volume doubling property if there exists a constant D_0 such that the volume growth function $V(x,r) = \mu(B(x,r))$ satisfies

$$\forall x \in X, r > 0, \quad V(x, 2r) \le D_0 V(x,r). \tag{2.2}$$

The doubling property has many simple useful consequences. For the convenience of the reader, we quote here two of them. They will be used throughout this work without further notice.

The first property we quote gives a controlled comparison between the volumes of balls of different centers and radii. Namely, if the doubling property holds then, for any $x, y \in X$ and $0 < r < s < \infty$, we have

$$V(x,s) \leq D_0^2 \left(\frac{\rho(x,y)+s}{r}\right)^A V(y,r),$$

with $A = \log_2 D_0$. In particular, this gives a polynomial upper bound on the growth of the volume as r tends to infinity.

The second property gives a power function lower bound on the volume growth. For this, we need to assume that the metric space (X,ρ) is a locally compact complete length metric space that is not compact. Under these assumptions, the doubling property yields the existence of positive constants C, α such that, for any $x \in X$ and $0 < s < r < \infty$, we have

$$V(x,s) \leq C \left(\frac{s}{r}\right)^\alpha V(x,r).$$

See, e.g., [**83**, Sec. 5.2.1]. Note that this last inequality says large balls have volume bounded below by a multiple of r^α and small balls have volume bounded above by a multiple s^α.

2.2.5. The Poincaré inequality. — This section introduces the Poincaré inequality that will play a key role in this monograph. In the present general setting, the relevant inequality is an L^2 inequality.

Definition 2.18. — Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet form on $L^2(X,\mu)$ which satisfies the qualitative hypothesis (A1)-(A2). We say that the Poincaré inequality holds if there exists a constant P_0 such that for any $x \in X$, $r > 0$ and $f \in \mathscr{F}_{\mathrm{loc}}(B(x,r))$,

$$\min_{\xi\in\mathbb{R}} \int_{B(x,r)} |f-\xi|^2 d\mu \leq P_0 r^2 \int_{B(x,r)} d\Gamma(f,f). \tag{2.3}$$

From a technical viewpoint, one of the most important properties related to the above Poincaré inequality is that, assuming that the doubling condition is satisfied, it is equivalent to the a priori weaker property that there exists a constant P_0' and a $k \geq 1$ such that for any $x \in X$, $r > 0$ and $f \in \mathscr{F}_{\mathrm{loc}}(B(x,kr))$,

$$\min_{\xi\in\mathbb{R}} \int_{B(x,r)} |f-\xi|^2 d\mu \leq P_0' r^2 \int_{B(x,kr)} d\Gamma(f,f). \tag{2.4}$$

See, e.g., [**83**, Sec. 5.3] and [**57**, Cor. 9.8]. In fact, [**57**, Cor. 9.8] shows that one can even weaken (2.4) further to (at least under the existence of a carré du champ)

$$\min_{\xi\in\mathbb{R}} \int_{B(x,r)} |f-\xi|\, d\mu \leq P_0'' r \left(\int_{B(x,kr)} d\Gamma(f,f)\right)^{1/2}.$$

We will not use this last fact in this monograph.

2.2.6. Lipschitz functions

Definition 2.19. — Given a metric space (X, ρ), a function f on X is said to be a Lipschitz function if there exists a constant C such that

$$\forall x, y \in X, \ \ |f(x) - f(y)| \le C\rho(x, y).$$

The smallest constant C for which this inequality holds is called the Lipschitz constant of f. We let $\mathrm{Lip}(X)$ be the space of Lipschitz functions on (X, ρ), $\mathrm{Lip}^\infty(X)$ be the space of bounded Lipschitz functions on (X, ρ), and $\mathrm{Lip}_c(X)$ be the space of Lipschitz funtions with compact support.

Remark 2.20. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$ satisfying (A1)-(A2). Let $f \in \mathcal{C}(X) \cap \mathcal{F}_{\mathrm{loc}}(X)$ be such that

$$d\Gamma(f, f)/d\mu \le 1, \ \ \text{a.e..}$$

Then f is Lipschitz in (X, ρ) with Lipschitz constant at most 1. See the definition of ρ.

Given any set A, the function $f_A(x) = \rho(x, A)$ is a Lipschitz function on (X, ρ). By Theorem 2.11, when ρ is the intrinsic distance associated with a strictly local regular Dirichlet space satisfying (A1)-(A2), f_A is in $\mathcal{C}(X) \cap \mathcal{F}_{\mathrm{loc}}(X)$. However, in general, there is no reason to believe that a Lipschitz function for ρ is necessarily in $\mathcal{F}_{\mathrm{loc}}(X)$. The following proposition addresses this issue. See [**63**, Corollary 3.6] and [**101**]. The reader should note that we use $\sup f$ to mean the essential supremum of f relative to the underlying measure μ unless stated otherwise.

Proposition 2.21. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$ which satisfies the qualitative conditions* (A1)-(A2) *and admits a carré du champ Υ. If $f \in \mathrm{Lip}(X)$ with Lipschitz constant k then $f \in \mathcal{F}_{\mathrm{loc}}(X)$ and*

$$k = \sup_X \sqrt{\Upsilon(f, f)}.$$

In particular, $\mathrm{Lip}(X)$ *coincides with the space*

$$\{f \in \mathcal{C}(X) \cap \mathcal{F}_{\mathrm{loc}}(X) : \sup_X \Upsilon(f, f) < \infty\}.$$

Corollary 2.22. — *Referring to the setting of* Proposition 2.21, *let U be an open subset of X. Every function on U which is Lipschitz with respect to ρ_U with Lipschitz constant k is in $\mathcal{F}_{\mathrm{loc}}(U)$ and satisfies*

$$k \ge \sup_U \sqrt{\Upsilon(f, f)}. \tag{2.5}$$

Proof. — Let f be k-Lipschitz in (U, ρ_U). For any open ball $B = B(x, r)$ in (X, ρ) such that $\rho(B, \partial U) \geq 2r$, the restriction $f|_B$ is Lipschitz with respect to ρ_U and thus with respect to ρ since $\rho = \rho_U$ in B. Therefore we can extend $f|_B$ to some compactly supported Lipschitz function f' on (X, ρ) with the same Lipschitz constant k. We have $f \equiv f'$ in B. Using Proposition 2.21 and the local property of $d\Gamma$, we see that $f' \in \mathcal{D}(\mathcal{E})$, $f \in \mathcal{F}(B)$ and

$$k = \sup_X \sqrt{\Upsilon(f', f')} \geq \sup_B \sqrt{\Upsilon(f, f)}.$$

This holds for any open ball $B = B(x, r)$ in (X, ρ) with $\rho(B, X \setminus U) \geq 2r$, therefore $f \in \mathcal{F}_{\text{loc}}(U)$ and (2.5) holds. □

The following result follows from the work of P. Hajłasz and his collaborators. See [**55, 56**]. We will actually give a version of the relevant argument, in a slightly different context, in Chapter 3. See Theorem 3.30.

Theorem 2.23. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$ which satisfies the qualitative hypotheses* (A1)-(A2) *and admits a carré du champ. Assume further that the doubling property and the Poincaré inequality are satisfied. Then the space* $\text{Lip}_c(X)$ *is a dense subspace of the Hilbert space $\mathcal{D}(\mathcal{E})$.*

2.2.7. The heat semigroup. — Any Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $\mathcal{D}(\mathcal{E}) \subset L^2(X, \mu)$, yields a strongly continuous self-adjoint semigroup of contractions, P_t, $t > 0$, acting on $L^2(X, \mu)$ which preserves positivity, i.e., satisfies $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$. The infinitesimal generator L of this semigroup can be obtained from the Dirichlet form as follows. Its domain $\mathcal{D}(L)$ is the subset of $\mathcal{D}(\mathcal{E})$ of those functions u for which there exists a constant C such that, for any $v \in \mathcal{D}(\mathcal{E})$, $\mathcal{E}(u, v) \leq C\|v\|_2$. On this domain, L is defined by $\langle Lu, v\rangle = \mathcal{E}(u, v)$. It is a self-adjoint operator and $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-L})$, $\mathcal{E}(u, v) = \langle\sqrt{-L}\, u, \sqrt{-L}\, v\rangle$. Moreover, the formula $P_t = e^{tL}$ can be understood in the sense of spectral theory and of the associated functional calculus, that is,

$$P_t = e^{tL} = \int_0^\infty e^{-t\lambda} dE_\lambda$$

where $-L = \int_0^\infty \lambda dE_\lambda$ is a spectral resolution of $-L$.

2.2.8. Local weak solutions of the Laplace and heat equation. — The notion of weak solution plays a crucial role in the result presented in this monograph. It is useful here to introduce these notions in the general context of a strictly local Dirichlet space $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$, without additional restrictive hypotheses.

For this purpose, for any open set $V \subset X$, note that the definition of $\mathcal{F}_{\text{loc}}(V)$ makes sense in this general context, and let $\mathcal{F}_c(V)$ be the set of functions in $\mathcal{D}(\mathcal{E})$ that have compact support in V (see Lemma 2.5). For $u \in \mathcal{F}_{\text{loc}}(V)$ and $v \in \mathcal{F}_c(V)$, note that we can define $\mathcal{E}(v, u)$ by setting

$$\mathcal{E}(v, u) = \mathcal{E}(v, u^{\#})$$

for any $u^\# \in \mathcal{D}(\mathcal{E})$ such that $u = u^\#$ a.e. in a neighborhood of the compact support of v (see the definition of $\mathcal{F}_{\rm loc}(V)$). This does not depend on the choice of $u^\#$ because of strict locality.

It is important to observe that the notions of local weak solution introduced below depends not only on the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ but also on the topology of X. This is because the test functions will be required to have compact support in some open subset of X.

2.2.9. Local weak solutions of the Laplace equation. — Identify $L^2(X,\mu)$ with its dual using the scalar product. Let V be a nonempty open subset of X. Consider the subspace $\mathcal{F}_c(V) \subset \mathcal{D}(\mathcal{E}) \subset L^2(X,\mu)$ and their duals $L^2(X,\mu) \subset \mathcal{D}(\mathcal{E})' \subset \mathcal{F}_c(V)'$. We will use the brackets $\langle\cdot,\cdot\rangle$ to denote duality pairing between these spaces.

Definition 2.24. — Let V be a nonempty open subset of X. Let $f \in \mathcal{F}_c(V)'$. A function $u : V \mapsto \mathbb{R}$ is a weak (local) solution of $Lu = f$ in V if

1. $u \in \mathcal{F}_{\rm loc}(V)$;
2. For any function $\phi \in \mathcal{F}_c(V)$, $\mathcal{E}(\phi, u) = \langle \phi, f\rangle$.

Remark 2.25. — If f can be represented by a locally integrable function in V and u is such that there exists a function $u^* \in \mathcal{D}(L)$ (the domain of the infinitesimal generator L) satisfying $u = u^*|_V$ then u is a weak local solution of $Lu = f$ if and only if $Lu^*|_V = f$ a.e in V.

Remark 2.26. — The notion of weak local solution defined above may contain implicitly a Neumann type boundary condition if X has a natural boundary. Consider for example the case when X is the closed upper-half plane $P_+ = \overline{\mathbb{R}^2_+}$ equipped with its natural Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^2_+} \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \right) d\lambda, \quad f \in W^1(\mathbb{R}^2_+).$$

Let $V = \{z = (x,y) : x^2 + y^2 < 1; y \geq 0\} \subset P_+$. Note that V is open in P_+. Let u be a local weak solution of $\Delta u = 0$ in V. Then u is smooth in V and must have vanishing normal derivative along the segment $(-1,1)$ of the real axis.

2.2.10. Local weak solutions of the heat equation. — This section introduces local weak solutions of the heat equation $\partial_t u = Lu$ in a time space cylinder $I \times V$ where I is a time interval and V is an nonempty open subset of X.

Given a Hilbert space H, let $L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ such that

$$\|v\|_{L^2(I\to H)} = \left(\int_I \|v(t)\|_H^2 dt \right)^{1/2} < \infty.$$

Let $W^1(I \to H) \subset L^2(I \to H)$ be the Hilbert space of those functions $v : I \mapsto H$ in $L^2(I \to H)$ whose distributional time derivative v' can be represented by functions in

$L^2(I \to H)$, equipped with the norm

$$\|v\|_{W^1(I\to H)} = \left(\int_I (\|v(t)\|_H^2 + \|v'(t)\|_H^2)dt\right)^{1/2} < \infty.$$

Given an open time interval I, set

$$\mathcal{F}(I \times X) = L^2(I \to \mathcal{D}(\mathcal{E})) \cap W^1(I \to \mathcal{D}(\mathcal{E})').$$

Given an open time interval I and an open set $V \subset X$ (both nonempty), let

$$\mathcal{F}_{\mathrm{loc}}(I \times V)$$

be the set of all functions $v : I \times V \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in I and any open subset V' relatively compact in V there exists a function $u^{\#} \in \mathcal{F}(I \times X)$ satisfying $u = u^{\#}$ a.e. in $I' \times V'$. Finally, let

$$\mathcal{F}_c(I \times V) = \{v \in \mathcal{F}(I \times X) : v(t, \cdot) \text{ has compact support in } V \text{ for a.e. } t \in I\}.$$

Definition 2.27. — Let I be an open time interval. Let V be an open subset in X and set $Q = I \times V$. A function $u : Q \mapsto \mathbb{R}$ is a weak (local) solution of the heat equation $(\partial_t - L)u = 0$ in Q if

1. $u \in \mathcal{F}_{\mathrm{loc}}(Q)$;
2. For any open interval J relatively compact in I and any $\phi \in \mathcal{F}_c(Q)$,

$$\int_J \int_V \phi \partial_t u d\mu dt + \int_J \mathcal{E}(\phi(t, \cdot), u(t, \cdot))dt = 0.$$

As noticed in the elliptic case, this definition may contain implicitly some Neumann-type boundary condition along a natural boundary of X.

As a first example, note that if f is in $L^2(X, \mu)$ then $P_t f = e^{tL} f$ is a weak solution of the heat equation in $I \times X$ for any bounded time interval $I \subset (0, \infty)$ (in the sense introduced above). The next lemma produces more interesting local solutions. Its origins are in the work of Aronson [**6**]. See the "Extension Principle" on page 621 of [**6**].

Lemma 2.28. — *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and strictly local. Let $\phi \in \mathcal{D}(\mathcal{E})$ and set*

$$v(t, x) = \begin{cases} P_t\phi(x) & \text{if } t > 0, \\ \phi(x) & \text{if } t \leq 0. \end{cases} \tag{2.6}$$

Then $v \in \mathcal{F}_{\mathrm{loc}}(\mathbb{R} \times X)$. Moreover, if V is an open set such that $\phi|_V = 1$ then v is a local weak solution of the heat equation in $\mathbb{R} \times V$.

Proof. — For the first assertion, it suffices to show that v is in $\mathcal{F}(I \times X)$ for any bounded time interval I. As $\mathcal{E}(P_t\phi, P_t\phi) \leq \mathcal{E}(\phi, \phi)$, it is clear that v is a bounded continuous function from $\mathbb{R}$ to $\mathcal{D}(\mathcal{E})$. Hence $v \in L^2(I \to \mathcal{D}(\mathcal{E}))$. The interesting part is to show that $v \in W^1(I \to \mathcal{D}(\mathcal{E})')$. Set

$$\psi(t,x) = \begin{cases} -LP_t\phi(x) & \text{if } t > 0, \\ 0 & \text{if } t \le 0 \end{cases}$$

We first show that ψ is the distributional time derivative of v, i.e., $v' = \psi$. The only difficulty is on the positive side, at $t = 0$. But, by spectral theory,

$$\int_0^s \psi(t,\cdot)dt = \int_0^s -LP_t\phi dt = \int_0^s \int_0^\infty -\lambda e^{-t\lambda} dE_\lambda \phi dt = P_s\phi - \phi = v(s,\cdot) - v(0,\cdot).$$

We are left to show that $v'(t,\cdot)$ is in $\mathcal{D}(\mathcal{E})'$ with uniformly bounded norm. But, for any $t > 0$ and $u \in \mathcal{D}(\mathcal{E})$,

$$\langle v'(t,\cdot), u\rangle = \int (-LP_t\phi) u d\mu = \mathcal{E}(P_t\phi, u)$$

and thus

$$|\langle v'(t,\cdot), u\rangle| \le \mathcal{E}(\phi,\phi)\mathcal{E}(u,u)$$

as desired. Hence, $v \in \mathcal{F}(I \times X)$ for any bounded time interval I. This proves the first assertion. Note that the hypothesis that the Dirichlet form is regular strictly local has not been used so far.

Assume now that ϕ is equal to 1 in the open set V. To show that v is a local weak solution in $\mathbb{R} \times V$ it suffices to show that for almost every t and any $q \in \mathcal{F}_c(\mathbb{R} \times V)$,

$$\int q v' d\mu + \int d\Gamma_V(q(t,\cdot), v(t,\cdot)) d\mu = 0.$$

Recall that $v' = \psi$ (see above). Both integrals are clearly 0 for negative t. For positive t, the equality above easily follows from the fact that $v(t,\cdot) = P_t\phi$ and $\phi, q \in \mathcal{D}(\mathcal{E})$. □

2.3. Harnack-type Dirichlet spaces

In this section we introduce the notion of Harnack-type Dirichlet space and discuss the main properties of these spaces.

2.3.1. Parabolic Harnack inequality.

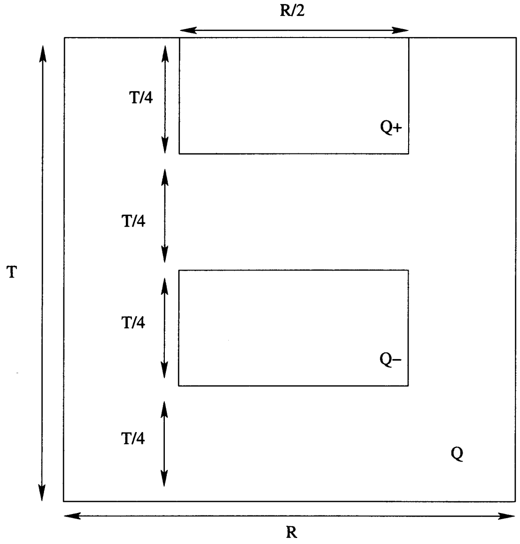
— The following is the main definition of this section. See Figure 1 below. We keep the notation introduced in the previous sections.

Definition 2.29. — A regular strictly local Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack type if the distance $\rho_{\mathcal{E}}$ satisfies the conditions (A1)-(A2) of Section 2.2.3, and the following uniform parabolic Harnack inequality holds. There exists a constant H_0 such that, for any $z \in X$, $r > 0$, and any non-negative weak solution u of the heat equation $\partial_t u - Lu = 0$ in $(0, r^2) \times B(z,r)$, we have

$$\sup_{(t,x)\in Q_-} u(t,x) \le H_0 \inf_{(t,x)\in Q_+} u(t,x), \tag{2.7}$$

FIGURE 1. The cylinders $Q_-, Q_+ \subset Q$, $\sqrt{T} = R = r$

On a Harnack-type Dirichlet space, any positive solution of the heat equation in Q satisfies $\sup_{Q_-}\{u\} \le H_0 \inf_{Q_+}\{u\}$.



where $Q_- = (r^2/4, r^2/2) \times B(z, r/2)$, $Q_+ = (3r^2/4, r^2) \times B(z, r/2)$ and both sup and inf are essential, i.e., computed up to sets of measure zero.

Any Harnack-type Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the following elliptic Harnack inequality (with the same constant H_0 as in (2.7)). For any $z \in X$ and $r > 0$ and any non-negative weak solution u of the equation $Lu = 0$ in $B(z, 2r)$, we have

$$\sup_{B(z,r)} u \le H_0 \inf_{B(z,r)} u. \tag{2.8}$$

This elliptic Harnack inequality is weaker than its parabolic counterpart. See [**58**] for a discussion of situations where they are equivalent.

The following variant of the Harnack inequality (2.7) is sometimes useful. We mention it for the record. Fix parameters $\tau, \theta \in (0,1)$ and $0 < \epsilon < \eta < \sigma < 1$. If $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack-type then there exists a constant H_0' such that, for any $z \in X$, $r > 0$, $s \in \mathbb{R}$ and any non-negative weak solution u of the heat equation $\partial_t u - Lu = 0$ in $(s - \tau r^2, s) \times B(z, r)$, we have

$$\sup_{(t,x)\in Q'_-} u(t,x) \le H_0' \inf_{(t,x)\in Q'_+} u(t,x), \tag{2.9}$$

where $Q'_- = (s - \sigma r^2, s - \eta r^2) \times B(z, \theta r)$, $Q_+ = (s - \epsilon r^2, s) \times B(z, \theta r)$. The constant H_0' depends on the parameter $\tau, \theta, \epsilon, \eta, \sigma$ fixed above as well as of the constant H_0.

One of the important consequences of the Harnack inequality (2.7) is the following quantitative Hölder continuity estimate (see, e.g., [**83**, Theorem 5.4.7]).

Theorem 2.30. — *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet form on $L^2(X,\mu)$. Fix $\tau > 0$. Then there exists $\alpha \in (0,1)$ and $H_1 \in (0,\infty)$ such that any local (weak) solution of $\frac{\partial}{\partial t}u + Lu = 0$ in $Q = (s - \tau r^2, s) \times B(x,r)$, $x \in X$, $r > 0$ has a continuous representative and satisfies*

$$\sup_{(t,y),(t',y')\in Q'} \left\{ \frac{|u(t,y) - u(t',y')|}{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha} \right\} \le \frac{H_1}{r^\alpha} \sup_Q |u|.$$

Here $Q' = (s - \frac{3}{4}\tau r^2, s - \frac{1}{2}\tau r^2) \times B(x, r/2)$ and $B(x,r)$ is a ball in $(X, \rho_{\mathcal{E}})$ centered at x.

A crucial consequence of this is that, on a Harnack-type Dirichlet space, local weak solutions of the Laplace or heat equation are continuous functions (in the sense that they admit a continuous representative).

2.3.2. Doubling, Poincaré, and Harnack inequality. — The main result concerning Harnack-type Dirichlet spaces is the following theorem of Sturm [**89, 92**] which generalizes works by Grigor'yan and by Saloff-Coste mentioned earlier.

Theorem 2.31. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space. Assume that the distance $\rho_{\mathcal{E}}$ satisfies the assumptions* (A1)-(A2) *of* Section 2.2.3. *Then the following properties are equivalent:*

- *The form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack-type, i.e., the uniform parabolic Harnack inequality* (2.7) *is satisfied.*
- *The volume doubling condition* (2.2) *and the Poincaré inequality* (2.3) *are satisfied.*
- *The heat semigroup P_t admits an integral kernel $p(t,x,y)$, $t > 0$, $x, y \in X$ and there exist constants c_i, $i = 1, \dots, 4$, such that*

$$\frac{c_1}{V(x,\sqrt{t})} \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_2 t}\right) \le p(t,x,y) \le \frac{c_3}{V(x,\sqrt{t})} \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_4 t}\right) \tag{2.10}$$

for all $x, y \in X$ and all $t > 0$.

For a complete discussion, see [**88, 89, 92**]. For a treatment in the case of Riemannian manifolds and comments on the literature, see [**83**]. For a discussion of related properties, see [**58, 82**] and the works by Grigor'yan including [**53**].

It is easy to see and often useful to observe that, under the doubling condition, the heat kernel estimate (2.10) is equivalent to the more symmetric estimate

$$\frac{c_1 \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_2 t}\right)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \le p(t,x,y) \le \frac{c_3 \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_4 t}\right)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}, \tag{2.11}$$

for all $x, y \in X$ and all $t > 0$.

Note that it is not hard to show, that, if the heat kernel exists, it must be a weak solution of the heat equation. By spectral theory, the time derivatives of any order of the heat kernel are also weak solutions. This is the starting point of the next theorem which gathers some properties of the heat kernel.

Theorem 2.32. — *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack-type. Then the heat semigroup $P_t = e^{tL}$ admits an integral kernel $p(t,x,y)$ (the heat kernel) which is a continuous function of $(t,x,y) \in (0,\infty) \times X \times X$, symmetric in (x,y). For each x, the function $(t,y) \mapsto p(t,x,y)$ is a weak solution of the heat equation on $(0,\infty) \times X$. Moreover, there exist $\alpha, \beta > 0$ and, for each $k = 0,1,2,\ldots$, a constant C_k such that*

$$|\partial_t^k p(t,x,y)| \leq \frac{C_k}{t^k V(x,\sqrt{t})} \left(1 + \frac{\rho_{\mathcal{E}}(x,y)^2}{t}\right)^{\beta+k} \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{4t}\right)$$

for all $t > 0$, $x, y \in X$, and

$$\left|\partial_t^k p(t,x,y) - \partial_t^k p(t,x,y')\right| \leq \frac{C_k}{t^k} \left(\frac{\rho_{\mathcal{E}}(x,y)}{\sqrt{t}}\right)^{\alpha} p(2t,x,y)$$

for all $t > 0$, $x, y, y' \in X$ with $\rho_{\mathcal{E}}(y,y') \leq \sqrt{t}$.

See, e.g., [**33, 58, 83**]. An even more basic property of Dirichlet forms of Harnack-type is that the semigroup P_t is conservative (we present here one of the several ways to show this. For a sharp criterion, see [**51**]).

Lemma 2.33. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet form on $L^2(X,\mu)$. For any $t > 0$ and any $x \in X$,*

$$\int_X p(t,x,y)d\mu(y) = 1, \tag{2.12}$$

in other words the semigroup P_t is conservative.

Proof. — Since the semigroup P_t is Markovian, we have

$$\int_X p(t,x,y)d\mu(y) \leq 1.$$

Fix $z \in X$ and $R > 0$. Let

$$\phi_R(x) = \min(1, \max(0, R + 1 - \rho(x,z))).$$

We know that the function ϕ_R is supported in $B(z,R+1)$ and is identically one on $B(z,R)$. Since $\rho(z,\cdot) \in \mathcal{F}_{\mathrm{loc}}(X)$ with $d\Gamma(\rho(z,\cdot),\rho(z,\cdot)) \leq d\mu$ by [**88**, Lemma 1], it follows that $\phi_R \in \mathcal{F}_c(X) \subset \mathcal{D}(\mathcal{E})$ and $d\Gamma(\phi_R,\phi_R) \leq d\mu$ on X.

Let ψ_R be the function ψ defined at (2.6) using the function $\phi = \phi_R$. Consider the function

$$v(t,x) = \begin{cases} \int_X p(t,x,y)dy, & \text{if } t > 0 \\ 1, & \text{if } t \leq 0 \end{cases}$$

which is the increasing limit of the functions ψ_R by the dominated convergence theorem. By Lemma 2.28, each of the functions ψ_R is a nonnegative weak solution in

$\mathbb{R} \times B(z,R)$ of the heat equation. Since $0 \le \psi_R \le 1$, by the Hölder estimates of Proposition 2.30, for for any t,t' with $|t-t'| \le R^2$ and $y,y' \in B(z,\frac{R}{2})$ we have

$$|\psi_R(y,t) - \psi_R(y',t')| \le A\frac{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha}{R^\alpha}.$$

Taking the limit as $R \to \infty$, we see that for all $y,y' \in X$ and $t,t' \in \mathbb{R}$

$$|v(y,t) - v(y',t')| \le \limsup_{R\to\infty} \left\{ A\frac{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha}{R^\alpha} \right\} = 0.$$

That, is $P_t 1 \equiv 1$. □

2.3.3. The associated Hunt process and harmonic sheaf. — Chapter 7 of [**47**] shows how one can associate (in an essentially unique way) a Markov process with continuous paths to any strictly local regular Dirichlet space. We will denote this process by $(X_t)_{t\ge0}$ and we will denote by $\mathbb{P}_x, \mathbb{E}_x$ the associated probability measure and expectation on continuous paths starting at x. This means that for any Borel set A and any continuous bounded function ϕ,

$$\mathbb{P}_x(X_t \in A) = P_t\mathbf{1}_A(x), \;\; \mathbb{E}_x(\phi(X_t)) = P_t\phi(x).$$

Let us observe that, in the case of a Harnack-type Dirichlet space, the Markov semigroup P_t is particularly nice. Its transition function admits a continuous density $p(t,x,y)$ and P_t has the strong Feller property, that is, P_t sends $\mathscr{C}_0(X)$ into itself and sends the space of bounded measurable functions into the space of bounded continuous functions. This follows immediately from Theorem 2.32.

Let V be an open subset of X. Then $\tau_V = \inf\{t > 0 : X_t \notin V\}$ is a stopping time and, assuming $X_0 \in V$ and $\tau_V < \infty$, $X_{\tau_V} \in \partial V$. Given a starting point $X_0 = x \in V$, the probability distribution of X_{τ_V} is a probability measure $\omega_V(x,\cdot)$ supported on ∂V called the harmonic measure. A continuous function u in an open set U is called harmonic (for the process $(X_t)_{t\ge0}$) if for any open subset V relatively compact in U,

$$\forall\, x \in V, \;\; u(x) = \omega_V(x,u) = \mathbb{E}_x(u(X_{\tau_V})).$$

References for these considerations are [**12, 13, 19, 20, 31, 86**] among others. The beginning sections of [**12, 13**] give a useful overview. The classical treaties [**37, 38**] contain much useful information concerning the classical setting. The classical references for axiomatic potential theory are [**11, 21, 31**].

This defines a sheaf of harmonic functions on X. When the underlying strictly local regular Dirichlet form is of Harnack-type, this sheaf is a Brelot sheaf. See [**11, 21**] and also [**12, 13**]. In particular, there is a basis of the topology made of regular sets, the Brelot convergence property is satisfied (on an open connected set, the increasing limit of a sequence of harmonic functions is harmonic if it is finite at one point) and all relatively compact open sets are resolutive.

A measurable lower semicontinuous function u is superharmonic in U if for any open subset V relatively compact in U,

$$\forall\, x \in V, \;\; \omega_V(x,u) \le u(x)$$

and $\omega_V(x,u)$ is harmonic in V. A function $p \geq 0$ which is superharmonic on X is called a potential if its only non-negative harmonic minorant is the constant function 0. The smallest closed set outside of which a potential p is harmonic is called the harmonic support of p and denoted by $S(p)$. On a strictly local Dirichlet space of Harnack-type, the axiom of proportionality is satisfied (two potentials on X that are harmonic on $X \setminus \{x\}$ are proportional). Moreover the axiom of domination holds (any locally bounded potential p which is continuous on $S(p)$ is continuous on X). It follows that semipolar sets and polar sets coincide.

Let us now observe that, assuming that our Dirichlet space is of Harnack-type, we can also consider the harmonic sheaf of the local weak solutions of $Lu = 0$. This is a sheaf of continuous functions because of Theorem 2.30. We claim that this sheaf coincides with the sheaf of harmonic functions relative to the process $(X_t)_{t\geq 0}$ as considered above. The fact that local weak solutions of $Lu = 0$ in an open set V are harmonic functions in V for the process (modulo the choice of the continuous representative that exists by Theorem 2.30) follows from [**47**, Theorem 4.3.2]. One way to see the converse, i.e., that a harmonic function for the process is a local solution (in particular, is locally in the domain of the Dirichlet form), is to use [**86**, Lemma 3.3] (and, more generally, [**86**, Chap. 3]).

2.4. Boundary conditions

Given an open set $U \subset X$, the aim of this section is to define the heat semigroup in U with Dirichlet boundary condition (Dirichlet-type heat semigroup) and the heat semigroup in U with Neumann boundary condition (Neumann-type heat semigroup). We will also discuss the corresponding notions of weak solutions. Throughout this section $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular strictly local Dirichlet form on $L^2(X,\mu)$ and U is an arbitrary nonempty open set in X.

2.4.1. The Dirichlet-type heat semigroup in U. — There are no difficulties to define the heat semigroup with Dirichlet boundary condition in an open set U. Consider the restriction of $\mathcal{E}$ to the space $\mathcal{F}_c(U)$ as a form on $L^2(U,\mu)$ (abusing notation, we write μ for $\mu|_V$, for any open set V). Clearly, this is a symmetric bilinear closable form.

Definition 2.34. — Let $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ be the closure of the form $(\mathcal{E}, \mathcal{F}_c(U))$ in $L^2(U,\mu)$. Set

$$\mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}_U^D)$$

and

$$\mathcal{F}^0_{\mathrm{loc}}(U) = \Big\{ f \in L^2_{\mathrm{loc}}(U,\mu) : \forall V \subset U, \text{ open, relatively compact in } \overline{U}, \\ \exists f^{\#} \in \mathcal{F}^0(U), f^{\#} = f \text{ a.e. in } V \Big\}.$$

Note that $\mathcal{F}^0(U)$ is the closure of $\mathcal{F}_c(U)$ in $(\mathcal{D}(\mathcal{E}), (\|\cdot\|_2^2 + \mathcal{E}(\cdot,\cdot))^{1/2})$. Because the closable form $(\mathcal{E}, \mathcal{F}_c(U))$ is Markovian (see [**47**]), its closure is a Dirichlet form [**47**, Theorem 3.1.1]. By definition, it is clear that $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ is regular with core $\mathcal{F}_c(U) \cap \mathcal{C}_c(U)$ on U. It is also strictly local.

Remark 2.35. — Note that $\mathcal{F}^0_{\text{loc}}(U)$ is not the local domain of $\mathcal{E}_U^D$ on U which, by definition, is

$$\begin{aligned}\mathcal{F}_{\text{loc}}^{\mathcal{E}_U^D}(U) &= \left\{ f \in L^2_{\text{loc}}(U,\mu) : \forall V \text{ compact } \subset U, \exists f^\# \in \mathcal{F}^0(U),\ f^\#\Big|_V = f \text{ a.e.}\right\}\\ &= \mathcal{F}_{\text{loc}}(U).\end{aligned}$$

Consider the two intrinsic (pseudo-)metrics $\rho_{\mathcal{E}_U^D}$, $\rho^*_{\mathcal{E}_U^D}$ associated with $(\mathcal{E}_U^D, \mathcal{F}^0(U))$ in Section 2.2.3. If U is a proper open set of X, these two metrics are distinct. By definition,

$$\rho_{\mathcal{E}_U^D}(x,y) \le \max\{\rho(x,U^c), \rho(y,U^c)\} - \min\{\rho(x,U^c), \rho(y,U^c)\}$$

whereas, using Definition 2.12 and Theorem 2.11(4),

$$\rho^*_{\mathcal{E}_U^D}(x,y) = \rho_U(x,y).$$

Note that hypothesis (A2) (i.e., completeness) fails in this case (for both metrics). If one considers the form $(\mathcal{E}_U^D, \mathcal{F}^0(U))$ on $\overline{U}$ then it is not regular and the distance $\rho_{\mathcal{E}_U^D}$ does not define the original topology of $\overline{U}$ (it does not distinguish between two boundary points). The equality $\rho^*_{\mathcal{E}_U^D} = \rho_U$ is useful because it provides us with the property that for any subset V of U, the function

$$f_{U,V}(x) = \rho_U(x,V)$$

is in $\mathcal{F}_{\text{loc}}(U)$ and satisfies

$$\frac{d\Gamma(f_{U,V}, f_{U,V})}{d\mu} \le 1 \text{ a.e..}$$

Indeed, $(\mathcal{E}_U^D, \mathcal{F}^0(U))$ is regular over U and [**88**, Lemma 1'] applies.

Lemma 2.36. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular strictly local Dirichlet form on $L^2(X,\mu)$ and U is an nonempty open set in X. Assume that the conditions* (A1)-(A2) *of* Section 2.2.3 *are satisfied. Then a function f defined on U belongs to $\mathcal{F}^0_{\text{loc}}(U)$ if and only if, for any continuous function ϕ defined on X with compact support such that $\phi \in \mathcal{D}(\mathcal{E})$ and $d\Gamma(\phi,\phi)/d\mu \le 1$ we have $\phi f \in \mathcal{F}^0(U)$.*

Proof. — Given a compact set K in $\overline{U}$, the intrinsic distance ρ can be used to construct a continuous compactly supported function $\phi_K \in D(\mathcal{E})$ with compact support in X, which is equal to 1 on a neighborhood of K and satisfies $d\Gamma(\phi_K, \phi_K) \le 1$. As $\phi_K f = f$ on K, if we know that $\phi_K f$ is in $\mathcal{F}^0(U)$ for any compact K then we must have $f \in \mathcal{F}^0_{\text{loc}}(U)$.

Conversely, assume that $f \in \mathcal{F}^0_{\text{loc}}(U)$. Let ϕ be as in the lemma with compact support $K \subset X$. As f is in $\mathcal{F}^0_{\text{loc}}(U)$, there exists a function $f^\# \in \mathcal{F}^0(U)$ which coincides with f on $\Omega = K \cap U$. Let f_n be functions in $\mathcal{F}_c(U) \cap \mathcal{C}_c(U)$ that approximate $f^{\#}$ in the Hilbert space $\mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}_U^D)$. Then $\phi f_n \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, \mu)$. Since these functions are compactly supported in U, they are in $\mathcal{F}_c(U)$. Now, it is clear that the sequence ϕf_n converges to $\phi f^{\#} = \phi f$ in $L^2(U, d\mu)$. Using Lemma 2.5, it is not hard to show that the sequence ϕf_n is Cauchy in the Hilbert space $\mathcal{F}_0(U) = \mathcal{D}(\mathcal{E}_U^D)$. Hence $\phi \widetilde{f} \in \mathcal{F}^0(U)$ as desired. □

Definition 2.37. — Let $P_{U,t}^D$ be the subMarkovian semigroup on $L^2(U, \mu)$ associated with $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$. Let L_U^D be the associated infinitesimal generator.

Note that L_U^D is self-adjoint and non-positive.

Remark 2.38. — The Dirichlet semigroup on U can be described in term of the process $(X_t)_{\{t \geq 0\}}$ and the exit time τ_U introduced earlier as follows. For any bounded continuous function ϕ in U, we have

$$P_{U,t}^D \phi(x) = \mathbb{E}_x[\phi(X_t)\mathbf{1}_{\{t < \tau_U\}}].$$

See [**47**, Theorem 4.4.2]. Let us assume that the original strictly local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ admits a locally bounded heat kernel

$$p(t, x, y), \ (t, x, y) \in (0, \infty) \times X \times X.$$

In this case it is well known that for any open subset U of X, the semigroup $P_{U,t}^D$ associated with the Dirichlet-type Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ also admits a locally bounded kernel $p_U^D(t, x, y)$, $(t, x, y) \in (0, \infty) \times U \times U$ and, moreover, we have (a.e.),

$$p_U^D(t, x, y) \leq p(t, x, y).$$

More generally the kernel $p_U^D(t, x, y)$ is a monotone increasing function of the domain U. When the heat kernel $p(t, x, y)$ is a continuous function of x, y (e.g., if the original Dirichlet space is of Harnack-type), one can express the Dirichlet heat kernel for any open set U using the well-known Dynkin-Hunt formula

$$p_U^D(t, x, y) = p(t, x, y) - \mathbb{E}_x[\mathbf{1}_{\{\tau_U \leq t\}} p(t - \tau_U, X_{\tau_U}, y)].$$

Note that this formula captures beautifully the monotonicity of p_U^D with respect to the open set U and that $p(t, x, y)$ can be replaced by $p_V^D(t, x, y)$, for any open set V, $U \subset V \subset X$.

Lemma 2.39. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space. Let U be an open set in X. The heat kernel $p_U^D(t, x, y)$ is a continuous function of (t, x, y) in $(0, \infty) \times U \times U$ and for any $x \in U$, the function $y \mapsto p_U^D(t, x, y)$ belongs to $\mathcal{F}^0(U)$.*

Proof. — Because of symmetry, we have

$$p_U^D(t + s, x, y) = \int p_U^D(t, x, z) p_U^D(s, y, z) d\mu(z).$$

In particular

$$P_U^D(2t,x,x) = \int |p_U^D(t,x,z)|^2 d\mu(z).$$

This explains why $f_{t,x} : y \mapsto p_U^D(t,x,y)$ is in $L^2(U,d\mu)$. Spectral theory and the fact that $p_U^D(t+s,x,y) = P_{U,t}^D f_{s,x}(y)$ show that $f_{t,x}$ is in $\mathcal{F}^0(U)$ (in fact, it is also in the domain of any power of the Dirichlet Laplacian Δ_U^D). □

The following Theorem is in part based on the previous remark and lemma. It gives estimates for the Dirichlet heat kernel p_B^D in an intrinsic ball B. See [**58**, Sect. 3.3-3.4].

Theorem 2.40. — *Let $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space.*

(i) *For any fixed $\epsilon \in (0,1)$ there are constants $c_1, C_1 \in (0,\infty)$ such that, for any $x,y,z \in X$, $t,R > 0$ such that $x,y \in B(z,(1-\epsilon)R)$ and $\epsilon t \le R^2$, the Dirichlet heat kernel p_B^D, $B = B(z,R)$, is bounded below by*

$$p_B^D(t,x,y) \ge \frac{c_1}{V(x,\sqrt{t})}\exp(-C_1\rho(x,y)^2/t). \tag{2.13}$$

(ii) *For any fixed $\epsilon \in (0,1)$ there are constants $c_2, C_2 \in (0,\infty)$ such that, for any $x,y,z \in X$, $t,R > 0$ such that $x,y \in B(z,R)$ and $t \ge (\epsilon R)^2$, the Dirichlet heat kernel p_B^D, $B = B(z,R)$, is bounded above by*

$$p_B^D(t,x,y) \le \frac{C_2}{V(z,R)}\exp\left(-c_2 t/R^2\right) \tag{2.14}$$

(iii) *There exist constants c_3, C_3 such that, for any $x,y,z \in X$, $t,R > 0$ such that $x,y \in B(z,R)$, the Dirichlet heat kernel p_B^D, $B = B(z,R)$, is bounded above by*

$$p_B^D(t,x,y) \le \frac{C_3}{V(x,\sqrt{t})}\exp\left(-c_3\rho(x,y)^2/t\right) \tag{2.15}$$

All the constants c_i, C_i above depend only on the constant H_0 appearing in (2.7).

Proof. — The upper bounds in (iii) follows by comparing the Dirichlet heat kernel p_B^D to the original heat kernel $p(t,x,y)$ in X. The lower bound in (i) follows from the parabolic Harnack inequality (2.7) and a classical chaining argument (see e.g., [**58, 83**]). Note that in (2.13), x,y stay away from the boundary of the ball $B = B(z,R)$ and t/R^2 stays bounded from above. The estimate (ii) follows by changing notation in [**58**, Lemma 3.9, part 3]. □

2.4.2. Weak solutions with Dirichlet boundary condition along ∂U. — We continue to consider a domain U in a strictly local regular Dirichlet space $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$. We assume that this underlying space satisfies the qualitative properties (A1)-(A2) of Section 2.2.3 and let ρ_U be the inner metric in U from Definition 2.12. Recall that $\mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}_U^D)$ is the closure of $\mathcal{F}_c(U)$ in $\mathcal{D}(\mathcal{E})$.

Definition 2.41. — Let V be an open subset of U. Set

$$\begin{aligned}\mathcal{F}^0_{\rm loc}(U,V) &= \{f \in L^2_{\rm loc}(V,\mu) : \forall \text{ open } \Omega \subset V \text{ rel. cpt. in } \overline{U} \text{ with}\\ &\qquad \rho_U(\Omega, U\setminus V) > 0,\ \exists f^\# \in \mathcal{F}^0(U) : f^\# = f\ \mu\text{-a.e. on } \Omega\}\end{aligned}$$

Let us give an alternative equivalent definition of $\mathcal{F}^0_{\rm loc}(U,V)$ that refers to the set U viewed as a subset of $\widetilde{U}$.

Definition 2.42. — Given an open set V in U, let $V^\#$ be the largest open set in $\widetilde{U}$ which is contained in the closure of V in $\widetilde{U}$ and whose intersection with U is V.

Remark 2.43. — This means that $V^\#$ is obtained by adding to V those points in $\widetilde{U}\setminus U$ that are in the interior of the closure of V in $\widetilde{U}$. For instance, if $x \in U$ and $V = U\setminus\{x\}$ then $V^\#$ is $\widetilde{U}\setminus\{x\}$. If $x\in U$ and

$$V = B_U(x,r) = \{y \in U : \rho(x,y) < r\}$$

then

$$V^\# = B_{\widetilde{U}}(x,r) \cup E(\widetilde{U},x,r)$$

where $E(\widetilde{U},x,r)$ is the "exceptional" set of those points $z \in \widetilde{U}\setminus U$ that are interior points of the closed ball $\{y\in\widetilde{U} : \rho_U(x,y)\le r\}$ and satisfy $\rho_U(x,z) = r$. Typically, $E(\widetilde{U},x,r)$ is empty.

Example 2.44. — Let $\mathbb{R}^n$ be equipped with its usual Euclidean structure and $U = \mathbb{R}^n_+\{x=(x_1,\dots,x_n) : x_n > 0\}$. Let $V = \{x=(x_1,\dots,x_n) : x_n>0,\ \|x\|<r\}$. Then $\widetilde{U} = \{x=(x_1,\dots,x_n) : x_n \ge 0\}$ and $V^\# = \{x=(x_1,\dots,x_n) : x_n\ge 0,\ \|x\|<r\}$. The added set is $\{x=(x_1,\dots,0) : \|x\|<r\}$.

Lemma 2.45. — *Let V be an open subset of U. A function f is in $\mathcal{F}^0_{\rm loc}(U,V)$ if and only if, for any open set $\Omega\subset V$ that is relatively compact in $V^\#$, there exists a function $f^\#\in\mathcal{F}^0(U)$ such that $f^\# = f$ on Ω.*

Proof. — To say that a set $A\subset V$ is relatively compact in $V^\#$ is to say that $\rho_U(A,\widetilde{U}\setminus V^\#) > 0$ and, by continuity, $\rho_U(A,\widetilde{U}\setminus V^\#) = \rho_U(A,U\setminus V)$. The lemma follows. □

The next lemma gives yet another equivalent definition of $\mathcal{F}^0_{\rm loc}(U,V)$.

Lemma 2.46. — *Let V be an open subset of U. A function $f\in\mathcal{F}_{\rm loc}(V)$ is in $\mathcal{F}^0_{\rm loc}(U,V)$ if and only if, for any bounded function $\phi\in\mathcal{F}(U)$ with compact support contained in $V^\#$ and such that $d\Gamma(\phi,\phi)/d\mu \in L^\infty(U,d\mu)$, we have $\phi f\in\mathcal{F}^0(U)$.*

Proof. — Suppose that $\phi f\in\mathcal{F}^0(U)$ for any ϕ as in the lemma. For any fixed open set $\Omega\subset V$ which is relatively compact in $V^\#$, we can pick ϕ to be an appropriate cut-off function with compact support in $V^\#$ and equal to 1 on Ω. Such cut-off function can easily be obtained using the inner distance function ρ_U. See Remark 2.35. Then, by assumption $f^\# = \phi f \in \mathcal{F}^0(U)$ and $f^\# = f$ on Ω. Hence $f\in\mathcal{F}^0_{\rm loc}(U,V)$.

In the other direction, assume that $f \in \mathcal{F}^0_{\mathrm{loc}}(U,V)$. Let ϕ be as in the lemma with compact support $K \subset V^\#$ in $\widetilde{U}$. As f is in $\mathcal{F}^0_{\mathrm{loc}}(U,V)$ and K is compact in $V^\#$ there exists a function $f^\# \in \mathcal{F}^0(U)$ which coincides with f on $\Omega = K \cap U$. Let f_n be functions in $\mathcal{F}_c(U) \cap \mathcal{C}_c(U)$ that approximate $f^\#$ in the Hilbert space $\mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}^D_U)$.

Let ϕ_n be functions in $\mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$ which coincide with ϕ on the support of f_n. Then $\phi_n f_n = \phi f_n \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$. Since these functions are compactly supported in U, they are in $\mathcal{F}_c(U)$. Now, it is clear that the sequence ϕf_n converges to $\phi f^\# = \phi f$ in $L^2(U, d\mu)$. Using Lemma 2.5, it is not hard to show that the sequence ϕf_n is Cauchy in the Hilbert space $\mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}^D_U)$. Hence $\phi f \in \mathcal{F}^0(U)$ as desired. □

Next we define the notion of a local weak solution of the elliptic equation $Lu = f$ with Dirichlet boundary conditions along ∂U.

Definition 2.47. — Let V be an open set in U. Let $f \in \mathcal{F}'_c(V)$. We say that a function $u : V \to \mathbb{R}$ is a local weak solution of the equation $Lu = f$ in V with weak Dirichlet boundary conditions along ∂U if

1. The function u belongs to $\mathcal{F}^0_{\mathrm{loc}}(U,V)$;
2. For any function $\phi \in \mathcal{F}_c(V)$, $\int_V d\Gamma(\phi, u) = \int_V \phi f d\mu$.

In other words, a function u is a local weak solution of the equation $Lu = f$ in V with weak Dirichlet boundary conditions along ∂U if it is a local weak solution of the elliptic equation $Lu = f$ in V which, in addition, belongs to $\mathcal{F}^0_{\mathrm{loc}}(U,V)$ (condition (1)). Condition (1) captures the vanishing of u along the portion of the boundary ∂U that touches V.

Next, we will define the notion of a weak solution of the heat equation in $Q = I \times V$ with Dirichlet boundary conditions along ∂U. Set

$$\mathcal{F}^0(I \times U) = L^2(I \to \mathcal{F}^0(U)) \cap W^1(I \to \mathcal{F}^0(U)').$$

Given an open interval I and an open set $V \subset U$, let $Q = I \times V$ and let

$$\mathcal{F}^0_{\mathrm{loc}}(U,Q)$$

be the set of all functions $v : Q \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in I and any open set $\Omega \subset V$ relatively compact in $\overline{U}$ with $\rho_U(\Omega, U \setminus V) > 0$, there exists a function $u^\# \in \mathcal{F}^0(I \times U)$ such that $u^\# = u$ a.e. in $I' \times \Omega$.

Definition 2.48. — Let I be an open time interval, V be an open set in U and let $Q = I \times V$. A function $u : Q \to \mathbb{R}$ is a weak solution of the heat equation in Q with weak Dirichlet boundary conditions along ∂U if the following two conditions are satisfied:

1. The function u belongs to $\mathcal{F}^0_{\mathrm{loc}}(U,Q)$.
2. For any open interval J relatively compact in I,

$$\forall\, \phi \in \mathcal{F}_c(Q), \quad \int_J \int_V d\Gamma(\phi(t,\cdot), u(t,\cdot))dt + \int_J \int_V \phi \partial_t u \, d\mu dt = 0.$$

2.4.3. The Neumann-type heat semigroup in U. — The definition of the Neumann-type heat semigroup in an arbitrary open set U is more subtle than the corresponding Dirichlet case. Consider first the case of an open set U in $\mathbb{R}^n$. Recall that $W^1(U)$ is the subspace of $L^2(U)$ of those functions whose first order partial derivatives in the sense of distributions can be represented by functions in $L^2(U)$. The space $W^1(U)$ equipped with the norm $(\int_U |f|^2 + |\nabla f|^2 d\lambda)^{1/2}$ is a Hilbert space and $\mathcal{E}_U^N(f,g) = \int_U \nabla f \cdot \nabla g d\lambda$, $f,g \in W^1(U)$ is a strictly local Dirichlet form. The subtle question in this case is whether or not this form is regular on $\overline{U}$. Indeed, in general, it is not true that functions that are smooth in $\overline{U}$ are dense in $W^1(U)$. For general Dirichlet spaces, the question of defining a Neumann-type Dirichlet form in subdomains has its origin in the work of Silverstein (e.g., [**86**]) and is discussed in [**27, 28, 68, 86**].

Definition 2.49. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X,\mu)$ with energy form Γ. Let U be an open set in X. Recall that

$$\mathcal{F}(U) = \left\{ f \in L^2(U,\mu) : \int_U d\Gamma_U(f,f) < \infty \right\}.$$

Set

$$\mathcal{E}_U^N(f,g) = \int_U d\Gamma(f,g), \quad f,g \in \mathcal{F}(U).$$

The main point of the following proposition is that this form is closed. It is quite clear that it is Markovian (see [**47**]) and strictly local. We note that the restrictive hypothesis that (A1)-(A2) hold are not necessary for the result to hold but that the proof given below makes use of this hypothesis. See [**27, 68**] for a discussion of the result in more general contexts.

Proposition 2.50. — *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a strictly local regular Dirichlet form on* $L^2(X,\mu)$ *such that the conditions* (A1)-(A2) *of* Section 2.2.3 *are satisfied. Then the form*

$$(\mathcal{E}_U^N, \mathcal{F}(U))$$

is a strictly local Dirichlet form.

Proof. — For any compact subset V of U, set

$$\psi_V(x) = \max\{0, 1 - 2\rho(x,V)/\rho(\partial U, V)\}.$$

This function is continuous, identically equal to 1 in V and compactly supported in U. Let V' be the compact support of ψ_V. By Theorem 2.11(1), it belongs to $\mathcal{F}_c(U)$ and satisfies $d\Gamma(\psi_V, \psi_V) \le C_V d\mu$ for some constant C_V. By Lemma 2.5, if $u \in \mathcal{F}_{\mathrm{loc}}(U)$ then $\psi_V u \in \mathcal{F}_c(U)$ and

$$\mathcal{E}(\psi_V u, \psi_V u) \le 2C_V \left(\int_{V'} |u|^2 d\mu + \int_{V'} d\Gamma(u,u) \right).$$

Let $(u_i)_0^\infty$ be a Cauchy sequence for the norm $N(u) = (\int_U |u|^2 d\mu + \mathcal{E}_U^N(u,u))^{1/2}$. Let $u \in L^2(U,\mu)$ be its limit in $L^2(U,\mu)$. For any compact set $V \subset U$, the sequence

$\psi_V u_i$ is Cauchy in $(\|\cdot\|_2^2 + \mathcal{E}(\cdot,\cdot))^{1/2}$. It follows that it converges to $\psi_V u$ in $\mathcal{D}(\mathcal{E})$. Since $\psi_V \equiv 1$ on V, it follows that $u \in \mathcal{F}_{\mathrm{loc}}(U)$. We claim that the Radon measures $\Gamma(u_i, u_i)$ form a Cauchy sequence in total variation. This easily follows from the fact that $\Gamma(u_i,u_i)-\Gamma(u_j,u_j) = \Gamma(u_i-u_j, u_i+u_j)$ and the hypothesis that $(u_i)_1^\infty$ is Cauchy for the norm N. Let ν be the Radon measure on U with finite total variation that is the limit of the sequence $\Gamma(u_i,u_i)$. Since ν coincides with $\Gamma(u,u)$ on any compact subset of U, we must have $\nu = \Gamma(u,u)$. Hence $\int_U d\Gamma(u,u) < \infty$ and $u \in \mathcal{F}(U)$. A similar argument shows that the sequence of Radon measures $\Gamma(u_i, u)$ is Cauchy in total variation and its limit coincides with $\Gamma(u,u)$. As $\Gamma(u-u_i, u-u_i) = \Gamma(u,u) - 2\Gamma(u,u_i) + \Gamma(u_i,u_i)$, it thus follows that

$$\lim_{i\to\infty} \int_U d\Gamma(u-u_i, u-u_i) = 0.$$

That is, u_i tends to u in the norm N. This proves that the form $(\mathcal{E}_U^N, \mathcal{F}(U))$ is closed as desired. □

Definition 2.51. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X,\mu)$ such that the conditions (A1)-(A2) of Section 2.2.3 are satisfied. Let $P_{U,t}^N$ be the reversible semigroup on $L^2(U,\mu)$ associated with the Dirichlet form $(\mathcal{E}_U^N, \mathcal{F}(U))$. We call this semigroup the Neumann-type heat semigroup in U.

It should be noted that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is not regular on U (when $\neq X$) and not always regular on $\overline{U}$, the closure of U. Later we will show that this form is regular on $\widetilde{U}$ under the hypothesis that X is of Harnack type and U is inner uniform in X.

Proposition 2.52. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type strictly local regular Dirichlet form on $L^2(X,\mu)$. Let U be an open subset of X. Then the Neumann-type heat semigroup $P_{U,t}^N$ admits a continuous kernel $p_U^N(t,x,y)$, $(t,x,y) \in (0,\infty)\times U \times U$.*

Proof. — It is not hard to see that for any $f \in L^2(U,\mu)$, $u(t,x) = P_{U,t}^N f(x)$ is a local weak solution of the heat equation (for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$) in U. As we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack-type, this implies that $u(t,x)$ is continuous in U and it easily follows that $P_{U,t}^N$ admits a locally bounded kernel which is itself a local weak solution of the heat equation in U. For details, see the arguments used in [**89**, Sect. 2.4]. Naturally, we will refer to $p_U^N(t,x,y)$ as the Neumann-type heat kernel in U. Please note that, without further assumption on the boundary of U, the continuous Neumann-type heat kernel may not extend continuously to the boundary of U and may be unbounded on U for fixed $t > 0$, i.e., it may happen that $P_{U,t}^N$ is not ultracontractive, even for bounded U. □

2.4.4. Weak solutions with Neumann boundary condition along ∂U. — In Section 2.2.8, we pointed out how the Neumann boundary condition may be contained in the definition of local weak solution. Namely, consider the example of the upper-half space $\mathbb{R}^n_+$. Consider its closure $X = \overline{\mathbb{R}^n_+} = \{x = (x_i)_1^n : x_n \geq 0\}$ equipped with the classical Dirichlet form

$$\int_{\mathbb{R}^n_+} |\nabla f|^2 d\lambda, \quad f \in W^1(\mathbb{R}^n_+).$$

Because we are working on $\overline{\mathbb{R}^n_+}$ (and not $\mathbb{R}^n_+$), this strictly local Dirichlet form is regular. Applying the notion of local weak solution of the heat equation to this example, say in $(0,\infty)\times V$ with $V = \{x = (x_i)_1^n \in X : \|x\| < 1\}$, the trace of the open Euclidean unit ball in $\overline{\mathbb{R}^n_+}$, we obtain solutions u of the heat equation in $(0,\infty)\times(\mathbb{R}^n_+ \cap V)$ which satisfy the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ along the boundary $\{x = (x_i)_1^n : x_n = 0\} \cap V$.

In order to generalize this idea in the case of a general open set U in a strictly local regular Dirichlet space, it is crucial to introduce the correct "closure" of U. Even for domains in Euclidean space, the usual closure is not, in general, the correct notion to use. A good simple example to keep in mind is the case of the slitted plane $\mathbb{R}^2 \setminus \{(x,y) \in \{0\}\times(-\infty,0]\}$. Instead of the usual closure $\overline{U}$ of U in (X,ρ) we will use the abstract completion $\widetilde{U}$ of (U,ρ_U) introduced in Definition 2.14.

Fix a regular strictly local Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$. Let U be an open subset of X. Note that for any open subset V of U and any open time interval I, the notion of local weak solution of the heat equation in $Q = I\times V$ is the same whether we consider the (global) Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on X or the Neumann-type Dirichlet form $(\mathcal{E}^N_U,\mathcal{F}(U))$ on U.

Now, things are different if we assume that the intrinsic distance ρ satisfies the conditions (A1)-(A2) of Section 2.2.3, that $(\widetilde{U},\rho_U)$ is locally compact, and consider the Dirichlet form $(\mathcal{E}^N_U,\mathcal{F}(U))$ as a Dirichlet form on $\widetilde{U}$. Then, given an open set $V \subset U$ as above, consider any open set V' in $\widetilde{U}$ such that $V' \cap U = V$. With this data, any local weak solution u of the heat equation for $(\mathcal{E}^N_U,\mathcal{F}(U))$ in $Q' = I\times V'$ is a local weak solution of the heat equation for $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ in Q but, in addition, must satisfy some additional property related to the "boundary" $\widetilde{U}\setminus U$. In general, it is not possible to describe this additional condition in any other terms than those used in the definition of a local weak solution in Q'. However, in the case when U is an open set in Euclidean space and $\widetilde{U}\setminus U$ is made of smooth enough pieces, this condition indeed requires that $\frac{\partial u}{\partial \nu} = 0$ along the relevant portions of $\widetilde{U}\setminus U$.

Although it is not strictly necessary, we introduce the following definition which captures what has just been said above. Recall from Definition 2.42 that, given an open set V in U, $V^\#$ denotes the largest open set in $\widetilde{U}$ which is contained in the closure of V in $\widetilde{U}$ and whose intersection with U is V. This means that $V^\#$ is obtained by adding to V those points in $\widetilde{U}\setminus U$ that are in the interior of the closure of V in $\widetilde{U}$.

Example 2.53. — For $x \in U$ and $V = B_U(x,r)$, the set $V^\# = B_U(x,r)^\#$ is the interior of the closure of $B_{\widetilde{U}}(x,r)$ in $\widetilde{U}$. If no points of $\{y \in \widetilde{U} : \rho_U(x,y) = r\}$ are interior in the closure of $B_{\widetilde{U}}(x,r)$ in $\widetilde{U}$ (in a sense, this is the generic case), then $B_U(x,r)^{\#} = B_{\widetilde{U}}(x,r)$.

Definition 2.54. — Fix a regular strictly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ satisfying the conditions (A1)-(A2) of Section 2.2.3. Let U be an open set in X. Let V be an open set in U and $V^{\#} \subset \widetilde{U}$ be as defined above. Let I be an open time interval and $Q = I \times V$, $Q^{\#} = I \times V^{\#}$.

A function $u : Q \to \mathbb{R}$ is called a local weak solution of the heat equation (for $(X, \mathcal{E}, \mathcal{D}(\mathcal{E}))$) in Q with Neumann condition along the boundary of U if it is a local weak solution of the heat equation for $(\widetilde{U}, \mathcal{E}^N_U, \mathcal{F}(U))$ in $Q^{\#}$.

Remark 2.55. — In this definition, we used the sentence "with Neumann boundary condition along the boundary of U". Similarly, in the title of this subsection we wrote "with Neumann boundary condition along ∂U". Nevertheless, the "boundary" that is relevant here is $\widetilde{U} \setminus U$ which is possibly very different than the boundary of U in X. For instance, on a slitted plane, the relevant Neumann boundary condition holds on both side of the slit.

CHAPTER 3

THE NEUMANN HEAT KERNEL IN INNER UNIFORM DOMAINS

This chapter contains some of the crucial technical material necessary to obtain our main results. The first section introduces the notion of an inner uniform domain which plays a key part in our results as already explained in the introduction. The second section contains the statements of our main results concerning Neumann-type Dirichlet forms, namely, that the Neumann-type Dirichlet form on any inner uniform domain of any Harnack-type Dirichlet space with a carré du champ is, itself, of Harnack-type. The third section proves that the Poincaré inequality holds true on the inner balls of an inner uniform domain in a Harnack-type Dirichlet space. This is the main technical result of this monograph and all our other results rest on it. The fourth and last section uses the Poincaré inequality of Section 3 to prove the statements announced in Section 2 concerning the Neumann-type Dirichlet forms and heat kernels in inner uniform domains.

3.1. Inner uniform domains

3.1.1. Uniform domains. — Recall that, in any metric space, the length of a continuous curve $\gamma : I = [a,b] \mapsto X$ is given by

$$L(\gamma) = \sup \left\{ \sum_{1}^{n} \rho(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, a \leq t_0 < \cdots < t_n \leq b \right\}.$$

In general, $L(\gamma) \geq \rho(\gamma(a), \gamma(b))$. A metric space is a length space if $\rho(x,y)$ is equal to the infimum of the lengths of continuous curves joining x to y.

A length space is a geodesic length space if, for any pair x, y there exists a continuous curve $\gamma : I = [0,1] \mapsto X$ with $\gamma(0) = x, \gamma(1) = y$ and

$$\forall\, s,t \in I, \quad \rho(\gamma(s), \gamma(t)) = |t-s| \rho(x,y).$$

Such a curve is called a minimal geodesic (parametrized by a multiple of arc length). One of the basic results in this context states that, in a complete locally compact length metric space, any two points can be joined by a minimal geodesic (e.g., [**22**, Theorem 2.5.23]).

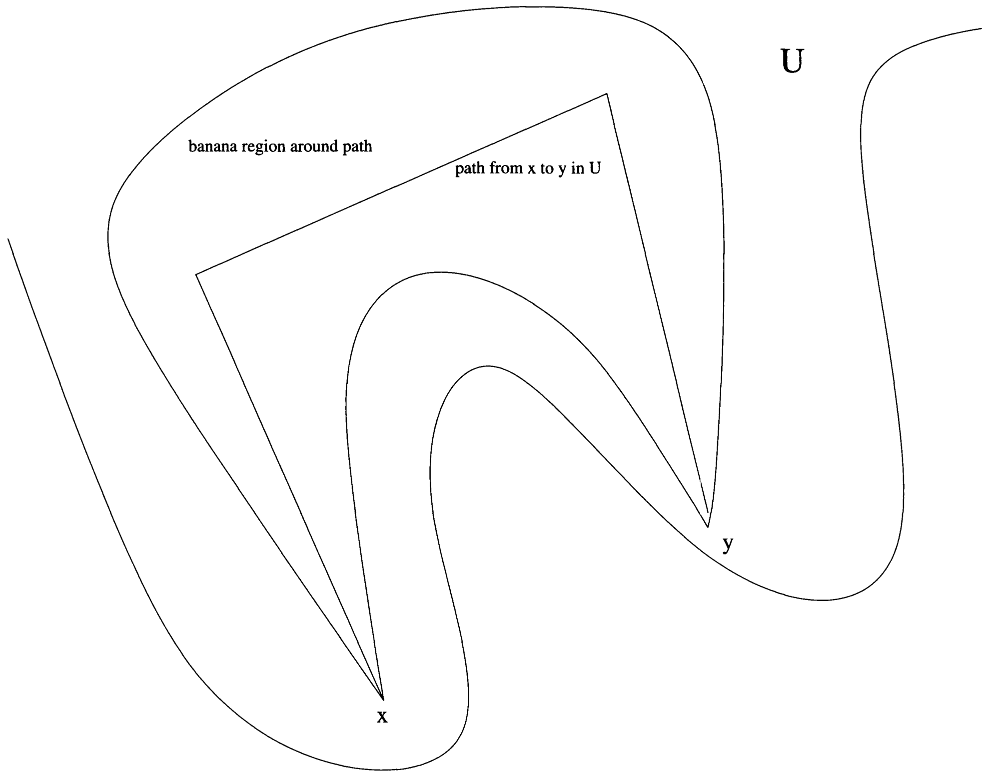


FIGURE 1. The banana condition

Definition 3.1. — Let U be an open connected subset of a length metric space (X,ρ). We say that U is *uniform* if there are constants $c_0, C_0 \in (0,\infty)$ such that, for any $x,y \in U$, there exists a continuous curve $\gamma_{x,y} : [0,1] \to U$ with $\gamma(0) = x$, $\gamma(1) = y$ and satisfying the following two properties:

1. The length $L(\gamma_{x,y})$ of $\gamma_{x,y}$ is at most $C_0\rho(x,y)$.
2. For any $z \in \gamma_{x,y}([0,1])$,

$$\rho(z,\partial U) \geq c_0 \frac{\rho(z,x)\rho(z,y)}{\rho(x,y)}. \tag{3.1}$$

Note that $\max\{\rho(z,x),\rho(z,y)\} \geq \rho(x,y)/2$. Hence (3.1) is equivalent to

$$\rho(z,\partial U) \geq c_0' \min\{\rho(z,x),\rho(z,y)\}.$$

Condition (3.1) is a banana-type (or cigar) condition.

There is another definition of uniform sets that appears in the literature and may actually be more common. The two definitions yield different classes of sets in general but it is known and we will show below that the two definitions are equivalent when the metric space (X,ρ) is equipped with a measure μ which is doubling. For our purpose and in order to distinguish the two definitions, we call the sets defined below length-uniform.

Definition 3.2. — Let U be an open connected subset of a length metric space (X, ρ). We say that U is *length-uniform* if there are constants $c_0, C_0 \in (0, \infty)$ such that, for any $x, y \in U$, there exists a continuous curve $\gamma_{x,y} : [0,1] \to U$ with $\gamma(0) = x$, $\gamma(1) = y$ and satisfying the following two properties:

1. The length $L(\gamma_{x,y})$ of $\gamma_{x,y}$ is at most $C_0 \rho(x, y)$.
2. For any $z \in \gamma_{x,y}([0,1])$,

$$\rho(z, \partial U) \geq c_0 \frac{L(\gamma_{[x,z]}) L(\gamma_{[z,y]})}{L(\gamma_{x,y})} \tag{3.2}$$

where for any $z = \gamma_{x,y}(s)$, $z' = \gamma_{x,y}(s')$, $0 \leq s \leq s' \leq 1$, $L(\gamma_{[z,z']}) = L(\gamma|_{[s,s']})$.

Proposition 3.3. — *Assume that (X, ρ) is a complete locally compact length metric space with the property that there exists a constant D such that, for any $r > 0$, the maximal number of disjoint balls of radius $r/4$ contained in any ball of radius r is bounded above by D. Then any connected open subset U of X is uniform if and only if it is length-uniform.*

Proof. — As $L(\gamma_{[z,z']}) \geq \rho(z, z')$ for any $z, z' \in \gamma_{x,y}([0,1])$, it is clear that a length-uniform domain is uniform. To prove the converse, we follow closely the straightening technique of [**70**, Lemma 2.7].

Let U be a uniform domain and $x, y \in U$. Let γ' be a path from x to y satisfying the conditions (1)-(2) of Definition 3.1. Let x_0 be a point on γ' such that $\rho(x, x_0) = \rho(y, x_0)$. Such a point exists by a continuity argument. Observe that we must have $\rho(x, y) \leq L(\gamma') \leq C_0 \rho(x, y)$

We are going to construct a path γ_1 from x_0 to x that will replace the portion of γ' between x_0 and x and a path γ_2 from x_0 to y that will replace the portion of γ' from x_0 to y.

Fix $\epsilon \in (0, c_0)$ to be chosen later (c_0 as in Definition 3.1(2)). Let t_0 be the time parameter such that $\gamma'(t_0) = x_0$. Define $t_1 \in [0, t_0]$, $x_1 \in \gamma'([0, t_0])$ by

$$t_1 = \inf\{t \in [0, t_0] : \rho(\gamma'(t), \gamma'(t_0)) \leq \epsilon \rho(x, x_0)/2\}, \quad x_1 = \gamma'(t_1)$$

and set $r_1 = \min\{\rho(x, x_0), \rho(x, x_1)\}$. Assuming that $t_j, x_j = \gamma'(t_j)$, $j = 0, \ldots, i$, have been constructed, set $r_i = \min\{r_{i-1}, \rho(x, x_i)\}$ and

$$t_{i+1} = \inf\{t \in [0, t_i] : \rho(\gamma'(t), \gamma'(t_i)) \leq \epsilon r_i/2\}, \quad x_{i+1} = \gamma'(t_{i+1}).$$

It is possible that $x_i = x$ in which case $x_j = x$, $r_j = 0$ for all $j \geq i$.

Observe that, by construction and because γ' satisfies condition (2) of Definition 3.1, each ball $B(x_i, \epsilon r_i)$ is contained in U. We claim that there exists N such that, for any i and any $k \geq N$

$$r_{i+k} \leq \frac{1}{2} r_i. \tag{3.3}$$

Suppose that for some i, k we have $r_j > r_i/2$ for $j = i+1, \cdots, i+k$. This implies that $x_j \neq x$, $i \leq j \leq i+k$, and that $x_n \notin B(x_j, \epsilon r_j/2)$ for $i \leq j < n \leq i+k$. It follows that the balls $B(x_j, \epsilon r_i/8)$ are disjoint. Moreover,

$$B(x_j, \epsilon r_j/2) \subset B(x, (1+\epsilon/2)r_j) \subset B(x, (1+\epsilon/2)r_i).$$

But the number of disjoint balls of radius $\epsilon r_i/8$ that can be contained in one ball of radius $(1+\epsilon/2)r_i$ is bounded by a constant N that depends only of ϵ. This proves (3.3) for some integer $N = N(\epsilon)$.

From (3.3), it follows that x_i converges to x and t_i converges to 0. Define a new path γ_1 from x_0 to x passing through all the points x_i constructed above and following a minimal geodesic from x_i to x_{i+1}. This path has finite total length bounded by

$$\sum_i \rho(x_i, x_{i+1}) \leq \epsilon N(\epsilon) \left(\sum_i 2^{-1} \right) \rho(x, x_0).$$

Moreover, if z is a point on γ_1 on the portion between x_{i+1} and x_i, we have

$$L(\gamma_{[x,z]}) \leq \sum_{j \geq i} \rho(x_j, x_{j+1}) \leq C_\epsilon r_i$$

and, by the triangle inequality and the fact that $B(x_i, \epsilon r_i) \subset U$,

$$\rho(z, X \setminus U) \geq \epsilon r_i/2.$$

As said earlier, a similar argument produces a path γ_2 from x_0 to y with the same properties (with x replaced by y). Putting the paths γ_1, γ_2 together yields a path γ satisfying the conditions (1)-(2) of Definition 3.2. □

It may be useful to state the conclusions obtained in the proof above in a slightly more precise way.

Theorem 3.4. — *Assume that (X, ρ) is a complete locally compact length metric space with the property that there exists a constant D such that, for any $r > 0$, the maximal number of disjoint balls of radius $r/4$ contained in any ball of radius r is bounded above by D. Let U be a open connected uniform subset of X. Then, there are constants $c_0', C_0' \in (0, \infty)$ such that, for any pair of points $x, y \in U$ there exist a continuous rectifiable path $\gamma = \gamma_{x,y} : [0,1] \to U$ parametrized by a constant multiple of arclength and joining x to y in U with the following properties:*

1. *$L(\gamma) \leq C_0' \rho(x, y)$ and, for any $t \in [0,1]$,*

$$\begin{array}{rcccl} L(\gamma_{[x,z]}) & = & tL(\gamma) & \leq & C_0' \rho(x, \gamma(t)), \\ L(\gamma_{[z,y]}) & = & (1-t)L(\gamma) & \leq & C_0' \rho(y, \gamma(t)). \end{array}$$

2. *The linearly growing banana-type set $b(\gamma)$ joining x to y and defined by*

$$b(\gamma) = \bigcup_{t \in [0,1]} B(\gamma(t), c_0' t(1-t)L(\gamma)) \tag{3.4}$$

is contained in U.

Remark 3.5. — A curve γ with $b(\gamma) \subset U$ is sometimes called a John curve. We will not use the notion of John domain here but recall that a John domain is a domain in which any two points can be joined by a John curve ([**93**, Ex. 2.18]). The difference with the definition of a uniform domain is that, for a John domain, there is no requirement that the length of the curve is bounded by $C_0\rho(x,y)$. Actually, the most common definition of a John domain in the literature is that of a centered John domain (this definition forces the domain to be bounded). See [**70**].

3.1.2. Inner uniform domains. — Recall the definition of the inner metric $\rho_U(x,y)$ in an open set U of a length metric space (X,ρ) given in Definition 2.12. Namely, $\rho_U(x,y)$ is the infimum of the lengths of continuous curves connecting x to y in U. The metric ρ_U is finite on the connected components of U (see [**22**, Exercise 2.1.3]). Note that condition (1) in Definition 3.1 implies that $\rho_U(x,y)$ is bounded from above by $C_0\rho(x,y)$. In the next definition, this requirement is dropped.

Definition 3.6. — Let U be an open connected subset of a length metric space (X,ρ). We say that U is *inner uniform* if there are constants $c_0, C_0 \in (0,\infty)$ such that, for any $x,y \in U$, there exists a continuous curve $\gamma_{x,y} : [0,1] \to U$ with $\gamma(0) = 1$, $\gamma(1) = y$ and satisfying the following two properties:

1. The length $L(\gamma_{x,y})$ of $\gamma_{x,y}$ is at most $C_0\rho_U(x,y)$.
2. For any $z \in \gamma_{x,y}([0,1])$,

$$\rho(z,\partial U) \geq c_0 \frac{\rho_U(z,x)\rho_U(z,y)}{\rho_U(x,y)}. \tag{3.5}$$

Again, note that, assuming that $\gamma_{x,y}$ has length at most $C_0\rho_U(x,y)$, the second condition (3.5) is equivalent to

$$\rho(z,\partial U) \geq c_0' \min\{\rho_U(z,x), \rho_U(z,y)\}.$$

Note also that one always has $\rho(z,\partial U) = \rho_U(z,\partial U)$. Using $\rho(z,\partial U)$ is natural since it might be quite difficult to compute the inner distance (e.g., for Euclidean domains). Observe that a domain U is uniform if and only if it is an inner uniform domain and, for all $x,y \in U$, $\rho_U(x,y) \leq C_0'\rho(x,y)$.

The proof of Proposition 3.3 and Theorem 3.4 applies almost verbatim to inner uniform domains and gives the following result.

Theorem 3.7. — *Assume that (X,ρ) is a complete locally compact length metric space with the property that there exists a constant D such that, for any $r > 0$, the maximal number of disjoint balls of radius $r/4$ contained in any ball of radius r is bounded above by D. Let U be a open connected inner uniform subset of X. Then, there are constants $c_0', C_0' \in (0,\infty)$ such that, for any pair of points $x,y \in U$ there exist a continuous rectifiable path $\gamma = \gamma_{x,y} : [0,1] \to U$ parametrized by a constant multiple of arclength and joining x to y in U with the following properties:*

1. *$L(\gamma) \le C_0' \rho_U(x,y)$ and, for any $t \in [0,1]$,*

$$\begin{aligned} L(\gamma_{[x,z]}) &= tL(\gamma) &\le C_0' \rho_U(x,\gamma(t)), \\ L(\gamma_{[z,y]}) &= (1-t)L(\gamma) &\le C_0' \rho_U(y,\gamma(t)). \end{aligned}$$

2. *For any $t \in [0,1]$ we have*

$$\begin{aligned} \rho_U(x,\gamma(t)) \le C_0' \rho(x,\gamma(t)) \quad &\text{if} \quad t \in [0,1/2] \\ \rho_U(y,\gamma(t)) \le C_0' \rho(y,\gamma(t)) \quad &\text{if} \quad t \in [1/2,1]. \end{aligned}$$

3. *The linearly growing banana-type set $b(\gamma)$ joining x to y and defined at (3.4) is contained in U.*

Remark 3.8. — Let U be a domain in a complete length metric space (X,ρ). Let ρ_U be the inner metric in U and let $\widetilde{U}$ be the (abstract) completion of (U,ρ_U). Then U is an inner uniform domain in (X,ρ) if and only if U is uniform in $(\widetilde{U},\rho_U)$. The main point to check is that the inner metric of U in $(\widetilde{U},\rho_U)$ is ρ_U itself.

Lemma 3.9. — *Assume that the measure μ on the metric space (X,ρ) satisfies the doubling condition (2.2). Let the open connected set $U \subset X$ be inner uniform. Then the measure $\mu|_U$ on $(\widetilde{U},\rho_U)$ satisfies the doubling condition and $(\widetilde{U},\rho_U)$ is locally compact.*

Proof. — Fix any $x \in U$ and $r > 0$. Without loss of generality assume that the ball $B_U(x,r)$ is not contained in any ball of center x and smaller radius. Then the ball $B_U(x,r)$ contains a point z with $\rho_U(x,z) \ge r/2$. As U is inner uniform, we see that there exists a continuous curve γ connecting x and z and satisfying (3.5). Take some point $y \in B_U(x,r)$ on the path γ such that $\rho_U(x,y) = r/4$. Such a point exists because the distance function $\rho_U(x,\cdot)$ is continuous and $\rho_U(x,z) \ge r/2$. By the uniform condition (3.5) and the triangle inequality, we have

$$\begin{aligned} \rho_U(y,\widetilde{U} \setminus U) &\ge c_0 \frac{\rho_U(x,y)\rho_U(y,z)}{\rho_U(x,z)} = \frac{c_0 r}{4} \frac{\rho_U(y,z)}{\rho_U(x,z)} \\ &\ge \frac{c_0 r}{4} \frac{\rho_U(x,z) - \rho_U(x,y)}{\rho_U(x,z)} \\ &\ge \frac{c_0}{4}\left(1 - \frac{r/4}{r/2}\right) r = \frac{c_1}{8} r. \end{aligned}$$

Therefore $B_U(y,c_0 r/8) = B(y,c_0 r/8) \subset B_U(x,r)$. On the other hand the ball $B_U(x,2r)$ is a subset of $B(y,4r)$. The doubling property (2.2) of the measure μ in (X,ρ) gives

$$\mu(B_U(x,2r)) \le \mu(B(y,4r)) \le C\mu(B(y,c_0 r/8)) \le C\mu(B_U(x,r))$$

for some constant C depending only on c_0 and the constant D_0 appearing in (2.2). This inequality extends to the balls in $(\widetilde{U},\rho_U)$ by an obvious limiting argument. The doubling property easily implies that the length space $(\widetilde{U},\rho_U)$ is locally compact. □

3.1.3. Basic examples and related notions. — Uniform domains in Euclidean space already form a very large class of domains, allowing for fractal type boundary. Inner uniform Euclidean domains form a yet wider class of domains. A slitted plane is perhaps the simplest example of an inner uniform domain that is not uniform (the Euclidean distance between points on opposite sides of the slit may be much smaller than the inner distance between those points). The set $U = \mathbb{R}^2 \setminus \{z = (0,y) : y \in [-2k, -2k+1], k = 0,1,2,\dots\}$ is an example of a John domain that is not inner uniform: for each k, the points $z_k^{\pm} = (\pm\sqrt{k}, -2k)$ are at inner distance $2\sqrt{k}$ of each other but any John curve joining them has length greater than $2k$.

The complement of a spiral in $\mathbb{R}^2 = \mathbb{C}$ given in parametric form by $z(t) = \exp(t + ic\pi t)$ for some $c > 0$, $t \in \mathbb{R}$, is inner uniform. The domain

$$U(\Phi) = \{x = (x_i)_1^n : x_n > \Phi(x_1, \dots, x_{n-1})\}$$

above the graph of a Lipschitz function Φ is always uniform (with constants c_0, c_1 depending on the Lipschitz constant of Φ). The complement of any closed convex set in Euclidean space is inner uniform (and, usually, not uniform). These examples are discussed further in Chapter 6.

John domains and uniform domains were introduced (together with the names) in 1979 by O. Martio and J. Sarvas [**70**]. The motivation came from problems in the area of quasi-conformal maps. Independently, P. Jones [**66**] introduced uniform domains (using an equivalent definition based on the quasi-hyperbolic metric and without giving them a name) in his study of BMO extension domains (Jones result is that a domain is a BMO extension domain if and only if it is a uniform domain). Inner uniform domains (and some variations) were introduced later on by various authors. The survey [**93**] contains a thorough discussion of these notions. In their influential work [**65**], Jerison and Kenig introduced a subclass of the class of uniform domains, the NTA (nontangentially accessible) domains. The NTA domains are closer to the model of a domain above the graph of a Lipschitz function because the NTA condition involves both the inside and the outside of the domain.

The literature treating these notions is somewhat unwieldy, probably because of the many variations and different contexts considered by various authors.

3.2. The Neumann-type Dirichlet form in inner uniform domains

Let U be an open connected subset of a strictly local regular Harnack-type Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$. Note that, in particular, this implies that the associated intrinsic distance ρ satisfies the basic hypothesis (A1)-(A2) and the volume doubling property is satisfied. In particular, (X, ρ, μ) is a locally compact complete length metric space satisfying the doubling volume property. It follows that, for any inner uniform domain U, the abstract completion $\widetilde{U}$ with respect to the intrinsic distance ρ_U is also a locally compact space.

In Section 2.4.3, we defined the Neumann-type heat semigroup $P^N_{U,t}$ in U associated with the strictly local Dirichlet form $(\mathcal{E}^N_U, \mathcal{F}(U))$. The main goal of this chapter is to prove the following result.

Theorem 3.10. — *Let U be an open connected subset of a Harnack-type Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$. Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ. If U is inner uniform then the form $(\mathcal{E}^N_U, \mathcal{F}(U))$ on $(\widetilde{U}, \rho_U, \mu)$ is a strictly local regular Dirichlet form. Moreover, it is of Harnack-type. In particular, the Neumann-type semigroup $P^N_{U,t}$, $t > 0$, admits a continuous kernel $p(t,x,y)$ defined on $(0,\infty) \times \widetilde{U} \times \widetilde{U}$ and there are constants c_i, $i = 1, \dots, 4$ such that*

$$\frac{c_1}{V_U(x,\sqrt{t})} e^{-c_2 \rho_U(x,y)^2/t} \leq p^N_U(t,x,y) \leq \frac{c_3}{V_U(x,\sqrt{t})} e^{-c_4 \rho_U(x,y)^2/t}.$$

Of course, further conclusions (e.g., Hölder continuity, etc) can be drawn from the fact that $(\mathcal{E}^N_U, \mathcal{F}(U))$ is of Harnack-type on $(\widetilde{U}, \rho_U, \mu)$. See Section 2.3.

Recall that, in Section 2.4.4, we introduced the notion of local weak solution of the heat equation with Neumann condition along ∂U. The following result is a simple translation of the Harnack inequality in $\widetilde{U}$ in those terms.

Corollary 3.11. — *Let U be an open connected inner uniform subset of a Harnack-type strictly local regular Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$. Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator. Then there exists a constant C such that, for any $z \in U$, $r > 0$ and any non-negative weak solution u of the heat equation in $(0, r^2) \times B_U(z,r)$ with Neumann boundary condition along ∂U, we have*

$$\sup_{(t,x)\in Q_-} u(t,x) \leq C \inf_{(t,x)\in Q_+} u(t,x)$$

where $Q_- = (r^2/4, r^2/2) \times B_U(z, r/2)$, $Q_+ = (3r^2/4, r^2) \times B_U(z, r/2)$.

Proof. — If no points in $\{y \in \widetilde{U} : \rho_U(z,y) = r\}$ are interior in the closure of $B_{\widetilde{U}}(z,r)$ in $\widetilde{U}$ then $B_U(z,r)^\# = B_{\widetilde{U}}(z,r)$ and the statement above is the direct translation of the Harnack inequality on $\widetilde{U}$ that holds true by Theorem 3.10. See Example 2.53. Otherwise, the result follows easily from Theorem 3.10 by a finite covering argument. □

3.3. The Poincaré inequality in inner uniform domains

This section contains one of the main technical results on which this monograph is based. It says that if U is an inner uniform domain in a Harnack-type Dirichlet space then the Neumann-type Dirichlet form $(\mathcal{E}^N_U, \mathcal{F}(U))$ on $(\widetilde{U}, \rho_U, \mu)$ satisfies the Poincaré inequality. This will be used to prove (a) that $(\mathcal{E}^N_U, \mathcal{F}(U))$ is regular on $(\widetilde{U}, \rho_U, \mu)$ and, (b) that $(\widetilde{U}, \mu, \mathcal{E}^N_U, \mathcal{F}(U))$ is Harnack-type. Furthermore, a simple extension of the same argument will be crucial for the treatment of the Dirichlet heat kernel in U.

3.3.1. Statements of the main inequalities

Theorem 3.12. — *Let $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space with energy measure Γ and intrinsic distance ρ. Let U be an inner uniform domain. Let ρ_U be the inner distance extended continuously to the completion $\widetilde{U}$ of U. Then there exists a constant P_U such that for any ball $B = B_{\widetilde{U}}(x,r) = \{y \in \widetilde{U} : \rho_U(x,y) < r\}$, $x \in \widetilde{U}$, $r > 0$, we have*

$$\forall\, f \in \mathcal{F}(B \cap U),\ \inf_{\xi \in \mathbb{R}} \int_B |f-\xi|^2 d\mu \le P_U r^2 \int_B d\Gamma(f,f).$$

Recall that on a Harnack-type Dirichlet space, the measure μ must have the doubling property and the L^2 Poincaré inequality (2.3) must be satisfied. Hence the theorem above is a corollary to the following more general result.

Theorem 3.13. — *Let $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a regular strictly local Dirichlet space with energy measure Γ and intrinsic distance ρ satisfying the condition* (A1)-(A2) *of* Section 2.2.3. *Let U be an inner uniform domain. Let ρ_U be the inner distance extended continuously to the completion $\widetilde{U}$ of U. Let ν be a Radon measure on U which satisfies the doubling condition* (2.2) *on $(\widetilde{U},\rho_U)$. Fix a constant $N>1$ and assume that there exists a constant P_0 such that for any ball $B = B(x,r)$ with $\rho(B, X\setminus U) > Nr$, the L^2 Poincaré inequality*

$$\forall f \in \mathcal{D}(\mathcal{E}),\ \inf_{\xi\in\mathbb{R}} \int_B |f-\xi|^2 d\nu \le P_0 r^2 \int_B d\Gamma(f,f) \tag{3.6}$$

holds true.

Then there exists a constant P_U such that for any ball $B = B_{\widetilde{U}}(x,r)$, $x\in\widetilde{U}$, $r>0$, we have

$$\forall\, f \in \mathcal{F}(B\cap U),\ \inf_{\xi\in\mathbb{R}} \int_B |f-\xi|^2 d\nu \le P_U r^2 \int_B d\Gamma(f,f). \tag{3.7}$$

Remark 3.14. — The main point of this theorem is that the balls involved in the hypothesis (3.6) are balls in (X,ρ) that are well inside U at the scale corresponding to their radius. The ball B appearing in the conclusion is any ball in $(\widetilde{U},\rho_U)$ and the domain of validity is the natural domain $\mathcal{F}(B\cap U)$. For the rest of this chapter, we will write $\mathcal{F}(V)$ for $\mathcal{F}(V\cap U)$ for any open set V in $\widetilde{U}$. This slight abuse of notation should bring no difficulties.

Remark 3.15. — Even if we assume that ρ and ρ_U are comparable in U (i.e., U is uniform) the Poincaré inequality (3.7) does not always hold if one replaces the inner balls $\{y\in U : \rho_U(x,y) < r\}$ by the trace on U of the ball in (X,ρ), i.e., $B' = \{y \in X : \rho(x,y) < r\} \cap U$. See Figure 2. For such balls, only the weaker inequality (3.9) below holds.

Proof. — We start by giving an outline of the proof and then give details for the main step of the proof.

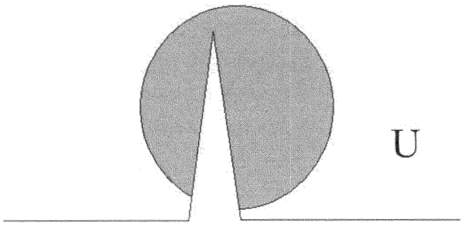


FIGURE 2. A bad Euclidean ball for U (large Poincaré constant)

Notice that the assumption $f \in \mathcal{D}(\mathcal{E})$ in (3.6) can be relaxed as follows. For any $\epsilon > 0$ and for any $f \in \mathcal{F}(B(x, r+\epsilon))$ there exists a function $\tilde{f} \in \mathcal{D}(\mathcal{E})$ coinciding with f on $B = B(x,r)$. Therefore $d\Gamma(\tilde{f}, \tilde{f}) = d\Gamma(f,f)$ on B by the local property of $d\Gamma$ (recall that, abusing notation, we do not distinguish between Γ_V and Γ restricted to V). Hence (3.6) implies that for any ball $B = B(x,r)$ such that $\rho(B, X \setminus U) > NR$, the following L^2 Poincaré inequality holds

$$\forall f \in \mathcal{F}(B(x, r+\epsilon)),\ \inf_{\xi \in \mathbb{R}} \int_{B(x,r)} |f-\xi|^2 d\nu \leq P_0' r^2 \int_{B(x,r)} d\Gamma(f,f). \tag{3.8}$$

We will prove (3.7) in two stages. First we will prove that there exists $k \geq 1$ such that

$$\forall f \in \mathcal{F}(B_{\widetilde{U}}(x, kr)),\ \inf_{\xi \in \mathbb{R}} \int_{B_{\widetilde{U}}(x,r)} |f-\xi|^2 d\nu \leq C r^2 \int_{B_{\widetilde{U}}(x,kr)} d\Gamma(f,f) \tag{3.9}$$

for each ball $B_{\widetilde{U}}(x,r)$, $x \in \widetilde{U}$, $r > 0$. We call this a weak Poincaré inequality because the ball on the right-hand side has been enlarged.

The second step consists of showing that the family of weak L^2 Poincaré inequalities (3.9) for $x \in \widetilde{U}$, $r > 0$ and functions $f \in \mathcal{F}(B_U(x, kr))$ implies the standard L^2 Poincaré inequality (3.7) for functions in $\mathcal{F}(B)$. This is a well established result, and we will omit the proof. See, e.g. [**83**, Chapter 5.3.2-5.3.3] and the references therein. In fact the proof of this second step is very similar to the proof of the first step. Note that it is essential that the requirement $f \in \mathcal{D}(\mathcal{E})$ in (3.6) can be relaxed to $f \in \mathcal{F}(B(x, kR))$ of (3.9) in step one, and henceforth similarly relaxed to the requirement $f \in \mathcal{F}(B)$ of (3.7) in step two. □

As explained above, using a well-known line of reasoning, (3.9) suffices to obtain Theorem 3.13 and its corollary, Theorem 3.12. The main technique used in the proof of (3.9) is the technique of Whitney covering.

3.3.2. Whitney covering. — In this section we use the following conventions. A ball B in $(\widetilde{U}, \rho_U)$ is always supposed to be given in the form $B = B_{\widetilde{U}}(x,r)$ with a specified center and a radius $r = r(B)$ which is minimal in the sense that

$$B_{\widetilde{U}}(x,s) \neq B \text{ if } s < r(B).$$

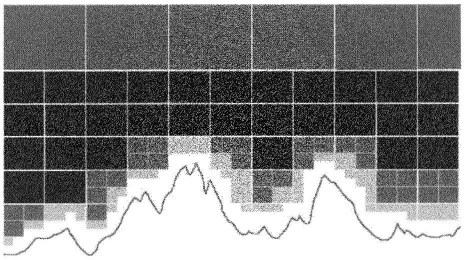


FIGURE 3. Typical cover by Whitney cubes

For any ball $B = B_{\widetilde{U}}(x, r)$ and $k > 0$, we set

$$kB = B_{\widetilde{U}}(x, kr).$$

Definition 3.16. — A strict ϵ-Whitney covering of an open set U in a metric space (X, ρ) is any set $\mathfrak{R}$ of disjoint balls $A = B(x, r) \subset U$ such that

1. The balls $3A$ cover U, that is, $\bigcup_{A \in \mathfrak{R}} 3A = U$;
2. The balls A are well inside U in the sense that

$$\forall\, A \in \mathfrak{R},\quad r(A) = \epsilon \rho(A, X \setminus U). \tag{3.10}$$

For small enough ϵ, e.g., $\epsilon \in (0, 1/3)$, such a Whitney covering always exists for any open set U by a general argument using Zorn's lemma. If, as it will be the case for us, (X, ρ) is equipped with a doubling Borel measure, the cover $\mathfrak{R}$ is always countable. We will use Whitney coverings as defined above but any reasonable variation on this definition would do as well. For instance, there is nothing special of importance to us about having exact equality in (3.10).

Remark 3.17. — For a domain in Euclidean space one can use a very neat Whitney covering using cubes instead of balls (see figure 3). Consider all the cubes of size length 2^k with edges parallel to the coordinate axes and each of the vertices having all coordinates of the form $n2^k$. A given cube Q is included into the covering $\mathfrak{R}$ if and only if its distance to the boundary is greater or equal to the fixed desired multiple of its side length and no larger cubes containing that cube have this property.

***Lemma 3.18* (Finite intersection property).** — *Let $\mathfrak{R}$ be a strict ε-Whitney covering of an open set U in some metric space (X, ρ) with $\varepsilon \in (0, \frac{1}{4})$. Assume that X is equipped with a Borel measure having the doubling property* (2.2)*. Then there is a finite constant a_1 such that*

$$\forall k < \frac{1}{10\varepsilon},\quad \sum_{A \in \mathfrak{R}} \chi_{kA} \le a_1.$$

Proof. — Pick any point $y \in U$. It must belong to some triple of a Whitney ball $B \in \mathfrak{R}$ with center z. If a k-multiple of a given Whitney ball $A = B(x,r)$ contains y, then $\rho(x,y) < kr$. By the Whitney covering condition (3.10)

$$\rho(x, X \setminus U) = \frac{r}{\varepsilon}.$$

By the triangle inequality, this means that by

$$\frac{r}{\varepsilon} - kr \leq \rho(y, X \setminus U) \leq \frac{r}{\varepsilon} + kr.$$

Applying the Whitney covering condition (3.10) and the triangle inequality again yields

$$r(B) = \varepsilon\rho(z, X \setminus U) \leq \varepsilon(3r(B) + \rho(y, X \setminus U))$$

since $y \in 3B$. Therefore

$$r(B) \leq \frac{\varepsilon}{1-\varepsilon}\rho(y, X \setminus U) \leq 2\varepsilon\left(\frac{r}{\varepsilon} + kr\right) = (2 + 2k\varepsilon)r \leq 3r(A).$$

Similarly $r(A) \leq 3r(B)$. Hence, $x \in 5kB$, and $A \subset 10kB \subset \frac{1}{\varepsilon}B$. By the doubling condition (2.2), there are only finitely many disjoint Whitney balls of radius at least $r(B)/3$ in the ball $\frac{1}{\varepsilon}B$ and the number of such balls is uniformly bounded from above. Thus the number of Whitney balls A with the property that a k-multiple of it covers y, is finite and bounded from above by a constant independent of y. □

3.3.3. The trace of $\mathfrak{R}$ on a fixed ball B in $\widetilde{U}$. — We return to the proof of (3.9). Set $\varepsilon = 10^{-4}/N$, where the constant N comes from the assumption of the Theorem 3.13. Let $\mathfrak{R}$ be a strict ε-Whitney covering of the set U in (X, ρ) and note that it is also a strict ε-Whitney covering of the set U in $(\widetilde{U}, \rho_U)$.

Definition 3.19. — For any ball $B = B_{\widetilde{U}}(x,r)$ in $(\widetilde{U}, \rho_U)$ define the collection $\mathfrak{R}(B)$ by

$$\mathfrak{R}(B) = \{A |\ A \in \mathfrak{R},\ 3A \cap B \neq \varnothing\} \tag{3.11}$$

Fix a ball $B = B_{\widetilde{U}}(x,r)$ in $(\widetilde{U}, \rho_U)$. Recall that we aim to prove (3.9) for the ball B. If B is relatively far from the boundary, i.e. $\rho(B, \widetilde{U} \setminus U) > Nr$, then the strong L^2-Poincaré inequality (3.8) holds. Hence assume that B is relatively large compared with $\rho(B, \widetilde{U} \setminus U)$, namely

$$Nr \geq \rho(B, \widetilde{U} \setminus U) \tag{3.12}$$

The ball B is covered by the triples of the balls in the collection $\mathfrak{R}(B)$. All the balls $A \in \mathfrak{R}$ have small radius compared to their distance to the boundary in the sense of (3.10), and the ball B is relatively large by assumption (3.12). Hence it is not hard to see that

$$B \subset \bigcup_{A \in \mathfrak{R}(B)} 3A \subset 2B. \tag{3.13}$$

Lemma 3.20. — *Let U be an inner uniform domain in (X,ρ). Then for every ball $B = B_{\widetilde{U}}(x,r)$ in $(\widetilde{U},\rho_U)$ there exists a point $y \in B$ with $\rho(y,\widetilde{U}\setminus U) \geq c_0 r/8$ and $\rho_U(y,x) = r/4$. Here c_0 is the constant appearing in* (3.5).

Proof. — Take some point $z \in B_U(x,r)\setminus B_U(x,r/2)$, which is a nonempty set because, by convention, the radius r of the ball B is minimal. Let γ be a path from x to z given by the uniform condition (3.5). Take some point $y \in B$ on the path γ such that $\rho_U(x,y) = r/4$. Such a point exists because the distance function $\rho_U(x,\cdot)$ is continuous and $\rho_U(x,z) \geq r/2$. By the uniform condition (3.5) and the triangle inequality, we have

$$\begin{aligned}\rho_U(y,\widetilde{U}\setminus U) &\geq c_0\frac{\rho_U(x,y)\rho_U(y,z)}{\rho_U(x,z)} = \frac{c_0 r}{4}\frac{\rho_U(y,z)}{\rho_U(x,z)}\\ &\geq \frac{c_0 r}{4}\frac{\rho_U(x,z)-\rho_U(x,y)}{\rho_U(x,z)}\\ &\geq \frac{c_0}{4}\left(1-\frac{r/4}{r/2}\right)r = \frac{c_0 r}{8}\end{aligned}$$

as desired. □

Definition 3.21. — Let B_0 be a ball from the Whitney cover $\mathfrak{R}(B)$ with the property that the point y constructed in Lemma 3.20 is inside $3B_0$. We call the ball B_0 *the central ball* in B.

Note that by construction, we have

$$\rho_U(B_0,\widetilde{U}\setminus U) \geq \frac{c_0 r}{16} \tag{3.14}$$

3.3.4. Decomposition using Whitney balls. — We proceed to estimate the left-hand side of (3.9), namely,

$$\min_{\xi\in\mathbb{R}}\int_B |f-\xi|^2 d\nu$$

for any function $f \in \mathcal{F}(kB)$, where the constant k will be chosen later. Choose

$$\xi = f_{4B_0} = \frac{1}{\nu(4B_0)}\int_{4B_0} f d\nu$$

where B_0 is the central ball in B introduced in Definition 3.21 and write

$$\begin{aligned}\inf_\xi \int_B |f-\xi|^2 d\nu &\leq \sum_{D\in\mathfrak{R}(B)}\int_{3D}|f-f_{4B_0}|^2 d\nu\\ &\leq 2\sum_{D\in\mathfrak{R}(B)}\left[\int_{4D}|f_{4D}-f_{4B_0}|^2 d\nu + \int_{4D}|f-f_{4D}|^2 d\nu\right].\end{aligned} \tag{3.15}$$

The second term is easily bounded. Namely, for every $D \in \mathfrak{R}(B)$, we have $4D \subset 2B$ and $4D$ is far from the boundary relative to its radius in the sense of (3.10). Hence the Poincaré inequality (3.8) is satisfied on $4D$ for any $f \in \mathcal{F}(3B)$. It follows that

$$\begin{aligned}\sum_{D\in\mathfrak{R}(B)}\int_{4D}|f-f_{4D}|^2d\nu &\leq P_0r^2\sum_{D\in\mathfrak{R}(B)}\int_{4D}d\Gamma(f,f)\\ &\leq P_0r^2\int_{2B}\left(\sum_{D\in(B)}\chi_{4D}\right)d\Gamma(f,f). \qquad (3.16)\end{aligned}$$

The sum of characteristic functions appearing in (3.16) is uniformly bounded from above by Lemma 3.18.

To estimate the first term of (3.15), we will use the following Lemma which estimates the differences of averages of a function on close Whitney balls via its energy integral.

Lemma 3.22. — *There exists a constant a_2 such that for any two neighboring Whitney balls, i.e., any balls $D, E \in \mathfrak{R}$ with $3D\cap 3E \neq \varnothing$, and for any $f \in \mathcal{F}(16D)\cap\mathcal{F}(16E)$ we have*

$$|f_{4D}-f_{4E}| \leq a_2 r(D)\left(\frac{1}{\nu(D)}\int_{16D}d\Gamma(f,f)\right)^{\frac{1}{2}}.$$

Proof. — Using the Poincaré inequality (3.8) we estimate

$$\begin{aligned}\nu(4D\cap 4E)|f_{4D}-f_{4E}|^2 &= \int_{4D\cap 4E}|f_{4D}-f_{4E}|^2d\nu\\ &\leq 2\int_{4D\cap 4E}|f-f_{4D}|^2d\nu+2\int_{4D\cap 4E}|f-f_{4E}|^2d\nu\\ &\leq 2\int_{4D}|f-f_{4D}|^2d\nu+2\int_{4E}|f-f_{4E}|^2d\nu\\ &\leq 32P_0r(D)^2\int_{4D}d\Gamma(f,f)+32P_0r(E)^2\int_{4E}d\Gamma(f,f)\end{aligned}$$

As the Whitney balls D, E are neighbors, their radii must be approximately equal, up to the multiple of $4/3$, by the Whitney condition (3.10) and the triangle inequality. Therefore the four multiple of E is contained inside the 16 multiple of D. Furthermore, by the doubling property (2.2) for the measure ν on $\widetilde{U}$, we have

$$\nu(4D\cap 4E) \geq c\nu(D),$$

where $c > 0$ depends only on the doubling constant of ν appearing in (2.2). The desired inequality follows. □

In order to estimate the first term on the right-hand side of (3.15), we will use the following construction. Recall that for each ball $D \in \mathfrak{R}(B)$, the inner uniform condition on the domain U produces a path γ of length at most $C_0\rho_U(B_0, D)$, connecting the closest points of B_0 and D. Let's choose a string of distinct balls $\mathbb{S}(D) = \{B_0^D, B_1^D, \dots B_l^D\}$ of length $l = l(D)$ with the following properties:

1. $\forall j,\ B_j^D \in \mathfrak{R}$;

2. $B_0 = B_0^D$ and $B_l^D = D$;
3. $3B_j^D \cap 3B_{j-1}^D \neq \varnothing$;
4. $3B_j^D \cap \gamma \neq \varnothing$.

In other words, the string $\mathbb{S}(D) = \{B_0^D, B_1^D, \dots B_l^D\}$ connects the two balls B_0 and D by Whitney balls along the path given by the uniform condition.

Lemma 3.23. — *There is a constant $a_3 \in (0, \infty)$ such that for any inner geodesic ball $B = B_{\widetilde{U}}(x, r)$ satisfying* (3.12) *and for any ball $D \in \mathfrak{R}(B)$, the sequence of Whitney balls $\mathbb{S}(D)$ constructed above satisfies:*

(i) For all j, $\rho_U(B_j^D, B_0) \leq C_0 \rho_U(B_0, D) < 2C_0 r$, so that $B_j^D \subset 4C_0 B$
(ii) For all j, $\rho_U(B_j^D, D) \leq \frac{a_3}{2} r(B_j^D)$, so that $D \subset a_3 B_j^D$.

Proof. — The inequality in (i) follows from the length estimate on the path given by the uniform condition (3.5) and from (3.13). To prove (ii), we use the Whitney condition (3.10), the uniform condition and the triangle inequality to obtain

$$\frac{2}{\varepsilon} r(B_j^D) \geq \rho_U(B_j^D, \widetilde{U} \setminus U) \geq c_0 \frac{\rho_U(B_j^D, B_0) \rho_U(B_j^D, D)}{\rho_U(B_0, D)} \tag{3.17}$$

and

$$\frac{2}{\varepsilon} r(B_j^D) \geq \rho_U(B_j^D, \widetilde{U} \setminus U) \geq \rho_U(B_0, \widetilde{U} \setminus U) - \rho_U(B_j^D, B_0). \tag{3.18}$$

One of these inequalities will give the desired result in each of the two cases below.
Case (a). Assume $2\rho_U(B_j^D, B_0) > \rho_U(B_0, \widetilde{U} \setminus U)$. Because $\rho_U(B_0, D) \leq 2r$, (3.14) and (3.17) give

$$r(B_j^D) \geq \frac{\varepsilon c_0}{2} \frac{c_0 r}{32} \frac{\rho_U(B_j^D, D)}{2r} = C \rho_U(B_j^D, D)$$

with $C = \varepsilon c_0^2 / 128$.
Case (b) Assume instead that $2\rho_U(B_j^D, B_0) \leq \rho_U(B_0, \widetilde{U} \setminus U)$, then (3.18) allows us to estimate $r(B_j^D)$ from below by

$$r(B_j^D) \geq \frac{\varepsilon}{2} \frac{1}{2} \rho_U(B_0, \widetilde{U} \setminus U) \geq \frac{\varepsilon c_0}{64} r \geq \frac{\varepsilon c_0}{128 C_0} \rho_U(B_j^D, D).$$

Here, to obtain the last inequality, we have used (3.5) to see that

$$\rho_U(B_j^D, D) \leq L(\gamma) \leq C_0 \rho_U(B_0, D) \leq 2C_0 r.$$

In both cases, we have $\rho_U(B_j^D, D) \leq \frac{a_3}{2} r(B_j^D)$ with $a_3 = \min\left(\frac{256}{\varepsilon c_0^2}, \frac{256 C_0}{\varepsilon c_0}\right)$ as desired. □

Definition 3.24. — Given U, $\mathfrak{R}$, $B = B_{\widetilde{U}}(x, r)$ and $\mathfrak{R}(B)$ as in (3.11), set

$$\mathfrak{R}_1(B) = \{A \in \mathfrak{R} : \exists D \in \mathfrak{R}(B),\ A \in \mathbb{S}(D)\}.$$

Note that the first part of Lemma 3.23 can be rephrased as

$$\bigcup_{A \in \mathfrak{R}_1(B)} A \subset 4C_0 B. \tag{3.19}$$

Returning now to estimating the first term of (3.15) for $f \in \mathcal{F}(kB)$, observe that the doubling property (2.2) for the measure ν gives

$$\sum_{D \in \mathfrak{R}(B)} \int_{4D} |f_{4D} - f_{4B_0}|^2 d\nu \le D_0^2 \int_U \sum_{D \in \mathfrak{R}(B)} |f_{4D} - f_{4B_0}|^2 \chi_D \, d\nu \tag{3.20}$$

Using, for each D, the string $\mathbb{S}(D) = \{B_j^D\}_{j=1}^{l(D)}$ and Lemma 3.22, write

$$|f_{4D} - f_{4B_0}| \le \sum_{j=1}^{l(D)} |f_{4B_j^D} - f_{4B_{j-1}^D}| \le \sum_{j=1}^{l(D)} a_2 r(B_j^D) \left(\frac{1}{\nu(B_j^D)} \int_{16B_j^D} d\Gamma(f,f) \right)^{\frac{1}{2}}.$$

By Lemma 3.23, $\chi_D = \chi_D \chi_{a_3 B_j^D}$, and thus

$$\begin{aligned} |f_{4D} - f_{4B_0}|\chi_D &\le \sum_{i=1}^{l(D)} a_2 r(B_j^D) \left(\frac{1}{\nu(B_j^D)} \int_{16B_j^D} d\Gamma(f,f) \right)^{\frac{1}{2}} \chi_D \, \chi_{a_3 B_j^D} \\ &\le a_2 \sum_{A \in \mathfrak{R}_1(B)} r(A) \left(\frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right)^{\frac{1}{2}} \chi_D \, \chi_{a_3 A} \end{aligned} \tag{3.21}$$

where we have extended the summation from the collection $\mathbb{S}(D)$ to the collection $\mathfrak{R}_1(B)$. We will need the following well known result.

***Lemma 3.25* (e.g., [83], Lemma 5.3.12).** — *Assume that the doubling condition* (2.2) *is satisfied for all balls in* $(\widetilde{U}, \rho_U)$ *with respect to the measure* ν. *Fix* $K \ge 1$. *There exists a constant* $C = C(K)$ *such that for any (possibly infinite) sequence of balls* $B_i = B_{\widetilde{U}}(x_i, r_i)$ *in* $(\widetilde{U}, \rho_U)$ *and any sequence of non-negative numbers* b_i, *we have*

$$\int_U \left(\sum_i b_i \chi_{KB_i} \right)^2 d\nu \le C \int_U \left(\sum_i b_i \chi_{B_i} \right)^2 d\nu. \tag{3.22}$$

To complete the estimation of the first term in (3.15), we continue the estimate (3.20) using the inequality (3.21) to get

$$\sum_{D\in\mathfrak{R}(B)}\int_{4D}|f_{4D}-f_{4B_0}|^2d\nu \le D_0^2\int\sum_{D\in\mathfrak{R}(B)}|f_{4D}-f_{4B_0}|^2\chi_D d\nu$$

$$\le (D_0a_2)^2\int\sum_{D\in\mathfrak{R}(B)}\left(\sum_{A\in\mathfrak{R}_1(B)}r(A)\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)^{\frac12}\chi_D\,\chi_{a_3A}\right)^2 d\nu$$

$$= (D_0a_2)^2\int\left(\sum_{D\in\mathfrak{R}(B)}\chi_D\right)\left(\sum_{A\in\mathfrak{R}_1(B)}r(A)\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)^{\frac12}\chi_{a_3A}\right)^2 d\nu$$

$$\le (D_0a_2)^2\int\left(\sum_{A\in\mathfrak{R}_1(B)}r(A)\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)^{\frac12}\chi_{a_3A}\right)^2 d\nu$$

where we used the fact that $\sum_{D\in\mathfrak{R}(B)}\chi_D\le 1$ (the balls $D\in\mathfrak{R}(B)$ are disjoint). Next, Lemma 3.25 with $K=a_3$ yields

$$\int\left(\sum_{A\in\mathfrak{R}_1(B)}r(A)\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)^{\frac12}\chi_{a_3A}\right)^2 d\nu$$

$$\le C(a_3)\int\left(\sum_{A\in\mathfrak{R}_1(B)}r(A)\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)^{\frac12}\chi_{A}\right)^2 d\nu$$

and, since the balls $A\in\mathfrak{R}_1(B)$ are disjoint, we finally get

$$\sum_{D\in\mathfrak{R}(B)}\int_{4D}|f_{4D}-f_{4B_0}|^2d\nu$$

$$\le D_0^2a_2^2C(a_3)\sum_{A\in\mathfrak{R}_1(B)}r(A)^2\left(\frac{1}{\nu(A)}\int_{16A}d\Gamma(f,f)\right)\nu(A)$$

$$\le D_0^2a_2^2C(a_3)r^2\int\left(\sum_{A\in\mathfrak{R}_1(B)}\chi_{16A}\right)d\Gamma(f,f)$$

$$\le a_1D_0^2a_2^2C(a_3)r^2\int_{64c_0B}d\Gamma(f,f)$$

For the last inequality, we used Lemma 3.18 and the fact that if $A\in\mathfrak{R}_1(B)$ then $A\subset 4c_0B$ by (3.19).

This completes the analysis of the first term in (3.15) and together with (3.16) establishes the weak Poincaré inequality (3.9) with $k=64c_0$. In view of the outline presented after the statement of Theorem 3.13, this completes the proof of Theorems 3.12 and 3.13. To complete the proof of Theorem 3.10, it remains to establish some properties of the Neumann-type Dirichlet form and the associated metric. Before we focus on this task, we explore how Theorem 3.13 extends to a symmetric form obtained from the original form by a simple change of measure.

3.3.5. Poincaré inequality with a different measure. — Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular strictly local Dirichlet space with energy measure Γ and intrinsic distance ρ satisfying the conditions (A1)-(A2) of Section 2.2.3. Assume that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X, \mu)$. Let $U \subset X$ be an open set and let $v \in L^\infty_{\mathrm{loc}}(U, \mu)$ be a locally uniformly positive and locally bounded measurable function on U.

Definition 3.26. — Given the data mentioned above, set

$$\mathcal{E}_U^{N,v}(f,g) = \int_U v d\Gamma(f,g) = \int_U \Upsilon(f,g) v d\mu, \quad f,g \in \mathcal{D}(\mathcal{E}_U^{N,v}) \tag{3.23}$$

where

$$\mathcal{D}(\mathcal{E}_U^{N,v}) = \mathcal{F}^v(U) = \left\{ f \in \mathcal{F}_{\mathrm{loc}}(U) \cap L^2(U, v d\mu) : \int_U \Upsilon(f,f) v d\mu < \infty \right\}.$$

We call such a form a weighted Neumann-type Dirichlet form on U.

Note that, if we take the function v to be the constant function $v \equiv 1$, the form defined in (3.23) becomes $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$. Using the argument of the proof of Proposition 2.50, it is not hard to see that $(\mathcal{E}_N^{N,v}, \mathcal{F}^v(U))$ is a strictly local Dirichlet form on $L^2(U, v d\mu)$ with carré du champ operator $\Upsilon^v(f,g) = \Upsilon(f,g)$.

The following straightforward corollary of Theorem 3.13 will yield heat kernel estimates for the heat semigroup associated with $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$.

Theorem 3.27. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space with energy measure Γ that admits a carré du champ Υ. Let ρ be its intrinsic distance and U be an inner uniform domain. Let ρ_U be the inner distance extended continuously to the completion $\widetilde{U}$ of U. Let v be a measurable function which is locally uniformly bounded and positive in U. Assume that the measure $v d\mu$ on U satisfies the doubling condition (2.2). Assume also that there exist positive constants C and N such that the function v satisfies*

$$\sup_B v \le C \inf_B v \tag{3.24}$$

on any ball $B = B_U(x,r)$ with $\rho_U(B, \widetilde{U} \setminus U) > Nr$.

Then there exists a constant P_1 such that, for any ball $B = B_{\widetilde{U}}(x,r)$ in $(\widetilde{U}, \rho_U)$, we have

$$\forall f \in \mathcal{F}^v(B), \ \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^2 v d\mu \le P_1 r^2 \int_B v d\Gamma(f,f), \tag{3.25}$$

that is, the Poincaré inequality holds true for the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ with reference measure $v d\mu$ on $\widetilde{U}$.

Proof. — It suffices to repeat the proof of Theorem 3.13. Note that the hypothesis implies that the inequality

$$\forall f \in \mathcal{F}^v(U), \ \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^2 v d\mu \leq P_0 r^2 \int_B v d\Gamma(f,f)$$

is satisfied for any ball $B = B(x,r)$ with $\rho(B, X \setminus U) > Nr$. This inequality, without the function v inserted, holds true because the underlying Dirichlet space is of Harnack-type hence satisfies the Poincaré inequality. Condition (3.24) makes it very easy to insert the function v. The condition that the measure $v d\mu$ is doubling replaces the condition that the measure ν is doubling in Theorem 3.13. The argument then goes through without notable changes. □

Remark 3.28. — For a continuous positive function v in U, the hypothesis that the original form admits a carré du champ can be dropped and the Theorem above holds for $\mathcal{E}_U^{N,v}(f,f) = \int_U v d\Gamma(f,f)$. An example of a function v satisfying the conditions of Theorem 3.27 is any positive power of the distance to the boundary,

$$v(x) = \delta_U(x)^\alpha, \quad \text{where} \quad \delta_U(x) = \rho_U(x, \widetilde{U} \setminus U).$$

Another very important example is when v is the square of a positive weak solution of L in U vanishing along ∂U. In later chapters, the heat kernel estimates for the forms $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ in this latter case will be used to obtain the heat kernel estimates for the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$.

3.4. Applications to Neumann-type Dirichlet forms

3.4.1. Regularity of Neumann-type Dirichlet forms. — It turns out that the Poincaré inequalities proved in the previous section can be used to show that the Neumann-type Dirichlet form $\mathcal{E}_U^N$ (or $\mathcal{E}_U^{N,v}$) is regular on $\widetilde{U}$ when U is inner uniform. Regularity is essential to be able to say that the form $(\mathcal{E}_U^N, \mathcal{F}(U))$ is of Harnack-type in $\widetilde{U}$ as stated in Theorem 3.10.

Definition 3.29. — Let $\mathrm{Lip}(\widetilde{U})$ be the space of Lipschitz functions on $(\widetilde{U}, \rho_U)$. Let $\mathrm{Lip}_c(\widetilde{U})$ be the space of Lipschitz functions on $(\widetilde{U}, \rho_U)$ which are compactly supported in $\widetilde{U}$. The Lipschitz constant k of a function f in $\mathrm{Lip}(\widetilde{U})$ is

$$k = \sup \left\{ \frac{f(x) - f(y)}{\rho_U(x,y)} : x, y \in \widetilde{U} \right\}.$$

Theorem 3.30. — *Assume that the local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the conditions* (A1-A2) *of* Section 2.2.3 *and admits a carré du champ operator Υ. Let $U \subset X$ be an open connected subset of X and let ϵ be any positive number. Let v be a locally bounded measurable function on $\widetilde{U}$ which is locally uniformly positive on U.*

Assume that the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ *on* $\widetilde{U}$ *satisfies the following family of Poincaré inequalities with respect to the metric* ρ_U*:* $\forall x \in \widetilde{U}$, $0 < r < \epsilon$,

$$\text{(3.26)} \qquad \forall\, u \in \mathcal{F}^v(B),\ \inf_{\xi} \int_B (u-\xi)^2 v d\mu \leq P_0' r^2 \int_B \Upsilon(u,u) v d\mu,\ B = B_{\widetilde{U}}(x,r).$$

Assume that the measure $vd\mu$ *satisfies the following doubling condition on* $\widetilde{U}$ *with respect to the metric* ρ_U*, that is,*

$$\text{(3.27)} \qquad \forall x \in \widetilde{U},\ 0 < r < \epsilon, \quad \int_{B_{\widetilde{U}}(x,2r)} v d\mu \leq D_0' \int_{B_{\widetilde{U}}(x,r)} v d\mu.$$

Then the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ *on* $L^2(\widetilde{U}, vd\mu)$ *is regular with core* $\mathrm{Lip}_c(\widetilde{U})$.

In order to prove Theorem 3.30 we will use the description of Lipschitz functions on X, proved in [**63**, Corollary 3.6] and stated in 2.21 and 2.22.

Proof of Theorem 3.30. — The space $\mathrm{Lip}_c(\widetilde{U})$ is dense in $C_0(\widetilde{U})$ with supremum norm by [**59**, Theorem 6.8]. To see that $\mathrm{Lip}_c(\widetilde{U})$ is dense in $\mathcal{D}(\mathcal{E}_U^{N,v})$, we follow [**56**, page 205], [**57**, page 13] and [**55**, Lemma 10]. Let

$$d\nu = v d\mu$$

denote the reference measure on U. Let g be any function in $\mathcal{D}(\mathcal{E}_U^{N,v}) = \mathcal{F}^v(U)$. We aim to prove that g can be approximated by functions in $\mathrm{Lip}_c(\widetilde{U})$. Because we can approximate the function g in $\mathcal{F}^v(U)$ by the bounded functions

$$g_n = \min(\max(g,-n),n),$$

without loss of generality, we can assume that the function g is bounded. Set

$$\phi_R(x) = R^{-1}\min\{(2R - \rho_U(x, \widetilde{U}\setminus U))_+, (2R - \rho_U(x,x_0))_+, R\}$$

where x_0 is a fixed point in U and $(t)_+ = \min\{0,t\}$. Since v is locally finite on $\widetilde{U}$, these compactly supported cut-off functions ϕ_R are in $\mathcal{F}^v(U) \cap L^\infty(U, vd\mu)$. Since $g \in \mathcal{F}^v(U) \cap L^\infty(U, vd\mu)$, we have $g\phi_R \in \mathcal{F}^v(U)$ by the energy estimate of Lemma 2.7. It is easy to see that $\phi_R g$ tends to g in $\mathcal{E}_U^{N,v}$-norm and in $L^2(\widetilde{U}, d\nu)$ when R tends to infinity. Thus, in the rest of the proof, we assume that g is a bounded function in $\mathcal{D}(\mathcal{E}_U^{N,v})$ with compact support in $\widetilde{U}$.

For any $r > 0$, set

$$g_r(y) = \frac{1}{\nu\left(B_{\widetilde{U}}(y,r)\right)} \int_{B_{\widetilde{U}}(y,r)} g d\nu.$$

Fix $r \in (0,\epsilon)$ and set $r_i = 2^{-i}r$, $B_i = B_{\widetilde{U}}(x, r_i)$. We say that $x \in \widetilde{U}$ is a Lebesgue point of g if

$$\lim_{i\to\infty} g_{r_i}(x) = g(x)$$

It is known that for every $g \in L^2(\widetilde{U}, \nu)$, the points in $\widetilde{U}$ that are not Lebesgue for a function g form a set of ν-measure zero.

For every Lebesgue point $x \in \widetilde{U}$, Jensen's inequality , the Poincaré inequality and the doubling property of the measure ν on $(\widetilde{U}, \rho_U)$ yield

$$\begin{aligned}
|g(x) - g_r(x)| &\leq \sum_{i=0}^{\infty} |g_{r_i} - g_{r_{i+1}}| \leq \sum_{i=0}^{\infty} \left(\frac{1}{\nu(B_{i+1})} \int_{B_i} |g(y) - g_{r_i}(x)|^2 d\nu(y) \right)^{\frac{1}{2}} \\
&\leq \sum_{i=0}^{\infty} r_i \left(\frac{C}{\nu(B_i)} \int_{B_i} \Upsilon(g,g) d\nu \right)^{\frac{1}{2}} \leq \sqrt{C} \sum_{i=0}^{\infty} r_i \sqrt{\mathcal{M}(\Upsilon(g,g))(x)} \\
&= r\sqrt{C}\sqrt{\mathcal{M}\Upsilon(g,g)(x)}. \qquad (3.28)
\end{aligned}$$

Here $\mathcal{M}(f)$ is the 2ϵ-maximal function of f with ϵ as in (3.26)-(3.27), i.e.,

$$\mathcal{M}f(x) = \mathcal{M}_{2\epsilon} f(x) = \sup_{0<s<2\epsilon} \frac{1}{\nu(B_{\widetilde{U}}(x,s))} \int_{B_{\widetilde{U}}(x,s)} f d\nu.$$

Similarly for any Lebesgue points $x, y \in \widetilde{U}$ with $\rho_U(x,y) \leq r$, the doubling property of $\nu|_U$ and the Poincaré inequality (3.26) yield

$$\begin{aligned}
|g_r(x) - g_r(y)| &\leq |g_r(x) - g_{2r}(x)| + |g_r(y) - g_{2r}(x)| \\
&\leq 2 \left(\frac{C}{\nu(B_U(x,2r))} \int_{B_U(x,2r)} |g(z) - g_{2r}(x)|^2 d\nu(z) \right)^{\frac{1}{2}} \\
&\leq 2r \left(\frac{C'}{\nu(B_U(x,2r))} \int_{B_U(x,2r)} \Upsilon_U(g,g) d\nu \right)^{\frac{1}{2}} \\
&\leq (2\sqrt{C'}r)\sqrt{\mathcal{M}\Upsilon(g,g)(x)}. \qquad (3.29)
\end{aligned}$$

Combining (3.28) and (3.29) we see that for any Lebesgue points $x, y \in \widetilde{U}$ with $\rho_U(x,y) \leq r$, there exists another constant C such that

$$|g(x) - g(y)| \leq Cr \left[\sqrt{\mathcal{M}\Upsilon(g,g)(x)} + \sqrt{\mathcal{M}\Upsilon(g,g)(y)} \right]. \qquad (3.30)$$

For any $\lambda > 0$, set

$$\begin{aligned}
E_\lambda &= \left\{ x \in \widetilde{U} : x \text{ is a Lebesgue point of } g,\ g(x)^2 \leq \lambda^2 \text{ and } \mathcal{M}\Upsilon(g,g)(x) \leq \lambda^2 \right\} \\
F_\lambda &= \widetilde{U} \setminus E_\lambda
\end{aligned}$$

Note that F_λ is precompact in $\widetilde{U}$ for λ large enough, say $\lambda \geq \lambda_0$, because g has compact support in $\widetilde{U}$. Furthermore, the restriction $g|_{E_\lambda}$ of g to E_λ is Lipschitz with constant $2C\lambda$ on E_λ by (3.30). Let f_λ be some Lipschitz extension of g from E_λ to $\widetilde{U}$ with the same Lipschitz constant (see, e.g., [**59**, Theorem 6.2]). Let $\lambda \geq \lambda_0$. For such λ, F_λ is precompact in $\widetilde{U}$. As $f_\lambda = g$ in E_λ, it follows that f_λ is a bounded function in $\widetilde{U}$ with compact support in $\widetilde{U}$ and with $\|f_\lambda\|_\infty \leq \lambda(1+2CR_0)$ where R_0 is the diameter of F_{λ_0}. Moreover, f_λ has compact support. Since $g \in \mathcal{D}(\mathcal{E}_U^{N,v})$, we have

$g \in L^2(U,\nu)$, and $\int_U \Upsilon(g,g)d\nu < \infty$. It is known that the maximal function $\mathcal{M}\Upsilon(g,g)$ satisfies

$$N\nu\{x \in U : \mathcal{M}\Upsilon(g,g) > N\} \to 0$$

as $N \to \infty$, see [**71**, Theorem 2.19]. Also, we have

$$\int_{F_\lambda} |g|^2 d\nu \to 0, \quad \text{as} \quad \lambda \to \infty.$$

Since non-Lebesgue points of g form a set of measure zero,

$$\begin{aligned} \lambda^2\nu\{F_\lambda\} &\le \lambda^2\nu\{x \in U : \mathcal{M}\Upsilon(g,g) > \lambda^2\} + \lambda^2\nu\{x \in U : g(x)^2 > \lambda^2\} \\ &\le \lambda^2\nu\{x \in U : \mathcal{M}\Upsilon(g,g) > \lambda^2\} + \int_{\{g^2>\lambda^2\}} |g|^2 d\nu \to 0 \end{aligned} \tag{3.31}$$

as $\lambda \to \infty$. The function f_λ is bounded by $\lambda(1+2CR_0)$ and Lipschitz with respect to ρ_U with Lipschitz constant $2C\lambda$. Therefore $\Upsilon(f,f) \le 4C^2\lambda^2$ by Corollary 2.22. Inequality (3.31) gives

$$\int_{F_\lambda} \left(|f_\lambda|^2 + \Upsilon(f_\lambda, f_\lambda)\right) d\nu \le \lambda^2 \left((1+2CR_0)^2 + 4C^2\right) \nu\{F_\lambda\} \to 0$$

as $\lambda \to \infty$. Now, since $f_\lambda = g$ on E_λ, we have

$$\int_U \left(|g-f_\lambda|^2 + \Upsilon(g-f_\lambda, g-f_\lambda)\right) d\nu \le 2\int_{F_\lambda} \left(|g|^2 + |f_\lambda|^2 + \Upsilon(g,g) + \Upsilon(f_\lambda, f_\lambda)\right) d\nu$$

and the right-hand side tends to 0 as λ tends to infinity. Thus f_λ tends to g in the Hilbert space $\mathcal{F}^v(U) = \mathcal{D}(\mathcal{E}_U^{N,v})$, as desired. □

Corollary 3.31. — *Referring to* Theorem 3.30, *the form* $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ *is a strictly local regular Dirichlet form on* $L^2(\widetilde{U}, vd\mu)$.

Lemma 3.32. — *Referring to* Theorem 3.30, *the metric* $\rho_{\mathcal{E}_U^{N,v}}$ *on* $\widetilde{U}$ *coincides with the inner metric* ρ_U.

Proof. — For any $x, y \in \widetilde{U}$ we have

$$\rho_{\mathcal{E}_U^{N,v}}(x,y) = \sup\left\{u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}_U^{N,v}) \cap C_0(\widetilde{U}), \Upsilon(u,u) \le 1 \text{ a.e. on } U\right\}.$$

To show $\rho_{\mathcal{E}_U^{N,v}}(x,y) \ge \rho_U(x,y)$ it suffices to notice that the function

$$\max\{\rho_U(x,y) - \rho_U(x,\cdot), 0\}$$

is a compactly supported Lipschitz function on $(\widetilde{U}, \rho_U)$ with $\Upsilon(u,u) \le 1$ a.e. on U.

The opposite inequality for points in U follows easily from statement (4) of Theorem 2.11. To show $\rho_{\mathcal{E}_U^{N,v}}(x,y) \le \rho_U(x,y)$ when at least one of the points x, y belongs to $\widetilde{U} \setminus U$, choose a sequence $\{x_i\}_{i=1}^\infty$ of points in U approximating $x \in \widetilde{U}$ and a sequence

$\{y_i\}_{i=1}^{\infty}$ of points in U approximating $y \in \widetilde{U}$. For any continuous function u on $\widetilde{U}$ used in the definition of $\rho_{\mathcal{E}_U^{N,v}}(x,y)$, write

$$\begin{aligned}|u(x)-u(y)| &\leq \inf_i \left[|u(x_i)-u(y_i)|+|u(x)-u(x_i)|+|u(y)-u(y_i)|\right] \\ &\leq \liminf_{i\to\infty} |u(x_i)-u(y_i)| \\ &\leq \liminf_{i\to\infty} \rho_{\mathcal{E}_U^{N,v}}(x_i,y_i) = \liminf_{i\to\infty} \rho_U(x_i,y_i) = \rho_U(x,y). \qquad \square\end{aligned}$$

Corollary 3.33. — *Referring to* Theorem 3.30, *the metric* $\rho_{\mathcal{E}_U^{N,v}}$ *is everywhere finite continuous and the topology given by this metric coincides with the original topology on* $\widetilde{U}$. *The conditions* (A1)-(A2) *of* Section 2.2.3 *are satisfied for the Dirichlet form* $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$.

3.4.2. Proof of the heat kernel estimates for the Neumann-type semigroups. — We now have all the ingredients to prove Theorem 3.10. In fact, we can now prove a more general result concerning the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ under some conditions on the function v.

Theorem 3.34. — *Let* U *be an inner uniform domain in a Harnack-type Dirichlet space* $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$. *Assume that* $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ *admits a carré du champ. Let* v *be a measurable locally uniformly positive and locally uniformly bounded function on* U. *Assume that the measure* $v d\mu$ *on* U *satisfies the doubling condition* (2.2) *on* $(\widetilde{U},\rho_U)$. *Assume further that there exist positive constants* C *and* N *such*

$$\sup_B v \leq C \inf_B v \tag{3.32}$$

on any ball $B = B(x,r)$ *with* $\rho(B, X\setminus U) > Nr$. *Then the Dirichlet space* $(\widetilde{U}, v d\mu, \mathcal{E}_U^{N,v}, \mathcal{F}^v(U))$ *is of Harnack-type. In particular, the associated Neumann-type semigroup admits a continuous kernel* $p_U^{N,v}(t,x,y)$ *which satisfies*

$$\frac{c_1 \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{V_v(x,\sqrt{t})} \leq p_U^{N,v}(t,x,y) \leq \frac{c_3 \exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{V_v(x,\sqrt{t})}, \tag{3.33}$$

for all $x,y\in\widetilde{U}$ *and all* $t>0$. *Here* V_v *denotes the volume* $V_v(x,r)=\int_{B_U(x,r)} v d\mu$.

Proof. — We already observed that the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is a strictly local Dirichlet form on $L^2(U,\mu)$. Theorem 3.30 shows that this Dirichlet form is regular on $\widetilde{U}$. Lemma 3.32 shows that the metric $\rho_{\mathcal{E}_U^{N,v}}$ associated with the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ coincides with ρ_U. In particular this implies conditions (A1)-(A2) of Section 2.2.3. By hypothesis, the measure $v d\mu$ satisfies the doubling condition (2.2) on $(\widetilde{U},\rho_U)$. Finally, Theorem 3.27 (which we already used above implicitly to prove regularity) shows that the Poincaré inequality (2.3) holds. Now, to obtain the desired heat kernel inequalities, it suffices to apply Theorem 2.31. $\square$

Note again that since $(\widetilde{U}, vd\mu, \mathscr{E}_U^{N,v}, \mathscr{F}^v(U))$ is of Harnack-type, all the nice properties (e.g., Hölder continuity, etc) associated with Harnack-type Dirichlet spaces hold. See Section 2.3.

Proof of Theorem 3.10. — To obtain Theorem 3.10 as a corollary of Theorem 3.27, we simply need to observe that the measure $\mu|_U$ is doubling when U is inner uniform. This was proved in Lemma 3.9. □

CHAPTER 4

THE HARMONIC PROFILE OF AN UNBOUNDED INNER UNIFORM DOMAIN

The aim of this chapter is to construct and study the basic properties of the harmonic profile $h = h_U$ of an unbounded inner uniform domain U in an Harnack-type Dirichlet space. The first section describes the main result we are after, in the simplest possible terms. The second and third section gives a proof in the case of uniform domains, together with many important additional results, most notably, an elliptic boundary Harnack principle. The outline of the proof is based on Aikawa's work [**1**]. The fourth section extends these results to the case of inner uniform domains. More precisely, it explains how Theorem 3.10 turns the case of inner uniform domains into a corollary of the case of uniform domains. Although we follow [**1**] closely, it is worth pointing out that Ancona's approach [**3**] could also be used for the same purpose. Details in the case of Euclidean uniform domains are given in [**4**] which contains many interesting additional results.

4.1. The harmonic profile

By definition, we call harmonic profile (or profile, for short) of an unbounded domain U in a local regular Dirichlet space X any function h such that:

1. h is a local weak solution of the Laplace equation in U;
2. $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$;
3. $h > 0$ in U.

We will only consider such function when the underlying Dirichlet space X is a Harnack-type Dirichlet space in which case condition (1) implies that h is continuous in U and can be rephrased as saying that h is harmonic (for the corresponding process) in U. Condition (2) is essential and captures the idea that h vanishes along the boundary of U (as the boundary of U may contain irregular points, it is not true in general that h vanishes continuously on ∂U).

Our technique places the existence and properties of a harmonic profile of U at the center of the study of the heat equation with Dirichlet boundary condition in U. In fact, the results concerning the heat equation with the Dirichlet boundary in an unbounded domain U obtained in this monograph strongly suggest that much further efforts should be dedicated to develop good estimates for harmonic profiles.

The goal of this chapter is to prove the following result concerning the profile of any unbounded inner uniform domain in a Harnack-type Dirichlet space.

Theorem 4.1. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type strictly local regular Dirichlet space with associated intrinsic distance ρ. Assume further that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ. Let U be an unbounded inner uniform domain in (X, ρ). The following properties holds:*

1. *The domain U admits a profile h;*
2. *Any two profiles h_1, h_2 of U are proportional, i.e., there exists a real $c \in (0, \infty)$ such that $h_2 = ch_1$.*
3. *If h is a profile of U then the measure $d\nu = h^2 d\mu$ on $(\widetilde{U}, \rho_U)$ satisfies the volume doubling property, that is, there is a constant $D \in (0, \infty)$ such that for any $x \in \widetilde{U}$ and $r > 0$, we have*
$$V_{\widetilde{U},h^2}(x, 2r) \leq DV_{\widetilde{U},h^2}(x, r)$$
where $V_{\widetilde{U},h^2}(x, r) = \int_{B_{\widetilde{U}}(x,r)} h^2 d\mu$.

Our interest in this result lies in the fact that the function h whose existence is stated in (1) above plays a key part in the statement and the proof of the two-sided Gaussian bounds for the Dirichlet heat kernel of the domain U. Property (3) is also essential in this respect. The interesting property (2) is not essential for our main purpose and could be obtained, ultimately, as an application of our final results. Indeed, we will later give examples where property (1) and (3) can be derived from rather elementary considerations. In such cases, property (2) can indeed be seen as an application of our main results. However, for general inner uniform domains, we will derive (1) and (3) from a powerful property (the boundary Harnack principle) that also implies (2).

It will be convenient to proceed in two stages to study the profile of an inner uniform domain. The first stage will consist in deriving result in the restricted case of uniform domains. In a second and easy stage, we will show that, as a corollary, the results extended to the case of inner uniform domains. For the second stage, we will need to assume the existence of a carré du champ.

4.2. The elliptic boundary Harnack principle in uniform domains

4.2.1. Boundary Harnack principle. — In the case of uniform domains, Theorem 4.1 will follow without to much difficulty from the following result known in the classical case as the boundary Harnack principle.

Theorem 4.2. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Let U be an unbounded uniform domain in (X, ρ). Then there exist constants $A_0, A_1 \in (1, \infty)$ such that for any $\xi \in \partial U$, any $r > 0$, and any positive* *local weak solutions u and v of $Lu = 0$ in $U \cap B(\xi, A_0 r)$, with weak Dirichlet boundary condition along ∂U, we have*

$$\forall\, x, x' \in U \cap B(\xi, r), \quad \frac{u(x)}{u(x')} \le A_1 \frac{v(x)}{v(x')}.$$

Moreover, the constants A_0, A_1 depend only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality (2.7) *on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$.*

For later purpose and clarity, we also state a second version of this result using the inner balls $B_{\widetilde{U}}(x, r)$ instead of the trace $U \cap B(x, r)$ of the balls of (X, ρ). The two results are equivalent.

Theorem 4.3. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Let U be unbounded uniform domain in (X, ρ). Then there exist constants $A_0, A_1 \in (1, \infty)$ such that for any $\xi \in \widetilde{U} \setminus U$, any $r > 0$, and any positive local weak solutions u and v of $Lu = 0$ in $U \cap B_{\widetilde{U}}(\xi, A_0 r)$ with weak boundary condition along ∂U, we have*

$$\forall\, x, x' \in U \cap B_{\widetilde{U}}(\xi, r), \quad \frac{u(x)}{u(x')} \le A_1 \frac{v(x)}{v(x')}.$$

Moreover, the constants A_0, A_1 depends only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality (2.7) *on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$.*

Sketch of the proof of Theorem 4.2. — Let u, v be positive harmonic functions as in Theorem 4.2 with A_0 large enough to be chosen later. Let $G^U_{\xi,R}$ denote the (restricted) Green function in the open set $U(\xi, R)$ i.e., the Green function corresponding to the Dirichlet-type Dirichlet form $(\mathcal{E}^D_{U(\xi,R)}, \mathcal{F}^0(U(\xi, R)))$ where $U(\xi, R) = U \cap B(\xi, R)$ (the same line of reasoning applies if one works with the inner balls $B_{\widetilde{U}}(\xi, R)$ instead of the ball $B(\xi, R)$ and set $U(\xi, R) = U \cap B_{\widetilde{U}}(\xi, R)$).

For any fixed $a_0 \in (1, A_0)$ (to be chosen later), by classical potential theoretic arguments, there exists a Borel measure ν_u supported on $U \cap \partial B(\xi, a_0 r)$ such that, for all $x \in U \cap B(\xi, a_0 r)$,

$$u(x) = \int_{U \cap \partial B(\xi, a_0 r)} G^U_{\xi, A_0 r}(x, y)\, d\nu_u(y) \tag{4.1}$$

and similarly for v.

Assume now that we know there exists a constant $A_1' \in (1, \infty)$ such that for all $x, x' \in U \cap B(\xi, r)$ and $y, y' \in U \cap \partial B(\xi, a_0 r)$, we have

$$\frac{G^U_{\xi, A_0 r}(x, y)}{G^U_{\xi, A_0 r}(x', y)} \le A_1' \frac{G^U_{\xi, A_0 r}(x, y')}{G^U_{\xi, A_0 r}(x', y')}, \tag{4.2}$$

with A_1' depending only on the constants c_0, c_1 of inner uniformity of U and on the constant C from the Harnack inequality (2.7) on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then it follows that for any (fixed) $y' \in U \cap \partial B(\xi, a_0 r)$ and all $x, x' \in U \cap B(\xi, r)$, we have

$$\frac{1}{A_1'} u(x) \leq \frac{G^U_{\xi,A_0 r}(x,y')}{G^U_{\xi,A_0 r}(x',y')} \int_{U \cap \partial B_{\widetilde{U}}(\xi, a_0 r)} G^U_{\xi,A_0 r}(x',y) d\nu_u(y) \leq A_1' u(x),$$

and similarly for v. Therefore, for all $x, x' \in U \cap B(\xi, r)$,

$$\frac{u(x)}{u(x')} \leq A_1' \frac{G^U_{\xi,A_0 r}(x,y')}{G^U_{\xi,A_0 r}(x',y')} \leq (A_1')^2 \frac{v(x)}{v(x')}.$$

This reduces the proof of Theorem 4.2 to the validity of (4.2). That (4.2) holds true under the hypothesis of Theorem 4.2 will be proved in the next subsection. □

Remark 4.4. — In Theorems 4.2 and 4.3, we stated the results for weak solutions satisfying the Dirichlet boundary condition in the weak sense. The outline of the proof given above shows that the boundary Harnack inequalities in question follow from Green functions estimates (namely, (4.2)) and the representation formula (4.1). So, these conclusions holds for any class of functions for which (4.1) holds. In classical potential theory, it is customary to work with local weak solution that are bounded and vanish quasi-everywhere on the relevant portion of the boundary (see, e.g., [**1**, Theorem 1]).

4.2.2. The boundary Harnack principle for the Green functions $G^U_{\xi,R}$. — As indicated by the proof of Theorem 4.2, it is important to study the (restricted) Green function $G^U_{\xi,R}$, i.e., the Green function relative to the open set $U \cap B$ where $B = B(x, R)$ is a ball of radius R in (X, ρ). The following theorem states the validity of the inequality (4.2) under the hypothesis of Theorem 4.2. It is the result needed to finish the proof of Theorem 4.2.

Theorem 4.5. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Let U be an unbounded uniform domain in (X, ρ). Then there exist constants $a_0, A_0, A_1' \in (1, \infty)$ with $a_0 < A_0$ such that, for any $\xi \in \partial U$ and any $r > 0$,*

$$\frac{G_{U \cap B(\xi, A_0 r)}(x,y)}{G_{U \cap B(\xi, A_0 r)}(x',y)} \leq A_1' \frac{G_{U \cap B(\xi, A_0 r)}(x,y')}{G_{U \cap B(\xi, A_0 r)}(x',y')},$$

for all $x, x' \in U \cap B(\xi, r)$ and $y, y' \in U \cap \partial B(\xi, a_0 r)$. That is, the inequality (4.2) is satisfied. Moreover the constants a_0, A_0, A_1' depend only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality (2.7) on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Remark 4.6. — Note that the hypothesis on the pairs (x, y) and on (x', y') are identical. Hence the stated one-sided bound yields, in fact, a two-sided inequality: the left-hand side and right-hand side ratios have comparable size.

The rest of this section is devoted to the proof of Theorem 4.5. The first subsection develops Green functions estimates that follow from some of the heat kernel estimates reviewed in earlier chapters. Later subsections adapt the work of Aikawa, who works in Euclidean uniform domains, to uniform domains in a Harnack-type Dirichlet space.

4.2.3. Basic Green functions estimates. — Throughout this section, we let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a fixed Harnack-type Dirichlet space with associated intrinsic distance ρ and we assume that $\sup \rho = \infty$, that is, X is not compact.

Given a relatively compact open set V, we let G_V be the green function

$$G_V(x, y) = \int_0^\infty p_V^D(t, x, y)dt, \quad x, y \in V, \ x \neq y.$$

As a function of V, for fixed $x \neq y$, it is an increasing function. It is a symmetric function of x, y which, for any fixed x, is a local weak solution (in y) of the Laplace equation $Lu = 0$ in $V \setminus \{x\}$. It follows that it is locally Hölder continuous in $V \setminus \{x\}$. In fact, using the heat kernel bounds available for $p_V^D(t, x, y)$, it is not hard to show that $y \mapsto G_V(x, y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(V, V \setminus \{x\})$. That is, $G_V(x, \cdot)$ is a local weak solution of the Laplace equation in $V \setminus \{x\}$ with Dirichlet boundary condition along ∂V. This is the content of the next lemma.

Lemma 4.7. — *For any relatively compact open set V in X, the Green function $y \mapsto G_V(x, y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(V, V \setminus \{x\})$ for any fixed $x \in U$.*

Proof. — We start by observing that the heat kernel $y \mapsto p_V^D(t, x, y)$ is in $\mathcal{F}^0(V)$. See Lemma 2.39. The heat kernel upper bounds of Theorem 2.40 easily imply that $\phi G_V(x, \cdot) \in L^2(X, \mu)$ for any continuous function ϕ with compact support K in $X \setminus \{x\}$. Indeed, $p_V^D(t, x, y) \leq p_B^D(t, x, y)$ with $B = B(x, R)$ and R is chosen large enough to insure that $V \subset B$. By Theorem 2.40, there are constants $c_1, C(B) \in (0, \infty)$ such that, for all $t \geq R^2$ and $x, y \in V$,

$$p_V^D(t, x, y) \leq C(B)e^{-c_1 t/R^2}; \tag{4.3}$$

and there are constants $c_2, C_2 \in (0, \infty)$ depending on V, K such that for all $t > 0$ and $x, y \in V \cap K$,

$$p_V^D(t, x, y) \leq C_2 e^{-c_2/t}. \tag{4.4}$$

This shows that the integral $\phi G_V(x, \cdot) = \int_0^\infty \phi p_V^D(t, x, \cdot)dt$ converges at 0 and ∞ in $L^2(X, \nu)$. Hence $\phi G_V(x, \cdot)$ is in $L^2(X, \mu)$.

Next, we use the same line of reasoning to show that the convergence is also in $\mathcal{F}^0(V)$. Let ϕ be as above with the additional property that $d\Gamma(\phi, \phi) \leq d\mu$ on X (recall that there are many such test functions). For fixed $0 < a < b < \infty$, set $\psi = \int_a^b p_V^D(t, x, \cdot)dt$ and observe that $\phi\psi, \phi^2\psi \in \mathcal{F}^0(V)$. Use the properties of Γ and integration by part (in V), to obtain

$$\begin{aligned}\int_V d\Gamma(\phi\psi,\phi\psi) &= \int_V |\psi|^2 d\Gamma(\phi,\phi) + \int_V d\Gamma(\psi,\phi^2\psi)\\ &= \int_V |\psi|^2 d\Gamma(\phi,\phi) + \int_V \psi(-L\psi)\phi^2 d\mu\\ &= \int_V |\psi|^2 d\Gamma(\phi,\phi) + \int_V \psi(p_V^D(a,x,\cdot) - p_V^D(b,x,\cdot))\phi^2 d\mu\\ &\le \sup\left\{\frac{d\Gamma(\phi,\phi)}{d\mu}\right\}\int_{K\cap V}\psi^2 d\mu + \sup\{\phi^2\}\int_{K\cap V} p_V^D(a,x,\cdot)\psi d\mu.\end{aligned}$$

Now, observe that (4.3)-(4.4) implies that $\int_{K\cap V}\psi^2 d\mu = \int_{K\cap V}\left(\int_a^b p_V^D(t,x,\cdot)dt\right)^2 d\mu$ tends to 0 when a, b tend to infinity or when a, b tend to 0 (this is indeed the argument we used above to show that $G_V(x,\cdot)$ is in $L^2(X, d\mu)$). The same estimates (4.3)-(4.4) implies that

$$\int_{K\cap V} p_V^D(b,x,\cdot)\psi d\mu = \int_{K\cap V} p_V^D(b,x,\cdot)\left(\int_a^b p_V^D(t,x,\cdot)dt\right)d\mu$$

tends to 0 when a, b tend to infinity or when a, b tend to 0. This implies that the integral $\phi G_V(x,y) = \int_0^\infty p_V^D(t,x,\cdot)dt$ converges in $\mathcal{F}^0(V)$ as desired. □

The following Lemma gives estimates for the Green function G_B of a Ball $B(z,R)$ in X.

Lemma 4.8. — *There is a constant C_1 depending only on the Harnack constant H_0 from condition* (2.7) *on X such that, for $z \in X$, $R > 0$, we have*

$$\forall\, x,y \in B(z,R),\ \ G_{B(z,R)}(x,y) \le C_1 \int_{\rho(x,y)^2/2}^{2R^2} \frac{ds}{V(x,\sqrt{s})}. \tag{4.5}$$

Fix $\theta \in (0,1)$. There is a constant c_1 depending only on the Harnack constant H_0 from condition (2.7) *on X such that, for $z \in X$, $R > 0$, we have*

$$\forall\, x,y \in B(z,\theta R),\ \ G_{B(z,R)}(x,y) \ge c_1 \int_{\rho(x,y)^2/2}^{2R^2} \frac{ds}{V(x,\sqrt{s})}. \tag{4.6}$$

Proof. — Set $B = B(z,R)$. Start with the lower bound in $B(z,\theta R)$. By definition,

$$G_B(x,y) = \int_0^\infty p_B^D(t,x,y)dt \ge \int_{\rho(x,y)^2/2}^{2R^2} p_B^D(t,x,y)dt.$$

In the time-space region $(\rho(x,y)^2/2, 2R^2) \times B(z,\theta R)$, the heat kernel lower bound (2.13) gives

$$p_B^D(s,x,y) \ge \frac{c_1'}{V(x,\sqrt{s})}.$$

The desired lower bound for G_B in $B(z,\theta R)$ follows.

The upper bound is somewhat more difficult as we have to take care of the whole time interval $(0, \infty)$. However, the upper bounds of Theorem 2.40 give the desired result. □

The next lemma gives bounds for the Green function in a set W of the form $U \cap B(z, R)$ where U is an open set and $B(z, R)$ is a ball in (X, ρ). We are mostly interested in the case when U is uniform in (X, ρ). The x, y range in these estimates is somewhat restricted.

Lemma 4.9. — *Fix $\theta \in (0, 1)$.*

1. *There is a constant C_1 depending only on θ and the Harnack constant H_0 from condition (2.7) on X such that, for $z \in X$, $R > 0$ and any open set U, we have*

$$G_{U \cap B(z,R)}(x, y) \leq C_1 \frac{R^2}{V(x, R)} \tag{4.7}$$

   for all $x, y \in U \cap (B(z, R)$ with $\rho(x, y) \geq \theta R$

2. *Assume that U is a uniform open set in (X, ρ). There is a constant ϵ_1 depending only on θ, the Harnack constant H_0 from condition (2.7) on X and the uniformity constants c_0, C_0 of U such that for $z \in X$ and $R > 0$ we have*

$$G_{U \cap B(z,R)}(x, y) \geq \epsilon_1 \frac{R^2}{V(x, R)} \tag{4.8}$$

   as long as $x, y \in U \cap B(z, R/(4C_0))$ and $\rho(x, X \setminus U), \rho(y, X \setminus U) \in (\theta R, \infty)$.

Proof. — Set $B = B(z, R)$, $W = U \cap B(z, R)$. The upper bound (4.7) follows easily from Lemma 4.8 and the monotonicity inequality $G_W \leq G_B$.

The uniformity of U imply that there is $\epsilon_1 > 0$ such that, for any x, y satisfying $\rho(x, X \setminus U), \rho(y, X \setminus U) > \epsilon R$, there is a path in U from x to y of length less than $C_0 \rho(x, y)$ that stays at distance at least $\epsilon_1 R$ of $X \setminus U$. Since $x, y \in B(z, R/(4C_0))$, this path is of length at most $R/2$ and thus is contained in

$$B(z, 3R/4) \cap \{\zeta \in U : \rho(\zeta, X \setminus U) > \epsilon_1 R\}.$$

Using this path, the Harnack inequality easily reduces the lower bound (4.8) to the case when y satisfies $\rho(x, y) = \eta R$ for some arbitrary fixed $\eta > 0$ small enough. Pick $\eta > 0$ so that, under the conditions of the Lemma, the ball $B_x = B(x, 2\eta R)$ is contained in $U \cap B(z, R)$. Then, the monotonicity property of Green functions implies that $G_W(x, y) \geq G_{B_x}(x, y)$. Lemma 4.8 and the volume doubling property then yield

$$G_W(x, y) \geq \int_{\eta^2 R^2/2}^{8\eta^2 R^2} \frac{ds}{V(x, \sqrt{s})} \geq \frac{cR^2}{V(x, R)}.$$

This is the desired lower bound. □

Proposition 4.10. — *Assume that U is a uniform open set U in (X, ρ). For any fixed $\epsilon \in (0, 1)$ small enough and $A > 1$ large enough there exists a constant $C \in (1, \infty)$ such that for any $z \in X$, $r > 0$,*

$$x \in \{\zeta \in U : \rho(\zeta, X \setminus U) > \epsilon r\} \cap B(z, r),$$

$$y \in \{\zeta \in U : \rho(\zeta, X \setminus U) > \epsilon r\} \cap (B(z, r) \setminus B(x, \epsilon r/2)),$$

and

$$y' \in U \cap B(z, Ar) \setminus B(x, \epsilon r/2)$$

the Green function $G_{U \cap B(z,Ar)}$ satisfies

$$\frac{G_{U \cap B(z,Ar)}(x, y')}{G_{U \cap B(z,Ar)}(x, y)} \leq C.$$

Proof. — The meaning of this estimate is that, in some rough sense, the Green function is non-decreasing as one goes away from the boundary of U. The stated inequality is an easy consequence of the upper and lower bounds stated in the previous Lemma (take $A = 4C_0$ and $R = Ar$). □

4.2.4. The work of Aikawa. — In Section 4.1, we have reduced the proof of Theorem 4.2 to Theorem 4.5. The following two subsections contain the key ingredients of the proof of Theorem 4.5. They follow closely the arguments developed by Hiroaki Aikawa in [**1**] for (locally) uniform Euclidean domain. We would like to emphasize here the importance of Aikawa's work in the development of the results presented in this monograph. Before reading [**1**], we had arrived at the conclusion that it would be very interesting to prove a (scale invariant!) boundary Harnack principle for inner uniform Euclidean domains. We, however, were far from having a proof. As pointed out in the introduction, uniformity (and inner uniformity) generalize very easily outside the setting of Euclidean spaces. We were delighted to discover in [**1**] the results and techniques we needed for our purpose. We encourage the reader to consult the introduction of [**1**] which give an excellent brief account of the history of the boundary Harnack principle.

The original line of reasoning used by Ancona in [**3**] to prove the boundary Harnack principle can also be extended to the case of Euclidean uniform domains. This is explained in detail in [**4**]. As the boundary Harnack principle is very important and useful, it seems worthwhile to develop Ancona's line of reasoning in the more general context considered in this monograph (i.e., for inner uniform domains in Harnack-type Dirichlet spaces). However, we will not discuss this here and leave it for future development.

4.2.5. Green functions and capacity width. — Throughout this section, we continue to work under the hypothesis that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a fixed Harnack-type Dirichlet space with associated intrinsic distance ρ and we assume that $\sup \rho = \infty$, that is, X is not compact.

Recall that the capacity of a compact set K in an open relatively compact set Ω is defined by

$$\mathrm{Cap}_\Omega(K) = \inf\{\mathcal{E}(u,u) : u \in \mathcal{F}^0(\Omega); u \geq 1 \text{ a.e. on K}\}.$$

Alternatively, this quantity can also be obtained as

$$\mathrm{Cap}_\Omega(K) = \sup\{\nu(K) : \nu \text{ a Borel measure supported on } K \text{ with } G_\Omega \nu \leq 1 \text{ on } \Omega\}.$$

The supremum is attained for a (non-negative) measure ν_K supported by K and called the equilibrium measure of K (in Ω). Moreover, the equilibrium potential $G_\Omega \nu_K$ satisfies $G_\Omega \nu_K = 1$ q.e. on K.

It is proved in [**54**] that, under the present hypotheses, there are constants $a, A \in (0,\infty)$ such that, for any $x \in X$ and $0 < r < R < \infty$, we have

$$a \int_r^R \frac{s}{V(x,s)} ds \leq \left(\mathrm{Cap}_{B(x,R)}(B(x,r))\right)^{-1} \leq A \int_r^R \frac{s}{V(x,s)} ds. \tag{4.9}$$

For the following crucial definition and lemmas, we follow [**1**] closely.

Definition 4.11. — For $\eta \in (0,1)$ and any open set $V \subset X$, define the capacity width $w_\eta(V)$ by

$$w_\eta(V) = \inf\left\{ r > 0 : \forall x \in V,\ \frac{\mathrm{Cap}_{B(x,2r)}(\overline{B(x,r)} \setminus V)}{\mathrm{Cap}_{B(x,2r)}(\overline{B(x,r)})} \geq \eta \right\}.$$

Note that $w_\eta(V)$ is a decreasing function of $\eta \in (0,1)$ and an increasing function of the set V.

Lemma 4.12. — *Under the hypotheses of* Theorem 4.5, *there are constants $A, \eta \in (0,\infty)$ depending only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality* (2.7) *on $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ such that*

$$\forall r > 0,\ \ w_\eta(\{x \in U : \rho(x, X \setminus U) < r\}) \leq Ar.$$

Proof. — Let $V_r = \{x \in U : \rho(x, X\setminus U) < r\}$. Using the uniformity and unboundedness of U, there is a constant A_1 such that for any point $y \in V_r$ there is a point $z \in U \cap B(y, A_1 r)$ with the property that $\rho(z, X\setminus U) \geq 2r$. Then $B(z,r) \subset B(y,Ar)$ if $A = (A_1+1)$. The capacity of $B(y,Ar)\setminus V_r$ in $B(y,2Ar)$ is larger than the capacity of $B(z,r)$ in $B(y,2Ar)$ which is larger than the capacity of $B(z,r)$ in $B(z,3Ar)$. By (4.9), this is of order r, proving that $w_\eta(V_r) \leq Ar$ for some $\eta > 0$. □

The following lemma relates harmonic measure and capacity width. We write $f \simeq g$ to signify that f is bounded above and below by different multiples of g where the constants involved depend only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality (2.7) on $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$.

We fix $\eta > 0$ once and for all, small enough that the conclusion of Lemma 4.12 applies and write $w(V) = w_\eta(V)$ for the capacity width of an open set V at the fixed level η. Recall that, for $z \in V$ and $E \subset \partial V$,

$$\omega_V(z,E) = \mathbb{P}_z(X_{\tau_V} \in E)$$

where $\tau_V = \inf\{t > 0 : X_t \in V\}$.

Lemma 4.13. — *Under the hypotheses of* Theorem 4.5, *there is constant* a_1 *depending only on the constant* H_0 *from the Harnack inequality* (2.7) *on* $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ *such that for any non-empty open set* $V \subset X$, $x \in V$ *and* $r > 0$, *we have*

$$\omega_{V\cap B(x,r)}(x, V \cap \partial B(x,r)) \leq \exp\left(2 - a_1 r/w(V)\right).$$

Proof. — For any $\kappa \in (0,1)$, we can pick s with $w(V) \leq s < w(V) + \kappa$ such that

$$\forall y \in V, \quad \frac{\mathrm{Cap}_{B(y,2s)}(\overline{B(y,s)} \setminus V)}{\mathrm{Cap}_{B(y,2s)}(\overline{B(y,s)})} \geq \eta.$$

Fix $y \in V$ and let $E = \overline{B(y,s)} \setminus V$. Let ν_E be the equilibrium measure of E in $\Omega = B(y, 2s)$. We claim that there exists $\epsilon > 0$ depending only on the constant H_0 from the Harnack inequality (2.7) on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ such that

$$G_\Omega \nu_E \geq \epsilon\eta \text{ on } B(y,s). \tag{4.10}$$

Let $F = \overline{B(y,s)}$ and ν_F be the equilibrium measure of F in Ω. Then, by Harnack inequality, for any z with $\rho(y,z) = 3s/2$, we have

$$\forall \zeta \in B(y,s), \quad G_\Omega(z,\zeta) \simeq G_\Omega(z,y).$$

Hence

$$G_\Omega \nu_F(z) = \int_F G_\Omega(z,\zeta)\nu_F(d\zeta) \sim G_\Omega(z,y)\nu_F(F)$$

and

$$G_\Omega \nu_E(z) = \int_E G_\Omega(z,\zeta)\nu_E(d\zeta) \sim G_\Omega(z,y)\nu_E(E).$$

Moreover, since $\nu_F(F) = \mathrm{Cap}_\Omega(F)$, the two-sided inequality (4.9) and Lemma 4.8 yield that $G_\Omega\nu_F(z) \sim 1$. Hence, by the definition of s, for any $z \in \partial B(y, 3s/2)$,

$$G_\Omega\nu_E(z) \simeq \frac{G_\Omega\nu_E(z)}{G_\Omega\nu_F(z)} \simeq \frac{\nu_E(E)}{\nu_F(F)} \simeq \frac{\mathrm{Cap}_\Omega(E)}{\mathrm{Cap}_\Omega(F)} \geq \eta.$$

This proves (4.10).

For simplicity, write $\omega(\cdot) = \omega_{V\cap B(x,r)}(\cdot, V \cap \partial B(x,r))$. To prove Lemma 4.13, we can assume that $r/w(V)$ is greater than 2. Let k be the integer such that $2kw(V) < r \leq 2(k+1)w(V)$ and pick $s > w(V)$ so close to $w(V)$ that $2ks < r$. We claim that

$$\sup_{V\cap B(x,r-2js)} \{\omega\} \leq (1-\epsilon\eta)^j \tag{4.11}$$

for $j = 0, 1, \ldots, k$ with ϵ, η as in (4.10). Note that for $j = k$, (4.11) yields the inequality stated in Lemma 4.13. So, we are left with the task of proving (4.11).

Inequality (4.11) is proved by induction, starting with the trivial case $j = 0$. Assume that (4.11) holds for $j - 1$. By the maximum principle, it suffices to prove

$$\sup_{V\cap \partial B(x,r-2js)} \{\omega\} \leq (1-\epsilon\eta)^j. \tag{4.12}$$

Let $y \in V \cap \partial B(x, r-2js)$. Then $B(y, 2s) \subset B(x, r-2(j-1)s)$ so that the induction hypothesis implies that

$$\omega \leq (1-\epsilon\eta)^{j-1} \text{ on } V \cap B(y, 2s).$$

Since ω vanishes (quasi-everywhere) on $\partial V \cap B(x, r) \supset \partial V \cap B(y, 2s)$, the maximum principle implies that

$$\omega \leq (1-\epsilon\eta)^{j-1}\omega_{V\cap B(y,2s)}(\cdot, V \cap \partial B(y, 2s)) \text{ on } V \cap B(y, 2s).$$

To estimate

$$u = \omega_{V\cap B(y,2s)}(\cdot, V \cap \partial B(y, 2s))$$

on $V \cap B(y, 2s)$, we compare it to

$$v = 1 - G_{B(y,2s)}\nu_E$$

where, as above, $E = \overline{B(y, s)} \setminus V$ and ν_E denotes the equilibrium measure of F in $B(y, 2s)$. Both functions are harmonic in $V \cap B(y, 2s)$ and $u \leq v$ on $\partial(V \cap B(y, 2s))$ q.e (in the limit sense). By (4.10), this implies

$$u = \omega_{V\cap B(y,2s)}(\cdot, V \cap \partial B(y, 2s)) \leq 1 - \epsilon\eta \text{ on } V \cap B(y, 2s).$$

Hence

$$\omega \leq (1-\epsilon\eta)^{j} \text{ on } V \cap B(y, 2s).$$

Since this holds for any $y \in V \cap \partial B(x, r-2js)$, (4.11) is proved. □

The next lemma is the crucial step in the proof of inequality (4.2) which, itself, is the key ingredient for the boundary Harnack principle of Theorem 4.2.

Before we state this crucial lemma, let us observe that if U is a uniform domain in (X, ρ), for any point ξ on ∂U and any $R > 0$ there exists a point ξ_R in U such that

$$\rho(\xi, \xi_R) = 4R \text{ and } \rho(\xi_R, X \setminus U) \geq 4c_0 R$$

where c_0 is the constant appearing in the condition (3.1) expressing the uniformity of U.

Lemma 4.14. — *Under the hypotheses of* Theorem 4.5, *there are constants $A_1, A_2 \in (0, \infty)$ depending only on the constants c_0, C_0 of uniformity of U and on the constant H_0 from the Harnack inequality* (2.7) *on $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ such that, for any $\xi \in \partial U$, $r > 0$, $k > A_3$ and $\zeta \in U \cap B(\xi, r)$, we have*

$$\omega_{U\cap B(\xi,2r)}(\zeta, U \cap \partial B(\xi, 2r)) \leq A_2 \frac{V(\xi, r)}{r^2} G_{U\cap B(\xi,kr)}(\zeta, \xi_r).$$

Here ξ_r is any point in U with $\rho(\xi, \xi_r) = 2r$ and $\rho(\xi_r, X \setminus U) \geq 4c_0 r$.

Proof. — We follow the structure of the proof of [**1**, Lemma 2]. All constants appearing in the proof below depend only on the uniformity constants c_0, C_0 of U and the Harnack constant for X. Choose $A_3 = 20(1 + C_0)$ so that, in particular, all the paths given by the uniform condition (3.1) connecting pairs of points in $U \cap B(\xi, 10r)$ stay in $U \cap B(\xi, A_3 R/2)$. By the set monotonicity of the Green function G_Ω, it suffices to prove the lemma with $k = A_3$. Let $\delta(\xi, r) = \rho(\xi_r, X\backslash U)$ and, for any $\zeta \in U \cap B(\xi, A_3 r)$, set

$$G'(\zeta) = G_{U\cap B(\xi, A_3 r)}(\zeta, \xi_r).$$

As

$$B(\xi_r, \delta(\xi, r)/2) \subset U \cap [B(\xi, 6r) \setminus B(\xi, 2r)] \subset U \cap [B(\xi, A_3 r) \setminus B(\xi, 2r)],$$

the maximum principle yields

$$\forall\, y \in U \cap B(\xi, 2r),\;\; G'(y) \leq \sup_{z:\rho(\xi_r, z) = \delta(\xi, r)/2} \{G'(z)\}.$$

By construction $\xi_r \in B(\xi, A_3 r/C_0)$ and $\partial B(\xi_r, \delta(\xi, r)/2) \subset B(\xi, A_3 r/C_0)$. Applying Lemma 4.9 thus yields that

$$\sup_{z:\rho(\xi_r, z) = \delta(\xi, r)/2} \{G'(z)\} \leq C_1 \frac{r^2}{V(\xi_r, r)}.$$

The volume doubling property implies that $V(\xi_r, r)$ is comparable to $V(\xi, r)$. Hence, the last two displayed inequalities imply that there exists $\epsilon_1 > 0$ such that

$$\forall\, y \in U \cap B(\xi, 2r), \quad \epsilon_1 r^{-2} V(\xi, r) G'(y) \leq e^{-1}.$$

Write

$$U \cap B(\xi, 2r) = \bigcup_{j \geq 0} U_j \cap B(\xi, 2r), \tag{4.13}$$

where

$$U_j = \left\{ x \in U : \exp(-2^{j+1}) \leq \epsilon_1 r^{-2} V(\xi, r) G'(x) < \exp(-2^j) \right\}.$$

Let $V_j = \left(\bigcup_{k \geq j} U_k\right) \cap B(\xi, 2r)$. We claim that

$$w_\eta(V_j) \leq A_4 r \exp\left(-\frac{2^j}{\lambda}\right) \tag{4.14}$$

for some constants $A_4, \lambda \in (0, \infty)$.

Suppose $x \in V_j$. Observe that for $z \in \partial B(\xi_r, \delta(\xi, r)/2)$, by the uniform condition (3.1), the length of the Harnack chain of balls in $U \cap B(\xi, A_3 r) \setminus \{\xi_r\}$ connecting x to z is at most $A_5 \log(1 + A_6 r/\rho(x, X \setminus U))$ for some constants $A_5, A_6 \in (0, \infty)$ and therefore, by Lemma 4.9, there are constants $\epsilon_2, \epsilon_3, \lambda \in (0, \infty)$ such that

$$\begin{aligned} \exp(-2^j) &> \epsilon_1 \frac{V(\xi, r)}{r^2} G'(x) \\ &\geq \epsilon_2 \frac{V(\xi, r)}{r^2} G'(z) \left(\frac{\rho(x, X \setminus U)}{r}\right)^\lambda \\ &\geq \epsilon_3 \left(\frac{\rho(x, X \setminus U)}{r}\right)^\lambda. \end{aligned}$$

Here, to apply Lemma 4.9, we note that both z and ξ_r are in $U \cap B(\xi, \frac{A_3 r}{4C_0})$ and that $\rho(z, \xi_r) \geq 2c_0 r$. The last displayed inequality implies that for any $x \in V_j$ we have

$$\rho(x, X \setminus U) \leq A_5 r \exp\left(\frac{-2^j}{\lambda}\right)$$

This together with Lemma 4.12 yields (4.14).

Let $R_0 = 2r$ and

$$R_j = \left(2 - \frac{6}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2}\right) r$$

for $j \geq 1$. Then $R_j \downarrow r$ and

$$\begin{aligned} (4.15) \qquad & \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{a_1 (R_{j-1} - R_j)}{A_4 r \exp(-2^j/\lambda)}\right) \\ &= \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{6a_1}{A_4 \pi^2} j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) < C < \infty. \end{aligned}$$

Let $\omega_0 = \omega(\cdot, U \cap \partial B(\xi, 2r), U \cap B(\xi, 2r))$ and

$$d_j = \begin{cases} \sup\{\frac{r^2 \omega_0(x)}{V(\xi,r) G'(x)} : x \in U_j \cap B(\xi, R_j)\} & \text{if } U_j \cap B(\xi, R_j) \neq \varnothing, \\ 0, & \text{if } U_j \cap B(\xi, R_j) = \varnothing. \end{cases}$$

Since the sets $U_j \cap B(\xi, 2r)$ cover $U \cap B(\xi, 2r)$ and $B(\xi, r) \subset B(\xi, R_k)$ for each k, to prove Lemma 4.14, it suffices to show that

$$\sup_{j \geq 0} d_j \leq A_2 < \infty$$

where A_2 is as in Lemma 4.14.

We proceed by iteration. Since $\omega_0 \leq 1$, we have by definition of U_0,

$$d_0 = \sup_{U_0 \cap B(\xi, 2r)} \frac{r^2 \omega_0(x)}{V(\xi, r) G'(x)} \leq \epsilon_1 e^2.$$

Let $j > 0$. For $x \in U_{j-1} \cap B(\xi, R_{j-1})$, because of the definition of d_{j-1}, we have

$$\omega_0(x) \leq d_{j-1} \frac{V(\xi, r)}{r^2} G'(x).$$

Also, $\omega_0 \leq 1$. Therefore the maximum principle yields that, for $x \in V_j \cap B(\xi, R_{j-1})$,

$$(4.16) \qquad \omega_0(x) \leq \omega(x, V_j \cap \partial B(\xi, R_{j-1}), V_j \cap B(\xi, R_{j-1})) + d_{j-1} \frac{V(\xi, r)}{r^2} G'(x).$$

If $x \in U \cap B(\xi, R_j)$, then $B(x, R_{j-1} - R_j) \cap \partial B(\xi, R_{j-1}) = \varnothing$, so that the first term on the right hand side is not greater than

$$\begin{aligned}\omega(x, V_j \cap \partial B(x, R_{j-1} - R_j), V_j \cap B(x, R_{j-1} - R_j))\\ &\leq \exp\left(2 - a_1 \frac{R_{j-1} - R_j}{w_\eta(V_j)}\right)\\ &\leq \exp\left(2 - \frac{a_1}{A_4} \exp\left(\frac{2^j}{\lambda}\right) \frac{R_{j-1} - R_j}{r}\right)\\ &= \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right)\end{aligned}$$

by Lemma 4.13 and (4.14). Here $\epsilon_6 = \frac{6a_1}{\pi^2 A_4}$. Moreover, by definition of U_j,

$$\epsilon_1 r^{-2} V(\xi, r) G'(x) \geq \exp(-2^{j+1})$$

for $x \in U_j$. Hence, for $x \in U_j \cap B(\xi, R_j)$, (4.16) becomes

$$\begin{aligned}\omega_0(x) &\leq \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1} \frac{V(\xi, r)}{r^2} G'(x)\\ &\leq \left(\epsilon_1 \exp\left(2 + 2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}\right) \frac{V(\xi, r)}{r^2} G'(x).\end{aligned}$$

Dividing both sides by $\frac{V(\xi,r)}{r^2} G'(x)$ and taking the supremum over $x \in U_j \cap B(\xi, R_j)$, we obtain

$$d_j \leq \epsilon_1 e^2 \exp\left(2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}$$

and hence for every integer $i > 0$

$$d_i \leq \epsilon_1 e^2 \left(1 + \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{6a_1}{\pi^2 A_4} j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right)\right) = \epsilon_1 e^2 (1 + C) < \infty$$

by (4.15). □

Proof of Theorem 4.5. — As above, U is an unbounded uniform domain in (X, ρ). Recall that Theorem 4.5, that we want to prove, asserts that there exist constants $a_0, A_0, A_1' \in (1, \infty)$ with $a_0 < A_0$ such that for, any $\xi \in \partial U$ and any $r > 0$,

$$\frac{G_{U \cap B(\xi, A_0 r)}(x, y)}{G_{U \cap B(\xi, A_0 r)}(x', y)} \leq A_1' \frac{G_{U \cap B(\xi, A_0 r)}(x, y')}{G_{U \cap B(\xi, A_0 r)}(x', y')},$$

for all $x, x' \in U \cap B(\xi, r)$ and $y, y' \in U \cap \partial B(\xi, a_0 r)$.

We set $A_0 = A_3 + 7 = 20(1 + C_0) + 7$. Then, clearly, all paths appearing in the uniform condition for U and joining two points in $B(\xi, 10r)$ stay in $U \cap B(\xi, A_0 r/2)$. Fix $x^* \in U \cap B(\xi, r)$, $y^* \in U \cap \partial B(\xi, 6r)$ such that $c_1 r \leq \rho(x^*, X \setminus U) \leq r$, $c_0(6r) \leq \rho(y^*, X \setminus U) \leq 6r$. It suffices to show that, for all $x \in U \cap B(\xi, r)$ and $y \in U \cap \partial B(\xi, 6r)$, we have

$$G_{U \cap B(\xi, A_0 r)}(x, y) \asymp \frac{G_{U \cap B(\xi, A_0 r)}(x^*, y)}{G_{U \cap B(\xi, A_0 r)}(x^*, y^*)} G_{U \cap B(\xi, A_0 r)}(x, y^*). \tag{4.17}$$

Set $U' = U \cap B(\xi, A_0 r)$. Fix $y \in U \cap \partial B(\xi, 6r)$ and call u (respectively, v) the left-hand (respectively, right-hand) side of (4.17), viewed as a function of x. Then u is positive harmonic in $U' \setminus \{y\}$ whereas v is positive harmonic in $U' \setminus \{y^*\}$. Both functions vanish quasi-everywhere on the boundary of U'.

Since $y^* \in U \cap \partial B(\xi, 6r)$ and $6c_0 r \leq \rho(y^*, X \setminus U) \leq 6r$, it follows that the ball $B(y^*, 3c_0 r)$ is contained in $U \cap (B(\xi, 9r) \setminus B(\xi, 3r))$. Let $z \in \partial B(y^*, c_0 r)$. By a repeated used of Harnack inequality (a finite bounded number of time, depending only on the uniform constants of U), one can compare the value of v at z and at x^* so that, by Lemma 4.9 and the doubling volume property,

$$v(z) \simeq v(x^*) = C_1 G_{U'}(x^*, y) \leq C_1' \frac{r^2}{V(\xi, r^2)}.$$

Now, if $y \in B(y^*, 2c_0 r)$ then, by Lemma 4.9 and the doubling volume property,

$$u(z) = G_{U'}(z, y) \geq \epsilon_1 \frac{r^2}{V(\xi, r^2)}$$

so that we have $u(z) \geq \epsilon_2 v(z)$ in this case. If instead $y \in U \setminus B(\xi, 2c_0 r)$ then we can connect z and x^* by a path of length comparable to r that stay away (at scale r) of both $X \setminus U$ and the point y. Hence, Harnack inequality implies that $u(x^*) \simeq u(z)$. As $u(x^*) = v(x^*)$, this proves that $u(z) \simeq v(z)$ in this case. This shows that we always have

$$u \geq \epsilon_3 v \text{ on } \partial B(y^*, c_0 r).$$

By the maximum principle, this implies that

$$u \geq \epsilon_3 v \text{ on } U' \setminus B(y^*, c_0 r).$$

Since $U \cap B(\xi, r) \subset U' \setminus B(y^*, c_0 r)$, we have proved that $u \geq \epsilon_3 v$ on $U \cap B(\xi, r)$, that is,

$$G_{U'}(x, y) \geq \epsilon_3 \frac{G_{U'}(x^*, y)}{G_{U'}(x^*, y^*)} G_{U'}(x, y^*) \tag{4.18}$$

for all $x \in U \cap B(\xi, r)$ and $y \in U \cap \partial B(\xi, 6r)$. This is one-half of (4.17).

We now focuses on the other half of (4.17), that is,

$$\epsilon_4 G_{U'}(x, y) \leq \frac{G_{U'}(x^*, y)}{G_{U'}(x^*, y^*)} G_{U'}(x, y^*), \tag{4.19}$$

with $x \in U \cap B(\xi, r)$ and $y \in U \cap \partial B(\xi, 6r)$.

For $x \in U \cap B(\xi, 2r)$ and $z \in U \cap B(\xi, 9r) \setminus B(\xi, 3r)$, Lemma 4.9 yields

$$G_{U'}(x, z) \leq C_1 \frac{r^2}{V(x, r)}.$$

Regarding $G_{U'}(x, z)$ as a harmonic function of x, the maximum principle gives

$$G_{U'}(\cdot, z) \leq C_1 \frac{r^2}{V(\xi, r)} \omega(\cdot, U \cap \partial B(\xi, 2r), U \cap B(\xi, 2r)) \text{ on } U \cap B(\xi, 2r).$$

Using Lemma 4.14 and Harnack inequality (to move from ξ_r to y^*), for $x \in U \cap B(\xi, r)$ and $z \in U \cap B(\xi, 9r) \setminus B(\xi, 3r)$, we get

$$G_{U'}(x,z) \le C_2 \frac{r^2}{V(\xi,r)} \frac{V(\xi,r)}{r^2} G_{U'}(x,\xi_r) \le C_3 G_{U'}(x,y^*). \tag{4.20}$$

Fix $x \in U \cap B(\xi, r)$ and $y \in U \cap \partial B(\xi, 6r)$. If $\rho(y, X \setminus U) \ge c_0 r/2$ then $G_{U'}(x,y) \asymp G_{U'}(x,y^*)$ and $G_{U'}(x^*,y) \asymp G_{U'}(x^*,y^*)$ by the Harnack inequality, so that (4.19) follows. Hence we now assume that $y \in U \cap \partial B(\xi, 6r)$ satisfies $\rho(y, X \setminus U) < c_0 r/2$. Let $\xi' \in \partial U$ be a point such that $\rho(y, \xi') < c_0 r/2$. Without loss of generality, we can assume that $c_0 \le 1$. Then it follows that $y \in B(\xi', r)$. Also

$$B(\xi', 2r) \subset B(y, 3r) \subset B(\xi, 9r) \setminus B(\xi, 3r).$$

Hence, (4.20) gives $G_{U'}(x,z) \le C_3 G_{U'}(x,y^*)$ for any $z \in B(\xi', 2r)$. In turn, this implies

$$G_{U'}(x,y) \le C_3 G_{U'}(x,y^*) \omega(y, U \cap \partial B(\xi', 2r), U \cap B(\xi', 2r)). \tag{4.21}$$

Let us apply Lemma 4.14 with ξ replaced by ξ' and observe that $U \cap B(\xi', A_3 r) \subset U \cap B(\xi, A_0 r) = U'$ since we have set $A_0 = A_3 + 7$ and $\rho(\xi, \xi') \le 7r$. This yields

$$\begin{aligned} \omega(y, U \cap \partial B(\xi', 2r), U \cap B(\xi', 2r)) &\le A_2 \frac{V(\xi', r)}{r^2} G_{U \cap B(\xi', A_3 r)}(y, \xi'_r) \\ &\le A_2' \frac{V(\xi, r)}{r^2} G_{U'}(\xi'_r, y) \end{aligned} \tag{4.22}$$

with $\xi_r \in U$ a point such that $\rho(\xi'_r, \xi') = 4r$ and $\rho(\xi'_r, X \setminus U) \ge 4c_0 r$. Note that we have used the doubling volume property as well as the set monotonicity and the symmetry of the Green function. Now, (4.21) and (4.22) give

$$G_{U'}(x,y) \le C_4 \frac{V(\xi,r)}{r^2} G_{U'}(\xi'_r, y) G_{U'}(x, y^*).$$

By construction, $\rho(\xi'_r, y) \ge \rho(\xi'_r, \xi') - \rho(\xi', y) \ge 2r$ and $\rho(x^*, y) \ge \rho(\xi, y) - \rho(\xi, x^*) \ge 5r$. Therefore, by the uniformity of U, there is a chain of balls of radius $\simeq r$ of length uniformly bounded in terms of the constants in the uniform condition for U and all contained in $U' \setminus \{y\}$ that goes from x^* to ξ'_r. Applying the Harnack inequality repeatedly thus yields $G_{U'}(\xi'_r, y) \simeq G_{U'}(x^*, y)$. As Lemma 4.9 gives $G_{U'}(x^*, y^*) \simeq r^2/V(\xi, r)$, the last displayed inequality implies (4.19). This completes the proof of (4.2) and thus of Theorem 4.5. □

4.3. Existence of a harmonic profile

The aim of this section is to prove that certain reasonable unbounded domains U in a non-compact Harnack-type Dirichlet space admit a harmonic profile, that is a function $h = h_U$ with the property that

1. h is a local weak solution of the Laplace equation in U;
2. $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$;
3. $h > 0$ in U.

4.3.1. A profile candidate. — Except in the rare case of domains where an explicit formula can be provided, the profile of a domain U is obtained through a limiting process involving the ratio of harmonic (or approximately harmonic) functions. In the present general context, we consider a fixed point $x_0 \in U$ and a sequence of inner balls $B_i = U \cap B_{\widetilde{U}}(x_0, r_i)$ with $r_i \uparrow \infty$. For each i, let x_i be a point such that $\rho_U(x_i, x_0) = r_i/2$ (the existence of such points follows from the fact that U is an unbounded domain). Thus x_i tends to ∞ (in the obvious sense) with i. Next, consider the restricted Green function $G_{B_i}(x,y)$ in B_i and set

$$h_i(x) = \frac{G_{B_i}(x_i, x)}{G_{B_i}(x_i, x_0)}. \tag{4.23}$$

Defining h_i to be 0 in $U \setminus B_i$, we obtain functions $h_i \geq 0$ that are defined on all of U. By construction, for each i, h_i belongs to $\mathcal{F}^0_{\mathrm{loc}}(B_i, B_i \setminus \{x_i\})$ and is a local weak solution of $Lu = 0$ in $B_i \setminus \{x_i\}$. Also, it is equal to 1 at $x = x_0$. Because we assume that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space, any family $\{u_i\}$ of local weak solutions in a domain V such that u_i is uniformly bounded at one point x_0 is equicontinuous on compact subsets of V. Indeed, the family $\{u_i\}$ is uniformly bounded and uniformly Hölder continuous (of exponent $\alpha > 0$) on any open relatively compact subset of V. See Theorem 2.30.

Applying this to the families $\{h_i : i \geq i_0\}$ on $U \cap B_{\widetilde{U}}(x_0, r_{i_0}/3)$, we see that there exists a subsequence of the sequence (h_i) defined at (4.23) that converges uniformly in any compact subset of U. Replacing the original sequence by such a subsequence, we set

$$\forall\, x \in U, \quad h(x) = \lim_{i \to \infty} h_i(x). \tag{4.24}$$

Let us show that h_i converge to h locally in $\mathcal{F}(U)$ and that h is a local weak solution of the Laplace equation in U. It is clear that h_i converges to h locally in $L^2(U)$. By the form of Leibniz rule that holds for the energy measure Γ, for any function $\phi \in \mathcal{C}_c(U) \cap \mathcal{D}(\mathcal{E})$ and i, j large enough, we have

$$\mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) = \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) + \mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)).$$

The last term on the right-hand side is 0 because $\phi^2(h_i - h_j)$ is in $\mathcal{C}_c(U) \cap \mathcal{D}(\mathcal{E})$ and $h_i - h_j$ is a local weak solution of $Lu = 0$ in an open subset of U containing the compact support of ϕ. Since h_i converges to h locally uniformly in U, this proves convergence locally in $\mathcal{F}(U)$. It easily follows that h is a local weak solution of the Laplace equation since, for any $\phi \in \mathcal{C}_c(U) \cap \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}(h, \phi) = \mathcal{E}(\lim h_i, \phi) = \lim \mathcal{E}(h_i, \phi) = 0.$$

By the Harnack inequality, it follows that $h > 0$ in U.

The one crucial property of the profile h missing at this point and the one property that requires further hypotheses on the domain U is the fact that, to be a harmonic profile, the function h at (4.24) must be in $\mathcal{F}^0_{\mathrm{loc}}(U)$, that is, must vanish along the boundary of U, in the appropriate sense.

4.3.2. For uniform domains, the candidate h is indeed a profile. — We will make use of the following definition.

Definition 4.15. — Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type strictly local regular Dirichlet space. Let U be a domain in X. We say that a local boundary Harnack principle holds in U if for any open set Ω and any compact $K \subset \Omega$ there exists a constant $C(\Omega, K) \in (0, \infty)$ such that for any two non-negative continuous functions u, v in $\mathcal{F}^0_{\mathrm{loc}}(U, U \cap \Omega)$ that are weak local solutions of the Laplace equation in $U \cap \Omega$ we have

$$\forall\, x, y \in U \cap K, \quad \frac{u(x)}{u(y)} \leq C(\Omega, K)\frac{v(x)}{v(y)}.$$

Note that this is a very weak form of the Harnack boundary principle in that it does not provide control on $C(\Omega, K)$ in terms of the relative geometry of Ω and K. Theorem 4.2 states that, in a Harnack-type Dirichlet space, uniform domains and even inner uniform domains (under the additional hypothesis that a carré du champ exists) satisfy a much more precise boundary Harnack principle.

Theorem 4.16. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space. Let U be an unbounded domain in (X, ρ) such that the local Harnack boundary principle holds in U. Then any function h obtained as in* (4.24) *is in $\mathcal{F}^0_{\mathrm{loc}}(U)$ and thus is a profile for U.*

Proof. — The first step is to show that the sequence h_i used in (4.24) to define h is in fact Cauchy in $L^2_{\mathrm{loc}}(\widetilde{U}, d\mu)$. Let K be a compact set in X and $V = K \cap U$. It suffices to show that (h_i) is Cauchy in $L^2(V, \mu|_V)$. Let Ω be a relatively compact open set in X containing K, $B = B(x_0, R)$ be a large ball in X such that $\Omega \subset B(\xi_0, R/2)$ and ξ_0' be a point in $[B(\xi_0, 2R) \setminus \Omega] \cap U$. Let $g(x) = G_{U\cap B}(\xi_0, x)$ be the corresponding Green function. Note that g is continuous positive in $\Omega \cap U$, belongs to $\mathcal{F}^0_{\mathrm{loc}}(U, U \cap \Omega)$ and is a local weak solution of the Laplace equation in $\Omega \cap U$. Applying the Harnack boundary principle to Ω, K and any of the pairs h_i, g with i large enough yields

$$\sup_{U\cap K} \{h_i/g\} \leq C(\Omega, K).$$

For any $\eta \in (0, 1)$ small enough, let $V_\eta = \{x \in V : \rho(x, X \setminus U) \geq \eta\} \subset U$ and note that V_η is a compact subset of U. For i, j large enough, we have

$$\int_V |h_i - h_j|^2 d\mu \leq \int_{V_\eta} |h_i - h_j|^2 d\mu + 2C(\Omega, K) \int_{V\setminus V_\eta} g^2 d\mu.$$

As $\int_{V\setminus V_\eta} g^2 d\mu$ tends to 0 with η and (h_i) converges to h in $L^2_{\mathrm{loc}}(U, \mu)$, this indeed shows that (h_i) is Cauchy $L^2(V, \mu)$.

The second and last step is to show that for any open subset V of U which is relatively compact in X, (h_i) is a Cauchy sequence in $\mathcal{F}(V)$. Since, for i large enough, $h_i \in \mathcal{F}^0(U, V)$, this implies that $h \in \mathcal{F}^0(U, V)$ and thus $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$ as desired. To this end, let $\phi(x) = (1 - \rho(x, V))_+ = \max\{1 - \rho(x, V), 0\}$. By Theorem 2.11, this function is in $\mathcal{F}_c(X)$ with $d\Gamma(\phi, \phi) \leq d\mu$ a.e. and $d\Gamma(\phi, \phi)|_V = 0$. By construction, $h_i \in \mathcal{F}^0_{\mathrm{loc}}(B_i, B_i \setminus \{x_i\})$ where $B_i = U \cap B_{\widetilde{U}}(x_0, r_i)$ and $x_i \in U$ is such that $\rho_U(x_i, x_0) =$

$r_i/2$. Hence, for i large enough, $\phi h_i \in \mathcal{D}(\mathcal{E}^D_{B_i}) \subset \mathcal{F}^0(U)$. To show that (h_i) is Cauchy in $\mathcal{F}(V)$, it suffices to show that ϕh_i is Cauchy in $\mathcal{F}^0(U)$. We have

$$\mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) = \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) + \mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)).$$

We claim that the last term on the right-hand side is 0. Indeed, $\phi^2(h_i - h_j)$ is in $\mathcal{F}^0(U)$ and can be approximated by functions $\psi_n \in \mathcal{F}_c(U)$ with compact supports all contained in an open set $\Omega \subset U$ with $\Omega \subset B_i \setminus \{x_i\}$ for all i large enough. As $h_i - h_j$ is a local weak solution of $Lu = 0$ in Ω,

$$\mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)) = \mathcal{E}(h_i - h_j, \lim_{n\to\infty} \psi_n) = \lim_{n\to\infty} \mathcal{E}(h_i - h_j, \psi_n) = 0.$$

Hence, setting $K = \{x \in \widetilde{U} : \rho(x, V) \le 1$,

$$\begin{aligned}\int_V d\Gamma(h_i - h_j, h_i - h_j) &\le \mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) \\ &= \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) \le \int_K (h_i - h_j)^2 d\mu.\end{aligned}$$

Since (h_i) is Cauchy in $L^2_{\mathrm{loc}}(\widetilde{U})$, this shows that it is also Cauchy in $\mathcal{F}(V)$. □

4.3.3. The doubling property of the profile. — By Theorem 4.2, if U is an unbounded domain in X and U is uniform in (X, ρ), then U satisfies a very strong form of the boundary Harnack principle. In particular, the function h defined at (4.24) is a profile for U. The following theorem states this fact together with the key additional information that the measure $h^2 d\mu$ is doubling on $(\widetilde{U}, \rho_U)$. In the case of uniform domains, it proves all the statement of Theorem 4.1 except for the uniqueness of the profile, up to a multiplicative factor. Uniqueness can be deduced from the global boundary Harnack principle by a classical argument (see, e.g., [**3**]).

Theorem 4.17. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Let U be unbounded uniform domain in (X, ρ). Then the function h defined at* (4.24) *is a profile for U, that is, h is a continuous positive function in U which belongs to $\mathcal{F}^0_{\mathrm{loc}}(U)$ and is a local weak solution of the Laplace equation in U. Moreover, the measure $d\nu = h^2 d\mu$ on $(\widetilde{U}, \rho_U)$ satisfies the volume doubling property, that is, there is a constant $D \in (0, \infty)$ such that*

$$\forall\, x \in \widetilde{U},\ \ r > 0,\ \ V_{h^2}(x, 2r) \le D V_{h^2}(x, r).$$

In fact,

$$\forall\, x \in \widetilde{U},\ \ r > 0,\ \ V_{h^2}(x, r) = \int_{B_{\widetilde{U}}(x,r)} h^2 d\mu \simeq h(x_r)^2 V(x, r) \tag{4.25}$$

where x_r is any point in U such that $\rho_U(x, x_r) \le r/4$ and $\rho(x_r, X \setminus U) \ge c_0 r/8$.

Proof. — By construction, we know that h is a local weak solution of the Laplace equation in U. Fix $x \in \overline{U}$ and $r > 0$. Let $x_r \in U$ be a point with

$$\rho_U(x_r, x) = \frac{r}{4} \text{ and } \rho(x_r, X \setminus U) \geq \frac{c_0 r}{8}.$$

Such a point always exists by Lemma 3.20. Furthermore, if x'_r is any point satisfying

$$\rho_U(x_r, x) \leq 10r \text{ and } \rho(x_r, X \setminus U) \geq \frac{r}{10}$$

then the basic Harnack inequality shows that $h(x'_r) \simeq h(x_r)$.

Note that it suffices to prove (4.25). Indeed, the volume doubling condition for V_{h^2} then follows from the doubling condition (2.2) for the measure μ and the fact that $h(x_r) \simeq h(x_{2r})$ which is, again, an easy consequence of the basic Harnack inequality used repeatedly along the curve γ between x_R and x_{2R} given by the uniform condition (3.1).

If $\rho(x, X \setminus U) > 2r$ then (4.25) follows immediately from the Harnack inequality for the function h, the doubling condition (2.2) for the measure μ and the uniform condition (3.1). In fact, it is an easy consequence of the Harnack inequality and the doubling condition in (X, ρ, μ) that there exists a constant $\epsilon_1 > 0$ such that

$$\int_{B_{\widetilde{U}}(x,r)} h^2 d\mu \geq \epsilon_1 h(x_r)^2 V(x, r). \tag{4.26}$$

Indeed, $B_{\widetilde{U}}(x, r)$ contains $B(x_r, c_0 r/8)$.

The matching upper bound

$$\int_{B_{\widetilde{U}}(x,r)} h^2 d\mu \leq C_1 h(x_r)^2 V(x, r). \tag{4.27}$$

is the more interesting result which immediately follows from the following claim. There exist a constant C such that

$$\forall\, x \in \widetilde{U},\ r > 0,\ \ y \in B_{\widetilde{U}}(x, r), \quad h(y) \leq C h(x_r). \tag{4.28}$$

To prove this claim, note that $B_{\widetilde{U}}(x, r) \subset B(x, r) \subset B(x, 4r)$. Fix a point $\xi'_r \in U$ such that $\rho(x, \xi'_r) = 2r$ and $\rho(\xi'_r, X \setminus U) \geq c_1 r$. Then, for any point $y \in U \cap B(x, r)$, $\rho(\xi'_r, y) \geq r$. Pick A to be a constant larger than required by Proposition 4.10 and larger than constant A_0 in the boundary Harnack principle of Theorem 4.2. Consider the Green function $G_{U \cap B(x,2Ar)}$. By the Boundary Harnack principle of Theorem 4.2, we have

$$\frac{h(y)}{h(x_r)} \leq C \frac{G_{U \cap B(x,2Ar)}(\xi'_r, y)}{G_{U \cap B(x,2Ar)}(\xi'_r, x_r)}$$

for all y in $U \cap B(x, r)$. By Proposition 4.10, we have

$$\frac{G_{U \cap B(x,2Ar)}(\xi'_r, y)}{G_{U \cap B(x,2Ar)}(\xi'_r, x_r)} \leq C$$

for all y in $U \cap B(x, r)$. The desired conclusion (4.28) thus follows. □

4.4. From uniform domains to inner uniform domains

It is quite possible that the arguments given above for uniform domains extend to inner uniform domains (with adequate changes of notation). Note however that, in the case of inner uniform domains, one must absolutely replace the sets $U \cap B(x,r)$ (trace of $B(x,r)$ on U) by the inner balls $U \cap B_{\widetilde{U}}(x,r)$ which may be very different indeed.

Instead of studying this possibility, we will now explain how the results for inner uniform domains can be obtained as a corollary of those for uniform domains. The key for this is Theorem 3.10 which asserts that if $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and U is an inner uniform domain in (X, ρ) then the space $(\widetilde{U}, \mu, \mathcal{E}_U^N, \mathcal{F}(U))$ is also a Harnack-type Dirichlet space (with ρ_U as intrinsic distance).

The next key observation is rather trivial in nature. Namely, if we now look at U as an open connected set in $(\widetilde{U}, \rho_U)$ instead of in (X, ρ) then the hypothesis that U is *inner uniform* in (X, ρ) translates immediately into the property that U is *uniform* in $(\widetilde{U}, \rho_U)$. Hence, we can apply to U the results obtained for unbounded uniform sets in Harnack-type Dirichlet spaces. The price we have to pay for this trick to work is the additional hypothesis that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ. Indeed, we need this hypothesis in order to apply Theorem 3.10.

We now state explicitly the two important results that follow from this line of reasoning. The first is the boundary Harnack principle. The second concerns the profile.

Theorem 4.18. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Assume that it admits a carré du champ. Let U be unbounded inner uniform domain in (X, ρ). Then there exist constants $A_0, A_1 \in (1, \infty)$ such that for any $\xi \in \widetilde{U} \setminus U$, any $r > 0$, and any positive local weak solutions u and v of $Lu = 0$ in $U \cap B_{\widetilde{U}}(\xi, A_0 r)$ with weak Dirichlet boundary condition on along the boundary of U, we have*

$$\forall\, x, x' \in U \cap B_{\widetilde{U}}(\xi, r), \quad \frac{u(x)}{u(x')} \le A_1 \frac{v(x)}{v(x')}.$$

Moreover, the constants A_0, A_1 depend only on the constants c_0, C_0 of inner uniformity of U and on the Harnack constant H_0 of $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Theorem 4.19. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Harnack-type Dirichlet space with associated intrinsic distance ρ. Assume that it admits a carré du champ. Let U be an unbounded inner uniform domain in (X, ρ). Then the function h defined at (4.24) is a profile for U, that is, h is a continuous positive function in U which belongs to $\mathcal{F}^0_{\mathrm{loc}}(U)$ and is a local weak solution of the Laplace equation in U. The measure $d\nu = h^2 d\mu$ on $(\widetilde{U}, \rho_U)$ satisfies the volume doubling property, that is, there is a constant $D \in (0, \infty)$ such that*

$$\forall\, x \in \widetilde{U}, \;\; r > 0, \;\; V_{h^2}(x, 2r) \le D V_{h^2}(x, r).$$

Moreover

$$\forall x \in \widetilde{U},\ \ r > 0,\ \ V_{h^2}(x,r) = \int_{B_{\widetilde{U}}(x,r)} h^2 d\mu \simeq h(x_r)^2 V(x,r) \tag{4.29}$$

where x_r is any point in U such that $\rho_U(x, x_r) \leq r/4$ and $\rho(x_r, X \setminus U) \geq c_0 r/8$.

Remark 4.20. — The estimates (4.29) is important in practice to control $V_{h^2}(x,r)$. Note that, for any fixed constant C, there exist constants $0 < a < A < \infty$ such that

$$ah(x'_r)^2 V(x,r) \leq V_{h^2}(x,r) \leq Ah(x'_r)^2 V(x,r)$$

for all $x, x'_r \in \widetilde{U}$ and $r > 0$ such that $\rho_U(x, x'_r) \leq Cr$ and $\rho(x'_r, X \setminus U) \geq C^{-1}r$. In other words, the precise choice of x_r in (4.29) is irrelevant as long as x_r is at distance at most Cr from x in U and at distance at least r/C from the boundary.

CHAPTER 5

THE DIRICHLET HEAT KERNEL IN INNER UNIFORM DOMAINS

As mentioned in the introduction, one of the main goals of this monograph is to present sharp two-sided Gaussian type estimates for the Dirichlet heat kernel in an inner uniform domain in a Harnack-type Dirichlet space. This chapter is devoted to this main goal. Even in the simplest and most studied case of Euclidean domains above the graph of a Lipschitz function, there is some novelty to the results we are going to describe.

5.1. The h-transform technique

5.1.1. Dirichlet-type Dirichlet forms. — Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space satisfying the proprieties (A1)-(A2) of Section 2.2.3. This section introduced a basic variant of Definition 2.34 and of the construction considered in Definition 3.26. Given a continuous function v on a domain $U \subset X$ (a weight), consider the bilinear form defined on $\mathcal{F}_c(U) \subset L^2(U, vd\mu)$ by $(f,g) \mapsto \int_U vd\Gamma(f,g)$. This form is well defined because v is continuous on U and $d\Gamma(f,g)$ is a signed Radon measure with compact support and finite total mass in U. We claim that this form is closable. One way to see that is to observe that the obvious extension of this form with domain (see also 3.23)

$$\mathcal{F}^v(U) = \left\{ f \in L^2(U, vd\mu) \cap \mathcal{F}_{\mathrm{loc}}(U) : \int_U vd\Gamma(f,f) < \infty \right\}$$

is closed. This follows from the argument used to prove Proposition 2.50.

Definition 5.1. — Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space satisfying the properties (A1)-(A2) of Section 2.2.3. Let U be a domain in X. Let v be a continuous function defined on U. Define

$$(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$$

to be the closure of the symmetric closable bilinear form

$$(f,g) \mapsto \int_U vd\Gamma(f,g), \quad f,g \in \mathcal{F}_c(U) \subset L^2(U, vd\mu).$$

The next proposition gathers properties of the form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ which easily follow by inspection.

Proposition 5.2. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space satisfying the properties* (A1)-(A2) *of* Section 2.2.3. *Let U be a domain in X. Let v be a continuous function defined on U. The form*

$$(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$$

is a strictly local regular Dirichlet form on $L^2(U, v d\mu)$ with energy measure

$$d\Gamma^v(f,g) = v d\Gamma(f,g), \quad f,g \in \mathcal{F}_{\mathrm{loc}}(U).$$

Remark 5.3. — In (3.23), we introduced the related Neumann-type Dirichlet form under slightly different circumstances. Namely, in (3.23), we assumed the existence of a carré du champ but only required that v be measurable, locally bounded and locally bounded away from 0 in U. Indeed, the Dirichlet form $\mathcal{E}_U^{D,v}$ can be defined without difficulty for such more general functions v if one assumes the existence of a carré du champ. In these circumstances, the form $\mathcal{E}_U^{D,v}$ itself admits a carré du champ given for $f, g \in \mathcal{F}_{\mathrm{loc}}(U)$ by

$$\Upsilon^v(f,g) = \frac{d\Gamma^v(f,g)}{v d\mu} = \frac{v d\Gamma(f,g)}{v d\mu} = \frac{d\Gamma(f,g)}{d\mu} = \Upsilon(f,g).$$

Remark 5.4. — Note the similarities and important differences between the above easy and very general proposition concerning the Dirichlet-type Dirichlet form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ and the rather difficult Theorem 3.30 which asserts the regularity of the Neumann-type Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ on $\widetilde{U}$ when the underlying Dirichlet space is Harnack-type and admits a carré du champ, and U is inner uniform.

5.1.2. The h-transform technique. — This subsection describes the basic ingredients of the well known technique of h-transform originally due to Doob and also known under the name of Doob's transform or Doob's h-transform. This technique is a key ingredient for our main results and we describe it in details in a context suitable for our purpose. We start with the following simple definition.

Definition 5.5. — Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(X, \mu)$ with associated semigroup $(P_t)_{t>0}$ and infinitesimal generator $(L, \mathcal{D}(L))$. Let h be a measurable positive function on X. Let H denote multiplication by h viewed as a unitary operator

$$H : L^2(U, h^2 d\mu) \to L^2(U, d\mu), \quad f \mapsto hf.$$

Define $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$, L_h and $P_{h,t}$ to be, respectively, the pulled back closed bilinear form, operator and semigroup on $L^2(X, h^2 d\mu)$ defined by

$$\mathcal{E}_h(f,g) = \mathcal{E}(hf, hg), \quad \mathcal{D}(\mathcal{E}_h) = H^{-1}\mathcal{D}(\mathcal{E}); \tag{5.1}$$

$$L_h = H^{-1} \circ L \circ H, \quad \mathcal{D}(L_h) = H^{-1}\mathcal{D}(L); \tag{5.2}$$

$$P_{h,t} = H^{-1} \circ P_t \circ H. \tag{5.3}$$

From this definition it immediately follows that $\mathcal{E}_h$ is a densely defined closed symmetric bilinear form on $L^2(X, h^2 d\mu)$ associated with the self-adjoint semigroup of contractions $P_{h,t}$ on $L^2(X, h^2 d\mu)$ which admits L_h as its (self-adjoint) infinitesimal generator. However, it must be observed that this form is not, in general, a Dirichlet form because it is not Markovian (i.e., normal contractions do not operate on $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$). This form is Markovian if and only if the positivity preserving semigroup $P_{h,t}$ is Markovian, i.e., if and only if $P_{h,t}\mathbf{1} \le 1$ a.e. for all $t > 0$. This happens if and only if the original semigroup P_t (extended to act on all functions f taking values in $[0, \infty]$) and the function h satisfy $P_t h \le h$, $t > 0$.

Lemma 5.6. — *Referring to* Definition 5.5, *if the semigroup $(P_t)_{t>0}$ admits a kernel $p(t,x,y)$, $(t,x,y) \in (0,\infty)\times X\times X$, with respect to the measure $d\mu$ then the semigroup $(P_{h,t})_{t>0}$ admits a kernel $p_h(t,x,y)$, $(t,x,y) \in (0,\infty) \times X \times X$, with respect to the measure $h^2 d\mu$ and these two kernels are related by*

$$p(t,x,y) = p_h(t,x,y)h(x)h(y), \quad (t,x,y) \in (0,\infty) \times X \times X.$$

Proof. — By definition, for $f \in L^2(X, h^2 d\mu)$, we have

$$\begin{aligned} P_{h,t}f(x) &= \frac{1}{h(x)} P_t[hf](x) \\ &= \frac{1}{h(x)} \int_X p(t,x,y)h(y)f(y)d\mu(y) \\ &= \int_X \frac{p(t,x,y)}{h(x)h(y)} f(y)h^2(y)d\mu(y). \end{aligned}$$

Hence the semigroup $P_{h,t}$ admits the kernel

$$p_h(t,x,y) = \frac{p(t,x,y)}{h(x)h(y)}$$

with respect to the measure $h^2 d\mu$. □

After these generalities, we now change our viewpoint and notation a bit and consider a domain U in an underlying strictly local regular Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfying the conditions (A1)-(A2) of Section 2.2.3. On U, we consider the strictly local regular Dirichlet form of Dirichlet type $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ introduced in Definition 2.34. Now, given a continuous positive function h on U, we constructed above two very different closed bilinear forms on $L^2(U, h^2 d\mu)$, namely:

— The form $\left(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2})\right)$ obtained by setting $v = h^2$ in Definition 5.1.

— The form $\left(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D)\right)$ with $\mathcal{D}(\mathcal{E}_{U,h}^D) = H^{-1}\mathcal{D}(\mathcal{E}_U^D)$ obtained by h-transform through setting $(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ in Definition 5.5.

Under some special circumstances, these two forms are actually equal. The following proposition spells out cases where that happens. This type of result is not new and can be found in various form in the literature. For instance, [**29**, Theorems 2.6 and 2.8] and [**45**] gives results in this direction. Note however that the results of [**29**] do not directly apply to our situation because the function h is only locally in the domain of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Since the following proposition is essential to us, we present a proof in the spirit of the present work.

Proposition 5.7. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space satisfying the conditions* (A1)-(A2) *of* Section 2.2.3. *Let U be a domain in X. Let h be a continuous positive function on U. Referring to the notation introduced above, we have:*

— *Assume that $h \in \mathcal{F}_{\mathrm{loc}}(U)$. Then the set $H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U, \mu))$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}^D_{U,h}) = H^{-1}\mathcal{D}(\mathcal{E}^D_U)$ and, in fact,*

$$H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U, \mu)) = \mathcal{F}_c(U) \cap L^\infty(U, \mu) = \mathcal{F}_c(U) \cap L^\infty(U, h^2 d\mu).$$

— *Assume that $h \in \mathcal{F}_{\mathrm{loc}}(U)$ and is a weak local solution of $Lu = 0$ in U. Then the forms $\left(\mathcal{E}^D_{U,h}, \mathcal{D}(\mathcal{E}^D_{U,h})\right)$ and $\left(\mathcal{E}^{D,h^2}_U, \mathcal{D}(\mathcal{E}^{D,h^2}_U)\right)$ coincide.*

Proof. — The set $\mathcal{F}_c(U) \cap L^\infty(U, \mu)$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$. Since, by definition, H is a unitary operator between the Hilbert spaces $\mathcal{D}(\mathcal{E}^D_U)$ and $\mathcal{D}(\mathcal{E}^D_{U,h})$, it follows that $H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U, \mu))$ is also dense in the Hilbert space $\mathcal{D}(\mathcal{E}^D_{U,h})$. Since $h, 1/h$ are both in $\mathcal{F}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U, \mu)$ the equality

$$H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U, \mu)) = \mathcal{F}_c(U) \cap L^\infty(U, \mu)$$

follows from the fact that $\mathcal{F}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U, \mu)$ is an algebra.

To prove the second statement, we observe that, according to what we just proved above, it suffices to compare the two forms on the common dense subset $\mathcal{F}_c(U) \cap L^\infty(U, \mu)$ of their respective domains. As $h \in \mathcal{F}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U, \mu)$, for any $g \in \mathcal{F}_c(U) \cap L^\infty(U, \mu)$, the functions g, g^2, gh, g^2h are all in $\mathcal{F}_c(U)$. Using the properties of the energy form of a strictly local Dirichlet form, i.e., the product rule and the chain rule and [**47**, Lemma 3.2.5], we have

$$\begin{aligned}
\mathcal{E}^D_{U,h}(g,g) &= \int_U d\Gamma(hg, hg) \\
&= \int g^2 d\Gamma(h,h) + 2\int gh d\Gamma(g,h) + \int h^2 d\Gamma(g,g) \\
&= \int d\Gamma(h, g^2h) + \int h^2 d\Gamma(g,g) \\
&= \int h^2 d\Gamma(g,g) = \mathcal{E}^{D,h^2}_U(g,g). \qquad (5.4)
\end{aligned}$$

To obtain the last line, we have used the fact that $\int d\Gamma(h, g^2h) = 0$ since h is a local weak solution and $g^2h \in \mathcal{F}_c(U)$. □

5.2. The h^2-weighted Dirichlet form

In the previous section, under very general circumstances, we discussed the two closed bilinear forms

$$\mathscr{E}_U^{D,h^2} \quad \text{and} \quad \mathscr{E}_{U,h}^D$$

associated with a domain U and a continuous positive function h on U. When h is a local weak solution of the Laplace equation in U, we proved that these two forms are equals. Note that, by definition, $\mathscr{E}_U^{D,h^2}$ is a strictly local regular Dirichlet form on U.

5.2.1. Regularity on $\widetilde{U}$. — Our next goal is to show that, under appropriate hypotheses, if h is a local weak solution of the Laplace equation in U and belongs to $\mathscr{F}^0_{\mathrm{loc}}(U)$ then the form $\mathscr{E}_U^{D,h^2} = \mathscr{E}_{U,h}^D$ is actually a strictly local regular Dirichlet form on $\widetilde{U}$. This is a bit surprising and is a key observation to develop our approach. Indeed, our aim is to show further that the Dirichlet space

$$\left(\widetilde{U}, h^2 d\mu, \mathscr{E}_U^{D,h^2}, \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)\right)$$

is a Harnack-type Dirichlet space. Once this is done, the desired results concerning the Dirichlet heat semigroup on U will easily follow.

As $\mathscr{E}_U^{D,h^2}$ is regular on U, the property that $\mathscr{E}_U^{D,h^2}$ is regular on $\widetilde{U}$ amounts to the fact that $\mathscr{C}_c(\widetilde{U}) \cap \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)$ is dense in $\mathscr{C}_c(\widetilde{U})$ for the sup norm. This property is perhaps more subtle than it first appears. The following proposition requires some hypotheses on the underlying space but nothing on the domain U except the local compactness of $\widetilde{U}$ and the existence of a harmonic profile, namely, h.

Proposition 5.8. — *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet space satisfying the conditions* (A1)-(A2) *of* Section 2.2.3 *and which admits a carré du champ operator. Let U be an unbounded domain in X such that $\widetilde{U}$ is locally compact. Let h be a continuous positive function on U which is a local weak solution of the Laplace equation in U and belongs to $\mathscr{F}^0_{\mathrm{loc}}(U)$ (i.e., a harmonic profile for U). Then the Dirichlet-type Dirichlet form*

$$\left(\mathscr{E}_U^{D,h^2}, \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)\right)$$

is regular on $(\widetilde{U}, h^2 d\mu)$ with core $\mathrm{Lip}_c(\widetilde{U})$.

Proof. — As explained above, to prove the desired regularity, it suffices to show that $\mathscr{C}_c(\widetilde{U}) \cap \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)$ is dense in $\mathscr{C}_c(\widetilde{U})$ for the sup norm. As $\mathrm{Lip}_c(\widetilde{U})$ is dense in $\mathscr{C}_c(\widetilde{U})$ in sup norm, it suffices to show that $\mathrm{Lip}_c(\widetilde{U}) \subset \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)$. Consider a function f in $\mathrm{Lip}_c(\widetilde{U})$. To show that $f \in \mathscr{D}\left(\mathscr{E}_U^{D,h^2}\right)$, we need to show that fh is in $\mathscr{F}^0(U)$. Now, as we assume the existence of a carré du champ, any function f in $\mathrm{Lip}_c(\widetilde{U})$ with Lipschitz constant k is in $\mathcal{C}_c(\widetilde{U}) \cap \mathcal{F}(U)$ (see Corollary 2.22) and satisfies

$$\Upsilon(f,f) = \frac{d\Gamma(f,f)}{d\mu} \leq k \text{ a.e.}$$

By Lemma 2.46, it follows that $fh \in \mathcal{F}^0(U)$ as desired. □

5.2.2. The h-transform on inner uniform domains. — Definition 3.26 provides us with an extension

$$\left(\mathcal{E}_U^{N,h^2}, \mathcal{D}\left(\mathcal{E}_U^{N,h^2}\right)\right) \text{ of } \left(\mathcal{E}_U^{D,h^2}, \mathcal{D}\left(\mathcal{E}_U^{D,h^2}\right)\right).$$

On the one hand, Proposition 5.8 asserts that $\mathrm{Lip}_c(\widetilde{U})$ is a core in the latter form (Dirichlet-type) under mild assumptions on the underlying Dirichlet space and on the domain U. On the other hand, Theorem 3.30, together with Theorem 4.1, implies that $\mathrm{Lip}_c(\widetilde{U})$ is a core in the former form (Neumann-type) under the strong assumptions that the underlying space is Harnack-type and admits a carré du champ, that the domain U is inner uniform, and that h is the profile provided by Theorem 4.1. Under these circumstances, it follows that, somewhat surprisingly, the Dirichlet-type and Neumann-type Dirichlet forms above coincide. This, of course, is a very special property of the weight $v = h^2$ when h is a profile for U. The next theorem captures these considerations and some of their far reaching consequences. As was the case for Proposition 5.7, this is not new and there are many similar developments in the literature. See, e.g., [**29**, **45**]. Here it is important to observe that the results in [**29**] do not strictly speaking apply to our setting because the harmonic profile h of an unbounded inner uniform domain U is only locally in the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. In particular, the apparant contradiction between the assertion regarding transience in Theorem 5.9 below and the statement regarding recurrence in [**29**, Theorem 2.6(ii)] is not a contradiction at all.

Theorem 5.9. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space which admits a carré du champ. Let U be an unbounded inner uniform domain in (X, ρ). Let h be the profile for U provided by* Theorem 4.1. *Then the Dirichlet forms*

$$\left(\mathcal{E}_U^{N,h^2}, \mathcal{D}\left(\mathcal{E}_U^{N,h^2}\right)\right) \text{ and } \left(\mathcal{E}_U^{D,h^2}, \mathcal{D}\left(\mathcal{E}_U^{D,h^2}\right)\right)$$

coincide and are regular on $\widetilde{U}$ with core $\mathrm{Lip}_c(\widetilde{U})$. Moreover, these forms also coincide with $\left(\mathcal{E}_{U,h}^D, \mathcal{D}\left(\mathcal{E}_{U,h}^D\right)\right)$ and the Dirichlet space

$$\left(\widetilde{U}, h^2 d\mu, \mathcal{E}_U^{D,h^2}, \mathcal{D}\left(\mathcal{E}_U^{D,h^2}\right)\right) = \left(\widetilde{U}, h^2 d\mu, \mathcal{E}_{U,h}^D, \mathcal{D}\left(\mathcal{E}_{U,h}^D\right)\right)$$

is a Harnack-type Dirichlet space. If h is not constant (i.e., the boundary of U has positive capacity), this Dirichlet space is transient (i.e., non-parabolic).

Proof. — Apply Proposition 5.8, Theorem 3.30 and Theorem 4.1 to show that the first two forms are regular with core $\mathrm{Lip}(\widetilde{U})$, hence coincide. Apply Proposition 5.7 to see that they also coincide with the form obtained by h transform. Then, apply

Theorem 4.1 and Theorem 3.34 to see that $\left(\widetilde{U}, \mu, \mathcal{E}_U^{D,h^2}, \mathcal{D}\left(\mathcal{E}_U^{D,h^2}\right)\right)$ is of Harnack-type. To see that this space is transient it suffices to observe that $1/h$ is a non-constant positive local weak solution of L_h in U. Indeed, it is well-known that the existence of such a non-trivial positive harmonic function implies transience. □

Corollary 5.10. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack-type Dirichlet space which admits a carré du champ. Let U be an unbounded inner uniform domain in (X, ρ). Let h be the profile for U provided by* Theorem 4.1. *Let $P_{U,h,t}^D$, $t > 0$, be the semigroup associated with the Harnack-type Dirichlet form*

$$\left(\mathcal{E}_{U,h}^D, \mathcal{D}\left(\mathcal{E}_{U,h}^D\right)\right) = \left(\mathcal{E}_U^{D,h^2}, \mathcal{D}\left(\mathcal{E}_U^{D,h^2}\right)\right) = \left(\mathcal{E}_U^{N,h^2}, \mathcal{D}\left(\mathcal{E}_U^{N,h^2}\right)\right)$$

on $(\widetilde{U}, h^2 d\mu)$. Then $P_{U,h,t}^D$, $t > 0$, admits a continuous kernel $p_{U,h}^D(t,x,y)$, $(t,x,y) \in (0,\infty) \times \widetilde{U} \times \widetilde{U}$ which is bounded by

$$\text{(5.5)} \qquad \frac{c_1 \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{V_{h^2}(x,\sqrt{t})} \leq p_{U,h}^D(t,x,y) \leq \frac{c_3 \exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{V_{h^2}(x,\sqrt{t})},$$

on $(0,\infty) \times \widetilde{U} \times \widetilde{U}$. Here V_{h^2} denotes the volume $V_{h^2}(x,r) = \int_{B_U(x,r)} h^2 d\mu$ of the inner ball $B_U(x,r)$ with respect to $h^2 d\mu$.

Proof. — Apply Theorem 2.31 to the Harnack-type Dirichlet form

$$\left(\mathcal{E}_{U,h}^D, \mathcal{D}\left(\mathcal{E}_{U,h}^D\right)\right)$$

provided by Theorem 5.9. □

5.3. The Dirichlet-type Dirichlet form on an inner uniform domain

This section contains the main results of this monograph concerning the Dirichlet heat kernel and solutions of the heat equation with Dirichlet boundary condition on inner uniform domains. Throughout, we assume that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. We let h be the profile for U provided to us by Theorem 4.1.

Recall that $P_{U,t}^D$, $t > 0$, denotes the Dirichlet heat semigroup on $L^2(U,\mu)$ associated with the Dirichlet-type Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ of Definition 2.34 and that $p_U^D(t,x,y)$, $(t,x,y) \in (0,\infty) \times U \times U$, denotes the associated Dirichlet heat kernel.

5.3.1. The Dirichlet heat kernel. — By Proposition 5.6 applied to the form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ and to the profile h, the Dirichlet heat kernel $p_U^D(t,x,y)$ is related to the heat kernel $p_{U,h}^D(t,x,y)$ of Corollary 5.10 by the formula

$$\text{(5.6)} \qquad p_U^D(t,x,y) = h(x)h(y)p_{U,h}^D(t,x,y).$$

This formula, together with Theorem 5.9, Corollary 5.10 and Theorem 2.32 yields the following fundamental result concerning the Dirichlet heat kernel in an unbounded inner uniform domain.

Theorem 5.11. — *Assume that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. Let h be the profile for U given by* Theorem 4.1. *Then there exists $c_1, c_2, \alpha, \beta > 0$ and, for each $k = 0, 1, 2, \dots$, a constant C_k such that, for all $t > 0$, $x, y \in U$, we have:*

$$p_U^D(t,x,y) \geq \frac{c_1 h(x)h(y)}{V_{h^2}(x,\sqrt{t})} \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right),$$

$$|\partial_t^k p_U^D(t,x,y)| \leq \frac{C_k h(x)h(y)}{t^k V_{h^2}(x,\sqrt{t})} \left(1 + \frac{\rho_U(x,y)^2}{t}\right)^{\beta+k} \exp\left(-\frac{\rho_U(x,y)^2}{4t}\right)$$

and for $y' \in U$ with $\rho_U(y,y') \leq \sqrt{t}$,

$$\left|\frac{\partial_t^k p(t,x,y)}{h(y)} - \frac{\partial_t^k p(t,x,y')}{h(y')}\right| \leq \frac{C_k}{t^k}\left(\frac{\rho_U(x,y)}{\sqrt{t}}\right)^{\alpha} \frac{p(2t,x,y)}{h(y)}.$$

To make these estimates even more explicit one can use (4.29) which provides a two-sided estimate for the volume function $V_{h^2}(x,r)$, namely,

$$V_{h^2}(x,r) \simeq h(x_r)^2 V(x,r)$$

in terms of the original volume $V(x,r)$ in (X,μ,ρ) and the value of h at a point x_r satisfying $\rho_U(x_r,x) \leq r$ and $\rho(x_r, X \setminus U) \geq c_0 r/8$.

Remark 5.12. — There are alternative estimates of the Dirichlet heat kernel, namely,

$$p_U^D(t,x,y) \geq \frac{c_1 h(x)h(y)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)$$

and

$$|\partial_t^k p_U^D(t,x,y)| \leq \frac{C_k h(x)h(y)}{t^k\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \left(1 + \frac{\rho_U(x,y)^2}{t}\right)^{\beta+k} \exp\left(-\frac{\rho_U(x,y)^2}{4t}\right).$$

These estimates (with different constants c_1, c_2, β, C_k) are equivalent to those given in Theorem 5.11 by symmetry and because $V_{h^2}(x,r)$ is doubling.

The estimates of Theorem 5.11 yield control of other important quantities such as the Dirichlet Green function in U and the probability

$$\mathbf{P}_U(t,x) = P_{U,t}^D \mathbf{1}_U(x) = \mathbb{P}_x(\tau_U > t) \tag{5.7}$$

that the associated Markov process started at x stays in U up to time t.

Theorem 5.13. — *Assume that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. Let h be the profile for U given by* Theorem 4.1 *and assume that h is not constant. Then there exist constants $c, C \in (0,\infty)$ such that the Dirichlet-type Green function $G_U^D(x,y)$ satisfies*

$$\forall\, x,y \in U, \quad c\int_{\rho_U(x,y)^2}^{\infty} \frac{1}{V_{h^2}(x,\sqrt{s})}ds \leq \frac{G_U^D(x,y)}{h(x)h(y)} \leq C\int_{\rho_U(x,y)^2}^{\infty} \frac{1}{V_{h^2}(x,\sqrt{s})}ds.$$

Theorem 5.14. — *Assume that $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. Let h be the profile for U given by* Theorem 4.1. *Then there exist constants $c, C \in (0,\infty)$ such that the function $\mathbf{P}_U$ defined via* (5.7) *is bounded by*

$$\forall\,(t,x)\in(0,\infty)\times U,\;\; c\frac{h(x)}{h(x_{\sqrt{t}})} \le \mathbf{P}_U(t,x) \le C\frac{h(x)}{h(x_{\sqrt{t}})}$$

where x_r is, again, an arbitrary point in U satisfying $\rho_U(x,x_r)\le r$ and $\rho(x_r, X\setminus U)\ge c_0 r/8$.

Proof. — For the lower bound, write $p_U^D(t,x,y) = p_{U,h}^D(t,x,y)h(x)h(y)$ and

$$\begin{aligned}\int_U p_U^D(t,x,y)d\mu(y) &\ge h(x)\int_{B_{\widetilde{U}}(x,\sqrt{t})} \frac{p_{U,h}^D(t,x,y)}{h(y)}h^2(y)d\mu(y)\\ &\ge c_1\frac{h(x)}{h(x_{\sqrt{t}})}.\end{aligned}$$

Here we have used the fact that there is a constant C such that

$$\forall\, y\in B_{\widetilde{U}}(x,r),\;\; \frac{h(y)}{h(x_{\sqrt{t}})}\le C$$

(see 4.28) and the estimate

$$\forall\, y\in B_{\widetilde{U}}(x,r),\;\; p_{U,h}^D(t,x,y)\ge \frac{c}{V_{h^2}(x,\sqrt{t})}.$$

For the upper bound, write (for the last step, we use (4.28) again)

$$\begin{aligned}\int_U p_U^D(t,x,y)d\mu(y) &\le C\frac{h(x)}{h(x_{\sqrt{t}})}\int_U \frac{h(y)}{h(y_{\sqrt{t}})}p(t,x,y)d\mu(y)\\ &\le C\frac{h(x)}{h(x_{\sqrt{t}})}\sup_y\left\{\frac{h(y)}{h(y_{\sqrt{t}})}\right\} \le C'\frac{h(x)}{h(x_{\sqrt{t}})}.\end{aligned}$$ □

Let us look more closely at the basic heat kernel estimate for $p_U^D(t,x,y)$, that is

$$\frac{c_1 h(x)h(y)e^{-\frac{\rho_U(x,y)^2}{c_2 t}}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \le p_U^D(t,x,y)\le \frac{c_3 h(x)h(y)e^{-\frac{\rho_U(x,y)^2}{c_4 t}}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}}.$$

Recall that, for any $z\in U$ and $r>0$, $V_{h^2}(z,r)\simeq h(z_{\sqrt{t}})V(z\sqrt{t})$ where z_r is any point in U such that $\rho_U(z,z_r)\le Kr$ and $\rho(z,X\setminus U)\ge \kappa r$ for some fixed $K,\kappa\in(0,\infty)$ (such a point always exist if $K/\kappa > 8/c_0$ with c_0 as in the definition of inner uniformity). Hence the factor

$$\frac{h(x)h(y)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}}$$

in the two-sided estimate above can be estimated by

$$\frac{h(x)h(y)}{h(x_{\sqrt{t}})h(y_{\sqrt{t}})\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}.$$

By Theorem 5.14, this factor is thus also comparable to

$$\frac{\mathbf{P}_U(t,x)\mathbf{P}_U(t,y)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}.$$

This leads to the following two interesting results.

Theorem 5.15. — *Assume that $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. Then there exist constants $c_1,c_2,c_3,c_4\in(0,\infty)$ such that for all $t>0$ and $x,y\in U$,*

$$c_1\frac{\mathbf{P}_U(t,x)\mathbf{P}_U(t,y)e^{-\frac{\rho_U(x,y)^2}{c_2t}}}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}\le p_U^D(t,x,y)\le\frac{c_3\mathbf{P}_U(t,x)\mathbf{P}_U(t,y)e^{-\frac{\rho_U(x,y)^2}{c_4t}}}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}.$$

Theorem 5.16. — *Assume that $(X,\mu,\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded uniform domain. Then there exist constants $c_1,c_2,c_3,c_4\in(0,\infty)$ such that for all $t>0$ and $x,y\in U$,*

$$c_1\mathbf{P}_U(t,x)\mathbf{P}_U(t,y)p(c_2t,x,y)\le p_U^D(t,x,y)\le c_3\mathbf{P}_U(t,x)\mathbf{P}_U(t,y)p(c_4t,x,y).$$

Proof. — This follows from the previous result and the classical estimates for the global heat kernel $p(t,x,y)$. The crucial point is that when U is uniform (as assumed in this theorem), the intrinsic distance ρ_U is comparable to the (global) distance ρ. □

The last theorem allows us to compare our results to some of those obtained in [**95, 96, 97**] for conical domains and [**98, 99**] for Euclidean domains above the graph of a Lipschitz function. Indeed, the heat kernel estimates of [**98, 99**] are stated in terms of the function $\mathbf{P}_U(t,x)$, $t>0$, $x\in U$. Since the results of [**98, 99**] as well as ours give sharp two-sided estimates for the Dirichlet heat kernel, it is clear, of course, that these estimates must be equivalent. However, deducing our estimates from those of [**98, 99**] is not a trivial task. The results obtained here are more general than those in [**98, 99**] as so far as they apply to inner uniform domains but [**98, 99**] consider a wider class of operators including non-divergence type operators.

5.3.2. The parabolic boundary Harnack principle. — We start this section with an elementary proposition which follows readily from the definitions of the various objects involved in the statement. We saw in the previous section that the Dirichlet heat kernel $p_U^D(t,x,y)$ is related to the heat kernel $p_{U,h}^D$ of the Dirichlet form $(\mathcal{E}_{U,h}^D,\mathcal{D}(\mathcal{E}_{U,h}^D))$ by the formula $p_U^D(t,x,y)=h(x)h(y)p_{U,h}^D(t,x,y)$, $x,y\in U$. See (5.6). A similar relation holds at the level of local weak solutions of the heat equation with Dirichlet boundary condition along ∂U when $(\mathcal{E}_{U,h}^D,\mathcal{D}(\mathcal{E}_{U,h}^D)=(\mathcal{E}_U^{D,h^2},\mathcal{D}(\mathcal{E}_U^{D,h^2}))$ and this form is considered on $\widetilde{U}$. This statement is more subtle because it involves the topology of $\widetilde{U}$ in a crucial way.

Proposition 5.17. — *Let $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet space satisfying the conditions* (A1)-(A2) *of* Section 2.2.3 *and which admits a carré du champ. Let U be an unbounded domain in X such that $\widetilde{U}$ is locally compact. Let h be a continuous positive function on U which is a local weak solution of the Laplace equation in U and belongs to $\mathcal{F}^0_{\mathrm{loc}}(U)$.*

Let V be an open subset in U and $V^\# \subset \widetilde{U}$ be as in Definition 2.42. *Let I be an open time interval and $Q = I \times V$, $Q^\# = I \times V^\#$. Then the following properties hold.*

1. *A function u defined on V is a local weak solution of the Laplace equation for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in V with weak Dirichlet boundary condition along ∂U if and only if the function u/h is a local weak solution of the Laplace equation for $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2}))$ in $V^\#$.*
2. *A function u defined on Q is a local weak solution of the heat equation for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in Q with weak Dirichlet boundary condition along ∂U if and only if the function u/h is a local weak solution of the heat equation for $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2}))$ in $Q^\#$.*

Proof. — We only prove the elliptic case (1). The additional details needed in the parabolic case are of a similar nature, involving the "transport" of various spaces and conditions from one form to the other using the maps H and H^{-1} of pointwise multiplication by h and h^{-1}.

Recall that $V^\#$ is the largest open set in $\widetilde{U}$ which is contained in the closure of V in $\widetilde{U}$ and whose intersection with U is V. This means that $V^\#$ is obtained from V by adding those points in $\widetilde{U} \setminus U$ that are interior points in the closure of V in $\widetilde{U}$. See Example 2.44.

By Definition 2.47, u is a weak solution of the Laplace equation in V with weak Dirichlet condition along ∂U if and only if (a) $u \in \mathcal{F}^0(U, V)$ and (b) $\mathcal{E}(u, \phi) = \int_V d\Gamma(u, \phi) = 0$ for any $\phi \in \mathcal{F}_c(V)$. Because ϕ has compact support in V, it is obvious that condition (*b*) is equivalent to (b') $\mathcal{E}_U^{D,h^2}(u/h, \psi) = 0$ for any $\psi \in \mathcal{F}_c(V)$. To see this, use the polarized version of (5.4) and the fact that $\phi \in \mathcal{F}_c(V)$ is equivalent to $\psi = \phi/h \in \mathcal{F}_c(V)$.

Concerning condition (a), Lemma 2.45 asserts that $u \in \mathcal{F}^0(U, V)$ if and only if, for any open set $\Omega \subset V$ that is relatively compact in $V^\#$, there exists a function $u^\# \in \mathcal{F}^0(U)$ such that $u^\# = u$ on Ω. By Proposition 5.7, we also know that $\mathcal{D}(\mathcal{E}_U^{D,h^2}) = H^{-1}\mathcal{F}^0(U)$.

If u is in $\mathcal{F}^0(U, V)$ and Ω is open relatively compact in $V^\#$, let $u^\# \in \mathcal{F}^0(U)$ such that $u^\# = u$ on $U \cap \Omega$ hence also on Ω (these equalities are understood a.e.). Then $u^\#/h = u/h$ on Ω and $u^\#/h \in \mathcal{D}(\mathcal{E}_U^{D,h^2})$. This means that $u/h \in \mathcal{F}_{\mathrm{loc}}^{\mathcal{E}_U^{D,h^2}}(V^\#)$.

Conversely, assume that $u/h \in \mathcal{F}_{\mathrm{loc}}^{\mathcal{E}_U^{D,h^2}}(V^{\#})$. Then for any open relatively compact set Ω in $V^{\#}$, there exists $v^{\#}$ such that $v^{\#} = u/h$ on Ω and $v^{\#} \in \mathcal{D}(\mathcal{E}_U^{D,h^2})$. Hence, $v^{\#}h = u$ on Ω and $v^{\#}h \in \mathcal{F}^0(U)$. This means that u is in $\mathcal{F}^0(U, V)$. □

Using the proposition above and Theorem 5.9, i.e., the fact that the form $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2})) \mp (\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$ is Harnack-type when U is inner uniform, we obtain the following parabolic version of the Harnack inequality and Hölder continuity up to the boundary for weak solutions with weak Dirichlet condition along ∂U.

Theorem 5.18. — *Assume that $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet space with a carré du champ and that U is an unbounded inner uniform domain. Let h be the profile for U given by* Theorem 4.1. *Fix $\tau, \theta \in (0,1)$ and $0 < \epsilon < \eta < \sigma < 1$. Then there exist constants $\alpha \in (0,1)$ and $H_1, H_2 \in (0,\infty)$ such that for any $x \in \widetilde{U}$, $r, s > 0$ and any weak solution u of the heat equation for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the cylinder $Q = (s - \tau r^2, s) \times B_{\widetilde{U}}(x, r)$ with weak Dirichlet boundary condition along ∂U, we have*

$$\sup_{(t,y),(t',y')\in Q'} \left\{ \frac{|u(t,y)/h(y) - u(t',y')/h(y')|}{(\sqrt{|t-t'|} + \rho_U(y,y'))^{\alpha}} \right\} \le H_1 \sup_Q \{|u/h|\} \tag{5.8}$$

where $Q' = (s - \sigma\tau r^2, s - \epsilon\tau r^2) \times B(x, \theta r)$. Moreover, if u is non-negative in Q then

$$\sup_{Q_-} \{u/h\} \le H_2 \inf_{Q_+} \{u/h\} \tag{5.9}$$

where $Q_- = (s - \sigma\tau r^2, s - \eta\tau r^2) \times B(x, \theta r)$, $Q_+ = (s - \epsilon\tau r^2, s) \times B(x, \theta r)$.

CHAPTER 6

EXAMPLES

This chapter discusses various examples in some details.

First, we point out constructions of the harmonic profile h in two special cases that are easier than and independent of the general construction given in Chapter 4. These two special cases are: domains that are defined as the domain above the graph of a function and domains that are the complement of a convex set. We also discuss a few more specific examples of uniform and inner uniform Euclidean domains.

Second, uniform domains in models of sub-Riemannian geometries have been studied by several authors including [**25, 26, 49, 50**]. This allows us to give applications of our results in this context and we present the simplest possible examples in the case of the $(2n+1)$-dimensional Heisenberg group.

Finally, we also give very simple examples in the context of Euclidean complexes since some of these objects where mentioned earlier as models of Harnack-type Dirichlet spaces.

6.1. Limits in $\mathcal{F}^0(U)$

This section spells out a very well known fact (essentially, the weak compactness of the unit ball in a Hilbert space) that is useful in constructing the profile function h on various domains.

The lemma below gives a version in the context of Dirichlet spaces.

Lemma 6.1. — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on (X, μ). Let U be a domain in X. Let f_i be a sequence in $L^2(U)$ that converges in $L^2(U)$ to a function f. Assume further that $f_i \in \mathcal{F}^0(U) = \mathcal{D}(\mathcal{E}_U^D)$ and*

$$\sup_i \{\mathcal{E}_U^D(f_i, f_i)\} = C < \infty.$$

Then f belongs to $\mathcal{F}^0(U)$.

Proof. — Using the identification of $L^2(U)$ with its dual, we have $\mathcal{F}^0(U) \subset L^2(U) \subset \mathcal{F}^0(U)'$. Since the set $\{g \in \mathcal{F}^0(U) : \|g\|_2^2 + \mathcal{E}_U^D(g, g) < C\}$ is weakly compact, it follows in particular that for any subsequence σ of (f_i) there is a further subsequence σ' such that $\int_U f_i g d\mu \to \int_U f(\sigma') g d\mu$ for some $f(\sigma') \in \mathcal{F}^0(U)$ all $g \in L^2(U)$. Since $\lim f_i = f$ in $L^2(U)$, we must have $f(\sigma') = f \in \mathcal{F}^0(U)$. □

The local version that is important to us reads as follows.

***Lemma* 6.2.** — *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on (X, μ) satisfying the conditions* (A1)-(A2) *of* Section 2.2.3. *Let U be a domain in X. Let f_i be a sequence in $L^2_{\mathrm{loc}}(\overline{U})$ that converges in $L^2_{\mathrm{loc}}(\overline{U})$ to a function f. Assume further that $f_i \in \mathcal{F}^0_{\mathrm{loc}}(U)$ and that, for any compact set $K \subset \overline{U}$*

$$\sup_i \left\{ \int_K d\Gamma(f_i, f_i) \right\} = C < \infty.$$

Then f belongs to $\mathcal{F}^0_{\mathrm{loc}}(U)$.

Proof. — Use Lemma 2.36 and the previous result. □

6.2. From classical solutions to weak solutions in Euclidean domains

In the general context of domains in a Harnack Dirichlet space, it is most natural to deal with solution of the Laplace and heat equations in the weak sense defined in Chapter 2, including possibly Neumann or Dirichlet boundary condition. Indeed, in this general context, the domain of the Laplace operator is a rather mysterious object. Things are a bit different when working on an Euclidean domain U, especially in the case of the Dirichlet boundary condition. The reason is that, inside U, any weak solution (of either the Laplace equation or the heat equation) is a smooth function and a local solution in the classical sense.

In the case of the Neumann boundary condition, it can be quite troublesome to turn the weak formulation of the boundary condition using $\widetilde{U}$ as in Section 2.4.4 into a Neumann boundary condition in a classical sense. The weak formulation of the Neumann boundary condition essentially reduces the boundary condition to the case of local weak solutions without boundary condition. However, verifying that a function is a weak solution might be (very) difficult when we do not have a good grasp on the space $\mathcal{F}_{\mathrm{loc}}(\widetilde{U})$. In this respect, the case of the von Koch snowflake is probably a good example to keep in mind. The interior V (and also the exterior) of the snowflake is a uniform Euclidean domain (see below) and thus $\widetilde{V} = \overline{V}$. Theorem 3.10 applies and shows that the Neumann type Dirichlet form

$$\mathcal{E}^N_V(f, f) = \int_V |\nabla f|^2 d\lambda, \quad f \in W^1(V),$$

on $(\overline{V}, \lambda)$ is regular and of Harnack type. This means that any local weak solution u of the heat equation for $(\overline{V}, \mathcal{E}^N_V, W^1(V))$ is Hölder continuous. The heat kernel and its time derivatives provide non-trivial examples of such weak solutions. However, giving classical type conditions that would imply that a function is a weak solution appears rather difficult.

A simple case worth discussing is the case of the interior V of a slitted disk, say $V = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1, (x,y) \notin \{0\} \times (-1,0]\}$. This is an inner uniform domain. Theorem 3.10 applies and shows that the Neumann type Dirichlet form

$$\mathcal{E}_V^N(f,f) = \int_V |\nabla f|^2 d\lambda, \quad f \in W^1(V),$$

on $(\widetilde{V}, \lambda)$ is regular and of Harnack type. This time, $\widetilde{V} \neq \overline{V}$. Namely, points of $\overline{V} \setminus V$ along the (open) negative y semi-axis must be split (in a topologically obvious way) into a left and a right copy to obtain $\widetilde{V} \setminus V$. However, we can clearly define the outward unit normal $\vec{\nu}$ to the boundary at any point along $\widetilde{V} \setminus V$, except at the 3 points $z_0 = (0,0)$, $z_l = (0,-1)_l$ and $z_r = (0,-1)_r$ (the subscripts denote the left and right copies mentioned above). In particular, at the points $(0,y)_l, (0,y)_r$, $y \in (-1,0)$, we set

$$\vec{\nu}((0,y)_\epsilon) = \begin{cases} -\vec{i} & \text{if } \epsilon = r \\ +\vec{i} & \text{if } \epsilon = l, \end{cases}$$

and

$$\frac{\partial f}{\partial \vec{\nu}}((0,y)_r) = -\lim_{x>0, x\to 0} \frac{\partial f}{\partial x}(x,y), \quad \frac{\partial f}{\partial \vec{\nu}}((0,y)_l) = \lim_{x<0, x\to 0} \frac{\partial f}{\partial x}(x,y).$$

Suppose that u is a smooth function in V satisfying

$$\sup_V \{|u| + |\partial u/\partial x| + |\partial u/\partial y|\} < \infty$$

with $\partial u/\partial x$, $\partial u/\partial y$ having limits at any $z \in \widetilde{V} \setminus V$, $z \notin \{z_0, z_l, z_r\}$ and

$$\forall\, z \in \widetilde{V} \setminus V,\ z \notin \{z_0, z_l, z_r\}, \quad \frac{\partial u}{\partial \vec{\nu}}(z) = 0.$$

Suppose further that, for some $f \in L^1(V)$,

$$\Delta u = f \text{ in } V.$$

Then we claim that u is a weak solution of the Laplace equation for the Dirichlet form $(\widetilde{V}, \mathcal{E}_V^N, W^1(V))$ with right-hand side f. To see this, we need to show that, for any $\phi \in W^1(V)$,

$$\int_V \nabla u \cdot \nabla \phi d\lambda = \int u \phi d\lambda.$$

Since the form is regular, it is enough to treat the case where $\phi \in \mathcal{C}(\widetilde{V}) \cap W^1(V)$. Now, the proof of the claim follows from integration by parts on a sequence of domains V_n with smooth boundary approximating V from the inside. The hypotheses made on u and the continuity of ϕ on $\widetilde{V}$ make it easy to see that the boundary term vanishes in the limit. This is a very simple example but this type of argument will work in a variety of similar examples with rectifiable boundary along which a reasonable notion of normal derivative can be defined.

We now consider the case of Dirichlet boundary condition. The following proposition is general enough to cover many cases of interest and its proof is quite standard. It shows in particular that classical solutions with Dirichlet boundary condition are indeed weak solution satisfying the Dirichlet condition in the weak sense. Recall that the reason this is not entirely obvious is that classical solution are not required, a priori, to be locally in $W^1(V)$. The fact that this is the case must be extracted from some energy estimates.

Proposition 6.3. — *Let U be a domain in $\mathbb{R}^n$ such that $\widetilde{U}$ is locally compact. Let h be a positive harmonic function in U that belongs to $\mathcal{F}^0_{\mathrm{loc}}(\widetilde{U})$. Set $d\nu = h^2 d\mu$. Let I be an open time interval, Ω be an open set in $\widetilde{U}$. Set $Q = I \times \Omega$. Let u be a continuous function on Q which vanishes on $I \times (\Omega \cap (\widetilde{U} \setminus U))$, is once continuously differentiable in time, twice continuously differentiable in space and satisfies $\partial_t u + \Delta u = 0$ in $\Omega \cap U$. Then the following conclusions hold:*

1. *The function u is a local weak solution of the heat equation for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in $I \times (\Omega \cap U)$ with weak Dirichlet boundary condition along ∂U;*
2. *The function $v = u/h$ is a local weak solution of the heat equation in $I \times \Omega$ in the sense of* Definition 2.27 *for the Dirichlet form $(\mathcal{E}^{D,h^2}_U, \mathcal{D}(\mathcal{E}^{D,h^2}_U))$ on $L^2(\widetilde{U}, d\nu)$.*

Proof. — This is essentially well known. For instance, [**44**, Corollary 2.3] is a very similar (essentially equivalent) statement. However, we do not know of a proper reference making use as we do here of the set $\widetilde{U}$. Note that Proposition 5.17 tells us that the conclusions (1) and (2) are actually equivalent so we only need to prove (1).

It is convenient to introduce the following notation. We let $\mathcal{C}^\infty_c(\widetilde{U})$ be the space of those continuous functions f in $\widetilde{U}$ that have compact support in $\widetilde{U}$ are smooth in U and verify, for any $k = (k_1, \dots, k_n)$, $\sup_U\{|\partial^k f/\partial^k x|\} < \infty$. For instance, the restriction to U of any smooth compactly supported function is in this space.

Without loss of generality, we can assume that u is bounded on Q (simply replace Q by an arbitrary $Q' = I' \times \Omega'$ relatively compact in $I \times \Omega$). For every $\epsilon \in (0,1)$, let G_ϵ be a smooth function of one real variable such that $G_\epsilon, G'_\epsilon, G''_\epsilon \geq 0$, G_ϵ vanishes on $(-\infty, \epsilon]$ and $G'_\epsilon \equiv 1$ on $(3\epsilon, \infty)$. Given u as above, set $u_\epsilon = G_\epsilon(\sqrt{u^2+\epsilon^2} - \epsilon)$ on Q. This function has the same smoothness property as u and vanishes on $\{u^2 \leq 3\epsilon^2\}$. Moreover, a simple computation shows that $\frac{\partial}{\partial t} u_\epsilon + \Delta u_\epsilon \leq 0$ on $\Omega \cap U$. Let $\phi \in C^\infty_c(\widetilde{U})$ with compact support in Ω and $0 \leq \phi \leq 1$. Note that ϕu_ϵ has compact support in $\Omega \cap \{u^2 > 3\epsilon^2\} \subset U$. Now, using the inequality satisfied by u_ϵ and integrating by parts, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\frac{1}{2}\int_\Omega |\phi u_\epsilon|^2 d\mu\Big) + \int |\nabla(\phi u_\epsilon)|^2 d\mu \\
&\quad \leq -\int_\Omega \phi^2 u_\epsilon \Delta u_\epsilon + \int \phi u_\epsilon \Delta(\phi u_\epsilon) d\mu \\
&\quad = \int_\Omega u_\epsilon^2 \phi \Delta \phi d\mu - \int_\Omega \phi u_\epsilon \nabla\phi \cdot \nabla u_\epsilon d\mu
\end{aligned}$$

$$= \int_\Omega u_\epsilon^2 \phi \Delta \phi d\mu - \int_\Omega u_\epsilon \nabla \phi \cdot \nabla(\phi u_\epsilon) d\mu + \int_\Omega u_\epsilon^2 |\nabla \phi|^2 d\mu$$
$$\leq C_\phi \int_\Omega u_\epsilon^2 d\mu + \frac{1}{2} \int_\Omega |\nabla(\phi u_\epsilon)|^2 d\mu.$$

Hence
$$\frac{\partial}{\partial t} \int_\Omega |\phi u_\epsilon|^2 d\mu + \int |\nabla(\phi u_\epsilon)|^2 d\mu \leq 2C_\phi \int_\Omega u_\epsilon^2 d\mu.$$

Multiplying ϕu_ϵ by an appropriate cutoff function in time and integrating in time yields (after some simple manipulations)

$$\sup_{I'} \left[\int_{\Omega'} |u_\epsilon|^2 d\mu \right] + \int_{Q'} |\nabla u_\epsilon|^2 dt d\mu \leq C(Q') \int_Q u_\epsilon^2 dt d\mu \tag{6.1}$$

for any $Q' = I' \times \Omega'$ relatively compact in Q. Next, observe that $\mathrm{sgn}(u) u_\epsilon$ tends to u in $L^2(Q)$ and that $|\nabla u_\epsilon|$ tends to $|\nabla u|$ pointwise in Q'. Hence we also have

$$\sup_{I'} \int_{\Omega'} |u|^2 d\mu + \int_{Q'} |\nabla u|^2 dt d\mu \leq C(Q') \int_Q |u|^2 dt d\mu \tag{6.2}$$

By a straightforward variant of Lemma 6.2 for functions of time and space it follows that, for any function $\phi \in C_c^\infty(\widetilde{U})$ with compact support in Ω, the function $w = \phi u(t, \cdot)$ is in $\mathcal{D}(\mathcal{E}_U^D)$ for a.e. $t \in I'$ and satisfies

$$\int_{Q'} |w|^2 + |\nabla w|^2 dt d\mu \leq C(\phi, Q') \int_Q |u|^2 dt d\mu.$$

Moreover, for any $\psi \in \mathcal{D}(\mathcal{E}_U^D)$ and a.e. $t \in I'$, we have

$$\begin{aligned} \left| \int_U \psi \frac{\partial}{\partial t} w d\mu \right| &= \left| \int_U \psi \phi \Delta u d\mu \right| \\ &\leq \left| \int_U \psi \Delta(\phi u) d\mu \right| + \int_U |\psi| [|u \Delta \phi| + |\nabla u \cdot \nabla \phi|] d\mu \\ &\leq \int_U |\nabla \psi \cdot \nabla(\phi u)| d\mu + \int_U |\psi| [|u \Delta \phi| + |\nabla u \cdot \nabla \phi|] d\mu \\ &\leq C_1(\phi, Q') \left(\int_{\Omega'} |u|^2 + |\nabla u|^2 d\mu \right)^{1/2} \|\psi\|_{\mathcal{D}(\mathcal{E}_U^D)}. \end{aligned}$$

for some constant C_1 depending on ϕ and Q'. It follows that $\frac{\partial}{\partial t} w$ belongs (for almost every t) to the dual $\mathcal{D}'(\mathcal{E}_U^D)$ of $\mathcal{D}(\mathcal{E}_U^D)$ and that

$$\int_{I'} \|\partial_t w\|^2_{\mathcal{D}'(\mathcal{E}_U^D)} dt \leq C_2(\phi, Q') \int_Q |u|^2 dt d\mu.$$

This and the other properties of u show that u is a weak solution as stated in part (1) of this proposition. □

We note that there are domains for which the proposition above is of no help at all because most solutions of the Dirichlet problem are not continuous up to the boundary. A typical example is provided by the famous Lebesgue spine. This difficulty can sometimes be resolved by considering bounded functions that vanish quasi-everywhere on the boundary. However, the treatment of this case require further hypotheses on the domain (e.g., inner uniformity) and much more sophisticated arguments than those used in the proposition above. The practical differences between the classical formulation and the formulation in term of weak solutions are less clear in such cases and we will not discuss this situation here.

6.3. The domain above the graph of a function

This section is devoted to Euclidean domains above the graph of a function. The main result concerns the case where the function is a Lipschitz function. This gives a concrete illustration of our general results but we indicate a much simpler proof of the existence and properties of the profile h in this very important case.

Recall that λ denotes Lebesgue measure. Also, in the case of an Euclidean domain U, $\mathcal{F}^0_{\mathrm{loc}}(U)$ is the set of all functions $f \in W^1_{\mathrm{loc}}(U)$ such that for any compactly supported smooth function ϕ on $\mathbb{R}^n$, the function ϕf is in $W^1_0(U)$. Another equivalent description is to say that $f \in W^1_{\mathrm{loc}}(\overline{U})$ and vanishes quasi-everywhere along ∂U.

Proposition 6.4. — *Let $U \subset \mathbb{R}^n$ be the domain above the graph of an upper semi-continuous function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ (we do not allow Φ to take the value ∞), that is,*

$$U = \{x = (\vec{y}, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : \Phi(\vec{y}) < z\}.$$

Then there exists a harmonic function $h : U \to (0, \infty)$ such that:

(1) $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$.

(2) *h is non-decreasing in U in the vertical direction.*

Proof. — Let Φ, U be as above. Let ϕ_i be a decreasing sequence of smooth functions approximating Φ pointwise.

For each integer i, let D_i be an open subset of U with smooth boundary of the form

$$D_i = \{x = (\vec{y}, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : \widetilde{\phi}_i(\vec{y}) < z < \widetilde{\psi}_i(\vec{y}),\ \ \|\vec{y}\| < 10i\}$$

where $\widetilde{\phi}_i, \widetilde{\psi}_i$ are smooth functions on $\{\|\vec{y}\| < 10i\}$ satisfying $\widetilde{\phi}_i(\vec{y}) = \phi_i(\vec{y})$ on $\{\|\vec{y}\| < 9i\}$, $\phi_i(\vec{y}) \le \widetilde{\phi}_i(\vec{y}) \le \phi_i(\vec{y}) + 1$ on $\{\|\vec{y}\| < 10i\}$, $\widetilde{\psi}_i(\vec{y}) = 10i + \phi_i(\vec{y})$ on $\{\|\vec{y}\| < 9i\}$, $10i + \phi_i(\vec{y}) - 1 \le \widetilde{\psi}_i(\vec{y}) \le 10i + \phi_i(\vec{y})$ on $\{\|\vec{y}\| < 10i\}$. Note that $D_i \subset D_j$ if $i \le j$ (for the boundary of D_i to be smooth, the functions $\widetilde{\psi}_i, \widetilde{\phi}_i$ must have infinite vertical derivatives along $\{\|\vec{y}\| = 10i\}$).

On the boundary of the domain D_i, consider a fixed smooth function f_i such that $f_i = 0$ on the part of the boundary that is below $2 + \widetilde{\phi}_i$, $f_i = 1$ on the part of the boundary that is above $-2 + \widetilde{\psi}_i$ and is an increasing function of z alone along the vertical walls of ∂D_i. The details of this construction are not very relevant and the exact regularity that is required for the boundary of D_i and for the function f_i reflect mostly the choice of what classical result concerning solutions of the Dirichlet problem one is willing to use for the next step.

Let u_i be the solution of the Dirichlet problem on D_i with boundary data f_i. The function u_i is a positive harmonic function in D_i bounded above by 1 and it belongs to $\mathscr{C}^\infty(\overline{D_i})$. It vanishes along the bottom boundary of D_i. The crucial further property of u_i is that it is increasing in the vertical direction. Indeed, $\partial_z u_i$ is a smooth function on $\overline{D_i}$ which, by construction, is non-negative on the boundary and harmonic on D_i. Hence it is non-negative on D_i.

Next we fix a point $x_0 \in D_1$ and consider the sequence

$$h_i = u_i/u_i(x_0)$$

This is a sequence of non-negative harmonic functions defined on the increasing sequence of domains D_i, $\cup_i = U$. As all h_i are equal to 1 at x_0, by the classical Harnack inequalities, we can extract a sequence that converges locally uniformly to a positive harmonic function h defined on U. Abusing notation, we let h_i denote a convergent subsequence. Obviously, the function h is increasing in the vertical direction.

Because of this property and the fact that h_i converges to h locally uniformly in U, it follows that h_i (extended by 0 outside D_i) converges to h in $L^2_{\mathrm{loc}}(\overline{U})$. Pick any ball $B(x_0, R)$ with some fixed large radius R and any smooth function ϕ with compact support in $B(x_0, R)$ and with $|\nabla\phi| \leq 1$. Consider the sequence of function ϕh_i, where i is taken so large that $B(x_0, R)$ intersect D_i only along its bottom portion where $h_i = 0$. It follows that $\phi h_i, \phi^2 h_i \in W^1_0(U)$. By Green's formula, $\int_U \nabla h_i \cdot \nabla(\phi^2 h_i) dx = 0$. Hence

$$\int_U \nabla(\phi h_i) \cdot \nabla(\phi h_i) dx = \int_U |\nabla\phi|^2 h_i^2 dx.$$

Since h_i converges in $L^2_{\mathrm{loc}}(\overline{U})$, it follows that

$$\sup_i \left\{ \int_U |\phi h_i|^2 dx \right\} < \infty.$$

Hence, by Lemma 6.2, $\phi h \in W^1_0(U)$, that is, $h \in \mathscr{F}^0_{\mathrm{loc}}(U)$. Note that using Lemma 6.2 is not necessary here since the argument above can be used to prove directly that ϕh_i is Cauchy in $W^1_0(U)$ and thus converges to ϕh in $W^1_0(U)$. □

Remark 6.5. — Proposition 6.4 takes advantage of the simple fact that the domain U is defined by the graph of a function to yield the existence of a harmonic profile. It applies, for instance, both to the inside and to the outside of a paraboloid, examples that are discussed in the introduction.

Another class of examples discussed in Chapter 1 is the collection of the slit upper-half planes $\mathbb{R}^2_{+\mathbf{f}}$ associated with any (finite or) countable family

$$\mathbf{f} = \{(x_i, y_i))\} \subset \mathbb{R}^2_+$$

where $\mathbb{R}^2_{+\mathbf{f}}$ is the upper-half plane with the vertical segments

$$s_i = \{z = (x_i, y) : 0 < y \leq y_i\}$$

deleted. Proposition 6.4 applies to this family of examples (take $\Phi = 0$ except at x_i where $\Phi(x_i) = y_i$). Note that these simple examples illustrate the fact that the property of h to be increasing in the vertical direction is very far from implying the validity of a boundary Harnack principle or the validity of the doubling condition for the measure with density h^2 on $(\widetilde{U}, \rho_U)$.

Proposition 6.6. — *Let $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz function with Lipschitz constant k. Let U be the domain in $\mathbb{R}^n$ above the graph of Φ. Then U is (C_0, c_o)-uniform with respect to the usual metric in $\mathbb{R}^n$, with $C_0 = 4k + 3$ and $c_0 = (2k + 2)^{-2}$.*

Proof. — Given any two points $x, y \in U$, let $R = \rho(x, y)$ be the Euclidean distance between x and y. Let $\vec{e}_n$ be the unit vector pointing 'up', in relationship to the graph of the function Φ. Consider the path γ consisting of three line segments:

$$(x, x'),\ (x', y') \text{ and } (y', y),$$

where

$$x' = x + (2k + 1)R\vec{e}_n, \text{ and } y' = y + (2k + 1)R\vec{e}_n$$

We have $\rho(x', \partial U) \geq 2R$ and $\rho(y', \partial U) \geq 2R$, while $\rho(x', y') = R$, so the second segment of the curve γ is at least R away from ∂U. The length of the path γ is at most $(4k + 3)R$. It remains to confirm that on the first segment of the path γ, for $z = x + t\vec{e}_n$ with $t \leq (2k + 1)R$,

$$\rho(z, \partial U) \geq c_1 t \frac{\rho(z, y)}{R}$$

Using the Lipschitz nature of the function Φ, after a simple trigonometry exercise we obtain

$$\rho(x + t\vec{e}_n, \partial U) \geq \frac{t}{\sqrt{k^2 + 1}} \geq \frac{t}{k + 1} \geq t \frac{\rho(z, y)}{(k + 1)(2k + 2)R} \geq c_1 t \frac{\rho(z, y)}{R}$$

as desired. □

Proposition 6.7. — *Let $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz function with Lipschitz constant k. Let U be the domain in $\mathbb{R}^n$ above the graph of Φ. Let h be the profile produced by* Proposition 6.4. *Then the measure with density h^2 on U satisfies the doubling volume property on $(\overline{U}, \rho_U)$.*

Proof. — Note that ρ_U is uniformly comparable to the Euclidean distance in U and that $\overline{U} = \widetilde{U}$. Pick $x \in \overline{U}$ and $r > 0$ and set $B = \{y \in U : \rho_U(x, y) < r\}$, $2B = \{y \in U : \rho_U(x, y) < 2r\}$. We need to bound $\int_{2B} h^2 d\lambda$ from above by $C \int_B h^2 d\lambda$. The two balls $B, 2B$ have comparable volume of order r^n (see Proposition 6.6!). Let x_r be the point at distance $8kr$ vertically above x. The Euclidean ball of center x_r and radius $6r$ is contained in U. By the classical Harnack inequality, the values of h is $B(x_r, 3r)$ are at most $H_0 h(x_r)$ for some constant H_0. As h is increasing, the values of h in $B(x, 2r)$ are also bounded by $H_0 h(x_r)$. Hence

$$\int_{2B} h^2 d\lambda \leq C r^n h(x_r).$$

Let x'_r be the point at distance $r/2$ vertically above x. The Euclidean ball $B(x'_r, r/(4k))$ is contained in B and the values of h on the ball $B(x'_r, r/(8k))$ are bounded below by $H_0^{-1}h(x'_r)$. Hence

$$\int_B h^2 dx \geq cr^n h(x'_r).$$

Finally, applying the Harnack inequality at most $(10k)^2$ times along the vertical segment joining x'_r to x_r shows that $h(x_r) \leq H_0^{100k^2} h(x'_r)$. Hence

$$\int_{2B} h^2 d\lambda \leq C' \int_B h^2 d\lambda$$

as desired. □

Theorem 6.8. — *Let $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz function. Let U be the domain in $\mathbb{R}^n$ above the graph of Φ. Let h be the profile produced by* Proposition 6.4. *Then the Dirichlet form*

$$\int_U |\nabla f|^2 h^2 d\lambda, \quad f \in W^1(U, h^2 d\lambda)$$

is a Harnack-type Dirichlet form on $(\overline{U}, h^2 d\lambda, \rho_U)$. Moreover, this form coincides with the form

$$\int_U |\nabla (hf)|^2 d\lambda, \quad f \in H^{-1} W_0^1(U, d\lambda)$$

where H denotes pointwise multiplication by h in U.

Proof. — Use Propositions 6.6 and 6.7 above to apply Theorem 3.34 of Chapter 3 together with Propositions 5.7 and 5.8 of Chapter 5. □

Remark 6.9. — This proof does not require the results obtained in Chapter 4 such as the Harnack boundary principle. Instead, it provide a new proof of these important results in the case of domains above the graph of a Lipschitz function.

In the following corollaries, we state direct consequences of the previous theorem. We state this consequences in classical terms, that is, in terms of classical solutions of the Laplace equation vanishing continuously on the boundary. It is well known that every point on the boundary of the domain above the graph of a Lipschitz function is regular. This implies that the profile $h \in W^1_{0,\mathrm{loc}}(U)$ belongs to $\mathscr{C}(\overline{U})$ and vanishes continuously along the boundary of U.

Our first corollary is the well-known Harnack boundary principle and the assorted Hölder regularity for harmonic functions vanishing along ∂U.

Corollary 6.10. — *Let $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz function. Let U be the domain in $\mathbb{R}^n$ above the graph of Φ. Let h be the profile produced by* Proposition 6.4.

1. *There are constants* $A, H_1 \in (0,\infty)$ *such that for any* $x \in \overline{U}, r > 0$*, and any positive function* u *continuous on* $\overline{U} \cap B(x, Ar)$*, harmonic in* $U \cap B(x, Ar)$ *and vanishing on* $\partial U \cap B(x, Ar)$*, we have*

$$\sup_{U\cap B(x,r)} \{u/h\} \leq H_1 \inf_{U\cap B(x,r)} \{u/h\}.$$

2. *There are constants* $A, H_2 \in (0,\infty)$ *and* $\alpha \in (0,1)$ *such that for any* $x \in \overline{U}, r > 0$*, and any function* u *continuous on* $\overline{U} \cap B(x, Ar)$*, harmonic in* $U \cap B(x, Ar)$ *and vanishing on* $\partial U \cap B(x, Ar)$*, we have*

$$\sup_{y,z\in U\cap B(x,r)} \left\{ \frac{|u(y)/h(y) - u(z)/h(z)|}{\|y-z\|^\alpha} \right\} \leq H_2 r^{-\alpha} \sup_{U\cap B(x,Ar)} \{|u|/h\}.$$

Remark 6.11. — One consequence of the Harnack inequality above is that any two profiles for such a domain U must be (positively) proportional. The results of this corollary are very well-known and were first proved in [**3**]. The approach used here is quite different and gives an alternative proof. This new approach has the advantage of connecting these results to more classical interior estimates (according to our approach, the boundary estimates are simply interior estimates for a different form on a different space).

Our next corollary gives a similar result in the parabolic case. Parabolic boundary Harnack inequalities in Lipschitz domains (for divergence and non-divergence form operators) are studied in [**40, 41, 42, 60, 61, 62, 80**]. The parabolic Harnack inequality stated below should be seen as a building block to obtain further results along these lines. See, e.g., [**42, 79**]. Let us note that, in a sense, our proof gives both the elliptic and parabolic results at once. Fix $\tau > 0$ and $0 < \epsilon < \eta < \sigma < 1$. For any $x \in \overline{U}$ any $r > 0$, set

$$\begin{aligned} Q_A &= (s-\tau r^2, s) \times [U \cap B(x, Ar)],\\ Q_- &= (s-\sigma\tau r^2, s-\eta\tau r^2) \times [U \cap B(x, r)],\\ Q_+ &= (s-\epsilon\tau r^2, s) \times [U \cap B(x, r)],\\ Q' &= (s-\sigma\tau r^2, s-\epsilon\tau r^2) \times [U \cap B(x, r)]. \end{aligned}$$

Corollary 6.12. — *Let* $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ *be Lipschitz function. Let* U *be the domain in* $\mathbb{R}^n$ *above the graph of* Φ*. Let* h *be the profile produced by* Proposition 6.4.

1. *Fix* $\tau \in (0,1)$*. There are constants* $A, H_1 \in (0,\infty)$ *such that for any* $x \in \overline{U}, r > 0$*, and any positive function* u *continuous on* $\overline{Q_A}$*, solution of the heat equation in* Q_A *and vanishing on* $(s-\tau r^2, s) \times \partial U \cap B(x, Ar)$*, we have*

$$\sup_{Q_-}\{u/h\} \leq H_1 \inf_{Q_+}\{u/h\}.$$

2. *There are constants* $A, H_2 \in (0,\infty)$ *and* $\alpha \in (0,1)$ *such that for any* $x \in \overline{U}, r > 0$*, and any function* u *continuous on* $\overline{Q}$*, solution of the heat equation in* Q_A *and*

vanishing on $\partial U \cap B(x, Ar)$, we have

$$\sup_{(t,y),(t',y')\in Q'} \left\{ \frac{|u(t,y)/h(y) - u(t',y')/h(y')|}{(\sqrt{|t-t'|} + \|y-y'\|)^{\alpha}} \right\} \leq H_2 r^{-\alpha} \sup_{Q_A}\{|u|/h\}.$$

Remark 6.13. — The role of the constant A in these two corollaries is important due to the fact that we have stated the result using Euclidean balls and their trace on U. The geometry of the domain (i.e., the Lipschitz constant) controls the size of $A > 1$ that is needed. The more "natural" statement uses the inner balls $B_U(x,r)$ and, in such statement, the constant $A > 1$ can be picked arbitrarily close to 1. See Theorem 5.18.

Finally, we state the basic Dirichlet heat kernel estimates in this particular situation.

Corollary 6.14. — *Let $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz function. Let U be the domain in $\mathbb{R}^n$ above the graph of Φ. Let h be the profile produced by* Proposition 6.4. *The Dirichlet heat kernel $h_U^D(t,x,y)$ satisfies*

$$\frac{c_1 h(x)h(y)}{t^{n/2}h(x_{\sqrt{t}})h(y_{\sqrt{t}})} e^{-C_1\rho_U(x,y)/t} \leq h_U^D(t,x,y) \leq \frac{C_2 h(x)h(y)}{t^{n/2}h(x_{\sqrt{t}})h(y_{\sqrt{t}})} e^{-c_2\rho_U(x,y)/t}$$

for all $t > 0$, $x, y \in U$, and with $z_r = (z_1, \dots, z_{n-1}, z_n + \sqrt{t})$ if $z = (z_1, \dots, z_n) \in U$.

Remark 6.15. — To put this section in perspective with respect to the more general results discussed in this monograph, it may be useful to consider the case when the Laplace operator is replaced by a uniformly elliptic divergence form operator (the domain U stays as above). The various results stated in this section still hold true in that case (and are well-known, at least in the elliptic case. What can be extracted from the literature in the parabolic case is less clear). However, the simple construction of the profile and the proof of its basic property (there is a constant C such that $h(x) \leq Ch(y)$ if x, y are on the same vertical line and x is below y) given in this section fall apart. This means that, to obtain the desired results in that case, we have to rely on the general results of Chapter 4. Please note that the profiles associated to two different uniformly elliptic operators may have rather different behaviors near the boundary and also at infinity. Because of this and as far as Dirichlet heat kernels are concerned, there is no easy way to pass from the results concerning the Laplacian to those for uniformly elliptic operators.

6.4. The complement of a convex set

The aim of this section is to treat the case of Euclidean domains that are the complement of a convex set. As in the previous section, we will give a specific construction of a profile having the desired property, making it possible to avoid using the general results of Chapter 4.

One interesting aspect of the class of sets considered in this section, i.e., complements of convex sets, is that those sets are not uniform but only inner uniform in general. This means that the inner distance and inner geometry are crucial even just to state the results in this case: Harnack inequalities and heat kernel bounds can be stated correctly only in terms of the inner distance ρ_U and the associated inner balls $B_U(x,r)$. This is in contrast with what happens in the case of the domain above the graph of a Lipschitz function where one can still work with Euclidean balls and their traces on U.

6.4.1. The complement of a convex set is inner uniform. — We start with a proof that the complement U in $\mathbb{R}^n$ of a closed convex set V is inner uniform. This shows that the general results developed in this monograph for inner uniform subsets of Harnack type Dirichlet spaces apply to this class of Euclidean subsets.

Proposition 6.16. — *Let U be a domain of the form $U = \mathbb{R}^n \setminus V$ where $V \subset \mathbb{R}^n$ is closed and convex. Then U is inner uniform with $c_0 = 21$, $c_1 = 1/462$.*

Proof. — This result is not as obvious as it may first appear and the proof is somewhat technical. We need some notation. For any $x \in U$, let $z(x)$ be the point of V closest to x. Set $\vec{u}(x) = (x - z(x))/|x - z(x)|$. Both $z(x)$ and $\vec{u}(x)$ are continuous functions of x. See, e.g., [**48**, pages 11–12].

Claim *For any two points $x, y \in U$ with $\min\{\rho_U(x,V), \rho_U(y,V)\} = r > 0$, there exists an absolutely continuous curve $\gamma \subset U$ joining x to y, of length at most $4(\rho_U(x,y)+2r)$ such that $\rho_U(\gamma, V) \geq r$.*

Proposition 6.16 easily follows form this claim. Indeed, let x, y be points in U with $R = \rho_U(x,y)$, $r = \min\{\rho_U(x,V), \rho_U(y,V)\}$. If $R \leq r$ the straight line segment $[x,y]$ from x to y is contained in U. Moreover, $[x,y]$ is contained in a half-space E contained in U (to see this, consider a point ξ of $[x,y]$ such that $\rho_U([x,y],V) = \rho_U(\xi,V)$). The semi-circle with diameter $[x,y]$ contained in E and orthogonal to the hyperplane bounding E yields a curve of length $\pi\rho_U(x,y) = \pi|x-y|$ such that

$$\rho_U(z,V) \geq \frac{|z-x||z-y|}{|x-y|} = \frac{\rho_U(z,x)\rho_U(z,y)}{\rho_U(x,y)}.$$

Consider now the case where $R > r$. Set

$$x_R = x + (R/2)\vec{u}(x), \quad y_R = y + (R/2)\vec{u}(y)$$

and let γ the curve joining x_R to y_R given by the claim. Note that

$$\min\{\rho_U(x_R,V), \rho_U(y_R,V)\} \in (R/2, 3R/2).$$

Hence, $\rho_U(\gamma,V) \geq R/2$ and γ has length at most $4(\rho_U(x_R,y_R) + 3R) \leq 20R$. Let γ' be the absolutely continuous curve that goes straight from x to x_R, then from x_R to

y_R following γ, and finally straight from y_R to y. By construction, the length of γ' is at most $21R$ and for any point z on γ',

$$\rho_U(z,V) \geq \begin{cases} \rho_U(z,x) & \text{if } z \in [x, x_R] \\ R/2 & \text{if } z \in \gamma \\ \rho_U(z,y) & \text{if } z \in [y_R, y]. \end{cases}$$

If $z \in [x, x_R]$ (resp. $z \in [y_R, y]$) then we have $\rho_U(z,y) \leq 3R/2$ (resp. $\rho_U(z,x) \leq 3R/2$) and thus

$$\rho(z,V) \geq \frac{2}{3} \frac{\rho_U(z,x)\rho_U(z,y)}{\rho_U(x,y)}.$$

If $z \in \gamma'$ then $\rho_U(z,x)\rho_U(z,y) \leq 231R^2$ and thus

$$\rho(z,V) \geq \frac{1}{462} \frac{\rho_U(z,x)\rho_U(z,y)}{\rho_U(x,y)}.$$

Hence, to finish the proof of Proposition 6.16 we are now left with the task of proving the claim made above.

For any $x \in U$, let H_x be the linear hyperplane orthogonal to $\vec{u}(x)$. By construction V is contained in the half-space $\{\xi : (\xi - z(x)) \cdot \vec{u}(x) \leq 0\}$ and we have

$$\rho_U((x + H_x), V) = \rho_U(x, V).$$

Fix two points $x, y \in U$ with $\min\{\rho_U(x,V), \rho_U(y,V)\} = r > 0$ and set

$$\alpha = \alpha(x,y) = \vec{u}(x) \cdot \vec{u}(y).$$

If $\alpha = 1$ we must have $H_x = H_y$ and it follows that the straight line segment $[x,y]$ satisfies the conditions required in the claim. Assume next that $\alpha \in (-\sqrt{2}/2, 1)$ and let P be the $(n-2)$ dimensional vector space $H_x \cap H_y$. The unit vectors

$$\vec{v}(y) = (1-\alpha^2)^{-1/2}(\vec{u}(x) - \alpha\vec{u}(y)) \in H_y, \quad \vec{v}(x) = (1-\alpha^2)^{-1/2}(\vec{u}(y) - \alpha\vec{u}(x)) \in H_x$$

are orthogonal to P and have scalar product

$$\vec{v}(x) \cdot \vec{v}(y) = (1-\alpha^2)^{-1}(-\alpha + \alpha^3) = -\alpha.$$

We can write (uniquely)

$$y - x = \vec{p} + a\vec{v}(x) + b\vec{v}(y), \quad \vec{p} \in P, a, b \in \mathbb{R}.$$

Since $x, y \in U$, we must have $(x-y) \cdot \vec{u}(y) > 0$ and $(y-x) \cdot \vec{u}(x) > 0$, that is, $a < 0$, $b > 0$. Thus if $\alpha \geq -\sqrt{2}/2$,

$$|y-x|^2 = |\vec{p}|^2 + a^2 + b^2 - 2ab\alpha \geq |\vec{p}|^2 + (1 + \min\{\alpha, 0\}(a^2+b^2) \geq |p|^2 + \frac{1}{4}(a^2+b^2).$$

Consider the curve γ made of the three straight line segments

$$\begin{cases} [x, x + a\vec{v}(x)] & \subset x + H_x, \\ [x + a\vec{v}(x), y - b\vec{v}(y)] & \subset x + a\vec{v}(x) + P \subset (x + H_x) \cap (y + H_y), \\ [y - b\vec{v}(y), y] & \subset y + H_y. \end{cases}$$

Its length is

$$|p| + |a| + |b| \leq \sqrt{3}\sqrt{|p|^2 + a^2 + b^2} \leq 2\sqrt{3}|y - x| \leq 2\sqrt{3}\rho_U(x, y)$$

and

$$\rho_U(\gamma, V) = \min\{\rho_U(x, V), \rho_U(y, V)\}.$$

Thus γ satisfies the conditions required in the claim.

Finally, consider the case when $\alpha \in [-1, -\sqrt{2}/2]$. For any fixed $\lambda \in (1, 2)$, let γ' be an absolutely continuous path in U from x to y of length $\lambda\rho_U(x, y)$. Let $[0, T] \ni t \mapsto \gamma(t)$ be the arclength parametrization of γ. and let $\alpha(t) = \vec{u}(x) \cdot \vec{u}(\gamma(t))$. The function $t \mapsto \alpha(t)$ is continuous and varies from $\alpha(0) = |\vec{u}(x)|^2 = 1$ to $\alpha(T) = \vec{u}(x) \cdot \vec{u}(y) = \alpha \in [-1, -\sqrt{2}/2]$. Hence there exists $t_0 \in [0, T]$ such that $\alpha(t_0) = 0$. Let $x_0 = \gamma(t_0)$. As the unit vectors $\vec{u}(x), \vec{u}(x_0), \vec{u}(y)$ satisfy $\vec{u}(x) \cdot \vec{u}(x_0) = 0$, $\vec{u}(x) \cdot \vec{u}(y) < -\sqrt{2}/2$, we must have $|\vec{u}(x_0) \cdot \vec{u}(y)| \leq \sqrt{2}/2$. Observe however that x_0 may be closer to V than x and y. Thus, let $x_0' = x_0 + r\vec{u}(x_0)$ so that

$$\vec{u}(x_0') = \vec{u}(x_0), \quad \rho_U(x_0', V) > r, \quad \rho_U(x, x'0) + \rho_U(x_0', y) \leq 2r + \lambda\rho_U(x, y)$$

As $\vec{u}(x) \cdot \vec{u}(x_0') = 0$ and $\vec{u}(x_0') \cdot \vec{u}(y) \geq -\sqrt{2}/2$, the argument above yields curves γ_1, γ_2 from x to x_0' and x_0' to y which stay at distance at least r away from V and have length at most $2\sqrt{3}\rho_U(x, x_0')$, $2\sqrt{3}\rho_U(x_0', y)$, respectively. Putting γ_1, γ_2 together we obtain a curve from x to y that stays at distance at least r away from V and has length at most $2\sqrt{3}(\lambda\rho_U(x, y) + 2r)$. Picking $\lambda > 1$ close enough to 1 proves the claim. This finishes the proof of Proposition 6.16. □

6.4.2. Exterior of a star-shaped domain.— In this section we will construct a positive harmonic function $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$ when U is a domain which is the exterior of a star-shaped closed set in $\mathbb{R}^n$. Let B_t be the standard Brownian motion process in $\mathbb{R}^n$ (driven by Δ, not $\frac{1}{2}\Delta$). Let

$$P(t, x) = P_U(t, x) = \mathbb{P}^x\{\forall s \leq t,\ B_s \in U\}$$

be the probability that the process B_t, killed at ∂U, is still alive at time t.

Lemma 6.17. — *For each $t > 0$, the function $P(t, x)$ is in $\mathcal{F}^0_{\mathrm{loc}}(U)$. Moreover, for any $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $t_0 > 0$ there exists $C(t_0, \phi)$ such that*

$$\forall\, t \in (t_0, \infty), \quad \|\phi P(t, \cdot)\|_{W^1(U)} \leq C(t_0, \phi).$$

Proof. — The Dirichlet heat kernel $P^D_U(t, x, y)$ is bounded by the heat kernel of $\mathbb{R}^n$ and thus is in $L^2(U)$ for each fixed t, y. By the semigroup property, it follows that

$$x \mapsto u_{t,}(x) = p^D_U(t, x, y)$$

is in the domain of all powers of the Dirichlet Laplacian. In particular, it is in $\mathcal{F}^0(U) = W^1_0(U)$. Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and set

$$v_{t,y}(x) = \phi(x)p^D_U(t, x, y) = \phi(x)u_{t,y}(x).$$

We claim that there are constants $C = C(t_0, \phi), c = c(t_0, \phi) \in (0, \infty)$ such that

$$\forall t \in (t_0, \infty),\ \ \forall y \in U,\ \ \|v_{t,y}\|^2_{W^1(U)} \le C \exp(-c|y|^2). \tag{6.3}$$

This shows that $\phi(x)P(t,x) = \int_U \phi(x)p_U^D(t,x,y)dy$ is in $W_0^1(U)$ for all $t > 0$ and $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with

$$\forall t \in (t_0, \infty),\ \ \|\phi P(t,\cdot)\|^2_{W^1(U)} \le C'(t_0, \phi).$$

In particular, $P(t,\cdot) \in \mathscr{F}^0_{\text{loc}}(U)$ as desired. To prove the claim above observe that

$$\forall t \in (t_0, \infty),\ \ \forall y \in U,\ \ \|v_{t,y}\|^2_{L^2(U)} \le C \exp(-c|y|^2)$$

by the classical Gaussian formula for the heat kernel in $\mathbb{R}^n$ and the fact that ϕ has compact support. Moreover, the same estimate holds for the function $w_{t,y}(x) = \phi(x)\partial_t p_U^D(t,x,y)$. See, e.g. Theorem 4 and Corollary 5 in [**33**]. Now, we have

$$\Delta v_{t,y} = u_{t,y}\Delta\phi + w_{t,y} + 2\nabla\phi\cdot\nabla u_{t,y}$$

and

$$\begin{aligned}
\|\nabla v_{t,y}\|_2^2 &= -\int_U v_{y,y}\Delta v_{t,y} d\mu \\
&= -\int_U v_{t,y}\left(u_{t,y}\Delta\phi + v_{t,y}w_{t,y} + 2u_{t,y}\nabla\phi\cdot\nabla(\phi u) - 2u_{t,y}^2|\nabla\phi|^2\right)d\mu \\
&\le \|v_{t,y}\|_2\|(\Delta\phi)u_{t,y}\|_2 + \|v_{t,y}\|_2\|w_{t,y}\|_2 \\
&\qquad\qquad + 2\||\nabla\phi|u_{t,y}\|_2\|\nabla v_{t,y}\|_2 + 2\||\nabla\phi|u_{t,y}\|_2^2 \\
&\le \|v_{t,y}\|_2\|(\Delta\phi)u_{t,y}\|_2 + \|v_{t,y}\|_2\|w_{t,y}\|_2 + (1/2)\|\nabla v_{t,y}\|_2^2 + 4\||\nabla\phi|u_{t,y}\|_2^2
\end{aligned}$$

The L^2 estimate for $v_{t,y} = \phi u_{t,y}$ obtained above applies as well to $(\Delta\phi)u_{t,y}$ and $|\nabla\phi|u_{t,y}$. Together with the estimate for $w_{t,y}$, this yields (with a different constants C, c depending only of ϕ and t_0)

$$\forall t \in (t_0, \infty),\ \ \|\nabla v_{t,y}\|_2^2 \le C \exp(-c|y|^2).$$

This proves the claim (6.3). □

Definition 6.18. — A set $V \subset \mathbb{R}^n$ is said to be star-shaped with center z if

$$\forall \alpha \in (0,1),\ \ S_\alpha^z(V) \subset V$$

where $S_\alpha^z : \mathbb{R}^n \to \mathbb{R}^n$ denotes a scaling transformation with center z:

$$S_\alpha^z(x) = z + \alpha(x - z)$$

Lemma 6.19. — *Let U be a domain such that $U = \mathbb{R}^n \setminus V$ for some closed star-shaped region V with center z. Then for any point $x_0 \in U$ there exists a sequence of times t_i tending to infinity such that the sequence of functions*

$$h_i(x) = \frac{\int_{t_i}^{t_i+1} P(s,x)ds}{P(t_i, x_0)} \tag{6.4}$$

converges in $\mathscr{C}^\infty(U)$ (equipped with its natural family of seminorms) to a positive harmonic function h.

Proof. — Observe that $P(t,x)$ is a positive bounded solution of the heat equation in U and is non-increasing in t for any x in U. Using classical parabolic Harnack inequalities, the sequence $\{h_n(x)\}_{n=1}^{\infty}$ is uniformly bounded on any compact subset of U. Indeed, for any $\eta > 0$, any compact subset K of U and any integers k, l, there exists a constant $C = C(\eta, K, k, l)$ such that, for any $t \geq \eta$,

$$\sup_{x\in K}\left|\frac{\partial^{k+l}}{\partial x_{i_1}\dots\partial x_{i_k}\partial^l t}P(t,x)\right| \leq CP(t+\eta,x_0) \leq CP(t,x_0) \tag{6.5}$$

and

$$\sup_{x\in K}\left\{-\frac{\partial}{\partial t}P(t,x)\right\} \leq C \inf_{(s,x)\in Q}\left\{-\frac{\partial}{\partial t}P(s,x)\right\},\ Q = [t+\eta, t+2\eta]\times K. \tag{6.6}$$

The bound (6.5) implies that there exists a subsequence

$$h_i = \int_{n_i}^{n_i+1} P(s,\cdot)ds/P(n_i,x_0)$$

of the sequence $\int_n^{n+1} P(s,\cdot)ds/P(n,x_0)$ that converges to some function h in $\mathscr{C}^\infty(U)$. Note that the functions h_i are not harmonic. In order to prove that h is harmonic, we shall analyze

$$-\Delta h_i = -\frac{\int_{n_i}^{n_i+1}\Delta P(s,x)ds}{P(n_i,x_0)} = \frac{P(n_i,x) - P(n_i+1,x)}{P(n_i,x_0)}$$

Using scaling with center z and the scaling properties of Brownian motion, for any $x \in U$, any $t > 0$ and any $\varepsilon > 0$, we have

$$\begin{aligned} P(t,x) &= \mathbb{P}^x\Big(\forall s \leq t, B_s \in U\Big) = \mathbb{P}^x\Big(\forall s \leq (1+\varepsilon)^2 t, B_{(1+\varepsilon)^{-2}s} \in U\Big) \\ &= \mathbb{P}^{x+\varepsilon(x-z)}\Big(\forall s \leq (1+\varepsilon)^2 t, B_s \in (1+\varepsilon)U\Big) \\ &\leq \mathbb{P}^{x+\varepsilon(x-z)}\Big(\forall s \leq (1+\varepsilon)^2 t, B_s \in U\Big) \\ &\leq P\big((1+\varepsilon)^2 t, x+\varepsilon(x-z)\big). \end{aligned} \tag{6.7}$$

Therefore,

$$0 \leq P(t,x) - P\big((1+\varepsilon)^2t,x\big) \leq P\big((1+\varepsilon)^2t, x+\varepsilon(x-z)\big) - P\big((1+\varepsilon)^2t,x\big). \tag{6.8}$$

Fix $x \in U$. Set $r(x) = \operatorname{dist}(x,\partial U)$, $R(x) = |x-z|$. Let $\varepsilon = 1/t$. Assume that t is larger than $\max\{1, 2R(x)/r(x)\}$ and set $K = \{y : |x-y| \leq r(x)/2\}$. By (6.5), we have

$$\begin{aligned} |P\big((1+\varepsilon)^2t, x+\varepsilon(x-z)\big) - P\big((1+\varepsilon)^2t,x\big)| &\leq \varepsilon C(1,K,1,0)R(x)P((1+\varepsilon)^2t,x_0) \\ &\leq C_1(x)t^{-1}P(t,x_0). \end{aligned} \tag{6.9}$$

As $t(1+\varepsilon)^2 > t+1$, we have

$$P(t,x) - P(t+1,x) \leq \big(P(t,x) - P(t(1+\epsilon)^2,x)\big) \leq C_1(x)t^{-1}P(t,x_0).$$

Since the sequence h_i converges to h in $\mathscr{C}^\infty(U)$, we have

$$|\Delta h(x)| = \lim_{i\to\infty} \frac{P(n_i,x) - P(n_i+1,x)}{P(n_i,x_0)} \le \lim_{i\to\infty} \frac{C_1(x)}{n_i} = 0$$

as desired. □

Lemma 6.20. — *Let U be a domain such that $U = \mathbb{R}^n \setminus V$ for some closed star-shaped region V with center z. For any $x \in U$ and any $\lambda \ge 1$, the function h constructed above satisfies*

$$h(S^z_\lambda(x)) \ge h(x)$$

Remark 6.21. — A region V can be star-shaped with respect to many points z. In such cases, Lemma 6.20 provides the property $h(S^z_\lambda(x)) \ge h(x)$ for any such z. This will be important for us when we discuss complements of convex sets. It is one of the advantages of the above construction to provide one function h which satisfies the desired property with respect to any center z.

Proof. — Proceeding as for (6.7), for all $\lambda \ge 1$ and $t > 0$, the scaling properties of B_t and U yields

$$\begin{aligned} P(t,x) &= \mathbb{P}^x\Big(\forall s \le t, B_s \in U\Big) = P^x\Big(\forall s \le \lambda^2 t, B_{\lambda^{-2}s} \in U\Big) \\ &= \mathbb{P}^{S^z_\lambda(x)}\Big(\forall s \le \lambda^2 t, B_s \in S^z_\lambda(U)\Big) \\ &\le \mathbb{P}^{S^z_\lambda(x)}\Big(\forall s \le \lambda^2 t, B_s \in U\Big) \le P(\lambda^2 t, S^z_\lambda(x)) \end{aligned} \tag{6.10}$$

By (6.10), for any $\lambda \ge 1$, we have

$$P(s, S^z_\lambda(x)) \ge P(\lambda^{-2}s, x) \ge P(s,x). \tag{6.11}$$

As

$$h(x) = \lim_{i\to\infty} \frac{\int_{n_i}^{n_i+1} P(s,x)ds}{P(n_i,x_0)},$$

integrating (6.11) over $s \in (n_i, n_i+1)$, dividing by $P(n_i,x_0)$ and taking the limit as $i \to \infty$ we obtain $h(S^z_\lambda(x)) \ge h(x)$ as desired. □

Proposition 6.22. — *Let $U = \mathbb{R}^n \setminus V$ for some closed star-shaped region $V \subset \mathbb{R}^n$. Assume that for any compact subset K of $\mathbb{R}^n$ there exist $t_0, C \in (0,\infty)$ and a compact subset K' of U such that*

$$\forall t > t_0,\ \sup_{x\in K}\{P(t,x)\} \le C \sup_{x\in K'}\{P(t,x)\}. \tag{6.12}$$

Then the function h constructed above is in $\mathscr{F}^0_{\mathrm{loc}}(U)$.

Proof. — We know that $P(t,x)$ belongs to $\mathscr{T}^0_{\mathrm{loc}}(U)$. For any $t \geq 1$ and any $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with support in a compact subset A of $\mathbb{R}^n$, the basic Caccioppoli type estimate applied to $P(t,x)$ gives

$$\int_t^{t+1} \int_U |\nabla(\varphi P(s,x))|^2 ds dx \leq C_\varphi \int_t^{t+1} \int_A |P(s,x)|^2 ds dx \leq C_\varphi \int_A |P(t,x)|^2 dx.$$

Hence

$$\int_t^{t+1} \int_U |\nabla(\varphi P(s,x))|^2 ds dx \leq C'_\varphi \sup_{x\in A} P(t,x)^2.$$

For any $t \geq \max\{t_0, 1\}$, using the hypothesis (6.12) and (6.5), we obtain

$$\int_t^{t+1} \int_U |\nabla(\varphi P(t,x))|^2 dx \leq C''_\varphi P(t,x_0)^2$$

Let t_i be an increasing sequence of times tending to infinity such that

$$h_i = \int_{t_i}^{t_i+1} P(s,\cdot) ds / P(t_i, x_0)$$

tends in $\mathscr{C}^\infty(U)$ to a harmonic function h. By the hypothesis (6.12), h_i also converges to h in $L^2_{\mathrm{loc}}(\overline{U})$. Furthermore, by the previous energy inequality, for any $\varphi \in \mathscr{C}^\infty_c(\widetilde{U})$, we have

$$\int_U |\nabla(\varphi h_i)|^2 \leq C''_\varphi.$$

Together with Lemma 6.2, this shows that the limit h of the h_i belongs to $\mathscr{T}^0_{\mathrm{loc}}(U)$ as desired. □

Proposition 6.23. — *Let U be a domain of the form $U = \mathbb{R}^n \setminus V$ for some closed set $V \subset \mathbb{R}^n$. Assume there exist $z_0 \in V$ and $r_0 > 0$ such that V contains the Euclidean ball $E_0 = \{z \in \mathbb{R}^n : |z - z_0| < r_0\}$ and is star-shaped with respect to each $z \in E_0$. Then there exists a positive harmonic function h in U which belongs to $\mathscr{T}^0_{\mathrm{loc}}(U)$ and satisfies $h(S^z_\lambda(x)) \geq h(x)$ for every $\lambda \geq 1$ and $z \in E_0$.*

Proof. — Apply Proposition 6.22 and Lemma 6.20. To verify (6.12) use the fact that for any fixed $\lambda > 1$ (for instance, we can use $\lambda = 2$) and for any $x \in U$ the Euclidean distance between $S^{z_0}_\lambda(x)$ and V is larger than $r_0/(2(\lambda-1))$ (the semi-axis with origin x and direction $x - x_0$ is surrounded by spherical cone with vertex x and aperture α with $\tan(\alpha) = r_0/|x - z_0|$ entirely contained in U). Hence, for any compact $K \subset \overline{U}$ and $\lambda > 1$, $S^{z_0}_\lambda(K)$ is compact in U. □

Example 6.24. — An interesting example worth mentioning here is the following. Let $U \subset \mathbb{R}^3$ be given by

$$U = \mathbb{R}^3 \setminus [C_1 \cup C_2 \cup C_3]$$

where C_i is the doubly infinite cylinder of radius 1 around the i-th axis, that is,

$$C_i = \{x = (x_j)_1^3 \in \mathbb{R}^3 : \sum_{j\neq i} x_j^2 \leq 1\}.$$

This is star shaped with respect to any point z in the unit ball around the origin. Proposition 6.23 provide the existence of a profile in this case but fails to indicate that the measure $h^2 d\lambda$ is doubling. It is easy to see that this domain is uniform so that our general results for uniform domains apply and the measure $h^2 d\lambda$ is indeed doubling in this case.

Proposition 6.25. — *Let U be a domain of the form $U = \mathbb{R}^n \setminus V$ for some closed convex set $V \subset \mathbb{R}^n$. Then there exists a positive harmonic function h in U which belongs to $\mathcal{F}^0_{\mathrm{loc}}(U)$ and satisfies $h(S^z_a(x)) \geq h(x)$ for every $a \geq 1$ and $z \in V$. Moreover the measure $d\nu = h^2 d\lambda$ satisfies the doubling condition* (2.2) *on* $(\widetilde{U}, \rho_U)$.

Proof. — To obtain the desired function $h \in \mathcal{F}^0_{\mathrm{loc}}(U)$ satisfying $h(S^z_\lambda(x)) \geq h(x)$ for every $\lambda \geq 1$ and $z \in V$, apply Proposition 6.22 and Lemma 6.20. This requires checking (6.12). For any $x \in U$, let $z(x)$ be the closest point to x in V. On the oriented semi-axis D_x with vertex $z(x)$ going through x, the function h is a non-decreasing function and for any point $y = z(x) + s(x - z(x)) \in D_x$, $\rho_U(y, V) = |y - z(x)| = s|x - z(x)|$, $\rho_U(x, y) = |s - 1|$. Thus (6.12) is satisfied with $K' = \{y \in U : \rho(y, V) \geq 1 \text{ and } \rho(y, K) \leq 1\}$ which is compact in U for any compact K in $\overline{U}$.

To prove that $d\nu = h^2 d\lambda$ is doubling, fix $x \in \widetilde{U}$ and $B = B_{\widetilde{U}}(x, r)$, $2B = B_{\widetilde{U}}(x, 2r)$. It is clear that $\lambda(B) \simeq \lambda(2B) \simeq r^n$. For any $y \in U$, let $z(y)$ be as above and set

$$y_r = z(y) + \frac{|y - z(y)| + 4r}{|y - z(y)|}(y - z(y)).$$

Set also

$$x'_r = z(x) + \frac{|x - z(x)| + r/2}{|x - z(x)|}(x - z(x)).$$

If $y \in 2B$ then $\rho_U(y_r, x) \leq 6r$. Moreover, by Proposition 6.16 and the classical Harnack inequality, it is clear that $h(y_r) \simeq h(x_r)$. Again, by the classical Harnack inequality, $h(x_r) \simeq h(x'_r)$. Finally, because of the monotonicity of h along any of the semiaxis with base $z(y)$ passing through y, $h(y) \leq h(y_r)$. □

Remark 6.26. — Three different cases occurs for the complement of a closed convex set V. The fist case is when V has non-empty interior. In that case, $\overline{U} = \widetilde{U}$ and the profile h vanishes continuously along ∂U. The second case is when V has no interior points but is $n-1$ dimensional. In that case V is a closed set with nonempty interior in a hyperplane. Call δV the boundary of V in that hyperplane. Then $\widetilde{U} \setminus U$ is equal to two copies of V glued along their $(n-2)$-dimensional boundary δV. The profile function h still vanishes continuously along V. The third and last case is when V has capacity 0 (equivalently, Hausdorff dimension less than $n-1$). In that case, V has no effect and $\widetilde{U} = \mathbb{R}^n$. The profile h is the constant function 1.

Our next corollary states the validity of the parabolic boundary Harnack inequalities in the complement of a convex set. Fix $\tau > 0$, $0 < \epsilon < \eta < \sigma < 1$ and $\theta \in (0, 1)$.

For any $x \in \overline{U}$ any $r > 0$, set

$$\begin{aligned} Q &= (s-\tau r^2, s) \times B_U(x,r), \\ Q_- &= (s-\sigma\tau r^2, s-\eta\tau r^2) \times B_U(x,\theta r), \\ Q_+ &= (s-\epsilon\tau r^2, s) \times B_U(x,r), \\ Q' &= (s-\sigma\tau r^2, s-\epsilon\tau r^2) \times B(x,\theta r). \end{aligned}$$

Corollary 6.27. — *Let U be a domain in $\mathbb{R}^n$ which is the complement of a closed convex set. Let h be the profile produced by* Proposition 6.25.

1. *There is a constants $H_1 \in (0,\infty)$ such that for any $x \in U, r > 0$, and any positive function u continuous on $\overline{Q}$, solution of the heat equation in Q and vanishing on $(s-\tau r^2, s) \times [\partial U \cap \overline{B_U(x,r)}]$, we have*

$$\sup_{Q_-}\{u/h\} \le H_1 \inf_{Q_+}\{u/h\}.$$

2. *There are constants $H_2 \in (0,\infty)$ and $\alpha \in (0,1)$ such that for any $x \in U, r > 0$, and any function u continuous on $\overline{Q}$, solution of the heat equation in Q and vanishing on $(s-\tau r^2, s) \times [\partial U \cap \overline{B_U(x,r)}]$, we have*

$$\sup_{(t,y),(t',y')\in Q'} \left\{ \frac{|u(t,y)/h(y) - u(t',y')/h(y')|}{(\sqrt{|t-t'|} + \|y-y'\|)^\alpha} \right\} \le H_2 r^{-\alpha} \sup_{Q_A}\{|u|/h\}.$$

Finally, we state the basic Dirichlet heat kernel estimates in this particular situation.

Corollary 6.28. — *Let U be a domain in $\mathbb{R}^n$ above the graph of a Lipschitz function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Let h be the profile produced by* Proposition 6.4. *The Dirichlet heat kernel $h_U^D(t,x,y)$ satisfies*

$$\frac{c_1 h(x)h(y)}{t^{n/2}h(x_{\sqrt{t}})h(y_{\sqrt{t}})} e^{-C_1\rho_U(x,y)/t} \le h_U^D(t,x,y) \le \frac{C_2 h(x)h(y)}{t^{n/2}h(x_{\sqrt{t}})h(y_{\sqrt{t}})} e^{-c_2\rho_U(x,y)/t}$$

for all $t > 0$, $x, y \in U$, and with $z_r = (z_1,\dots,z_{n-1}, z_n + \sqrt{t})$ if $z = (z_1,\dots,z_n) \in U$.

Remark 6.29. — Again, one may want to consider the case when the Laplace operator is replaced by a uniformly elliptic divergence form operator (the domain U stays as above). The various results stated in this section still hold true in that case but the simple construction of the profile and the proof of its basic properties given in this section do not easily extend to this case. This means that, to obtain the desired results, we have to rely on the general results of Chapter 4.

6.5. Miscellaneous examples

6.5.1. The Von Koch snowflake. — Consider the von Koch snowflake domain V and the complement U of its closure K. The compact set K can be constructed by starting with an equilateral triangle, then recursively altering each line segment obtain at level n via the following well known procedure:

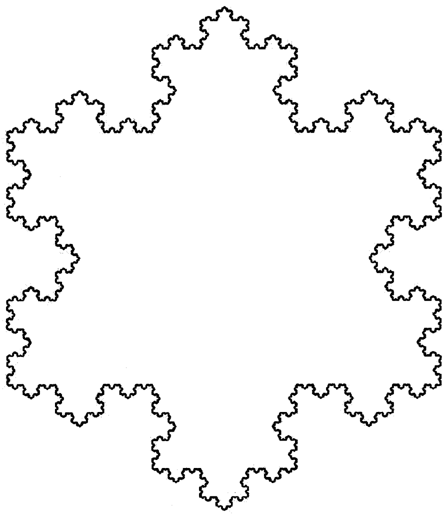


FIGURE 1. Von Koch snowflake - a domain in $\mathbb{R}^2$ with fractal boundary.

1. Divide the line segment into three segments of equal length.
2. Draw an equilateral triangle that has the middle segment from step 1 as its base and points towards the outside.
3. Remove the line segment that is the base of the triangle from step 2.

The von Koch curve is the limit as the above steps are followed over and over again. These domains and other domains with fractal boundaries were studied from the point of view of heat equation in [**15, 32, 34**]. See also [**46, 69**].

Proposition 6.30. — *Both the interior V and the exterior U of a von Koch snowflake domain of* Figure 6.5.1 *constructed above are uniform domains in* $\mathbb{R}^2$.

This is well know and follows, for instance, from [**70**] and the fact that the boundary of the snowflake is a quasi-circle. We outline a direct argument below.

Proof. — Let V denote the interior of von Koch snowflake. Let x and y be any two points in V. First we note that the Euclidean distance $\|x-y\|$ and the inner distance $\rho_V(x,y)$ are comparable. The points x and y each belongs to one of the triangles that were part of the iterative construction. Say, $x \in T_0$ for some triangle T_0 which was constructed on the n-th iteration. Consider the sequence $\{T_i\}_{i=1}^k$ of triangles constructed in the following way. Let T_1 be the triangle which side serves as the base $b(T_0)$ of the triangle T_0, let T_2 be the triangle which side serves as the base $b(T_1)$ of the triangle T_1, etc., until T_k is the main triangle T of the von Koch snowflake. Let $\{T_i'\}_{i=1}^l$ be the similar sequence for the point $y \in T_0'$.

Let 1 be the side length of the main triangle in the von Koch snowflake, and let $R = \rho(x, y)$ be the Euclidean distance between x and y. Without loss of generality we can assume that $T_{k-1} \neq T'_{l-1}$, or otherwise we can zoom in and consider the triangle T_{k-1} as the main triangle of von Koch snowflake.

Since x and y are located in different triangles and since the Euclidean distance ρ is comparable to the inner geodesic distance ρ_V in the interior of von Koch snowflake, we know that

$$\rho_V(x, b(T_{k-1})) \leq CR, \quad \text{and} \quad \rho_V(y, b(T'_{l-1})) \leq CR$$

for some positive constant C. Let γ' be the geodesic curve in V connecting x to the base $b(T_{k-1})$ and let $x'_i = \gamma' \cap b(T_i)$, $i = 0, \dots, k-1$. Let $|T_i|$ denote the length of the edge of the triangle T_i. Let x_i be the closest point in the base $b(T_i)$ to x'_i with

$$\rho_U(x_i, \partial V) \geq \min\left(\frac{R}{8}, \frac{|T_i|}{4}\right), \tag{6.13}$$

so that

$$\rho_V(x_i, x'_i) \leq \min\left(\frac{R}{4}, \frac{|T_i|}{2}\right)$$

and the sequence $\{x_i\}_{i=0}^{k-1}$ of points $x_i \in b(T_i)$ satisfies

$$\begin{aligned} \sum_{j=1}^{k-1} \rho_V(x_i, x_{i-1}) &\leq L(\gamma') + \sum_{i=0}^{k-1} 2\rho_V(x_i, x'_i) \\ &\leq CR + \sum_{i\,:\,|T_i| \leq R/2} |T_i| + \sum_{i\,:\,|T_i| > R/2} R/2 \\ &\leq CR + \frac{R}{2}\left(1 + \frac{1}{3} + \frac{1}{9} + \dots\right) + \frac{R}{2} \cdot N \end{aligned} \tag{6.14}$$

where N is the number of triangles in the family $\{T_i\}_{i=0}^{k-1}$ with $|T_i| > \frac{R}{2}$. The diameters of the triangles in the sequence $\{T_i\}_{i=0}^{k-1}$ are growing at least exponentially and there is at most one triangle in this sequence with $|T_i| > \rho_V(x, y)$ because for any index $i < k - 1$, we have

$$\rho_V(x, y) \geq \rho_V(b(T_i), b(T_{i+1})) \geq |T_i|.$$

Therefore the constant N in (6.14) is uniformly bounded, and so there exists a constant C' such that

$$\sum_{j=1}^{k-1} \rho_V(y_i, y_{i-1}) \leq C'R.$$

Similarly consider a sequence $\{y_j\}_{j=0}^{l-1}$ of points $y_j \in b(T_j')$ in the base of the triangle T_j' with

$$\begin{aligned} \rho_V(y_j, \partial V) &\geq \min\left(\frac{R}{8}, \frac{|T_j'|}{4}\right), \\ \sum_{j=1}^{l-1} \rho_V(y_i, y_{i-1}) &\leq C'R. \end{aligned}$$

Let z be the point in $T_k = T_l'$ with $\rho_V(z, \partial V) \geq \frac{R}{8}$, $\rho_V(z, x_{k-1}) \leq 2CR$ and $\rho_V(z, y_{l-1}) \leq 2CR$. The path γ consisting of line segments connecting the points

$$x, x_0, x_1, \dots, x_{k-1}, z, y_{l-1}, y_{l-2}, \dots, y_0, y$$

in this order is a desired path satisfying the uniform condition (3.1).

Similarly we can prove that the exterior of von Koch snowflake is a uniform domain in $\mathbb{R}^2$, because it can be represented as a union of countably many triangles constructed via a similar procedure. □

6.5.2. Cones. — In dimension 2, any positive connected cone $U = \{x \in \mathbb{R}^2 : x = r\theta,\ \theta \in (\theta_1, \theta_2)\}$ is, obviously, a uniform (convex if $|\theta_1 - \theta_2| \leq \pi$) subset of $\mathbb{R}^2$.

In higher dimension, a cone

$$U = \{x \in \mathbb{R}^n : x = r\theta,\ \theta \in \Omega \subset \mathbb{S}^{n-1}\}.$$

is uniform (inner uniform) whenever Ω is a uniform (inner uniform) domain in $\mathbb{S}^{n-1}$. In particular, it is uniform whenever Ω is a domain with smooth boundary in $\mathbb{S}^{n-1}$.

The following well known proposition gives a formula (in terms of an auxiliary function) for the profile of any positive cone.

***Proposition 6.31* (see, e.g., [10]).** — *Let $U = \mathbb{R}_+ \times \Omega \subset \mathbb{R}^n$ be the positive cone in $\mathbb{R}^n$ based on the spherical domain $\Omega \subset S^{n-1}$, where a sphere S^{n-1} is the unit sphere in $\mathbb{R}^n$. Let ϕ be the first Dirichlet eigenfunction of the spherical Laplacian with eigenvalue λ. Then in polar coordinates,*

$$h(x) = |x|^\alpha \phi(x/|x|)$$

with

$$\alpha = \frac{\sqrt{(n-2)^2 + 4\lambda} - (n-2)}{2} > 0$$

(so that $\alpha(\alpha + n - 2) = \lambda$) is a positive harmonic function in U vanishing on ∂U.

Proof. — This result follows from the positivity of the first Dirichlet eigenfunction and the representation of Δ in polar coordinates via the spherical Laplacian $L_{S^{n-1}}$,

$$\Delta = \frac{1}{r^2} L_{S^{n-1}} + \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\right).$$

The details are left to the reader. □

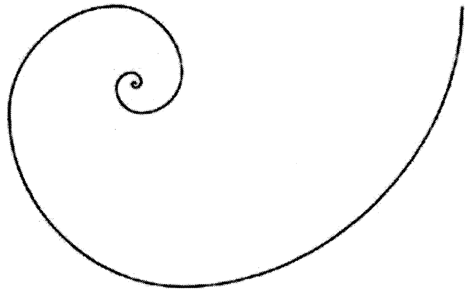


FIGURE 2. The Fibonacci spiral in $\mathbb{R}^2$.

Obviously, in dimension greater than 2, not all cones are inner-uniform but a cone is inner-uniform whenever its "base" Ω is inner-uniform has a domain in $\mathbb{S}^{n-1}$. This produce many interesting examples. Dirichlet heat kernel estimates in conical domains are the subject of a series of interesting papers by Varopoulos [**95, 96, 97**], motivated in part by heat kernel estimates on Lie groups. Even in such example, the local behavior of the profile h near the boundary of the cone can be very non-trivial (e.g., the set Ω could be the interior of a small enough Von Koch snow-flake drawn on the surface of the 2-sphere).

6.5.3. The Fibonacci spiral. — The proof of the following simple proposition is left to the reader.

Proposition 6.32. — *The complement U in $\mathbb{R}^2 = \mathbb{C}$ of the spiral S given in the parametric form by $z(t) = \exp(t + ic\pi t)$ (see* Figure 2*) for some constant $c > 0$ is inner uniform.*

Note, of course, that the set U is not uniform!

Proposition 6.33. — *Let $U \subset \mathbb{R}^2 = \mathbb{C}$ be the complement of the infinitely winding spiral S defined above. Then the function*

$$h(x) = \operatorname{Im}\left[\exp\left(\frac{1 - ic\pi}{2}\log(x_1 + icx_2)\right)\right], \quad x = (x_1, x_2)$$

is a positive harmonic function in U vanishing on ∂U. Here the function log *is any branch of a complex logarithm function in the simply connected domain $\mathbb{C} \setminus S$.*

Proof. — For this result we constructed the function h as the imaginary part of the combination of conformal maps,

$$h = \operatorname{Im} \circ \phi^{-1} \circ \psi,$$

where

$$\phi : \{z \in \mathbb{C} : 0 \leq \operatorname{Im}(z) \leq \frac{2\pi}{1 + c^2\pi^2}\} \to \mathbb{C} \setminus S, \quad z \to \exp\left(z + ic\pi z\right)$$

and

$$\psi : \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \frac{2\pi}{1+c^2\pi^2}\} \to \mathbb{H}, \quad z \to \exp\left(\frac{1+c^2\pi^2}{2} z\right).$$

The function h is the imaginary part of the conformal map from $\mathbb{C} \setminus S$ to the set $\mathbb{H}$ of complex numbers with positive real part. Therefore h is harmonic, positive and vanishes on ∂U. □

Remark 6.34. — For points in the complement of the spiral S of the form $x = \exp(t + ic\pi t - \theta)$ with fixed $\theta \in (0, \frac{2}{c})$, we have

$$\begin{aligned} h(x) &= \mathrm{Im}\left[\exp\left(\frac{1-ic\pi}{2}(t + ic\pi t - \theta)\right)\right] \\ &= \exp\left(\frac{1+c^2\pi^2}{2} t\right) e^{-\frac{\theta}{2}} \sin\left(\frac{\theta c\pi}{2}\right) \asymp |x|^{(1+c^2\pi^2)/2}. \end{aligned}$$

This shows the growth of the function h in $\mathbb{C} \setminus S$ resembles that in the cone with angle $\frac{2\pi}{1+c^2\pi^2}$.

6.6. Examples in sub-Riemannian geometry

6.6.1. The canonical sub-Riemannian geometry of the Heisenberg group.

— The Heisenberg group $\mathbb{H}_n$ of dimension $2n+1$ can be viewed as $\mathbb{R}^{2n+1}$ equipped with the product

$$gg' = (x, y, z)(x', y', z') = (x + x', y + y', z + z' + (1/2)(y \cdot x' - x \cdot y'))$$

where

$$g = (x, y, z) \in \mathbb{R}^{2n+1}, \quad x = (x_i)_1^n \in \mathbb{R}^n, \quad y = (y_i)_1^n \in \mathbb{R}^n, \quad z \in \mathbb{R}$$

and similarly for g'. On the right-hand side, $x \cdot y' = \sum_1^n x_i y_i'$ denotes the usual scalar product in $\mathbb{R}^n$. The Lebesgue measure λ is a Haar measure (left and right) on $\mathbb{H}_n$.

The Lie algebra, viewed as the space of left-invariant vector fields, has basis $(X_1, \cdot, X_n, Y_1, \cdot, Y_n, Z)$ satisfying $[X_i, Y_i] = Z$ whereas all other brackets $[X_i, Y_j]$, $i \neq j$, $[X_i, X_j]$, $[Y_i, Y_j]$, $i, j \in \{1, \dots n\}$, and those involving Z vanish. This Lie algebra is thus generated by the set

$$\{X_1, \dots, X_n, Y_1, \dots, Y_n\}.$$

By Hörmander's theorem [**64**], this makes the sum of squares

$$\Delta = \sum_1^n (X_i^2 + Y_i^2)$$

a hypoelliptic operator. It is, perhaps, the most studied model of a hypoelliptic sum of squares. In many ways, as far as the Heisenberg group is concerned, this sub-Laplacian is more natural and canonical than any Riemannian Laplacian. One reason is that $\mathbb{H}_n$ admits a dilation structure given by

$$\delta_r(x, y, z) = (rx, ry, r^2 z)$$

that commutes with the group law. The operator Δ is homogeneous of degree 2 for that structure whereas no Riemannian Laplacian can have this property. In the present global coordinate system, the fields X_i, Y_i, $1 \le i \le n$, and Z are given by

$$X_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2}\frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} - \frac{x_i}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

The Dirichlet structure that yields Δ as its infinitesimal generator is simply

$$\mathcal{E}(f,f) = \int_{\mathbb{H}_n} \sum_1^n |X_i f|^2 + |Y_i f|^2 d\lambda$$

with domain $\mathcal{D}(\mathcal{E}) = W^1(\mathbb{H}_n)$. Here $W^1(\mathbb{H}_n)$ denotes the subspace of $L^2(\mathbb{H}_n) = L^2(\mathbb{H}_n, \lambda)$ of those functions f whose derivatives $X_i f, Y_i f$ in the sense of distributions can be represented by L^2 functions. This space, equipped with the norm $(\|f\|_2^2 + \sum_1^n \|X_i f\|_2^2 + \|Y_i f\|_2^2)^{1/2}$ is a Hilbert space. The space $(\mathbb{H}_n, \mathcal{E}, W^1(\mathbb{H}_n))$ is a regular strictly local Dirichlet space.

The intrinsic distance ρ associated with this Dirichlet form structure is often called the Carnot-Carathéodory metric on $\mathbb{H}_n$ and can be computed using the length of horizontal curves, that is, curves that stay tangent to the linear span of $\{X_1, \dots, Y_n\}$. It defines the usual topology and $(\mathbb{H}_n, \rho)$ is a complete metric space. This distance is homogeneous of degree 1 with respect to the dilations δ_r, $r > 0$, and it follows that the volume of a ball of radius r is equal to $c_n r^{2n+2}$, $r > 0$, for some $c_n > 0$ (c_n is the volume of the ball of radius 1). Verifying that the global Poincaré inequality holds is not very difficult. See [**83, 94**]. Thus

$$(\mathbb{H}_n, \rho, \mathcal{E}, W^1(\mathbb{H}_n))$$

is a Harnack-type Dirichlet space.

In a series of papers including [**49, 50**], Greshnov has studied the notion of uniform domains, as well as further related notions such as NTA domains, in this context. See also [**25, 26**]. In particular, Greshnov proved that the following subset of $\mathbb{H}_n$ are uniform domains in $(\mathbb{H}_n, \rho)$:

1. The lateral half-spaces $U_i = \{(x,y,z) \in \mathbb{R}^{2n+1} : x_i > 0\}$ (of course, x_i can be replaced by y_i);
2. The upper half-space $U_+ = \{(x,y,z) \in \mathbb{R}^{2n+1} : z > 0\}$.
3. The cube $Q = \{(x,y,z) \in \mathbb{R}^{2n+1} : \max_i\{|x_i|, |y_i|, |z|\} < 1\}$. Hence, by dilation, also $Q(r) = \{(x,y,z) \in \mathbb{R}^{2n+1} : \max_i\{|x_i|, |y_i|, |z|^{1/2}\} < r\}$.
4. The ball $B(e,1)$ (hence, by translation and dilation, $B(g,r)$, $g \in \mathbb{H}_n$, $r > 0$, all with the same constants of uniformity).

See [**49**] for the ball and [**50**] for the other cases. Further examples are described in [**23, 24, 26**].

In all these cases, Theorem 3.10 applies and shows that the associated Neumann type Dirichlet form is of Harnack type on $\overline{U}$ when U is one of the domains above. The domains U_+ and U_i are unbounded and their harmonic profiles are given respectively by $h(x,y,z) = z$ and $h(x,y,z) = x_i$. We now describe the bounds produced by

Theorem 5.11 in these cases. It is useful to recall first the two sided Gaussian bounds for the global heat kernel on $\mathbb{H}_n$. Let $h(t,x,y)$ be the kernel of the heat semigroup $e^{t\Delta}$ on $\mathbb{H}_n$. Then, for any $\epsilon \in (0,1)$, we have

$$\frac{c(\epsilon)}{t^{n+1}} \exp\left(-\frac{\rho(x,y)^2}{4(1+\epsilon)t}\right) \le h(t,x,y) \le \frac{C}{t^{n+1}}\left(1+\frac{\rho(x,y)^2}{t}\right)^{n+1} \exp\left(-\frac{\rho(x,y)^2}{4t}\right).$$

See [**100**]. Actually, in the special case under consideration here, there are exact formulas for the distance function and integral formula for the heat kernel.

The case of the lateral half-space U_1 (obviously, it is enough to treat $i=1$) is quite easy since

$$\rho((x,y,z),\mathbb{H}_1 \setminus U_1) \simeq |x_1| \text{ and } h_1(x,y,z) = x_1$$

where h_1 denotes a profile for U_1. It follows that

$$V_{h_1^2}(g,r) \simeq r^{2n+2}(|x_1|^2 + r^2).$$

Now, Theorem 5.11 yields upper and lower bounds of the type

$$\frac{c_1|x_1||x_1'|}{t^{n+1}(\sqrt{t}+|x_1|)(\sqrt{t}+|x_1'|)} \exp\left(-c_2\frac{\rho(g,g')^2}{t}\right)$$

for the Dirichlet heat kernel $h^D_{U_1}(t,g,g')$, $t>0$, $g=(x,y,z), g'=(x',y',z') \in U_1$. Of course the constants c_1, c_2 differ in the lower bound and in the upper bound.

The case of the upper-half space U_+ is a bit more subtle. The function $h_+((x,y,z)) = z$ is obviously a profile for U_+ but the distance of $g=(x,y,z)$ to U_+ is controlled by

$$\rho((x,y,z),\mathbb{H}_n \setminus U_+) \simeq \min\left\{\sqrt{z}, z/\max_i\{|x_i|,|y_i|\}\right\}.$$

Indeed, it is enough to treat the case when $g=(x,y,z)$ with $x=(x_1,0,\dots,0)$, $y=0$. For such g, we have

$$\begin{aligned}\rho(g,\mathbb{H}_n\setminus U_+) &= \inf\{\rho(g,g') : g'=(x',y',0)\}\\ &\simeq \inf\left\{|y_1'| + \sqrt{|z-x_1y_1'/2|} : y_1' \in \mathbb{R}\right\} \simeq \inf\left\{\sqrt{z}, z/|x_1|\right\}.\end{aligned}$$

So, for instance, $\rho((0,0,z),\mathbb{H}_n\setminus U_+) \simeq \sqrt{z}$ whereas $\rho((z,0,\dots,0),0,z),\mathbb{H}_n\setminus U_+) \simeq 1$. Hence, the behavior of the profile is not a function of the distance to the boundary.

In any case, we have

$$V_{h_+^2}(g,r) \simeq r^{2n+2}(z+r^2+r\max_i\{|x_i|,|y_i|\})^2, \quad g=(x,y,z).$$

Indeed, if $g=((x_1,0,\dots,0),(0,\dots,0),z)$, $x_i,z>0$, we can multiply on the right by $((0,0,\dots,0),(-r,0,\dots,0),r^2)$ which has length about r to get to $g_r = ((x_1,0,\dots,0),(-r,0,\dots,0),z+r^2+x_1r/2)$ which has $h(g_r) = z+r^2+x_1r/2$. Hence Theorem 5.11 yields upper and lower bounds of type

$$\frac{c_1zz'}{t^{n+1}(z+t+\sqrt{t}\max_i\{|x_i|,|y_i|\})(z'+t+\sqrt{t}\max_i\{|x_i'|,|y_i'|\})} \exp\left(-c_2\frac{\rho(g,g')^2}{t}\right)$$

for the Dirichlet heat kernel $h^D_{U_+}(t,g,g')$, $t>0$, $g=(x,y,z), g'=(x',y',z') \in U_+$.

6.7. Examples in the context of Euclidean complexes

In Chapter 2, we mentioned that Euclidean complexes such as the k-dimensional complex naturally associated with the square lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, when equipped with their natural Dirichlet form, are examples of Harnack-type Dirichlet spaces. In this section, we briefly illustrate our results concerning the Dirichlet heat kernel in this context. We focus on very basic examples.

Our first example is the upper-half space $U_+ = \{(x, y, z) \in X : z > 0\}$ in the two dimensional square complex

$$X = \{(x, y, z) \in \mathbb{R}^3 : \{x, y, z\} \cap \mathbb{Z} \neq \varnothing\}.$$

associated with the integer lattice $\mathbb{Z}^3 \in \mathbb{R}^3$. The space X is equipped with the Dirichlet structure discussed at the end of Chapter 2, with its natural distance function and its natural Lebesgue measure. This space is a Harnack-type Dirichlet space. See, e.g., [**78**]. The domain U_+ admits an obvious profile for this structure, the function $h((x, y, z)) = z$. We encourage the reader to check that this is, indeed, a local weak solution of the Laplace equation in U_+ for the relevant Dirichlet form. It is clear that h satisfies the other properties required to be a profile for U_+. The heat kernel $h(t, \xi, \xi')$ on X is bounded above and below by expressions of the form

$$\frac{c_1}{V(\sqrt{t})} \exp\left(-c_2 \frac{\rho(\xi, \xi')^2}{t}\right)$$

with $V(r) = \max\{r^2, r^3\}$. The set U_+ is obviously uniform and the Dirichlet heat kernel on U_+, $h^D_{U_+}(t, \xi, \xi')$ is bounded above and below by

$$\frac{c_1 z z'}{V(\sqrt{t})(\sqrt{t} + z)(\sqrt{t} + z')} \exp\left(-c_2 \frac{\rho(\xi, \xi')^2}{t}\right).$$

Our second example is $U = X \setminus \{(0, 0)\}$ with X being the metric graph of $\mathbb{Z}^2$, that is

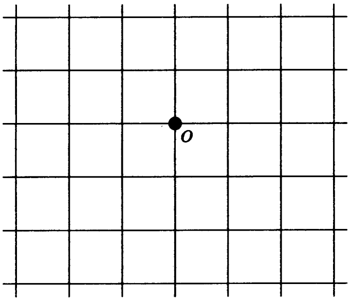
$$X = \{(x, y) \in \mathbb{R}^2 : \{x, y\} \cap \mathbb{Z} \neq \varnothing\}.$$

Note that, since a metric graph is a 1-dimensional object, the point $(0, 0)$ has positive capacity. The profile h of U is harmonic hence linear on each edge. It is continuous and vanishes (linearly) at $(0, 0)$. Applying the methods of [**54**], it is not hard to see that $h(\xi) \simeq \log(1 + \rho(\xi))$, $\rho(\xi) = \rho((0, 0), \xi)$, $\xi \in X$. In this situation, the volume growth is described by $V(\xi, r) \simeq V(r)$ with $V(r) = \max\{r, r^2\}$. The set U is not uniform (because of what happens in a neighborhood of $(0, 0)$) but is obviously inner uniform. The Dirichlet heat kernel on U, $h^D_U(t, \xi, \xi')$ is bounded above and below by

$$\frac{c_1 \log(1 + \rho(\xi)) \log(1 + \rho(\xi'))}{V(\sqrt{t})(\log(1 + \rho(\xi) + \sqrt{t}))(\log(1 + \rho(\xi') + \sqrt{t}))} \exp\left(-c_2 \frac{\rho_U(\xi, \xi')^2}{t}\right).$$

Note that the distance ρ_U is used in the exponential. The only meaningful difference between ρ and ρ_U is in a neighborhood of $(0, 0)$.

FIGURE 3. The domain $U = X \setminus \{o\}$ in the metric graph X of $\mathbb{Z}^2$



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