Astérisque

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Mirzakharni's recursion formula is equivalent to the Witten-Kontsevich theorem

MIRZAKHARNI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersections numbers involving higher degree κ classes.

Résumé (La formule de récurrence de Mirzakhani est équivalente au théorème de Witten-Kontsevich)

Dans cet article, nous démontrons la formule de récurrence de Mirzakhani sur les volumes de Weil-Petersson des espaces de module de courbes en utilisant le théorème de Witten-Kontsevich. Nous donnons aussi des propriétés des nombres d'intersection associées aux classes κ de degré supérieur.

1. Introduction

Following the notation of Mulase and Safnuk [**21**], let $\mathcal{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \ldots, L_n)$ and let $\mathrm{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [**19**] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I \coprod J}} \int_0^{L_1} \int_0^\infty \int_0^\infty xy H(t, x+y)$$

$$\times \mathrm{Vol}_{g_1,n_1}(x, \mathbf{L}_I) \mathrm{Vol}_{g_2,n_2}(y, \mathbf{L}_J) dx dy dt$$

$$+ \frac{1}{2L_1} \int_0^{L_1} \int_0^\infty \int_0^\infty xy H(t, x+y) \mathrm{Vol}_{g-1,n+1}(x, y, L_2, \ldots, L_n) dx dy dt$$

$$+ \frac{1}{2L_1} \sum_{j=2}^{n} \int_0^{L_1} \int_0^{\infty} x \big(H(x, L_1 + L_j) + H(x, L_1 - L_j) \big)$$

$$\times \mathrm{Vol}_{g,n-1}(x, L_2, \ldots, \hat{L}_j, \ldots, L_n) dx dt,$$

where the kernel function

$$H(x, y) = \frac{1}{1 + e^{(x+y)/2}} + \frac{1}{1 + e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [20] showed the following relation

$$\frac{\mathrm{Vol}_{g,n}(2\pi \mathbf{L})}{(2\pi^2)^{3g+n-3}} = \frac{1}{(3g+n-3)!} \int_{\mathcal{M}_{g,n}} \left(\kappa_1 + \sum_{i=1}^{n} L_i^2 \psi_i \right)^{3g+n-3}$$

$$= \sum_{\substack{d_0 + \cdots + d_n \\ = 3g+n-3}} \prod_{i=0}^{n} \frac{1}{d_i!} \langle \kappa_1^{d_0} \prod \tau_{d_i} \rangle_{g,n} \prod_{i=1}^{\infty} L_i^{2d_i}.$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [20] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L} = (L_1, \ldots, L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [21] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani's recursion.

$$(2d_1 + 1)!! \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g$$

$$= \sum_{j=2}^{n} \sum_{b=0}^{a} \frac{a!}{(a-b)!} \frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{b=0}^{a} \sum_{r+s=b+d_1-2} \frac{a!}{(a-b)!} (2r+1)!!(2s+1)!! \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{b=0}^{a} \sum_{\substack{c+c'=a-b \\ I \coprod J = \{2, \ldots, n\}}} \sum_{r+s=b+d_1-2} \frac{a!}{c!c'!} (2r+1)!!(2s+1)!! \beta_b$$

$$\times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

where

$$\beta_b = (2^{2b+1} - 4) \frac{\zeta(2b)}{(2\pi^2)^b} = (-1)^{b-1} 2^b (2^{2b} - 2) \frac{B_{2b}}{(2b)!}.$$

Safnuk [23] gave a proof of the above differential form of Mirzakhani's recurson formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani's recurson and matrix integrals has been studied by Eynard-Orantin [7] and Eynard [6].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [14, 24] in the form of DVV recursion relation [4]. There are several other new proofs of Witten-Kontsevich theorem [3, 12, 13, 22] besides Mirzakhani's proof [20].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [5, 10, 18, 26].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [11] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

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2. Proof of Mirzakhani's recursion formula

We first give three lemmas. The following lemma can be found in [21].

Lemma 2.1. — *The constants β_b in Mirzakhani's recursion satisfy the following:*

$$\sum_{k=0}^{\infty} \beta_k x^k = \frac{\sqrt{2x}}{\sin\sqrt{2x}}.$$

And its inverse:

$$\left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1} = \frac{\sin\sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k+1)!} x^k.$$

Proof. — Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x}{2} \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \frac{x}{2i} \cot \frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x}\left(\cot\sqrt{\frac{x}{2}} - \cot\sqrt{2x}\right) = \frac{\sqrt{2x}}{\sin\sqrt{2x}}. \qquad \square$$

The following elementary result is crucial to our proof.

Lemma 2.2. — *Let $F(m,n)$ and $G(m,n)$ be two functions defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = \left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1}.$$

Then the following two identities are equivalent:

$$G(m,n) = \sum_{k=0}^{m} \alpha_k F(m-k, n+k), \quad \forall \, (m,n) \in \mathbb{N} \times \mathbb{N},$$

$$F(m,n) = \sum_{k=0}^{m} \beta_k G(m-k, n+k), \quad \forall \, (m,n) \in \mathbb{N} \times \mathbb{N}.$$

Proof. — Assume the first identity holds, then we have

$$\sum_{i=0}^{m} \beta_i G(m-i, n+i) = \sum_{i=0}^{m} \beta_i \sum_{j=0}^{m-i} \alpha_j F(m-i-j, n+i+j)$$

$$= \sum_{k=0}^{m} \sum_{i+j=k} (\beta_i \alpha_j) F(m-k, n+k)$$

$$= \sum_{k=0}^{m} \delta_{k0} F(m-k, n+k)$$

$$= F(m,n).$$

So we proved the second identity. The proof of the other direction is the same. □

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [2], Faber [8] and Kaufmann-Manin-Zagier [11] into a nice combinatorial formalism.

Lemma 2.3 ([11]). — *For $m > 0$,*

$$\langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^m \rangle_g = \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1 + \cdots + m_k = m \\ m_i > 0}} \binom{m}{m_1, \ldots, m_k} \langle \prod_{j=1}^{n} \tau_{d_j} \prod_{j=1}^{k} \tau_{m_j+1} \rangle_g.$$

Proof. — (sketch) Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \ldots \psi_{n+p}^{a_p+1})$ by $R(a_1, \ldots, a_p)$, then we have the formula from [2]

$$R(a_1, \ldots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\substack{\text{each cycle } c \\ \text{of } \sigma}} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

A formal combinatorial argument [11] leads to the following inversion equation

$$\kappa_{a_1} \cdots \kappa_{a_p} = \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\ldots,p\} = S_1 \coprod \cdots \coprod S_k \\ S_k \neq \varnothing}} R(\sum_{j \in S_1} a_j, \ldots, \sum_{j \in S_k} a_j),$$

from which the result follows easily. □

Proposition 2.4. — *We have*

$$\sum_{b=0}^{a}(-1)^{b}\binom{a}{b}\frac{(2(d_1+b)+1)!!}{(2b+1)!!}\langle\tau_{d_1+b}\prod_{i=2}^{n}\tau_{d_i}\kappa_1^{a-b}\rangle_g$$

$$=\sum_{j=2}^{n}\frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!}\langle\kappa_1^{a}\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$

$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^{a}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\ldots,n\}}}\binom{a}{c}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Proof. — Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^{a}\rangle_g$$

$$=(2d_1+1)!!\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\ldots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g$$

$$=\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\ldots,m_k}$$

$$\times\left(\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g\right.$$

$$+\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g$$

$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\ldots,n\}\\ I'\coprod J'=\{1,\ldots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!$$

$$\left.\times\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{m_i+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{m_i+1}\rangle_{g-g'}\right)$$

$$= \sum_{j=2}^{n} \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \coprod J = \{2,\ldots,n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}$$

$$+ \sum_{k=0}^{a} \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \cdots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \ldots, m_k}$$

$$\times \sum_{j=1}^{k} \frac{(2(d_1 + m_j) + 1)!!}{(2m_j + 1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^{n} \tau_{d_i} \prod_{i \neq j} \tau_{m_i+1} \rangle_g$$

$$= RHS + \sum_{k \geq 0} \frac{(-1)^{a-k-1}}{(k+1)!} \sum_{b=1}^{a} \sum_{\substack{m_1 + \cdots + m_k = a-b \\ m_i > 0}} \binom{a}{b} \binom{a-b}{m_1, \ldots, m_k}$$

$$\times (k+1) \frac{(2(d_1 + b) + 1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_g$$

$$= RHS - \sum_{b=1}^{a} (-1)^b \binom{a}{b} \frac{(2(d_1 + b) + 1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \kappa_1^{a-b} \rangle_g$$

$$= RHS - LHS + (2d_1 + 1)!! \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g.$$

So we have proved $RHS = LHS$. $\square$

Proposition 2.4 is also implicitly contained in the arguments of Mulase and Safnuk [21].

Theorem 2.5. — *We have*

$$\frac{(2d_1 + 1)!!}{a!} \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g$$

$$= \sum_{b=0}^{a} \sum_{j=2}^{n} \frac{(2(b + d_1 + d_j) - 1)!!}{(a-b)!(2d_j - 1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{b=0}^{a} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{(a-b)!} \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{b=0}^{a} \sum_{\substack{c+c'=a-b \\ I \coprod J=\{2,\ldots,n\}}} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'!} \beta_b$$

$$\times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

where the constants β_k are given by

$$(\sum_{k=0}^{\infty} \beta_k x^k)^{-1} = \frac{\sin\sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)!!} x^k.$$

Proof. — Denote the LHS by $F(a, d_1)$. Let

$$G(a, d_1) = \sum_{j=2}^{n} \frac{(2(d_1+d_j)-1)!!}{a!(2d_j-1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1,j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{a!} \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \coprod J=\{2,\ldots,n\}}} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'!} \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

Note that Proposition 2.4 is just

$$\sum_{b=0}^{a} \frac{(-1)^b}{b!(2b+1)!!} F(a-b, d_1+b) = G(a, d_1).$$

By Lemmas 2.1 and 2.2, we have

$$F(a, d_1) = \sum_{b=0}^{a} \beta_b G(a-b, d_1+b) = RHS.$$

So we conclude the proof.   □

3. Higher Weil-Petersson volumes

Mirzakhani's formula provides a recursive way of computing the following Weil-Petersson volumes of moduli spaces of curves

$$WP(g) := \int_{\overline{\mathscr{M}_{g,n}}} \kappa_1^{3g-3+n}.$$

Mirzakhani's formula resorts to intersection numbers of mixed ψ and κ classes.

A natural question is whether there exist an explicit formula expressing $WP(g)$ in terms of those $WP(g')$ with $g' < g$. Recall the following beautiful formula due to Itzykson-Zuber [9].

Proposition 3.1 (**Itzykson-Zuber**). — *Let* $g \geq 0$. *Then*

$$\phi_{g+1} = \frac{25g^2 - 1}{24}\phi_g + \frac{1}{2}\sum_{m=1}^{g}\phi_{g+1-m}\phi_m,$$

where $\phi_0 = -1, \phi_1 = \frac{1}{24}$ *and*

$$\phi_g = \frac{(5g-5)(5g-3)}{2^g(3g-3)!}\langle\tau_2^{3g-3}\rangle_g, \quad g \geq 2.$$

By projection formula, we have

$$\langle\tau_2^{3g-3}\rangle_g = \langle\kappa_1^{3g-3}\rangle_g + \cdots,$$

where $\cdots$ denote terms involving higher degree kappa classes. Also note that $\langle\kappa_1^{3g-3}\rangle_g$ is conjecturally [16] the largest term in the right hand side.

To our disappointment, so far, all recursion formulae for $WP(g)$ stemming from the Witten-Kontsevich theorem involve either ψ class or higher degree κ classes inevitably.

Mirzakhani, Mulase and Safnuk's arguments use Wolpert's formula [25]

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [11]. Consider the semigroup N^{∞} of sequences $\mathbf{m} = (m(1), m(2), \ldots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a_1}, \ldots, \mathbf{a_n} \in N^{\infty}$, $\mathbf{m} = \sum_{i=1}^{n}\mathbf{a_i}$, $\mathbf{m} \geq \mathbf{t}$ and $\mathbf{s} := (s_1, s_2, \ldots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i\geq 1}im(i), \quad ||\mathbf{m}|| := \sum_{i\geq 1}m(i), \quad \mathbf{s^m} := \prod_{i\geq 1}s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i\geq 1}m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i\geq 1}\binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a_1}, \ldots, \mathbf{a_n}} := \prod_{i\geq 1}\binom{m(i)}{a_1(i), \ldots, a_n(i)}.$$

Let $\mathbf{b} \in N^{\infty}$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1}\kappa_i^{b(i)}.$$

We are interested in the following intersection numbers

$$\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}}\kappa(\mathbf{b})\psi_1^{d_1}\cdots\psi_n^{d_n}.$$

When $d_1 = \cdots = d_n = 0$, these intersection numbers are called higher Weil-Petersson volumes of moduli spaces of curves. The details of the following discussions are contained in [17].

The following lemma is a direct generalization of Lemma 2.2.

Lemma 3.2. — *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. Let $\alpha_\mathbf{L}$ and $\beta_\mathbf{L}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_\mathbf{0}\beta_\mathbf{0} = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_\mathbf{L}\beta_{\mathbf{L}'} = 0, \qquad \mathbf{b} \neq 0.$$

Then the following two identities are equivalent:

$$G(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_\mathbf{L} F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall\, (\mathbf{b}, n) \in N^\infty \times \mathbb{N},$$

$$F(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_\mathbf{L} G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall\, (\mathbf{b}, n) \in N^\infty \times \mathbb{N}.$$

We may generalize Mirzakhani's recursion formula to include higher degree κ classes.

Theorem 3.3. — *There exist (uniquely determined) rational numbers $\alpha_\mathbf{L}$ depending only on $\mathbf{L} \in N^\infty$, such that for any $\mathbf{b} \in N^\infty$ and $d_j \geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.*

$$(2d_1 + 1)!!\langle \kappa(\mathbf{b}) \prod_{j=1}^{n} \tau_{d_j} \rangle_g$$

$$= \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_\mathbf{L} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_\mathbf{L} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \bigsqcup J = \{2, \ldots, n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_\mathbf{L} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r+1)!!(2s+1)!!$$

$$\times \langle \kappa(\mathbf{e})\tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.$$

These tautological constants $\alpha_\mathbf{L}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{\|\mathbf{L}\|}\alpha_\mathbf{L}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b} \neq 0,$$

namely

$$\alpha_\mathbf{b} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}' \neq 0}} \frac{(-1)^{\|\mathbf{L}'\|-1}\alpha_\mathbf{L}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b} \neq 0,$$

with the initial value $\alpha_\mathbf{0} = 1$.

Theorem 3.4. — *We have*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1 + 2|\mathbf{L}| + 1)!!}{(2|\mathbf{L}| + 1)!!} \langle \kappa(\mathbf{L}')\tau_{d_1+|\mathbf{L}|} \prod_{j=2}^{n} \tau_{d_j} \rangle_g$$

$$= \sum_{j=2}^{n} \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{b})\tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{r+s=|d_1|-2} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{b})\tau_r\tau_s \prod_{i\neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \coprod J=\{2,\ldots,n\}}} \sum_{r+s=d_1-2} \binom{\mathbf{b}}{\mathbf{e}} (2r+1)!!(2s+1)!!$$

$$\times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J} \tau_{d_i} \rangle_{g-g'}.$$

Theorem 3.3 and Theorem 3.4 implies each other through Lemma 3.2.

Both Theorems 3.3 and 3.4 are effective recursion formulae for computing higher Weil-Petersson volumes with the three initial values

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \qquad \langle \tau_0^3 \rangle_0 = 1, \qquad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

From the following Proposition 3.4, we have

$$\langle \kappa(\mathbf{b}) \rangle_g = \frac{1}{2g-2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \kappa(\mathbf{L}') \rangle_g.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a Maple program [1] implementing Theorems 3.3 and 3.4.

In fact, we find that ψ and κ classes are compatible in the sense that recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes by the same proof as Proposition 2.4. In view of Theorem 3.8 below, this can be rephrased as differential equations governing generating functions of ψ classes also govern generating functions of mixed ψ and κ classes.

We present some examples below.

Proposition 3.5. — *Let* $\mathbf{b} \in N^\infty$ *and* $d_j \geq 0$. *Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g-2+n)\langle \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.$$

The above proposition is a generalization of the dilaton equation. In the special case $\mathbf{b} = (m, 0, 0, \ldots)$, it has been proved by Norman Do and Norbury [5].

Proposition 3.6. — *Let* $\mathbf{b} \in N^\infty$. *Then*

$$\langle \tau_0 \tau_1 \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g = \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \underline{n}=I \coprod J}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}.$$

The above proposition, together with the projection formula, can be used to derive an effective recursion formula for higher Weil-Petersson volumes [**17**] (without ψ classes).

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m}, \mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \cdots \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

Following Mulase and Safnuk [**21**], we introduce the following family of differential operators for $k \geq -1$,

$$V_k = -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}| + 1)!!} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}}$$

$$+ \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}}$$

$$+ \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}.$$

Theorem 3.7 ([**17, 21**]). — *The recursion of Theorem 3.4 implies*

$$V_k \exp(G) = 0.$$

Moreover, we can check directly that the operators V_k, $k \geq -1$ satisfy the Virasoro relations

$$[V_n, V_m] = (n - m) V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy.

Theorem 3.8 ([**17, 21**]). — *We have*

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

where p_k are polynomials in **s** *given by*

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of **s**, *$G(\mathbf{s}, \mathbf{t})$ is a τ-function for the KdV hierarchy.*

At a final remark, it would be interesting to prove that $\alpha_{\mathbf{L}}$ in Theorem 3.3 are positive for all $\mathbf{L} \in N^{\infty}$. This problem is kindly pointed out to us by a referee.

More generally the question can be formulated as following: two sequences $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ with $\alpha_{\mathbf{0}} = \beta_{\mathbf{0}} = 1$ are said to be inverse to each other if they satisfy

$$\left(\sum_{\mathbf{L}} \alpha_{\mathbf{L}} \mathbf{s}^{\mathbf{L}}\right) \cdot \left(\sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}}\right) = 1.$$

Find sufficient conditions on $\beta_{\mathbf{L}}$ such that $\alpha_{\mathbf{L}} > 0$ for all $\mathbf{L}$.

We conjecture that $\alpha_{\mathbf{L}}$ are positive when $\sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}}$ equals any of the following.

$$\sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!} \mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|-1)!!} \mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!|\mathbf{L}|!} \mathbf{s}^{\mathbf{L}}.$$

The latter two arise when we consider Hodge integrals involving λ classes [17].

For works on the positivity criteria of coefficients of reciprocal power series of a single variable, see for example [15]. However it seems there is no literature dealing with the coefficients of reciprocal series of several variables.

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