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SMOOTH DENSITY OF CANONICAL STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS

by

Hiroshi Kunita

Dedicated to Professor J.-M. Bismut for his sixtieth birthday

Abstract. — We consider jump diffusion process ξ_t on $\mathbf{R}^d$ determined by a canonical SDE:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

where $Z_t = (Z_t^1, ..., Z_t^m)$ is an m-dimensional Lévy process and $V_0, ..., V_m$ are smooth vector fields. We prove that the law of the solution ξ_t has a C^∞-density under the following two conditions. (1) The Lévy process Z_t is nondegenerate. (2) $\{V_0, V_1, ..., V_m\}$ can be degenerate but satisfies a uniform Hörmander condition (H). For the proof we make use of the Malliavin calculus on the Wiener-Poisson space studied by Ishikawa-Kunita.

***Résumé* (Densité lisse pour les solutions d'équations différentielles stochastiques avec sauts)**

Nous considérons un processus de diffusion à sauts ξ_t dans $\mathbf{R}^d$ déterminé par une EDS canonique:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

où $Z_t = (Z_t^1, ..., Z_t^m)$ est un processus de Lévy m-dimensionnel et $V_0, ..., V_m$ sont des champs de vecteurs. Nous montrons que la loi de ξ_t a une densité C^∞ si les conditions suivantes sont satisfaites. (1) Le processus de Lévy Z_t est non dégénéré. (2) La distribution $\{V_0, V_1, ..., V_m\}$ peut être dégénérée mais elle satisfait à une condition de Hörmander uniforme (H). Pour la démonstration, nous utilisons le calcul de Malliavin sur l'espace de Wiener-Poisson étudié par Ishikawa-Kunita.

1. Introduction and main results

Let $V_0, V_1, \cdots V_m$ be smooth vector fields on $\mathbf{R}^d$ whose derivatives (including higher orders) are all bounded. Let $Z_t = (Z_t^1, ..., Z_t^m), t \geq 0$ be an m-dimensional nondegenerate Lévy process. In this paper, we consider a jump diffusion determined by a

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canonical SDE based on $\{V_0, V_1, \cdots, V_m\}$ and Z_t;

$$d\xi_t = \sum_{i=1}^{m} V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt. \tag{1.1}$$

Canonical SDE's are studied in mathematical finance. Let Z_t be a one dimensional Lévy process. We consider a one dimensional linear canonical SDE.

$$dS_t = S_t \diamond dZ_t.$$

The solution starting from S_0 at time 0 is unique and it is written as $S_t := S_0 \exp Z_t$ (See Section 2). It is called a *geometric Lévy process.* The solution S_t describes the movement of a stock. If Z_t is a Lévy process with finite Lévy measure (a compound Poisson process), the process S_t is the *Merton model* or the *Kou model*, according as the normalized Lévy measure is a Gaussian distribution or a double exponential distribution, respectively. See [**16**],[**8**]. The precise definition of the canonical SDE will be given at Section 2.

The main purpose of this paper is to show the existence of the smooth density for the law of the random variable ξ_t that is a solution of equation (1.1). For this purpose we need to assume suitable nondegenerate conditions both for the Lévy process Z_t and the family of vector fields $\{V_0, ..., V_m\}$.

We first consider the Lévy process. The Lévy process Z_t is represented for arbitrary $\delta > 0$, by

$$Z_t = \sigma W_t + \int_0^t \int_{0<|z|\le\delta} z\tilde{N}(drdz) + \int_0^t \int_{|z|>\delta} zN(drdz) + b_\delta t,$$

where σ is an $m \times m$-matrix, W_t is an m-dimensional standard Brownian motion. $N(dtdz)$ is a Poisson random measure which is independent of W_t with intensity $\hat{N}(dtdz) = dt\nu(dz)$, ν being the Lévy measure. Further, $\tilde{N}(dtdz) = N(dtdz) - \hat{N}(dtdz)$ and $b_\delta = (b_\delta^1, ..., b_\delta^m)$ is a drift vector. Set $A = (a_{ij}) = \sigma\sigma^T$. It is a covariance of the Gaussian part σW_1 (Lévy-Itô decomposition). Throughout this paper, we assume that the Lévy measure ν has finite moments of any order. Set $v(\rho) := \int_{|z|<\rho} |z|^2 \nu(dz)$. If there exists $\alpha \in (0, 2)$ such that

$$\liminf_{\rho\to 0} \frac{v(\rho)}{\rho^\alpha} > 0,$$

then the Lévy measure is said to satisfy an *order condition.* Note that the Lévy measure ν satisfying an order condition is an infinite measure: Indeed, we have $\nu(\{z; 0 < |z| < \delta\}) = \infty$ for any $\delta > 0$. In case of one dimensional Lévy process, the above order condition is known as a sufficient condition for the existence of the smooth density of the law of the Lévy process (Orey's theorem. See Sato [**20**], Proposition 28.3). Then the law of the geometric Lévy process S_t has a smooth density if the order condition is satisfied.

Now we set $b_{ij}(\rho) = \int_{|z|\le\rho} z^i z^j \nu(dz)/v(\rho)$ and $B(\rho) = (b_{ij}(\rho))$. The *infinitesimal covariance* B is a symmetric and nonnegative definite matrix, which coincides with the greatest lower bound of the matrix $B(\rho)$ as $\rho \to 0$. If the Lévy measure satisfies an order condition and the matrix $A + B$ is nondegenerate (invertible), then we say that the Lévy process is *nondegenerate*. In this paper, *we assume that the Lévy process Z_t is nondegenerate.*

We will next consider nondegenete properties for the family of vector fields $\{V_0, ..., V_m\}$. In Ishikawa-Kunita [**6**], we studied the case where the family of vector fields $\{V_1, ..., V_m\}$ is *uniformly nondegenerate*, i.e., there exists a positive constant C such that the inequality

$$\sum_{i=1}^{m} |l^T V_i(x)|^2 \ge C|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d$$

holds valid, where l^T is the transpose of l and $l^T V(x)$ denotes the inner product of two vectors l and $V(x)$. We showed the existence of the smooth density of its law by applying Malliavin calculus on the Wiener-Poisson space.

In this paper we want to relax the above uniformly nondegenerate condition. Let $V_0, ..., V_m$ be C^∞-vector fields such that their derivatives (including higher orders) are all bounded. Then Lie brackets $[V_{i_1}[\cdots[V_{i_{n-1}}, X_{i_n}]\cdots]$, $i_1, ..., i_n \in \{0, 1, ..., m\}$ are bounded vector fields. We introduce families of vector fields. Let $\Sigma_0 = \{V_1, ..., V_m\}$ be a linear space of vector fields spanned by $V_1, ..., V_m$. Given $\delta > 0$, we set

$$\hat{V}_0^\delta = V_0 + \sum_{i=1}^{m} b_\delta^i V_i.$$

Set $\Sigma_0^\delta = \Sigma_0$ and define for $k = 1, 2, ...$

$$\Sigma_k^\delta = \Big\{[\hat{V}_0^\delta, V] + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}[V_i, [V_j, V]],\ [V_i, V], i = 1, ..., m, V \in \Sigma_{k-1}^\delta\Big\}.$$

Theorem 1.1. — *Assume that for the family of vector fields $\{V_0, ..., V_m\}$ there exist a positive integer N_0 and a positive number δ_0 such that for any $0 < \delta < \delta_0$ the inequality*

$$\sum_{k=0}^{N_0} \sum_{V \in \Sigma_k^\delta} |l^T V(x)|^2 \ge C(\delta)|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.2}$$

holds valid, where $C(\delta)$ are positive numbers satisfying

$$\liminf_{\delta \to 0} C(\delta)/v(\delta)^2 = \infty.$$

Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

The condition required for vector fields in the above theorem is complicated, since δ's are involved. We can replace it by a simpler one if we restrict the Lévy process Z_t to a simpler one, namely if we assume

$$b_0 = \lim_{\delta \to 0} b_\delta \quad \text{exists and is finite.} \tag{1.3}$$

The existence of b_0 is equivalent to that of $\lim_{\delta\to 0}\int_{\delta<|z|\le 1} z\nu(dz)$. In this case, it holds $b_0 = b_1 - \lim_{\delta\to 0}\int_{\delta<|z|\le 1} z\nu(dz)$. In particular, if the integral $\int_{0<|z|\le 1}|z|\nu(dz)$ is finite, b_0 exists and is finite. Hence for any stable process whose exponent is less than 1, b_0 exists. Further, if the Lévy measure ν is symmetric, b_0 exists and is equal to b_1 even if $\int_{0<|z|\le 1}|z|\nu(dz)$ is infinite. Hence for any symmetric stable process, b_0 exists and is equal to b_1.

Now, assume (1.3) and let $\delta \to 0$ in the Lévy-Itô decomposition of Z_t. Then we obtain

$$Z_t = \sigma W_t + \int_0^t \int_{|z|>0} zN(drdz) + b_0 t.$$

Hence b_0 can be regarded as the drift vector of the Lévy process Z_t. We define a new drift vector field $\hat{V}_0$ by

$$\hat{V}_0 = V_0 + \sum_{i=1}^m b_0^i V_i,$$

and introduce families of vector fields by $\Sigma_0 = \{V_1, ..., V_m\}$ and for $k = 1, ...$

$$\Sigma_k = \Big\{[\hat{V}_0, V] + \frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]],\ [V_i,V], i = 1,...,m, V \in \Sigma_{k-1}\Big\}.$$

Theorem 1.2. — *Assume (1.3) for the Lévy process Z_t. Assume further that the family of vector fields $\{\hat{V}_0, V_1, ..., V_m\}$ satisfy the uniform Hörmander condition (H), i.e., there exists a positive integer N_0 and a positive constant C such that*

$$\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k}|l^T V(x)|^2 \ge C|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.4}$$

holds valid. Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

Observe that Theorem 1.2 indicates that both the canonical SDE with jumps and Stratonovich SDE (diffusion) have the common local criterion (Hörmander' condition) for the existence of the smooth density of their laws. This is partly because that we restrict our attention to small jumps of the SDE, ignoring the effect of big jumps. Loosely speaking, under an order condition, the solution of equation (1.1) could behave like a diffusion if sizes of jumps are small.

Perhaps, Bismut [**2**] is the first work toward the smooth density of the law of the solution of SDE with jumps, where he developed the Malliavin calculus for jump processes. After this fundamental work, the similar problem has been discussed in some different contexts by Léandre [**13**],[**14**],[**15**], Bichteler-Gravreau-Jacod [**1**], Komatsu-Takeuchi [**7**] and others. A common feature in the above works might be that they assumed for the Lévy measure ν the existence of a smooth density and an asymptotic of the density as $z \to 0$. Furthermore, a formula of integration by parts holds valid in these cases, which are shown through Girsanov's theorem for jump diffusion.

In our discussion any Lévy measure (singular or not) is allowed, as far as it satisfies an order condition. Then no formula of integration by parts is known. We take another approach to the Malliavin calculus, developed in Ishikawa-Kunita [**6**]. It will be presented in the next section.

2. Malliavin calculus for canonical SDE

Let $Z_t, t \geq 0$ be an m-dimensional Lévy process admitting the Lévy-Itô decomposition and let ξ_0 be an $\mathbf{R}^d$-valued random variable independent of Z_t. By the solution of equation (1.1) starting from ξ_0 at time 0, we mean a cadlag $\mathbf{R}^d$-valued semimartingale $\{\xi_t; t \geq 0\}$ adapted to $\mathcal{F}_t = \sigma(\xi_0, Z_r; r \leq t)$ satisfying

$$
\begin{aligned}
\xi_t &= \xi_0 + \sum_{i=1}^m \int_0^t V_i(\xi_r) \diamond dZ_r^i + \int_0^t V_0(\xi_r)dr \\
&= \xi_0 + \sum_{i,k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k + \int_0^t \hat{V}_0^\delta(\xi_r)dr. \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\} \tilde{N}(drdz) \\
&\quad + \int_0^t \int_{|z|\geq\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\} N(drdz) \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_r) - \xi_r - \sum_{i=1}^m z^i V_i(\xi_r)\} \hat{N}(drdz).
\end{aligned} \tag{2.1}
$$

Here " $\circ$ " denotes the Stratonovitch integral. Using Itô integral, it holds

$$
\begin{aligned}
&\sum_{k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k \\
&\quad = \sum_{k=1}^m \int_0^t V_i(\xi_{r-})\sigma_{ik} dW_r^k + \frac{1}{2}\sum_{j=1}^m a_{ij} \int_0^t \Big(\sum_{l=1}^d \frac{\partial V_i}{\partial x^l} V_j^l\Big)(\xi_{r-})dr.
\end{aligned}
$$

Further, for $z = (z^1, ..., z^m) \in \mathbf{R}^m$ $\phi_s^z, s \in \mathbf{R}$ is the one parameter group of diffeomorphisms generated by the vector field $\sum_{i=1}^m z^i V_i$, i.e., $\phi_s^z = \exp s(\sum_i z^i V_i)$.

The equation has a unique solution ξ_t^δ. It holds $\xi_t^\delta = \xi_t^{\delta'}$ for any $\delta > 0$ and $\delta' > 0$. Hence the common solution is denoted by ξ_t. In the case where $\xi_0 = x$, we denote the solution by $\xi_{0,t}(x)$. Then it has a modification such that the maps $\xi_{0,t}; \mathbf{R}^d \to \mathbf{R}^d$ are onto diffeomorphisms a.s. and further the Jacobian matrix $\nabla\xi_{0,t}(x)$ is invertible for any x a.s. It defines a stochastic flow of diffeomorphisms (Fujiwara-Kunita [**3**]). We have $\xi_t = \xi_{0,t}(\xi_0)$.

We will consider a one dimensional linear SDE $dS_t = S_t \diamond dZ_t$. In this case we have $V_1(x) = x$. Then it holds $(\exp szV_1)(x) = e^{sz}x$. Hence equation (2.1) is written by

$$\begin{aligned} S_t &= S_0 + \sigma\int_0^t S_{r-}dW_r + \frac{1}{2}\sigma^2\int_0^t S_{r-}dr + b_\delta\int_0^t S_{r-}dr \\ &\quad + \int_0^t\int_{0<|z|\le\delta}(e^z-1)S_{r-}\tilde{N}(drdz) + \int_0^t\int_{|z|>\delta}(e^z-1)S_{r-}N(drdz) \\ &\quad + \int_0^t\int_{0<|z|\le\delta}(e^z-1-z)S_{r-}dr\nu(dz). \end{aligned}$$

The solution is given by $S_t = S_0 \exp Z_t$. Indeed apply Itô's formula to the function $F(x) = e^x$ and the semimartingale Z_t (Theorem 2.5 in [**10**]). Then we find that $S_t := \exp Z_t$ satisfies the above equation.

Now, for the proof of theorems stated in Section 1, we need the Malliavin calculus on the Wiener-Poisson space studied in Ishikawa-Kunita [**6**]. We will quickly recall it. Let T_0 be an arbitrarily fixed positive number and let $U = [0, T_0] \times \mathbf{R}^m$. Elements of U are denoted by $u = (t, z)$. Let ε_u^+ be a perturbation of the Poisson random measure N such that $N(A) \circ \varepsilon_u^+ = N(A \cap \{u\}^c) + 1_A(u)$. If we apply $\varepsilon_{(t_1,z_1)}^+$ to the solution ξ_t of SDE (2.1), we have $\xi_t \circ \varepsilon_{(t_1,z_1)}^+ = \xi_t$ if $t_1 > t$ and $\xi_t \circ \varepsilon_{(t_1,z_1)}^+ = \xi_{t_1,t} \circ \phi_1^{z_1} \circ \xi_{t_1-}$ if $t_1 \le t$, where $\xi_{s,t} := \xi_{0,t} \circ \xi_{0,s}^{-1}$ are diffeomorphisms of $\mathbf{R}^d$, a.s.

For $\mathbf{u} = (u_1, ..., u_n)$, we set $\varepsilon_{\mathbf{u}}^+ = \varepsilon_{u_1}^+ \circ \cdots \circ \varepsilon_{u_n}^+$. Let $\mathbf{u} = ((t_1, z_1), ..., (t_n, z_n))$ where $t_1 < t_2 < \cdots < t_n$. Then $\xi_t^{\mathbf{u}} := \xi_t \circ \varepsilon_{\mathbf{u}}^+$ is represented by

$$\xi_t^{\mathbf{u}} = \xi_{t_i,t} \circ \phi_1^{z_i} \circ \xi_{t_{i-1},t_i-} \circ \cdots \circ \phi_1^{z_1} \circ \xi_{t_1-}, \quad \text{if } \ t_i \le t < t_{i+1}.$$

Malliavin covariances R and $\tilde{K}$ of the random variable ξ_{T_0} with respect to the Wiener space and the Poisson space are defined by

$$\begin{aligned} R &= \int_0^{T_0} \nabla\xi_{t,T_0}(\xi_{t-})C(\xi_{t-})AC(\xi_{t-})^T\nabla\xi_{t,T_0}(\xi_{t-})^T dt, \\ \tilde{K} &= \int_0^{T_0} \nabla\xi_{t,T_0}(\xi_{t-})C(\xi_{t-})BC(\xi_{t-})^T\nabla\xi_{t,T_0}(\xi_{t-})^T dt, \end{aligned}$$

respectively. Here $\nabla\xi_{t,T_0}(x)$ is the Jacobian matrix of the map $\xi_{t,T_0}(x)$. The $d \times m$ matrix $C(x)$ is given by $C(x) = (V_1(x), ..., V_m(x))$.

We set $Q = R + \tilde{K}$ and call it as the *Malliavin covariance of* ξ_{T_0}. Set $Q^{\mathbf{u}} = Q \circ \varepsilon_{\mathbf{u}}^+$. Then $Q^{\mathbf{u}}$ is the Malliavin covariance of $\xi_{T_0}^{\mathbf{u}}$.

Now consider $\hat{Q} = \nabla\xi_{0,T_0}(\xi_0)^{-1}Q(\nabla\xi_{0,T_0}(\xi_0)^T)^{-1}$ (*modified Malliavin covariance* of ξ_{T_0}). It is written as

$$\hat{Q} = \int_0^{T_0} (\nabla\xi_{t-})^{-1}C(\xi_{t-})(A+B)C(\xi_{t-})^T(\nabla\xi_{t-}^T)^{-1}dt,$$

where $\nabla\xi_t = \nabla\xi_{0,t}(\xi_0)$. Then the modified Malliavin covariance $\hat{Q}^{\mathbf{u}}$ of $\xi_{T_0}^{\mathbf{u}}$ equals $\hat{Q} \circ \varepsilon_{\mathbf{u}}^{+}$.

A criterion for the existence of the smooth density of the law of ξ_{T_0} is given by the following.

Lemma 2.1. — *Assume that*

$$\sup_{\mathbf{u}\in A(1)^n} \sup_{l\in S_{d-1}} E[(l^T\hat{Q}^{\mathbf{u}}l)^{-p}] < \infty \tag{2.2}$$

holds for any positive integer n and $p > 1$. Then the law of ξ_{T_0} has a C^∞-density.

Proof. — It is shown in [**6**], Proposition 6.1 that if $Q^{\mathbf{u}}$ is invertible a.s. and

$$\sup_{\mathbf{u}\in A(1)^n} \sup_{l\in S_{d-1}} E[(l^TQ^{\mathbf{u}}l)^{-p}] < \infty \tag{2.3}$$

is satisfied for any positive integer n and $p > 1$, then the law of ξ_{T_0} has a C^∞-density. Here, we set $A(1) = \{(t,z); t \in (0,T_0), |z| \le 1\}$ and $S_{d-1} = \{l \in \mathbf{R}^d; |l| = 1\}$.

We will show that condition (2.2) implies condition (2.3). Note that (2.2) implies $\sup_{\mathbf{u}\in A(1)^n} E[\sup_{l\in S_{d-1}}(l^T\hat{Q}^{\mathbf{u}}l)^{-p}] < \infty$. Then the minimum eigenvalue $\Lambda_1^{\mathbf{u}}$ of the matrix $\hat{Q}^{\mathbf{u}}$ satisfies $\sup_{\mathbf{u}\in A(1)^n} E[(\Lambda_1^{\mathbf{u}})^{-p}] < \infty$ for any $p > 1$. Since the equality $(Q^{\mathbf{u}})^{-1} = \nabla\xi_{0,T_0}^T(\hat{Q}^{\mathbf{u}})^{-1}\nabla\xi_{0,T_0}$ holds and $\nabla\xi_{0,T_0} \in L^p$ holds for any $p > 1$,

$$\{(l^TQ^{\mathbf{u}}l)^{-1}, l \in S_{d-1}, \mathbf{u} \in A(1)^n\}$$

is also L^p bounded for any $p > 1$. Thus we have (2.3). □

Theorem 2.2. — *Assume that for any $l \in S_{d-1}$ and $\mathbf{u} \in A(1)^n$, the random variable*

$$\sum_{V\in\Sigma_0} \int_0^{T_0} \left|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})\right|^2 dt$$

is strictly positive a.s. Assume further that for any $p > 1$ and positive integer n there exists a positive constant $C_{n,p}$ such that

$$E\left[\left(\sum_{V\in\Sigma_0} \int_0^{T_0} \left|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})\right|^2 dt\right)^{-p}\right] < C_{n,p}, \tag{2.4}$$

for any $l \in S_{d-1}$ and $\mathbf{u} \in A(1)^n$. Then the law of ξ_{T_0} has a C^∞-density.

Proof. — Let $\lambda_1 > 0$ be the minimum eigen value of the matrix $A+B$. Then we have

$$l^T \hat{Q}^{\mathbf{u}} l \geq \lambda_1 \sum_{V \in \Sigma_0} \int_0^{T_0} |l^T (\nabla \xi_t^{\mathbf{u}})^{-1} V(\xi_t^{\mathbf{u}})|^2 dt.$$

Therefore the assertion follows from Lemma 2.1. □

The proof of our main theorem will be completed by checking the above criterion (2.4). However its process will be quite long. Our program for the proof is as follows. In Section 4, instead of the uniform Hörmander condition (H), we will present another criterion that ensures the existence of the smooth density of the law of ξ_{T_0} (Theorem 4.1). Sections 3,4 and 6 are devoted to the proof of Theorem 4.1. Section 3 is a preliminary part. We will discuss SDE governed by semimartingales $(\nabla \xi_t)^{-1} V(\xi_t)$, where V is a vector field. In Theorem 6.1 (Appendix), we obtain an estimate for probabilities of events concerned with these semimartingales, where "Komatsu-Takeuchi's key lemma" plays an important role. The estimate is analogous to the one obtained by Kusuoka-Stroock [**12**] or Norris [**17**] in case of diffusion process. The proof of Theorem 4.1 will be completed by proving criterion (2.4) through these estimates.

In Section 5 we show that the uniform Hörmander condition fulfills the criterion of Theorem 4.1 and then we give the proof of our main theorems (Theorems 1.1-1.2).

3. SDE's for derivatives of stochastic flow

Let $V(x)$ be a vector field. We begin by studying the SDE which governs $(\nabla \xi_t)^{-1} V(\xi_t)$.

Lemma 3.1. — *We have a.s.*

$$
\begin{aligned}
(\nabla\xi_t)^{-1}V(\xi_t) &= V(\xi_0) + \sum_{i,j=1}^{m}\int_0^t (\nabla\xi_{s-})^{-1}[V_i,V](\xi_{s-})\sigma_{ij}dW^j(s) \\
&+\frac{1}{2}\sum_{i,j=1}^{m} a_{ij}\int_0^t (\nabla\xi_{s-})^{-1}[V_i,[V_j,V]](\xi_{s-})ds \\
&+\int_0^t (\nabla\xi_{s-})^{-1}[\tilde{V}_0^{\delta},V](\xi_{s-})ds \\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-})\}\tilde{N}(dsdz) \\
&+\int_0^t\int_{|z|\geq\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-})\}N(dsdz) \\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\Big\{\nabla\phi_1^z(\xi_{s-})^{-1}V(\phi_1^z\circ\xi_{s-}) - V(\xi_{s-}) \\
&\qquad -\sum_i z^i[V_i,V](\xi_{s-})\Big\}\hat{N}(dsdz),
\end{aligned}
$$

where $\nabla\phi_1^z(x)$ is the Jacobian matrix of $\phi_1^z(x); \mathbf{R}^d \to \mathbf{R}^d$ and $\nabla\phi_1^z(x)^{-1}$ is its inverse matrix.

Proof. — It is shown in Ishikawa-Kunita [**6**] that the inverse matrix $(\nabla\xi_t)^{-1}$ satisfies a.s.

$$
\begin{aligned}
(\nabla\xi_t)^{-1} &= I - \sum_{i,j}\int_0^t (\nabla\xi_{s-})^{-1}\nabla V_i(\xi_{s-})\sigma_{i,j}\circ dW^j(s) \\
&-\int_0^t (\nabla\xi_{s-})^{-1}\nabla\tilde{V}_0^{\delta}(\xi_{s-})ds \\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I\}\tilde{N}(dsdz) \\
&+\int_0^t\int_{|z|\geq\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I\}N(drdz) \\
&+\int_0^t\int_{|z|<\delta} (\nabla\xi_{s-})^{-1}\{\nabla\phi_1^z(\xi_{s-})^{-1} - I \\
&\qquad +\sum_i z^i\nabla V_i(\xi_{s-})\}\hat{N}(dsdz).
\end{aligned}
$$

On the other hand, in view of Itô's formula for semimartingale with jumps, we have

$$
\begin{aligned}
V(\xi_t) = V(\xi_0) &+ \sum_{i,j} \int_0^t \nabla V(\xi_{s-}) V_i(\xi_{s-}) \sigma_{ij} \circ dW^j(s) \\
&+ \int_0^t \nabla V(\xi_{s-}) \tilde{V}_0^\delta(\xi_{s-}) ds \\
&+ \int_0^t \int_{|z|<\delta} (V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-})) \tilde{N}(dsdz) \\
&+ \int_0^t \int_{|z|\geq\delta} (V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-})) N(dsdz) \\
&+ \int_0^t \int_{|z|<\delta} \{V(\phi_1^z \circ \xi_{s-}) - V(\xi_{s-}) \\
&\qquad - \sum_i z^i \nabla V(\xi_{s-}) V_i(\xi_{s-})\} \hat{N}(dsdz).
\end{aligned}
$$

For the product of two semimartingales $X_t = (\nabla\xi_t)^{-1}$ and $Y_t = V(\xi_t)$, we have the formula

$$
\begin{aligned}
X_t Y_t &= X_0 Y_0 + \int_0^t X_s \circ dY_s^c + \int_0^t (\circ dX_s^c) Y_s \\
&\quad + \int_0^t X_{s-} dY_s^d + \int_0^t dX_s^d Y_{s-} + [X^d, Y^d]_t,
\end{aligned}
$$

where X_t^c, Y_t^c are continuous parts of semimartingales X_t, Y_t, respectively and X_t^d, Y_t^d are discontinuous parts of X_t, Y_t, respectively. A direct application of the above formula implies the equation of the lemma. □

Now define

$$
\begin{aligned}
\Psi_0^\delta V(x) &= \frac{1}{2} \sum_{i,j=1}^m a_{ij} [V_i, [V_j, V]](x) \\
&\quad + [\hat{V}_0^\delta, V](x) + \int_{0<|z|\leq\delta} \Big(\nabla\phi_1^z(x)^{-1} V(\phi_1^z(x)) - V(x) \\
&\quad - \sum_{i=1}^m [V_i, V](x) z^i \Big) \nu(dz),
\end{aligned}
\tag{3.1}
$$

and set

$$
\Phi_s(z) V(x) := \nabla\phi_s^z(x)^{-1} V(\phi_s^z(x)) - V(x). \tag{3.2}
$$

To simplify notations, we introduce the following. We set $\mathcal{J} = \hat{\mathbf{R}}^m \cup \mathbf{R}^m \cup \{\Delta\}$, where $\hat{\mathbf{R}}^m$ is an m-dimensional Euclidean space. Elements of $\hat{\mathbf{R}}^m$ and $\mathbf{R}^m$ are denoted by $y = (y^1, ..., y^m)$ and $z = (z^1, ..., z^m)$, respectively. We define stochastic process $Y_{l,V}^{(1)}(t, v)$ with parameter $l \in S_{d-1}$, vector field V and $v \in \mathcal{J}$ by

$$\begin{aligned} Y_{l,V}^{(1)}(t,\Delta) &= l^T(\nabla\xi_t)^{-1}\Psi_0^\delta V(\xi_t), \\ Y_{l,V}^{(1)}(t,y) &= \sum_{i=1}^m l^T(\nabla\xi_t)^{-1}[V_i,V](\xi_t)\frac{y^i}{|y|}, \\ Y_{l,V}^{(1)}(t,z) &= l^T(\nabla\xi_t)^{-1}\frac{\Phi_1(z)}{|z|}V(\xi_t). \end{aligned}$$

Let $W(dsdy)$ be a Gaussian orthogonal random measure on $[0, T_0] \times \hat{\mathbf{R}}^m$ such that $E[W(dsdy)] = 0$ and $\sigma W_t = \int_0^t \int_{\hat{\mathbf{R}}^m} yW(dsdy)$. Then the intensity measure $E(W(dsdy)^2) = dsw(dy)$ satisfies $(\int_{\hat{\mathbf{R}}^m} y^i y^j w(dy)) = A$. We set $\hat{w}(dy) = |y|^2 w(dy)$.

Then, setting $Y_{l,V}(t) = l^T \nabla\xi_t V(\xi_t)$, the equation of Lemma 3.1 is written as

$$\begin{aligned} Y_{l,V}(t) &= l^T V(\xi_0) + \int_0^t Y_{l,V}^{(1)}(s-,\Delta)ds \\ &\quad + \int_0^t \int_{\hat{\mathbf{R}}^m} Y_{l,V}^{(1)}(s-,y)|y|dW \\ &\quad + \int_0^t \int_{|z|\le\delta} Y_{l,V}^{(1)}(s-,z)|z|d\tilde{N} \\ &\quad + \int_0^t \int_{|z|>\delta} Y_{l,V}^{(1)}(s-,z)|z|dN. \end{aligned} \tag{3.3}$$

We will continue the above argument inductively. Let $k \geq 1$. We will define a family of k-th step semimartingales with spatial parameter associated with a given vector field V. We set $\Psi(\Delta)V = \Psi_0^\delta V$, $\Psi(y)V = \sum_k [V_k, V]y^k/|y|$ and $\Psi(z)V = \Phi_1(z)V/|z|$. Define for $v_k, ..., v_1 \in \mathcal{J}$

$$\Psi(v_k, ..., v_1)V = \Psi(v_k) \circ \cdots \circ \Psi(v_1)V. \tag{3.4}$$

Apply equality (3.3) to the vector field $\Psi(v_k, ..., v_1)V$ in place of V. Then, setting

$$Y_{l,V}^{(k)}(t, v_k, ..., v_1) = l^T(\nabla\xi_t)^{-1}\Psi(v_k, ..., v_1)V(\xi_t),$$

equality (3.3) is written as

$$
\begin{aligned}
(3.5)\qquad Y^{(k)}_{l,V}(t,v_k,\dots,v_1) &= Y^{(k)}_{l,V}(0,v_k,\dots,v_1) \\
&+\int_0^t Y^{(k+1)}_{l,V}(s-,\Delta,v_k,\dots,v_1)ds \\
&+\int_0^t\int Y^{(k+1)}_{l,V}(s-,y_{k+1},v_k,\dots,v_1)|y_{k+1}|W(dsdy_{k+1}) \\
&+\int_0^t\int_{|z_{k+1}|\le\delta} Y^{(k+1)}_{l,V}(s-,z_{k+1},v_k,\dots,v_1)|z_{k+1}|\tilde N(dsdz_{k+1}) \\
&+\int_0^t\int_{|z_{k+1}|>\delta} Y^{(k+1)}_{l,V}(s-,z_{k+1},v_k,\dots,v_1)|z_{k+1}|N(dsdz_{k+1}).
\end{aligned}
$$

4. Alternative criterion for the smooth density

We will now study the existence of the smooth density of the law of ξ_t. In this section we present an alternative criterion which ensures the existence of the smooth density. The condition will be given at Theorem 4.1. In the next section we will study how the condition given in this section is related to Hörmander's condition in Theorem 1.1.

Let $\epsilon > 0$. Associated with the Lévy measure ν, we define a probability measure $\hat\mu_\epsilon$ on $\mathbf{R}^m$ by

$$\hat\mu_\epsilon(dz) = \frac{1}{v(\epsilon)}|z|^2 1_{[0,\epsilon]}(|z|)\nu(dz),$$

where $v(\rho) = \int_{|z|<\rho}|z|^2\nu(dz)$. We denote by μ_ϵ the measure on $\mathcal{J}$ such that it is equal to $\hat\mu_\epsilon$ on $\mathbf{R}^m$, equals to $\hat\omega$ on $\hat{\mathbf{R}}^m$ and equals to $\delta_{\{\Delta\}}$ on Δ.

Keeping Theorem 6.1 (in Appendix) in mind, we introduce some positive constants. Let α be the exponent of the order condition of ν and let β and r be positive numbers such that $\frac{3}{2} < \alpha(1+\beta) < 2$ and $r > (2-\alpha(1+\beta))^{-1}$. Let $q > 4r$ and $q(k) = (1+\beta)rq^{-k}$. For a positive integer N_0 and $\varepsilon,\delta > 0$, define $L^{N_0}_{\epsilon,\delta}(w,x), w,x\in\mathbf{R}^d$ by

$$
\begin{aligned}
L^{N_0}_{\epsilon,\delta}(w,x) = \sum_{V\in\Sigma_0}\Big\{&|w^TV(x)|^2+ \\
&+\sum_{k=1}^{N_0}\int\cdots\int |w^T\Psi(v_k,\dots,v_1)V(x)|^2\mu_{\varepsilon^{q(k)}}(dv_k)\cdots\mu_{\varepsilon^{q(1)}}(dv_1)\Big\}.
\end{aligned}
$$

($\Psi(v_k,\dots,v_1)$ may depend on δ).

Theorem 4.1. — *For the canonical SDE (1.1), assume that there exists a positive integer N_0, a nonnegative integer n_0, $\delta_0,\epsilon_0 > 0$ and a positive number C such that*

$$L^{N_0}_{\varepsilon,\delta_0}(w,x) \ge \frac{C|w|^2}{(1+|x|)^{n_0}} \tag{4.1}$$

holds for any $0 < \epsilon < \epsilon_0$ *and* $w, x \in \mathbf{R}^d$. *Then for any initial random variable* ξ_0 *and* $0 < T_0 < \infty$, *the law of the solution* ξ_{T_0} *has a* C^∞*-density.*

For the proof of the above theorem, we need Norris' type estimate stated in Theorem 6.1 in Appendix. We fix δ_0 satisfying (4.1). We define events (with parameter $l \in S_{d-1}$ and $\varepsilon > 0$) by

$$E = \left\{ \sum_{V \in \Sigma_0} \int_0^{T_0} |Y_{l,V}(t-)|^2 dt < \varepsilon \right\}.$$

We want to prove that for any $p > 1$ there exists $C_p > 0$ such that $P(E) \le C_p \epsilon^p$ holds for any $0 < \epsilon < \epsilon_0$ and $l \in S_{d-1}$. In order to prove this, associated with the vector field V we introduce a sequence of events $E_V^{(k)}$ (with parameter $l \in S_{d-1}$ and ε) by

$$\left\{ \int_0^{T_0} \left(\int |Y_{l,V}^{(k)}(t-, v_k, ..., v_1)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \right) dt < \varepsilon^{q^{-k}} \right\},$$

for $k = 0, 1, 2, ...$, where $Y_{l,V}^{(0)} = Y_{l,V}$. Then we have $E \subset \cap_{V \in \Sigma^0} E_V^{(0)}$ and the set $E_V^{(0)}$ is included in

$$(E_V^{(0)} \cap (E_V^{(1)})^c) \cup (E_V^{(1)} \cap (E_V^{(2)})^c) \cup \cdots \cup (E_V^{(N_0-1)} \cap E_V^{(N_0)})^c) \cup G_V,$$

where

$$G_V = E_V^{(0)} \cap E_V^{(1)} \cap \cdots \cap E_V^{(N_0)}.$$

Consequently, in order to prove that $P(E)$ is small, it is sufficient to prove that both $P(E_V^{(k)} \cap (E_V^{(k+1)})^c)$ and $P(\cap_{V \in \Sigma_0} G_V)$ are small. These two assertions will be shown in the following two lemmas.

Lemma 4.2. — *For any* $p > 1$ *there exists a positive constant* C_p *such that*

$$P(E_V^{(k)} \cap (E_V^{(k+1)})^c) \le C_p \varepsilon^p, \quad k = 0, 1, .., N^0 - 1 \tag{4.2}$$

holds for all $0 < \varepsilon < \varepsilon_0$ *and* $l \in S_{d-1}$.

Proof. — We first consider the case $k = 0$. We want to apply Theorem 6.1 in Appendix to the semimartingale $Y_{l,V}(t)$. The integrand functions of the right hand side of (3.3) have finite moments of any order ([**3**]), i.e.,

$$E[\sup_t |Y_{l,V}^{(1)}(t)|^{p'} + \sup_{t,y} |Y_{l,V}^{(1)}(t, y)|^{p'} + \sup_{t,z} |Y_{l,V}^{(1)}(t, z)|^{p'}] < \infty.$$

Therefore the functional θ^γ defined by (6.2) satisfies $E[(\sup_\gamma \theta^\gamma)^{p'}] < \infty$ for any p'. Then we can apply Theorem 6.1 and we get

$$P(E_V^{(0)} \cap (E_V^{(1)})^c) \le C_{p'} \varepsilon^{p'}$$

for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$.

We want to apply Theorem 6.1 again to $Y_{l,V}^{(k)}(t, v_k, ..., v_1)$, which is written by (3.5). Set $\gamma = (v_k, ..., v_1)$ and $\pi(d\gamma) = \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1)$. It can be shown that for any $p' > 1$, $E[(\sup_\gamma \theta^\gamma)^{p'}] < \infty$ holds. Then the inequality

$$P(E_V^{(k)} \cap (E_V^{(k+1)})^c) \le C_{p'} \varepsilon^{p' q^{-(k+1)}}, \quad k = 1, 2, ...$$

holds for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$ by Theorem 6.1. Set $p = p' q^{-N_0}$. Then (4.2) holds valid for any k. □

Lemma 4.3. — *Assume (4.1). Then for any $p > 1$ there exists a positive constant C'_p such that*

$$P(\cap_{V \in \Sigma_0} G_V) < C'_p \varepsilon^p, \tag{4.3}$$

for all $0 < \varepsilon < 1$ and $l \in S_{d-1}$.

Proof. — Set

$$K_\epsilon = \sum_{k=0}^{N_0} \sum_{V \in \Sigma_0} \int_0^{T_0} \left(\int |Y_{l,V}^{(k)}(t-, v_k, ..., v_1)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \right) dt.$$

Then, if $\omega \in G := \cap_{V \in \Sigma_0} G_V$, we have the inequality

$$K_\varepsilon(\omega) < m \sum_{k=0}^{N_0} \varepsilon^{q^{-k}} < m(N_0+1)\varepsilon^{q^{-N_0}}$$

if $\varepsilon^{1/q} < 1$. Therefore, we have $G \subset \{K_\varepsilon < m(N_0+1)\varepsilon^{q^{-N_0}}\}$. Thus, the problem is reduced to getting the estimate of $P(K_\varepsilon < m(N_0+1)\varepsilon^{q^{-N_0}})$.

Observe that K_ε is written as

$$K_\varepsilon = \int_0^{T_0} L_{\varepsilon,\delta_0}^{N_0}((\nabla \xi_{t-})^{-1} l, \xi_{t-}) dt.$$

Inequality (4.1) implies

$$K_\varepsilon \ge C \int_0^{T_0} \frac{|(\nabla \xi_{t-})^{-1} l|^2}{(1 + |\xi_{t-}|)^{n_0}} dt.$$

Further, for any $l \in S_{d-1}$, we have the inequality

$$\left(\int_0^{T_0} \frac{|(\nabla \xi_{t-})^{-1} l|^2}{(1 + |\xi_{t-}|)^{n_0}} dt \right)^{-1} \le \frac{1}{T_0^2} \int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt,$$

by using Jensen's inequality. Therefore

$$G \subset \left\{ \int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt > \frac{C T_0^2}{m(N_0+1)\varepsilon^{q^{-N_0}}} \right\}.$$

Then we get by Chebyschev's inequality, $P(G) \le C'_p \varepsilon^p$ where

$$C'_p = \left(\frac{m(N_0+1)}{C T_0^2} \right)^{\frac{p}{q^{-N_0}}} E\left[\left(\int_0^{T_0} |\nabla \xi_{t-}|^2 (1 + |\xi_{t-}|)^{n_0} dt \right)^{\frac{p}{q^{-N_0}}} \right].$$

We have thus obtained the estimate (4.3) for all $0 < \varepsilon < 1$ and $l \in S_{d-1}$. □

Proof of Theorem 4.1. — It suffices to prove (2.4). Inequalities of Lemmas 4.1 and 4.2 imply

$$P\left(\sum_{V\in\Sigma_0}\int_0^{T_0}|Y_{l,V}(t-)|^2dt<\epsilon\right)<C_p''\epsilon^p$$

for all $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$. Consequently we obtain

$$\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_0^{T_0}|l^T(\nabla\xi_t)^{-1}V(\xi_t)|^2dt\right)^{-p}\right]\le C_p$$

for any $p > 1$.

Consider next the case where $\mathbf{u} \neq 0$. Let $\mathbf{u} = \{(t_1, z_1), ..., (t_n, z_n)\}$, where we have $0 < t_1 < \cdots < t_n < T_0$. We set $\xi_t^{\mathbf{u}} = \xi_t \circ \varepsilon_{\mathbf{u}}^+$ and $Y_{l,V}^{\mathbf{u}}(t) = l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})$. Then there exists an interval $[t_i, t_{i+1}]$ such that its length is greater than or equal to $T_0/(n+1)$. Choose $t_i' < t_{i+1}'$ such that $[t_i', t_{i+1}'] \subset [t_i, t_{i+1}]$ and $t_{i+1}' - t_i' = T_0/(n+1)$. Then $\xi_t^{\mathbf{u}}, t \in [t_i', t_{i+1}']$ is a solution of SDE (1.1) with the initial data $\xi_{t_i}^{\mathbf{u}}$. We can apply the argument of this section to the process $Y_{l,V}^{\mathbf{u}}(t), t \in [t_i', t_{i+1}']$. Then we have

$$\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_{t_i'}^{t_{i+1}'}|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})|^2dt\right)^{-p}\right]< C_{p,\mathbf{u}}.$$

Note that the family of initial data satisfies

$$\sup_{\mathbf{u}\in A(1)^n} E[|\xi_{t_i}^{\mathbf{u}}|^p]\le c(n,\rho_0,p)<\infty.$$

Then we can choose a positive constant $C_{n,p}$ such that it dominates all $C_{p,\mathbf{u}}$. Therefore,

$$\sup_{\mathbf{u}\in A(1)^n}\sup_{l\in S_{d-1}} E\left[\left(\sum_{V\in\Sigma_0}\int_0^{T}|l^T(\nabla\xi_t^{\mathbf{u}})^{-1}V(\xi_t^{\mathbf{u}})|^2dt\right)^{-p}\right]< C_{n,p}$$

for any n and p. □

5. Relation with Lie algebra

In this section we want to prove the following.

Theorem 5.1. — *Under the same condition as in Theorem 1.1, there exists $\delta_0', \epsilon_0' > 0$ and $C' > 0$ such that the inequality*

$$L_{\varepsilon,\delta_0'}^{N_0}(w,x)\ge C'|w|^2,\quad \forall w,x\in\mathbf{R}^d \tag{5.1}$$

holds for all $0 < \varepsilon < \varepsilon_0'$.

If the above theorem is established, Theorem 1.1 follows from Theorem 4.1 and Theorem 5.1, immediately. Theorem 1.2 is an easy consequence of Theorem 4.1. Indeed, it is verified as follows.

Proof of Theorem 1.2. — Since b_0 exists by the assumption of the theorem, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, the inequality

$$\sum_{k=0}^{N_0} \sum_{V \in \Sigma_k^\delta} |l^T V(x)|^2 \geq \frac{1}{2} \sum_{k=0}^{N_0} \sum_{V \in \Sigma_k} |l^T V(x)|^2 \geq \frac{C}{2}$$

holds. Then Theorem 1.2 follows from Theorem 1.1. □

Before we proceed to the proof of Theorem 5.1, we shall approximate the vector field $\Psi(v_k, ..., v_1)V$ given by (3.4) by a linear sum of vector fields of the form $\Psi_{k_k}\Psi_{k_{k-1}} \cdots \Psi_{k_1} V$ where Ψ_{k_i} are such that $\Psi_0 = \Psi_0^\delta V$ or $\Psi_i V = [V_i, V]$, $i = 1, ..., m$, in the case where $v_1, ..., v_k \in \hat{\mathbf{R}}^m \cup \mathbf{R}^m$ are small.

We first consider the case $k = 1$. We have $\Psi(\Delta)V = \Psi_0^\delta V$, $\Psi(y)V = \sum_i [V_i, V] y^i / |y|$ and $\Psi(z)V = \Phi_1(z)V/|z|$. Set $z = (z^1, ..., z^m)$. Then $\Phi_s(z)$ given by (3.2) satisfies the differential equation

$$\frac{\Phi_s(z)V(x)}{ds} = (\nabla \phi_s^z(x))^{-1} \Big(\sum_{i=1}^m [V_i, V](\phi_s^z(x)) z^i \Big).$$

Hence $\Phi_1(z)V(x)$ is written as

$$\begin{aligned} &\Phi_1(z)V(x) - \sum_{i=1}^m [V_i, V](x) z^i \\ &= \frac{1}{2} (\nabla \phi_\theta^z(x))^{-1} \sum_{i,j} [V_j, [V_i, V]](\phi_\theta^z(x)) z^i z^j, \end{aligned}$$

where $0 \leq \theta \leq 1$, by the mean value theorem. Consequently we obtain

$$\Big| \Phi_1(z)V(x) - \sum_{i=1}^m [V_i, V](x) z^i \Big| \leq c_1 |z|^2.$$

Since $\Psi(z) = \Phi_1(z)/|z|$, we get

$$\Big| \Psi(z)V(x) - \sum_{i=1}^m [V_i, V](x) \frac{z^i}{|z|} \Big| \leq c_1 |z|$$

for sufficiently small z.

We next consider the case $k \geq 2$. Suppose $v_k = z_k, ..., v_1 = z_1$. We can show similarly that there exists $\delta_0 > 0$ such that the inequality

$$\Big| \Psi(z_k, ..., z_1)V(x) - \sum_{i_k, ..., i_1 = 1}^m \Psi_{i_k} \cdots \Psi_{i_1} V(x) \frac{z_k^{i_k}}{|z_k|} \cdots \frac{z_1^{i_1}}{|z_1|} \Big| \leq c_2 \sum_{i=1}^k |z_i|$$

holds for $|z_i| \leq \delta_0, i = 1, ..., k$, where $z_i = (z_i^1, ..., z_i^m)$. For the general $v_k, ..., v_1$, we have

$$\left|\Psi(v_k, ..., v_1)V(x) - \sum_{i_k,\ldots,i_1=0}^{m} \Psi_{i_k} \cdots \Psi_{i_1} V(x) \varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1)\right| \leq c_2\Big(\sum_{i \in \{k; v_k = z_k\}} |z_i|\Big). \tag{5.2}$$

Here $\varphi_0(\Delta) = 1, \varphi_k(\Delta) = 0, k = 1, ..., m$ and $\varphi_0(y) = \varphi_0(z) = 0, \varphi_k(z) = \frac{z^k}{|z|}$ and $\varphi_k(y) = \frac{y^k}{|y|}, k = 1, ..., m$.

We claim:

Lemma 5.2. — *For any $\delta > 0$ and $c > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta, c) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $l \in S_{d-1}$, we have*

$$\int \cdots \int |l^T \Psi(v_k, ..., v_1)V(x)|^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \geq \frac{\hat{\lambda}_1^{\,k}}{2} \left(\sum_{i_k,\ldots,i_1=0}^{m} |l^T \Psi_{i_k} \cdots \Psi_{i_1} V(x)|^2 \right) - c, \tag{5.3}$$

where λ_1 is the minimal eigen value of the matrix $A + B$ and $\hat{\lambda}_1 = \lambda \wedge 1$.

Proof. — Let us consider F_ϵ given by

$$\int\int \Big(\sum_{i_k,\ldots,i_1=0}^{m} l^T \Psi_{i_k} \cdots \Psi_{i_1} V \varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1)\Big)^2 \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1).$$

Since

$$\lim_{\varepsilon \to 0} \int \int \varphi_{i_k}(v_k) \cdots \varphi_{i_1}(v_1) \varphi_{i'_k}(v_k) \cdots \varphi_{i'_1}(v_1) \mu_{\varepsilon^{q(k)}}(dv_k) \cdots \mu_{\varepsilon^{q(1)}}(dv_1) \geq \prod_{j=1}^{k} (a_{i_j i'_j} + b_{i_j i'_j} + c_{i_j i'_j}),$$

(where $c_{ij} = 1$ if $i = j = 0$ and $= 0$ otherwise), the inferior limit of F_ϵ is greater than or equal to

$$\sum_{i_k,\ldots,i_1=0}^{m} \sum_{i'_k,\ldots,i'_1=0}^{m} l^T \Psi_{i_k} \cdots \Psi_{i_1} V l^T \Psi_{i'_k} \cdots \Psi_{i'_1} V \times (a_{i_k,i'_k} + b_{i_k,i'_k} + c_{i_k i'_k}) \cdots (a_{i_1 i'_1} + b_{i_1,i'_1} + c_{i_1 i'_1}).$$

The above has the lower bound $\hat{\lambda}_1^k \sum_{i_k,\ldots,i_1=0}^m |l^T \Psi_{i_k} \cdots \Psi_{i_1} V|^2$. Therefore, we have

$$F_\epsilon \geq \hat{\lambda}_1^k \left(\sum_{i_k,\ldots,i_1=0}^{m} |l^T \Psi_{i_k} \cdots \Psi_{i_1} V(x)|^2 \right) - \frac{c}{2},$$

for sufficiently small ε.

On the other hand, we have from (5.2) the inequality

$$\int\int\Big(l^T\Psi(v_k,...,v_1)V-\textstyle\sum_{i_k,...,i_1=0}^m l^T\Psi_{i_k}\cdots\Psi_{i_1}V\varphi_{i_k}(v_k)\cdots\varphi_{i_1}(v_1)\Big)^2$$

$$\times\mu_{\varepsilon^{q(k)}}(dv_k)\cdots\mu_{\varepsilon^{q(1)}}(dv_1)\le c_2\sum{}' \varepsilon^{2q(i)}\le\frac{1}{2}c$$

for sufficiently small ε, where $\sum'$ is the summation for $i\in\{k;v_k=z_k\}$. Consequently we get the inequality (5.3). □

Proof of Theorem 5.1. — We shall first introduce another family of vector fields. Given $\delta>0$, we define a linear transformation Ψ_0^δ of vector fields by (3.1). We may consider $\Psi_0^\delta V$ as a modification of the vector field $[\hat{V}_0^\delta,V]$. We define

$$\Gamma_0^\delta=\Sigma_0,\cdots,\Gamma_k^\delta=\{\Psi_0^\delta V,[V_i,V],i=1,...,m,V\in\Gamma_{k-1}^\delta\}.$$

These can be regarded as a modification of Σ_k^δ of Section 1.

Now, apply (5.3) to each term of $L_{\varepsilon,\delta}(l,x)$. Then for any $0<\varepsilon<\varepsilon_0(\delta,c)$ and $l\in S_{d-1}$, $L_{\varepsilon,\delta}(l,x)$ is greater than or equal to

$$\text{(5.4)}\qquad \sum_{V\in\Sigma_0}|l^TV(x)|^2+\frac{\hat{\lambda}_1^{N_0}}{2}\sum_{k=1}^{N_0}\sum_{V\in\Sigma_0}\sum_{i_k=0}^{m}\cdots\sum_{i_1=0}^{m}|l^T\Psi_{i_k}\cdots\Psi_{i_1}V(x)|^2$$

$$-(m+1)^{N_0}N_0c\ge\frac{\hat{\lambda}_1^{N_0}}{2}\Bigg\{\sum_{V\in\cup_{k=0}^{N_0}\Gamma_k^\delta}|l^TV(x)|^2\Bigg\}-(m+1)^{N_0}N_0c.$$

We want to rewrite the right hand side of the above by using vector fields in Σ_k^δ. We set

$$\Phi_0^\delta V=[\hat{V}_0^\delta,V]+\frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]].$$

Then we have

$$\begin{aligned}&\Big|l^T\Psi_0^\delta V(x)-l^T\Phi_0^\delta V(x)\Big|^2\\ =&\ \left|\int_{|z|\le\delta}\Big(\Phi_1(z)V(x)-\sum_i[V_i,V](x)z^i\Big)\nu(dz)\right|^2\\ \le&\ c_1v(\delta)^2.\end{aligned}$$

We can show by induction

$$\Big|l^T(\Psi_0^\delta)^kV(x)-l^T(\Phi_0^\delta)^kV(x)\Big|^2\le 2^kc_1v(\delta)^2.$$

Therefore,

$$|l^T(\Psi_0^\delta)^kV(x)|^2\ge\frac{1}{2}|l^T(\Phi_0^\delta)^kV(x)|^2-2^{k+1}c_1v(\delta)^2.$$

Summing up these inequalities, we obtain

$$\sum_{k=0}^{N_0}\sum_{V\in\Gamma_k^\delta}|l^TV(x)|^2\geq\frac{1}{2}\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k^\delta}|l^TV(x)|^2-N2^{N_0+1}c_1v(\delta)^2,$$

where N is the number of terms of the sum $\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k^\delta}$. Therefore, assuming (1.2), the right hand side of (5.4) dominates

$$C':=\left\{\frac{\hat\lambda_1^{N_0}}{4}\Big(\frac{C(\delta)}{2}-N2^{N_0+1}c_1v(\delta)^2\Big)-(m+1)^{N_0}N_0c\right\}.$$

The above constant C' becomes positive if we choose δ, c sufficiently small, say $\delta=\delta_0'$ and $c=c_0'$. Set $\varepsilon_0'=\varepsilon_0(\delta_0',c_0')$. Then we get the inequality (5.1) for $l\in S_{d-1}$ and $x\in\mathbf{R}^{\mathbf{d}}$. The inequality is extended to any $w,x\in\mathbf{R}^d$. □

6. Appendix. An analogue of Norris' estimate

In this section, we will consider semimartingales with parameter γ, which is directly related to the solution of an SDE. We consider a semimartingale $Y_t^\gamma, 0\leq t\leq T_0$ defined by

$$\begin{aligned}Y_t^\gamma &= y^\gamma+\int_0^t a^\gamma(s)ds+\sum_i\int_0^t f_i^\gamma(s)dW_s^i \\ &\quad +\int_0^t\int_{|z|\leq\delta}g^\gamma(s,z)d\tilde N+\int_0^t\int_{|z|>\delta}g^\gamma(s,z)dN,\end{aligned}\tag{6.1}$$

where $a^\gamma(s), f^\gamma(s), g^\gamma(s,z)$ are left continuous predictable processes, continuous with respect to parameters $z\in\mathbf{R}^m, \gamma\in\Gamma$. Here Γ is a compact space. We assume further that $a^\gamma(t)$ is a semimartingale represented by

$$\begin{aligned}a^\gamma(t+) &= a^\gamma+\int_0^t b^\gamma(s)ds+\sum_i\int_0^t e_i^\gamma(s)dW_s^i \\ &\quad +\int_0^t\int_{|z|\leq\delta}h^\gamma(s,z)d\tilde N+\int_0^t\int_{|z|>\delta}h^\gamma(s,z)dN,\end{aligned}$$

where $b^\gamma(s), e_i^\gamma(s), h^\gamma(s,z), s\geq 0$ are left continuous predictable processes continuous with respect to z and γ. We set

$$\begin{aligned}\theta^\gamma &= \|(a^\gamma)^2+(b^\gamma)^2\|+\sum_i\|(f_i^\gamma)^2+(e_i^\gamma)^2\| \\ &\quad +\int_{|z|\leq\delta}\|g^\gamma(z)^2+h^\gamma(z)^2\|\nu(dz)+\sup_{|z|>\delta}\|h^\gamma(z)^2\|,\end{aligned}\tag{6.2}$$

where $\|F\|=\sup_{0\leq t\leq T_0}|F(t)|$. Set further

$$\hat g^\gamma(t,z)=\frac{g^\gamma(t,z)}{|z|},\quad \hat\mu_\epsilon(dz)=\frac{1}{v(\epsilon)}|z|^2 1_{[0,\epsilon]}(|z|)\nu(dz).$$

We shall consider two events for given $r > 0, q > 4r, \beta > 0$ and $\epsilon > 0$

$$A(\epsilon) = \Big\{ \int_\Gamma \Big(\int_0^{T_0} |Y_{t-}^\gamma|^2 dt \Big) \pi(d\gamma) < \varepsilon^q \Big\},$$

$$B(\epsilon) =$$

$$\Big\{ \int_\Gamma \int_0^{T_0} \Big\{ a^\gamma(t)^2 + \sum_i |f_i^\gamma(t)|^2 + \int \hat{g}^\gamma(t,z)^2 \hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz) \Big\} \pi(d\gamma) dt > \varepsilon \Big\}.$$

We will show that the probability where both $A(\epsilon)$ and $B(\epsilon)$ occur simultaneously is small if ϵ is small.

Theorem 6.1. — *Let α be the exponent of the order condition of the Lévy measure ν. Let $\beta > 0$ be a number such that $3/2 < \alpha(1+\beta) < 2$. Let $r > \frac{1}{2-\alpha(1+\beta)}$ and $q > 4r$. Assume $E[(\sup_\gamma \theta^\gamma)^p] < \infty$ holds for any $p > 1$. Then for any $p > 1$, there exists a positive constant C_p such that the inequality*

$$P(A(\epsilon) \cap B(\epsilon)) < C_p \epsilon^p \tag{6.3}$$

holds for any semimartingale Y_t^γ represented by (6.1), any probability measure π on Γ and any $0 < \varepsilon < \varepsilon_0$, where $0 < \varepsilon_0 < 1$ is a positive number independent of p.

In order to prove the above theorem, we need the following. Let Y_t^γ be the process of (6.1) and let λ be an arbitrary positive number.

Komatsu-Takeuchi's estimate. *([7], Theorem 3) For any $0 < v < \frac{1}{4}$, there exist a positive random variable $\mathcal{E}(\lambda,\gamma)$ with $E[\mathcal{E}(\lambda,\gamma)] \le 1$ and positive constants C, C_0, C_1, C_2 such that the inequality*

$$\begin{aligned} \lambda^4 \int_0^{T_0} |Y_t^\gamma|^2 \wedge \frac{1}{\lambda^2} dt + \lambda^{-v} \log \mathcal{E}(\lambda,\gamma) + C \ge \\ C_0 \lambda^{1-4v} \int_0^{T_0} |a^\gamma(t)|^2 dt + C_1 \lambda^{2-2v} \sum_i \int_0^{T_0} |f_i^\gamma(t)|^2 dt \\ + C_2 \lambda^{2-2v} \int_0^{T_0} \int_{\mathbf{R}^m} |g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2} dt \nu(dz) \end{aligned} \tag{6.4}$$

holds on the set $\{\theta^\gamma \le \lambda^{2v}\}$ for all $\lambda > 1$ and Y^γ.

Remark 6.2. — In Theorem 3 in [7], the assertion is stated in the case where $Y_t^\gamma, a^\gamma(t)$ etc. do not depend on the parameter γ. Further the Lévy measure is assumed to be of the form $\nu(dz) = |z|^{-m-\alpha} dz$. However their result can be applied to the present case.

Proof of Theorem 6.1. — By the choice of β and r, it holds $0 < 2-\alpha(1+\beta)-\frac{1}{r}$. We will choose v such that $0 < v < (2-\alpha(1+\beta)-\frac{1}{r}) \wedge \frac{1}{8}$. We want to rewrite inequality (6.4) in order to apply it for the estimate (6.3). Our aim is to get (6.5) below on the set $\{\sup_\gamma \theta^\gamma \leq e^{-vr}\}$. We first consider the last term of (6.4). It holds for any $0 < \kappa < \lambda$

$$\begin{aligned}\int_{\mathbf{R}^m}\Big(|g^\gamma(t,z)|^2 \wedge \frac{1}{\lambda^2}\Big)\nu(dz) &\geq \int_{|z|<\frac{\kappa}{\lambda}}\Big(|\hat{g}^\gamma(t,z)|^2 \wedge \frac{1}{|z|^2\lambda^2}\Big)|z|^2\nu(dz)\\ &\geq v\Big(\frac{\kappa}{\lambda}\Big)\int_{|z|<\frac{\kappa}{\lambda}}\Big(|\hat{g}^\gamma(t,z)|^2 \wedge \frac{1}{\kappa^2}\Big)\hat{\mu}_{\frac{\kappa}{\lambda}}(dz).\end{aligned}$$

Now set $\lambda = \varepsilon^{-r}$ and $\kappa = \varepsilon^{\beta r}$. Then $\frac{\kappa}{\lambda} = \varepsilon^{(1+\beta)r}$ and $v(\frac{\kappa}{\lambda}) \geq C_4\varepsilon^{\alpha(1+\beta)r}$ by the order condition for $v(\rho)$. Therefore, (6.4) is rewritten by

$$\begin{aligned}&\varepsilon^{-4r}\int_0^{T_0}|Y_t^\gamma|^2 \wedge \varepsilon^{2r}dt + \varepsilon^{vr}\log\mathscr{E}(\varepsilon^{-r},\gamma) + C\\ \geq\ & C_0\varepsilon^{-r(1-4v)}\int_0^{T_0}|a^\gamma(t)|^2dt + C_1\varepsilon^{-r(2-2v)}\sum_i\int_0^{T_0}|f_i^\gamma(t)|^2dt\\ &+C_2C_4\varepsilon^{-r(2-2v)+\alpha(1+\beta)r}\int_0^{T_0}\int_{\mathbf{R}^m}|\hat{g}^\gamma(t,z)|^2 \wedge \varepsilon^{-2\beta r}dt\hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz).\end{aligned}$$

Now set $\rho = \min\{r(1-4v), r(2-2v)-\alpha(1+\beta)\} - 1$. In view of the choice of v, we have $\rho > 0$. Set $C_5 = \min\{C_0, C_2, C_4\}$. Then the above inequality yields

$$\begin{aligned}&\varepsilon^{-4r}\int_0^{T_0}|Y_t^\gamma|^2 \wedge \varepsilon^{2r}dt + \varepsilon^{vr}\log\mathscr{E}(\varepsilon^{-r},\gamma) + C \geq\\ &C_5\varepsilon^{-(\rho+1)}\int_0^{T_0}\Big\{|a^\gamma(t)|^2 + \sum_i|f_i^\gamma(t)|^2\\ &+\int|\hat{g}^\gamma(t,z)|^2 \wedge \varepsilon^{-2\beta r}\hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz)\Big\}dt,\end{aligned}$$

on the set $\{\theta^\gamma \leq \epsilon^{-vr}\}$.

Next, integrate each term of the above by the measure π with respect to the parameter γ. We have by Jensen's inequality $\int\log\mathscr{E}(\lambda,\gamma)\pi(d\gamma) \leq \log\mathscr{E}(\lambda)$, where $\mathscr{E}(\lambda) = \int\mathscr{E}(\lambda,\gamma)\pi(d\gamma)$ is a positive random variable such that $E[\mathscr{E}(\lambda)] \leq 1$. Therefore we have

$$\begin{aligned}&\varepsilon^{-4r}\int_\Gamma\Big(\int_0^{T_0}|Y_t^\gamma|^2 \wedge \varepsilon^{2r}dt\Big)\pi(d\gamma) + \varepsilon^{vr}\log\mathscr{E}(\varepsilon^{-r}) + C \geq\\ &C_5\varepsilon^{-(\rho+1)}\int_\Gamma\int_0^{T_0}\Big\{|a^\gamma(t)|^2 + \sum_i|f_i^\gamma(t)|^2\\ &+\int|\hat{g}^\gamma(t,z)|^2 \wedge \varepsilon^{-2\beta r}\hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz)\Big\}dt\pi(d\gamma)\end{aligned}\tag{6.5}$$

on the set $\{\sup_\gamma \theta^\gamma \le \epsilon^{-vr}\}$.

We can now give the proof of (6.3). We define three events by

$$
\begin{aligned}
A_1(\varepsilon) &= \Big\{\sup_\gamma \theta^\gamma > \epsilon^{-vr}\Big\},\\
A_2(\varepsilon) &= \Big\{\sup_\gamma \theta^\gamma \le \epsilon^{-vr}\Big\} \bigcap \Big\{\int_\Gamma \int_0^{T_0} |Y_{t-}^\gamma|^2 \wedge \varepsilon^{2r} dt\pi(d\gamma) < \varepsilon^q\Big\}\\
&\quad \bigcap \Big\{\sup_\gamma \|\hat{g}^\gamma\| \le \varepsilon^{-2\beta r}\Big\} \bigcap \Bigg\{\int_\Gamma \int_0^{T_0} \Big(a^\gamma(t)^2 + \sum_i |f_i^\gamma(t)|^2 +\\
&\quad + \int \hat{g}^\gamma(t,z)^2 \wedge \varepsilon^{-2\beta r} \hat{\mu}_{\varepsilon^{(1+\beta)r}}(dz)\Big) dt\pi(d\gamma) > \varepsilon\Bigg\},\\
A_3(\varepsilon) &= \Big\{\sup_\gamma \|\hat{g}^\gamma\| > \varepsilon^{-2\beta r}\Big\}.
\end{aligned}
$$

Then it holds $A(\varepsilon) \cup B(\varepsilon) \subset A_1(\varepsilon) \cup A_2(\varepsilon) \cup A_3(\varepsilon)$ for any γ. Therefore, the probability of (6.3) is dominated by $P(A_1(\varepsilon)) + P(A_2(\varepsilon)) + P(A_3(\varepsilon))$. We shall get estimates of $P(A_i(\varepsilon)), i = 1, 2, 3$. In view of our assumption of the theorem, the first one is estimated as

$$P(A_1(\varepsilon)) \le \varepsilon^p E\Big[(\sup_\gamma \theta^\gamma)^{p/r}\Big] \le c_p \varepsilon^p.$$

A similar estimate is valid for $P(A_3(\varepsilon))$. For the estimate of $P(A_2(\varepsilon))$, we remark that (6.5) implies

$$A_2(\varepsilon) \subset \Big\{\mathscr{E}(\varepsilon^{-r})^{\varepsilon^{vr}} \ge \exp\left(-\varepsilon^{q-4r} + C_5 \varepsilon^{-\rho} - C\right)\Big\}.$$

Therefore, by Chebyschev's inequality

$$P(A_2(\varepsilon)) \le e^C \exp\left(\varepsilon^{q-4r} - C_5 \varepsilon^{-\rho}\right) E\left[\mathscr{E}(\varepsilon^{-r})^{\varepsilon^{vr}}\right].$$

Further $\varepsilon^{q-4r} < \frac{C_5}{2}\varepsilon^{-\rho}$ holds for $\varepsilon < \varepsilon_0$, where $\varepsilon_0^{q-4r} = C_5/2$. Therefore,

$$P(A_2(\varepsilon)) \le e^C \exp\left(-\frac{C_5}{2}\varepsilon^{-\rho}\right) \le c_p' \varepsilon^p$$

for $\varepsilon < \varepsilon_0$. □

References

[1] K. Bichteler, J.-B. Gravereaux & J. Jacod – *Malliavin calculus for processes with jumps*, Stochastics Monographs, vol. 2, Gordon and Breach Science Publishers, 1987.

[2] J.-M. Bismut – "Calcul des variations stochastique et processus de sauts", *Z. Wahrsch. Verw. Gebiete* **63** (1983), p. 147–235.

[3] T. Fujiwara & H. Kunita – "Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group", *J. Math. Kyoto Univ.* **25** (1985), p. 71–106.

[4] ———, "Canonical SDE's based on semimartingales with spatial parameters. I. Stochastic flows of diffeomorphisms", *Kyushu J. Math.* **53** (1999), p. 265–300.

[5] N. IKEDA & S. WATANABE – *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., 1989.

[6] Y. ISHIKAWA & H. KUNITA – "Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps", *Stochastic Process. Appl.* **116** (2006), p. 1743–1769.

[7] T. KOMATSU & A. TAKEUCHI – "On the smoothness of PDF of solutions to SDE of jump type", *Int. J. Differ. Equ. Appl.* **2** (2001), p. 141–197.

[8] S. G. KOU & J. WANG – "Option pricing under double exponential jump diffusion model", *Management Science* **50** (2004), p. 1178–1192.

[9] H. KUNITA – *Stochastic flows and stochastic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 24, Cambridge University Press, 1990.

[10] ______, "Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms", in *Real and stochastic analysis*, Trends Math., Birkhäuser, 2004, p. 305–373.

[11] H. KUNITA & S. WATANABE – "On square integrable martingales", *Nagoya Math. J.* **30** (1967), p. 209–245.

[12] S. KUSUOKA & D. STROOCK – "Applications of the Malliavin calculus. I", in *Stochastic analysis (Katata/Kyoto, 1982)*, North-Holland Math. Library, vol. 32, North-Holland, 1984, p. 271–306.

[13] R. LÉANDRE – "Régularité de processus de sauts dégénérés", *Ann. Inst. H. Poincaré Probab. Statist.* **21** (1985), p. 125–146.

[14] ______, "Régularité de processus de sauts dégénérés. II", *Ann. Inst. H. Poincaré Probab. Statist.* **24** (1988), p. 209–236.

[15] ______, "Malliavin calculus of Bismut type for Poisson processes without probability", to appear in the "Fractional systems", a special issue of *Journal européen des systèmes automatisés*, J. Sabatier *et al.* eds., 2008.

[16] R. MERTON – "Option pricing when underlying stock returns are discontinuous", *J. Financial Economics* **3** (1976), p. 125–144.

[17] J. NORRIS – "Simplified Malliavin calculus", in *Séminaire de Probabilités, XX, 1984/85*, Lecture Notes in Math., vol. 1204, Springer, 1986, p. 101–130.

[18] D. NUALART – *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer, 1995.

[19] J. PICARD – "On the existence of smooth densities for jump processes", *Probab. Theory Related Fields* **105** (1996), p. 481–511.

[20] K. SATO – *Lévy processes and infinitely divisible distributions*, Cambridge Univ. Press, 1999.

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