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DIRECT IMAGE FOR SOME SECONDARY K-THEORIES

by

Alain Berthomieu

This article is dedicated to J.-M. Bismut, for his sixtieth birthday

Abstract. — The real counterpart of relative K-theory (considered in the complex setting in [**4**]) is considered here, some direct image under proper submersion is constructed, and a Grothendieck-Riemann-Roch theorem for Johnson-Nadel-Chern-Simons classes is proved. Metric properties are also studied.

This needs to revisit the construction of η-forms in the case where the direct image is provided by the vertical Euler (de Rham) operator. A direct image under proper submersions of some "non hermitian smooth" or "free multiplicative" K-theory is deduced (in the same context).

Double submersions are also studied to establish some functoriality properties of these direct images.

***Résumé* (Image directe pour certaines K-théories secondaires).** — On construit un morphisme d'image directe par submersion propre pour la version réelle de la K-théorie relative (considérée dans [**4**] dans un contexte holomorphe), et un théorème de type Grothendieck-Riemann-Roch est établi pour les classes de Johnson-Nadel-Chern-Simons. On étudie aussi des propriétés métriques.

Ceci nécessite de construire des formes η (de transgression du théorème d'indice des familles) dans le cas où l'image directe est définie par l'operateur d'Euler (de Rham) des fibres. On en déduit également un morphisme d'image directe pour une K-théorie « lisse non hermitienne » ou « multiplicative libre ».

La question de la fonctorialité de ces images directes pour des doubles submersions est également abordée.

1. Introduction

In [**35**], Nadel proposed characteristic classes (also considered by Johnson [**23**], see infra) for triples (E, F, f) where E and F are holomorphic vector bundles on

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some Kähler manifold X, and $f\colon E \xrightarrow{\sim} F$ is a C^∞ vector bundle isomorphism. He conjectured that if X is projective, his classes, which take their values in $H^{(0,\mathrm{odd})}(X)$, were projections of the image by the Abel-Jacobi map of the difference of the Chow group valued Chern classes of E and F. Inspired by [**26**] §6, I developped in [**4**] a notion of relative K-theory which appeared as suitably adapted to describe such triples considered by Nadel. This theory measures the kernel of the forgetful map from the K^0-theory of holomorphic vector bundles on X to the usual topological K^0-theory. As such, if X is projective, any pointed fine moduli space of vector bundles on X naturally maps to this relative K-theory. Moreover, it is rationally isomorphic to the Chow subgroup of homologically trivial cycles.

In [**4**], Johnson-Nadel classes were extended by considering a suitable projection of the Chern-Simons transgression form associated to compatible connections on E and F. The obtained characteristic class was proved to solve a generalised form of Nadel's conjecture.

I realised very recently that D. Johnson already obtained partial results in this direction: in [**24**] it seems that the same classes as considered by Nadel were defined, and in [**23**] some weaker version (than in [**4**]) of the classes were constructed and a weaker version of the "generalized Nadel conjecture" was proved.

[**4**] also contains direct images and Grothendieck-Riemann-Roch type results for relative K-theory and its characteristic class, for submersions and immersions of smooth projective varieties.

One of the goals of this article is to study the counterpart of this theory in the context of complex flat vector bundles over some real smooth manifold M. The corresponding relative K-theory was defined by Karoubi [**26**] §6 and studied by Karoubi and Dupont [**17**]. It is here described from objects of the form $(E, \nabla_E, F, \nabla_F, f)$ where f is a smooth vector bundle isomorphism between complex vector bundles E and F endowed with flat connections ∇_E and ∇_F (see Definition 4). If M is compact, the pointed algebraic variety $\mathcal{V}_F$ of flat vector bundle structures on some fixed topological vector bundle on M naturally maps to this relative K-theory.

If $\pi\colon M \longrightarrow B$ is a proper submersion, I construct here (see Definition 26 and Theorem 27) a direct image morphism $\pi_*\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$. The main technical problem consists in finding a vector bundle isomorphism (or something equivalent) between representatives of $\pi_! E$ and $\pi_! F$ as virtual flat vector bundles on B in such a way that the direct image becomes natural and functorial.

The counterpart here of Johnson-Nadel classes is simply given by Chern-Simons transgression forms in odd degree de Rham cohomology:

$$\mathcal{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f) = \big[\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big] \in H^{\mathrm{odd}}_{dR}(X). \tag{1}$$

Because of its rigidity properties, this Chern-Simons class may essentially detect different connected components of the above algebraic variety $\mathcal{V}_F$, and the class of the determinant line bundle (see §2.3).

A Grothendieck-Riemann-Roch type theorem for $\mathcal{N}_{\mathrm{ch}}$ (Theorem 29) is obtained as a by-product of the constructions performed in pursuing the second goal of the article, namely the study of "free multiplicative" or "non hermitian smooth" K-theory. This K-theory, denoted by $\widehat{K}_{\mathrm{ch}}$ is generated by triples of the form (E, ∇, α) where ∇ is a connection on the complex vector bundle E over M and α is an odd degree differential form defined modulo exact forms. Relations are direct sum and if $f\colon E \longrightarrow F$ is any smooth vector bundle isomorphism:

$$(E, \nabla_E, \alpha) = \big(F, \nabla_F, \alpha + \widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big) \tag{2}$$

(Here $\widetilde{\mathrm{ch}}$ is again a Chern-Simons transgression form). K^0_{rel} and $\widehat{K}_{\mathrm{ch}}$ are related by a commutative diagram whose lines are exact sequences (see Proposition 10):

$$\begin{array}{ccccccc} K^1_{\mathrm{top}}(M) & \longrightarrow & K^0_{\mathrm{rel}}(M) & \longrightarrow & K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\ \big\downarrow \| & & \big\downarrow \mathcal{N}_{\mathrm{ch}} & & \big\downarrow & & \big\downarrow \| \\ K^1_{\mathrm{top}}(M) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \end{array} \tag{3}$$

In this diagram, $\Omega^\bullet(M)$ denotes differential forms, K_{top} denotes ordinary K-theory, and K^0_{flat} denotes the K^0 theory of the category of flat bundles modulo exact sequences. For any vector bundle E on M endowed with a flat connection ∇_E, the image in $\widehat{K}_{\mathrm{ch}}(M)$ of $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$ is the triple $(E, \nabla_E, 0)$.

On one hand, Karoubi's multiplicative K-theory [**26**] [**27**] [**28**] consists of quotients (the form α being defined modulo greater subgroups than only exact forms) of subgroups (defined by restrictions on the Chern-Weil character form $\mathrm{ch}(\nabla)$) of this theory. These subgroups and constraints stemm from natural filtrations of the de Rham complex of M suitably adapted to the geometry studied. In [**28**], Karoubi studies foliations for which he constructs generalisations of the Godbillon-Vey invariant, and holomorphic and algebraic varieties for which known characteristic classes for holomorphic or algebraic vector bundles are shown to factor through the suitable multiplicative K-theory. Poutriquet [**36**] studies the context of conical singularities. The corresponding multiplicative K-theory he constructs shows interesting similarities with intersection cohomology. Felisatti and Neumann [**18**] generalise the concept of multiplicative K-theory to simplicial manifolds with applications to classifying spaces of Lie groups and Lie groupoids.

As an example, the multiplicative K-theory adapted to the study of flat bundles is the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) such that

$$\mathrm{ch}(\nabla) - d\alpha \in \mathbb{Z} \subset \Omega^{\mathrm{even}}(M) \tag{4}$$

Removing this constraint would justify the name "free multiplicative" K-theory. Direct image results for $\widehat{K}_{\mathrm{ch}}$ should have corollaries for "nonfree" multiplicative K-theories under mild compatibility conditions on the filtrations of the de Rham complex used to define them.

On the other hand, Bunke and Schick [**14**] defined a smooth (hermitian) K-theory, which coincides with the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) where α is a real form and ∇ respects some hermitian metric on E. Bunke and Schick's smooth K-theory is motivated by quantum field theory considerations [**19**] and it fits in the general framework of smooth extensions of generalized cohomology theories [**20**] [**21**]. Among other examples, Bunke and Schick construct interesting smooth K-theory canonical classes on homogeneous spaces and generalisations of parametrized ρ-invariants [**14**] §5.

Allowing nonunitary connections (and nonreal forms) would justify the name "non hermitian smooth K-theory". Anyway, the hermitian restriction would prevent from obtaining a natural morphism $K^0_{\mathrm{flat}}(M) \longrightarrow \widehat{K}_{\mathrm{ch}}(M)$ because of the existence of nonunitary flat vector bundles.

The obstruction for a flat bundle (E, ∇_E) to be unitary can be detected by characteristic classes similar to $\mathcal{N}_{\mathrm{ch}}(E, \nabla_E, E, \nabla_E^*, \mathrm{Id}_E)$ where ∇_E^* is the adjoint connection of ∇_E with respect to any hermitian metric on E (22). Such classes were first considered by Kamber and Tondeur [**25**], they correspond to the imaginary part of Chern-Cheeger-Simons classes [**15**], (see [**11**] Proposition 1.14). Karoubi proved in [**26**] §6.31 that they could detect some Borel generators of algebraic K-theory of integer rings in number fields [**12**]. See also [**11**] §I(g) for an interpretation as stable characteristic classes arising from stable continuous cohomology of $GL(\mathbb{C})$.

Here this Borel-Kamber-Tondeur class is extended to $\widehat{K}_{\mathrm{ch}}$. It is not always a cohomology class, but rather a purely imaginary differential form defined modulo exact forms (see Definition 16).

Moreover, a direct image morphism for $\widehat{K}_{\mathrm{ch}}$ under proper submersions is constructed (Theorem 31), which is compatible with the usual (sheaf theoretic) direct image of flat vector bundles (using fiberwise twisted de Rham cohomology, see Definition 22). This is performed from the families analytic index of the fiberwise twisted Euler operator together with a suitable η-form which is a non hermitian generalisation of that of Bunke [**13**] (Theorem 28). Functoriality is established only for the "nonfree" multiplicative subgroup of $\widehat{K}_{\mathrm{ch}}$ subject to the constraint (4), using some universal characterisation of the η-form.

Finally the symmetries induced by the fiberwise Hodge star operator are studied. Reality (resp. vanishing) properties of the pushforwards are established in the even (resp. odd) dimensional fibre case (Theorems 32 and 33).

The paper is organized as follows: the definitions of K-theories and characteristic classes, and their mutual relations are given in §2, the pushforward morphisms are defined and all the theorems are stated in §3, the construction of the direct image for relative K-theory is performed in §4, the construction of the η-form and all its consequences are detailed in §5, and §6 is devoted to results about symmetries induced by the fiberwise Hodge star operator. Finally, double fibrations are studied in §7. This paper is a reformulation of previously diffused preprints. I apologize for some changes of title, names and notations between earlier versions and this one.

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2. Various K-theories

After recalling some facts about Chern-Simons transgression in §2.1, the definitions of all the K-theory groups considered here are given in §2.2. §2.3 is devoted to the counterpart of Johnson-Nadel's classes defined in [**4**], §2.4 to the diagrams and exact sequences in which these K-groups enter, §2.5 and §2.6 to hermitian metrics and the extended Borel-Kamber-Tondeur class on $\widehat{K}_{\mathrm{ch}}$.

2.1. Preliminaries

2.1.1. *Connections and vector bundle morphisms.* — Let M be a smooth manifold. Let E and F be two vector bundles on M. Two vector bundles isomorphisms f and $g\colon E \overset{\sim}{\longrightarrow} F$ are called isotopic if there exists a smooth family $(f_t)_{t\in[0,1]}$ of isomorphisms $f_t\colon E \overset{\sim}{\longrightarrow} F$ such that $f_0 = f$ and $f_1 = g$. Suppose that E and F are endowed with connections ∇_E and ∇_F respectively (which need not be flat). A vector bundle morphism (which does not need to be an isomorphism) $f\colon E \longrightarrow F$ is parallel if $\nabla_F \circ f = f \circ \nabla_E$. For three vector bundles E', E and E'' endowed with connections $\nabla_{E'}$, ∇_E and $\nabla_{E''}$, the short exact sequence

$$0 \longrightarrow E' \overset{i}{\longrightarrow} E \overset{p}{\longrightarrow} E'' \longrightarrow 0 \tag{5}$$

is parallel if the morphisms i and p are parallel with respect to $\nabla_{E'}$, ∇_E and $\nabla_{E''}$. Parallel longer exact sequences or complexes of vector bundles are defined in a similar obvious way. In such parallel long exact sequences (or complexes), the kernel or image subbundles are respected by the connections of their ambient bundles (which are not supposed to be flat), so that cokernel or coimage bundles inherit natural connections (which need not be flat). Thus, longer parallel exact sequences (or complexes) can be decomposed, in the classical way, in several short exact sequences (see (56) and (57)) which turn out to be themselve parallel.

2.1.2. *Chern-Simons transgression forms.* — For any vector bundle G on M the vector space of smooth differential forms on M with values in G will be denoted by $\Omega^\bullet(M, G)$. A connection ∇_E on the smooth vector bundle E on M gives rise to an exterior differential operator d^{∇_E} on $\Omega^\bullet(M, E)$. Its square is the exterior product with an element of $\Omega^2(M, \mathrm{End}E)$ (in particular, it does not differentiate). This element of $\Omega^2(M, \mathrm{End}E)$ is the curvature of ∇_E and will be denoted by ∇_E^2. Chern-Weil theory associates to E and ∇_E the following complex differential form on M

$$\mathrm{ch}(\nabla_E) = \mathrm{Tr}\,\exp\left(-\frac{1}{2\pi i}\nabla_E^2\right) = \phi\mathrm{Tr}\exp(-\nabla_E^2) \tag{6}$$

where ϕ is the operator on even degree differential forms which divides $2k$-degree forms by $(2\pi i)^k$. This form is closed, its de Rham cohomology class is independent of ∇_E and equals the image of the Chern character of E in $H^{\mathrm{even}}(M, \mathbb{C})$.

Consider $p_1 \colon M \times [0,1] \to M$ (the projection on the first factor) and the bundle $\widetilde{E} = p_1^* E$ on $M \times [0,1]$, choose any connection $\widetilde{\nabla}_E$ on $\widetilde{E}$, denote for all $t \in [0,1]$ by $\nabla_{E,t}$ the restriction $\widetilde{\nabla}_E|_{M\times\{t\}}$. Extend ϕ to odd forms by deciding that ϕ divides $(2k-1)$-degree forms by $(2\pi i)^k$, and define

$$\begin{aligned}\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) &= \int_{[0,1]} \mathrm{ch}(\widetilde{\nabla}_E) = -\int_0^1 \phi\mathrm{Tr}\left(\frac{\partial \nabla_{E,t}}{\partial t}\exp\left(-\nabla_{E,t}^2\right)\right) dt \\ &= -\frac{1}{2\pi i}\int_0^1 \mathrm{Tr}\left(\frac{\partial \nabla_{E,t}}{\partial t}\exp\left(-\frac{1}{2\pi i}\nabla_{E,t}^2\right)\right) dt.\end{aligned} \tag{7}$$

Modifying $\widetilde{\nabla}_E$ (without changing $\nabla_{E,0}$ nor $\nabla_{E,1}$) changes $\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1})$ by addition of an exact form. This form is a "transgression" form in the sense that:

$$d\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) = \mathrm{ch}(\nabla_{E,1}) - \mathrm{ch}(\nabla_{E,0}) \tag{8}$$

Its class in $\Omega^{\mathrm{odd}}(M, \mathbb{C})/d\Omega^{\mathrm{even}}(M, \mathbb{C})$ is functorial by pull-backs, and locally gauge invariant, which means that $\widetilde{\mathrm{ch}}(\nabla, g^*\nabla)$ is an exact form if g is a global smooth automorphism of E isotopic to the identity.

If $\nabla_{E,2}$ is a third connection on E, Chern-Simons forms verify the following cocycle equality (modulo exact forms):

$$\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,2}) = \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) + \widetilde{\mathrm{ch}}(\nabla_{E,1}, \nabla_{E,2}). \tag{9}$$

In particular $\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) = -\widetilde{\mathrm{ch}}(\nabla_{E,1}, \nabla_{E,0})$.

Let $\nabla_{E,i} \oplus \nabla_{F,i}$ be the canonical direct sum connections on $E \oplus F$ associated to $\nabla_{E,i}$ and $\nabla_{F,i}$, the additivity of the Chern character form (6) for such direct sum connections yields the following equality (modulo exact forms):

$$\widetilde{\mathrm{ch}}(\nabla_{E,0} \oplus \nabla_{F,0}, \nabla_{E,1} \oplus \nabla_{F,1}) = \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) + \widetilde{\mathrm{ch}}(\nabla_{F,0}, \nabla_{F,1}). \tag{10}$$

Consider a short exact sequence as in (5), and a bundle morphism $s\colon E \to E'$ such that $s \circ i$ is the identity of E'. Then $s \oplus p\colon E \xrightarrow{\sim} E' \oplus E''$ is an isomorphism.

Lemma 1. — *$\widetilde{\mathrm{ch}}(\nabla_E, (s\oplus p)^*(\nabla_{E'} \oplus \nabla_{E''}))$ vanishes if the exact sequence is parallel with respect to $\nabla_{E'}$, ∇_E and $\nabla_{E''}$.*

Proof. — The fact that i and p are parallel means that with respect to the decomposition $E \cong E' \oplus E''$ (provided by the isomorphism $s \oplus p$), the connections ∇_E and $(s\oplus p)^*(\nabla_{E'} \oplus \nabla_{E''})$ differ from a one-form ω with values in $\mathrm{Hom}(E'', E')$.

Consider the path of connections $\nabla_t = \nabla_E - t\omega$. Then, ω is upper triangular with respect to the decomposition $E \cong E' \oplus E''$, and thus ∇_t^2 too. But ω has vanishing diagonal terms. Consequently, the trace vanishes in the Formula (7) applied to this situation, and this proves the lemma. □

2.2. Definitions of the considered K-groups

2.2.1. *Topological K-theory*

Definition 2. — *The topological K^0-group $K^0_{\mathrm{top}}(M)$ is the free abelian group generated by isomorphism classes of smooth complex vector bundles on M modulo direct sum.*

Let $p_1 : M \times S^1 \to M$ be the projection on the first factor, the topological K^1-group $K^1_{\mathrm{top}}(M)$ is the quotient group $K^0_{\mathrm{top}}(M \times S^1)/p_1^* K^0_{\mathrm{top}}(M)$.

$K^1_{\mathrm{top}}(M)$ is isomorphic to the kernel of the restriction map $\iota^*\colon K^0_{\mathrm{top}}(M \times S^1) \to K^0_{\mathrm{top}}(M \times \{pt\})$ where pt is some point in S^1 and $\iota\colon pt \to S^1$ the inclusion map. One can also describe $K^1_{\mathrm{top}}(M)$ as generated by global smooth automorphisms g_E of any vector bundle E on M; the corresponding element of $K^0_{\mathrm{top}}(M \times S^1)$ is the formal difference of the vector bundle obtained by gluing using g_E the restrictions to $M \times \{1\}$ and $M \times \{0\}$ of the pull-back of E on $M \times [0,1]$, minus the pull-back of E on $M \times S^1$. Any element of $K^1_{\mathrm{top}}(M)$ can be represented in this way with some trivial vector bundle as E.

2.2.2. *K^0-theory of the category of flat bundles.* — The connection ∇_E on the vector bundle E on M is said to be flat if its curvature $\nabla_E^2 \in \Omega^2(M, \mathrm{End}E)$ vanishes. The couple (E, ∇_E) is then called a flat vector bundle. Two flat vector bundles (E, ∇_E) and (F, ∇_F) are isomorphic if there exists some vector bundle isomorphism $f\colon E \xrightarrow{\sim} F$ which is parallel with respect to ∇_E and ∇_F.

Definition 3. — *The group $K^0_{\mathrm{flat}}(M)$ is the quotient of the free abelian group generated by isomorphism classes of flat vector bundles, by the following relation:*

$$(E, \nabla_E) = (E', \nabla_{E'}) + (E'', \nabla_{E''}) \qquad \textit{if} \qquad 0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0 \tag{11}$$

is a parallel exact sequence.

If a flat vector bundle (E, ∇_E) admits some subbundle which is respected by ∇_E, then the subbundle and the quotient bundle inherit connections, which are both flat (a similar result is proved in [**11**] Proposition 2.5). Following the comment of the end of §2.1.1, longer parallel exact sequences (or complexes) of flat vector bundles can be decomposed in short parallel exact sequences of flat vector bundles (see (56) and (57)).

2.2.3. *Relative K-theory.* — Consider now on M quintuples $(E, \nabla_E, F, \nabla_F, f)$ where (E, ∇_E) and (F, ∇_F) are flat vector bundles on M, and $f\colon E \xrightarrow{\sim} F$ is a smooth isomorphism. Two objects $(E, \nabla_E, F, \nabla_F, f)$ and $(G, \nabla_G, H, \nabla_H, h)$ are isomorphic if there are parallel isomorphisms $\varphi_E\colon E \xrightarrow{\sim} G$ and $\varphi_F\colon F \xrightarrow{\sim} H$ which verify that $h = \varphi_F \circ f \circ \varphi_E^{-1}$.

Definition 4. — *$K^0_{\mathrm{rel}}(M)$ is the quotient of the free abelian group generated by such isomorphism classes of quintuples modulo the following relations:*

(i) $(E, \nabla_E, F, \nabla_F, f) = 0$ *if f is isotopic to some parallel isomorphism.*

(ii) $$(E, \nabla_E, F, \nabla_F, f) + (G, \nabla_G, H, \nabla_H, h) = \\ = (E \oplus G, \nabla_E \oplus \nabla_G, F \oplus H, \nabla_F \oplus \nabla_H, f \oplus h)$$

(iii) $(E, \nabla_E, E' \oplus E'', \nabla_{E'} \oplus \nabla_{E''}, s \oplus p)$ *vanishes in $K^0_{\mathrm{rel}}(M)$ if there is a short exact sequence of flat bundles as in* (11) *above and if $s\colon E \to E'$ is a smooth bundle map such that $s \circ i$ is the identity of E'.*

Remark 5. — Note that $(E, \nabla_E, F, \nabla_F, f) = (E, \nabla_E, F, \nabla_F, g)$ if f and g are isotopic, that $(E, \nabla_E, F, \nabla_F, f) + (F, \nabla_F, G, \nabla_G, g) = (E, \nabla_E, G, \nabla_G, g \circ f)$, and that $(E, \nabla_E, F, \nabla_F, f) = (E', \nabla_{E'}, F', \nabla_{F'}, f') + (E'', \nabla_{E''}, F'', \nabla_{F''}, f'')$ if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array} \tag{12}$$

is a commutative diagram whose lines are short exact sequences in the category of flat vector bundles (on M).

In fact the first one and the third one of these three relations are together equivalent to (i), (ii) and (iii) so that they can be used to provide an alternative definition of $K^0_{\mathrm{rel}}(M)$ (see [**4**] §2.1 for details).

Independently, relation (iii) above is equivalent to the following

(iii)′ $(E' \oplus E'', \nabla_{E'} \oplus \nabla_{E''}, E, \nabla_E, i + j)$ vanishes in $K^0_{\mathrm{rel}}(M)$ if there is a short exact sequence of flat bundles as in (11) above and if $j\colon E'' \to E$ is a smooth bundle map such that $p \circ j$ is the identity of E''.

In fact, $i + j$ is isotopic to $(s \oplus p)^{-1}$.

2.2.4. *"Free multiplicative" or "non hermitian smooth" K-theory.* — Consider some triple (E, ∇_E, α) where E is a smooth complex vector bundle on M, ∇_E a connection on E and α an odd degree differential form defined modulo exact forms. Two such objects $(E_1, \nabla_{E_1}, \alpha_1)$ and $(E_2, \nabla_{E_2}, \alpha_2)$ will be equivalent if there is some smooth vector bundle isomorphism $f \colon E_1 \xrightarrow{\sim} E_2$ such that

$$\alpha_2 = \alpha_1 + \widetilde{\mathrm{ch}}(\nabla_{E_1}, f^*\nabla_{E_2}). \tag{13}$$

This is compatible with iterated changes of connections (see (9)).

Definition 6. — *The group $\widehat{K}_{\mathrm{ch}}(M)$ is the quotient of the free abelian group generated by such equivalence classes of triples modulo direct sum (of the vector bundles, with direct sum connection and sum of the differential forms).*

The Chern character on $\widehat{K}_{\mathrm{ch}}(M)$ is the map

$$\dddot{\mathrm{ch}} \colon (E, \nabla_E, \alpha) \in \widehat{K}_{\mathrm{ch}}(M) \longmapsto \mathrm{ch}(\nabla_E) - d\alpha \in \Omega^{\mathrm{even}}(M, \mathbb{C}). \tag{14}$$

Equations (8) and (10) ensure that $\dddot{\mathrm{ch}}$ is well defined.

The kernel of $\dddot{\mathrm{ch}}$ will be denoted $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ following [**30**] Definition 3. The preimage $MK^0(M)$ of $\mathbb{Z}$ by $\dddot{\mathrm{ch}}$ was considered by Karoubi in [**26**] §7.5 and [**28**] EXEMPLE 3. Of course, $MK^0(M) \cong \mathbb{Z} \oplus K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ is the subgroup of $\widehat{K}_{\mathrm{ch}}(M)$ generated by the triples (E, ∇_E, α) as above, but subjected to the extra condition:

$$d\alpha = \mathrm{ch}(\nabla_E) - \mathrm{rk}E. \tag{15}$$

This is why $\widehat{K}_{\mathrm{ch}}$ is considered as "unrestricted" with respect to $MK^0(M)$, and called "free" multiplicative K-theory. The relation with the smooth K-theory considered by Bunke and Schick in [**14**] will be explained in §2.6.

2.3. Chern-Simons class on relative K-theory

Definition 7. — *The Chern-Simons class on $K^0_{\mathrm{rel}}(M)$ is defined as*

$$\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f) = \left[\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\right] \in H^{\mathrm{odd}}(M, \mathbb{C}) \tag{16}$$

(of course $\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)$ is closed since $\mathrm{ch}(\nabla_E)$ and $\mathrm{ch}(f^*\nabla_F)$ both equal $\mathrm{rk}E$).

Arguments as in [**4**] Theorem 3.5 and its corollary allow to prove the

Proposition 8. — *$\mathscr{N}_{\mathrm{ch}}$ induces a group morphism from $K^{\mathrm{rel}}_0(M)$ to $H^{\mathrm{odd}}(M, \mathbb{C})$.*

Arguments as in [**4**] §5.1 and 5.2 or [**35**] allow to prove the following facts:

- Let Φ multiply $2k$ and $(2k-1)$-degree forms by $k!$, then $\Phi\mathrm{ch}$ is the Chern character without denominators. The nonintegrality of $\Phi\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f)$ detects the fact that $(E, \nabla_E) \neq (F, \nabla_F) \in K^{\mathrm{flat}}_0(M)$.

- The nonintegrality of the degree≥ 3 components of $\Phi\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ detects the fact that (F,∇_F) cannot be obtained from (E,∇_E) through a deformation of flat bundles, where a deformation of flat bundles on M is a smooth vector bundle $\widetilde{E}$ on $M\times[0,1]$ with a connection $\widetilde{\nabla}$ whose restriction to $E_t=\widetilde{E}|_{M\times\{t\}}$ is flat for any point $t\in[0,1]$ and such that $(E_0,\widetilde{\nabla}|_{M\times\{0\}})\cong(E,\nabla_E)$ and $(E_1,\widetilde{\nabla}|_{M\times\{1\}})\cong(F,\nabla_F)$
- The nonnullity of the degree≥ 3 components of $\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ detects the fact that (F,∇_F) cannot be obtained from (E,∇_E) through a deformation of flat bundles, for which the parallel transport along $[0,1]$ would be isotopic to f.
- If (F,∇_F) can be obtained from (E,∇_E) by deformation of flat bundles, then the degree 1 component of $\mathscr{N}_{\mathrm{ch}}$ modulo integral cohomology detects the variation of the determinant line.

The third statement is known as the rigidity of higher classes of flat bundles.

Remark 9. — Let $(E,\nabla_E,F,\nabla_F,f)\in K^0_{\mathrm{rel}}(M)$. Define $\omega=f^*\nabla_F-\nabla_E$ (then of course $\omega\in\Omega^1(M,\mathrm{End}E)$). It can be proved as in [**4**] Lemma 4.3 that in fact

$$\widetilde{\mathrm{ch}}(\nabla_E,f^*\nabla_F)=-\sum_{r=1}^{\left[\frac{\dim M}{2}\right]}\left(\frac{1}{2\pi i}\right)^r\frac{(r-1)!}{(2r-1)!}\mathrm{Tr}(\omega^{2r-1}).\tag{17}$$

(This is of course a particular property of flat connections and cannot be generalised to any connections). Thus $\mathscr{N}_{\mathrm{ch}}(E,\nabla_E,F,\nabla_F,f)$ can be computed in the same way as the classes studied in [**35**] and [**11**].

2.4. Relations between the preceding K-groups. — The Chern character $\mathrm{ch}\colon K^0_{\mathrm{top}}(M)\longrightarrow H^{\mathrm{even}}(M,\mathbb{C})$ is obtained by considering the de Rham cohomology class of the form of (6).

Consider some element β of $K^1_{\mathrm{top}}(M)$. Represent it by some vector bundle over $M\times S^1$. Integrate along S^1 the Chern character of this bundle, the obtained class in $H^{\mathrm{odd}}(M,\mathbb{C})$ is the Chern character of β. If β is represented by some global automorphism g_E of some vector bundle E on M as in the construction after Definition 2, then it follows from (7) that for any connection ∇ on E

$$\mathrm{ch}(\beta)=\widetilde{\mathrm{ch}}(\nabla,g_E^*\nabla).\tag{18}$$

If $E=\mathbb{C}^N$ is trivial (then denote g_E by $g_{\mathbb{C}^N}$), let $d_{\mathbb{C}^N}$ be its canonical trivial flat connection, the formula

$$\beta\in K^1_{\mathrm{top}}(M)\longmapsto(\mathbb{C}^N,d_{\mathbb{C}^N},\mathbb{C}^N,d_{\mathbb{C}^N},g_{\mathbb{C}^N})\tag{19}$$

defines a group morphism φ (see [**4**] Proposition 2.2 for a proof).

$K^0_{\mathrm{rel}}(M)$, $K^0_{\mathrm{flat}}(M)$ and $\widehat{K}_{\mathrm{ch}}(M)$ are related by the following morphisms:

$$
\begin{array}{ccc}
K^0_{\mathrm{rel}}(M) \xrightarrow{\partial} K^0_{\mathrm{flat}}(M) & & K^0_{\mathrm{flat}}(M) \xrightarrow{\kappa} \widehat{K}_{\mathrm{ch}}(M) \\
(E,\nabla_E,F,\nabla_F,f) \longmapsto (F,\nabla_F)-(E,\nabla_E) & \text{and} & (E,\nabla_E) \longmapsto (E,\nabla_E,0)
\end{array} \tag{20}
$$

κ is well defined thanks to Lemma 1, and takes its values in $MK^0(M)$.

Let $\gamma \in \Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})$. It is easily checked that the following element $a(\gamma) = (E,\nabla_E,\alpha+\gamma) - (E,\nabla_E,\alpha)$ of $\widehat{K}_{\mathrm{ch}}(M)$ is independent on the choice of (E,∇_E,α) of $\widehat{K}_{\mathrm{ch}}(M)$ used to compute it. $a(\gamma) \in MK^0(M)$ if and only if γ is closed. Consider the obvious forgetful maps from K^0_{flat} or $\widehat{K}_{\mathrm{ch}}$ to K^0_{top}:

Proposition 10. — *This diagram commutes. Its lines are exact sequences:*

$$
\begin{array}{ccccccc}
K^1_{\mathrm{top}}(M) & \xrightarrow{\varphi} & K^0_{\mathrm{rel}}(M) & \xrightarrow{\partial} & K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\
\downarrow \| & & \downarrow \mathcal{N}_{\mathrm{ch}} & & \downarrow \kappa & & \downarrow \| \\
K^1_{\mathrm{top}}(M) & \xrightarrow{\mathrm{ch}} & H^{\mathrm{odd}}(M,\mathbb{C}) & \xrightarrow{a} & MK^0(M) & \longrightarrow & K^0_{\mathrm{top}}(M).
\end{array} \tag{21}
$$

In this diagram, the part "$H^{\mathrm{odd}}(M,\mathbb{C}) \xrightarrow{a} MK^0(M)$" can be replaced by "$\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C}) \xrightarrow{a} \widehat{K}_{\mathrm{ch}}(M)$" without losing the commutativity nor the exactness of the second line.

Proof. — A proof of the exactness of the first line can be found (in the holomorphic setting) in [**4**]. A proof of the exactness of the second line can be found in [**27**] Théorème 5.3. This proof generalises easily to the proposed modified second line. The commutativity of the right square is tautological. The commutativity of the left square follows from (16), (18) and (19). The commutativity of the central square is a consequence of the compatibility of (20) and (13) with the definitions of a and of $\mathcal{N}_{\mathrm{ch}}$. The proposed replacement in the middle of the second line has no influence on the commutativity of the squares. □

2.5. Symmetries associated to hermitian metrics. — For any complex vector bundle E on M endowed with a hermitian metric h^E and a connection ∇_E, the adjoint connection ∇^*_E of ∇_E is defined as follows:

$$
h^E(\nabla^*_{E\mathrm{v}}\sigma,\theta) = \mathrm{v}.h^E(\sigma,\theta) - h^E(\sigma,\nabla_{E\mathrm{v}}\theta) \tag{22}
$$

where σ and θ are local sections of E, v is a tangent vector, v.f is the derivative of the function f along v, $\nabla^*_{E\mathrm{v}}\sigma$ is the derivative of σ along v with respect to the connection ∇^*_E and accordingly for $\nabla_{E\mathrm{v}}\theta$. Of course $(\nabla^*_E)^* = \nabla_E$, (and $\nabla_E = \nabla^*_E$ if and only if ∇_E respects the hermitian metric h^E).

Adjoint connections allow to define conjugation involutions on the above considered K-groups (on the model of complex conjugation):

Definition 11. — *The conjugate elements of* $(E, \nabla_E) \in K^0_{\text{flat}}(M)$*, or* $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\text{rel}}(M)$*, or* $(E, \nabla_E, \alpha) \in \widehat{K}_{\text{ch}}(M)$ *are defined by:*

$$(23)\qquad \begin{aligned}(E, \nabla_E)^c &= (E, \nabla^*_E) \in K^0_{\text{flat}}(M)\\ (E, \nabla_E, F, \nabla_F, f)^c &= (E, \nabla^*_E, F, \nabla^*_F, f) \in K^0_{\text{rel}}(M)\\ (E, \nabla_E, \alpha)^c &= (E, \nabla^*_E, \overline{\alpha}) \in \widehat{K}_{\text{ch}}(M).\end{aligned}$$

Lemma 12. — *The above formulae define involutive group automorphisms. Moreover*

$$(24)\qquad \begin{aligned}\mathscr{N}_{\text{ch}}(E, \nabla^*_E, F, \nabla^*_F, f) &= \overline{\mathscr{N}_{\text{ch}}(E, \nabla_E, F, \nabla_F, f)}\\ \dddot{\text{ch}}(E, \nabla^*_E, \overline{\alpha}) &= \overline{\dddot{\text{ch}}(E, \nabla_E, \alpha)}.\end{aligned}$$

Proof. — The curvatures ∇^2_E and ∇^{*2}_E are mutually skew adjoint, so that ∇^*_E is flat if and only if ∇_E is. Thus (E, ∇^*_E) and $(E, \nabla^*_E, F, \nabla^*_F, f)$ really define classes in $K^0_{\text{flat}}(M)$ and $K^0_{\text{rel}}(M)$ respectively.

If h^E_1 and h^E_2 are two different hermitian metrics on E, define the global automorphism g_E of E by the following formula, valid for any local sections σ and θ of E:

$$(25)\qquad h^E_2(\sigma, \theta) = h^E_1(g_E(\sigma), \theta).$$

Call $\nabla^*_{E,1}$ and $\nabla^*_{E,2}$ the adjoint of ∇_E relatively to h^E_1 and h^E_2 respectively, then $\nabla^*_{E,1}$ and $\nabla^*_{E,2} = g_E^{-1}\nabla^*_{E,1}g_E$ are gauge equivalent. This proves that in the first and third cases, the K-theory classes of the proposed elements are independent of the hermitian metrics used to define them.

Note that g_E is isotopic to the identity of E. If one chooses two hermitian metrics on F and compute in the same way the corresponding automorphism g_F, then f and $g_F \circ f \circ g_E^{-1}$ are isotopic. This proves the independence on the hermitian metrics on E and on F of the class of $(E, \nabla^*_E, F, \nabla^*_F, f)$ in $K^0_{\text{rel}}(M)$.

Consider a parallel exact sequence of the form (11) where E', E and E'' are endowed with hermitian metrics. Its transpose

$$(26)\qquad 0 \to E'' \xrightarrow{p^*} E \xrightarrow{i^*} E' \to 0$$

turns out to be a parallel exact sequence with respect to the adjoint connections on E'', E' and E. This proves the first statement (on K^0_{flat}) of the lemma. The second formula of the lemma associates to any quintuple of the same form as in relation (*iii*) in Definition 4 a quintuple of the form appearing in relation (*iii*)′ in Remark 5. This proves the second statement (on K^0_{rel}) of the lemma.

The fact that the curvatures of ∇_E and ∇^*_E are mutually skew adjoint has the following consequence (which proves the last statement (on $\dddot{\text{ch}}$) of the lemma):

$$(27)\qquad \text{ch}(\nabla^*_E) = \overline{\text{ch}(\nabla_E)}.$$

Finally, considering the adjoint connection of $\widetilde{\nabla}_E$ in Formula (7) yields (using (27)) the following relation modulo exact forms:

$$\widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla^*_{E,1}) = \overline{\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1})} \tag{28}$$

where $\nabla^*_{E,0}$ and $\nabla^*_{E,1}$ are adjoint of $\nabla_{E,0}$ and $\nabla_{E,1}$ with respect to possibly different hermitian metrics on E. The compatibility of the third line of (23) with relation (13) follows. This proves the third statement (on $\widehat{K}_{\mathrm{ch}}$) of the lemma. The statement on $\mathcal{N}_{\mathrm{ch}}$ is a direct consequence of (28). □

(27) and (28) imply that $\mathrm{ch}(\nabla_{E,0})$ and $\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1})$ are real forms if $\nabla_{E,0}$ and $\nabla_{E,1}$ respect (possibly different) hermitian metrics on E.

Elements of $K^0_{\mathrm{rel}}(M)$ of the form $(E, \nabla_E, E, \nabla^*_E, \mathrm{Id}_E)$ are purely imaginary with respect to this conjugation; conversely, the subgroup of $K^0_{\mathrm{rel}}(M)$ generated by such elements is equal to, or of index 2 in, the purely imaginary part of $K^0_{\mathrm{rel}}(M)$. This is because (see the beginning of Remark 5)

$$\begin{aligned}(E, \nabla_E, F, \nabla_F, f) - (E, \nabla^*_E, F, \nabla^*_F, f) &= \\ &= (F, \nabla_F, F, \nabla^*_F, \mathrm{Id}_F) - (E, \nabla_E, E, \nabla^*_E, \mathrm{Id}_E).\end{aligned} \tag{29}$$

2.6. Borel-Kamber-Tondeur class on $\widehat{K}_{\mathrm{ch}}$. — In the notation of (25), the fact that g_E is isotopic to the identity proves that ∇^{*0}_E and ∇^{*1}_E are locally gauge invariant. Thus

Lemma 13. — *Let E be a vector bundle with connection ∇_E. Let ∇^*_E be the adjoint of ∇_E with respect to any hermitian metric on E. The class of $\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E)$ in $\Omega^{\mathrm{odd}}(M, \mathbb{C})/d\Omega^{\mathrm{even}}(M, \mathbb{C})$ is independent of the metric h^E.*

Moreover, it is a purely imaginary form, since:

$$\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E) = -\widetilde{\mathrm{ch}}(\nabla_E, \nabla^*_E) = -\overline{\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E)}. \tag{30}$$

Consider the connection $\nabla^u_E = \frac{1}{2}(\nabla_E + \nabla^*_E)$; it respects h^E, and

$$\widetilde{\mathrm{ch}}(\nabla^*_E, \nabla_E) = \widetilde{\mathrm{ch}}(\nabla^*_E, \nabla^u_E) + \widetilde{\mathrm{ch}}(\nabla^u_E, \nabla_E) = 2i\mathfrak{Im}\big(\widetilde{\mathrm{ch}}(\nabla^u_E, \nabla_E)\big). \tag{31}$$

Finally, if $\nabla_{E,0}$ and $\nabla_{E,1}$ are connections on E, then the cocycle condition (9) produces the following relation modulo exact forms

$$\begin{aligned}\widetilde{\mathrm{ch}}(\nabla^*_{E,1}, \nabla_{E,1}) - \widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla_{E,0}) &= \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) - \widetilde{\mathrm{ch}}(\nabla^*_{E,0}, \nabla^*_{E,1}) \\ &= 2i\,\mathfrak{Im}\,\widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}).\end{aligned} \tag{32}$$

Remark 14. — It is proved in [**11**], proof of Proposition 1.14, that $\widetilde{\mathrm{ch}}(\nabla_E, \nabla_E^u)$ is purely imaginary if ∇_E is flat. In this case, (31) holds without $i\,\mathfrak{Im}$. Moreover

$$\widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E) = \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(E) \tag{33}$$

is the degree decomposition of $\widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E)$, where the $c_k(E)$ are the classes considered by Bismut and Lott (see [**11**] formulae (0.2) and (1.34)). These are exactly the imaginary part of the Chern-Cheeger-Simons classes of flat complex vector bundles ([**15**] and [**11**] Proposition 1.14).

Lemma 15. — *If $E_{\mathbb{R}}$, is a real vector bundle on M with connections $\nabla_{E_{\mathbb{R}},0}$ and $\nabla_{E_{\mathbb{R}},1}$, and if E is its complexification with associated connections ∇_0 and ∇_1, then, up to exact forms, $\widetilde{\mathrm{ch}}(\nabla_0, \nabla_1)$ is real in degrees $4k+3$ and purely imaginary in degrees $4k+1$. In particular, $\widetilde{\mathrm{ch}}(\nabla_0^*, \nabla_0)$ vanishes in degrees $4k+3$.*

Proof. — Suppose that E is endowed with a hermitian form which is the complexification of a real scalar product on $E_{\mathbb{R}}$, and use the path of connections $\nabla_t = (1-t)\nabla^* + t\nabla$, then the lemma follows from formulae (7) and (30) by counting the i, and from Lemma 13. □

Definition 16. — *For any $(E, \nabla_E, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$, its Borel-Kamber-Tondeur class $\mathfrak{B}(E, \nabla_E, \alpha)$ is the class in $\Omega^{\mathrm{odd}}(M, \mathbb{C})/d\Omega^{\mathrm{even}}(M, \mathbb{C})$ of the differential form:*

$$\mathfrak{B}(E, \nabla_E, \alpha) = \widetilde{\mathrm{ch}}(\nabla_E^*, \nabla_E) - \alpha + \overline{\alpha} \tag{34}$$

where ∇_E^* is the adjoint of ∇_E for any hermitian metric on E.

Relations (28) and (13) imply that $\mathfrak{B}$ is a morphism from $\widehat{K}_{\mathrm{ch}}(M)$ to the subgroup of purely imaginary forms in $\Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M)$. Moreover, from (32):

$$d\mathfrak{B}(E, \nabla_E, \alpha) = 2i\,\mathfrak{Im}\big(\dddot{\mathrm{ch}}(E, \nabla_E, \alpha)\big).$$

It follows from Lemma 15 that if E is the complexification of a real bundle $E_{\mathbb{R}}$ on M with connection ∇_E coming from a connection on $E_{\mathbb{R}}$, then $\mathfrak{B}(E, \nabla, 0)$ vanishes in degrees $4k+3$ for any integer k.

Any vector bundle admits some hermitian metric and some connection which respects it, so that using relation (13), one checks that $\mathfrak{B}$ is twice the operation of taking the imaginary part with respect to the conjugation defined in Lemma 12, i.e. twice the projection on the second factor of

$$\widehat{K}_{\mathrm{ch}}(M) = \mathrm{Ker}\mathfrak{B} \oplus i\Omega^{\mathrm{odd}}(M, \mathbb{R})/d\Omega^{\mathrm{even}}(M, \mathbb{R}). \tag{35}$$

$\mathrm{Ker}\mathfrak{B}$ coincides with the smooth K-theory $\widehat{K}^0(M)$ considered by Bunke and Schick in [**14**]. In fact any vector bundle V on M endowed with some hermitian metric h^V and unitary connection ∇_V defines some geometric family with zero-dimensional fibre

$\mathcal{V} = (V, h^V, \nabla_V)$ (see [**14**] §2.1.4), then $(V, \nabla_V, \alpha) \longmapsto (\mathcal{V}, \alpha)$ defines a map from $\operatorname{Ker}\mathfrak{B}$ to $\widehat{K}(B)$ which can be proved to be an isomorphism by the five lemma.

$\mathfrak{B}$ sends $MK^0(M)$ into $iH^{\mathrm{odd}}(M, \mathbb{R})$. From Remark 14, one sees that the imaginary part of Cheeger-Chern-Simons classes [**15**] studied by Bismut and Lott in [**11**] factor through $\mathfrak{B}$ and the second morphism defined in (20). This justifies the interest of adding the part $i\Omega^{\mathrm{odd}}(M, \mathbb{R})/d\Omega^{\mathrm{even}}(M, \mathbb{R})$ to Bunke and Schick's smooth K^0-theory, in order to take into account all flat connections.

Finally, the K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ considered by Lott in [**30**] Definition 7 is $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) = \operatorname{Ker}\mathfrak{B} \cap K^{-1}_{\mathbb{C}/\mathbb{Z}}(M) = \operatorname{Ker}\mathfrak{B} \cap \operatorname{Ker} \dddot{\mathrm{ch}}$.

3. Direct images for K-groups

Let M and B be smooth real manifolds possibly with boundary and $\pi\colon M \to B$ a smooth proper submersion. The goal of this part is to define direct images morphisms from K-theories on M to K-theories on B in each case precedingly reviewed, and to state all the theorems proved in this paper.

The direct image $\pi_!$ for K^0_{flat} is constructed from fiberwise twisted de Rham cohomology (see Definition 22). This is compatible with the forgetful map $K^0_{\mathrm{flat}} \longrightarrow K^0_{\mathrm{top}}$ and the pushforward π^{Eu}_* on K^0_{top} associated to the fiberwise twisted Euler operator (Definitions 17 and 20 and Lemma 23). The notion of "link", which is a generalisation of the concept of vector bundle isomorphism (see Definition 24) is used to solve the problem of defining a pushforward $\pi_*\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$. (As stated in the introduction, it consists for any $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\mathrm{rel}}(M)$ in finding some link naturally associated to f between $\pi_!(E, \nabla_E)$ and $\pi_!(F, \nabla_F)$ on B). Finally for $\widehat{K}_{\mathrm{ch}}$, the ingredient is the Chern-Simons analog for transgressing the families index theorem, known as η-form (see Theorem 28). As in the case of topological K-theory, the pushforward $\pi^{\mathrm{Eu}}_!$ is here associated to the fiberwise Euler operator.

In some cases, some more preliminary is needed to be able to state the entire definitions. The proofs are delayed to the subsequent sections.

The fibres of π are supposed to be compact without boundary, orientable, and modelled on the closed manifold Z. For $y \in B$, $\pi^{-1}(y)$ will be denoted Z_y.

3.1. The case of topological K-theory

3.1.1. *Preliminary: construction of family index bundles.* — Let ξ be a smooth complex vector bundle on M. Let TZ^* be the dual of TZ. For any $y \in B$, the infinite dimensional spaces

$$\mathcal{E}^{\pm}_y = \mathcal{C}^{\infty}\left(Z_y, \Lambda^{\substack{\mathrm{even}\\ \mathrm{odd}}} T^*Z \otimes \xi\right) = \Omega^{\substack{\mathrm{even}\\ \mathrm{odd}}}(Z_y, \xi) \tag{36}$$

are fibres over y of infinite rank vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ on B such that

$$\mathscr{C}^\infty(B,\mathcal{E}^\pm)\cong\mathscr{C}^\infty\left(M,\wedge^{\text{even}}_{\text{odd}}T^*Z\otimes\xi\right) \tag{37}$$

(see [**11**] (3.1) to (3.6)). Choose some connection ∇_ξ on ξ, the vertical exterior differential operator $d^{\nabla_\xi}: \Omega^\bullet(Z_y,\xi)\longrightarrow\Omega^{\bullet+1}(Z_y,\xi)$ will be considered as an odd endomorphism of the $\mathbb{Z}_2$-graded vector bundle $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ (d^{∇_ξ} depends only on the restriction of ∇_ξ to the fibres of π). Choose some smooth hermitian metric h^ξ on ξ and euclidean metric g^Z on TZ, from which a volume form $d\mathrm{Vol}_Z$ along the fibres of π, and an inner product $(\ |\ )_Z$ on $\wedge^\bullet T^*Z\otimes\xi$ are deduced. One obtains on $\mathcal{E}$ the L^2 scalar product (where α and $\beta\in\mathcal{E}^+_y\oplus\mathcal{E}^-_y$):

$$\langle\alpha,\beta\rangle_{L^2}=\int_{Z_y}(\alpha|\beta)_Z d\mathrm{Vol}_Z. \tag{38}$$

Let $(d^{\nabla_\xi})^*$ be the formal adjoint of d^{∇_ξ} for this metric.

Let μ^+ and μ^- be complex vector bundles on B with hermitian metrics h^+ and h^-. For any bundle map ψ: $(\mathcal{E}^+\oplus\mu^+)\to(\mathcal{E}^-\oplus\mu^-)$ of everywhere finite rank, call ψ^* the adjoint of ψ with respect to $h^\pm$ and $\langle\ ,\ \rangle_{L^2}$, and set

$$\begin{aligned}\mathcal{D}^{\nabla_\xi+}_\psi&=\left(d^{\nabla_\xi}+(d^{\nabla_\xi})^*\right)+\psi\colon\mathcal{E}^+\oplus\mu^+\longrightarrow\mathcal{E}^-\oplus\mu^-,\\ \mathcal{D}^{\nabla_\xi-}_\psi&=\left(d^{\nabla_\xi}+(d^{\nabla_\xi})^*\right)+\psi^*\colon\mathcal{E}^-\oplus\mu^-\longrightarrow\mathcal{E}^+\oplus\mu^+.\end{aligned} \tag{39}$$

These are elliptic operators on Z_y so that their kernels are finite dimensional.

Definition 17. — *A triple (μ^+,μ^-,ψ) as above such that* $\dim\operatorname{Ker}\mathcal{D}^{\nabla_\xi\pm}_\psi$ *are constant (independent of $y\in B$) will be called "suitable" in the sequel.*

In that case, the kernels of $\mathcal{D}^{\nabla_\xi\pm}_\psi$ are vector bundles $\mathcal{H}^\pm$ on B, they will be called kernel bundles. The couple $(\mathcal{H}^+\oplus\mu^-,\mathcal{H}^-\oplus\mu^+)$ is called a couple of family index bundles for ξ.

If ζ is another vector bundle on M with hermitian metric and connection ∇_ζ, and if (ν^+,ν^-,φ) is a suitable triple for ζ, call $\mathcal{K}^\pm$ the kernel bundles $\operatorname{Ker}\mathcal{D}^{\nabla_\zeta\pm}_\varphi$ *then the couple $(\mathcal{H}^+\oplus\mu^-\oplus\mathcal{K}^-\oplus\nu^+,\mathcal{H}^-\oplus\mu^+\oplus\mathcal{K}^+\oplus\nu^-)$ will be called a couple of family index bundles for $\xi-\zeta$.*

3.1.2. *Definition of the direct image morphism for K^0_{top} and K^1_{top}*

Proposition 18. — *If B is compact, then for any ξ on M endowed with any connection ∇_ξ and any hermitean metric h^ξ, there exists suitable data (μ^+,μ^-,ψ).*

This is proved in [**2**] Proposition (2.2) (see also [**3**] Lemma 9.30). The following classical result will be precised in Theorem 25 below.

Lemma 19. — *If $(\mathcal{G}^+_1,\mathcal{G}^-_1)$ and $(\mathcal{G}^+_2,\mathcal{G}^-_2)$ are couples of family index bundles for the same vector bundle ξ on M (for different metrics or connections or suitable data),* *then*

$$[\mathcal{G}_1^+] - [\mathcal{G}_1^-] = [\mathcal{G}_2^+] - [\mathcal{G}_2^-] \in K^0_{\mathrm{top}}(B). \tag{40}$$

The same holds if $(\mathcal{G}_1^+, \mathcal{G}_1^-)$ *and* $(\mathcal{G}_2^+, \mathcal{G}_2^-)$ *are couples of family index bundles for the couples of vector bundle* $\xi_1 - \zeta_1$ *and* $\xi_2 - \zeta_2$ *such that*

$$[\xi_1] - [\zeta_1] = [\xi_2] - [\zeta_2] \in K^0_{\mathrm{top}}(M). \tag{41}$$

Definition 20. — *If B is compact, then for any vector bundle ξ on M, take any couple* $(\mathcal{G}^+, \mathcal{G}^-)$ *of family index bundles for ξ and put*

$$\pi_*^{\mathrm{Eu}}([\xi]) = [\mathcal{G}^+] - [\mathcal{G}^-] \in K^0_{\mathrm{top}}(B). \tag{42}$$

If B is not compact, $\pi_*^{\mathrm{Eu}}([\xi])$ *is defined in the same way on compact subsets of B and by inductive limit (or using the stability properties of vector bundles* **[22]** *§8 Theorems 1.2 and 1.5) on the whole B.*

The above lemma proves that π_*^{Eu} is a morphism from $K_0^{\mathrm{top}}(M)$ to $K_0^{\mathrm{top}}(B)$. It is the one associated to the fiberwise Euler operator (see **[2]** Definition 2.3: if d^Z is the as above constructed d^{∇_ξ} in the case where ξ is the trivial rank one complex vector bundle with canonical connection and metric, then the fiberwise Euler operator is $d^Z + d^{Z*}$ acting on vertical differential forms $\mathbb{Z}_2$-graded by the parity of their degree). This is in contrast with the case of **[8]**, **[14]**, **[30]** and **[7]** §1, where the direct image is associated to the fiberwise *Spin* or *Spin*^{c} Dirac operator, but compatible with **[11]**, **[31]**, **[32]** and **[7]** §§2 and 3.

Fiberwise twisted Euler operators of the form $\mathcal{D}^{\nabla_\xi}$ can be pulled back on fibered products (here $\widetilde{B} \longrightarrow B$ is any differentiable map):

$$\begin{array}{ccc} \widetilde{B} \times_B M & \longrightarrow & M \\ \downarrow & & \downarrow \\ \widetilde{B} & \longrightarrow & B \end{array} \tag{43}$$

(the model of the fibre may not change). The additional data (μ^+, μ^-, ψ) used to construct the direct image can also be pulled back in such situations, and this makes the construction of families index bundles functorial. Thus π_*^{Eu} is also functorial by pullbacks on fibered products. This justifies the following

Definition 21. — *The direct image morphism* $\pi_*^{\mathrm{Eu}}\colon K^1_{\mathrm{top}}(M) \longrightarrow K^1_{\mathrm{top}}(B)$ *is the morphism induced (on quotients) by* $\pi_*^{\mathrm{Eu}}\colon K^0_{\mathrm{top}}(M \times S^1) \longrightarrow K^0_{\mathrm{top}}(B \times S^1)$.

3.2. The case of the K^0-theory of flat bundles. — Consider some flat vector bundle (E, ∇_E) on M. The de Rham cohomology $H^\bullet(Z, E)$ of the fibres of π with coefficients in E provides ($\mathbb{Z}$-graded) vector bundles on B, which are endowed with flat connections in a canonical way, see [**11**] §III (f). Put $\pi_!^+ E = H^{\mathrm{even}}(Z, E)$ and $\pi_!^- E = H^{\mathrm{odd}}(Z, E)$ (they are smooth vector bundles on B, whose definition depends on ∇_E). and call $\nabla_{\pi_!^+ E}$ and $\nabla_{\pi_!^- E}$ their canonical flat connections.

Definition 22. — *$(\pi_!^+ E, \nabla_{\pi_!^+ E})$ and $(\pi_!^- E, \nabla_{\pi_!^- E})$ will be called the sheaf theoretic direct images of (E, ∇_E). The direct image morphism $\pi_! \colon K^0_{\mathrm{flat}}(M) \to K^0_{\mathrm{flat}}(B)$ is given by:*

$$(E, \nabla_E) \longmapsto (\pi_!^+ E, \nabla_{\pi_!^+ E}) - (\pi_!^- E, \nabla_{\pi_!^- E}).$$

The definition of $\pi_!$ is justified by the following fact: for a parallel short exact sequence of flat bundles as in (11)

$$0 \longrightarrow (E', \nabla_{E'}) \xrightarrow{\ i\ } (E, \nabla_E) \xrightarrow{\ p\ } (E'', \nabla_{E''}) \longrightarrow 0 \tag{44}$$

the long exact sequence in cohomology reads

$$\begin{array}{ccccc} \pi_!^+ E' & \xrightarrow{[i]} & \pi_!^+ E & \xrightarrow{[p]} & \pi_!^+ E'' \\ \uparrow & & & & \downarrow \\ \pi_!^- E'' & \xleftarrow{[p]} & \pi_!^- E & \xleftarrow{[i]} & \pi_!^- E' \end{array} \tag{45}$$

and all the morphisms in (45) are parallel. This diagram decomposes in several short parallel exact sequences of flat vector bundles as was remarked at the ends of §2.1.1 and §2.2.2. Thus $\pi_!(E, \nabla_E) = \pi_!(E', \nabla_{E'}) + \pi_!(E'', \nabla_{E''}) \in K^0_{\mathrm{flat}}(B)$. This proves that the above definition of $\pi_!$ fits with relation (11).

The following result is needed to define the direct image for K^0_{rel}:

Lemma 23. — *The following diagram commutes:*

$$\begin{array}{ccc} K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\ {\scriptstyle \pi_!}\downarrow & & \downarrow{\scriptstyle \pi_*^{\mathrm{Eu}}} \\ K^0_{\mathrm{flat}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B). \end{array} \tag{46}$$

Proof. — Let (E, ∇_E) be any flat vector bundle over M. By the Hodge theory of the fibres of π, the $H^{\pm}(Z_y, E)$ are isomorphic to $\mathrm{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)^{\pm}$ on Z_y. (They are of constant dimension, whatever the riemannian metric on M and the hermitian metric on E may be). Thus $(\{0\}, \{0\}, 0)$ is a suitable triple in this situation. The couple $(\pi_!^+ E, \pi_!^- E)$ is thus isomorphic to a couple of family index bundles for E, so that $[\pi_!^+ E] - [\pi_!^- E] = \pi_*^{\mathrm{Eu}}[E] \in K^0_{\mathrm{top}}(B)$. □

3.3. The case of relative K-theory

3.3.1. *The notion of "link".* — For four smooth vector bundles E, F, G, and H on M such that

$$[E]-[F]=[G]-[H]\in K^0_{\mathrm{top}}(M)$$

there exists some vector bundle K on M and some $\mathscr{C}^\infty$ isomorphism

$$\ell\colon E\oplus H\oplus K\xrightarrow{\sim}F\oplus G\oplus K. \tag{47}$$

Definition 24. — *These (K,ℓ) will be called a "link between $E-F$ and $G-H$".*

Two such links (K_1,ℓ_1) and (K_2,ℓ_2) are equivalent if there exists some vector bundle L on M such that the two following isomorphisms are isotopic

$$E\oplus H\oplus K_1\oplus K_2\oplus L\xrightarrow[\ell_2\oplus\mathrm{Id}_{K_1}\oplus\mathrm{Id}_L]{\ell_1\oplus\mathrm{Id}_{K_2}\oplus\mathrm{Id}_L}F\oplus G\oplus K_1\oplus K_2\oplus L. \tag{48}$$

The equivalence class of a link (K,ℓ) will be denoted by $[\ell]$. The set of equivalence classes of links between $E-F$ and $G-H$ will be denoted by $\mathscr{L}^{G-H}_{E-F}$.

Of course a link between $E-F$ and $G-H$ is also a link between $E-G$ and $F-H$ or $H-G$ and $F-E$ or $H-F$ and $G-E$. Any link is equivalent to some other one with a trivial vector bundle as K. Moreover, if (K,ℓ) is a link between $E-F$ and $G-H$, then (K,ℓ^{-1}) will be a link between $G-H$ and $E-F$ (or $F-H$ and $E-G$ and so on). Its equivalence class will be denoted either by $[\ell^{-1}]$ or $[\ell]^{-1}$.

The identity of $K\oplus K$ is isotopic to the switch of the two copies of K, thus (K,ℓ) as in (47) is equivalent to itself. It is also obviously equivalent to $(K\oplus L,\ell\oplus\mathrm{Id}_L)$ (for any vector bundle L). It follows that any link is equivalent to a link of the form (47) where K is a trivial vector bundle.

Links can be pulled back, and added (for direct sum of data). Moreover, two links (L,ℓ) between $E-F$ and $G-H$, and (M,ℓ') between $G-H$ and $J-K$ can be composed as $(L\oplus M\oplus G\oplus H,\ell\oplus\ell')$ between $E-F$ and $J-K$; this composition is easily checked to be associative. The equivalence class of the composed link will be denoted by $[\ell'\circ\ell]$ or $[\ell']\circ[\ell]$.

$K^1_{\mathrm{top}}(M)$ acts freely transitively on $\mathscr{L}^{G-H}_{E-F}$. The element β of $K^1_{\mathrm{top}}(M)$ represented by the global smooth automorphism g_N of the vector bundle N maps the equivalence class α of (K,ℓ) to the equivalence class $\beta\alpha$ of $(K\oplus N,\ell\oplus g_N)$.

3.3.2. *Definition of the direct image for K^0_{rel}*

Theorem 25. — *Let ξ be any vector bundle on M, let $(\mathscr{F}^+,\mathscr{F}^-)$ and $(\mathscr{G}^+,\mathscr{G}^-)$ be two couples of family index bundles for (the same) ξ, then there exists a canonical element $[\ell^{\mathscr{G}}_{\mathscr{F}}]\in\mathscr{L}^{\mathscr{G}^+-\mathscr{G}^-}_{\mathscr{F}^+-\mathscr{F}^-}$. It is canonical in the sense of the following global compatibility property: if $(\mathscr{H}^+,\mathscr{H}^-)$ is another couple of family index bundles for ξ, then one has $[\ell^{\mathscr{H}}_{\mathscr{F}}]=[\ell^{\mathscr{H}}_{\mathscr{G}}]\circ[\ell^{\mathscr{G}}_{\mathscr{F}}]$.*

This extends trivially to couples of family index bundles for $\xi - \zeta$ (for any vector bundles ξ and ζ on M).

Moreover, if $[\xi_1] - [\zeta_1] = [\xi_2] - [\zeta_2] \in K^0_{\mathrm{top}}(M)$ and if $(\mathcal{F}_1^+, \mathcal{F}_1^-)$ and $(\mathcal{F}_2^+, \mathcal{F}_2^-)$ are couples of family index bundles for $\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$ respectively, then there exists a canonical map $\pi_\ell \colon \mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1} \longrightarrow \mathcal{L}^{\mathcal{F}_2^+-\mathcal{F}_2^-}_{\mathcal{F}_1^+-\mathcal{F}_1^-}$. It is canonical in the sense of the following global compatibility property: if $[\xi_3] - [\zeta_3] = [\xi_1] - [\zeta_1] \in K^0_{\mathrm{top}}(M)$ and if $(\mathcal{F}_3^+, \mathcal{F}_3^-)$ is a couple of family index bundles for $\xi_3 - \zeta_3$, then for any $\alpha \in \mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1}$ and $\beta \in \mathcal{L}^{\xi_3-\zeta_3}_{\xi_2-\zeta_2}$, one has $\pi_\ell(\beta \circ \alpha) = \pi_\ell(\beta) \circ \pi_\ell(\alpha) \in \mathcal{L}^{\mathcal{F}_3^+-\mathcal{F}_3^-}_{\mathcal{F}_1^+-\mathcal{F}_1^-}$.

If $\xi_1 = \xi_2$ and $\zeta_1 = \zeta_2$, then $[\ell^{\mathcal{F}_2}_{\mathcal{F}_1}] = \pi_\ell(\mathrm{Id}_{\xi\oplus\zeta})$.

π_ℓ is compatible with the actions by K^1_{top} in the following sense: if $\alpha \in \mathcal{L}^{\xi_2-\zeta_2}_{\xi_1-\zeta_1}$ and $\beta \in K^1_{\mathrm{top}}$, then $\pi_\ell(\beta\alpha) = \pi_*^{\mathrm{Eu}}(\beta)\pi_\ell(\alpha)$.

If (E, ∇_E), (F, ∇_F), (G, ∇_G) and (H, ∇_H) are flat vector bundles on M, and if $\ell \colon E \oplus H \oplus K \xrightarrow{\sim} F \oplus G \oplus K$ is a link between $E - F$ and $G - H$, it is possible to find a link $\ell' \colon E \oplus H \oplus \mathbb{C}^n \xrightarrow{\sim} F \oplus G \oplus \mathbb{C}^n$ equivalent to ℓ (by adding $\mathrm{Id}_{K'} \colon K' \xrightarrow{\sim} K'$ for some K' such that $K \oplus K' \cong \mathbb{C}^n$). The obtained element

$$(E \oplus H \oplus \mathbb{C}^n, \nabla_E \oplus \nabla_H \oplus d_{\mathbb{C}^n}, F \oplus G \oplus \mathbb{C}^n, \nabla_F \oplus \nabla_G \oplus d_{\mathbb{C}^n}, \ell') \in K^0_{\mathrm{rel}}(M)$$

does not depend on the choice of ℓ' and depends on ℓ only through its equivalence class (this can be checked using (48) with L replaced in the same way by some trivial bundle).

For some element $(E, \nabla_E, F, \nabla_F, f)$ of $K_0^{\mathrm{rel}}(M)$, Consider the sheaf theoretic direct images $(\pi_!^+ E, \nabla_{\pi_!^+ E})$ and $(\pi_!^- E, \nabla_{\pi_!^- E})$ of (E, ∇_E), and $(\pi_!^+ F, \nabla_{\pi_!^+ F})$ and $(\pi_!^- F, \nabla_{\pi_!^- F})$ of (F, ∇_F). Following the proof of Lemma 23, $(\pi_!^+ E, \pi_!^- E)$ and $(\pi_!^+ F, \pi_!^- F)$ are couples of family index bundles for E and F respectively. Using the above Theorem 25 (especially π_ℓ), one obtains an equivalence class of links between $\pi_!^+ E - \pi_!^- E$ and $\pi_!^+ F - \pi_!^- F$ as image by π_ℓ of the equivalence class of $f \colon E \to F$ (which is a link between $E - \{0\}$ and $F - \{0\}$).

Definition 26. — *We define*

$$\pi_*(E, \nabla_E, F, \nabla_F, f) = \big(\pi_!^+ E \oplus \pi_!^- F, \nabla_{\pi_!^+ E} \oplus \nabla_{\pi_!^- F}, \pi_!^- E \oplus \pi_!^+ F, \nabla_{\pi_!^- E} \oplus \nabla_{\pi_!^+ F}, \pi_\ell([f])\big).$$

Theorem 27. — *This defines a morphism $K_0^{\mathrm{rel}}(M) \xrightarrow{\pi_*} K_0^{\mathrm{rel}}(B)$ which enters in the following commutative diagram (with lines modeled on the first line of (21)):*

$$\begin{array}{ccccccc} K^1_{\mathrm{top}}(M) & \xrightarrow{\varphi} & K^0_{\mathrm{rel}}(M) & \xrightarrow{\partial} & K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\ {\scriptstyle \pi_*^{\mathrm{Eu}}}\downarrow & & {\scriptstyle \pi_*}\downarrow & & {\scriptstyle \pi_!}\downarrow & & \downarrow{\scriptstyle \pi_*^{\mathrm{Eu}}} \\ K^1_{\mathrm{top}}(B) & \xrightarrow{\varphi} & K^0_{\mathrm{rel}}(B) & \xrightarrow{\partial} & K^0_{\mathrm{flat}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B). \end{array} \tag{49}$$

3.4. The case of multiplicative, or smooth, K^0-theory

3.4.1. *Transgression of the family index theorem.* — Let $F_{\mathbb{R}}$ be a real vector bundle over M endowed with a euclidean metric and a unitary connection $\nabla_{F_{\mathbb{R}}}$. The curvature $\nabla^2_{F_{\mathbb{R}}}$ is a two-form with values in antisymmetric endomorphisms of $F_{\mathbb{R}}$. Define $e(\nabla_{F_{\mathbb{R}}})$ to be zero if $F_{\mathbb{R}}$ is of odd rank (as real vector bundle) and to be the Pfaffian of $\frac{1}{2\pi}\nabla^2_{F_{\mathbb{R}}}$ if $F_{\mathbb{R}}$ is of even rank. One obtains a closed real differential form whose degree equals the rank of $F_{\mathbb{R}}$, whose de Rham cohomology class $e(F_{\mathbb{R}})$ is independent on $\nabla_{F_{\mathbb{R}}}$ (and on the euclidean metric on $F_{\mathbb{R}}$) and coincides with the image of the Euler class of $F_{\mathbb{R}}$ in $H^\bullet(M,\mathbb{C})$. (This is the Chern-Weil version of the Euler class).

The vertical tangent bundle TZ of the submersion π, which is the subbundle of TM consisting of vectors tangent to the fibres of π, will be supposed to be globally orientable along M. If ξ is a vector bundle on M and F^+ and F^- are vector bundles on B such that $[F^+]-[F^-]=\pi_*^{\mathrm{Eu}}[\xi]\in K^0_{\mathrm{top}}(B)$, the cohomological counterpart of the families index theorem asserts that

$$\mathrm{ch}(F^+)-\mathrm{ch}(F^-)=\int_Z e(TZ)\mathrm{ch}(\xi)\ \in H^{\mathrm{even}}(B,\mathbb{C}) \tag{50}$$

where $\int_Z$ stands for integration along the fibres of π.

Choose any smooth complementary subbundle T^HM of TZ in TM. Of course $T^HM\cong\pi^*TB$. Let P^{TZ} be the projection of TM onto TZ with kernel T^HM. Endow TZ with some riemannian metric g^Z. All riemannian metrics on M which coincide with g^Z on TZ and make TZ and T^HM orthogonal give rise to Levi-Civita connections ∇_{LC} on TM which all project to the same connection $\nabla_{TZ}=P^{TZ}\nabla_{LC}$ on TZ.

Let ∇_ξ, ∇_{F^+} and ∇_{F^-} be connections on ξ, F^+ and F^- respectively. It follows from (50) that $\mathrm{ch}(\nabla_{F^+})-\mathrm{ch}(\nabla_{F^-})$ and $\int_Z e(\nabla_{TZ})\wedge\mathrm{ch}(\nabla_\xi)$ are cohomologous differential forms on B. The following theorem is a non hermitian analogue of results of Bunke [**13**]:

Theorem 28. — *Let $[\ell]=([\ell_K])_{K\text{ compact}\subset B}$ be any collection of mutually compatible equivalence classes of links between restrictions of F^+-F^- and couples of family index bundles for ξ on compact subsets of B. There exists a way to associate to such data (ξ, ∇_ξ, g^Z, T^HM, F^+, ∇_{F^+}, F^-, ∇_{F^-} and $[\ell]$) an element $\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])$ of $\Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C})$ with properties*

(a) $d\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])=\int_Z e(\nabla_{TZ})\mathrm{ch}(\nabla_\xi)-\mathrm{ch}(\nabla_{F^+})+\mathrm{ch}(\nabla_{F^-})$

(b) *η is natural by pullbacks on fibered products as in* (43).

(c) *η is additive for direct sums of vector bundles ξ and $F^\pm$ with direct sum connections (and direct sum of links).*

(d) $\eta(\nabla_E, \nabla_{TZ}, \nabla_{\pi_!^+E}, \nabla_{\pi_!^-E}, [\mathrm{Id}]) = 0$ *if* (E, ∇_E) *is a flat bundle on* M *with sheaf theoretic direct images* $(\pi_!^+E, \nabla_{\pi_!^+E})$ *and* $(\pi_!^-E, \nabla_{\pi_!^-E})$.

Moreover η *with these properties is unique for vector bundles* ξ *with vanishing rational Chern classes on* M.

In statement (d), [Id] stands for the trivial link between $\pi_!^+E - \pi_!^-E$ and itself, when using $(\pi_!^+E, \pi_!^-E)$ as couple of family index bundles for E (see the proof of Lemma 23). In statement (b), the vertical tangent bundle $\widetilde{TZ}$ of $\widetilde{B} \times_B M$ is naturally isomorphic to the pullback of the vertical tangent bundle TZ of M, the connection on $\widetilde{TZ}$ is supposed to be the pullback connection of ∇_{TZ}. The statement (a) is seen as a Chern-Simons like transgression of the family index Theorem (50). As a first consequence of this:

Theorem 29. — *For any* $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\mathrm{rel}}(M)$, *one has*

$$\mathscr{N}_{\mathrm{ch}}\big(\pi_*(E, \nabla_E, F, \nabla_F, f)\big) = \int_Z e(TZ)\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f).$$

This “Riemann-Roch-Grothendieck” theorem for K^0_{rel} is a cohomological formula, it does not need the Chern-Weil version of the Euler class in its expression.

3.4.2. *Direct image for multiplicative/smooth* K^0*-theory*

Definition 30. — *Let* $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$, *take any vector bundles* F^+ *and* F^- *such that* $[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K^0_{\mathrm{top}}(B)$, *choose any connections* ∇_{F^+} *on* F^+ *and* ∇_{F^-} *on* F^-, *take any collection of equivalence classes of links* $[\ell]$ *between* $F^+ - F^-$ *and any families index bundles for* ξ *on compact subsets of* B, *and define the direct image of* $(\xi, \nabla_\xi, \alpha)$ *by*

(51)
$$\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = \Big(F^+, \nabla_{F^+}, \int_Z e(\nabla_{TZ})\alpha\Big) - \big(F^-, \nabla_{F^-}, \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])\big).$$

This definition is intended to obtain the following property:

$$\dddot{\mathrm{ch}}\big(\pi_!(\xi, \nabla_\xi, \alpha)\big) = \int_Z e(\nabla_{TZ}) \wedge \dddot{\mathrm{ch}}(\xi, \nabla_\xi, \alpha) \qquad (52)$$

which implies that $\pi_!^{\mathrm{Eu}}$ sends $MK_0(M)$ to $MK_0(B)$, and $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ to $K^{-1}_{\mathbb{C}/\mathbb{Z}}(B)$.

Theorem 31. — $\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)$ *as defined above does not depend on the choices of* F^+, F^-, ∇_{F^+}, ∇_{F^-} *nor* $[\ell]$.

(51) defines a morphism $\pi_!^{\mathrm{Eu}}\colon \widehat{K}_{\mathrm{ch}}(M) \longrightarrow \widehat{K}_{\mathrm{ch}}(B)$. *The following diagrams commute (the lines of* (54) *are modeled on the modified second line of* (21)*):*

$$\begin{array}{ccc} K^0_{\mathrm{flat}}(M) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(M) \\ {\scriptstyle \pi_!}\big\downarrow & & \big\downarrow{\scriptstyle \pi_!^{\mathrm{Eu}}} \\ K^0_{\mathrm{flat}}(B) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(B), \end{array} \tag{53}$$

(54)
$$\begin{array}{ccccccc} K^1_{\mathrm{top}}(M) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(M,\mathbb{C})/d\Omega^{\mathrm{even}}(M,\mathbb{C}) & \xrightarrow{a} & \widehat{K}_{\mathrm{ch}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\ {\scriptstyle \pi_*^{\mathrm{Eu}}}\big\downarrow & & {\scriptstyle \int_Z e(\nabla_{TZ})\wedge\bullet}\big\downarrow & & \big\downarrow{\scriptstyle \pi_!^{\mathrm{Eu}}} & & \big\downarrow{\scriptstyle \pi_*^{\mathrm{Eu}}} \\ K^1_{\mathrm{top}}(B) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(B,\mathbb{C})/d\Omega^{\mathrm{even}}(B,\mathbb{C}) & \xrightarrow{a} & \widehat{K}_{\mathrm{ch}}(B) & \longrightarrow & K^0_{\mathrm{top}}(B). \end{array}$$

Moreover $\mathfrak{B}\big(\pi_!(\xi,\nabla_\xi,\alpha)\big) = \int_Z e(\nabla_{TZ}) \wedge \mathfrak{B}(\xi,\nabla_\xi,\alpha) \in \Omega^{\mathrm{odd}}(B)/d\Omega^{\mathrm{even}}(B)$.

Here the morphism denoted by $\int_Z e(\nabla_{TZ}) \wedge \bullet$ is integration along the fibre after product with $e(\nabla_{TZ})$ (i.e. $\alpha \mapsto \int_Z e(\nabla_{TZ}) \wedge \alpha$). It vanishes if $\dim Z$ is odd.

The relation concerning $\mathfrak{B}$ implies that $\pi_!^{\mathrm{Eu}}$ sends $\widehat{K}^0(M)$ to $\widehat{K}^0(B)$ (Bunke and Schick's K-theory, see the end of §2.6 after (35)) and $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ to $K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$.

3.5. Hermitian symmetry and functoriality results

3.5.1. *Direct images and symmetries.* — The conjugations on K^0_{flat}, K^0_{rel} and $\widehat{K}_{\mathrm{ch}}$ were defined in Definition 11.

Theorem 32. — *If* $\dim Z$ *is even, then* $\pi_!$ *on* K^0_{flat}, π_* *on* K^0_{rel} *and* $\pi_!^{\mathrm{Eu}}$ *on* $\widehat{K}_{\mathrm{ch}}$ *are all real in the sense that:*

$$\begin{aligned} \pi_!\big((E,\nabla_E)^c\big) &= \big(\pi_!(E,\nabla_E)\big)^c \in K^0_{\mathrm{flat}}(B), \\ \pi_*\big((E,\nabla_E,F,\nabla_F,f)^c\big) &= \big(\pi_!(E,\nabla_E,F,\nabla_F,f)\big)^c \in K^0_{\mathrm{rel}}(B), \\ \pi_!^{\mathrm{Eu}}\big((\xi,\nabla_\xi,\alpha)^c\big) &= \big(\pi_!^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)\big)^c \in \widehat{K}_{\mathrm{ch}}(B). \end{aligned}$$

In fact the last statement of this theorem is a consequence of the last statement (about $\mathfrak{B}$) of the preceding one, and of the facts stated just before (35).

Theorem 33. — *If* $\dim Z$ *is odd, then* π_*^{Eu} *on* K_{top} *and* $\pi_!^{\mathrm{Eu}}$ *on* $\widehat{K}_{\mathrm{ch}}$ *both vanish.*

If $\dim Z$ *is odd, then there exists a map* $\pi_\leftarrow\colon K^0_{\mathrm{flat}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$ *such that* $\pi_! = \partial \circ \pi_\leftarrow$ *(on* K^0_{flat}*) and* $\pi_* = \pi_\leftarrow \circ \partial$ *(on* K^0_{rel}*).*

$\pi_\leftarrow$ *is purely imaginary in the sense that if* $(E,\nabla_E) \in K^0_{\mathrm{flat}}$*:*

$$\pi_\leftarrow\big((E,\nabla_E)^c\big) = -\big(\pi_\leftarrow(E,\nabla_E)\big)^c \in K^0_{\mathrm{rel}}(B).$$

Moreover, $\mathcal{N}_{\mathrm{ch}} \circ \pi_\leftarrow$ *vanishes.*

3.5.2. *Double fibrations.* — Consider two submersions $\pi_1\colon M \longrightarrow B$ and $\pi_2\colon B \longrightarrow S$ and the composed submersion $\pi_2 \circ \pi_1\colon M \longrightarrow S$. The following classical results

$$(55)\qquad \begin{aligned}(\pi_2\circ\pi_1)_*^{\mathrm{Eu}} = \pi_{2*}^{\mathrm{Eu}}\circ\pi_{1*}^{\mathrm{Eu}}\colon K^\bullet_{\mathrm{top}}(M) \longrightarrow K^\bullet_{\mathrm{top}}(S)\\ (\pi_2\circ\pi_1)_! = \pi_{2!}\circ\pi_{1!}\colon K^0_{\mathrm{flat}}(M) \longrightarrow K^0_{\mathrm{flat}}(S)\end{aligned}$$

will be reproved or explained during the proof of the following

Theorem 34. — $(\pi_2\circ\pi_1)_* = \pi_{2*}\circ\pi_{1*}\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(S)$.

Only a partial result is obtained for multiplicative K-theory:

Theorem 35. — *The restriction to $MK^0(M)$ of $\pi_{2!}^{\mathrm{Eu}}\circ\pi_{1!}^{\mathrm{Eu}}$ and $(\pi_2\circ\pi_1)_!^{\mathrm{Eu}}$ coincide.*

4. Proof of Theorems 25 and 27

4.1. Proof of Theorem 25. — The link between any two couples of family index bundles for the same vector bundle ξ is obtained by an intermediary link with some special couple of ("positive kernel") family index bundles (see Definition 37 in §4.1.2). It is proved in §4.1.2 that any couple can be linked with some special one, and that all these links are mutually compatible, the general link is then obtained in two steps by a homotopy technique in §4.1.3 and §4.1.4. B is supposed to be compact in §4.1.2 and §4.1.3.

4.1.1. *Links and exact sequences of vector bundles.* — Consider a short exact sequence of complex vector bundles on M:

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ p\ } E'' \longrightarrow 0.$$

Take any morphisms $s\colon E \longrightarrow E'$ and $j\colon E'' \longrightarrow E$ such that $s\circ i = \mathrm{Id}_{E'}$ and $p\circ j = \mathrm{Id}_{E''}$, then, as was remarked just after Remark 5, $i+j$ and $(s\oplus p)^{-1}$ are isotopic isomorphisms from $E'\oplus E''$ to E. They thus provide the same equivalence class of link between $(E'\oplus E'') - \{0\}$ and $E - \{0\}$, or any equivalent combination. Take any hermitian metrics on E', E and E'', and consider the adjoints i^* and p^* with respect to these metrics. Then $s\oplus p$ and $i^*\oplus p$ are isotopic, and so are $i+j$ and $i+p^*$. This is because autoadjoint automorphisms (here $i^*\circ i$ and $p\circ p^*$) are always isotopic to the identity.

Consider now a longer complex of vector bundles on M:

$$(56)\qquad 0 \longrightarrow E^0 \xrightarrow{v_0} E^1 \xrightarrow{v_1} \cdots \xrightarrow{v_{k-1}} E^k \longrightarrow 0.$$

It may not be an exact sequence, but the v_i are supposed to be of everywhere constant rank. Call H^k the cohomology of this complex in degree k. The H^k are vector bundles on M. Choose some hermitian metrics h^i on the E^i, and consider the associated adjoints v_i^* of the v_i. By finite dimensional Hodge theory one has canonical isomorphisms $H^i \cong \mathrm{Ker}(v_i + v_{i+1}^*)$. Let $\iota_i : H^i \hookrightarrow E^i$ be induced by the inclusion of $\mathrm{Ker}(v_i + v_i^*)$ and $p_i : E^i \longrightarrow H^i$ by the orthogonal projection on $\mathrm{Ker}(v_i + v_i^*)$. Denote by E^+ and E^- the direct sums $\oplus_{i\mathrm{even}} E^i$ and $\oplus_{i\mathrm{odd}} E^i$ respectively, and accordingly for H^+, H^-, v_+, v_-, v_+^*, v_-^*, ι_+, ι_-, p_+ and p_-. The isomorphism $v_+ + v_-^* + p_+ + \iota_- : E^+ \oplus H^- \xrightarrow{\sim} E^- \oplus H^+$ is isotopic to $\big(v_- + v_+^* + p_- + \iota_+\big)^{-1}$. This is because $p_\pm$ and $\iota_\pm$ are mutually adjoint.

Definition 36. — *The equivalence class of links between $E^+ - E^-$ and $H^+ - H^-$ (or any equivalent combinations) associated to the complex* (56) *is the common class defined by anyone of these two isomorphisms.*

This definition is justified by the independence on the hermitian metrics. This class of link is not modified by isotopy of the complex, i.e. smooth homotopy of the morphisms such that any of them stays of same constant rank. This class of links can be described in the same terms from the following exact sequence

$$0 \longrightarrow H^+ \xrightarrow{\iota_+} E^+ \xrightarrow{v_+ + v_-^*} E^- \xrightarrow{p_-} H^- \longrightarrow 0.$$

It is left as an exercise to check that it is the same class as the one obtained from the composition of links associated to the following short exact sequences

(57)
$$0 \longrightarrow \mathrm{Ker} v_i \longrightarrow E^i \xrightarrow{v_i} \mathrm{Im} v_i \longrightarrow 0, \qquad 0 \longrightarrow \mathrm{Im} v_{i-1} \longrightarrow \mathrm{Ker} v_i \longrightarrow H^i \longrightarrow 0$$

which enter in the canonical decomposition of (56) in short exact sequences.

4.1.2. *Link with "positive kernel" family index bundles.* — Consider as in §3.1.1 some vector bundle ξ on M with hermitian metric h^ξ and connection ∇_ξ. Take some vertical riemannian metric g^Z on TZ and consider some triple (μ^+, μ^-, ψ) as in §3.1.1, with which a vertical modified de Rham operator $\mathcal{D}_\psi^{\nabla_\xi \pm}$ is computed. The triple (μ^+, μ^-, ψ) may be not suitable.

If B is compact, there exists some vector bundle λ on B and some bundle morphism $\varphi : \lambda \to \mathcal{E}^- \oplus \mu^-$ such that $\mathcal{D}_{\psi+\varphi}^{\nabla_\xi +}$ is surjective, as can be proved in exactly the same way as in [**2**] Proposition 2.2, or [**3**] Lemma 9.30 or [**29**] Lemma 8.4 of chapter III. This proves the existence of suitable triples in general.

Definition 37. — *A (suitable) triple which has the same surjectivity property as $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ will hereafter be called a "positive kernel" triple; the obtained couple of family index bundles $((\mathrm{Ker}\mathcal{D}_{\psi+\varphi}^{\nabla_\xi +} \oplus \mu^-, \mu^+ \oplus \lambda)$ in the above example) will be called "couple of positive kernel family index bundles".*

Suppose now that (μ^+, μ^-, ψ) were suitable and gave rise to kernel bundles $\mathcal{H}^\pm$. Choose λ and φ as above. Let $P^{\mathcal{H}^-}$ be the projector from $\mathcal{E}^- \oplus \mu^-$ onto $\mathcal{H}^-$ with kernel $\mathrm{Im}\mathscr{D}_{\psi}^{\nabla_\xi +}$. The following sequence of vector bundles on B is exact:

$$(58)\qquad 0 \longrightarrow \mathscr{H}^+ \longrightarrow \mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} \longrightarrow \lambda \xrightarrow{p^{\mathscr{H}^-}\circ\varphi} \mathscr{H}^- \longrightarrow 0$$
$$(\sigma, v, w) \longmapsto w$$

($\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +}$ is a subbundle of $\mathscr{E}^+ \oplus \mu^+ \oplus \lambda$ on which its elements are decomposed). This provides an equivalence class of links between $\mathscr{H}^+ - \mathscr{H}^-$ and $\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} - \lambda$ as in Definition 36. An equivalence class of links between $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$ and $(\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} \oplus \mu^-) - (\lambda \oplus \mu^+)$ is trivially deduced.

Lemma 38. — *Classes of links obtained in this way are mutually compatible.*

Proof. — Suppose that λ' and φ' satisfy the same surjectivity hypothesis as λ and φ with respect to $\mu^\pm$ and ψ. Then $\lambda \oplus \lambda'$ and $\varphi \oplus \varphi'$ also do. On the other hand, the same construction can be performed starting from the triple $(\mu^+ \oplus \lambda, \mu^-, \psi + \varphi)$ and using λ' and φ', or starting from the triple $(\mu^+ \oplus \lambda', \mu^-, \psi + \varphi')$ and using λ and φ. One obtains in each case some equivalence class of links between two of the couples $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$, $(\mathrm{Ker}\mathscr{D}_{\psi+\varphi}^{\nabla_\xi +} \oplus \mu^-) - (\lambda \oplus \mu^+)$, $(\mathrm{Ker}\mathscr{D}_{\psi+\varphi'}^{\nabla_\xi +} \oplus \mu^-) - (\lambda' \oplus \mu^+)$ or $(\mathrm{Ker}\mathscr{D}_{\psi+\varphi+\varphi'}^{\nabla_\xi +} \oplus \mu^-) - (\lambda \oplus \lambda' \oplus \mu^+)$.

These links are all compatible (in the sense of composition of links) as can be checked by considering the exact sequence (58) associated either to $\lambda \oplus \lambda'$ and $\varphi + t\varphi'$ with t varying along $[0, 1]$ or to $\lambda \oplus \lambda'$ and $s\varphi + \varphi'$ with $s \in [0, 1]$. □

4.1.3. *Deformation of ψ, h^ξ and ∇_ξ.* — Consider two triples (μ^+, μ^-, ψ_0) and (μ^+, μ^-, ψ_1) with same μ^+ and μ^-. Take the product with the interval $[0, 1]$ and consider some everywhere finite rank $\widetilde{\psi}\colon \mathscr{E}^+ \oplus \mu^+ \longrightarrow \mathscr{E}^- \oplus \mu^-$ over $B \times [0, 1]$ with restrictions $\widetilde{\psi}|_{B\times\{0\}} = \psi_0$ and $\widetilde{\psi}|_{B\times\{1\}} = \psi_1$. The pullback of ξ on $B \times [0, 1]$ is endowed with any (not necessarily pullback) hermitian metric and connection.

If B is compact, one can perform the above construction over $B \times [0, 1]$, finding some positive kernel triple $\big(\mu^+ \oplus \lambda, \mu^-, \widetilde{\psi} + \widetilde{\varphi}\big)$ over $B \times [0, 1]$. An isotopy class of bundle isomorphism $\mathrm{Ker}\mathscr{D}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}}^{\nabla_\xi +} \cong \mathrm{Ker}\mathscr{D}_{\psi_1+\widetilde{\varphi}|_{B\times\{1\}}}^{\nabla_\xi +}$ is obtained by parallel transport along $[0, 1]$. This produces an equivalence class of links between the couples $\big(\mathrm{Ker}\mathscr{D}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}}^{\nabla_\xi +} \oplus \mu^-\big) - (\mu^+ \oplus \lambda)$ and $\big(\mathrm{Ker}\mathscr{D}_{\psi_1+\widetilde{\varphi}|_{B\times\{1\}}}^{\nabla_\xi +} \oplus \mu^-\big) - (\mu^+ \oplus \lambda)$.

Suppose (μ^+, μ^-, ψ_0) and (μ^+, μ^-, ψ_1) are both suitable triples with associated kernel bundles $\mathscr{H}_0^\pm$ and $\mathscr{H}_1^\pm$ (and with respect to not necessarily same metric and connection on ξ). On $B \times \{0\}$, the construction of the preceding paragraph produces an equivalence class of links between $(\mathscr{H}_0^+ \oplus \mu^-) - (\mathscr{H}_0^- \oplus \mu^+)$ and $\mathrm{Ker}\mathscr{D}_{\psi_0+\widetilde{\varphi}|_{B\times\{0\}}}^{\nabla_\xi +} - \lambda$, and similarly on $B \times \{1\}$. This three links compose to produce an equivalence class of links between $(\mathscr{H}_0^+ \oplus \mu^-) - (\mathscr{H}_0^- \oplus \mu^+)$ and $(\mathscr{H}_1^+ \oplus \mu^-) - (\mathscr{H}_1^- \oplus \mu^+)$.

Lemma 39. — *This equivalence class of links does not depend on λ, $\widetilde{\varphi}$ and $\widetilde{\psi}$.*

Proof. — The independence on λ and $\widetilde{\varphi}$ follows from Lemma 38 (and the fonctoriality of links by pullbacks). The independence on the choice of $\widetilde{\psi}$ can be proved by deforming it to any other choice (with fixed boundary values), and make the above construction on $B \times [0,1] \times [0,1]$. □

4.1.4. *General construction (and proof of Theorem 25).* — If two suitable triples $(\mu_0^+, \mu_0^-, \psi_0)$ and $(\mu_1^+, \mu_1^-, \psi_1)$ give rise to couples of family index bundles $(\mathcal{F}^+, \mathcal{F}^-)$ and $(\mathcal{G}^+, \mathcal{G}^-)$, one performs the preceding construction starting from the bundles $\mu_0^+ \oplus \mu_1^+$ and $\mu_0^- \oplus \mu_1^-$ and the two morphisms ψ_0 extended by 0 on μ_1^+ and ψ_1 extended by 0 on μ_1^-. One obtains an equivalence class of links between the couples $(\mathcal{F}^+ \oplus \mu_1^+ \oplus \mu_1^-) - (\mathcal{F}^- \oplus \mu_1^- \oplus \mu_1^+)$ and $(\mathcal{G}^+ \oplus \mu_0^+ \oplus \mu_0^-) - (\mathcal{G}^- \oplus \mu_0^- \oplus \mu_0^+)$. On compact subsets of B, one defines $[\ell_{\mathcal{F}}^{\mathcal{G}}]$ as the composition of this class of link with the trivial links between $\mathcal{F}^+ - \mathcal{F}^-$ and $(\mathcal{F}^+ \oplus \mu_1^+ \oplus \mu_1^-) - (\mathcal{F}^- \oplus \mu_1^- \oplus \mu_1^+)$ and between $(\mathcal{G}^+ \oplus \mu_0^+ \oplus \mu_0^-) - (\mathcal{G}^- \oplus \mu_0^- \oplus \mu_0^+)$ and $\mathcal{G}^+ - \mathcal{G}^-$. One obtains a projective family of equivalence classes of links on compact subsets of B. Stability properties of vector bundles [**22**] §8 Proposition 1.4 can be used to prove that these links can be described with isomorphisms of the form $\mathcal{F}^+ \oplus \mathcal{G}^- \oplus \mathbb{C}^N \xrightarrow{\sim} \mathcal{F}^- \oplus \mathcal{G}^+ \oplus \mathbb{C}^N$ with some fixed N, and such that two such isomorphisms are always isotopic. It is then possible to obtain a global link by inductive limit on an exhaustion by compact subsets with an iterative deformation procedure to fix the isomorphism at finite distance.

Definition 40. — *$[\ell_{\mathcal{F}}^{\mathcal{G}}]$ is the equivalence class of links between $\mathcal{F}^+ - \mathcal{F}^-$ and $\mathcal{G}^+ - \mathcal{G}^-$ obtained in this way.*

The independence on the various choices follows from Lemma 39.

For three suitable triples, the construction (for compact B) can be adapted so that the restriction to $B \times \{\frac{1}{2}\}$ corresponds to the third data, this proves the compatibility of these links with respect to mutual composition. Now the equivalence of links on compacts propagates in the inductive limit along an exhaustion by compact subsets.

If ξ_1, ζ_1, ξ_2 and ζ_2 are vector bundles on M such that $[\xi_1] - [\zeta_1] = [\xi_2] - [\zeta_2]$ in $K^0_{\mathrm{top}}(M)$, consider some vector bundle isomorphism $\ell \colon \xi^+ \oplus \zeta^- \oplus L \xrightarrow{\sim} \xi^- \oplus \zeta^+ \oplus L$ as in (47). Let $(\mathcal{F}_i^+, \mathcal{F}_i^-)$, $(\mathcal{G}_i^+, \mathcal{G}_i^-)$ and $(\mathcal{L}^+, \mathcal{L}^-)$ be couples of family index bundles for ξ_i, ζ_i and L respectively for $i = 1$ and 2. (It is always possible to choose L such that it admits family index bundles on the whole B: it suffices to take L trivial with canonical metric and connection). Thus $(\mathcal{F}_1^+ \oplus \mathcal{G}_2^+ \oplus \mathcal{L}^+, \mathcal{F}_1^- \oplus \mathcal{G}_2^- \oplus \mathcal{L}^-)$ and $(\mathcal{F}_2^+ \oplus \mathcal{G}_1^+ \oplus \mathcal{L}^+, \mathcal{F}_2^- \oplus \mathcal{G}_1^- \oplus \mathcal{L}^-)$ are couples of family index bundles for the same vector bundle modulo the isomorphism ℓ.

Definition 41. — *$\pi_\ell([\ell])$ is the equivalence class of links obtained between these couples using Definition 40, and interpreted as an equivalence class of links between $(\mathcal{F}_1^+ \oplus \mathcal{G}_1^-) - (\mathcal{F}_1^- \oplus \mathcal{G}_1^+)$ and $(\mathcal{F}_2^+ \oplus \mathcal{G}_2^-) - (\mathcal{F}_2^- \oplus \mathcal{G}_2^+)$.*

The fact that $[\ell_{\mathcal{F}}^{\mathcal{G}}] = \pi_\ell([\mathrm{Id}])$ is tautological.

But if one takes a different link from the identity, and the same couples of family index bundles at both boundaries, one obtains a realisation of the direct image π_*^{Eu} on K_{top}^1 by gluing the ends and applying Definition 21. The last statement of the theorem is a consequence of this fact and the obvious compatibility of the whole construction with direct sums.

The independence of $\pi_\ell([\ell])$ on the choice of L and ℓ (in some same equivalence class of links see (48)) is due to the above facts and to the invariance of $\pi_\ell([\ell])$ under isotopy of ℓ. The canonicity of $\pi_\ell([\ell])$ is a direct consequence of the corresponding property of $\ell_{\mathcal{F}}^{\mathcal{G}}$.

4.2. Proof of Theorem 27. — This result is a consequence of a compatibility result (Proposition 43) of some canonical link obtained from Theorem 25 and another one obtained from Definition 36 applied to some long exact sequence in cohomology. This second link is computed in §4.2.2 as a composition of two pieces. The compatibility proof then uses a geometric deformation, in which the canonical link is proved to decompose in two pieces too. The fit of each piece of one link with its counterpart in the other one is proved in §4.2.3 and §4.2.4.

As remarked just before Definition 40, an equivalence of links on compact subsets propagates in the inductive limit along an exhaustion by compacts. So, in this whole section, B can be supposed to be compact without restriction.

4.2.1. *Reduction of the problem*

Lemma 42. — *Suppose that (E^i, ∇_{E^i}) are flat vector bundles on M entering in the following parallel complex:*

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^k \longrightarrow 0. \tag{59}$$

Take the same notations E^+, E^-, H^+ and H^- as in Definition 36 and define the connections $\nabla_{E^+} = \bigoplus_{i \text{ even}} \nabla_{E^i}$ and $\nabla_{E^-} = \bigoplus_{i \text{ odd}} \nabla_{E^i}$ and accordingly for ∇_{H^+} and ∇_{H^-}. Let $[\ell]$ be the equivalence class of links between $E^+ - E^-$ and $H^+ - H^-$ associated to (59) *from Definition 36. Then*

$$(E^+ \oplus H^-, \nabla_{E^+} \oplus \nabla_{H^-}, E^- \oplus H^+, \nabla_{E^-} \oplus \nabla_{H^+}, [\ell]) = 0 \in K_{\mathrm{rel}}^0(M).$$

Proof. — The decomposition of the complex (59) in several short exact sequences as in (57) gives rise to short exact sequences of flat vector bundles as was remarked at the ends of §2.1.1 and §2.2.2. Through this decomposition, $[\ell]$ is reduced to canonical links associated to short exact sequences as in relation (*iii*) of Definition 4 (as was remarked after Definition 36) and the lemma follows. □

Consider now the exact sequence (44). Denote by $\pi_!^i E$ the i^{th} degree de Rham cohomology of the fibres of π with coefficients in the restriction of (E, ∇_E) (to the fibres) and similarly for E' and E''. The associated long exact sequence in cohomology (45) also reads:

$$(60) \qquad 0 \longrightarrow \pi_!^0 E' \xrightarrow{[i]} \pi_!^0 E \xrightarrow{[p]} \pi_!^0 E'' \longrightarrow \pi_!^1 E' \xrightarrow{[i]} \cdots \xrightarrow{[p]} \pi_!^{\dim Z} E'' \longrightarrow 0.$$

Let $[i+j]$ be the equivalence class of links corresponding to (44) constructed at the beginning of §4.1.1.

Proposition 43. — *$\pi_\ell([i+j])$ coincides with the equivalence class of links between $(\pi_!^+ E' \oplus \pi_!^+ E'') - (\pi_!^- E' - \pi_!^- E'')$ and $\pi_!^+ E - \pi_!^- E$ associated to* (60).

The proof of this proposition is delayed in the following paragraphs.

We are now in position to prove Theorem 27 using Proposition 43.

The definition of π_* on K^0_{rel} is clearly compatible with the isotopy of f. If f is parallel, then $\pi_\ell([f])$ is itself a parallel isomorphism between $\pi_!^+ E \oplus \pi_!^- F$ and $\pi_!^- E \oplus \pi_!^+ F$. This proves the compatibility of π_* with relation (*i*) of Definition 4. π_* is also obviously compatible with direct sums as in relation (*ii*) of Definition 4. The compatibility of π_* with relation (*iii*) is a direct consequence of the above proposition and Lemma 42.

The commutativity of the right square of diagram (49) was proved in Lemma 23. The commutativity of the central square of diagram (49) is tautological. The commutativity of the left square of diagram (49) is a consequence of the last statement of Theorem 25.

4.2.2. *Sheaf theoretic direct images and short exact sequences.* — Back to the model exact sequence (44), consider E' as a subbundle of E. The vertical exterior differential operator d^{∇_E} respects the subbundle (over B) $\Omega^\bullet(Z, E')$ of the vertical de Rham complex $(\Omega^\bullet(Z,E), d^{\nabla_E})$. This filtration $0 \subset \Omega(M,E') \subset \Omega(M,E)$ gives rise to some spectral sequence, and to some filtration $0 \subset FH^\bullet(Z,E) \subset H^\bullet(Z,E)$ of the fiberwise cohomology of E. The (E_0, d_0)-term of this spectral sequence is the direct sum of the fiberwise de Rham complexes of E' and of E''; consequently, the E_1-term is the direct sum $\pi_! E' \oplus \pi_! E''$ of the fiberwise cohomology of E' and of E''.

Let $s\colon E \to E'$ be a smooth vector bundle morphism such that $s \circ i$ is the identity of E', then E'' will be identified with the subbundle $\operatorname{Ker} s$ of E so that E will be identified with $E' \oplus E''$. Thus E inherits two flat connections ∇_E and $\nabla_{E'} \oplus \nabla_{E''}$, whose difference is (as was used in Lemma 1 in a nonflat context) a one form ω with values in $\operatorname{Hom}(E'', E')$. On any closed E''-valued form, d^{∇_E} applies as $\omega\wedge$ so that the operator d_1 of the spectral sequence is given by

(61) $$d_1 = [\omega\wedge] \colon H^\bullet(Z, E'') \longrightarrow H^{\bullet+1}(Z, E').$$

This is exactly the linking maps of the exact diagram (45), and the spectral sequence converges at E_2 which is the filtrated fiberwise cohomology of E.

Thus the exact diagram (45) decomposes in two exact sequences:

(62) $$0 \longrightarrow FH^{\pm}(Z,E) \longrightarrow \pi_!^{\pm}E' \xrightarrow{[\omega]} \pi_!^{\mp}E'' \longrightarrow H^{\mp}(Z,E)/FH^{\mp}(Z,E) \longrightarrow 0.$$

The canonical link associated to (45)-(60) is the direct sum of the two canonical links of these two exact sequences modulo the canonical isotopy class of isomorphism between (graded) cohomology and (graded) filtrated cohomology.

(63) $$\pi_!^\bullet E = H^\bullet(Z,E) \cong FH^\bullet(Z,E) \oplus \big(H^\bullet(Z,E)/FH^\bullet(Z,E)\big).$$

4.2.3. *"Adiabatic" limit of harmonic forms.* — Put $\nabla_\theta = (\nabla_{E'} \oplus \nabla_{E''}) + \theta\omega$ for any $\theta \in [0,1]$, then $\nabla_E = \nabla_1$, and ∇_θ is flat for any $\theta \in [0,1]$. Moreover, the flat bundles (E, ∇_θ) and (E, ∇_E) are isomorphic for any $\theta > 0$ through the automorphism $\mathrm{Id}_{E'} \oplus \theta \mathrm{Id}_{E''}$ of E. For any $\theta > 0$, d^{∇_θ} (as d^{∇_E}) also respects the subbundle $\Omega(Z, E')$ of $\Omega(Z, E)$, and the associated spectral sequence is isomorphic to the preceding one if θ is positive, so that the considerations of the preceding paragraph apply verbatim for $\theta \in (0, 1]$.

Put any riemanian metric on M, and endow $E \cong E' \oplus E''$ with a direct sum hermitian metric. The Hodge theory of the fibres of π provides for any $\theta \in (0,1]$ an isomorphism between the (graded) kernel $\mathscr{H}_\theta^\bullet$ of the fiberwise Euler-de Rham operator $\mathscr{D}_\theta = d^{\nabla_\theta} + (d^{\nabla_\theta})^*$ and the cohomology of the de Rham complex associated with d^{∇_θ}. In particular, the dimension of $\mathscr{H}_\theta^i$ is constant for any i when θ goes over $(0,1]$. The isomorphism class provided by parallel transport along $(0,1]$ of $\mathscr{H}_\theta$ is isotopic to the twist of the de Rham cohomology by $\mathrm{Id}_{E'} \oplus \theta \mathrm{Id}_{E''}$.

Let $d^{\nabla_{E'}}$ and $d^{\nabla_{E''}}$ be the fiberwise exterior differential operators on $\Omega(Z, E')$ and $\Omega(Z, E'')$ respectively obtained from $\nabla_{E'}$ and $\nabla_{E''}$, and define $\mathscr{D}' = d^{\nabla_{E'}} + (d^{\nabla_{E'}})^*$ and $\mathscr{D}'' = d^{\nabla_{E''}} + (d^{\nabla_{E''}})^*$. Then $\mathscr{D}_\theta = \mathscr{D}' + \mathscr{D}'' + \theta(\omega + \omega^*)$ so that one has a continuous family of elliptic operators on $B \times [0,1]$. Suppose that B is compact, this ensures the positivity of the minimum positive eigenvalue of $\mathscr{D}' + \mathscr{D}''$ along all B, which will be denoted by $\lambda_{\min}$. There exists $\varepsilon > 0$ such that $\theta\omega$ is bounded by $\frac{1}{5}\lambda_{\min}$ in L^2 norm for all $\theta \leq \varepsilon$. Then for any $y \in B$ and any $\theta \leq \varepsilon$, $\mathscr{D}_\theta$ has no eigenvalue equal to $\pm\frac{\lambda_{\min}}{2}$. Thus the (graded) direct sum $\mathscr{F}_\theta^\bullet$ of eigenspaces of $\mathscr{D}_\theta$ corresponding to eigenvalues belonging to $[-\frac{\lambda_{\min}}{2}, \frac{\lambda_{\min}}{2}]$ is a finite rank vector bundle on $B \times [0, \varepsilon]$ whose restriction $\mathscr{F}_0$ to $B \times \{0\}$ equals $\mathrm{Ker}\,\mathscr{D}' \oplus \mathrm{Ker}\,\mathscr{D}''$.

As θ converges to 0, $\frac{1}{\theta} P^{\mathscr{F}_\theta} d^{\nabla_\theta} P^{\mathscr{F}_\theta}$ converges to $P^{\mathscr{F}_0}(\omega\wedge)P^{\mathscr{F}_0}$ and this is the image of $[\omega\wedge]$ through the Hodge isomorphism $\mathscr{F}_0 \cong \pi_! E' \oplus \pi_! E''$.

This proves that $\mathscr{H}_\theta$ converges to the kernel $\mathscr{H}_0$ of $P^{\mathscr{F}_0}(\omega\wedge)P^{\mathscr{F}_0}$ as θ converges to 0, because the dilation factor $\frac{1}{\theta}$ does not modify the kernels. This limit subspace $\mathscr{H}_0$ is identified by Hodge isomorphism $\mathscr{F}_0 \cong H(Z,E') \oplus H(Z,E'')$ with the filtrated fiberwise cohomology of E as seen around Equation (61). Consequently, the parallel transport along $[0,1]$ for $\mathscr{H}$ is, modulo the Hodge isomorphisms, in the same isotopy class as the isomorphism (63) between fiberwise cohomology of E and its filtrated counterpart.

4.2.4. *End of proof of Proposition 43.* — Clearly

$$[\mathscr{H}_\theta^+] - [\mathscr{H}_\theta^+] = [\mathscr{F}_0^+] - [\mathscr{F}_0^-] = \pi_*^{\mathrm{Eu}}([E]) \in K^0_{\mathrm{top}}(B)$$

for any positive θ. Following the construction of canonical links, the equivalence class of links $\pi_\ell([i+j])$ is isomorphic (modulo Hodge isomorphisms at the boundaries) to the class of links between $[\mathscr{F}_0^+] - [\mathscr{F}_0^-]$ and $[\mathscr{H}_1^+] - [\mathscr{H}_1^-]$ obtained by parallel transport along $[0,1]$ of some kernel bundle on $B \times [0,1]$ associated to the above model deformation of d^{∇_E} and canonical links at the boundaries.

However we will cut at some $\theta \in (0,\varepsilon]$ to perform the construction. In fact over $B \times (0,1]$, the triple $(\{0\},\{0\},0)$ is suitable (because of fiberwise Hodge theory). Over $B \times [0,\varepsilon]$, one has $\mathbb{Z}$-graded vector subbundles $\mathscr{F}_\theta$ and $\mathscr{H}_\theta$ of $\Omega(Z,E)$ which are all respected by d^{∇_θ} and $\mathscr{D}_\theta$. Let $\mathscr{P}^{\mathscr{F}_\theta}$ be the orthogonal projection onto $\mathscr{F}_\theta$, the triple $(\{0\},\{0\},-\mathscr{P}^{\mathscr{F}_\theta}\mathscr{D}_\theta)$ is suitable, with associated kernel bundles $\mathscr{F}_\theta^\pm$.

To describe the canonical link between $\mathscr{F}_\theta^+ - \mathscr{F}_\theta^-$ and $\mathscr{H}_\theta^+ - \mathscr{H}_\theta^-$ of Definition 40 over $B \times (0,\varepsilon]$, one observes that we are in the special case studied in §4.1.3. On $B \times (0,\varepsilon] \times [0,1]$, one puts (following the notations of §4.1.3) $\widetilde{\psi} = -(1-t)\mathscr{P}^{\mathscr{F}_\theta \mathscr{D}_\theta}$, $\lambda = \mathscr{F}_\theta^-$ and $\widetilde{\varphi} = \mathrm{Id}_{\mathscr{F}_\theta^-}$. The obtained kernel bundle is the kernel of

$$t\mathscr{D}_\theta + \mathrm{Id}_{\mathscr{F}_\theta^-} : \mathscr{F}_\theta^+ \oplus \mathscr{F}_\theta^- \longrightarrow \mathscr{F}_\theta^-$$

i.e. $\mathscr{K}_t = \{(\sigma, -t\mathscr{D}_\theta\sigma)/\sigma \in \mathscr{F}_\theta^+\}$. For $t=0$ the link between $\mathscr{F}_\theta^+ - \mathscr{F}_\theta^-$ and itself is tautological. For $t=1$ the link is associated to the exact sequence (58)

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathscr{H}_\theta^+ & \longrightarrow & \mathscr{K}_1 & \longrightarrow & \mathscr{F}_\theta^- & \longrightarrow \mathscr{H}_\theta^- \longrightarrow 0 \\ & & (\sigma, -\mathscr{D}_\theta\sigma) & \longmapsto & -\mathscr{D}_\theta\sigma & \end{array}$$

which is isotopic through the obvious isomorphism $\mathscr{K}_t^+ \cong \mathscr{F}_\theta^+$ (obtained for any t by parallel transport along $[0,t]$) to the exact sequence

$$0 \longrightarrow \mathscr{H}_\theta^+ \longrightarrow \mathscr{F}_\theta^+ \xrightarrow{-\mathscr{D}_\theta} \mathscr{F}_\theta^- \longrightarrow \mathscr{H}_\theta^- \longrightarrow 0. \tag{64}$$

It follows that $\pi_\ell([i+j])$ is the composition of the Hodge isomorphism $\pi_! E \cong \mathscr{H}_1$, the parallel transport $\mathscr{H}_1 \cong \mathscr{H}_\theta$, the canonical equivalence class of links between $\mathscr{H}_\theta^+ - \mathscr{H}_\theta^-$ and $\mathscr{F}_\theta^+ - \mathscr{F}_\theta^-$ associated to (64) as in Definition 36, the parallel transport again $\mathscr{F}_\theta \cong \mathscr{F}_0$ and the Hodge isomorphism again $\mathscr{F}_0 \cong \pi_! E' \oplus \pi_! E''$.

The convergence of $\frac{1}{\theta}P^{\mathcal{F}_\theta}d^{\nabla_\theta}P^{\mathcal{F}_\theta}$ to $P^{\mathcal{F}_0}(\omega\wedge)P^{\mathcal{F}_0}$ as θ converges to 0 proves that the equivalence class of links between $\mathcal{H}_\theta^+ - \mathcal{H}_\theta^-$ and $\mathcal{F}_\theta^+ - \mathcal{F}_\theta^-$ converges to the equivalence class of links between $\mathcal{H}_0^+ - \mathcal{H}_0^-$ and $\mathcal{F}_0^+ - \mathcal{F}_0^-$ provided via the Hodge isomorphisms $\mathcal{F}_0 \cong \pi_! E' \oplus \pi_! E''$ and $\mathcal{H}_0 \cong FH(Z,E) \oplus H(Z,E)/FH(Z,E)$ by Definition 36 and (62).

In the case of compact B, Proposition 43 follows from this convergence and the compatibility of the adiabatic limit of harmonic forms with (63) checked in the preceding paragraph. In the case of noncompact B one concludes using the fact that K^1_{top} is stable by inductive limit along an exhaustion of compact sets, so that two equivalence classes of links whose restrictions to any compact subset agree are equal.

5. η-forms

The goal of this section is to prove Theorems 28, 29 and 31. The construction of η-forms occupies three paragraphs: preliminaries of algebraic nature are given in §5.1, the adaptation to suitable triples of the construction of family index transgression forms is performed in §5.2. In §5.3, the construction is completed, and the existence part of Theorem 28 is proved. The anomaly formulae obtained in (91) and (92) allow to complete the proof of Theorem 28 and to prove Theorems 29 and 31 in §5.4.

5.1. $\mathbb{Z}_2$-graded theory

5.1.1. *$\mathbb{Z}_2$-graded bundles and superconnections.* — Consider a complex vector space V which decomposes as $V = V^+ \oplus V^-$, with a $\mathbb{Z}_2$-graduation operator $\tau|_{V^\pm} = \pm\mathrm{Id}|_{V^\pm}$. The supertrace of $a \in \mathrm{End}V$ is defined by $\mathrm{Tr}_s a = \mathrm{Tr}(\tau \circ a)$, (this is the trace on V^+ minus the trace on V^-). End$V$ is also $\mathbb{Z}_2$-graded (even endomorphisms respect both parts V^+ and V^- and odd ones exchange them). The supercommutator in End$V$ is defined for pure degree objects as

$$[a,b] = ab - (-1)^{\deg a \deg b} ba$$

and bilinearly extended to End$V$. This is such that the supertrace vanishes on supercommutators.

Suppose now that E is a $\mathbb{Z}_2$ graded vector bundle on M, that is $E = E^+ \oplus E^-$ where E^+ and E^- are complex vector bundles themselve. The supertrace is defined as above and extends naturally on End$E$-valued differential forms, with values in ordinary differential forms. End$E$-valued differential forms inherit a global $\mathbb{Z}_2$-graduation, ordinary differential forms being $\mathbb{Z}_2$-graduated by the parity of their degree. They act on E-valued differential forms and multiply in the following way

$$(\alpha \mathbin{\widehat{\otimes}} a)(\beta \mathbin{\widehat{\otimes}} b) = (-1)^{\deg a \deg \beta}(\alpha \wedge \beta)\widehat{\otimes}(ab) \tag{65}$$

where $\alpha\widehat{\otimes}a$ and $\beta\widehat{\otimes}b$ are decomposed tensors in the graded tensor product of differential forms with either End$E$ or E. The supercommutator of End$E$-valued differential forms is defined in the same way as above but by considering the global graduation. With this convention, the supertrace allways vanishes on supercommutators.

A superconnection A on E is the sum of a connection ∇ which respects the decomposition of E and of a globally odd End$E$-valued differential form ω. Its curvature is its square $A^2 = (\nabla+\omega)^2 = \nabla^2 + [\nabla,\omega] + \omega^2$, a global even End$E$-valued differential form (A^2 is not a differential operator).

Following (6), denote by $\mathrm{ch}(A) = \phi\mathrm{Tr}_s \exp -A^2$ the Chern-Weil form representing the Chern character of any superconnection A. It is an even degree differential form on M. The space of superconnections on E is convex (and of course contains ordinary connections) so that the preceding Chern-Weil and Chern-Simons theory also works for superconnections (especially Formula (7)). Thus $\mathrm{ch}(A)$ is closed and its cohomology class is the same as the Chern character of E in complex cohomology (this means $\mathrm{ch}(E^+) - \mathrm{ch}(E^-)$ because of the $\mathbb{Z}_2$-graduation).

5.1.2. *Special adjunction.* — A hermitian metric on $E = E^+ \oplus E^-$ will be supposed to make this decomposition orthogonal. Let β be a differential form and $a \in \mathrm{End}E$, the adjoint of a will be denoted by a^*. For End$E$-valued differential forms, there are two notions of adjunction: the ordinary adjoint of $\beta\,\widehat{\otimes}\,a$ is $\overline{\beta}\,\widehat{\otimes}\,a^*$, while its special adjoint is

$$(\beta\,\widehat{\otimes}\,a)^S = (-1)^{\frac{\deg\beta(\deg\beta-1)}{2}+\deg\beta\deg a}\,\overline{\beta}\,\widehat{\otimes}\,a^* \tag{66}$$

following the convention implicitely used in [**11**] §I(c) and (d). If ω_1 and ω_2 are any (multidegree) End$E$-valued differential forms, denote by ω_1^S and ω_2^S their special adjoints, then for the product (65):

$$(\omega_1\omega_2)^S = \omega_2^S\omega_1^S. \tag{67}$$

Denote by ω^* the usual adjoint of ω, the relations between usual and special adjunctions and the supertrace is as follows:

$$\mathrm{Tr}_s(\omega^*) = \overline{\mathrm{Tr}_s(\omega)} \qquad \text{and} \qquad \phi\mathrm{Tr}_s(\omega^S) = \overline{\phi\mathrm{Tr}_s(\omega)} \tag{68}$$

in particular, $\phi\mathrm{Tr}_s(\omega)$ is real if ω is a special autoadjoint (multidegree) End$E$-valued differential form.

Let ω be some globally odd End$E$-valued differential form, and $A = \nabla + \omega$ a superconnection on E. Then the adjoint of A is defined by

$$A^S = \nabla^* + \omega^S. \tag{69}$$

Thus $\frac{1}{2}(A + A^S)$ is the sum of $\nabla^u = \frac{1}{2}(\nabla + \nabla^*)$ (which respects the hermitian metric of E) and of some special autoadjoint End$E$-valued differential form of globally odd degree. The following adjunction and commutation rules

$$(70) \qquad [\nabla^*, \omega^S] = [\nabla, \omega]^S \qquad \text{and} \qquad (\nabla^*)^2 = -(\nabla^2)^* = (\nabla^2)^S$$

have the following consequences

$$(71) \qquad (A^S)^2 = (A^2)^S \qquad \text{and} \qquad \mathrm{ch}(A^S) = \overline{\mathrm{ch}(A)}.$$

In particular, $\mathrm{ch}\big(\frac{1}{2}(A + A^S)\big)$ is a real form. Finally, Lemma 13 and formulae (28) and (31) are also valid in the context of superconnections.

5.2. Adaptation of Bismut's superconnection

5.2.1. *Definition of Bismut and Lott's Levi-Civita superconnection.* — Remember the definitions of P^{TZ} and T^HM from §3.4.1. Let $y \in B$. For any vector $\mathrm{u} \in T_yB$, its horizontal lift u^H is a global section of the restriction of T^HM to $Z_y = \pi^{-1}(y)$ such that at any point of Z_y one has $\pi_*\mathrm{u}^H = \mathrm{u}$.

Consider some vector bundle ξ on M with a connection ∇_ξ and hermitian metric h^ξ. ∇_ξ is not supposed to be flat nor to respect h^ξ. Remember the definition of $\mathcal{E}$ from (37). The flow associated to vector fields of the form u^H send fibres of π to fibres of π diffeomorphically, so that there is some fiberwise Lie differentiation operator $\mathcal{L}^{\nabla_\xi}_{\mathrm{u}^H}$ which acts on ξ-valued vertical differential forms $\mathcal{E}$ (it is defined using the connection ∇_ξ). Put then for any local section σ of $\mathcal{E}$ (see [**11**] Definition 3.2)

$$(72) \qquad \overline{\nabla}_{\mathrm{u}}\sigma = \mathcal{L}^{\nabla_\xi}_{\mathrm{u}^H}\sigma.$$

$\overline{\nabla}$ is a connection on $\mathcal{E}$ as can be proved following [**11**] (3.8) to (3.10).

If u and v are vector fields defined on a neighbourhood of $y \in B$, then the vector field $P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$ on $Z_y = \pi^{-1}(y)$ depends on the values of u and v at y only. Let $\iota_T \colon \wedge^2 TB \longrightarrow \mathrm{End}^{\mathrm{odd}}(\mathcal{E})$ be the operator which to u and $\mathrm{v} \in T_yB$ associates the interior product by $-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$ in $\wedge^\bullet T^*Z \otimes \xi$. ι_T can be extended to a globally odd $\mathrm{End}\mathcal{E}$-valued differential form (of differential form degree 2) on B.

$\overline{\nabla} + d^{\nabla_\xi} + \iota_T$ is a superconnection on $\mathcal{E}$ in the sense of §5.1.1 and also of [**37**], [**3**] Definitions 1.37 and 9.12 and [**6**]. It can be proved to coincide with the total exterior differential operator d^M on ξ-valued differential forms (defined using ∇_ξ) on M through the identification (37) as in [**3**] Proposition 10.1 (the proof of [**11**] §III (b) cannot be adapted here because $(d^M)^2 \neq 0$ if ∇_ξ is not flat).

Remember the definition of metric data g^Z, h^ξ, $(\ |\ )_Z$ and $\langle\ ,\ \rangle_{L^2}$ from §3.1.1 and (38). Define the adjoint connection $\overline{\nabla}^S$ of $\overline{\nabla}$ as in (22) by the following formula, valid for any element u of the tangent bundle of B and any local sections σ and θ of $\mathcal{E}$:

$$(73) \qquad \langle \overline{\nabla}^S_{\mathrm{u}}\sigma, \theta\rangle_{L^2} = \mathrm{u}.\langle\sigma, \theta\rangle_{L^2} - \langle\sigma, \overline{\nabla}_{\mathrm{u}}\theta\rangle_{L^2}.$$

Let $T\wedge \colon \wedge^2 TB \longrightarrow \mathrm{End}^{\mathrm{odd}}(\mathcal{E})$ be the operator which associates to u and $\mathrm{v} \in T_yB$ the exterior product in $\mathcal{E}_y$ by the one form $(-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H])^\flat$ (the dual through g^Z

to the vector field $-P^{TZ}[\mathrm{u}^H, \mathrm{v}^H]$) on $\pi^{-1}(y)$. Before and after being extended to a globally odd End$\mathcal{E}$-valued differential form on B (of differential form degree 2), $T\wedge$ is the adjoint of ι_T, so that $\iota_T - T\wedge$ is a special autoadjoint End$\mathcal{E}$-valued differential form in the sense of paragraph 5.1.

d^{∇_ξ} and its adjoint $(d^{\nabla_\xi})^*$ as defined in §3.1.1 are also mutually special adjoint as End$\mathcal{E}$-valued differential forms (with differential form degree 0). The superconnection $\overline{\nabla}^S + (d^{\nabla_\xi})^* - T\wedge$ is the adjoint of the superconnection $\overline{\nabla} + d^{\nabla_\xi} + \iota_T$ in the sense of [**11**] §I(b) and Proposition 3.7, (and (69) above).

The relevant Bismut-Levi-Civita superconnection in this context is defined for any $t > 0$ as in [**11**] (3.50) (and also (3.49), (3.30) and Proposition 3.4) by:

$$C_t = \frac{1}{2}(\overline{\nabla} + \overline{\nabla}^S) + \frac{\sqrt{t}}{2}(d^{\nabla_\xi} + (d^{\nabla_\xi})^*) + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge). \tag{74}$$

In the case of a fibered product of the form (43), the construction of C_t is functorial if the horizontal subspace $T^H(\widetilde{B} \times_B M)$ is taken to be the subspace of $T(\widetilde{B} \times_B M)$ consisting of vectors which are sent to T^HM by the tangent map of $\widetilde{B} \times_B M \longrightarrow M$. (It is not always isomorphic to the pullback of T^HM).

5.2.2. *Properties and asymptotics of the Chern character of C_t.* — C_t^2 is a fiberwise positive second order elliptic differential operator so that its heat kernel $\exp -C_t^2$ is trace class. The Chern character of C_t is defined to be

$$\mathrm{ch}(C_t) = \phi \mathrm{Tr}_s \exp -C_t^2.$$

Lemma 44. — *$\mathrm{ch}(C_t)$ is a real form. It is a constant integer if ∇_ξ is flat.*

Proof. — The superconnection C_t is for any t the half sum of $\overline{\nabla} + \sqrt{t}d^{\nabla_\xi} + \frac{1}{\sqrt{t}}\iota_T$ and its adjoint $\overline{\nabla}^S + \sqrt{t}(d^{\nabla_\xi})^* - \frac{1}{\sqrt{t}}T\wedge$. The reality of $\mathrm{ch}(C_t)$ follows from (71) and the comment after it. The case of flat connection ∇_ξ is treated in [**11**] Theorem 3.15. □

Remember the definition of the Euler form e and the connection ∇_{TZ} from paragraph 3.4.1, and put $\nabla_\xi^u = \frac{1}{2}(\nabla_\xi + \nabla_\xi^*)$ as in (31).

Proposition 45. — *As t tends to 0, $\mathrm{ch}(C_t)$ has for any $k \geq 1$ an asymptotic of the form*

$$\begin{cases} \mathrm{ch}(C_t) = \sum_{j=0}^{k-1} t^{j+\frac{1}{2}} A_j + \mathcal{O}(t^{k+\frac{1}{2}}) & \textit{if } \dim Z \textit{ is odd} \\ \mathrm{ch}(C_t) = \sum_{j=0}^{k-1} t^j B_j + \mathcal{O}(t^k) & \textit{if } \dim Z \textit{ is even} \end{cases}$$

in either case:

$$\lim_{t \to 0} \mathrm{ch}(C_t) = \int_Z e(\nabla_{TZ}) \wedge \mathrm{ch}(\nabla_\xi^u). \tag{75}$$

Proof. — The asymptotics with $\sum_{j=-\frac{1}{2}\dim Z}^{k-1}$ are classical results on heat kernels (see [**3**] §§2.5 and 2.6 and appendix after §9.7).

The limit formula (and thus the vanishing of the terms A_j and/or B_j for negative j) is a consequence of [**11**] (3.76). The connection ∇_ξ is supposed to be flat in [**11**], which is not the case here: thus formula [**11**] (3.52) does not hold true here. However, consider $\mathscr{R}$ defined as in [**11**] (3.56) without taking [**11**] (3.52) into account, then the $z = 0$ case of the Lichnerowicz-type formula of [**11**] Theorem 3.11 holds true here. Thus the rescaling formula [**11**] (3.75) and its consequence [**11**] (3.76) remain true here. (This is only a matter of Clifford degrees which has nothing to do with the fact that ∇_ξ be flat or not).

In particular, if $\dim Z$ is odd, then the same argument as in [**11**] (3.79) applies, and both sides of the equality (75) vanish. □

5.2.3. *Calculating C_t for the product with the real line.* — Consider now the product manifold $\widetilde{M} = \mathbb{R} \times M$ and its obvious submersion $\widetilde{\pi} = \mathrm{Id}_{\mathbb{R}} \times \pi$ onto $\widetilde{B} = \mathbb{R} \times B$. Extend ξ tautologically to $\widetilde{M}$ with constant (with respect to s) hermitian metric and connection $d_{\mathbb{R}} + \nabla_\xi$ (where $d_{\mathbb{R}} = ds\frac{\partial}{\partial s}$ is the trivial canonical differential along $\mathbb{R}$). Consider any smooth real positive function f on $\mathbb{R}$ such that $f(1) = 1$ and endow the vertical tangent bundle of $\widetilde{\pi}$ with the metric $\frac{1}{f(s)}g^Z$. Choose $T^H\widetilde{M} = T\mathbb{R} \oplus T^HM$ as horizontal bundle of $\widetilde{\pi}$. Let's calculate the Bismut-Lott Levi-Civita superconnection $\widetilde{C}_t$ in this context.

The equivalent here of the connection $\overline{\nabla}$ defined in (72) is simply equal to $d_{\mathbb{R}} + \overline{\nabla}$. The vertical exterior differential operator d^{∇_ξ} is unchanged, and so is the operator ι_T (defined at the beginning of §5.2).

The volume form of the fibres of $\widetilde{\pi}$ on $\{s\} \times B$ is equal to $f(s)^{-\frac{\dim Z}{2}}$ times the corresponding volume form of the fibres on $\{1\} \times B$. The punctual scalar product between vertical differential forms of degree k on $\{s\} \times B$ is equal to the one on $\{1\} \times B$ multiplied by $f(s)^k$. Call $\widetilde{\mathscr{E}}$ the infinite rank vector bundle on $\widetilde{B}$ of ξ-valued vertical differential forms, and define $N_V \in \mathrm{End}\mathscr{E}$ or $\mathrm{End}\widetilde{\mathscr{E}}$ to be the operator which multiplies vertical differential forms by their degree. The global L^2 scalar product on the restriction of $\widetilde{\mathscr{E}}$ to $\{s\} \times B$ is thus equal to $f(s)^{N_V - \frac{\dim Z}{2}} \langle\ ,\ \rangle_{L^2}$ (where $\langle\ ,\ \rangle_{L^2}$ defined in (38) is the one on $\{1\} \times B$).

It follows that the adjoint of d^{∇_ξ} is $f(s)(d^{\nabla_\xi})^*$ (if $(d^{\nabla_\xi})^*$ is its adjoint on $\{1\} \times B$) and the adjoint of ι_T is $\frac{1}{f(s)}T\wedge$ (if $T\wedge$ is its adjoint on $\{1\} \times B$). In the same way, following (22), one has $(d_{\mathbb{R}} + \overline{\nabla})^S = d_{\mathbb{R}} + ds\frac{f'(s)}{f(s)}(N_V - \frac{\dim Z}{2}) + \overline{\nabla}^S$.

Thus if $C_{t,s}$ denotes the Bismut-Lott Levi-Civita superconnection on $\{s\} \times B$:

$$\begin{aligned} C_{t,s} &= \tfrac{1}{2}\left(\overline{\nabla} + \overline{\nabla}^S\right) + \tfrac{\sqrt{t}}{2}\left(d^{\nabla_\xi} + f(s)(d^{\nabla_\xi})^*\right) + \tfrac{1}{2\sqrt{t}}\left(\iota_T - \tfrac{1}{f(s)}T\wedge\right), \\ \widetilde{C}_t &= C_{t,s} + d_{\mathbb{R}} + \tfrac{1}{2}ds\tfrac{f'(s)}{f(s)}\left(N_V - \tfrac{\dim Z}{2}\right). \end{aligned}$$

One then computes:

$$[d_{\mathbb{R}}, C_{t,s}] = \frac{\sqrt{t}}{2} f'(s) ds (d^{\nabla_\xi})^* + \frac{f'(s)}{2\sqrt{t} f(s)^2} ds\, T\wedge$$

$$[N_V, C_{t,s}] = \frac{\sqrt{t}}{2}(d^{\nabla_\xi} - f(s)(d^{\nabla_\xi})^*) + \frac{1}{2\sqrt{t}}\left(-\iota_T - \frac{1}{f(s)} T\wedge\right)$$

$$\begin{aligned}\left[d_{\mathbb{R}} + \frac{1}{2} ds \frac{f'(s)}{f(s)} N_V\,,\, C_{t,s}\right] &= \frac{\sqrt{t}}{4} ds \frac{f'(s)}{f(s)} (d^{\nabla_\xi} + f(s)(d^{\nabla_\xi})^*) + \\ &\quad + ds \frac{f'(s)}{4\sqrt{t} f(s)} \left(-\iota_T + \frac{1}{f(s)} T\wedge\right) \\ &= t\, ds \frac{f'(s)}{f(s)} \frac{\partial C_{t,s}}{\partial t}\end{aligned}$$

$$\begin{aligned}\widetilde{C}_t^2 &= C_{t,s}^2 + \left[d_{\mathbb{R}} + \frac{1}{2} ds \frac{f'(s)}{f(s)} \left(N_V - \frac{\dim Z}{2}\right)\,,\, C_{t,s}\right] \\ &= C_{t,s}^2 + t\, ds \frac{f'(s)}{f(s)} \frac{\partial C_{t,s}}{\partial t}\end{aligned}$$

$$\mathrm{Tr}_s \exp(-\widetilde{C}_t^2) = \mathrm{Tr}_s \exp(-C_{t,s}^2) - t\, ds \frac{f'(s)}{f(s)} \mathrm{Tr}_s \left(\frac{\partial C_{t,s}}{\partial t} \exp(-C_{t,s}^2)\right). \tag{76}$$

5.2.4. *$t \longrightarrow 0$ asymptotics of the infinitesimal transgression form.* — The transgression Formula (7) yields here

$$\frac{d}{dt} \mathrm{ch}(C_t) = -d\left[\phi \mathrm{Tr}_s \left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right)\right]$$

so that for any $0 < S < T < +\infty$

$$\mathrm{ch}(C_S) - \mathrm{ch}(C_T) = d\left[\int_S^T \phi \mathrm{Tr}_s \left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) dt\right]. \tag{77}$$

Proposition 46. — *One has the following estimate*

$$\text{as} \quad t \to 0: \qquad \phi \mathrm{Tr}_s \left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) = \begin{cases} \mathcal{O}(1) & \textit{if } \dim Z \textit{ is even,} \\ \mathcal{O}(t^{-\frac{1}{2}}) & \textit{if } \dim Z \textit{ is odd.}\end{cases} \tag{78}$$

Proof. — This will be proved with the technique proposed in [**3**] Theorem 10.32: apply Proposition 45 on $\widetilde{M}$, one obtains because of the factor t appearing in (76) an asymptotic of the form

$$\phi \mathrm{Tr}_s \left(\frac{\partial C_t}{\partial t} \exp -C_t^2\right) = \begin{cases} \displaystyle\sum_{j=-1}^{k-1} E_j t^j + \mathcal{O}(t^k) & \text{if } \dim Z \text{ is even,} \\ \displaystyle\sum_{j=0}^{k-1} E_j t^{j-\frac{1}{2}} + \mathcal{O}(t^{k-\frac{1}{2}}) & \text{if } \dim Z \text{ is odd.}\end{cases}$$

This proves the assertion for odd dimensional fibres.

Let $\widetilde{\nabla}_{TZ}$ be the Levi-Civita connection on the vertical tangent bundle of the submersion $\widetilde{\pi}$ over $\widetilde{M}$ (as defined at §3.4.1). If $\dim Z$ is even, let $\int_Z$ denote the integral along the fibres of $\widetilde{\pi}$, then E_{-1} is the factor of ds in the decomposition of the form $\int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\nabla_\xi^u)$ with respect to $\Omega(\widetilde{B},\mathbb{C}) = C^\infty(\mathbb{R},\Omega(B,\mathbb{C})) \oplus ds \wedge C^\infty(\mathbb{R},\Omega(B,\mathbb{C}))$. This is because the Chern character is functorial by pullbacks. However, $\widetilde{\nabla}_{TZ}$ is not the pullback of ∇_{TZ}. A direct calculation from the classical formula for Levi-Civita connections (see [**3**] formula (1.18)) yields

$$\widetilde{\nabla}_{TZ} = d_{\mathbb{R}} + \nabla_{TZ} + \frac{f'(s)}{2f(s)}ds$$

so that $\widetilde{\nabla}_{TZ}^2 = \nabla_{TZ}^2$ because $d_{\mathbb{R}}$ and ds both commute with ∇_{TZ}. Thus the curvature of $\widetilde{\nabla}_{TZ}$ is the pullback of the one of ∇_{TZ} and neither $e(\widetilde{\nabla}_{TZ})$ nor $\mathrm{ch}(\nabla_\xi^u)$ have a ds component. This proves the vanishing of E_{-1}. □

5.2.5. *Adapting C_t to some suitable triple.* — Let χ be a smooth real increasing function on $\mathbb{R}_+$ which vanishes on $[0,\frac{1}{2}]$ and equals 1 on $[1,+\infty)$. Consider some suitable triple (μ^+,μ^-,ψ) with respect to ξ, h^ξ, ∇_ξ and g^Z in the sense of Definition 17. Put some hermitian metrics $h^\pm$ on $\mu^\pm$ and some connection ∇_μ on $\mu^+\oplus\mu^-$ which respects the decomposition. Denote $\frac{1}{2}(\overline{\nabla}+\overline{\nabla}^*)$ by $\overline{\nabla}^u$. Consider the following t-depending superconnection on $(\mathscr{E}^+\oplus\mu^+)\oplus(\mathscr{E}^-\oplus\mu^-)$:

$$B_t = \overline{\nabla}^u \oplus \nabla_\mu + \frac{\sqrt{t}}{2}\mathscr{D}^{\nabla_\xi}_{\chi(t)\psi} + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge) = C_t \oplus \nabla_\mu + \frac{\sqrt{t}}{2}\chi(t)\,(\psi+\psi^*)\,. \tag{79}$$

B_t^2 is as C_t^2 a fiberwise positive second order elliptic operator, so that its heat kernel is trace class. Its Chern character is defined as is $\mathrm{ch}(C_t)$, the supertrace being the trace on $\mathrm{End}(\mathscr{E}^+\oplus\mu^+)$ minus the trace on $\mathrm{End}(\mathscr{E}^-\oplus\mu^-)$.

Lemma 47. — *$\mathrm{ch}(B_t)$ is real if ∇_μ respects h^+ and h^-. For $t\leq\frac{1}{2}$, one has*

$$\mathrm{ch}(B_t) = \mathrm{ch}(C_t) + \mathrm{ch}(\nabla_\mu). \tag{80}$$

Proof. — The equality is obvious. ψ is of differential form degree 0 so that ψ^* is the special adjoint of ψ. The reality follows from (71) (as does Lemma 44). □

Call $\mathscr{H}^\pm = \mathrm{Ker}\mathscr{D}_\psi^{\nabla_\xi\pm}$ and $P^{\mathscr{H}^\pm}$ the orthogonal projection $\mathscr{E}^\pm\oplus\mu^\pm \longrightarrow \mathscr{H}^\pm$, (and $P^{\mathscr{H}} = P^{\mathscr{H}^+}\oplus P^{\mathscr{H}^-}$). The associated connection on $\mathscr{H}=\mathscr{H}^+\oplus\mathscr{H}^-$ is

$$\nabla_{\mathscr{H}} = P^{\mathscr{H}}\left(\overline{\nabla}^u\oplus\nabla_\mu\right)P^{\mathscr{H}}. \tag{81}$$

This connection respects the decomposition $\mathscr{H}^+\oplus\mathscr{H}^-$, and it also respects the hermitian metric on $\mathscr{H}$ obtained by restriction of $\langle\ ,\ \rangle_{L^2}\oplus h^\pm$ provided ∇_μ respects $h^\pm$ (this can be proved by a direct elementary computation).

It is proved in [**3**] Theorem 9.26 that:

$$\lim_{t\to+\infty} \mathrm{ch}(B_t) = \mathrm{ch}(\nabla_{\mathcal{H}}) \tag{82}$$

in the sense of any $\mathscr{C}^\ell$ norm on any compact subset of B.

Both B_t and its Chern character are functorial by pullbacks on fibered products as in (43) (if the horizontal subspace of the source manifold is taken as described at the end of §5.2). Note also that the construction can be performed with any smooth function χ on $B \times \mathbb{R}_+$ which vanishes on $B \times [0,\varepsilon]$ and equals 1 on $B \times [A,+\infty)$ for any $0<\varepsilon<A$, and which is increasing with respect to the variable in $\mathbb{R}_+$. This is of course not essential, but will be useful to prove some independence of the constructed forms on the choice of the function χ.

5.2.6. $t \longrightarrow +\infty$ *asymptotics of the infinitesimal transgression form.* — For any $0<S<T<+\infty$, the counterpart of (77) for B_t is here

$$\mathrm{ch}(B_S) - \mathrm{ch}(B_T) = d\left[\int_S^T \phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)dt\right]. \tag{83}$$

Lemma 48. — *$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)$ is a real form if ∇_μ respects $h^\pm$ (the hermitian metrics on $\mu^\pm$). If not, this form is changed into its complex conjugate if ∇_μ is changed into its adjoint connection with respect to $h^\pm$.*

If ∇_ξ is flat and if the suitable triple used in the construction of B_t is the trivial one $(\{0\},\{0\},0)$, *then:*

$$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right) = \phi\mathrm{Tr}_s\left(\frac{\partial C_t}{\partial t}\exp -C_t^2\right) = 0.$$

Proof. — The second assertion is proved in [**32**]. It is reproved here as a direct consequence of (76), of the last assertion of Lemma 44 (and the fact that if ∇_ξ is flat on ξ over M, then $d_{\mathbb{R}} + \nabla_\xi$ is also flat on the pullback of ξ over $\widetilde{M}$).

In general, $\exp -B_t^2$ is a globally even $\mathrm{End}\mathscr{E}$-valued differential form, so that its supercommutator with $\frac{\partial B_t}{\partial t}$ is their usual commutator; and it is special autoadjoint if ∇_μ respects $h^\pm$ on $\mu^\pm$ (if not, the two forms obtained from mutually adjoint connections on μ are mutually special adjoint).

On the other hand, $\frac{\partial B_t}{\partial t}$ is for any t a special autoadjoint $\mathrm{End}\mathscr{E}$-valued differential form, so that the product $\frac{\partial B_t}{\partial t}\exp -B_t^2$ is the special adjoint of $(\exp -B_t^2)\frac{\partial B_t}{\partial t}$ (if ∇_μ respect $h^\pm$). Thus

$$\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right) = \phi\mathrm{Tr}_s\left((\exp -B_t^2)\frac{\partial B_t}{\partial t}\right) = \overline{\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right)}$$

and the reality follows (the case when ∇_μ does not respect $h^\pm$ is similar). □

Proposition 49. — *One has the following estimate:*

$$\text{as} \quad t \to +\infty : \qquad \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) = \mathcal{O}(t^{-\frac{3}{2}}).$$

Proof. — The $t \to +\infty$ asymptotic is proved by the adaptation of [**3**] Theorem 9.23 which is proposed (though not detailed) at the end of §9.3 of [**3**]. (Here $\chi(t)$ is constant on a neighbourhood of $+\infty$, so that the arguments of the proof of Theorems 9.7 and 9.23 of [**3**] apply). □

This estimate together with formulae (80) and (78) prove the convergence of the integral $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt$. It follows from (82), (83), (80), and Proposition 45 that this integral is a transgression form in the following sense:

$$(84) \quad d \left[\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt \right] = \int_Z e(\nabla_{TZ}) \wedge \mathrm{ch}(\nabla_\xi^u) + \mathrm{ch}(\nabla_\mu) - \mathrm{ch}(\nabla_{\mathcal{H}})$$

(where $\mathrm{ch}(\nabla_\mu) = \mathrm{ch}(\nabla_{\mu^+}) - \mathrm{ch}(\nabla_{\mu^-})$ and accordingly for $\mathrm{ch}(\nabla_{\mathcal{H}})$). The preceding considerations about functoriality apply here, so that this transgression form is functorial by pullbacks on fibered products as in (43) (if the horizontal subspace of the source manifold is taken as described at the end of §5.2).

5.3. Proof of the first part of Theorem 28

5.3.1. *Chern-Simons transgression and links.* — Let E, F, G and H be vector bundles on M with connections ∇_E, ∇_F, ∇_G and ∇_H. Suppose there exists some link (K, ℓ) between $E - F$ and $G - H$ as in (47). One associates to (K, ℓ) the differential form (defined modulo exact forms)

$$(85) \qquad \widetilde{\mathrm{ch}}([\ell]) = \widetilde{\mathrm{ch}}\big(\nabla_E \oplus \nabla_H \oplus \nabla_K, \ell^*[\nabla_F \oplus \nabla_G \oplus \nabla_K]\big)$$

for some connection ∇_K on K. It is easily checked from (9) and (10) that the class of this form modulo exact forms does not depend on the choice of ∇_K and is not modified by changing (K, ℓ) by an equivalent link. It is possible to choose a unitary ∇_K, so that $\widetilde{\mathrm{ch}}([\ell])$ is a real form (modulo exact forms) if it happens that ∇_E, ∇_F, ∇_G and ∇_H are all unitary connections. And of course

$$(86) \qquad d\widetilde{\mathrm{ch}}([\ell]) = \mathrm{ch}(\nabla_E) + \mathrm{ch}(\nabla_H) - \mathrm{ch}(\nabla_F) - \mathrm{ch}(\nabla_G).$$

For the composition of two links ℓ and ℓ', and any connections on the considered bundles one obtains (modulo exact forms and always from (9) and (10)):

$$(87) \qquad \widetilde{\mathrm{ch}}([\ell' \circ \ell]) = \widetilde{\mathrm{ch}}([\ell]) + \widetilde{\mathrm{ch}}([\ell']).$$

5.3.2. *Definition of the η-form and check of its properties.* — Consider now some vector bundle ξ with connection ∇_ξ and hermitian metric h^ξ on M, some horizontal tangent vector space $T^H M$ and vertical metric g^Z for the submersion $\pi\colon M \to B$, and vector bundles F^+ and F^- on B such that

$$[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K^0_{\mathrm{top}}(B).$$

Put any connections ∇_{F^+} on F^+ and ∇_{F^-} on F^- and choose some equivalence class of links $[\ell]$ between $F^+ - F^-$ and some family index bundles $(\mathcal{H}^+ \oplus \mu^-) - (\mathcal{H}^- \oplus \mu^+)$ provided by any suitable triple (μ^+, μ^-, ψ) (with connections $\nabla_{\mathcal{H}}$ and ∇_μ, $\mathcal{H}^\pm$ being the kernel bundles).

Definition 50. — *The families Chern-Simons transgression form is the (inductive limit of the) class modulo exact forms of the following differential form on (compact subsets of) B:*

$$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt +$$
$$+ \int_Z e(\nabla_{TZ}) \wedge \widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi) + \widetilde{\mathrm{ch}}([\ell])$$

where $\widetilde{\mathrm{ch}}([\ell])$ is computed with the connections ∇_μ, $\nabla_{\mathcal{H}}$ and $\nabla_{F^\pm}$.

If B is noncompact, the above construction produces some projective collection of elements of $\Omega^{\mathrm{odd}}(K, \mathbb{C})/d\Omega^{\mathrm{even}}(K, \mathbb{C})$ on compact submanifolds (with boundary and of the same dimension as B. In fact, this will be fully established in Proposition 51 below. The properties checked just hereafter are local and will also be valid for a noncompact B). This gives rise to an unambiguous object in $\Omega^{\mathrm{odd}}(B, \mathbb{C})/d\Omega^{\mathrm{even}}(B, \mathbb{C})$ (which can be constructed by an analogue procedure to the one which was sketched just before Definition 40).

It follows from (84), (8) and (86) that the form $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ verifies the transgression formula stated as property (a) in Theorem 28.

The $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2 \right) dt$ part of η is functorial by pullback on fibered products as in (43) as was remarked at the end of subSection 5.2.6 just after the proof of Proposition 49. The $\widetilde{\mathrm{ch}}$ are both functorial, as was remarked just before Equation (9), and $e(\nabla_{TZ})$ too, under the assumption on horizontal subspaces of the end of §5.2. This proves the naturality property (b) for η.

$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ is additive in the following sense: let ξ_1 and ξ_2 be bundles on M with connections ∇_{ξ_1} and ∇_{ξ_2}, let F_1^+, F_1^-, F_2^+ and F_2^- be bundles with connections on B such that $[F_1^+] - [F_1^-] = \pi_*^{\mathrm{Eu}}[\xi_1]$ and $[F_2^+] - [F_2^-] = \pi_*^{\mathrm{Eu}}[\xi_2]$ in $K^0_{\mathrm{top}}(B)$. Let $[\ell_1]$ be some link between $F_1^+ - F_1^-$ and some (couple of) family index bundles for ξ_1 on B, and correspondingly for $[\ell_2]$. The additivity (for direct sums) of the topological direct image construction ensures that $\ell_1 \oplus \ell_2$ provides an equivalence class of link between $(F_1^+ \oplus F_2^+) - (F_1^- \oplus F_2^-)$ and bundles on B which form a couple of family index bundles for $\xi_1 \oplus \xi_2$. Then

$$\eta(\nabla_{\xi_1} \oplus \nabla_{\xi_2}, \nabla_{TZ}, \nabla_{F_1^+} \oplus \nabla_{F_2^+}, \nabla_{F_1^-} \oplus \nabla_{F_2^-}, [\ell_1 \oplus \ell_2]) = \\ = \eta(\nabla_{\xi_1}, \nabla_{TZ}, \nabla_{F_1^+}, \nabla_{F_1^-}, [\ell_1]) + \eta(\nabla_{\xi_2}, \nabla_{TZ}, \nabla_{F_2^+}, \nabla_{F_2^-}, [\ell_2]).$$

This additivity is a direct consequence of the fact that the Chern character and the supertrace entering the construction of $\int_0^{+\infty} \phi \mathrm{Tr}_s \left(\frac{\partial B_t}{\partial t} \exp -B_t^2\right) dt$ are additive for direct sums, and accordingly for Chern-Simons transgressions (10). Property (c) of Theorem 28 is thus established for η.

The vanishing of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{\pi_!^+\xi}, \nabla_{\pi_!^-\xi}, [\mathrm{Id}])$ for any flat bundle (ξ, ∇_ξ) is a consequence of the first statement of Lemma 48 and of [**11**] Proposition 3.14 and Theorem 3.17: Lemma 48 proves that the integrand of the first term in the definition of η vanishes for all $t > 0$ (if it is computed using the trivial suitable triple $(\{0\}, \{0\}, 0)$). In particular, the link $[\mathrm{Id}]$ in the third term $\widetilde{\mathrm{ch}}([\mathrm{Id}])$ is trivial as link, but it links $\pi_!^+\xi - \pi_!^-\xi$ endowed with their sheaf theoretic direct image flat connections $\nabla_{\pi_!^+\xi}$ and $\nabla_{\pi_!^-\xi}$, with $\pi_!^+\xi - \pi_!^-\xi$ endowed with their metric connections $\nabla_{\mathcal{H}^+}$ and $\nabla_{\mathcal{H}^-}$ obtained by the projection on the kernel of the fiberwise Dirac operator (82).

It is proved in [**11**] Proposition 3.14 that $\nabla_{\mathcal{H}^+} = \nabla^u_{\pi_!^+\xi}$ and accordingly on F^-, and in [**11**] Theorem 3.17 (see also Remark 14 above) that, up to exact forms

$$\widetilde{\mathrm{ch}}(\nabla_{\pi_!^+\xi}, \nabla^u_{\pi_!^+\xi}) - \widetilde{\mathrm{ch}}(\nabla_{\pi_!^-\xi}, \nabla^u_{\pi_!^+\xi}) = \int_Z e(\nabla_{TZ})\widetilde{\mathrm{ch}}(\nabla_\xi, \nabla^u_\xi).$$

Thus the two last terms in the definition of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{\pi_!^+\xi}, \nabla_{\pi_!^-\xi}, [\mathrm{Id}])$ mutually compensate, and the property (d) of Theorem 28 is established for η.

5.3.3. *Invariance properties of η.* — The proof of the first part of Theorem 28 is thus reduced to the following

Proposition 51. — *$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ does not depend on h^ξ, nor on the function χ nor on the construction of topological direct image and the choice of data used in it, provided the class of link $[\ell]$ is modified by composition with the canonical link between the obtained representatives of the topological direct image when they are changed.*

$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ of course depends on the other data in a way which will be precised later in §5.4.1.

Proof. — This will be proved in two steps.

First step: independence on h^ξ, χ, and on deformation of ψ. — Consider the submersion $\pi \times \mathrm{Id}_{[0,1]} : M \times [0,1] \longrightarrow B \times [0,1]$. The vertical tangent space of $\pi \times \mathrm{Id}_{[0,1]}$ is simply the pullback to $M \times [0,1]$ of the one of π, and it will be supposed to be endowed with a pullback metric. Choose some horizontal subspace T^HM for π and pull it back on $M \times [0,1]$, where it is a suitable horizontal subspace with respect to $\pi \times \mathrm{Id}_{[0,1]}$. These choices of horizontal subspaces verify the conditions of the end of §5.2 with respect to the maps $B \times \{0\} \hookrightarrow B \times [0,1]$ and $B \times \{1\} \hookrightarrow B \times [0,1]$. Call $\widetilde{\nabla}_{TZ}$ the associated pullback connection on the vertical tangent bundle of $\pi \times \mathrm{Id}_{[0,1]}$.

Consider some vector bundle ξ on M, with connection ∇_ξ, and any pair of bundles F^+ and F^- on B with connections ∇_{F^+} and ∇_{F^-} such that $[F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi]$ in $K^0_{\mathrm{top}}(B)$, and some equivalence class of link $[\ell]$ between $F^+ - F^-$ and some couple of family index bundles for ξ. Pull back ξ on $M \times [0,1]$ and F^+ and F^- on $B \times [0,1]$ and call $\widetilde{\xi}$, $\widetilde{F}^+$ and $\widetilde{F}^-$ the pullbacks. Call $\widetilde{\nabla}_\xi$, $\widetilde{\nabla}_{F^+}$ and $\widetilde{\nabla}_{F^-}$ the pullback connections on them. Endow $\widetilde{\xi}$ with some not necessarily pullback hermitian metric $\widetilde{h}^\xi$ and choose any suitable data $(\widetilde{\mu}^+, \widetilde{\mu}^-, \widetilde{\psi})$ with respect to $\pi \times \mathrm{Id}_{[0,1]}$ providing kernel bundles $\widetilde{\mathscr{H}}^\pm = \mathrm{Ker}\,\mathscr{D}_{\widetilde{\psi}}^{\nabla_\xi \pm}$ on $B \times [0,1]$. Of course one has

$$[\widetilde{F}^+] - [\widetilde{F}^-] = [\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-] - [\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+] = (\pi \times \mathrm{Id}_{[0,1]})_*^{\mathrm{Eu}}[\widetilde{\xi}] \in K^0_{\mathrm{top}}(B \times [0,1]).$$

$[\ell]$ naturally provides an equivalence class of link between $F^+ - F^-$ and the restrictions to $B \times \{0\}$ of $(\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-) - (\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+)$, which can be extended (by parallel transport along $[0,1]$) to an equivalence class of link $[\widetilde{\ell}]$ on the whole $B \times [0,1]$ between $\widetilde{F}^+ - \widetilde{F}^-$ and $(\widetilde{\mathscr{H}}^+ \oplus \widetilde{\mu}^-) - (\widetilde{\mathscr{H}}^- \oplus \widetilde{\mu}^+)$.

Construct the differential form $\widetilde{\eta} = \eta(\widetilde{\nabla}_\xi, \widetilde{\nabla}_{TZ}, \widetilde{\nabla}_{F^+}, \widetilde{\nabla}_{F^-}, [\widetilde{\ell}])$ in the same way as in Definition 50 with respect to all these data on $M \times [0,1]$. This must be made using a smooth function $\widetilde{\chi}$ on $B \times [0,1] \times \mathbb{R}_+$ vanishing on $B \times [0,1] \times [0,\varepsilon]$, equal to 1 on $B \times [0,1] \times [A, +\infty)$ and increasing with respect to the variable in $\mathbb{R}^+$ as was sketched at the end of §5.2.5. The obtained form $\widetilde{\eta}$ verifies (a):

$$d\widetilde{\eta} = \int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\widetilde{\nabla}_\xi) - \mathrm{ch}(\widetilde{\nabla}_{F^+}) + \mathrm{ch}(\widetilde{\nabla}_{F^-})$$

where $\int_Z$ stands for integration along the fibres of $\pi \times \mathrm{Id}_{[0,1]}$. Call η_0 and η_1 the restrictions of $\widetilde{\eta}$ to $B \times \{0\}$ and $B \times \{1\}$ respectively. Integrating this formula along $[0,1]$ provides that the following differential form on B is exact:

$$(88)\quad d\Big(\int_{[0,1]} \widetilde{\eta}\Big) = \eta_1 - \eta_0 + \int_{[0,1]}\int_Z e(\widetilde{\nabla}_{TZ})\mathrm{ch}(\widetilde{\nabla}_\xi) - \int_{[0,1]} \mathrm{ch}(\widetilde{\nabla}_{F^+}) + \int_{[0,1]} \mathrm{ch}(\widetilde{\nabla}_{F^-})$$

but $\widetilde{\nabla}_{TZ}$ and $\widetilde{\nabla}_\xi$ are pullback connections on $M \times [0,1]$ for the projection on the second factor $M \times [0,1] \longrightarrow M$ and accordingly for $\widetilde{\nabla}_{F^+}$ and $\widetilde{\nabla}_{F^-}$ on $B \times [0,1]$, so that their Chern characters or Euler form are pullback forms, and their integral along $[0,1]$ vanish. It follows that η_0 and η_1 are equal modulo exact forms.

Now η_0 and η_1 are both regular definitions of $\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ as in Definition 50, because the class of link between $F^+ - F^-$ and the restrictions to $B \times \{1\}$

of $(\widetilde{\mathcal{H}}^{+}\oplus\widetilde{\mu}^{-})-(\widetilde{\mathcal{H}}^{-}\oplus\widetilde{\mu}^{+})$ is in the equivalence class of $[\ell]$ (it can be deformed along $[0,1]$ to the one between the restrictions on $B\times\{0\}$) (and because of the functoriality property of η). This proves the independence of the class of η modulo exact forms on h^{ξ} and χ, and also that a deformation of the suitable triple does not modify the class of η modulo exact forms.

Second step: general independence on the suitable triple used. — First remark that if (μ^{+},μ^{-},ψ) is a suitable triple, then $(\mu^{+}\oplus\zeta^{+},\mu^{-}\oplus\zeta^{-},\psi)$ is also suitable (ζ^{+} and ζ^{-} are inert excess vector bundles) and gives rise to the same η. The same is true for $(\mu^{+}\oplus\zeta,\mu^{-}\oplus\zeta,\psi\oplus\mathrm{Id}_{\zeta})$ because the extra term due to Id_{ζ} appearing in $\phi\mathrm{Tr}_{s}\left(\frac{\partial B_{t}}{\partial t}\exp -B_{t}^{2}\right)$ is the supertrace on $\zeta\oplus\zeta$ of some $\mathrm{End}(\zeta\oplus\zeta)$-valued differential form whose diagonal terms are equal.

For some suitable triple (μ^{+},μ^{-},ψ), giving rise to kernel bundles $\mathcal{H}^{\pm}$, one associates to it some positive kernel triple $(\mu^{+}\oplus\lambda,\mu^{-},\psi+\varphi)$ as just before Definition 37. One puts on $B\times[0,1]$ the bundles $\widetilde{\mu}^{+}=\mu^{+}\oplus\lambda\oplus\mathcal{H}^{-}$, $\widetilde{\mu}^{-}=\mu^{-}$ and $\widetilde{\psi}=\psi+\cos(\frac{\pi}{2}t)\varphi+\sin(\frac{\pi}{2}t)\iota_{\mathcal{H}^{-}}$ where $\iota_{\mathcal{H}^{-}}$ is the obvious embedding of $\mathcal{H}^{-}$ into $\mathcal{E}^{-}\oplus\mu^{-}$. The obtained triple $(\widetilde{\mu}^{+},\widetilde{\mu}^{-},\widetilde{\psi})$ is a positive kernel triple with respect to $\pi\times\mathrm{Id}_{[0,1]}$. Its kernel bundle restricts to $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi+\varphi})\oplus\mathcal{H}^{-}$ on $M\times\{0\}$ and $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi})\oplus\lambda$ on $M\times\{1\}$. Thus applying the above considerations to this case, proves that η_{1} constructed using $(\mu^{+}\oplus\mathcal{H}^{-},\mu^{-},\psi+\iota_{\mathcal{H}}^{-})$ (corresponding to $M\times\{1\}$ with an inert copy of λ added to $\mu^{+}\oplus\mathcal{H}^{-}$) and η_{0} constructed using $(\mu^{+}\oplus\lambda,\mu^{-},\psi+\varphi)$ (corresponding to $M\times\{0\}$ with an inert copy of $\mathcal{H}^{-}$ added to $\mu^{+}\oplus\lambda$) differ from an exact form; the parallel transport along $[0,1]$ from $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi+\varphi})\oplus\mathcal{H}^{-}$ to $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi})\oplus\lambda$ (following $\mathrm{Ker}(\mathcal{D}^{\nabla_{\xi}+}_{\widetilde{\psi}}|_{M\times\{t\}})$) is easily checked to lie in the equivalence class of the link between $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi})-\mathcal{H}^{-}$ and $(\mathrm{Ker}\mathcal{D}^{\nabla_{\xi}+}_{\psi+\varphi})-\lambda$ obtained from (58) and Definition 36. The lemma is thus proved in full generality. □

5.4. Anomaly formulae and their consequences

5.4.1. *Anomaly formulae.* — The Chern-Simons theory (7) also applies for the Euler class: for any real vector bundle $F_{\mathbb{R}}$ on M, consider $p_{1}\colon M\times[0,1]\to M$ (the projection on the first factor) and the bundle $\widetilde{F_{\mathbb{R}}}=p_{1}^{*}F_{\mathbb{R}}$ on $M\times[0,1]$, choose any euclidean metric and unitary connection $\widetilde{\nabla}_{F_{\mathbb{R}}}$ on $\widetilde{F_{\mathbb{R}}}$, denote by $\nabla_{F_{\mathbb{R}},t}=\widetilde{\nabla}_{F_{\mathbb{R}}}|_{M\times\{t\}}$ the restrictions of $\widetilde{\nabla}_{F_{\mathbb{R}}}$ to $M\times\{t\}$ for all $t\in[0,1]$, and define

$$\widetilde{e}(\nabla_{F_{\mathbb{R}},0},\nabla_{F_{\mathbb{R}},1})=\int_{[0,1]}e(\widetilde{\nabla}_{F_{\mathbb{R}}}).\tag{89}$$

The class of $\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1})$ in $\Omega(M,\mathbb{C})/d\Omega(M,\mathbb{C})$ only depends on the limiting connections $\nabla_{F_{\mathbb{R}},0}$ and $\nabla_{F_{\mathbb{R}},1}$ and $\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1})$ verifies the following transgression formula

$$d\widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) = e(\nabla_{F_{\mathbb{R}},1}) - e(\nabla_{F_{\mathbb{R}},0}).$$

It is also functorial by pull-backs, and locally gauge invariant, and verifies a similar cocycle property (9) as does $\widetilde{\mathrm{ch}}$. Moreover, making the product of $e(\widetilde{\nabla}_{F_{\mathbb{R}}})$ and $\mathrm{ch}(\widetilde{\nabla}_E)$ yields the following equality modulo exact forms:

(90)
$$\begin{aligned}\int_{[0,1]} e(\widetilde{\nabla}_{F_{\mathbb{R}}}) \wedge \mathrm{ch}(\widetilde{\nabla}_E) &= \widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) \wedge \mathrm{ch}(\nabla_{E,0}) + e(\nabla_{F_{\mathbb{R}},1}) \wedge \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) \\ &= e(\nabla_{F_{\mathbb{R}},0}) \wedge \widetilde{\mathrm{ch}}(\nabla_{E,0}, \nabla_{E,1}) + \widetilde{e}(\nabla_{F_{\mathbb{R}},0}, \nabla_{F_{\mathbb{R}},1}) \wedge \mathrm{ch}(\nabla_{E,1}).\end{aligned}$$

Take now the same model as in the first step of the proof of Proposition 51, but with not necessarily pullback connections $\widetilde{\nabla}_\xi$ nor fiberwise riemannian metric $\widetilde{g}^Z$ nor horizontal space $\widetilde{T^H M}$. The obtained connection $\widetilde{\nabla}_{TZ}$ is of course not a pullback connection. Denote by ∇^0_ξ and ∇^0_{TZ} the connections on ξ and on TZ corresponding to data on $M \times \{0\}$ and by ∇^1_ξ and ∇^1_{TZ} their counterpart on $M \times \{1\}$. Consider pullbacks on $B \times [0,1]$ of some couple (F^+, F^-) of bundles on B with connections ∇_{F^+} and ∇_{F^-} such that $[F^+] - [F^-] = \pi^{\mathrm{Eu}}_*[\xi] \in K^0_{\mathrm{top}}(B)$ with pullback connections, the counterpart of (88) in this setting is

$$\begin{aligned}&\eta(\nabla^1_\xi, \nabla^1_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) - \eta(\nabla^0_\xi, \nabla^0_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \\ &= \int_Z \Big[e(\nabla^0_{TZ}) \wedge \widetilde{\mathrm{ch}}(\nabla^0_\xi, \nabla^1_\xi) + \widetilde{e}(\nabla^0_{TZ}, \nabla^1_{TZ}) \wedge \mathrm{ch}(\nabla^1_\xi)\Big]\end{aligned} \tag{91}$$

where the integrand can be modified as in (90).

Now one also can change the bundles on B in the following way: take suitable (μ^+, μ^-, ψ) and call $\mathscr{H}^\pm = \mathrm{Ker}\mathscr{D}^{\nabla_\xi \pm}_\psi$, endow $\mathscr{H}^+ \oplus \mu^-$ and $\mathscr{H}^- \oplus \mu^+$ with any connections $\nabla^\uparrow$ and $\nabla^\downarrow$. Consider vector bundles F^+, F^-, G^+ and G^- on B such that $[F^+] - [F^-] = [G^+] - [G^-] = \pi^{\mathrm{Eu}}_*[\xi] \in K^0_{\mathrm{top}}(B)$, choose some connections ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-} on them, and some links $[\ell_F]$ and $[\ell_G]$ between $F^+ - F^-$ or $G^+ - G^-$ respectively and $(\mathscr{H}^+ \oplus \mu^-) - (\mathscr{H}^- \oplus \mu^+)$. Then from the construction of η it follows that

$$\begin{aligned}\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell_F]) &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla^\uparrow, \nabla^\downarrow, [\mathrm{Id}]) + \widetilde{\mathrm{ch}}([\ell_F]) \\ &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) - \widetilde{\mathrm{ch}}([\ell_G]) + \widetilde{\mathrm{ch}}([\ell_F]) \\ &= \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) + \widetilde{\mathrm{ch}}([\ell_F \circ \ell_G^{-1}])\end{aligned} \tag{92}$$

where of course $\widetilde{\mathrm{ch}}([\ell_F])$ and $\widetilde{\mathrm{ch}}([\ell_G])$ are computed with $\nabla_{F^\pm}$ or $\nabla_{G^\pm}$ respectively, and $\nabla^\uparrow$ and $\nabla^\downarrow$.

Formulae (91) and (92) give all the dependence of η on its data.

5.4.2. *End of proof of Theorem 28.* — If ξ has vanishing rational Chern classes, then some finite direct sum $\xi \oplus \xi \oplus \cdots \oplus \xi$ is topologically trivial on X. The anomaly formulae (which are consequences of properties (a) and (b)) then relate η for $\nabla_\xi \oplus \nabla_\xi \oplus \cdots \oplus \nabla_\xi$ (and any direct sum of copies of direct image representatives) and η for the canonical flat connection on the trivial bundle with corresponding flat direct image, which vanishes because of (d). Property (c) allows to simply divide by the number of copies of ξ to obtain the desired η, which is thus obtained using only (a), (b), (c) and (d).

Remark. — One could generalise to bundles ξ whose restrictions to the fibers of π have vanishing rational Chern classes by adding some property linking η for ξ and η for $\xi \otimes \pi^*\zeta$ where ζ is any bundle on B. Some more axioms are needed to obtain a general caracterisation.

One could hope to obtain a caracterisation of η modulo the image of $K^1_{\mathrm{top}}(B)$ by the Chern character, with no care of links of bundles on B with someones obtained by analytic families index construction. However, the fact that one must consider a not controlled finite number of copies of ξ would prevent to obtain more than a caracterisation modulo rational cohomology.

5.4.3. *Proof of Theorem 29.* — The anomaly formulae (91) and (92) yield in the situation of Theorem 29 that

$$\eta(\nabla_E, \nabla_{TZ}, \nabla_{\pi_!^+ E}, \nabla_{\pi_!^- E}, [\mathrm{Id}]) - \eta(\nabla_F, \nabla_{TZ}, \nabla_{\pi_!^+ F}, \nabla_{\pi_!^- F}, [\mathrm{Id}]) = \\ = \int_Z e(\nabla_{TZ})\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F) - \widetilde{\mathrm{ch}}\big(\pi_\ell([f])\big).$$

Both η vanish (this is property (d)), and that the right hand side vanishes is exactly the desired result in view of Definitions 7 and 26.

5.4.4. *Proof of Theorem 31.* — Let $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. If F^+, F^-, G^+ and G^- are vector bundles on B such that $[G^+] - [G^-] = [F^+] - [F^-] = \pi_*^{\mathrm{Eu}}[\xi] \in K_0^{\mathrm{top}}(B)$. Consider any connections ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-} on them. It follows from (92) that

$$\eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell_F]) - \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{G^+}, \nabla_{G^-}, [\ell_G]) = \widetilde{\mathrm{ch}}([\ell_F \circ \ell_G^{-1}]).$$

Formula (51) written with G^+ and G^- (with their connections) instead of F^+ and F^- thus provides the same class in $\widehat{K}_{\mathrm{ch}}(B)$ (see (10), (13) and (85)). $\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)$ is thus a well defined element in $\widehat{K}_{\mathrm{ch}}(B)$.

Suppose that $(\xi, \nabla_\xi, \alpha) = (\xi', \nabla_{\xi'}, \alpha') \in \widehat{K}_{\mathrm{ch}}(M)$, and that $f\colon \xi \to \xi'$ is some smooth vector bundle isomorphism, then

$$\alpha' = \alpha + \widetilde{\mathrm{ch}}(\nabla_\xi, f^*\nabla_{\xi'}) + \beta$$

where β is a closed form lying in the image of $K^1_{\text{top}}(M)$ by the Chern character. Thus if $[F^+]-[F^-]=\pi^{\text{Eu}}_*[\xi]\in K^0_{\text{top}}(B)$ with connections ∇_{F^+} on F^+ and ∇_{F^-} on F^-, one has from (91) and (92) (for any suitable links $[\ell_\xi]$ and $[\ell_{\xi'}]$):

$$\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell_\xi])-\eta(\nabla_{\xi'},\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell_{\xi'}])=$$
$$=\int_Z e(\nabla_{TZ})\wedge\widetilde{\text{ch}}(\nabla_\xi,f^*\nabla_{\xi'})+\widetilde{\text{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1}).$$

Remember the definition of $a\colon\Omega^{\text{odd}}(M,\mathbb{C})/d\Omega^{\text{even}}(M,\mathbb{C})\longrightarrow\widehat{K}_{\text{ch}}(M)$ given just before Proposition 10. One obtains from the preceding equation:

$$\pi^{\text{Eu}}_!(\xi,\nabla_\xi,\alpha)-\pi^{\text{Eu}}_!(\xi',\nabla_{\xi'},\alpha')=$$
$$=a\Big(\int_Z e(\nabla_{TZ})\wedge\Big(\widetilde{\text{ch}}(\nabla_\xi,f^*\nabla_{\xi'})+\alpha-\alpha'\Big)+\widetilde{\text{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\Big)$$
$$=a\Big(\int_Z e(\nabla_{TZ})\wedge\beta\Big)+a\left(\widetilde{\text{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\right)$$

which vanishes in $\widehat{K}_{\text{ch}}(B)$, because $\widetilde{\text{ch}}(\ell_\xi\circ\ell_{\xi'}^{-1})\in\text{ch}\big(K^1_{\text{top}}(B)\big)\subset H^{\text{odd}}(B,\mathbb{C})$ and so does $\int_Z e(\nabla_{TZ})\wedge\beta$ by virtue of the cohomological version of Atiyah-Singer families index theorem for K^1_{top}.

Moreover the additivity of η for direct sums (property (c)) yields

$$\pi^{\text{Eu}}_!(\xi_1\oplus\xi_2,\nabla_{\xi_1}\oplus\nabla_{\xi_2},\alpha_1+\alpha_2)=\pi^{\text{Eu}}_!(\xi_1,\nabla_{\xi_1},\alpha_1)+\pi^{\text{Eu}}_!(\xi_2,\nabla_{\xi_2},\alpha_2).$$

$\pi^{\text{Eu}}_!$ is thus well defined as a morphism from $\widehat{K}_{\text{ch}}(M)$ to $\widehat{K}_{\text{ch}}(B)$.

The commutativity of diagram (53) is a consequence of property (d) of η.

The commutativity of the right and the central squares of diagram (54) are tautological. The commutativity of the left square of (54) is a consequence of the cohomological version of Atiyah-Singer families index theorem for K^1_{top}.

In the same way one has the following equality modulo exact forms:

$$\mathfrak{B}\big(\pi^{\text{Eu}}_*(\xi,\nabla_\xi,\alpha)\big)=\widetilde{\text{ch}}(\nabla^*_{F^+},\nabla_{F^+})-2i\mathfrak{Im}\Big(\int_Z e(\nabla_{TZ})\wedge\alpha\Big)$$
$$-\widetilde{\text{ch}}(\nabla^*_{F^-},\nabla_{F^-})+2i\mathfrak{Im}\big(\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])\big).$$

Of course the connections on F^+ and on F^- can be supposed to respect some hermitian metrics on F^+ and F^- without changing the formula, and this makes and vanish the terms $\widetilde{\text{ch}}(\nabla^*_{F^+},\nabla_{F^+})$ and $\widetilde{\text{ch}}(\nabla^*_{F^-},\nabla_{F^-})$.

The reality considerations for $\widetilde{\text{ch}}([\ell])$ between (85) and (86) and the last statement of Lemma 48 imply that $\eta(\nabla_\xi,\nabla_{TZ},\nabla_{F^+},\nabla_{F^-},[\ell])$ is a real form (modulo exact forms) if it happens that ∇_ξ, ∇_{F^+} and ∇_{F^-} respect some hermitian metrics on their bundles. Consider any connection ∇^u_ξ which respects some hermitian metrics on ξ. The reality of $\eta(\nabla_\xi^u, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])$ and Formula (91) yield

$$\eta(\nabla_\xi,\nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \\ = \eta(\nabla_\xi^u, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) + \int_Z e(\nabla_{TZ}) \wedge \widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi)$$

$$2i\mathfrak{Im}\big(\eta(\nabla_\xi,\nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell])\big) = \int_Z e(\nabla_{TZ}) \wedge 2i\mathfrak{Im}\big(\widetilde{\mathrm{ch}}(\nabla_\xi^u, \nabla_\xi)\big).$$

Now using (31)) one gets:

$$\mathfrak{B}\big(\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)\big) = \int_Z e(\nabla_{TZ}) \wedge \big(\widetilde{\mathrm{ch}}(\nabla_\xi^*, \nabla_\xi) - 2i\mathfrak{Im}\alpha\big) \\ = \int_Z e(\nabla_{TZ}) \wedge \mathfrak{B}(\xi, \nabla_\xi, \alpha)$$

and the last statement of Theorem 31 is proved.

5.4.5. *Influence of the vertical metric and the horizontal distribution.* — If geometric data are changed on M, namely the vertical riemannian metric g^Z and/or the horizontal subspace T^HM, this changes the connection ∇_{TZ}, and this also changes the morphism $\pi_!^{\mathrm{Eu}}$.

Lemma 52. — *Let ∇_{TZ} and $\pi_!^{\mathrm{Eu}}$ be associated to data g^Z and T^HM, let $g^{Z'}$ and $T^{H'}M$ be other data and call ∇'_{TZ} and $\pi_!^{\mathrm{Eu}'}$ the associated connection on TZ and morphism from $\widehat{K}_{\mathrm{ch}}(M)$ to $\widehat{K}_{\mathrm{ch}}(B)$. Then, for any $(\xi, \nabla_\xi, \alpha)$ one has*

$$\pi_!^{\mathrm{Eu}'}(\xi, \nabla_\xi, \alpha) - \pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = -a\left(\int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})\dddot{\mathrm{ch}}(\xi, \nabla_\xi, \alpha)\right).$$

Proof. — If $\dim Z$ is odd, $\pi_!^{\mathrm{Eu}}$ and $\pi_!^{\mathrm{Eu}'}$ will be proved to vanish in §6.3. $\widetilde{e}$ also vanishes. If $\dim Z$ is even, it successively follows from (91) that

$$\eta(\nabla_\xi, \nabla'_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) - \eta(\nabla_\xi, \nabla_{TZ}, \nabla_{F^+}, \nabla_{F^-}, [\ell]) = \int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \mathrm{ch}(\nabla_\xi)$$

$$\pi_!^{\mathrm{Eu}'}(\xi, \nabla_\xi, \alpha) - \pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = \\ = a\left(\int_Z \big(e(\nabla'_{TZ}) - e(\nabla_{TZ})\big)\alpha - \int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \mathrm{ch}(\nabla_\xi)\right) = \\ = a\left(\int_Z \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ}) \wedge \big(- \mathrm{ch}(\nabla_\xi) + d\alpha\big)\right)$$

this last equality is valid modulo exact forms because

$$d\big(\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})\alpha\big) = e(\nabla'_{TZ})\alpha - e(\nabla_{TZ})\alpha + (-1)^{\deg \widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})}\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})d\alpha$$

and $\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})$ is of degree $\dim Z - 1$, with $\dim Z$ even. □

If $\dim Z$ is even, and since $\widetilde{e}(\nabla_{TZ}, \nabla'_{TZ})$ is of degree $\dim Z - 1$, it follows that MK_0 is the biggest subgroup of $\widehat{K}_{\mathrm{ch}}$ on which there is no variation of $\pi_!^{\mathrm{Eu}}$ when geometric data g^Z and T^HM are changed. This gives a topological significance to the direct image morphism $\pi_!^{\mathrm{Eu}}$ on MK_0.

In the language of [**14**], the geometry of the fibration should be encoded into some smooth refinement of the used K-orientation, (here it is the one associated to the fiberwise Euler operator) and the restriction of $\pi_!^{\mathrm{Eu}}$ to $MK_0(M)$ would be independent of the choice of this smooth K-orientation.

6. Fiberwise Hodge symmetry

The goal of this part is to prove Theorems 32 and 33. All these results are consequences of symmetries induced by the fiberwise Hodge star operator. Paragraph §6.1 is essentially devoted to technical computations dealing with relations of this star operator with various geometrical features of the theory.

6.1. Symmetries induced on family index bundles

6.1.1. *The fiberwise Hodge $*$ operator.* — Here we will make constant use of the notations introduced in §3.1.1, §3.4.1 and §5.2.1. For any vertical tangent vector $\mathtt{w} \in TZ$, consider its dual one-form $\mathtt{w}^\flat$ (through the fiberwise riemannian metric g^Z), and its Clifford action on $\wedge^\bullet T^*Z \otimes \xi$

$$c(\mathtt{w}) = (\mathtt{w}^\flat \wedge) - \iota_{\mathtt{w}} \tag{93}$$

($\iota_{\mathtt{w}}$ denotes the interior product by $\mathtt{w}$); $c(\mathtt{w})$ is skewadjoint with respect to $(\ |\ )_Z$ and verifies $c(\mathtt{w})^2 = -g^Z(\mathtt{w}, \mathtt{w})$, it is an isometry if $g^Z(\mathtt{w}, \mathtt{w}) = 1$.

Consider the vertical Hodge operator $*_Z = c(\mathrm{e}_1)c(\mathrm{e}_2)\cdots c(\mathrm{e}_{\dim Z})$ for any orthonormal direct base $\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_{\dim Z}$ of TZ. It is an isometry of $\mathscr{E}$ (endowed with $\langle\ ,\ \rangle_{L^2}$), and it has the same parity as $\dim Z$ (with respect to the $\mathbb{Z}_2$ grading of $\mathscr{E}$). Its inverse $*_Z^{-1} = (-1)^{\frac{1}{2}\dim Z(\dim Z+1)} *_Z$ is also its adjoint with respect to both $(\ |\ )_Z$ and $\langle\ ,\ \rangle_{L^2}$. Define the metrized exterior product of ξ-valued vertical differential forms by the following formula on decomposed tensors:

$$(\alpha \mathbin{\widehat{\otimes}} a) \underset{h^\xi}{\wedge} (\beta \mathbin{\widehat{\otimes}} b) = (\alpha \wedge \overline{\beta}) h^\xi(a, b)$$

(a sign $(-1)^{\deg a \deg \beta}$ should be put on the right side if ξ would be $\mathbb{Z}_2$-graded, but this case will not be considered in the sequel, note also that this operation is independent of the riemannian vertical metric g^Z). Then for any $\gamma \in \mathscr{E}$ whose differential form degree is $\leq \deg\alpha$:

$$(\alpha \mathbin{\widehat{\otimes}} a) \underset{h^\xi}{\wedge} (*_Z\gamma) = (-1)^{\frac{1}{2}\deg\alpha(\deg\alpha - 1) + \dim Z \deg\alpha} \big((\alpha \mathbin{\widehat{\otimes}} a) \big| \gamma\big)_Z d\mathrm{Vol}_Z. \tag{94}$$

6.1.2. *Symmetry induced by $*_Z$ on fiberwise twisted Euler operators.* — For any vector $w \in TZ$, $c(w)$ commutes with $*_Z$ if $\dim Z$ is odd and it anticommutes with $*_Z$ if $\dim Z$ is even. It follows from the two preceding formulae that if ∇^*_ξ is associated to ∇_ξ and h^ξ as in (22), then for any γ and γ' in $\mathscr{E}$:

$$d^Z(\gamma \underset{h^\xi}{\wedge} \gamma') = (d^{\nabla_\xi}\gamma) \underset{h^\xi}{\wedge} \gamma' + (-1)^{\deg\gamma}\gamma \underset{h^\xi}{\wedge} (d^{\nabla^*_\xi}\gamma') \tag{95}$$
$$\text{so that} \qquad (d^{\nabla_\xi})^* = (-1)^{1+\frac{1}{2}\dim Z(\dim Z-1)} *_Z d^{\nabla^*_\xi} *_Z$$

from which one deduces that

$$d^{\nabla_\xi} + (d^{\nabla_\xi})^* = -(-1)^{\dim Z} *_Z^{-1} \left(d^{\nabla^*_\xi} + (d^{\nabla^*_\xi})^*\right) *_Z . \tag{96}$$

This formula can also be checked from [**11**] formulae (3.36), (1.30), (1.31) and the last sentence at the end of the first alinea of §III(d).

Suppose that (μ^+, μ^-, ψ) is a suitable triple with respect to ξ endowed with h^ξ and ∇_ξ, and produce kernel bundles $\mathscr{H}^+$ and $\mathscr{H}^-$. If $\dim Z$ is even, $*_Z$ respects the parity of vertical forms while $*_Z$ exchanges this parity if $\dim Z$ is odd. It then follows from (96) that:

Proposition 53. — *If* $\dim Z$ *is odd,* $\left(\mu^-, \mu^+, (*_Z \oplus \mathrm{Id}_{\mu^+}) \circ \psi^* \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^-})\right)$ *is a suitable triple for* ξ *endowed with* h^ξ *and* ∇^*_ξ. *It produces kernel bundles* $(*_Z \oplus \mathrm{Id}_{\mu^-})\mathscr{H}^-$ *and* $(*_Z \oplus \mathrm{Id}_{\mu^+})\mathscr{H}^+$.

If $\dim Z$ *is even. The triple* $\left(\mu^+, \mu^-, -(*_Z \oplus \mathrm{Id}_{\mu^-}) \circ \psi \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^+})\right)$ *is suitable with respect to* ξ *endowed with* h^ξ *and* ∇^*_ξ. *It produces kernel bundles* $(*_Z \oplus \mathrm{Id}_{\mu^+})\mathscr{H}^+$ *and* $(*_Z \oplus \mathrm{Id}_{\mu^-})\mathscr{H}^-$.

Indeed denote in both cases by Ψ the third element of the proposed triples, then the triple (μ^+, μ^-, Ψ) or (μ^-, μ^+, Ψ) for ∇^*_ξ is chosen so that (96) reads

$$\begin{cases} \mathscr{D}_\psi^{\nabla_\xi \pm} = -(*_Z \oplus \mathrm{Id}_{\mu^\mp})^{-1} \mathscr{D}_\Psi^{\nabla^*_\xi \pm} (*_Z \oplus \mathrm{Id}_{\mu^\pm}) & \text{if } \dim Z \text{ is even,} \\ \mathscr{D}_\psi^{\nabla_\xi \pm} = (*_Z \oplus \mathrm{Id}_{\mu^\mp})^{-1} \mathscr{D}_\Psi^{\nabla^*_\xi \mp} (*_Z \oplus \mathrm{Id}_{\mu^\pm}) & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{97}$$

6.1.3. *Odd dimensional fibre case.* — Suppose B is compact and the fibres of π are odd dimensional. Consider some positive kernel triple $(\lambda, \{0\}, \varphi)$ for ξ, which is supposed to be endowed with a connection ∇_ξ which respects the hermitian metric h^ξ. (There exists some as was mentioned just before Definition 37). It is here needed that φ vanishes on $\mathscr{E}^+$ (which is in fact the case in the above cited references [**2**] Proposition 2.2, or [**3**] Lemma 9.30 or [**29**] Lemma 8.4 of chapter III). Call $\mathscr{K}^+$ the associated kernel bundle. It follows from Proposition 53 that $\left(\{0\}, \lambda, (*_Z \oplus \mathrm{Id}_\lambda) \circ \varphi^* \circ *_Z^{-1}\right)$ is suitable and gives rise to kernel bundles $\{0\}$ and $(*_Z \oplus \mathrm{Id}_\lambda)\mathscr{K}^+ \subset \mathscr{E}^- \oplus \lambda$.

Lemma 54. — *The triple* $\left(\lambda, \lambda, \varphi + (*_Z \oplus \mathrm{Id}_\lambda) \circ \varphi^* \circ *_Z^{-1} + i^{1+\frac{1}{2}\dim Z(\dim Z+1)}\mathrm{Id}_\lambda\right)$ *is suitable with kernel bundles* $\{0\}$ *and* $\{0\}$.

The point about the factor of Id_λ is that it should be nonvanishing and purely imaginary if $*_Z^2 = \mathrm{Id}$ but real (and nonvanishing) if $*_Z^2 = -\mathrm{Id}$.

The vanishing of π_*^{Eu} on K^0_{top} in the case of odd dimensional fibres and compact B follows. If B is noncompact, one concludes using the fact that any element of $K^0_{\mathrm{top}}(B)$ whose restriction to any compact subset vanishes, is itself trivial. The vanishing of π_*^{Eu} on K^1_{top} is a consequence of its vanishing on K^0_{top}.

Proof. — Consider any element $(v, \sigma) \in \lambda \oplus \mathscr{E}^+$ belonging to the kernel bundle. The corresponding condition reads

$$\begin{cases} \left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right)\sigma = -\varphi v \\ \qquad -\varphi^*(*_Z^{-1}\sigma) = i^{1+\frac{1}{2}\dim Z(\dim Z+1)}v. \end{cases}$$

Writing $\sigma = *_Z\sigma'$, one obtains the following

$$\left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right) *_Z \sigma' = -i^{-1-\frac{1}{2}\dim Z(\dim Z+1)}\varphi\varphi^*\sigma'$$

$$\left\langle \left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right) *_Z \sigma', \sigma' \right\rangle_{L^2} = -i^{-1-\frac{1}{2}\dim Z(\dim Z+1)}\langle \varphi^*\sigma', \varphi^*\sigma'\rangle_\lambda$$

where $\langle\ ,\ \rangle_\lambda$ is the scalar product on λ. It follows from (96), and the fact that ∇_ξ respects the hermitian metric, that $\left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right)*_Z$ is selfadjoint if $*_Z^2 = \mathrm{Id}$ and antiselfadjoint if $*_Z^2 = -\mathrm{Id}$. Thus the right hand side of this equality is real whenever the left hand side is purely imaginary and conversely. In any case this proves that $\varphi^*\sigma'$ vanishes. Thus v vanishes, thus $\left(d^{\nabla_\xi} + (d^{\nabla_\xi})^*\right)\sigma$ vanishes. It follows that σ belongs to the (positive) kernel bundle associated to the triple $\left(\{0\}, \lambda, (*_Z \oplus \mathrm{Id}_\lambda) \circ \varphi^* \circ *_Z^{-1}\right)$. But this kernel bundle vanishes, and so does σ.

The proof of the vanishing of the cokernel is similar. □

Let $(\mathscr{F}^+, \mathscr{F}^-)$ be any couple of family index bundles for ξ. It follows from the preceding lemma and Theorem 25 that there exists a canonical link $\ell_{\mathscr{F}}^{\{0\}}$ between $\mathscr{F}^+ - \mathscr{F}^-$ and $\{0\} - \{0\}$. These canonical links are all compatible, this means that if $(\mathscr{G}^+, \mathscr{G}^-)$, and $(\mathscr{K}^+, \mathscr{K}^-)$ are couples of family index bundles for $\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$ which are linked through some link ℓ, then

(98) $$\pi_\ell([\ell]) = [\ell_{\mathscr{F}}^{\{0\}}] \circ [\ell_{\mathscr{G}}^{\{0\}}]^{-1}.$$

This is because the same construction as in the proof of Lemma 54 can be performed on $M \times [0,1]$ compatibly with a deformation as was used in §4.1.4.

That $\pi_\ell([\ell])$ is constant, (i.e. $\pi_\ell([\ell])$ does not depend on $[\ell]$) is compatible with the action of K^1_{top} on links and the vanishing of π_*^{Eu} on K^1_{top}.

6.1.4. *Symmetry on canonical links.* — B is no longer supposed compact. Let ξ_1, ξ_2, ζ_1 and ζ_2 be bundles on M with connections ∇_{ξ_1}, ∇_{ξ_2}, ∇_{ζ_1} and ∇_{ζ_2} such that $[\xi_1] - [\zeta_1] = [\xi_2] - [\zeta_2] \in K^0_{\mathrm{top}}(M)$. Let $[\ell]$ be some equivalence class of link between

$\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$. One supposes that $\xi_1 - \zeta_1$ and $\xi_2 - \zeta_2$ admit respective couples of family index bundles $(\mathcal{F}^+, \mathcal{F}^-)$ and $(\mathcal{G}^+, \mathcal{G}^-)$.

Call $(\mathcal{F}^{*+}, \mathcal{F}^{*-})$ and $(\mathcal{G}^{*+}, \mathcal{G}^{*-})$ the respectively associated family index bundles for $\xi_1 - \zeta_1$ endowed with $\nabla^*_{\xi_1}$ and $\nabla^*_{\zeta_1}$ or for $\xi_2 - \zeta_2$ endowed with $\nabla^*_{\xi_2}$ and $\nabla^*_{\zeta_2}$ obtained using the symmetric triples of Proposition 53. There are isomorphisms (of the form $(*_Z \oplus \mathrm{Id}_{\mu^\pm})$)

$$\mathcal{F}^\pm \cong \mathcal{F}^{*\pm} \quad \text{and} \quad \mathcal{G}^\pm \cong \mathcal{G}^{*\pm} \quad \text{if } \dim Z \text{ is even,}$$
$$\mathcal{F}^\pm \cong \mathcal{F}^{*\mp} \quad \text{and} \quad \mathcal{G}^\pm \cong \mathcal{G}^{*\mp} \quad \text{if } \dim Z \text{ is odd.}$$

This provides a link $\ell^*_{\mathcal{F}}$ between $\mathcal{F}^+ - \mathcal{F}^-$ and $\mathcal{F}^{*+} - \mathcal{F}^{*-}$ if $\dim Z$ is even or between $\mathcal{F}^+ - \mathcal{F}^-$ and $\mathcal{F}^{*-} - \mathcal{F}^{*+}$ if $\dim Z$ is odd. And a link $\ell^*_{\mathcal{G}}$ accordingly.

Remember the definition of $\pi_\ell([\ell])$ as an equivalence class of links between $\mathcal{F}^+ - \mathcal{F}^-$ and $\mathcal{G}^+ - \mathcal{G}^-$ from Definition 41. Denote by $\pi_\ell([\ell]\check{})$ the corresponding equivalence class of links between $\mathcal{F}^{*+} - \mathcal{F}^{*-}$ and $\mathcal{G}^{*+} - \mathcal{G}^{*-}$.

Proposition 55. — *These classes of links are compatible in the sense that*

$$\begin{cases} \pi_\ell([\ell]\check{}) = [\ell^*_{\mathcal{F}}]^{-1} \circ \pi_\ell([\ell]) \circ [\ell^*_{\mathcal{G}}] & \textit{if } \dim Z \textit{ is even,} \\ [\ell^{\{0\}}_{\mathcal{F}^*}] = [\ell^*_{\mathcal{F}}] \circ [\ell^{\{0\}}_{\mathcal{F}}]^{-1} & \textit{if } \dim Z \textit{ is odd.} \end{cases}$$

Proof. — The symmetry of family index bundles of Proposition 53 is valid on a deformation on $B \times [0,1]$ as was performed in §4.1.3 and used in §4.1.4 for the general construction of $\ell^{\mathcal{G}}_{\mathcal{F}}$. In the even dimensional fibre case, one obtains two constructions of $\ell^{\mathcal{G}}_{\mathcal{F}}$ and $\ell^{\mathcal{G}^*}_{\mathcal{F}^*}$ in exactly the same terms as in §4.1.3 and §4.1.4 which are mutually isomorphic through $*_Z$. Thus $[\ell^{\mathcal{G}^*}_{\mathcal{F}^*}] = [\ell^*_{\mathcal{F}}]^{-1} \circ [\ell^{\mathcal{G}}_{\mathcal{F}}] \circ [\ell^*_{\mathcal{G}}]$ and the first statement of the proposition follows from the fact that $\pi_\ell([\ell])$ is constructed as a particular case of some (inductive limit of) $[\ell^{\mathcal{G}}_{\mathcal{F}}]$.

In the odd dimensional fibre case, first remark that the links of type $[\ell^{\{0\}}_{\mathcal{F}}]$, though constructed under a compacity hypothesis, are globally valid for globally defined couple of family index bundles (if there exists some. This is because locally defined links between global objects yield global links, as was sketched just before Definition 40). One may then suppose that $\mathcal{G}^+ = \mathcal{G}^- = \{0\}$ (see (98)). The point is now that $*_Z$ exchanges the parity, so that a link obtained through some couple of positive kernel family index bundles (see Definition 37) is mapped by $*_Z$ to a link obtained through a couple of "negative kernel" family index bundles. The counterpart of (58) in this situation reads

$$0 \longrightarrow \mathcal{H}^+ \xrightarrow{\varphi^*} \lambda \longrightarrow \mathrm{Ker}\,\mathcal{D}^{\nabla_\xi -}_{\psi \oplus \varphi^*} \longrightarrow \mathcal{H}^- \longrightarrow 0$$

where the two last maps are orthogonal projections (after inclusion of λ in $\lambda \oplus \mathcal{E}^-$). The proposition is reduced to prove that the equivalence class of links associated to this exact sequence (from Definition 36) equals $[\ell_{\mathscr{H}^+-\mathscr{H}^-}^{\lambda-\mathrm{Ker}\mathscr{D}_{\psi\oplus\varphi^*}^{\nabla_\xi-}}]$. This is a consequence of the fact that this link can be realised as a deformation, by an analogue construction to what was made in the second step of the proof of Proposition 51. □

6.1.5. *Symmetry on connections on the infinite rank bundle $\mathscr{E}$.* — Remember the definitions of the infinite rank bundle $\mathscr{E}$ from (36) and (37), and of the connections $\overline{\nabla}$ and $\overline{\nabla}^S$ on $\mathscr{E}$ from (72) and (73)

Consider the adjoint ∇_ξ^* of ∇_ξ, and the connection $\overline{\nabla}\check{}$ on $\mathscr{E}$ which is associated to ∇_ξ^* in the same way as $\overline{\nabla}$ is associated to ∇_ξ through (72). Call $\overline{\nabla}^{\check{}S}$ the adjoint connection of $\overline{\nabla}\check{}$ defined in the same way as was $\overline{\nabla}^S$ with respect to $\overline{\nabla}$ in (73). The reader is warned that the connection denoted here by $\overline{\nabla}^S$ corresponds to the connection denoted by $(\nabla^W)^*$ in [**11**] Proposition 3.7, and that $\overline{\nabla}\check{}$ and $\overline{\nabla}^{\check{}S}$ here have no counterpart in [**11**].

Lemma 56. — *For any vector* u *tangent to B, and any local section σ of $\mathscr{E}$*

$$\overline{\nabla}_{\mathrm{u}}\check{}\,\sigma = *_Z^{-1}\big(\overline{\nabla}_{\mathrm{u}}^S(*_Z\sigma)\big) \qquad \textit{and} \qquad \overline{\nabla}_{\mathrm{u}}^{\check{}S}\sigma = *_Z^{-1}\big(\overline{\nabla}_{\mathrm{u}}(*_Z\sigma)\big).$$

Proof. — Remember the definition of ∇_{TZ} from §3.4.1. Denote allways by ∇_{TZ} the associated connection on $\wedge^\bullet T^*Z$, it is compatible with the Clifford action (93), so that its associated covariant derivative commutes with $*_Z$. Let $\nabla_{TZ\otimes\xi}$ be the connection on $\wedge^\bullet T^*Z\otimes\xi$ associated to ∇_{TZ} and ∇_ξ, its adjoint $\nabla_{TZ\otimes\xi}^*$ with respect to $(\ |\ )_Z$ is nothing but the connection on $\wedge^\bullet T^*Z\otimes\xi$ associated to ∇_{TZ} and ∇_ξ^*. Then the covariant derivatives associated to both $\nabla_{TZ\otimes\xi}$ and $\nabla_{TZ\otimes\xi}^*$ commute with $*_Z$.

Let u be some vector tangent to B, and u^H its horizontal lift. For any vector y tangent to the fibre, the vertical projection $P^{TZ}\nabla_{LC_{\mathrm{y}}}\mathrm{u}^H$ of the covariant derivative of u^H along y for the connection ∇_{LC} is independent of the global riemannian metric defining ∇_{LC}. Moreover, if v is another vertical tangent vector at the same point as y, then the scalar product $g^Z(P^{TZ}\nabla_{LC_{\mathrm{y}}}\mathrm{u}^H,\mathrm{v})$ is symmetric in y and v. As proved in [**11**] (3.27) and (3.32), if $(\mathrm{e}_1,\mathrm{e}_2,\dots,\mathrm{e}_{\dim Z})$ is an orthonormal base of TZ, then for any local section σ of $\mathscr{E}$, the connections $\overline{\nabla}$ and $\overline{\nabla}^S$ express locally on M as:

$$\begin{aligned}\overline{\nabla}_{\mathrm{u}}\sigma &= \nabla_{TZ\otimes\xi_{\mathrm{u}^H}}\sigma + \sum_{i \text{ and } k} g^Z(P^{TZ}\nabla_{LC_{\mathrm{e}_i}}\mathrm{u}^H,\mathrm{e}_k)\mathrm{e}_i^\flat\wedge(\iota_{\mathrm{e}_k}\sigma),\\ \overline{\nabla}_{\mathrm{u}}^S\sigma &= \nabla_{TZ\otimes\xi_{\mathrm{u}^H}}^*\sigma - \sum_{i \text{ and } k} g^Z(P^{TZ}\nabla_{LC_{\mathrm{e}_i}}\mathrm{u}^H,\mathrm{e}_k)\mathrm{e}_i^\flat\wedge(\iota_{\mathrm{e}_k}\sigma).\end{aligned} \tag{99}$$

The lemma follows from the obvious corresponding formulae for $\overline{\nabla}\check{}$ and $\overline{\nabla}^{\check{}S}$, the fact that $\nabla_{TZ\otimes\xi}$ and $\nabla_{TZ\otimes\xi}^*$ commute with $*_Z$, the fact that

$$(\mathrm{e}_i^\flat\wedge)\iota_{\mathrm{e}_k}*_Z = -*_Z(\mathrm{e}_k^\flat\wedge)\iota_{\mathrm{e}_i}$$

for any i and k and the symmetry in e_i and e_k of $g^Z(P^{TZ}\nabla_{LC_{\mathrm{e}_i}}\mathrm{u}^H,\mathrm{e}_k)$. □

6.2. Proof of results about K^0_{flat} and K^0_{rel}

6.2.1. *End of proof of Theorem 32.* — Suppose that E is a vector bundle on M with a flat connection ∇_E (and hermitian metric h^E), and construct the associated objects $\mathcal{E}$, $\overline{\nabla}$ and $\overline{\nabla}^S$ as above. Let $P\colon \mathcal{E} \longrightarrow \operatorname{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)$ be the orthogonal projection, then it is proved in [**11**] Proposition 3.14 that $\nabla_{\pi_! E} \cong P\overline{\nabla}P$ and $\nabla^*_{\pi_! E} \cong P\overline{\nabla}^S P$ through the fiberwise Hodge isomorphism $\pi_! E \cong \operatorname{Ker}\big(d^{\nabla_E} + (d^{\nabla_E})^*\big)$.

Consider on E the adjoint connection ∇^*_E (which is flat). The direct image of the flat bundle (E, ∇^*_E) will be denoted by $\pi_!\check{~}E$ and the flat connection on it by $\nabla_{\pi_!\check{~}E}$ so that $\pi_!(E, \nabla^*_E) = (\pi_!\check{~}E, \nabla_{\pi_!\check{~}E})$.

As precedingly, call $\overline{\nabla}\check{~}$ and $\overline{\nabla}\check{~}^S$ the connections on $\mathcal{E}$ constructed from ∇^*_E as in (72) and (73), let $P\check{~}\colon \mathcal{E} \longrightarrow \operatorname{Ker}\big(d^{\nabla^*_E} + (d^{\nabla^*_E})^*\big)$ be the orthogonal projection, then from [**11**] Proposition 3.14 again, $\nabla_{\pi_!\check{~}E} \cong P\check{~}\overline{\nabla}\check{~}P\check{~}$ and $\nabla^*_{\pi_!\check{~}E} \cong P\check{~}\overline{\nabla}\check{~}^S P\check{~}$ through the fiberwise Hodge isomorphism $\pi_!\check{~}E \cong \operatorname{Ker}\big(d^{\nabla^*_E} + (d^{\nabla^*_E})^*\big)$.

It follows from (96) that $P\check{~} = *_Z P *_Z^{-1}$ so that $*_Z$ directly provides a smooth isomorphism $\pi_! E \cong \pi_!\check{~}E$. It then follows from the preceding Lemma 56 that through this isomorphism $\nabla^*_{\pi_! E} \cong \nabla_{\pi_!\check{~}E}$ and $\nabla_{\pi_! E} \cong \nabla^*_{\pi_!\check{~}E}$. Now $*_Z$ respects the $^+$ and $^-$ parts of $\mathcal{E}$ if $\dim Z$ is even, and exchanges them if $\dim Z$ is odd so that

$$\pi_!(E, \nabla^*_E) = (-1)^{\dim Z}(\pi_! E, \nabla^*_{\pi_! E}). \tag{100}$$

In particular, the first equation of Theorem 32 is proved.

Suppose now that $\dim Z$ is even. Then if $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\text{rel}}(M)$

$$\begin{aligned}
&\pi_*(E, \nabla^*_E, F, \nabla^*_F, f) = \\
&\quad = \big(\pi_!^{+\check{~}}E \oplus \pi_!^{-\check{~}}F, \nabla_{\pi_!^{+\check{~}}E} \oplus \nabla_{\pi_!^{-\check{~}}F}, \pi_!^{-\check{~}}E \oplus \pi_!^{+\check{~}}F, \nabla_{\pi_!^{-\check{~}}E} \oplus \nabla_{\pi_!^{+\check{~}}F}, \pi_\ell([f])\big)\\
&\quad = \big(\pi_!^{+}E \oplus \pi_!^{-}F, \nabla^*_{\pi_!^{+}E} \oplus \nabla^*_{\pi_!^{-}F}, \pi_!^{-}E \oplus \pi_!^{+}F, \nabla^*_{\pi_!^{-}E} \oplus \nabla^*_{\pi_!^{+}F},\\
&\qquad\qquad [\ell^*_{\pi_! E} \oplus (\ell^*_{\pi_! F})^{-1}] \circ \pi_\ell([f]) \circ [(\ell^*_{\pi_! E})^{-1} \oplus \ell^*_{\pi_! F}]\big).
\end{aligned}$$

The reality of π_* in the case of even dimensional fibres (second statement of Theorem 32) follows from this, the first statement of Proposition 55 and the obvious compatibility of links of the form $\ell^*_{\mathcal{G}}$ with direct sums.

The last equation of Theorem 32 was proved just after its statement.

The proof of Theorem 32 is thus completed.

6.2.2. *Results on $\pi_\leftarrow$.* — If $\dim Z$ is odd, consider some $(E, \nabla_E) \in K^0_{\text{flat}}(M)$, there is a link $[\ell^{\{0\}}_{\pi_! E}]$ between $\pi_!^+E - \pi_!^-E$ and $\{0\} - \{0\}$ as defined at the end of §6.1.2 (see also the proof of Proposition 55),

Definition 57. — *For $(E, \nabla_E) \in K^0_{\text{flat}}(M)$, one defines*

$$\pi_\leftarrow(E, \nabla_E) = \big(\pi_!^-E, \nabla_{\pi_!^-E}, \pi_!^+E, \nabla_{\pi_!^+E}, [\ell^{\{0\}}_{\pi_! E}]^{-1}\big) \in K^0_{\text{rel}}(B). \tag{101}$$

Proposition 58. — *$\pi_{\leftarrow}$ defines a morphism from $K^0_{\mathrm{flat}}(M)$ to $K^0_{\mathrm{rel}}(B)$.*

The relation $\pi_! = \partial \circ \pi_{\leftarrow}$ is then tautological.

The last but one statement of Theorem 33 (that $\pi_{\leftarrow}$ is purely imaginary) is a direct consequence of the second statement of Proposition 55, since through the isomorphism induced by $*_Z$ one has

$$\begin{aligned}\pi_{\leftarrow}(E, \nabla^*_E) &= \big(\pi_!^{-\check{}}E, \nabla_{\pi_!^{-\check{}}E}, \pi_!^{+\check{}}E, \nabla_{\pi_!^{+\check{}}E}, [\ell^{\{0\}}_{\pi_!\check{}E}]^{-1}\big)\\ &= \big(\pi_!^{+}E, \nabla^*_{\pi_!^+E}, \pi_!^{-}E, \nabla^*_{\pi_!^-E}, [\ell^*_{\pi_!E}] \circ [\ell^{\{0\}}_{\pi_!E}]^{-1}\big)\end{aligned}$$

(and in general $\ell^*_{\mathscr{G}} = \ell^*_{\mathscr{G}^*}$ in the odd dimensional fibre case). In particular, if ∇_E respects some hermitian metric on E then

$$\begin{aligned}\pi_{\leftarrow}(E, \nabla_E) &= \big(\pi_!^+E, \nabla^*_{\pi_!^+E}, \pi_!^+E, \nabla_{\pi_!^+E}, [\mathrm{Id}]\big)\\ &= -\big(\pi_!^-E, \nabla^*_{\pi_!^-E}, \pi_!^-E, \nabla_{\pi_!^-E}, [\mathrm{Id}]\big)\end{aligned}$$

from which one deduces using (33), (34), Remark 14, [**11**] Theorem 0.1 and the vanishing of the Euler class of odd rank real vector bundles that:

$$\begin{aligned}\mathscr{N}_{\mathrm{ch}} \circ \pi_{\leftarrow}(E, \nabla_E) &= \frac{1}{2}\big(\mathfrak{B}(E, \nabla_E, 0) - \mathfrak{B}(E, \nabla^*_E, 0)\big)\\ &= \frac{1}{2}\int_Z e(TZ) \wedge \mathfrak{B}(E, \nabla_E, 0) = 0.\end{aligned} \tag{102}$$

The relation $\pi_* = \pi_{\leftarrow} \circ \partial$ on K^0_{rel} is proved by the following computation, which uses (98) and the compatibility of links of the form $\ell^{\{0\}}_{\mathscr{G}}$ with direct sums:

$$\begin{aligned}&\pi_*(E, \nabla_E, F, \nabla_F, f) =\\ &= \big(\pi_!^+E \oplus \pi_!^-F, \nabla_{\pi_!^+E} \oplus \nabla_{\pi_!^-F}, \pi_!^-E \oplus \pi_!^+F, \nabla_{\pi_!^-E} \oplus \nabla_{\pi_!^+F}, \pi_\ell([f])\big)\\ &= \big(\pi_!^+E, \nabla_{\pi_!^+E}, \pi_!^-E, \nabla_{\pi_!^-E}, [\ell^{\{0\}}_{\pi_!E}]\big) + \big(\pi_!^-F, \nabla_{\pi_!^-F}, \pi_!^+F, \nabla_{\pi_!^+F}, [\ell^{\{0\}}_{\pi_!F}]^{-1}\big).\end{aligned}$$

One deduces from this, Theorem 29 and the vanishing of the Euler class of odd rank real vector bundles that $\mathscr{N}_{\mathrm{ch}} \circ \pi_{\leftarrow}(E, \nabla_E)$ depends only on the topological K-theory class of E. Its vanishing for general $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$ follows this, (102), the additivity of $\pi_{\leftarrow}$ and of $\mathscr{N}_{\mathrm{ch}}$ and the fact that there is some integer k such that the direct sum of k copies of E is topologically trivial on M.

The last statement remaining unproved in Theorem 33 is the vanishing of $\pi_!^{\mathrm{Eu}}$ on $\widehat{K}_{\mathrm{ch}}$. It is delayed to §6.3. Let us now prove the above proposition:

Proof. — The point to check is that $\pi_{\leftarrow}\big((E', \nabla_{E'}) + (E'', \nabla_{E''}) - (E, \nabla_E)\big)$ vanishes in $K^0_{\mathrm{rel}}(B)$ if E', E'' and E come from a parallel exact sequence like (11).

$(\pi_!^+E' \oplus \pi_!^+E'', \pi_!^-E' \oplus \pi_!^-E'')$ and $(\pi_!^+E, \pi_!^-E)$ are both couples of family index bundles for E (as topological vector bundle). They are thus canonically linked by

$[\ell^{\pi_! E}_{\pi_! E'\oplus\pi_! E''}]$. It follows from Lemma 42 that

(103)
$$\Big(\pi_!^+ E' \oplus \pi_!^+ E'' \oplus \pi_!^- E, \nabla_{\pi_!^+ E'} \oplus \nabla_{\pi_!^+ E''} \oplus \nabla_{\pi_!^- E},$$
$$\pi_!^- E' \oplus \pi_!^- E'' \oplus \pi_!^+ E, \nabla_{\pi_!^- E'} \oplus \nabla_{\pi_!^- E''} \oplus \nabla_{\pi_!^+ E}, [\ell^{\pi_! E}_{\pi_! E'\oplus\pi_! E''}]\Big) = 0 \in K^0_{\mathrm{rel}}(B).$$

But it follows from (98) (which is also valid on noncompact B for globally defined couples of family index bundles as in the proof of Proposition 55) that

$$[\ell^{\pi_! E}_{\pi_! E'\oplus\pi_! E''}] = \big([\ell^{\{0\}}_{\pi_! E'}] \oplus [\ell^{\{0\}}_{\pi_! E''}]\big) \circ [\ell^{\{0\}}_{\pi_! E}]^{-1} = [\ell^{\{0\}}_{\pi_! E'}] \oplus [\ell^{\{0\}}_{\pi_! E''}] \oplus [\ell^{\{0\}}_{\pi_! E}]^{-1}$$

so that the right hand side of (103) is easily recognized (using relation *(ii)* of Definition 4) to be equal to $\pi_{\leftarrow}\big((E', \nabla_{E'}) + (E'', \nabla_{E''}) - (E, \nabla_E)\big)$. □

6.3. End of proof of Theorem 33. — The Formulas (99) and their obvious counterpart for $\overline{\nabla}^{\check{}}$ and $\overline{\nabla}^{\check{} S}$ prove that $\frac{1}{2}(\overline{\nabla}^{\check{}} + \overline{\nabla}^{\check{} S}) = \frac{1}{2}(\overline{\nabla} + \overline{\nabla}^{S}) = \overline{\nabla}^{u}$. Denote by $\check{C_t}$ the superconnection on $\mathcal{E}$ constructed from ∇_ξ^* as C_t is from ∇_ξ:

$$\check{C_t} = \overline{\nabla}^{u} + \frac{\sqrt{t}}{2}(d^{\nabla_\xi^*} + (d^{\nabla_\xi^*})^*) + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge).$$

Remember the definition of B_t from (79), and let $\check{B_t}$ be the modified superconnection constructed as in (79) from $\check{C_t}$ (or ∇_ξ^*) and the suitable triple of Proposition 53 then

Lemma 59. — *We have*
$$\phi\mathrm{Tr}_s\left(\frac{\partial \check{B_t}}{\partial t}\exp -\check{B_t}^2\right) = (-1)^{\dim Z}\phi\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp -B_t^2\right).$$
In particular, $\phi\mathrm{Tr}_s\left(\frac{\partial C_t}{\partial t}\exp -C_t^2\right)$ *vanishes if* ∇_ξ *respects* h^ξ *and* $\dim Z$ *is odd.*

Proof. — For any $\mathtt{w} \in TZ$, $c(\mathtt{w})$ commutes with $*_Z$ if $\dim Z$ is odd and it anticommutes with $*_Z$ if $\dim Z$ is even. Thus $\iota_T - T\wedge = -c(T)$ also does. Then it follows from (96) that:

$$\frac{\sqrt{t}}{2}(d^{\nabla_\xi} + (d^{\nabla_\xi})^*) + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge) =$$
$$= -(-1)^{\dim Z} *_Z^{-1}\left(\frac{\sqrt{t}}{2}(d^{\nabla_\xi^*} + (d^{\nabla_\xi^*})^*) + \frac{1}{2\sqrt{t}}(\iota_T - T\wedge)\right) *_Z .$$

Lemma 56 has the consequence that the covariant derivative with respect to $\overline{\nabla}^{u}$ commutes with $*_Z$. For $\mathbb{Z}_2$-graduation reasons, this proves that the exterior derivative associated with $\overline{\nabla}^{u}$ on End$E$-valued differential forms on B supercommutes with $*_Z$.

Let N_H be the graduation operator on $\wedge T^*B$ which multiplies k-degree differential forms by k, the properties above give the following formula:

(104)
$$C_t = -(-1)^{\dim Z}(-1)^{N_H} *_Z^{-1} \check{C_t} *_Z (-1)^{N_H}.$$

Put $\mathrm{Id}_\mu = \mathrm{Id}_{\mu^+} \oplus \mathrm{Id}_{\mu^-}$. Then, using (97) instead of (96) one obtains

$$B_t = -(-1)^{\dim Z}(-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} B_t^{\check{}}(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}. \tag{105}$$

Now it successively follows that

$$\begin{aligned}
B_t^2 &= (-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} B_t^{\check{}2}(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}\\
\exp(-B_t^2) &= (-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1} \exp(-B_t^{\check{}2})(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}\\
\frac{\partial B_t}{\partial t}\exp(-B_t^2) &=\\
&= -(-1)^{\dim Z}(-1)^{N_H}(*_Z \oplus \mathrm{Id}_\mu)^{-1}\frac{\partial B_t^{\check{}}}{\partial t}\exp(-B_t^{\check{}2})(*_Z \oplus \mathrm{Id}_\mu)(-1)^{N_H}.
\end{aligned}$$

In this context of infinite rank vector bundles, it remains true that the supertrace of the supercommutator of two L^2-bounded End$\mathcal{E}$-valued differential forms, one of which is trace class, vanishes. Using the fact that Id_μ has the same parity as $*_Z$, one can apply this to $[(*_Z \oplus \mathrm{Id}_\mu)^{-1}, \omega(*_Z \oplus \mathrm{Id}_\mu)]$ to obtain

$$\mathrm{Tr}_s(\omega) = \mathrm{Tr}_s\big((*_Z \oplus \mathrm{Id}_\mu)^{-1}\omega(*_Z \oplus \mathrm{Id}_\mu)\big)$$

which is valid for any globally odd End$(\mathcal{E} \oplus \mu)$-valued trace-class differential form ω, in particular for $\frac{\partial B_t^{\check{}}}{\partial t}\exp(-B_t^{\check{}2})$. Thus

$$\mathrm{Tr}_s\left(\frac{\partial B_t}{\partial t}\exp(-B_t^2)\right) = -(-1)^{\dim Z}(-1)^{N_H}\mathrm{Tr}_s\left(\frac{\partial B_t^{\check{}}}{\partial t}\exp(-B_t^{\check{}2})\right)(-1)^{N_H}.$$

The fact that this form and $\frac{\partial B_t}{\partial t}\exp(-B_t^2)$ and $\frac{\partial B_t^{\check{}}}{\partial t}\exp(-B_t^{\check{}2})$ are globally odd, implies that only their odd differential form degree parts contribute to their supertrace. The equation of the lemma follows. □

Suppose now that $\dim Z$ is odd. Denote by 0 the connection on the null rank vector bundle $\{0\}$ on B. Consider any element $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. Choose any suitable data (μ^+, μ^-, ψ) giving rise to family index bundles $\mathcal{K}^+$ and $\mathcal{K}^-$. Let $[\ell_{\mathcal{K}}^{\{0\}}]$ be the canonical class of links between $\mathcal{K}^+ - \mathcal{K}^-$ and $\{0\} - \{0\}$ obtained just before (98). Then

$$\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) = \Big(\{0\}, 0, \int_Z e(\nabla_{TZ}) \wedge \alpha\Big) - \big(\{0\}, 0, \eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathcal{K}}^{\{0\}}]^{-1})\big). \tag{106}$$

$e(\nabla_{TZ})$ vanishes. Theorem 33 is then reduced to the following lemma. Note that all arguments were local, but (106) and the following lemma have global meaning, so that the arguments also work for noncompact B.

Lemma 60. — *We have* $\eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathcal{K}}^{\{0\}}]^{-1}) = 0$.

Proof. — Compare $\eta(\nabla_\xi, \nabla_{TZ}, 0, 0, [\ell_{\mathcal{K}}^{\{0\}}]^{-1})$ with the form η computed from the "adjoint" triple $\big(\mu^-, \mu^+, (*_Z \oplus \mathrm{Id}_{\mu^+}) \circ \psi^* \circ (*_Z^{-1} \oplus \mathrm{Id}_{\mu^-})\big)$. They are of course equal because $\pi_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)$ can be written with the same formula by simply replacing η by the other one. The terms of the form $\int_Z e(\nabla_{TZ})\widetilde{\mathrm{ch}}(\nabla^u_\xi, \nabla_\xi)$ vanish in both cases. The terms of the form $\int_0^1 \frac{\partial B_t}{\partial t} \exp(-B_t^2)$ are mutually opposite from the preceding lemma. Denote by $\mathscr{K}^{*\pm}$ the "adjoint" family index bundles, it follows from (87) and the second equation of Proposition 55 that

$$\widetilde{\mathrm{ch}}([\ell_{\mathscr{K}}^{\{0\}}]^{-1}) + \widetilde{\mathrm{ch}}([\ell_{\mathscr{K}}^{\{0\}}]^{-1}) = -\widetilde{\mathrm{ch}}([\ell^*_{\mathscr{K}}]).$$

But $\widetilde{\mathrm{ch}}([\ell^*_{\mathscr{K}}])$ vanishes because the $(*_Z + \mathrm{Id}_\mu)$ isomorphism respect the connections (81) on kernel bundles. Thus both η forms are mutually opposite. □

7. Double fibrations

Consider two proper submersions $\pi_1 : M \longrightarrow B$ and $\pi_2 : B \longrightarrow S$ and the composed submersion $\pi_2 \circ \pi_1 : M \longrightarrow S$. The goal of this section is to compare direct image with respect to $\pi_2 \circ \pi_1$ and the composition of the two direct images relative to π_1 and π_2 for topological, relative and multiplicative/smooth K-theories, and then to prove Theorems 34 and 35. Unless otherwise stated, S is supposed to be compact.

7.1. Topological K-theory. — Consider some vector bundle ξ on M, some couples of family index bundles $(\mathscr{H}^+, \mathscr{H}^-)$ and $(\mathscr{F}^+, \mathscr{F}^-)$ for ξ relatively to π_1 and to $\pi_2 \circ \pi_1$ respectively and some couple $(\mathscr{G}^+, \mathscr{G}^-)$ of family index bundles for $\mathscr{H}^+ - \mathscr{H}^-$ (with respect to π_2).

Theorem 61. — *There exists some canonical equivalence class of links* $[\ell^{\mathscr{F}}_{\mathscr{G}}]$ *between* $\mathscr{G}^+ - \mathscr{G}^-$ *and* $\mathscr{F}^+ - \mathscr{F}^-$.

This implies the functoriality of π^{Eu} for double submersions of compact manifolds, and hence in full generality (see Definition 20).

The canonicity is to understand in the same sense that in Theorem 25. The construction of $[\ell^{\mathscr{F}}_{\mathscr{G}}]$ uses the convergence of Euler operators under adiabatic limits. The point is to obtain some spectral convergence which allows to understand the behaviour of the kernel and of eigenvalues converging to 0 in this limit. We closely follow the analysis performed in [**5**] §5 with some analogue of [**5**] Theorem 5.28 and formula (5.118) as goal. In fact we want to connect family index bundles on M for ξ and on B for $\mathscr{H}^\pm$. We will combine spectral convergence with the fact that if $a > 0$ is such that the Euler operator along the fibres of $\pi_2 \circ \pi_1$ has no eigenvalue equal to a nor $-a$ along S, then the eigenspaces corresponding to eigenvalues lying in $[-a, +a]$ form vector bundles on S which are themselves family index bundles. (This was already used in §4.2.3).

7.1.1. *Fiberwise exterior differentials:*— We precise (in (108)) the decomposition of the exterior differential along horizontal and vertical differential form degrees corresponding to [**5**] Theorem 5.1.

The respective vertical tangent vector bundles associated with π_1, π_2 and $\pi_2 \circ \pi_1$ will be denoted by TM/B, TB/S and TM/S. Choose some connection ∇_ξ on ξ along M. Call $\mathcal{E}^{\pm}_{M/S}$ (resp. $\mathcal{E}^{\pm}_{M/B}$) the infinite rank vector bundles on S (resp. B) of even/odd degree differential forms along the fibres of $\pi_2 \circ \pi_1$ (resp. π_1) with values in ξ. Choose any smooth supplementary subbundle T^HM of TM/B in TM/S. Of course $T^HM \cong \pi_1^* TB/S$. On the fibre of $\pi_2 \circ \pi_1$ over any point s of S, one obtains for ξ-valued differential forms an isomorphism analogous to (37):

$$(107) \qquad \mathcal{E}_{M/S} \cong \Omega\big(\pi_2^{-1}(\{s\}), \mathcal{E}_{M/B}\big).$$

For any $b \in B$ and any tangent vector $U \in T_bB/S$, call U^H its horizontal lift as a section of T^HM over $\pi_1^{-1}(b)$. For any $s \in S$, the construction of (72) produces a connection on the restriction of $\mathcal{E}_{M/B}$ over $\pi^{-1}(\{s\})$ which will be denoted $\overline{\nabla}$. We will denote by d^H the exterior differential operator on $\Omega\big(\pi_2^{-1}(\{s\}), \mathcal{E}_{M/B}\big) \cong \mathcal{E}_{M/S}$ associated to $\overline{\nabla}$.

The "vertical" differential operator d^{∇_ξ} will be denoted by $d^{M/B}$ on $\mathcal{E}_{M/B}$ and $d^{M/S}$ on $\mathcal{E}_{M/S}$. As was remarked at the beginning of §5.2,

$$(108) \qquad d^{M/S} = d^{M/B} + d^H + \iota_T \qquad \text{(through the identification (107))}$$

where ι_T here stands for the restrictions to the fibres of $\pi_2 \circ \pi_1$ of the operator $\iota_T \in \wedge^2(TB/S)^* \otimes \mathrm{End}^{\mathrm{odd}}(\mathcal{E}_{M/B})$ of §5.2. We will consider this ι_T as an element of $\mathrm{End}\mathcal{E}_{M/S}$ (through the identification (107)).

7.1.2. *Fiberwise Euler operators.* — Here we precise (in (109) and 110) the dependence of the Euler operator on the parameter θ of the adiabatic limit. This corresponds to [**5**] Definition 5.5.

Endow TM/B with some (riemannian) metric $g^{M/B}$ and ξ with some hermitian metric h^ξ. Take some riemannian metric $g^{B/S}$ on TB/S. Put on TM/S the riemannian metric for which the decomposition $TM/S = TM/B \oplus T^HM$ is orthogonal and which coincides with $g^{M/B}$ and $\pi_1^* g^{B/S}$ on either parts. The adjoints in $\mathrm{End}\mathcal{E}_{M/S}$ will be considered with respect to the L^2 scalar product on $\mathcal{E}_{M/S}$ obtained from h^ξ and this riemannian metric (as in (38)). These are not the adjoints (neither usual nor special) considered on $\Omega\big(\pi_2^{-1}(\{s\}), \mathcal{E}_{M/B}\big)$ in §5.1. For instance, the adjoint ι_T^* of ι_T here is not $T\wedge$ as it was in §5.2.

Let $d^{M/B*}$ be the adjoint of $d^{M/B}$ with respect to $g^{M/B}$ and h^ξ as endomorphisms of $\mathcal{E}_{M/B}$, then $d^{M/B}$ and $d^{M/B*}$ are also adjoint as endomorphisms of $\mathcal{E}_{M/S}$ because of the choice of a riemannian submersion metric on M. Call d^{H*} the adjoint of d^H as endomorphism of $\mathcal{E}_{M/S}$ and put:

$$(109)\qquad \begin{aligned} d^\theta &= d^H + \frac{1}{\theta} d^{M/B} + \theta\iota_T \quad \text{and} \quad d^{\theta *} = d^{H*} + \frac{1}{\theta} d^{M/B*} + \theta\iota_T^* \\ \mathcal{D}^H &= d^H + d^{H*} \quad \text{and} \quad \mathcal{D}^V = d^{M/B} + d^{M/B*} \\ \mathcal{D}^\theta &= d^\theta + d^{\theta*} = \mathcal{D}^H + \frac{1}{\theta}\mathcal{D}^V + \theta(\iota_T + \iota_T^*). \end{aligned}$$

Let N_V be the endomorphism of $\mathcal{E}_{M/B}$ defined as in §5.2.3. N_V multiplies k-degree vertical forms by k, "vertical" meaning forms along the fibres of π_1. Using T^HM, N_V extends to an operator on $\mathcal{E}_{M/S}$ through the identification (107). Let g_θ be the riemannian metric on M such that TM/B and T^HM are orthogonal and which restricts to $g^{M/B}$ and $\frac{1}{\theta^2}\pi_1^* g^B$ on either part, the observation here is that

$$(110)\qquad \mathcal{D}^\theta = \theta^{N_V}(d^{M/S} + d_\theta^{M/S*})\theta^{-N_V}$$

where $d_\theta^{M/S*}$ is the adjoint of $d^{M/S}$ with respect to g_θ and h^ξ. The riemanian submersion metric chosen here simplifies considerably the form of $\mathcal{D}^\theta$ with respect to the case of [**5**] where such a choice is not allowed and forces more complicated conjugations than by θ^{N_V} (see [**5**] §5(a)).

7.1.3. *Introducing some intermediate suitable triple.* — In the adiabatic limit, the ξ-twisted Euler operator on M should converge to the Euler operator on B twisted by the kernel bundles on B for ξ with respect to π_1. In the general setting considered here, this forces to introduce some suitable triple with respect to π_1 in the global Euler operator. This is performed here, the induced 2×2-matrix decomposition of the modified $\mathcal{D}_\theta$ is presented and the first estimates on the matrix elements are obtained by analogy with [**5**] §5.

Consider some suitable triple (μ^+, μ^-, ψ) with respect to π_1 (and ξ with ∇_ξ and h^ξ and $g^{M/B}$). $\mu^\pm$ are endowed with some hermitian metrics. Choose some connection ∇_μ on $\mu^\pm$ (which respects either part) and consider the associated Euler operator $\mathcal{D}^{\nabla_\mu} = d^{\nabla_\mu} + (d^{\nabla_\mu})^*$ on $\Omega(B/S, \mu^\pm)$. Take some function χ as in §5.2.5. For $\theta \in (0, 1]$, one puts

$$(111)\qquad \begin{aligned} \mathcal{D}_\psi^\theta &= \mathcal{D}^\theta + \mathcal{D}^\mu + \frac{1-\chi(\theta)}{\theta}(\psi + \psi^*) \\ &= \mathcal{D}^H + \mathcal{D}^\mu + \frac{1}{\theta}\mathcal{D}^V_{(1-\chi(\theta))\psi} + \theta(\iota_T + \iota_T^*) \in \operatorname{End}^{\mathrm{odd}}\left(\mathcal{E}_{M/S} \oplus \Omega(B/S, \mu)\right) \end{aligned}$$

where $\mathcal{D}^V_{(1-\chi(\theta))\psi}$ is obtained from $\mathcal{D}^V$ and $(1 - \chi(\theta))\psi$ as $\mathcal{D}_\psi^{\nabla_\xi}$ is from $\mathcal{D}^{\nabla_\xi}$ and ψ in (39). Here ψ is extended to forms on B/S through the isomorphism (107). The choice of a riemannian submersion metric on TM ensures the compatibility of the adjunctions of ψ before and after extending it to forms on B/S. This result (111) corresponds to [**5**] Proposition 5.9 with $\theta = \frac{1}{T}$ and with the extra term $\theta(\iota_T + \iota_T^*)$.

There is a double decomposition associated to $\mathcal{D}_\psi^V$.

$$\mathcal{E}_{M/B} \oplus \mu^\pm = \operatorname{Ker}\mathcal{D}_\psi^V \oplus (\operatorname{Ker}\mathcal{D}_\psi^V)^\perp \tag{112}$$

which gives a double decomposition of infinite rank vector bundles on S:

$$\mathcal{E}_{M/S} \oplus \Omega(B/S, \mu^\pm) = \Omega\big(B/S, \operatorname{Ker}\mathcal{D}_\psi^V\big) \oplus \Omega\big(B/S, (\operatorname{Ker}\mathcal{D}_\psi^V)^\perp\big).$$

The choice of a riemannian submersion metric induces that the second one is orthogonal: let $p\colon \mathcal{E}_{M/S} \oplus \Omega(B/S, \mu^\pm) \longrightarrow \Omega\big(B/S, \operatorname{Ker}\mathcal{D}_\psi^V\big)$ be the orthogonal projection, it is the tensor product of the identity in $\Omega(B/S)$ and the orthogonal projection on the first factor of (112). Put $p^\perp = \operatorname{Id} - p$. For any positive θ one decomposes the operator $\mathcal{D}_\psi^\theta$ as a 2×2 matrix:

$$\mathcal{D}_\psi^\theta = \begin{pmatrix} p\mathcal{D}_\psi^\theta p & p\mathcal{D}_\psi^\theta p^\perp \\ p^\perp \mathcal{D}_\psi^\theta p & p^\perp \mathcal{D}_\psi^\theta p^\perp \end{pmatrix} =: \begin{pmatrix} A_1^\theta & A_2^\theta \\ A_3^\theta & A_4^\theta \end{pmatrix}.$$

As in §5.2.5, the vector bundle $\operatorname{Ker}\mathcal{D}_\psi^V$ is endowed with the restriction of the metric on $\mathcal{E}_{M/B} \oplus \mu^\pm$, and with the connection $\nabla_{\mathcal{H}}$ obtained by projecting the connection on $\mathcal{E}_{M/B} \oplus \mu$ onto it (in fact $p(\overline{\nabla} \oplus \nabla_\mu)p$, see [**5**] Theorem 5.1 and formula (5.34)). Because of the compatibility of orthogonal projections, the exterior differential operator on $\Omega\big(B/S, \operatorname{Ker}\mathcal{D}_\psi^V\big)$ associated to this connection is $d^{\nabla_{\mathcal{H}}} = p(d^H \oplus d^{\nabla_\mu})p$. Clearly $(d^{\nabla_{\mathcal{H}}})^* = p\big(d^{H*} \oplus (d^{\nabla_\mu})^*\big)p$. Define then $\mathcal{D}^{\nabla_{\mathcal{H}}} = d^{\nabla_{\mathcal{H}}} + (d^{\nabla_{\mathcal{H}}})^*$. For any $\theta \leq \frac{1}{2}$ (to ensure that $\chi(\theta) = 0$), one has

$$A_1^\theta = \mathcal{D}^{\nabla_{\mathcal{H}}} + \theta p(\iota_T + \iota_T^*)p. \tag{113}$$

Of course $p(\iota_T + \iota_T^*)p$ is a bounded operator in the L^2-topology, and this remark with the above equation replaces here equation (5.35) of Theorem 5.1 in [**5**].

In the same way, for $\theta \in [0, \frac{1}{2}]$

$$\begin{aligned} & A_2^\theta = p\big((d^H + d^{H*}) \oplus (d^{\nabla_\mu} + (d^{\nabla_\mu})^*)\big)p^\perp + \theta p(\iota_T + \iota_T^*)p^\perp \\ \text{and}\quad & A_3^\theta = p^\perp\big((d^H + d^{H*}) \oplus (d^{\nabla_\mu} + (d^{\nabla_\mu})^*)\big)p + \theta p^\perp(\iota_T + \iota_T^*)p \end{aligned} \tag{114}$$

are uniformly bounded operators in the L^2-topology. This is because of the choice of the riemannian submersion metric and is a simplification with respect to the corresponding result Proposition 5.18 of [**5**].

7.1.4. *Estimates on the operator A_4^θ.* — First one wants to obtain results analogous to [**5**] Theorems 5.19 and 5.20. There are three differences between the situation here and [**5**]. The absence of conjugation (by C_T in [**5**] Definition 5.4 and (5.10)) due to the choice of a riemannian submersion metric is a simplification and does not create any obstacle; the presence here of the term $\theta(\iota_T + \iota_T^*)$ does not change these results because of the fact that $\iota_T + \iota_T^*$ is a bounded operator in the L^2-topology and because of its factor θ; more seriously, the commutator $[D^Z_\infty, D^H_\infty]$ in [5] which corresponds to $[\mathcal{D}^V, \mathcal{D}^H]$ in the notations here, is to be replaced by

$$[\mathcal{D}^V_\psi, \mathcal{D}^H + \mathcal{D}^\mu] = [\mathcal{D}^V, \mathcal{D}^H] + [\psi + \psi^*, \mathcal{D}^H + \mathcal{D}^\mu].$$

Of course the first term has the required majoration property [5] (5.67). The operator $\psi + \psi^*$ is a fiberwise kernel operator (along the fibres of π_1), and its kernel is smooth along the fibered double $M \times_B M$. Thus, if v is a smooth vector field on B, the commutator $[\psi+\psi^*, \nabla^{\wedge^\bullet T^*M/B\otimes\xi}_{\mathrm{v}^H} \oplus \nabla^\mu_{\mathrm{v}}]$ (where v^H is the horizontal lift of v, a section of T^HM), is a fiberwise kernel operator with globally smooth kernel. In particular, it is bounded in L^2-topology, and so is the (super)commutator $[\psi + \psi^*, \mathcal{D}^H + \mathcal{D}^\mu]$. The estimate [5] (5.67) then follows from [5] (5.61) (whose equivalent here holds true).

The conclusions of Theorems 5.19 and 5.20 of [5] remain thus valid here, namely the existence of some constant C such that for any $\theta \leq \frac{1}{2}$ and any section s of $\mathcal{E}_{M/S} \oplus \mu^\pm$

$$\|A^\theta_4(p^\perp s)\|_{L^2} \geq C\left(\|p^\perp s\|_{H^1} + \frac{1}{\theta}\|p^\perp s\|_{L^2}\right). \tag{115}$$

where $\| \ \|_{H^1}$ stands for the usual Sobolev H^1-norm.

Secondly, one needs some equivalent of [5] Proposition 5.22, particularly the estimate (5.71) contained in it. But the proof here is in fact easier than in [5] because Equation (114) provides a simplification of the corresponding Proposition 5.18 in [5], the extra term $\theta(\iota_T + \iota^*_T)$ is a uniformly bounded operator, $(\psi+\psi^*)$ too, and $\frac{1}{\theta}(\psi+\psi^*)$ is part of A^θ_4, it does not disable the ellipticity of A^θ and it is taken into account in the obtained estimates: there exist constants c, C and θ_0 such that for any $\theta \leq \theta_0$, $\lambda \in \mathbb{C}$ such that $|\lambda| \leq \frac{c}{2\theta}$ and any $s \in \mathcal{E}_{M/S} \oplus \mu^\pm$,

$$\begin{aligned} &\|(\lambda - A^\theta_4)^{-1} p^\perp s\|_{L^2} \leq C\theta \|p^\perp s\|_{L^2} \\ &\|(\lambda - A^\theta_4)^{-1} p^\perp s\|_{H^1} \leq C \|p^\perp s\|_{L^2}. \end{aligned} \tag{116}$$

7.1.5. *Spectral convergence of Euler operators.* — Our goal is to follow kernel bundles. This makes us now introduce some suitable triple $(\zeta^+, \zeta^-, \varphi)$ for $\mathcal{D}^{\nabla_{\mathcal{H}}}$. We extend p and $p^\perp$ to $\mathcal{E}_{M/S} \oplus \zeta^\pm$ in the following way: p induces the identity on $\zeta^\pm$ and $p^\perp$ induces the null map on $\zeta^\pm$. Consider then

$$\begin{aligned} \mathcal{D}^\theta_{\psi,\varphi} &= \begin{pmatrix} p(\mathcal{D}^\theta_\psi + (1-\chi(\theta))(\varphi+\varphi^*))p & p\mathcal{D}^\theta_\psi p^\perp \\ p^\perp \mathcal{D}^\theta_\psi p & p^\perp \mathcal{D}^\theta_\psi p^\perp \end{pmatrix} \\ &= \begin{pmatrix} A^\theta_1 + (1-\chi(\theta))(\varphi+\varphi^*) & A^\theta_2 \\ A^\theta_3 & A^\theta_4 \end{pmatrix}. \end{aligned}$$

(It is not essential that the same function χ appears here and in (111)). Equation (113) obviously leads to the following equality for $\theta \in [0, \frac{1}{2}]$:

$$A^\theta_1 + (1-\chi(\theta))(\varphi+\varphi^*) = \mathcal{D}^{\nabla_{\mathcal{H}}} + \varphi + \varphi^* + \theta p(\iota_T + \iota^*_T)p = \mathcal{D}^{\nabla_{\mathcal{H}}}_\varphi + \theta p(\iota_T + \iota^*_T)p \tag{117}$$

with the same remark (as after (113)) that $p(\iota_T + \iota_T^*)p$ is bounded.

Using this, the remark after (114) above and (116) instead of [**5**](5.35), (5.49) and (5.71) respectively, the analysis performed in [**5**] §§5(d) and (g) applies here. The only difference is that the following equivalent here of the first line of [**5**] (5.89) (for the usual norm of bounded operators in L^2-topology) is NOT true

$$\left\| \left(A_1^\theta + (1-\chi(\theta))(\varphi+\varphi^*) - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}}\right)(\lambda - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}})^{-1} \right\| \leq C\theta^2(1+|\lambda|). \tag{118}$$

The set U_T (or $U_{\frac{1}{\theta}}$) where λ is supposed to lie, defined in [**5**] (5.76), is such that $|\lambda| \leq c_1 T$ (or $\frac{c_1}{\theta}$) and $\|(\lambda - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}})^{-1}\| \leq \frac{c_2}{4}$ for some constants c_1 and c_2. But only the following consequence of (118)

$$\left\| \left(A_1^\theta + (1-\chi(\theta))(\varphi+\varphi^*) - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}}\right)(\lambda - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}})^{-1} \right\| \leq C\theta$$

is needed for establishing the equivalent of [**5**] (5.90). This last estimate can be obtained here directly from the remark after (117) and the properties of $U_{\frac{1}{\theta}}$.

One obtains firstly the convergence of the resolvent of $\mathcal{D}_{\psi,\varphi}^\theta$ to any great enough positive integral power $(\lambda - \mathcal{D}_{\psi,\varphi}^\theta)^{-k}$ to $p(\lambda - \mathcal{D}_\varphi^{\nabla_{\mathcal{H}}})^{-k}p$ in the sense of the norm $\|A\|_1 = \operatorname{tr}(A^*A)^{\frac{1}{2}}$ ([**5**] Theorem 5.28), and secondly the convergence of the spectral projector of $\mathcal{D}_{\psi,\varphi}^\theta$ with respect to eigenvalues of absolute value bounded by some suitable positive constant a to the orthogonal projector onto the kernel of $\mathcal{D}_\varphi^{\nabla_{\mathcal{H}}}$ ([**5**] equation (5.118)). Thus

Theorem 62. — *There exists some $\varepsilon_2 > 0$ and $a > 0$, and a vector bundle $\mathcal{K}$ on $S \times [0,\varepsilon_2]$ such that $\mathcal{K}|_{S\times\{0\}} \cong \operatorname{Ker}\mathcal{D}_\varphi^{\nabla_{\mathcal{H}}}$ and $\mathcal{K}|_{S\times\{t\}}$ identifies with the direct sum of eigenspaces of $\mathcal{D}_{\psi,\varphi}^\theta$ corresponding to (all) eigenvalues lying in $[-a,+a]$.*

This is because all the used estimates are uniform along S, which is compact.

7.1.6. *Construction of the canonical link (proof of Theorem 61).* — The above eigenspaces are also eigenspaces for the squared operator $\left(\mathcal{D}_{\psi,\varphi}^\theta\right)^2$, they are thus naturally $\mathbb{Z}_2$-graded, and for any nonzero eigenvalue, $\mathcal{D}_{\psi,\varphi}^\theta$ exchanges bijectively the positive and negative degree parts. (In particular, there is no nonzero eigenvalue in $[-a,a]$ if $(\zeta^+,\zeta^-,\varphi)$ is a positive kernel triple).

In any case, on $S\times\{0\}$, $(\mathcal{K}^+,\mathcal{K}^-)$ are kernel bundles so that in $K^0_{\mathrm{top}}(S)$:

$$\begin{aligned}[\mathcal{K}^+|_{S\times\{0\}} \oplus \zeta^-] - [\mathcal{K}^-|_{S\times\{0\}} \oplus \zeta^+] &= \pi_{2*}^{\mathrm{Eu}}\big([\operatorname{Ker}\mathcal{D}_\psi^{V+}] - [\operatorname{Ker}\mathcal{D}_\psi^{V-}]\big)\\ &= \pi_{2*}^{\mathrm{Eu}}\big(\pi_{1*}^{\mathrm{Eu}}([\xi]) + [\mu^+] - [\mu^-]\big).\end{aligned}$$

The constructions of §4.1.2, §4.1.3 and §4.1.4 can be applied to $\mathcal{D}_\psi^\theta$ on any compact subset of $S\times(0,1]$. This is because $\mathcal{D}_\psi^\theta$ is the sum of the fiberwise elliptic operator $\mathcal{D}^\theta \oplus \mathcal{D}^\mu$ and an order 0 pseudo-differential operator, which does not destroy the ellipticity. (Only ellitpic regularity is needed to construct suitable triples). This does not work on $[0,1]$ because of the explosion of A_4^θ (115).

Choose $\varepsilon_1 \in (0,\varepsilon_2)$ and some suitable triple $(\lambda^+,\lambda^-,\phi)$ for $\mathcal{D}_\psi^\theta$ with respect to the submersion $(\pi_2\circ\pi_1)\times \mathrm{Id}_{[\varepsilon_1,1]}$. One then obtains kernel bundles $\mathcal{L}^\pm$ on $S\times[\varepsilon_1,1]$ which verify the following equality in $K^0_{\mathrm{top}}(S)$ for any $\theta\in[\varepsilon_1,1]$

$$[(\mathcal{L}^+\oplus\lambda^-)|_{S\times\{\theta\}}]-[(\mathcal{L}^-\oplus\lambda^+)|_{S\times\{\theta\}}]=(\pi_2\circ\pi_1)_*^{\mathrm{Eu}}[\xi]+\pi_{2*}^{\mathrm{Eu}}([\mu^+]-[\mu^-]).$$

This is clear on $S\times\{1\}$ and spreads by parallel transport along $[\varepsilon_1,1]$.

One obtains a class of links between $(\mathcal{K}^+|_{S\times\{0\}}\oplus\zeta^-)-(\mathcal{K}^-|_{S\times\{0\}}\oplus\zeta^+)$ and $((\mathcal{L}^+\oplus\lambda^-)|_{S\times\{1\}})-((\mathcal{L}^-\oplus\lambda^+)|_{S\times\{1\}})$ by composing the parallel transport along $[0,\varepsilon_2]$ for $\mathcal{K}$, the canonical link between $(\mathcal{K}^+|_{S\times\{t\}}\oplus\zeta^-)-(\mathcal{K}^-|_{S\times\{t\}}\oplus\zeta^+)$ and $((\mathcal{L}^+\oplus\lambda^-)|_{S\times\{t\}})-((\mathcal{L}^-\oplus\lambda^+)|_{S\times\{t\}})$ of Theorem 25 (which may be applied to $\mathcal{D}_\psi^\theta$) for any $t\in[\varepsilon_1,\varepsilon_2]$ and parallel transport again along $[\varepsilon_1,1]$ for $\mathcal{L}$.

Choose any couple of family index bundles (ν^+,ν^-) for $\mu^+-\mu^-$.

Definition 63. — *The canonical equivalence class of links $[\ell_{\mathcal{G}}^{\mathcal{F}}]$ of Theorem 61 is obtained by composing the above link with the canonical links of Theorem 25 between $(\mathcal{G}^+\oplus\nu^+)-(\mathcal{G}^-\oplus\nu^-)$ and $(\mathcal{K}^+|_{S\times\{0\}}\oplus\zeta^-)-(\mathcal{K}^-|_{S\times\{0\}}\oplus\zeta^+)$ on one hand, and $((\mathcal{L}^+\oplus\lambda^-)|_{S\times\{1\}})-((\mathcal{L}^-\oplus\lambda^+)|_{S\times\{1\}})$ and $(\mathcal{F}^+\oplus\nu^+)-(\mathcal{F}^-\oplus\nu^-)$ on the other hand.*

This class of links is clearly independent of the choice of ν^+ or ν^-, or of the triples $(\lambda^+,\lambda^-,\phi)$ or $(\zeta^+,\zeta^-,\varphi)$ because of the global compatibility of links obtained from Theorem 25.

Now take two systems of suitable data (μ_1^+,μ_1^-,ψ_1) and (μ_2^+,μ_2^-,ψ_2) with respect to π_1 (and ξ with ∇_ξ and h^ξ and $g^{M/B}$). There is a link (as constructed in §4.1.3) between $(\mathrm{Ker}\,\mathcal{D}_{M/B}^{\psi_1+}\oplus\mu_1^-)-(\mu_1^+\oplus\mathrm{Ker}\,\mathcal{D}_{M/B}^{\psi_1-})$ and $(\mathrm{Ker}\,\mathcal{D}_{M/B}^{\psi_2+}\oplus\mu_2^-)-(\mu_2^+\oplus\mathrm{Ker}\,\mathcal{D}_{M/B}^{\psi_2-})$. This link is obtained by constructing a families index map for a submersion of the form $\pi_1\times\mathrm{Id}_{[0,1]}: M\times[0,1]\longrightarrow B\times[0,1]$. This construction can be extended to the case of a double submersion in the following form $M\times[0,1]\overset{\pi_1\times\mathrm{Id}_{[0,1]}}{\longrightarrow}B\times[0,1]\overset{\pi_2\times\mathrm{Id}_{[0,1]}}{\longrightarrow}S\times[0,1]$, and the compatibility of canonical links either for linked data or for one and for two submersions follows.

7.2. Flat and relative K-theory. — The first goal of this section is to explain why $(\pi_2\circ\pi_1)_!=\pi_{2!}\circ\pi_{1!}$ on K^0_{flat}: this is a by-product of the Leray spectral sequence (see §7.2.1). It is well known that the Leray spectral sequence fits with the adiabatic limit of the preceding section, the goal of §7.2.2 is to explain how this traduces in the language of links. This is needed in to prove Theorem 34 in §7.2.3.

7.2.1. *Leray spectral sequence.* — Consider some flat vector bundle (F, ∇_F) on M, let $G^\bullet = \pi_{1!}^\bullet F$ with flat connections $\nabla_{G^\bullet}$, and $H^\bullet = (\pi_2 \circ \pi_1)_!^\bullet F$ with flat connections $\nabla_{H^\bullet}$. Note that here the full $\mathbb{Z}$-graduation is needed and not only the parity $\mathbb{Z}_2$-graduation.

The vertical F-valued de Rham complex $\Omega^\bullet(M/S, F)$ along M/S is filtrated by the horizontal degree: for any p, $F^p\Omega^\bullet(M/S, F)$ consists of F-valued differential forms whose interior product with more than p elements of TM/B vanishes. Thus $H^\bullet$ is also filtrated from this filtration: $F^pH^\bullet$ consists of classes which can be represented by some element in $F^p\Omega^\bullet(M/S, F)$. This filtration is compatible with the flat connections of $H^\bullet$, so that for any p and k,

$$0 \longrightarrow F^{p+1}H^k \longrightarrow F^pH^k \longrightarrow F^pH^k/F^{p+1}H^k \longrightarrow 0 \tag{119}$$

is a parallel exact sequence of flat bundles. The corresponding flat connections will be respectively denoted by $\nabla_{F^{p+1}H^k}$, $\nabla_{F^pH^k}$ and $\nabla_{p/k}$.

It is proved in [**31**] Proposition 3.1 that the associated spectral sequence gives rise to flat vector bundles $(E_r^{p,q}, \nabla_r^{p,q})$ on S with flat (parallel) spectral sequence morphisms $d_r \colon E_r^{p,q} \longrightarrow E_r^{p-r,q+r+1}$ (and $E_{r+1}^{p,q} = \mathrm{Ker} d_r|_{E_r^{p,q}}/(\mathrm{Im} d_r \cap E_r^{p,q})$).

It is a classical fact (see [**31**] Theorem 2.1) that this spectral sequence is isomorphic to the Leray spectral sequence, and thus $E_2^{p,q} \cong H^p(B/S, G^q)$ while for all sufficiently great r one has $E_r^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$.

Put $E_r^+ = \bigoplus_{p+q \text{ even}} E_r^{p,q}$ and $E_r^- = \bigoplus_{p+q \text{ odd}} E_r^{p,q}$, and denote their direct sum (flat) connections by ∇_r^+ and ∇_r^-. Applying Lemma 42 to the complexes

$$\cdots \xrightarrow{d_r} E_r^{p+r,q-r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p-r,q+r+1} \xrightarrow{d_r} \cdots \tag{120}$$

proves in particular that $[E_r^+, \nabla_r^+] - [E_r^-, \nabla_r^-] \in K^0_{\mathrm{flat}}(S)$ is independent of r.

For $r = 2$, this is nothing but $\pi_{2!}([G^+, \nabla_{G^+}] - [G^-, \nabla_{G^-}]) = \pi_{2!}(\pi_{1!}[F, \nabla_F])$.

On the other hand, it follows from (119) that the element

$$[F^pH^\bullet, \nabla_{F^pH^\bullet}] + \sum_{i=0}^{p-1}[F^iH^\bullet/F^{i+1}H^\bullet, \nabla_{i/\bullet}] \in K^0_{\mathrm{flat}}(S)$$

is independent of p. For $p = 0$, it equals $[H^\bullet, \nabla_H] = (\pi_2 \circ \pi_1)_![F, \nabla_F]$, while for sufficienly great p and r, it equals $[E_r^+, \nabla_r^+] - [E_r^-, \nabla_r^-]$. Thus

Proposition. — *We have* $\pi_{2!} \circ \pi_{1!} = (\pi_2 \circ \pi_1)_! \; : \; K^0_{\mathrm{flat}}(M) \longrightarrow K^0_{\mathrm{flat}}(S)$.

7.2.2. *Compatibility of topological and sheaf theoretic links*

One has now two classes of links between $E_2^+ - E_2^- \cong \pi_{2!}(\pi_{1!}[F, \nabla_F])$ and $H^+ - H^- \cong (\pi_2 \circ \pi_1)_![F, \nabla_F]$: the link $[\ell_{\mathrm{top}}]$ constructed in subSection 7.1.6 and the sheaf theoretic one $[\ell_{\mathrm{flat}}]$ obtained by combining links obtained using Definition 36 from (120) and (119). The geometric setting of adiabatic limit is here the same as in §7.1. The three triples (μ^+, μ^-, ψ), $(\zeta^+, \zeta^-, \varphi)$ and $(\lambda^+, \lambda^-, \phi)$ are taken trivial.

Proposition 64. — *We have* $[\ell_{\text{top}}] = [\ell_{\text{flat}}]$.

Proof. — First step: Hodge theoretic version of the Leray spectral sequence. — Such a theory was studied by various authors in various contexts [**33**] [**16**] [**5**] [**31**], the version corresponding to the situation here in explained in [**31**] §2 and §3. It can be summarized as follows: E_0 is nothing but $\mathcal{E}_{M/S}$ (see (107)) as global infinite rank vector bundle over S. Then there exists a nested sequence of vector subbundles $\widetilde{E}_r$ of $E_0 = \widetilde{E}_0$ which are for all $r \geq 2$ of finite rank and endowed with canonical flat connections $\widetilde{\nabla}_r$. This sequence stabilizes for sufficiently great r. For any r, there is some canonical isomorphism $E_r \cong \widetilde{E}_r$ with the corresponding term of the Leray spectral sequence, for $r \geq 2$ it makes ∇_r and $\widetilde{\nabla}_r$ correspond to each other. All the $\widetilde{E}_r$ are naturally endowed with the restriction of the L^2 hermitian inner product on $\mathcal{E}_{M/S}$ (which needs here to be obtained from some riemannian submersion metric). Finally for any r, let $\widetilde{d}_r^*$ be the adjoint of the bundle endomorphism $\widetilde{d}_r$ corresponding to the operator d_r of the spectral sequence, and define $\widetilde{\mathcal{D}}_r = \widetilde{d}_r + \widetilde{d}_r^*$, then $\widetilde{E}_{r+1} := \operatorname{Ker}\widetilde{\mathcal{D}}_r$.

For $r = 0$, $d_0 \cong d^{M/B}$, so that E_1 identifies through the Hodge theory of the fibres of π_1 with $\widetilde{E}_1 = \Omega(B/S, \operatorname{Ker}\mathcal{D}^V) \cong \Omega(B/S, G^\bullet)$ in the notations of the preceding paragraph. Thus $\widetilde{E}_1$ identifies with vertical differential forms with values in $\pi_{1!}F$, where "vertical" is to understand with respect to the fibration π_2. Let $\widetilde{p}_1$ be the orthogonal projection of E_0 onto $\widetilde{E}_1$, then $\widetilde{d}_1 = \widetilde{p}_1 d^H$ acting on $\widetilde{E}_1$. It follows that $\widetilde{E}_2 = \operatorname{Ker}(p_1\mathcal{D}^H|_{\widetilde{E}_1})$ identifies with vertical harmonic $G^\bullet$-valued differential forms, hence with $\pi_{2!}(\pi_{1!}F)$.

For any $r \geq 2$, $\widetilde{E}_r$ can be described as follows ([**31**] Proposition 2.1):

$$
\begin{aligned}
\widetilde{E}_r = \big\{ s_0 \in \mathcal{E}_{M/S} &\text{ such that there exists } s_1, s_2, \dots, s_{r-1} \in \mathcal{E}_{M/S} \text{ verifying} \\
&\mathcal{D}^V s_0 = 0,\ \mathcal{D}^H s_0 + \mathcal{D}^V s_1 = 0 \ \text{ and} \\
&(\iota_T + \iota_T^*) s_{i-2} + \mathcal{D}^H s_{i-1} + \mathcal{D}^V s_i = 0 \ \text{ for any } 2 \leq i \leq r-1 \big\}.
\end{aligned} \tag{121}
$$

Then in this description $\widetilde{\mathcal{D}}_r s_0 = \widetilde{p}_r((\iota_T + \iota_T^*)s_{r-2} + \mathcal{D}^H s_{r-1})$, where $\widetilde{p}_r$ is the orthogonal projection of $\mathcal{E}_{M/S}$ onto $\widetilde{E}_r$. One can then prove along the same lines as in [**5**] §VI (a) (especially formulae (6.13) and (6.15)) that

$$
\begin{aligned}
&d^{M/B} s_0 = 0,\ d^H s_0 + d^{M/B} s_1 = 0, \\
&\qquad \iota_T s_{i-2} + d^H s_{i-1} + d^{M/B} s_i = 0 \ \text{ for any } 2 \leq i \leq r-1 \\
&\qquad\quad \text{and } \ \widetilde{d}_r s_0 = \widetilde{p}_r(\iota_T s_{r-2} + d^H s_{r-1}).
\end{aligned}
$$

Second step: convergence of harmonic forms. — Use now the convergence of the resolvent $\big(\lambda - (\frac{1}{\theta})^{r-1}\mathcal{D}^\theta\big)^{-1}$ (here both μ and ψ vanish) to $\widetilde{p}_r(\lambda - \widetilde{\mathcal{D}}_r)^{-1}\widetilde{p}_r$ ([**31**] Theorem 2.2) for sufficiently large r. One can deduce that the orthogonal projection p_θ of $\mathcal{E}_{M/S}$ onto $\operatorname{Ker}\mathcal{D}^\theta$ converges at $\theta = 0$ to $\widetilde{p}_r$. In other words $\operatorname{Ker}\mathcal{D}^\theta$ is the restriction to $S \times (0,1]$ of some vector bundle on $S \times [0,1]$ whose restriction to $S \times \{0\}$ is $\widetilde{E}_\infty$. There is a bigrading on $\mathcal{E}_{M/S}$, from (107) according to horizontal (i.e. corresponding to $\Omega^\bullet$) and vertical (corresponding to the grading of $\mathcal{E}_{M/B}$) degrees. $\widetilde{E}_\infty$ decomposes with respect to this bigrading [5] Theorem 6.1. Consider some $s_0 \in \widetilde{E}_\infty^{p,q}$ and call $s_i^{p+i,q-i}$ for any i the corresponding component of the s_i introduced in (121). The above description of $\widetilde{d}_r$ proves that for any sufficiently large r the differential form $s_0 + s_1^{p+1,q-1} + \dots s_r^{p+r,q-r}$ is closed. According to the scaling appearing in (109) the section $p_\theta(s_0 + \theta s_1^{p+1,q-1} + \theta^2 s_2^{p+2,q-2} \dots \theta^r s_r^{p+r,q-r})$ is the rescaled harmonic form corresponding to some fixed cohomology class. Its convergence to s_0 at $\theta = 0$ proves that the isomorphism between $\operatorname{Ker}\mathcal{D}^1$ ad $\widetilde{E}_\infty$ provided by the parallel transport along $[0,1]$ exactly corresponds to the isomorphism $[H^\bullet, \nabla_H] \cong [E_r^\bullet, \nabla_r]$ obtained at the end of §7.2.1 from (119).

Third step: eigenvalues converging to 0. — The convergence of the resolvent $\big(\lambda - (\frac{1}{\theta})^{r-1}\mathcal{D}^\theta\big)^{-1}$ to $\widetilde{p}_r(\lambda - \widetilde{\mathcal{D}}_r)^{-1}\widetilde{p}_r$ ([31] Theorem 2.2) for any r gives the following description of the vector bundle $\mathcal{K}$ of Theorem 62 over $S \times [0,\varepsilon_2]$: its restriction to $S \times \{\theta\}$ is the direct sum of eigenspaces of $\mathcal{D}_\theta$ corresponding to "little" modulus eigenvalues while its restriction to $S \times \{0\}$ is the direct sum of the $\widetilde{E}_r$, each $\widetilde{E}_r$ corresponding to eigenspaces associated to eigenvalues of order less than or equal to θ^{r-1}. For any positive θ, $(\mathcal{K}, d^\theta)$ form a complex whose cohomology is $\mathcal{L}$. The convergence of the resolvents also prove that the operator $(\frac{1}{\theta})^{r-1}\mathcal{D}^\theta$ on the suitable eigensubspace converges to $\widetilde{\mathcal{D}}_r$, and accordingly for $(\frac{1}{\theta})^{r-1}d^\theta$ and $\widetilde{d}_r$.

By proceeding exactly as in §4.2.4, one obtains that the canonical class of links between $\mathcal{K}^\pm$ and $\mathcal{L}^\pm$ equals the canonical class of links associated by Definition 36 to the Leray spectral sequence ((120) for all r).

One may use the limit $t \to 0$ or $\varepsilon_1 \to 0$ in the construction of ℓ_{top}. The two remaining components of the construction of ℓ_{top} (parallel transport along $[0,1]$ and $\ell_{\mathcal{K}}^{\mathcal{L}}$) were shown to be equal to the two components of $[\ell_{\mathrm{flat}}]$ (the links coming from filtration of cohomology and from the spectral sequence respectively). □

7.2.3. *Proof of Theorem 34.* — Consider some $(E, \nabla_E, F, \nabla_F, f) \in K^0_{\mathrm{rel}}(M)$, then $\pi_{2*} \circ \pi_{1*}(E, \nabla_E, F, \nabla_F, f)$ equals $\big(\pi_{2!} \circ \pi_{1!}(E, \nabla_E), \pi_{2!} \circ \pi_{1!}(F, \nabla_F), \pi_{2\ell}(\pi_{1\ell}([f]))\big)$ while $(\pi_2 \circ \pi_1)_*(E, \nabla_E, F, \nabla_F, f)$ equals $\big((\pi_2 \circ \pi_1)_!(E, \nabla_E), (\pi_2 \circ \pi_1)_!(F, \nabla_F), (\pi_2 \circ \pi_1)_\ell([f])\big)$.

Consider the pull-back $\widetilde{E}$ of E to $M \times [0,1]$ with some connection $\widetilde{\nabla}$ whose restrictions on $M \times \{0\}$ and $M \times \{1\}$ respectively equal ∇_E and $f^*\nabla_F$. There is a canonical (topological) class of link $[\widetilde{\ell}]$ between one-step and two-step direct images of $\widetilde{E}$ whose restrictions to $M \times \{0\}$ and $M \times \{1\}$ coincide with $[\ell^E_{\mathrm{top}}]$ and $[\ell^F_{\mathrm{top}}]$ (with obvious notations from the preceding subsection, this is because of the naturality of $[\ell_{\mathrm{top}}]$). Now $\pi_{2\ell}(\pi_{1\ell}([f]))$ and $(\pi_2 \circ \pi_1)_\ell([f])$ both correspond to the parallel transport along

[0, 1]. Thus

$$\begin{aligned} \pi_{2*}\circ\pi_{1*}(E,\nabla_E,F,\nabla_F,f)-(\pi_2\circ\pi_1)_*(E,\nabla_E,F,\nabla_F,f) &= \\ = \big(\pi_{2!}\circ\pi_{1!}(E,\nabla_E),(\pi_2\circ\pi_1)_!(E,\nabla_E),[\ell^E_{\mathrm{top}}]\big) & \\ -\big(\pi_{2!}\circ\pi_{1!}(F,\nabla_F),(\pi_2\circ\pi_1)_!(F,\nabla_F),[\ell^F_{\mathrm{top}}]\big). & \end{aligned} \tag{122}$$

But in both cases $[\ell_{\mathrm{top}}]=[\ell_{\mathrm{flat}}]$ and ℓ_{flat} is only obtained from parallel complexes of flat vector bundles (from either (120) or (119)). It follows from Lemma 42 that both terms in the right hand side of (122) vanish and this proves the theorem. The case of noncompact S follows directly from the fact that links of the form $[\ell_{\mathrm{flat}}]$ are globally defined.

7.3. Multiplicative and smooth K-theory

7.3.1. *Calculation of* $\pi_{2!}^{\mathrm{Eu}}\circ\pi_{1!}^{\mathrm{Eu}}-(\pi_2\circ\pi_1)_!^{\mathrm{Eu}}$. — Consider the vector bundles ξ on M, F^+ and F^- on B and G^+ and G^- on S (with connections ∇_ξ, ∇_{F^+}, ∇_{F^-}, ∇_{G^+} and ∇_{G^-}) such that

$$[F^+]-[F^-]=\pi_{1*}^{\mathrm{Eu}}[\xi]\in K^0_{\mathrm{top}}(B)\quad\text{and}\quad[G^+]-[G^-]=(\pi_2\circ\pi_1)_*^{\mathrm{Eu}}[\xi]\in K^0_{\mathrm{top}}(S).$$

Choose some smooth supplementary subbundle T^HM/S of TM/S in TM, such that $T^HM/S\cap TM/B=T^HM$; then $\pi_{1*}T^HM/S$ is a smooth supplementary subbundle of TB/S in TB. One can define connections $\nabla_{TM/B}$, $\nabla_{TM/S}$ and $\nabla_{TB/S}$ on TM/B, TM/S and TB/S as at the beginning of the proof Lemma 56 from the choices of horizontal subspaces T^HM, T^HM/S and $\pi_{1*}T^HM/S$ respectively. Let $[\ell_F]$ and $[\ell_G]$ be equivalence classes of links between either F^+-F^- or G^+-G^- and couples of family index bundles (as in Definition 50), and denote $\eta_1=\eta(\nabla_\xi,\nabla_{TM/B},\nabla_{F^+},\nabla_{F^-},[\ell_F])$ and $\eta_{12}=\eta(\nabla_\xi,\nabla_{TM/S},\nabla_{G^+},\nabla_{G^-},[\ell_G])$:

$$\pi_{1!}^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)=\left(F^+,\nabla_{F^+},\int_{M/B}e(\nabla_{TM/B})\alpha\right)-(F^-,\nabla_{F^-},\eta_1)\in\widehat{K}_{\mathrm{ch}}(B)$$

$$\text{and}\quad(\pi_2\circ\pi_1)_!^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)=$$

$$=\left(G^+,\nabla_{G^+},\int_{M/S}e(\nabla_{TM/S})\alpha\right)-(G^-,\nabla_{G^-},\eta_{12})\in\widehat{K}_{\mathrm{ch}}(S).$$

Take vector bundles H^{++}, H^{+-}, H^{-+} and H^{--} on S (with connections ∇_{++}, ∇_{+-}, ∇_{-+} and ∇_{--}) such that $\pi_{2*}^{\mathrm{Eu}}[F^\pm]=[H^{\pm+}]-[H^{\pm-}]\in K^0_{\mathrm{top}}(S)$. Consider some classes of links $[\ell_+]$ and $[\ell_-]$ between $H^{\pm+}-H^{\pm-}$ and couples of families index bundles and denote by $\eta_\pm$ the forms $\eta(\nabla_{F^\pm}, \nabla_{TB/S}, \nabla_{\pm +}, \nabla_{\pm -}, [\ell_\pm])$:

$$\pi_{2!}^{\mathrm{Eu}}(\pi_{1!}^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)) = \left(H^{++}, \nabla_{++}, \int_{B/S} e(\nabla_{TB/S}) \int_{M/B} e(\nabla_{TM/B})\alpha\right)$$
$$-(H^{+-},\nabla_{+-},\eta_+) - \left(H^{-+},\nabla_{-+}, \int_{B/S} e(\nabla_{TB/S})\eta_1\right) + (H^{--},\nabla_{--},\eta_-).$$

Now $G^+ - G^-$ and $(H^{++} \oplus H^{--}) - (H^{+-} \oplus H^{-+})$ are linked through $[\ell_G]$, $[\ell_+]$, $[\ell_-]$ and the construction of §7.1.6. Call $[\ell_{\mathrm{top}}]$ the resulting link and $\widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}])$ the associated Chern-Simons form as in §5.3.1, then

$$\pi_{2!}^{\mathrm{Eu}}(\pi_{1!}^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)) = \left(G^+, \nabla_{G^+}, \int_{M/S} \pi_1^*\big(e(\nabla_{TB/S})\big)e(\nabla_{TM/B})\alpha\right)$$
$$-\left(G^-, \nabla_{G^-}, \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1\right).$$

Choose any supplementary subbundle of T^HM in T^HM/S, it then identifies with π_1^*TB/S and is endowed with the connection $\pi_1^*\nabla_{TB/S}$. Denote by $\widetilde{e}_{M/B/S}$ the form $\widetilde{e}(\nabla_{TM/S}, \nabla_{TM/B} \oplus \pi_1^*\nabla_{!TB/S})$ defined in §89, then the following form

$$\widetilde{e}_{M/B/S}d\alpha + \big(e(\nabla_{TM/S}) - \pi_1^*\big(e(\nabla_{TB/S})\big)e(\nabla_{TM/B})\big)\alpha$$

is exact so that in $\widehat{K}_{\mathrm{ch}}(S)$:

$$\pi_{2!}^{\mathrm{Eu}}(\pi_{1!}^{\mathrm{Eu}}(\xi,\nabla_\xi,\alpha)) = \left(G^+, \nabla_{G^+}, \int_{M/S} e(\nabla_{TM/S})\alpha\right) - (G^-, \nabla_{G^-}, \widetilde{\eta}_{12})$$

$$\text{with} \qquad \widetilde{\eta}_{12} = \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1 - \int_{M/S} \widetilde{e}_{M/B/S}d\alpha.$$

For any $(\xi,\nabla_\xi,\alpha) \in MK_0(M)$, one has $d\alpha = \mathrm{ch}(\nabla_\xi) - \mathrm{rk}\xi$ but for degree reasons $\int_{M/S}\widetilde{e}_{M/B/S}$ vanishes (the degree of this form equals $\dim M - \dim S - 1$). Thus

Proposition 65. — $(\pi_{2!}^{\mathrm{Eu}} \circ \pi_{1!}^{\mathrm{Eu}} - (\pi_2\circ\pi_1)_!^{\mathrm{Eu}})(\xi,\nabla_\xi,\alpha) = a(\eta_{12} - \approx{\eta}_{12})$ *where*

$$\overset{\approx}{\eta}_{12} = \eta_+ - \eta_- - \widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}]) + \int_{B/S} e(\nabla_{TB/S})\eta_1 + \int_{M/S} \widetilde{e}_{M/B/S}\mathrm{ch}(\nabla_\xi).$$

An argument similar to just before Lemma 60 yields that this equality also holds true in the case of noncompact S.

7.3.2. *Proof of Theorem 35.* — It is easily verified that $\overset{\approx}{\eta}_{12}$ is additive in the sense of property (c) of Theorem 28, is functorial by pullbacks over fibered products (with double fibration structure!); a direct calculation proves that it verifies the same transgression formula (property (a) of Theorem 28) as η_{12}. In the case of a flat bundle (ξ,∇_ξ), $F^\pm$ here correspond to $G^\pm$ in §7.2.1, $G^\pm$ here correspond to $H^\pm$ of §7.2.1, and $H^{\pm\pm}$ here correspond to $E_2^{\pm\pm}$ of §7.2.1: in any case, the suitable data are taken trivial because all bundles are flat, and thus all the forms η_+, η_- and η_1 vanish (property (d) of Theorem 28). Finally $\mathrm{ch}(\nabla_\xi) = \mathrm{rk}\xi$ so that the integral involving $\widetilde{e}_{M/B/S}$ vanishes, and $\widetilde{\mathrm{ch}}([\ell_{\mathrm{top}}])$ also vanishes, because of Proposition 64 and Lemma 1 ($[\ell_{\mathrm{flat}}]$ of Proposition 64 is obtained by using Definition 36 from parallel exact sequences of flat bundles).

The coincidence of η_{12} and $\approx{\eta}_{12}$ for elements of $MK_0(M)$ is then obtained from the second statement of Theorem 28.

Remark. — It is likely that $\overset{\approx}{\eta}_{12} = \eta_{12}$ in any case, so that one would have

$$(\pi_2 \circ \pi_1)_!^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha) - \pi_{2!}^{\mathrm{Eu}}(\pi_{1!}^{\mathrm{Eu}}(\xi, \nabla_\xi, \alpha)) = a\left(\int_{M/S} \widetilde{e}_{M/B/S}\, \dddot{\mathrm{ch}}(\xi, \nabla_\xi, \alpha)\right)$$

for any $(\xi, \nabla_\xi, \alpha) \in \widehat{K}_{\mathrm{ch}}(M)$. This formula would be compatible with Theorem 28 and with anomaly formulae (91) and (92). A corresponding result is proved in [**14**], where the above discrepancy is compensated by a suitable composition of smooth K-orientations in the double fibration.

References

[1] M. F. Atiyah, V. K. Patodi & I. M. Singer – "Spectral asymmetry and Riemannian geometry. III", *Math. Proc. Cambridge Philos. Soc.* **79** (1976), p. 71–99.

[2] M. F. Atiyah & I. M. Singer – "The index of elliptic operators. IV", *Ann. of Math.* **93** (1971), p. 119–138.

[3] N. Berline, E. Getzler & M. Vergne – *Heat kernels and Dirac operators*, Grund. Math. Wiss., vol. 298, Springer, 1992.

[4] A. Berthomieu – "Proof of Nadel's conjecture and direct image for relative K-theory", *Bull. Soc. Math. France* **130** (2002), p. 253–307.

[5] A. Berthomieu & J.-M. Bismut – "Quillen metrics and higher analytic torsion forms", *J. reine angew. Math.* **457** (1994), p. 85–184.

[6] J.-M. Bismut – "The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs", *Invent. Math.* **83** (1985), p. 91–151.

[7] ______, "Eta invariants, differential characters and flat vector bundles", *Chinese Ann. Math. Ser. B* **26** (2005), p. 15–44.

[8] J.-M. Bismut & J. Cheeger – "η-invariants and their adiabatic limits", *J. Amer. Math. Soc.* **2** (1989), p. 33–70.

[9] J.-M. Bismut, H. Gillet & C. Soulé – "Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms", *Comm. Math. Phys.* **115** (1988), p. 79–126.

[10] J.-M. Bismut & K. Köhler – "Higher analytic torsion forms for direct images and anomaly formulas", *J. Algebraic Geom.* **1** (1992), p. 647–684.

[11] J.-M. BISMUT & J. LOTT – "Flat vector bundles, direct images and higher real analytic torsion", *J. Amer. Math. Soc.* **8** (1995), p. 291–363.

[12] A. BOREL – "Stable real cohomology of arithmetic groups", *Ann. Sci. École Norm. Sup.* **7** (1974), p. 235–272.

[13] U. BUNKE – "Index theory, eta forms, and Deligne cohomology", *Mem. Amer. Math. Soc.* **198** (2009).

[14] U. BUNKE & T. SCHICK – "Smooth K-theory", *Astérisque* **328** (2010), p. 45–134.

[15] J. CHEEGER & J. SIMONS – "Differential characters and geometric invariants", in *Geometry and topology (College Park, Md., 1983/84)*, Lecture Notes in Math., vol. 1167, Springer, 1985, p. 50–80.

[16] X. DAI – "Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence", *J. Amer. Math. Soc.* **4** (1991), p. 265–321.

[17] J. L. DUPONT – "Characteristic classes for flat bundles and their formulas", *Topology* **33** (1994), p. 575–590.

[18] M. FELISATTI & F. NEUMANN – "Secondary theories for simplicial manifolds and classifying spaces", in *Proceedings of the School and Conference in Algebraic Topology*, Geom. Topol. Monogr., vol. 11, Geom. Topol. Publ., Coventry, 2007, p. 33–58.

[19] D. S. FREED – "Dirac charge quantization and generalized differential cohomology", in *Surveys in differential geometry*, Surv. Differ. Geom., VII, Int. Press, Somerville, MA, 2000, p. 129–194.

[20] D. S. FREED & M. HOPKINS – "On Ramond-Ramond fields and K-theory", *J. High Energy Phys.* **5** (2000), paper 44, 14.

[21] M. HOPKINS & I. M. SINGER – "Quadratic functions in geometry, topology, and M-theory", *J. Differential Geom.* **70** (2005), p. 329–452.

[22] D. HUSEMOLLER – *Fibre bundles*, McGraw-Hill Book Co., 1966.

[23] D. L. JOHNSON – "Secondary characteristic classes and intermediate Jacobians", *J. reine angew. Math.* **347** (1984), p. 134–145.

[24] ______, "Smooth moduli and secondary characteristic classes of analytic vector bundles", seemingly unpublished previous work to [**23**].

[25] F. W. KAMBER & P. TONDEUR – "Characteristic invariants of foliated bundles", *Manuscripta Math.* **11** (1974), p. 51–89.

[26] M. KAROUBI – "Homologie cyclique et K-théorie", *Astérisque* **149** (1987).

[27] ______, "Théorie générale des classes caractéristiques secondaires", *K-Theory* **4** (1990), p. 55–87.

[28] ______, "Classes caractéristiques de fibrés feuilletés, holomorphes ou algébriques", in *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part II (Antwerp, 1992)*, vol. 8, 1994, p. 153–211.

[29] H. B. J. LAWSON & M.-L. MICHELSOHN – *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton Univ. Press, 1989.

[30] J. LOTT – "$\mathbf{R}/\mathbf{Z}$ index theory", *Comm. Anal. Geom.* **2** (1994), p. 279–311.

[31] X. MA – "Functoriality of real analytic torsion forms", *Israel J. Math.* **131** (2002), p. 1–50.

[32] X. MA & W. ZHANG – "Eta-invariants, torsion forms and flat vector bundles", *Math. Ann.* **340** (2008), p. 569–624.

[33] R. R. MAZZEO & R. B. MELROSE – "The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration", *J. Differential Geom.* **31** (1990), p. 185–213.

[34] R. B. MELROSE & P. PIAZZA – "An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary", *J. Differential Geom.* **46** (1997), p. 287–334.

[35] A. M. NADEL – "Invariants for holomorphic vector bundles", *Math. Ann.* **309** (1997), p. 37–52.

[36] D. POUTRIQUET – "K-théorie des singularités coniques isolées", Ph.D. Thesis, Université Paul Sabatier, Toulouse, 2006.

[37] D. QUILLEN – "Superconnections and the Chern character", *Topology* **24** (1985), p. 89–95.

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