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PURE SPINORS ON LIE GROUPS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday.

Abstract. — For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a natural inner product given by the pairing of vectors and covectors. Differential forms on M may be viewed as spinors for the corresponding Clifford bundle, and in particular there is a notion of *pure spinor*. In this paper, we study pure spinors and Dirac structures in the case when $M = G$ is a Lie group with a bi-invariant pseudo-Riemannian metric, e.g. G semi-simple. The applications of our theory include the construction of distinguished volume forms on conjugacy classes in G, and a new approach to the theory of quasi-Hamiltonian G-spaces.

***Résumé* (Spineurs purs sur les groupes de Lie).** — Pour toute variété lisse M, le fibré $\mathbb{T}M = TM \oplus T^*M$ est muni d'un produit scalaire naturel défini par la dualité entre vecteurs et co-vecteurs. Les formes différentielles sur M sont des spineurs pour le fibré de Clifford correspondant. On définit alors les *spineurs purs*. Dans cet article, nous étudions les spineurs purs et les structures de Dirac dans le cas où M est un groupe de Lie G muni d'une métrique pseudo-riemannienne bi-invariante, par exemple un groupe semi-simple. Comme applications de notre théorie, nous définissons une forme volume distinguée sur les classes de conjugaison de G, et nous proposons une nouvelle approche de la théorie des G-espaces quasi-hamiltoniens.

0. Introduction

For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a non-degenerate symmetric bilinear form, extending the pairing between vectors and covectors. There is a natural Clifford action ϱ of the sections $\Gamma(\mathbb{T}M)$ on the space $\Omega(M) = \Gamma(\wedge T^*M)$ of differential forms, where vector fields act by contraction and 1-forms by exterior multiplication. That is, $\wedge T^*M$ is viewed as a spinor module over the Clifford bundle $\mathrm{Cl}(\mathbb{T}M)$. A form $\phi \in \Omega(M)$ is called a *pure spinor* if the solutions $w \in \Gamma(\mathbb{T}M)$ of

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$\varrho(w)\phi = 0$ span a Lagrangian subbundle $E \subset \mathbb{T}M$. Given a closed 3-form $\eta \in \Omega^3(M)$, a pure spinor ϕ is called *integrable* (relative to η) [**9, 28**] if there exists a section $w \in \Gamma(\mathbb{T}M)$ with

$$(\mathrm{d}+\eta)\phi = \varrho(w)\phi.$$

In this case, there is a generalized foliation of M with tangent distribution the projection of E to TM. The subbundle E defines a *Dirac structure* [**20, 50**] on M, and the triple (M, E, η) is called a *Dirac manifold.*

The present paper is devoted to the study of Dirac structures and pure spinors on Lie groups G. We assume that the Lie algebra $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form B, and take $\eta \in \Omega^3(G)$ as the corresponding Cartan 3-form. Let $\bar{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the opposite bilinear form $-B$. We will describe a trivialization

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}}),$$

under which any Lagrangian Lie subalgebra $\mathfrak{s} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ defines a Dirac structure on G. There is also a similar identification of spinor bundles

$$\mathscr{R}\colon G \times \mathrm{Cl}(\mathfrak{g}) \xrightarrow{\cong} \wedge T^*G,$$

taking the standard Clifford action of $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ on $\mathrm{Cl}(\mathfrak{g})$, where the first summand acts by left (Clifford) multiplication and the second summand by right multiplication, to the Clifford action ϱ. This isomorphism takes the *Clifford differential* d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$, given as Clifford commutator by a cubic element [**4, 38**], to the the differential $\mathrm{d}+\eta$ on $\Omega(G)$. As a result, pure spinors $x \in \mathrm{Cl}(\mathfrak{g})$ for the Clifford action of $\mathrm{Cl}(\mathfrak{g} \oplus \bar{\mathfrak{g}})$ on $\mathrm{Cl}(\mathfrak{g})$ define pure spinors $\phi = \mathscr{R}(x) \in \Omega(G)$, and the integrability condition for ϕ is equivalent to a similar condition for x. The simplest example $x = 1$ defines the *Cartan-Dirac structure* E_G [**14, 50**], introduced by Alekseev, Ševera and Strobl in the 1990's. In this case, the resulting foliation of G is just the foliation by conjugacy classes. We will study this Dirac structure in detail, and examine in particular its behavior under group multiplication and under the exponential map. When G is a complex semi-simple Lie group, it carries another interesting Dirac structure, which we call the *Gauss-Dirac structure.* The corresponding foliation of G has a dense open leaf which is the 'big cell' from the Gauss decomposition of G.

The main application of our study of pure spinors is to the theory of q-Hamiltonian actions [**2, 3**]. The original definition of a q-Hamiltonian G-space in [**3**] involves a G-manifold M together with an invariant 2-form ω and a G-equivariant map $\Phi\colon M \to G$ satisfying appropriate axioms. As observed in [**14, 15**], this definition is equivalent to saying that the 'G-valued moment map' Φ is a suitable morphism of Dirac manifolds (in analogy with classical moment maps, which are morphisms $M \to \mathfrak{g}^*$ of Poisson manifolds). In this paper, we will carry this observation further, and develop all the basic results of q-Hamiltonian geometry from this perspective. A conceptual advantage of this alternate viewpoint is that, while the arguments in [**3**] required G to be compact, the Dirac geometry approach needs no such assumption, and in fact works in the complex (holomorphic) category as well. This is relevant for applications:

For instance, the symplectic form on a representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ (for Σ a closed surface) can be obtained by q-Hamiltonian reduction, and there are many interesting examples for noncompact G. (For instance, the case $G = \mathrm{PSL}(2,\mathbb{R})$ gives the symplectic form on Teichmüller space.) Complex q-Hamiltonian spaces appear e.g. in the work of Boalch [**13**] and Van den Bergh [**11**].

The organization of the paper is as follows. Sections 1 and 2 contain a review of Dirac geometry, first on vector spaces and then on manifolds. The main new results in these sections concern the geometry of Lagrangian splittings $\mathbb{T}M = E \oplus F$ of the bundle $\mathbb{T}M$. If $\phi, \psi \in \Omega(M)$ are pure spinors defining E, F, then, as shown in [**17, 19**], the top degree part of $\phi^\top \wedge \psi$ (where $\top$ denotes the standard anti-involution of the exterior algebra) is nonvanishing, and hence defines a volume form μ on M. Furthermore, there is a bivector field $\pi \in \mathfrak{X}^2(M)$ naturally associated with the splitting, which satisfies

$$\phi^\top \wedge \psi = e^{-\iota(\pi)}\mu.$$

We will discuss the properties of μ and π in detail, including their behavior under Dirac morphisms.

In Section 3 we specialize to the case $M = G$, where G carries a bi-invariant pseudo-Riemannian metric, and our main results concern the isomorphism $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$ and its properties. Under this identification, the Cartan-Dirac structure $E_G \subset \mathbb{T}G$ corresponds to the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, and hence it has a natural Lagrangian complement $F_G \subset \mathbb{T}G$ defined by the anti-diagonal. We will show that the exponential map gives rise to a Dirac morphism $(\mathfrak{g}, E_{\mathfrak{g}}, 0) \to (G, E_G, \eta)$ (where $E_{\mathfrak{g}}$ is the graph of the linear Poisson structure on $\mathfrak{g} \cong \mathfrak{g}^*$), but this morphism does not relate the obvious complements $F_{\mathfrak{g}} = T\mathfrak{g}$ and F_G. The discrepancy is given by a 'twist', which is a solution of the *classical dynamical Yang-Baxter equation.* For G complex semi-simple, we will construct another Lagrangian complement of E_G, denoted by $\widehat{F_G}$, which (unlike F_G) is itself a Dirac structure. The bivector field corresponding to the splitting $E_G \oplus \widehat{F_G}$ is then a Poisson structure on G, which appeared earlier in the work of Semenov-Tian-Shansky [**49**].

In Section 4, we construct an isomorphism $\wedge T^*G \cong G \times \mathrm{Cl}(\mathfrak{g})$ of spinor modules, valid under a mild topological assumption on G (which is automatic if G is simply connected). This allows us to represent the Lagrangian subbundles E_G, F_G and $\widehat{F_G}$ by explicit pure spinors ϕ_G, ψ_G, and $\widehat{\psi}_G$, and to derive the differential equations controlling their integrability. We show in particular that the Cartan-Dirac spinor satisfies

$$(\mathrm{d} + \eta)\phi_G = 0.$$

Section 5 investigates the foundational properties of q-Hamiltonian G-spaces from the Dirac geometry perspective. Our results on the Cartan-Dirac structure give a direct construction of the fusion product of q-Hamiltonian spaces. On the other hand, we use the bilinear pairing of spinors to show that, for a q-Hamiltonian space (M, ω, Φ), the top degree part of $e^\omega \Phi^* \psi_G \in \Omega(M)$ defines a volume form μ_M. This volume form was discussed in [**8**] when G is compact, but the discussion here applies equally well to non-compact or complex Lie groups. Since conjugacy classes in G are examples of q-Hamiltonian G-spaces, we conclude that for any simply connected Lie group G with bi-invariant pseudo-Riemannian metric (e.g. G semi-simple), *any conjugacy class in G carries a distinguished invariant volume form.* If G is complex semi-simple, one obtains the same volume form μ_M if one replaces ψ_G with the Gauss-Dirac spinor $\widehat{\psi}_G$. However, the form $e^{\omega}\Phi^*\widehat{\psi}_G$ satisfies a nicer differential equation, which allows us to compute the volume of M, and more generally the measure $\Phi_*|\mu_M|$, by Berline-Vergne localization [**12**]. We also explain in this Section how to view the more general q-Hamiltonian q-Poisson spaces [**2**] in our framework.

Lastly, in Section 6, we revisit the theory of K^*-valued moment maps in the sense of Lu [**42**] and its connections with P-valued moment maps [**3**, Sec. 10] from the Dirac geometric standpoint.

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Notation. — Our conventions for Lie group actions are as follows: Let G be a Lie group (not necessarily connected), and $\mathfrak{g}$ its Lie algebra. A G-action on a manifold M is a group homomorphism $\mathcal{A}\colon G \to \mathrm{Diff}(M)$ for which the action map $G\times M \to M$, $(g,m) \mapsto \mathcal{A}(g)(m)$ is smooth. Similarly, a $\mathfrak{g}$-action on M is a Lie algebra homomorphism $\mathcal{A}\colon \mathfrak{g} \to \mathfrak{X}(M)$ for which the map $\mathfrak{g} \times M \to TM$, $(\xi, m) \mapsto \mathcal{A}(\xi)_m$ is smooth. Given a G-action $\mathcal{A}$, one obtains a $\mathfrak{g}$-action by the formula $\mathcal{A}(\xi)(f) = \frac{\partial}{\partial t}\big|_{t=0}\mathcal{A}(\exp(-t\xi))^*f$, for $f \in C^\infty(M)$ (here vector fields are viewed as derivations of the algebra of smooth functions).

1. Linear Dirac geometry

The theory of Dirac manifolds was initiated by Courant and Weinstein in [**20, 21**]. We briefly review this theory, developing and expanding the approach via pure spinors advocated by Gualtieri [**28**] (see also Hitchin [**32**] and Alekseev-Xu [**9**]). All vector spaces in this section are over the ground field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We begin with some background material on Clifford algebras and spinors (see e.g. [**19**] or [**47**].)

1.1. Clifford algebras. — Suppose V is a vector space with a non-degenerate symmetric bilinear form B. We will sometimes refer to such a bilinear form B as an *inner product* on V. The *Clifford algebra* over V is the associative unital algebra generated by the elements of V, with relations

$$vv' + v'v = B(v, v')\,1.$$

It carries a compatible $\mathbb{Z}_2$-grading and $\mathbb{Z}$-filtration, such that the generators $v \in V$ are odd and have filtration degree 1. We will denote by $x \mapsto x^\top$ the canonical anti-automorphism of exterior and Clifford algebras, equal to the identity on V. For any $x \in \mathrm{Cl}(V)$, we denote by $l^{\mathrm{Cl}}(x), r^{\mathrm{Cl}}(x)$ the operators of graded left and right multiplication on $\mathrm{Cl}(V)$:

$$l^{\mathrm{Cl}}(x)x' = xx', \quad r^{\mathrm{Cl}}(x)x' = (-1)^{|x||x'|}x'x.$$

Thus $l^{\mathrm{Cl}}(x) - r^{\mathrm{Cl}}(x)$ is the operator of graded commutator $[x, \cdot]_{\mathrm{Cl}}$.

The *quantization map* $q\colon \wedge V \to \mathrm{Cl}(V)$ is the isomorphism of vector spaces defined by $q(v_1 \wedge \cdots \wedge v_r) = v_1 \cdots v_r$ for pairwise orthogonal elements $v_i \in V$. Let

$$\mathrm{str}\colon \mathrm{Cl}(V) \to \det(V) := \wedge^{\mathrm{top}}(V)$$

be the *super-trace*, given by q^{-1}, followed by taking the top degree part. It has the property $\mathrm{str}([x, x']_{\mathrm{Cl}}) = 0$.

A *Clifford module* is a vector space S together with an algebra homomorphism $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$. If S is a Clifford module, one has a dual Clifford module given by the dual space S^* with Clifford action $\varrho^*(x) = \varrho(x^\top)^*$.

Recall that $\mathrm{Pin}(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$ generated by all $v \in V$ whose square in the Clifford algebra is $vv = \pm 1$. It is a double cover of the orthogonal group $\mathrm{O}(V)$, where $g \in \mathrm{Pin}(V)$ takes $v \in V$ to $(-1)^{|g|}gvg^{-1}$, using Clifford multiplication. The *norm homomorphism* for the Pin group is the group homomorphism

$$\text{(1)} \qquad \mathsf{N}\colon \mathrm{Pin}(V) \to \{-1, +1\}, \quad \mathsf{N}(g) = g^\top g = \pm 1.$$

Let $\{\cdot,\cdot\}$ be the graded Poisson bracket on $\wedge V$, given on generators by $\{v_1, v_2\} = B(v_1, v_2)$. Then $\wedge^2 V$ is a Lie algebra under the Poisson bracket, isomorphic to $\mathfrak{o}(V)$ in such a way that $\varepsilon \in \wedge^2 V$ corresponds to the linear map $v \mapsto \{\varepsilon, v\}$. The Lie algebra $\mathfrak{pin}(V) \cong \mathfrak{o}(V)$ is realized as the Lie subalgebra $q(\wedge^2(V)) \subset \mathrm{Cl}(V)$.

A subspace $E \subset V$ is called *isotropic* if $E \subset E^\perp$ and *Lagrangian* if $E = E^\perp$. The set of Lagrangian subspaces is non-empty if and only if the bilinear form is *split*. If $\mathbb{K} = \mathbb{C}$, this just means that $\dim V$ is even, while for $\mathbb{K} = \mathbb{R}$ this requires that the bilinear form has signature (n, n). From now on, we will reserve the letter W for a vector space with split bilinear form $\langle\cdot,\cdot\rangle$. We denote by $\mathrm{Lag}(W)$ the Grassmann manifold of Lagrangian subspaces of W. It carries a transitive action of the orthogonal group $\mathrm{O}(W)$.

Remark 1.1. — Suppose $\mathbb{K} = \mathbb{R}$, and identify $W \cong \mathbb{R}^{2n}$ with the standard bilinear form of signature (n, n). The group $\mathrm{O}(W) \cong \mathrm{O}(n, n)$ has maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$. Already the subgroup $\mathrm{O}(n) \times \{1\}$ acts transitively on $\mathrm{Lag}(W)$, and in fact the action is free. It follows that $\mathrm{Lag}(W)$ is diffeomorphic to $\mathrm{O}(n)$. Further details may be found in [**46**].

1.2. Pure spinors. — An irreducible module S over the Clifford algebra $\mathrm{Cl}(W)$ is called a *spinor module.* Any $E \in \mathrm{Lag}(W)$ defines a spinor module $\mathsf{S} = \mathrm{Cl}(W)/\mathrm{Cl}(W)E$. The choice of a Lagrangian complement F to E identifies $\mathsf{S} = \wedge E^*$, where the generators in $E \subset W$ act by contraction and the generators in $F \subset W$ act by exterior multiplication. (Here F is identified with E^*, using the pairing defined by $\langle\cdot,\cdot\rangle$.) The dual spinor module is $\mathsf{S}^* = \wedge E$, with generators in E acting by exterior multiplication and those in F by contraction.

For any non-zero element $\phi \in \mathsf{S}$ of a spinor module, its null space

$$N_\phi = \{w \in W |\, \varrho(w)\phi = 0\}$$

is easily seen to be isotropic. The element $\phi \in \mathsf{S}$ is a *pure spinor* [**17**] provided N_ϕ is Lagrangian. One can show that any Lagrangian subspace $E \in \mathrm{Lag}(W)$ arises in this way: in fact, $\mathsf{S}^E = \{\phi \in \mathsf{S}|\, \varrho(E)\phi = 0\}$ is a one-dimensional subspace, with non-zero elements given by the pure spinors defining E. Any spinor module S admits a $\mathbb{Z}_2$-grading (unique up to parity inversion) compatible with the Clifford action. Pure spinors always have a definite parity, either even or odd.

Example 1.2. — Let V be a vector space with inner product B. We denote by $\overline{V}$ the same vector space with the opposite bilinear form $-B$. Then $W = V \oplus \overline{V}$ is a vector space with split bilinear form. The space $\mathsf{S} = \mathrm{Cl}(V)$ is a spinor module over $\mathrm{Cl}(W) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(\overline{V})$, with Clifford action given on generators by $\varrho(v \oplus v') = l^{\mathrm{Cl}}(v) - r^{\mathrm{Cl}}(v')$. The element $1 \in \mathrm{Cl}(V)$ is a pure spinor, with corresponding Lagrangian subspace the diagonal $V_\Delta \subset V \oplus \overline{V}$.

1.3. The bilinear pairing of spinors. — For any two spinor modules $\mathsf{S}_1, \mathsf{S}_2$ over $\mathrm{Cl}(W)$, the space $\mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}_1, \mathsf{S}_2)$ of intertwining operators is one-dimensional. Given a spinor module S, let

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}^*, \mathsf{S})$$

be the *canonical line.* There is a bilinear pairing [**17**]

$$\mathsf{S} \otimes \mathsf{S} \to K_{\mathsf{S}},\ \ \phi \otimes \psi \mapsto (\phi, \psi)_{\mathsf{s}},$$

defined by the isomorphism $\mathsf{S} \otimes \mathsf{S} \cong \mathsf{S} \otimes \mathsf{S}^* \otimes \mathrm{Hom}_{\mathrm{Cl}(W)}(\mathsf{S}^*, \mathsf{S})$ followed by the duality pairing $\mathsf{S} \otimes \mathsf{S}^* \to \mathbb{K}$. The pairing satisfies

$$(\varrho(x^\top)\phi, \psi)_{\mathsf{s}} = (\phi, \varrho(x)\psi)_{\mathsf{s}}, \quad x \in \mathrm{Cl}(W), \tag{2}$$

and is characterized by this property up to a scalar. (2) implies the following invariance property under the action of the group $\mathrm{Pin}(V)$, involving the norm homomorphism (1),

$$(g\phi, g\psi)_{\mathsf{s}} = \mathsf{N}(g)(\phi, \psi)_{\mathsf{s}}, \quad g \in \mathrm{Pin}(V).$$

Theorem 1.3 **(E. Cartan [17]).** — *Let* S *be a spinor modules over* $\mathrm{Cl}(W)$, *and let* $\phi, \psi \in \mathsf{S}$ *be pure spinors. Then the corresponding Lagrangian subspaces* N_ϕ, N_ψ *are transverse if and only if* $(\phi, \psi)_{\mathsf{s}} \neq 0$.

A simple proof of this result is given in Chevalley's book [**19**, III.2.4], see also [**47**, Section 3.5].

Example 1.4. — Suppose V is a space with inner product B, and take $\mathsf{S} = \mathrm{Cl}(V)$ as a spinor module over $\mathrm{Cl}(V \oplus \overline{V})$ (cf. Example 1.2). Then $K_{\mathsf{S}} = \det(V)$, with bilinear pairing on spinors given as

$$(x, x')_{\mathrm{Cl}(V)} = \mathrm{str}(x^\top x') \in \det(V). \tag{3}$$

Using the isomorphism $q : \wedge(V) \to \mathrm{Cl}(V)$ to identify $\mathsf{S} \cong \wedge(V)$, the bilinear pairing becomes

$$(y, y')_{\wedge(V)} = (y^\top \wedge y')^{[\mathrm{top}]} \in \det(V). \tag{4}$$

1.4. Contravariant spinors. — For any vector space V, the direct sum $\mathbb{V} := V \oplus V^*$ carries a split bilinear form given by the pairing between V and V^*:

$$\langle w_1, w_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle, \qquad w_i = v_i \oplus \alpha_i \in \mathbb{V}. \tag{5}$$

Every vector space W with split bilinear form is of this form, by choosing a pair of transverse Lagrangian subspaces V, V', and using the bilinear form to identify $V' = V^*$. Then $\mathsf{S} = \wedge V^*$, with Clifford action given on generators $w = v \oplus \alpha \in \mathbb{V}$ by

$$\varrho(w) = \epsilon(\alpha) + \iota(v)$$

(where $\epsilon(\alpha) = \alpha \wedge \cdot$), is a natural choice of spinor module for $\mathrm{Cl}(\mathbb{V})$. The restriction of ϱ to $\wedge V^* \subset \mathrm{Cl}(\mathbb{V})$ is given by exterior multiplication, while the restriction to $\wedge V \subset \mathrm{Cl}(\mathbb{V})$ is given by contraction [(1)]. The line $K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(\mathbb{V})}(\mathsf{S}^*, \mathsf{S})$ is canonically isomorphic to $\det(V^*) = \wedge^{\mathrm{top}} V^*$, and the bilinear pairing on spinors is simply

$$(\phi, \psi)_{\wedge(V^*)} = (\phi^\top \wedge \psi)^{[\mathrm{top}]} \in \det(V^*),$$

similar to Example 1.4. Theorem 1.3 shows that if ϕ, ψ are pure spinors for transverse Lagrangian subspaces, the pairing $(\phi, \psi)_{\wedge(V^*)}$ defines a *volume form* on V.

Remarks 1.5. — We mention the following two facts for later reference.

a. We have the identity

$$(-1)^{|\phi|}(-(\varrho(w)\phi)^\top \wedge \psi + \phi^\top \wedge (\varrho(w)\psi)) = \iota(v)(\phi^\top \wedge \psi), \quad w = v \oplus \alpha \in \mathbb{V},$$

which refines property (2) of the bilinear pairing.

b. One can also consider the *covariant spinor module* $\wedge(V)$, obtained by reversing the roles of V and V^*. Suppose $\mu \in \det(V)$ is non-zero, and let $\star\colon \wedge(V^*) \to \wedge(V)$ be the corresponding star operator, defined by $\star\phi = \iota(\phi)\mu$. Let μ^* be the dual generator defined by $\star((\mu^*)^\top) = 1$. Then $\star$ is an isomorphism of $\mathrm{Cl}(\mathbb{V})$-modules. Furthermore, using μ, μ^* to trivialize $\det(V), \det(V^*)$, the isomorphism intertwines the bilinear pairings:

$$(\phi, \psi)_{\wedge(V^*)} = (\star\phi, \star\psi)_{\wedge(V)}, \quad \phi, \psi \in \wedge(V^*).$$

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(1) We are using the convention that $\iota\colon \wedge(V) \to \mathrm{End}(\wedge V^*)$ is the extension of the map $v \mapsto \iota(v)$ as an *algebra homomorphism*. Note that some authors use the extension as an algebra anti-homomorphism.

Any 2-form $\omega \in \wedge^2 V^*$ defines a pure spinor $\phi = e^{-\omega}$, with N_ϕ the graph of ω:

$$\mathrm{Gr}_\omega = \{v \oplus \alpha |\ v \in V,\ \alpha = \iota(v)\omega\}.$$

Note that, in accordance with Theorem 1.3, $\mathrm{Gr}_\omega \cap V = \{0\}$ if and only if ω is non-degenerate, if and only if $(e^\omega)^{[\mathrm{top}]}$ is non-zero. The most general pure spinor $\phi \in \wedge V^*$ can be written in the form

$$\phi = e^{-\omega_Q} \wedge \theta, \tag{6}$$

where $\omega_Q \in \wedge^2 Q^*$ is a 2-form on a subspace $Q \subset V$ and $\theta \in \det(\mathrm{Ann}(Q))\backslash\{0\}$ is a volume form on V/Q. To write (6), we have chosen an extension of ω_Q to a 2-form on V. (Clearly, ϕ does not depend on this choice.) The corresponding Lagrangian subspace is

$$N_\phi = \{v \oplus \alpha |\ v \in Q,\ \alpha|_Q = \iota(v)\omega_Q\}.$$

The triple (Q, ω_Q, θ) is uniquely determined by ϕ, see e.g. [**19**, III.1.9]. A simple consequence is that any pure spinor has definite parity, that is, ϕ is either even or odd depending on the parity of $\dim(V/Q)$. For any $E \in \mathrm{Lag}(\mathbb{V})$ we define subspaces $\ker(E) \subset \mathrm{ran}(E) \subset V$ by

$$\ker(E) = E \cap V, \qquad \mathrm{ran}(E) = \mathrm{pr}_V(E),$$

where $\mathrm{pr}_V \colon \mathbb{V} \to V$ is the projection along V^*. For any pure spinor ϕ, written in the form (6), we have $\mathrm{ran}(E_\phi) = Q$ and $\ker(E_\phi) = \ker(\omega_Q)$. In particular, $\phi^{[\mathrm{top}]}$ is non-zero if and only if $\ker(E_\phi) = 0$. Similarly, $\mathrm{ran}(E_\phi) = V$ if and only if $\phi^{[0]}$ is non-zero, if and only if $\phi = e^{-\omega}$ for a global 2-form ω.

1.5. Action of the orthogonal group. — Recall the identification $\wedge^2(W) \cong \mathfrak{o}(W)$ (see Section 1.1). For any Lagrangian subspace $E \subset W$, the space $\wedge^2(E)$ is embedded as an Abelian subalgebra of $\mathfrak{o}(W)$. The inclusion map exponentiates to an injective group homomorphism,

$$\wedge^2(E) \to \mathrm{O}(W),\ \ \varepsilon \mapsto A^\varepsilon, \quad A^\varepsilon(v \oplus \alpha) = v \oplus (\alpha - \iota(v)\varepsilon), \tag{7}$$

with image the orthogonal transformations fixing E pointwise. The subgroup $\wedge^2(E)$ acts freely and transitively on the subset of $\mathrm{Lag}(W)$ of Lagrangian subspaces transverse to E, which therefore becomes an affine space. Observe that A^ε has a distinguished lift $\widetilde{A}^\varepsilon = \exp(\varepsilon) \in \mathrm{Pin}(W)$ (exponential in the subalgebra $\wedge(E) \subset \mathrm{Cl}(W)$).

For any spinor module S over $\mathrm{Cl}(W)$, the induced representation of the group $\mathrm{Pin}(W) \subset \mathrm{Cl}(W)^\times$ preserves the set of pure spinors, and the map $\phi \mapsto N_\phi$ is equivariant. That is, if $\widetilde{A} \in \mathrm{Pin}(W)$ lifts $A \in \mathrm{O}(W)$, then

$$N_{\varrho(\widetilde{A})\phi} = A(N_\phi).$$

Consider again the case $W = \mathbb{V}$. Then 2-forms $\omega \in \wedge^2 V^*$ and bivectors $\pi \in \wedge^2(V)$ define orthogonal transformations

$$A^{-\omega}(v \oplus \alpha) = v \oplus (\alpha + \iota_v\omega), \quad A^{-\pi}(v \oplus \alpha) = (v + \iota_\alpha\pi) \oplus \alpha.$$

Their lifts act in the spin representation as follows:

$$\varrho(\widetilde{A}^{-\omega})\phi = e^{-\omega}\,\phi,\quad \varrho(\widetilde{A}^{-\pi})\phi = e^{-\iota(\pi)}\phi. \tag{8}$$

1.6. Morphisms. — It is easy to see that the group of orthogonal transformations of $\mathbb{V}$ preserving the 'polarization'

$$0 \longrightarrow V^* \longrightarrow \mathbb{V} \longrightarrow V \longrightarrow 0 \tag{9}$$

(i.e., taking the subspace V^* to itself) is the semi-direct product $\wedge^2 V^* \rtimes \mathrm{GL}(V) \subset \mathrm{O}(\mathbb{V})$, where $\omega \in \wedge^2 V^*$ acts as $A^{-\omega}$ and $\mathrm{GL}(V)$ acts in the natural way on V and by the conjugate transpose on V^*.

More generally, for vector spaces V and V', we define the set of *morphisms from* $\mathbb{V}$ *to* $\mathbb{V}'$ [**33**] to be

$$\mathrm{Hom}(V, V') \times \wedge^2 V^*,$$

with the following composition law:

$$(\Phi_1, \omega_1) \circ (\Phi_2, \omega_2) = (\Phi_1 \circ \Phi_2,\ \omega_2 + \Phi_2^* \omega_1). \tag{10}$$

Given $w = v \oplus \alpha \in \mathbb{V}$ and $w' = v' \oplus \alpha' \in \mathbb{V}'$, we write

$$w \sim_{(\Phi,\omega)} w' \quad \Leftrightarrow \quad v' = \Phi(v),\ \Phi^*\alpha' = \alpha + \iota_v\omega.$$

In particular, taking $V' = V$ and $\Phi = \mathrm{id}$ we have $w \sim_{(\mathrm{id},\omega)} w'$ if and only $w' = A^{-\omega}(w)$. The *graph of a morphism* (Φ, ω) is the subspace

$$\Gamma_{(\Phi,\omega)} = \{(w', w) \in \mathbb{V}' \times \mathbb{V} \mid w \sim_{(\Phi,\omega)} w'\}. \tag{11}$$

We have $\Gamma_{(\Phi_1,\omega_1)\circ(\Phi_2,\omega_2)} = \Gamma_{(\Phi_1,\omega_1)} \circ \Gamma_{(\Phi_2,\omega_2)}$ under composition of relations. The morphisms (Φ, ω) are 'isometric', in the sense that

$$w_1 \sim_{(\Phi,\omega)} w_1',\quad w_2 \sim_{(\Phi,\omega)} w_2' \Rightarrow \langle w_1, w_2\rangle = \langle w_1', w_2'\rangle. \tag{12}$$

Equivalently, $\Gamma_{(\Phi,\omega)}$ is Lagrangian in $\mathbb{V}' \oplus \overline{\mathbb{V}}$. We write

$$\begin{aligned}\ker(\Phi,\omega) &= \{w \in \mathbb{V} \mid w \sim_{(\Phi,\omega)} 0\},\\ \mathrm{ran}(\Phi,\omega) &= \{w' \in \mathbb{V}' \mid \exists w \in \mathbb{V}\colon w \sim_{(\Phi,\omega)} w'\}.\end{aligned}$$

Thus $\ker(\Phi,\omega) = \{(v, -\iota_v\omega) \mid v \in \ker(\Phi)\}$ while $\mathrm{ran}(\Phi,\omega) = \mathrm{ran}(\Phi) \oplus (V')^*$.

Definition 1.6. — Let $(\Phi,\omega)\colon \mathbb{V} \to \mathbb{V}'$ be a morphism, and $E \in \mathrm{Lag}(\mathbb{V})$. We define the *forward image* $E' \in \mathrm{Lag}(\mathbb{V}')$ to be the Lagrangian subspace

$$E' := \Gamma_{(\Phi,\omega)} \circ E = \{w' \in \mathbb{V}' \mid \exists\, w \in E\colon w \sim_{(\Phi,\omega)} w'\}.$$

Similarly, for $F' \in \mathrm{Lag}(\mathbb{V}')$ the *backward image* is defined as the Lagrangian subspace

$$F := F' \circ \Gamma_{(\Phi,\omega)} = \{w \in \mathbb{V} \mid \exists\, w' \in F'\colon\ w \sim_{(\Phi,\omega)} w'\}.$$

The proof that forward and backward images of Lagrangian subspaces are Lagrangian is parallel to the similar statement in the symplectic category of Guillemin-Sternberg [**30**] (see also Weinstein [**54**]). It is simple to check that the composition $E' = \Gamma_{(\Phi,\omega)} \circ E$ is transverse if and only if $\ker(\Phi,\omega) \cap E = \{0\}$. Similarly, the composition $F = F' \circ \Gamma_{(\Phi,\omega)}$ is transverse if and only if $\mathrm{ran}(\Phi,\omega) + F' = \mathbb{V}'$ (equivalently, if and only if $\mathrm{ran}(\Phi) + \mathrm{ran}(F') = V'$).

Remark 1.7. — As in the symplectic category [**30, 54**], one could consider morphisms given by arbitrary *Lagrangian relations*, i.e. Lagrangian subspaces $\Gamma \subset \mathbb{V}' \oplus \overline{\mathbb{V}}$ (see e.g. [**16**]). The graphs (11) of morphisms (Φ,ω) are exactly those Lagrangian relations preserving the 'polarization' (9), in the sense that $\Gamma \circ V^* = (V')^*$ (where the composition is transverse), see [**33**].

The (Φ,ω)-relation may also be interpreted in terms of the spinor representations of $\mathrm{Cl}(\mathbb{V})$ and $\mathrm{Cl}(\mathbb{V}')$:

Lemma 1.8. — *Suppose $(\Phi,\omega)\colon \mathbb{V} \to \mathbb{V}'$ is a morphism, and $w \in \mathbb{V}$, $w' \in \mathbb{V}'$. Then*

$$w \sim_{(\Phi,\omega)} w' \quad \Leftrightarrow \quad \varrho(w)(e^{\omega}\Phi^*\psi') = e^{\omega}\Phi^*(\varrho(w')\psi'), \qquad \psi' \in \wedge(V')^*. \tag{13}$$

Proof. — This follows from $(\epsilon(\alpha)+\iota_v)(e^{\omega}\Phi^*\psi') = e^{\omega}(\epsilon(\alpha+\iota_v\omega)+\iota_v)\Phi^*\psi'$, for $v\oplus\alpha \in \mathbb{V}$. □

Lemma 1.9. — *Suppose $(\Phi,\omega)\colon \mathbb{V} \to \mathbb{V}'$ is a morphism, and ψ' is a pure spinor defining a Lagrangian subspace F'. Then $\psi = e^{\omega}\Phi^*\psi'$ is non-zero if and only if the composition $F = F' \circ \Gamma_{(\Phi,\omega)}$ is transverse, and in that case it is a pure spinor defining F.*

Proof. — Suppose $w \in F$, i.e. $w \sim_{(\Phi,\omega)} w'$ with $w' \in F' = N_{\psi'}$. Then $w \in N_{\psi}$ by Equation (13). Thus $F \subset N_{\psi}$. For $\psi \neq 0$, this is an equality since F is Lagrangian. □

Example 1.10. — Suppose $E, F \subset \mathbb{V}$ are Lagrangian, with defining pure spinors ϕ, ψ. Let $E^{\top}$ be the image of E under the map $v \oplus \alpha \mapsto v \oplus (-\alpha)$. Then $\phi^{\top}$ is a pure spinor defining $E^{\top}$. Consider the diagonal inclusion $\mathrm{diag}\colon V \to V \times V$, so that $\mathrm{diag}^*(\phi^{\top} \otimes \psi) = \phi^{\top} \wedge \psi$ is just the wedge product. The wedge product is non-zero if and only if the composition $E^{\top} \wedge F := (E^{\top} \times F) \circ \Gamma_{\mathrm{diag}}$ is transverse. This is the case, for instance, if E and F are transverse (since the top degree part of $\phi^{\top} \wedge \psi$ is non-zero in this case). Explicitly,

$$E^{\top} \wedge F = \{v \oplus \alpha \,|\, \exists v \oplus \alpha_1 \in E,\ v \oplus \alpha_2 \in F\colon \alpha = \alpha_2 - \alpha_1\}.$$

Note that $\mathrm{ran}(E^{\top} \wedge F) = \mathrm{ran}(E) \cap \mathrm{ran}(F)$, with 2-form the difference of the restrictions of the 2-forms on $\mathrm{ran}(E)$ and $\mathrm{ran}(F)$. Note also that $(A^{-\omega}(E))^{\top} \wedge (A^{-\omega}(F)) = E^{\top} \wedge F$ for all $\omega \in \wedge^2 V^*$.

This "wedge product" operation of Lagrangian subspaces was noticed independently by Gualtieri, see [**29**].

1.7. Dirac spaces. — A *Dirac space* is a pair (V, E), where V is a vector space and $E \subset \mathbb{V}$ is a Lagrangian subspace. As remarked in Section 1.4, E determines a subspace $Q = \mathrm{ran}(E) = \mathrm{pr}_V(E) \subset V$ together with a 2-form $\omega_Q \in \wedge^2 Q^*$,

$$\omega_Q(v, v') = \langle \alpha, v' \rangle = -\langle \alpha', v \rangle \tag{14}$$

for arbitrary lifts $v \oplus \alpha, v' \oplus \alpha' \in E$ of $v, v' \in Q$. The kernel of ω_Q is the subspace $\ker(E) = E \cap V$. Conversely, any subspace Q equipped with a 2-form ω_Q determines a Lagrangian subspace $E = \{v \oplus \alpha \in \mathbb{V} |\ v \in Q,\ \alpha|_Q = \omega_Q(v, \cdot)\}$. The gauge transformation $A^{-\omega}$ by a 2-form $\omega \in \wedge^2 V^*$ preserves Q, while ω_Q changes by the pull-back of ω to Q.

Definition 1.11. — Let $(\mathbb{V}, E)$ and $(\mathbb{V}', E')$ be Dirac spaces. A *Dirac morphism* $(\Phi, \omega) \colon (V, E) \to (V', E')$ is a morphism (Φ, ω) with $E' = \Gamma_{(\Phi,\omega)} \circ E$. It is called a *strong Dirac morphism*(2) if this composition is transverse, i.e.,

$$\ker(\Phi, \omega) \cap E = \{0\}.$$

Clearly, the composition of strong Dirac morphisms is again a strong Dirac morphism. Note that the definition of a Dirac morphism $(\Phi, \omega) \colon (V, E) \to (V', E')$ amounts to the existence of a linear map $\widehat{\mathfrak{a}} \colon E' \to E$, assigning to each $w' \in E'$ an element of E to which it is (Φ, ω)-related:

$$\widehat{\mathfrak{a}}(w') \sim_{(\Phi,\omega)} w' \qquad \forall w' \in E'. \tag{15}$$

The map $\widehat{\mathfrak{a}}$ is completely determined by its V-component

$$\mathfrak{a} = \mathrm{pr}_V \circ \widehat{\mathfrak{a}} \colon E' \to V,$$

since $\widehat{\mathfrak{a}}(v' \oplus \alpha') = v \oplus (\Phi^* \alpha' + \iota_v \omega)$ where $v = \mathfrak{a}(v' \oplus \alpha')$. Hence (Φ, ω) is a Dirac morphism if and only if there exists a map $\mathfrak{a} \colon E' \to V$, such that the corresponding map $\widehat{\mathfrak{a}}$ takes values in E.

Lemma 1.12. — *For a strong Dirac morphism $(\Phi, \omega) \colon (V, E) \to (V', E')$, the map $\widehat{\mathfrak{a}}$ satisfying* (15) *is unique. Its range is given by*

$$\mathrm{ran}(\widehat{\mathfrak{a}}) = E \cap \ker(\Phi, \omega)^\perp. \tag{16}$$

Proof. — The map $\widehat{\mathfrak{a}}$ associated to a Dirac morphism is unique up to addition of elements in $E \cap \ker(\Phi, \omega)$. Hence, it is unique precisely if the Dirac morphism is strong. Its range consists of all $w \in E$ which are (Φ, ω)-related to some element of $w' \in E'$. By (12), the subspace $\{w \in \mathbb{V} \,|\, \exists w' \in \mathbb{V}' \colon w \sim_{(\Phi,\omega)} w'\}$ is orthogonal to $\ker(\Phi, \omega)$. Hence, by a dimension count it coincides with $\ker(\Phi, \omega)^\perp$. On the other hand, if $w \in E$ lies in this subspace, it is automatic that $w' \in E'$ since $E' = \Gamma_{(\Phi,\omega)} \circ E$. □

(2) In the particular case when $\omega = 0$, Dirac morphisms are also called *forward Dirac maps* [**15, 16**], and strong Dirac morphisms are called *Dirac realizations* [**14**].

Example 1.13. — Let $E \subset \mathbb{V}$ be a Lagrangian subspace, and let ω_Q be the corresponding 2-form on $Q = \mathrm{ran}(E)$. Let $\iota_Q : Q \to V$ be the inclusion. Then $(\iota_Q, \omega_Q) : (Q, Q) \to (V, E)$ is a strong Dirac morphism. Equivalently $(\iota_Q, 0) : (Q, \mathrm{Gr}_{\omega_Q}) \to (V, E)$ is a strong Dirac morphism. Here $\mathfrak{a}(v \oplus \alpha) = \iota_Q(v)$.

Example 1.14. — Suppose $\pi \in \wedge^2 V$ and $\pi' \in \wedge^2 V'$. Then $(\Phi, 0) : (V, \mathrm{Gr}_\pi) \to (V', \mathrm{Gr}_{\pi'})$ is a Dirac morphism if and only if $\Phi(\pi) = \pi'$. It is automatically strong (since $\ker(\mathrm{Gr}_\pi) = 0$), with $\mathfrak{a}(v' \oplus \alpha') = \pi^\sharp(\Phi^* \alpha')$.

Proposition 1.15. — *Suppose $(\Phi, \omega) : (V, E) \to (V', E')$ is a Dirac morphism, and that F' is a Lagrangian subspace transverse to E'. Let ϕ be a pure spinor defining E, and ψ' a pure spinor defining F'. Then $\psi := e^\omega \Phi^* \psi'$ is non-zero, and is a pure spinor defining the backward image $F = F' \circ \Gamma_{(\Phi,\omega)}$. Moreover, the following are equivalent:*

a. (Φ, ω) *is a* strong *Dirac morphism,*
b. *the backward image F is transverse to E,*
c. *The pairing $(\phi, \psi)_{\wedge(V^*)} \in \det(V^*)$ is non-zero, that is, it is a volume form on V.*

Proof. — By (6), we may write $\psi' = e^{-\omega_{Q'}} \theta'$, where $\omega_{Q'}$ is a 2-form on $Q' = \mathrm{ran}(F')$, and $\theta' \in \wedge^{\mathrm{top}}(V'/\mathrm{ran}(F'))^*$. Identifying $(V'/\mathrm{ran}(F'))^*$ with the annihilator of $\mathrm{ran}(F')$, this gives

$$\begin{aligned}\psi \neq 0 &\Leftrightarrow \Phi^* \theta' \neq 0 \\ &\Leftrightarrow \ker(\Phi^*) \cap \mathrm{ann}(\mathrm{ran}(F')) = 0 \\ &\Leftrightarrow \{w' \in F' |\ 0 \sim_{(\Phi,\omega)} w'\} = \{0\}.\end{aligned}$$

(Indeed, $0 \sim_{(\Phi,\omega)} w'$ if and only if $w' = 0 \oplus \alpha'$ with $\Phi^* \alpha' = \{0\}$. Moreover $w' \in F' = (F')^\perp$ if and only if $\alpha' \in \mathrm{ann}(\mathrm{ran}(F'))$.) But the condition $0 \sim_{(\Phi,\omega)} w'$ implies that $w' \in E'$. Since $E' \cap F' = 0$ it follows that $\{w' \in F' |\ 0 \sim_{(\Phi,\omega)} w'\} = \{0\}$, hence $\psi \neq 0$. Lemma 1.9 shows that it is a pure spinor defining the backward image F.

$(a) \Leftrightarrow (b)$. By definition, $E \cap F$ consists of all $w \in E$ such that $w \sim_{(\Phi,\omega)} w'$ for some $w' \in F'$. Since $E' = \Gamma_{(\Phi,\omega)} \circ E$, this element w' also lies in E', and hence $w' = 0$. Thus,

$$E \cap F = E \cap \ker(\Phi, \omega),$$

which is zero precisely if the Dirac morphism (Φ, ω) is strong. $(b) \Leftrightarrow (c)$ is immediate from Theorem 1.3. □

1.8. Lagrangian splittings. — Suppose W is a vector space with split bilinear form. By a *Lagrangian splitting* of W we mean a direct sum decomposition $W = E \oplus F$ into transverse Lagrangian subspaces.

Lemma 1.16. — *Let W be a vector space with split bilinear form $\langle \cdot, \cdot \rangle$. There is a 1-1 correspondence between projection operators $\mathsf{p} \in \mathrm{End}(W)$ with the property $\mathsf{p} + \mathsf{p}^t = 1$, and Lagrangian splittings $W = E \oplus F$. (Here p^t is the transpose with respect to the inner product on W.)*

Proof. — A Lagrangian splitting of W into transverse Lagrangian subspaces is equivalent to a projection operator whose kernel and range are isotropic. For any projection operator $\mathsf{p} = \mathsf{p}^2$, the range $\mathrm{ran}(\mathsf{p})$ is isotropic if and only if $\mathsf{p}^t\mathsf{p} = 0$, while $\ker(\mathsf{p}) = \mathrm{ran}(1-\mathsf{p})$ is isotropic if and only if $(1-\mathsf{p})^t(1-\mathsf{p}) = 0$. If both the kernel and the range of p are isotropic, then

$$1 - (\mathsf{p} + \mathsf{p}^t) = (1-\mathsf{p})^t(1-\mathsf{p}) - \mathsf{p}^t\mathsf{p} = 0.$$

Conversely, if p is a projection operator with $\mathsf{p} + \mathsf{p}^t = 1$, then $\mathsf{p}^t\mathsf{p} = (1-\mathsf{p})\mathsf{p} = 0$, and similarly $(1-\mathsf{p})^t(1-\mathsf{p}) = 0$. □

Again, we specialize to the case $W = \mathbb{V}$. Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, with associated projection operator p. The property $\mathsf{p} + \mathsf{p}^t = 1$ implies that there is a bivector $\pi \in \wedge^2 V$ defined by

$$\pi^\sharp(\alpha) = -\,\mathrm{pr}_V(\mathsf{p}(\alpha)), \quad \alpha \in V^*, \tag{17}$$

that is, $\pi(\alpha, \beta) = -\langle \mathsf{p}(\alpha), \beta\rangle = \langle \alpha, \mathsf{p}(\beta)\rangle$, $\alpha, \beta \in V^*$. If $\{e_i\}$ is a basis of E and $\{f^i\}$ is the dual basis of F, then

$$\pi = \tfrac{1}{2}\,\mathrm{pr}_V(e_i) \wedge \mathrm{pr}_V(f^i). \tag{18}$$

The graph of the bivector π was encountered in Example 1.10 above:

Proposition 1.17. — *The graph of the bivector π is given by*

$$\mathrm{Gr}_\pi = E^\top \wedge F. \tag{19}$$

In particular, $\mathrm{ran}(\pi^\sharp) = \mathrm{ran}(E) \cap \mathrm{ran}(F)$, *and the symplectic 2-form on* $\mathrm{ran}(\pi^\sharp)$ *is the difference of the restrictions of the 2-forms on* $\mathrm{ran}(E), \mathrm{ran}(F)$. *If ϕ, ψ are pure spinors defining E, F, then*

$$\phi^\top \wedge \psi = e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\mathrm{top}]}.$$

Proof. — Since both sides of (19) are Lagrangian subspaces, it suffices to prove the inclusion $\supset$. Let $v \oplus \alpha \in E^\top \wedge F$. Hence, there exist α_1, α_2 with $\alpha = \alpha_2 - \alpha_1$ and $v \oplus \alpha_1 \in E$, $v \oplus \alpha_2 \in F$. Thus $v \oplus \alpha_1 = -\mathsf{p}(\alpha)$, which implies that $\pi^\sharp(\alpha) = -\,\mathrm{pr}_V\,\mathsf{p}(\alpha) = v$. The description of $\mathrm{ran}\,\pi^\sharp = \mathrm{ran}(\mathrm{Gr}_\pi)$ is immediate from (19), see the discussion in Example 1.10. The formula for $\phi^\top \wedge \psi$ follows since both sides are pure spinors defining the Lagrangian subspace Gr_π, with the same top degree part. □

Proposition 1.18. — *Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, defining a bivector π. If $\varepsilon \in \wedge^2 E$, so that $F_\varepsilon = A^{-\varepsilon}F$ is a new Lagrangian complement to E, the bivector π_ε for the splitting $E \oplus F_\varepsilon$ is given by*

$$\pi_\varepsilon = \pi + \mathrm{pr}_V(\varepsilon),$$

where $\mathrm{pr}_V\colon \wedge E \to \wedge V$ *is the algebra homomorphism extending the projection to V.*

Proof. — Let ϕ, ψ be pure spinors defining E, F. Then F_ε is defined by the pure spinor $\psi_\varepsilon = \varrho(e^{-\varepsilon})\psi$. Using Remark 1.5(a), we obtain

$$\phi^\top \wedge \psi_\varepsilon = \phi^\top \wedge \varrho(e^{-\varepsilon})\psi = e^{-\iota(\mathrm{pr}_V(\varepsilon))}\phi^\top \wedge \psi.$$

The claim now follows from (1.17). □

Proposition 1.19. — *Let $(\Phi,\omega)\colon (V,E) \to (V',E')$ be a strong Dirac morphism. Suppose $F' \in \mathrm{Lag}(\mathbb{V}')$ is transverse to E', and F is its backward image under (Φ,ω). Then the bivectors for the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$ are Φ-related:*

$$\Phi(\pi) = \pi'.$$

Proof. — To prove $\Phi(\pi) = \pi'$, we have to show that $(\Phi, 0)\colon (V, \mathrm{Gr}_\pi) \to (V', \mathrm{Gr}_{\pi'})$ is a Dirac morphism:

$$\Gamma_{(\Phi,0)} \circ (E^\top \wedge F) = (E')^\top \wedge F'.$$

Since both sides are Lagrangian, it suffices to prove the inclusion $\supset$. If $v' \oplus \alpha' \in (E')^\top \wedge F'$, then $\alpha' = \alpha_2' - \alpha_1'$, where $v' \oplus \alpha_1' \in E'$ and $v' \oplus \alpha_2' \in F'$. Since (Φ,ω) is a strong Dirac morphism for E, E', there is a unique element $v \oplus \alpha_1 \in E$ such that $v' = \Phi(v)$, $\Phi^*(\alpha_1') = \alpha_1 + \iota_v\omega$. Let $\alpha_2 = \Phi^*(\alpha_2') - \iota_v\omega$. Then $v \oplus \alpha_2 \in F$ since $v \oplus \alpha_2 \sim_{(\Phi,\omega)} v' \oplus \alpha_2$. Hence $v \oplus \Phi^*(\alpha') = v \oplus (\alpha_2 - \alpha_1) \in E^\top \wedge F$, proving that $v' \oplus \alpha' \in \Gamma_{(\Phi,0)} \circ (E^\top \wedge F)$. □

We next explain how a splitting $\mathbb{V}' = E' \oplus F'$ may be 'pulled back' under a linear map $\Phi\colon V \to V'$, given a bivector $\pi \in \wedge^2 V$ and a linear map $\mathfrak{a}\colon E' \to V$ satisfying suitable compatibility relations.

Theorem 1.20. — *Suppose that $\Phi\colon V \to V'$ is a linear map and $\omega \in \wedge^2 V^*$ a 2-form. Given a Lagrangian splitting $\mathbb{V}' = E' \oplus F'$, with associated projection $\mathsf{p}' \in \mathrm{End}(\mathbb{V}')$, there is a 1-1 correspondence between*

(i) *Lagrangian subspaces $E \subset \mathbb{V}$ such that $(\Phi,\omega)\colon (V,E) \to (V',E')$ is a strong Dirac morphism, and*
(ii) *Bivectors $\pi \in \wedge^2 V$ together with linear maps $\mathfrak{a}\colon E' \to V$, satisfying $\Phi \circ \mathfrak{a} = \mathrm{pr}_{V'}|_{E'}$ and*

$$\pi^\sharp \circ \Phi^* = -\mathfrak{a} \circ \mathsf{p}'|_{(V')^*}. \tag{20}$$

Under this correspondence, π is the bivector defined by the splitting $\mathbb{V} = E \oplus F$, where F is the backward image of F', and $\mathfrak{a}$ is the linear map defined by the strong Dirac morphism (Φ,ω) (see (15)).

Proof. — "$(i) \Rightarrow (ii)$". By Proposition 1.15, we know that the backward image F of F' is transverse to E. Let p and p' be the projections defined by the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$, and π, π' the corresponding bivectors. As in (15), the strong Dirac morphism (Φ,ω) defines a linear map $\widehat{\mathfrak{a}}\colon E' \to E$, taking

$w' \in E'$ to the unique element $w \in E$ such that $w \sim_{(\Phi,\omega)} w'$. We claim that for all $w \in \mathbb{V}$, $w' \in V'$,

$$w \sim_{(\Phi,\omega)} w' \Rightarrow \mathsf{p}(w) = \widehat{\mathfrak{a}}(\mathsf{p}'(w')). \tag{21}$$

Indeed, let $w_1 = \mathsf{p}(w) \in E$, so that $w_2 = w - w_1 \in F$. There is a (unique) element $w_2' \in F'$ with $w_2 \sim_{(\Phi,\omega)} w_2'$, so let $w_1' = w' - w_2'$. Since $w_2 \sim_{(\Phi,\omega)} w_2'$, it follows that $w_1 \sim_{(\Phi,\omega)} w_1'$. Hence $w_1' \in E'$ by definition of E'. It follows that $\mathsf{p}(w) = w_1 = \widehat{\mathfrak{a}}(w_1') = \widehat{\mathfrak{a}}(\mathsf{p}'(w'))$, as claimed. In particular, since $\Phi^*\alpha' \sim_{(\Phi,\omega)} \alpha'$ for $\alpha' \in V'$, (21) implies that

$$\pi^\sharp(\Phi^*\alpha') = -\operatorname{pr}_V(\mathsf{p}(\Phi^*\alpha')) = -\operatorname{pr}_V(\widehat{\mathfrak{a}}(\mathsf{p}'(\alpha'))) = -\mathfrak{a}(\mathsf{p}'(\alpha')), \quad \alpha' \in (V')^*$$

where $\mathfrak{a} = \operatorname{pr}_V \circ \widehat{\mathfrak{a}}$.

"$(i) \Leftarrow (ii)$". Our aim is to construct the projection p with kernel $F := F' \circ \Gamma_{(\Phi,\omega)}$ and range E. We define p by the following equations, for $v, v_1, v_2 \in V$ and $\alpha, \alpha_1, \alpha_2 \in V^*$:

$$\begin{aligned}
\langle \mathsf{p}(v_1), v_2 \rangle &= \langle \mathsf{p}'(\Phi(v_1)), \Phi(v_2) \rangle, \\
\langle \mathsf{p}(\alpha_1), \alpha_2 \rangle &= -\pi(\alpha_1, \alpha_2), \\
\langle \mathsf{p}(v), \alpha \rangle &= \langle \mathfrak{a}^*\alpha,\ \Phi(v) \rangle + \pi(\iota_v\omega, \alpha), \\
\langle \mathsf{p}(\alpha), v \rangle &= \langle \alpha, v \rangle - \langle \mathfrak{a}^*\alpha,\ \Phi(v) \rangle - \pi(\iota_v\omega, \alpha),
\end{aligned}$$

where $\mathfrak{a}^* \colon V^* \to (E')^* = F'$ is the dual map to $\mathfrak{a}$. The linear map p defined in this way has the property $\mathsf{p} + \mathsf{p}^t = 1$. We claim that this linear map satisfies (21), where $\widehat{\mathfrak{a}} \colon E' \to \mathbb{V}$ is defined as follows,

$$\widehat{\mathfrak{a}}(w') = \mathfrak{a}(w') \oplus (\Phi^* \operatorname{pr}_{(V')^*}(w') - \iota_{\mathfrak{a}(w')}\omega).$$

For $w = v \oplus \iota_v\omega$, $w' = \Phi(v) \oplus 0$, (21) is easily checked using the definition of p. Hence it suffices to consider the case $w = \Phi^*\alpha'$, $w' = \alpha'$ with $\alpha' \in (V')^*$. For all $v \in V$, using the definition of p and $\Phi \circ \mathfrak{a} = \operatorname{pr}_{V'}|_{E'}$, i.e., $\mathfrak{a}^* \circ \Phi^* = (\mathsf{p}')^t|_{(V')^*}$, we have:

$$\begin{aligned}
\langle \mathsf{p}(\Phi^*\alpha'), v \rangle &= \langle \alpha', \Phi(v) \rangle - \langle (\mathsf{p}')^t\alpha', \Phi(v) \rangle - \pi(\iota_v\omega, \Phi^*\alpha') \\
&= \langle \mathsf{p}'\alpha', \Phi(v) \rangle + \pi(\Phi^*\alpha', \iota_v\omega) \\
\langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), v \rangle &= \langle \Phi^* \operatorname{pr}_{(V')^*} \mathsf{p}'(\alpha'), v \rangle - \omega(\mathfrak{a}(\mathsf{p}'(\alpha')), v) \\
&= \langle \mathsf{p}'\alpha', \Phi(v) \rangle + \omega(\pi^\sharp(\Phi^*\alpha'), v)
\end{aligned}$$

which shows $\langle \mathsf{p}(\Phi^*\alpha'), v \rangle = \langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), v \rangle$. Similarly, for $\beta \in V^*$ we have, by (20),

$$\langle \mathsf{p}(\Phi^*\alpha'), \beta \rangle = -\langle \pi^\sharp(\Phi^*\alpha'), \beta \rangle = \langle \widehat{\mathfrak{a}}(\mathsf{p}'(\alpha')), \beta \rangle.$$

This proves (21). Equation (21) applies in particular to all elements $w \in F$, since these are by definition (Φ, ω)-related to elements $w' \in F'$. We hence see that $\mathsf{p}(w) = 0$ for all $w \in F$. This proves that $F \subset \ker(\mathsf{p})$. Taking orthogonals, $\operatorname{ran}(\mathsf{p}^t) \subset F$. In particular, the range of p^t is isotropic, i.e. $\mathsf{p}\mathsf{p}^t = 0$, and hence $\mathsf{p} - \mathsf{p}^2 = \mathsf{p}(1 - \mathsf{p}) = \mathsf{p}\mathsf{p}^t = 0$. Thus p is a projection. As before, we see that $\ker \mathsf{p} = \operatorname{ran}(1 - \mathsf{p})$ is isotropic as well, hence $F = \ker(\mathsf{p})$ since F is maximal isotropic. It remains to show that the Lagrangian subspace $E := \operatorname{ran}(\mathsf{p})$ satisfies $\Gamma_{(\Phi,\omega)} \circ E \subset E'$. Suppose $w \sim_{(\Phi,\omega)} w'$ for some $w \in E$.

By (21), we also have $w = \mathsf{p}(w) \sim_{(\Phi,\omega)} \mathsf{p}'(w')$. Thus $0 \sim_{(\Phi,\omega)} (w'-\mathsf{p}'(w')) = (\mathsf{p}')^t(w')$. Observe that $\operatorname{ran}(\Phi) \supset \Phi(\mathfrak{a}(E')) = \operatorname{ran}(E')$. Hence $\ker(\Phi^*) \subset \operatorname{ann}(\operatorname{ran}(E'))$. Since $E' \cap F' = 0$, it follows that

$$\ker(\Phi^*) \cap \operatorname{ann}(\operatorname{ran}(F')) = 0. \tag{22}$$

Using Equation (22), the relation $0 \sim_{(\Phi,\omega)} (\mathsf{p}')^t(w') \in F'$ implies that $(\mathsf{p}')^t(w') = 0$, i.e. $w' \in E'$. □

The proof shows that $\mathsf{p}|_V = \widehat{\mathfrak{a}} \circ \mathsf{p}' \circ \Phi$, whereas $h := \mathsf{p}|_{V^*} : V^* \to E$ is given by

$$h(\alpha) = (-\pi^\sharp(\alpha)) \oplus (\alpha - \Phi^* \operatorname{pr}_{(V')^*} \mathfrak{a}^*(\alpha) - \iota(\pi^\sharp(\alpha))\omega). \tag{23}$$

It follows that $E = \operatorname{ran}(\widehat{\mathfrak{a}}) + \operatorname{ran}(h)$. Projecting to V, it follows in particular that

$$\operatorname{ran}(E) = \operatorname{ran}(\mathfrak{a}) + \operatorname{ran}(\pi^\sharp). \tag{24}$$

2. Pure spinors on manifolds

A pure spinor on a manifold is simply a differential form whose restriction to any point is a pure spinor on the tangent space. The following discussion is carried out in the category of real manifolds and C^∞ vector bundles, but works equally well for complex manifolds with holomorphic vector bundles.

2.1. Dirac structures. — For any manifold M, we denote by $\mathbb{T}M = TM \oplus T^*M$ the direct sum of the tangent and cotangent bundles, with fiberwise inner product $\langle \cdot, \cdot \rangle$. The fiberwise Clifford action defines a bundle map

$$\varrho \colon \mathrm{Cl}(\mathbb{T}M) \to \operatorname{End}(\wedge T^*M). \tag{25}$$

The same symbol will denote the action of sections of $\mathrm{Cl}(\mathbb{T}M)$ on sections of $\wedge T^*M$, i.e. differential forms. The bilinear pairing will be denoted by

$$(\cdot, \cdot)_{\wedge T^*M} \colon \wedge T^*M \otimes \wedge T^*M \to \det(T^*M), \tag{26}$$

and the same notation will be used for sections. Thus $(\phi, \phi')_{\wedge T^*M} = (\phi^\top \wedge \phi')^{[\mathrm{top}]}$ for differential forms $\phi, \phi' \in \Gamma(\wedge T^*M) = \Omega(M)$. An *almost Dirac structure* on M is a smooth Lagrangian subbundle $E \subset \mathbb{T}M$. The pair (M, E) is called an almost Dirac manifold. A *pure spinor defining* E is a nonvanishing differential form $\phi \in \Omega(M)$ such that $\phi|_m$ is a pure spinor defining E_m, for all m. Equivalently, ϕ is a nonvanishing section of the line bundle $(\wedge T^*M)^E$. Thus E is globally represented by a pure spinor if and only if the line bundle $(\wedge T^*M)^E$ is orientable. (Otherwise, one may still use pure spinors to describe E locally.)

Let $\eta \in \Omega^3(M)$ be a closed 3-form. A direct computation shows that the spinor representation defines a bilinear bracket $[\![\cdot, \cdot]\!]_\eta \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ by the condition:

$$\varrho([\![x_1, x_2]\!]_\eta)\psi = [[\mathrm{d} + \eta, \varrho(x_1)], \varrho(x_2)]\psi, \quad \psi \in \Omega(M),\ x_i \in \Gamma(\mathbb{T}M), \tag{27}$$

where the brackets on the right-hand side are graded commutators of operators on $\Omega(M)$. The bracket $[\![\cdot,\cdot]\!]_\eta$ is the *η-twisted Courant bracket* [**35, 50**].(3) (For more on the definition of $[\![\cdot,\cdot]\!]_\eta$ as a 'derived bracket', see e.g. [**9, 36, 48**].) The operator on $\Omega(M)$ defined by

$$[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\mathrm{d}+\eta]]]$$

is multiplication by a function

$$\Upsilon(x_1,x_2,x_3) = -\langle [\![x_3,x_2]\!]_\eta, x_1\rangle \in C^\infty(M). \tag{28}$$

Given an almost Dirac structure $E\subset \mathbb{T}M$, let Υ^E denote the restriction of the trilinear form $(x_1,x_2,x_3)\mapsto \Upsilon(x_1,x_2,x_3)$ to the sections of E. In contrast to Υ, the trilinear form Υ^E is *tensorial* and *skew-symmetric.* The resulting element

$$\Upsilon^E \in \Gamma(\wedge^3 E^*)$$

is called the *η-twisted Courant tensor* of E.

Definition 2.1. — A *Dirac structure* on a manifold M is an almost Dirac structure E together with a closed 3-form η such that its η-twisted Courant tensor vanishes: $\Upsilon^E=0$. The triple (M,E,η) is called a *Dirac manifold.*

For E an almost Dirac structure one can always choose a complementary almost Dirac structure F such that $E\oplus F=\mathbb{T}M$. (This is parallel to a well-known fact from symplectic geometry [**51**, Proposition 8.2], with a similar proof.) As a vector bundle, $F\cong E^*$ with pairing induced by the inner product on $\mathbb{T}M$. We have:

Proposition 2.2. — *Let E be an almost Dirac structure on M, and F be a complementary almost Dirac structure. Suppose E is represented by a pure spinor $\phi\in\Omega(M)$. Then there is a unique section $\sigma^E\in\Gamma(E^*)$ (depending on ϕ) such that*

$$(d+\eta)\phi = \varrho(-\Upsilon^E+\sigma^E)\phi.$$

Here we view Υ^E and σ^E as sections of $\wedge F\subset \mathrm{Cl}(\mathbb{T}M)$.

Proof. — Choose a Lagrangian subbundle F complementary to E. Since

$$\Gamma(\wedge F)\to\Omega(M),\ x\mapsto \varrho(x)\phi$$

is an isomorphism, there is a unique odd element $x\in\Gamma(\wedge F)\subset\Gamma(\mathbb{T}M)$ such that $(\mathrm{d}+\eta)\phi=\varrho(x)\phi$. To see that x has filtration degree 3, let x_1,x_2,x_3 be three sections of E. Since $\varrho(x_i)\phi=0$, it follows that

$$\begin{aligned}\varrho([x_1,[x_2,[x_3,x]]])\phi &= [[[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\varrho(x)]]]\phi = \varrho(x_1x_2x_3)\varrho(x)\phi\\ &= \varrho(x_1x_2x_3)(\mathrm{d}+\eta)\phi = [[[\varrho(x_1),[\varrho(x_2),[\varrho(x_3),\mathrm{d}+\eta]]]\phi = \Upsilon^E(x_1,x_2,x_3)\phi,\end{aligned}$$

(3) This definition agrees with the non-skew symmetric version of the Courant bracket [**40, 50**], called the *Dorfman bracket* in [**28**]; the η-term in the bracket, however, differs from the one in [**50**] by a sign.

proving that the Clifford commutator $[x_1,[x_2,[x_3,x]]] = \iota(x_1)\iota(x_2)\iota(x_3)x$ (contraction of $x \in \Gamma(\wedge(E^*))$ with sections of E) is a scalar. This implies that x has filtration degree 3, and that the degree 3 part of x is $-\Upsilon^E$. □

We hence see that an almost Dirac structure $E \subset \mathbb{T}M$ is integrable if and only if

$$(\mathrm{d}+\eta)\phi \in \varrho(\mathbb{T}M)\phi,$$

for any pure spinor $\phi \in \Omega(M)$ (locally) representing E. The characterization of the integrability condition $\Upsilon^E = 0$ in terms of pure spinors was observed by Gualtieri [**28**], see also [**9**].

Examples of Dirac structures (for a given η) include graphs of 2-forms $\omega \in \Omega(M)$ with $\mathrm{d}\omega = \eta$, as well as graphs of bivector fields $\pi \in \mathfrak{X}^2(M)$ defining *η-twisted Poisson structures* [**35, 50**] in the sense that $\frac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + \pi^\sharp(\eta) = 0$. One may also consider *complex* Dirac structures on M, given by complex Lagrangian subbundles $E \subset \mathbb{T}M^{\mathbb{C}}$ satisfying $\Upsilon^E = 0$. The defining pure spinors are complex-valued differential forms ϕ on M, given as nonvanishing sections of $(\wedge T^*M^{\mathbb{C}})^E$. If E is a Dirac structure, then its image E^c under the complex conjugation mapping is a Dirac structure defined by the complex conjugate spinor ϕ^c. E is called a *generalized complex structure* [**28, 32**] if $E \cap E^c = 0$.

Suppose $E \subset \mathbb{T}M$ is a Dirac structure. The vanishing of the Courant tensor implies that E is a *Lie algebroid*, with anchor given by the natural projection on TM, and Lie bracket $[\cdot,\cdot]_E$ on $\Gamma(E)$ given by the restriction of the Courant bracket $[\![\cdot,\cdot]\!]_\eta$. From the theory of Lie algebroids, it follows that the generalized distribution $\mathrm{ran}(E)$ is integrable (in the sense of Sussmann) [**24**]. The generalized foliation having $\mathrm{ran}(E)$ as its tangent distribution is called the *Dirac foliation*. For any leaf $Q \subset M$ of the Dirac foliation, the collection of 2-forms on T_mQ (defined as in (14)) defines a smooth 2-form $\omega_Q \in \Omega^2(Q)$ with

$$\mathrm{d}\omega_Q = i_Q^*\eta,$$

where $i_Q\colon Q \to M$ is the inclusion (for a proof, see e.g. [**47**, Proposition 6.10]). If E is the graph of a Poisson bivector π (with $\eta = 0$), this is the usual symplectic foliation.

2.2. Dirac morphisms. — Suppose $\Phi\colon M \to M'$ is a smooth map, and $\omega \in \Omega^2(M)$ is a 2-form. As in the linear case, we view the pair (Φ,ω) as a 'morphism', with composition rule (10). Given sections $x \in \Gamma(\mathbb{T}M)$ and $x' \in \Gamma(\mathbb{T}M')$, we will write

$$x \sim_{(\Phi,\omega)} x' \;\Leftrightarrow\; \forall m \in M\colon x_m \sim_{((d\Phi)_m,\omega_m)} x'_{\Phi(m)}.$$

In terms of the spinor representation, this is equivalent to the condition

$$e^\omega\Phi^*(\varrho(x')\psi') = \varrho(x)(e^\omega\Phi^*(\psi')),\quad \psi' \in \Omega(M').$$

Using the definition (27) of the Courant bracket as a derived bracket, one obtains:

Lemma 2.3 **(Stienon-Xu).** — [**53**, Lemma 2.2] *Let M, M' be manifolds with closed 3-forms η,η', $\Phi\colon M \to M'$ a smooth map, and $\omega \in \Omega^2(M)$ a 2-form such that* $\Phi^*\eta' =$

$\eta + d\omega$. *Then*

$$x_i \sim_{(\Phi,\omega)} x_i',\ i = 1,2 \ \Rightarrow [\![x_1, x_2]\!]_\eta \sim_{(\Phi,\omega)} [\![x_1', x_2']\!]_{\eta'}.$$

That is, the morphism $(\Phi, \omega)\colon M \to M'$ intertwines both the inner product and the (η- resp. η'-twisted) Courant brackets on $\mathbb{T}M$ and $\mathbb{T}M'$.

Definition 2.4. — a. Suppose (M, E) and (M', E') are almost Dirac manifolds. A morphism $(\Phi, \omega)\colon M \to M'$ is called a *(strong) almost Dirac morphism* $(\Phi, \omega)\colon (M, E) \to (M, E')$ if $((\mathrm{d}\Phi)_m, \omega_m)\colon (T_m M, E_m) \to (T_{\Phi(m)}M', E'_{\Phi(m)})$ is a linear (strong) Dirac morphism for all $m \in M$.

b. Suppose (M, E, η) and (M', E', η') are Dirac manifolds. A (strong) almost Dirac morphism $(\Phi, \omega)\colon M \to M'$ is called a *(strong) Dirac morphism* $(\Phi, \omega)\colon (M, E, \eta) \to (M', E', \eta')$ if $\eta + \mathrm{d}\omega = \Phi^*\eta'$.

For $\omega = 0$, strong Dirac morphisms coincide with the *Dirac realizations* of [**14**].

Example 2.5. — If (M, E, η) is a Dirac manifold, then so is $(M, A^{-\omega}(E), \eta + \mathrm{d}\omega)$, for any 2-form ω, and (id_M, ω) is a Dirac morphism between the two. The Dirac structures E and $A^{-\omega}(E)$ are isomorphic as Lie algebroids; in particular, they define the same Dirac foliation. However, the 2-forms on the leaves of this foliation change by the pull-back of ω.

Example 2.6. — Any manifold M can be trivially viewed as a Dirac manifold $M = (M, TM, 0)$. A strong Dirac morphism from M to pt is then the same thing as a symplectic 2-form on M. More generally, strong Dirac morphisms $M \to N$ are (special types of) symplectic fibrations.

Example 2.7. — If (M, E, η) is a Dirac manifold, and $Q \subset M$ is a leaf of the associated foliation of M, then the inclusion map defines a strong Dirac morphism $(\iota_Q, \omega_Q)\colon (Q, TQ, 0) \to (M, E, \eta)$.

From the linear case, it follows that a strong almost Dirac morphism gives rise to a bundle map

$$\widehat{\mathfrak{a}}\colon \Phi^* E' \to E.$$

This is indeed a smooth bundle map: the projection $\mathbb{T}M \oplus \Phi^*\mathbb{T}M' \to \Phi^*\mathbb{T}M'$ restricts to a bundle isomorphism $\Gamma_\Phi \cap (E \oplus \Phi^*\mathbb{T}M') \to \Phi^*E'$, and $\widehat{\mathfrak{a}}$ is the inverse of this bundle isomorphism followed by the projection to $\mathbb{T}M$. We let

$$\mathfrak{a} = \mathrm{pr}_{TM} \circ \widehat{\mathfrak{a}}\colon\ \Phi^*E' \to \mathrm{ran}(E) \subset TM \tag{29}$$

Proposition 2.8. — *Suppose $(\Phi, \omega)\colon (M, E, \eta) \to (M', E', \eta')$ is a strong Dirac morphism. Then the induced bundle map $\widehat{\mathfrak{a}}\colon \Phi^*E' \to E$ is a* comorphism of Lie algebroids [**43**]. *That is, it is compatible with the anchor maps in the sense that*

$$d\Phi \circ \mathfrak{a} = \mathrm{pr}_{\Phi^*TM'}|_{\Phi^*E'},$$

and the induced map on sections

$$\widehat{\mathfrak{a}}\colon \Gamma(E') \to \Gamma(E),\ \ (\widehat{\mathfrak{a}}(x'))_m = \widehat{\mathfrak{a}}(x'_{\Phi(m)})$$

preserves brackets.

Proof. — Compatibility with the anchor is obvious. If x'_1, x'_2 are section of E', then (using Lemma 2.3) both $\widehat{\mathfrak{a}}(\Phi^*[x'_1, x'_2]_{E'})$ and $[\widehat{\mathfrak{a}}(\Phi^* x'_1), \widehat{\mathfrak{a}}(\Phi^* x'_2)]_E$ are sections of E which are (Φ, ω)-related to $[x'_1, x'_2]_{E'}$. Hence their difference is (Φ, ω)-related to 0. Since (Φ, ω) is a strong Dirac morphism, it follows that the difference is in fact 0. □

The second part of Proposition 2.8 shows that (29) defines a Lie algebra homomorphism $\mathfrak{a}\colon \Gamma(E') \to \mathfrak{X}(M)$. That is, the strong Dirac morphism defines an '*action*' of the Lie algebroid E' on the manifold M.

2.3. Bivector fields. — From the linear theory, we see that any Lagrangian splitting $\mathbb{T}M = E \oplus F$ defines a bivector field π on M. Furthermore,

$$e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\mathrm{top}]} = \phi^\top \wedge \psi$$

for any pure spinors ϕ, ψ defining E, F. Recall that $(\phi^\top \wedge \psi)^{[\mathrm{top}]}$ is a volume form on M.

For an arbitrary volume form μ on M, and any bivector field $\pi \in \mathfrak{X}^2(M)$, one has the formula [**26**]

$$\mathrm{d}(e^{-\iota(\pi)}\mu) = \iota\big(-\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + X_\pi\big)(e^{-\iota(\pi)}\mu). \tag{30}$$

Here $[\cdot,\cdot]_{\mathrm{Sch}}$ is the Schouten bracket on multivector fields, and X_π is the vector field on M defined by $\mathrm{d}\iota(\pi)\mu = -\iota(X_\pi)\mu$. If π is a Poisson bivector field, then $X_\pi \in \mathfrak{X}(M)$ is called the *modular vector field* of π with respect to the volume form μ [**56**]. (See [**37**] for modular vector fields for *twisted* Poisson structures.)

Theorem 2.9. — *Let π be the bivector field defined by the Lagrangian splitting $\mathbb{T}M = E \oplus F$. Let $\Upsilon^E \in \Gamma(\wedge^3 F)$ and $\Upsilon^F \in \Gamma(\wedge^3 E)$ be the Courant tensor fields of E, F.*

a) *The Schouten bracket of π with itself is given by the formula*

$$\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} = \mathrm{pr}_{TM}(\Upsilon^E) + \mathrm{pr}_{TM}(\Upsilon^F),$$

where $\mathrm{pr}_{TM}\colon \wedge E \to \wedge TM$ is the algebra homomorphism extending the projection $E \to TM$, and similarly for $\mathrm{pr}_{TM}\colon \wedge F \to \wedge TM$.

b) *Given pure spinors $\phi, \psi \in \Omega(M)$ defining E, F, let $\sigma^E \in \Gamma(F)$ and $\sigma^F \in \Gamma(E)$ be the unique sections such that*

$$(d+\eta)\phi = \varrho(-\Upsilon^E + \sigma^E)\phi, \qquad (d+\eta)\psi = \varrho(-\Upsilon^F + \sigma^F)\psi.$$

Then the vector field X_π defined using the volume form $\mu = (\phi^\top \wedge \psi)^{[\mathrm{top}]}$ is given by

$$X_\pi = \mathrm{pr}_{TM}(\sigma^F) - \mathrm{pr}_{TM}(\sigma^E).$$

Proof. — We may assume that E, F are globally defined by pure spinors ϕ, ψ. Using Remark 1.5(a), we have

$$\begin{aligned}\mathrm{d}(\phi^\top \wedge \psi) &= (-1)^{|\phi|}(\phi^\top \wedge \mathrm{d}\psi + (\mathrm{d}\phi)^\top \wedge \psi)\\ &= (-1)^{|\phi|}(\phi^\top \wedge (\mathrm{d}+\eta)\psi + ((\mathrm{d}+\eta)\phi)^\top \wedge \psi)\\ &= (-1)^{|\phi|}(\phi^\top \wedge (\varrho(-\Upsilon^F + \sigma^F)\psi) + (\varrho(-\Upsilon^E + \sigma^E)\phi)^\top \wedge \psi)\\ &= \iota(\mathrm{pr}_{TM}(-\Upsilon^F + \sigma^F) + \mathrm{pr}_{TM}(-\Upsilon^E - \sigma^E))(\phi^\top \wedge \psi).\end{aligned}$$

On the other hand, $\phi^\top \wedge \psi = e^{-\iota(\pi)}\mu$ gives, by (30),

$$\mathrm{d}(\phi^\top \wedge \psi) = \iota(-\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + X_\pi)(\phi^\top \wedge \psi).$$

Applying the star operator $\star$ for μ, and using that $\star(\phi^\top \wedge \psi)$ is invertible, it follows that

$$\mathrm{pr}_{TM}(-\Upsilon^F + \sigma^F) + \mathrm{pr}_{TM}(-\Upsilon^E - \sigma^E) = -\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} + X_\pi. \qquad \square$$

As a special case, if both E, F are Dirac structures (i.e. integrable), then the corresponding bivector field π satisfies $[\pi,\pi]_{\mathrm{Sch}} = 0$, i.e., it is a Poisson structure. The symplectic leaves of π are the intersections of the leaves of the Dirac structures E with those of F. The fact that transverse Dirac structures (or equivalently *Lie bialgebroids*) define Poisson structures goes back to Mackenzie-Xu [**44**].

Proposition 2.10. — *Suppose $(\Phi,\omega)\colon (M,E) \to (M',E')$ is an almost Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Lagrangian subbundle complementary to E'. Then there is a smooth Lagrangian subbundle $F \subset \mathbb{T}M$ complementary to E, with the property that for all $m \in M$, F_m is the backward image of $F'_{\Phi(m)}$ under $(d_m\Phi, \omega_m)$. Furthermore:*

a. *The bivector fields π, π' defined by the splittings $\mathbb{T}M = E \oplus F$ and $\mathbb{T}M' = E' \oplus F'$ satisfy*

$$\pi \sim_\Phi \pi',$$

i.e. $(d\Phi)_m \pi_m = \pi'_{\Phi(m)}$ for all $m \in M$.

b. *The Courant tensors $\Upsilon^F \in \Gamma(\wedge^3 E)$ and $\Upsilon^{F'} \in \Gamma(\wedge^3 E')$ are related by*

$$\Upsilon^F = \widehat{\mathfrak{a}}(\Phi^*\Upsilon^{F'}),$$

using the extension of $\widehat{\mathfrak{a}}\colon \Gamma(\Phi^*E') \to \Gamma(E)$ to the exterior algebras.

c. *The bivector field π satisfies*

$$\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} = \mathfrak{a}(\Phi^*\Upsilon^{F'}) + \mathrm{pr}_{TM}(\Upsilon^E),$$

using the extension of $\mathfrak{a}\colon \Gamma(\Phi^*E') \to \Gamma(TM)$ to the exterior algebras.

d.

$$\pi^\sharp \circ \Phi^* = -\mathfrak{a} \circ \mathsf{p}'\colon T^*M' \to TM,$$

where $\mathsf{p}'\colon \mathbb{T}M' \to E'$ is the projection along F'.

e. *If ψ' is a pure spinor defining F', and $\psi = e^\omega \Phi^* \psi'$ the corresponding pure spinor defining F, the sections σ^F, $\sigma^{F'}$ are related by $\sigma_F = \widehat{\mathfrak{a}}(\Phi^* \sigma_F)$, that is,*

$$\sigma^F \sim_{(\Phi,\omega)} \sigma^{F'}.$$

Proof. — Let $\psi' \in \Omega(M')$ be a pure spinor (locally) representing F'. From the linear case (Proposition 1.15), it follows that $\psi = e^\omega \Phi^* \psi'$ is non-zero everywhere, and is a pure spinor representing a Lagrangian subbundle $F \subset \mathbb{T}M$ transverse to E. Now (a) follows from the linear case, see Proposition 1.19. We next verify (b), at any given point $m \in M$. Let $m' = \Phi(m)$. Given $(x_i)_m \in F_m$ for $i = 1, 2, 3$, let $(x_i')_{m'} \in F'_{m'}$ with

$$(x_i)_m \sim_{((d\Phi)_m,\ \omega_m)} (x_i')_{m'}.$$

Choose sections $x_i \in \Gamma(F)$, $x_i' \in \Gamma(F')$ extending the given values at m, m'. We have to show $\Upsilon^F(x_1, x_2, x_3)|_m = \Upsilon^{F'}(x_1', x_2', x_3')|_{m'}$. We calculate:

$$\Upsilon^F(x_1, x_2, x_3)\ \psi = \varrho(x_1 x_2 x_3)\,(\mathrm{d} + \eta)(e^\omega \Phi^* \psi') = \varrho(x_1 x_2 x_3)\, e^\omega \Phi^* (\mathrm{d} + \eta')\psi'$$

On the other hand,

$$(\Phi^* \Upsilon^{F'}(x_1', x_2', x_3'))\ \psi = e^\omega \Phi^* \Upsilon^{F'}(x_1', x_2', x_3')\ \psi' = e^\omega \Phi^* \varrho(x_1' x_2' x_3')\,(\mathrm{d} + \eta')\psi'$$

These two expressions coincide at m, proving (b). Theorem 2.9 together with (b) implies the statement (c). Part (d) follows from Proposition 1.20. Part (e) follows from (b) together with the definition of σ^F, $\sigma^{F'}$. □

Part (b) shows in particular that if F' is a Dirac structure, transverse to E', then its backward image is again a Dirac structure.

2.4. Dirac cohomology. — In this Section, we will discuss certain cohomology groups associated with any pair of transverse Dirac structures $E, F \subset \mathbb{T}M$ and a given volume form μ on M. We assume that E, F are given by pure spinors ϕ, ψ, normalized by the condition $(\phi, \psi)_{\wedge T^*M} = \mu$. Let $\sigma^E \in \Gamma(F), \sigma^F \in \Gamma(E)$ be sections defined as in Theorem 2.9, and denote

$$\sigma = \sigma^F - \sigma^E \in \Gamma(\mathbb{T}M).$$

Replacing ϕ, ψ with $\tilde{\phi} = f\phi$, $\tilde{\psi} = f^{-1}\psi$, for f a nonvanishing function on M, this section changes by a closed 1-form:

$$\tilde{\sigma} = \sigma - f^{-1}\mathrm{d}f. \tag{31}$$

Indeed, letting let p be the projection from $\mathbb{T}M$ to E along F we have $\tilde{\sigma}^F = \sigma^F - \mathsf{p}(f^{-1}\mathrm{d}f)$, $\tilde{\sigma}^E = \sigma^E + (I - \mathsf{p})(f^{-1}\mathrm{d}f)$.

We define the *Dirac cohomology groups* associated to a triple (E, F, μ) as the cohomology of the operators

$$\not{\partial}_+ = d + \eta + \varrho(\sigma), \quad \not{\partial}_- = d + \eta - \varrho(\sigma)$$

on $\Omega(M)$, restricted to the subspace on which they square to zero:

$$H_\pm(E, F, \mu) := \ker(\not{\partial}_\pm)/\ker(\not{\partial}_\pm) \cap \operatorname{im}(\not{\partial}_\pm) \equiv H(\ker \not{\partial}_\pm^2, \not{\partial}_\pm). \tag{32}$$

The pure spinors ϕ, ψ define classes in $H_+(E,F,\mu)$ and $H_-(E,F,\mu)$, respectively, since $\not{\partial}_+\phi = 0$ and $\not{\partial}_-\psi = 0$. The Dirac cohomology groups are independent of the choice of defining spinors ϕ, ψ: Changing the pure spinors by a function f as above, (31) shows that the operators $\not{\partial}_\pm$ change by conjugation, $\tilde{\not{\partial}}_+ = f\not{\partial}_+ f^{-1}$ and $\tilde{\not{\partial}}_- = f^{-1}\not{\partial}_- f$.

Example 2.11. — Let M be a manifold with volume form μ. Consider transverse Dirac structures $E = \mathrm{Gr}_\omega$ for some closed 2-form ω, and $F = T^*M$. In this case, one can choose $\phi = e^{-\omega}, \psi = \mu$. We obtain $\eta = 0$, $\sigma = 0$, $\not{\partial}_\pm = \mathrm{d}$, and the Dirac cohomology groups $H_\pm(TM, T^*M, \mu)$ coincide with the de Rham cohomology of M.

Example 2.12. — Let M be a manifold with volume form μ and with a Poisson bivector π. Let $E = TM, F = \mathrm{Gr}_\pi$. The choice $\phi = 1, \psi = e^{-\iota(\pi)}\mu$ gives $\not{\partial}_- = d - \iota(X_\pi)$, where X_π is the modular vector field. The operator $\not{\partial}_-^2 = -\mathscr{L}(X_\pi)$ vanishes on differential forms invariant under the flow generated by X_π. The Dirac cohomology $H_-(TM, \mathrm{Gr}_\pi, \mu) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi))$ resembles the Cartan model of equivariant cohomology for circle actions.

Let π be the Poisson structure defined by the splitting $\mathbb{T}M = E \oplus F$, and $X_\pi = \mathrm{pr}_{TM}\,\sigma$ the modular vector field. Let

$$H_\pi(M) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi)). \tag{33}$$

By Remark 1.5(a) there is a pairing

$$H_+(E,F,\mu) \otimes H_-(E,F,\mu) \to H_\pi(M)$$

given on representatives by the formula $u \otimes v \mapsto u^\top \wedge v$. The pure spinors ϕ, ψ define cohomology classes $[\phi] \in H_+(E,F,\mu), [\psi] \in H_-(E,F,\mu)$, and $[\phi^\top \wedge \psi] \in H_\pi(M)$. If M is compact, the integration map $\int_M \colon \Omega(M)^{X_\pi} \to \mathbb{R}$ descends to $H_\pi(M)$. Hence

$$\int_M \phi^\top \wedge \psi = \int_M \mu > 0$$

shows that the cohomology classes $[\phi] \in H_+(E,F,\mu), [\psi] \in H_-(E,F,\mu)$ are both nonzero.

There is the following version of functoriality with respect to strong Dirac morphisms for Dirac cohomology.

Proposition 2.13. — *Let $(\Phi, \omega)\colon (M, E, \eta) \to (M', E', \eta')$ be a strong Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Dirac structure transverse to E', with backward image F. Assume that E, E' are defined by pure spinors ϕ, ϕ' such that the corresponding sections σ^E and $\sigma^{E'}$ vanish. Let ψ' and $\psi = e^\omega\Phi^*\psi'$ be pure spinors defining F' and F, and let μ' and μ be the resulting volume forms. Then $e^\omega \circ \Phi^*$ intertwines $\not{\partial}_-$ and $\not{\partial}'_-$, and hence induces a map in Dirac cohomology $e^\omega\Phi^*\colon H_-(E', F', \mu') \to H_-(E, F, \mu)$ taking $[\psi']$ to $[\psi]$.*

Proof. — Since $\sigma^E, \sigma^{E'}$ vanish we have $\sigma = \sigma^F$ and $\sigma' = \sigma^{F'}$. By Proposition 2.10 (e), the map $e^{\omega}\Phi^*$ intertwines the Clifford actions of σ^F and $\sigma^{F'}$, while on the other hand this map also intertwines $\mathrm{d}+\eta$ with $\mathrm{d}+\eta'$. Hence it intertwines $\not{\partial}_-$ with $\not{\partial}'_-$. □

2.5. Classical dynamical Yang-Baxter equation. — The following result describes the Courant tensor of Lagrangian subbundles defined by elements in $\Gamma(\wedge^2 E)$.

***Proposition 2.14* (Liu-Weinstein-Xu [40]).** — *Let $\mathbb{T}M = E \oplus F$ be a splitting into Lagrangian subbundles, where both E, F are integrable relative to the closed 3-form η, and let us identify $F^* = E$. Given a section $\varepsilon \in \Gamma(\wedge^2 E)$, defining a section $A^{-\varepsilon} \in \Gamma(\mathrm{O}(\mathbb{T}M))$, let $F_\varepsilon = A^{-\varepsilon}(F)$ be the Lagrangian subbundle spanned by the sections $x + \iota_x\varepsilon$ for $x \in \Gamma(F) = \Gamma(E^*)$. Then the Courant tensor $\Upsilon_\varepsilon \in \Gamma(\wedge^3 E)$ of F_ε is given by the formula:*

$$\Upsilon_\varepsilon = d_F\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_E.$$

Here $[\cdot,\cdot]_E$ is the Lie algebroid bracket of E, and $d_F\colon \Gamma(\wedge^\bullet F^*) \to \Gamma(\wedge^{\bullet+1}F^*)$ is the Lie algebroid differential of F.

Remark 2.15. — The result in [**40**] is stated only for $\eta = 0$. However, since the statement is local, one may use a gauge transformation by a local primitive of η to reduce to this case.

We are interested in the following special case: Let $M = \mathfrak{g}^*$, with its standard linear Poisson structure $\pi_{\mathfrak{g}^*} \in \Gamma(\wedge^2 T\mathfrak{g}^*) = C^\infty(\mathfrak{g}^*) \otimes \wedge^2\mathfrak{g}^*$, and put $F = T\mathfrak{g}^*$ and $E = \mathrm{Gr}_{\pi_{\mathfrak{g}}^*}$. The bundle E is spanned by sections $\mathscr{C}_0(\xi) \oplus \langle\theta_0, \xi\rangle$ for $\xi \in \mathfrak{g}$, where $\mathscr{C}_0(\xi)$ is the generating vector fields for the co-adjoint action, and $\langle\theta_0, \xi\rangle \in \Omega^1(\mathfrak{g}^*)$ is the 'constant' 1-form defined by ξ. The trivialization $E = \mathfrak{g}^* \times \mathfrak{g}$ defined by these sections identifies E with the action algebroid for the co-adjoint action: The bracket on $\Gamma(E) = C^\infty(\mathfrak{g}^*, \mathfrak{g})$ is defined by the Lie bracket on $\mathfrak{g}$ via the Leibniz rule, and the anchor map is given by the action map $\mathscr{C}_0\colon \mathfrak{g} \to T\mathfrak{g}^*$. For $\varepsilon \in \Gamma(\wedge^2 E)$, the bracket $[\varepsilon, \varepsilon]_E$ is given by the Schouten bracket on $\wedge\mathfrak{g}$. On the other hand we may view $\varepsilon \in C^\infty(\mathfrak{g}^*, \wedge^2\mathfrak{g})$ as a 2-form on $\mathfrak{g}^*$, and then $\mathrm{d}\varepsilon = \mathrm{d}_F\varepsilon$ is just its exterior differential. The resulting equation reads

$$\mathrm{d}\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}} = \Upsilon_\varepsilon.$$

If Υ_ε is a multiple of the structure constants tensor, this is a special case of the *classical dynamical Yang-Baxter equation* (CDYBE) [**5, 25**]. We will see below how a solution arises from the Cartan-Dirac structure on G.

For more information on the relation between Dirac structures and the CDYBE, see the work of Liu-Xu [**41**] and Bangoura-Kosmann-Schwarzbach [**10**].

3. Dirac structures on Lie groups

In this Section, we will study Dirac structures over Lie groups G with bi-invariant pseudo-Riemannian metrics. This will be based on the existence of a canonical isomorphism

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$$

preserving scalar products and Courant brackets. In the subsequent section, we will describe a corresponding isomorphism of spinor modules.

3.1. The isomorphism $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$. — Let G be a Lie group (not necessarily connected), and let $\mathfrak{g}$ be its Lie algebra. We denote by $\xi^L, \xi^R \in \mathfrak{X}(G)$ the left-, right-invariant vector fields on G which are equal to $\xi \in \mathfrak{g} = T_eG$ at the group unit. Let $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ be the left-, right-Maurer-Cartan forms, i.e. $\iota(\xi^L)\theta^L = \iota(\xi^R)\theta^R = \xi$. They are related by $\theta^R_g = \mathrm{Ad}_g(\theta^L_g)$, for all $g \in G$. The adjoint action of G on itself will be denoted $\mathcal{A}_{\mathsf{ad}}$ (or simply $\mathcal{A}$, if there is no risk of confusion). The corresponding infinitesimal action is given by the vector fields

$$\mathcal{A}_{\mathsf{ad}}(\xi) = \xi^L - \xi^R.$$

Suppose that the Lie algebra $\mathfrak{g}$ of G carries an *invariant inner product.* By this we mean an Ad-invariant, non-degenerate symmetric bilinear form B, not necessarily positive definite. Equivalently, B defines a bi-invariant pseudo-Riemannian metric on G. Given B, we can define the bi-invariant 3-form $\eta \in \Omega^3(G)$,

$$\eta := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L])$$

Since η is bi-invariant, it is closed, and so it defines an η-twisted Courant bracket $[\![\cdot,\cdot]\!]_\eta$ on G. The conjugation action $\mathcal{A}_{\mathsf{ad}}$ extends to an action of $D = G \times G$ on G, by

$$\mathcal{A}\colon D \to \mathrm{Diff}(G), \quad \mathcal{A}(a, a') = l_{a'} \circ r_{a^{-1}}, \tag{34}$$

where $l_a(g) = ag$ and $r_a(g) = ga$. The corresponding infinitesimal action

$$\mathcal{A}\colon \mathfrak{d} \to \mathfrak{X}(G), \quad \mathcal{A}(\xi, \xi') = \xi^L - (\xi')^R$$

lifts to a map

$$\mathsf{s}\colon \mathfrak{d} \to \Gamma(\mathbb{T}G), \qquad \mathsf{s}(\xi, \xi') = \mathsf{s}^L(\xi) + \mathsf{s}^R(\xi'), \tag{35}$$

where

$$\mathsf{s}^L(\xi) = \xi^L \oplus \tfrac{1}{2} B(\theta^L, \xi), \quad \mathsf{s}^R(\xi') = -(\xi')^R \oplus \tfrac{1}{2} B(\theta^R, \xi').$$

Let us equip $\mathfrak{d}$ with the bilinear form $B_{\mathfrak{d}}$ given by $+B$ on the first $\mathfrak{g}$-summand and $-B$ on the second $\mathfrak{g}$-summand. Thus $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is an example of a Lie algebra with invariant split bilinear form.

Proposition 3.1. — *The map* $\mathsf{s}\colon \mathfrak{d} \to \Gamma(\mathbb{T}G)$ *is D-equivariant, and satisfies*

$$\langle \mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2) \rangle = B_{\mathfrak{d}}(\zeta_1, \zeta_2), \quad [\![\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2)]\!]_\eta = \mathsf{s}([\zeta_1, \zeta_2]) \tag{36}$$

for all $\zeta_1, \zeta_2 \in \mathfrak{d}$. *Furthermore,*

$$\Upsilon(\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2), \mathsf{s}(\zeta_3)) = B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]) \tag{37}$$

for all $\zeta_i \in \mathfrak{d}$, *where* $\Upsilon\colon \Gamma(\mathbb{T}G)^{\otimes 3} \to C^\infty(G)$ *was defined in* (28).

Proof. — The D-equivariance of the map s is clear. Let ϱ be the Clifford action of $\mathbb{T}G$ on $\wedge T^*G$. We have $[\varrho(\mathsf{s}^L(\xi)), \mathrm{d} + \eta] = \mathscr{L}(\xi^L)$ and $[\varrho(\mathsf{s}^R(\xi)), \mathrm{d} + \eta] = -\mathscr{L}(\xi^R)$, thus

$$[\mathrm{d} + \eta,\ \varrho(\mathsf{s}(\zeta))] = \mathscr{L}(\mathscr{A}(\zeta))$$

for all $\zeta \in \mathfrak{d}$. This proves the second Equation in (36), while the first Equation is obvious. Finally, (37) follows from (36) and the definition of Υ. Hence,

$$\varrho([\![\mathsf{s}(\zeta_1), \mathsf{s}(\zeta_2)]\!]_\eta) = [[\mathrm{d} + \eta, \varrho(\mathsf{s}(\zeta_1)],\ \varrho(\mathsf{s}(\zeta_2))] = \varrho(\mathsf{s}([\zeta_1, \zeta_2])).$$

□

Put differently, the map s defines a D-equivariant isometric isomorphism

$$\mathbb{T}G \cong G \times \mathfrak{d}, \tag{38}$$

identifying the η-twisted Courant bracket on $\mathbb{T}G$ with the unique Courant bracket on $G \times \mathfrak{d}$ which agrees with the Lie bracket on $\mathfrak{d}$ on constant sections.

3.2. η-twisted Dirac structures on G. — Using (38), we see that any Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$ defines a Lagrangian subbundle

$$E^{\mathfrak{s}} \cong G \times \mathfrak{s},$$

spanned by the sections $\mathsf{s}(\zeta)$ with $\zeta \in \mathfrak{s}$. The Lagrangian subbundle $E^{\mathfrak{s}}$ is invariant under the action of the subgroup of D preserving $\mathfrak{s}$. Let $\Upsilon^{\mathfrak{s}} \in \wedge^3\mathfrak{s}^*$ be defined as

$$\Upsilon^{\mathfrak{s}}(\zeta_1, \zeta_2, \zeta_3) = B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]),\ \ \zeta_i \in \mathfrak{s}. \tag{39}$$

By (37), the Courant tensor $\Upsilon^{E^{\mathfrak{s}}}$ is just $\Upsilon^{\mathfrak{s}}$, using the sections s to identify $(E^{\mathfrak{s}})^* \cong G \times \mathfrak{s}^*$. In particular, we see that $\mathfrak{s}$ defines a Dirac structure if and only if $\mathfrak{s}$ is a Lie subalgebra. To summarize:

Any Lagrangian subalgebra $\mathfrak{s} \subset \mathfrak{d}$ *defines an* η*-twisted Dirac structure* $E^{\mathfrak{s}}$.

The Dirac structure $E^{\mathfrak{s}}$ is invariant under the action of any Lie subgroup normalizing $\mathfrak{s}$, and in particular under the action of the subgroup $S \subset D$ integrating $\mathfrak{s}$. As a Lie algebroid, $E^{\mathfrak{s}}$ is just the action algebroid for this S-action. In particular, its leaves are just the components of the S-orbits on G. The 2-form on the orbit $\mathcal{O} = \mathscr{A}(S)g$ of an element $g \in G$ is the S-invariant form $\omega_{\mathcal{O}}$ given as follows: for $\zeta_i = (\xi_i, \xi_i') \in \mathfrak{s}$,

$$\begin{aligned}\omega_{\mathcal{O}}(\mathscr{A}(\zeta_1), \mathscr{A}(\zeta_2))|_g &= \tfrac{1}{2}\langle B(\theta^L, \xi_1) + B(\theta^R, \xi_1'),\ \xi_2^L - (\xi_2')^R\rangle \\ &= \tfrac{1}{2}B(\xi_2 - \mathrm{Ad}_{g^{-1}}\xi_2', \xi_1 + \mathrm{Ad}_{g^{-1}}\xi_1') \\ &= \tfrac{1}{2}(B(\mathrm{Ad}_g\xi_2, \xi_1') - B(\xi_2', \mathrm{Ad}_g\xi_1)),\end{aligned} \tag{40}$$

using $B(\xi_1, \xi_2) = B(\xi_1', \xi_2')$ since $\mathfrak{s}$ is Lagrangian. By the general theory from Section 2.1, these 2-forms satisfy $\mathrm{d}\omega_{\mathcal{O}} = \iota_{\mathcal{O}}^*\eta$, where $\iota_{\mathcal{O}}\colon \mathcal{O} \to G$ is the inclusion. The kernel of ω_θ equals $\ker(E^{\mathfrak{s}})$, i.e. it is spanned by all $\mathscr{A}(\zeta)$ such that the T^*G-component of $\mathsf{s}(\zeta)$ is zero:

$$\ker(\omega_\theta|_g) = \{\mathscr{A}(\zeta)|_g \mid \zeta = (\xi, \xi') \in \mathfrak{s},\ \ \mathrm{Ad}_g\,\xi + \xi' = 0\}. \tag{41}$$

Remark 3.2. — For $\mathfrak{g}$ a complex semi-simple Lie algebra, a complete classification of Lagrangian subalgebras of $\mathfrak{d}$ was obtained by Karolinsky [**34**]. The Poisson geometry of the variety of Lagrangian subalgebras of $\mathfrak{d}$ was studied in detail by Evens–Lu [**27**].

Remark 3.3. — If $\mathfrak{d} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is a splitting into two Lagrangian subalgebras (i.e., $(\mathfrak{d}, \mathfrak{s}_1, \mathfrak{s}_2)$ is a Manin triple), one obtains two transverse Dirac structures $E^{\mathfrak{s}_1}, E^{\mathfrak{s}_2}$. As discussed after Theorem 2.9, such a pair of transverse Dirac structures gives rise to a Poisson structure on G, with symplectic leaves the intersections of the orbits of S_1, S_2. For $\mathfrak{g}$ a complex semi-simple Lie algebra, the Manin triples were classified by Delorme [**22**]. See Evens–Lu [**27**] for a wealth of information regarding Poisson structures obtained from Lagrangian subalgebras. An example will be worked out in Section 3.6 below.

Remark 3.4. — We may also use this construction to obtain generalized complex (and Kähler) structures [**28**] on even-dimensional real Lie groups K, with complexification $G = K^{\mathbb{C}}$. Indeed, let $\mathfrak{s} \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ be a Lagrangian subalgebra such that

$$\mathfrak{s} \cap \mathfrak{s}^c = \{0\}, \tag{42}$$

where $\mathfrak{s}^c$ denotes the complex conjugate of $\mathfrak{s}$. Then the associated Dirac structure $E^{\mathfrak{s}} \subset \mathbb{T}G$ satisfies $E^{\mathfrak{s}} \cap (E^{\mathfrak{s}})^c = \{0\}$ along K. Hence it defines a generalized complex structure on K. For a concrete example, suppose K is compact, and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ be a triangular decomposition. (That is, $\mathfrak{t} = \mathfrak{t}_K^{\mathbb{C}}$ is the complexification of a maximal Abelian subalgebra, and $\mathfrak{n}_+, \mathfrak{n}_-$ are the sums of the positive, negative root spaces). Then

$$\mathfrak{s} = (\mathfrak{n}^+ \oplus 0) \oplus \mathfrak{l} \oplus (0 \oplus \mathfrak{n}^-) \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$$

has the desired property, for any Lagrangian subspace $\mathfrak{l} \subset \mathfrak{t} \oplus \bar{\mathfrak{t}}$ with $\mathfrak{l} \cap \mathfrak{l}^c = \{0\}$ (i.e., $\mathfrak{l}$ is a linear generalized complex structure on the vector space $\mathfrak{t}_K$). The generalized complex structures on Lie groups considered in Gualtieri [**28**, Example 6.39] are examples of this construction.

3.3. The Cartan-Dirac structure. — The simplest example of a Lagrangian subalgebra is the diagonal $\mathfrak{s} = \mathfrak{g}_\Delta \hookrightarrow \mathfrak{d}$, with corresponding S the diagonal subgroup $G_\Delta \subset D$. The associated Dirac structure E_G is spanned by the sections $\mathsf{e}(\xi) := \mathsf{s}(\xi, \xi)$:

$$E_G = \mathrm{span}\,\{\mathsf{e}(\xi) \mid \xi \in \mathfrak{g}\} \subset \mathbb{T}G, \tag{43}$$

$$\mathsf{e}(\xi) = (\xi^L - \xi^R, B(\tfrac{\theta^L + \theta^R}{2}, \xi)).$$

We call E_G the *Cartan-Dirac structure*, see [**15, 39, 50**]. This Dirac structure was introduced independently by Alekseev, Ševera, and Strobl in the mid-1990's. The $G_\Delta \cong G$-action is just the action by conjugation on G, hence the Dirac foliation is given by the conjugacy classes $\mathscr{C} \subset G$. The formula (40) specializes to the 2-form on conjugacy classes introduced in [**31**]:

$$\omega_{\mathscr{C}}(\mathscr{A}_{\mathsf{ad}}(\xi_1), \mathscr{A}_{\mathsf{ad}}(\xi_2)) = -\tfrac{1}{2}B((\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi_1, \xi_2),$$

The kernel at $g \in \mathscr{C}$ is the span of vector fields $\mathscr{A}_{\mathsf{ad}}(\xi)|_g$ with $\mathrm{Ad}_g\,\xi + \xi = 0$. The anti-diagonal in $\mathfrak{g} \oplus \overline{\mathfrak{g}}$ is a G-invariant Lagrangian complement to the diagonal, and hence defines a G-invariant Lagrangian subbundle F_G complementary to E_G, spanned by $\mathsf{f}(\xi) = \mathsf{s}(\xi/2, -\xi/2)$:

$$F_G = \mathrm{span}\{\mathsf{f}(\xi) |\ \xi \in \mathfrak{g}\} \subset \mathbb{T}G, \tag{44}$$

$$\mathsf{f}(\xi) = \big(\tfrac{\xi^L + \xi^R}{2}, B(\tfrac{\theta^L - \theta^R}{4}, \xi)\big).$$

The 1/2 factors in the definition of $\mathsf{f}(\xi)$ are introduced so that $\langle \mathsf{e}(\xi), \mathsf{f}(\xi')\rangle = B(\xi, \xi')$.

Let $\Xi \in \wedge^3(\mathfrak{g})$ be the structure constants tensor of $\mathfrak{g}$, normalized as follows:

$$\iota(\xi_3)\iota(\xi_2)\iota(\xi_1)\Xi = \tfrac{1}{4}\,B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}}). \tag{45}$$

Let $\mathsf{e}\colon \wedge\mathfrak{g} \to \Gamma(\wedge E_G)$ be the extension of $\mathsf{e}\colon \mathfrak{g} \to \Gamma(E_G)$ as an algebra homomorphism. Thus $\mathsf{e}(\Xi)$ is a section of $\wedge^3(E_G)$.

Lemma 3.5. — *The Courant tensor of F_G is given by :*

$$\Upsilon^{F_G} = \mathsf{e}(\Xi).$$

Proof. — This follows from (37) since $B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]_{\mathfrak{d}}) = \frac{1}{4}B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}})$ for $\zeta_i = (\xi_i/2,\ -\xi_i/2)$. □

The element Ξ also defines a trivector field, $\mathscr{A}_{\mathsf{ad}}(\Xi) \in \mathfrak{X}^3(G)$. Theorem 2.9 implies that the bivector field $\pi_G \in \mathfrak{X}^2(G)$ defined by the Lagrangian splitting $\mathbb{T}G = E \oplus F$ satisfies

$$\tfrac{1}{2}[\pi_G, \pi_G]_{\mathrm{Sch}} = \mathscr{A}_{\mathsf{ad}}(\Xi).$$

To give an explicit formula for π_G, let v_i, v^i be B-dual bases of $\mathfrak{g}$, i.e. $B(v_i, v^j) = \delta_i^j$.

Proposition 3.6. — *The bivector field π_G is given by*

$$\pi_G = \tfrac{1}{2}\sum_i v^{i,L} \wedge v_i^R. \tag{46}$$

Proof. — By (18), we have

$$\pi_G = \tfrac{1}{2}\sum_i \big((v_i)^L - (v_i)^R\big) \wedge \frac{(v^i)^L + (v^i)^R}{2}.$$

Since $\sum_i v^{i,L} \wedge v_i^L = \sum_i v^{i,R} \wedge v_i^R$, this simplifies to the expression in (46). □

The bivector field π_G was first considered in [**1, 2**].

3.4. Group multiplication. — In this Section, we will examine the behavior of the Cartan-Dirac structure under group multiplication,

$$\mathrm{Mult}\colon G\times G\to G,\quad (a,b)\mapsto ab.$$

For any differential form $\beta\in\Omega(G)$, we will denote by $\beta^i\in\Omega(G\times G)$ its pull-back to the i'th factor, for $i=1,2$. We will use similar notation for vector fields on $G\times G$, and for sections of the bundle $\mathbb{T}(G\times G)$. Let $\varsigma\in\Omega^2(G\times G)$ denote the 2-form

$$\varsigma=-\tfrac{1}{2}B(\theta^{L,1},\theta^{R,2}). \tag{47}$$

A direct computation shows that

$$\mathrm{Mult}^*\eta \;=\; \eta^1+\eta^2+\mathrm{d}\varsigma, \tag{48}$$

hence we have a multiplication morphism

$$(\mathrm{Mult},\varsigma)\colon (G,\eta)\times(G,\eta)=(G\times G,\eta^1+\eta^2)\to(G,\eta).$$

Remark 3.7. — This is expressed more conceptually in terms of the simplicial model $B_pG=G^p$ of the classifying space BG. Let $\partial_i\colon G^p\to G^{p-1}$, $0\le i\le p$ be the 'face maps' given as $\partial_i(g_1,\dots,g_p)=(g_1,\dots,g_ig_{i+1},\dots,g_p)$, while ∂_0 omits the first entry g_1, and ∂_p omits the last entry g_p. Let $\delta=\sum_{i=0}^p\partial_i^*\colon\Omega^\bullet(G^{p-1})\to\Omega^\bullet(G^p)$. Then δ commutes with the de-Rham differential, turning $\bigoplus_{p,q}\Omega^q(G^p)$ into a double complex. The total differential on $\Omega^q(G^p)$ is $\mathrm{d}+(-1)^q\delta$. Then $\eta\in\Omega^3(G)$ and $\varsigma\in\Omega^2(G^2)$ define a cocycle of degree 4 (see [**55**]):

$$\mathrm{d}\eta=0,\quad \partial\eta=-\mathrm{d}\varsigma,\quad \partial\varsigma=0. \tag{49}$$

(If G is compact, simple, and simply connected, and B the basic inner product, this pair is the Bott-Shulman representative of the generator of $H^4(BG)\cong H^3(G)$.) The second condition is just the property (48) used above. Using the third property, one may verify that the multiplication morphism is associative, in the sense that

$$(\mathrm{Mult},\varsigma)\circ\big((\mathrm{Mult},\varsigma)\times(\mathrm{id}_G,0)\big)=(\mathrm{Mult},\varsigma)\circ\big((\mathrm{id}_G,0)\times(\mathrm{Mult},\varsigma)\big).$$

We will compare the morphism $(\mathrm{Mult},\varsigma)$ with the groupoid multiplication of $\mathfrak{d}$, viewed as the pair groupoid over $\mathfrak{g}$: writing $\zeta=(\xi,\xi')$, $\zeta_i=(\xi_i,\xi_i')$, $i=1,2$, the groupoid multiplication is

$$\zeta=\zeta_2\circ\zeta_1\;\Leftrightarrow\;\xi=\xi_2,\;\xi'=\xi_1',\;\xi_2'=\xi_1.$$

Proposition 3.8. — *The isomorphism $G\times\mathfrak{d}\to\mathbb{T}G$ defined by s intertwines the groupoid multiplication of $\mathfrak{d}$ with the morphism $(\mathrm{Mult},\varsigma)$, in the sense that*

$$\zeta_2\circ\zeta_1=\zeta\quad\Leftrightarrow\quad \mathsf{s}^1(\zeta_1)+\mathsf{s}^2(\zeta_2)\sim_{(\mathrm{Mult},\varsigma)}\mathsf{s}(\zeta), \tag{50}$$

for $\zeta,\zeta_1,\zeta_2\in\mathfrak{d}$.

Proof. — Spelling out the relations (50), we have to show that, for all $\xi \in \mathfrak{g}$,

$$\begin{aligned} &\mathsf{s}^{R,1}(\xi) \sim_{(\mathrm{Mult},\varsigma)} \mathsf{s}^{R}(\xi), \quad \mathsf{s}^{L,2}(\xi) \sim_{(\mathrm{Mult},\varsigma)} \mathsf{s}^{L}(\xi), \\ &\mathsf{s}^{L,1}(\xi) + \mathsf{s}^{R,2}(\xi) \sim_{(\mathrm{Mult},\varsigma)} 0. \end{aligned} \tag{51}$$

The equivariance properties

$$\begin{aligned} &\mathrm{Mult}(ga,b) = g\,\mathrm{Mult}(a,b), \quad \mathrm{Mult}(a,bg^{-1}) = \mathrm{Mult}(a,b)g^{-1}, \\ &\mathrm{Mult}(ag^{-1},gb) = \mathrm{Mult}(a,b) \end{aligned}$$

of the multiplication map imply the following relations of generating vector fields:

$$-\xi^{R,1} \sim_{\mathrm{Mult}} -\xi^{R}, \quad \xi^{L,2} \sim_{\mathrm{Mult}} \xi^{L}, \quad \xi^{L,1} - \xi^{R,2} \sim_{\mathrm{Mult}} 0.$$

This proves the 'vector field part' of the relations (51). The 1-form part is equivalent to the following three identities, which are verified by a direct computation:

$$\begin{aligned} &\tfrac{1}{2}B(\theta^{R,1},\xi) + \iota(-\xi^{R,1})\varsigma = \tfrac{1}{2}\,\mathrm{Mult}^*\,B(\theta^{R},\xi), \\ &\tfrac{1}{2}B(\theta^{L,2},\xi) + \iota(\xi^{L,2})\varsigma = \tfrac{1}{2}\,\mathrm{Mult}^*\,B(\theta^{L},\xi)\,, \\ &\tfrac{1}{2}B(\theta^{L,1}+\theta^{R,2},\xi) + \iota(\xi^{L,1}-\xi^{R,2})\varsigma = 0. \end{aligned}$$

□

Theorem 3.9. — *The multiplication map* Mult: $G \times G \to G$ *extends to a strong Dirac morphism*

$$(\mathrm{Mult},\varsigma)\colon (G,E_G,\eta) \times (G,E_G,\eta) \to (G,E_G,\eta),$$

with $\varsigma \in \Omega^2(G\times G)$ *as defined above. In terms of the trivialization* $E_G = G \times \mathfrak{g}$, *the map* $\widehat{\mathfrak{a}}$: $\mathrm{Mult}^* E_G \to E_G \times E_G$ *associated with the strong Dirac morphism is given by the diagonal embedding* $\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}$. *Similarly, the inversion map* Inv: $G \to G$, $g \mapsto g^{-1}$ *extends to a Dirac morphism*

$$(\mathrm{Inv},0)\colon (G,E_G,\eta) \to (G,E_G^{\top},-\eta).$$

Proof. — By Proposition 3.8, the sections $\mathsf{e}(\xi) = \mathsf{s}(\xi,\xi)$ satisfy

$$\mathsf{e}^1(\xi) + \mathsf{e}^2(\xi) \sim_{(\mathrm{Mult},\varsigma)} \mathsf{e}(\xi).$$

This shows that $(\mathrm{Mult},\varsigma)$ is a Dirac morphism. For any given point $(a,b) \in G\times G$, no non-trivial linear combination of $\mathsf{e}^1(\xi)|_a$, $\mathsf{e}^2(\xi')|_b$ is $(\mathrm{Mult},\varsigma)$-related to 0. Hence, the Dirac morphism $(\mathrm{Mult},\varsigma)$ is strong.

We have $\mathrm{Inv}^*\,B(\theta^L+\theta^R,\xi) = -B(\theta^L+\theta^R,\xi)$ and $\xi^L - \xi^R \sim_{\mathrm{Inv}} (\xi^L - \xi^R)$. Hence

$$\mathsf{e}(\xi) \sim_{(\mathrm{Inv},0)} \mathsf{e}(\xi)^{\top}$$

where $\mathsf{e}(\xi)^{\top}$ is the image of $\mathsf{e}(\xi)$ under the map $(v,\alpha) \to (v,-\alpha)$. Since $\mathrm{Inv}^*\,\eta = -\eta$, this shows that $(\mathrm{Inv},0)\colon (G,E_G,\eta) \to (G,E_G^{\top},-\eta)$ is a Dirac morphism. □

Remark 3.10. — More generally, suppose that $\mathfrak{s} \subset \mathfrak{d}$ is a Lagrangian subalgebra, defining a Dirac structure $E^{\mathfrak{s}}$. Since $\mathfrak{g}_\Delta \circ \mathfrak{s} = \mathfrak{s}$, the same argument as in the proof above shows that $(\mathrm{Mult},\varsigma)$ is a strong Dirac morphism from $(G,E_G,\eta) \times (G,E^{\mathfrak{s}},\eta)$ to $(G,E^{\mathfrak{s}},\eta)$.

Let $\widetilde{F}_{G\times G} \subset \mathbb{T}(G\times G)$ be the backward image of the Lagrangian subbundle F_G under $(\mathrm{Mult},\varsigma)$. Since F_G is spanned by the sections $\mathsf{f}(\xi)=\frac{1}{2}(\mathsf{s}^L(\xi)-\mathsf{s}^R(\xi))$, (51) shows that $\widetilde{F}_{G\times G}$ is spanned by the sections

$$\tag{52} \tfrac{1}{2}(\mathsf{s}^{L,2}(\xi)-\mathsf{s}^{R,1}(\xi)),\quad \tfrac{1}{2}(\mathsf{s}^{L,1}(\xi)+\mathsf{s}^{R,2}(\xi)).$$

Since F_G is a complement to E_G, its backward image $\widetilde{F}_{G\times G}$ is a complement to $E_G^1\oplus E_G^2$ (see Proposition 1.15). Let us describe the element of $\wedge^2(E_G^1\oplus E_G^2)$ relating $\widetilde{F}_{G\times G}$ to the standard complement $F_G^1\oplus F_G^2$. Let $v_i\in\mathfrak{g}$ and $v^i\in\mathfrak{g}$ be B-dual bases, and put

$$\tag{53} \gamma=\tfrac{1}{2}(v_i)^1\wedge(v^i)^2\in\wedge^2(\mathfrak{g}\oplus\mathfrak{g}).$$

Let

$$\tag{54} \mathsf{e}(\gamma)=\tfrac{1}{2}\sum_i \mathsf{e}^1(v_i)\wedge\mathsf{e}^2(v^i)\in\Gamma(\wedge^2(E_G^1\oplus E_G^2))$$

be the corresponding section.

Proposition 3.11. — *The Lagrangian complement $\widetilde{F}_{G\times G}=F_G\circ\Gamma_{(\mathrm{Mult},\varsigma)}$ is obtained from $F_G^1\oplus F_G^2$ by the bivector $\mathsf{e}(\gamma)$:*

$$\widetilde{F}_{G\times G}=A^{-\mathsf{e}(\gamma)}(F_G^1\oplus F_G^2).$$

Proof. — We compute $\iota(\mathsf{f}^1(\xi))\mathsf{e}(\gamma)=\mathsf{e}(\iota^1(\xi)\gamma)=\frac{1}{2}\mathsf{e}^2(\xi)=\frac{1}{2}(\mathsf{s}^{L,2}(\xi)+\mathsf{s}^{R,2}(\xi))$. Thus

$$\mathsf{f}^1(\xi)+\iota(\mathsf{f}^1(\xi))\mathsf{e}(\gamma)=\tfrac{1}{2}(\mathsf{s}^{L,1}(\xi)-\mathsf{s}^{R,1}(\xi)+\mathsf{s}^{L,2}(\xi)+\mathsf{s}^{R,2}(\xi))$$

is the sum of the sections in (52). Similarly, we find that $\mathsf{f}^2(\xi)+\iota(\mathsf{f}^2(\xi))\mathsf{e}(\gamma)$ is the difference of the sections in (52). □

The bivector field on $G\times G$ corresponding to the splitting $(E_G^1\times E_G^2)\oplus A^{-\mathsf{e}(\gamma)}(F_G^1\times F_G^2)$ of $\mathbb{T}(G\times G)$ is given by (see Proposition 1.18(i)),

$$\tag{55} \widetilde{\pi}=\pi_G^1+\pi_G^2+\mathscr{A}_{\mathsf{ad}}^{12}(\gamma),$$

where π_G is the bivector field for the splitting $\mathbb{T}G=E_G\oplus F_G$, and $\mathscr{A}_{\mathsf{ad}}^{12}=\mathscr{A}_{\mathsf{ad}}^1\oplus\mathscr{A}_{\mathsf{ad}}^2\colon\mathfrak{g}\oplus\mathfrak{g}\to\mathfrak{X}(G\times G)$. By Proposition 2.10(c) we have $\widetilde{\pi}\sim_{\mathrm{Mult}}\pi$. Furthermore, Proposition 2.10(c) and Lemma 3.5, imply that the Schouten bracket $\frac{1}{2}[\widetilde{\pi},\widetilde{\pi}]_{\mathrm{Sch}}$ equals the trivector field $\mathscr{A}_{\mathsf{ad}}^{\mathrm{diag}}(\Xi)$, where $\mathscr{A}_{\mathsf{ad}}^{\mathrm{diag}}$ is the diagonal action on $G\times G$.

3.5. Exponential map. — We will now discuss the behavior of the Cartan-Dirac structure under the exponential map,

$$\exp\colon\mathfrak{g}\to G.$$

Let $\mathfrak{g}_\natural\subset\mathfrak{g}$ denote the set of regular points of the exponential map, that is, all points where $d\exp$ is an isomorphism. We begin with some preliminaries concerning $\mathbb{T}\mathfrak{g}^*$, not using the inner product on $\mathfrak{g}$ for the time being. Let $\mathscr{A}_0$ be the action of $D_0:=\mathfrak{g}^*\rtimes G$ on $\mathfrak{g}^*$ by

$$\mathscr{A}_0(\beta,g)\nu=(\mathrm{Ad}_{g^{-1}})^*\nu-\beta.$$

This action lifts to an action by automorphisms of $\mathbb{T}\mathfrak{g}^*$, preserving the inner product as well as the (untwisted) Courant bracket. Let $\mathfrak{d}_0 = \mathfrak{g}^* \rtimes \mathfrak{g}$ be the Lie algebra of D_0, equipped with the canonical inner product defined by the pairing, and let $\mathcal{A}_0 \colon \mathfrak{d}_0 \to \mathfrak{X}(\mathfrak{g}^*)$ be the infinitesimal action. To simplify notation, we denote the constant vector field defined by $\beta \in \mathfrak{g}^*$ by $\beta_0 = \mathcal{A}_0(\beta, 0)$, and write $\mathcal{A}_0(\xi) = \mathcal{A}_0(0, \xi)$. Let $\theta_0 \in \Omega^1(\mathfrak{g}^*) \otimes \mathfrak{g}^*$ be the tautological 1-form, defined by $\iota(\beta_0)\theta_0 = \beta$. Consider the D_0-equivariant map

$$\mathsf{s}_0 \colon \mathfrak{d}_0 \to \Gamma(\mathbb{T}\mathfrak{g}^*), \quad \mathsf{s}_0(\beta, \xi) = \mathcal{A}_0(\beta, \xi) \oplus \langle \theta_0, \xi \rangle. \tag{56}$$

Then $\langle \mathsf{s}_0(\zeta), \mathsf{s}_0(\zeta') \rangle = B_{\mathfrak{d}_0}(\zeta, \zeta')$, showing that s_0 defines a D_0-equivariant isometric isomorphism

$$\mathbb{T}\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{d}_0.$$

A direct computation shows that this isomorphism is compatible with the Courant bracket $[\![\cdot,\cdot]\!]_0$ on $\mathbb{T}\mathfrak{g}^*$ and the Lie bracket on $\mathfrak{d}_0$.

Since $\mathfrak{g} \subset \mathfrak{d}_0$ is a Lagrangian Lie subalgebra, the sections $\mathsf{e}_0(\xi) := \mathsf{s}_0(0, \xi)$ span a Dirac structure $E_{\mathfrak{g}^*} \subset \mathbb{T}\mathfrak{g}^*$. Since $E_{\mathfrak{g}^*} \cap T\mathfrak{g}^* = 0$, this Dirac structure is of the form $E_{\mathfrak{g}^*} = \mathrm{Gr}_{\pi_{\mathfrak{g}^*}}$ for a Poisson bivector field $\pi_{\mathfrak{g}^*}$ satisfying

$$\iota(\langle \theta_0, \xi \rangle)\pi_{\mathfrak{g}^*} = \mathcal{A}_0(\xi), \quad \xi \in \mathfrak{g}. \tag{57}$$

The Poisson structure $\pi_{\mathfrak{g}^*}$ is just the standard linear Poisson structure on $\mathfrak{g}^*$. Similarly, the sections $\mathsf{f}_0(\beta) := \mathsf{s}_0(\beta, 0)$ span the Lagrangian subspace $F_{\mathfrak{g}^*} = T\mathfrak{g}^*$, which is complementary to $E_{\mathfrak{g}^*}$.

Let us now use the invariant inner product B on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let

$$\varpi \in \Omega^2(\mathfrak{g}), \quad \mathrm{d}\varpi = \exp^* \eta \tag{58}$$

be the primitive of $\exp^* \eta \in \Omega^3(\mathfrak{g})$ defined by the de Rham homotopy operator for the radial homotopy.

Proposition 3.12. — *The sections* $\mathsf{e}_0(\xi)$ *and* $\mathsf{e}(\xi)$ *are* $(\exp, \varpi)$*-related:*

$$\mathsf{e}_0(\xi) \sim_{(\exp, \varpi)} \mathsf{e}(\xi). \tag{59}$$

Similarly, over the subset $\mathfrak{g}_\natural \subset \mathfrak{g}$*, one has*

$$\mathsf{f}_0(\xi) + \mathsf{e}_0(C\,\xi) \sim_{(\exp, \varpi)} \mathsf{f}(\xi), \tag{60}$$

where $C \colon \mathfrak{g}_\natural \to \mathrm{End}(\mathfrak{g})$ *is given by the formula:*

$$C|_\nu = \big(1/2 \coth(z/2) - 1/z\big)\big|_{z = \mathrm{ad}_\nu}, \quad \nu \in \mathfrak{g}_\natural. \tag{61}$$

Proof. — Recall that β_0 denotes the 'constant vector field' $\mathcal{A}_0(\beta, 0)$. We extend the notation $(\cdot)_0$ to $\mathfrak{g}^* \cong \mathfrak{g}$-valued functions on $\mathfrak{g}^* \cong \mathfrak{g}$: For instance, the vector field corresponding to the function $\nu \mapsto -\mathrm{ad}_\xi\, \nu = \mathrm{ad}_\nu\, \xi$ is $(\mathrm{ad}_\nu\, \xi)_0 = \mathcal{A}_0(\xi)$.

The vector field part of the relation (59) says that $\mathcal{A}_0(\xi) \sim_{\exp} \xi^L - \xi^R = \mathcal{A}_{\mathsf{ad}}(\xi)$, which follows by the G-equivariance of exp. The 1-form part of (59) is equivalent to the following property [3] of ϖ:

$$\iota(\mathcal{A}_0(\xi))\varpi = \tfrac{1}{2} \exp^* B(\theta^L + \theta^R, \xi) - B(\theta_0, \xi).$$

Since exp is a local diffeomorphism over $\mathfrak{g}_\natural$, the section $\mathsf{f}(\xi)$ of $\mathbb{T}G$ is $(\exp,\varpi)$-related to a unique section $\widetilde{\mathsf{f}}(\xi)$ of $\mathbb{T}\mathfrak{g}|_{\mathfrak{g}_\natural}$. Since inner products are preserved under the $(\exp,\varpi)$-relation (see (12)) we have

$$\langle \mathsf{e}_0(\xi'),\widetilde{\mathsf{f}}_0(\xi)\rangle = \langle \mathsf{e}(\xi'),\mathsf{f}_0(\xi)\rangle = B(\xi',\xi) = \langle \mathsf{e}_0(\xi'),\mathsf{f}_0(\xi)\rangle$$

for all $\xi'\in\mathfrak{g}$, showing that the $F_\mathfrak{g}$-component of $\widetilde{\mathsf{f}}_0(\xi)$ is equal to $\mathsf{f}_0(\xi)$. It follows that $\widetilde{\mathsf{f}}_0(\xi) = \mathsf{f}_0(\xi) + \mathsf{e}_0(C(\xi))$, where C is defined by $B(\xi',C(\xi)) = \langle \mathsf{f}_0(\xi'),\widetilde{\mathsf{f}}_0(\xi)\rangle$. To compute C, we re-write (60) in the equivalent form (using (59)):

$$\mathsf{f}_0(\xi) \sim_{(\exp,\varpi)} \mathsf{f}(\xi) - \mathsf{e}(C(\xi))$$

Again, we write out the vector field and 1-form parts of this relation:

$$\text{(62)}\qquad \begin{aligned} \xi_0 &= \tfrac{1}{2}\exp^*(\xi^L+\xi^R) - \mathscr{E}_0(C(\xi)),\\ \iota(\xi_0)\varpi &= \tfrac{1}{4}\exp^* B(\theta^L-\theta^R,\xi) - \tfrac{1}{2}\exp^* B(\theta^L+\theta^R,C(\xi)). \end{aligned}$$

We now verify that C given by (61) satisfies these two equations. Let $T,U^L,U^R\colon\mathfrak{g}\to\mathrm{End}(\mathfrak{g})$ be the functions defined by

$$\iota(\xi_0)\varpi = B(\theta_0,T\,\xi),\quad \exp^*\theta^L = U^L\,\theta_0,\quad \exp^*\theta^R = U^R\,\theta_0.$$

It is known that (for the first identity, see e.g. [**45**])

$$T|_\nu = \Big(\frac{\sinh(z)-z}{z^2}\Big)\Big|_{z=\mathrm{ad}_\nu},\quad U^L|_\nu = \Big(\frac{1-e^{-z}}{z}\Big)\Big|_{z=\mathrm{ad}_\nu},\quad U^R|_\nu = \Big(\frac{e^z-1}{z}\Big)\Big|_{z=\mathrm{ad}_\nu}.$$

Note that U^L and U^R are transposes relative to the inner product on $\mathfrak{g}$, and that they are invertible over $\mathfrak{g}_\natural$. Their definitions imply that

$$\exp^*\xi^L = ((U^L)^{-1}\xi)_0,\quad \exp^*\xi^R = ((U^R)^{-1}\xi)_0.$$

The first equation in (62) becomes

$$\xi_0 = \Big(\Big(\frac{(U^L)^{-1}+(U^R)^{-1}}{2} - \mathrm{ad}_\nu\, C\Big)\xi\Big)_0$$

which follows from the identity

$$1 = \frac{1}{2}\Big(\frac{z}{1-e^{-z}}+\frac{z}{e^z-1}\Big) - z\Big(\frac{1}{2}\coth\Big(\frac{z}{2}\Big)-\frac{1}{z}\Big).$$

In a similar fashion, the second equation in (62) follows from the identity

$$\frac{\sinh(z)-z}{z^2} = \frac{1}{4}\Big(\frac{e^z-1}{z}-\frac{1-e^{-z}}{z}\Big) - \frac{1}{2}\Big(\frac{e^z-1}{z}+\frac{1-e^{-z}}{z}\Big)\Big(\frac{1}{2}\coth(\frac{z}{2})-\frac{1}{z}\Big).\ \square$$

As an immediate consequence of (59), we obtain

Theorem 3.13. — *The exponential map and the 2-form ϖ define a Dirac morphism*

$$(\exp,\varpi)\colon(\mathfrak{g},E_\mathfrak{g},0)\to(G,E_G,\eta).$$

It is a strong Dirac morphism over the open subset $\mathfrak{g}_\natural\subset\mathfrak{g}$.

Let $\widetilde{F}_{\mathfrak{g}}$ be the backward image (defined over $\mathfrak{g}_\natural$) of F_G under $(\exp, \varpi)$, and let

$$\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^2\mathfrak{g})$$

be the unique map such that the associated orthogonal transformation $A^{-\mathsf{e}_0(\varepsilon)} \in \Gamma(\mathrm{O}(\mathbb{T}\mathfrak{g}_\natural))$ takes $F_{\mathfrak{g}}$ to $\widetilde{F}_{\mathfrak{g}}$. By (60), this section is given by $\iota_\xi\varepsilon = C(\xi)$, with C given by (61).

Let $[\varepsilon, \varepsilon]_{\mathrm{Sch}} \in C^\infty(\mathfrak{g}_\natural, \wedge^3\mathfrak{g})$ be defined using the Schouten bracket on $\wedge\mathfrak{g}$, and $\mathrm{d}\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^3\mathfrak{g})$ the exterior differential of ε, viewed as a 2-form on $\mathfrak{g}_\natural$.

Proposition 3.14. — *The map ε satisfies the classical dynamical Yang-Baxter equation:*

$$d\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}} = \Xi. \tag{63}$$

Proof. — Proposition 2.14 and the discussion following it show that $\mathrm{d}\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}}$ equals the Courant tensor of $\widetilde{F}_{\mathfrak{g}}$ (relative to the complementary subbundle $E_{\mathfrak{g}}$). By Lemma 3.5, together with Proposition 2.10, $\Upsilon^{\widetilde{F}_{\mathfrak{g}}} = \Xi$. □

This solution of the classical dynamical Yang-Baxter equation was obtained in [**5**], using a different argument. As a special case of Proposition 1.18, the map ε relates the linear Poisson bivector $\pi_{\mathfrak{g}}$ on $\mathfrak{g} \cong \mathfrak{g}^*$ with the pull-back $\exp^* \pi_G \in \mathfrak{X}^2(\mathfrak{g}_\natural)$ of the bivector field (46) on G:

$$\exp^* \pi_G = \pi_{\mathfrak{g}} + \mathscr{R}_0(\varepsilon).$$

3.6. The Gauss-Dirac structure. — In this Section we assume that $G = K^{\mathbb{C}}$ is a complex Lie group, given as the complexification of a compact, connected Lie group K of rank l. Thus the Cartan-Dirac structure E_G will be regarded as a holomorphic Dirac structure on the complex Lie group G. We will show that G carries another interesting Dirac structure besides the Cartan-Dirac structure. An important feature of this Dirac structure is that the corresponding Dirac foliation has an open dense leaf.

Take the bilinear form B on $\mathfrak{g}$ to be the complexification of a positive definite invariant inner product on $\mathfrak{k}$. Let T_K be a maximal torus in K, with complexification $T = T_K^{\mathbb{C}}$. Let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+ \tag{64}$$

be the triangular decomposition relative to some choice of positive Weyl chamber, where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the nilpotent subalgebra given as the sum of positive (resp. negative) root spaces. For every root α, let e_α be a corresponding root vector, with the normalization $B(\overline{e}_\alpha, e_\alpha) = 1$ and $e_{-\alpha} = \overline{e}_\alpha$. The unipotent subgroups corresponding to $\mathfrak{n}_\pm$ are denoted $N_\pm$. Recall that the multiplication map

$$j\colon N_- \times T \times N_+ \to G,\ (g_-, g_0, g_+) \mapsto g_-g_0g_+ \tag{65}$$

is a diffeomorphism onto its image $\mathcal{O} \subset G$, called the *big Gauss cell.* The big Gauss cell is open and dense in G, and the inverse map $j^{-1}\colon \mathcal{O} \to N_- \times T \times N_+$ is known as the Gauss decomposition. Consider $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ as Section 3.1. Then

$$\mathfrak{s} = \{(\xi_+ + \xi_0) \oplus (\xi_- - \xi_0) \in \mathfrak{d} \mid \ \xi_- \in \mathfrak{n}_-,\ \xi_0 \in \mathfrak{t},\ \xi_+ \in \mathfrak{n}_+\} \tag{66}$$

is a Lagrangian subalgebra of $\mathfrak{d}$, corresponding to the subgroup

$$S = \{(g_+ t,\ g_- t^{-1}) \in G \times G \,|\, g_- \in N_-,\ t \in T,\ g_+ \in N_+\}$$

of $D = G \times G$. Since $\mathfrak{s}$ is transverse to the diagonal $\mathfrak{g}_\Delta$, the corresponding Lagrangian subbundle $\widehat{F}_G := E^{\mathfrak{s}}$ is transverse to the Cartan-Dirac structure E_G:

$$\mathbb{T}G = E_G \oplus \widehat{F}_G.$$

We shall refer to it as to *Gauss-Cartan* splitting.

Unlike the complement F_G defined by the anti-diagonal, $\widehat{F}_G$ is integrable (since $\mathfrak{s}$ is a subalgebra), and it defines a Dirac manifold $(G, \widehat{F}_G, \eta)$. We refer to $\widehat{F}_G$ as the *Gauss-Dirac structure.* Its leaves are the orbits of S as a subgroup of D,

$$\mathcal{A}(g_+ t, g_- t)(g) = g_- t^{-1} g t^{-1} g_+^{-1}. \tag{67}$$

The S-orbit of the group unit is exactly the big Gauss cell. Let $\omega_{\mathcal{O}}$ be the 2-form on $\mathcal{O}$, and $j^*\omega_{\mathcal{O}}$ its pull-back to $N_- \times T \times N_+$.

Proposition 3.15. — *The pull-back of the 2-form $\omega_{\mathcal{O}}$ on the big Gauss cell $N_- \times T \times N_+$ is given by:*

$$j^*\omega_{\mathcal{O}} = -\tfrac{1}{2} B(\theta_-^L, \mathrm{Ad}_{g_0} \theta_+^R). \tag{68}$$

Here $\theta_\pm^L$, $\theta_\pm^R$ are the Maurer-Cartan-forms on $N^\pm$, and g_0 is the T-component (i.e. projection of $N_- \times T \times N_+$ to the middle factor).

Proof. — Let $\omega \in \Omega^2(N_- \times T \times N_+)$ denote the 2-form on the right hand side of (68). Since both ω and $\omega_{\mathcal{O}}$ are S-invariant, it suffices to check that $j^*\omega_{\mathcal{O}} = \omega$ at the group unit $g = e$. At the group unit, the formula (40) for $\omega_{\mathcal{O}}$ simplifies to

$$\omega_{\mathcal{O}}(\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2))|_e = \tfrac{1}{2}(B(\xi_1', \xi_2) - B(\xi_2', \xi_1)), \tag{69}$$

for $\zeta_1 = (\xi_1, \xi_1')$, $\zeta_2 = (\xi_2, \xi_2') \in \mathfrak{s} \subset \mathfrak{d}$. Its kernel is

$$\ker(\omega_{\mathcal{O}}|_e) = \{\mathcal{A}(\zeta)\big|_e \mid \zeta = (\xi_0, -\xi_0), \xi_0 \in \mathfrak{t}\} = T_e(T)$$

which coincides with the kernel of $-\frac{1}{2}B(\theta_-^L, \theta_+^R)|_e$. Moreover, it is clear that $T_e(N_+)$ and $T_e(N_-)$ are isotropic subspaces for both 2-forms. Hence it is enough to compare on tangent vectors $\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2)$ for ζ_i of the form $\zeta_1 = (0, \xi_-)$ with $\xi_- \in \mathfrak{n}_-$, and $\zeta_2 = (\xi_+, 0)$ with $\xi_+ \in \mathfrak{n}_+$. (69) gives,

$$\omega_{\mathcal{O}}(\mathcal{A}(0, \xi_-), \mathcal{A}(\xi_+, 0))|_e = \tfrac{1}{2} B(\xi_+, \xi_-).$$

Since $j^*\mathcal{A}(\xi_+, 0)|_e = (0, 0, \xi_+) \in \mathfrak{n}_+ \subset \mathfrak{g} = T_e G$ and $j^*\mathcal{A}(0, \xi_-)|_e = (-\xi_-, 0, 0)$, the right hand side of (68) gives exactly the same answer. □

Since F_G and $\widehat{F}_G$ are both complements to the Cartan-Dirac structure E_G, they are related by an element in $\Gamma(\wedge^2 E_G)$. To compute this element, let $\mathfrak{p}$ be the anti-diagonal in $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$, and let $\mathfrak{g}_\Delta \cong \mathfrak{g}$ be the diagonal. Let

$$\mathfrak{r} = \sum e_{-\alpha} \wedge e_\alpha \in \wedge^2 \mathfrak{g} \tag{70}$$

be the classical r-matrix.

Lemma 3.16. — *The bivector taking $\mathfrak{p}$ to $\mathfrak{s}$ is the image $\mathfrak{r}_\Delta \in \wedge^2 \mathfrak{g}_\Delta$ of the classical $\mathfrak{r}$-matrix under the diagonal embedding $\mathfrak{g} \to \mathfrak{g}_\Delta \subset \mathfrak{d}$.*

Proof. — Let $\mathfrak{g} \oplus \mathfrak{g}^*$ carry the bilinear form defined by the pairing, and consider the isometric isomorphism

$$\mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}, \quad \xi \oplus \mu \mapsto (\xi + \tfrac{B^\sharp(\mu)}{2}) \oplus (\xi - \tfrac{B^\sharp(\mu)}{2}).$$

This isomorphism takes $\mathfrak{g} = \mathfrak{g} \oplus 0$ to the diagonal $\mathfrak{g}_\Delta$, and $\mathfrak{g}^*$ to the anti-diagonal, $\mathfrak{p}$. The graph $\mathrm{Gr}_\mathfrak{r} \subset \mathfrak{g} \oplus \mathfrak{g}^*$ of the bivector $\mathfrak{r}$ is spanned by vectors of the form

$$0 \oplus B^\flat(\xi_0), \quad e_\alpha \oplus B^\flat(e_\alpha), \quad e_{-\alpha} \oplus (-B^\flat(e_{-\alpha})),$$

for $\xi_0 \in \mathfrak{t}$ and positive roots α. The isomorphism $\mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{d}$ takes these vectors to

$$\xi_0/2 \oplus (-\xi_0/2), \quad 0 \oplus e_{-\alpha}, \quad e_\alpha \oplus 0.$$

Hence, it defines an isomorphism $\mathrm{Gr}_\mathfrak{r} \cong \mathfrak{s}$. □

Corollary 3.17. — *The orthogonal transformation $A^{-\mathsf{e}(\mathfrak{r})} \in \Gamma(\mathrm{O}(\mathbb{T}G))$ takes F_G to $\widehat{F}_G$.*

Proof. — This follows from Lemma 3.16 and the isomorphism $\mathbb{T}G \cong G \times \mathfrak{d}$. □

The Gauss-Cartan splitting $\mathbb{T}G = E_G \oplus \widehat{F}_G$ also defines a bivector field $\widehat{\pi}_G$, and Proposition1.18 implies that it is related to the bivector field π_G (46) by

$$\widehat{\pi}_G = \pi_G + \mathscr{A}_{\mathsf{ad}}(\mathfrak{r}).$$

Since $\widehat{F}_G$ is integrable, this bivector field is in fact a Poisson structure on G – see the remarks before Proposition 2.10. (On the other hand, unlike π_G, the Poisson structure is not invariant under the full adjoint action, but is only T-invariant.)

Proposition 3.18. — *The Poisson structure $\widehat{\pi}_G$ associated with the Gauss-Cartan splitting $\mathbb{T}G = E_G \oplus \widehat{F}_G$ is given by the formula:*

$$\widehat{\pi}_G = \tfrac{1}{2} \sum_i e_i^L \wedge (e^i)^R - \sum_{\alpha \succ 0} e_{-\alpha}^L \wedge e_\alpha^R + \tfrac{1}{2}\mathfrak{r}^L + \tfrac{1}{2}\mathfrak{r}^R.$$

Here e_i is a basis of $\mathfrak{t}$, with B-dual basis e^i, and $\mathfrak{r}^L$, $\mathfrak{r}^R$ are the left-, right-invariant bivector fields defined by $\mathfrak{r}$. The symplectic leaves of this Poisson structure are the connected components of the intersections of conjugacy classes in G with the orbits of the action (67).

This Poisson structure was first defined by Semenov-Tian-Shansky, see [**49**].

Proof. — The vectors

$$\tfrac{1}{2}(e_i \oplus (-e_i)),\ 0 \oplus (-e_{-\alpha}),\ e_\alpha \oplus 0$$

form basis of $\mathfrak{s}$ that is dual (relative to the bilinear form on $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$) to the basis

$$e^i \oplus e^i,\ e_\alpha \oplus e_\alpha,\ e_{-\alpha} \oplus e_{-\alpha}$$

of the diagonal. Using the formula (18) for the bivector field, we obtain

$$\begin{aligned}\widehat{\pi}_G &= \tfrac{1}{2}\sum_i ((e^i)^L - (e^i)^R) \wedge \tfrac{e_i^L + e_i^R}{2} + \tfrac{1}{2}\sum_{\alpha \succ 0}(e_\alpha^L - e_\alpha^R) \wedge (-e_{-\alpha}^L) + \tfrac{1}{2}\sum_{\alpha \succ 0}(e_{-\alpha}^L - e_{-\alpha}^R) \wedge (-e_\alpha)^R \\ &= \tfrac{1}{2}\sum_i e_i^L \wedge (e^i)^R - \sum_{\alpha \succ 0} e_{-\alpha}^L \wedge e_\alpha^R + \tfrac{1}{2}\mathfrak{r}^L + \tfrac{1}{2}\mathfrak{r}^R.\end{aligned}$$

Here we have used that the left- and right-invariant bivector fields generated by

$$\sum_i e_i \wedge e^i = \sum_i e_i \wedge e^i + \sum_{\alpha \succ 0} e_{-\alpha} \wedge e_\alpha + \sum_{\alpha \succ 0} e_\alpha \wedge e_{-\alpha}$$

coincide. □

Remark 3.19. — The Lagrangian subalgebra $\mathfrak{s}$ defines a Manin triple $(\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}, \mathfrak{g}_\Delta, \mathfrak{s})$, which induces a Poisson-Lie group structure on the double $D = G \times G$. The Poisson structure $\widehat{\pi}_G$ is the push-forward image of this Poisson-Lie structure under the natural projection $D \to D/G \cong G$, see e.g. [**1**, Sec. 3.6].

4. Pure spinors on Lie groups

In the previous section we identified $\mathbb{T}G \cong G \times \mathfrak{d}$ as Courant algebroids. In particular, we have an identification $\mathrm{Cl}(\mathbb{T}G) \cong G \times \mathrm{Cl}(\mathfrak{d})$ of Clifford algebra bundles. In this section, we will complement this isomorphism of Clifford bundles by an isomorphism of spinor modules,

$$\wedge T^*G \cong G \times \mathrm{Cl}(\mathfrak{g}),$$

where $\mathrm{Cl}(\mathfrak{g})$ is given the structure of a spinor module over $\mathrm{Cl}(\mathfrak{d})$. The differential $\mathrm{d} + \eta$ on $\Omega(G)$ intertwines with a certain differential d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$. Hence, given a pure spinor $x \in \mathrm{Cl}(\mathfrak{g})$ defining a Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, one directly obtains a pure spinor $\phi^{\mathfrak{s}} \in \Omega(G)$ defining $E^{\mathfrak{s}}$. We will also obtain expressions for $(\mathrm{d} + \eta)\phi^{\mathfrak{s}}$ from the properties of x.

4.1. $\mathrm{Cl}(\mathfrak{g})$ as a spinor module over $\mathrm{Cl}(\mathfrak{g} \oplus \overline{\mathfrak{g}})$.

— Recall from Examples 1.2 and 1.4 that for any vector space V with inner product B, the Clifford algebra $\mathrm{Cl}(V)$ may be viewed as a spinor module over $\mathrm{Cl}(V \oplus \overline{V})$. In the special case that $V = \mathfrak{g}$ is a Lie algebra, with B an invariant inner product, there is more structure that we now discuss.

Let $\lambda\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ be the map defined by the condition $-\iota(\xi_2)\lambda(\xi_1) = [\xi_1, \xi_2]_{\mathfrak{g}}$ (see Section 1.1), and let $\Xi \in \wedge^3\mathfrak{g}$ be the structure constants tensor (45). Then

$$\{\lambda(\xi_1), \lambda(\xi_2)\} = \lambda([\xi_1, \xi_2]_{\mathfrak{g}}), \quad \{\lambda(\xi_1), \xi_2\} = [\xi_1, \xi_2]_{\mathfrak{g}}$$
$$\{\Xi, \xi\} = -\frac{1}{4}\lambda(\xi), \qquad \{\Xi, \Xi\} = 0$$

for all $\xi_1, \xi_2, \xi \in \mathfrak{g}$. The quantizations of these elements have similar properties: Let

$$\tau\colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g}), \quad \tau(\xi) = q(\lambda(\xi)). \tag{71}$$

Then

$$[\tau(\xi_1), \tau(\xi_2)]_{\mathrm{Cl}} = \tau([\xi_1, \xi_2]_{\mathfrak{g}}), \quad [\tau(\xi_1), \xi_2]_{\mathrm{Cl}} = [\xi_1, \xi_2]_{\mathfrak{g}},$$
$$[q(\Xi), \xi]_{\mathrm{Cl}} = -\frac{1}{4}\tau(\xi), \qquad [q(\Xi), q(\Xi)]_{\mathrm{Cl}} \in \mathbb{K}.$$

(One can show (cf. [**4**]) that the constant $[q(\Xi), q(\Xi)]_{\mathrm{Cl}}$ is $\frac{1}{24}$ times the trace of the Casimir operator in the adjoint representation.) This last identity implies that the derivation

$$\mathrm{d}^{\mathrm{Cl}} = -4[q(\Xi), \cdot]_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}) \tag{72}$$

squares to 0. We call d^{Cl} the *Clifford differential* [**4, 38**].

For the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, with bilinear form $B \oplus (-B)$, the corresponding elements $\Xi_{\mathfrak{d}}$ and $\lambda_{\mathfrak{d}}$ in $\wedge\mathfrak{d} = \wedge\mathfrak{g} \otimes \wedge\mathfrak{g}$ are given by

$$\Xi_{\mathfrak{d}} = \Xi \otimes 1 + 1 \otimes \Xi, \qquad \lambda_{\mathfrak{d}}(\xi, \xi') = \lambda(\xi) \otimes 1 - 1 \otimes \lambda(\xi'), \quad \text{for } (\xi, \xi') \in \mathfrak{d}.$$

Note also that $q(\Xi_{\mathfrak{d}})^2 = 0$. Consider the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ as a spinor module over $\mathrm{Cl}(\mathfrak{d})$, with Clifford action given on generators $\zeta = (\xi, \xi') \in \mathfrak{d}$ by

$$\varrho^{\mathrm{Cl}}(\xi, \xi') = l^{\mathrm{Cl}}(\xi) - r^{\mathrm{Cl}}(\xi').$$

Then the Clifford differential d^{Cl} is implemented as a Clifford action:

$$\mathrm{d}^{\mathrm{Cl}} = -4\varrho^{\mathrm{Cl}}(q(\Xi_{\mathfrak{d}})).$$

The elements $\tau_{\mathfrak{d}}(\zeta) = q(\lambda_{\mathfrak{d}}(\zeta))$ generate a $\mathfrak{d}$-action on $\mathrm{Cl}(\mathfrak{g})$, with generators

$$\mathscr{L}^{\mathrm{Cl}}(\zeta) = l^{\mathrm{Cl}}(\tau(\xi)) - r^{\mathrm{Cl}}(\tau(\xi')) = \varrho^{\mathrm{Cl}}(\tau(\zeta)).$$

Note that

$$\mathscr{L}^{\mathrm{Cl}}(\zeta) = [\varrho^{\mathrm{Cl}}(\zeta), \mathrm{d}^{\mathrm{Cl}}], \tag{73}$$

which implies that

$$[\varrho^{\mathrm{Cl}}(\zeta_1), [\varrho^{\mathrm{Cl}}(\zeta_2), \mathrm{d}^{\mathrm{Cl}}]] = \varrho^{\mathrm{Cl}}([\zeta_1, \zeta_2]).$$

Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, and recall the definition of $\Upsilon^{\mathfrak{s}}$ given in (39). Given a Lagrangian complement $\mathfrak{p}$ to $\mathfrak{s}$, let $\mathrm{pr}_{\mathfrak{s}} : \mathfrak{d} \to \mathfrak{s}$ be the projection along $\mathfrak{p}$, and define a linear functional $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ by

$$\langle \sigma^{\mathfrak{s}}, \ \xi \rangle = \tfrac{1}{2}\,\mathrm{trace}(\mathrm{pr}_{\mathfrak{s}} \circ \mathrm{ad}_{\xi}\,\big|_{\mathfrak{s}}), \quad \xi \in \mathfrak{s}. \tag{74}$$

If $\mathfrak{s}$ is a Lagrangian subalgebra (i.e. $\Upsilon^{\mathfrak{s}} = 0$), we may omit $\mathrm{pr}_{\mathfrak{s}}$ in this formula; in this case $\sigma^{\mathfrak{s}}$ equals $-\frac{1}{2}$ times the modular character of the Lie algebra $\mathfrak{s}$.

Proposition 4.1. — *Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, with defining pure spinor $x \in \mathrm{Cl}(\mathfrak{g})$. Choose a Lagrangian complement $\mathfrak{p} \cong \mathfrak{s}^*$ to $\mathfrak{s}$ to view $\Upsilon^{\mathfrak{s}}$ as an element of the Clifford algebra $\mathrm{Cl}(\mathfrak{d})$. Then*

$$d^{\mathrm{Cl}} x = \varrho^{\mathrm{Cl}}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}})x.$$

In particular, $\mathfrak{s}$ is a Lie subalgebra if and only if the defining pure spinor x is 'integrable', in the sense that

$$d^{\mathrm{Cl}}\, x \in \varrho^{\mathrm{Cl}}(\mathfrak{d})x.$$

Proof. — The choice of a Lagrangian complement identifies $\mathfrak{d} = \mathfrak{s} \oplus \mathfrak{s}^*$, with bilinear form given by the pairing. Using a basis e_i of $\mathfrak{s}$ and a dual basis f^i of $\mathfrak{s}^*$, we have

$$\begin{aligned}4\Xi_{\mathfrak{d}} =& \tfrac{1}{6}\textstyle\sum_{ijk} B_{\mathfrak{d}}([e_i,e_j],e_k) f^i \wedge f^j \wedge f^k + \tfrac{1}{2}\sum_{ijk} B_{\mathfrak{d}}([e_j,e_k],f^i)\; e_i \wedge f^j \wedge f^k \\ &+ \tfrac{1}{2}\sum_{ijk} B_{\mathfrak{d}}([f^j,f^k],e_i)\; e_j \wedge e_k \wedge f^i + \tfrac{1}{6}\textstyle\sum_{ijk} B_{\mathfrak{d}}([f^j,f^k],f^i)\; e_j \wedge e_k \wedge e_i.\end{aligned}$$

The quantization map takes the last two terms into the left ideal $\mathrm{Cl}(\mathfrak{d})\mathfrak{s}$, and it takes the second term to

$$\tfrac{1}{2}\sum_{ik} B_{\mathfrak{d}}([e_i,e_k],f^i)\; f^k + \tfrac{1}{2}\sum_{ijk} B_{\mathfrak{d}}([e_j,e_k],f^i)\; f^j f^k e_i = -\sigma^{\mathfrak{s}} \mod \mathrm{Cl}(\mathfrak{d})\mathfrak{s}.$$

This gives

$$-4q(\Xi_{\mathfrak{d}}) = -\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}} \mod \mathrm{Cl}(\mathfrak{d})\mathfrak{s},$$

from which the result is immediate. □

Let us now assume that the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$ lifts to a group homomorphism

$$\tau\colon G \to \mathrm{Pin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g}) \tag{75}$$

to the double cover $\mathrm{Pin}(\mathfrak{g}) \to \mathrm{O}(\mathfrak{g})$. If G is connected, this is automatic if $\pi_1(G)$ is torsion free. Note that (75) is consistent with our previous notation $\tau(\xi) = q(\lambda(\xi))$, since [4]

$$\tau(\xi) = \frac{d}{dt}\Big|_{t=0} \tau(\exp t\xi).$$

We will write $\mathsf{N}(g) = \mathsf{N}(\tau(g)) = \pm 1$ for the image under the norm homomorphism, and $|g| = |\tau(g)|$ for the parity of $\tau(g)$. Since $\tau(g)$ lifts Ad_g, one has $(-1)^{|g|} = \det(\mathrm{Ad}_g)$. The definition of the Pin group implies that conjugation by $\tau(g)$ is the *twisted adjoint action*,

$$\tau(g)x\tau(g^{-1}) = \widetilde{\mathrm{Ad}}_g(x) := (-1)^{|g||x|}\,\mathrm{Ad}_g(x) \tag{76}$$

(using the extension of $\mathrm{Ad}_g \in \mathrm{O}(\mathfrak{g})$ to an automorphism of the Clifford algebra). This twisted adjoint action extends to an action of the group D on $\mathrm{Cl}(\mathfrak{g})$,

$$\mathscr{A}^{\mathrm{Cl}}(a,a')(x) = \tau(a)x\tau((a')^{-1}). \tag{77}$$

4.2. The isomorphism $\wedge T^*G \cong G \times \mathrm{Cl}(\mathfrak{g})$. — Let us now fix a generator $\mu \in \det(\mathfrak{g})$, and consider the corresponding star operator $\star\colon \wedge\, \mathfrak{g}^* \to \wedge\mathfrak{g}$, see Remark 1.5(b). The star operator satisfies

$$\mathrm{Ad}_g \circ \star = (-1)^{|g|} \star \circ \mathrm{Ad}^*_{g^{-1}}\,. \tag{78}$$

We use the trivialization by left-invariant forms to identify $\wedge T^*G \cong G\times\wedge\mathfrak{g}^*$. Applying $\star$ pointwise, we obtain an isomorphism $q\circ\star\colon \wedge\, T^*_g G \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{g})$ for each $g\in G$. Let us define the linear map

$$\mathcal{R}\colon \mathrm{Cl}(\mathfrak{g}) \to \Omega(G), \quad \mathcal{R}(x)|_g = (q\circ\star)^{-1}(x\tau(g)). \tag{79}$$

We denote by $\mu^*\in\det(\mathfrak{g}^*)$ the dual generator, defined by $\iota((\mu^*)^\top)\mu = 1$, and let μ_G be the left-invariant volume form on G defined by μ^*.

Proposition 4.2. — *The map* (79) *has the following properties:*

a. *$\mathcal{R}$ intertwines the Clifford actions, in the sense that*

$$\mathcal{R}(\varrho^{\mathrm{Cl}}(\zeta)x) = \varrho(\mathsf{s}(\zeta))\mathcal{R}(x), \quad \forall x\in \mathrm{Cl}(\mathfrak{g}),\ \zeta\in\mathfrak{d}.$$

Up to a scalar function, $\mathcal{R}$ is uniquely characterized by this property.

b. *$\mathcal{R}$ intertwines differentials:*

$$\mathcal{R}(d^{\mathrm{Cl}}(x)) = (d+\eta)\mathcal{R}(x), \quad \forall x\in\mathrm{Cl}(\mathfrak{g}).$$

c. *$\mathcal{R}$ satisfies has the following D-equivariance property: For any $h=(a,a')\in D$, and at any given point $g\in G$,*

$$\mathcal{A}(h^{-1})^*\mathcal{R}(x) = (-1)^{|a|(|g|+|x|)}\mathcal{R}(\mathcal{A}^{\mathrm{Cl}}(h)x).$$

d. *$\mathcal{R}$ relates the bilinear pairings on the Clifford modules $\mathrm{Cl}(\mathfrak{g})$ and $\Omega(G)$ as follows: At any given point $g\in G$, and for all $x,x'\in\mathrm{Cl}(\mathfrak{g})$,*

$$(\mathcal{R}(x),\mathcal{R}(x'))_{\wedge T^*G} = (-1)^{|g|(\dim G+1)}\ \mathsf{N}(g)\ (x,x')_{\mathrm{Cl}(\mathfrak{g})}\ \mu_G. \tag{80}$$

Here the pairing $(\cdot,\cdot)_{\mathrm{Cl}(\mathfrak{g})}$ is viewed as scalar-valued, using the trivialization of $\det(\mathfrak{g})$ *defined by μ. (Cf. Remark 1.5.)*

Notice that the signs in part (c), (d) disappear if G is connected.

Proof. — (a) Given $\xi\in\mathfrak{g}$, let $\epsilon(\xi):\wedge\mathfrak{g}\to\wedge\mathfrak{g}$ be defined by $\epsilon(\xi)\xi' = \xi\wedge\xi'$. Then

$$l^{\mathrm{Cl}}(\xi)\circ q = q\circ(\epsilon(\xi)+\tfrac12\iota(B^\flat(\xi))),\quad r^{\mathrm{Cl}}(\xi)\circ q = q\circ(\epsilon(\xi)-\tfrac12\iota(B^\flat(\xi))).$$

Since the star operator exchanges exterior multiplication and contraction, we have

$$\star^{-1}\circ q^{-1}\circ\varrho^{\mathrm{Cl}}(\xi,\xi') = \Big(\iota(\xi-\xi')+\epsilon\Big(B^\flat\Big(\tfrac{\xi+\xi'}{2}\Big)\Big)\Big)\circ\star^{-1}\circ q^{-1}.$$

On the other hand,

$$(\varrho^{\mathrm{Cl}}(\xi,\xi')x)\tau(g) = (\xi x-(-1)^{|x|}x\xi')\tau(g) = \varrho^{\mathrm{Cl}}(\xi,\mathrm{Ad}_{g^{-1}}\xi')(x\tau(g)).$$

This implies that, at $g\in G$,

$$\mathcal{R}(\varrho^{\mathrm{Cl}}(\xi,\xi')x) = \Big(\iota(\xi-\mathrm{Ad}_{g^{-1}}\xi')+\epsilon\Big(B^\flat\Big(\tfrac{\xi+\mathrm{Ad}_{g^{-1}}\xi'}{2}\Big)\Big)\Big)\mathcal{R}(x),$$

which is precisely the Clifford action of $\mathsf{s}(\xi,\xi')$ since

$$\mathsf{s}(\xi,\xi')=(\xi-\mathrm{Ad}_{g^{-1}}\xi')\oplus B^\flat\left(\tfrac{\xi+\mathrm{Ad}_{g^{-1}}\xi'}{2}\right)$$

under left-trivialization $\mathbb{T}G\cong G\times(\mathfrak{g}\oplus\mathfrak{g}^*)$. This shows that $\mathscr{R}$ intertwines the Clifford actions of $\mathrm{Cl}(\mathfrak{d})\cong\mathrm{Cl}(\mathbb{T}_gG)$. By the uniqueness properties of spinor modules, $\mathscr{R}$ is uniquely characterized by this property up to a scalar.

(b) From the global equivariance property in (c), verified below, we obtain the infinitesimal equivariance: $\mathscr{L}(\mathscr{e}(\zeta))\mathscr{R}(x)=\mathscr{R}(\mathscr{L}^{\mathrm{Cl}}(\zeta)x)$. Since $[\varrho(\mathsf{s}(\zeta)),\mathrm{d}+\eta]=\mathscr{L}(\mathscr{e}(\zeta))$ and $[\varrho^{\mathrm{Cl}}(\zeta),\mathrm{d}^{\mathrm{Cl}}]=\mathscr{L}^{\mathrm{Cl}}(\zeta)$, this gives

$$\begin{aligned}\varrho(\mathsf{s}(\zeta))\big((\mathrm{d}+\eta)\mathscr{R}(x)-\mathscr{R}(\mathrm{d}^{\mathrm{Cl}}x)\big)&=\mathscr{L}(\mathscr{e}(\zeta))\mathscr{R}(x)-\mathscr{R}(\varrho^{\mathrm{Cl}}(\zeta)\mathrm{d}^{\mathrm{Cl}}x)\\&=\mathscr{L}(\mathscr{e}(\zeta))\mathscr{R}(x)-\mathscr{R}(\mathscr{L}^{\mathrm{Cl}}(\zeta)x).\end{aligned}$$

That is, the map $(\mathrm{d}+\eta)\circ\mathscr{R}-\mathscr{R}\circ\mathrm{d}^{\mathrm{Cl}}\colon\mathrm{Cl}(\mathfrak{g})\to\Gamma(\mathbb{T}G)$ intertwines the Clifford actions, and hence agrees with $\mathscr{R}$ up to a scalar function. Since its parity is opposite to that of $\mathscr{R}$, that function is zero.

(c) We have to show that for all $a\in G$,

$$l_a^*\mathscr{R}(x)=\mathscr{R}(x\tau(a)),\quad r_a^*\mathscr{R}(x)=(-1)^{|a|(|g|+|x|)}\mathscr{R}(\tau(a)x).\tag{81}$$

In terms of the left-trivialization $\wedge T^*G=G\times\wedge\mathfrak{g}^*$,

$$(l_a^*\mathscr{R}(x))|_g=\mathscr{R}(x)\big|_{ag},\quad(r_a^*\mathscr{R}(x))\big|_g=\mathrm{Ad}^*_{a^{-1}}(\mathscr{R}(x)\big|_{ga}).$$

(Here $\mathrm{Ad}^*_{a^{-1}}$ stands for the contragredient action on $\wedge\mathfrak{g}^*$, not for a pull-back on $\Omega(G)$.) We compute, using (76) and (78):

$$\begin{aligned}\mathrm{Ad}^*_{a^{-1}}\left(\mathscr{R}(x)\big|_{ga}\right)&=(-1)^{|a|}\star^{-1}q^{-1}\mathrm{Ad}_a(x\tau(ga))\\&=(-1)^{|a|}(-1)^{|a|(|x|+|g|+|a|)}\star^{-1}q^{-1}(\tau(a)x\tau(g))\\&=(-1)^{|a|(|x|+|g|)}\mathscr{R}(\tau(a)x)\big|_g\end{aligned}$$

The equivariance property with respect to left translations is immediate from the definition.

(d) Use the generator $\mu\in\det(\mathfrak{g})$ and μ_G to trivialize both $\det(\mathfrak{g})$ and $\det(\wedge T^*G)$. By Remark 1.5(b) and Example 1.4 we have, at $g\in G$,

$$(\mathscr{R}(x),\mathscr{R}(x'))_{\wedge T^*G}=(x\tau(g),x'\tau(g))_{\mathrm{Cl}(\mathfrak{g})}.$$

This is computed as follows:

$$\begin{aligned}\mathrm{str}(\tau(g)^\top x^\top x'\tau(g))&=(-1)^{|g|(|g|+|x|+|x'|)}\mathrm{str}(\tau(g)\tau(g)^\top x^\top x')\\&=\mathsf{N}(g)\,(-1)^{|g|(1+|x|+|x'|)}\mathrm{str}(x^\top x')\end{aligned}$$

Finally, replace $|x|+|x'|$ with $\dim G$, using that $(x,x')_{\mathrm{Cl}(\mathfrak{g})}$ vanishes unless $|x|+|x'|=\dim G \mod 2$. □

As an immediate consequence of Propositions 4.1 and 4.2, we have

Corollary 4.3. — *If $x \in \mathrm{Cl}(\mathfrak{g})$ is a pure spinor defining a Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, then the differential form $\phi^{\mathfrak{s}} := \mathcal{R}(x) \in \Omega(G)$ is a pure spinor defining the Lagrangian subbundle $E^{\mathfrak{s}}$. It satisfies the differential equation*

$$(d + \eta)\phi^{\mathfrak{s}} = \varrho(\mathsf{s}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}}))\phi^{\mathfrak{s}}, \tag{82}$$

where $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ is defined as in (74) *(using a complementary Lagrangian subspace $\mathfrak{p} \cong \mathfrak{s}^* \subset \mathfrak{d}$). Let $H \subset D$ be a subgroup preserving $\mathfrak{s}$, and define the character $u^{\mathfrak{s}} \colon H \to \mathbb{K}^\times$ by $\mathcal{A}^{\mathrm{Cl}}(h)x = u^{\mathfrak{s}}(h)x$. Then*

$$\mathcal{A}(h^{-1})^*\phi^{\mathfrak{s}} = (-1)^{|a|(|g|+|x|)}\, u^{\mathfrak{s}}(h)\phi^{\mathfrak{s}} \tag{83}$$

for all $h = (a, a') \in H$, and at any given point $g \in G$.

We are mainly interested in pure spinors defining the Cartan-Dirac structure E_G and its Lagrangian complement F_G. These are obtained by taking $x = 1$ and $x = q(\mu)$ in the above:

Proposition 4.4. — *Let $\phi_G, \psi_G \in \Omega(G)$ be the differential forms*

$$\phi_G = \mathcal{R}(1), \quad \psi_G = \mathcal{R}(q(\mu)). \tag{84}$$

Then ϕ_G, ψ_G are pure spinors defining the Lagrangian subbundles E_G, F_G. They satisfy the differential equations,

$$(d + \eta)\phi_G = 0, \quad (d + \eta)\psi_G = -\varrho(\mathsf{e}(\Xi))\psi_G. \tag{85}$$

The equivariance properties under the adjoint action of G read

$$\mathcal{A}_{\mathsf{ad}}(a^{-1})^*\phi_G = (-1)^{|a||g|}\phi_G, \quad \mathcal{A}_{\mathsf{ad}}(a^{-1})^*\psi_G = (-1)^{|a|(|g|+1)}\psi_G.$$

We will refer to ϕ_G as the *Cartan-Dirac spinor.*

Proof. — It is clear that the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{d}$ is defined by the pure spinor $x = 1$. Similarly, the anti-diagonal $\mathfrak{p} \subset \mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ is defined by the pure spinor $q(\mu) \in \mathrm{Cl}(\mathfrak{g})$:

$$\varrho^{\mathrm{Cl}}(\xi, -\xi)q(\mu) = \xi q(\mu) + (-1)^{\dim G} q(\mu)\xi = 0.$$

Hence ϕ_G, ψ_G are pure spinors defining E_G, F_G. The equivariance properties are special cases of (83), since both $\mathfrak{g}_\Delta$ and $\mathfrak{p}$ are preserved under G_Δ. Here we are using $|1| = 0$, $|q(\mu)| = \dim G \mod 2$, while $u^{\mathfrak{p}}(a) = (-1)^{|a|(1+\dim G)}$ by the calculation:

$$\tau(a)q(\mu)\tau(a^{-1}) = (-1)^{|a| \dim G} q(\mathrm{Ad}_a(\mu)) = (-1)^{|a|(1+\dim G)} q(\mu).$$

The differential equation for ϕ_G follows since $\mathrm{d}^{\mathrm{Cl}}(1) = 0$. It remains to check the differential equation for ψ_G. Since the anti-diagonal satisfies $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{d}} \subset \mathfrak{g}_\Delta$, the element $\sigma^{\mathfrak{p}} \in \mathfrak{p}^*$ is just zero. On the other hand, the element $\Upsilon^{\mathfrak{p}}$ is given by Ξ_Δ, the image of Ξ under the the map $\wedge\mathfrak{g} \xrightarrow{\sim} \wedge\mathfrak{g}_\Delta$. Hence $\mathsf{s}(\Xi_\Delta) = \mathsf{e}(\Xi)$, confirming that ψ_G satisfies (85). □

Remarks 4.5. — a. The map $\mathcal{R}$ depends on the choice of generator $\mu \in \det(\mathfrak{g})$, via the star operator: Replacing μ with $t\mu$ changes $\mathcal{R}$ to $t^{-1}\mathcal{R}$. Hence, the definition of $\psi_G = \mathcal{R}(q(\mu))$ is independent of the choice of μ.

b. Since $(1, q(\mu))_{\mathrm{Cl}(\mathfrak{g})} = \mu$, the bilinear pairing between ϕ_G, ψ_G equals the volume form, up to a sign:

$$\big(\phi_G, \psi_G\big)_{\wedge T^*G} = \mathsf{N}(g)(-1)^{|g|(\dim G+1)}\,\mu_G.$$

Proposition 4.6. — *Over the open subset $\mathcal{U}$ of G where $1+\mathrm{Ad}_g$ is invertible, the pure spinor ψ_G is given by the formula:*

$$\psi_G = \det{}^{1/2}\big(\tfrac{1+\mathrm{Ad}_g}{2}\big)\exp\Big(\tfrac{1}{4}\,B\Big(\tfrac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\,\theta^L, \theta^L\Big)\Big),$$

at any given point $g \in \mathcal{U}$. (The square root depends on the choice of lift $\tau\colon G \to \mathrm{Pin}(\mathfrak{g})$.)

Note that the exponent in this formula becomes singular where $1 + \mathrm{Ad}_g$ fails to be invertible, but these singularities are compensated by the zeroes of the factor $\det^{1/2}\big(\frac{1+\mathrm{Ad}_g}{2}\big)$. One proof of this formula is given in [**47**]; here is an outline of an alternative approach.

Sketch of proof. — One easily checks that over $\mathcal{U}$, F_G coincides with the graph of the 2-form $\omega_F := -\frac{1}{4}\,B\Big(\frac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\,\theta^L, \theta^L\Big)$. Hence $\psi_G|_{\mathcal{U}} = f\exp(-\omega_F)$ for some nonvanishing function $f \in C^\infty(\mathcal{U})$, with $f(e) = 1$. Equation (85) reads, after dividing by $f\exp(-\omega_F)$,

$$\mathrm{d}\log(f) + \eta + \exp(\omega_F)\varrho(\mathsf{e}(\Xi))\big(\exp(-\omega_F)\big) = 0.$$

Taking the form degree 1 parts of both sides of this equation, one obtains the following condition on f:

$$\mathrm{d}\log(f) + \Big(\exp(\omega_F)\varrho(\mathsf{e}(\Xi))\big(\exp(-\omega_F)\big)\Big)_{[1]} = 0.$$

f is uniquely determined by this Equation with the initial condition $f(e) = 1$. It is straightforward (though slightly cumbersome) to verify that $f(g) = \det^{1/2}\big(\frac{1+\mathrm{Ad}_g}{2}\big)$ solves this equation. □

If G is connected, one has $\det(1+\mathrm{Ad}_g) \neq 0$ on a dense open subset of G. However, for a disconnected group G it vanishes on the components with $\det(\mathrm{Ad}_g) = -1$.

Example 4.7. — Let $G = \mathrm{O}(2)$. Here $\mathrm{O}(\mathfrak{g}) = \mathbb{Z}_2$ and $\mathrm{Pin}(\mathfrak{g}) = \mathbb{Z}_4$. There are two possible lifts $\mathrm{O}(\mathfrak{g}) \to \mathrm{Pin}(\mathfrak{g})$. Let $\theta \in \Omega^1(G)$ be the left-invariant Maurer-Cartan-form (using the isomorphism $\mathfrak{g} = \mathbb{R}$ defined by a generator $\mu \in \det(\mathfrak{g}) = \mathfrak{g}$). One finds that on $\mathrm{SO}(2) \subset \mathrm{O}(2)$, $\phi_G = \theta$, while $\psi_G = 1$. On the non-identity component $\mathrm{O}(2)\backslash\mathrm{SO}(2)$, the roles are reversed: $\psi_G = \pm\theta$ and $\phi_G = \pm 1$. (The signs depend on the choice of lift.) Observe that ϕ_G, ψ_G given by these formulas have the correct equivariance properties.

4.3. Group multiplication. — In this section, we will examine the composition of the map $\mathscr{R}\colon \mathrm{Cl}(\mathfrak{g}) \to \Omega(G)$ with the pull-back under group multiplication. It will be convenient to work with the element $\Lambda \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G)$, defined by the property

$$\mathscr{R}(x) = \mathrm{str}(x\Lambda)$$

where we have extended $\mathrm{str}\colon \mathrm{Cl}(\mathfrak{g}) \to \wedge^{[\mathrm{top}]}(\mathfrak{g}) = \mathbb{K}$ to the tensor product with $\Omega(G)$. The properties of $\mathscr{R}$ under the Clifford action translate into

$$(l^{\mathrm{Cl}}(\xi) + \varrho(s^R(\xi)))\Lambda = 0, \quad (-r^{\mathrm{Cl}}(\xi) + \varrho(s^L(\xi)))\Lambda = 0.$$

Thus Λ is itself a pure spinor for the action of $\mathrm{Cl}(\mathfrak{d}) \times \mathrm{Cl}(\mathbb{T}G)$, defining a Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}G$. The equivariance properties (81) of $\mathscr{R}$ translate into

$$l_a^*\Lambda = \tau(a^{-1})\Lambda, \qquad r_{a^{-1}}^*\Lambda = \Lambda\tau(a)$$

The first identity is immediate, while for the second identity is obtained by the calculation:

$$\begin{aligned}\mathrm{str}(x\, r_{a^{-1}}^*\Lambda) &= r_{a^{-1}}^*\mathscr{R}(x) = (-1)^{|a|(|x|+|g|)}\mathscr{R}(\tau(a)x)\\ &= (-1)^{|a|(|x|+|g|)}\,\mathrm{str}(\tau(a)x\Lambda)\\ &= \mathrm{str}(x\Lambda\tau(a)).\end{aligned}$$

(Note that $|\Lambda| = |g|$ at $g \in G$.) We finally observe that the pull-back of Λ to the group unit is simply

$$i_e^*\Lambda = 1 \in \mathrm{Cl}(\mathfrak{g}). \tag{86}$$

Let $\Lambda^1, \Lambda^2 \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$ be the pull-back to the first, second G-factor, and recall the 2-form $\varsigma \in \Omega^2(G \times G)$ from (47).

Proposition 4.8. — *The pull-back of* Λ *under group multiplication satisfies*

$$e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda = \Lambda^1\Lambda^2, \tag{87}$$

using the product in the algebra $\mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$.

Proof. — Using (51), we find that both sides of (87) are annihilated by the following operators:

$$l^{\mathrm{Cl}}(\xi) + \varrho(s^{R,1}(\xi)), \ \ -r^{\mathrm{Cl}}(\xi) + \varrho(s^{L,2}(\xi)), \qquad \varrho(s^{L,1}(\xi) + s^{R,2}(\xi)).$$

Hence the two sides of (87) are pure spinors, defining the same Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}(G \times G)$. So the two sides agree up to a scalar function.

The 2-form ς is invariant under $l_{a,1}$ (left multiplication by a on the first factor) and $r_{a^{-1},2}$ (right multiplication by a^{-1} on the second factor). From the equivariance of Λ, and since $\mathrm{Mult} \circ l_{a,1} = l_a \circ \mathrm{Mult}$ and $\mathrm{Mult} \circ r_{a^{-1},2} = r_{a^{-1}} \circ \mathrm{Mult}$, we obtain the following equivariance property of $e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda$:

$$\begin{aligned}(l_{a,1})^*(e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda) &= \tau(a^{-1})\ (e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda),\\ (r_{a^{-1},2})^*(e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda) &= (e^{\varsigma}\,\mathrm{Mult}^*\,\Lambda)\tau(a).\end{aligned}$$

The product $\Lambda^1\Lambda^2$ has a similar equivariance property. Hence, to verify (87) it suffices to compare the two sides at $(e,e) \in G \times G$. But by (86), both sides of (87) pull back to 1 at (e,e). □

We will use Proposition 4.8 to obtain a formula for the pull-back of $\psi_G = \mathscr{R}(q(\mu))$, the pure spinor defining the Lagrangian subbundle $F_G \subset \mathbb{T}G$. Recall the element $\gamma \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ from (53).

Theorem 4.9. — *The pull-back of ψ_G under group multiplication is given by the formula*

$$e^{\varsigma}\,\mathrm{Mult}^*\,\psi_G = \varrho(\exp(-\mathsf{e}(\gamma)))\,(\psi_G^1 \otimes \psi_G^2)$$

Note that *up to a scalar function*, this identity follows from Proposition 3.11.

Proof. — The element γ enters the following formula (cf. [**4**, Lemma 3.1]) , relating the product $\mathrm{Mult}^{\mathrm{Cl}}$ in $\mathrm{Cl}(\mathfrak{g})$ with the wedge product $\mathrm{Mult}^{\wedge}$ in $\wedge(\mathfrak{g})$:

$$q^{-1} \circ \mathrm{Mult}^{\mathrm{Cl}} = \mathrm{Mult}^{\wedge} \circ \exp(-\iota^{\wedge}(\gamma)) \circ q^{-1} \colon \mathrm{Cl}(\mathfrak{g} \oplus \mathfrak{g}) \to \wedge(\mathfrak{g}).$$

Since $\mathrm{str} \circ l^{\mathrm{Cl}}(q(\mu)) \circ q \colon \wedge\,\mathfrak{g} \to \mathbb{K}$ is simply the augmentation map, we have

$$\psi_G = \mathscr{R}(q(\mu)) = \mathrm{str}(q(\mu)\Lambda) = q^{-1}(\Lambda)_{[0]},$$

where the subscript indicates the degree 0 part with respect to $\wedge\mathfrak{g}$. Using (87), we calculate:

$$\begin{aligned} e^{\varsigma}\,\mathrm{Mult}^*\,\psi_G &= q^{-1}(\Lambda^1\Lambda^2)_{[0]} \\ &= q^{-1} \circ \big(\mathrm{Mult}^{\mathrm{Cl}}(\Lambda^1 \otimes \Lambda^2)\big)_{[0]} \\ &= \big(\mathrm{Mult}^{\wedge} \circ \exp(-\iota^{\wedge}(\gamma)) \circ q^{-1}(\Lambda^1 \otimes \Lambda^2)\big)_{[0]} \\ &= \exp(-\mathsf{e}(\gamma)) \circ \big(\mathrm{Mult}^{\wedge} \circ q^{-1}(\Lambda^1 \otimes \Lambda^2)\big)_{[0]} \\ &= \exp(-\mathsf{e}(\gamma)) \circ (\psi_G^1 \otimes \psi_G^2). \end{aligned}$$

Here we used that $(\iota^{\mathrm{Cl}}(\xi)+\varrho(\mathsf{e}(\xi)))\Lambda = 0$, hence $(\iota^{\wedge}(\gamma)-\varrho(\mathsf{e}(\gamma)))q^{-1}(\Lambda^1\otimes\Lambda^2) = 0$. □

4.4. Exponential map. — Let us return to our description (Section 3.5) $\mathbb{T}\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{d}_0$ of the Courant algebroid over $\mathfrak{g}^*$, where $\mathfrak{d}_0 = \mathfrak{g}^* \rtimes \mathfrak{g}$.

Let $\wedge\mathfrak{g}^*$ be the contravariant spinor module over $\mathrm{Cl}(\mathfrak{d}_0)$ (cf. Section 1.4), with Clifford action denoted $\varrho^{\wedge}$. Let $\mathrm{d}^{\wedge}$ be the exterior algebra differential. For all $w = (\beta,\xi) \in \mathfrak{d}_0$ one has

$$L^{\wedge}(w) := [\mathrm{d}^{\wedge}, \varrho^{\wedge}(w)] = \mathrm{d}^{\wedge}\beta - (\mathrm{ad}_{\xi})^*.$$

One easily checks that $L^{\wedge}(w)$ defines an action of the Lie algebra $\mathfrak{d}_0$. This action exponentiates to an action of the group D_0, given as

$$\mathscr{A}^{\wedge}(\beta,g)y = \exp(\mathrm{d}^{\wedge}\beta) \wedge (\mathrm{Ad}_{g^{-1}})^*y,$$

The function

$$\tau_0 \colon \mathfrak{g}^* \to \wedge\mathfrak{g}^*, \quad \tau_0(\beta) = \exp(\mathrm{d}^{\wedge}\beta) \in \wedge\mathfrak{g}^*$$

is the counterpart to the function $\tau\colon G \to \mathrm{Cl}(\mathfrak{g})$. The D_0-action commutes with the differential, and it is straightforward to check that the Clifford action is D_0-equivariant:

$$\mathcal{A}^\wedge(\beta,g)\big(\varrho^\wedge(w)y\big) = \varrho^\wedge(\mathrm{Ad}_{(\beta,g)}\,w)\big(\mathcal{A}^\wedge(\beta,g)y\big),$$

for $w \in \mathfrak{d}_0$, $(\beta,g) \in D_0$, $y \in \wedge\mathfrak{g}^*$.

Choose a generator $\mu \in \det(\mathfrak{g}^*)$, and let $\star\colon \wedge\,\mathfrak{g} \to \wedge\mathfrak{g}^*$ be the associated star operator [4]. Let X_π denote the modular vector field of the Kirillov-Poisson structure $\pi_{\mathfrak{g}^*}$, relative to the translation-invariant volume form $\mu_{\mathfrak{g}^*} \in \Gamma(\det(T^*\mathfrak{g}^*))$ defined by the dual generator $\mu^* \in \det(\mathfrak{g})$. (Recall that $X_\pi = 0$ if $\mathfrak{g}$ is unimodular.) Define a linear map

$$\mathcal{R}_0\colon \wedge\,\mathfrak{g}^* \to \Omega(\mathfrak{g}^*),$$

given at any point $\nu \in \mathfrak{g}^*$ by

$$\mathcal{R}_0(y) = \star^{-1}(y \wedge \tau_0(\nu)) \in \wedge\mathfrak{g} = \wedge T^*_\nu\mathfrak{g}^*.$$

Parallel to Proposition 4.2, we have,

Proposition 4.10. — a. *The map $\mathcal{R}_0$ intertwines the Clifford actions of $\mathfrak{d}_0$:*

$$\mathcal{R}_0 \circ \varrho^\wedge(w) = \varrho(\mathsf{s}_0(w)) \circ \mathcal{R}_0, \quad w \in \mathfrak{d}_0.$$

It is uniquely determined by this property, up to a scalar function.

b. *The map $\mathcal{R}_0$ intertwines the differentials, up to contraction by the modular vector field:*

$$\mathcal{R}_0 \circ d^\wedge = (d - \iota(X_\pi)) \circ \mathcal{R}_0.$$

c. *$\mathcal{R}_0$ has the equivariance property, for all $h = (\beta, a) \in D_0 = \mathfrak{g}^* \rtimes G$,*

$$(\mathcal{A}_0(h^{-1}))^*\mathcal{R}_0(y) = \det(\mathrm{Ad}_a)\ \mathcal{R}_0(\mathcal{A}^\wedge(h)y).$$

d. *$\mathcal{R}_0$ preserves the bilinear pairings on the spinor modules $\wedge\mathfrak{g}^*$, $\Omega(\mathfrak{g}^*)$, in the sense that*

$$(\mathcal{R}_0(y), \mathcal{R}_0(y'))_{\wedge T^*\mathfrak{g}^*} = (y,\, y')_{\wedge\mathfrak{g}^*}\ \mu_{\mathfrak{g}^*}$$

for all $y, y' \in \wedge\mathfrak{g}^*$.

Proof. — Each of the statements (a),(c),(d) is proved by a direct computation, parallel to those in Proposition 4.2. To prove (b), we first note that (c) implies the infinitesimal equivariance, for $(\beta,\xi) \in \mathfrak{d}_0$,

$$\big(\mathcal{L}(\mathcal{A}_0(\beta,\xi)) - \mathrm{tr}(\mathrm{ad}_\xi)\big)\mathcal{R}_0(y) = \mathcal{R}_0(\mathcal{L}_0^\wedge(\beta,\xi)y). \tag{88}$$

Since $\iota(X_\pi)\langle\theta_0,\xi\rangle = \mathrm{tr}(\mathrm{ad}_\xi)$, we have

$$\mathcal{L}(\mathcal{A}_0(\beta,\xi)) - \mathrm{tr}(\mathrm{ad}_\xi) = [(\mathrm{d} - \iota(X_\pi)), \varrho(\mathsf{s}_0(\beta,\xi))].$$

(4) Note that in the previous Section, μ denoted a generator of $\det(\mathfrak{g})$, and hence the star operator went from $\wedge\mathfrak{g}^* \to \wedge\mathfrak{g}$. This change in notation is intended, since our aim is to compare the Poisson manifold $\mathfrak{g}^*$ with the Dirac manifold G.

Hence we can re-write (88) as

$$[(\mathrm{d}-\iota(X_\pi)),\varrho(\mathsf{s}_0(\beta,\xi))]\ \mathscr{R}_0(y)=\mathscr{R}_0\Big([\mathrm{d}^\wedge,\varrho^\wedge(\beta,\xi)]\Big).$$

Together with (a), this implies that the linear map

$$(\mathrm{d}-\iota(X_\pi))\circ\mathscr{R}_0-\mathscr{R}_0\circ\mathrm{d}^\wedge\colon\ \wedge\,\mathfrak{g}^*\to\Omega(\mathfrak{g}^*) \tag{89}$$

intertwines the Clifford actions of $\mathfrak{d}_0$. Since the parity of this map is opposite to that of $\mathscr{R}_0$, the uniqueness assertion in (a) implies that (89) is zero. □

As before, we may use this map to construct pure spinors $\mathscr{R}_0(y)\in\Omega(\mathfrak{g}^*)$ from pure spinors $y\in\wedge\mathfrak{g}^*$.

The element $y=1$ is the pure spinor defining the Lagrangian subspace $\mathfrak{g}\subset\mathfrak{d}_0$, and its image $\phi_{\mathfrak{g}^*}=\mathscr{R}_0(1)$ defines the Lagrangian subbundle $E_{\mathfrak{g}^*}$ (spanned by the sections $\mathsf{e}_0(\xi)$). The pure spinor $y=\mu\in\wedge\mathfrak{g}^*$ defines a Lagrangian complement $\mathfrak{g}^*\subset\mathfrak{d}_0$, and its image $\psi_{\mathfrak{g}^*}=\mathscr{R}_0(\mu)=1$ defines the Lagrangian subbundle $F_{\mathfrak{g}^*}=T\mathfrak{g}^*$ (spanned by the sections $\mathsf{f}_0(\beta)$). For the bilinear pairing between these pure spinors, we obtain

$$(\phi_{\mathfrak{g}^*},\psi_{\mathfrak{g}^*})_{\wedge T^*\mathfrak{g}^*}=\mu_{\mathfrak{g}^*}.$$

since $(1,\mu)_{\wedge\mathfrak{g}^*}=\mu$.

Lemma 4.11. — *The pure spinor $\phi_{\mathfrak{g}^*}$ is given by the formula*

$$\phi_{\mathfrak{g}^*}=(-1)^{n(n-1)/2}e^{-\iota(\pi_{\mathfrak{g}^*})}\mu_{\mathfrak{g}^*}$$

where $n=\dim G$.

Proof. — The Kirillov-Poisson bivector on $\mathfrak{g}^*$ is given by $\pi_{\mathfrak{g}^*}|_\nu=-\mathrm{d}^\wedge\nu\in\wedge^2\mathfrak{g}^*=\wedge^2T_\nu\mathfrak{g}^*$. That is, $\tau_0=\exp(-\pi_{\mathfrak{g}^*})$. The Lemma follows since $\star$ intertwines exterior product with contractions, and since $\star^{-1}(1)=(\mu^*)^\top=(-1)^{n(n-1)/2}\mu^*$. □

Let us now return to our original setting where $\mathfrak{g}$ carries an invariant inner product B, used to identify $\mathfrak{g}\cong\mathfrak{g}^*$. We take the generators $\mu\in\det(\mathfrak{g})$ (from the last section) and $\mu\in\det(\mathfrak{g}^*)$ (from the present section) to be equal under this identification.

Let $\mu_\mathfrak{g}$ be the translation invariant volume form on $\mathfrak{g}\cong\mathfrak{g}^*$, and μ_G the corresponding left-invariant volume form on G. Let $J\in C^\infty(\mathfrak{g})$ be the Jacobian of the exponential map, defined by $\exp^*\mu_G=J\,\mu_\mathfrak{g}$. Recall that $\mathfrak{g}_\natural\subset\mathfrak{g}$ is the dense open subset where exp is a local diffeomorphism, i.e where $J\neq 0$. With $\varpi\in\Omega^2(\mathfrak{g})$ as in Section 3.5, we have:

Proposition 4.12. — *Over the subset $\mathfrak{g}_\natural$, the maps $\mathscr{R}_0\colon\wedge\,\mathfrak{g}\to\Omega(\mathfrak{g})$ and $\mathscr{R}\colon\mathrm{Cl}(\mathfrak{g})\to\Omega(G)$ are related as follows:*

$$\exp^*(\mathscr{R}(x))=J^{1/2}e^{-\varpi}\varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(\mathscr{R}_0(y)), \tag{90}$$

for $x=q(y)$. Here $\varepsilon\in C^\infty(\mathfrak{g}_\natural,\wedge^2\mathfrak{g})$ is the solution of the classical dynamical Yang-Baxter equation, cf. Proposition 3.14, and $J^{1/2}\in C^\infty(\mathfrak{g})$ is a smooth square root of J, equal to 1 at the origin.

Proof. — The map $\widetilde{\mathcal{R}}_0\colon \wedge\mathfrak{g} \to \Omega(\mathfrak{g}_\natural)$ given as

$$\widetilde{\mathcal{R}}_0(y) = e^{-\varpi}\varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})\exp^*\mathcal{R}(q(y))$$

intertwines the $\mathrm{Cl}(\mathfrak{d}_0)$-actions, hence it coincides with $\widetilde{\mathcal{R}}_0 = f\,\mathcal{R}_0$ for a scalar function. To find f, we consider bilinear pairings. Note that

$$\begin{aligned}(\widetilde{\mathcal{R}}_0(y), \widetilde{\mathcal{R}}_0(y'))_{\wedge T^*\mathfrak{g}} &= (\exp^*\mathcal{R}(q(y)), \exp^*\mathcal{R}(q(y')))_{\wedge T^*\mathfrak{g}}\\ &= \exp^*\big(\mathcal{R}(q(y)),\ \mathcal{R}(q(y'))\big)_{\wedge T^*G}.\end{aligned}$$

Taking $y' = 1,\ y = \mu$ we obtain

$$f^2\mu_{\mathfrak{g}} = f^2\,(\mathcal{R}_0(\mu),\ \mathcal{R}_0(1))_{\wedge T^*\mathfrak{g}} = (\widetilde{\mathcal{R}}_0(\mu), \widetilde{\mathcal{R}}_0(1))_{\wedge T^*\mathfrak{g}} = \exp^*\mu_G = J\,\mu_{\mathfrak{g}}.$$

This shows that $f^2 = J$. □

Remark 4.13. — Of course, $\exp^*(R(x))$ is defined globally on all of $\mathfrak{g}$, not only on $\mathfrak{g}_\natural$. It follows from the Proposition that $J^{1/2}\exp(\mathsf{e}_0(\varepsilon))$ extends smoothly to all of $\mathfrak{g}$. Hence, the expression

$$J^{1/2}\exp(\varepsilon)$$

extends smoothly to a global function $\mathfrak{g} \to \wedge\mathfrak{g}$. For a direct proof, see [**5**].

Applying the proposition to $y = 1$ and $y = \mu$, we find in particular that

$$\begin{aligned}\exp^*\phi_G &= J^{1/2}\,e^{-\varpi}\phi_{\mathfrak{g}},\\ \exp^*\psi_G &= J^{1/2}\,e^{-\varpi}\varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(1).\end{aligned} \tag{91}$$

4.5. The Gauss-Dirac spinor. — We return to the set-up of Section 3.6, with $G = K^{\mathbb{C}}$ denoting the complexification of a compact Lie group, with Cartan subgroup $T = T_K^{\mathbb{C}}$. Recall that the Gauss-Dirac structure $\widehat{F}_G$ is defined by the Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, with basis the collection of all $e_\alpha \oplus 0,\ 0 \oplus e_{-\alpha},\ e_i \oplus (-e_i)$ where $\alpha \succ 0$ are positive roots and $i = 1, \dots, l = \mathrm{rank}(G)$. The element

$$x = \prod_{\alpha \succ 0} e_\alpha e_{-\alpha}\ \prod_i e_i \in \mathrm{Cl}(\mathfrak{g}) \tag{92}$$

is non-zero and is annihilated by the Clifford action of $\mathfrak{s}$; hence it is a pure spinor defining $\mathfrak{s}$. Note that x satisfies

$$\tau(h_+)x = x,\quad x\tau(h_-^{-1}) = x,\quad \tau(h_0)x\tau(h_0) = h_0^{2\rho}\,x$$

for all $h_+ \in N_+$, $h_- \in N_-$, $h_0 \in T$. Here $\rho = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$, and $t \mapsto t^{2\rho} \in \mathbb{C}^\times$ is the character of T defined by the weight 2ρ. Hence,

$$\widehat{\psi}_G = \mathcal{R}(x) \in \Omega(G)$$

is a pure spinor defining $\widehat{F}_G$. We refer to $\widehat{\psi}_G$ as the *Gauss-Dirac spinor*. Its equivariance properties are:

$$l_{h_+}^*\widehat{\psi}_G = \widehat{\psi}_G,\quad r_{h_-^{-1}}^*\widehat{\psi}_G = \widehat{\psi}_G,\quad l_{h_0}^* r_{h_0}^*\widehat{\psi}_G = h_0^{2\rho}\widehat{\psi}_G.$$

That is, $\widehat{\psi}_G$ is invariant up to the character, given by the group homomorphism $S \to T$ followed by the 2ρ-character.

Since the big Gauss cell $\mathcal{O} = N_-TN_+ \subset G$ is dense in G, the equivariance property, together with the fact that the pull-back of ψ_G to the group unit is equal to $\mathrm{str}(x) = 1$, completely characterizes the pure spinor ψ_G, and allows us to give an explicit formula. Recall the 2-form $\omega_{\mathcal{O}}$ on the big Gauss cell, given by (68):

Proposition 4.14. — *The restriction of the pure spinor $\widehat{\psi}_G$ to the big Gauss cell $\mathcal{O} = j(N_- \times T \times N_+)$ is given by the formula,*

$$\widehat{\psi}_G\big|_{\mathcal{O}} = g_0^{\rho} \exp(-\omega_{\mathcal{O}}).$$

Here $g_0 \colon \mathcal{O} \to T$ is the composition of the Gauss decomposition $j^{-1} \colon \mathcal{O} \to N_- \times T \times N_+$ with projection to the middle factor.

Proof. — Both sides are pure spinors defining the Gauss-Dirac structure over $\mathcal{O}$, with the same equivariance property under S, and both sides pull back to 1 at the group unit e. □

We now compare the Gauss-Dirac spinor $\widehat{\psi}_G$ with the pure spinor ψ_G from Proposition 4.4.

Proposition 4.15. — *The pure spinors $\psi_G, \widehat{\psi}_G$ are related by a twist by the r-matrix $\mathfrak{r}$:*

$$\widehat{\psi}_G = \varrho(\exp(-\mathsf{e}(\mathfrak{r}))\psi_G.$$

Proof. — Let $\mathfrak{r}_\Delta \in \wedge^2\mathfrak{d}$ be the image of $\mathfrak{r}$ under the diagonal inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d}$. We will show that

$$x = \varrho^{\mathrm{Cl}}(\exp(-\mathfrak{r}_\Delta))q(\mu). \tag{93}$$

The proposition follows from this identity by applying the map $\mathcal{R}$. Up to a scalar, (93) holds since both sides are pure spinors defining the same Lagrangian subspace. To determine the scalar, we apply the super-trace to both sides. Recall that the spinor action of elements $\xi_\Delta \in \mathfrak{g}_\Delta \subset \mathfrak{g}$ is given by Clifford commutator with the corresponding element $\xi \in \mathfrak{g}$. Since the super-trace vanishes on Clifford commutators, it follows that

$$\mathrm{str}(\varrho^{\mathrm{Cl}}(\exp(-\mathfrak{r}))q(\mu)) = \mathrm{str}(q(\mu)) = 1 = \mathrm{str}(x).$$ □

Let us next compute the Clifford differential $\mathrm{d}^{\mathrm{Cl}} = -4[q(\Xi), \cdot]$ of the element (92). Let $\rho = \frac{1}{2}\sum_{\alpha \succ 0} \alpha \in \mathfrak{t}^*$ be the half-sum of positive (real) roots.

Lemma 4.16. — *The quantization of the structure constant tensor satisfies,*

$$-4q(\Xi) = 2\pi\sqrt{-1}\rho \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

Here B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — By definition,

$$-4q(\Xi) = \frac{1}{6}\sum B([e^a, e^b], e^c)e_a e_b e_c,$$

using a basis e_a of $\mathfrak{g}$, with B-dual basis e^a. Take this basis to be the Cartan-Weil basis, and use the Clifford relations to write factors $e_{-\alpha}$ to the left and factors e_α to the right. Then

$$-4q(\Xi) \in \mathrm{Cl}(\mathfrak{g})^T \subset \mathfrak{n}_- \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \oplus \mathrm{Cl}(\mathfrak{t}).$$

(For a T-equivariant element in $\mathrm{Cl}(\mathfrak{g})$, the T-weight of the $\mathfrak{n}_-$-factors must be compensated by the T weights of the $\mathfrak{n}_+$-factors.) Since $-4q(\Xi)$ is an odd element of filtration degree 3, and since Ξ has no component in $\wedge^3\mathfrak{t}$, it follows that

$$-4q(\Xi) \in \mathfrak{t} \oplus \mathfrak{n}_- \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

To compute the $\mathfrak{t}$-component, we calculate the constant component of

$$[\xi, -4q(\Xi)]_{\mathrm{Cl}} = \mathrm{d}^{\mathrm{Cl}}\xi = q(\lambda(\xi))$$

for any $\xi \in \mathfrak{t}$. We have

$$\lambda(\xi) = -\sum_{\alpha \succ 0}[\xi, e_{-\alpha}] \wedge e_\alpha = 2\pi\sqrt{-1}\sum_{\alpha \succ 0}\langle \alpha, \xi\rangle e_{-\alpha} \wedge e_\alpha,$$

hence (see Sternberg [**52**, Equation (9.25)])

$$q(\lambda(\xi)) = 2\pi\sqrt{-1}\sum_{\alpha \succ 0}\langle \alpha, \xi\rangle e_{-\alpha}e_\alpha + 2\pi\sqrt{-1}\ \langle \rho, \xi\rangle.$$ □

As a consequence, we obtain,

Proposition 4.17. — *The element $x = \prod_{\alpha \succ 0} e_\alpha e_{-\alpha} \prod_i e_i$ satisfies,*

$$\left(d^{\mathrm{Cl}} - 2\pi\sqrt{-1}\iota^{\mathrm{Cl}}(\rho)\right)x = 0.$$

Proof. — d^{Cl} is given as the Clifford commutator with $-4q(\Xi)$. Since x is annihilated under both left and right multiplication by elements of $\mathfrak{n}_- \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$, it follows that

$$\mathrm{d}^{\mathrm{Cl}}(x) = 2\pi\sqrt{-1}[\rho, x]_{\mathrm{Cl}}.$$ □

As a consequence, the Gauss-Dirac spinor satisfies the differential equation:

$$(\mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\rho)))\widehat{\psi}_G = 0. \tag{94}$$

In fact, there is a more general version of this Equation, stated in the following Proposition. For any (real) dominant weight λ of G (not to be confused with the map λ above), let $\Delta_\lambda \in C^\infty(G)$ be the function

$$\Delta_\lambda(g) = \frac{\langle v_\lambda, g \cdot v_\lambda\rangle}{\langle v_\lambda, v_\lambda\rangle},$$

where v_λ is a highest weight vector in the irreducible unitary representation $(V_\lambda, \langle\cdot,\cdot\rangle)$ of highest weight λ. The function Δ_λ is invariant under the left-action of N_-, under the right-action of N_+, and under the T-action it satisfies

$$\Delta_\lambda(tg) = \Delta_\lambda(gt) = t^\lambda \Delta_\lambda(g). \tag{95}$$

Since $\Delta_\lambda(e) = 1$, it follows that $\Delta_\lambda \neq 0$ on the big Gauss cell. We are interested in the product $\Delta_\lambda \widehat{\psi}_G$. Away from the zeroes of Δ_λ, this is a pure spinor defining the Gauss-Dirac structure. Similar to $\widehat{\psi}_G$, it is invariant under the left-action of N_- and the right-action of N_+, and satisfies

$$l_t^*(\Delta_\lambda\widehat{\psi}_G) = r_t^*(\Delta_\lambda\widehat{\psi}_G) = t^{\lambda+\rho}(\Delta_\lambda\widehat{\psi}_G) \tag{96}$$

for all $t \in T$.

Proposition 4.18. — *For any dominant weight λ, the product $\Delta_\lambda\widehat{\psi}_G$ satisfies the differential equation:*

$$\big(d + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\big)\Delta_\lambda\widehat{\psi}_G = 0, \tag{97}$$

where B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — Let $\mathfrak{s} \subset \mathfrak{d}$ be the Lagrangian subalgebra (66) defining the Gauss-Dirac structure. We have, for all $\zeta = (\xi, \xi') \in \mathfrak{s}$,

$$\begin{aligned}
&\varrho(\mathsf{s}(\zeta))\big(\mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\big)\,\Delta_\lambda\widehat{\psi}_G\\
&= \Big[\varrho(\mathsf{s}(\zeta)),\ \mathrm{d} + \eta - 2\pi\sqrt{-1}\varrho(\mathsf{e}(\lambda+\rho))\Big]\Delta_\lambda\widehat{\psi}_G\\
&= \big(\mathscr{L}(\xi^L - (\xi')^R)) - 2\pi\sqrt{-1}B(\xi-\xi', \lambda+\rho)\big)\Delta_\lambda\widehat{\psi}_G = 0,
\end{aligned}$$

where the last equality follows from the equivariance properties (96) of $\Delta_\lambda\widehat{\psi}_G$. (Note that for the elements of the form $\zeta = (\xi, 0)$ with $\xi \in \mathfrak{n}_+$ or $\zeta = (0, \xi)$ with $\xi \in \mathfrak{n}_-$, the inner product with $\lambda + \rho \in \mathfrak{t}$ vanishes.) Hence, the left hand side of (97) is annihilated by all $\mathsf{s}(\zeta)$, for $\zeta \in \mathfrak{s}$. Hence it is a function times $\widehat{\psi}_G$, and thus vanishes since it has parity opposite to that of $\widehat{\psi}_G$. □

Remark 4.19. — The holomorphic Dirac structure $\widehat{F}_G$ on $G = K^{\mathbb{C}}$ restricts to a complex Dirac structure $\widehat{F}_G|_K = \widehat{F}_K$ on the real Lie group K, with defining pure spinor the pull-back (restriction) $\widehat{\psi}_K$ of $\widehat{\psi}_G$. On the other hand, $E_G|_K = (E_K)^{\mathbb{C}}$. In the notation of Section 2.4, applied to the Gauss-Cartan-splitting $(\mathbb{T}K)^{\mathbb{C}} = E_K^{\mathbb{C}} \oplus \widehat{F}_K$, we have $\sigma = 2\pi\sqrt{-1}e(\rho) \in \Gamma((\mathbb{T}K)^{\mathbb{C}})$, thus $\eth_\pm = \mathrm{d} + \eta \pm 2\pi\sqrt{-1}\varrho(\mathsf{e}(\rho))$. As usual, $\eth_+\phi_K = 0$, $\eth_-\widehat{\psi}_K = 0$ (the second equation is the pull-back of (94) to K). Let μ be the bi-invariant (real) volume form on K defined by $\phi_K, \widehat{\psi}_K$. Since $\eth_\pm^2 = \pm 2\pi\sqrt{-1}\,\mathscr{L}(\mathscr{C}_{\mathrm{ad}}(\rho))$, the Dirac cohomology groups $H_\pm(E_K^{\mathbb{C}}, \widehat{F}_K, \mu)$ are the cohomology groups of $\eth_\pm$ on the space of $\mathscr{C}_{\mathrm{ad}}(\rho)$-invariant complex-valued differential forms on K. These may be computed by the standard localization argument ([**12**], see also [**33**]): The set of zeroes of the vector field $\mathscr{C}_{\mathrm{ad}}(\rho)$ on K is just the maximal torus T_K, and the pull-back to T_K intertwines $\eth_\pm$ with $d \pm 2\pi\sqrt{-1}B(\theta_T, \rho))$, with θ_T

the Maurer-Cartan form on T_K. Hence, by localization the pull-back to T_K induces an isomorphism,

$$H_{\pm}(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(\Omega(T_K)^{\mathbb{C}}, \mathrm{d} \pm 2\pi\sqrt{-1}B(\theta_T, \rho))$$

Since ρ is a weight, it defines a T_K-character t^ρ, and the operators $\mathrm{d}\pm 2\pi\sqrt{-1}B(\theta_T,\rho)$ are obtained from d by conjugation by $t^{\pm\rho}$. Hence $H_{\pm}(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(T_K)^{\mathbb{C}}$.

5. q-Hamiltonian G-manifolds

In this section, we use the techniques developed in this paper to extend the theory of group-valued moment maps, as developed in [**3, 8**] for the case of compact Lie groups, to more general settings.

5.1. Dirac morphisms and group-valued moment maps.

— We briefly recall the definitions.

Definition 5.1. — A *quasi-Hamiltonian* $\mathfrak{g}$*-manifold* (or simply *q-Hamiltonian* $\mathfrak{g}$*-manifold*) is a manifold M with a Lie algebra action $\mathcal{A}_M\colon \mathfrak{g} \to \mathfrak{X}(M)$, a 2-form ω, and a $\mathfrak{g}$-equivariant *moment map* $\Phi\colon M \to G$ such that

$$\begin{aligned} \mathrm{d}\omega &= \Phi^*\eta \\ \iota(\mathcal{A}_M(\xi))\omega &= \Phi^* B(\xi, \tfrac{\theta^L+\theta^R}{2}) &&\text{(moment map condition)} \\ \ker(\omega_m) &= \{\mathcal{A}_M(\xi)_m |\ \mathrm{Ad}_{\Phi(m)}\xi = -\xi\} &&\text{(minimal degeneracy condition).} \end{aligned} \tag{98}$$

If the action of $\mathfrak{g}$ extends to an action of the Lie group G, and if ω and Φ are equivariant for the action of G, we speak of a *q-Hamiltonian* G*-manifold.*

The first two conditions in (98) imply that ω is $\mathfrak{g}$-invariant (see [**3**]). As shown by Bursztyn-Crainic [**14**], the definition of a q-Hamiltonian space may be restated in Dirac geometric terms (see also Xu [**57**] for another interpretation).

Theorem 5.2. — *There is a 1-1 correspondence between q-Hamiltonian* $\mathfrak{g}$*-manifolds, and manifolds* M *together with a strong Dirac morphism*

$$(\Phi,\omega)\colon (M, TM, 0) \to (G, E_G, \eta). \tag{99}$$

More precisely, $(M, \mathcal{A}_M, \omega, \Phi)$ *satisfies the first two conditions if and only if* (Φ,ω) *is a Dirac morphism, and in this case the third condition is equivalent to this Dirac morphism being* strong.

Proof. — Let $(M, \mathcal{A}_M, \omega, \Phi)$ be a q-Hamiltonian $\mathfrak{g}$-space. Given $m \in M$, let $E'_{\Phi(m)}$ be the forward image of T_mM under $((\mathrm{d}\Phi)_m, \omega_m)$:

$$E'_{\Phi(m)} = \{(\mathrm{d}\Phi(v), \alpha) |\ v \in T_mM,\ (\mathrm{d}\Phi)_m^*\alpha = \iota(v)\omega_m\}.$$

Taking v of the form $\mathscr{A}_M(\xi)_m$ for $\xi \in \mathfrak{g}$, and using the moment map condition, we see $E'_{\Phi(m)} \supset (E_G)_{\Phi(m)}$. In fact, one has equality since both are Lagrangian subspaces. This shows that (Φ, ω) is a Dirac morphism. In particular,

$$(\mathrm{d}\Phi)_m(\ker(\omega_m)) = \ker((E_G)_{\Phi(m)}) = \{\mathscr{A}_{\mathsf{ad}}(\xi)_{\Phi(m)} \,|\, \mathrm{Ad}_{\Phi(m)}\,\xi = -\xi\}.$$

Hence, the minimal degeneracy condition holds if and only $(\mathrm{d}\Phi)_m$ restricts to an isomorphism on $\ker(\omega_m)$, i.e. if and only if (Φ, ω) is a strong Dirac morphism. Conversely, given a strong Dirac morphism (99), the associated map $\mathfrak{a}$ defines a $\mathfrak{g}$-action $\mathscr{A}_M(\xi) = \mathfrak{a}(\Phi^*\mathsf{e}(\xi))$ on M, for which the map Φ is $\mathfrak{g}$-equivariant. The above argument then shows that $(M, \mathscr{A}_M, \omega, \Phi)$ is a q-Hamiltonian $\mathfrak{g}$-space. □

Remark 5.3. — As a consequence of this result (or rather its proof), we see that if $(M, \mathscr{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98), then the third condition (minimal degeneracy) is equivalent to the transversality property [**15, 57**]

$$\ker(\omega) \cap \ker(\mathrm{d}\Phi) = \{0\}.$$

Remark 5.4. — There is a similar result for q-Hamiltonian G-manifolds. Here, it is necessary to assume the existence of a G-action on M for which the Dirac morphism (Φ, ω) is equivariant, and such that the infinitesimal action coincides with that defined by $\mathfrak{a}$.

Example 5.5. — By Example 2.7, the inclusion of the conjugacy classes $\mathscr{C}$ in G, with 2-forms defined by the Cartan-Dirac structure, defines a strong Dirac morphism $(\iota_{\mathscr{C}}, \omega_{\mathscr{C}})$. Thus, conjugacy classes are q-Hamiltonian G-manifolds.

Using our results on the Cartan-Dirac structure, it is now straightforward to deduce the basic properties of q-Hamiltonian spaces $(M, \mathscr{A}_M, \omega, \Phi)$. In contrast with the original treatment in [**3**], the discussion works equally well for non-compact Lie groups, and also in the holomorphic category.

***Theorem 5.6* (Fusion).** — *Let $(M, \mathscr{A}_M, \Phi, \omega)$ be a q-Hamiltonian $G \times G$-manifold. Let $\mathscr{A}_{\mathrm{fus}}$ be the diagonal G-action, $\Phi_{\mathrm{fus}} = \mathrm{Mult} \circ \Phi$, and $\omega_{\mathrm{fus}} = \omega + \Phi^*\varsigma$, with $\varsigma \in \Omega^2(G^2)$ the 2-form defined in (47). Then $(M, \mathscr{A}_{\mathrm{fus}}, \Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}})$ is a q-Hamiltonian G-manifold. (An analogous statement holds for q-Hamiltonian $\mathfrak{g} \times \mathfrak{g}$-manifolds.)*

Proof. — Since

$$(\Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}}) = (\mathrm{Mult}, \varsigma) \circ (\Phi, \omega)$$

is a composition of two strong Dirac morphism, it is itself a strong Dirac morphism from $(M, TM, 0)$ to (G, E_G, η). The induced map $M \times \mathfrak{g} = \Phi^*_{\mathrm{fus}} E_G \to TM$ is a composition of the map $\mathrm{Mult}^*\, E_G \to E_{G\times G}$ defined by the strong Dirac morphism $(\mathrm{Mult}, \varsigma)$, with the map $\Phi^* E_G \times E_G \to TM$ given by the strong Dirac morphism (Φ, ω). If we use the sections $\mathsf{e}(\xi)$ to identify $E_G \cong G \times \mathfrak{g}$, the latter map is the $\mathfrak{g} \times \mathfrak{g}$-action on M, while the former is the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. This confirms that the resulting action is just the diagonal action. □

If $M = M_1 \times M_2$ is a direct product of two q-Hamiltonian manifolds, the quadruple $(M, \mathcal{A}_{\mathrm{fus}}, \Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}})$ is called the *fusion product* of M_1, M_2. In particular we obtain products of conjugacy classes as new examples of q-Hamiltonian G-spaces.

Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$-manifold: That is, ω_0 is symplectic, and $\Phi_0 \colon M \to \mathfrak{g}^*$ is the moment map for a Hamiltonian $\mathfrak{g}$-action on M. As is well-known, this is equivalent to Φ_0 being a Poisson map from the symplectic manifold (M, ω_0) to the Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$. But this is also equivalent to

$$(\Phi_0, \omega_0) \colon (M, TM, 0) \to (\mathfrak{g}^*, \mathrm{Gr}_\pi, 0)$$

being a strong Dirac morphism. A *Hamiltonian G-manifold* comes with a G-action on M integrating the $\mathfrak{g}$-action, and such that the Dirac morphism (Φ_0, ω_0) is equivariant. Given an invariant inner product B on $\mathfrak{g}$, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$, we may compose the Dirac morphism (Φ_0, ω_0) with the Dirac morphism $(\exp, \varpi)$ from Theorem 3.13, and obtain:

***Theorem 5.7* (Exponentials).** — *Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian G-manifold, and let $\omega = \omega_0 + \Phi_0^* \varpi$, $\Phi = \exp \circ \Phi_0$. Then $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98). On $M_\natural = \Phi_0^{-1}(\mathfrak{g}_\natural)$, the third condition (minimal degeneracy) holds as well, thus $(M_\natural, \mathcal{A}_M, \omega, \Phi)$ is a q-Hamiltonian G-manifold. (Similar statements hold for q-Hamiltonian $\mathfrak{g}$-manifolds.)*

5.2. Volume forms.

— Any symplectic manifold (M, ω) carries a distinguished volume form, given as the top degree component $\exp(\omega)^{[\dim M]} = \frac{1}{n!}\omega^n$. For a q-Hamiltonian G-manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the 2-form ω is usually degenerate, hence $\exp(\omega)^{[\mathrm{top}]}$ will have zeroes. Nevertheless, any q-Hamiltonian G-manifold carries a distinguished volume form, provided the adjoint action $\mathrm{Ad} \colon G \to \mathrm{O}(\mathfrak{g})$ lifts to $\mathrm{Pin}(\mathfrak{g})$:

***Theorem 5.8* (Volume forms).** — *Suppose the adjoint action $\mathrm{Ad} \colon G \to \mathrm{O}(\mathfrak{g})$ lifts to $\mathrm{Pin}(\mathfrak{g})$, and let $\psi_G \in \Omega(G)$ be the pure spinor defined by this lift. For any q-Hamiltonian G-manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the differential form*

$$\mu_M = (\exp(\omega) \wedge \Phi^* \psi_G)^{[\dim M]} \tag{100}$$

is a volume form. It has the equivariance property $\mathcal{A}_M(g)^* \mu_M = \det(\mathrm{Ad}_g)\, \mu_M$. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98), the form μ_M is non-zero exactly at those points where ω satisfies the minimal degeneracy condition.

Of course, the factor $\det(\mathrm{Ad}_g) = \pm 1$ is trivial if G is connected.

Proof. — Since ψ_G is a pure spinor defining the complementary Lagrangian subbundle F_G, and since (Φ, ω) is a strong Dirac morphism, the pull-back $\Phi^* \psi_G$ is non-zero everywhere. Furthermore, $\exp(\omega) \Phi^* \psi_G$ is a pure spinor defining the backward image F of F_G under the Dirac morphism (Φ, ω). Since F is transverse to TM (see Proposition 1.15), the top degree part of $\exp(\omega) \Phi^* \psi_G$ is nonvanishing. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ only satisfies the first two conditions in (98), then the above argument applies at all points of M where (Φ, ω) is a strong Dirac morphism. But these are exactly the points where $\Phi^*\psi_G$ is non-zero.

The equivariance property of μ_M is a direct consequence of the equivariance properties of ϕ_G and ψ_G described in Proposition 4.4. □

The volume form μ_M is called the *Liouville volume form* of the q-Hamiltonian G-manifold $(M, \mathscr{A}_M, \omega, \Phi)$. Let $|\mu_M|$ be the associated measure. If the moment Φ is proper, the push-forward $\Phi_*|\mu_M|$ is a well-defined measure on G, called the *Duistermaat-Heckman measure.*

Remark 5.9. — For the case of compact Lie groups, the q-Hamiltonian Liouville forms and Duistermaat-Heckman measures were introduced in [**8**]. The fact that μ_M is a volume form was verified by 'direct computation'. However, the argument in [**8**] does not extend to non-compact Lie groups.

Remark 5.10. — The expression $\exp(\omega)\Phi^*\psi_G$ entering the definition of the volume form μ_M satisfies the differential equation

$$(\mathrm{d} + \iota(\mathscr{A}_M(\Xi)))\Big(\exp(\omega)\Phi^*\psi_G\Big) = 0. \tag{101}$$

This follows from the differential equation (85) for ψ_G together with Remark 1.5(a).

Proposition 5.11. — *Suppose $(M, \mathscr{A}_M, \omega, \Phi)$ is a q-Hamiltonian G-manifold, and that* Ad *lifts to the Pin group. Then M is even-dimensional if* $\det(\mathrm{Ad}_\Phi) = +1$*, and odd-dimensional if* $\det(\mathrm{Ad}_\Phi) = -1$*. In particular, it is even-dimensional when G is connected, and in this case M carries a canonical orientation.*

Proof. — The construction of ψ_G in terms of the map $\mathscr{R}$ (see Proposition 4.4) shows that the form ψ_G has even degree at points $g \in G$ with $\det(\mathrm{Ad}_g) = 1$, and odd degree at points with $\det(\mathrm{Ad}_g) = -1$. Hence, the parity of the volume form μ_M is determined by the parity of $\det(\mathrm{Ad}_\Phi)$. If G is connected, the lift of Ad (which exists by assumption) is unique, and $\det(\mathrm{Ad}_g) \equiv 1$. □

Without the existence of a lift to $\mathrm{Pin}(\mathfrak{g})$, the form ψ_G is only defined locally, up to sign. That is, we still obtain a G-invariant *measure* on M, given locally as $(e^\omega \Phi^*\psi_G)^{[\mathrm{top}]}$. It is interesting to specialize these results to conjugacy classes:

Theorem 5.12. — *Suppose G is a connected Lie group, whose Lie algebra carries an invariant inner product B. Then:*

a. *Every conjugacy class $\mathscr{C} \subset G$ carries a distinguished invariant measure (depending only on B).*
b. *The conjugacy class $\mathscr{C}$ of $g \in G$ is even-dimensional if and only if* $\det(\mathrm{Ad}_g) = +1$.
c. *If the adjoint action $G \to \mathrm{O}(\mathfrak{g})$ lifts to* $\mathrm{Pin}(\mathfrak{g})$, *then every conjugacy class carries a distinguished orientation.*

Example 5.13. — Consider the conjugacy classes of $G = \mathrm{O}(2)$: If $g \in \mathrm{SO}(2)$, the conjugacy class of g is zero-dimensional, consisting of either one or two points. On the other hand, the circle $\mathrm{O}(2)\backslash \mathrm{SO}(2) \cong S^1$ forms a single conjugacy class. Similarly, for $G = \mathrm{O}(3)$, the elements $g \in G$ with $\det(g) = -1$ have $\det(\mathrm{Ad}_g) = 1$. Each of these form a single 2-dimensional conjugacy class. The group SO(3) is the simplest example where the adjoint action $G \to \mathrm{SO}(\mathfrak{g})$ (which in this case is just the identity map) does not lift to the spin group. Indeed the conjugacy class of rotations by 180^o is isomorphic to $\mathbb{R}P(2)$, hence non-orientable.

Example 5.14. — Suppose G carries an involution σ, such that the corresponding involution of $\mathfrak{g}$ preserves B. Form the semi-direct product $G \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is generated by the involution σ. The $G \rtimes \mathbb{Z}_2$-conjugacy class of the element (e, σ) is isomorphic to the homogeneous space $M = G/G^\sigma$, which therefore is an example of a q-Hamiltonian $G \rtimes \mathbb{Z}_2$-space. The 2-form on M is just zero. Let us compute the Liouville measure on M, for the case that the restriction of B to $\mathfrak{g}^\sigma = \ker(\sigma - 1)$ is still non-degenerate. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$, with $B(e_i, e_j) = \pm\delta_{ij}$, such that $e_1 \ldots, e_k$ are a basis of $\mathfrak{g}^\sigma$. Then

$$\widetilde{\sigma} = 2^{(n-k)/2} e_{k+1} \cdots e_n \in \mathrm{Pin}(\mathfrak{g})$$

is a lift of σ. Note that $\widetilde{\sigma}^2 = \pm 1$, with sign depending on $n-k$. Taking $\mu = e_1 \wedge \cdots \wedge e_n$ as the Riemannian volume form on $\mathfrak{g}$, we have

$$\widetilde{\sigma} q(\mu) = \pm 2^{(n-k)/2} e_1 \ldots e_k$$

so $\star q^{-1}(\widetilde{\sigma} q(\mu)) = \pm 2^{(n-k)/2} e_{k+1} \wedge \cdots \wedge e_n$. We conclude that the Liouville measure on $M = G/G^\sigma$ coincides with the G-invariant measure defined by the metric on $(\mathfrak{g}^\sigma)^\perp \subset \mathfrak{g}$.

***Proposition 5.15* (Volume form for 'fusions').** — *The volume form of a q-Hamiltonian $G \times G$-manifold $(M, \mathscr{C}_M, \omega, \Phi)$ (as in Theorem 5.6) coincides with the volume form of its fusion $(M, \mathscr{C}_{\mathrm{fus}}, \omega_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$:*

$$(\exp(\omega)\, \Phi^* \psi_{G \times G})^{[\dim M]} = (\exp(\omega_{\mathrm{fus}})\, \Phi_{\mathrm{fus}}^* \psi_G)^{[\dim M]}.$$

Proof. — Using Theorem 4.9, we have

$$\begin{aligned} \exp(\omega_{\mathrm{fus}})\, \Phi_{\mathrm{fus}}^* \psi_G &= \exp(\omega + \Phi^* \varsigma)\, \Phi^*\, \mathrm{Mult}^*\, \psi_G \\ &= \exp(\omega)\, \Phi^* \big(\varrho(\exp(-\mathrm{e}(\gamma)))\psi_G^1 \otimes \psi_G^2\big) \\ &= \exp(-\iota(\mathscr{C}_M(\gamma)))(\exp(\omega)\Phi^* \psi_{G \times G}), \end{aligned}$$

where we used Remark 1.5(a) for the last equality. Since the operator $\exp(-\iota(\mathscr{C}_M(\gamma)))$ does not affect the top degree part, the proof is complete. □

Example 5.16. — An important example of a q-Hamiltonian G-space is the double $D(G) = G \times G$, with moment map the commutator $\Phi(a, b) = aba^{-1}b^{-1}$. As explained in [**8**] the double is obtained by fusion, as follows: Start by viewing the Lie group G as a homogeneous space $G = G \times G/G_\Delta$, where G_Δ is the diagonal subgroup. Since G_Δ is the fixed point set for the involution σ of $G \times G$ switching the two factors, we see as in Example 5.14 that G is a q-Hamiltonian $(G \times G) \rtimes \mathbb{Z}_2$-space, with moment map $a \mapsto (a, a^{-1}, \sigma)$. The Liouville measure is simply the Haar measure on G. Fusing two copies, the direct product $G \times G$ becomes a q-Hamiltonian $G \times G$-space. Finally, passing to the diagonal action one arrives at the double $D(G)$. By Proposition 5.15, the resulting Liouville measure on $D(G)$ is just the Haar measure.

***Proposition 5.17* (Volume form for 'exponentials').** — *Let $(M, \mathscr{A}_M, \Phi_0, \omega_0)$ be a Hamiltonian G-space, and $(M, \mathscr{A}_M, \Phi, \omega)$ its 'exponential', as in Theorem 5.7. Then*

$$(\exp(\omega)\Phi^*\psi_G)^{[\dim M]} = \Phi_0^* J^{1/2} \ \exp(\omega_0)^{[\dim M]}.$$

Proof. — Using the relation (91) between $\exp^* \psi_G$ and $\psi_{\mathfrak{g}} = 1$, we find

$$\begin{aligned} \exp(\omega)\, \Phi^*\psi_G &= \exp(\omega_0 + \Phi_0^*\varpi)\Phi_0^* \exp^* \psi_G \\ &= \exp(\omega_0)\Phi_0^* J^{1/2} \varrho(\widetilde{A}^{-\mathsf{e}_0(\varepsilon)})(1) \\ &= \Phi_0^* J^{1/2} \exp(-\iota(\mathscr{A}_M(\varepsilon))) \exp(\omega_0). \end{aligned}$$

Since $\exp(-\iota(\mathscr{A}_M(\varepsilon)))$ does not affect the top degree part, the proof is complete. □

5.3. The volume form in terms of the Gauss-Dirac spinor. — Suppose now that K is a *compact* Lie group, with complexification $G = K^{\mathbb{C}}$, and let $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the complexification of a positive definite inner product on $\mathfrak{k}$. In this case, as discussed in Section 3.6, E_G has a second Lagrangian complement $\widehat{F}_G$, defined by the Gauss-Dirac spinor $\widehat{\psi}_G$. Its pull-back to $K \subset G$, denoted by $\widehat{\psi}_K$, is thus a complex-valued pure spinor defining a (complex) Lagrangian complement $\widehat{F}_K \subset (\mathbb{T}K)^{\mathbb{C}}$.

Given a q-Hamiltonian K-space $(M, \mathscr{A}_M, \Phi, \omega)$, the complex differential form $\exp(\omega)\Phi^*\widehat{\psi}_K$ is related to $\exp(\omega)\Phi^*\psi_K$ by the r-matrix,

$$\exp(\omega)\Phi^*\widehat{\psi}_K = \exp(-\iota(\mathscr{A}_M(\mathfrak{r})))\Big(\exp(\omega)\Phi^*\psi_K\Big).$$

Since $\exp(-\iota(\mathscr{A}_M(\mathfrak{r})))$ does not affect the top degree part, it follows that we can write our volume form also in terms of $\widehat{\psi}_K$:

$$\mu_M = \Big(\exp(\omega)\Phi^*\widehat{\psi}_K\Big)^{[\dim M]}.$$

Remark 5.18. — Let $\tilde{F}_M$ be the backward image of $\widehat{F}_K$ under the strong Dirac morphism $(\Phi, \omega)\colon (M, TM, 0) \to (K, E_K, \eta)$. Since $\tilde{F}_M$ is transverse to $TM^{\mathbb{C}}$, it is given by a graph of a (complex-valued) bivector π, and $H_-(TM^{\mathbb{C}}, \tilde{F}_M, \mu_M) \cong H_\pi(M) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi))$. A simple calculation shows that $X_\pi = 2\pi\sqrt{-1}\mathscr{A}_M(\rho)$ (where B is used to identify $\mathfrak{k}^* \cong \mathfrak{k}$).

The pure spinors $\phi_M = 1$ and ϕ_K satisfy $d\phi_M = 0$ and $(d + \eta)\phi_K = 0$. Hence, by Proposition 2.13 the map $e^{\omega}\Phi^*$ descends to Dirac cohomology, $H_-(E_K^{\mathbb{C}}, \widehat{F}_K, \mu_K) \to H_\pi(M)$. In particular, $\eth_-\widehat{\psi}_K = 0$ implies that $\exp(\omega)\Phi^*\widehat{\psi}_K$ is closed under $\mathrm{d} - 2\pi\sqrt{-1}\iota(\mathscr{A}_M(\rho))$. For M is compact, the class $[e^{\omega}\Phi^*\widehat{\psi}_K]$ in $H_\pi(M)$ is nonvanishing because its integral is $\int_M \mu_M > 0$.

Let $\Delta_\lambda \colon G \to \mathbb{C}$ be the holomorphic functions introduced in Section 4.5.

Proposition 5.19. — *For any dominant weight* λ, *the complex differential form* $\exp(\omega)\Phi^*(\Delta_\lambda \widehat{\psi}_K)$ *satisfies the differential equation*

$$(d - 2\pi\sqrt{-1}\iota(\mathcal{C}_M(\lambda+\rho)))\Big(\exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K)\Big) = 0. \tag{102}$$

Her B_K *is used to identify* $\mathfrak{k}^* \cong \mathfrak{k}$.

Proof. — This follows from the differential equation for the Gauss-Dirac spinor, Proposition 4.18, together with Remark 1.5(a). □

As remarked in [**6**], the orthogonal projection of $\dim V_\lambda \Delta_\lambda|_K$ to the K-invariant functions on K coincides with the irreducible character χ_λ of highest weight λ. Thus,

$$\begin{aligned}\int_M \exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K) &= \int_M |\mu_M|\Phi^*\Delta_\lambda \\ &= \int_K \Phi_*|\mu_M|\,\Delta_\lambda = (\dim V_\lambda)^{-1}\int_K \chi_\lambda\,\Phi_*|\mu_M|.\end{aligned}$$

On the other hand, by (102) the integral may be computed by localization [**12**] to the zeroes of the vector field $\mathcal{C}_M(\lambda+\rho)$. As shown in [**6**], the 2-form ω pulls back to symplectic forms $\omega_Z = \iota_Z^*\omega$ on the components Z of the zero set, and the restriction $\Phi_Z = \iota_Z^*\Phi$ takes values in T. Since $\iota_T^*(\Delta_\lambda\widehat{\psi}_K)(t) = t^{\lambda+\rho}$ for $t \in T$, one obtains the following formula for the Fourier coefficients of the q-Hamiltonian Duistermaat-Heckman measure:

$$\int_K \chi_\lambda \Phi_*|\mu_M| = \dim V_\lambda \sum_{Z \subset \mathcal{C}_M(\lambda+\rho)^{-1}(0)} \int_Z \frac{\exp(\omega_Z)(\Phi_Z)^{\lambda+\rho}}{\mathrm{Eul}(\nu_Z, 2\pi\sqrt{-1}(\lambda+\rho))}.$$

Here $\mathrm{Eul}(\nu_Z,\cdot)$ is the T-equivariant Euler form of the normal bundle. This formula was proved in [**6**], using a more elaborate argument. Taking $\lambda = 0$, one obtains a formula for the volume $\int_M |\mu_M|$ of M.

5.4. q-Hamiltonian q-Poisson $\mathfrak{g}$-manifolds. — Just as any symplectic 2-form determines a Poisson bivector π, any q-Hamiltonian G-manifold carries a distinguished bivector field π. However, since ω is not non-degenerate π is not simply obtained as an inverse, and also π is not generally a Poisson structure.

Suppose $(M, \mathcal{C}_M, \omega, \Phi)$ is a q-Hamiltonian $\mathfrak{g}$-manifold, or equivalently that (Φ,ω) is a strong Dirac morphism $(M, TM, 0) \to (G, E_G, \eta)$. Let $\widetilde{F} \subset \mathbb{T}M$ be the backward image of F_G under this Dirac morphism. It is a complement to TM, hence it is of the form $\widetilde{F} = \mathrm{Gr}_\pi$ for some $\mathfrak{g}$-invariant bivector field $\pi \in \mathfrak{X}^2(M)$. By Proposition 2.10(c), the Schouten bracket of this bivector field with itself satisfies

$$\tfrac{1}{2}[\pi,\pi]_{\mathrm{Sch}} = \mathcal{C}_M(\Xi). \tag{103}$$

Let $\mathsf{p}' \colon \mathbb{T}G \to E_G$ be the projection along F_G. Let $\{v_a\}$ and $\{v^a\}$ be bases of $\mathfrak{g}$ with $B(v_a, v^b) = \delta_a^b$. Then $\mathsf{p}'(x') = \sum_a \langle x, \mathsf{f}(v_a)\rangle \mathsf{e}(v^a)$ for all $x' \in \Gamma(\mathbb{T}G)$. For

$\alpha' \in \Omega^1(G) \subset \Gamma(\mathbb{T}G)$, we have $\langle \alpha', \mathsf{f}(v^a)\rangle = \frac{1}{2}\langle \alpha', v_a^L + v_a^R\rangle\ \mathsf{e}(v^a)$. Hence, (20) shows that

$$\pi^\sharp \Phi^* \alpha' = -\sum_a \Phi^* \langle \alpha', \tfrac{v_a^L + v_a^R}{2}\rangle\ \mathcal{A}_M(v^a), \quad \alpha' \in \Omega^1(G), \tag{104}$$

and, by (24), we have:

$$\operatorname{ran}(\mathcal{A}_M) + \operatorname{ran}(\pi^\sharp) = TM. \tag{105}$$

This last condition can be viewed as a counterpart to the invertibility of a Poisson bivector defined by a symplectic form. Dropping this condition, one arrives at the following definition:

***Definition* 5.20.** — [**1, 2**] A *q-Hamiltonian q-Poisson* $\mathfrak{g}$*-manifold* is a manifold M, together with a Lie algebra action $\mathcal{A}_M\colon \mathfrak{g} \to \mathfrak{X}(M)$, a $\mathfrak{g}$-invariant bivector field π, and a $\mathfrak{g}$-equivariant *moment map* $\Phi\colon M \to G$, such that conditions (103) and (104) are satisfied. If the $\mathfrak{g}$-action on M integrates to a G-action, such that π, Φ are G-equivariant, we speak of a *q-Hamiltonian q-Poisson* G*-manifold.*

***Example* 5.21.** — The basic example of a Hamiltonian Poisson G-manifold is provided by the coadjoint action on $M = \mathfrak{g}^*$, with $\pi = \pi_{\mathfrak{g}^*}$ the Kirillov bivector and moment map the identity map. Similarly, the quadruple $(G, \mathcal{A}_{\mathrm{ad}}, \pi_G, \mathrm{id})$, with π_G the bivector field (46), is a q-Hamiltonian q-Poisson G-manifold.

The techniques in this paper allow us to give a much simpler proof to the following theorem from [**14**]:

***Theorem* 5.22.** — *There is a 1-1 correspondence between q-Hamiltonian q-Poisson* $\mathfrak{g}$*-manifolds* $(M, \mathcal{A}_M, \pi, \Phi)$*, and Dirac manifolds* (M, E_M, η_M) *equipped with a strong Dirac morphism*

$$(\Phi, 0)\colon (M, E_M, \eta_M) \to (G, E_G, \eta). \tag{106}$$

Under this correspondence, $\operatorname{ran}(E_M) = \operatorname{ran}(\mathcal{A}_M) + \operatorname{ran}(\pi^\sharp)$.

Proof. — Suppose $(\Phi, 0)\colon (M, E_M, \eta_M) \to (G, E_G, \eta)$ is a strong Dirac morphism. Consider the bundle map $\mathfrak{a}\colon \Phi^* E_G \to TM$ defined by Φ (see Section 2.2). By Proposition 2.10(c), the vector fields $\mathcal{A}_M(\xi) = \mathfrak{a}(\mathsf{e}(\xi)) \in \mathfrak{X}(M)$ define a Lie algebra action of $\mathfrak{g}$ on M for which Φ is equivariant. Note also that since $\operatorname{ran}(\mathfrak{a}) \subset \operatorname{ran}(E_M)$, this action preserves the leaves $Q \subset M$ of E_M. In fact, the bundle E_M is $\mathfrak{g}$-invariant: If $E_M = \mathrm{Gr}_\omega$ this follows from the $\mathfrak{g}$-invariance of ω (see comment after Def. 5.1), and in the general case it follows since $E_M|_Q$ is invariant, for any leaf Q. Let F_M be the backward image of F_G under $(\Phi, 0)$, and $\pi \in \mathfrak{X}^2(M)$ be the bivector field defined by the splitting $\mathbb{T}M = E_M \oplus F_M$. Then π is $\mathfrak{g}$-invariant (since E_M, F_M are). Equation (103) follows from Proposition 2.10(d), while Equation (104) is a consequence of Theorem 1.20, Equation (20).

Conversely, given a quasi-Poisson $\mathfrak{g}$-manifold $(M, \mathcal{A}_M, \pi, \Phi)$, let $\mathfrak{a}\colon \Phi^* E_G \to TM$ be the bundle map given on sections by $\Phi^* \mathsf{e}(\xi) \mapsto \mathcal{A}_M(\xi)$. The $\mathfrak{g}$-equivariance of Φ

implies that $\Phi \circ \mathfrak{a} = \mathrm{pr}_{\Phi^*TG}|_{\Phi^*E_G}$. Theorem 1.20 provides a Lagrangian splitting $\mathbb{T}M = E_M \oplus F_M$ such that F_M is the backward image of F_G and E_G is the forward image of E_M. It remains to check the integrability condition of E_M relative to the 3-form $\eta_M = \Phi^*\eta$. Let $\Upsilon^E \in \Gamma(\wedge^3 F_M)$ be the Courant tensor of E_M. We have to show that $\Upsilon^E = 0$, or equivalently that $\Gamma(E_M)$ is closed under the η_M-twisted Courant bracket. Recall that E_M is spanned by the sections of two types:

$$\widehat{\mathscr{A}}_M(\xi) := \widehat{\mathfrak{a}}(\Phi^*\mathsf{e}(\xi)) = \mathscr{A}_M(\xi) \oplus \Phi^* B(\tfrac{\theta^L+\theta^R}{2}, \xi)$$

for $\xi \in \mathfrak{g}$, and sections $h(\alpha)$, for $\alpha \in \Omega^1(M)$, where the map h is defined as in (23), with $\mathbb{V}$ replaced with $\mathbb{T}M$, and with $\omega = 0$. Since $\widehat{\mathfrak{a}}$ is a comorphism of Lie algebroids (cf. Proposition 2.8), we have

$$[\![\widehat{\mathscr{A}}_M(\xi_1), \widehat{\mathscr{A}}_M(\xi_2)]\!]_{\eta_M} = \widehat{\mathscr{A}}_M([\xi_1, \xi_2]). \tag{107}$$

Furthermore, since π is $\mathfrak{g}$-invariant, it follows from (23) that the map h is $\mathfrak{g}$-equivariant, and therefore

$$[\varrho(h(\alpha)), [\varrho(\widehat{\mathscr{A}}_M(\xi)), \mathrm{d} + \Phi^*\eta]] = [\varrho(h(\alpha)), \mathscr{L}(\mathscr{A}_M(\xi))] = -\varrho(h(\mathscr{L}(\mathscr{A}_M(\xi)\alpha)).$$

Thus

$$[\![\widehat{\mathscr{A}}_M(\xi), h(\alpha)]\!]_{\eta_M} = h(\mathscr{L}(\mathscr{A}_M(\xi))\alpha) \tag{108}$$

by definition of the Courant bracket. Equations (107) and (108) show that $[\![\widehat{\mathscr{A}}_M(\xi), \cdot]\!]_{\eta_M}$ preserves $\Gamma(E_M)$. Thus $\Upsilon^E(x_1, x_2, x_3)$ vanishes if one of the three sections $x_i \in \Gamma(E_M)$ lies in the range of $\widehat{\mathscr{A}}_M$. It remains to show that $\Upsilon^E(h(\alpha_1), h(\alpha_2), h(\alpha_3)) = 0$ for all 1-forms α_i, or equivalently that $h^*\Upsilon^E = 0$, where $h^* \colon F_M \to TM$ is the dual map to $h \colon T^*M \to E_M = F_M^*$. Since $h = \mathsf{p}|_{T^*M}$, where $\mathsf{p} : \mathbb{T}M \to E_M$ is the projection along F_M (see (23)), we have $h^* = \mathrm{pr}_{TM}|_{F_M}$. Thus, we must show that $\mathrm{pr}_{TM}\,\Upsilon^E = 0$. By Proposition 2.10(b), and the defining property of q-Hamiltonian q-Poisson spaces, we have

$$\mathrm{pr}_{TM}(\Upsilon^F) = \mathfrak{a}(\Phi^*\Upsilon^{F_G}) = \mathscr{A}_M(\Xi) = \tfrac{1}{2}[\pi, \pi]_{\mathsf{Sch}}.$$

On the other hand, Theorem 2.9(a) gives $\mathrm{pr}_{TM}(\Upsilon^E) + \mathrm{pr}_{TM}(\Upsilon^F) - \frac{1}{2}[\pi, \pi]_{\mathsf{Sch}} = 0$. Taking the two results together, we obtain $\mathrm{pr}_{TM}(\Upsilon^E) = 0$ as desired. □

As an immediate consequence, the data $(M, \mathscr{A}_M, \pi, \Phi)$ defining a q-Hamiltonian q-Poisson G-manifold are equivalent to the data of a G-equivariant Dirac manifold (M, E_M, η_M), equipped with a G-equivariant Dirac morphism $(\Phi, 0)$, for which the G-action on M integrates the $\mathfrak{g}$-action defined by the Dirac morphism.

***Proposition 5.23* (Fusion).** — *Suppose $(M, \mathscr{A}_M, \pi, \Phi)$ is a q-Hamiltonian q-Poisson $\mathfrak{g} \times \mathfrak{g}$-manifold. Let $\mathscr{A}_{\mathrm{fus}}$ be the diagonal $\mathfrak{g}$-action, $\Phi_{\mathrm{fus}} = \mathrm{Mult} \circ \Phi$, and $\pi_{\mathrm{fus}} = \pi + \mathscr{A}_M(\gamma)$. Then $(M, \mathscr{A}_{\mathrm{fus}}, \pi_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$ is a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold.*

Proof. — By Theorem 5.22, the given q-Poisson $\mathfrak{g} \times \mathfrak{g}$-manifold corresponds to a Dirac manifold (M, E_M, η_M) such that $(\Phi, 0)$ is a Dirac morphism into $(G, E_G, \eta) \times (G, E_G, \eta)$. Thus, $\eta_M = \Phi^*(\eta_G^1 + \eta_G^2)$. The bivector field π is defined by the Lagrangian splitting $\mathbb{T}M = E_M \oplus F_M$, where F_M is the backward image of $F_G^1 \oplus F_G^2$ under $(\Phi, 0)$. Composing with $(\mathrm{Mult}, \varsigma)$ (cf. Thm. 3.9), we obtain a strong Dirac morphism,

$$(\Phi_{\mathrm{fus}}, \Phi^*\varsigma)\colon (M, E_M, \eta_M) \to (G, E_G, \eta),$$

which in turn defines a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold. Let $\widetilde{F}_M$ be the backward image of F_G under this Dirac morphism. By Proposition 3.11, $\widetilde{F}$ is related to F by the section $\widehat{\mathscr{A}}_M(\gamma) \in \Gamma(\wedge^2 E_M)$, where $\widehat{\mathscr{A}}_M\colon \mathfrak{g} \times \mathfrak{g} \to E_M$ is the map defined by the Dirac morphism $(\Phi, 0)$. Hence, by Proposition 1.18, the bivector for the new splitting $\mathbb{T}M = E_M \oplus \widetilde{F}_M$ is $\pi_{\mathrm{fus}} = \pi + \mathscr{A}_M(\gamma)$. □

***Proposition 5.24* (Exponentials).** — *Suppose $(M, \mathscr{A}_M, \pi_0, \Phi_0)$ is a Hamiltonian Poisson $\mathfrak{g}$-manifold. That is, $\mathscr{A}_M$ is a $\mathfrak{g}$-action on M, π_0 is a $\mathfrak{g}$-invariant Poisson structure, and $\Phi_0\colon M \to \mathfrak{g}$ is a $\mathfrak{g}$-equivariant moment map generating the given action on M. Assume that $\Phi_0(M) \subset \mathfrak{g}_\natural$, and let*

$$\Phi = \exp \circ \Phi_0, \qquad \pi = \pi_0 + \mathscr{A}_M(\Phi_0^*\varepsilon)$$

where $\varepsilon \in C^\infty(\mathfrak{g}_\natural, \wedge^2\mathfrak{g})$ is the solution of the CDYBE defined in Section 3.5. Then $(M, \mathscr{A}_M, \pi, \Phi)$ is a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold.

Proof. — It is well-known that $(M, \mathscr{A}_M, \pi_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$-manifold if and only if $\Phi_0\colon M \to \mathfrak{g}^*$ is a Poisson map, i.e., if and only if

$$(\Phi_0, 0)\colon (M, E_M, 0) \to (\mathfrak{g}^*, E_{\pi_{\mathfrak{g}^*}}, 0)$$

is a strong Dirac morphism, with $E_M = \mathrm{Gr}_{\pi_0}$ and $E_{\mathfrak{g}^*} = \mathrm{Gr}_{\pi_{\mathfrak{g}^*}}$. Using B to identify $\mathfrak{g}^* \cong \mathfrak{g}$, and composing with the strong Dirac morphism $(\exp, \varpi)$, one obtains the strong Dirac morphism

$$(\Phi, \Phi_0^*\varpi)\colon (M, E_M, 0) \to (G, E_G, \eta),$$

which in turn gives rise to a q-Hamiltonian q-Poisson $\mathfrak{g}$-manifold $(M, \mathscr{A}_M, \pi, \Phi)$. The backward image $\widetilde{F}_M \subset \mathbb{T}M$ of F_G under the Dirac morphism $(\Phi, \Phi_0^*\varpi)$ is a Lagrangian complement to $E_M = \mathrm{Gr}_\pi$. Let $\widehat{\mathfrak{a}}\colon \Phi_0^*E_{\mathfrak{g}} \to E_M$ be defined by the Dirac morphism $(\Phi_0, 0)$, and put $\widehat{\mathscr{A}}_M(\xi) = \widehat{\mathfrak{a}} \circ \Phi_0^*\mathsf{e}_0(\xi)$. As explained in Section 3.5, $\widetilde{F}_M$ is related the Lagrangian complement $F_M = TM$ by the section $\widehat{\mathscr{A}}_M(\Phi_0^*\varepsilon)$. Hence, $\pi = \pi_0 + \mathscr{A}_M(\Phi_0^*\varepsilon)$. □

5.5. $\mathfrak{k}^*$-valued moment maps. — Let K be any Lie group. An ordinary Hamiltonian Poisson K-manifold is a triple (M, π, Φ) where M is a K-manifold, $\pi \in \mathfrak{X}^2(M)$ is an invariant Poisson structure, and $\Phi\colon M \to \mathfrak{k}^*$ is a K-equivariant map satisfying the moment map condition,

$$\pi^\sharp(\mathrm{d}\langle\Phi, \xi\rangle) = \mathscr{A}_M(\xi).$$

The moment map condition is equivalent to Φ being a Poisson map. The following result implies that $\mathfrak{k}^*$-valued moment maps can be viewed as special cases of $G = \mathfrak{k}^* \rtimes K$-valued moment maps. Let $\mathfrak{g} = \mathfrak{k}^* \rtimes \mathfrak{k}$ carry the invariant inner product given by the pairing.

Proposition 5.25. — *The inclusion map $j\colon \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K = G$ is a strong Dirac morphism $(j, 0)$, as well as a backward Dirac morphism, relative to the Kirillov-Poisson structure on $\mathfrak{k}^*$ and the Cartan-Dirac structure on G. The backward image of F_G under this Dirac morphism is $F_{\mathfrak{k}^*} = T\mathfrak{k}^*$. The pure spinor ψ_G on $G = \mathfrak{k}^* \rtimes K$ satisfies*

$$j^*\psi_G = 1.$$

Proof. — The Cartan-Dirac structure E_G is spanned by the sections $\mathsf{e}(w)$ for $w = (\beta, \xi) \in \mathfrak{g}$, while $E_{\mathfrak{k}^*}$ is spanned by the sections $\mathsf{e}_0(\xi)$ for $\xi \in \mathfrak{k}$. The first part of the Proposition will follow once we show that $\mathsf{s}_0(\beta, \xi) \sim_{(j,0)} \mathsf{s}(\beta, \xi)$, i.e.

$$\mathsf{e}_0(\xi) \sim_{(j,0)} \mathsf{e}(\beta, \xi), \quad \mathsf{f}_0(\beta) \sim_{(j,0)} \mathsf{f}(\beta, \xi). \tag{109}$$

The vector field part of the first relation follows since the inclusion $j\colon \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K$ is equivariant for the conjugation action of $G = \mathfrak{k}^* \rtimes K$. (Here, the $\mathfrak{k}^*$-component of G acts trivially on $\mathfrak{k}^*$, while the K-component acts by the co-adjoint action.) For the 1-form part, we note that the pull-back of the Maurer-Cartan forms $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ to the subgroup $\mathfrak{k}^* \subset G$ is the Maurer-Cartan form for additive group $\mathfrak{k}^*$, i.e.

$$j^*\theta^L = j^*\theta^R = \theta_0$$

where the 'tautological 1-form' $\theta_0 \in \Omega^1(\mathfrak{k}^*) \otimes \mathfrak{k}^*$ is defined as in Section 3.5. Thus

$$j^*B\big(\tfrac{\theta_G^L + \theta_G^R}{2}, (\beta, \xi)\big) = B(\theta_0, (\beta, \xi)) = \langle \theta_0, \xi \rangle.$$

This verifies the first relation in (109); the second one is checked similarly.

Since the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$ is trivial over $\mathfrak{k}^*$, the lift $\tau\colon G \to \mathrm{Pin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})$ satisfies $\tau|_{\mathfrak{k}^*} = 1$. It follows that the pure spinor $\psi_G = \mathscr{R}(q(\mu))$ satisfies $j^*\psi_G = 1$. □

Corollary 5.26. — *Let (M, π) be a Poisson manifold. Then $\Phi\colon M \to \mathfrak{k}^*$ is a Poisson map if and only if the composition $j \circ \Phi\colon M \to G$ is a strong Dirac morphism*

$$(j \circ \Phi, 0)\colon (M, \mathrm{Gr}_\pi, 0) \to (G, E_G, \eta).$$

Put differently, Hamiltonian Poisson K-manifolds are q-Hamiltonian q-Poisson $\mathfrak{k}^* \rtimes K$-manifolds for which the moment map happens to take values in $\mathfrak{k}^*$.

As a special case, a Hamiltonian K-manifold (M, ω, Φ) (with ω a symplectic 2-form, and Φ satisfying the moment map condition $\iota(\mathscr{C}_M(\xi))\omega = \mathrm{d}\langle \Phi, \xi \rangle$) is equivalent to a q-Hamiltonian $G = \mathfrak{k}^* \rtimes K$-space for which the moment map takes values in $\mathfrak{k}^*$. Since $j^*\psi_G = 1$, its q-Hamiltonian volume form coincides with the usual Liouville form $(\exp \omega)^{[\mathrm{top}]}$.

6. K^*-valued moment maps

For a Poisson Lie group K, J.-H. Lu [**42**] introduced another type of group-valued moment map, taking values in the dual Poisson Lie group K^*. For a compact Lie group K, with its standard Poisson structure, this moment map theory turns out to be equivalent to the usual $\mathfrak{k}^*$-valued one. In this Section, we will re-examine this equivalence using the techniques developed in this paper.

6.1. Review of K^*-valued moment maps. — The theory of Poisson-Lie groups were introduced by Drinfeld in [**23**], see e.g. [**18**] for an overview and bibliography. Suppose K is a connected Poisson Lie group, with Poisson structure defined by a Manin triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}')$. (That is, $\mathfrak{g}$ is a Lie algebra with an invariant split inner product, and $\mathfrak{k}, \mathfrak{k}'$ are complementary Lagrangian subalgebras.) Use the paring to identify $\mathfrak{k}' = \mathfrak{k}^*$, and let K^* be the associated dual Poisson Lie group. We assume that $\mathfrak{g}$ integrates to a Lie group G (the double) such that K, K^* are subgroups and the product map $K \times K^* \to G$ is a diffeomorphism. The left action of K on G descends to a *dressing action* $\mathscr{A}_{K^*}$ on K^* (viewed as a homogeneous space G/K). The Poisson structure on K^*, or equivalently its graph $E_{K^*} = \mathrm{Gr}_{\pi_{K^*}} \subset \mathbb{T}K^*$, may be expressed in terms of the infinitesimal dressing action, as the span of sections

$$e_{K^*}(\xi) = \mathscr{A}_{K^*}(\xi) \oplus \langle \theta^R_{K^*}, \xi \rangle$$

for $\xi \in \mathfrak{k}$. Here $\theta^R_{K^*} \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ is the right-invariant Maurer-Cartan form for K^*. Note that as a Lie algebroid, E_{K^*} is just the action algebroid.

For the remainder of this Section 6, we will assume that K is a compact real Lie group. The *standard Poisson structure* on K is described as follows. Let $G = K^{\mathbb{C}}$ be the complexification, with Lie algebra $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad G = KAN$$

be the Iwasawa decompositions. Here $\mathfrak{a} = \sqrt{-1}\mathfrak{t}_K$, $A = \exp \mathfrak{a}$ and $N = N_+$ (using the notation from Section 3.6). We denote by B_K an invariant inner product on $\mathfrak{k}$, and let $\langle \cdot, \cdot \rangle$ be the imaginary part of $2B_K^{\mathbb{C}}$. Then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a} \oplus \mathfrak{n})$ (where $\mathfrak{g}$ is viewed as a real Lie algebra) is a Manin triple. Thus K becomes a Poisson Lie group, with dual Poisson Lie group $K^* = AN$.

A *K^*-valued Hamiltonian $\mathfrak{k}$-manifold*, as defined by Lu [**42**], is a symplectic manifold (M, ω) together with a Poisson map $\Phi \colon M \to K^*$. Equivalently, $(\Phi, \omega) \colon (M, TM, 0) \to (K^*, E_{K^*}, 0)$ is a strong Dirac morphism. The Poisson map Φ induces a $\mathfrak{k}$-action on M, and if this action integrates to an action of K we speak of a *K^*-valued Hamiltonian K-manifold.* An interesting feature is that ω is not K-invariant, in general: Instead, the action map $K \times M \to M$ is a Poisson map. Accordingly, the volume form $(\exp \omega)^{[\mathrm{top}]}$ is not K-invariant. However, let $\Phi^A \colon M \to A$ be the composition of Φ with projection $K^* = AN \to A$, and $(\Phi^A)^{2\rho} \colon M \to \mathbb{R}_{>0}$ its image under the homomorphism $T \to \mathbb{C}^\times$, $t \mapsto t^{2\rho}$ defined by the sum of positive roots. By [**7**, Theorem 5.1], the product

$$(\Phi^A)^{2\rho}\, (\exp \omega)^{[\mathrm{top}]} \tag{110}$$

is a K-invariant volume form. The proof in [**7**] uses a tricky argument; one of the goals of this Section is to give a more conceptual explanation.

6.2. P-valued moment maps.— To explain the origin of the volume form (110), we will use the notion of a P-valued moment map introduced in [**3**, Section 10]. Let $g \mapsto g^c$ denote the complex conjugation map on G, and let

$$I(g) \equiv g^\dagger = (g^{-1})^c.$$

On the Lie algebra level, let $\xi \mapsto \xi^c$ denote conjugation, and $\xi^\dagger = -\xi^c$. We have $K = \{g \in G|\ g^\dagger = g^{-1}\}$. Let

$$P = \{g^\dagger g|\ g \in G\}$$

denote the subset of 'positive definite' elements in G. Then P is a submanifold fixed under I, and the product map defines the Cartan decomposition $G = KP$. Let E_G be the (holomorphic) Dirac structure on G defined by the inner product

$$B := \tfrac{1}{\sqrt{-1}} B_K^{\mathbb{C}}.$$

Since $(\theta^L)^\dagger = I^*\theta^R,\ (\theta^R)^\dagger = I^*\theta^L$, the Cartan 3-form on G satisfies satisfies $\eta^c = I^*\eta$, thus $\eta_P := \iota_P^*\eta$ is real-valued. Similarly, the pull-backs of the 1-forms $B(\frac{\theta^L+\theta^R}{2}, \xi)$ for $\xi \in \mathfrak{k}$ are real-valued. It follows that the sections

$$\mathsf{e}_P(\xi) := \mathsf{e}(\xi)|_P$$

are real-valued. Letting $E_P \subset \mathbb{T}P$ be the subbundle spanned by these sections, it follows that (P, E_P, η_P) is a real Dirac manifold, with $(E_P)^{\mathbb{C}} = E_G|_P$. As a Lie algebroid, E_P is just the action algebroid for the K-action on P. Similarly, the sections $\mathsf{f}_P(\xi) := \mathsf{f}(\xi)|_P$ are real-valued, defining a complement F_P to E_P. The bundle F_P is defined by the (real-valued) pure spinor, $\psi_P := \iota_P^*\psi_G \in \Omega(P)$.

Remark 6.1. — Since $\det(\mathrm{Ad}_g + 1) > 0$ for $g \in P$ (all eigenvalues of Ad_g are strictly positive), one finds that $\ker(E_P) = \{0\}$. Hence E_P is the graph of a bivector π_P with $\frac{1}{2}[\pi_P, \pi_P] = \pi_P^\sharp(\eta_P)$.

A *P-valued Hamiltonian $\mathfrak{k}$-manifold* [**3**, Section 10] is a manifold M together with a strong Dirac morphism $(\Phi_1, \omega_1)\colon (M, TM, 0) \to (P, E_P, \eta_P)$. For any such space we obtain, as for the q-Hamiltonian setting, an invariant volume form

$$(\exp(\omega_1) \wedge \Phi_1^*\psi_P)^{[\mathrm{top}]}. \tag{111}$$

Here ψ_P may be replaced by $\widehat{\psi}_P$, the pull-back of the Gauss-Dirac spinor.(5) By Proposition 5.19, the expression $\exp\omega_1 \wedge \Phi_1^*(\Delta_\lambda\ \widehat{\psi}_P)$ is closed under the differential $\mathrm{d} - 2\pi\iota(\mathscr{C}_M(\lambda+\rho)))$, for any dominant weight λ.

(5) In Section 5.3, B was taken as the complexification of B_K, while here we have an extra factor $\sqrt{-1}$. This amounts to a simple rescaling of the bilinear form $B_K^{\mathbb{C}}$, not affecting any of the results.

6.3. Equivalence between K^*-valued and P-valued moment maps. — To relate the K^*-valued theory with the P-valued theory, we use the K-equivariant diffeomorphism

$$\kappa\colon K^* \to P,\ g \mapsto g^\dagger g.$$

Note that this map takes values in the big Gauss cell, $\mathcal{O} = N_-KN \subset G$. Let $\varpi_{\mathcal{O}}$ denote the (complex) 2-form on the big Gauss cell, and $\varpi_{K^*} = \kappa^*\omega_{\mathcal{O}}$. It is easy to check that ϖ_{K^*} is real-valued. One can check that

$$e_{K^*}(\xi) \sim_{(\kappa,\varpi_{K^*})} e_P(\xi)$$

for all $\xi \in \mathfrak{k}$: The vector field part of this relation is equivalent to the $\mathfrak{k}$-equivariance, while the 1-form part is verified in [**3**, Section 10]. It follows that (κ, ϖ_{K^*}) is a Dirac isomorphism from $(K^*, E_{K^*}, 0)$ onto (P, E_P, η_P).

Thus, if (M, ω, Φ) is a K^*-valued Hamiltonian $\mathfrak{k}$-manifold, then (M, ω_1, Φ_1) with $\omega_1 = \omega + \Phi^*\varpi_{K^*}$ and $\Phi_1 = \kappa \circ \Phi$ is a P-valued Hamiltonian $\mathfrak{k}$-manifold. In particular, we obtain an invariant volume form on M,

$$\big(\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\widehat{\psi}_P\big)^{[\mathrm{top}]}.$$

Using the explicit formula (Proposition 4.14) for the Gauss-Dirac spinor, we obtain

$$\kappa^*\widehat{\psi}_P = a^{2\rho}\exp(-\varpi_{K^*}),$$

where $a\colon K^* \to A$ is projection to the A-factor. Hence,

$$\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\psi_P = (\Phi^A)^{2\rho}\exp(\omega),$$

identifying the volume form for the associated P-valued space with the volume form (110).

Proposition 6.2. — *For any K^*-valued Hamiltonian $\mathfrak{k}$-space (M, ω, Φ), the volume form $(\Phi^A)^{2\rho}(\exp\omega)^{[\mathrm{top}]}$ is $\mathfrak{k}$-invariant. Moreover, for all dominant weights λ the differential form*

$$(\Phi^A)^{2(\lambda+\rho)}\exp(\omega)$$

is closed under the differential $d - 2\pi\mathcal{E}_M(B_K^\sharp(\lambda+\rho))$.

Proof. — Invariance follows from the identification with the volume form for the associated P-valued space. The second claim follows from Proposition 5.19, since the function Δ_λ from Section 4.5 satisfies $\kappa^*\Delta_\lambda = a^{2\lambda}$. □

The differential equation permits a computation of the integrals $\int_M (\Phi^A)^{2(\lambda+\rho)}(\exp(\omega))^{[\mathrm{top}]}$ by localization [**12**] to the zeroes of the vector field $\mathcal{E}_M(B_K^\sharp(\lambda+\rho))$, similar to the formula in 5.3.

6.4. Equivalence between P-valued and $\mathfrak{k}^*$-valued moment maps. — Finally, let us express the correspondence [**3**, Section 10] between P-valued moment maps and $\mathfrak{k}^*$-valued moment maps in terms of Dirac morphisms. The exponential map for $G = K^{\mathbb{C}}$ restricts to a diffeomorphism

$$\exp_{\mathfrak{p}}\colon \mathfrak{p} := \sqrt{-1}\mathfrak{k} \to P := \exp(\sqrt{-1}\mathfrak{k}).$$

Let $\varpi \in \Omega^2(\mathfrak{g})$ be the primitive of $\exp^*\eta$ defined in (58), and $\varpi_{\mathfrak{p}}$ its pull-back to $\mathfrak{p}$. Since η_P is real-valued, so is $\varpi_{\mathfrak{p}}$, and $d\varpi_{\mathfrak{p}} = (\exp|_{\mathfrak{p}})^*\eta_P$. Similarly, $J_{\mathfrak{p}} := J|_{\mathfrak{p}} > 0$. The formulas for $\varpi_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ are similar to those for the Lie algebra $\mathfrak{k}$, but with sinh functions replaced by sin functions. Use $B^\sharp = \sqrt{-1}B_K^\sharp$ to identify $\mathfrak{k}^* \cong \mathfrak{p}$. By Proposition 3.12,

$$\mathsf{e}_0(\xi) \sim_{(\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}})} \mathsf{e}_P(\xi), \quad \xi \in \mathfrak{k}.$$

Hence $(\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}})$ is a Dirac (iso)morphism from $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$ to (P, E_P, η_P). This sets up a 1-1 correspondence between P-valued and $\mathfrak{k}^*$-valued Hamiltonian $\mathfrak{k}$-spaces. Thinking of the latter as given by strong Dirac morphisms (Φ_0, ω_0) to $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$, the correspondence reads

$$(\Phi_1, \omega_1) = (\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}}) \circ (\Phi_0, \omega_0).$$

The volume forms are related by $(\exp(\omega_1) \wedge \Phi_1^*\psi_P)^{[\mathrm{top}]} = J_{\mathfrak{p}}^{1/2} \exp(\omega_0)^{[\mathrm{top}]}$.

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