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Mots-clefs. — Pseudo-caractère, lieu de réductibilité, représentation galoisienne, trianguline, déformation p-adique, variété de Hecke, groupe de Selmer, conjectures de Bloch-Kato, groupe unitaire, représentation automorphe, conjecture d'Arthur.

Ce livre est dédié à la mémoire de Serge Bellaïche, frère et ami.

FAMILIES OF GALOIS REPRESENTATIONS AND SELMER GROUPS

Joël BELLAÏCHE & Gaëtan CHENEVIER

Abstract. — This book presents an in-depth study of the families of Galois representations carried by the p-adic eigenvarieties attached to unitary groups. The study encompasses some general algebraic aspects (properties of the space of representations of a group in the neighbourhood of a point, reducibility loci, pseudocharacters), and other aspects more specific to Galois groups of local or number fields. In particular, we define and study certain deformation functors of crystalline representations of the absolute Galois group of $\mathbb{Q}_p$, namely *trianguline deformations*, which are naturally associated to the families above. As an application, we show how the geometry of these eigenvarieties at "classical" points is related to the dimension of certain Selmer groups. This, combined with conjectures of Langlands and Arthur on the discrete automorphic spectrum of unitary groups, allows us to prove, amongst other things, new cases of the Bloch-Kato conjectures (in any dimension).

***Résumé* (Familles de représentations galoisiennes et groupes de Selmer).** — Ce livre présente une étude approfondie des familles de représentations galoisiennes portées par les variétés de Hecke p-adiques des groupes unitaires. Cette étude comprend des aspects algébriques généraux (propriétés de l'espace des représentations d'un groupe au voisinage d'un point, lieux de réductibilité, pseudo-caractères), et d'autres plus spécifiques aux groupes de Galois des corps locaux ou des corps de nombres. Nous définissons et étudions notamment certains foncteurs de déformations des représentations cristallines du groupe de Galois absolu de $\mathbb{Q}_p$ (*déformations triangulines*) qui sont naturellement associés aux familles ci-dessus. En guise d'application, nous montrons comment la géométrie de ces variétés de Hecke aux points « classiques » est reliée à la dimension de certains groupes de Selmer. Ceci, conjugué aux conjectures de Langlands et Arthur sur le spectre automorphe discret des groupes unitaires, nous permet entre autres de démontrer de nouveaux cas des conjectures de Bloch-Kato (en toute dimension).

CONTENTS

INTRODUCTION

This book [1] takes place in a now thirty years long trend of researches, initiated by Ribet ([**96**]) aiming at constructing "arithmetically interesting" non trivial extensions between global Galois representations (either on finite p^∞-torsion modules, or on p-adic vector spaces) or, as we shall say, non-zero elements of Selmer groups, by studying congruences or variations of automorphic forms. As far as we know, despite of its great successes (to name one: the proof of Iwasawa's main conjecture for totally real fields by Wiles [**123**]), this current of research has never established, in any case, the existence of *two linearly independent* elements in a Selmer group—although well-established conjectures predict that sometimes such elements should exist. [2] The final aim of the book is, focusing on the characteristic zero case, to understand the conditions under which, by this kind of method, existence of two or more independent elements in a Selmer space could be proved.

To be somewhat more precise, let G be reductive group over a number field. We assume, to fix ideas, that the existence of the p-adic rigid analytic eigenvariety $\mathcal{E}$ of G, as well as the existence and basic properties of the Galois representations attached to algebraic automorphic forms of G are known [3]. Thus $\mathcal{E}$ carries a family of p-adic Galois representations. Our main result takes the form of a numerical relation between the dimension of the tangent space at suitable points $x \in \mathcal{E}$ and the dimension of the

(1) During the elaboration and writing of this book, Joël Bellaïche was supported by the NSF grant DMS-05-01023. Gaëtan Chenevier would like to thank the C.N.R.S. for their support, as well as the I.H.É.S. for their hospitality during part of this work.

(2) By a very different approach, let us mention here that the parity theorem of Nekovar [**89**] shows, in the *sign +1* case and for p-ordinary modular forms, that the rank of the Selmer group is at least 2 if nonzero (see also [**90**, Chap. 12] for a more general result concerning potentially ordinary Hilbert modular forms).

(3) Besides GL_2 over a totally real field and its forms, the main examples in the short term of such G would be suitable unitary groups and suitable forms of GSp_4. Concerning unitary groups in $m \geq 4$ variables, it is one of the goals of the book project of the GRFA seminar [**60**] to construct the expected Galois representations, which makes the assumption relevant. All our applications to unitary eigenvarieties for such groups (hence to Selmer groups) will be conditional to their work. However, thanks to Rogawski's work and [**78**], everything concerning $U(3)$ will be unconditional.

part of Selmer groups of components of adρ_x that are "seen by $\mathcal{E}$", where ρ_x is the Galois representation carried by $\mathcal{E}$ at the point x.

Such a result can be used both ways: if the Selmer groups are known, and small, it can be used to study the geometry of $\mathcal{E}$ at x, for example (see [**73**], [**9**]) to prove its smoothness. On the other direction, it can be used to get a lower bound on the dimension of some interesting Selmer groups, lower bound that depends on the dimension of the tangent space of $\mathcal{E}$ at x. An especially interesting case is the case of unitary groups with $n+2$ variables, and of some particular points $x \in \mathcal{E}$ attached to non-tempered automorphic forms (4). These forms were already used in [**5**] for a unitary group with three variables, and later for GSp_4 in [**112**], and for U(3) again in [**8**]. At those points, the Galois representation ρ_x is a sum of an irreducible n-dimensional representation (5) ρ, the trivial character, and the cyclotomic character χ. The representations ρ we could obtain this way are, at least conjecturally, all irreducible n-dimensional representations satisfying some selfduality condition, and such that the order of vanishing of $L(\rho, s)$ at the center of its functional equation is odd. Our result then gives a lower bound on the dimension of the Selmer group of ρ. Let us call Sel(ρ) this Selmer group (6). This lower bound implies, in any case, that Sel(ρ) is non zero (which is predicted by the Bloch-Kato conjecture), and if $\mathcal{E}$ is non smooth at x, that the dimension of Sel(ρ) is at least 2.

This first result (the non-triviality of Sel(ρ), proved in chapter 8) extends to any dimension n a previous work of the authors [**8**] in which they proved that Sel(ρ) $\neq 0$ in the case $n = 1$, i.e. $G = \mathrm{U}(3)$, and the work of Skinner-Urban [**112**] in the case $n = 2$ and ρ ordinary. Moreover, the techniques developed in this paper shed also much light on those works. For example, the arguments in [**8**] to produce a non trivial element in Sel(ρ) involved some arbitrary choice of a germ of irreducible curve of $\mathcal{E}$ at x, and it was not clear in which way the resulting element depended on this choice. With our new method, we do not have to make such a choice and we construct directly a canonical subvector space of Sel(ρ). Let us mention here that while the second half of this book was being written, other special cases of Bloch-Kato's conjectures have been announced by Skinner-Urban in [**114**].

In order to prove our second, main, result (the lower bound on dim(Sel(ρ)), see chapter 9) we study the reducibility loci of the family of Galois representations on $\mathcal{E}$. An original feature of the present work is that we focus on points $x \in \mathcal{E}$ at which the Galois deformation at p is as non trivial as possible (we call some of them *anti-ordinary*

(4) In general, their existence is predicted by Arthur's conjectures and known in some cases.
(5) Say, of the absolute Galois group of a quadratic imaginary field.
(6) Precisely, this is the group usually denoted by $H^1_f(E, \rho)$.

points) [7]. We discovered that at these points, the local Galois deformation is highly irreducible, that is not only generically irreducible [8], but even irreducible on every proper artinian thickening on the point x inside $\mathcal{E}$ (recall however that $\rho_x = 1 \oplus \chi \oplus \rho$ is reducible). In other words, the reducibility locus of the family is schematically equal to the point x. It should be pointed out here that the situation is quite orthogonal to that for Iwasawa's main conjecture (see [**86**], [**123**]), for which there is a big known part in the reducibility locus at x (the Eisenstein part), and this locus cannot be controlled *a priori*. In our case, this fact turns out to be one of the main ingredients in order to get some geometric control on the size of the subspace we construct in $\mathrm{Sel}(\rho)$, and it is maybe the main reason why our points x are quite susceptible to produce independent elements in $\mathrm{Sel}(\rho)$.

The question of whether we should expect that this method constructs the full Selmer group of ρ at x remains a very interesting mystery, whatever the answer may be. Although it might not be easy to decide this even in explicit examples (say with $L(\rho, s)$ vanishing at order > 1 at its center), our geometric criterion reduces this question to some computations of spaces of p-adic automorphic forms on explicit definite unitary groups, which should be feasible. Last but not least, we hope that it may be possible to relate the geometry of $\mathcal{E}$ at x (which is built on spaces of p-adic automorphic forms) to the L-function (or rather a p-adic L-function) of ρ, so that our results could be used in order to prove the "lower bound of Selmer group" part of the Bloch-Kato conjecture. However, this is beyond the scope of this book.

The four first chapters form a detailed study of p-adic families of Galois representations, especially near reducible points, and how their behavior is related to Selmer groups. There are no references to automorphic forms in them, in contrast with the following chapters 5 to 9 which are devoted to the applications to eigenvarieties. In what follows, we very briefly describe the contents of each of the different chapters by focusing on the way they fit in the general theme of the book. As they contain a number of results of independent interest, we invite the reader to then consult their respective introductions for more details.

When we deal with families of representations $(\rho_x)_{x \in X}$ of a group G (or an algebra) over a "geometric" space X, there are two natural notions to consider. The most obvious one is the datum of a "big" representation of G on a locally free sheaf of $\mathcal{O}_X$-modules whose evaluation at each $x \in X$ is ρ_x. Another one, visibly weaker, is

(7) A bit more precisely, among the (finite number of) points of $x \in \mathcal{E}$ having the same Galois representation ρ_x, we choose one which is *refined* in quite a special way.

(8) Note that although we do assume in the applications of this paper to Selmer groups the existence of Galois representations attached to algebraic automorphic forms on $\mathrm{U}(m)$ with $m \geq 4$, we do not assume that the expected ones are irreducible, but instead our arguments prove this irreducibility for some of them.

the datum of a "trace map" $G \longrightarrow \mathcal{O}(X)$ whose evaluation at each $x \in X$ is $\operatorname{tr}(\rho_x)$; these abstract traces are then called *pseudocharacters* (or pseudorepresentations). As a typical example, the parameter space of isomorphism classes of semisimple representations of G usually only carries a universal family in the sense of traces. This is what happens also for the family of Galois representations on the eigenvarieties. When all the ρ_x are irreducible, the two definitions turn out to be essentially the same, but the links between them are much more subtle around a reducible ρ_x and they are related to the extensions between the irreducible constituents of ρ_x, our object of interest.

Thus our first chapter is a general study of pseudocharacters T over a henselian local ring A (having in view that the local rings of a rigid analytic space are henselian). There is no mention of a Galois group in all this chapter, and those results can be applied to any group or algebra. Most of our work is based on the assumption that the residual pseudocharacter $\bar{T}$ (that is, the pseudocharacter one gets after tensorizing T by the residue field of A) is without multiplicity, so it may be reducible, which is fundamental, but all its components appear only once. Under this hypothesis, we prove a precise structure theorem for T, describe the groups of extensions between the constituents of $\bar{T}$ we can construct from T, and define and characterize the *reducibility loci* of T (intuitively the subscheme of $\operatorname{Spec} A$ where T has a given reducibility structure). We also discuss conditions under which T is, or cannot be, the trace of a true representation. This chapter provides the framework for many of our subsequent results.

In the second chapter we study infinitesimal (that is, artinian) families of p-adic local Galois representations, and their Fontaine theory, with the purpose of characterizing abstractly those coming from eigenvarieties. A key role is played by the theory of (φ, Γ)-modules over the Robba ring and Colmez' notion of *trianguline representation* [**46**]. We generalize some results of Colmez to any dimension and with artinian coefficients, giving in particular a fairly complete description of the trianguline deformation space of a *non critical* trianguline representation (of any rank). For the applications to eigenvarieties, we also give a criterion for an infinitesimal family to be trianguline in terms of crystalline periods.

In the third chapter, we generalize a recent result of Kisin in [**73**] on the analytic continuation of crystalline periods in a family of local Galois representations. This result was proved there for the strong definition of families, namely for true representations of $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a locally free $\mathcal{O}$-module, and we prove it more generally for any torsion free coherent $\mathcal{O}$-module. Our main technical tool is a method of descent by blow-up of crystalline periods (which turns out to be rather general) and a reduction to Kisin's case by a flatification argument.

In the fourth chapter, we give our working definition of "p-adic families of refined Galois representations", motivated by the families carried by eigenvarieties, and we apply to them the results of chapters 2 and 3. In particular, we are able in favorable cases to understand their reducibility loci in terms of the Hodge-Tate-Sen weight maps, and to prove that they are infinitesimally trianguline.

In the fifth chapter we discuss our main motivating conjecture relating the dimension of Selmer groups of geometric semi-simple Galois representations to the order of the zeros of their L-functions at integers. We are mainly interested in "one half" of this conjecture, namely, giving a lower bound on the dimension of the Selmer groups, as well as in a very special case of it that we call the *sign conjecture*. As was explained in [**5**], an important feature of the method we use is that we need as an input some results (supposedly simpler) about upper bounds of other Selmer groups. For the sign conjecture, we only need the vanishing of $\mathrm{Sel}(\chi)$ (for a quadratic imaginary field) which is elementary. However, we need more "upper bounds results" for our second main theorem, and we cannot prove all of them in general. Thus we formulate as hypotheses the results we shall need, which will appear as assumptions in the results of chapter 9. Using results of Kisin and Kato, we are able to prove all that we need in most cases when $n = 2$, and in all the cases for $n = 1$.

The sixth chapter contains all the results we need about the unitary groups, their automorphic forms, and the Galois representations attached to them. In particular, we formulate there the two hypotheses ($\mathrm{AC}(\pi)$) and ($\mathrm{Rep}(m)$) that we use in chapters 8 and 9. This chapter may be read in conjonction with the appendix of this book, which is a detailled discussion of Langlands and Arthur's conjectures.

In the seventh chapter, we introduce and study in details the eigenvarieties of definite unitary groups and we prove the basic properties of the (sometimes conjectural) family of Galois representations that they carry. We essentially rely on the thesis of one of us [**36**] and actually go a bit further on several respects. Eigenvarieties furnish a lot of interesting examples where all the concepts studied in this book occur, and provides also an important tool for the applications to Selmer groups. The first half of this chapter only concerns eigenvarieties and may be read independently, whereas the second one depends on chapters 1 to 4.

Finally, in chapters 8 and 9 we prove our main results, and we refer to those chapters for precise statements.

The first four chapters of this book appeared as a preprint on the ArXiV on February 2006. The book was made available in full, there, on January 2007. The few important additions made in the final revision completed in October 2008 are explicitely mentionned in the core of the text.

We made considerable efforts all along the redaction of this book to develop concepts and techniques adapted to study eigenvarieties. We hope that the reader will enjoy playing with them as much as we did.

The Leitfaden is as follows:

$$\begin{tikzcd}
 & 1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] & \\
5 \arrow[ddrr] & & 4 \arrow[dr] & & 6 \arrow[dl] \\
 & & & 7 \arrow[dl] & \\
 & & 8 \arrow[d] & & \\
 & & 9 & &
\end{tikzcd}$$

It is a pleasure to thank Laurent Berger, Laurent Clozel, Pierre Colmez, Brian Conrad, Jean-Francois Dat, Michael Harris, Guy Henniart, Kiran Kedlaya, Colette Moeglin and Claudio Procesi for many useful conversations. We are especially grateful to Laurent Clozel for his careful reading of the whole manuscript and innumerable remarks. We thank Barry Mazur and Richard Taylor for giving us the opportunity to explain the results of this book during the "Eigenvarieties semester" of Harvard University in spring 2006, as well as the Fondation Peccot and the Collège de France, where the second author exposed a part of these results on March-April 2008.

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CHAPTER 1

PSEUDOCHARACTERS, REPRESENTATIONS AND EXTENSIONS

1.1. Introduction

This section is devoted to the local study (in the sense of the étale topology) of pseudocharacters T satisfying a residual multiplicity freeness hypothesis. Two of our main objectives are to determine when those pseudocharacters come from a true representation and to prove the optimal generalization of "Ribet's lemma" for them.

Let us specify our main notations and hypotheses. Throughout this section, we will work with a pseudocharacter $T : R \longrightarrow A$ of dimension d, where A is a *local henselian* commutative ring of residue field k where $d!$ is invertible and R a (not necessarily commutative) A-algebra (1). To formulate our residual hypothesis, we assume (2) that $T \otimes k : R \otimes k \longrightarrow k$ is the sum of r pseudocharacters of the form $\mathrm{tr}\,\bar{\rho}_i$ where the $\bar{\rho}_i$'s are absolutely irreducible representations of $R \otimes k$ defined over k. Our *residually multiplicity free* hypothesis is that the $\bar{\rho}_i$'s are two by two non isomorphic. In this context, "Ribet's lemma" amounts to determining how much we can deduce about the existence of non-trivial extensions between the representations $\bar{\rho}_i$ from the existence and irreducibility properties of T. Before explaining our work and results in more details, let us recall the history of those two interrelated themes: pseudocharacters and the generalizations of "Ribet's lemma".

We begin with the original Ribet's lemma ([**96**, Prop. 2.1]). Ribet's hypotheses are that $d = r = 2$, and that A is a complete discrete valuation ring. He works with a representation $\rho : R \longrightarrow M_2(A)$, but that is no real supplementary restriction since every pseudocharacter over a complete strictly local (3) discrete valuation ring is the

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(1) See § 1.2.1. for the definition of a pseudocharacter. In the applications, R will be the group algebra $A[G]$ where G a group, especially a Galois group. However, it is important to keep this degree of generality as most of the statements concerning pseudocharacters are ring theoretic.

(2) Since we are not interested here in the field of coefficients of our representations and extensions, we may replace k by a separable extension, so this assumption is actually not a restriction.

(3) A local ring is said to be *strictly* local if its residue field is separably closed.

trace of a true representation [4]. Ribet proves that if $\rho \otimes K$ (K being the fraction field of A) is irreducible, then a non-trivial extension of $\bar{\rho}_1$ by $\bar{\rho}_2$ (resp. of $\bar{\rho}_2$ by $\bar{\rho}_1$) arises as a subquotient of ρ. This seminal result suggests numerous generalizations: we may wish to weaken the hypotheses on the dimension d, the number of residual factors r, the ring A, and for more general A, to work with general pseudocharacters instead of representations. We may also wonder if we can obtain, under suitable hypotheses, extensions between deformations ρ_1 and ρ_2 of $\bar{\rho}_1$ and $\bar{\rho}_2$ over some suitable artinian quotient of A, not only over k.

A big step forward is made in the papers by Mazur-Wiles and Wiles ([**86**], [**123**]) on Iwasawa's main conjecture. As their work is the primary source of inspiration for this section, let us explain it with some details (our exposition owes much to [**58**]; see also [**9**, §2]). They still assume $d = r = 2$, but the ring A now is any finite flat reduced local A_0-algebra A, where A_0 is a complete discrete valuation ring. Though the notion of pseudocharacter at that time was still to be defined, their formulation amounts to considering a pseudocharacter (not necessarily coming from a representation) $T : R \longrightarrow A$, where R is the group algebra of a global Galois group. The pseudocharacter is supposed to be *odd*, which implies our multiplicity free hypothesis. They introduce an ideal I of A, which turns out to be the smallest ideal of A such that $T \otimes A/I$ is the sum of two characters $\rho_1, \rho_2 : R \longrightarrow A/I$ deforming respectively $\bar{\rho}_1$ and $\bar{\rho}_2$. Assuming that I has cofinite length l, their result is the construction of a finite A/I-module of length at least l in $\mathrm{Ext}^1_{R/IR}(\rho_1, \rho_2)$. We note that it is not possible to determine the precise structure of this module, so we do not know if their method constructs, for example, l independent extensions over k of $\bar{\rho}_2$ by $\bar{\rho}_1$ or, on the contrary, one "big" extension of ρ_2 by ρ_1 over the artinian ring A/I, that would generate a free A/I-module in $\mathrm{Ext}^1_{R/IR}(\rho_1, \rho_2)$.

The notion of *pseudocharacter* was introduced soon after by Wiles in dimension 2 ([**123**]), and by Taylor in full generality ([**117**]), under the name of *pseudorepresentation*. Besides their elementary properties, the main question that has been studied until now is whether they arise as the trace of a true representation. Taylor showed in 1990, relying on earlier results by Procesi, that the answer is always yes in the case where A is an algebraically closed field of characteristic zero; this result was extended, with a different method, to any algebraically closed field (of characteristic prime to $d!$) by Rouquier. The question was settled affirmatively in 1996 for any local henselian ring A, in the case where the residual pseudocharacter $\bar{T}$ is *absolutely irreducible*, independently by Rouquier ([**102**]) and Nyssen ([**91**]).

We now return to the progresses on Ribet's lemma.

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[4] We leave the proof of this assertion to the interested reader (use the fact that the Brauer group of any finite extension of K is trivial, e.g. by [**110**, XII §2, especially exercise 2]).

Urban's work ([**119**]) deals with the question of obtaining, using the notations of the paragraph describing Mazur-Wiles modules, a *free* A/I-module of extensions of ρ_1 by ρ_2. His hypotheses are as follows: the dimension d is arbitrary, but the number r of residual factors is still 2. The ring A/I is an arbitrary artinian local ring, and the pseudocharacter T is (over A/I) equal to $\operatorname{tr}\rho_1 + \operatorname{tr}\rho_2$, but he also assumes that T comes from a true representation ρ (at least over A/I), which moreover is modulo the maximal ideal of A a non-trivial extension of $\bar{\rho}_1$ by $\bar{\rho}_2$. Then he proves that ρ is indeed a non trivial extension of ρ_1 by ρ_2. Thus he obtains a more precise result than Mazur and Wiles, but with the much stronger assumption that his pseudocharacter comes from a representation that already gives the searched extension modulo the maximal ideal. Our work (see § 1.7) will actually show that the possibility of producing a free A/I-module of extensions as he does depends fundamentally on that hypothesis, which is very hard to check in practice excepted when A is a discrete valuation ring, or when T allows to construct only one extension of $\bar{\rho}_1$ by $\bar{\rho}_2$.

One of us studied ([**6**]) the case of an arbitrary number of residual factors r (and an arbitrary d) but like Ribet with A a complete discrete valuation ring. The main feature here is that the optimal result about extensions becomes more combinatorially involved. Assuming that ρ is generically irreducible, we can say nothing about the vanishing of an individual space of extensions $\operatorname{Ext}^1_{R\otimes k}(\bar{\rho}_i, \bar{\rho}_j)$. What we can say is that there are enough couples (i,j) in $\{1,\dots,r\}^2$ with non-zero $\operatorname{Ext}^1_{R\otimes k}(\bar{\rho}_i, \bar{\rho}_j)$ for the graph drawn by the oriented edges (i,j) to be connected. This result was soon after extended to deal with extensions over A/I assuming the residual multiplicity one hypothesis, in a joint work with P. Graftieaux in [**12**]. The combinatorial description of extensions we will obtain here is reminiscent of the results of that work.

Let us conclude those historical remarks by noting that two basic questions are not answered by all the results mentioned above: about Ribet's lemma, is it possible to find reasonable hypotheses so that two independent extensions of $\bar{\rho}_1$ by $\bar{\rho}_2$ over k exist? About pseudocharacters (over a strictly local henselian ring A, say), for which conceptual reasons might a pseudocharacter not be the trace of a true representation?

In this chapter, we will obtain the most general form of Ribet's lemma (for any A and T, and implying all the ones above) as well as a satisfactory answer to both questions above, and others. Indeed we will derive a precise structure theorem for residually multiplicity free pseudocharacters, and using this result we will be able to understand precisely and to provide links (some expected, others rather surprising) between the questions of when does a pseudocharacter come from a representation, how many extensions it defines, and how its (ir)reducibility behaves with respect to changing the ring A by a quotient.

We now explain our work, roughly following the order of the subparts of this section.

The first subpart § 1.2 deals with generalities on pseudocharacters. There A is not local henselian but can be any commutative ring. Though this part is obviously influenced by [**102**], we have tried to make it self-contained, partly for the convenience of the reader, and partly because we needed, in any case, to improve and generalize most of the arguments of Rouquier. We begin by recalling Rouquier's definition of a pseudocharacter of dimension d. We then introduce the notion of *Cayley-Hamilton* pseudocharacter T: it means that every x in R is killed by its "characteristic polynomial" whose coefficients are computed from the $T(x^i)$, $i = 1, \dots, d$. This notion is weaker than the notion of faithfulness that was used by Taylor and Rouquier, but it is stable by many operations, and this fact allows us to give more general statements with often simpler proofs. This notion is also closely related to the *Cayley-Hamilton trace algebras* studied by Procesi (see [**93**]). Every A-algebra R with a pseudocharacter T has a bunch of quotients on which T factors and becomes Cayley-Hamilton, the smallest of those being the unique faithful quotient $R/\mathrm{Ker}\, T$. We also prove results concerning idempotents, and the radical of an algebra with a Cayley-Hamilton pseudocharacter, that will be useful in our analysis of residually multiplicity free pseudocharacters. Finally, we define and study the notion of Schur functors of a pseudocharacter.

In § 1.3 and § 1.4, we study the structure of the residually multiplicity free pseudocharacters over a local henselian ring A. We introduce the notion of *generalized matrix algebra*, or briefly *GMA*, over A. Basically, a GMA over A is an A-algebra whose elements are square matrices (say, of size d) but where we allow the non diagonal entries to be elements of arbitrary A-modules instead of A - say the (i, j)-entries are elements of the given A-module $A_{i,j}$. Of course, to define the multiplications of such matrices, we need to suppose given some morphisms $A_{i,j} \otimes_A A_{j,k} \longrightarrow A_{i,k}$ satisfying suitable rules. The result motivating the introduction of GMA is our main structure result (proved in § 1.4), namely: if $T : R \longrightarrow A$ is a residually multiplicity free pseudocharacter, then every Cayley-Hamilton quotient of R is a GMA. Conversely, we prove that the trace function on any GMA is a Cayley-Hamilton pseudocharacter, which is residually multiplicity free if we assume that $A_{i,j}A_{j,i} \subset m$ (the maximal ideal of A) for every $i \neq j$, which provides us with many non trivial examples of such pseudocharacters. This result is a consequence of the main theorem of our study of GMA's (§ 1.3) which states that any GMA over A can be embedded, compatibly with the traces function, in an algebra $M_d(B)$ for some explicit commutative A-algebra B. Those two results take place in the long-studied topic of embedding an abstract algebra in a matrix algebra. It should be compared to a result of Procesi ([**93**]) on embeddings of trace algebras in matrix algebras: our results deal with less general algebras R, but with more general A, since we avoid the characteristic zero hypothesis that was fundamental in Procesi's invariant theory methods.

In § 1.5, we get the dividends of our rather abstract work on the structure of residually multiplicity free pseudocharacters. Firstly, for such a pseudocharacter, and for every partition of $\{1, \dots, r\}$ of cardinality k, we prove that there exists a greatest subscheme of $\operatorname{Spec} A$ on which T is a sum of k pseudocharacters, each of which being residually the sum of $\operatorname{tr} \bar{\rho}_i$ for i belonging to an element of the partition. We also show that on that subscheme, this decomposition of T as a sum of k such pseudocharacters is unique, and that that subscheme of $\operatorname{Spec}(A)$ does not change if R is changed into a quotient through which T factors. That subscheme is called the *reducibility loci* [(5)] attached to the given partition, and it will become one of our main object of study in Section 4. Moreover, if S is any Cayley-Hamilton quotient of R, hence a GMA defined by some A-modules $A_{i,j}$'s, we give a very simple description of the ideals of the reducibility loci in terms of the $A_{i,j}$.

Secondly, we construct submodules (explicitly described in terms of the modules $A_{i,j}$) of the extensions modules $\operatorname{Ext}_R(\rho_j, \rho_i)$. This is our version of "Ribet's lemma", as it provides a link between non-trivial extensions of ρ_j by ρ_i and the irreducibility properties of T encoded in its reducibility loci, and we show that it is in any reasonable sense optimal.

Nevertheless, and despite its simplicity, this result may not seem perfectly satisfactory, as it involves the unknown modules $A_{i,j}$'s. It may seem desirable to get a more direct link between the module of extensions we can construct and the reducibility ideals, solving out the modules $A_{i,j}$. However, this is actually a very complicated task, that has probably no nice answer in general, as it involves in the same times combinatorial and ring-theoretical difficulties: for the combinatorial difficulties, and how they can be solved (at a high price in terms of simplicity of statements) in a context that is ring-theoretically trivial (namely A a discrete valuation ring), we refer the reader to [**12**]; for the ring-theoretical difficulties in a context that is combinatorially trivial we refer the reader to § 1.7. In this subpart, we make explicit in the simple case $r = 2$ the subtle relations our results implies between, for a given pseudocharacter T, the modules of extensions that T allows to construct, the existence of a representation whose trace is T, the reducibility ideal of T and the ring-theoretic properties of A. We also give some criteria for our method to construct several *independent* extensions. Finally, let us say that the final sections of this paper will show that our version of Ribet's lemma, as stated, can actually easily be used in practice.

In § 1.6, we determine the local henselian rings A on which *every* residually multiplicity free pseudocharacter comes from a representation. The answer is surprisingly

(5) We stress that we could not define those loci without the assumption of residually multiplicity one (see [**10**]).

simple, if we restrict ourselves to noetherian A. Those A's are exactly the unique factorization domains. The proof relies on our structure result and its converse.

Finally, in § 1.8, we study pseudocharacters having a property of symmetry of order two (for example, selfdual pseudocharacters). It is natural to expect to retrieve this symmetry on the modules of extensions we have constructed, and this is what this subsection elucidates. Our main tool is a (tricky) lemma about lifting idempotents "compatibly with an automorphism or an anti-automorphism of order two" which may be of independent interest.

It is a pleasure to acknowledge the influence of all the persons mentioned in the historical part of this introduction. Especially important to us have been the papers and surveys of Procesi, as well as a few but illuminating discussions with him, either at Rome, the ENS, or by email.

1.2. Some preliminaries on pseudocharacters

1.2.1. Definitions. — Let A be a commutative ring [(6)] and R an A-algebra (not necessarily commutative). Let us recall the definition of an *A-valued pseudocharacter on R* introduced by R. Taylor in [**117**, §1]. Let $T : R \longrightarrow A$ be an A-linear map which is central, that is such that $T(xy) = T(yx)$ for all x, $y \in R$. For each integer $n \geq 1$, define a map $S_n(T) : R^n \longrightarrow A$ by

$$S_n(T)(x) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) T^\sigma(x),$$

where $T^\sigma : R^n \longrightarrow A$ is defined as follows. Let $x = (x_1, \dots, x_n) \in R^n$. If σ is a cycle, say $(j_1, \dots, j_m)$, then set $T^\sigma(x) = T(x_{j_1} \cdots x_{j_m})$, which is well defined. In general, we let $T^\sigma(x) = \prod_{i=1}^r T^{\sigma_i}(x)$, where $\sigma = \prod_{i=1}^r \sigma_i$ is the decomposition in cycles of the permutation σ (including the cycles with 1 element). We set $S_0(T) := 1$.

The central function T is called a *pseudocharacter* on R if there exists an integer n such that $S_{n+1}(T) = 0$, and such that $n!$ is invertible in A. The smallest such n is then called the dimension of T, and it satisfies $T(1) = n$ (see Lemma 1.2.5 (2)) [(7)].

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(6) In all the book, *rings* and *algebras* are associative and have a unit, and a ring homomorphism preserves the unit.

(7) The definition of a pseudocharacter of dimension n used here looks slightly more restrictive than the one introduced in [**117**] or [**102**], as we assume that $n!$ is invertible. This assumption on $n!$ is first crucial to express the Cayley-Hamilton theorem from the trace, which is a basic link between pseudocharacters and true representations, and also to avoid a strange behavior of the dimension of pseudocharacters with base change. Note that Taylor's theorem is only concerned by the case where A is a field of characteristic 0, hence $n!$ is invertible. Moreover, Lemma 2.13 of [**102**] is false when $n!$ is not invertible (and correct if it is), hence this hypothesis should be added in the hypothesis of Lemma 4.1 and Theorem 5.1 there (see Remark 1.2.6) below.

These notions apply in the special case where $R = A[G]$ for some group (or monoid) G. In this case, T is uniquely determined by the data of its restriction to G (central, and satisfying $S_{n+1}(T) = 0$ on G^{n+1}).

If $T : R \longrightarrow A$ is an A-valued pseudocharacter on R of dimension d and if A' is a commutative A-algebra, then the induced linear map $T \otimes A' : R \otimes A' \longrightarrow A'$ is an A'-valued pseudocharacter on R of dimension d.

1.2.2. Main example. — Let $V := A^d$ and $\rho : R \longrightarrow \mathrm{End}_A(V)$ be a morphism of A-algebras. For each $n \geq 1$, $V^{\otimes_A n}$ carries an A-linear representation of $\mathfrak{S}_n$ and a diagonal action of the underlying multiplicative monoid of R^n. If $e = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma \in A[\mathfrak{S}_n]$, then a computation [(8)] shows that for $x \in R^n$,

$$\mathrm{tr}\,(xe|V^{\otimes_A n}) = S_n(\mathrm{tr}\,(\rho))(x).$$

As e acts by 0 on $V^{\otimes_A n}$ if $n > d$, the central function $T := \mathrm{tr}\,(\rho)$ is a pseudocharacter of dimension d (assuming that $d!$ is invertible in A). Moreover, when ρ is an isomorphism, an easy computation using standard matrices shows that T is the unique A-valued pseudocharacter of dimension d of $R = M_d(A)$. By faithfully flat descent, these results also hold when $\mathrm{End}_A(V)$ is replaced by any Azumaya algebra of rank d^2 over A, and when tr is its reduced trace.

Let us now recall the main known converse results. If $T : R \longrightarrow k$ is a pseudocharacter, where k is a separably closed field, then T is the trace of a unique semi-simple representation $\rho : R \longrightarrow \mathrm{End}_k(V)$. This is [**117**, Thm. 1.2] in characteristic 0, who relies on the work of Procesi in [**92**], and [**102**, Thm. 4.2] in general. When k is a field, but not necessarily separably closed, then [**102**, Thm. 4.2] proves that if T is absolutely irreducible, that is $T \otimes k^{\mathrm{sep}}$ is not the sum of two non trivial pseudocharacters, then T is the reduced trace of a surjective k-algebra morphism $\rho : R \longrightarrow S$ for some central simple algebra S over k (S and ρ are even unique up to isomorphism). More generally, for any commutative ring A, if $T : R \longrightarrow A$ is a pseudocharacter such that $T \otimes A/m$ is absolutely irreducible for all $m \in \mathrm{Specmax}(A)$, then T is the reduced trace of a surjective A-algebra homomorphism $\rho : R \longrightarrow S$, where S is an Azumaya algebra over A ([**102**, Thm. 5.1], [**91**] when A is local henselian). When A is strictly local henselian, any Azumaya algebra over A is isomorphic to a matrix algebra $M_d(A)$, so the above theorem implies that over a stricly local henselian ring, any pseudocharacter $T : R \to A$ such that $T \otimes A/m$ is (absolutely) irreducible is the trace of a true representation $\rho : R \longrightarrow M_d(A)$.

(8) For instance, reducing to the case where $R = \mathrm{End}_A(V)$ and using the polarization identity for symmetric multi-linear forms, it suffices to check it when $x = (y, y, \ldots, y)$ (see also [**92**, §1.1], [**102**, prop. 3.1]).

One main goal of this section is to study the new case where A is local henselian and $T\otimes A/m$ is *reducible* (but satisfies a multiplicity one hypothesis (def. 1.4.1, § 1.6)).

1.2.3. The Cayley-Hamilton identity and Cayley-Hamilton pseudocharacters. — Let $T : R \longrightarrow A$ be a pseudocharacter of dimension d. For $x \in R$, let

$$P_{x,T}(X) := X^d + \sum_{k=1}^{d} \frac{(-1)^k}{k!} S_k(T)(x,\dots,x)X^{d-k} \in A[X].$$

In the example given in § 1.2.2, $P_{x,T}(X)$ is the usual characteristic polynomial of x. We will say that T is *Cayley-Hamilton* if it satisfies the *Cayley-Hamilton identity*, that is if

$$\text{for all } x \in R, \quad P_{x,T}(x) = 0.$$

In this case, R is integral over A. The algebra R equipped with T is then a *Cayley-Hamilton algebra* in the sense of C. Procesi [**93**, def. 2.6].

An important observation is that for a general pseudocharacter $T : R \longrightarrow A$ of dimension d, the map $R \longrightarrow R$, $x \mapsto P_{x,T}(x)$, is the evaluation at $(x,\dots,x)$ of a d-linear symmetric map $\mathrm{CH}(T) : R^d \longrightarrow R$, explicitly given by:

$$\mathrm{CH}(T)(x_1,\dots,x_d) := \frac{(-1)^d}{d!} \sum_{I,\sigma} (-1)^{|I|} S_{d-|I|}(T)(\{x_i, i \notin I\}) x_{\sigma(1)} \cdots x_{\sigma(|I|)},$$

where I is a subset of $\{1,\dots,d\}$ and σ a bijection from $\{1,\dots,|I|\}$ to I. A first consequence of the polarization identity ([**28**, Alg., Chap. I, §8, prop. 2], applied to the ring $\mathrm{Symm}_A^d(R)$) is that T is Cayley-Hamilton if and only if $\mathrm{CH}(T) = 0$. In particular, if T is Cayley-Hamilton then for any A-algebra A', $T \otimes A'$ is also Cayley-Hamilton.

In the same way, we see that for $x_1,\dots,x_{d+1} \in R$, we have

$$S_{d+1}(T)(x_1,\dots,x_{d+1}) = d!T(\mathrm{CH}(T)(x_1,\dots,x_d)x_{d+1}), \tag{1}$$

hence a good way to think about the identity $S_{d+1}(T) = 0$ defining a pseudocharacter is to see it as a polarized, A-valued, form of the Cayley-Hamilton identity.

1.2.4. Faithful pseudocharacters, the kernel of a pseudocharacter. — We recall that the *kernel* of T is the two-sided ideal $\mathrm{Ker}\, T$ of R defined by

$$\mathrm{Ker}\, T := \{x \in R,\ \forall y \in R, T(xy) = 0\}.$$

T is said to be *faithful* when $\mathrm{Ker}\, T = 0$. If $R \longrightarrow S$ is a surjective morphism of A-algebras whose kernel is included in $\mathrm{Ker}\, T$, then T factors uniquely as a pseudocharacter $T_S : S \longrightarrow A$, which is still of dimension d, and which will be often denoted by T. In particular, T induces a faithful pseudocharacter on $R/\mathrm{Ker}\, T$.

If T is faithful, then T is Cayley-Hamilton: indeed, for any $x_1, \ldots, x_{d+1} \in R$, we have $S_{d+1}(T)(x_1, \ldots, x_{d+1}) = 0$ by definition of a pseudocharacter of dimension d, hence $T(\mathrm{CH}(T)(x_1, \ldots, x_d)x_{d+1}) = 0$ by formula (1) above and the fact that $d!$ is invertible in A; since this holds for all $x_{d+1} \in A$, and T is faithful, we deduce that $\mathrm{CH}(T)(x_1, \ldots, x_d) = 0$.

More generally, let $(T_i)_{i=1}^r$ be a family of pseudocharacters $R \longrightarrow A$ such that $d!$ is invertible in A where $d = \dim T_1 + \cdots + \dim T_r$. Then $T := \sum_i T_i$ is a pseudocharacter of dimension d, and for all $x \in R$,

$$P_{x,T} = \prod_{i=1}^{r} P_{x,T_i}$$

(we may assume that $r = 2$, in which case this follows from [**102**, Lemme 2.2]). As a consequence, $P_{x,T}(x) \in (\mathrm{Ker}\, T_1)(\mathrm{Ker}\, T_2) \cdots (\mathrm{Ker}\, T_r) \subset \bigcap_i \mathrm{Ker}\, T_i$, hence $T : R/(\cap_i \mathrm{Ker}\, T_i) \longrightarrow A$ is Cayley-Hamilton. The following lemma is obvious from the formula of $P_{x,T}(X)$, but useful.

Lemma 1.2.1. — *Let $T : R \longrightarrow A$ be a Cayley-Hamilton pseudocharacter of dimension d, then for each $x \in \mathrm{Ker}\, T$ we have $x^d = 0$. In particular* $\mathrm{Ker}\, T$ *is a nil ideal, and is contained in the Jacobson radical of R.*

Remark 1.2.2. — If A' is an A-algebra and T is faithful, it is not true in general that $T \otimes A'$ is still faithful. Although we will not need it in what follows, let us mention that this is however the case when A' is projective as an A-module (so e.g. when A is a field), or when A' is flat over A and either R is of finite type over A (see [**102**, prop. 2.11]) or A is noetherian (mimic the proof *loc. cit.* and use that A^X is flat over A for any set X). This is also the case when $A' = S^{-1}A$ is a fraction ring of A such that $A \to S^{-1}A$ is injective, as for such an S the natural map

$$S^{-1}\mathrm{Ker}\, T \longrightarrow \mathrm{Ker}\, (T \otimes S^{-1}A)$$

is an isomorphism for any pseudocharacter $T : R \longrightarrow A$. Indeed, the natural map $S^{-1}\mathrm{Ker}\, T \to S^{-1}R$ is injective as $S^{-1}A$ is flat over A, and its image is obviously included in $\mathrm{Ker}\, (T \otimes S^{-1}A)$. Moreover, if $x \in \mathrm{Ker}\, (T \otimes S^{-1}A)$, then we may write $x = y/s$ with $y \in R$ and $s \in S$. For any $z \in R$, then $T(yz) = 0$ as its image in $S^{-1}A$ is $T(sxz) = sT(xz) = 0$, so $y \in \mathrm{Ker}\, T$ and the surjectivity follows.

1.2.5. Cayley-Hamilton quotients

Definition 1.2.3. — Let $T : R \longrightarrow A$ be a pseudocharacter of dimension d. Then a quotient S of R by a two-sided ideal of R which is included in $\mathrm{Ker}\, T$, and such that the induced pseudocharacter $T : S \longrightarrow A$ is Cayley-Hamilton, is called a *Cayley-Hamilton quotient* of (R, T).

Example 1.2.4. — (i) $R/\mathrm{Ker}\, T$ is a Cayley-Hamilton quotient of (R, T).

(ii) Let I be the two-sided ideal of R generated by the elements $P_{x,T}(x)$ for all $x \in R$. Then $S_0 := R/I$ is a Cayley-Hamilton quotient of (R,T). Indeed, $I \subset \mathrm{Ker}\, T$ by formula (1) (which, applied to $x_1 = \cdots = x_d = x$ and $x_{d+1} = y$, gives, using that its right hand side is zero by definition of a pseudocharacter, that $T(P_{x,T}(x)y) = 0$ for all $x, y \in R$, so that $P_{x,T}(x) \in \mathrm{Ker}\, T$ for all $x \in R$) and T is obviously Cayley-Hamilton on S_0.

(iii) Let B be a commutative A-algebra and $\rho : R \longrightarrow M_d(B)$ be a representation such that $\mathrm{tr} \circ \rho = T$. Then $\rho(R)$ is a Cayley-Hamilton quotient of T. Indeed, $\mathrm{Ker}\, \rho$ is obviously included in $\mathrm{Ker}\, T$ and tr is Cayley-Hamilton on $\rho(R)$ by the usual Cayley-Hamilton theorem.

The Cayley-Hamilton quotients of (R,T) form in a natural way a category: morphisms are A-algebra morphisms which are compatible with the morphism from R. Thus in that category any morphism $S_1 \longrightarrow S_2$ is surjective, and has kernel $\mathrm{Ker}\, T_{S_1}$ which is a nil ideal by Lemma 1.2.1. Note that S_0 is the initial object and $R/\mathrm{Ker}\, T$ the final object of that category.

1.2.6. Two useful lemmas on pseudocharacters. — Let $T : R \longrightarrow A$ be a pseudocharacter of dimension d. Recall that an element $e \in R$ is said to be idempotent if $e^2 = e$. The subset $eRe \subset R$ is then an A-algebra whose unit element is e.

Lemma 1.2.5. — *Assume that* $\mathrm{Spec}\,(A)$ *is connected.*

(1) *For each idempotent* $e \in R$, $T(e)$ *is an integer less than or equal to* d.

(2) *We have* $T(1) = d$. *Moreover, if* A' *is any* A*-algebra, the pseudocharacter* $T \otimes A'$ *has dimension* d.

(3) *If* $e \in R$ *is an idempotent, the restriction* T_e *of* T *to the* A*-algebra* eRe *is a pseudocharacter of dimension* $T(e)$.

(4) *If* T *is Cayley-Hamilton (resp. faithful), then so is* T_e.

(5) *Assume that* T *is Cayley-Hamilton. If* $e_1, \ldots, e_r$ *is a family of (nonzero) orthogonal idempotents of* R, *then* $r \leq d$. *Moreover, if* $T(e) = 0$ *for some idempotent* e *of* R, *then* $e = 0$.

Proof. — Let us prove (1). By definition of $S_{d+1}(T)$ and [**102**, cor. 3.2],

$$(2) \quad S_{d+1}(T)(e, e, \ldots, e) = \sum_{\sigma \in \mathfrak{S}_{d+1}} \varepsilon(\sigma) T(e)^{|\sigma|} = T(e)(T(e)-1)\cdots(T(e)-d) = 0$$

in A, where $|\sigma|$ is the number of cycles of σ. The discriminant of the split polynomial $X(X-1)\cdots(X-d) \in A[X]$ is $d!$, hence is invertible in A. As $\mathrm{Spec}\,(A)$ is connected, we get that $T(e) = i$ for some $i \leq d$. This proves (1).

To prove (2), apply (1) to $e = 1$. We see that $T(1) = i$ is an integer less than or equal to d. But following the proof of [**102**, Prop. 2.4], there is an $x \in A - \{0\}$ such that $x(T(1) - d) = 0$. Then $x(i - d) = 0$, and because $i - d$ is invertible if non zero, we must have $i = d = T(1)$. In particular, $S_d(T)(1, 1, \ldots, 1) = T(1)(T(1)-1)\ldots(T(1)-d+1) = d!$ is invertible, hence $S_d(T \otimes A')(1, \ldots, 1)$ is non zero, which proves (2).

Let $T_e := T_{|eRe} : eRe \longrightarrow A$. For all n, we have $S_n(T_e) = S_n(T)_{|(eRe)^{n+1}}$, so that T_e is a pseudocharacter of dimension $\leq d$. As e is the unit of eRe, and $T(e)!$ is invertible in A by (1), part (2) implies that $\dim T_e = T(e)$.

If $x \in eRe$ and $y \in R$, then $T(xy) = T(exey) = T(xeye) = T_e(xeye)$, hence T_e is faithful if T is. Assume now that T is Cayley-Hamilton and fix $x \in eRe$. Let us compute

$$e\mathrm{CH}(T)(x, \ldots, x, (1-e), \ldots, (1-e))$$

where x appears $r := T(e)$ times. As $x(1 - e) = e(1 - e) = 0$, we see that the only nonvanishing terms defining the sum above are the ones with (I, σ) satisfying $|I| \leq r$ and $\sigma(\{1, \ldots, |I|\}) \subset \{1, \ldots, r\}$. For such a term, it follows from [**102**, Lemme 2.5] that

$$S_{d-|I|}(T)(\{x_i, i \notin I\}) = S_{r-|I|}(T)(x, \ldots, x) S_{d-r}(T)(1-e, \ldots, 1-e).$$

As we have seen in proving part (2) above, and by (3), $S_{d-r}(T)(1-e, \ldots, 1-e) = S_{d-r}(T_{1-e})(1-e, \ldots, 1-e) = (d-r)!$ is invertible. We proved that:

$$e\mathrm{CH}(T)(x, \ldots, x, (1-e), \ldots, (1-e)) = \frac{(d-r)!^2}{d!}\mathrm{CH}(T_e)(x, \ldots, x)e,$$

hence T_e is Cayley-Hamilton if T is.

Let us prove (5), we assume that T is Cayley-Hamilton. Let e be an idempotent of R. If e satisfies $T(e) = 0$, then we see that $P_{e,T}(X) = X^d$, hence $e^d = e = 0$ by the Cayley-Hamilton identity. Thus if e is nonzero then $T(e)$ is non zero, hence by (1) is an integer between 1 and d, so is invertible in A. Assume now by contradiction that $e_1, \ldots, e_{d+1}$ is a family of orthogonal nonzero idempotents of R. Then we get that $S_{d+1}(e_1, \ldots, e_{d+1}) = T(e_1) \cdots T(e_{d+1})$, which has to be invertible and zero, a contradiction. □

Remark 1.2.6. — Lemma 2.14 of [**102**] is obviously incorrect as stated, and must be replaced by the part (5) of the above lemma (it is used in the proofs of Lemma 4.1 and Theorem 5.1 there).

We conclude by computing the Jacobson radical of R when T is Cayley-Hamilton. In what follows, A is a local ring with maximal ideal m and residue field $k := A/m$. We will denote by $\bar{R}$ the k-algebra $R \otimes_A k = R/mR$, and by $\bar{T}$ the pseudocharacter $T \otimes k : \bar{R} \longrightarrow k$.

Lemma 1.2.7. — *Assume that T is Cayley-Hamilton. Then the kernel of the canonical surjection $R \longrightarrow \overline{R}/\mathrm{Ker}\,\overline{T}$ is the Jacobson radical* $\mathrm{rad}(R)$ *of R.*

Proof. — Let J denote the kernel above, it is a two-sided ideal of R. By [**102**, Lemma 4.1] (see precisely the sixth paragraph of the proof there), $\overline{R}/(\mathrm{Ker}\,\overline{T})$ is a semisimple k-algebra, hence $\mathrm{rad}(R) \subset J$.

Let $x \in J$; we will show that $1 + x \in R^*$. We have $T(xy) \in m, \forall y \in R$, hence $T(x^i) \in m$ for all i, so that by the Cayley-Hamilton identity $x^d \in m(A[x])$. Let us consider the commutative finite A-algebra $B := A[x]$. Then B is local as B/mB is, and its maximal ideal is (m, x). As a consequence, $1 + x$ is invertible in B, hence in R. As J is a two-sided ideal of R such that $1 + J \subset R^*$, we have $J \subset \mathrm{rad}(R)$. □

1.2.7. Tensor operations on pseudocharacters. — In this section we assume that A is a $\mathbb{Q}$-algebra. All the tensor products involved below are assumed to be over A.

Let R be an A-algebra, $T : R \longrightarrow A$ be a pseudocharacter of dimension d, and m a positive integer. We define $T^{\otimes m} : R^{\otimes m} \longrightarrow A$ as the A-linear form that satisfies

$$T(x_1 \otimes \cdots \otimes x_m) = T(x_1) \ldots T(x_m). \tag{3}$$

Let us denote by $R^{\otimes m}[\mathfrak{S}_m]$ the twisted group algebra of $\mathfrak{S}_m$ over $R^{\otimes m}$ satisfying

$$\sigma \cdot x_1 \otimes \cdots \otimes x_m = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)} \cdot \sigma.$$

We can then extend $T^{\otimes m}$ to an A-linear map $R^{\otimes m}[\mathfrak{S}_m] \longrightarrow A$ by setting

$$U(x_1 \otimes \cdots \otimes x_m \cdot \sigma) := T^{\sigma}(x_1, \ldots, x_m)$$

(see § 1.2.1, note that this map coincides with $T^{\otimes m}$ on the subalgebra $R^{\otimes m}$).

Proposition 1.2.8. — *$T^{\otimes m}$ and U are both pseudocharacters of dimension d^m.*

Proof. — By [**93**], there is a commutative A-algebra B, with $A \subset B$, and a morphism $\rho : R \longrightarrow M_d(B) = \mathrm{End}_B(B^d)$ of A-algebras such that $\mathrm{tr}\,\rho(x) = T(x)1_B$ for every $x \in R$. Let $\rho^{\otimes m} : R^{\otimes m} \longrightarrow \mathrm{End}_B((B^d)^{\otimes m}) = M_{d^m}(B)$ be the mth tensor power of ρ. The equality $\mathrm{tr}\,\rho^{\otimes m}(z) = T^{\otimes m}(z)1_B$ follows from (3) for pure tensors $z \in R^{\otimes m}$ and then by A-linearity for all z. We deduce $\mathrm{tr}\,\rho^{\otimes m} = T$ as $A \subset B$. Thus $T^{\otimes m}$ is a pseudocharacter, being the trace of a representation.

We can extend the morphism $\rho^{\otimes m} : R^{\otimes m} \longrightarrow \mathrm{End}_B((B^d)^{\otimes m})$ into a morphism $\rho' : R^{\otimes m}[\mathfrak{S}_m] \longrightarrow \mathrm{End}_B((B^d)^{\otimes m})$ by letting $\mathfrak{S}_m$ act by permutations on the m tensor components of $(B^d)^{\otimes m}$. It is an easy computation to check that the trace of ρ' is U. So U is a pseudocharacter. □

Remark 1.2.9. — It should be true that more generally, if for $i = 1,2$, $T_i : R_i \longrightarrow A$ is a pseudocharacter of dimension d_i, and if $T : R_1 \otimes R_2 \longrightarrow A$ is the A-linear map defined by

$$T(x_1 \otimes x_2) = T_1(x_1)T_2(x_2),$$

then T is a pseudocharacter of dimension $d_1 d_2$. It is probably possible to deduce directly the formula $S_{d_1 d_2 + 1}(T) = 0$ from the formulas $S_{d_i+1}(T_i) = 0$, $i = 1,2$, but we have not written down a proof[9].

To conclude this paragraph, we give an application of the preceding proposition to the construction of the Schur functors of a given pseudocharacter in the case when $R := A[G]$ with G a group or a monoid.

Let $T : A[G] \longrightarrow A$ be a pseudocharacter and let $m \geq 1$ be an integer. There is a natural A-algebra embedding

$$\iota_m : A[G] \longrightarrow R^{\otimes m}[\mathfrak{S}_m] = A[G^m \rtimes \mathfrak{S}_m]$$

extending the diagonal map $G \longrightarrow G^m$. Let $e \in \mathbb{Q}[\mathfrak{S}_m]$ be any central idempotent. As the image of ι_m commutes with $\mathfrak{S}_m$, the map

$$T^e : A[G] \longrightarrow A, \ x \mapsto U(\iota_m(x)e),$$

is a pseudocharacter by Proposition 1.2.8.

Remark 1.2.10. —
(i) In the special case when $e = \frac{2}{m!}(\sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma)\sigma)$, then we set as usual $\Lambda^m(T) := T^e$. Note that $T^e(g) = \frac{2}{m!} S_m(T)(g, \ldots, g)$ for $g \in G$.
(ii) It follows easily from the definitions that when T (resp. T_1 and T_2) is the trace of a representation $G \longrightarrow \mathrm{GL}(V)$ (resp. of some representations V_1 and V_2), then T^e (resp. $T_1 T_2$) is the trace of the representation of G on $e(V^{\otimes m})$ (resp. on $V_1 \otimes V_2$).

1.3. Generalized matrix algebras

Let $d_1, \ldots, d_r$ be nonzero positive integers, and $d := d_1 + \cdots + d_r$.

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(9) Note also that the proof of the proposition above would break on the fact that if $\rho_i : R_i \longrightarrow M_d(B_i)$ are representations of trace T_i given by [**93**], it does not follow from $A \subset B_i$ that the map $A \longrightarrow B_1 \otimes_A B_2$ is injective, so that we cannot find a representation whose trace is T, but only a representation whose trace coincides with T after reduction to the image of A in $B_1 \otimes_A B_2$. However this line of reasoning would imply that T is a pseudocharacter in two cases: if A is reduced, because in that case, we can take $B_1 = B_2$ equal to the product of algebraic closures of residue fields of all points of $\mathrm{Spec}\,(A)$, and $\rho_i : R_i \longrightarrow M_d(B)$ be the "diagonal" representation deduced from T_i; and if A is local henselian, T_i residually multiplicity free (see § 1.4.1), since in this case we may use Proposition 1.3.13 to produce representations $\rho_i : R \longrightarrow M_d(B_i)$ of trace T_i such that A is a *direct factor* of B_i, so that we know that $A \subset B_1 \otimes_A B_2$.

1.3.1. Definitions, notations and examples

Definition 1.3.1. — Let A be a commutative ring and R an A-algebra. We will say that R is a *generalized matrix algebra (GMA)* of type $(d_1, \dots, d_r)$ if R is equipped with:

(i) a family of orthogonal idempotents $e_1, \dots, e_r$ of sum 1,
(ii) for each i, an A-algebra isomorphism $\psi_i : e_i R e_i \longrightarrow M_{d_i}(A)$,

such that the *trace* map $T : R \longrightarrow A$, defined by $T(x) := \sum_{i=1}^r \mathrm{tr}\,(\psi_i(e_i x e_i))$, satisfies $T(xy) = T(yx)$ for all $x, y \in R$. We will call $\mathcal{E} = \{e_i, \psi_i, i = 1, \dots, r\}$ the *data of idempotents* of R.

Remark 1.3.2. — If R is a GMA as above, then R equipped with the map $T(\bullet)1_R$ is a *trace algebra* in the sense of Procesi [**93**].

Notation 1.3.3. — If $(R, \mathcal{E})$ is a GMA as above, we shall often use the following notations. For $1 \leq i \leq r, 1 \leq k, l \leq d_i$, there is a unique element $E_i^{k,l} \in e_i R e_i$ such that $\psi_i(E_i^{k,l})$ is the elementary matrix of $M_{d_i}(A)$ with unique nonzero coefficient at row k and column l. These elements satisfy the usual relations

$$E_i^{k,l} E_{i'}^{k',l'} = \delta_{i,i'} \delta_{l,k'} E_i^{k,l'},$$

$e_i = \sum_{1 \leq k \leq d_i} E_i^{k,k}$, and $A E_i^{1,1}$ is free of rank one over A. Clearly, the data of the $E_i^{k,l}$ satisfying these last three conditions is equivalent to condition (ii) in the definition of R. For each i, we also set $E_i := E_i^{1,1}$.

Example 1.3.4. — Let A be a commutative ring, and B be a commutative A-algebra. Let $A_{i,j}$, $1 \leq i, j \leq r$, be a family of A-submodules of B satisfying the following properties:

$$\text{For all } i,\ j,\ k, \qquad A_{i,i} = A, \qquad A_{i,j} A_{j,k} \subset A_{i,k} \tag{4}$$

Then the following A-submodule R of $M_d(B)$

$$\begin{pmatrix} M_{d_1}(A_{1,1}) & M_{d_1,d_2}(A_{1,2}) & \dots & M_{d_1,d_r}(A_{1,r}) \\ M_{d_2,d_1}(A_{2,1}) & M_{d_2}(A_{2,2}) & \dots & M_{d_2,d_r}(A_{2,r}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r,d_1}(A_{r,1}) & M_{d_r,d_2}(A_{r,2}) & \dots & M_{d_r}(A_{r,r}) \end{pmatrix} \tag{5}$$

is an A-subalgebra. Let $e_i \in M_d(B)$ be the matrix which is the identity in the i^{th} diagonal block (of size d_i) and 0 elsewhere. As $A_{i,i} = A$, e_i belongs to R, and in R we have a decomposition in orthogonal idempotents

$$1 = e_1 + e_2 + \dots + e_r.$$

We also have canonical isomorphisms $\psi_i : e_iRe_i \xrightarrow{\sim} M_{d_i}(A)$. Hence R together with $\{e_i, \psi_i, i = 1, \dots, r\}$ is a GMA, and the trace T is the restriction of the trace of $M_d(B)$. Note that assuming $d!$ invertible in A, § 1.2.2 shows that T is a pseudocharacter of dimension d over R, which is Cayley-Hamilton (see § 1.2.3).

The GMA R is called the *standard GMA of type* $(d_1, \dots, d_r)$ *associated to the* A*-submodules* $A_{i,j}$ *of* B.

1.3.2. Structure of a GMA. — Let R be a GMA of type $(d_1, \dots, d_r)$. We will attach to it a canonical family of A-modules $\mathcal{A}_{i,j} \subset R$, $1 \leq i, j \leq r$, as follows. Set

$$\mathcal{A}_{i,j} := E_iRE_j.$$

For each triple $1 \leq i, j, k \leq r$, we have

$$\mathcal{A}_{i,j}\mathcal{A}_{j,k} \subset \mathcal{A}_{i,k}$$

in R, hence the product in R induces a map

$$\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k}.$$

Moreover, T induces an A-linear isomorphism

$$\mathcal{A}_{i,i} \xrightarrow{\sim} A.$$

By Morita equivalence, the map induced by the product [(10)] of R

$$e_iRE_i \otimes \mathcal{A}_{i,j} \otimes E_jRe_j \longrightarrow e_iRe_j$$

is an isomorphism of $e_iRe_i \otimes e_jR^{\mathrm{opp}}e_j$-modules. In particular, with the help of ψ_i and ψ_j, we get a canonical identification

$$e_iRe_j = M_{d_i,d_j}(\mathcal{A}_{i,j}),$$

as a module over $e_iRe_i \otimes e_jR^{\mathrm{opp}}e_j = M_{d_i}(A) \otimes M_{d_j}(A)^{\mathrm{opp}}$. Moreover, in terms of these identifications, the natural map induced by the product in R, $e_iRe_j \otimes e_jRe_k \longrightarrow e_iRe_k$, is the map $M_{d_i,d_j}(\mathcal{A}_{i,j}) \otimes M_{d_j,d_k}(\mathcal{A}_{j,k}) \longrightarrow M_{d_i,d_k}(\mathcal{A}_{i,k})$ induced by $\varphi_{i,j,k}$.

To summarize all of this, there is a canonical isomorphism of A-algebras

$$R \simeq \begin{pmatrix} M_{d_1}(\mathcal{A}_{1,1}) & M_{d_1,d_2}(\mathcal{A}_{1,2}) & \dots & M_{d_1,d_r}(\mathcal{A}_{1,r}) \\ M_{d_2,d_1}(\mathcal{A}_{2,1}) & M_{d_2}(\mathcal{A}_{2,2}) & \dots & M_{d_2,d_r}(\mathcal{A}_{2,r}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r,d_1}(\mathcal{A}_{r,1}) & M_{d_r,d_2}(\mathcal{A}_{r,2}) & \dots & M_{d_r}(\mathcal{A}_{r,r}) \end{pmatrix}, \tag{6}$$

where the right hand side is a notation for the algebra that is $\bigoplus_{i,j} M_{d_i,d_j}(\mathcal{A}_{i,j})$ as an A-module, and whose product is defined by the usual matrix product formula, using the $\varphi_{i,j,k}$'s to multiply entries. Moreover, we have canonical isomorphisms $\mathcal{A}_{i,i} \xrightarrow{\sim} A$.

(10) All the tensor products below are assumed to be over A.

By an abuse of language, we will often write this precise isomorphism as an equality $\mathcal{A}_{i,i} = A$.
Let us consider the following sets of conditions on the $\varphi_{i,j,k}$'s:

(UNIT) For all i, $\mathcal{A}_{i,i} = A$ and for all i, j, $\varphi_{i,i,j} : A \otimes \mathcal{A}_{i,j} \longrightarrow \mathcal{A}_{i,j}$ (resp. $\varphi_{i,j,j} : \mathcal{A}_{i,j} \otimes A \longrightarrow \mathcal{A}_{i,j}$) is the A-module structure of $\mathcal{A}_{i,j}$.

(ASSO) For all i, j, k, l, the two natural maps $\mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \otimes \mathcal{A}_{k,l} \longrightarrow \mathcal{A}_{i,l}$ coincide.

(COM) For all i, j and for all $x \in \mathcal{A}_{i,j}$, $y \in \mathcal{A}_{j,i}$, we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.

Lemma 1.3.5. — *The $\varphi_{i,j,k}$'s above satisfy the conditions (UNIT), (ASSO) and (COM). The $\varphi_{i,j,i}$'s are all nondegenerate if and only if $T : R \otimes R \longrightarrow A$, $x \otimes y \mapsto T(xy)$, is nondegenerate.*

Proof. — First, (ASSO) follows from the associativity of the product in R. To check (UNIT), we must show that for all i, j, and for all $x, y \in R$, then $E_i x E_i y E_j = T(E_i x E_i) E_i y E_j$ and $E_i x E_j y E_j = T(E_j y E_j) E_i x E_j$. As $T(R) = A$ is in the center of R, it suffices to check that for all i, and for all $x \in R$,

$$E_i x E_i = T(E_i x E_i) E_i,$$

but this is obvious. The property (COM) holds as $T(xy) = T(yx)$ for all $x, y \in R$,

Note that if $x \in R$ and $i \neq j$, $T(e_i x e_j) = T(e_j e_i x) = 0$. Hence for $x \in e_i R e_j$ and $y \in e_{i'} R e_{j'}$ we have $T(xy) = 0$, except in the case $j = i'$ and $i = j'$. As an A-module, $R \simeq \oplus_{i,j} M_{d_i,d_j}(\mathcal{A}_{i,j})$, and by the above remark, the pairing $T : R \otimes R \longrightarrow A$ is the direct sum of the pairings $T_{i,j} : e_i R e_j \otimes e_j R e_i \longrightarrow A$, for all ordered pairs (i, j). Thus T is nondegenerate if and only if all the pairings $T_{i,j}$ are non degenerate. But $T_{i,j}$ is isomorphic to the pairing $M_{d_i,d_j}(\mathcal{A}_{i,j}) \otimes M_{d_j,d_i}(\mathcal{A}_{j,i}) \longrightarrow A$, induced by $\psi_{i,j,i} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,i} \longrightarrow A$ and the trace. By Morita's equivalence, this pairing is nondegenerate if and only if $\psi_{i,j,i}$ is, hence the last assertion of the lemma. □

Reciprocally, if we have a family of A-modules $\mathcal{A}_{i,j}$, $1 \leq i, j \leq r$, equipped with A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k}$ satisfying (UNIT), (ASSO) and (COM), then we leave as an exercise to the reader to check that $R := \oplus_{i,j} M_{d_i,d_j}(\mathcal{A}_{i,j})$ has a unique structure of GMA of type $(d_1, \dots, d_r)$ such that for all i, j, $E_i R E_j = \mathcal{A}_{i,j}$.

1.3.3. Representations of a GMA. — If R is an A-algebra, we will call *representation* of R any morphism of A-algebras $\rho : R \longrightarrow M_n(B)$, where B is a commutative A-algebra. If R is equipped with a central function $T : R \longrightarrow A$, we will say that ρ is a *trace representation* if $\mathrm{tr} \circ \rho(x) = T(x) 1_B$ for any $x \in R$.

Let $(R, \mathcal{E})$ be a GMA of type $(d_1, \dots, d_r)$. We will be interested by the trace representations of R, and especially by those that are compatible with the structure $\mathcal{E}$, as follows:

Definition 1.3.6. — Let B be a commutative A-algebra. A representation $\rho : R \longrightarrow M_d(B)$ is said to be *adapted to* $\mathcal{E}$ if its restriction to the A-subalgebra $\oplus_{i=1}^r e_i R e_i$ is the composite of the representation $\oplus_{i=1}^r \psi_i$ by the natural "diagonal" map $M_{d_1}(A) \oplus \cdots \oplus M_{d_r}(A) \longrightarrow M_d(B)$.

Obviously, an adapted representation is a trace representation. In the other direction we have:

Lemma 1.3.7. — *Let B be a commutative A-algebra and $\rho : R \longrightarrow M_d(B)$ be a trace representation. Then there is a commutative ring C containing B and a $P \in \mathrm{GL}_d(C)$ such that $P\rho P^{-1} : R \longrightarrow M_d(C)$ is adapted to $\mathcal{E}$. Moreover, if every finite type projective B-module is free, then we can take $C = B$.*

Proof. — As $\mathrm{tr} \circ \rho = T$, the $\rho(E_i^{k,k})$'s form an orthogonal family of d idempotents of trace 1 of $M_d(B)$ whose sum is 1. As a consequence, in the B-module decomposition

$$B^d = \oplus_{i,k} \rho(E_i^{k,k})(B^d),$$

the modules $\rho(E_i^{k,k})(B^d)$'s are projective, hence become free (necessarily of rank 1) over a suitable ring C containing B (and of course we can take $C = B$ if those modules are already free). We now define a C-basis $f_1, \dots, f_d$ of C^d as follows. For each $1 \leq i \leq r$, choose first g_i a C-basis of $\rho(E_i^{1,1})(C^d)$. Then for $1 \leq k \leq d_i$,

$$f_{d_1+\cdots+d_{i-1}+k} := \rho(E_i^{k,1})(g_i)$$

is a C-basis of $\rho(E_1^{i,i})(C^d)$. By construction, in this new C-basis, ρ is adapted to $\mathcal{E}$. □

Let us call G the natural covariant functor from commutative A-algebras to sets such that for a commutative A-algebra B, $G(B)$ is the set of representations (not considered "up to isomorphism") $\rho : R \longrightarrow M_d(B)$ adapted to $\mathcal{E}$.

Let B be a commutative algebra and $\rho \in G(B)$. By a slight abuse of language we set $E_i := \rho(E_i) \in M_d(B)$. By definition, for each i, j, $\rho(E_i R E_j) = E_i \rho(R) E_j$, hence it falls into the B-module of matrices whose coefficients are 0 everywhere, except on line $d_1 + \cdots + d_{i-1} + 1$ and row $d_1 + \cdots + d_{j-1} + 1$. We get this way an A-linear map $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow B$, whose image is an A-submodule of B which we denote by $A_{i,j}$. Hence

Proposition 1.3.8. — *The subalgebra $\rho(R)$ of $M_d(B)$ is the standard GMA of type $(d_1, \dots, d_r)$ associated to the A-submodules $A_{i,j}$ of B (see example 1.3.4).*

Moreover, the $f_{i,j}$'s have the two following properties:

(i) $f_{i,i}$ is the structural map $A \longrightarrow B$,

(ii) the product $\cdot : B \otimes B \longrightarrow B$ induces the $\varphi_{i,j,k}$'s, i.e.

$$\forall i, j, k, \quad f_{i,k} \circ \varphi_{i,j,k} = f_{i,j} \cdot f_{j,k}.$$

This leads us to introduce the following new functor. If B is a commutative A-algebra, let $F(B)$ be the set of $(f_{i,j})_{1\le i,j\le r}$, where $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow B$ is an A-linear map, satisfying conditions (i) and (ii) above. It is easy to check that F is a covariant functor from commutative A-algebras to sets. In the discussion above, we attached to each $\rho \in G(B)$ an element $f_\rho = (f_{i,j}) \in F(B)$.

Proposition 1.3.9. — *$\rho \mapsto f_\rho$ induces an isomorphism of functors $G \xrightarrow{\sim} F$. Both those functors are representable by a commutative A-algebra B^{univ}.*

Proof. — Let B be a commutative A-algebra and $f := (f_{i,j}) \in F(B)$. Then f induces coefficient-wise a natural map

$$\rho_f : R = \oplus_{i,j} M_{d_i,d_j}(\mathcal{A}_{i,j}) \longrightarrow \oplus_{i,j} M_{d_i,d_j}(B) = M_d(B).$$

This map is by definition a morphism of A-algebras which is adapted to $\mathcal{E}$. We get this way a morphism $F \longrightarrow G$ which is obviously an inverse of $\rho \mapsto f_\rho$ constructed above.

To prove the second assertion, it suffices to prove that F is representable. If M is an A-module, we will denote by $\mathrm{Symm}(M) := \oplus_{k\ge 0}\mathrm{Symm}^k(M)$ the symmetric A-algebra of M. We set

$$\mathcal{B} := \mathrm{Symm}(\bigoplus_{i\neq j} \mathcal{A}_{i,j}).$$

Let J be the ideal of $\mathcal{B}$ generated by all the elements of the form $b \otimes c - \varphi(b \otimes c)$, where $b \in \mathcal{A}_{i,j}, c \in \mathcal{A}_{j,k}$ and $\varphi = \varphi_{i,j,k}$, for all i, j and k in $\{1, \dots, r\}$. It is obvious that $B^{\mathrm{univ}} := \mathcal{B}/I$, equipped with the canonical element $(f_{i,j} : \mathcal{A}_{i,j} \to B^{\mathrm{univ}})_{i,j} \in F(B^{\mathrm{univ}})$, is the universal object we are looking for. □

1.3.4. An embedding problem. — It is a natural question to ask when a trace algebra (R, T) has an *injective* trace representation of dimension d, that is, when it can be embedded trace compatibly in a matrix algebra over a commutative ring. A beautiful theorem of Procesi [**93**] gives a very satisfactory answer when A is a $\mathbb{Q}$-algebra: (R, T) has an *injective* trace representation of dimension d if and only if T satisfies the d-th Cayley-Hamilton identity (see [**93**] and § 1.2.3).

Assume that $(R, \mathcal{E})$ is a GMA. Then we may ask two natural questions:

(1) Is there an injective d-dimensional trace representation of R?

(2) Is there an injective d-dimensional adapted representation of R?

Actually, it turns out that those questions are equivalent. Indeed, if $\rho : R \longrightarrow M_d(B)$ is an injective trace representation, then Lemma 1.3.7 gives an injective adapted representation $R \to M_d(C)$ for some ring $C \supset B$. By elementary reasoning, question (2) is equivalent to the following questions (3) or (4).

(3) Is the universal adapted representation $\rho : R \longrightarrow M_d(B^{\mathrm{univ}})$ injective?

(4) Are the universal maps $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow B^{\mathrm{univ}}$ injective?

For a GMA for which we know *a priori* that T is a Cayley-Hamilton pseudocharacter of dimension d (residually multiplicity free Cayley-Hamilton pseudocharacters over local henselian rings are examples of such a situation - see § 1.4), Procesi's result gives a positive answer to question (1), hence to questions (2) to (4) as well, in the case where A is a $\mathbb{Q}$-algebra. We shall give below a positive answer in the general case to those questions. As a consequence, by Proposition 1.3.8, any GMA is isomorphic to some standard GMA of Example 1.3.4, and its trace is a Cayley-Hamilton pseudocharacter of dimension d. Note that it does not seem much easier to prove first this last fact.

This result (the positive answer to questions (1) to (4)) will be used in its full generality only in the proof of the Theorem 1.6.3 (and here only for $r = 2$), and also to prove the converse of Theorem 1.4.4 (i) (see Example 1.4.2). In particular, it is not needed for the Galois theoretic applications of the following sections. However, we shall use several times this result in a special case (see § 1.3.5 below) where there is a much simpler proof, and where more precise results are available. Hence, for the convenience of the reader, we first give the proof in this special case.

1.3.5. Solution of the embedding problem in the reduced and nondegenerate case. — Let $I = \{1, \dots, r\}$ and assume that we are given a family of A-modules $\mathcal{A}_{i,j}$, $i, j \in I$, and for each i, j, k in I an A-linear map [11]

$$\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k},$$

which satisfy (UNIT), (ASSO) and (COM). We denote by F again the functor from commutative A-algebras to sets which is associated to this data, as defined in § 1.3.3.

Lemma 1.3.10. — (i) *Assume that the $\mathcal{A}_{i,j}$'s are free of rank* 1 *over A, and that the $\varphi_{i,j,k}$ are isomorphisms. Then there is a $(f_{i,j}) \in F(A)$ such that the $f_{i,j}$'s are isomorphisms.*

(ii) *The relation $i \sim j$ if, and only if, $\mathcal{A}_{i,j}$ is free of rank one and $\varphi_{i,j,i}$ is an isomorphism, is an equivalence relation on I. Moreover, if $i \sim j \sim k$, then $\varphi_{i,j,k}$ is an isomorphism.*

Proof. — We first show (i). Let $e_{i,j}$ be an A-basis of $\mathcal{A}_{i,j}$. As $\varphi_{i,j,k}$ is an isomorphism, there exists a unique $\lambda_{i,j,k} \in A^*$ such that $\varphi_{i,j,k}(e_{i,j} \otimes e_{j,k}) = \lambda_{i,j,k} e_{i,k}$. Let us fix some $i_0 \in I$. For all i, j, set $\mu_{i,j} := \lambda_{i,i_0,j}$. We claim that the A-linear isomorphisms $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow A$ defined by $f_{i,j}(e_{i,j}) = \mu_{i,j}$ satisfy $(f_{i,j}) \in F(A)$. It suffices to check that for all i, j, k, we have $\mu_{i,j}\mu_{j,k} = \lambda_{i,j,k}\mu_{i,k}$. But this is the hypothesis (ASSO) applied to i, i_0, j and k.

[11] All the tensor products below are assumed to be over A.

Let us show (ii). By (UNIT) we have $i \sim i$, and by (COM) $i \sim j$ implies $j \sim i$. If $i \sim j$ and $j \sim k$ we claim that $\varphi_{i,j,k}$ and is an isomorphism. It will imply that $\mathcal{A}_{i,k}$ and $\mathcal{A}_{k,i}$ is free of rank 1 over A, and that $\varphi_{i,k,i}$ is an isomorphism by (ASSO), hence $i \sim k$. Using (UNIT) and (ASSO), we check easily [12] the equality of linear maps

$$\varphi_{i,k,i} \circ (\varphi_{i,j,k} \otimes \varphi_{k,j,i}) = \varphi_{j,k,j} \cdot \varphi_{i,j,i} : \ \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \otimes \mathcal{A}_{k,j} \otimes \mathcal{A}_{j,i} \longrightarrow A.$$

As $i \sim j$ and $j \sim k$, it implies that $\varphi_{i,j,k}$ is injective. The surjectivity of $\varphi_{i,j,k}$ comes from the fact that the natural map

$$\mathcal{A}_{i,k} \otimes \mathcal{A}_{k,j} \otimes \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k}$$

is an isomorphism (as $j \sim k$) whose image is contained in $\mathrm{Im}\,(\varphi_{i,j,k})$ by (ASSO). □

Before stating the main proposition of this subsection, we need to recall some definitions from commutative algebra. If A is a commutative ring, recall that the *total fraction ring* of A is the fraction ring $\mathrm{Frac}(A) := S^{-1}A$ where $S \subset A$ is the multiplicative subset of nonzerodivisors of A, that is $f \in S$ if and only if the map $g \mapsto gf$, $A \to A$, is injective. We check at once that the natural map $A \to S^{-1}A$ is injective and flat, and that each nonzerodivisor of $S^{-1}A$ is invertible. Of course, $S^{-1}A$ is the fraction field of A if A is a domain.

Proposition 1.3.11. — *Assume A is reduced. The following properties are equivalent:*

(i) *A has a finite number of minimal prime ideals,*
(ii) *A embeds into a finite product of fields,*
(iii) *$S^{-1}A$ is a finite product of fields.*

If they are satisfied, $S^{-1}A = \prod_P A_P$ where the product is over the finite set of minimal prime ideals of A.

Proof. — It is clear that (i) and (ii) are each equivalent to the following assertion: "there exist a finite number of prime ideals $P_1, \ldots, P_r$ of A such that $P_1 \cap \cdots \cap P_r = 0$". In particular, (i) and (ii) are equivalent.

Note that $\mathrm{Spec}(S^{-1}A) \subset \mathrm{Spec}(A)$ is the subset of prime ideals that do not meet S. For P any minimal prime ideal of A, and $f \in S$, remark that the image of f in $A_P = \mathrm{Frac}(A/P)$ is not a zero divisor of this latter ring by flatness of A_P over A, so $S \cap P = \varnothing$. In particular, A and $S^{-1}A$ have the same minimal prime ideals, and (iii)

[12] If $(x, y, z, t) \in (\mathcal{A}_{i,j} \times \mathcal{A}_{j,k} \times \mathcal{A}_{k,j} \times \mathcal{A}_{j,i})$, using (ASSO), (ASSO) again, and (UNIT), we have with the obvious notations: $(xy)(zt) = x(y(zt)) = x((yz)t) = (yz)(xt)$. In general, to check this kind of identities with values in some $\mathcal{A}_{k,l}$, it suffices to do it in the GMA of type $(1, 1, \ldots, 1)$ defined by the $\mathcal{A}_{i,j}$, which might be a bit easier (e.g. in the proof of Proposition 1.3.13).

implies (i). Moreover, if A has a finite number of minimal prime ideals, say $P_1, \dots, P_r$, then we have an injection

$$A \longrightarrow \prod_{i=1\dots r} A_{P_i},$$

so

$$S = A \setminus (P_1 \cup \dots \cup P_r). \tag{7}$$

Assume now that (i) holds, we will show (iii) as well as the last assertion of the statement. As A and $S^{-1}A$ have the same minimal prime ideals, we may assume that $S^{-1}A = A$, *i.e.* that each nonzerodivisor of A is invertible. By (7), we get that for each maximal ideal m of A, $m \subset \cup_{i=1\dots r} P_i$. By [**29**, Chap. II, §1.1, Prop. 2], this implies that each P_i is maximal, hence

$$A \xrightarrow{\sim} \prod_{i=1\dots r} A_{P_i}$$

and we are done. □

An A-module M is said to be *torsion free* if the multiplication by each $f \in S$ on M is injective, *i.e.* if the natural map $M \to S^{-1}M$ is injective. An A-submodule M of $S^{-1}A$ is said to be a *fractional ideal of* $S^{-1}A$ if $fM \subset A$ for some $f \in A$ which is not a zerodivisor. Assume that A is reduced and that $S^{-1}A = \prod_s K_s$ is a finite product of fields. Note that if $A_s = \mathrm{Im}\,(A \longrightarrow K_s)$, then $\prod_s A_s$ is a fractional ideal of K. As a consequence, $M \subset K$ is a fractional ideal if, and only if, for each s, $\mathrm{Im}\,(M \longrightarrow K_s)$ is a fractional ideal of K_s. We will often denote by K the total fraction ring $S^{-1}A$.

Proposition 1.3.12. — *Assume that A is reduced and that its total fraction ring K is a finite product of fields. Assume moreover that the maps $\varphi_{i,j,i} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,i} \longrightarrow A$ are nondegenerate* [(13)].

Then there exists $(f_{i,j}) \in F(K)$ such that each $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow K$ is an injection whose image is a fractional ideal of K. Moreover, if $A = K$ is a field, the relation $i \sim j$ if, and only if, $\mathcal{A}_{i,j} \neq 0$ coincides with the one of Lemma 1.3.10.

Proof. — Write $K = \prod_s K_s$ as a finite product of fields. As $\mathcal{A}_{i,j}$ embeds into $\mathrm{Hom}_A(\mathcal{A}_{j,i}, A)$ by assumption, it is torsion free over A, hence embeds into $\mathcal{A}_{i,j} \otimes K$. As $A \to K$ is an injection into a fraction ring, we check easily that $\varphi_{i,j,i} \otimes K$ is again nondegenerate [(14)], hence so are the $\varphi_{i,j,i} \otimes K_s$'s. By (ASSO) applied to i, j, i, j, and by (COM) and (UNIT), we have:

$$\forall x, x' \in \mathcal{A}_{i,j}, \forall y \in \mathcal{A}_{j,i}, \quad \varphi_{i,j,i}(x', y)x = \varphi_{i,j,i}(x, y)x'.$$

(13) That is, that the induced maps $\mathcal{A}_{i,j} \longrightarrow \mathrm{Hom}_A(\mathcal{A}_{j,i}, A)$ are injective.

(14) If A is any commutative ring with total fraction ring $S^{-1}A$, and M any A-module (not necessarily of finite type), then the natural map $S^{-1}\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}A)$ is injective.

hence $\mathcal{A}_{i,j} \otimes K_s$ has K_s-dimension ≤ 1 and $\mathcal{A}_{i,j}$ is isomorphic to a fractional ideal of K. It remains only to construct the injections $f_{i,j}$ of the statement. By what we have just seen, we can assume that $A = K$ is a field, and in this case each $\mathcal{A}_{i,j}$ is either 0 or one dimensional over K, and the $\varphi_{i,j,i}$'s are nondegenerate, hence isomorphisms.

For $i, j \in I$, say $i \sim j$ if $\mathcal{A}_{i,j} \neq 0$. As the $\varphi_{i,j,i}$ are isomorphisms, this relation coincides with the one defined in Lemma 1.3.10 (ii). On each equivalence class of the relation $\sim$, we define some $f_{i,j}$'s by Lemma 1.3.10 (i), and we set $f_{i,j} := 0$ if $i \not\sim j$. $\square$

1.3.6. Solution of the embedding problem in the general case. — Same notations as in § 1.3.5. We recall that B^{univ} is the universal A-algebra representing F (see Proposition 1.3.9).

Proposition 1.3.13. — *The universal maps $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow B^{\mathrm{univ}}$ are A-split injections.*

Proof. — We use the notations of the proof of Proposition 1.3.9. Recall that $I = \{1, \ldots, r\}$ and set $\Omega := \{(i,j),\ i,\ j \in I,\ i \neq j\}$; if $x = (i',j') \in \Omega$ we will write $i(x) := i'$ and $j(x) := j'$.

If $\gamma = (x_1, \ldots, x_s)$ is a sequence of elements of Ω such that for all $k \in \{1, \ldots, s-1\}$ we have $j(x_k) = i(x_{k+1})$, then we will say that γ is a *path* from $i(x_1)$ to $j(x_s)$, and we will set $\mathcal{A}_\gamma := \mathcal{A}_{i(x_1),j(x_1)} \otimes \cdots \otimes \mathcal{A}_{i(x_s),j(x_s)}$. If moreover $i(x_1) = j(x_s)$, we will say that γ is a *cycle*. In this case, $\mathrm{rot}(\gamma) := (x_s, x_1, \ldots, x_{s-1})$ is again a cycle. Let $i, j \in I$, γ a path from i to j, and $c_1, \ldots, c_n$ a sequence of cycles (which can be empty). We will call the sequence of paths $\Gamma = (c_1, \ldots, c_n, \gamma)$ an *extended path* from i to j. If Γ is such a sequence and $(i',j') \in \Omega$, we denote by $\Gamma_{i',j'}$ the total number of times that (i',j') appears in the c_k's or in γ. It will be convenient to identify $\mathbb{N}^\Omega$ with the set of oriented graphs [15] with set of vertices I, by associating to $\tau = (\tau_{i,j})_{(i,j)\in\Omega}$ the graph with $\tau_{i,j}$ edges from i to j. If Γ is an extended path from i to j, we shall say that $\tau(\Gamma) := (\Gamma_{i',j'}) \in \mathbb{N}^\Omega$ is the *underlying graph* of Γ.

Let $\deg : \mathbb{N}^\Omega \longrightarrow \mathbb{Z}^I$ be the map such that, for $\tau \in \mathbb{N}^\Omega, i \in I$, $\deg(\tau)_i$ is the number of arrows in τ arriving at i minus the number of arrows departing from i. If $(i,j) \in \Omega$, let $\tau(i,j)$ be the graph with a unique arrow, which goes from i to j. If $i \in I$, set $\tau(i,i) = 0$. The following lemma is easily checked.

Lemma 1.3.14. — *Let $i, j \in I$.*

(i) *If Γ is an extended path from i to j, then* $\deg(\tau(\Gamma)) = \deg(\tau(i,j))$.
(ii) *If τ is a graph such that* $\deg(\tau) = \deg(\tau(i,j))$, *then $\tau = \tau(\Gamma)$ for some extended path Γ from i to j. If moreover $\tau_{i',j'} \neq 0$ and $\tau_{j',k'} \neq 0$ for some $i', j', k' \in I$,*

[15] In an oriented graph, we authorize multiple edges between two vertices i and j, with $i \neq j$, but we do not authorize edges from a vertex to itself.

then we can assume that the sequence Γ *has a path containing* $((i',j'),(j',k'))$ *as a subpath.*

By (ASSO), for each path γ from i to j, we have a canonical *contraction* map $\varphi_\gamma : \mathcal{A}_\gamma \longrightarrow \mathcal{A}_{i,j}$. If γ is a cycle, φ_γ goes from $\mathcal{A}_\gamma$ to A by (UNIT), and the assumption (COM) implies that $\varphi_{\mathrm{rot}(\gamma)} = \varphi_\gamma \circ \mathrm{rot}$, where $\mathrm{rot} : \mathcal{A}_{\mathrm{rot}(\gamma)} \longrightarrow \mathcal{A}_\gamma$ is the canonical circular map. We claim now that the following property holds:

(SYM) For any cycle c having some $(i',j') \in \Omega$ in common with some path γ', the map $\varphi_c \otimes \mathrm{id} : \mathcal{A}_c \otimes \mathcal{A}_{\gamma'} \longrightarrow \mathcal{A}_{\gamma'}$ is symmetric in that two $\mathcal{A}_{i',j'}$'s.

Indeed, by the rotation property we can assume that c begins with (i',j'), and by (ASSO) and (UNIT) that $\gamma' = (i',j')$. By (ASSO) and (UNIT) again, we can assume then that $c = ((i',j'),(j',i'))$, in which case it is an easy consequences of (ASSO) (applied with i,j,i,j), (UNIT) and (COM).

Fix $i,j \in I$. Let $\Gamma = (c_1,\dots,c_n,\gamma)$ be an extended path from i to j. We can consider the following A-linear map $\varphi_\Gamma : \mathcal{A}_{c_1} \otimes \cdots \otimes \mathcal{A}_{c_n} \otimes \mathcal{A}_\gamma \longrightarrow \mathcal{A}_{i,j}$,

$$\left(\bigotimes_{k=1}^{n} x_k\right) \otimes y \mapsto \left(\prod_{k=1}^{n} \varphi_{c_k}(x_k)\right) \varphi_\gamma(y).$$

By the property (SYM), φ_Γ factors canonically through a map

$$\overline{\varphi}_\Gamma : \bigotimes_{(k,l)\in\Omega} \mathrm{Symm}^{\Gamma_{k,l}}(\mathcal{A}_{k,l}) \longrightarrow \mathcal{A}_{i,j}.$$

It is clear that:

(i) for any permutation $\sigma \in \mathfrak{S}_n$, $\overline{\varphi}_{(c_{\sigma(1)},\dots,c_{\sigma(n)},\gamma)} = \overline{\varphi}_{(c_1,\dots,c_n,\gamma)}$,

(ii) as the φ_{c_k}'s are invariant under rotation, $\overline{\varphi}_{(\mathrm{rot}(c_1),\dots,c_n,\gamma)} = \overline{\varphi}_{(c_1,\dots,c_n,\gamma)}$.

Let $\gamma = (x_1,\dots,x_s)$ be a path from i to j and $c = (y_1,\dots,y_{s'})$ be a cycle. We will say that γ and c are *linked at* $i' \in I$ if there exists $x_k \in \gamma$ and $y_{k'} \in c$ with same origin, that is such that $i(x_k) = i(y_{k'}) = i'$. Then can consider the path $\gamma \cup c := (x_1,\dots,x_{k-1},y_{k'},\dots,y_{s'},y_1,\dots,y_{k'-1},x_k,\dots,x_s)$, which still goes from i to j. Then we see that $\overline{\varphi}_{\gamma\cup c} = \overline{\varphi}_{c,\gamma}$, and it does not depend in particular on the i' such that γ and c are linked at i'. As a consequence, going back to the notation of the paragraph above, we have:

(iii) if γ and c_1 (resp. c_1 and c_2) are linked, then $\overline{\varphi}_{(c_1,\dots,c_n,\gamma)} = \overline{\varphi}_{(c_2,\dots,c_n,\gamma\cup c_1)}$ (resp. $\overline{\varphi}_{(c_1,c_2,\dots,c_n,\gamma)} = \overline{\varphi}_{(c_1\cup c_2,c_3,\dots,c_n,\gamma)}$).

Let now Γ' be another extended path from i to j. Then using several times the "moves" (i), (ii) and (iii), we check at once that $\overline{\varphi}_\Gamma = \overline{\varphi}_{\Gamma'}$. Let $\tau \in \mathbb{N}^\Omega$ satisfies $\deg(\tau) = \deg(\tau(i,j))$. By Lemma 1.3.14 (ii), we can choose an extended path Γ from

i to j with underlying graph τ, and define

$$\overline{\varphi}_\tau := \overline{\varphi}_\Gamma, \quad \bigotimes_{(k,l)\in\Omega} \mathrm{Symm}^{\tau_{k,l}}(\mathcal{A}_{k,l}) \longrightarrow \mathcal{A}_{i,j},$$

which does not depend on Γ (whose associated graph is τ) by what we said above.

Let us finish the proof of the proposition. The A-algebras $\mathcal{B}$ and is naturally graded by the additive monoid $\mathbb{N}^\Omega$. We have $\mathcal{B} = \oplus_{\tau\in\mathbb{N}^\Omega}\mathcal{B}_\tau$, where $\mathcal{B}_\tau = \bigotimes_{i\neq j}\mathrm{Symm}^{\tau_{i,j}}(\mathcal{A}_{i,j})$. The map $\deg : \mathcal{G} \longrightarrow \mathbb{Z}^I$ is additive, hence we get a $\mathbb{Z}^I$-graduation [16] on $\mathcal{B}$. Obviously, if $n \in \mathbb{Z}^I$, then $\mathcal{B}_n = \bigoplus_{\tau\in\mathbb{N}^\Omega,\deg(\tau)=n}\mathcal{B}_\tau$. For this latter graduation, the ideal $J \subset \mathcal{B}$ is homogeneous, hence B^{univ} is also graded by $\mathbb{Z}^I$.

Fix now $i, j \in I$, and let $n := \deg(\tau(i,j))$. If $\deg(\tau) = n$, we constructed above a map $\overline{\varphi}_\tau : \mathcal{B}_\tau \longrightarrow \mathcal{A}_{i,j}$. By summing all of them we get an A-linear map:

$$\overline{\varphi}_n : \mathcal{B}_n \longrightarrow \mathcal{A}_{i,j}.$$

We claim that $\overline{\varphi}_n(I_n) = 0$. Assuming that, $\overline{\varphi}_n$ factors through a map

$$\psi_n : (B^{\mathrm{univ}})_n \longrightarrow \mathcal{A}_{i,j}.$$

Let $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow (B^{\mathrm{univ}})_n$ denote the canonical map. Then by construction, $\psi_n \circ f_{i,j} = \overline{\varphi}_{\tau(i,j)}$ is the identity map. It concludes the proof.

Let us check the claim. Let $b \in \mathcal{A}_{i',j'}$, $c \in A_{j',k'}$ and $\varphi = \varphi_{i',j',k'}$, for some (i',j'), $(j',k') \in \Omega$. By A-linearity, is suffices to show that $\overline{\varphi}_n$ vanishes on the elements of the form $x = f \otimes (b \otimes c - \varphi(b \otimes c))$, where f is in $\mathcal{B}_\tau$ for some graph τ satisfying $\deg(\tau + \tau(i',k')) = n$. By Lemma 1.3.14 (ii), we can find an extended path Γ from i to j with underlying graph $\tau + \tau(i',j') + \tau(j',k')$, such that some path γ' of Γ contains $((i',j'),(j',k'))$ as a subpath. Let Γ' be the extended path from i to j obtained from Γ by replacing $\gamma' = (\dots,(i',j'),(j',k'),\dots)$ by $(\dots,(i',k'),\dots)$. By construction, $\overline{\varphi}_\Gamma(f \otimes c \otimes b) = \overline{\varphi}_{\Gamma'}(f \otimes \varphi(b \otimes c))$, hence $\overline{\varphi}(x) = 0$. □

Remark 1.3.15. — When $r = 2$, a slight modification of the above proof shows that the A-linear map $A \oplus \bigoplus_{n\geq 1}(\mathrm{Symm}^n(\mathcal{A}_{1,2}) \oplus \mathrm{Symm}^n(\mathcal{A}_{2,1})) \longrightarrow B^{\mathrm{univ}}$, induced by $f_{1,2}$ and $f_{2,1}$, is an isomorphism. This describes B^{univ} completely in this case.

As we have noted in § 1.3.4, we have:

Corollary 1.3.16. — *If $(R,\mathcal{E})$ is a GMA of type $(d_1,\dots,d_r)$, and if $d!$ is invertible in A (where $d = d_1 + \dots + d_r$), then the trace T of R is a Cayley-Hamilton pseudocharacter of dimension d.*

[16] Actually, it is even graded by the subgroup of $\mathbb{Z}^I$ whose elements (n_i) satisfy $\sum_i n_i = 0$.

1.4. Residually multiplicity-free pseudocharacters

1.4.1. Definition. — In all this section, A is a local henselian ring (see [**94**]), m is the maximal ideal of A, and $k := A/m$. The henselian property will be crucial in what follows because it implies strong results on *lifting idempotents*. Let R be an A-algebra and let $T : R \longrightarrow A$ be a pseudocharacter of dimension d. We recall that this implies that $d!$ is invertible in A. Let $\bar{R} := R \otimes_A k$ and $\bar{T} := T \otimes_A k : \bar{R} \longrightarrow k$ be the reductions mod m of R and T.

Definition 1.4.1. — We say that T is residually multiplicity free if there are representations $\bar{\rho}_i : R \longrightarrow M_{d_i}(k)$, $i = 1, \dots, r$, which are absolutely irreducible and pairwise nonisomorphic, such that $\bar{T} = \sum_{i=1}^r \operatorname{tr} \bar{\rho}_i$.

We set $d_i := \dim \bar{\rho}_i$, we have $\sum_{i=1}^r d_i = d$.

Example 1.4.2. — Let us give an important example. Let $(R, \mathcal{E})$ be a GMA (§ 1.3.1), then its trace $T : R \longrightarrow A$ is a Cayley-Hamilton pseudocharacter by Corollary 1.3.16. We use the notations of § 1.3.2. Assume moreover that for all $i \neq j$, we have

$$T(\mathcal{A}_{i,j}\mathcal{A}_{j,i}) \subset m.$$

Now, for each i, let $\bar{\rho}_i : R \longrightarrow M_{d_s}(k)$, $r \mapsto (\psi_i(e_i r e_i) \bmod m)$. Then the $\bar{\rho}_i$ are surjective representations[(17)] which are pairwise non isomorphic since $\bar{\rho}_i(e_i) \neq 0$ while $\bar{\rho}_i(e_j) = 0$ for $j \neq i$, and $\overline{T} = \sum_{s=1}^r \operatorname{tr} \bar{\rho}_i$, hence T is residually multiplicity free. The main result of this section shows that this example is the general case.

1.4.2. Lifting idempotents. — Let A, R and T be as in § 1.4.1, and assume that T is residually multiplicity free. In particular, we have some representations $\bar{\rho}_i : R \longrightarrow M_{d_i}(k)$ as in definition 1.4.1.

Lemma 1.4.3. — *Suppose T Cayley-Hamilton. There are orthogonal idempotents $e_1, \dots, e_r$ in R such that*

(1) $\sum_{i=1}^r e_i = 1$.
(2) *For each i, $T(e_i) = d_i$*
(3) *For all $x \in R$, we have $T(e_i x e_i) \equiv \operatorname{tr} \bar{\rho}_i(x) \pmod{m}$*
(4) *If $i \neq j$, $T(e_i x e_j y e_i) \in m$ for any $x, y \in R$.*
(5) *There is an A-algebra isomorphism $\psi_i : e_i R e_i \longrightarrow M_{d_i}(A)$ lifting $(\bar{\rho}_i)_{|e_i R e_i} : e_i R e_i \longrightarrow M_{d_i}(k)$, and such that for all $x \in e_i R e_i$, $T(x) = \operatorname{tr}(\psi_i(x))$.*

Moreover, if $e'_1, \dots, e'_r$ is another family of orthogonal idempotents of R satisfying (3), then there exists $x \in 1 + \operatorname{rad}(R)$ such that for all i, $e'_i = x e_i x^{-1}$.

(17) Note that the maps $f_{i,j} : \mathcal{A}_{i,j} \longrightarrow k$, defined to be 0 if $i \neq j$, and $A \xrightarrow{\text{can}} k$ if $i = j$, define an element of $F(k)$.

Proof. — Let $\bar{\rho} : \bar{R} \longrightarrow M_d(k)$ be the product of the $\bar{\rho}_i$'s. We obviously have $\operatorname{Ker} \bar{\rho} \subset \operatorname{Ker} \bar{T}$. Because the $\bar{\rho}_i$'s are pairwise distinct, the image of $\bar{\rho}$ is $\prod_{i=1}^r M_{d_i}(k)$. In particular, the image J of $\operatorname{Ker} \overline{T}$ in this latter product is a two-sided ideal. But J is a nil ideal by Lemma 1.2.1, so $J = 0$ and

$$\operatorname{Ker} \bar{\rho} = \operatorname{Ker} \bar{T}.$$

We have the following diagram

$$\begin{tikzcd} \bar{R}/\operatorname{Ker} \bar{T} \arrow[rr, "\bar{\rho}=\prod_{i=1}^r \bar{\rho}_i"] \arrow[dr, "\bar{T}"'] & & \prod_{i=1}^r M_{d_i}(k) \arrow[dl, "tr"] \\ & k & \end{tikzcd} \tag{8}$$

which commutes by assumption on T, and whose first row is an isomorphism. Let us call ϵ_i, for $i = 1, \dots, r$, the central idempotents of $\bar{R}/\operatorname{Ker} \bar{T}$ corresponding to the unit of $M_{d_i}(k)$ in this decomposition.

By the Cayley-Hamilton identity, and following [**29**, chap. III, §4, exercice 5(b)] (18), there exists a family of orthogonal idempotents $e_i \in R$, $i = 1, \dots, r$, with e_i lifting the ϵ_i. The element $1 - \sum_{i=1}^r e_i$ is an idempotent which is in the Jacobson radical of R by Lemma 1.2.7, hence it is 0, which proves (1). By Lemma 1.2.5(1) $T(e_i)$ is an integer less than or equal to d, and because $\overline{T(e_i)} = \bar{T}(\epsilon_i) = d_i$, we have $T(e_i) = d_i$, which is (2).

The assertion (3) follows from the diagram (8). In order to prove (5) it suffices to show that the image of $e_i x e_j y e_i$ is zero in $\bar{R}/\operatorname{Ker} \bar{T}$. But this image is $\epsilon_i \bar{x} \epsilon_j \bar{y} \epsilon_i$ which is zero by the diagram (8), and we are done.

Now consider the restriction T_i of T to the subalgebra $e_i R e_i$ (with unit element e_i) of R. By Lemma 1.2.5(3-4), T_i is a pseudocharacter of dimension $d_i = T(e_i)$, faithful if T is. By (3), T_i is moreover residually absolutely irreducible. If we had assumed T faithful, we could have applied [**102**, Thm. 5.1 or cor. 5.2] to get (5). As we assume only T Cayley-Hamilton, we have to argue a bit more. By Lemma 1.2.5 (4), T_i is Cayley-Hamilton, hence we may assume that $r = 1$, and we have to prove that $R = M_d(A)$. By Lemma 1.2.7 and [**29**, chap. III, §4, exercice 5(c)], we can lift the basic matrices of $R/\operatorname{Ker}(\bar{\rho}_1) = M_{d_1}(k)$, i.e. find elements $(E^{k,l})_{1 \leq k,l \leq d}$ in R satisfying the relations $E^{k,l} E^{k',l'} = \delta_{l,k'} E^{k,l'}$. By Lemma 1.2.5 (1), for each $k \in \{1, \dots, d\}$, $T(E^{k,k})$ is an integer in $\{1, \dots d\}$. As this integer is furthermore congruent to 1 modulo m by definition of $E^{k,k}$, and as $d! \in A^*$, we have $T(E^{k,k}) = 1$. By Lemma 1.2.5 (4),

(18) The statement is that if A is a henselian local ring, R an A-algebra which is integral over A, and I a two-sided ideal of R, then any family of orthogonal idempotents of R/I lifts to R. Note that is stated there with R a finite A-algebra, but the same proof holds in the integral case.

$T_k : E^{k,k}RE^{k,k} \longrightarrow A$ is Cayley-Hamilton of dimension 1, hence T_k is an isomorphism and $E^{k,k}RE^{k,k} = AE^{k,k}$ is free of rank 1 over A. Now, if $x \in E^{k,k}RE^{l,l}$, then

$$x = E^{k,l}(E^{l,k}x) = E^{k,l}(T(E^{l,k}x)E^{l,l}) = T(E^{l,k}x)E^{k,l} \in AE^{k,l},$$

hence $R = \sum_{k,l} AE^{k,l}$. This concludes the proof of (5) (we even showed that Rouquier's Theorem 5.1 holds when faithful is replaced by Cayley-Hamilton).

To prove the last assertion, note first that the hypothesis on the e'_i means that $\overline{e'_i} = \varepsilon_i$, hence by the work above properties (1) to (5) also hold for the e'_i's. As $e_iRe_i \simeq M_{d_i}(A)$ is a local ring, the Krull-Schmidt-Azumaya Theorem [**49**, Thm. (6.12)] (see the remark there, [**49**, prop. 6.6] and [**49**, chap. 6, exercise 14]), there exists an $x \in R^*$ such that for each i, $xe_ix^{-1} = e'_i$. Up to conjugation by an element in $\sum_i(e_iRe_i)^*$, we may assume that $x \in 1 + \mathrm{rad}(R)$. □

1.4.3. The structure theorem. — Let A, R, T be as in § 1.4.1.

Theorem 1.4.4. — (i) *Let S be a Cayley-Hamilton quotient of (R,T). Then there is a data $\mathcal{E} = \{e_i, \psi_i, 1 \le i \le r\}$ on S for which S is a GMA and such that for each i, $\psi_i \otimes k = (\bar\rho_i)_{|e_iSe_i}$. Two such data on S are conjugate under S^*. Every such data defines A-submodules $\mathcal{A}_{i,j}$ of S that satisfy*

$$\mathcal{A}_{i,j}\mathcal{A}_{j,k} \subset \mathcal{A}_{i,k}, \quad T : \mathcal{A}_{i,i} \xrightarrow{\sim} A, \quad T(\mathcal{A}_{i,j}\mathcal{A}_{j,i}) \subset m$$

and

$$S \simeq \begin{pmatrix} M_{d_1}(\mathcal{A}_{1,1}) & M_{d_1,d_2}(\mathcal{A}_{1,2}) & \dots & M_{d_1,d_r}(\mathcal{A}_{1,r}) \\ M_{d_2,d_1}(\mathcal{A}_{2,1}) & M_{d_2}(\mathcal{A}_{2,2}) & \dots & M_{d_2,d_r}(\mathcal{A}_{2,r}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r,d_1}(\mathcal{A}_{r,1}) & M_{d_r,d_2}(\mathcal{A}_{r,2}) & \dots & M_{d_r}(\mathcal{A}_{r,r}) \end{pmatrix}$$

(ii) *Assume that A is reduced, and that its total fraction ring K is a finite product of fields. Take $S = R/\mathrm{Ker}\,T$. Choose a data $\mathcal{E}$ on S as in* (i). *Then there exists an adapted injective representation $\rho : S \longrightarrow M_d(K)$ whose image has the form*

$$\begin{pmatrix} M_{d_1}(A_{1,1}) & M_{d_1,d_2}(A_{1,2}) & \dots & M_{d_1,d_r}(A_{1,r}) \\ M_{d_2,d_1}(A_{2,1}) & M_{d_2}(A_{2,2}) & \dots & M_{d_2,d_r}(A_{2,r}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r,d_1}(A_{r,1}) & M_{d_r,d_2}(A_{r,2}) & \dots & M_{d_r}(A_{r,r}) \end{pmatrix}$$

where the $A_{i,j}$ are fractional ideals of K that satisfy

$$A_{i,j}A_{j,k} \subset A_{i,k}, \quad A_{i,i} = A, \quad A_{i,j}A_{j,i} \subset m.$$

Moreover the $A_{i,j}$'s are isomorphic to the $\mathcal{A}_{i,j}$'s of part (i)*, in such a way that the map $A_{i,j} \otimes_A A_{j,k} \longrightarrow A_{i,k}$ given by the product in K and the map $\mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k}$ given by the product in R coincide.*

(iii) *Let $P \in \mathrm{Spec}(A)$, $L := \mathrm{Frac}(A/P)$, and assume that $T \otimes L$ is irreducible*[19]*. If S is any Cayley-Hamilton quotient of (R, T), then $S \otimes L$ is trace isomorphic to $M_d(L)$. In particular, $T \otimes L$ is faithful and absolutely irreducible.*

Proof. — As S is Cayley-Hamilton, Lemma 1.4.3 gives us a data $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq r\}$ satisfying (i).

Assume now moreover that A is as in (ii), and set $S := R/\mathrm{Ker}\, T$. Since T is faithful on S, Lemma 1.3.5 proves that the $\varphi_{i,j,i}$'s are nondegenerate. Then Proposition 1.3.12 gives us a family of injections $f_{ij} : \mathcal{A}_{i,j} \longrightarrow L$, $(f_{i,j}) \in F(L)$ whose image are fractional ideals. Set $A_{i,j} := f_{i,j}(\mathcal{A}_{i,j})$. By Proposition 1.3.9, $(f_{i,j})$ defines an adapted representation $\rho : S \longrightarrow M_d(L)$ that satisfies (ii).

Let us prove (iii). Note that A/P is still local henselian and that $S \otimes A/P$ is a Cayley-Hamilton quotient of $(R \otimes A/P, T \otimes A/P)$, hence we may assume that A is a domain and that $P = 0$. By Remark 1.2.2, the natural map

$$(\mathrm{Ker}\, T) \otimes L \longrightarrow \mathrm{Ker}\,(T \otimes L)$$

is an isomorphism as L is the fraction field of A. By this and by (i) applied to $T : S/\mathrm{Ker}\, T \longrightarrow A$, we see that $S' := (S \otimes L)/(\mathrm{Ker}\, T \otimes L)$ is a GMA of type $(d_1, \dots, d_r)$ over L whose trace $T \otimes L$ is faithful. As $T \otimes L$ is irreducible by assumption, Proposition 1.3.12 implies that S' is trace isomorphic to $M_d(L)$, as the equivalence relation there may only have one class. Let us consider now the surjective map

$$\psi : S \otimes L \longrightarrow (S \otimes L)/(\mathrm{Ker}\, T \otimes L) \xrightarrow{\sim} M_d(L).$$

By Lemma 1.2.1, its kernel is in $\mathrm{rad}(S \otimes L)$. By an argument already given in part (5) of Lemma 1.4.3 (using the lifting of the $E^{k,l}$'s of $M_d(L)$ to $S \otimes L$, and checking that they span $S \otimes L$ by Lemma 1.2.5 (1) and (4)), ψ is an isomorphism, which concludes the proof. □

Remark 1.4.5. — If A is reduced and noetherian, it satisfies the conditions of (ii), hence the $A_{i,j}$'s and $R/\mathrm{Ker}\, T$ are finite type torsion free A-modules.

1.5. Reducibility loci and Ext-groups

1.5.1. Reducibility loci. — Let A be an henselian local ring, R an A-algebra and $T : R \longrightarrow A$ a residually multiplicity free pseudocharacter of dimension d. We shall use the notations of § 1.4.1.

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[19] This means that $T \otimes L$ is not the sum of two L-valued pseudocharacters on $S \otimes L$.

Proposition 1.5.1. — *Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_s)$ be a partition of $\{1, \dots, r\}$. There exists an ideal $I_{\mathcal{P}}$ of A such that for each ideal J of A, the following property holds if and only if $I_{\mathcal{P}} \subset J$:*
(dec$_{\mathcal{P}}$) *There exists pseudocharacters $T_1, \dots, T_s : R \otimes A/J \longrightarrow A/J$ such that*

(i) $T \otimes A/J = \sum_{l=1}^{s} T_l$,
(ii) *for each* $l \in \{1, \dots, s\}$, $T_l \otimes k = \sum_{i \in \mathcal{P}_l} \operatorname{tr} \bar{\rho}_i$.

If this property holds, then the T_l's are uniquely determined and satisfy $\operatorname{Ker} T_l \subset \operatorname{Ker}(T \otimes A/J)$.

Moreover, if S is a Cayley-Hamilton quotient of (R, T) then, using the notations of Theorem 1.4.4, we have (for any choice of the data $\mathcal{E}$ on S)

$$I_{\mathcal{P}} = \sum_{\substack{(i,j) \\ i,j \text{ are not in the same } \mathcal{P}_l}} T(\mathcal{A}_{i,j}\mathcal{A}_{j,i})$$

Proof. — Let S be a Cayley-Hamilton quotient of (R, T). We can then choose a GMA data $\mathcal{E}$ for S as in Theorem 1.4.4 (i), and consider the structural modules $\mathcal{A}_{i,j} = E_i S E_j$. We set

$$I_{\mathcal{P}}(T, S, \mathcal{E}) := \sum_{i,j \text{ are not in the same } \mathcal{P}_l} T(\mathcal{A}_{i,j}\mathcal{A}_{j,i}).$$

By Theorem 1.4.4 (i), $I_{\mathcal{P}}(T, S, \mathcal{E})$ does not depend on the choice of the data $\mathcal{E}$ used to define it. We claim that it does not depend on S. Indeed, we check at once that the image of $\mathcal{E}$ under the surjective homomorphism $\psi : S \longrightarrow R/\operatorname{Ker} T$ is a data of idempotents for $R/\operatorname{Ker} T$ (and even that ψ is an isomorphism on $\oplus_{i=1}^{r} e_i S e_i$). As $T \circ \psi = T$, we have that

$$T(\psi(\mathcal{A}_{i,j})\psi(\mathcal{A}_{j,i})) = T(\mathcal{A}_{i,j}\mathcal{A}_{j,i}),$$

which proves the claim. We can now set without ambiguity $I_{\mathcal{P}} := I_{\mathcal{P}}(T)$. As a first consequence of all of this, we see that if $J \subset A$ is an ideal, then $I_{\mathcal{P}}(T \otimes A/J)$ is the image in A/J of $I_{\mathcal{P}}(T)$.

To prove the proposition we are reduced to showing the following statement:

$T : R \longrightarrow A$ satisfies $I_{\mathcal{P}} = 0$ if and only if we can write $T = T_1 + \cdots + T_s$ as a sum of pseudocharacters satisfying assumption (ii) *in* (dec$_{\mathcal{P}}$).

Let us prove first the "only if" part of the statement above. Let $S = R/\operatorname{Ker} T$ and fix a GMA data $\mathcal{E}$ as in Theorem 1.4.4 (i). Set

(9) $$f_l := \sum_{i \in \mathcal{P}_l} e_i \in S,$$

then $1 = f_1 + \cdots + f_s$ is a decomposition in orthogonal idempotents. In this setting, the condition $I_{\mathcal{P}} = 0$ means that for each l,

$$T(f_l S(1-f_l)Sf_l) = 0. \tag{10}$$

As a consequence, the two-sided ideal $f_lS(1-f_l)Sf_l$ of the ring f_lSf_l is included in the kernel of the pseudocharacter $T_{f_l} = T_{|f_lRf_l} : f_lRf_l \longrightarrow A$ (see Lemma 1.2.5 (3)). The map $T_l : R \longrightarrow A$ defined by $T_l(x) := T(f_lxf_l)$ is then the composite of the A-algebra homomorphism

$$S \longrightarrow f_lSf_l/(f_lS(1-f_l)Sf_l), \ \ x \mapsto f_lxf_l + f_lS(1-f_l)Sf_l, \tag{11}$$

by T_{f_l}, hence it is a pseudocharacter. As $1 = f_1 + \cdots + f_s$, we have $T = T_1 + \cdots + T_s$, and the T_l's satisfy (ii) of ($\mathrm{dec}_{\mathcal{P}}$) by Lemma 1.4.3 (3), hence we are done. In particular, we have shown that $I_{\mathcal{P}}$ always satisfies ($\mathrm{dec}_{\mathcal{P}}$).

Let us prove now the "if" part of the statement. Let $K = \bigcap_i \operatorname{Ker} T_i$, by assumption $K \subset \operatorname{Ker} T$. By § 1.2.4, $T : R/K \longrightarrow A$ is Cayley-Hamilton, hence we can choose a data $\mathcal{E}$ for $S := R/K$ and consider again the $f_l \in S$'s defined from the e_i's as in formula (9) above. To check that $I_{\mathcal{P}} = 0$, it suffices to check that $I_{\mathcal{P}}(T, S, \mathcal{E}) = 0$ or, which is the same, that $T(f_lSf_{l'}Sf_l) = 0$ for $l \neq l'$. As $T = T_1 + \cdots + T_s$, it suffices to show that for all $x \in S$, $T_l(f_{l'}x) = 0$ if $l \neq l'$. But if $l \neq l'$, $T_l(f_{l'})$ is in the maximal ideal m by assumption (ii) of ($\mathrm{dec}_{\mathcal{P}}$) and Lemma 1.4.3 (3). By Lemma 1.2.5 (1), it implies that $T_l(f_{l'}) = 0$. By Lemma 1.2.5 (5), we get $f_{l'} \in \operatorname{Ker} T_l$, what we wanted.

In particular, we proved that for all $x \in S$, $T_l(x) = T(f_lx)$. As a consequence, $\operatorname{Ker} T_l \subset \operatorname{Ker} T$, $K = \operatorname{Ker} T$, $S = R/\operatorname{Ker} T$, and the T_l's are unique. □

Definition 1.5.2. — We call $I_{\mathcal{P}}$ the *reducibility ideal of T for the partition $\mathcal{P}$*. We call the closed subscheme $\operatorname{Spec}(A/I_{\mathcal{P}})$ of $\operatorname{Spec} A$ the *reducibility locus of T for the partition $\mathcal{P}$*. When $\mathcal{P}$ is the total partition $\{\{1\}, \{2\}, \ldots, \{r\}\}$, we call $I_{\mathcal{P}}$ the *total* reducibility ideal and $\operatorname{Spec}(A/I_{\mathcal{P}})$ the total reducibility locus of T.

Note that $I_{\mathcal{P}} \subset I_{\mathcal{P}'}$ if $\mathcal{P}'$ is a finer partition than $\mathcal{P}$.

1.5.2. The representation ρ_i. — We keep the assumptions of §1.5.1, and we assume now that $\{i\} \in \mathcal{P}$. Then for each ideal J containing $I_{\mathcal{P}}$, there is by Proposition 1.5.1 a unique pseudocharacter $T_i : R \otimes A/J \longrightarrow A/J$ with $T_i \otimes k = \operatorname{tr} \bar\rho_i$ and $T = T_i + T'$ with $T' \otimes k = \sum_{j \neq i} \operatorname{tr} \bar\rho_i$. If $J \subset J'$, the pseudocharacter $T_i : R \otimes A/J' \longrightarrow J'$ is just $T_i \otimes_{R/J} R/J'$, hence it is not dangerous to forget the ideal J in the notation. As $\bar\rho_i$ is irreducible, we know that there is a (surjective, unique up to conjugation) representation $\rho_i : R/JR \longrightarrow M_{d_i}(A/J)$ of trace T_i which reduces to $\bar\rho_i$ modulo m.

Definition 1.5.3. — If $\{i\} \in \mathcal{P}$ and $J \supset I_{\mathcal{P}}$, we let $\rho_i : R/JR \longrightarrow M_{d_i}(A/J)$ be the surjective representation defined above.

As usual, by a slight abuse of notation, we will denote also by ρ_i the R-module $(A/J)^{d_i}$ on which R acts via ρ_i. It will be useful for the next section to collect here the following facts which are easy consequences of the proof of Proposition 1.5.1:

Lemma 1.5.4. — *Let S be a Cayley-Hamilton quotient of (R,T), $\mathcal{E} = (e_i, \psi_i)$ an associated data of idempotents for S, and $\mathcal{P}$ a partition of $\{1,\dots,r\}$ such that $\{i\} \in \mathcal{P}$ and $J \supset I_{\mathcal{P}}$.*

(i) *If $j \neq i$, $e_i(S/J)e_j(S/J)e_i = 0$.*

(ii) *The canonical projection*

$$a_{i,i} : S/JS \longrightarrow e_i(S/JS)e_i \simeq M_{d_i}(A/J), \quad x \mapsto e_i x e_i,$$

is an A/J-algebra homomorphism and satisfies $T \circ a_{i,i} = T_i$. As a consequence, ρ_i factors through S/JS, $a_{i,i} \simeq \rho_i$, and $\rho_i(e_k) = \delta_{i,k}\mathrm{id}$.

(iii) *Assume moreover that $\{j\} \in \mathcal{P}$ for some $j \neq i$, then we have*

$$a_{i,j}(xy) - (a_{i,i}(x)a_{i,j}(y) + a_{i,j}(x)a_{j,j}(y)) \in \sum_{k\neq i,j} e_i(S/J)e_k(S/J)e_j, \ \forall x, y \in R,$$

where $a_{i,j} : S/JS \longrightarrow e_i(S/JS)e_j$, $x \mapsto e_i x e_j$, is the canonical projection.

Proof. — The idempotent f_l corresponding to $\{i\}$ is then e_i. Note that $e_i(S/JS)(1-e_i)(S/JS)e_i$ is a two-sided ideal of $e_i(S/JS)e_i \simeq M_{d_i}(A/J)$ whose trace is 0 by assumption and formula (10), which implies that $e_i(S/JS)(1-e_i)(S/JS)e_i = 0$. Since for $j \neq i$ we have $(1-e_i)e_j = e_j$, we have

$$e_i(S/JS)e_j(S/JS)e_i \subset e_i(S/JS)(1-e_i)(S/JS)e_i = 0,$$

which shows (i). As a consequence, $a_{i,i}$ coincides with the map in formula (11) (with of course S replaced by S/JS), which proves (ii). The last assertion is immediate from the fact that $e_i xy e_j - (e_i x(e_i + e_j)ye_j)$ lies in

$$e_i(S/JS)(1-(e_i+e_j))(S/JS)e_j = \sum_{k\neq i,j} e_i(S/JS)e_k(S/JS)e_j. \qquad \square$$

1.5.3. An explicit construction of extensions between the ρ_i's. — First let us recall that if R is an A-algebra and $\rho_i : R \to M_{d_i}(A)$, $i = 1, 2$, are two A-algebra representations (that we identify with the R-modules A^{d_i} they define), *an extension of ρ_2 by ρ_1* is an R-module V and an exact sequence of R-modules

$$0 \longrightarrow \rho_1 \longrightarrow V \longrightarrow \rho_2 \longrightarrow 0.$$

Note that the exactness of this sequence implies that V is a free A-module of rank $d_1 + d_2$. Such an extension defines an element in the module $\mathrm{Ext}^1_R(\rho_2, \rho_1)$, and two extensions V and V' define the same element if and only if there exists an isomorphism $V \to V'$ of R-modules that induces the identity on ρ_1 and ρ_2. We will make constant use of the following simple remark: if R' is another A-algebra with a surjective map of A-algebras $R' \longrightarrow R$, then the natural map

$$\mathrm{Ext}^1_R(\rho_2, \rho_1) \longrightarrow \mathrm{Ext}^1_{R'}(\rho_2, \rho_1)$$

is injective (of course, for the right hand side Ext^1 we view each ρ_i as a representation of R' via the given map $R' \to R$). Indeed, an R-module extension of ρ_2 by ρ_1 which is split as an extension of R'-modules is a fortiori split as an extension of R-modules since $R' \to R$ is surjective.

We keep the assumptions of § 1.5.1, and *we fix a Cayley-Hamilton quotient S of (R, T)*. We fix a data $\mathcal{E}$ on S, using Theorem 1.4.4 (i), such that $(S, \mathcal{E})$ is a GMA and set

$$\mathcal{A}'_{i,j} = \sum_{k \neq i,j} \mathcal{A}_{i,k}\mathcal{A}_{k,j}.$$

We have by definition $\mathcal{A}'_{i,j} \subset \mathcal{A}_{i,j}$.

Fix $i \neq j \in \{1, \dots, r\}$. Let $\mathcal{P}$ be any partition of $\{1, \dots, r\}$ such that the singletons $\{i\}$ and $\{j\}$ belong to $\mathcal{P}$, and J an ideal containing $I_{\mathcal{P}}$. By Definition 1.5.3, for $k = i, j$, we have a representation $\rho_k : R/JR \longrightarrow M_{d_k}(A/J)$. By an extension of ρ_j by ρ_i we mean a representation $R/JR \longrightarrow \mathrm{End}_{A/J}(V)$ together with an exact sequence of R/JR-module $0 \longrightarrow \rho_i \longrightarrow V \longrightarrow \rho_j \longrightarrow 0$. Hence V is in particular a free A/J-module of rank $d_1 + d_2$. Such an extension defines an element in the module $\mathrm{Ext}^1_{R/JR}(\rho_j, \rho_i)$.

Theorem 1.5.5. — *There exists a natural injective map of A/J-modules*

$$\iota_{i,j} : \mathrm{Hom}_A(\mathcal{A}_{i,j}/\mathcal{A}'_{i,j}, A/J) \hookrightarrow \mathrm{Ext}^1_{R/JR}(\rho_j, \rho_i).$$

Proof. — The map $\iota_{i,j}$ is constructed as follows. Pick an $f \in \mathrm{Hom}_A(\mathcal{A}_{i,j}/\mathcal{A}'_{i,j}, A/J)$. We see it as a linear form $f : \mathcal{A}_{i,j} \longrightarrow A/J$, trivial on $\mathcal{A}'_{i,j}$. It induces a linear application, still denoted by f: $M_{d_i,d_j}(\mathcal{A}_{i,j}) \longrightarrow M_{d_i,d_j}(A/J)$. We consider the following A-linear application $R \longrightarrow S \longrightarrow M_{d_i+d_j}(A/J)$,

$$x \mapsto \begin{pmatrix} a_{i,i}(x) \pmod J & f(a_{i,j}(x)) \\ 0 & a_{j,j}(x) \pmod J \end{pmatrix} \tag{12}$$

We claim that the map (12) is an A/J-algebra homomorphism which is an extension of ρ_j by ρ_i. By Lemma 1.5.4(ii), the upper and lower diagonal blocks are respectively $\rho_i(x)$ and $\rho_j(x)$, so the only thing to check is that this map is multiplicative. Since $a_{i,i}$ mod $J = \rho_i$ and $a_{j,j}$ mod $J = \rho_j$ are A-algebra morphisms, we only have to check that for all $x, y \in S$

$$f(a_{i,j}(xy)) = a_{i,i}(x)f(a_{i,j}(y)) + a_{j,j}(y)f(a_{i,j}(y)). \tag{13}$$

But by assumption, f is trivial on $\sum_{k\neq i,j} M_{d_i,d_k}(\mathcal{A}_{i,k})M_{d_k,d_j}(\mathcal{A}_{k,j}) \subset M_{d_i,d_j}(\mathcal{A}'_{i,j})$, hence on the right hand side 1.5.4 (iii). The vanishing of the left hand side is exactly (13).

As a consequence, the map (12) defines an element $\iota_{i,j}(f)$ in $\mathrm{Ext}^1_{R/JR}(\rho_j, \rho_i)$.

It is clear by the Yoneda interpretation of the addition in Ext^1 that the map $\iota_{i,j}(f)$ is linear. Let us prove that $\iota_{i,j}$ is injective. Assume $\iota_{i,j}(f) = 0$. This means that the extension is split. As it factors by construction through S/JS, it is certainly split when restricted to any subalgebra of S/JS. Let us restrict it to the subalgebra e_iS/JSe_j (without unit, but we can add $A/J(e_i + e_j)$ if we like). The restricted extension is

$$x \quad \mapsto \quad \begin{pmatrix} 0 & f(a_{i,j}(x)) \\ 0 & 0 \end{pmatrix}$$

and such an extension is split if and only if $f = 0$. □

The construction above is a generalization of the one of Mazur and Wiles, directly giving the matrices of the searched extensions. We will give a second construction in the next subsection, more in the spirit of Ribet's one, which will realize the extensions constructed before as subquotient of some explicit R-modules. Our second aim is to characterize the image of $\iota_{i,j}$ and to verify that this image is the biggest possible subset of the above Ext-group seen by S.

1.5.4. The projective modules M_i and a characterization of the image of $\iota_{i,j}$.

— We keep the assumptions and notations of § 1.5.3. For each i, we define the A-modules

$$M_i := SE_i = \oplus_{j=1}^r e_jSE_i.$$

Note that M_i is a left ideal of S, hence an S-module. It is even a projective S-module as $S = M_i \oplus S(1 - E_i)$.

Theorem 1.5.6. — *Let $j \in \{1, \dots, r\}$, $\mathcal{P}$ a partition containing $\{j\}$ and J an ideal containing $I_{\mathcal{P}}$, then*

(0) *there is a surjective map of S-modules $M_j/JM_j \longrightarrow \rho_j$ whose kernel has the property that any of its simple S-subquotients is isomorphic to $\bar\rho_k$ for some $k \neq j$. Moreover M_j is the projective hull of ρ_j (and of $\bar\rho_j$) in the category of S-modules.*

Let $i \neq j \in \{1, \dots, r\}$, $\mathcal{P}$ a partition containing $\{i\}$ and $\{j\}$, and J is an ideal containing $I_{\mathcal{P}}$. Then moreover:

(1) *the image of the map $\iota_{i,j}$ of Theorem 1.5.5 is exactly $\mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i) \subset \mathrm{Ext}^1_{R/JR}(\rho_j, \rho_i)$,*
(2) *any S/JS-extension of ρ_j by ρ_i is a quotient of $M_j/JM_j \oplus \rho_i$ by an S-submodule, every simple S-subquotient of which is isomorphic to some $\bar\rho_k$ for $k \neq j$.*

Proof. — First note that we may replace A by A/J and S by S/JS, that is we may assume that $J = 0$ in A (which simplifies the notations). Indeed, $(S/JS)E_j \simeq M_j \otimes_A A/J = M_j \otimes_S S/JS = M_j/JM_j$. Hence assertions (1) and (2) are automatically proved for A once they are proved for A/J. As for assertion (0), if we know the corresponding assertion for A/J, namely"The S/JS-module M_j/JM_j is the projective hull of ρ_j", then (0) follows, because the map of S-modules $M_j \longrightarrow M_j/JM_j \longrightarrow \rho_j$ is essential as $JS \subset mS \subset \mathrm{rad}(S)$, and because M_j is projective over S.

Assume that $\mathcal{P}$ contains $\{j\}$ and that $J \supset I_{\mathcal{P}}$. Let us consider the natural exact (split) sequence of A-modules

$$0 \longrightarrow N_j := \oplus_{i \neq j} e_i S E_j \longrightarrow M_j \longrightarrow e_j S E_j \longrightarrow 0. \tag{14}$$

We claim that N_j is an S-submodule of M_j, and that $M_j/N_j \simeq \rho_j$. It suffices to show that for $k \neq j$, $e_j S e_k N_j \subset N_j$. But this follows from Lemma 1.5.4 (i), as $e_j S e_k S e_j = 0$. As a consequence, $M_j/N_j \simeq e_j S E_j$ is an S-module, which is isomorphic to ρ_j by Lemma 1.5.4 (ii).

Let us prove the first assertion in (0). Recall that by Lemma 1.2.7, we have

$$S/\mathrm{rad}(S) \simeq \prod_{i=1}^{r} \mathrm{End}_k(\bar{\rho}_i)$$

(see the formula (8) in the proof of Lemma 1.4.3). So if U is a simple S-subquotient of N_j, then $U \simeq \bar{\rho}_k$ for some $k \in \{1, \ldots, r\}$. But by construction, $e_j N_j = 0$, hence $e_j U = 0$, and $\rho_j(e_j) = 1$ by Lemma 1.4.3, so $k \neq j$ and we are done.

We prove now that $M_j \longrightarrow \rho_j$ is a projective hull. We just have to show that this surjection is essential. If $Q \subset M_j$ is a S-submodule which maps surjectively to $M_j/N_j = e_j(M_j/N_j)$, then $e_j Q \subset e_j S E_j$ maps also surjectively to M_j/N_j, hence $e_j Q = e_j S E_j$. But then $E_j \in Q$, hence $Q = M_j$, and we are done.

Now we suppose that $\mathcal{P}$ contains $\{i\}$ and $\{j\}$. Let us apply $\mathrm{Hom}_S(-, \rho_i)$ to the exact sequence (14). As M_j is a projective S-module, it takes the form:

$$0 \longrightarrow \mathrm{Hom}_S(\rho_j, \rho_i) \longrightarrow \mathrm{Hom}_S(M_j, \rho_i) \longrightarrow \mathrm{Hom}_S(N_j, \rho_i) \xrightarrow{\delta} \mathrm{Ext}_S(\rho_j, \rho_i) \longrightarrow 0$$

We claim first that δ is an isomorphism. We have to show that any S-morphism $M_j \longrightarrow \rho_i$ vanishes on N_j. But by Lemma 1.5.4 (ii), if $k \neq j$ we have $e_k \rho_j = 0$. We are done as $N_j = \sum_{k \neq j} e_k N_j$ by definition.

It is well known that if $f \in \mathrm{Hom}_S(N_j, \rho_i)$, we have the following commutative diagram defining $\delta(f)$:

$$\begin{tikzcd} & 0 \arrow[r] & N_j \arrow[r] \arrow[d, "f"'] & M_j \arrow[r] \arrow[d, "x \mapsto (x,0)"] & \rho_j \arrow[r] \arrow[d, "id"] & 0 \\ \delta(f): & 0 \arrow[r] & \rho_i \arrow[r, "x \mapsto (0,x)"'] & \frac{M_j \oplus \rho_i}{Q} \arrow[r] & \rho_j \arrow[r] & 0 \end{tikzcd} \tag{15}$$

where Q is the image of the S-linear map $u : N_j \longrightarrow M_j \oplus \rho_i,\ \ x \mapsto (x,0) - (0, f(x))$. This will prove (2) if we can show that each simple subquotient of Q is isomorphic to some $\bar{\rho}_k$ with $k \neq j$. But as in the proof of (0), this follows from the fact that $e_j Q = u(e_j N_j) = 0$.

We claim now that we have a sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_S(N_j, \rho_i) &\xrightarrow{\sim} \mathrm{Hom}_{e_i S e_i}(e_i S E_j / (\textstyle\sum_{k \neq j,i} e_i S e_k S E_j), \rho_i) \\ &\xrightarrow{\sim} \mathrm{Hom}_A(\mathcal{A}_{i,j} / (\textstyle\sum_{k\neq j,i} \mathcal{A}_{i,k}\mathcal{A}_{k,j}), A) \end{aligned}$$

The first one is induced by the restriction map, the fact that it is an isomorphism is a simple matter of orthogonal idempotents, using that $e_k \rho_i = 0$ if $k \neq i$ and that $N_j = \oplus_{k\neq j} e_k S E_j$. The second one is induced by the Morita equivalence $A = E_i A \longrightarrow e_i S e_i = M_{d_i}(A)$.

It is now easy, using the diagram (15) and the fact that $(M_j \oplus \rho_i)/Q$ is naturally isomorphic as A-module to $e_j S E_j \oplus e_i S E_i$, to check that in terms of the isomorphisms above, the map δ is exactly the map $\iota_{i,j}$ given by formula (12), which proves (1). Note that the inclusion property between the two Ext^1 of the statement of assertion (1) has already been discussed in the beginning of § 1.5.3. □

Remark 1.5.7. — By the same method, we could give an expression for the higher Ext-groups $\mathrm{Ext}^n_{S/JS}(\rho_j, \rho_i)$ in terms of the $\mathcal{A}_{i,j}$'s. For example, when $r = 2$, the exact sequence (14) implies that $\mathrm{Ext}^2_{S/JS}(\rho_j, \rho_i) \xrightarrow{\sim} \mathrm{Ext}^1_{A/J}(\mathcal{A}_{i,j} \otimes A/J, A/J)$. However, for the usual applications of pseudocharacters, this is less interesting because when $n \geq 2$, the natural map

$$\mathrm{Ext}^n_{S/JS}(\rho_j, \rho_i) \longrightarrow \mathrm{Ext}^n_{R/JR}(\rho_j, \rho_i)$$

is not in general injective, and we usually only care about the extensions between the ρ_i's in the category of representations of R, not of its auxiliary quotient S. (Compare with the discussion in the beginning of § 1.5.3.)

***Remark 1.5.8* (Dependence on S).** — All the constructions of § 1.5.3 and § 1.5.4 depend on the choice of a Cayley-Hamilton quotient S of (R, T). If $S_1 \longrightarrow S_2$ is a morphism in the category of Cayley-Hamilton quotients (cf. 1.2.5), then it is surjective and we have obviously $\mathrm{Ext}^1_{S_2/IS_2}(\rho_j, \rho_i) \subset \mathrm{Ext}^1_{S_1/IS_1}(\rho_j, \rho_i)$. Thus our methods construct the biggest group of extensions when working with $S = S_0$, and the smallest when $S = R/\mathrm{Ker}\, T$. We stress that even in the most favorable cases, the inclusion above may be strict: an example will be given in Remark 1.6.5 below. However, we will not be able to get much information about the $\mathrm{Ext}^1_{S/JS}$ except when $S = R/\mathrm{Ker}\, T$. On the other hand, as proved in Proposition 1.5.1, the reducibility ideals do not depend on S.

Remark 1.5.9. — Assume we are under the assumptions of Theorem 1.5.6. We claim that for $i \neq j$, the natural inclusion

$$\mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i) \longrightarrow \mathrm{Ext}^1_S(\rho_j, \rho_i) \tag{16}$$

is an isomorphism. Indeed, it is injective since $S \longrightarrow S/JS$ is surjective, and the image of (16) is exactly the subspace of S-extensions of ρ_j by ρ_i in which the ideal $JS \subset S$ acts by 0. Let U be an S-extension of ρ_j by ρ_i, we have to show that $JU = 0$. But for $f \in J$, the multiplication by f induces an S-linear map

$$\rho_j \longrightarrow \rho_i,$$

which is necessarily 0 as $\mathrm{Hom}_S(\rho_j, \rho_i) = \mathrm{Hom}_{S/JS}(\rho_j, \rho_i) = 0$ by Lemma 1.5.4 (i).

1.5.5. Complement: Topology. — We keep the hypotheses of §1.5.1. We assume moreover that A is a Hausdorff topological ring such that the natural functor from the category of topological Hausdorff finite type A-modules to the category of A-modules has a section endowing A with its topology. We fix such a section, hence every finite type A-module is provided with an Hausdorff A-module topology, and any A-linear morphism between two of them is continuous with closed image. For example, this is well known to be the case when A is a complete noetherian local rings, and it holds also when A is the local ring of a rigid analytic space at a closed point (see [**9**, §2.4]).

Proposition 1.5.10. — *Assume that R is a topological A-algebra and that $T : R \longrightarrow A$ is continuous.*

(i) *Let I be an ideal containing $I_{\mathcal{P}}$ where $\mathcal{P}$ is a partition containing $\{i\}$. Then the representation $\rho_i : R/IR \longrightarrow M_{d_i}(A/I)$ is continuous.*
(ii) *Let I be an ideal containing $I_{\mathcal{P}}$ where $\mathcal{P}$ is a partition containing $\{i\}$ and $\{j\}$, $i \neq j$. If A is reduced and $S = R/\mathrm{Ker}\, T$, then the image of $\iota_{i,j}$ of Theorem 1.5.5 falls into the A-submodule of continuous extensions $\mathrm{Ext}^1_{R,\mathrm{cont}}(\rho_i, \rho_j)$.*

Proof. — By Lemma 1.5.4 (ii), we can find $e \in R$ such that for all $x \in R$, $T_i(x) = T(ex)$ (any lift of the element $e_i \in S/JS$ *loc. cit.* works for e), which proves (i). Let us show (ii). Fix $f \in \mathrm{Hom}_A(\mathcal{A}_{i,j}/\mathcal{A}'_{i,j}, A/I)$. By the formula (12) defining $\iota_{i,j}(f)$, it suffices to show that the natural maps $\pi_{i,j} : R \longrightarrow \mathcal{A}_{i,j}$, $x \mapsto E_i x E_j$, are continuous. Note that this makes sense because by Theorem 1.3.2 (iii), the $\mathcal{A}_{i,j}$'s are finite type A-modules. Let us choose a family of A-generators $x_1, \dots, x_n$ of $\mathcal{A}_{j,i}$. As $T : S \longrightarrow A$ is faithful by assumption, and by Lemma 1.3.5, the map

$$\mu : \mathcal{A}_{i,j} \longrightarrow \prod_{s=1}^{n} A, \quad x \mapsto \mu(x) = (T(xx_s))_s,$$

is injective. By assumption on the topology of finite type A-modules, the map above is an homeomorphism onto its image. It suffices then to prove that $\mu \circ \pi_{i,j}$ is continuous, which we can check componentwise. But for each s, $(\mu \circ \pi_{i,j})_s$ is the map $x \mapsto T(exfg)$, where $(e,f,g) \in R^3$ denotes any lift of $(E_i, E_j, x_s) \in S^3$. This concludes the proof. $\square$

1.6. Representations over A

We keep the notations and hypotheses of § 1.4.1: A is local henselian and $d!$ is invertible in A. In this subsection we are mainly concerned with the following natural question which is a converse to Example § 1.2.2: if $T : R \longrightarrow A$ is a residually multiplicity free pseudocharacter of dimension d, does T arise as the trace of a true representation $R \longrightarrow M_d(A)$?

When T is residually absolutely irreducible, the theorem of Nyssen and Rouquier ([**91**], [**102**, corollaire 5.2]) we recalled in § 1.2.2 shows that the answer is yes. Although for a given residually multiplicity free pseudocharacter, it may be difficult to determine if it arises as the trace of a representation (see next subsection for interesting particular cases), it turns out that there is a simple sufficient and (almost) necessary condition on A for this to be true for every residually multiplicity free pseudocharacter of dimension d on A.

Proposition 1.6.1. — *Assume that A is a factorial domain (that is, a UFD). Then any residually multiplicity-free pseudocharacter $T : R \longrightarrow A$ of dimension d is the trace of a representation $R \longrightarrow M_d(A)$.*

Proof. — We use the notations of § 1.4.1 for T. As A is a domain, its total fraction ring is a field K. By the point (i) of Theorem 1.4.4, there is a data $\mathcal{E}$ on $R/\operatorname{Ker} T$ that makes it a GMA, and by the point (ii) of the same theorem, there is an adapted (to $\mathcal{E}$) representation $\rho : R/\operatorname{Ker} T \longrightarrow M_d(K)$ whose image is the standard GMA (see example 1.3.4) attached to some fractional ideals $A_{i,j}$ of K, $i,j \in \{1, \dots, r\}$.

Let v be an essential valuation of A. Recall that since A is a UFD, every essential valuation is discrete, attached to an irreducible element of A. Let $v_{i,j}$ be the smallest integer of the form $v(x)$ for a nonzero $x \in A_{i,j}$—this makes sense since $A_{i,j}$ is a fractional ideal. Because $A_{i,i} = A$, $A_{i,j}A_{j,i} \subset A$ and $A_{i,j}A_{j,k} \subset A_{i,k}$, we have

$$v_{i,i} = 0, \quad v_{i,j} + v_{j,i} \geq 0, \quad v_{i,j} + v_{j,k} \geq v_{i,k} \tag{17}$$

Moreover, all of the $v_{i,j}$ are zero except for a finite number of essential valuations.

Because A is factorial, there exists for each i an element $x_i \in K^*$ such that $v(x_i) = v_{i,1}$ for every essential valuation v. Let P be the following diagonal matrix

$$\begin{pmatrix} x_1\mathrm{Id}_{d_1} & & & \\ & x_2\mathrm{Id}_{d_2} & & \\ & & \ddots & \\ & & & x_r\mathrm{Id}_{d_r} \end{pmatrix}$$

Let $\rho' := P^{-1}\rho P$. Then ρ' is adapted to $\mathcal{E}$ and its image is the standard GMA attached to the modules $A'_{i,j} = x_j x_i^{-1} A_{i,j}$.

If $x \in A'_{i,j}$, and v is an essential valuation on A, we have $v(x) \geq v(x_j) - v(x_i) + v_{i,j} = v_{j,1} - v_{i,1} + v_{i,j}$ which is nonnegative by (17). Hence $x \in A$ since A is factorial, and $A'_{i,j} \subset A$. That is, ρ' is a representation $R \longrightarrow M_d(A)$ of trace T. □

Remark 1.6.2. — Let A be a valuation ring, with field of fractions K, and valuation $v : K^* \longrightarrow \Gamma$, where Γ is a totally ordered group. Assume $v(K^*) = \Gamma$.

Then the proof above shows that the result of Proposition 1.6.1 holds also for this ring A if the ordered group Γ admits infima. Indeed, it suffices to define $v_{i,j}$ to be the infimum of the $v(x)$ with $x \in \mathcal{A}_{i,j}$ nonzero, and to choose $x_i \in K^*$ such that $v(x_i) = v_{1,i}$, which is possible by the assumption $v(K^*) = \Gamma$.

Consider for example a valuation ring A as above, with $\Gamma = \mathbb{R}$ (such a valuation ring exists by [**29**, chapitre VI, §3, n^o 4, example 6]). Then the result of Proposition 1.6.1 holds for A, though A is not a UFD (A has no irreducible elements!). Note however that A is not noetherian.

If on the contrary we do not assume that Γ admits infima, the result fails as showed for the ring $\mathcal{O}_{\mathbb{C}_p}$ in [**11**, remark 1.14].

We are now interested in the converse of Proposition 1.6.1. Because of the remark above, we shall assume that A is noetherian.

Theorem 1.6.3. — *Assume $d \geq 2$ and A noetherian (in addition of being local henselian). If each residually multiplicity free pseudocharacter of dimension d is the trace of a representation $R \longrightarrow M_d(A)$, then A is factorial.*

Proof. — We claim first that the hypothesis implies the following purely module-theoretical assertion on A:

For any A-modules B and C, and every morphism of A-modules $\phi : B \otimes C \longrightarrow m$ such that

(18) $$\phi(b,c)b' = \phi(b',c)b, \text{ for any } b, b' \in B,\ c \in C,$$

there exist two morphisms $f : B \longrightarrow A$, and $g : C \longrightarrow A$ such that $\phi(b \otimes c) = f(b)g(c)$ for any $b \in B$, $c \in C$.

Let us prove the claim. Let B, C be two A-modules with a morphism $\phi : B \otimes C \longrightarrow m$ satisfying the property above. Set $\mathcal{A}_{1,2} := B$, $\mathcal{A}_{1,2} = C$, $\mathcal{A}_{i,j} = 0$ for $i \neq j$ and $\{i,j\} \neq \{1,2\}$, $\mathcal{A}_{i,i} = A$ for $i = 1, \ldots, d$, $\phi_{1,2,1} := \phi$, $\phi_{2,1,2}(c \otimes b) = \phi(b \otimes c)$, and $\phi_{i,i,j}$, $\phi_{i,j,j}$ be the structural morphism. Then we check at once that these $\mathcal{A}_{i,j}$'s and $\phi_{i,j,k}$'s satisfy the properties (COM), (UNIT), and (ASSO) (see § 1.3.2), and thus defines a GMA $(R, \mathcal{E})$ whose they are the structural modules and morphisms. As $\phi(B \otimes C) \subset m$, we are in the case of Example 1.4.2, and the trace function $T : R \longrightarrow A$ of $(R, \mathcal{E})$ is a residually multiplicity free pseudocharacter of dimension d.

The hypothesis of the theorem then implies that there is a trace representation $R \longrightarrow M_d(A)$. Because A is local, every finite-type projective A-module is free and by Lemma 1.3.7, there is an adapted (to $\mathcal{E}$) representation $\rho : R \longrightarrow M_d(A)$, that is an element of $G(A)$ where the functor $G = G_{R,\mathcal{E}}$ is the one defined in § 1.3.3. By Proposition 1.3.9, $F(A)$ is not empty. If $(f_{i,j}) \in F(A)$, then by definition $(f, g) := (f_{1,2}, f_{2,1})$ satisfies the claim, and we are done.

Using the assertion above, we will now prove in three steps that A is a factorial domain.

First step. — A is a domain.
Choose an $x \in A$, $x \neq 0$, and let I be its annihilator. Set $B = A/xA$, $C = I$ and let $\phi : B \otimes C \longrightarrow A$ be the morphism induced by the multiplication in A. Then $\phi(B \otimes C) = I \subset m$ and the property (18) is obvious. Thus there exist $f : B \longrightarrow A$ and $g : C \longrightarrow A$ such that $\phi(b \otimes c) = f(b)g(c)$ for any $b \in B$, $c \in C$. As $xC = xI = 0$, we have $xg(C) = 0$ hence $g(C) \subset I$. As $xA/xA = xB = 0$ we also have $f(B) \subset I$. Hence $I = \phi(B \otimes C) = f(B)g(C) \subset I^2$. Because A is local and noetherian, this implies $I = 0$. Hence A is a domain.

Second step. — If A is a domain, then A is normal.
Let K be the fraction field of A. Assume, by contradiction, that A is not normal, and let $B \subset K$ be a finite A-algebra containing A, but different from A. Let $C = \{x \in K,\ \ xB \subset A\}$. We have then:

i. by definition, C is a B-submodule of K (hence an A-module too).
ii. $C \subset A$, because $1 \in B$. Hence C is an A-ideal.
iii. We have $C \subset m$. Indeed, C is an A-ideal by ii. As A is local, we only have to see that $C \neq A$. But if $1 \in C$, $B \subset C \subset A$ by i. and ii. , which is absurd.
iv. C is non zero: if (p_i/q_i) is a finite family of generators of B as an A-module, with $p_i, q_i \in A$, $q_i \neq 0$, then $0 \neq \prod_i q_i \in C$.

Now let $\phi : B \otimes_A C \longrightarrow K$ be the map induced by the multiplication in K. By iii. $\phi(B \otimes C) \subset m$. Moreover, hypothesis (18) is obviously satisfied. Thus there exist two morphisms $f : B \longrightarrow A$ and $g : C \longrightarrow A$ such that $\phi(b \otimes c) = f(b)g(c)$ for any $b \in B$, $c \in C$. Since $B \otimes_A K = K$, $f \otimes K : K \longrightarrow K$ is the multiplication by some element

$x \in K^*$, and so is f. As $C \otimes_A K = K$ by iv. , g has to be the multiplication by x^{-1}. We thus get

$$xB \subset A, \quad \text{and} \quad x^{-1}C \subset A. \tag{19}$$

The first relation implies $x \in C$, so $1 \in x^{-1}C$. As $x^{-1}C$ is a B-module, $B \subset x^{-1}C$ and by the second relation, $B \subset A$, which is absurd. (The reader may notice that this step does not use the noetherian hypothesis).

Third step. — If A is a normal domain, then A is factorial.
We may assume that the Krull dimension of A is at least 2, because a normal noetherian domain of dimension ≤ 1 is a discrete valuation ring, hence factorial. Let C be an invertible ideal of A, and set $B = mC^{-1} \subset K$. Let $\phi : B \otimes_A C \longrightarrow m$ be induced by the multiplication in K. Then reasoning as in the second step above, we see that there is an $x \in K^*$ such that $xmC^{-1} \subset A$ and $x^{-1}C \subset A$, as in (19).

Now, since A is normal and noetherian, it is completely integrally closed, and even a Krull ring ([**29**, chapter VII, §1, n^0 3, corollary]). Recall from [**29**, chapter VII, §1, n^0 2, Theorem 1] the ordered group $D(A)$ of divisorial fractional ideals of A, and the projection div from the set of all fractional ideals of A to $D(A)$. Since $x^{-1}C \subset A$, we have (using [**29**, chapter VII, §1, n^0 2, formula (2)])

$$\operatorname{div} x^{-1} + \operatorname{div} C = \operatorname{div}(x^{-1}C) \geq 0,$$

that is $\operatorname{div} C \geq \operatorname{div} x$. From $xmC^{-1} \subset A$ we have

$$\operatorname{div} x + \operatorname{div} m + \operatorname{div} C^{-1} \geq 0,$$

but since m has height greater than or equal to 2, $\operatorname{div} m = 0$ by [**29**, chapter VII, §1, n^0 6, corollary (1)], and since A is completely integrally closed, $\operatorname{div} C^{-1} = -\operatorname{div} C$ by [**29**, chapter VII, §1, n^0 2, corollary]. Hence $\operatorname{div} x \geq \operatorname{div} C$. Thus $\operatorname{div} x = \operatorname{div} C$, and if C is divisorial, then $C = Ax$ is principal. But a Krull ring where every divisorial ideal is principal is factorial, cf. [**29**, chapter VII, §3, n^0 1]. □

When a trace representation $\rho : R \longrightarrow M_d(A)$ does exist, we may ask what its kernel and image are. In some favorable cases, we can give a satisfactory answer:

Proposition 1.6.4. — *Assume A is reduced with total fraction ring K a finite product of fields K_s. Let $T : R \longrightarrow A$ be a residually multiplicity free pseudocharacter and assume $T \otimes K_s$ irreducible for each s. If $\rho : R \longrightarrow M_d(A)$ is a trace representation then* $\operatorname{Ker} \rho = \operatorname{Ker} T$ *and* $\rho(R) \otimes K = K[\rho(R)] = M_d(K)$.

Proof. — We obviously have $\operatorname{Ker} \rho \subset \operatorname{Ker} T$. Set $S := \rho(R) \subset M_d(A)$, which is a Cayley-Hamilton quotient of (R, T). To show that $T : S \longrightarrow A$ is faithful, it suffices to show the last statement. By the irreducibility assumption and Theorem 1.4.4 (iii),

$S \otimes K$ is (trace) isomorphic to $M_d(K)$. As a consequence, the injective map $\rho \otimes K : S \otimes K \longrightarrow M_d(K)$ is an isomorphism, which concludes the proof. □

Remark 1.6.5. — The proof above shows in particular that under the hypotheses of the proposition, the only Cayley-Hamilton quotient of R that is torsion free as an A-module is $R/\operatorname{Ker} T$. We cannot omit the hypothesis "torsion free". Here is a counter-example: with the notations of the proof of Theorem 1.6.3, take $A = \mathbb{Z}_p$, and set $B = \mathbb{Z}_p$, $C = \mathbb{Z}_p \oplus \mathbb{Z}/p\mathbb{Z}$ and let $\phi : B \otimes C \longrightarrow \mathbb{Z}_p$ be defined by $\phi(b \otimes (c, c')) = pbc$. As it is clear that ϕ satisfies (18), those data define a GMA R of type $(1, 1)$. Its trace function T is a Cayley-Hamilton residually multiplicity free pseudocharacter. Hence R is Cayley-Hamilton, we have $R = S_0$ in the notation of § 1.2.5, but $R \neq R/\operatorname{Ker} T$ because $\operatorname{Ker} T \simeq \mathbb{Z}/p\mathbb{Z}$. Moreover this example provides a case where $\operatorname{Ext}^1_{S_0/pS_0}(\bar\rho_1, \bar\rho_2)$ has dimension 2 whereas $\operatorname{Ext}^1_{(R/\operatorname{Ker} T)/p(R/\operatorname{Ker} T)}(\bar\rho_1, \bar\rho_2)$ has dimension 1.

1.7. An example: the case $r = 2$

Let A be a reduced, noetherian, henselian local ring and $T : R \longrightarrow A$ be a multiplicity free, d-dimensional, pseudocharacter. As before, K is the total fraction ring of A, which is a finite product of fields K_s. In this subsection, we investigate the consequences of our general results in the simplest case where $\bar T$ is the sum of only two irreducible pseudocharacters $\operatorname{tr} \bar\rho_1$ and $\operatorname{tr} \bar\rho_2$. Note that in this case, the only reducibility locus is the total one, of ideal $I_{\mathcal{P}}$ with $\mathcal{P} = \{\{1\}, \{2\}\}$.

Let S be a given Cayley-Hamilton quotient of (R, T) We are first interested in giving a lower bound on the dimension of $\operatorname{Ext}^1_{S/mS}(\rho_1, \rho_2)$, hence of $\operatorname{Ext}^1_{R/mR}(\rho_1, \rho_2)$.

Proposition 1.7.1. — *Let n be the minimal number of generators of the ideal $I_{\mathcal{P}}$. Then*

$$(\dim_k \operatorname{Ext}^1_{S/mS}(\bar\rho_1, \bar\rho_2))(\dim_k \operatorname{Ext}^1_{S/mS}(\bar\rho_2, \bar\rho_1)) \geq n$$

Proof. — By Remark 1.5.8, we may and do assume $S = R/\operatorname{Ker} T$. Let $\rho : R/\operatorname{Ker} T \longrightarrow M_d(K)$ be as in Theorem 1.4.4 whose we use notations. If $i \neq j$, let $n_{i,j}$ be the minimal number of generators of the finite A-module $A_{i,j}$.

By Theorems 1.5.5 and 1.5.6(1), we have

$$\dim_k \operatorname{Hom}_A(A_{i,j}, k) = \dim_k \operatorname{Ext}^1_{S/mS}(\bar\rho_j, \bar\rho_i).$$

But $\operatorname{Hom}_A(A_{i,j}, k) = \operatorname{Hom}_k(A_{i,j}/mA_{i,j}, k)$ and by the theory of duality on vector spaces, this space has the same dimension as $A_{i,j}/mA_{i,j}$; by Nakayama's lemma, this dimension is $n_{i,j}$. Thus $n_{i,j} = \dim_k \operatorname{Ext}^1_{S/mS}(\bar\rho_j, \bar\rho_i)$.

On the other hand, since $I_{\mathcal{P}} = A_{1,2}A_{2,1}$, we have $n_{1,2}n_{2,1} \geq n$, and the proposition follows. □

This easy observation is one of the main theme of the book: to produce many extensions of $\bar{\rho}_1$ by $\bar{\rho}_2$ we shall not only construct a pseudocharacter over a local ring A lifting $\operatorname{tr}\bar{\rho}_1 + \operatorname{tr}\bar{\rho}_2$, but do it sufficiently non trivially so that the reducibility locus of that pseudocharacter has a big codimension. The most favorable case occurs of course when $I_{\mathcal{P}}$ is the maximal ideal m of A. In this case, the above result writes

$$(\dim_k \operatorname{Ext}^1_{S/mS}(\bar{\rho}_1, \bar{\rho}_2))(\dim_k \operatorname{Ext}^1_{S/mS}(\bar{\rho}_2, \bar{\rho}_1)) \geq \dim_k m/m^2 \geq \dim A.$$

When moreover T is the trace of a true representation, we can say more:

Proposition 1.7.2. — *Assume that each $T \otimes K_s$ is irreducible, that $I_{\mathcal{P}}$ is the maximal ideal and that there is a trace representation $R \longrightarrow M_d(A)$, then*

$$\max(\dim_k \operatorname{Ext}^1_{S/mS}(\bar{\rho}_1, \bar{\rho}_2), \dim_k \operatorname{Ext}^1_{S/mS}(\bar{\rho}_2, \bar{\rho}_1)) \geq \dim_k m/m^2$$

Proof. — Again we may and do assume that $S = R/\operatorname{Ker} T$. Moreover we also have $\rho(R) = R/\operatorname{Ker} T = S$ by Proposition 1.6.4. By Lemma 1.3.7, and Lemma 1.3.8 we may assume that the image of ρ is a standard GMA attached to ideals $A_{1,2}$, $A_{2,1}$ of A. Then $A_{1,2}$ and $A_{2,1}$ are ideals of A such that $A_{1,2}A_{2,1} = I_{\mathcal{P}} = m$. Hence $m \subset A_{1,2}$ and $m \subset A_{2,1}$, but we cannot have $A_{1,2} = A_{2,1} = A$, hence one of those ideals is m. The proposition follows. □

Remark 1.7.3. — The inequality above does not hold when T has no representation over A. Indeed, let k be any field and set $A = k[[x,y,z]]/(xy - z^2)$ which is a complete noetherian normal local domain, but not factorial. Let K be its fraction field, and $A_{1,2} = yA + zA$, $A_{2,1} = \frac{x}{z}A + A$ in K, $A_{1,1} = A_{2,2} = A$. Let R be the standard GMA of type $(1,1)$ associated to these $A_{i,j} \subset K$. As $A_{1,2}A_{2,1} = m$, the trace T of R is an A-valued residually multiplicity free pseudocharacter. Its reducibility locus $I_{\mathcal{P}} = A_{1,2}A_{2,1} = (x,y,z) = m$ is the maximal ideal of A, and $T \otimes K$ is obviously irreducible but m/m^2 has dimension 3, whereas $\dim_k \operatorname{Ext}^1_{R/mR}(\bar{\rho}_1, \bar{\rho}_2) = \dim_k \operatorname{Ext}^1_{R/mR}(\bar{\rho}_2, \bar{\rho}_1) = 2$.

We now give a result relating the Ext groups and the existence of a trace representation over A:

Proposition 1.7.4. — *Assume that each $T \otimes K_s$ is irreducible. The two following assertions are equivalent:*

(i) *There is a representation $\rho : R \longrightarrow M_d(A)$ whose trace is T, and whose reduction modulo m is a non split extension of $\bar{\rho}_1$ by $\bar{\rho}_2$,*
(ii) $\operatorname{Ext}^1_{(R/\operatorname{Ker} T)/m(R/\operatorname{Ker} T)}(\bar{\rho}_1, \bar{\rho}_2)$ *has k-dimension 1.*

Moreover, if those properties hold, then the representation ρ in (i) *is unique up to isomorphism.*

Proof. — Let us prove first (i) $\Rightarrow$ (ii). Fix ρ as in (i). By reasoning as in the proof of the proposition above, we can assume that $\rho(R)$ has the standard GMA attached to some ideals $A_{1,2}$, $A_{2,1}$ of A for image, and has $\operatorname{Ker} T$ for kernel. Hence

$$(\rho \otimes k)(R \otimes k) = \begin{pmatrix} M_{d_1}(k) & M_{d_1,d_2}(\overline{A_{1,2}}) \\ M_{d_2,d_1}(\overline{A_{2,1}}) & M_{d_2}(k) \end{pmatrix}$$

where $\overline{A_{i,j}}$ is the image of the ideal $A_{i,j}$ in $A/m = k$. The hypothesis tells us that $\overline{A_{1,2}} = 0$ and $\overline{A_{2,1}} \neq 0$, hence $A_{1,2} \subset m$ and $A_{2,1} = A$. But by Theorems 1.5.5 and 1.5.6(1),

$$\operatorname{Ext}^1_{(R/\operatorname{Ker} T)/m(R/\operatorname{Ker} T)}(\bar\rho_1, \bar\rho_2) \simeq \operatorname{Hom}_k(A_{2,1}, k) = k,$$

which is (ii).

Let us prove (ii) $\Rightarrow$ (i). Let $\rho : R \longrightarrow M_d(K)$ be a representation as in Theorem 1.4.4, (ii), whose kernel is $\operatorname{Ker} T$ and whose image is the standard GMA of type (d_1, d_2) attached to fractional ideals $A_{1,2}$, $A_{2,1}$ of A. Since

$$k \simeq \operatorname{Ext}^1_{(R/\operatorname{Ker} T)/m(R/\operatorname{Ker} T)}(\bar\rho_1, \bar\rho_2) \simeq \operatorname{Hom}_k(A_{2,1}, k),$$

we have $A_{2,1}/mA_{2,1} \simeq k$ hence by Nakayama's lemma $A_{2,1} = fA$ for some $f \in K$. By Theorem 1.4.4 (iii), $A_{2,1}K = K$, hence $f \in K^*$. Then, if we change the basis of A^d, keeping the d_1 first vectors and multiplying the d_2 last vectors by f, we get a new representation $\rho' : R \longrightarrow \mathrm{GL}_2(A)$ whose image is the standard GMA attached to $A'_{i,j}$, with $A'_{2,1} = A_{2,1}/f = A$, hence $A'_{1,2} \subset m$. It is then clear that the reduction modulo m of that representation is a non split extension of $\bar\rho_1$ by $\bar\rho_2$. We leave the last assertion as an exercise to the reader. □

In the same spirit, we have

Proposition 1.7.5. — *Assume that each $T \otimes K_s$ is irreducible. The two following assertions are equivalent:*

(i) $\operatorname{Ext}^1_{(R/\operatorname{Ker} T)/m(R/\operatorname{Ker} T)}(\bar\rho_1, \bar\rho_2)$ *and* $\operatorname{Ext}^1_{(R/\operatorname{Ker} T)/m(R/\operatorname{Ker} T),T}(\bar\rho_2, \bar\rho_1)$ *have k-dimension 1.*

(ii) *The reducibility ideal $I_{\mathcal{P}}$ is principal, with a non-zero divisor generator.*

Proof. — We will use the notations ρ and $A_{1,2}$, $A_{2,1}$ of the part (ii) $\Rightarrow$ (i) of the proof of the above proposition.

Proof of (i) $\Rightarrow$ (ii). Reasoning as in the proof of the proposition above, we see that $A_{1,2} = fA$ and $A_{2,1} = f'A$ with $f, f' \in K^*$. Hence $I_{\mathcal{P}} = A_{1,2}A_{2,1} = ff'A$ with $ff' \in K^* \cap A$. Hence the ideal $I_{\mathcal{P}}$ is generated by ff' which is not a zero divisor.

Proof of (ii) $\Rightarrow$ (i). By hypothesis, $A_{1,2}A_{2,1} = fA$ with f not a zero divisor. Hence there is a family of $a_i \in A_{1,2}$, $b_i \in A_{2,1}$ such that $\sum_{i=1}^n a_i b_i = f$. Let $x \in A_{1,2}$, then

$xb_i \in fA$ so we can write $ab_i = fx_i$ for a unique $x_i \in A$. Hence

$$fx = \sum_i (a_i b_i)x = \sum_i a_i(xb_i) = \sum_i a_i f x_i.$$

Because f is not a zero divisor, $x = \sum a_i x_i$. This shows that the a_i generate $A_{1,2}$, and the morphism $A^n \to A_{1,2}$, $(x_1, \ldots, x_n) \mapsto \sum a_i x_i$ has a section $x \mapsto (x_1, \ldots, x_n)$. Hence $A_{1,2}$ is projective of finite type, hence free, and since $A_{1,2} \subset K$, it is free of rank one. The same argument holds of course for $A_{2,1}$, and we conclude by Theorems 1.5.5 and 1.5.6(1) applied to $J = m$ and $S = R/\operatorname{Ker} T$. □

1.8. Pseudocharacters with a symmetry

1.8.1. The set-up. — In this section we return to the hypotheses of § 1.4.1: A is a local henselian ring where $d!$ is invertible, $T : R \longrightarrow A$ is a d-dimensional pseudocharacter residually multiplicity free.

Moreover, in this section, we suppose given an automorphism of A-module $\tau : R \longrightarrow R$, which is either a morphism or an anti-morphism of A-algebra and such that $\tau^2 = \mathrm{id}_{\mathrm{R}}$. We note that in both cases $T \circ \tau$ is a pseudocharacter on R of dimension d, and we assume

$$T \circ \tau = T. \tag{20}$$

If B is any A-algebra, and $\rho : R \longrightarrow M_n(B)$ is any representation, then we shall denote by $\rho^\perp$ the representation $\rho \circ \tau : R \longrightarrow M_n(B)$ if τ is a morphism of algebra, and ${}^t(\rho \circ \tau)$ if τ is an anti-morphism of algebra. Note that $\rho^\perp$ is a representation whose trace is $(\operatorname{tr} \rho) \circ \tau$. If $\rho : R \longrightarrow M_d(K)$ is a semisimple representation of trace T, where K is a field, then the hypothesis (20) is equivalent to

$$\rho^\perp \simeq \rho \tag{21}$$

The hypothesis (20) also implies that $\bar{T} \circ \tau = \bar{T}$, hence $\bar{\rho}^\perp \simeq \bar{\rho}$. Thus there is a permutation σ of $\{1, \ldots, r\}$ of order two, such that for each $i \in \{1, \ldots, r\}$, we have $\bar{T}_i \circ \tau = \bar{T}_{\sigma(i)}$, and equivalently, $\bar{\rho}_i \circ \tau \simeq \bar{\rho}_{\sigma(i)}$. This implies $d_i = d_{\sigma(i)}$.

Remark 1.8.1. — (i) We check at once that the ideal $\operatorname{Ker} T \subset R$ is stable by τ, hence τ induces an automorphism, or an anti-automorphism, on $R/\operatorname{Ker} T$ which we will still denote by τ.

(ii) In the same vein, we have for each $x \in R$ an equality of characteristic polynomials

$$P_{x,T} = P_{\tau(x),T},$$

hence τ factors also through the maximal Cayley-Hamilton quotient of R (see § 1.2.5).

1.8.2. Lifting idempotents. — In the following lemma, A is a local henselian ring in which 2 is invertible.

Lemma 1.8.2. — *Let S be an integral A-algebra, τ an A-linear involution of S which is either a morphism or an anti-morphism of algebra, and let $I \subset \operatorname{rad}(S)$ be a two-sided ideal of S such that $\tau(I) = I$.*

Let (ϵ_i), $i = 1, \ldots, k$, be a family of orthogonal idempotents in S/I, and assume that the set $\{\epsilon_i, i = 1, \ldots, k\} \subset S/I$ is stable by τ. Then there is a family of orthogonal idempotents (e_i) in S, $i = 1, \ldots, k$, lifting (ϵ_i) and such that $\{e_i, i = 1, \ldots, k\}$ is stable by τ.

Proof. — We prove the lemma by induction on k. It is obvious for $k = 0$. Assume it is true for any $k' < k$. We will consider two cases.

First case. — $\tau(\epsilon_1) = \epsilon_1$. Let x be any lifting of ϵ_1 in S. Set $y = (x + \tau(x))/2$. Then $\tau(y) = y$. Let S_1 be the A-subalgebra of S generated by y. It is a commutative, finite A-algebra on which $\tau = \mathrm{Id}$. Set $I_1 := I \cap S_1$. Then $S_1/I_1 \subset S/I$ and S_1/I_1 contains the reduction of y which is ϵ_1. As A is henselian, there exists $e_1 \in S_1$ an idempotent lifting ϵ_1. Then $\tau(e_1) = e_1$.

The A-subalgebra [20] $S_2 := (1-e_1)S(1-e_1)$ is stable by τ, and if $I_2 := I \cap S_2$, then $S_2/I_2 \subset S/I$ contains the family $\epsilon_2, \ldots, \epsilon_k$ that is stable by τ. By induction hypothesis, this family can be lifted as an orthogonal family of idempotents $e_2, \ldots, e_k$, stable by τ, in S_2, and then $e_1, \ldots, e_k$ is an orthogonal family of idempotents lifting $\epsilon_1, \ldots, \epsilon_k$ in S that is stable by τ. The lemma is proved in this case.

Second case. — $\tau(\epsilon_1) \neq \epsilon_1$. Then up to renumbering, we may assume that $\tau(\epsilon_1) = \epsilon_2$. We claim that

there are two orthogonal idempotents e_1 and e_2 in S
lifting ϵ_1 and ϵ_2 respectively, such that $\tau(e_1) = e_2$.

This claim implies the lemma since we may apply the induction hypothesis to lift the family $\epsilon_3, \ldots, \epsilon_k$ in $(1 - (e_1 + e_2)S(1 - (e_1 + e_2))$ by the same reasoning as above. Moreover, in order to prove the claim, we may assume that $\varepsilon_1 + \varepsilon_2 = 1$. Indeed, set $\epsilon = \epsilon_1 + \epsilon_2$. This is an idempotent of S/I stable by τ. By the first case above, there is an idempotent e in S lifting ϵ and such that $\tau(e) = e$. Replacing S with eSe, and I with $I \cap eSe$, we have now $\epsilon_1 + \epsilon_2 = 1$, and we are done. To prove our claim, we have to distinguish again two cases:

[20] Recall that if $e \in S$ is an idempotent and I a two-sided ideal of S, then $eIe = I \cap eSe$ and $\operatorname{rad}(eSe) = e\operatorname{rad}(S)e$.

First subcase. — τ is an automorphism of algebra. Let $f \in S$ be any idempotent lifting ϵ_1. Set $f' := f(1-\tau(f))$. Then $f'\tau(f') = f(1-\tau(f))\tau(f)(1-f) = 0$ and $\tau(f')f' = \tau(f)(1-f)f(1-\tau(f)) = 0$. Hence the subalgebra S_1 of S generated by f' and $\tau(f')$ is commutative and stable by τ. Moreover, the reduction of f' modulo $I_1 := I \cap S_1$ is $\epsilon_1(1-\tau(\epsilon_1)) = \epsilon_1(1-\epsilon_2) = \epsilon_1$ and the reduction of $\tau(f')$ is $\tau(\epsilon_1) = \epsilon_2$.

Now, let g be an idempotent in S_1 lifting ϵ_1, and again let $g' = g(1-\tau(g))$. The same computation as above shows that $g'\tau(g') = \tau(g')g' = 0$, but now, since S_1 is commutative, g' is an idempotent. Set $e_1 := g'$, $e_2 =: \tau(g')$, and the claim is proved, hence the lemma in this subcase (we could also have concluded by using the fact that the lemma is easy if S is a finite commutative A-algebra).

Second subcase. — τ is an anti-automorphism. Let $f \in S$ be any idempotent lifting ϵ_1. Set $x := f\tau(f)$. Then $x \in I$ and $\tau(x) = x$. Let S_1 be the A-subalgebra of S generated by x, $I_1 := I \cap S_1$. This is a finite commutative A-algebra stable by τ. Note that $I_1 \subset \mathrm{rad}(S_1)$. Indeed, $I_1 \subset \mathrm{rad}(S)$, hence for all $y \in I_1$, $1+y$ is invertible in S, hence in S_1 as it is integral over A. We conclude as I_1 is a two-sided ideal of S_1. In particular, $x \in \mathrm{rad}(S_1)$. Since A is henselian and 2 is invertible in A, there exists a unique element $u \in 1 + \mathrm{rad}(S_1)$ such that $u^2 = 1-x$. Such an element u is invertible in S_1 and satisfies $\tau(u) = u$. Set $g = u^{-1}fu$. Then g is an idempotent lifting ϵ_1 and from $u\tau(u) = u^2 = 1 - f\tau(f)$ we get

$$g\tau(g) \quad = u^{-1}f(1-f\tau(f))\tau(f)u^{-1} = 0.$$

Finally, we set $e_1 = g - \frac{1}{2}\tau(g)g$ and $e_2 = \tau(e_1) = \tau(g) - \frac{1}{2}\tau(g)g$. Then e_i lifts ε_i and we claim that $e_i^2 = e_i$ and $e_1e_2 = e_2e_1 = 0$. Indeed, this follows at once from the following easy fact:

Let R be a ring in which 2 is invertible, and let e, f be two idempotents of R such that $ef = 0$. If we set $e' = (1-\frac{f}{2})e$ and $f' = f(1-\frac{e}{2})$, then e' and f' are orthogonal idempotents.

To check this fact, note that $e'^2 = (1-\frac{f}{2})e(1-\frac{f}{2})e = (1-\frac{f}{2})ee = e'$ as $ef = 0$ and $e^2 = e$. Similarly, $f'^2 = f'$. Moreover, it is clear that $e'f' = 0$, and

$$f'e' = f\left(1-\frac{e}{2}\right)\left(1-\frac{f}{2}\right)e = f\left(1-\frac{e+f}{2}\right)e = fe - \frac{fe+fe}{2} = 0,$$

which concludes the proof. □

Lemma 1.8.3. — *Assume that T is Cayley-Hamilton. There are idempotents $e_1, \dots, e_r$ in R and morphisms $\psi : e_iRe_i \longrightarrow M_{d_i}(A)$ satisfying properties (1) to (5) of Lemma 1.4.3 of prop 1.4.3 and moreover*

(6) *For $i \in \{1, \dots, r\}$, $\tau(e_i) = e_{\sigma(i)}$.*

Proof. — We call ϵ_i, $i = 1, \dots, r$ the central idempotents of $\bar{R}/\mathrm{Ker}\,\bar{T}$. Note that we have $\tau(\epsilon_i) = \epsilon_{\sigma(i)}$. Applying the preceding lemma to $S = R$ and $I := \mathrm{Ker}\,(R \longrightarrow$

$\bar{R}/\mathrm{Ker}\,\bar{T}) = \mathrm{rad}R$ (Lemma 1.2.7), there exists a family of orthogonal idempotents $e_1, \ldots, e_r$ lifting $\epsilon_1, \ldots, \epsilon_r$ that is stable by τ. Hence $\tau(e_i) = e_{\sigma(i)}$, which is (6), and the other properties are proved exactly as in Lemma 1.4.3. □

1.8.3. Notations and choices. — From now we let S be a Cayley-Hamilton quotient of R which is stable by τ. For example, by Remark 1.8.1 the faithful quotient $R/\mathrm{Ker}\,T$ has this property. As $\sigma^2 = \mathrm{Id}$, we may cut the set $I = \{1, \ldots, r\}$ into three parts

$$I = I_0 \coprod I_1 \coprod I_2$$

with $i \in I_0$ if and only if $\sigma(i) = i$, and with $\sigma(I_1) = I_2$. If $i \in I_1$ and $j = \sigma(i)$, we definitely choose $\bar{\rho}_j := \bar{\rho}_i^{\perp}$, which is permitted since $\bar{\rho}_j$ and $\bar{\rho}_i^{\perp}$ are isomorphic.

We now choose in a specific way a GMA datum on S taking into account the symmetry τ. First, Lemma 1.8.3 provides us with a family of idempotents e_i such that

$$\tau(e_i) = e_{\sigma(i)}, \forall i \in I.$$

Moreover, by property (5) (actually Lemma 1.4.3 (5)) we also have isomorphisms $\psi_i : e_i S e_i \longrightarrow M_{d_i}(A)$ for $i \in I$. We are happy with the ψ_i for $i \in I_0 \cup I_1$, but for $j \in I_2$, $j = \sigma(i)$ with $i \in I_1$ we forget about the ψ_j given by (5) by setting

$$\psi_j = \psi_{\sigma(i)} := \psi_i^{\perp}. \tag{22}$$

Of course, we also have $\psi_i = \psi_j^{\perp}$ as $\tau^2 = \mathrm{id}$. From now on, we fix a choice of e_i's and ψ_i's on S as above, and this choice makes S a GMA.

Let $i \in I_0$. Note that the two morphisms ψ_i and $\psi_i^{\perp} : e_i S e_i \longrightarrow M_{d_i}(A)$ have the same trace and are residually irreducible. Hence by Serre and Carayol's result ([**33**]), that is also the uniqueness part of the Nyssen and Rouquier's result, there exists a matrix $P_i \in \mathrm{GL}_{d_i}(A)$ such that $\psi_i = P_i \psi_i^{\perp} P_i^{-1}$. Note that P_i is determined up to the multiplication by an element of A^*. We fix the choice of such a matrix P_i for each $i \in I_0$. For $i \in I_1 \coprod I_2$ we set $P_i := \mathrm{Id}$. Note that obviously $P_i = P_{\sigma(i)}$. We have, for any $i \in I$,

$$\psi_{\sigma(i)} = P_i \psi_i^{\perp} P_i^{-1}, \quad \psi_i = P_i \psi_{\sigma(i)}^{\perp} P_i^{-1}. \tag{23}$$

Lemma 1.8.4. — *If τ is an automorphism (resp. an anti-automorphism) P_i^2 (resp. ${P_i}^t P_i^{-1}$) is a scalar matrix $x_i \mathrm{Id}_{d_i}$ where $x_i \in A^*$ (resp. $x_i \in \{\pm 1\}$).*

Proof. — Assume that ψ is an anti-automorphism (we leave the other, simpler, case to the reader). Using the two equalities of (23) we get

$$\psi_i = {P_i}^t P_i^{-1} \psi_i ({P_i}^t P_i^{-1})^{-1},$$

hence ${P_i}^t P_i^{-1}$ is a scalar matrix $x_i Id$ with $x_i \in A^*$ and we have $x_i {}^t P_i = P_i$ hence $x_i^2 = 1$. The result follows since A is local and 2 is invertible in A. □

1.8.4. Definition of the morphisms $\tau_{i,j}$. — Recall from § 1.3 that the idempotent E_i of S is defined as $\psi_i^{-1}(E_{1,1})$ and that $\mathcal{A}_{i,j}$ is the A-module E_iSE_j. Set $p_i = \psi_i^{-1}(P_i) \in e_iSe_i$. This is an invertible element in the algebra e_iSe_i and we denote its inverse *in this algebra* by p_i^{-1}.

Applying (23) to $\tau(E_i)$ we get easily

$$\tau(E_i) = p_{\sigma(i)}E_{\sigma(i)}p_{\sigma(i)}^{-1}$$

Assume first that τ is an automorphism. We have

$$\tau(\mathcal{A}_{i,j}) = \tau(E_i)\,S\,\tau(E_j) = p_{\sigma(i)}E_{\sigma(i)}p_{\sigma(i)}^{-1}\,S\,p_{\sigma(j)}E_{\sigma(j)}p_{\sigma(j)}^{-1}.$$

Hence we may define a morphism of A-modules $\tau_{i.j} : \mathcal{A}_{i,j} \longrightarrow \mathcal{A}_{\sigma(i),\sigma(j)}$ by setting

$$\tau_{i,j} = p_{\sigma(i)}^{-1}\tau_{|\mathcal{A}_{i,j}}p_{\sigma(j)}.$$

Assume now that τ is an anti-automorphism. We define similarly a morphism $\tau_{i,j} : \mathcal{A}_{i,j} \longrightarrow \mathcal{A}_{\sigma(j),\sigma(i)}$ by setting

$$\tau_{i,j} = p_{\sigma(j)}^{-1}\tau_{|\mathcal{A}_{i,j}}p_{\sigma(i)}.$$

Lemma 1.8.5. — *Assume τ is an automorphism (resp. an anti-automorphism).*

(i) *For all i,j, the A-linear endomorphism $\tau_{\sigma(i),\sigma(j)} \circ \tau_{i,j}$ (resp. $\tau_{\sigma(j),\sigma(i)} \circ \tau_{i,j}$) of $\mathcal{A}_{i,j}$ is the multiplication by an element of A^*.*

(ii) *For all i,j, $\tau_{i,j}$ is an isomorphism of A-modules.*

(iii) *For all i,j,k and $x \in \mathcal{A}_{i,j}$, $y \in \mathcal{A}_{j,k}$ we have $\tau_{i,j}(x)\tau_{j,k}(y) = \tau_{i,k}(xy)$ in $\mathcal{A}_{\sigma(i),\sigma(k)}$ (resp. $\tau_{j,k}(y)\tau_{i,j}(x) = \tau_{i,k}(xy)$ in $\mathcal{A}_{\sigma(k),\sigma(i)}$).*

(iv) *We have $\tau_{i,j}(\mathcal{A}'_{i,j}) = \mathcal{A}'_{\sigma(i),\sigma(j)}$ (resp. $\tau_{i,j}(\mathcal{A}'_{i,j}) = \mathcal{A}'_{\sigma(j),\sigma(i)}$).*

Proof. — The assertion (i) is an easy computation using Lemma 1.8.4. The assertion (ii) follows immediately from (i). The assertion (iii) is a straightforward computation and (iv) follows from (iii), (ii) and the definition of the $\mathcal{A}'_{i,j}$ (see § 1.5.3). □

1.8.5. Definition of the morphisms $\perp_{i,j}$. — Let $\mathcal{P}$ be a partition of $\{1,\ldots,r\}$ such that the singletons $\{i\}$ and $\{j\}$ belong to $\mathcal{P}$. Let $I_{\mathcal{P}}$ be the corresponding reducibility ideal. Note that by Lemma 1.8.5, $I_{\mathcal{P}} = I_{\sigma(\mathcal{P})}$ so that we may assume without changing $I_{\mathcal{P}}$ that the singletons $\{\sigma(i)\}$ and $\{\sigma(j)\}$ belong to $\mathcal{P}$. Let J be an ideal of A containing $I_{\mathcal{P}}$.

Recall that we defined a representation $\rho_i : R/JR \longrightarrow M_d(A/J)$ in Def. 1.5.3. By point (ii) of Lemma 1.5.4, ρ_i is the reduction mod J of the composite of the morphism ψ_i with the surjection $R \longrightarrow S \longrightarrow e_iSe_i$. Hence we have

$$\rho_{\sigma(i)} = P_i\rho_i^{\perp}P_i^{-1}. \tag{24}$$

Let c be an extension in $\mathrm{Ext}^1_{R/JR}(\rho_j,\rho_i)$. We can see it as a morphism of algebra

$$\begin{array}{rcl} \rho_c : R/JR & \longrightarrow & M_{d_i+d_j}(A/J) \\ x & \mapsto & \begin{pmatrix} \rho_i(x) & c(x) \\ 0 & \rho_j(x) \end{pmatrix}, \end{array}$$

where $c(x) \in M_{d_i,d_j}(A/J)$. Then setting $Q_{i,j} = \mathrm{diag}(P_i, P_j) \in M_{d_i+d_j}(A/J)$ we see using (24) that if τ is an automorphism,

$$(25) \quad Q_{i,j}\rho_c^\perp(x)Q_{i,j}^{-1} = \begin{pmatrix} \rho_{\sigma(i)}(x) & c'(x) \\ 0 & \rho_{\sigma(j)}(x) \end{pmatrix}, \text{ where } c'(x) = P_i\ c(\tau(x))\ P_j^{-1},$$

and that if τ is an anti-automorphism,

$$(26) \quad Q_{i,j}\rho_c^\perp(x)Q_{i,j}^{-1} = \begin{pmatrix} \rho_{\sigma(i)}(x) & 0 \\ c'(x) & \rho_{\sigma(j)}(x) \end{pmatrix}, \text{ where } c'(x) = P_j\ {}^t c(\tau(x))\ P_i^{-1}.$$

Hence $Q_{i,j}\rho_c^\perp Q_{i,j}^{-1}$ represents an element c' in

$$\mathrm{Ext}^1_{R/JR}(\rho_{\sigma(j)},\rho_{\sigma(i)}) \text{ (resp. in } \mathrm{Ext}^1_{R/JR}(\rho_{\sigma(i)},\rho_{\sigma(j)})\text{)}$$

and we set

$$\perp_{i,j}(c) := c',$$

thus defining a morphism

$$\begin{array}{rl} \perp_{i,j} : & \mathrm{Ext}^1_{R/JR}(\rho_j,\rho_i) \longrightarrow \mathrm{Ext}^1_{R/JR}(\rho_{\sigma(j)},\rho_{\sigma(i)}) \\ \text{(resp. } \perp_{i,j} : & \mathrm{Ext}^1_{R/JR}(\rho_j,\rho_i) \longrightarrow \mathrm{Ext}^1_{R/JR}(\rho_{\sigma(i)},\rho_{\sigma(j)})\text{).} \end{array}$$

Note that all we have done also works when R is replaced by its τ-stable Cayley-Hamilton quotient S, and that the morphisms $\perp_{i,j}$ thus defined on the $\mathrm{Ext}^1_{S/JS}$'s are simply the restriction of the morphisms $\perp_{i,j}$ on $\mathrm{Ext}^1_{R/JR}$.

1.8.6. The main result

Proposition 1.8.6. — *If τ is an automorphism, the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathcal{A}_{i,j}/\mathcal{A}'_{i,j}, A/J) & \xrightarrow{\iota_{i,j}} & \mathrm{Ext}^1_{S/JS}(\rho_j,\rho_i) \\ \Big\downarrow{\scriptstyle (\tau_{i,j}^{-1})^*} & & \Big\downarrow{\scriptstyle \perp_{i,j}} \\ \mathrm{Hom}_A(\mathcal{A}_{\sigma(i),\sigma(j)}/\mathcal{A}'_{\sigma(i),\sigma(j)}, A/J) & \xrightarrow{\iota_{\sigma(i),\sigma(j)}} & \mathrm{Ext}^1_{S/JS}(\rho_{\sigma(j)},\rho_{\sigma(i)}) \end{array}$$

If τ is an anti-automorphism, the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_A(\mathcal{A}_{i,j}/\mathcal{A}'_{i,j}, A/J) \arrow[r, "\iota_{i,j}"] \arrow[d, "(\tau_{i,j}^{-1})^*"] & \mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i) \arrow[d, "\perp_{i,j}"] \\
\mathrm{Hom}_A(\mathcal{A}_{\sigma(j),\sigma(i)}/\mathcal{A}'_{\sigma(j),\sigma(i)}, A/J) \arrow[r, "\iota_{\sigma(j),\sigma(i)}"] & \mathrm{Ext}^1_{S/JS}(\rho_{\sigma(i)}, \rho_{\sigma(j)})
\end{tikzcd}$$

Proof. — This follows immediately from the definitions of the morphisms $\tau_{i,j}$ (see § 1.8.4), $\perp_{i,j}$ (see § 1.8.5, especially (25) and (26)) and $\iota_{i,j}$'s (see § 1.5.3). □

1.8.7. A special case. — We keep the assumptions of § 1.8.1 and the notations above, but we assume that

(i) the ring A is reduced, of total fraction ring a finite product of fields $K = \prod_{s=1}^n K_s$,
(ii) the pseudocharacters $T \otimes K_s$ are irreducible,
(iii) τ is an anti-automorphism.

Let $\rho : S := R/\mathrm{Ker}\, T \longrightarrow M_d(K)$ be a representation as in Theorem 1.4.4 (ii). By assumption (ii) above and Theorem 1.4.4 (iii), ρ induces an isomorphism

(27) $$S \otimes_A K \xrightarrow{\sim} M_d(K).$$

For $s \in \{1, \dots, n\}$, denote by ρ_s the composite $S \xrightarrow{\rho} M_d(K) \longrightarrow M_d(K_s)$.

Lemma 1.8.7. — *For each $s \in \{1, \dots, n\}$ there exists a matrix $Q_s \in \mathrm{GL}_d(K_s)$ such that*

(28) $$\rho_s^\perp = Q_s \rho_s Q_s^{-1},$$

and there is a well-determined sign $\epsilon_s = \pm 1$ such that ${}^tQ_s = \epsilon_s Q_s$. If d is odd then $\epsilon_s = 1$.

Proof. — The representations ρ_s and $\rho_s^\perp$ are irreducible by hypothesis (ii) and have the same trace hence are isomorphic. Moreover ρ_s is absolutely irreducible by (27), hence the existence of a Q_s such that $\rho_s^\perp = Q_s \rho_s Q_s^{-1}$, and its uniqueness up to the multiplication by an element on K_s^*. Using that $(\rho_s^\perp)^\perp = \rho_s$, we see that ${}^tQ_s Q_s^{-1}$ centralizes ρ_s, hence is a scalar matrix. Thus ${}^tQ_s = \epsilon_s Q_s$ and $\epsilon_s = \pm 1$. The last assertion holds because there is no antisymmetric invertible matrix in odd dimension. □

We will now relate these signs ϵ_s to other signs, and prove that they are actually equal in many cases. Recall that if $k \in \{1, \dots, r\}$ is such that $\sigma(k) = k$, we fixed in § 1.8.3 a $P_k \in GL_{d_k}(A)$ such that $\psi_k = P_k \psi_k^\perp P_k^{-1}$, and we showed that ${}^tP_k P_k^{-1} = \pm 1 \in A^*$ is a sign in Lemma 1.8.4. As explained there, P_k is uniquely determined up to an element of A^*, so this sign is well defined, let us call it $\epsilon(k)$. By reducing those equalities mod m, $\epsilon(k)$ is also "*the sign*" of the residual representation $\bar{\rho}_k \simeq \bar{\rho}_k^{\perp}$ in the obvious sense.

Lemma 1.8.8. — *Assume that $\sigma(k) = k$ for some $k \in \{1, \dots, r\}$. Then for each s, $\epsilon_s = \epsilon(k)$ is the sign of $\bar{\rho}_k$.*

Proof. — As $\tau(e_k) = e_{\sigma(k)} = e_k$, we have $\tau(e_k S e_k) = e_k S e_k$. Recall that by the assumptions in § 1.8.3, we have

$$\rho_{|e_k S e_k} = \psi_k : e_k R e_k \xrightarrow{\sim} M_{d_k}(A)$$

with $\psi_k^{\perp} = P_k^{-1} \psi_k P_k$. For each $s \in \{1, \dots, n\}$, we also have

$$e_k = \rho_s^{\perp}(e_k) = Q_s \rho_s(e_k) Q_s^{-1} = Q_s e_k Q_s^{-1},$$

so Q_s commutes with $e_k = {}^t e_k$, and $e_k Q_s$ and Q_s are both symmetric or antisymmetric. Since $\psi_k : e_k(S \otimes_A K)e_k \xrightarrow{\sim} M_d(K)$ is an isomorphism, we get that for some $\lambda_s \in K_s^*$,

$$e_k Q_s = \lambda_s P_k^{-1}.$$

In particular, the three matrices $e_k Q_s$, Q_s and P_k (which does not depend on s) are simultaneously symmetric or antisymmetric, and we are done. □

Let us fix now $i \neq j$ two integers in $\{1, \dots, r\}$ such that $\sigma(i) = j$. Under hypothesis (iii), the morphism $\perp_{i,j}$ is an endomorphism of the A-module $\mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i)$ and is canonically defined. We will study it using Proposition 1.8.6 and in terms of the signs above. Recall that we also defined some A-linear isomorphism $\tau_{i,j}$ of $\mathcal{A}_{i,j} = \mathcal{A}_{\sigma(j),\sigma(i)}$.

Lemma 1.8.9. — *The morphism $\tau_{i,j} : \mathcal{A}_{i,j} \longrightarrow \mathcal{A}_{i,j}$ is the multiplication by the element $(\epsilon_1, \dots, \epsilon_n)$ of K^*.*

Proof. — Let $Q \in \mathrm{GL}_d(K)$ be the matrix whose image in $\mathrm{GL}_d(K_s)$ is Q_s for $s = 1, \dots, n$. The representation ρ identifies S with a standard GMA $\rho(S)$ in $M_d(K)$ and it follows from (28) that the anti-automorphism τ on $\rho(S)$ is the restriction of the anti-automorphism $M \mapsto Q {}^t M Q^{-1}$. Remember that $\rho(E_i)$ is the diagonal matrix whose all entries are zero but the $(d_1 + \dots + d_{i-1} + 1)^{\text{th}}$ which is one, and similarly for $\rho(E_j)$. Remember also that ρ identifies $\mathcal{A}_{i,j} = E_i S E_j$ with $A_{i,j} = \rho(E_i)\rho(S)\rho(E_j)$. Since $\tau(E_i) = E_j$ we have $\rho(E_j) = Q {}^t\rho(E_i) Q^{-1} = Q\rho(E_i)Q^{-1}$. Thus the 2 by 2 submatrix of Q, keeping only the $(d_1 + \dots + d_{i-1} + 1)^{\text{th}}$ and $(d_1 + \dots + d_{j-1} + 1)^{\text{th}}$ lines and row, is antidiagonal:

$$\begin{pmatrix} \rho(E_i)Q\rho(E_i) & \rho(E_i)Q\rho(E_j) \\ \rho(E_j)Q\rho(E_i) & \rho(E_j)Q\rho(E_j) \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_2(K)$$

But by the lemma we have ${}^t Q = (\epsilon_1, \dots, \epsilon_n) Q$, hence

$$b = (\epsilon_1, \dots, \epsilon_n) a.$$

Now $\tau_{i,j} : A_{i,j} \longrightarrow A_{i,j}$ is by definition the restriction of $M \mapsto QMQ^{-1}$ to $A_{i,j} = \rho(E_i)S\rho(E_j)$. By the formula above, this map is the multiplication by ab^{-1}, that is by the element $(\epsilon_1, \ldots, \epsilon_n)$ of K^*. □

Thus, by Proposition 1.8.6 and the lemmas above:

Proposition 1.8.10. — (i) *If all the signs ϵ_s are equal, then for each pair $i \neq j$ with $j = \sigma(i)$ the endomorphism*

$$\perp_{i,j} : \mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i) \longrightarrow \mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i)$$

is the multiplication by $\epsilon_1 = \pm 1$.

(ii) *If σ has a fixed point k, then all the ϵ_s are equal to the sign of $\bar{\rho}_k \simeq \bar{\rho}_k^{\perp}$.*

(iii) *If d is odd, all these signs are $+1$.*

Remark 1.8.11. — Note that the hypothesis of the corollary holds obviously when A is a domain. Note also that the fact that $\perp_{i,j}$ is the multiplication by ± 1 implies (and in fact is equivalent to) that every extension ρ in $\mathrm{Ext}^1_{S/JS}(\rho_j, \rho_i)$ is isomorphic to $\rho^{\perp}$ *as a representation* (not necessarily as an extension).

CHAPTER 2

TRIANGULINE DEFORMATIONS OF REFINED CRYSTALLINE REPRESENTATIONS

2.1. Introduction

The aim of this section is twofold. First, we study the d-dimensional trianguline representations of

$$G_p := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

for any $d \geq 1$ and with artinian ring coefficients, extending some results of Colmez in [**46**]. Then, we use them to define and study some deformation problems of the d-dimensional crystalline representations of G_p.

These deformation problems are motivated by the theory of p-adic families of automorphic forms and the wish to understand the family of Galois representations carried by eigenvarieties. They have been extensively studied in the special case of ordinary deformations (*e.g.* Hida families), however the general case is more subtle. When $d = 2$, it was first dealt with by Kisin in [**73**]. He proved that the local p-adic Galois representation attached to any finite slope overconvergent modular eigenform f admits a non trivial crystalline period on which the crystalline Frobenius acts through a_p if $U_p(f) = a_p f$, and also that this period "varies analytically" on the eigencurve. These facts lead him to define and study some deformation problem he called D^h in *loc. cit.* §8. In favorable cases, he was then able to show that the Galois deformations coming from Coleman's families give examples of such "h-deformations" (see §10, 11 *loc. cit*). In this section, we define and study a deformation problem for the d-dimensional case via the theory of (φ, Γ)-modules. It turns out to be isomorphic to Kisin's one when $d = 2$ but in a non trivial way. We postpone to chapters 3 and 4 the question of showing that higher rank eigenvarieties produce such deformations.

The approach we follow to define these problems is mainly suggested by Colmez's interpretation of the first result of Kisin recalled above in [**46**]. Precisely, Colmez proves that for a 2-dimensional p-adic representation V of G_p, a twist of V admits a non trivial crystalline period if, and only if, the (φ, Γ)-module of V over the

Robba ring [(1)] is triangulable ([**46**, Prop. 5.3]). For instance, the (φ,Γ)-module of a 2-dimensional crystalline representation is always trigonalisable (with non étale graded pieces in general) even if the representation is irreducible (that is non ordinary), which makes things interesting. This also led Colmez to define a *trianguline representation* as a representation whose (φ,Γ)-module over $\mathcal{R}$ is a successive extension of rank 1 (φ,Γ)-modules. Although this has not yet been proved, it is believed (and suggested by Kisin's work) that the above triangulation should vary analytically on the eigencurve, so that the general "finite slope families" should look pretty much like ordinary families from this point of view [(2)].

In what follows, we define and study in detail the *trianguline deformation functors* of a given d-dimensional crystalline representation for any d, establishing an "infinitesimal version" of the above program, that is working with artinian $\mathbb{Q}_p$-algebras as coefficients (instead of general $\mathbb{Q}_p$-affinoids which would require extra work). This case will be enough for the applications in the next sections and contains already quite a number of subtleties, mainly related to the notion "non criticality". We prove also a number of results of independent interest on triangular (φ,Γ)-modules, some of them generalizing to the d-dimensional case some results of Colmez in [**46**]. Let us describe now more precisely what we show.

In § 2.2, we collect the fundamental facts we shall use of the theory of (φ,Γ)-modules over the Robba ring $\mathcal{R}$. We deduce from Kedlaya's theorem that an extension between two étale (φ,Γ)-modules is itself étale (Lemma 2.2.5). A useful corollary is the fact that it is the same to deform the (φ,Γ)-module over $\mathcal{R}$ of a representation or to deform the representation itself (Proposition 2.3.13). We prove also in this part some useful results on modules over the Robba ring with coefficients in an artinian $\mathbb{Q}_p$-algebra.

In § 2.3, we study the triangular (φ,Γ)-modules over $\mathcal{R}_A := \mathcal{R}\otimes_{\mathbb{Q}_p} A$ where A is an artinian $\mathbb{Q}_p$-algebra. They are defined as (φ,Γ)-modules D, finite free over $\mathcal{R}_A$, equipped with a strictly increasing filtration (a *triangulation*)

$$(\mathrm{Fil}_i(D))_{i=0,\ldots,d},\ \ d := \mathrm{rk}_{\mathcal{R}_A}(D),$$

of (φ,Γ)-submodules which are free and direct summand over $\mathcal{R}_A$ [(3)]. When D has rank 1 over $\mathcal{R}_A$, we show that it is isomorphic to a "basic" one $\mathcal{R}_A(\delta)$ for some

(1) Recall that the category of (φ,Γ)-modules over the Robba ring $\mathcal{R}$ is strictly bigger than the category of $\mathbb{Q}_p$-representations of G_p, which occurs as its full subcategory of étale objects.

(2) A related question is to describe the A-valued points, A being any $\mathbb{Q}_p$-affinoid algebra, of the parameter space $\mathcal{S}$ of triangular (φ,Γ)-modules defined by Colmez in [**46**, §0.2]. The material of this part would be e.g. enough to answer the case where A is an artinian $\mathbb{Q}_p$-algebra, at least for "non critical" triangulations. See also our results in chapter 4.

(3) It is important here not to restrict to the étale D, even if in some important applications this would be the case. Indeed, most of the proofs use an induction on d and the $\mathrm{Fil}_i(D)\subset D$ will not even be isocline in general.

unique continuous character $\mathrm{W} \longrightarrow A^*$ (Proposition 2.3.1) of the Weil group W of $\mathbb{Q}_p$, hence the graded pieces of $\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)$ have the form $\mathcal{R}_A(\delta_i)$ in general. The *parameter* $(\delta_i)_{i=1,\dots,d}$ of D defined this way turns out to refine the datum of the Sen polynomial of D (Proposition 2.3.3). A first important result of this part is a *weight criterion* ensuring that such a (φ,Γ)-module is de Rham (Proposition 2.3.4); this criterion is a generalization to trianguline representations of Perrin-Riou's criterion "ordinary representations are semistable" ([**1**, Exposé IV, Théorème]). In the last paragraphs, we define and study the functor of triangular deformations of a given triangular (φ,Γ)-module D_0 over $\mathcal{R}$: its A-points are simply the triangular (φ,Γ)-modules deforming D_0 and whose triangulation lifts the fixed triangulation of D_0. In the same vein, a *trianguline deformation* of a trianguline representation V_0 is a triangular deformation of its (φ,Γ)-module [(4)] D_0 (it depends on the triangulation of D_0 we choose). The main result here is a complete description of these functors under some explicit conditions on the parameter of the triangulation of D_0 (Proposition 2.3.10).

In § 2.4, we show that crystalline representations are trianguline and study the different possible triangulations of the (φ,Γ)-module of a given crystalline representation [(5)] V. We show that they are in natural bijection with the *refinements* of V in Mazur's sense [**85**], that is the full φ-stable filtrations of $D_{\mathrm{crys}}(V)$. More importantly, we introduce a notion of *non critical refinement* in § 2.4.3 by asking that the φ-stable filtration is in general position compared to the Hodge filtration on $D_{\mathrm{crys}}(V)$. We interpret this condition in terms of the associated triangulation of the (φ,Γ)-module (Proposition 2.4.7), and compare it to other related definitions in the literature (Remark 2.4.6). This notion turns out to be the central one in all the subsequent results. The main ingredient for this part is Berger's paper [**13**].

In § 2.5, we apply all the previous parts to define and study the trianguline deformation functor of a refined crystalline representation. It should be understood as follows: the choice a refinement of V defines, by the previous results, a triangulation of its (φ,Γ)-module, and we can study the associated trianguline deformation problem defined above. When the chosen refinement is non critical, we can explicitly describe the trianguline deformation functor (Theorem 2.5.10), and also describe the crystalline locus inside it. A striking result is that "a trianguline deformation of a non critically refined crystalline representation is crystalline if and only if it is Hodge-Tate" (Theorem 2.5.1). This fact may be viewed as an infinitesimal local version of Coleman's "small slope forms are classical" result; it will play an important role in the

(4) In this section, all the (φ,Γ)-modules are understood with coefficients in the Robba ring $\mathcal{R}$.

(5) For simplicity, we restrict there to crystalline representations with distinct Hodge-Tate weights. In fact, the results of this part could be extended to the representations becoming semi-stable over an abelian extension of $\mathbb{Q}_p$, and even to all the de Rham representations in a weaker form.

applications to Selmer groups of the subsequent chapters (see e.g. Corollary 4.4.5). In the last paragraph, we give a criterion ensuring that a deformation satisfying some conditions in Kisin's style is in fact trianguline (Theorem 2.5.6). Combined with the extensions of Kisin's work studied in chapter 3, this result will be useful to prove that the Galois deformations living on eigenvarieties are trianguline in many interesting cases [(6)].

In a last § 2.6, we discuss some applications of these results to global deformation problems. Recall that a consequence of the Bloch-Kato conjecture for adjoint pure motives (see Remark 5.2.4) is that a geometric, irreducible, p-adic Galois representation V (say crystalline above p) admits no non trivial crystalline deformation [(7)]. Admitting this, we obtain that the trianguline deformations of V for a non critical refinement $\mathcal{F}$ (and with good reduction outside p say) have Krull-dimension at most $\dim(V)$ (Corollary 2.6.1). This "explains" for example why the eigenvarieties of reductive rank d have dimension at most d, and in general it relates the dimension of the tangent space of eigenvarieties of $\mathrm{GL}(n)$ at classical points (about which we know very few) to an "explicit" Selmer group. As another good indication about the relevance of the objects above, let us just say that when such a $(V, \mathcal{F})$ appears as a classical point x on a unitary eigenvariety X (say of "minimal level outside p"), standard conjectures imply that

$$R \simeq T \overset{\kappa}{\xrightarrow{\sim}} L[[X_1, \dots, X_d]],$$

where R prorepresents the trianguline deformation functor of $(V, \mathcal{F})$, T is the completion of X at x, and κ is the morphism of the eigenvariety to the weight space.

The authors are grateful to Laurent Berger and Pierre Colmez for very helpful discussions during the preparation of this section. We started working on the infinitesimal properties of the Galois representations on eigenvarieties in September 2003, and since we have been faced with an increasing number of questions concerning non de Rham p-adic representations which were fundamental regarding the arithmetic applications. We warmly thank them for taking the time to think about our questions during this whole period. As will be clear to the reader, Colmez's paper [**46**] has been especially influential to us. We also thank Denis Benois and an anonymous referee for their remarks.

(6) However not in all cases; part of this result may be viewed as a trick allowing one to circumvent the study of a theory of families of triangular (φ, Γ)-modules alluded above.

(7) As $\mathbb{Q}$-motives are countable, it is certainly expected that there is no non trivial 1-parameter p-adic family of motives, but the infinitesimal assertion is stronger.

2.2. Preliminaries of p-adic Hodge theory and (φ,Γ)-modules

2.2.1. Notations and conventions. — In all this section,

$$G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),$$

is equipped with its Krull topology. Let A be a finite dimensional local commutative $\mathbb{Q}_p$-algebra equipped with its unique Banach $\mathbb{Q}_p$-algebra topology, m its maximal ideal, $L := A/m$.

By an *A-representation of G_p*, we shall always mean an A-linear, continuous, representation of G_p on a finite type A-module. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, equipped with its canonical valuation v, and norm $|.|$, extending the one of $\mathbb{Q}_p$ (so $v(p) = 1$ and $|p| = 1/p$), and we denote by $\mathbb{C}_p$ its completion. We denote by B_{crys}, B_{dR}, $D_{\mathrm{crys}}(-)$, $D_{\mathrm{DR}}(-)$ etc. the usual rings and functors defined by Fontaine ([**1**, Exposés II et III]).

We denote by $\mathbb{Q}_p(1)$ the $\mathbb{Q}_p$-representation of G_p on $\mathbb{Q}_p$ defined by the cyclotomic character

$$\chi : G_p \longrightarrow \mathbb{Z}_p^*.$$

If V is an A-representation of G_p and $m \in \mathbb{Z}$, then we set $V(m) := V \otimes \chi^m$.

Our convention on the sign of the Hodge-Tate weights, and on the Sen polynomial, is that $\mathbb{Q}_p(1)$ has weight -1 and Sen polynomial $T + 1$. With this convention, the Hodge-Tate weights (without multiplicities) of a de Rham representation V are the jumps of the Hodge filtration on $D_{\mathrm{DR}}(V)$, that is the integers i such that $\mathrm{Fil}^{i+1}(D_{\mathrm{DR}}(V)) \subsetneq \mathrm{Fil}^i(D_{\mathrm{DR}}(V))$, and also the roots of the Sen polynomial of V.

2.2.2. (φ,Γ)-modules over the Robba ring $\mathcal{R}_A$. — It will be convenient for us to adopt the point of view of (φ,Γ)-modules over the Robba ring, for which we refer to [**54**], [**45**], [**71**], and [**14**].

Let $\mathcal{R}_A$ be the Robba ring with coefficients in A, *i.e.* the ring of power series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-1)^n, \;\; a_n \in A$$

converging on some annulus of $\mathbb{C}_p$ of the form $r(f) \leq |z-1| < 1$, equipped with its natural A-algebra topology. If we set

$$\mathcal{R} := \mathcal{R}_{\mathbb{Q}_p},$$

we have $\mathcal{R}_A = \mathcal{R} \otimes_{\mathbb{Q}_p} A$. Recall that $\mathcal{R}_A$ is equipped with commuting, A-linear, continuous actions of φ and of the group

$$\Gamma := \mathbb{Z}_p^*$$

defined by

$$\varphi(f)(z) = f(z^p), \;\; \gamma(f)(z) = f(z^\gamma).$$

To get a picture of these actions, note that if $z \in \mathbb{C}_p$ satisfies $|z-1| < 1$, we have $|z^n - 1| = |z-1|$ for $n \in \mathbb{Z}_p^*$, whereas $|z^p - 1| = |z-1|^p$ when $|z-1| > p^{-\frac{1}{p-1}}$.

Definition 2.2.1. — A (φ, Γ)-module over $\mathcal{R}_A$ is a finitely generated $\mathcal{R}_A$-module D which is free over $\mathcal{R}$ and equipped with commuting, $\mathcal{R}_A$-semilinear, continuous [(8)] actions of φ and Γ, and such that $\mathcal{R}\varphi(D) = D$.

Of course, the (φ, Γ)-modules over $\mathcal{R}_A$ form a category in the obvious way: if D_1 and D_2 are two such (φ, Γ)-modules, we define a homomorphism $D_1 \to D_2$ as a $\mathcal{R}_A$-linear map commuting with the actions of φ and Γ. We shall call $(\varphi, \Gamma)/A$ this category; it is obviously additive, and even A-linear, but it is not an abelian category.

2.2.3. Some algebraic properties of $\mathcal{R}_A$. — In the first part of this section, we assume that $A = L$ is a field, and we will now recall some algebraic properties of modules over $\mathcal{R}_L$.

A first remark is that $\mathcal{R}_L$ is a domain. Moreover, although it is not noetherian, a key property is that $\mathcal{R}_L$ is an *adequate Bezout domain* (this is essentially due to Lazard [**79**], see also [**13**, prop. 4.12] in these terms), hence the theory of finitely presented $\mathcal{R}_L$-modules is similar to the one for principal rings:

(B1) Finitely generated, torsion free, $\mathcal{R}_L$-modules are free.

(B2) For any finite type $\mathcal{R}_L$-submodule $M \subset \mathcal{R}_L^n$, there is a basis (e_i) of $\mathcal{R}_L^n$, and elements $(f_i)_{1 \leq i \leq d} \in (\mathcal{R}_L \backslash \{0\})^d$, such that $M = \oplus_{i=1}^d f_i \mathcal{R}_L e_i$. The f_i may be chosen such that f_i divides f_{i+1} in $\mathcal{R}_L$ for $1 \leq i \leq d-1$, and are unique up to units of $\mathcal{R}_L$ if this is satisfied (they are called the *elementary divisors* of M in $\mathcal{R}_L^n$).

Let $M \subset \mathcal{R}_L^n$ be a $\mathcal{R}_L$-submodule, the *saturation* of M in $\mathcal{R}_L^n$ is

$$M^{\mathrm{sat}} := \{m \in \mathcal{R}_L^n,\ \exists f \in \mathcal{R}_L \backslash \{0\}, fm \in M\} = (M \otimes_{\mathcal{R}_L} \mathrm{Frac}(\mathcal{R}_L)) \cap \mathcal{R}_L^n$$

and we say that M is *saturated* if $M^{\mathrm{sat}} = M$, or which is the same if $\mathcal{R}_L^n/M$ is torsion free [(9)]. By (B1) (resp. (B2)) such an M is saturated if, and only if, it is a direct summand as $\mathcal{R}_L$-module (resp. if its elementary divisors are units). Note also that by property (B2), if $M \subset \mathcal{R}_L^n$ is finite type over $\mathcal{R}_L$, then so is M^{sat}.

It turns out that in a (φ, Γ)-module situation, we can say much more. Let $t := \log(z) \in \mathcal{R}$ be the usual "$2i\pi$-element". It satisfies $\varphi(t) = pt$ and $\gamma(t) = \gamma t$ for all $\gamma \in \mathbb{Z}_p^*$. Note that t is not an irreducible element of $\mathcal{R}$.

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(8) It means that for any choice of a free basis $e = (e_i)_{i=1,\ldots,d}$ of D as $\mathcal{R}$-module, the matrix map $\gamma \mapsto M_e(\gamma) \in \mathrm{GL}_d(\mathcal{R})$, defined by $\gamma(e_i) = M_e(\gamma)(e_i)$, is a continuous function on Γ. If $P \in \mathrm{GL}_d(\mathcal{R})$, then $M_{P(e)}(\gamma) = \gamma(P) M_e(\gamma) P^{-1}$, hence it suffices to check it for a single basis.

(9) As $\mathrm{Frac}\mathcal{R}_L = \mathrm{Frac}\mathcal{R} \otimes_{\mathbb{Q}_p} L$, an $\mathcal{R}_L$-module is torsion free over $\mathcal{R}_L$ if, and only if, it is torsion free over $\mathcal{R}$.

Proposition 2.2.2. — *Let D be a (φ,Γ)-module over $\mathcal{R}_L$ and D' a (φ,Γ)-submodule.*

(i) $D'^{\mathrm{sat}} = D'[1/t] \cap D$.

(ii) *If D' has rank 1 over $\mathcal{R}_L$, then $D' = t^k D'^{\mathrm{sat}}$, $k \in \mathbb{N}$.*

Proof. — Part (ii) is [**46**, rem. 4.5]. To prove part (i), it suffices to show that the product of the elementary divisors of D' is a power of t. But this follows from (ii) applied to $\Lambda^j(D') \subset \Lambda^j(D)$ with $j = \mathrm{rk}_{\mathcal{R}_L}(D')$. □

We end this section by establishing some basic but useful facts when working with artinian rings; in what follows A is not supposed to be a field any more.

Lemma 2.2.3. — (i) *Let E be a free A-module and $E' \subset E$ a free submodule, then E' is a direct summand of E.*

(ii) *Let E be a $\mathcal{R}_A$-module (resp. $\mathcal{R}_A[1/t]$-module) which is free of finite type as $\mathcal{R}$-module (resp. $\mathcal{R}[1/t]$-module), and free as A-module. Then E is free of finite type over $\mathcal{R}_A$ (resp. $\mathcal{R}_A[1/t]$).*

(iii) *Let D be a finite free $\mathcal{R}_A$-module. Assume that D contains a submodule D' free of rank 1 such that D'/mD' is saturated in D/mD as $\mathcal{R}$-module. Then D' is a direct summand as $\mathcal{R}_A$-submodule of D.*

Proof. — Let $n \geq 1$ denote the smallest integer such that $m^n = 0$. As m is nilpotent, the following version of Nakayama's lemma holds for all A-modules F: F is zero (resp. free) if, and only if, $F/mF = 0$ (resp. $\mathrm{Tor}_1^A(F, A/m) = 0$). For a proof, see [**29**][Chap. II, §3, no.2, Prop. 4, and Cor. 2 Prop. 5].

To prove (i), consider the natural exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(E/E', A/m) \longrightarrow E'/mE' \longrightarrow E/mE.$$

We have to show that the Tor above is zero, *i.e.* that $mE \cap E' = mE'$. But this follows from the fact that for a free A-module F,

$$mF = F[m^{n-1}] := \{f \in F, m^{n-1} f = 0\}.$$

Let us show (ii) now, set $\mathcal{R}' = \mathcal{R}[1/t]$ or $\mathcal{R}$. As E is free over A we have $mE = E[m^{n-1}]$ hence mE is a saturated $\mathcal{R}'$-submodule of E. As a consequence, E/mE is a torsion free, finite type, $\mathcal{R}'_L = \mathcal{R}'_A/m\mathcal{R}'_A$-module, so it is free over $\mathcal{R}'_L$ by property (B1). As E is free over A, Nakayama's lemma shows that any $\mathcal{R}'_A$-lift $(\mathcal{R}'_A)^d \longrightarrow E$ of an $\mathcal{R}'_L$-isomorphism $(\mathcal{R}'_L)^d \xrightarrow{\sim} E/mE$ is itself an isomorphism.

Before we prove (iii), let us do the following remark.

Remark 2.2.4. — Let D be a (φ,Γ)-module free over $\mathcal{R}_A$ and $D' \subset D$ be a submodule also free over $\mathcal{R}_A$. Then part (i) shows that D' is a direct summand of D as A-module. In particular, for any ideal $I \subset A$, the natural map $D'/ID' \longrightarrow D/ID$ is injective, and $D' \cap ID = ID'$. Note that this remark gives sense to part (iii) of the proposition (take $I = m$).

To prove (iii), let us argue by induction on the length of A to show that D' is $\mathcal{R}$-saturated in D. We are done by assumption if A is a field. Let $I \subset A$ be a proper ideal, then $D'/ID' \subset D/ID$ satisfies the induction hypothesis by the Remark 2.2.4 above, hence D'/ID' is a saturated $\mathcal{R}$-submodule of D/ID. As ID is a direct summand as $\mathcal{R}$-module, it is saturated in D, hence we only have to show that $D' \cap ID = ID'$ is saturated in ID. We may and do choose an ideal I of length 1. But in this case, $ID' \subset ID$ is $D'/mD' \subset D/mD$, which is saturated by assumption. We proved that D' is $\mathcal{R}$ saturated in D. As a consequence, D/D' is $\mathcal{R}$-torsion free, and free over A by part (i), hence it is free over $\mathcal{R}_A$ by part (ii). □

2.2.4. Étale and isocline φ-modules. — Assume again that $A = L$ is a field. Let D be a φ-module over $\mathcal{R}_L$, i.e. a free of finite rank $\mathcal{R}_L$-module with a $\mathcal{R}_L$-semilinear action of φ such that $\mathcal{R}_L\varphi(D) = D$. Recall that Kedlaya's work (see [**71**, Theorem 6.10]) associates to D a sequence of rational numbers $s_1 \leq \cdots \leq s_d$ (where d is the rank of D) called the *slopes* of D. The φ-module D is said *isocline* of slope s if $s_1 = \cdots = s_d = s$ and *étale* if it is isocline of slope 0. A (φ, Γ)-module is étale (resp. isocline) if its underlying φ-module (forgetting the action of Γ) is. For more details see [**71**], especially part 4 and 6, [**72**] or, for a concise review, [**46**], part 2.

Lemma 2.2.5. — *Let $0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0$ be an exact sequence of φ-modules free of finite rank over $\mathcal{R}_L$. If D_1 and D_2 are isocline of the same slope s, then D is also isocline of slope s.*

Proof. — Up to a twist (after enlarging L if necessary) we may assume that $s = 0$, that is D_1 and D_2 étale, and we have to prove that D is étale as well.

Assume that D is not étale, so it has by Kedlaya's Theorem ([**71**, Thm 6.10]) a saturated φ-submodule N which is isocline of slope $s < 0$. Note n the rank of N and consider the φ-module $\Lambda^n D$ which contains as a saturated φ-submodule the φ-module $\Lambda^n N$, of slope $ns < 0$. By assumption, $\Lambda^n D$ is a successive extension of φ-modules of the form

$$\Lambda^a(D_1) \otimes_{\mathcal{R}_L} \Lambda^b(D_2),\ a + b = n,$$

which are all étale (see [**71**, Prop. 5.13]). Since $\Lambda^n N$ has rank one, it is isomorphic to a submodule of one of those étale φ-modules $\Lambda^a(D_1) \otimes_{\mathcal{R}_L} \Lambda^b(D_2)$. But by [**72**, Prop. 4.5.14], an étale φ-module has no rank one φ-submodule of slope < 0, a contradiction. □

2.2.5. Cohomology of (φ,Γ)-modules. — As in the context of Fontaine's (φ,Γ)-modules (see e.g. Herr's paper [**64**]), we define the cohomology groups $H^i(D)$ of a (φ,Γ)-module D over $\mathcal{R}$ as the cohomology of the 3-terms complex of $\mathbb{Q}_p$-vector spaces:

$$0 \longrightarrow D \xrightarrow{d_0} D \oplus D \xrightarrow{d_1} D \longrightarrow 0,$$

where γ is a topological generator [10] of Γ, $d_0(x) = ((\gamma-1)x, (\varphi-1)x)$ and $d_1(x,y) = (\varphi-1)x - (\gamma-1)y$. We refer to Colmez's paper [**46**, §3.1] for a discussion of this definition and for its basic properties. Let us simply say that by definition, $H^i(D)$ vanishes for $i \notin \{0,1,2\}$,

$$H^0(D) := D^{\Gamma=1,\varphi=1} = \{x \in D, \varphi(x) = x, \gamma(x) = x\ \forall \gamma \in \Gamma\},$$

and we have a long exact sequence of cohomology groups associated to any short exact sequence of (φ,Γ)-modules. When D is a (φ,Γ)-module over $\mathcal{R}_A$, then the $H^i(D)$ are A-modules in a natural way.

As usual, it turns out that $H^1(D)$ parameterizes the isomorphism classes of extensions of $\mathcal{R}$ by D. To be a little more precise, if D_2 and D_1 are two (φ,Γ)-modules over $\mathcal{R}_A$, by an *extension of D_1 by D_2 in $(\varphi,\Gamma)/A$* we mean a complex of (φ,Γ)-modules over $\mathcal{R}_A$

$$0 \longrightarrow D_2 \longrightarrow D \longrightarrow D_1 \longrightarrow 0$$

which is exact (hence split) as $\mathcal{R}_A$-module. As usual, two such sequences are said *equivalent* if there is a morphism between them which is the identity on D_1 and D_2. The set of equivalence classes of such extensions form an A-module in the usual way (Baer), that we shall denote by

$$\mathrm{Ext}_{(\varphi,\Gamma)/A}(D_1, D_2).$$

One checks at once (see [**46**, §3.1]) that for any (φ,Γ)-module D over $\mathcal{R}_A$ there is a natural A-linear isomorphism

$$H^1(D) \xrightarrow{\sim} \mathrm{Ext}_{(\varphi,\Gamma)/A}(\mathcal{R}_A, D).$$

2.2.6. (φ,Γ)-modules and representations of G_p. — Works of Fontaine, Cherbonnier-Colmez, and Kedlaya, allow to define a $\otimes$-equivalence D_{rig} between the category of $\mathbb{Q}_p$-representations of G_p and étale (in the sense of § 2.2.4) (φ,Γ)-modules over $\mathcal{R}$ ([**46**, prop. 2.7]). By [**13**, §3.4], $D_{\mathrm{rig}}(V)$ can be defined in Fontaine's style:

[10] When $p = 2$ there is no such generator, and the definition has to be modified as follows: let $\Delta \subset \Gamma$ be the torsion subgroup and choose $\gamma \in \Gamma$ a topological generator of Γ/Δ, then replace each D in the complex above by its subspace $D^{\Delta} \subset D$ of Δ-invariants.

there exists a topological $\mathbb{Q}_p$-algebra B (denoted $B^{\dagger,\mathrm{rig}}$ there) equipped with commuting actions of G_p and φ such that $B^{\mathrm{Ker}\,\chi} = \mathcal{R}$ (with its induced actions of φ and of Γ via χ^{-1}), and we have

$$D_{\mathrm{rig}}(V) := (V \otimes_{\mathbb{Q}_p} B)^{\mathrm{Ker}\,\chi}.$$

Some properties of these constructions are summarized in the following proposition.

Proposition 2.2.6. — (i) *The functor D_{rig} induces an $\otimes$-equivalence of categories between A-representations of G_p and étale (φ,Γ)-modules over $\mathcal{R}_A$. We have $\mathrm{rk}_{\mathbb{Q}_p}(V) = \mathrm{rk}_{\mathcal{R}}(D_{\mathrm{rig}}(V))$.*

(ii) *For an A-representation V of G_p, $D_{\mathrm{rig}}(-)$ induces an isomorphism*

$$\mathrm{Ext}^1_{A[G_p],\mathrm{cont}}(A,V) \xrightarrow{\sim} \mathrm{Ext}_{(\varphi,\Gamma)/A}(\mathcal{R}_A, D_{\mathrm{rig}}(V)) = H^1(D_{\mathrm{rig}}(V)).$$

Proof. — Part (i) is [**46**, prop. 2.7], part (ii) follows from (i) and Proposition 2.2.5. □

Lemma 2.2.7. — *An A-representation V of G_p is free over A if, and only if, $D_{\mathrm{rig}}(V)$ is free over $\mathcal{R}_A$.*

Proof. — Assume first that V is free over A. Let M be any finite length A-module M, and fix a presentation $A^n \longrightarrow A^m \longrightarrow M \longrightarrow 0$. As the functor $D_{\mathrm{rig}}(-)$ is exact by Proposition 2.2.6 (i), and by left exactness of $- \otimes_A D_{\mathrm{rig}}(V)$, we deduce from this presentation that we have a canonical A-linear (hence $\mathcal{R}_A$-linear) isomorphism

$$D_{\mathrm{rig}}(V) \otimes_A M \xrightarrow{\sim} D_{\mathrm{rig}}(V \otimes_A M).$$

From the special case $M = A/m$, we obtain that $D_{\mathrm{rig}}(V)/mD_{\mathrm{rig}}(V)$ is generated by $d := \mathrm{rk}_A(V)$ elements as $\mathcal{R}_A$-module, hence so is $D_{\mathrm{rig}}(V)$ by Nakayama's lemma since $m\mathcal{R}_A$ is nilpotent. In other words, there is a $\mathcal{R}_A$-linear surjection $\mathcal{R}_A^d \longrightarrow D_{\mathrm{rig}}(V)$. As

$$\mathrm{rk}_{\mathcal{R}} D_{\mathrm{rig}}(V) = \dim_{\mathbb{Q}_p}(V) = d \dim_{\mathbb{Q}_p}(A) = \mathrm{rk}_{\mathcal{R}}(\mathcal{R}_A^d)$$

by Proposition 2.2.6 (i), any such surjection is an isomorphism by property (B1).

The proof is the same in the other direction using the natural inverse functor of D_{rig}. □

2.2.7. Berger's theorem. — We will need to recover the usual Fontaine functors from $D_{\mathrm{rig}}(V)$, which is achieved by Berger's work [**13**] and [**14**] that we recall now. Let us introduce, for $r > 0 \in \mathbb{Q}$, the $\mathbb{Q}_p$-subalgebra

$$\mathcal{R}_r = \{f(z) \in \mathcal{R},\ f \text{ converges on the annulus } p^{-\frac{1}{r}} \leq |z-1| < 1\}.$$

Note that $\mathcal{R}_r$ is stable by Γ, and that φ induces a $\mathbb{Q}_p$-algebra homomorphism $\mathcal{R}_r \longrightarrow \mathcal{R}_{pr}$ when $r > \frac{p-1}{p}$ which is finite of degree p. The following lemma is [**14**, Thm 1.3.3]:

Lemma 2.2.8. — *Let D be a (φ, Γ)-module over $\mathcal{R}$. There exists a $r(D) > \frac{p-1}{p}$ such that for each $r > r(D)$, there exists a unique finite free, Γ-stable, $\mathcal{R}_r$-submodule D_r of D such that $\mathcal{R} \otimes_{\mathcal{R}_r} D_r \xrightarrow{\sim} D$ and that $\mathcal{R}_{pr} D_r$ has a $\mathcal{R}_{pr}$-basis in $\varphi(D_r)$. In particular, for $r > r(D)$,*

(i) *for $s \geq r$, $D_s = \mathcal{R}_s D_r \xrightarrow{\sim} \mathcal{R}_s \otimes_{\mathcal{R}_r} D_r$,*

(ii) *φ induces an isomorphism $\mathcal{R}_{pr} \otimes_{\mathcal{R}_r, \varphi} D_r \xrightarrow{\sim} D_{pr} \xrightarrow{\sim} \mathcal{R}_{pr} \otimes_{\mathcal{R}_r} D_r$.*

If $n(r)$ is the smallest integer n such that $p^{n-1}(p-1) \geq r$, then for $n \geq n(r)$ the primitive p^n-th roots of unity lie in the annulus $p^{-\frac{1}{r}} < |z-1| < 1$. For such a root ζ, the evaluation map $\mathcal{R}_r \to \mathbb{Q}_p(\zeta)$ is surjective and its kernel is independent of ζ: it is the ideal of $\mathcal{R}_r$ generated by $t_n = (z^{p^n} - 1)/(z^{p^{n-1}} - 1)$. Set

$$K_n = \mathcal{R}_r/(t_n) = \mathbb{Q}_p(\sqrt[p^n]{1}).$$

As $t \in \mathcal{R}_r$ is a uniformizer at each primitive p^n-th root of unity, the complete local ring $\lim_j \mathcal{R}_r/(t_n)^j$ is naturally isomorphic to [11] $K_n[[t]]$, and we obtain a natural map

$$\iota_{r,n} : \mathcal{R}_r \longrightarrow K_n[[t]], \ n \geq n(r), \ r > r(D), \tag{29}$$

which is injective with t-adically dense image. As the action of Γ on $\mathcal{R}_r$ preserves $t_n\mathcal{R}_r$, we have a natural action of Γ on $K_n[[t]]$ for which $\iota_{r,n}$ is equivariant. [12] For any (φ, Γ)-module D over $\mathcal{R}$, we can then form for $r > r(D)$ and $n \geq n(r)$ the space

$$D_r \otimes_{\mathcal{R}_r} K_n[[t]],$$

the tensor product being over the map $\iota_{r,n}$. It is a free $K_n[[t]]$-module of rank $\mathrm{rk}_{\mathcal{R}}(D)$ equipped with a natural semi-linear action of Γ. By Lemma 2.2.8 (i), this space does not depend on the choice of r such that $n \geq n(r)$. Moreover, for a fixed r, the same lemma part (ii) shows that φ induces a Γ-equivariant, $K_{n+1}[[t]]$-linear, isomorphism

$$(D_r \otimes_{\mathcal{R}_r} K_n[[t]]) \otimes_{t \mapsto pt} K_{n+1}[[t]] \longrightarrow D_r \otimes_{\mathcal{R}_r} K_{n+1}[[t]].$$

(Note that the map $\varphi : \mathcal{R}_r \longrightarrow \mathcal{R}_{pr}$ induces the inclusion $K_n[[t]] \longrightarrow K_{n+1}[[t]]$ such that $t \mapsto pt$.)

We use this to define functors $\mathcal{D}_{\mathrm{Sen}}(D)$ and $\mathcal{D}_{\mathrm{dR}}(D)$, as follows. Let $K_\infty = \bigcup_{n \geq 0} K_n$, it is equipped with a natural action of Γ identified with $\mathrm{Gal}(K_\infty/\mathbb{Q}_p)$ via the cyclotomic character χ. For $n \geq n(r)$ and $r > r(D)$, we define a K_∞-vector space with a semi-linear action of Γ by setting

$$\mathcal{D}_{\mathrm{Sen}}(D) := (D_r \otimes_{\mathcal{R}_r} K_n) \otimes_{K_n} K_\infty.$$

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(11) As K_n is finite étale over $\mathbb{Q}_p$, it lifts canonically to a subfield of this local ring, that we still denote by K_n.

(12) For the convenience of the reader, let us explicit this $\mathbb{Q}_p$-algebra action. First, Γ acts on K_n through the natural surjection $\mathbb{Z}_p^* \to (\mathbb{Z}/p^n\mathbb{Z})^* = \mathrm{Gal}(K_n/\mathbb{Q}_p)$. Moreover, the action of Γ on $K_n[[t]]$ is continuous for the t-adic topology, coincides with the one just defined on K_n, and satisfies $\gamma(t) = \gamma t$ for all $\gamma \in \mathbb{Z}_p^*$.

(Of course, the first tensor product is over the map $\mathcal{R}_r \longrightarrow K_n = \mathcal{R}_r/(t_n)$.) By the discussion above, this space does not depend on the choice of n, r. In the same fashion, the $\mathbb{Q}_p$-vector spaces

$$\mathcal{D}_{\mathrm{dR}}(D) := (K_\infty \otimes_{K_n} K_n((t)) \otimes_{\mathcal{R}_r} D_r)^\Gamma,$$

$$\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(D)) := (K_\infty \otimes_{K_n} t^i K_n[[t]] \otimes_{\mathcal{R}_r} D_r)^\Gamma \subset \mathcal{D}_{\mathrm{dR}}(D), \ \forall i \in \mathbb{Z},$$

are independent of $n \geq n(r)$ and $r > r(D)$. As $K_\infty((t))^\Gamma = \mathbb{Q}_p$, $\mathcal{D}_{\mathrm{dR}}(D)$ so defined is a finite dimensional $\mathbb{Q}_p$-vector-space whose dimension is less than or equal to $\mathrm{rk}_{\mathcal{R}}(D)$, and $(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(D)))_{i\in\mathbb{Z}}$ is a decreasing, exhausting, and saturated, filtration on $\mathcal{D}_{\mathrm{dR}}(D)$. This filtration is called the *Hodge filtration.*

We end by the definition of $\mathcal{D}_{\mathrm{crys}}(D)$. Let

$$\mathcal{D}_{\mathrm{crys}}(D) := D[1/t]^\Gamma.$$

It has an action of $\mathbb{Q}_p[\varphi]$ induced by the one on $D[1/t]$. It has also a natural filtration defined as follows. Choose $r > r(D)$ and $n \geq n(r)$, there is a natural inclusion

$$\mathcal{D}_{\mathrm{crys}}(D) \longrightarrow \mathcal{D}_{\mathrm{dR}}(D)$$

and we denote by $(\varphi^n(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D)))_{i\in\mathbb{Z}}$ the filtration induced from the one on $\mathcal{D}_{\mathrm{dR}}(D)$. By the analysis above, this defines a unique filtration $(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D)))_{i\in\mathbb{Z}}$, independent of the above choices of n and r, and called the Hodge filtration of $\mathcal{D}_{\mathrm{crys}}(D)$. We summarize some of Berger's results ([**13**], [**14**], [**45**, prop. 5.6])) in the following proposition.

Proposition 2.2.9. — *Let V be a $\mathbb{Q}_p$-representation of G_p, and*

$$* \in \{\mathrm{crys}, \mathrm{dR}, \mathrm{Sen}\}.$$

Then $\mathcal{D}_*(D_{\mathrm{rig}}(V))$ is canonically isomorphic to $D_*(V)$.

Definition 2.2.10. — We will say that a (not necessarily étale) (φ, Γ)-module D over $\mathcal{R}$ is crystalline (resp. de Rham) if $\mathcal{D}_{\mathrm{crys}}(D)$ (resp. $\mathcal{D}_{\mathrm{dR}}(D)$) has rank $\mathrm{rk}_{\mathcal{R}}(D)$ over $\mathbb{Q}_p$. The Sen polynomial of D is the one of the semi-linear Γ-module $\mathcal{D}_{\mathrm{Sen}}(D)$.

Due to the lack of references we have to include the following lemma.

Lemma 2.2.11. — *Let $0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$ be an exact sequence of (φ, Γ)-modules over $\mathcal{R}$. If $r \geq r(D), r(D'), r(D'')$ is big enough, then $D''_r = \mathrm{Im}\,(D_r \longrightarrow D'')$ and $D'_r = D_r \cap D'$.*

Proof. — Fix $r_0 > r(D), r(D'), r(D'')$. By Lemma 2.2.8,

$$X = \bigcup_{r \geq r_0} X_r, \ \text{for X} \in \{D, D', D''\}. \tag{30}$$

We can then find $r_1 \geq r_0$ such that $D'_{r_0} \subset D_{r_1}$ and $Im(D_{r_0} \longrightarrow D'') \subset D''_{r_1}$. As $D \longrightarrow D''$ is surjective, we can choose moreover some $r_2 \geq r_1$ such that $Im(D_{r_2} \longrightarrow D'')$ contains a $\mathcal{R}_{r_1}$-basis of D''_{r_1}. The exact sequence of the statement induces then for $r \geq r_2$ an exact sequence of $\mathcal{R}_r$-modules

$$0 \longrightarrow K_r := D_r \cap D' \longrightarrow D_r \longrightarrow D''_r \longrightarrow 0, \tag{31}$$

with $K_r \supset D'_r$. As D''_r is free, this sequence splits, hence remains exact when base changed to $\mathcal{R}_s$, $s \geq r$. Using Lemma 2.2.8 (i), this implies that $\mathcal{R}_s K_r = K_s$ for $s \geq r \geq r_2$. Moreover, K_r is finite type over $\mathcal{R}_r$. Indeed, D_r is (free of) finite type over $\mathcal{R}_r$ and the sequence (31) splits. By formula (30), we can then choose $r_3 \geq r_2$ such that $K_{r_2} \subset D'_{r_3}$, and we get that $K_r = \mathcal{R}_r K_{r_2} = D'_r$ for $r \geq r_3$. □

2.3. Triangular (φ,Γ)-modules and trianguline representations over artinian $\mathbb{Q}_p$-algebras

In all this subsection, A is a finite dimensional commutative local $\mathbb{Q}_p$-algebra with maximal ideal m and residue field $L := A/m$.

2.3.1. (φ,Γ)-modules of rank one over $\mathcal{R}_A$. — We begin by classifying all the (φ,Γ)-modules which are free of rank 1 over $\mathcal{R}_A$. Let $\delta : \mathbb{Q}_p^* \longrightarrow A^*$ be a continuous character. In the spirit of Colmez [**46**, §0.1], we define the (φ,Γ)-module $\mathcal{R}_A(\delta)$ which is $\mathcal{R}_A$ as $\mathcal{R}_A$-module but equipped with the $\mathcal{R}_A$-semi-linear actions of φ and Γ defined by

$$\varphi(1) := \delta(p), \;\; \gamma(1) := \delta(\gamma), \forall \gamma \in \Gamma.$$

Let $\mathrm{W} \subset G_p$ be the Weil group of $\mathbb{Q}_p$ and let $\theta : \mathrm{W}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Q}_p^*$ be the isomorphism of local class field theory normalized so that geometric Frobeniuses correspond to uniformizers. We may associate to any δ as above the continuous homomorphism $\delta \circ \theta : \mathrm{W} \longrightarrow A^*$. Such a homomorphism extends continuously to G_p if, and only if, $v(\delta(p) \bmod m)$ is zero, and in this case we see that

$$\mathcal{R}_A(\delta) = D_{\mathrm{rig}}(\delta \circ \theta).$$

When δ is the character defined by $\delta(p) = 1$ and $\delta_{|\mathbb{Z}_p^*} = \mathrm{id}$, then $\delta \circ \theta = \chi_{|W^{\mathrm{ab}}}$. Note that if $I \subsetneq A$ is an ideal, it is clear from the definition that $\mathcal{R}_A(\delta) \otimes_A A/I \xrightarrow{\sim} \mathcal{R}_{A/I}(\delta \bmod I)$. Moreover, if D is a (φ,Γ)-module over $\mathcal{R}_A$, we will set $D(\delta) := D \otimes_{\mathcal{R}_A} \mathcal{R}_A(\delta)$, and

$$D^{\delta} = \{x \in D, \varphi(x) = \delta(p)x, \, \gamma(x) = \delta(\gamma)x \; \forall \gamma \in \Gamma\} = H^0(D(\delta^{-1})).$$

Proposition 2.3.1. — *Any (φ,Γ)-module free of rank 1 over $\mathcal{R}_A$ is isomorphic to $\mathcal{R}_A(\delta)$ for a unique δ. Such a module is isocline of slope $v(\delta(p) \bmod m)$.*

Proof. — By Lemma 2.2.5, a (φ, Γ)-module D of rank 1 over $\mathcal{R}_A$ is automatically isocline of same slope as D/mD. As $\mathcal{R}_L(\delta \bmod m)$ has slope $v(\delta(p) \bmod m)$, and as $\mathcal{R}_A(\delta)(\delta') = \mathcal{R}_A(\delta\delta')$, we may assume that D is étale. But in this case, the result follows from the equivalence of categories of Proposition 2.2.6 (i), Lemma 2.2.7, and the fact that the continuous Galois characters $G_p \longrightarrow A^*$ correspond exactly to the δ such that $v(\delta(p) \bmod m) = 0$. □

2.3.2. Definitions

Definition 2.3.2. — Let D be a (φ, Γ)-module which is free of rank d over $\mathcal{R}_A$ and equipped with a strictly increasing filtration $(\mathrm{Fil}_i(D))_{i=0\ldots d}$:

$$\mathrm{Fil}_0(D) := \{0\} \subsetneq \mathrm{Fil}_1(D) \subsetneq \cdots \subsetneq \mathrm{Fil}_i(D) \subsetneq \cdots \subsetneq \mathrm{Fil}_{d-1}(D) \subsetneq \mathrm{Fil}_d(D) := D,$$

of (φ, Γ)-submodules which are free and direct summand as $\mathcal{R}_A$-modules. We call such a D a *triangular* (φ, Γ)-module over $\mathcal{R}_A$, and the filtration $\mathcal{T} := (\mathrm{Fil}_i(D))$ *a triangulation of D over $\mathcal{R}_A$*.

Following Colmez, we shall say that a (φ, Γ)-module which is free of rank d over $\mathcal{R}_A$ is *triangulable* if it can be equipped with a triangulation $\mathcal{T}$; we shall say that an A-representation V of G_p which is free of rank d over A is *trianguline* if $D_{\mathrm{rig}}(V)$ (which is free of rank d over $\mathcal{R}_A$ by Lemma 2.2.7) is triangulable.

Let D be a triangular (φ, Γ)-module. By Lemma 2.3.1, for each $i \in \{1, \ldots, d\}$

$$\mathrm{gr}_i(D) := \mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)$$

is isomorphic to $\mathcal{R}_A(\delta_i)$ for some unique $\delta_i : \mathrm{W} \longrightarrow A^*$. It makes then sense to define the *parameter of the triangulation* to be the continuous homomorphism

$$\delta := (\delta_i)_{i=1,\ldots,d} : \mathbb{Q}_p^* \longrightarrow (A^*)^d.$$

2.3.3. Weights and Sen polynomial of a triangular (φ, Γ)-module. — As the following proposition shows, the parameter of a triangular (φ, Γ)-module refines the data of its Sen polynomial. It will be convenient to introduce, for a continuous character $\delta : \mathbb{Q}_p^* \longrightarrow A^*$, its *weight*[13]:

$$\omega(\delta) := -\left(\frac{\partial \delta_{|\Gamma}}{\partial \gamma}\right)_{\gamma=1} = -\frac{\log(\delta(1+p))}{\log(1+p)} \in A.$$

[13] Let $A^1 \subset A^*$ be the subgroup of elements a such that $a^{p^n} \to 1$ when $n \to \infty$; it is also the direct product of its subgroups $1+m_A$ and $A^1 \cap L^*$. For any finite set $F \subset A^1$, we can find a submultiplicative norm $|.|$ on A such that $|a-1| < 1$ for all $a \in F$. In particular, $\log(a) := \sum_{n \geq 1} (-1)^{n+1}(a-1)^n/n$ converges absolutely in A for $a \in A^1$ and we check easily that $\log : (A^1, \cdot) \to (A, +)$ is a continuous group homomorphism. Moreover, for any $a \in A^1$, $\frac{a^{p^n}-1}{p^n} \to \log(a)$ when $n \to \infty$. This shows the second equality in the expression defining $\omega(\delta)$, as $\delta(1+p) \in A^1$.

Proposition 2.3.3. — *Let D be a triangular (φ,Γ)-module over $\mathcal{R}_A$ and δ the parameter of a triangulation of D. Then the Sen polynomial of D is*

$$\prod_{i=1}^{d}(T-\omega(\delta_i)).$$

Proof. — Assume first that $d=1$, i.e. that $D=\mathcal{R}_A(\delta)$. We see that we may take $r(D)=(p-1)/p$ and that $D_r=A\mathcal{R}_r$ for $r>r(D)$. But then $\mathcal{D}_{\mathrm{Sen}}(D)$ has a K_∞-basis on which Γ acts through $\delta_{|\Gamma}$, and the result follows. The general case follows by induction on d from the case $d=1$ and Lemma 2.2.11. □

2.3.4. De Rham triangular (φ,Γ)-modules.— We now give a sufficient condition on a triangular (φ,Γ)-module D over $\mathcal{R}_A$ to be de Rham (see Definition 2.2.10). A necessary condition is that the $\mathcal{R}_A(\delta_i)$ are themselves de Rham, *i.e.* that each $s_i:=\omega(\delta_i)$ is an integer (see the proof below).

Proposition 2.3.4. — *Let D be a triangular (φ,Γ)-module of rank d over $\mathcal{R}_A$, and let δ be its parameter. Assume that $s_i:=\omega(\delta_i)\in\mathbb{Z}$, and that $s_1<s_2<\cdots<s_d$; then D is de Rham.*

If, moreover, $D_0:=D/mD$ is crystalline and satisfies $\mathrm{Hom}(D_0,D_0(\chi^{-1}))=0$ *(resp. is semi-stable), then D is crystalline (resp. semi-stable).*

Proof. — In this proof, $K_\infty[[t]]$ will always mean $\bigcup_{n\geq 1}K_n[[t]]$.

Assume first that $d=1$ and $D=\mathcal{R}_A(\delta)$. If $s:=\omega(\delta)\in\mathbb{Z}$, then $\delta_{|\Gamma}\chi^s$ is a finite order character of Γ. We see easily that

$$(K_\infty[[t]][1/t]\otimes_{\mathbb{Q}_p}\delta_{|\Gamma})^\Gamma=(t^sK_\infty\otimes_{\mathbb{Q}_p}\delta_{|\Gamma})^\Gamma,$$

and the latter A-module is free of rank 1 by Hilbert 90. This concludes the case $d=1$.

Let us show now that D is de Rham by induction on d. Let $\gamma\in\Gamma$ be a topological generator [14], and consider the cyclic subgroup $\Gamma_0:=\langle\gamma\rangle\subset\Gamma$. By Lemma 2.2.11 we have for r large enough and $i\in\mathbb{Z}$ an exact sequence

$$\begin{aligned}0\longrightarrow\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(\mathrm{Fil}_{d-1}(D)))\longrightarrow\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(D))\longrightarrow\mathrm{Fil}^i(\mathcal{R}_A(\delta_d))\longrightarrow&\\ H^1(\Gamma_0,(\mathrm{Fil}_{d-1}(D)_r)\otimes K_\infty[[t]]t^i).&\end{aligned}$$

By the induction hypothesis applied to $\mathrm{Fil}_{d-1}(D)$, $\mathrm{Fil}^{s_1}(\mathcal{D}_{\mathrm{dR}}(\mathrm{Fil}_{d-1}(D)))$ has dimension $(d-1)\dim_{\mathbb{Q}_p}(A)$ and $\mathrm{Fil}^i(\mathcal{D}_{\mathrm{dR}}(\mathrm{Fil}_{d-1}(D)))=0$ for $i>s_{d-1}$. By the case $d=1$ studied above, it suffices then to show that

$$H^1(\Gamma_0,\mathrm{Fil}_{d-1}(D)_r\otimes_{\mathcal{R}_r}K_\infty[[t]]t^{s_d})=0.$$

[14] If $p=2$, choose γ such that γ is a topological generator of Γ modulo its torsion subgroup.

But the Γ_0-module $\mathrm{Fil}_{d-1}(D)_r \otimes_{\mathcal{R}_r} K_\infty[[t]]t^{s_d}$ is a successive extension of terms of the form

$$t^{s_d} K_\infty[[t]] \otimes_{\mathbb{Q}_p} \delta_i.$$

But the cohomology group $H^1(\Gamma_0, -)$ of each of these terms vanishes. Indeed, $s_d > s_i$ if $i < d$, and if $X = K_\infty[[t]]t^j$ with $j > 0$, we see at once that

$$H^1(\Gamma_0, X) = X/(\gamma - 1)X = 0.$$

This concludes the proof that D is de Rham.

Recall that Berger's Theorem [**14**, théorème A] associates canonically to any de Rham (φ, Γ)-module D over $\mathcal{R}$ a filtered (φ, N, G_p)-module $\mathcal{X}(D)$. We see as in Lemma 2.2.7 that $\mathcal{X}(D)$ carries an action of A and is free as A-module if (and only if) D is free over $\mathcal{R}_A$. Moreover we have

$$\mathcal{X}(D_0) = \mathcal{X}(D)/m\mathcal{X}(D), \quad \text{where} \quad D_0 := D/mD.$$

But if D_0 is semi-stable, the action of the inertia group $I_p \subset G_p$ on $\mathcal{X}(D)/m\mathcal{X}(D)$ is trivial, hence so is its action on the $m^i\mathcal{X}(D)/m^{i+1}\mathcal{X}(D) \xrightarrow{\sim} \mathcal{X}(D)/m\mathcal{X}(D)$ for $i \geq 1$. As the action of I_p on $\mathcal{X}(D)$ has finite image by definition, it is semisimple and the I_p-module $\mathcal{X}(D)$ is also trivial. If moreover

$$\mathrm{Hom}(D_0, D_0(\chi^{-1})) = 0,$$

then we get by induction on $i \geq 1$ that

$$N : \mathcal{X}(D) \longrightarrow \mathcal{X}(D(\chi^{-1}))/m^i\mathcal{X}(D(\chi^{-1}))$$

is zero, hence $N = 0$ and so D is crystalline. □

Remark 2.3.5. — (i) The proposition above may be viewed as a generalization of the fact that ordinary representations are semi-stable (Perrin-Riou's Theorem [**1**, Exposé IV]).

(ii) There exist triangulable étale (φ, Γ)-modules of rank 2 over $\mathcal{R}_L$ which are Hodge-Tate of integral weights $0 < k$, but which are not de Rham (hence they have no triangulation whose parameter δ satisfies the assumption of the Proposition 2.3.4. Instead, we have $s_1 = k$ and $s_2 = 0$ with the notation of that proposition). For example, this is the case of the (φ, Γ)-module of the restriction at p of the Galois representation attached to any finite slope, overconvergent, modular eigenform of integral weight $k > 1$ and U_p-eigenvalue a_p such that $v(a_p) > k - 1$.

(iii) It would be easy to show that a de Rham triangulable (φ, Γ)-module over $\mathcal{R}_L$ becomes semi-stable over a finite abelian extension of $\mathbb{Q}_p$, because it is true in rank 1. Reciprocally, a (φ, Γ)-module which becomes semi-stable over a finite abelian extension of $\mathbb{Q}_p$ is triangulable over $\mathcal{R}_L$, where L contains all the eigenvalues of φ (to see this, mimic the proof of Proposition 2.4.1).

2.3.5. Deformations of triangular (φ,Γ)-modules

Let D be a fixed (φ,Γ)-module free of rank d over $\mathcal{R}_L$ and equipped with a triangulation $\mathcal{T} = (\mathrm{Fil}_i(D))_{i=0,\dots,d}$ with parameter (δ_i). We denote by $\mathcal{C}$ the category of local artinian $\mathbb{Q}_p$-algebras A equipped with a map $A/m \xrightarrow{\sim} L$, and local homomorphisms inducing the identity on L.

Let $X_D : \mathcal{C} \longrightarrow \mathrm{Set}$ and $X_{D,\mathcal{T}} : \mathcal{C} \longrightarrow \mathrm{Set}$ denote the following functors. For an object A of $\mathcal{C}$, $X_D(A)$ is the set of isomorphism classes of couples (D_A, π) where D_A is a (φ,Γ)-module free over $\mathcal{R}_A$ and $\pi : D_A \longrightarrow D$ is a $\mathcal{R}_A$-linear (φ,Γ)-morphism inducing an isomorphism $D_A \otimes_A L \xrightarrow{\sim} D$; $X_{D,\mathcal{T}}(A)$ is the set of isomorphism classes of triples $(D_A, \pi, (\mathrm{Fil}_i(D_A)))$ where :

(i) $(D_A, (\mathrm{Fil}_i(D_A)))$ is a triangular (φ,Γ)-module of rank d over $\mathcal{R}_A$,

(ii) $\pi : D_A \longrightarrow D$ is a $\mathcal{R}_A$-linear (φ,Γ)-morphism inducing an isomorphism $D_A \otimes_A L \xrightarrow{\sim} D$ such that $\pi(\mathrm{Fil}_i(D_A)) = \mathrm{Fil}_i(D)$.

There is a natural "forgetting the triangulation" morphism of functors $X_{D,\mathcal{T}} \longrightarrow X_D$ that makes in favorable cases $X_{D,\mathcal{T}}$ a subfunctor of X_D (as in [**46**], we denote by x the identity character $\mathbb{Q}_p^* \longrightarrow \mathbb{Q}_p^*$).

Proposition 2.3.6. — *Assume that for all $i < j$, $\delta_i\delta_j^{-1} \notin x^{\mathbb{N}}$. Then $X_{D,\mathcal{T}}$ is a subfunctor of X_D.*

Proof. — We have to show that if A is an object of $\mathcal{C}$, and $(D_A, \pi) \in X_D(A)$ is a deformation of D, then D_A has at most one triangulation that satisfies (ii) above. That is to say, we have to prove that if $\mathcal{T} = (\mathrm{Fil}_i(D_A))$ is a triangulation of D_A satisfying (ii), then $\mathrm{Fil}_1(D_A)$ is uniquely determined as a submodule of D_A, $\mathrm{Fil}_2(D_A)/\mathrm{Fil}_1(D_A)$ is uniquely determined as a submodule of $D_A/\mathrm{Fil}_1(D_A)$, and so on. For this note that $\mathrm{Fil}_j(D)/\mathrm{Fil}_{j-1}(D) \simeq \mathcal{R}_L(\delta_j)$, and that $D/\mathrm{Fil}_j(D)$ is a successive extension of $\mathcal{R}_L(\delta_i)$ with $i > j$, so that

$$\mathrm{Hom}(\mathcal{R}_L(\delta_j), D/\mathrm{Fil}_j(D)) = 0$$

by the hypothesis on the δ_i and Proposition 2.2.2 (ii). So we can apply the following lemma, and we are done. □

Lemma 2.3.7. — *Let $(D_A, \pi) \in X_D(A)$, $\delta : \mathbb{Q}_p^* \longrightarrow A^*$ be a continuous character and $\bar{\delta} = \delta \pmod{m}$. Assume D has a saturated, rank 1, (φ,Γ)-submodule $D_0 \simeq \mathcal{R}_L(\bar{\delta})$ such that $(D/D_0)^{\bar{\delta}} = 0$. Assume moreover that D_A contains a $\mathcal{R}$-saturated (φ,Γ)-module D' isomorphic to $\mathcal{R}_A(\delta)$. Then δ is the unique character of $\mathbb{Q}_p^*$ having this property, and D' the unique such submodule.*

Proof. — We may assume by twisting that $\delta = 1$ (hence $\bar{\delta} = 1$ also). Let $\delta' : \mathbb{Q}_p^* \longrightarrow A^*$ lifting 1, and assume that D_A has some $\mathcal{R}$-saturated submodules $D_1 \xrightarrow{\sim} \mathcal{R}_A$ and

$D_2 \xrightarrow{\sim} \mathcal{R}_A(\delta')$. By assumption, $H^0(D/D_0) = 0$, and $D_i/mD_i = D_0$ for $i = 1, 2$ (see Remark 2.2.4). A dévissage and the left exactness of the functor $H^0(-)$ show that

$$H^0(D_A/D_i) = 0, \ \ i = 1, 2.$$

This implies that the inclusion $H^0(D_i) \longrightarrow H^0(D_A)$ is an equality, hence

$$H^0(D_1) = H^0(D_2) \subset D_2.$$

As $D_1 = \mathcal{R}H^0(D_1)$, we have $D_1 \subset D_2$, and $D_1 = D_2$ since D_1 and D_2 are saturated and have the same $\mathcal{R}$-rank. We conclude that $\delta' = 1$ by Proposition 2.3.1. □

We will give below a criterion for the relative representability of $X_{D,\mathcal{T}} \longrightarrow X_D$, but we need before to make some preliminary remarks. Let $F(-)$ be the functor on (φ, Γ)-modules over $\mathcal{R}_L$ defined by

$$F(E) = \{v \in E, \exists n \geq 1 \mid \forall \gamma \in \Gamma, (\gamma - 1)^n v = 0, \ (\varphi - 1)^n v = 0\}.$$

This is a left-exact functor, and $F(E)$ inherits a commuting continuous action of φ and Γ, hence of $\mathbb{Q}_p^*$, as well as a commuting action of A if E does.

Lemma 2.3.8. — *We have:*

(i) *For any (φ, Γ)-module E over $\mathcal{R}_L$, $F(E) \neq 0 \Leftrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L, E) \neq 0$.*
(ii) *$F(\mathcal{R}_L(\delta)) = 0$ if $\delta \notin x^{-\mathbb{N}}$, and $F(\mathcal{R}_L) = L$.*
(iii) *Let $A \in \mathcal{C}$ and $\delta : \mathbb{Q}_p^* \longrightarrow A^*$ a continuous homomorphism such that $\bar{\delta} = 1$. The natural inclusion $A \subset \mathcal{R}_A(\delta)$ induces a $\mathbb{Q}_p^*$-equivariant[15] isomorphism $A \xrightarrow{\sim} F(\mathcal{R}_A(\delta))$, as well as an isomorphism $F(\mathcal{R}_A(\delta)) \otimes_{\mathbb{Q}_p} \mathcal{R} \xrightarrow{\sim} \mathcal{R}_A(\delta)$.*

Proof. — Assertion (i) is an immediate consequence of the definition and of the fact that φ and Γ commute. Let us check assertion (ii). We may assume that $\delta = 1$ by (i) and Prop. 2.2.2 (ii). Let $\gamma \in \Gamma$ be a nontorsion element. We claim that for $f \in \mathcal{R}_L$ and $n \geq 1$

$$(\gamma - 1)^n f = 0 \Rightarrow f \in L.$$

Assume first $n = 1$, then $f(z^\gamma) = f(z)$ so f is constant on each circle $|z - 1| = r$ with $r \geq r(f)$, hence constant because γ is nontorsion and an analytic function on a 1-dimensional affinoid has only a finite number of zeros. Assume now $n = 2$, by the previous case $(\gamma - 1)(f)$ is a constant C, which means that

$$f(z^\gamma) = f(z) + C$$

on $r(f) \leq |z - 1| < 1$. Let us choose a p^m-th root of unity ζ in the annulus $r(f) \leq |\zeta - 1| < 1$ which is sufficiently close to the outer boundary so that the (finite) orbit of ζ under $\langle\gamma\rangle$, namely $\{\zeta^{\gamma^k}, k \in \mathbb{Z}\}$, is non-trivial. Let M be the cardinality of this

(15) Note that $A \subset \mathcal{R}_A(\delta)$ has a natural A-linear action of (φ, Γ), hence of $\mathbb{Q}_p^*$, namely via the character δ by definition.

orbit, we obtain by applying M times the previous equation that $f(\zeta) = f(\zeta) + MC$, so $C = (\gamma-1)(f) = 0$ and f is constant by the case $n = 1$. Assume now $n \geq 3$. If $(\gamma-1)^n(f) = 0$, then $(\gamma-1)^{n-2}(f)$ is constant by the previous case, so $(\gamma-1)^{n-1}(f) = 0$ and we conclude that f is constant by induction.

Let us check assertion (iii). It is clear that $A \subset F(\mathcal{R}_A(\delta))$. Moreover, as $F(\mathcal{R}_L) = L$ by (ii), the left-exactness of F shows that the length of $F(\mathcal{R}_A(\delta))$ is less than or equal to the length of A. In particular, the previous inclusion is an equality, and the last assertion of the stament holds by definition of $\mathcal{R}_A(\delta)$. □

Proposition 2.3.9. — *Assume that for all $i < j$, $\delta_i\delta_j^{-1} \notin x^{\mathbb{N}}$. Then $X_{D,\mathcal{T}} \longrightarrow X_D$ is relatively representable.*

Proof. — By Prop. 2.3.6, we already know that $X_{D,\mathcal{T}}$ is a subfunctor of X_D. By [**84**, §23], we have to check three conditions (see also §19 of *loc.cit.*).

First condition: if $A \longrightarrow A'$ is a morphism in $\mathcal{C}$ and if $(D_A, \pi) \in X_{D,\mathcal{T}}(A)$, then

$$(D_A \otimes_A A', \pi \otimes_A A') \in X_{D,\mathcal{T}}(A').$$

This is obviously satisfied as $(\mathrm{Fil}_i(D_A) \otimes_A A')$ is a triangulation of $D_A \otimes_A A'$ lifting D.

Second condition:[16] if $A \longrightarrow A'$ is an *injective* morphism in $\mathcal{C}$, and if $(D_A, \pi) \in X_D(A)$, then

$$(D_A \otimes_A A', \pi \otimes_A A') \in X_{D,\mathcal{T}}(A') \implies (D_A, \pi) \in X_{D,\mathcal{T}}(A).$$

Arguing by induction on $d = \dim_{\mathcal{R}_L} D$, it is enough to show that D_A has a (φ,Γ)-submodule E which is free of rank 1 over $\mathcal{R}_A$, saturated, and such that the natural map

$$\pi : E \longrightarrow D$$

surjects onto $\mathrm{Fil}_1(D)$ (the fact that E is a direct summand as $\mathcal{R}_A$-module will follow then from Lemma 2.2.3). By twisting if necessary, we may assume that $\delta_1 = 1$.

By (ii) of Lemma 2.3.8, the left-exactness of F, and the assumption on the δ_i, we have

$$F(D) = F(\mathrm{Fil}_1(D)) = L. \tag{32}$$

Let $D_{A'} = D_A \otimes_A A'$, $T_1' := \mathrm{Fil}_1(D_{A'})$ and $T_1 := T_1' \cap D_A$. Lemma 2.3.8 (ii) again and a dévissage show that $F(D_A/T_1) \subset F(D_{A'}/T_1') = 0$, so the natural inclusions

$$F(T_1) \xrightarrow{\sim} F(D_A) \quad \textit{and} \quad F(T_1') \xrightarrow{\sim} F(D_{A'}) \tag{33}$$

[16] This is actually called condition (3) in *loc.cit.*

are isomorphisms. Moreover, the fact that $F(D) = L$ and another dévissage [17] show that for each ideal I of A, and each finite length A-module M, if l denotes the length function (so that L has length 1) then

$$l(F(ID_A)) \leq l(I) \quad \text{and} \quad l(F(D_A \otimes_A M)) \leq l(M).$$

The inequalities above combined with the exact sequences

$$0 \longrightarrow F(D_A) \longrightarrow F(D_{A'}) \longrightarrow F(D_A \otimes_A A'/A),$$

$$0 \longrightarrow F(mD_A) \longrightarrow F(D_A) \longrightarrow F(D),$$

show then that

$$(34) \qquad l(F(D_A)) = l(A), \quad \text{and} \quad F(D_A) \otimes_A L \xrightarrow{\neq 0} F(D) = F(\mathrm{Fil}_1(D)) = L.$$

In particular, there is an element $v \in F(D_A) \subset D_A$ whose image is nonzero in $F(D) \subset D_A/mD_A$, thus this element v generates a free [18] A-submodule of $F(D_A)$. By (34) we get that $F(D_A) = Av$ is free of rank 1 over A and the nonzero map there is actually an isomorphism. Of course, the same assertion holds if we replace the A's in it by A', as $A' = F(T_1') = F(D_{A'})$ by Lemma 2.3.8 (iii) and (33). As a consequence, the natural map

$$(35) \qquad F(T_1) \otimes_A A' \longrightarrow F(T_1')$$

is an isomophism, at it is so modulo the maximal ideal [19]. Set

$$E := \mathcal{R}F(T_1) \subset D_A.$$

We claim that E has the required properties to conclude. Recapitulating, we have a sequence of maps

$$F(T_1) \otimes_A \mathcal{R}_A \hookrightarrow F(T_1) \otimes_A \mathcal{R}_{A'} \xrightarrow{\sim} F(T_1') \otimes_{A'} \mathcal{R}_{A'} \xrightarrow{\sim} T_1'.$$

As E is the image of the composition of all the maps above, we get that

$$F(T_1) \otimes_A \mathcal{R}_A \xrightarrow{\sim} E$$

is free of rank 1 over $\mathcal{R}_A$. We already showed that $\pi(E) = \mathrm{Fil}_1(D)$, hence it only remains to check that E is saturated in D_A. But this holds as E is saturated in T_1', which is saturated in D'_A, and we are done [20].

---

(17) For more details about this dévissage, the reader can have a look at Lemma 3.2.9 of the next section, in which it is studied in a more general situation.

(18) The claim here is that for $A \in \mathcal{C}$ (or more generally for any commutative local ring A), any free A-module M and any $v \in M$, if the image of v in M/mM is nonzero, then $Av \simeq A$. Indeed, if (e_i) is an A-basis of M, then v writes as a finite sum $\sum_i \lambda_i e_i$. If $v \notin mM$, then $\lambda_i \in A^* = A\backslash m$ (as A is local) for some i, thus the map $a \mapsto av$, $A \to M$, is injective.

(19) In particular, if we write $T_1' = \mathcal{R}_{A'}(\delta)$, the isomorphism above and Lemma 2.3.8 (iii) show that $\delta(\mathbb{Q}_p^*) \subset A^*$.

(20) Actually, using Lemma 7.8.7 we even see that $E = T_1$.

Third condition:[21] for A and A' in $\mathcal{C}$, if $(D_A,\pi) \in X_{D,\mathcal{T}}(A)$ and $(D_{A'},\pi') \in X_{D,\mathcal{T}}(A')$, then for $B = A \times_L A'$, the natural object

$$(D_B = D_A \times_D D_{A'}, \pi_B = \pi \circ \mathrm{pr}_1 = \pi' \circ \mathrm{pr}_2)$$

lies in $X_{D,\mathcal{T}}(B)$. But it is clear that the filtration $(\mathrm{Fil}_i(D_A) \times_D \mathrm{Fil}_i(D_{A'}))$ is a triangulation of D_B lifting $\mathcal{T}$, and we are done. □

Let us consider the natural morphism

$$\mathrm{diag} : X_{D,\mathcal{T}} \longrightarrow \prod_{i=1}^{d} X_{\mathrm{gr}_i(D)}.$$

Recall that x is the identity character $\mathbb{Q}_p^* \longrightarrow \mathbb{Q}_p^*$; recall also that $\chi = x|x|$ is the cyclotomic character.

Proposition 2.3.10. — *Assume that for all $i < j$, $\delta_i\delta_j^{-1} \notin \chi x^{\mathbb{N}}$, then*

(i) *$X_{D,\mathcal{T}}$ is formally smooth,*

(ii) *for each $A \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{diag}(A)$ is surjective.*

Proof. — Recall that (i) means that for $A \in \mathcal{C}$ and $I \subset A$ an ideal such that $I^2 = 0$, the natural map $X_{D,\mathcal{T}}(A) \longrightarrow X_{D,\mathcal{T}}(A/I)$ is surjective.

Assume first that $d = 1$, so the assumption is empty. The maps $\mathrm{diag}(A)$ are bijective (hence (ii) is satisfied), and by Proposition 2.3.1, $X_{D,\mathcal{T}}$ is isomorphic the functor $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}, -)$ which is easily seen to be formally smooth (and even pro-representable by $\mathrm{Spf}(L[[X,Y]])$), hence (i) is satisfied.

Let us show now (i) and (ii), we fix $A \in \mathcal{C}$ and $I \subset m$ an ideal of length 1. Let $U \in X_{D,\mathcal{T}}(A/I)$, and let $V = (V_i) \in \prod_i (X_{\mathrm{gr}_i(D)}(A))$ be any lifting of $\prod_i \mathrm{gr}_i(U)$. We are looking for an element $U' \in X_{D,\mathcal{T}}(A)$ with graduation V and reducing to U modulo I. We argue by induction on d. By the paragraph above, we already know the result when $d = 1$. By the case $d = 1$ again, we may assume that $\mathrm{gr}_d(U)$ is the trivial (φ,Γ)-module over $\mathcal{R}_{A/I}$ (note that the assumption on the δ_i is invariant under twisting), and we have to find a U' whose $\mathrm{gr}_d(U')$ is also trivial. Let $\mathcal{T}'$ denote the triangulation $(\mathrm{Fil}_i(D))_{i=0,\dots,d-1}$ of $\mathrm{Fil}_{d-1}(D)$. By induction, $X_{\mathrm{Fil}_{d-1}(D),\mathcal{T}'}$ is formally smooth and satisfies (ii), hence we can find an element $U'' \in X_{\mathrm{Fil}_{d-1}(D),\mathcal{T}'}(A)$ lifting $\mathrm{Fil}_{d-1}(U)$ and such that $\mathrm{gr}_i(U'')$ lifts V_i for $i = 1,\dots,d-1$. It suffices then to show that the natural map

$$H^1(U'') \longrightarrow H^1(\mathrm{Fil}_{d-1}(U))$$

[21] Precisely, this is what is left to check condition (2) in *loc. cit.* once (1) and (3) are known to hold, because with the notations there $A \times_C B \subset A \times_L B$ if L is the residue field of C. This reduction is also explained in the proof of [**73**, Prop. 8.7].

is surjective. But by the cohomology exact sequence, its cokernel injects into

$$H^2(\mathrm{Fil}_{d-1}(D(\delta_d^{-1}))).$$

But this cohomology group is 0 by assumption and by Lemma 2.3.11. □

Lemma 2.3.11. — *$H^2(\mathcal{R}_A(\delta)) = 0$ if $(\delta \bmod m) \notin \chi x^{\mathbb{N}}$.*

Proof. — By the cohomology exact sequence, we may assume that $A = L$. But then $H^2(\mathcal{R}_L(\delta)) = H^0(\mathcal{R}_L(\chi\delta^{-1})) = 0$ by [**46**, prop. 3.1]. The fact that for any (φ, Γ) module D over $\mathcal{R}$, we have

$$H^0(D) = \mathrm{Hom}_L(H^2(D^*(\chi), L)) \tag{36}$$

should hold by mimicking Herr's original argument. We warn the reader that at the moment, there is unfortunately no written reference for that result. [22] □

Remark 2.3.12. — Under the assumptions of Prop. 2.3.10, and if we assume moreover that $\delta_i\delta_j^{-1} \notin x^{\mathbb{N}}$ for $i \neq j$, then we can show that

$$\dim_L X_{D,\mathcal{T}}(L[\varepsilon]) = \frac{d(d+1)}{2} + n$$

where $n = \dim_L \mathrm{End}_{(\varphi,\Gamma)/L,\mathcal{T}}(D)$ and $\mathrm{End}_{(\varphi,\Gamma)/L,\mathcal{T}}(D)$ is the subspace of elements $u \in \mathrm{Hom}_{(\varphi,\Gamma)/L,\mathcal{T}}$ such that $u(\mathrm{Fil}_i(D)) \subset \mathrm{Fil}_i(D)$ for all i (we always have $n \leq d$, and for instance $n = 1$ when D is irreducible). A proof of this result will be given elsewhere.

2.3.6. Trianguline deformations of trianguline representations. — The notions of the last paragraph have their counterpart in terms of trianguline representations. Let V be a trianguline representation over L, and suppose we are given a triangulation $\mathcal{T}$ on $D := D_{\mathrm{rig}}(V)$.

We define the functor

$$X_V : \mathcal{C} \longrightarrow \mathrm{Set}$$

as follows: for $A \in \mathrm{Ob}(\mathcal{C})$, $X_V(A)$ is the set of equivalence classes of *deformations* of V over A, that is, A-representations V_A of G_p which are free over A and equipped with an $A[G_p]$-morphism $\pi : V_A \longrightarrow A$ inducing an isomorphism $V_A \otimes_A L \longrightarrow V$. In the same way, we define a functor

$$X_{V,\mathcal{T}} : \mathcal{C} \longrightarrow \mathrm{Set}$$

such that $X_{V,\mathcal{T}}(A)$ is the set of equivalence classes of *trianguline deformations of* $(V, \mathcal{T})$, that is couples $(V_A, \pi) \in X_V(A)$ together with a triangulation Fil_i of $D_{\mathrm{rig}}(V_A)$ which makes $(D_{\mathrm{rig}}(V_A), D_{\mathrm{rig}}(\pi), \mathrm{Fil}_i)$ an element of $X_{D,\mathcal{T}}(A)$.

The main fact here is that those functors are not new:

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[22] Added in proofs: (36) is now a theorem of Liu [**80**].

Proposition 2.3.13. — *The functor D_{rig} induces natural isomorphisms of functors $X_V \simeq X_D$ and $X_{V,\mathcal{T}} \simeq X_{D,\mathcal{T}}$.*

Proof. — The second assertion follows immediately from the first one, since $X_{V,\mathcal{T}}(A) = X_V(A) \times_{X_D(A)} X_{D,\mathcal{T}}(A)$ for any A in $\mathcal{C}$.

To see that D_{rig} induces a bijection $X_V(A) \longrightarrow X_D(A)$, we note that the injectivity is obvious because of the full faithfulness of D_{rig}, and that the surjectivity follows from the fact that if (D_A, π) is an element of $X_D(A)$, D_A is a successive extension of D as a (φ, Γ)-module over $\mathcal{R}$, hence it is étale by Lemma 2.2.5; so D_A is $D_{\mathrm{rig}}(V_A)$ for some representation V_A over L which is free over A by Lemma 2.2.7. □

2.4. Refinements of crystalline representations

(See [**85**, §3], [**36**, §7.5], [**8**, §6].)

2.4.1. Definition. — Let V be a finite, d-dimensional, continuous, L-representation of G_p. We assume that V is crystalline and that the crystalline Frobenius φ acting on $D_{\mathrm{crys}}(V)$ has all its eigenvalues in L^*.

By a *refinement* of V (see [**85**, §3]) we mean the datum of a full φ-stable L-filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0,\dots,d}$ of $D_{\mathrm{crys}}(V)$:

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_d = D_{\mathrm{crys}}(V).$$

We remark now that any refinement $\mathcal{F}$ determines *two orderings*:

(Ref1) It determines an ordering $(\varphi_1, \dots, \varphi_d)$ of the eigenvalues of φ, defined by the formula

$$\det(T - \varphi_{|\mathcal{F}_i}) = \prod_{j=1}^{i} (T - \varphi_j).$$

Obviously, if all these eigenvalues are distinct such an ordering conversely determines $\mathcal{F}$.

(Ref2) It determines also an ordering $(s_1, \dots, s_d)$ on the set of Hodge-Tate weights of V, defined by the property that the jumps of the Hodge filtration of $D_{\mathrm{crys}}(V)$ induced on $\mathcal{F}_i$ are $(s_1, \dots, s_i)$ [(23)].

More generally, the definition above still makes sense when D is any crystalline (φ, Γ)-module over $\mathcal{R}_L$ (see Definition 2.2.10), *i.e. not necessarily étale*, such that φ acting on $\mathcal{D}_{\mathrm{crys}}(D) = D[1/t]^{\Gamma}$ has all its eigenvalues in L^*. It will be convenient for us to adopt this degree of generality.

(23) As $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, the weights of $\mathcal{F}_i$ are also weights of $\mathcal{F}_{i+1}$, hence the definition of the s_i makes sense.

2.4.2. Refinements and triangulations of (φ,Γ)-modules. — The theory of refinements has a simple interpretation in terms of (φ,Γ)-modules that we now explain. Let D be a crystalline (φ,Γ)-module as above and let $\mathcal{F}$ be a refinement of D. We can construct from $\mathcal{F}$ a filtration $(\mathrm{Fil}_i(D))_{i=0,\dots,d}$ of D by setting

$$\mathrm{Fil}_i(D) := (\mathcal{R}[1/t]\mathcal{F}_i) \cap D,$$

which is a finite type saturated $\mathcal{R}$-submodule of D by Lemma 2.2.2.

Proposition 2.4.1. — *The map defined above $(\mathcal{F}_i) \mapsto (\mathrm{Fil}_i(D))$ induces a bijection between the set of refinements of D and the set of triangulations of D, whose inverse is $\mathcal{F}_i := \mathrm{Fil}_i(D)[1/t]^{\Gamma}$.*

In the bijection above, for $i = 1,\dots,d$, the graded piece $\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)$ is isomorphic to $\mathcal{R}_L(\delta_i)$ where $\delta_i(p) = \varphi_i p^{-s_i}$ and $\delta_{i|\Gamma} = \chi^{-s_i}$, where the φ_i and s_i are defined by (Ref1) and (Ref2).

Proof. — We have $\mathrm{Frac}(\mathcal{R}_L)^{\Gamma} = L$, hence the natural (φ,Γ)-map

$$D[1/t]^{\Gamma} \otimes_L \mathcal{R}_L[1/t] \longrightarrow D[1/t]$$

is injective. But it is also surjective because D is assumed to be crystalline, hence it is an isomorphism. We deduce from this that any (φ,Γ)-submodule D' of $D[1/t]$ over $\mathcal{R}_L[1/t]$ can be written uniquely as $\mathcal{R}_L[1/t] \otimes_L F = \mathcal{R}_L[1/t]F$, where $F = D'^{\Gamma}$ is a $L[\varphi]$-submodule of $D[1/t]^{\Gamma}$. This proves the first part of the proposition.

Let us prove the second part. By what we have just said, the eigenvalue of φ on the rank one L-vector space, $(\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)[1/t])^{\Gamma}$ is φ_i. As a consequence, the rank one (φ,Γ)-module $\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)$, which has the form $\mathcal{R}_L(\delta_i)$ for some δ_i by Proposition 2.3.1, satisfies

$$\delta_i(p) = \varphi_i p^{-t_i}, \delta_{i|\Gamma} = \chi^{-t_i}$$

for some $t_i \in \mathbb{Z}$ by Proposition 2.2.2 (ii). By Proposition 2.3.3, the t_i are (with multiplicities) the Hodge-Tate weights of V, and it remains to show that $t_i = s_i$. We need the following essential lemma.

Lemma 2.4.2. — *Let D be a (φ,Γ)-module over $\mathcal{R}_A$, $\lambda \in A^*$, and $v \in \mathcal{D}_{\mathrm{crys}}(D)^{\varphi=\lambda}$. Then $v \in \mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D))$ if, and only if, $v \in t^i D$.*

Proof. — For any $r > 0$ and any $i \in \mathbb{Z}$, Lazard's theorem [**79**] shows that

$$\mathcal{R}_r[1/t] \cap \bigcap_{n \geq n(r)} K_n[[t]]t^i = t^i\mathcal{R}_r. \tag{37}$$

Let D be a (φ,Γ)-module over $\mathcal{R}_A$ and fix some $r > r(D)$ such that $\mathcal{D}_{\mathrm{crys}}(D) \subset D_r[1/t]$. By definition of the filtration on $\mathcal{D}_{\mathrm{crys}}(D)$ (§ 2.2.7), we have for any $n \geq n(r)$

$$\varphi^n(\mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D))) = \mathcal{D}_{\mathrm{crys}}(D) \cap t^i(D_r \otimes_{\mathcal{R}_r} K_n[[t]]), \tag{38}$$

both terms in the intersection above being viewed inside $D_r \otimes_{\mathcal{R}_r} K_n[[t]][1/t]$. Let $v \in \mathcal{D}_{\mathrm{crys}}(D)$ be such that $\varphi(v) = \lambda v, \lambda \in A^*$; for each integer n, we have

$$v = \frac{1}{\lambda^n}\,\varphi^n(v).$$

Let $e_1, \dots, e_d$ be an $\mathcal{R}_r$-basis of D_r and write $v = \sum_{j=1}^{d} v_j e_j$ for some $v_j \in \mathcal{R}_r[1/t]$; for $i \in \mathbb{Z}$, $v \in t^i D_r$ if, and only if, $v_j \in t^i \mathcal{R}_n$ for all j. Relations (37) and (38) show then that $v \in \mathrm{Fil}^i(\mathcal{D}_{\mathrm{crys}}(D))$ if, and only if, $v \in t^i D$. □

We now show that $t_i = t_i(D, \mathcal{F})$ coincides with $s_i = s_i(D, \mathcal{F})$ by induction on d. Let $v \neq 0 \in \mathcal{F}_1$. As

$$v \in \mathrm{Fil}^{s_1}(\mathcal{D}_{\mathrm{crys}}(D)) \backslash \mathrm{Fil}^{s_1+1}(\mathcal{D}_{\mathrm{crys}}(D))$$

by assumption, Lemma 2.4.2 above shows that $t^{-s_1}v \in D \backslash tD$. By Proposition 2.2.2 (ii), this shows that $\mathcal{R}\mathcal{F}_1 t^{-s_1}$ is saturated in D, hence is $\mathrm{Fil}_1(D)$, and $s_1 = t_1$. Let us consider now the (φ, Γ)-module

$$D' = D/\mathrm{Fil}_1(D).$$

It is crystalline with $\mathcal{D}_{\mathrm{crys}}(D') = \mathcal{D}_{\mathrm{crys}}(D)/\mathcal{F}_1$, with Hodge-Tate weights (with multiplicities) the ones of D deprived of s_1, and has also a natural refinement defined by $\mathcal{F}'_i = \mathcal{F}_{i+1}/\mathcal{F}_1$. The Hodge filtration on $\mathcal{D}_{\mathrm{crys}}(D')$ is the quotient filtration $((\mathrm{Fil}^j(\mathcal{D}_{\mathrm{crys}}(D)) + \mathcal{F}_1/\mathcal{F}_1))_{j \in \mathbb{Z}}$. As a consequence, $s_i(D', \mathcal{F}') = s_{i+1}(D, \mathcal{F})$ if $i = 1, \dots, d-1$. But by construction, for $i = 1, \dots, d-1$ we have also $t_i(D', \mathcal{F}') = t_{i+1}(D, \mathcal{F})$. Hence $t_i = s_i$ for all i by the induction hypothesis. □

Remark 2.4.3. — In particular, Proposition 2.4.1 shows that crystalline representations are trianguline, and that the set of their triangulations is in natural bijection with the set of their refinements.

Definition 2.4.4. — Let $\mathcal{F}$ be a refinement of D (resp. of V). The *parameter* of $(D, \mathcal{F})$ (resp. $(V, \mathcal{F})$) is the parameter of the triangulation of D (resp. $D_{\mathrm{rig}}(V)$) associated to $\mathcal{F}$, *i. e.* the continuous character

$$\delta := (\delta_i)_{i=1,\dots,d} : \mathbb{Q}_p^* \longrightarrow (L^*)^d$$

defined by Proposition 2.4.1.

2.4.3. Non critical refinements. — Let $(D, \mathcal{F})$ be a refined crystalline (φ, Γ)-module as in § 2.4. We assume that its Hodge-Tate weights are distinct, and denote them by

$$k_1 < \cdots < k_d.$$

Definition 2.4.5. — We say that $\mathcal{F}$ is *non critical* if $\mathcal{F}$ is in general position compared to the Hodge filtration on $\mathcal{D}_{\mathrm{crys}}(D)$, *i.e.* if for all $1 \leq i \leq d$, we have a direct sum

$$\mathcal{D}_{\mathrm{crys}}(D) = \mathcal{F}_i \oplus \mathrm{Fil}^{k_i+1}(\mathcal{D}_{\mathrm{crys}}(D)).$$

Remark 2.4.6. — Assume that $D = D_{\mathrm{rig}}(V)$ for a crystalline V in what follows.

(i) If $d = 1$, the unique refinement of V is always non critical. If $d = 2$, all refinements of D are non-critical, excepted when V is a direct sum of two characters.

(ii) Another natural definition of non criticality would be the condition

$$\begin{cases} v(\varphi_1) < k_2, \\ \forall i \in \{2, \dots, d-1\}, \quad v(\varphi_1) + \dots + v(\varphi_i) < k_1 + \dots + k_{i-1} + k_{i+1}. \end{cases} \tag{39}$$

We call a refinement satisfying this condition *numerically non critical*[24]. The weak admissibility of $D_{\mathrm{crys}}(V)$ shows that a numerically non critical refinement is non critical in our sense, but the converse is false. However, as the following example shows, it may be very hard in practice to prove that a refinement is non critical when it is not already numerically non critical.

Assume $d = 2$. The numerical non criticality condition (39) reduces to $v(\varphi_1) < k_2$; note that this is the hypothesis appearing in the weak form of Coleman's *classicity of small slope U_p-eigenforms* result ([**42**]).

Assume $(V, \mathcal{F})$ is the non ordinary refinement of the restriction at p of a p-adic Galois representation V_f attached to a classical, ordinary, modular eigenform f of level prime to p. Then $\mathcal{F}$ satisfies $v(\varphi_1) = k_2$, so it is not numerically non-critical. On the other hand $\mathcal{F}$ is non critical if and only if V is not split:

- When f is a non CM cuspform, it is expected that V is not split but this is an open problem.
- When f is an Eisenstein series of level 1, and for a well-chosen V_f (globally non semisimple and geometric), this problem and its relations with the properties of the eigencurve have been studied by the authors in [**9**]. In particular, it is equivalent to a standard conjecture on the nonvanishing of certain values of the Kubota-Leopold p-adic zeta function.[25]

In any case, note that from the existence of overconvergent companion forms [**30**, Thm. 1.1.3], the numerically critical refinement $\mathcal{F}$ of V is non critical if and only if the evil twin of f is not in the image of the Theta operator, which

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(24) When $d = 2$ some authors call such a refinement *of non-critical slopes*. Mazur introduced in [**85**] a variant of this condition, namely $v(\varphi_{i-1}) < k_i < v(\varphi_{i+1})$ for $i = 2, \dots, d-1$ which is equivalent to ours for $d \leq 3$.

(25) This last statement is also a consequence of Iwasawa main conjecture. The main result of *loc.cit.* is actually that the tame level 1 eigencurve is smooth at the critical Eisenstein points. The higher level case, as well as the CM case, can also be studied by the method of the paper.

is exactly the condition found by Coleman in his study of the boundary case of his "classicity criterion". We take this as an indication of the relevance of our definition of non criticality.

(iii) If φ is semisimple (which is conjectured to occur in the geometric situations), we see at once that V always admits a non critical refinement in our sense. However, all the refinements of V may be numerically critical. Examples occurs already when $d = 3$. Indeed, (39) is equivalent in this case to $v(\varphi_1) < k_2 < v(\varphi_3)$ (use that $v(\varphi_1) + v(\varphi_2) + v(\varphi_3) = k_1 + k_2 + k_3$). Thus any V with weights $0, 1, 2$, semisimple φ, $v(\varphi_i) = 1$ for $i = 1, 2, 3$, and with generic Hodge filtration, is weakly admissible, hence gives such an example.

The following proposition is an immediate consequence of Proposition 2.4.1.

Proposition 2.4.7. — *$\mathcal{F}$ is non critical if, and only if, the sequence of Hodge-Tate weights (s_i) associated to $\mathcal{F}$ by Proposition 2.4.1 is increasing, i.e. if $s_i = k_i$ $\forall i$.*

It is easy to see that non criticality is preserved under crystalline twists and duality. However, we have to be more careful with tensor operations, for even the notion of refinement is not well behaved with respect to tensor products. We content ourselves with the following trivial results, that we state for later use.

Lemma 2.4.8. — (i) *Assume that $(D, \mathcal{F})$ is a non-critically refined crystalline (φ, Γ)-module over $\mathcal{R}_L$. Then the weight of $\Lambda^i(\mathcal{F}_i) \subset \mathcal{D}_{\mathrm{crys}}(\Lambda^i(D))$ is the smallest Hodge-Tate weight of $\Lambda^i(D)$.*

(ii) *Let D_1 and D_2 be two (φ, Γ)-modules over $\mathcal{R}_L$ with integral Hodge-Tate-Sen weights, equipped with a one dimensional L-vector space $W_i \subset \mathcal{D}_{\mathrm{crys}}(D_i)$. If the weight of W_i is the smallest integral weight of D_i for $i = 1$ and 2, then the weight of $W_1 \otimes_L W_2$ is the smallest integral weight of $D_1 \otimes_{\mathcal{R}_L} D_2$.*

2.5. Deformations of non critically refined crystalline representations

An essential feature of non critically refined crystalline representations is that they admit a nicer deformation theory.

Let $(V, \mathcal{F})$ be a refined crystalline representation. Let us call $\mathcal{T}$ the triangulation of $D_{\mathrm{rig}}(V)$ corresponding to $\mathcal{F}$ as in Proposition 2.4.1. Recall from § 2.3.6 the functors $X_V = X_{D_{\mathrm{rig}}(V)} : \mathcal{C} \longrightarrow \mathrm{Set}$ (resp. $X_{V,\mathcal{T}} = X_{D_{\mathrm{rig}}(V),\mathcal{T}}$) parameterizing the deformations of V (resp. the trianguline deformations of $(V, \mathcal{T})$). We shall use also the notation $X_{V,\mathcal{F}}$ for $X_{V,\mathcal{T}}$ and we call a trianguline deformation of $(V, \mathcal{T})$ a trianguline deformation of $(V, \mathcal{F})$.

2.5.1. A local and infinitesimal version of Coleman's classicity theorem

Theorem 2.5.1. — *Let V be a crystalline L-representation of G_p with distinct Hodge-Tate weights and such that* $\mathrm{Hom}_{G_p}(V, V(-1)) = 0$. *Let $\mathcal{F}$ be a non critical refinement of V and V_A a trianguline deformation of $(V, \mathcal{F})$. Then V_A is Hodge-Tate if, and only if, V_A is crystalline.*

Proof. — Assume that V_A is Hodge-Tate, we have to show that $D_{\mathrm{rig}}(V_A)$ is crystalline by Proposition 2.2.9. By assumption, $D_{\mathrm{rig}}(V_A) \in X_{D_{\mathrm{rig}}(V),\mathcal{T}}(A)$ for the triangulation $\mathcal{T}$ of $D_{\mathrm{rig}}(V)$ induced by $\mathcal{F}$, which has strictly increasing weights s_i as $\mathcal{F}$ is non critical and by Proposition 2.4.7. As V_A is Hodge-Tate, $D_{\mathrm{rig}}(V_A)$ satisfies by Proposition 2.3.3 the hypothesis of Proposition 2.3.4, hence the conclusion. □

This result has interesting global consequences, some of which will be explained in § 2.6 below. It is most useful when combined with the main result of the following paragraph, which gives a sufficient condition, *à la* Kisin, for a deformation to be trianguline.

2.5.2. A criterion for a deformation of a non critically refined crystalline representation to be trianguline. — Let D be a (φ, Γ)-module free over $\mathcal{R}_A$, we first give below a criterion to produce a (φ, Γ)-submodule of rank 1 over $\mathcal{R}_A$. This part may be seen as an analogue of [**46**, prop. 5.3].

Lemma 2.5.2. — *Let D be a (φ, Γ)-module free over $\mathcal{R}_A$, $\delta : \mathbb{Q}_p^* \longrightarrow A^*$ be a continuous character, and $\overline{\delta} := \delta \bmod m$.*

(i) *Let $M \subset D^{\delta}$ be a free A-module of rank 1. Then $\mathcal{R}_A[1/t]M$ is a (φ, Γ)-submodule of $D[1/t]$ which is free of rank 1 over $\mathcal{R}_A[1/t]$, and a direct summand.*

(ii) *Same assumption as in* (i), *but assume moreover that*
$$\mathrm{Im}\left(M \longrightarrow (D/mD)^{\overline{\delta}}\right) \not\subset t(D/mD).$$
Then $\mathcal{R}_A M$ is a (φ, Γ)-submodule of D which is free of rank 1 over $\mathcal{R}_A$ and a direct summand.

(iii) *Assume that $k \in \mathbb{Z}$ is the smallest integral root of the Sen polynomial of D/mD and let $\lambda \in A^*$. Let $M \subset \mathcal{D}_{\mathrm{crys}}(D)^{\varphi=\lambda}$ be free of rank 1 over A such that*
$$\mathrm{Fil}^{k+1}(M/mM) = 0.$$
Then $\mathcal{R}t^{-k}M$ is a (φ, Γ)-submodule of D which is free of rank 1 over $\mathcal{R}_A$ and a direct summand.

Proof. — The natural map
$$\mathcal{R} \otimes_{\mathbb{Q}_p} M = \mathcal{R}_A \otimes_A M \longrightarrow \mathcal{R}_A M \subset D$$

is injective by a standard argument as $(\mathrm{Frac}(\mathcal{R}))^{\Gamma} = \mathbb{Q}_p$. As a consequence, $\mathcal{R}_A M$ is free of rank 1 over $\mathcal{R}_A$. In particular, $\mathcal{R}_A[1/t]M$ is free of rank 1 over $\mathcal{R}_A[1/t]$, hence a $\mathcal{R}_A[1/t]M$ direct summand of $D[1/t]$ as $\mathcal{R}$-module by Proposition 2.2.2, and we conclude (i) by Lemma 2.2.3 (i) and (ii). To prove (ii), it suffices by Lemma 2.2.3 (iii) to show that $\mathrm{Im}\,(\mathcal{R}_A M \longrightarrow D/mD)$, which is $\mathcal{R}_{A/m}(M/mM)$ by Remark 2.2.4, is $\mathcal{R}$-saturated in D/mD. But this is the assumption.

For part (iii), write $M = Av$, $\varphi(v) = \lambda v$. Lemma 2.4.2 shows that $v \in t^k D$ and that $\overline{v} \notin t^{k+1}D/mD$. Part (ii) above applied to $M' := t^{-k}M$ concludes the proof. □

Remark 2.5.3. — When $A = L$ is a field, a (φ, Γ)-module D over $\mathcal{R}_A$ is triangulable over $\mathcal{R}_A$ if and only if $D[1/t]$ is triangulable over $\mathcal{R}_A[1/t]$ (with the obvious definition). However, this is no longer true for a general A.

An immediate consequence of Lemma 2.5.2 (iii) is the following proposition, which could also be proved without the help of (φ, Γ)-modules (see § 2.3.5 for the definition of $\mathcal{C}$).

***Proposition 2.5.4* (The "constant weight lemma").** — *Let V be any L-representation of G_p and $\lambda \in L^*$. Assume that $D_{\mathrm{crys}}(V)^{\varphi=\lambda}$ has L-dimension 1 and that its induced Hodge filtration admits the smallest integral Hodge-Tate weight k of V as jump. Let $A \in \mathrm{Ob}(\mathcal{C})$, $\lambda' \in A^*$ a lift of λ, and V_A a deformation of V such that $D_{\mathrm{crys}}(V_A)^{\varphi=\lambda'}$ is free of rank 1 over A.*

Then the Hodge filtration on $D_{\mathrm{crys}}(V_A)^{\varphi=\lambda'}$ has k as unique jump. In other words, k is a constant Hodge-Tate weight of V_A.

We are now able to give a criterion on a deformation V_A of a refined crystalline representation $(V, \mathcal{F})$ ensuring that it is a trianguline deformation. We need the following definition.

Definition 2.5.5. — We say that the refinement $\mathcal{F}$ of V is regular if the ordering $(\varphi_1, \dots, \varphi_d)$ of the eigenvalues of φ it defines has the property

(REG) for all $1 \leq i \leq d$, $\varphi_1\varphi_2 \cdots \varphi_i$ is a simple eigenvalue of $\Lambda^i(\varphi)$.

In particular, the φ_i are distinct, and $\mathcal{F}$ is the unique refinement such that $\Lambda^i(\mathcal{F}_i) = (\Lambda^i(D_{\mathrm{crys}}(V)))^{\varphi=\varphi_1\cdots\varphi_i}$.

The next theorem is the bridge between this chapter and chapter 3.

Theorem 2.5.6. — *Assume that $\mathcal{F} = \{\varphi_1, \dots, \varphi_d\}$ is a non critical, regular, refinement of a crystalline L-representation V of dimension d. Let V_A be a deformation of V, and assume that we are given continuous homomorphisms $\delta = (\delta_i)_{i=1,\dots,d} : \mathbb{Q}_p^* \longrightarrow A^*$ such that for all i,*

(i) *$D_{\mathrm{crys}}(\Lambda^i(V_A)(\delta_1 \cdots \delta_i)^{-1}_{|\Gamma})^{\varphi=\delta_1(p)\cdots\delta_i(p)}$ is free of rank 1 over A.*

(ii) $\delta_{i|\Gamma} \bmod m = \chi^{-t_i}$ *for some* $t_i \in \mathbb{Z}$, *and* $\varphi_i = (\delta_i(p) \bmod m)p^{t_i}$.

Then V_A *is a trianguline deformation of* $(V, \mathcal{F})$ *whose parameter is* $(\delta_i x^{t_i - k_i})_{i=1,\dots,d}$.

Proof. — Note that the assumptions and conclusions do not change if we replace each δ_i by $\delta_i x^{m_i}$ for $m_i \in \mathbb{Z}$. Thus we can assume that $t_i = 0$ for all i, *i.e.* $\delta_{i|\Gamma} \equiv 1 \bmod m$.

Fix $1 \leq i \leq d$ and set $W_i := \Lambda^i(V) \otimes_A A(\delta_1 \cdots \delta_i)^{-1}_{|\Gamma}$. By assumption (ii), W_i/mW_i is crystalline with smallest Hodge-Tate weight $w_i := k_1 + \cdots + k_i$. Moreover, by the regularity property of $\mathcal{F}$, $\Lambda^i(\mathcal{F}_i) = D_{\mathrm{crys}}(W_i/mW_i)^{\varphi = \varphi_1 \cdots \varphi_i}$. As $\mathcal{F}$ is non critical, $\Lambda^i(\mathcal{F}_i) \cap \mathrm{Fil}^{w_i+1}(D_{\mathrm{crys}}(W_i/mW_i)) = 0$. Lemma 2.5.2 (iii) shows that

$$t^{-w_i} \mathcal{R}_L \mathcal{D}_{\mathrm{crys}}(W_i)^{\varphi = \delta_1(p) \cdots \delta_i(p)} \subset D_{\mathrm{rig}}(W_i)$$

is free of rank one over $\mathcal{R}_A$ and direct summand. As a consequence, if we set $D := D_{\mathrm{rig}}(V_A)$, then $\Lambda^i(D) = D_{\mathrm{rig}}(W_i(\delta_1 \cdots \delta_i)_{|\Gamma})$ contains a (φ, Γ)-submodule L_i which is free of rank 1 and direct summand as $\mathcal{R}_A$-module, and such that

$$L_i \xrightarrow{\sim} \mathcal{R}_A(\delta_1 \cdots \delta_i x^{-w_i}). \tag{40}$$

Set $D_0 = 0$. We claim that for $i = 1, \dots, d$, there exists a (φ, Γ)-submodule $D_i \subset D$ over $\mathcal{R}_A$ which is free of rank i and direct summand as $\mathcal{R}_A$-module, and such that:

(a) $D_{i-1} \subset D_i$,

(b) $\Lambda^i(D_i) = L_i$.

This would conclude the proof. Indeed, for each i we necessarily have $D_i/mD_i = \mathrm{Fil}_i(D_{\mathrm{rig}}(V))$ by (b), (40), and the regularity of $\mathcal{F}$, so (D_i) is an A-triangulation of D lifting $(\mathrm{Fil}_i(D_{\mathrm{rig}}(V)))$. Moreover, ($a$) and ($b$) imply that for $i = 2, \dots, d$

$$L_i = \Lambda^i(D_i) \xrightarrow{\sim} \Lambda^{i-1}(D_{i-1}) \otimes_{\mathcal{R}_A} (D_i/D_{i-1}) = L_{i-1} \otimes_{\mathcal{R}_A} (D_i/D_{i-1}),$$

and (40) forces then the parameter of the triangulation (D_i) to be $(\delta_i x^{-k_i})$.

We now prove the claim by induction on i. Of course, we set $D_1 := L_1 \subset D$. Let $i \in \{2, \dots, d\}$ and assume that D_1, D_2, ... have been constructed up to D_{i-1}. Consider the natural exact sequence of (φ, Γ)-modules over $\mathcal{R}_A$:

$$0 \longrightarrow \Lambda^{i-1}(D_{i-1}) \otimes_{\mathcal{R}_A} (D/D_{i-1}) \longrightarrow \Lambda^i(D) \longrightarrow Q_i \longrightarrow 0, \tag{41}$$

where Q_i is defined as the cokernel of the first map. By Lemma 2.5.7 below applied to $B = \mathcal{R}_A$, $M = D$, $P = D_{i-1}$ and $r = i - 1$, there exists a $\mathcal{R}_A$-direct summand $D_i := P' \subset D$ containing D_{i-1} and such that $L_i = \Lambda^i(D_i)$ if, and only if,

$$L_i \subset \Lambda^{i-1}(D_{i-1}) \otimes_{\mathcal{R}_A} (D/D_{i-1})$$

inside $\Lambda^i(D)$. If it is so, note that D_i is necessarily (φ, Γ)-stable. Indeed, inside the (φ, Γ)-module $\Lambda^i(D)$ we have

$$L_i = L_{i-1} \otimes_{\mathcal{R}_A} D_i/D_{i-1} \subset L_i \otimes_{\mathcal{R}_A} D/D_{i-1}$$

and L_i, L_{i-1} are (φ, Γ)-stable by definition, as well as D_{i-1} by induction on i.

Therefore we only need to show that the image of L_i in Q_i is zero. Note that the natural map $\Lambda^i(D) \to Q_i$ is (φ, Γ)-equivariant. By a dévissage, it is enough to show that any (φ, Γ)-homomorphism

$$L_i \otimes_A A/m = \mathcal{R}_L(\varphi_1 \cdots \varphi_i x^{-w_i}) \to Q_i \otimes_A A/m$$

vanishes, or better that $\varphi_1\varphi_2 \cdots \varphi_i$ is not an eigenvalue of φ on $D_{\mathrm{crys}}(Q_i \otimes_A A/m)$. But by definition (41) these eigenvalues have the form $\varphi_{j_1}\varphi_{j_2} \cdots \varphi_{j_i}$ for some integers $j_1 < j_2 < \cdots < j_i$ in $\{1, \dots, d\}$ such that $(j_1, \dots, j_{i-1}) \neq (1, 2, \dots, i-1)$, and we are done by (REG). □

In the following lemma, it is understood that all tensor and exterior products are taken over the ring B.

Lemma 2.5.7. — *Let B be a commutative ring, M a free B-module of finite type, $P \subset M$ a B-submodule which is a direct summand and free of rank r, and let $L \subset \Lambda^{r+1}(M)$ be a B-submodule which is a direct summand and free of rank 1. The following conditions are equivalent:*

(i) *$L \subset \Lambda^r(P) \otimes M/P$,*

(ii) *there exists a B-submodule $P' \subset M$ containing P, which is a direct summand in M and free of rank $r+1$, such that $L = \Lambda^{r+1}(P')$.*

If they are satisfied, the submodule P' satisfying (ii) *is unique.*

Proof. — Choose a B-submodule $Q \subset M$ such that $P \oplus Q = M$. We have a natural decomposition

$$\Lambda^{r+1}(M) = \bigoplus_{i+j=r+1} \Lambda^i(P) \otimes \Lambda^j(Q).$$

Assume that (i) holds, that is, $L \subset \Lambda^r(P) \otimes Q$. As L is a direct summand in $\Lambda^{r+1}(M)$, it is also a direct summand in $\Lambda^r(P) \otimes Q$. As L and $\Lambda^r(P)$ are free of rank 1, we may write $L = \Lambda^r(P) \otimes Be$ for some $e \in Q$ such that Be is a direct summand in Q. This shows (ii) where $P' = P \oplus Be$. Conversely, it is obvious that (ii) $\Rightarrow$ (i). As $P' = P \oplus (P' \cap Q)$ and $(P' \cap Q) = \Lambda^r(P)^{-1} \otimes L$, P' is uniquely determined by L. □

2.5.3. Properties of the deformation functor $X_{V,\mathcal{F}}$. — In fact, we can in many cases describe quite simply $X_{V,\mathcal{F}}$ when $\mathcal{F}$ is non critical. The following results will not be needed in the remaining sections, but are interesting in their own. Recall that by definition we have a natural transformation

$$X_{V,\mathcal{F}} \longrightarrow X_V.$$

Proposition 2.5.8. — *Assume that the eigenvalues of φ on $D_{\mathrm{crys}}(V)$ are distinct. Then $X_{V,\mathcal{F}}$ is a subfunctor of X_V and $X_{V,\mathcal{F}} \longrightarrow X_V$ is relatively representable. Moreover,* *if $\mathcal{F}$ is non critical, the subfunctor $X_{V,\mathrm{crys}} \subset X_V$ of crystalline deformations factors through $X_{V,\mathcal{F}}$.*

Proof. — As the eigenvalues of φ are distinct, the characters δ_i of the parameter δ of $D_{\mathrm{rig}}(V)$ associated to $\mathcal{F}$ satisfy $\delta_i\delta_j^{-1} \notin x^{\mathbb{Z}}$ for $i \neq j$ (see Prop. 2.4.1). The first sentence thus follows from Prop. 2.3.6, Prop. 2.3.9 using Prop. 2.3.13.

Assume that $\mathcal{F}$ is non critical and let V_A be a crystalline deformation of V. We have to show that $D_A := D_{\mathrm{rig}}(V_A)$ admits a (necessarily unique) triangulation lifting the one associated to $\mathcal{F}$. As the φ_i are distinct, the characteristic polynomial of φ on $\mathcal{D}_{\mathrm{crys}}(D_A)$ writes uniquely as $\prod_i(T-\lambda_i) \in A[T]$ with $\lambda_i \equiv \varphi_i \bmod m$. As V_A is Hodge-Tate with smallest Hodge-Tate weight k_1, and as $\mathcal{F}$ is non critical, Lemma 2.5.2 (iv) shows that

$$\mathcal{R}t^{-k_1}\mathcal{D}_{\mathrm{crys}}(D_A)^{\varphi=\lambda_1}$$

is a submodule of D_A which is a direct summand as $\mathcal{R}_A$-module. We construct this way by induction the required triangulation of D_A. □

Remark 2.5.9. — In the last part of the statement of Prop.2.5.8, it is necessary to assume that $\mathcal{F}$ is non-critical. Indeed, let $V = \mathbb{Q}_p(1-k) \oplus \mathbb{Q}_p$ with $k > 1$ an integer and let $\mathcal{F} = (p^{k-1}, 1)$ be its critical refinement. Consider the filtered φ-module

$$N = Ae_1 \oplus Ae_2,\ A = \mathbb{Q}_p[\varepsilon],$$

whose Hodge-Tate weights are 0 and $k-1$, with $\mathrm{Fil}^{k-1}(N) = Ae_1$, and with the following A-linear φ-action: $\varphi(e_2) = e_2$ and $\varphi(v) = \lambda v$ where $v = e_1 + \varepsilon e_2$ and $\lambda = p^{k-1}(1+\varepsilon) \in A^*$. Then N is weakly admissible, hence it is the D_{crys} of a crystalline deformation V_A of V, but this deformation does not belong to $X_{V,\mathcal{F}}(A)$: the Hodge filtration induced on $Av \subset N$ admits the two jumps 0 and $k-1$. This example also shows that the assumption of non-criticality of $\mathcal{F}$ is necessary in the statement of Theorem 2.5.6.

The main theorem concerning non critical refinements is then the following, which may be viewed as a d-dimensional generalization of some computations of Kisin in [**73**, §7] (giving a different proof of his results when $d = 2$). Recall that $(V, \mathcal{F})$ is a refined crystalline L-representation of dimension d.

Theorem 2.5.10. — *Assume that $\mathcal{F}$ is non critical, that $\varphi_i\varphi_j^{-1} \notin \{1, p^{-1}\}$ if $i < j$, and that $\mathrm{Hom}_{G_p}(V, V(-1)) = 0$. Then $X_{V,\mathcal{F}}$ is formally smooth of dimension $\frac{d(d+1)}{2} + n$, where $n = \dim_L(\mathrm{End}_{G_p}(V))$. Moreover, the parameter map induces an exact sequence of L-vector spaces:*

$$0 \longrightarrow X_{V,\mathrm{crys}}(L[\varepsilon]) \longrightarrow X_{V,\mathcal{F}}(L[\varepsilon]) \longrightarrow \mathrm{Hom}(\mathbb{Z}_p^*, L^d) \longrightarrow 0.$$

Proof. — Let (δ_i) be the parameter of $(V,\mathcal{F})$. If $i \neq j$, then $\delta_i\delta_j^{-1} \notin x^{\mathbb{Z}}$ since $\varphi_i \neq \varphi_j$. Moreover if $i < j$, then $k_i < k_j$ and $\delta_i\delta_j^{-1} \notin \chi x^{\mathbb{N}}$ as $\varphi_i \neq p^{-1}\varphi_j$ by assumption. Except for the dimension assertion, the result follows from Propositions 2.3.10, 2.4.7, 2.3.4 and 2.5.8. It only remains to show that

$$\dim_L(X_{V,\mathrm{crys}}(L[\varepsilon])) = \frac{d(d-1)}{2} + n. \tag{42}$$

One way to prove this equality is to reduce to a linear algebra problem via the equivalence of category between crystalline representations and filtered φ-modules proved by Colmez and Fontaine. We give another proof based on results of Bloch and Kato [**23**] (actually, there would be a third one, based on Remark 2.3.12 and Prop. 2.5.8).

If U is any crystalline L-representation, we denote by $H^1_f(G_p,U) \subset H^1(G_p,U)$ the subset parameterizing extensions of 1 by U which are crystalline (following Bloch-Kato [**23**]). By a classical result of Fontaine, the category of crystalline representations of G_p is stable by subquotients, so $H^1_f(G_p,U)$ is actually an L-subvector space. For the same reason, we have a natural isomorphism

$$X_{V,\mathrm{crys}}(L[\varepsilon]) \xrightarrow{\sim} H^1_f(G_p,\mathrm{End}_L(V)).$$

As a consequence of their exponential map, Bloch and Kato show ([**23**, Cor. 3.8.4]) that for any U:

$$\dim_L(H^1_f(G_p,U)) = \dim_L(H^0(G_p,U)) + \dim_L(D_{\mathrm{DR}}(U)/\mathrm{Fil}^0(D_{\mathrm{DR}}(U))), \tag{43}$$

which shows (42) for $U = \mathrm{End}_L(V)$. □

2.6. Some remarks on global applications

We now derive some consequences of these results in a global situation.

Let V be a finite dimensional L-vector space equipped with a geometric continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and assume that $V_p := V_{|G_p}$ is crystalline, with distinct Hodge-Tate weights and distinct Frobenius eigenvalues. Let $X_V : \mathcal{C} \to \mathrm{Ens}$ (resp. X_{V_p}) denote the deformation functor of the representation V (resp. of the G_p-representation V_p), as in § 2.3.6.

Let us choose a refinement $\mathcal{F}$ of V_p, and consider the trianguline deformation functor $X_{V_p,\mathcal{F}}$ of $(V_p,\mathcal{F})$, it is a subfunctor of X_{V_p} by Prop. 2.5.8. Let $X_{V,\mathcal{F}}$ denote the subfunctor of X_V consisting of the deformations whose restriction at p is in $X_{V_p,\mathcal{F}}$ (that is, $\mathcal{F}$-trianguline), and whose restriction at an inertia group at each $l \neq p$ is constant (this is the usual condition f, for instance these deformations are unramified at l if V is, see § 7.6 for a more complete discussion). By Prop. 2.3.9, $X_{V,\mathcal{F}}$ is prorepresentable by a complete local noetherian L-algebra when V has no nontrival L-endomorphism. Theorems 2.5.1 and 2.5.10 imply:

Corollary 2.6.1. — *If $\mathcal{F}$ is non critical, then there is a natural exact sequence*

$$0 \longrightarrow H^1_f(\mathbb{Q}, \mathrm{ad}(V)) \longrightarrow X_{V,\mathcal{F}}(L[\varepsilon]) \xrightarrow{\kappa} \mathrm{Hom}(\mathbb{Z}_p^*, L).$$

In particular, if $H^1_f(\mathbb{Q}, \mathrm{ad}(V)) = 0$ (which is conjectured to be the case if V is absolutely irreducible), then $\dim_L(X_{V,\mathcal{F}}(L[\varepsilon])) \leq \dim_L(V)$.

In this setting, the question of the determination of $\dim_L(X_{V,\mathcal{F}}(L[\varepsilon]))$ seems to be quite subtle, even conjecturally. Among many other things, it is linked to the local dimension of the eigenvarieties of GL_d, which are still quite mysterious (see the work of Ash-Stevens [4] and of M. Emerton [**53**]).

However, there are similar questions for which the theory of p-adic families of automorphic forms suggests a nice answer [(26)]. As an example, let us consider now an analogous case where V is an irreducible, d-dimensional, geometric L-representation of $\mathrm{Gal}(\overline{E}/E)$, $E/\mathbb{Q}$ a quadratic imaginary field, satisfying $V^{c,*} \simeq V(d-1)$. Assume that $p = vv'$ splits in E, fix an identification $G_p \xrightarrow{\sim} \mathrm{Gal}(\overline{E_v}/E_v)$, and assume that $V_p := (V_{|G_p}, \mathcal{F})$ is crystalline and provided with a refinement $\mathcal{F}$, with distinct Hodge-Tate weights. Let $X_{V,\mathcal{F}}$ denote the subfunctor of the full deformation functor of V consisting of deformations whose restriction at v is in $X_{V_p,\mathcal{F}}$, satisfying ${V_A}^{c,*} = V_A(d-1)$ and the f condition outside p.

Conjecture. — Assume that $\mathcal{F}$ is non critical, then $X_{V,\mathcal{F}}$ is prorepresented by

$$\mathrm{Spf}(L[[X_1, \ldots, X_d]])$$

and κ is an isomorphism.

In the subsequent paragraph § 7.6, we will give more details about the proofs of the facts alluded here and we will explain how we can deduce this conjecture in many cases assuming the conjectured vanishing of $H^1_f(E, \mathrm{ad}(V))$, and using freely the results predicted by Langlands philosophy on the correspondence between automorphic forms for suitable unitary groups G (in d variables) attached to the quadratic extension $E/\mathbb{Q}$. As we will explain, we can even get an "$R = T$" statement for $\mathrm{Spf}(R) = X_{\mathcal{F}}$ and T the completion of a well chosen eigenvariety of G at the point corresponding to $(V, \mathcal{F})$.

To sum up, the eigenvariety of G at *irreducible, classical, non-critical* points should be smooth, and neatly related to deformation theory. By contrast, a much more complicated (but interesting) situation is expected at reducible, critical points, and this is the main object of subsequent sections of this book.

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(26) We consider here a simplified setting, an appropriate condition on the Mumford-Tate group of V should suffice in general.

CHAPTER 3

GENERALIZATION OF A RESULT OF KISIN ON CRYSTALLINE PERIODS

3.1. Introduction

In this section, we solve, generalizing earlier results of Kisin, some questions of "Fontaine's theory in families" concerning the continuation of crystalline periods.

Let X be a reduced rigid analytic space over $\mathbb{Q}_p$ and $\mathcal{M}$ a family of p-adic representations of $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over X, that is, in this section, a coherent torsion-free sheaf of $\mathcal{O}_X$-modules with a continuous action of the group G_p. Note that we do not assume that $\mathcal{M}$ is locally free. [(1)] For each point $x \in X$ of residue field $k(x)$ the $k(x)$-vector space $\overline{\mathcal{M}}_x$ is then a continuous representation of G_p, to which we can apply the p-adic Hodge theory of Tate and Sen and all its generalizations by Fontaine. The questions concerning "Hodge-Tate theory in families" were completely solved by Sen: in particular he shows in our context that there exist d analytic functions $\kappa_1, \dots, \kappa_d$ on X, where d is the generic rank of $\mathcal{M}$, such that $\kappa_1(x), \dots, \kappa_d(x)$ are the Hodge-Tate-Sen weights of $\overline{\mathcal{M}}_x$ for a Zariski-dense open set of X. We shall assume in this introduction, to simplify the discussion, that $\kappa_1 = 0$. We shall also assume that in our family the other weight functions $\kappa_2, \dots, \kappa_d$ move widely (in a technical sense we do not want to make precise here, but see 3.3.2 below), as it happens for families supported by eigenvarieties. In particular, the families we work with are quite different from the families with constant weights studied by Berger and Colmez.

Suppose we know that $\overline{\mathcal{M}}_z$ is crystalline with positive Hodge-Tate weights for a Zariski dense subset Z of points of X and that for all $z \in Z$ it has a crystalline period, that is, an eigenvector of the crystalline Frobenius φ with eigenvalue $F(z)$, F being a fixed rigid analytic function on X. In other words, assume that $D_{\mathrm{crys}}(\overline{\mathcal{M}}_z)^{\varphi=F(z)}$ is non zero for $z \in Z$. Can we deduce from this that

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(1) Indeed, we will apply the results of this section to modules associated to pseudocharacters in the neighborhood of a reducible point, and it is one of the main results of Section 1 that they do not in general come from representations over free modules.

(1) for each x in X, $\overline{\mathcal{M}}_x$ has a crystalline period, which is moreover an eigenvector for φ with eigenvalue $F(x)$?

Or, more generally, that

(2) for each $x \in X$, and $\operatorname{Spec} A$ a thickening of x, (i.e. A an artinian quotient of the rigid analytic local ring $\mathcal{O}_x$ of x at X) $\mathcal{M}$ has a non-torsion crystalline period over A which is an eigenvector for φ with eigenvalue the image $\bar{F}$ of F in A ? In other words, is it true that $D_{\mathrm{crys}}(\mathcal{M} \otimes A)^{\varphi=\bar{F}}$ has a free A-submodule of rank one?

Kisin was the first to deal with those questions and most of his works in [**73**] is an attempt to answer them in the case where $\mathcal{M}$ is a free $\mathcal{O}_X$-module. Under this freeness assumption (plus some mild technical hypothesis on Z that we will not state nor mention further in this introduction), he proves question (1) and also many cases of question (2), although these results are scattered along his paper and sometimes not explicitly stated. If we collect them all, we get that Kisin proved that question (2) has a positive answer (when $\mathcal{M}$ is free) for those x that satisfy two conditions:

(a) The representation $\overline{\mathcal{M}}_x$ is indecomposable,
(b) $D_{\mathrm{crys}}(\overline{\mathcal{M}}_x^{\mathrm{ss}})^{\varphi=F(x)}$ has dimension 1.

Condition (b) is probably necessary. But condition (a) is not, and appears because of the use by Kisin of some universal deformation arguments. In § 3.3, using mostly arguments of Kisin, but simplifying and reordering them, we prove that when $\mathcal{M}$ is a locally free module, question (2) has a positive answer for all x satisfying the condition (b) above. We hope that our redaction may clarify the beautiful and important results of Kisin.

But our main concern here is to generalize those results to the case of an arbitrary torsion-free coherent sheaf $\mathcal{M}$. We are able to prove that question (2) (hence also question (1)) still has a positive answer in this case, provided that x satisfies hypothesis (b) above. This is done in § 3.4.

Let us now explain the idea of the proof: basically we do a reduction to the case where $\mathcal{M}$ is locally free. To do this we use a rigid analytic version of a "flatification" result of Gruson-Raynaud which gives us a blow-up X' of X such that the *strict transform* $\mathcal{M}'$ of $\mathcal{M}$ on X' is locally free. Hence we know the (positive) answers to questions (1) and (2) for $\mathcal{M}'$ and the problem is to "descend" them to $\mathcal{M}$. This is the aim of § 3.2.3.

For this the difficulty is twofold. The first difficulty is that if x' is a point of X' above x (let us say to fix ideas with the same residue field, since a field extension here would not harm) then $\overline{\mathcal{M}}'_{x'}$ is not isomorphic to $\overline{\mathcal{M}}_x$ but to a quotient of it. Since the functor $D_{\mathrm{crys}}(-)^{\varphi=F}$ is only left-exact, the positive answer to question (1) for x' does not imply directly the positive answer for x—and of course, neither for question (2).

The second difficulty arises only when dealing with question (2): it is not possible in general to lift the thickening Spec(A) of x in X to a thickening of x' in X', whatever x' above x we may choose [(2)]. So the direct strategy of descending a positive answer to question (2) from a Spec(A) in X' to Spec(A) in X can not work. To circumvent the second difficulty, we use a lemma of Chevalley to construct a suitable thickening Spec(A') of Spec(A) in X', and then some rather involved arguments of lengths to deal with the first one as well as the difference between Spec(A') and Spec(A). As Chevalley's lemma requires to work at the level of complete noetherian ring, and as we have to use rigid analytic local rings when dealing with interpolation arguments, we need also at some step of the proof to compare various diagrams with their completion. For all these reasons, the total argument in § 3.2.3 is rather long.

Finally let us say that the idea of using a blow-up was already present in Kisin's argument in the free case [(3)], and is still present in the locally free case in § 3.3. This is why our descent result of § 3.2.3 is used twice, once in § 3.3 and once in § 3.4. However, were it to be used only in the locally free case, the descent method could be much simpler [(4)].

3.2. A formal result on descent by blow-up

3.2.1. Notations. — Let X be a reduced, separated, rigid analytic space over $\mathbb{Q}_p$, $\mathcal{O}_X$ (or simply $\mathcal{O}$) its structural sheaf, and let $\mathcal{M}$ be a coherent $\mathcal{O}$-module on X. For x a point of X, we shall note $\mathcal{O}_x$ the rigid analytic local ring of X at x, m_x its maximal ideal, and $k(x) = \mathcal{O}_x/m_x$ its residue field. Moreover, we denote by $\mathcal{M}_x$ the rigid analytic germ of $\mathcal{M}$ at x, that is $\mathcal{M}_x = \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}_x$ where U is any open affinoid containing x, and by $\overline{\mathcal{M}}_x := \mathcal{M}_x \otimes_{\mathcal{O}_x} k(x)$ the fiber of $\mathcal{M}$ at x.

Let G be a topological group and assume that $\mathcal{M}$ is equipped with a continuous $\mathcal{O}$-linear action of G. This means that for each open affinoid $U \subset X$, we have a continuous morphism $G \longrightarrow \mathrm{Aut}_{\mathcal{O}(U)}(\mathcal{M}(U))$, whose formation is compatible with the restriction to any open affinoid $V \subset U$. For x a point of X, $\mathcal{M}_x$ (resp. $\overline{\mathcal{M}}_x$) is then a continuous $\mathcal{O}_x[G]$-module (resp. $k(x)[G]$-module) in a natural way.

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(2) The reader may convince himself of this assertion by looking at the case where $X = \mathrm{Spec}\, L[[T^2, T^3]]$ is the cusp and $X' = \mathrm{Spec}\,(L[[T]])$ the blow up of X at its maximal ideal (that is, its normalization). The principal ideal $T^2L[[T^2,T^3]]$ has not the form $L[[T^2,T^3]] \cap T^nL[[T]]$ for $n \geq 0$, hence $A = L[[T^2,T^3]]/(T^2)$ is a counter-example.

(3) There Kisin does not use the blow-up to make $\mathcal{M}$ free, since it already is, but instead to make an ideal of crystalline periods locally principal. He does not prove a direct descent result as ours, using instead a comparison of universal deformation rings.

(4) The first difficulty above vanishes, since in that case the strict transform of $\mathcal{M}$ is simply its pull-back, and the second may be dealt with much more easily.

Remark 3.2.1. — (*On torsion free modules*) In this section and the subsequent ones, we will sometimes have to work with torsion free modules. Recall that a module M over a *reduced ring* A is said to be torsion free if the natural map $M \longrightarrow M \otimes_A K$ is injective where $K = \mathrm{Frac}(A)$ is the total fraction ring of A (see § 1.3.5).

If X is a reduced affinoid and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, then $\mathcal{M}(X)$ is torsion free over $\mathcal{O}(X)$ if, and only if, $\mathcal{M}_x$ is torsion free over $\mathcal{O}_x$ for all $x \in X$. Indeed, this follows at once from the faithful flatness of the maps $\mathcal{O}(X)_x \longrightarrow \mathcal{O}_x$ and the following lemma.

Lemma 3.2.2. — *Let A be a reduced noetherian ring and M an A-module of finite type. The following properties are equivalent:*

(i) *M is torsion free over A,*
(ii) *M is a submodule of a K-module,*
(iii) *M is a submodule of a finite free A-module,*
(iv) *M_x is torsion free over A_x for all $x \in \mathrm{Specmax}(A)$,*
(v) *there is a faithfully flat A-algebra B such that $M \otimes_A B$ is a B-submodule of a finite free B-module.*

Proof. — It is clear that (i), (ii) and (iii) are equivalent (for (ii) $\Rightarrow$ (iii) note that any K-module embeds into a free K-module as K is a finite product of fields). The equivalence between (i) and (iv) follows now from the injection $M \longrightarrow \prod_{x \in \mathrm{SpecmaxA}} M_x$, and the fact that $\mathrm{Frac}(A_x)$ is a factor ring of K: namely the product of the fraction fields of the irreducible component of $\mathrm{Spec}(A)$ containing x.

Note that condition (iii) is equivalent to ask that the natural map $M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is injective. But this can be checked after any faithfully flat extension B of A as the formation of the Hom's commute with any flat base change when the source is finitely presented, thus (i) $\Leftrightarrow$ (v). $\square$

3.2.2. The left-exact functor D. — Fix a point $x \in X$ and let $\mathrm{FL}(\mathcal{O}_x)$ denote the category of finite lenght $\mathcal{O}_x$-modules. Any such $\mathcal{O}_x$-module is a finite dimensional $\mathbb{Q}_p$-vector-space, hence a topological $\mathbb{Q}_p$-vector-space in a canonical way. Let $\mathrm{FL}_G(\mathcal{O}_x)$ be the category of finite length $\mathcal{O}_x$-modules equipped with a continuous $\mathcal{O}_x$-linear action of G and fix

$$D : \mathrm{FL}_G(\mathcal{O}_x) \longrightarrow \mathrm{FL}(\mathcal{O}_x)$$

an $\mathcal{O}_x$-linear left-exact functor. If M is an object of $\mathrm{FL}(\mathcal{O}_x)$, we shall denote by $l(M)$ its length as an $\mathcal{O}_x$-module. If an object $M \in \mathrm{FL}_G(\mathcal{O}_x)$ is annihilated by the maximal ideal of $\mathcal{O}_x$, then so is $D(M)$, and $l(M) = \dim_{k(x)}(D(M))$.

Here are some interesting examples:

(i) Let $G := G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and let B be any topological $\mathbb{Q}_p$-algebra equipped with a continuous action of G_p. Assume that B is G_p-regular in the sense of

Fontaine [**1**, Exposé III, §1.4]. For Q any $\mathcal{O}_x$-module of finite length equipped with a continuous G_p-action (hence a finite dimensional $\mathbb{Q}_p$-representation of G_p), we let

$$D(Q) := (Q \otimes_{\mathbb{Q}_p} B)^G.$$

The functor D satisfies our assumptions by *loc. cit.*

(ii) Fix $F \in \mathcal{O}_x^*$. For any Q as above, then

$$D(Q) := D^+_{\mathrm{crys}}(Q)^{\varphi=F} = \{v \in (Q \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{G_p}, \varphi(v) = Fv\},$$

where B^+_{crys} is the subring of B_{crys} defined by Fontaine in [**1**, Exposé II, §2.3], satisfies again our assumptions.

In the sequel, we will be mainly interested in the case (ii) above.

3.2.3. Statement of the result. — Assume that $\mathcal{M}_x$ is torsion free over $\mathcal{O}_x$ (recall that $\mathcal{O}_x$ is reduced). Let $\pi : X' \longrightarrow X$ be a proper and birational morphism of rigid spaces with X' reduced. Here *birational morphism* means that π is a morphism such that for some coherent sheaf of ideals $H \subset \mathcal{O}_X$,

- $U := X - V(H)$ is Zariski dense in X (where $V(H)$ is the closed subspace defined by H),
- π induces an isomorphism $\pi^{-1}(U) \longrightarrow U$,
- and $\pi^{-1}(U) \subset X'$ is Zariski dense in X'.

As an important example, we may take for π the blow-up [(5)] of H. Let $\mathcal{M}'$ be the strict transform of $\mathcal{M}$ by this morphism (see below).

Proposition 3.2.3. — *Assume that for all $x' \in \pi^{-1}(x)$ and for every ideal I' of $\mathcal{O}_{x'}$ of cofinite length, we have*

$$l(D(\mathcal{M}'_{x'} \otimes \mathcal{O}_{x'}/I')) = l(\mathcal{O}_{x'}/I').$$

Assume moreover that

$$\dim_{k(x)}(D(\overline{\mathcal{M}}_x^{\mathrm{ss}})) \leq 1$$

Then we also have, for every ideal I of cofinite length of $\mathcal{O}_x$:

$$l(D(\mathcal{M}_x \otimes \mathcal{O}_x/I)) = l(\mathcal{O}_x/I).$$

Remark 3.2.4. — (i) More precisely, we shall show that Proposition 3.2.3 holds when we replace the assumption on $\overline{\mathcal{M}}_x^{\mathrm{ss}}$ by the following slightly more general one: for any $k(x)[G]$-quotient U of $\overline{\mathcal{M}}_x$, $\dim_{k(x)}(D(U)) \leq 1$ (see the proof of Lemma 3.2.9, which is the only place where the assumption is used).

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(5) We refer to [**95**, §5.1] for the basics on blow-ups and to [**48**, §2.3, §4.1] for the notion of relative Spec and blow-ups in the context of rigid geometry.

(ii) As will be clear, the analogue of Proposition 3.2.3 in the context of schemes instead of rigid analytic spaces would hold by the same proof.

This whole subsection is devoted to the proof of the proposition. Let us fix a coherent sheaf of ideals $H \subset \mathcal{O}_X$ such that $U := X - V(H)$ is Zariski dense in X, that π is an isomorphism over U, and that $\pi^{-1}(U) \subset X'$ is Zariski dense in X'. The strict transform $\mathcal{M}'$ of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ is defined as follows: it is a coherent $\mathcal{O}_{X'}$-module which is locally the quotient of the coherent sheaf $\pi^*\mathcal{M}$ by its submodule of sections whose support is in the fiber of π over $V(H) \subset X$. In other words, if H' is a coherent sheaf of ideals of $\mathcal{O}_{X'}$ defining the closed subset $\pi^{-1}(V(H)) \subset X'$, then $\mathcal{M}'$ is the quotient of $\pi^*\mathcal{M}$ by its H'^∞-torsion. Note that it depends on the choice of H in general. This description makes clear that the action of G on the $\mathcal{O}_X$-module $\mathcal{M}$ defines an $\mathcal{O}_{X'}$-linear continuous action of G on $\mathcal{M}'$, and that the natural map $\pi^*\mathcal{M} \longrightarrow \mathcal{M}'$ is G-equivariant. A useful fact about the notion of strict transform is that the subsheaf of H'^∞-torsion of $\pi^*\mathcal{M}$ is precisely the kernel of the natural morphism [6] $\pi^*\mathcal{M} \longrightarrow j_*(\pi^*\mathcal{M}_{|\pi^{-1}(U)})$, where j is the open immersion of $\pi^{-1}(U)$ into X'. As a simple application, if $\mathcal{M}$ is torsion free then $\mathcal{M}'$ is torsion free as well and does not depend on the choice of H as above.

Since $\mathcal{M}_x$ is torsion free over $\mathcal{O}_x$, it can be embedded in a free of finite rank $\mathcal{O}_x$-module, so we can choose an injection

$$i : \mathcal{M}_x \longrightarrow \mathcal{O}_x^n.$$

Fix $x' \in \pi^{-1}(x)$; i induces a morphism $i' : \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x'} \longrightarrow \mathcal{O}_{x'}^n$. We check easily using the aforementionned useful fact that the kernel of i' is the submodule of $\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x'}$ whose elements are killed by a power of $H'_{x'}$ so the image of i' is $\mathcal{M}'_{x'}$. We thus have a commutative diagram of $\mathcal{O}_x$-modules (and even of $\mathcal{O}_{x'}$-modules for the half right of the diagram)

(44)
$$\begin{tikzcd}
\mathcal{M}_x \arrow[r] \arrow[d, hook, "i"] & \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x'} \arrow[r, two heads] \arrow[dr, "i'"] & \mathcal{M}'_{x'} \arrow[d, hook] \\
\mathcal{O}_x^n \arrow[rr] & & \mathcal{O}_{x'}^n .
\end{tikzcd}$$

---

[6] We are grateful to Brian Conrad for pointing this to us. Here is the general statement: if S is a rigid space, $I \subset \mathcal{O}_S$ a coherent sheaf of ideals, $j : U := S - V(I) \hookrightarrow S$ the inclusion of the complement of $V(I)$ and $\mathcal{F}$ a coherent $\mathcal{O}_S$-module, then the I^∞-torsion of $\mathcal{F}$ is the kernel of the natural map $\mathcal{F} \to j_*\mathcal{F}_{|U}$. Indeed, we may assume that S is affinoid. Set $F = \mathcal{F}(S)$ and take $m \in F$ such that $m_s = 0 \in F \otimes \mathcal{O}_s$ for all $s \in U$; we want to show that m is killed by a power of $I(S)$. The faithfull flatness of $\mathcal{O}(S)_s \to \mathcal{O}_s$ shows that the closed points of the support of m lie in $V(I(S))$, and we conclude as $\mathcal{O}(S)$ is a Jacobson ring.

We call $\widehat{\mathcal{O}}_x$ (resp. $\widehat{\mathcal{O}}_{x'}$) the completion of the local ring $\mathcal{O}_x$ (resp. of $\mathcal{O}_{x'}$) for the m_x-adic (resp. $m_{x'}$-adic) topology, and $\widehat{\mathcal{M}_x} = \mathcal{M}_x \otimes_{\mathcal{O}_x} \widehat{\mathcal{O}}_x$ (resp. $\widehat{\mathcal{M}'_{x'}} = \mathcal{M}'_{x'} \otimes_{\mathcal{O}_{x'}} \widehat{\mathcal{O}}_{x'}$) the completion of $\mathcal{M}_x$ (resp. of $\mathcal{M}'_{x'}$).

As $\mathcal{O}_x \longrightarrow \mathcal{O}_{x'}$ is a local morphism, it is continuous for the m_x-adic topology at the source and the $m_{x'}$-adic topology at the goal. This is also true for any morphism form a finite type $\mathcal{O}_x$-module to a finite type $\mathcal{O}_{x'}$-module. Hence such a morphism can be extended in a unique continuous way to their completion. We get this way morphisms $\widehat{\mathcal{O}}_x \longrightarrow \widehat{\mathcal{O}}_{x'}$ and

$$\widehat{\mathcal{M}_x} \longrightarrow \widehat{\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x'}} = (\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x'}) \otimes_{\mathcal{O}_{x'}} \widehat{\mathcal{O}}_{x'} = \widehat{\mathcal{M}_x} \otimes_{\widehat{\mathcal{O}}_x} \widehat{\mathcal{O}}_{x'}$$

the last equality being obtained by applying twice the transitivity of the tensor product. We thus have a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{M}_x} \arrow[r] \arrow[d, hook] & \widehat{\mathcal{M}_x} \otimes_{\widehat{\mathcal{O}}_x} \widehat{\mathcal{O}}_{x'} \arrow[r, two heads] \arrow[dr] & \widehat{\mathcal{M}'_x} \arrow[d, hook] \\ \widehat{\mathcal{O}}^n_x \arrow[rr] & & \widehat{\mathcal{O}}^n_{x'}. \end{tikzcd} \tag{45}$$

The injectivity of the vertical maps comes from the injectivity of the analogue maps in (44) and the flatness of $\widehat{\mathcal{O}}_x$ over $\mathcal{O}_x$ and of $\widehat{\mathcal{O}}_{x'}$ over $\mathcal{O}_{x'}$. The surjectivity of the upright horizontal map comes directly from the surjectivity of the analogue map from (44).

To simplify notations, we shall note A the local ring $\mathcal{O}_x$, k its residue field and M the A-module $\mathcal{M}_x$. We set

$$\widehat{A'} := \prod_{x' \in \pi^{-1}(x)} \widehat{\mathcal{O}}_{x'},$$

and we will see it with the product topology. We call $\widehat{M}$ the completion of M, that is also $M \otimes_A \widehat{A}$. By definition, it is $\widehat{\mathcal{M}_x}$. We set

$$\widehat{M'} := \prod_{x' \in \pi^{-1}(x)} \widehat{\mathcal{M}'_{x'}}.$$

Note that $\widehat{M'}$ is an $\widehat{A'}$-module.

Lemma 3.2.5. — *For each open (hence cofinite length) ideal $\widehat{J'}$ of $\widehat{A'}$,*

$$l(D(\widehat{M'}/\widehat{J'}\widehat{M'})) = l(\widehat{A'}/\widehat{J'}).$$

Proof. — Since $\widehat{J'}$ is open, $\widehat{A'}/\widehat{J'}$ is a finite product of finite length rings of the form $\widehat{\mathcal{O}}_{x'_i}/\widehat{J'_i}$. For each such i, $\widehat{J'_i}$ is open in $\widehat{\mathcal{O}}_{x'_i}$ so the ideal $J'_i := \widehat{J'_i} \cap \mathcal{O}_{x'_i}$ of $\mathcal{O}_{x'_i}$ satisfies $\mathcal{O}_{x'_i}/J'_i = \widehat{\mathcal{O}}_{x'_i}/\widehat{J'_i}$. By the hypothesis of the proposition we are proving, we thus have $l(D(\widehat{M'_{x'_i}}/\widehat{J'_i}\widehat{M'_{x'_i}})) = l(\widehat{\mathcal{O}}_{x'_i}/\widehat{J'_i})$. The lemma then results from the additivity of the functor D and of l. □

Lemma 3.2.6. — ([**73**, Lemma 10.7]) [7] *The morphism $\widehat{A} \longrightarrow \widehat{A}'$ is injective.*

By (45), we have a commutative diagram

(46)
$$\begin{tikzcd} \widehat{M} \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left] & \widehat{M} \otimes_{\widehat{A}} \widehat{A}' \arrow[r, two heads] \arrow[dr] & \widehat{M}' \arrow[d, hook] \\ \widehat{A}^n \arrow[rr, hook] & & \widehat{A}'^n \end{tikzcd}$$

The injectivity of the vertical maps is obvious from (45) and the injectivity of the horizontal lower map is Lemma 3.2.6. The injectivity of the upper horizontal map follows.

The following lemma is an application of Chevalley's Theorem (cf. [**83**, ex 8.6]) which we recall: let $\widehat{A}$ be a complete noetherian local ring, $\widehat{M}$ a finite type $\widehat{A}$-module, $\widehat{I}$ a cofinite length ideal of $\widehat{A}$ and $\widehat{M}_n$ a decreasing, exhaustive (that is $\cap_n \widehat{M}_n = (0)$) sequence of submodules of $\widehat{M}$. Then for n big enough, $\widehat{M}_n \subset \widehat{IM}$.

Now we go back to the proof of Proposition 3.2.3. Let I be a cofinite length ideal of A, and note $\widehat{I} \subset \widehat{A}$ its completion. We recall also that $\widehat{M} \subset \widehat{M}'$ by diagram (46).

Lemma 3.2.7. — *There exist a cofinite length ideal $\widehat{J} \subset \widehat{I}$ of $\widehat{A}$ and an open ideal $\widehat{J}'$ of $\widehat{A}'$ such that*

(i) $\widehat{J} = \widehat{J}' \cap \widehat{A}$,

(ii) $(\widehat{J}'\widehat{M}' \cap \widehat{M}) \subset \widehat{IM}$.

Proof. — We let $\widehat{J}'_n := (\prod_{x' \in \pi^{-1}(x)} \widehat{m}^n_{x'}) \subset \widehat{A}'$. By Krull's theorem, $\cap_n \widehat{J}'_n = 0$ and $\cap_n(\widehat{J}'_n \widehat{M}') = 0$.

We set $\widehat{J}_n := \widehat{J}'_n \cap \widehat{A}$, the intersection being in $\widehat{A'}$. Similarly, we set $\widehat{M}_n = (\widehat{J}'_n \widehat{M}') \cap \widehat{M}$, the intersection being in $\widehat{M}'$. Then $\cap_n \widehat{J}_n = 0$ and $\cap_n \widehat{M}_n = 0$.

By Chevalley's theorem, applied twice, once to the finite module $\widehat{M}$ over the local complete noetherian ring $\widehat{A}$, and once to $\widehat{A}$ as a module over itself, we know that for n big enough, $\widehat{M}_n \subset \widehat{IM}$, and $\widehat{J}_n \subset \widehat{I}$.

We fix such an n. We set $\widehat{J} := \widehat{J}_n$. It is clear that $\widehat{J}$ is of cofinite length since it contains $\widehat{m}^n_x$. We thus have by construction $\widehat{J} \subset \widehat{I}$, $\widehat{J} = \widehat{J}'_n \cap \widehat{A}$ and $(\widehat{J}'_n \widehat{M}') \cap \widehat{M} \subset \widehat{IM}$.

However, $\widehat{J}'_n$ is not open. If F is a finite subset of $\pi^{-1}(x)$, we let $\widehat{J}'_F$ be the ideal $\prod_{x' \in F} \widehat{m}^n_{x'} \times \prod_{x' \in \pi^{-1}(x) - F} \widehat{\mathcal{O}}_{x'}$ of A'. It is clear that $\widehat{J}'_n = \cap_F \widehat{J}'_F$, and that the $\widehat{J}'_F$ are open ideals of A'. Because $\widehat{A}/\widehat{J}$ and $\widehat{M}/\widehat{M}_n$ are artinian, there is a finite F such that $\widehat{J}'_F \cap \widehat{A} = \widehat{J}$ and $\widehat{J}'_F \widehat{M}' \cap \widehat{M} = \widehat{M}_n$. We set $\widehat{J}'$ equal to this $\widehat{J'}_n$ and we are done. □

[7] As stated there, the lemma assumes that π is a blow-up, but the proof only uses that π is proper and birational.

By (i) of this lemma, the morphism of $\widehat{A}[G]$-modules $\widehat{M} \longrightarrow \widehat{M'}$ induces a morphism of $(\widehat{A}/\widehat{J})[G]$-modules

$$f : \widehat{M}/\widehat{J}\widehat{M} \longrightarrow \widehat{M'}/\widehat{J'}\widehat{M'}.$$

Indeed, the image of $\widehat{J}\widehat{M} \subset \widehat{M}$ in $\widehat{M'}$ is included in $\widehat{J}\widehat{M'}$ which is included in $\widehat{J'}\widehat{M'}$.

We shall denote by K, C and Q the kernel, cokernel and image of f, respectively. Thus we have two exact sequences of $(\widehat{A}/\widehat{J})[G]$-modules:

$$0 \longrightarrow K \longrightarrow \widehat{M}/\widehat{J}\widehat{M} \longrightarrow Q \longrightarrow 0 \tag{47}$$

$$0 \longrightarrow Q \longrightarrow \widehat{M'}/\widehat{J'}\widehat{M'} \longrightarrow C \longrightarrow 0 \tag{48}$$

Note that the five modules involved here are all of finite length as $\widehat{A}/\widehat{J}$-modules.

Lemma 3.2.8. — *As an $\widehat{A}[G]$-module, C is a quotient of $(\widehat{M}/\widehat{J}\widehat{M}) \otimes_{\widehat{A}} (\widehat{A'}/\widehat{J'})/(\widehat{A}/\widehat{J})$.*

Proof. — This is formal. Indeed, we have a commutative diagram, where the vertical arrows are surjective:

$$\begin{tikzcd}
\widehat{M} \arrow[r] \arrow[d] & \widehat{M} \otimes_{\widehat{A}} \widehat{A'} \arrow[r, "s"] \arrow[d] & \widehat{M'} \arrow[d] \\
\widehat{M}/\widehat{J}\widehat{M} \arrow[r, "g"] \arrow[rr, bend right, "f"'] & \widehat{M} \otimes_{\widehat{A}} \widehat{A'}/\widehat{J'} \arrow[r, "h"] & \widehat{M'}/\widehat{J'}\widehat{M'}
\end{tikzcd}$$

This diagram makes clear that h is surjective, since s is. Hence the cokernel C of f is a quotient of the cokernel of

$$g : \widehat{M}/\widehat{J}\widehat{M} \longrightarrow \widehat{M} \otimes_{\widehat{A}} \widehat{A'}/\widehat{J'} = (\widehat{M}/\widehat{J}\widehat{M}) \otimes_{\widehat{A}} \widehat{A'}/\widehat{J'}$$

and this cokernel is, by right-exactness of the tensor product by $\widehat{M}/\widehat{J}\widehat{M}$, the module $(\widehat{M}/\widehat{J}\widehat{M}) \otimes_{\widehat{A}} (\widehat{A'}/\widehat{J'})/(\widehat{A}/\widehat{J})$. $\square$

We now prove an abstract lemma concerning the left-exact functor D and length of modules.

Lemma 3.2.9. — *Let V be an A-module of finite length with a continuous action of G, such that*

$$l(D((V \otimes k)^{\mathrm{ss}}) \leq 1.$$

Let N be an A-module of finite length[8] *and $\pi : V \otimes_A N \longrightarrow Q$ a surjective $A[G]$-linear map.*

(i) $l(D(Q)) \leq l(N)$.

[8] We view it as a G-module for the trivial action.

(ii) *Assume that equality holds in* (i), *and that there is a surjective map of A-modules $N \longrightarrow N'$ such that the natural induced surjection $V \otimes_A N \longrightarrow V \otimes_A N'$ factors through π. Then $l(D(V \otimes_A N')) = l(N')$.*

(iii) *Let J be a cofinite length ideal of A. If $l(D(V/JV)) = l(A/J)$, then for each ideal $J' \supset J$, $l(D(V/J'V)) = l(A/J')$.*

Proof. — First remark that the hypothesis $l(D((V \otimes k)^{\mathrm{ss}}) \leq 1$ implies, by left exactness of D, that $l(D(U)) \leq 1$ for any $k[G]$-module U which is a subquotient of $V \otimes k$. Indeed, U^{ss} is a $k(x)[G]$-submodule of $(V \otimes k)^{\mathrm{ss}}$, so

$$l(D(U)) \leq l(D(U^{\mathrm{ss}})) \leq l(D((V \otimes k)^{\mathrm{ss}})) \leq 1.$$

Let us prove (i). There is a filtration $N_0 \subset \cdots \subset N_i \subset \cdots \subset N_{l(N)} = N$ of N such that $N_i/N_{i-1} \simeq k$. We denote by VN_i the image of $V \otimes_A N_i$ into $V \otimes_A N$ and by Q_i the image of VN_i in Q. It is clear that VN_i/VN_{i-1} is a quotient of $V \otimes k$, and that Q_i/Q_{i-1} is a quotient of VN_i/VN_{i-1}, hence we have $l(D(Q_i/Q_{i-1})) \leq 1$ by the remark beginning the proof. By left exactness of D, this proves (i). Note also that if $l(D(Q)) = l(N)$, all the inequalities above have to be equalities, so that $l(D(Q_i)) = i$ for each i.

Let us prove (ii). In the proof of (i) above, we can certainly choose the N_i such that one of them, say N_k, is the kernel of the surjection $N \longrightarrow N'$. Then $k = l(N') - l(N)$. We have an exact sequence $0 \longrightarrow VN_k \longrightarrow V \otimes N \longrightarrow V \otimes N' \longrightarrow 0$, hence (using the hypothesis) an exact sequence

$$0 \longrightarrow Q_k \longrightarrow Q \longrightarrow V \otimes N' \longrightarrow 0.$$

Because D is left exact, we have $l(D(V \otimes N')) \geq l(D(Q)) - l(D(Q_k))$. But by hypothesis, we have $l(D(Q)) = l(N)$, which implies by the remark at the end of the proof of (i) that $l(D(Q_k)) = k$. Hence

$$l(D(V \otimes N')) \geq l(N) - k = l(N) - (l(N) - l(N')) = l(N').$$

The other equality follows from (i), hence $l(D(V \otimes N')) = l(N')$.

The assertion (iii) is a special case of (ii): apply (ii) to $N = A/J$, $Q = V \otimes_A N = V/JV$, $\pi = \mathrm{Id}$ and $N' = A/J'$. □

Going back to the proof of the Proposition 3.2.3 we get the following lemma.

Lemma 3.2.10. — *We have*

(i) $l(D(C)) \leq l(\widehat{A}'/\widehat{J}') - l(\widehat{A}/\widehat{J})$,

(ii) $l(D(Q)) = l(\widehat{A}/\widehat{J})$,

(iii) $l(D(\widehat{M}/\widehat{I}\widehat{M})) = l(\widehat{A}/\widehat{I})$.

Proof. — Lemma 3.2.8 tells us that C is a quotient of the module

$$(\widehat{M}/\widehat{J}\widehat{M}) \otimes_{\widehat{A}} (\widehat{A}'/\widehat{J}')/(\widehat{A}/\widehat{J}).$$

We now apply the point (i) of Lemma 3.2.9 to $V = \widehat{M}/\widehat{J}\widehat{M}$ and $N = (\widehat{A}'/\widehat{J}')/(\widehat{A}/\widehat{J})$. We note that $V \otimes_{\widehat{A}} k$, that is $\widehat{M} \otimes_{\widehat{A}} k = M \otimes_A k$, satisfies the hypothesis of Lemma 3.2.9 by hypothesis. So $l(D(C)) \leq l(D(N))$, hence (i).

To prove (ii) note that by the exact sequence (48),

$$\begin{aligned} l(D(Q)) &\geq l(D(\widehat{M}'/\widehat{J}'\widehat{M}')) - l(D(C)) \\ &\geq l(D(\widehat{M}'/\widehat{J}'\widehat{M}')) - l(\widehat{A}'/\widehat{J}') + l(\widehat{A}/\widehat{J}), \text{ by (i).} \end{aligned}$$

Since $l(D(\widehat{M}'/\widehat{J}'\widehat{M}')) = l(\widehat{A}'/\widehat{J}')$ by the Lemma 3.2.5 we get

$$l(D(Q)) \geq l(\widehat{A}/\widehat{J}).$$

To get the other inequality, recall that Q is by construction a quotient of $\widehat{M}/\widehat{J}\widehat{M} = \widehat{M} \otimes_{\widehat{A}} \widehat{A}/\widehat{J}$, so by point (i) of Lemma 3.2.9 we have $l(D(Q)) \leq l(\widehat{A}/\widehat{J})$.

Let us prove (iii). Assertion (ii) of Lemma 3.2.7 tells that $\widehat{M}/\widehat{J}\widehat{M} \longrightarrow \widehat{M}/\widehat{I}\widehat{M}$ factors through the canonical surjection $\widehat{M}/\widehat{J}\widehat{M} \longrightarrow Q$. We apply point (ii) of Lemma 3.2.9 to Q, with $V = \widehat{M}/\widehat{J}\widehat{M}$, $N = \widehat{A}/\widehat{J}$, $N' = \widehat{A}/\widehat{I}$. This is possible because $l(D(Q)) = l(N)$ by (ii) above, and that gives us $l(D(V \otimes N')) = l(N')$, which is (iii). □

Now recall that since I is of cofinite length, $A/I \simeq \widehat{A}/\widehat{I}$ and $M/IM \simeq \widehat{M}/\widehat{I}\widehat{M}$. Hence by (iii) of Lemma 3.2.10 above,

$$l(D(M/IM)) = l(A/I).$$

The proof of Proposition 3.2.3 is complete.

3.3. Direct generalization of a result of Kisin

3.3.1. Notations and definitions. — We keep the general notations of paragraph 3.2.1. We fix p a prime number and set

$$G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Recall that a subset $Z \subset X$ is said to be Zariski-dense if the only analytic subset of X containing Z is X_{red} itself. We shall need below some arguments involving the notion of irreducible component of a rigid analytic space, for which we refer to [**47**].

We will say that a subset $Z \subset X$ *accumulates* at $x \in X$, or that x *is an accumulation point of* Z, if there is a basis of affinoid neighborhoods U of x such that $U \cap Z$ is Zariski-dense in U. We say moreover that a subset $Z \subset X$ is an *accumulation subset* if for any $z \in Z$ then Z accumulates at z.

We shall use the notation $X(\overline{\mathbb{Q}}_p)$ as a shortcut for the union of the $X(K)$ for all $K \subset \overline{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$. By a slight abuse of language, we shall sometimes say that a subset $Z \subset X(\overline{\mathbb{Q}}_p)$ is Zariski-dense, or accumulates at some point. By this we shall always mean that the subset $|Z| \subset X$ consisting of the underlying closed points of Z has this property.

3.3.2. Hypotheses. — We assume that we are given a couple of functions $(F, \kappa) \in \mathcal{O}(X)^* \times \mathcal{O}(X)$, and a Zariski-dense subset $Z \subset X$ satisfying the following conditions.

(CRYS) For $z \in Z$, $\overline{\mathcal{M}}_z$ is a crystalline representation of G_p whose smallest Hodge-Tate weight is $\kappa(z) \in \mathbb{Z}$, and that satisfies $D_{\mathrm{crys}}(\overline{\mathcal{M}}_z)^{\varphi=p^{\kappa(z)}F(z)} \neq 0$.

(HT) For any non-negative integer C, if Z_C denotes the subset of $z \in Z$ such that the Hodge-Tate weights of $\overline{\mathcal{M}}_z$ other than $\kappa(z)$ are bigger that $\kappa(z) + C$, then Z_C accumulates at any point of Z.

Remark 3.3.1. — The assumption (HT) together with the Zariski-density of Z in X imply that Z accumulates at each $z \in Z$. This stronger density condition on Z, introduced in [**37**] under the terminology "Z is *very Zariski-dense* in X", turns out to be rather well-behaved and allows to avoid some pathological Zariski-dense subsets. [9]

For some technical reasons, we shall also need to know that:

(*) There exists a continuous character $\mathbb{Z}_p^* \longrightarrow \mathcal{O}(X)^*$ whose derivative at 1 is the map κ and whose evaluation at any point $z \in Z$ is the elevation to the $\kappa(z)$-th power.

Condition (*) allows us to define by composition with the cyclotomic character χ a continuous character

$$\psi : G_p \xrightarrow[\chi]{\sim} \mathbb{Z}_p^* \longrightarrow \mathcal{O}(X)^*$$

whose evaluation of at any point $z \in Z$ is then the $\kappa(z)$-th power of the cyclotomic character (whence crystalline).

Definition 3.3.2. — We shall often denote by $\kappa : G_p \longrightarrow \mathcal{O}(X)^*$ the character ψ defined above, and if N is any sheaf of $\mathcal{O}[G_p]$-modules on X, we will also denote by $N(\kappa)$ the $\mathcal{O}$-module N whose G_p-action is twisted by the character ψ.

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(9) As an exercise, the reader can check that there are Zariski-dense subsets of $\mathbb{A}^2$ whose intersection with any affinoid subdomain $V \subset \mathbb{A}^2$ is not Zariski-dense in V. However, if Z is a very Zariski dense subset of a rigid space X, then for any irreducible component T of X there is an open affinoid of T in which Z is Zariski-dense.

3.3.3. The finite slope subspace X_{fs}. — The arguments in this part will follow closely Kisin's paper [**73**, §5]. Under the extra assumption that $\mathcal{M}$ is a free $\mathcal{O}_X$-module, Kisin defines in [**73**, Prop. 5.4] a canonical Zariski closed subspace

$$X_{fs} \subset X,$$

that he calls *"the finite slope subspace"*, which is attached to the $\mathcal{O}_X$-module $\mathcal{M}(\kappa)$ and the function $F \in \mathcal{O}(X)^*$. The properties of X_{fs} are rather technical and we shall not repeat them here (see [**73**, Prop. 5.4]).

Under the weaker assumption that $\mathcal{M}$ is locally free, we claim that there exists a unique Zariski closed subspace $Y \subset X$ such that for any admissible open $U \subset X$ on which $\mathcal{M}$ is free we have $Y \cap U = U_{fs}$, where U_{fs} is the finite slope subspace of U attached as above to $(\mathcal{M}(\kappa)_{|U}, F_{|U})$. As X is admissibly covered by such admissible open subspaces, Y is unique if it exists. For the same reason, Y exists if and only if for any pair of admissible open $U, V \subset X$ on which $\mathcal{M}$ is free, we have

$$U_{fs} \cap V = (U \cap V)_{fs}.$$

Replacing V by $U \cap V$, we may assume that $V \subset U$, and then this equality follows from the last assertion of [**73**, Prop. 5.4] applied to the (flat) open immersion $V \to U$. This shows the existence of Y. When $\mathcal{M}$ is free then $Y = X_{fs}$ by definition, hence it is harmless to set $X_{fs} := Y$ in the general case as well. Our first aim will be to show that $X_{fs} = X$.

Theorem 3.3.3. — *Assume $\mathcal{M}$ is locally free.*

(i) *For all $x \in X$, then $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x(\kappa(x)))^{\varphi=F(x)}$ is non zero. Moreover, $X_{fs} = X$.*

(ii) *Let $x \in X$ and assume that $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}^{\mathrm{ss}}_x(\kappa(x)))^{\varphi=F(x)}$ has $k(x)$-dimension 1. Then for all ideal I of cofinite length of $\mathcal{O}_x$, $D^+_{\mathrm{crys}}((\mathcal{M}_x/I\mathcal{M}_x)(\kappa))^{\varphi=F}$ is free of rank 1 over $\mathcal{O}_x/I$.*

Remark 3.3.4. — Part (i) of this theorem is a combination of results of Kisin in [**73**]. Moreover, he proved *loc. cit.* some cases of part (ii), essentially the cases where $\overline{\mathcal{M}}_x$ is an indecomposable $k(x)[G_p]$-module (although it is not stated explicitly, this is done during the proof of Proposition 10.6 of [**73**], page 444 and 445). The proof we give here simplifies a bit some arguments of [**73**, section 8] and avoids all use of universal deformation rings, using some length arguments and our lemma of descent by blow-up instead. It also paves the way for the proof of Theorem 3.4.1 below.

Proof. — By replacing $\mathcal{M}$ by $\mathcal{M}(\kappa)$, we may assume that $\kappa = 0$. Let

$$TQ(T) \in \mathcal{O}(X)[T]$$

be the Sen polynomial of $\mathcal{M}$ (see [**108**]), whose roots at $x \in X$ are the generalized Hodge-Tate weights of $\overline{\mathcal{M}}_x$. Let $W \subset X$ denote the subset consisting of the points

$x \in X$ such that the Sen polynomial of $\overline{\mathcal{M}}_x$ has 0 as unique root in the integers $\mathbb{N}$ (and which is a simple root).

Lemma 3.3.5. — *For each admissible open U of X, $W \cap U$ is Zariski-dense in U.*

Proof. — For each $k \geq 0$, and $U \subset X$ admissible open, let U_k denotes the (reduced) zero locus of $Q(k)$ in U, so

$$W \cap U = U - \bigcup_{k \geq 0} U_k.$$

Let T be a closed analytic subset of U such that $U = T \cup \bigcup_{k\geq 0} U_k$. Let T' be any irreducible component of U. If $T' \not\subset T$, then $T' \subset U_k$ for some k by [**73**, Lemma 5.7]. Let T'' be an irreducible component of X such that $T'' \cap U \supset T'$, then $T'' \subset X_k$, which is not possible by (HT) applied to $C = k+1$. Hence $T = U$, which proves the lemma. □

To prove that $X_{fs} = X$ it suffices to show (as Kisin does to prove his Theorem 6.3) that

Lemma 3.3.6. — *The set $\{x \in W,\ D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x)^{\varphi = F(x)} \neq 0\}$ is Zariski-dense in X.*

Indeed, by Tate's computation of the cohomology of $\mathbb{C}_p(i)$ for $i \in \mathbb{Z}$, the natural map

$$D^+_{\mathrm{DR}}(\overline{\mathcal{M}}_x) \longrightarrow (\overline{\mathcal{M}}_x \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_p}$$

is an isomorphism between $k(x)$-vector-spaces of dimension 1 when $x \in W$. In particular, if x is in the subset of Lemma 3.3.6, the natural injection

$$D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x)^{\varphi = F(x)} \longrightarrow D^+_{\mathrm{DR}}(\overline{\mathcal{M}}_x)$$

is an isomorphism, hence $x \in X_{fs}$.

Proof. — Let us fix first some $z \in Z$ and choose an open affinoid $U \subset X$ containing z which is small enough so that $\mathcal{M}$ is free over U, U is F-small ([**73**, (5.2)]), and such that Z is Zariski-dense in U (it exists by (HT)). Assumption (HT) implies then that $Z_C \cap U$ is Zariski-dense in U for any C.

We now apply [**73**, Prop. 5.14] and its corollary [**73**, Cor 5.15] to $\mathcal{R} := \mathcal{O}(U)$, $M := \mathcal{M}_{|U}$, $I := Z \cap U$, $\mathcal{R}_i := k(i)$ and $I_k := Z_{k+\sup_U |F|+1}$. Note that we just checked condition (3) there (that is, I_k is Zariski-dense in U) and that condition (2) follows from our assumption (ii). Moreover, condition (1) follows from (CRYS) and the weak admissibility of $D_{\mathrm{crys}}(\overline{\mathcal{M}}_x)$, $x \in I_k$, applied to the filtered φ-submodule $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x)^{\varphi=F(x)}$. As a consequence, [**73**, cor. 5.15] tells that for all $x \in U$, $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x)^{\varphi = F(x)} \neq 0$. We conclude the proof by Lemma 3.3.5. □

Applying now [**73**, cor. 5.16], we first get the point (i) of our Theorem 3.3.3.

Remark 3.3.7. — (i) We note the extreme indirectness of this method of proof (which is entirely Kisin's): to prove that $D^+_{\mathrm{crys}}(\overline{\mathcal{M}_x})^{\varphi=F(x)} \neq 0$ for every $x \in X$, knowing that this is true for the points of Z, we use the closed set X_{fs}, which by definition contains the points satisfying this properties provided that they are in the set W - in particular, not in Z!

(ii) The proof of Lemma 3.3.6 shows that if X is an affinoid space, F-small, on which $\mathcal{M}$ is free, then in the proof of point (i) of Theorem 3.3.3 condition (HT) may be replaced by the weaker condition

(HT') : for every non negative C, Z_C is Zariski-dense in X.

We now prove point (ii) of our Theorem 3.3.3. Let us fix some $x \in X$ (but not necessarily in Z) and choose an F-small open affinoid neighborhood U of x such that $\mathcal{M}$ is free over U. As $U \subset X = X_{fs}$, by the corollary *loc. cit.* we get that $D^+_{\mathrm{crys}}(\mathcal{M}(U))^{\varphi=F}$ is generically free of rank 1 over $\mathcal{O}(U)$. More precisely, if $H \subset \mathcal{O}(U)$ denotes the smallest ideal such that [10]

$$D^+_{\mathrm{crys}}(\mathcal{M}(U))^{\varphi=F} \subset H(\mathcal{M}(U)\widehat{\otimes}_{\mathbb{Q}_p}B^+_{\mathrm{crys}}),$$

then $U - V(H)$ is Zariski-dense in U. Let

$$\pi : U' \longrightarrow U,$$

be the blow-up of the ideal H and $\mathcal{M}'$ the pullback of $\mathcal{M}$ on U'.

Lemma 3.3.8. — *Let $x' \in U'$ and let $V \subset U'$ be a sufficiently small open affinoid containing x'.*

(i) *The ideal of $\mathcal{O}(V)$ generated by all the coefficients (see the footnote 10) of $D^+_{\mathrm{crys}}(\mathcal{M}'(V))^{\varphi=F} \subset \mathcal{M}'(V)\widehat{\otimes}_{\mathbb{Q}_p}B^+_{\mathrm{crys}}$ is $\mathcal{O}(V)$ itself.*

(ii) *If I' is a cofinite length ideal of $\mathcal{O}_{x'}$ then $D^+_{\mathrm{crys}}(\mathcal{M}'_{x'}/I'\mathcal{M}'_{x'})^{\varphi=F}$ is free of rank 1 over $\mathcal{O}_{x'}/I'$.*

Proof. — By the universal property of blow-ups, for V sufficiently small $H\mathcal{O}(V)$ is a principal ideal generated by a non zero divisor f_V of $\mathcal{O}(V)$. As a consequence, the ideal of the statement is $\mathcal{O}(V)$ itself, as it contains $H\mathcal{O}(V)/f_V$. Indeed, it is clear that if $D^+_{\mathrm{crys}}(\mathcal{M}'(V))^{\varphi=F}$ contains fv for some non zero divisor $f \in \mathcal{O}(V)$ and $v \in \mathcal{M}'(V)\widehat{\otimes}_{\mathbb{Q}_p}B^+_{\mathrm{crys}}$, it contains v. This proves (i).

[10] The Banach $\mathcal{O}(U)$-module $\mathcal{M}(U)\widehat{\otimes}_{\mathbb{Q}_p}B^+_{\mathrm{crys}}$ is ON-able, H is the ideal of $\mathcal{O}(U)$ generated by all the coefficients in a given ON-basis of all the elements of $D^+_{\mathrm{crys}}(\mathcal{M}(U))^{\varphi=F}$. It does not depend on the choice of the ON-basis as the ideals of $\mathcal{O}(U)$ are all closed.

It follows that the natural map $D^+_{\mathrm{crys}}(\mathcal{M}'_{x'}/I'\mathcal{M}'_{x'})^{\varphi=F} \longrightarrow D^+_{\mathrm{crys}}(\overline{\mathcal{M}}'_{x'})^{\varphi=F}$ is non-zero. Moreover $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}'_{x'})^{\varphi=F} = D^+_{\mathrm{crys}}(\overline{\mathcal{M}}_x)^{\varphi=F} \otimes_{k(x)} k(x')$, hence it has $k(x')$-dimension 1 by assumption on $\overline{\mathcal{M}}_x$ and part (i) of Theorem 3.3.3. So the first assertion of the following lemma (applied to $D = D^+_{\mathrm{crys}}(-)^{\varphi=F}$, $A = \mathcal{O}_{x'}$, $J = I'$, $V = \mathcal{M}'_{x'}/I'\mathcal{M}'_{x'}$) implies the result. □

Lemma 3.3.9. — *Let J be a cofinite length ideal of A, V a continuous $(A/J)[G_p]$-module that is free of finite rank over A/J and such that $D(V \otimes_A k)$ has k-dimension 1. Assume moreover that one of the following two conditions holds:*

(i) *$D(V) \longrightarrow D(V \otimes_A k)$ is non-zero,*

(ii) *$l(D(V)) = l(A/J)$.*

Then $D(V)$ is free of rank one over A/J.

Proof. — Under assertion (i), the lemma is exactly [**73**, Lemma 8.6]. Under assertion (ii), it can be proved using similar ideas: we prove that $D(V \otimes_A A/J')$ is free of rank one over A/J' for any ideal J' containing J, by induction on the length of A/J'. There is nothing to prove for $J' = m$. Assume the result known for ideals of colength $< k$, and let J' be an ideal containing J of colength k. Let J'' be an ideal such that $J' \subset J'' \subset m$, the first inclusion being proper and of colength one. We have (since $V \otimes_A A/J'$ is free over A/J') an exact sequence:

$$0 \longrightarrow D(V \otimes_A k) \otimes_k J''/J' \longrightarrow D(V \otimes_A A/J') \longrightarrow D(V \otimes_A A/J'').$$

By (iii) of Lemma 3.2.9, $l(D(V \otimes_A A/J')) = l(A/J')$ and similarly for J''. Hence the last morphism of the exact sequence above is surjective. So we have $D(V \otimes_A A/J') \otimes_A A/J'' = D(V \otimes_A A/J'')$, hence $D(V \otimes_A A/J') \otimes_A k = D(V \otimes_A A/J'') \otimes_A k$. By induction, the latter has k-dimension 1. Hence by Nakayama's lemma, the A/J'-module $D(V \otimes_A A/J')$ is generated by a single element and since its length is $l(A/J')$, it is free of rank one over A/J'. □

We can now use our "descent result" (Proposition 3.2.3) for the blow-up $\pi : U' \longrightarrow U$. Assertion (ii) of Lemma 3.3.8 shows that for every $x' \in \pi^{-1}(x)$, and every cofinite length ideal I' of $\mathcal{O}_{x'}$,

$$l\left(D^+_{\mathrm{crys}}(\mathcal{M}'_{x'} \otimes_{\mathcal{O}_{x'}} \mathcal{O}_{x'}/I')^{\varphi=F}\right) = l(\mathcal{O}_{x'}/I').$$

Thus by Proposition 3.2.3, we have for every cofinite length ideal I of $\mathcal{O}_x$,

$$l\left(D^+_{\mathrm{crys}}(\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/I)^{\varphi=F}\right) = l(\mathcal{O}_x/I).$$

To conclude that $D^+_{\mathrm{crys}}(\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/I)^{\varphi=F}$ is free of rank one over $\mathcal{O}_x/I$ we simply invoke Lemma 3.3.9 (ii) with $I = J, V = \mathcal{M}_x/J\mathcal{M}_x$. The proof of Theorem 3.3.3 is now complete. □

3.4. A generalization of Kisin's result for non-flat modules

In this subsection we keep the assumptions of § 3.3.2, but we do not assume that $\mathcal{M}$ is locally free, but only that $\mathcal{M}$ is torsion-free.

Theorem 3.4.1. — *Let $x \in X$ and assume that*[11] $D^+_{\mathrm{crys}}(\overline{\mathcal{M}}^{\mathrm{ss}}_x(\kappa(x)))^{\varphi=F(x)}$ *has $k(x)$-dimension* 1. *Then for all ideal I of cofinite length of $\mathcal{O}_x$,*

$$l\left(D^+_{\mathrm{crys}}(\mathcal{M}_x/I\mathcal{M}_x(\kappa))^{\varphi=F}\right) = l(\mathcal{O}_x/I).$$

We will rely on the following flatification result whose scheme theoretic analogue is an elementary case of a result of Gruson-Raynaud ([**95**, Thm. 5.2.2]). Recall that X is reduced and separated.

Lemma 3.4.2. — *There exists a proper and birational morphism $\pi : X' \longrightarrow X$ (with X' reduced) such that the strict transform of $\mathcal{M}$ by π is a locally free coherent sheaf of modules $\mathcal{M}'$ on X'. More precisely, we may choose π to be the blow-up of the normalization $\tilde{X}$ of X along a nowhere dense closed subspace.*

Proof. — Let $f : \tilde{X} \longrightarrow X$ be the normalization of X (see [**47**, §2.1]), then $\tilde{X}$ is reduced, f is finite (hence proper), and f is birational by [**47**, Thm. 2.1.2]. Moreover, the strict transform $\mathcal{M}'$ of $\mathcal{M}$ by f is torsion free as $\mathcal{M}$ is, hence by replacing $(X, \mathcal{M})$ by $(\tilde{X}, \mathcal{M}')$ we may assume that X is normal. We may also assume that X is connected.

We claim that there is an integer $r \geq 0$ such that for each open affinoid $U \subset X$, $\mathcal{M}(U)$ is generically free of rank r over $\mathcal{O}(U)$. If U is connected (hence irreducible), let us denote by r_U this generic rank. There is an injective $\mathcal{O}(U)$-linear map $\mathcal{M}(U) \longrightarrow \mathcal{O}(U)^{r_U}$ which is an isomorphism after inverting some $f \neq 0 \in \mathcal{O}(U)$. In particular, for each x in a Zariski-open subset of U, we have $\mathcal{M}_x \xrightarrow{\sim} \mathcal{O}_x^{r_U}$. As a consequence, for each open affinoid $U' \subset U$, the $\mathcal{O}_x$-module $\mathcal{M}_x$ is free of rank r_U on a Zariski open and dense subset of U', thus $r_{U'} = r_U$ if U' is connected. A connectedness argument shows then that r_U is independent of $U \subset X$, and the claim follows. In particular, for all $x \in X$ the torsion free $\mathcal{O}_x$-module $\mathcal{M}_x$ has also generic rank r.

Let us recall now some facts about the Fitting ideals (see [**77**, XIX,§2], [**95**, §5.4]). For each open affinoid $U \subset X$ it makes sense to consider the r-th Fitting ideal $F_r(\mathcal{M}(U))$ of the finite $\mathcal{O}(U)$-module $\mathcal{M}(U)$. Its formation commutes with any affinoid open immersion so those $\{F_r(\mathcal{M}(U))\}$ glue to a coherent sheaf of ideals $F_r(\mathcal{M}) \subset \mathcal{O}_X$. A point $x \in X$ lies in $V(F_r(\mathcal{M}))$ if and only if $\dim_{k(x)}(\overline{\mathcal{M}_x}) > r$ and $X - V(F_r(\mathcal{M}))$ is the biggest admissible open subset of X on which $\mathcal{M}$ can be locally generated (on stalks) by r elements. By what we saw in the paragraph above, $X - V(F_r(\mathcal{M}))$ is

[11] In fact, the result holds more generally under the assumption of Remark 3.2.4, but we state it as such for short.

actually Zariski dense in X. Moreover, if $x \in X - V(F_r(\mathcal{M}))$ then $\mathcal{M}_x$ is free of rank r over $\mathcal{O}_x$. Indeed, it can be generated by r elements and we saw that

$$\mathcal{M}_x \subset \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathrm{Frac}(\mathcal{O}_x) \xrightarrow{\sim} \mathrm{Frac}(\mathcal{O}_x)^r$$

for each $x \in X$, and we are done.

Let $\pi : X' \longrightarrow X$ be the blow-up of $F_r(\mathcal{M})$, we will eventually prove that π has all the required properties. Note that X' is reduced as X is and that π is birational as $X - V(F_r(\mathcal{M}))$ is Zariski dense in X. As a general fact, the coherent sheaf of ideals $F_r(\mathcal{M})\mathcal{O}_{X'}$ coincides with the r-th sheaf of Fitting ideals $F_r(\pi^*\mathcal{M})$ of $\pi^*\mathcal{M}$, and it is an invertible sheaf by construction. Let $Q \subset \pi^*\mathcal{M}$ be the coherent subsheaf of $F_r(\mathcal{M})\mathcal{O}_{X'}$-torsion of $\pi^*\mathcal{M}$. We claim that $(\pi^*\mathcal{M})/Q$ is locally free of rank r. This can be checked on the global sections on an open affinoid $U \subset X'$. But if A is a reduced noetherian ring and M a finite type A-module such that M is generically free of rang r and whose r-th Fitting ideal $F_r(M)$ is invertible, then $M/\mathrm{Ann}_M(F_r(M))$ is locally free of rank r by [**95**, Lemma 5.4.3]. This proves the claim if we take $A = \mathcal{O}(U)$ and $M = \pi^*(\mathcal{M})(U)$.

By definition, the strict transform $\mathcal{M}'$ of $\mathcal{M}$ is the quotient of $\pi^*\mathcal{M}$ by its $(F_r(\mathcal{M})\mathcal{O}_{X'})^\infty$-torsion. The natural surjective morphism $\pi^*\mathcal{M} \to \mathcal{M}'$ factors then through $(\pi^*\mathcal{M})/Q$, which is locally free of rank r by what we just proved, so $(\pi^*\mathcal{M})/Q \xrightarrow{\sim} \mathcal{M}'$ is locally free of rank r, and we are done. □

Proof. — (of Theorem 3.4.1) Let us choose a π as in Lemma 3.4.2, as well as a coherent sheaf of ideals $H \subset \mathcal{O}_X$ attached to π as in § 3.2.3. As $X - V(H)$ is Zariski dense in X, and as Z accumulates at Z by assumption (HT), $Z \cap (X - V(H))$ is Zariski-dense in X. Moreover (CRYS), (HT) and (*) are still satisfied when we replace Z by $Z \cap (X - V(H))$ in their statement, so we may assume that $Z \cap V(H) = \varnothing$.

Let us denote by Z' the set of $z' \in X'$ such that $\pi(z') \in Z$. Since $X' - \pi^{-1}(V(H)) \xrightarrow{\sim} X - V(H)$ is Zariski-dense in X', Z' is Zariski dense in X'. Note that for $z' \in Z'$, we have $\overline{\mathcal{M}'_{z'}} = \overline{\mathcal{M}_z}$ if $z = \pi(z')$. Define κ' and F' on X' as $\kappa \circ \pi$ and $F \circ \pi$. Then it is obvious that $X', Z', \mathcal{M}', F', \kappa'$ satisfy the hypothesis (CRYS), (HT) and (*). Because $\mathcal{M}'$ is locally free we may apply to it Theorem 3.3.3 at any $x' \in X'$. This implies Theorem 3.4.1 by our descent Proposition 3.2.3. □

Remark 3.4.3. — (i) In the applications of Theorem 3.4.1 to Section 4, we will use some coherent sheaves $\mathcal{M}$ on an affinoid X which are in fact direct sums of coherent torsion-free $\mathcal{O}$-modules of generic ranks ≤ 1, for which Lemma 3.4.2 is obvious.

(ii) As Brian Conrad pointed out to us, there is an alternative proof of the first assertion of Lemma 3.4.2 using rigid analytic Quot spaces (see [**48**, Thm. 4.1.3]).

CHAPTER 4

RIGID ANALYTIC FAMILIES OF REFINED p-ADIC REPRESENTATIONS

4.1. Introduction

In this section, we define and study the notion of *p-adic families of refined Galois representations.* As explained in the general introduction, the general framework is the data of a continuous d-dimensional pseudocharacter

$$T : G \longrightarrow \mathcal{O}(X),$$

where X is a reduced, separated, rigid analytic space. Here G is a topological group equipped with a continuous map $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow G$, and we shall be mainly interested in the properties of the restriction of T to G_p. The presence of the group G is an extra structure that will only play a role when discussing the reducibility properties of T, and we invite the reader to assume that $G = G_p$ at a first reading.

We assume that for all z in a *Zariski-dense* subset $Z \subset X$, the semisimple continuous representation $\bar{\rho}_z$ of G whose trace is the evaluation T_z of T at z (see § 1.2.2), has the following properties after restriction to G_p (see § 4.2.3):

(i) $\bar{\rho}_z$ is crystalline,

(ii) its Hodge-Tate weights are distinct, and if we order them by $\kappa_1(z) < \cdots < \kappa_d(z)$, then the maps $z \mapsto \kappa_n(z)$ extend to analytic functions on X and each difference $\kappa_{n+1} - \kappa_n$ varies a lot on Z.

(iii) its crystalline eigenvalues $\varphi_1(z), \dots, \varphi_d(z)$ are distinct, and their *normalized versions* $z \mapsto F_n(z) := \varphi_n(z)p^{-\kappa_n(z)}$ extend to analytic functions on X.

These hypotheses may seem a little bit complicated, but this is because we want them to encode all the aspects of the families of Galois representations arising on eigenvarieties. We refer to § 4.2.3 for a detailed discussion of each assumption. Let us just mention two things here. First, although families with "constant Hodge-Tate weights" have been studied by several people, the study of the kind of families above has been comparatively quite poor, except for works of Sen and Kisin. A reason may be that the very fact that the weights are moving implies that the generic member of such a family is not even a Hodge-Tate representation, and in particular lives outside the De Rham world. Second, each $\bar{\rho}_z$ is equipped by assumption (iii) with a natural ordering of its crystalline Frobenius eigenvalues, that is with a *refinement* $\mathcal{F}_z$ of $\bar{\rho}_z$ (hence the name of the families).

Our aim is the following: we want to give a schematic upper bound of the reducibility loci at the points $z \in Z$ and to prove that the infinitesimal deformations of the ρ_z inside their reducibility loci (that we defined in Section 1) are trianguline, and in favorables cases even Hodge-Tate or crystalline. Let us describe now precisely our results.

Assume first that $z \in Z$ is such that $\bar{\rho}_z$ is irreducible [(1)] and that $\mathcal{F}_z$ is a *non critical regular* refinement of $\bar{\rho}_z$. Then on each thickening A of z in X, we show that $T \otimes A$ is the trace of a unique trianguline deformation of $(\bar{\rho}_z, \mathcal{F}_z)$ to the artinian ring A (Theorem 4.4.1).

When $\bar{\rho}_z$ is reducible, the situation turns out to be much more complicated, but still rather nice in some favorable cases [(2)]. Assume that $\bar{\rho}_z = \oplus_{i=1}^r \bar{\rho}_i$ is multiplicity-free. The relevant combinatorical information contained in the data of that decomposition of $\bar{\rho}_z$ and the refinement $\mathcal{F}_z$ is summarized in a permutation $\sigma \in \mathfrak{S}_d$ that we construct in § 4.4.3. Assume again that $\mathcal{F}_z$ is regular, but not that $\mathcal{F}_z$ is non critical. Instead, we assume only that the refinement $\mathcal{F}_{z,i}$ induced by $\mathcal{F}_z$ on each of its sub-representation $\bar{\rho}_i$ is a non critical refinement, and that each $\mathcal{F}_{z,i}$ is a "subinterval" of $\mathcal{F}_z$ (see § 4.4.4). Our main result concerns then the total reducibility locus, say Red_z, of T at the point z. We show that each difference of weights

$$\kappa_n - \kappa_{\sigma(n)}$$

is constant on this reducibility locus Red_z. We stress here that this result is schematic, it means that the closed subscheme Red_z lies in the schematic fiber of each map $\kappa_n - \kappa_{\sigma(n)}$ at z. Moreover, on each thickening A of z lying in the reducibility locus Red_z, we show that $T \otimes A$ can be written uniquely as the sum of traces of true representations ρ_i over A, each ρ_i being a trianguline deformation of $(\bar{\rho}_i, \mathcal{F}_{z,i})$ (Theorem 4.4.4). We end the section by giving another proof of the assertion above on the weights on the reducibility locus under some slightly different kind of assumptions (Theorem 4.4.6).

As an example of application of the results above, let us assume that σ acts transitively on $\{1, \ldots, d\}$ (in which case we say that $\mathcal{F}_z$ is an *anti-ordinary refinement*), so each difference of weights $\kappa_n - \kappa_m$ is constant on Red_z. If some κ_m is moreover constant (what we can assume up to a twist), we get that all the weights κ_i are constant

(1) This irreducibility assumption applies for $\bar{\rho}_z$ viewed as a representation of G, which is weaker than being irreducible as a representation of G_p.

(2) In the applications to Selmer groups, we will "luckily" be in that case.

on the total reducibility locus at z, hence are distinct integers. In particular the deformations ρ_i above of $\bar{\rho}_i$ are Hodge-Tate representations, and our work on trianguline deformations shows then that they are even crystalline (under some mild conditions on the $\bar{\rho}_i$, see Corollary 4.4.5). This fact will be very important in the applications to eigenvarieties and global Selmer groups of the last section, as it will allow us to prove that the scheme Red_z coincides with the *reduced point* z there.

We end this introduction by discussing some aspects of the proofs and other results. We fix $z \in Z$ as above, let $A := \mathcal{O}_z$ and we consider the composed pseudocharacter

$$T : G \longrightarrow \mathcal{O}(X) \longrightarrow A$$

again denoted by T. It is residually multiplicity free and A is henselian, hence T fulfills the assumptions of our work in Section 1. Some important role is played by some specific $A[G]$-modules called M_j (introduced § 1.5.4) whose quite subtle properties turn out to be enough to handle the difficulties coming from the fact that T may not be the trace of a representation over A. We extend those modules, with the action of G, to torsion free coherent $\mathcal{O}$-modules in an affinoid neighborhood U of z in X (§ 4.3.3) to which we apply the results of Section 3.

However, this only gives us a part of the information, namely the one concerning the first eigenvalue φ_1 of the refinement. Indeed, this eigenvalue is the only one that varies analytically (if κ_1 is normalized to zero say) and therefore the only one to which we can apply the results of Section 3. To deal with the other eigenvalues as well, we will work not only with the family T, but with all its exterior powers $\Lambda^k T$. Some inconvenience of using these exterior products however appears in the fact that our definition of a refined family is not stable under exterior powers (see § 4.2.4, and the last paragraph of this introduction). This leads us to introducing the notion of p-adic family of *weakly refined* Galois representations, which is a modification of the one given above where we only care about κ_1 and F_1 (see Definition § 4.2.7). Any exterior power of a refined family is then a *weakly refined family.* Let us note here that an important tool to get the trianguline assertion at the end is Theorem 2.5.6 of Section 2.

In fact, our results mentioned above have analogues in the context of weakly refined families (in which case they hold for every $x \in X$), that we prove in Theorems 4.3.2 and 4.3.4. Another interesting result here is the proof (Theorem 4.3.6) that there exists a non-torsion crystalline period attached to the eigenvalue φ_1 in the infinitesimal extensions between the ρ_i constructed in Section 1 (that is, in the image of $\iota_{i,j}$).

In the last subsection of this chapter (§ 4.5), we give much weaker results concerning *any* reducibility locus, not only the total reducibility locus.

Though the trick of using exterior powers is not at all unfamiliar in the context of Fontaine's theory, we have the feeling that it is not the best thing to do here, and that the use of exterior products is responsible for some technical hypotheses that appear later in this section (e.g. assumptions (REG) and (MF′) in § 4.4.1). But we have not found a way to avoid it. Actually, by using only arguments similar to the ones in [**73**], it seems quite hard to argue inductively (as we would like to) by "dividing modulo the families of eigenvectors for φ_1". Among other things, a difficulty is that although the points in Z belong to Kisin's X_{fs}, they do for quite indirect reasons (see e.g. [**73**, Remark 5.5 (4)]), which makes many arguments there—and here also—quite delicate. As a possible solution, our work in this section and in Section 2 confirms Colmez's idea that the construction of X_{fs} in [**73**] should be reworked from the point of view of (φ, Γ)-modules over the Robba ring [(3)] and suggests that X_{fs} should directly contain the points of Z which are non critically refined. As this would have led us quite away from our initial aim, we did not follow this approach. We hope however that the present work sheds lights on aspects of this interesting problem.

4.2. Families of refined and weakly refined p-adic representations

4.2.1. Notations. — As in sections 2 and 3, we set $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Moreover we suppose given a topological group G together with a continuous morphism $G_p \longrightarrow G$.

Example 4.2.1. — The main interesting examples [(4)] are

(a) $G = G_p$ and the morphism is the identity.
(b) $G = G_{K,S} = \mathrm{Gal}(K_S/K)$ where K is a number field, S a set of places of K, and $K_S \subset \bar{K}$ the maximal extension which is unramified outside S; the morphism sending G_p to a decomposition group of K at some prime $\mathfrak{P}$ of K such that $K_{\mathfrak{P}} = \mathbb{Q}_p$.

If ρ is a representation of G, it induces a representation of G_p that we shall denote by $\rho_{|G_p}$. We will replace $\rho_{|G_p}$ by ρ without further comments when the context prevents any ambiguity, for example in assertions such as "ρ is Hodge-Tate", or "ρ is crystalline".

(3) E.g. for any x in a refined family X, $\bar{\rho}_x$ should be trianguline.

(4) Actually, our result in the case (a) would implies our result in the case (b), were there not the technical, presumably unnecessary, irreducibility hypothesis (MF) in § 4.4.1 below, that we can sometimes verify in case (b) and not in case (a).

4.2.2. Rigid analytic families of p-adic representations

Definition 4.2.2. — A *(rigid analytic) family of p-adic representations* is the data of a reduced and separated rigid analytic space $X/\mathbb{Q}_p$ and a continuous [5] pseudocharacter $T : G \longrightarrow \mathcal{O}(X)$.

The *dimension* of the family is the dimension of T; it will usually be denoted by d in the sequel. For each point $x \in X$, we call *evaluation* of T at x and note

$$T_x : G \longrightarrow k(x),$$

the composition of T with the evaluation map: $\mathcal{O}(X) \longrightarrow k(x)$ at the residue field $k(x)$ of x. Then T_x is a continuous $k(x)$-valued pseudocharacter. By a theorem of Taylor, it is the trace of a (unique up to isomorphism) continuous semisimple representation

$$\bar{\rho}_x : G \longrightarrow \mathrm{GL}_d(\overline{k(x)}),$$

which is actually defined over a finite extension of $k(x)$.

In other words, a family of p-adic representations parameterized by the rigid space X is a collection of representations $\{\bar{\rho}_x, x \in X\}$ for which we assume that the trace maps $T(g) : x \mapsto \mathrm{tr}\,(\rho_x(g))$ are analytic functions on X for each $g \in G$, and such that $g \mapsto T(g)$ is continuous. Examples are given by the continuous representations of G on locally free $\mathcal{O}$-modules on X, but our definition is more general as we showed in Section 1.6. In particular, the families of p-adic Galois representations parameterized by Eigenvarieties turn out to be families in this "weak" sense only in general.

4.2.3. Refined and weakly refined families of p-adic representations

Definition 4.2.3. — A *(rigid analytic) family of refined p-adic representations* (or shortly, a *refined family*) of dimension d is a family of p-adic representations (X, T) of dimension d together with the following data

(a) d analytic functions $\kappa_1, \ldots, \kappa_d \in \mathcal{O}(X)$,
(b) d analytic functions $F_1, \ldots, F_d \in \mathcal{O}(X)$,
(c) a Zariski dense subset Z of X;

subject the following requirements.

(i) For every $x \in X$, the Hodge-Tate-Sen weights of $\bar{\rho}_x$ are, with multiplicity, $\kappa_1(x), \ldots, \kappa_d(x)$.
(ii) If $z \in Z$, $\bar{\rho}_z$ is crystalline (hence its weights $\kappa_1(z), \ldots, \kappa_d(z)$ are integers).
(iii) If $z \in Z$, then $\kappa_1(z) < \kappa_2(z) < \cdots < \kappa_d(z)$.

---

[5] We recall that for each admissible open $U \subset X$ (not necessarily affinoid, e.g $U = X$), $\mathcal{O}(U)$ is equipped with the coarsest locally convex topology (see [**105**]) such that the restriction maps $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$, $V \subset U$ an open affinoid (equipped with is Banach algebra topology), are continuous. This topology is the Banach-algebra topology when U is affinoid.

(iv) For $z \in Z$, the eigenvalues of the crystalline Frobenius acting on $D_{\mathrm{crys}}(\bar{\rho}_z)$ are distinct and are $(p^{\kappa_1}(z)F_1(z), \dots, p^{\kappa_d}(z)F_d(z))$.

(v) For C a non-negative integer, let Z_C be the set of $z \in Z$ such that

$$\kappa_{n+1}(z) - \kappa_n(z) > C\,(\kappa_n(z) - \kappa_{n-1}(z)) \text{ for all } n = 2, \dots, d-1,$$

and $\kappa_2(z) - \kappa_1(z) > C$. Then for all C, Z_C accumulates at any point of Z. In other words, for all $z \in Z$ and $C > 0$, there is a basis of affinoid neighborhoods U of z such that $U \cap Z$ is Zariski-dense in U (see § 3.3.1).

(*) For each n, there exists a continuous character $\mathbb{Z}_p^* \longrightarrow \mathcal{O}(X)^*$ whose derivative at 1 is the map κ_n and whose evaluation at any point $z \in Z$ is the elevation to the $\kappa_n(z)$-th power.

The data (a) to (c) are called *a refinement* of the family (X, T).

Definition 4.2.4. — Fix a refined family as above and let $z \in Z$. The (distinct) eigenvalues of φ on $D_{\mathrm{crys}}(\bar{\rho}_z)$ are naturally ordered by setting

$$\varphi_n(z) := p^{\kappa_n(z)} F_n(z), \quad n \in \{1, \dots, d\},$$

which defines a refinement $\mathcal{F}_z$ of the representation $\bar{\rho}_z$ in the sense of § 2.4.

Example 4.2.5. — The main examples of refined families arise from eigenvarieties [(6)]. A refined family is said to be *ordinary* if $|F_n(x)| = 1$ for each $x \in X$ and $n \in \{1, \dots, d\}$. Many ordinary families (in the context of example 4.2.1 (b)) have been constructed by Hida. In this case we could show that $T_{|G_p}$ is a sum of 1-dimensional families. Non ordinary refined families of dimension 2 have been first constructed by Coleman in [**43**] (see also [**44**], [**85**]), and in this case $T_{|G_p}$ is in general irreducible. Examples of non ordinary families of any dimension $d > 2$ have been constructed by one of us in [**36**].

Let us do some remarks about Definition 4.2.3.

Remark 4.2.6. — (i) (*Weights*) If (X, T) is a family of p-adic representations, and Z a Zariski-dense subset of X that satisfies condition (ii) of the definition of a refined family, Sen's theory implies that, after replacing X by a finite cover, there exist functions $\kappa_1, \dots, \kappa_n$ satisfying the condition (i) (and obviously (ii)) of the definition of a refined family; but it does not imply that, even after a suitable reordering, the κ_n's satisfy condition (iii).

Condition (v) imposes that the Hodge-Tate weights κ_n (and their successive differences) vary a lot on Z. Condition (*) appears for the same reason as in § 3.3.2.

(6) In particular, their construction if mostly *global* at the moment.

(ii) (*Frobenius eigenvalues*) Assumption (iv) means that the eigenvalues of the crystalline Frobenius φ acting on $D_{\text{crys}}(\rho_z)$ do not vary analytically on Z, but rather that they do when appropriately normalized. Note that when $d > 1$, even if some eigenvalue varies analytically, i.e. if some κ_n is constant, then the others do not by assumption (v). Moreover, because of the fixed ordering on the κ_n by assumption (iii), $(\{\kappa_n\}, \{F_{\sigma(n)}\}, Z)$ is not a refinement of the family (X, T) when $\sigma \neq 1 \in \mathfrak{S}_d$.

(iii) (*Generic non criticality*) Let $Z_{\text{num}} \subset Z$ be the subset consisting of points $z \in Z$ such that $\mathcal{F}_z$ is numerically non critical in the sense of Remark 2.4.6 formula (39). Then Z_{num} is Zariski-dense in X (use (v) and the fact that around each point of X, each $|F_n|$ is bounded). In particular, the $\mathcal{F}_z$ are "generically" non critical in the sense of § 2.4.3.

(iv) (*Subfamilies*) If (X, T) is a refined family, and if T is the sum of two pseudocharacters T_1 and T_2, then under mild conditions (X, T_1) and (X, T_2) are also refined families. See Prop. 4.5.3 below.

It will also be useful to introduce the notion of *weakly refined* families (resp. of *weak refinement* of a family).

Definition 4.2.7. — A *weak refinement* of a family (X, T) of dimension d is the data of

(a) analytic functions $\kappa_n \in \mathcal{O}(X)$ for $n = 1, \ldots, d$,
(b) an analytic function $F \in \mathcal{O}(X)$,
(c) a Zariski dense subset $Z \subset X$.

subject to the following requirements

(i), (ii) as in Definition 4.2.3.
(iii) If $z \in Z$, then $\kappa_1(z)$ is the smallest Hodge-Tate weight of $\bar{\rho}_z$.
(iv) For C a non-negative integer, let $Z_C = \{z \in Z,\quad \forall n \in \{2, \ldots, d\},\ \kappa_n(x) > \kappa_1(z) + C\}$. Then Z_C accumulates at any point of Z for all C.
(v) For $z \in Z$, $\varphi_1(z) := p^{\kappa_1}(z) F_1(z)$ is a multiplicity-one eigenvalue of the crystalline Frobenius acting on $D_{\text{crys}}(\bar{\rho}_z)$.
(∗) There exists a continuous character $\mathbb{Z}_p^* \longrightarrow \mathcal{O}(X)^*$ whose derivative at 1 is the map κ_1 and whose evaluation at any point $z \in Z$ is the elevation to the $\kappa_1(z)$-th power. As in Def. 3.3.2, we denote also by $\kappa_1 : G_p \longrightarrow \mathcal{O}(X)^*$ the associated continuous character.

Of course, if $(X, T, \{\kappa_n\}, \{F_n\}, Z)$ is a refined family, then $(X, T, \{\kappa_n\}, F_1, Z)$ is a weakly refined family.

Remark 4.2.8. — The conditions (i) to (v) and ($*$) are invariant by any permutation in the order of the weights $\kappa_2, \dots, \kappa_d$ (not κ_1). Two weak refinements differing only by such a permutation should be regarded as equivalent.

4.2.4. Exterior powers of a refined family are weakly refined. — Let (X, T) be a family of p-adic representations of dimension d. For $k \leq d$, then $(X, \Lambda^k T)$ is a family of p-adic representations of dimension $\binom{d}{k}$ (see § 1.2.7), and we have $(\Lambda^k T)_x = \mathrm{tr}\,(\Lambda^k \bar{\rho}_x)$ for any $x \in X$.

Assume that $(X, T, \{\kappa_n\}, \{F_n\}, Z)$ is refined. The Hodge-Tate-Sen weights of $\Lambda^k T$ are then the

$$\kappa_I := \sum_{j \in I} \kappa_j$$

where I runs among the subsets of cardinality k of $\{1, \dots, d\}$. Moreover, the $\Lambda^k \bar{\rho}_z$ are crystalline for $z \in Z$. However, there is no natural refinement on $(X, \Lambda^k T)$ in general [7]. We set

$$F := \prod_{j \in \{1, \dots, k\}} F_j, \ \kappa'_1 := \kappa_{\{1, \dots, k\}} = \kappa_1 + \dots + \kappa_k,$$

and $\kappa'_2, \dots, \kappa'_{\binom{d}{k}}$ any numbering of the κ_I for I running among subsets of $\{1, \dots, d\}$ of cardinality k which are different from $\{1, \dots, k\}$. The following lemma is clear.

Lemma 4.2.9. — *The data $(\kappa'_1, \dots, \kappa'_{\binom{d}{k}}, F, Z)$ is a weak refinement of the family $(X, \Lambda^k T)$.*

4.3. Existence of crystalline periods for weakly refined families

4.3.1. Hypotheses. — In this subsection, $(X, T, \kappa_1, \dots, \kappa_d, F, Z)$ is a family of dimension d of *weakly* refined p-adic representations.

Fix $x \in X$. As in Section 3 we shall denote by A the rigid analytic local ring $\mathcal{O}_x$, by m its maximal ideal, and by $k = A/m = k(x)$ its residue field. We still denote by T the composite pseudocharacter $G \longrightarrow \mathcal{O}(X) \longrightarrow A$. Our aim in this section is to prove that the infinitesimal pseudocharacters $T : G \longrightarrow A/I$, $I \subset A$ an ideal of cofinite length, have some crystalline periods in a sense we explain below. For this, we will have to make the following three hypotheses on x, that will stay in force during all § 4.3.

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[7] For one thing, there is no natural order on the set of subsets I of $\{1, \dots, d\}$ of cardinality k that makes the application $I \mapsto \kappa_I(z)$ increasing for all $z \in Z$. Compare with Remark 4.2.6(i).

(ACC) The set Z accumulates at x [8].

(MF) T is residually multiplicity free [9].

(REG) $D^+_{\rm crys}(\bar{\rho}_x(\kappa_1(x)))^{\varphi=F(x)}$ has $k(x)$-dimension 1.

Recall from Definition 1.4.1 that (MF) means that

$$\bar{\rho}_x = \oplus_{i=1}^r \bar{\rho}_i$$

where the $\bar{\rho}_i$ are absolutely irreducible, defined over $k(x)$, and two by two nonisomorphic [10]. In particular, this holds of course when $\bar{\rho}_x$ is irreducible and defined over $k(x)$. As in § 1.4.1 we shall note $d_i = \dim_k \bar{\rho}_i$, so that $\sum_{i=1}^r d_i = d$. Note that A is a henselian ring ([**16**, §2.1]) and a $\mathbb{Q}$-algebra. In particular, $d!$ is invertible in A, and $T : A[G] \longrightarrow A$ satisfies the hypothesis of § 1.4.1.

Note moreover that hypothesis (REG) (for "regularity") is, as (MF), a kind of multiplicity free hypothesis. Indeed, Theorem 3.4.1 implies easily (see below) that for any x satisfying (ACC), $D^+_{\rm crys}(\bar{\rho}_x(\kappa_1(x)))^{\varphi=F(x)}$ has $k(x)$-dimension *at least* 1.

Remark 4.3.1. — The assumptions above define a $j \in \{1, \dots, r\}$ as follows. By property (REG), $F(x)$ is a multiplicity one eigenvalue of φ on

$$D^+_{\rm crys}(\bar{\rho}_x(\kappa_1(x))) = D^+_{\rm crys}(\bar{\rho}_1(\kappa_1(x))) \oplus \cdots \oplus D^+_{\rm crys}(\bar{\rho}_r(\kappa_1(x))).$$

Hence this is an eigenvalue of φ on one (and only one) of the $D^+_{\rm crys}(\bar{\rho}_i(\kappa_1(x)))$ say $D^+_{\rm crys}(\bar{\rho}_j(\kappa_1(x)))$, which defines a unique $j \in \{1, \dots, r\}$.

4.3.2. The main results. — We will use below some notations and concepts introduced in Section 1. Let K be the total fraction ring of A and let $\rho : A[G] \longrightarrow M_d(K)$ be a representation whose trace is T and whose kernel is $\operatorname{Ker} T$. It exists by Theorem 1.4.4 (ii) and Remark 1.4.5 as A is reduced and noetherian. Fix a GMA structure on $S := A[G]/\operatorname{Ker} T$ given by the theorem cited above, j as defined in Remark 4.3.1, and let $M_j \subset K^d$ the "column" S-submodule defined in § 1.5.4. It is of finite type over A by construction and Remark 1.4.5.

[8] This hypothesis is probably unnecessary but to remove it would require quite a big amount of supplementary work, such as a global generalization of what was done in Section 1 (that is on X instead of A). Note that any $z \in Z$ satisfies (ACC). Moreover, in the applications to eigenvarieties, (ACC) will be satisfied for all the x's corresponding to p-adic finite slope eigenforms whose weights are in $\mathbb{Z}_p$, which is more than sufficient for our needs.

[9] This hypothesis is imposed to us by our reliance on chapter 1. However, though we did not write down the details, we are certain that all the results in this subsection hold, with essentially the same proofs, with the weaker assumption than only the representation denoted by $\bar{\rho}_j$ below appears with multiplicity one in $T \otimes k$.

[10] The results of this section will apply also in the case where the $\bar{\rho}_i$ are not defined over $k(x)$. Indeed, it suffices to apply them to the natural weakly refined family on $X \times_{\mathbb{Q}_p} L$, L any finite extension of $\mathbb{Q}_p$ over which the $\bar{\rho}_i$ are defined.

Let moreover $\mathcal{P}$ be a partition of $\{1,\dots,r\}$. Recall that if $\mathcal{P}$ contains $\{j\}$, then for every ideal I containing the reducibility ideal $I_{\mathcal{P}}$ (see § 1.5.1), there is a unique continuous representation

$$\rho_j : G \longrightarrow \mathrm{GL}_{d_j}(A/I),$$

whose reduction mod m is $\bar{\rho}_j$ and such that $T\otimes A/I = \operatorname{tr}\rho_j + T'$, where $T' : G \longrightarrow A/I$ is a pseudocharacter of dimension $d - d_j$ (see Definition 1.5.3, Proposition 1.5.10).

Theorem 4.3.2. — *Assume that $\mathcal{P}$ contains $\{j\}$ and let I be a cofinite length ideal of A containing $I_{\mathcal{P}}$. Then $D^+_{\mathrm{crys}}(\rho_j(\kappa_1))^{\varphi=F}$ and $D^+_{\mathrm{crys}}(M_j/IM_j(\kappa_1))^{\varphi=F}$ are free of rank one over A/I.*

Proof. — We will prove the theorem assuming the following crucial lemma, whose proof is postponed to the next subsection.

Lemma 4.3.3. — *Let I be a cofinite length ideal of A, then*

(i) *the Sen operator of $D_{\mathrm{Sen}}(M_j/IM_j)$ is annihilated by $\prod_{n=1}^{d}(T-\kappa_n)$,*
(ii) $l\left(D^+_{\mathrm{crys}}(M_j/IM_j(\kappa_1))^{\varphi=F}\right) = l(A/I)$.

By Theorem 1.5.6(0), there is an exact sequence of $(A/I)[G]$-modules

$$0 \longrightarrow K \longrightarrow M_j/IM_j \longrightarrow \rho_j \longrightarrow 0$$

where K has a Jordan-Holder sequence, all subquotients of which are isomorphic to $\bar{\rho}_i$ for some $i \neq j$. If X is a finite length A-module equipped with a continuous A-linear action of G_p we set $D(X) := D^+_{\mathrm{crys}}(X(\kappa_1))^{\varphi=F}$. As $D(\bar{\rho}_i) = 0$ for $i \neq j$ by (REG), we have $D(K) = 0$, hence applying the left exact functor D to the above sequence, we get an injection

$$D(M_j/IM_j) \hookrightarrow D(\rho_j).$$

Thus by Lemma 4.3.3 (ii) we have $l(D(\rho_j)) \geq l(A/I)$. Applying Lemma 3.2.9(i) to the A/I-representation ρ_j gives $l(D(\rho_j)) = l(A/I)$, hence an isomorphism

$$D(M_j/IM_j) \simeq D(\rho_j),$$

and case (2) of Lemma 3.3.9 gives that $D(\rho_j)$ is free of rank 1 over A/J. Hence the result. □

Theorem 4.3.4. — *Assume that $\bar{\rho}_x$ has distinct Hodge-Tate-Sen weights and that the weight k of $D^+_{\mathrm{crys}}(\bar{\rho}_j(\kappa_1(x)))^{\varphi=F(x)}$ is the smallest integral Hodge-Tate weight of $\bar{\rho}_j(\kappa_1(x))$. Then for the unique l such that $\kappa_l(x) - \kappa_1(x) = k$, we have that κ_l is a weight of ρ_j and*

$$(\kappa_l - \kappa_1) - (\kappa_l(x) - \kappa_1(x)) \in I_{\mathcal{P}} \quad \textit{if } \{j\} \in \mathcal{P}.$$

Proof. — Let $I \supset I_{\mathcal{P}}$ be a cofinite length ideal of A. By Theorem 4.3.2,

$$D^+_{\mathrm{crys}}(\rho_j(\kappa_1))^{\varphi=F}$$

is free of rank one over A/I. Moreover, k is the smallest integral Hodge-Tate weight of $\bar{\rho}_j(\kappa_1(x))$. Thus we can apply Proposition 2.5.4 to $V := \bar{\rho}_j(\kappa_1)$ which shows that V has a constant weight k, *i.e.* that $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H_p}$ contains a free A/I-submodule of rank 1 on which the Sen operator acts as the multiplication by k. By Lemma 4.3.3(i), this implies that

$$\prod_{n=1}^{d}(k-(\kappa_n-\kappa_1)) = 0 \quad \text{in } A/I.$$

The difference of any two distinct terms of the product above is invertible in (the local ring) A/I as $\kappa_n(x) \neq \kappa_{n'}(x)$ if $n \neq n'$. Hence one, and only one, of the factor $k-(\kappa_n-\kappa_1)$, $n = 1, \dots, d$, of the above product is zero, and reducing mod m gives that this factor is the one with $n = l$. In particular, κ_l is a Hodge-Tate-Sen weight of ρ_j and

$$k = \kappa_l - \kappa_1 = \kappa_l(x) - \kappa_1(x) \quad \text{in } A/I.$$

We conclude the proof as $I_{\mathcal{P}}$ is the intersection of the I of cofinite length containing it, by Krull's theorem. □

Remark 4.3.5. — (i) The conclusion of the theorem can be rephrased as: $\kappa_l - \kappa_1$ is constant on the reducibility locus corresponding to $\mathcal{P}$, if $\mathcal{P}$ contains $\{j\}$.

(ii) The hypothesis that k is the smallest weight is satisfied in many cases. For one thing, it is obviously satisfied when k is the only integral Hodge-Tate weight of $\bar{\rho}_j(\kappa_1(x))$, which is the generic situation. More interestingly, it is also satisfied for crystalline $\bar{\rho}_x$ whenever $v(F(x))$ is smaller than the *second* (in the increasing order) Hodge-Tate weight of $\bar{\rho}_j(\kappa_1)$ since, by weak admissibility, $k \leq v(F(x))$. This is always true when $\bar{\rho}_j$ has dimension ≤ 2, since by admissibility, the second (that is, the greatest) weight is greater than or equal to the valuation of *any* eigenvalue of the Frobenius.

(iii) The assumption that $\bar{\rho}_x$ has distinct Hodge-Tate weights implies that $\bar{\rho}_x$ has no multiple factors, hence (MF) if these factors are defined over $k(x)$.

Now let $i \neq j$ be an integer in $\{1, \dots, r\}$. Recall that if $\mathcal{P}$ contains $\{i\}$ and $\{j\}$, and if I contains $I_{\mathcal{P}}$, then there is a map $\iota_{i,j}$ whose image is $\mathrm{Ext}^1_{S/JS,\mathrm{cont}}(\rho_j, \rho_i)$ (see Theorem 1.5.3 , Theorem 1.5.6(1) and Proposition 1.5.10).

Theorem 4.3.6. — *Assume that $\mathcal{P}$ contains $\{i\}$ and $\{j\}$ and let I be a cofinite length ideal of A containing $I_{\mathcal{P}}$. Let $\rho_c : G \longrightarrow \mathrm{GL}_{d_i+d_j}(A/I)$ be an extension of ρ_j by ρ_i which belongs to the image of $\iota_{i,j}$. Then $D^+_{\mathrm{crys}}(\rho_c(\kappa))^{\varphi=F}$ is free of rank one over A/I.*

Proof. — The proof is exactly the same as the proof of Theorem 4.3.2 except that we start using point (2) of Theorem 1.5.6 instead of point (0). □

4.3.3. Analytic extension of some $A[G]$-modules, and proof of Lemma 4.3.3

We keep the assumptions and notations of § 4.3.2. Let $M \subset K^d$ by any S-submodule which is of finite type over A.

Lemma 4.3.7. — *There is an open affinoid subset U of X containing x in which Z is Zariski-dense and a torsion-free coherent sheaf $\mathcal{M}$ on U with a continuous action of G such that $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} A \simeq M$ as $A[G]$-modules and topological A-modules.*

If moreover $K.M = K^d$, we may choose U and $\mathcal{M}$ such that $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathrm{Frac}(\mathcal{O}(U))$ is free of rank d over $\mathrm{Frac}(\mathcal{O}(U))$, and carries a semisimple representation of G with trace $T \otimes_{\mathcal{O}(X)} \mathcal{O}(U)$.

Proof. — By (ACC), we may choose a basis of open affinoid neighbourhoods $(V_i)_{i\in I}$ of $x \in X$ such that Z is Zariski-dense in V_i for each i. We may view I as a directed set if we set $j \geq i$ if $V_j \subset V_i$, and then $\operatorname{indlim}_i \mathcal{O}(V_i) = A$.

By construction we have $\operatorname{tr}(\rho(G)) \subset \mathcal{O}(X)$. As each $\mathcal{O}(V_i)$ is reduced and noetherian, a standard argument implies that the $\mathcal{O}(V_i)$-module

$$\mathcal{O}(V_i)[G]/\mathrm{Ker}\,(T \otimes_{\mathcal{O}(X)} \mathcal{O}(V_i))$$

is of finite type (see e.g. [**8**, Lemma 7.1 (i)]). As a consequence, its quotient $\mathcal{O}(V_i)[\rho(G)] \subset M_d(K)$ is also of finite type over $\mathcal{O}(V_i)$.

As M is of finite type over A, we can find an element $0 \in I$ and a finite type $\mathcal{O}(V_0)$-submodule M_0 of M such that $AM_0 = M$. We define now N_0 as the smallest $\mathcal{O}(V_0)$-submodule of M containing M_0 and stable by G. It is finite type over $\mathcal{O}(V_0)$ as we just showed that $\mathcal{O}(V_0)[\rho(G)] \subset M_d(K)$ is. Moreover, the map $G \longrightarrow \mathrm{Aut}_{\mathcal{O}(V_0)}(N_0)$ (resp. $G \longrightarrow \mathrm{Aut}_A(M)$) is continuous by [**8**, Lemma 7.1 (v)] (resp. by its proof).

For $i \geq 0$, we set $N_i = \mathcal{O}(V_i)N_0 \subset M$. The following abstract lemma implies that for i big enough, the morphism $N_i \otimes_{\mathcal{O}(V_i)} A \longrightarrow M$ is an isomorphism. We fix such an i, set $U = V_i$ and define $\mathcal{M}$ as the coherent sheaf on V_i whose global sections are N_i. It is torsion free over $\mathcal{O}(V_i)$ as $N_i \subset M \subset K^d$, which concludes the proof of the first assertion.

Assume moreover that $K.M = K^d$ and let $N_i \subset K^d$ the module constructed above, so $K.N_i = K^d$. The kernel of the natural map

$$N_i \otimes_{\mathcal{O}(V_i)} \mathrm{Frac}(\mathcal{O}(V_i)) \longrightarrow N_i \otimes_{\mathcal{O}(U_i)} K = K^d$$

is exactly supported by the minimal primes of the irreducible components of $\mathcal{O}(V_i)$ that do not contain x, and at the other minimal primes N_i is free of rank d with trace T, and it is semisimple because so is its scalar extension to K by construction and Lemma 4.3.9 (i) below. Let $U' \subset V_i$ be the Zariski open subset of V_i whose complement is the (finite) union of irreducible components of V_i not containing x. Choose $j \geq i$ such that $V_j \subset U'$, then $U := V_j$ and $\mathcal{M}(U) := N_i \otimes_{\mathcal{O}(V_i)} \mathcal{O}(V_j)$ have all the required properties. □

Lemma 4.3.8. — *Let $(A_i)_{i \in I}$ be a directed family of commutative rings and let A be the inductive limit of (A_i). Assume A is noetherian. Let M be a finite type A-module and N_0 a finite-type A_0-submodule of M such that $A N_0 = M$. For $i \geq 0$, set $N_i := A_i N_0 \subset M$.*

Then for i big enough, the natural morphism $N_i \otimes_{A_i} A \longrightarrow M$ is an isomorphism.

Proof. — Define K_i by the following exact sequence:

$$0 \longrightarrow K_i \longrightarrow N_i \otimes_{A_i} A \longrightarrow M \longrightarrow 0.$$

For $i \leq j$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & N_i \otimes_{A_i} A & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_j & \longrightarrow & N_j \otimes_{A_j} A & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

The horizontal lines are exact sequences, the right vertical arrow is the identity and the middle one is surjective by the associativity of the tensor product. Hence the left vertical arrow $K_i \longrightarrow K_j$ is surjective. Because K_0 is a finite type A-module, and A is noetherian, there is an i such that for each $j \geq i$, $K_i \longrightarrow K_j$ is an isomorphism.

Let $x \in K_i$. We may write $x = \sum_k n_k \otimes a_k$ with $n_k \in N_i$ and $a_k \in A$, and $\sum_k n_k a_k = 0$ in M. Take $j \geq i$ such that all the a_k's are in A_j. Then the image of x in $N_j \otimes_{A_j} A$ is 0, and x is 0 in K_j. But then $x = 0$ in K_i, which proves that $K_i = 0$ and the lemma. □

Lemma 4.3.9. — (i) *$S \otimes_A K$ is a semisimple K-algebra.*

(ii) *There exists a finite type S-module $N \subset K^d$ such that $(N \oplus M_j)K = K^d$ and that $(N \otimes_A k)^{\mathrm{ss}}$ is isomorphic to a sum of copies of $\bar{\rho}_i$ with $i \neq j$.*

Proof. — Recall from § 4.3.2 that $S = A[G]/\mathrm{Ker}\, T$. Since $K \supset A$ is a fraction ring of A, we have $\mathrm{Ker}\,(T \otimes_A K) = K.\mathrm{Ker}\, T$ in $K[G]$. As a consequence, the natural map $S \otimes_A K \longrightarrow K[G]/\mathrm{Ker}\,(T \otimes_A K)$ is an isomorphism, and Lemma 1.2.7 proves (i).

Let us show (ii). By (i) we can chose a $K.S$-module $N' \subset K^d$ such that $K.M_j \oplus N' = K^d$. As S is finite type over A by Remark 1.4.5, we can find a S-submodule $N \subset N'$ such that N is finite type over A and $K.N = N'$. We claim that N has the required property. By construction we only have to prove the assertion about $(N \otimes_A k)^{\mathrm{ss}}$. Arguing as in the proof of Theorem 1.5.6 (0), it suffices to show that $e_j N = 0$, where e_j is as before the idempotent in the fixed GMA structure of S. But $e_j(K^d) = e_j(K.M_j)$

by definition of M_j and Theorem 1.4.4 (ii). So $e_j(K.N) = 0 = e_jN$, and we are done. □

We are now ready to prove Lemma 4.3.3.

Proof. — (of Lemma 4.3.3). Let us show (ii) first. We set $M = N \oplus M_j$, where N is given by Lemma 4.3.9.

By Proposition 1.5.6(0) and Lemma 4.3.9 (ii),

$$(M \otimes k)^{\mathrm{ss}} \simeq \oplus_{i=1}^{r} n_i \bar{\rho}_i \text{ where } n_i \geq 1 \text{ for all } i, \text{ and } n_j = 1. \tag{49}$$

But by (REG) $D^+_{\mathrm{crys}}(\bar{\rho}_i(\kappa_1(x))^{\varphi=F(x)}$ has dimension $\delta_{i,j}$. In particular,

$$\dim_k(D^+_{\mathrm{crys}}((M \otimes k)(\kappa(x))^{\mathrm{ss}})^{\varphi=F(x)}) = 1. \tag{50}$$

Moreover, $D^+_{\mathrm{crys}}(M/IM(\kappa_1)) = D^+_{\mathrm{crys}}(M_j/IM_j(\kappa_1)) \oplus D^+_{\mathrm{crys}}(N/IN(\kappa_1))$, and

$$D^+_{\mathrm{crys}}(N/IN(\kappa_1))^{\varphi=F} = 0 \tag{51}$$

by a dévissage and the same argument as above.

We claim now that the equality follows directly from Theorem 3.4.1 applied to the module $\mathcal{M}$ over U associated to M given by Lemma 4.3.7 (applied in the case $K.M = K^d$). By formula (51), we just have to verify that $\mathcal{M}$ satisfies the hypotheses (CRYS), (HT) and $(*)$ of § 3.3.2, and we already checked that $D^+_{\mathrm{crys}}((M \otimes k)(\kappa(x))^{\mathrm{ss}})^{\varphi=F(x)}$ has length one in (50).

By assumption (iv) of weakly refined families, $Z_C \cap U$ accumulates at every point of $Z \cap U$. As $\mathcal{M}(U)$ is torsion free of generic rank d and with trace T, and by the generic flatness theorem, there is a proper Zariski closed subspace F of U such that for $y \in U - F$, $\overline{\mathcal{M}}_y^{\mathrm{ss}} = \bar{\rho}_y$. Recall that the $\mathrm{Frac}(\mathcal{O}(U))[G]$-module $\mathcal{M} \otimes_{\mathcal{O}(U)} \mathrm{Frac}(\mathcal{O}(U))$ is semisimple. So enlarging F is necessary, we have that for $y \in U - F$, $\overline{\mathcal{M}}_y = \overline{\mathcal{M}}_y^{\mathrm{ss}}$, hence $\overline{\mathcal{M}}_y = \bar{\rho}_y$. We replace Z by $(Z \cap U) - (F \cap Z \cap U)$, so by (ACC) Z is a Zariski dense subset of U and still has the property that Z_C accumulates at any point of Z. Property (CRYS) follows then from (ii) and (v) of the definition of a weak refinement, and property (HT) from (iii) and (iv). This concludes the proof.

Let us show (i) now. If E is a $\mathbb{Q}_p$-Banach space, we set [11] $E_{\mathbb{C}_p} := E\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. Recall that Sen's theory [**108**] attaches in particular to any continuous morphism $\tau : G_p \longrightarrow \mathcal{B}^*$, $\mathcal{B}$ any Banach $\mathbb{Q}_p$-algebra, an element $\varphi \in \mathcal{B}_{\mathbb{C}_p}$ whose formation commutes with any continuous Banach algebra homomorphism $\mathcal{B} \longrightarrow \mathcal{B}'$. When τ is

(11) All the $\mathbb{Q}_p$-Banach spaces of this proof to which we apply the functor $-_{\mathbb{C}_p}$ are discretely normed. We use freely the fact that any continuous closed injection $E \longrightarrow F$ between such spaces induces an exact sequence $0 \longrightarrow E_{\mathbb{C}_p} \longrightarrow F_{\mathbb{C}_p} \longrightarrow (F/E)_{\mathbb{C}_p} \longrightarrow 0$ by [**109**, 1.2], and also that any submodule of a finite type module over an affinoid algebra is closed.

a finite dimensional $\mathbb{Q}_p$-representation of G_p, this element is the usual Sen operator. Applying this to the Banach algebra

$$\mathcal{B} := \mathrm{End}_{\mathcal{O}(U)}(\mathcal{M}(U))$$

we get such an element φ.

We claim that φ is killed by the polynomial

$$P := \prod_{n=1}^{d}(T - \kappa_n).$$

Indeed, arguing as in the proof of (ii) above me may assume that for all $z \in Z$ we have $\overline{\mathcal{M}}_z \simeq \bar{\rho}_z$ and $\mathcal{B}/m_z\mathcal{B} \simeq \mathrm{End}_{k(z)}(\overline{\mathcal{M}}_z)$. As a consequence, using the evaluation homomorphism $\mathcal{B} \longrightarrow \mathcal{B}/m_z\mathcal{B}$ and assumption (i) in Definition 4.2.7, we get that $P(\varphi) \in m_z\mathcal{B}_{\mathbb{C}_p}$. But $\mathcal{O}(U)_{\mathbb{C}_p}$ is reduced by [**47**, Lemma 3.2.1(1)], so $\mathcal{B}_{\mathbb{C}_p}$ is a (finite type) torsion free $\mathcal{O}(U)_{\mathbb{C}_p}$-module. Since Z is Zariski-dense in U, hence in $U(\mathbb{C}_p)$, and since affinoid algebras are Jacobson rings, we obtain that $P(\varphi) = 0$ in $\mathcal{B}_{\mathbb{C}_p}$. We conclude the proof as the operator of the statement of Lemma 4.3.3(i) is the image of φ under $\mathcal{B}_{\mathbb{C}_p} \longrightarrow \mathrm{End}_{A/I}(M_j/IM_j)_{\mathbb{C}_p}$. □

We now present a variant [(12)] of Lemma 4.3.3. Suppose we keep hypotheses (ACC) and (REG) from § 4.3.1 but release hypothesis (MF). Instead we assume that

(FM) There exists a free A-module M of rank d with an A-linear action of G whose trace is T, and such that $M \otimes_A K$ is a semisimple $K[G]$-module.

For example, (FM) holds if $\bar{T}$ is absolutely irreducible (by Rouquier-Nyssen's theorem), or, under (MF), if A is a UFD (by Proposition 1.6.1).

Under those hypotheses, we claim that Lemma 4.3.3, and even a little bit more, holds with M_j replaced by the module M. More precisely, we have

Lemma 4.3.10. — *Let I be a cofinite length ideal of A, then*

(i) *the Sen operator of $D_{\mathrm{Sen}}(M/IM)$ is annihilated by $\prod_{n=1}^{d}(T-\kappa_n)$,*
(ii) $l\left(D^+_{\mathrm{crys}}(M/IM(\kappa_1))^{\varphi=F}\right) = l(A/I)$,
(iii) $D^+_{\mathrm{crys}}(M/IM(\kappa_1))^{\varphi=F}$ *is free of rank one over A/I.*

Proof. — First, note that the natural A-algebra homomorphism $\rho : A[G] \longrightarrow \mathrm{End}_A(M)$ factors through $S := A[G]/\mathrm{Ker}\,(T)$. Indeed, $M \otimes_A K$ is a semisimple $K[G]$-module, so $\rho(K[G]) \subset \mathrm{End}_K(M \otimes_A K)$ is a semisimple artinian ring, of which $K.\rho(\mathrm{Ker}\,(T))$ is a nilpotent 2-sided ideal by Lemma 1.2.1, hence vanishes. As a consequence, $\rho_{|G}$ is continuous, as T is.

(12) Added in 2008.

We claim that the proof of Lemma 4.3.3 holds with M_j replaced everywhere by M, it is actually only easier. Indeed, the reader may observe that there are only two points where the specific nature of M_j is used, besides being an S-module of finite type over A.

The first one is in the proof of Lemma 4.3.9(ii). But that assertion holds trivially for M with $N := 0$, since M has rank d so $K.M = K^d$. The second one is for the proof of formula (51) (note that (50) is irrelevant here, and in any case obviously satisfied as $N = 0$). But since M is free, $M \otimes_A k$ is a d-dimensional representation of trace $\bar{T} = T_z$, hence its semi-simplification is exactly $\bar{\rho}_z$, so that formula (51) is nothing more than assumption (REG) at x. Hence the proof of Lemma 4.3.3 holds for M, giving (i) [13] and (ii). Then (iii) follows from (ii) and Lemma 3.3.9(ii). □

4.4. Refined families at regular crystalline points

4.4.1. Hypotheses. — In this subsection, $(X, T, \kappa_1, \dots, \kappa_d, F_1, \dots, F_d, Z)$ is a family of dimension d of refined p-adic representations. We fix $z \in Z$ (and not only in X). As in § 4.3.1 we write $A = \mathcal{O}_z$ and still denote by T the composite pseudocharacter $G \longrightarrow \mathcal{O}(X) \longrightarrow A$. We assume moreover that T is residually multiplicity free, and we use the same notation as before:

$$\bar{\rho}_z = \oplus_{i=1}^{r} \bar{\rho}_i, \;\; d_i = \dim \bar{\rho}_i.$$

Recall from Definition 4.2.4 that ρ_z is equipped with a refinement

$$\mathcal{F}_z = (\varphi_1(z), \dots, \varphi_d(z))$$

satisfying $\varphi_n(z) = p^{\kappa_n(z)} F_n(z)$. As $D_{\text{crys}}(\bar{\rho}_z) = \oplus_{i=1}^{r} D_{\text{crys}}(\bar{\rho}_i)$ this refinement induces for each i a refinement of $\bar{\rho}_i$ that we will denote by $\mathcal{F}_{z,i}$. We will make the following hypotheses on z.

(REG) The refinement $\mathcal{F}_z$ is *regular* (see Example 2.5.5): for all $n \in \{1, \dots, d\}$, $p^{\kappa_1(z)+\cdots+\kappa_n(z)} F_1(z) \dots F_n(z)$ is an eigenvalue of φ on $D_{\text{crys}}(\Lambda^n \bar{\rho}_z)$ of multiplicity one.

(NCR) For every $i \in \{1, \dots, r\}$, $\mathcal{F}_{z,i}$ is a non-critical refinement (cf. § 2.4.3) of $\bar{\rho}_i$.

Note that the hypothesis (NCR) does not mean at all that the refinement of $\bar{\rho}_z$ is noncritical: if for example $d = r$, that is the $\bar{\rho}_i$ are characters, any refinement of $\bar{\rho}_z$ satisfies (NCR).

---

(13) Actually, $D_{\text{Sen}}(M/IM)$ is free over A/I and its Sen polynomial coincides with $\prod_{n=1}^{d}(T - \kappa_n)$ (note that for our M, we may choose in Lemma 4.3.7 an $\mathcal{M}$ which is free over U).

4.4.2. The residually irreducible case ($r = 1$). — We keep the hypotheses above [14]. We first deal with the simplest case for which $\bar{\rho}_z$ is irreducible and defined over $k(z)$. In this case (REG) and (NCR) mean that $\mathcal{F}_z$ is a regular non critical refinement of $\bar{\rho}_z$.

Recall that in this residually irreducible case, there exists a unique continuous representation $\rho : G \longrightarrow \mathrm{GL}_d(A)$ whose trace is T by the theorem of Rouquier and Nyssen (the continuity follows from Proposition 1.5.10 (i)). We define a continuous character $\delta : \mathbb{Q}_p^* \longrightarrow (A^*)^d$ by setting:

$$\delta(p) := (F_1, \dots, F_d), \quad \delta_{|\mathbb{Z}_p^*} = (\kappa_1^{-1}, \dots, \kappa_d^{-1}).$$

Recall that each κ_n may be viewed as a character $\mathbb{Z}_p^* \longrightarrow A^*$ in the same way as in Definition 3.3.2, using property (*) of Definition 4.2.3.

Theorem 4.4.1. — *For any ideal $I \subsetneq A$ of cofinite length, $\rho \otimes A/I$ is a trianguline deformation of $(\bar{\rho}_z, \mathcal{F}_z)$ whose parameter is $\delta \otimes A/I$.*

Proof. — Fix I as in the statement and $V := \rho \otimes A/I$. By Theorem 2.5.6, it suffices to show that for each $1 \le k \le d$, $D_{\mathrm{crys}}(\Lambda^k V(\kappa_1 \cdots \kappa_k))^{\varphi = F_1 \cdots F_k}$ is free of rank 1 over A/I. Indeed, by definition of the characters κ_i and of the t_i *loc. cit.* , we have $t_i = k_i$ for each i.

Fix $1 \le k \le d$ and consider the family $(X, \Lambda^k T)$. As seen in § 4.2.4, this family is naturally weakly refined, with same set Z,

$$F = \prod_{n=1}^{k} F_n,$$

and first weight

$$\kappa = \sum_{n=1}^{k} \kappa_n.$$

We check that this family satisfies the hypotheses of Lemma 4.3.10. Namely, (ACC) comes from the fact that z is in Z, (REG) from (REG), and (FM) is clear with $M := \Lambda^k \rho$ except maybe the fact that $M \otimes_A K$ is a semisimple $K[G]$-module. But this follows from the irreducibility of $\rho \otimes_A K$ and a well-known result of Chevalley: over a field of characteristic zero, a tensor product of two finite dimensional semisimple representations is again semisimple. Thus, we can apply (iii) of Lemma 4.3.10, and we are done. □

[14] In 2006, Theorem 4.4.1 was proved with the additional hypothesis (MF′)—see § 4.4.4 below.

4.4.3. A permutation. — In order to study the reducible cases we need to define a permutation σ of $\{1,\dots,d\}$ that mixes up the combinatorial data of the refinement of $\bar{\rho}_x$ and of its decomposition $\bar{\rho}_x = \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_r$.

The refinement $\mathcal{F}_z$ together with the induced refinements $\mathcal{F}_{z,i}$ of the $\bar{\rho}_i$'s define a partition $R_1 \coprod \dots \coprod R_r$ of $\{1,\dots,d\}$: R_i is the set of n such that $p^{\kappa_n(z)}F_n(z)$ is a φ-eigenvalue on $D_{\mathrm{crys}}(\bar{\rho}_i)$. In the same way, we define a partition $W_1 \coprod \dots \coprod W_r$ of $\{1,\dots,d\}$: W_i is the set of integers n such that $\kappa_n(z)$ is a Hodge-Tate weight of $\bar{\rho}_i$. This is a partition as the $\kappa_n(z)$ are two-by-two distinct.

Definition 4.4.2. — We define σ as the unique bijection that sends R_i onto W_i and that is increasing on each R_i.

Note that σ does not depend on the chosed ordering on the $\bar{\rho}_i$.

Example 4.4.3. — (*Refined deformations of ordinary representations*) Assume that $r = d$, so $\bar{\rho}_z$ is a sum of characters $\bar{\rho}_1,\dots,\bar{\rho}_d$. Since there is an obvious bijection between this set of characters and the set of eigenvalues of φ on $D_{\mathrm{crys}}(\bar{\rho}_z)$, the refinement determines an ordering of those characters. We may assume up to renumbering that this order is $\bar{\rho}_1,\dots,\bar{\rho}_d$. By definition of the permutation above, the weights of $\bar{\rho}_1,\dots,\bar{\rho}_d$ are respectively $\kappa_{\sigma(1)}(z),\dots,\kappa_{\sigma(d)}(z)$. Note that in this case, σ determines the refinement. We refer to this situation by saying that the *representation* $\bar{\rho}_z$ is *ordinary*.

Assume that $\bar{\rho}_z$ is ordinary. We say that the point z (and the refinement $\mathcal{F}_z$) is *ordinary* if moreover $\sigma = \mathrm{Id}$, that is if the valuation of the eigenvalues in the refinement are increasing. For example, the families constructed by Hida (see Example 4.2.5) are ordinary in this strong sense: each $z \in Z$ is ordinary.

When, on the contrary, σ is transitive on $\{1,\dots,d\}$ we call the corresponding refinement, and the point z, *anti-ordinary*. For $d = 3$, examples of families with such z have been constructed and studied in [**8**]. Intermediary cases are also interesting. For example, Urban and Skinner consider in [**112**] a refined family of dimension $d = 4$ with a point $z \in Z$ where $\bar{\rho}_z$ is ordinary and σ is a transposition. They call such a point *semi-ordinary*.

In general, let us just say that we expect that any ordinary representation and any permutation σ should occur as a member of a refined family in the above way.

4.4.4. The total reducibility locus. — Keep the assumptions and notations of § 4.4.3 and § 4.4.1. In particulat we keep assumption (REG) and (NCR). In addition, we shall need

(MF′) For every family of integers $(a_i)_{i=1,\dots,r}$ with $1 \leq a_i \leq d_i$, the representation $\bar{\rho}_{(a_i)} := \bigotimes_{i=1}^r \Lambda^{a_i}\bar{\rho}_i$ is absolutely irreducible. Moreover, if (a_i) and (a'_i) are two distinct sequences as above with $\sum_{i=1}^r a_i = \sum_{i=1}^r a'_i$, then $\bar{\rho}_{(a_i)} \not\simeq \bar{\rho}_{(a'_i)}$.

The assumptions (REG), (NCR), and (MF′) remain in force during all § 4.4.4. Although it does not seem possible to weaken significantly the hypotheses (REG), (NCR) in order to prove Theorem 4.4.6 below, hypothesis (MF′) is probably unnecessary. It is equivalent to the assertion that for all $k \in \{1,\dots,d\}$, $\Lambda^k T$ is a residually multiplicity free pseudocharacter with residual irreducible component the traces of the representations $\bar{\rho}_{(a_i)}$ with $\sum_{i=1}^r a_i = k$.

We will use again some notations and concepts introduced in Section 1, applied to the residually multiplicity free pseudocharacter $T : A[G] \longrightarrow A$. Let $\mathcal{P}$ be the finest partition $\{\{1\},\dots,\{r\}\}$ of $\{1,\dots,r\}$, so $I_{\mathcal{P}}$ is the total reducibility ideal of T. Recall that for every ideal $I \subsetneq A$ containing $I_{\mathcal{P}}$ (see § 1.5.1), there is for each i a unique continuous representation

$$\rho_i : G \longrightarrow \mathrm{GL}_{d_i}(A/I)$$

whose reduction mod m is $\bar{\rho}_i$ and such that $T \otimes A/I = \sum_{i=1}^r \operatorname{tr}\rho_i$ (see Definition 1.5.3, Proposition 1.5.10).

Let $1 \leq i \leq r$ and write $R_i = \{j_1,\dots,j_{d_i}\}$ with $s \mapsto j_s$ increasing. We define a continuous character $\delta_i : \mathbb{Q}_p^* \longrightarrow (A^*)^{d_i}$ by setting:

$$\delta_i(p) := (F_{j_1}p^{\kappa_{j_1}(z)-\kappa_{\sigma(j_1)}(z)},\dots,F_{j_{d_i}}p^{\kappa_{j_{d_i}}(z)-\kappa_{\sigma(j_i)}(z)}),$$

$$\delta_{i|\mathbb{Z}_p^*} = (\kappa_{\sigma(j_1)}^{-1},\dots,\kappa_{\sigma(j_{d_i})}^{-1}).$$

We will need to consider the following further assumption on the partition R_i defined in § 4.4.3:

(INT) Each R_i is a subinterval of $\{1,\dots,d\}$.

Theorem 4.4.4. — *Assume (INT) and let $I_{\mathcal{P}} \subset I \subsetneq A$ be any cofinite length ideal. Then for each i, ρ_i is a trianguline deformation of $(\bar{\rho}_i, \mathcal{F}_{z,i})$ whose parameter is δ_i.*

Moreover, for each $n \in \{1,\dots,d\}$, we have

$$\kappa_{\sigma(n)} - \kappa_n = \kappa_{\sigma(n)}(z) - \kappa_n(z) \ \text{ in } A/I_{\mathcal{P}}.$$

Proof. — We argue as in the proof of Theorem 4.4.1 taking into account the extra difficulties coming from the reducible situation. By (INT), we have for each i that $R_i = \{x_i+1, x_i+2,\dots,x_i+d_i\}$ for some $x_i \in \{1,\dots,r\}$. Up to renumbering the $\bar{\rho}_i$, we may assume that $x_1 = 0$ and that $x_i = d_1 + \cdots + d_{i-1}$ if $i > 1$.

We fix I as in the statement. We will prove below that each ρ_i is a trianguline deformation of $(\bar{\rho}_i, \mathcal{F}_{z,i})$ whose parameter δ'_i coincides with δ_i on p, but satisfies

$$\delta'_{i|\mathbb{Z}_p^*} = (\kappa_{j_1}^{-1}\chi^{\kappa_{j_1}(z)-\kappa_{\sigma(j_1)}(z)},\dots,\kappa_{j_{d_i}}^{-1}\chi^{\kappa_{j_{d_i}}(z)-\kappa_{\sigma(j_{d_i})}(z)}).$$

As the Sen polynomial of ρ_i is

$$\prod_{s=1}^{d_i}(T-\kappa_{\sigma(j_s)})$$

by Lemma 4.3.3 and by definition of σ (use the fact that the $\kappa_n(z)$ are distinct), Proposition 2.3.3 will then conclude the second part of the statement (argue as in the proof of Theorem 4.3.4 to go from I to $I_{\mathcal{P}}$).

Let us prove now the result mentioned above. Fix $j \in \{1,\dots,r\}$ and if $j > 1$ assume by induction that for each $i < j$, ρ_i is a trianguline deformation of $(\bar{\rho}_i, \mathcal{F}_{z,i})$ whose parameter is δ'_i defined above. Note that $\mathcal{F}_{z,i}$ is regular by (INT) and (REG) (see the proof below for more details about this point), and non critical by (NCR). So by Proposition 2.5.6, it suffices to prove that for $h=1,\dots,d_j$,

(52) $D_{\mathrm{crys}}((\Lambda^h\rho_j)(\kappa_{x_j+1}+\cdots+\kappa_{x_j+h}))^{\varphi=F_{x_j+1}\cdots F_{x_j+h}}$ is free of rank 1 over A/I,

what we do now.

For $k=x_j+h$ any number in R_j, let $a_i(k)=|R_i\cap\{1,\dots,k\}|$ for $i\in\{1,\dots,r\}$. In other words, we have $a_i(k)=d_i$ (resp. $a_i(k)=0$) for all $i\in\{1,\dots,j-1\}$ (resp. for $i>j$), and $a_j(k)=h$. We want to apply Theorem 4.3.2 to the weakly refined families Λ^kT, $k\in R_j$, as in the proof of Theorem 4.4.1. We set again $F=\prod_{n=1}^k F_n$ and $\kappa=\sum_{k=1}^n\kappa_n$. As already explained in the proof of Theorem 4.4.1, the family Λ^kT satisfies the assumption of §4.3.1.

We note first that the (unique by (REG)) irreducible subrepresentation of $\Lambda^k\bar{\rho}_z$ that has the φ-eigenvalue $p^{\kappa(z)}F(z)$ in its D_{crys} is $\bar{\rho}_{(a_i(k))}$ with the notations of (MF′). This representation is exactly $\Lambda^h(\bar{\rho}_j)$ twisted by each $\det(\bar{\rho}_i)$ with $i<j$ (twisted by nothing if $j=1$). With the obvious definition for the $\rho_{(a_i)}$ when (a_i) is any sequence as in (MF′), we have a decomposition

$$\Lambda^kT\otimes A/I=\sum_{(a_i)}\mathrm{tr}\,(\rho_{(a_i)}),$$

hence I contains the total reducibility ideal of Λ^kT (Λ^kT is multiplicity free by (MF′)). Theorem 4.3.2 implies then that

$$D_{\mathrm{crys}}(\rho_{(a_i(k))}(\kappa))^{\varphi=F} \tag{53}$$

is free of rank one over A/I.

By induction, we know that ρ_i is a trianguline deformation of $(\bar{\rho}_i,\mathcal{F}_{z,i})$ whose parameter is δ'_i for each $i<j$. In particular, for any such i,

$$\det(\rho_i)(\kappa_{x_i+1}+\cdots+\kappa_{x_i+d_i})$$

is a crystalline character of G_p whose Frobenius eigenvalue is $F_{x_i+1}\cdots F_{x_i+d_i}$. As

$$\rho_{(a_i(k))}(\kappa)=\Lambda^h(\rho_j)(\kappa_{x_j+1}+\cdots+\kappa_{x_j+h})\bigotimes_{i<j}\det(\rho_i)(\kappa_{x_i+1}+\cdots+\kappa_{x_i+d_i}),$$

we get from formula (53) that

$$D_{\mathrm{crys}}((\Lambda^h \rho_j)(\kappa_{x_j+1} + \cdots + \kappa_{x_j+h}))^{\varphi = F_{x_j+1} \cdots F_{x_j+h}}$$

is free of rank 1 over A/I for $h = 1, \dots, d_j$, which is the assertion (52) that we had to prove. □

Note that the theorem implies that $\kappa_n - \kappa_m$ is constant on the total reducibility locus whenever n and m are in the same σ-orbit.

Corollary 4.4.5. — *Assume (INT) and that the permutation σ is transitive.*

(i) *Every difference of weights $\kappa_n - \kappa_m$ is constant on the total reducibility locus.*

(ii) *Assume moreover that one weight $\kappa_m \in A/I$ is constant, and that for some i we have $\mathrm{Hom}_{G_p}(\bar{\rho}_i, \bar{\rho}_i(-1)) = 0$. Then ρ_i is crystalline.*

Proof. — The assertion (i) follows immediately from the second assertion of Theorem 4.4.4.

As a consequence, if κ_m is constant for some m, every κ_n is constant on the total reducibility locus. By Theorem 4.4.4 and Proposition 2.3.3, this means that each ρ_j, seen as a representation $\rho_j : G \longrightarrow \mathrm{GL}_{d_j}(A/I)$, $I_{\mathcal{P}} \subset I \subsetneq A$, is Hodge-Tate. On the other hand, each ρ_j is a trianguline deformation of the non critically refined representation $(\bar{\rho}_j, \mathcal{F}_{z,j})$ again by Theorem 4.4.4, hence ρ_i is crystalline by Proposition 2.5.1. □

It turns out that the "non-trianguline" part of Theorem 4.4.4, namely that the $\kappa_n - \kappa_{\sigma(n)}$ are constant on the total reducibility locus, can be also proved even if we do not assume (INT), but instead the different kind of assumption:

(HT') For each $k \in \{1, \dots, d\}$, $\Lambda^k \bar{\rho}_z$ has distinct Hodge-Tate weights.

Theorem 4.4.6. — *Assume (HT') (or (INT)). Then for all $n = 1, \dots, d$,*

$$(\kappa_{\sigma(n)} - \kappa_n) - (\kappa_{\sigma(n)}(z) - \kappa_n(z)) \in I_{\mathcal{P}}$$

In other words, $\kappa_{\sigma(n)} - \kappa_n$ is constant on the total reducibility locus.

Of course, part (i) of Corollary 4.4.5 also holds assuming (HT') instead of (INT).

Proof. — It is obviously sufficient to prove that for all k in $\{1, \dots, r\}$, we have

$$\sum_{n=1}^{k} \left(\kappa_{\sigma(n)} - \kappa_n - (\kappa_{\sigma(n)}(z) - \kappa_n(z))\right) \in I_{\mathcal{P}}. \tag{54}$$

We consider the family $(X, \Lambda^k T)$. As seen in § 4.2.4, this family is naturally weakly refined, with same set Z,

$$F = \prod_{n=1}^{k} F_n, \tag{55}$$

and first weight

$$\kappa = \sum_{n=1}^{k} \kappa_n. \tag{56}$$

As already explained in the proof of Theorem 4.4.1 this family satisfies all the hypotheses of § 4.3.1, and we want to apply to it Theorem 4.3.4.

For this, we note first that the (unique by (REG)) irreducible subrepresentation of $\Lambda^k \bar{\rho}_z$ that has the φ-eigenvalue $p^{\kappa(z)}F(z)$ in its D_{crys} is the one denoted $\bar{\rho}_{(a_i)}$ above, with a_i being, for $i = 1, \dots, r$, the numbers of $n \leq k$ such that $p^{\kappa_n(z)}F_n(z)$ is an eigenvalue of $D_{\mathrm{crys}}(\bar{\rho}_i)$. In other words, a_i is the number of $n \in \{1, \dots, k\}$ such that $n \in R_i$, that is $a_i = |R_i \cap \{1, \dots, k\}|$.

It follows from (NCR) and Lemma 2.4.8 that $D_{\mathrm{crys}}(\bar{\rho}_{(a_i)}(\kappa(z)))^{\varphi=F(z)}$ has weight $\kappa'(z) - \kappa(z)$, where $\kappa'(z)$ is the smallest weight of $\bar{\rho}_{(a_i)}$. Hence $\kappa'(z)$ is the sum, for $n = 1, \dots, k$ of the sum of the a_n smallest weights of $\bar{\rho}_n$. In other words,

$$\kappa'(z) = \sum_{n=1}^{k} \kappa_{\sigma(n)}(z). \tag{57}$$

We now are in position to apply Theorem 4.3.4, which tells us that

$$\kappa' - \kappa - (\kappa'(z) - \kappa(z)) \in I$$

where I is the total reducibility ideal for the pseudocharacter $\Lambda^k T$. But it follows immediately from the definition of reducibility ideals and from hypothesis (MF′) that $I \subset I_{\mathcal{P}}$, the total irreducibility ideal of T. So

$$\kappa' - \kappa - (\kappa'(z) - \kappa(z)) \in I_{\mathcal{P}},$$

which, using (56) and (57) is the formula (54) we wanted to prove. □

4.5. Results on other reducibility loci

It would be nice, and certainly useful, to have a result analogous to Theorem 4.4.6 for arbitrary reducibility ideals $I_{\mathcal{P}}$, not only the total reducibility ideal. This result should probably be that certain differences of weights $\kappa_i - \kappa_j$, for suitable couples (i, j) combinatorically defined in terms of the permutation σ and the partition $\mathcal{P}$, should be constant of the reducibility locus attached to $\mathcal{P}$.

But when we try to apply the methods used above, we get into trouble because there does not exist in general a module M_I for I a subset of $\{1, \dots, r\}$, analogous to the module M_j for $j \in \{1, \dots, r\}$, in the sense that for J a cofinite-length ideal of A, the isotypic component of the $\bar{\rho}_j$, $j \in I$ in M_I/JM_I is free over A/J. This lack of freeness prevents to apply the "constant weight lemma" to this module, and more generally any of our main results of section 2. This may be a strong motivation to extend the results of section 2 to the non-free case, but this does not seem to be easy, and we postpone this question to subsequent works (of us or others).

However, we can still get an interesting although much coarser result on arbitrary reducibility loci by the method of our Theorem 9.1 in [8]. We shall give a sufficient condition for the other (non-trivial) reducibility ideals at a point z to be torsion free. This is equivalent to saying that the pseudocharacter T is generically irreducible over every irreducible component of X through z.

As our result is coarse, we do not need for it our hypotheses of § 4.4.1, so we release (NCR), and (MF′), and we only assume below that z is a point of Z that satisfies (REG). In that context the definitions of the subsets W_i and R_i (for $i = 1, \dots, r$) of $\{1, \dots, d\}$ in § 4.4.3 still make sense. For every $P \subset \{1, \dots, r\}$ we define the subset $W_P := \coprod_{i \in P} W_i$ and $R_P := \coprod_{i \in P} R_i$.

Theorem 4.5.1. — *Let $\mathcal{P} = \{P, Q\}$ be a non-trivial partition of $\{1, \dots, r\}$. Assume that $W_P \neq R_P$. Then $I_\mathcal{P}$ is a non-zero torsion-free ideal of A.*

Remark 4.5.2. — In particular, if the permutation σ of § 4.4.3 is transitive, then the hypothesis of this theorem holds for all P since $\sigma(R_P) = W_P$. In this case, the conclusion may be rephrased as: T is generically irreducible on each irreducible component of X through z.

When $\bar\rho_z$ is ordinary, the hypothesis of the theorem, for all P, is equivalent to the transitivity of σ. In general, the transitivity is a stronger assumption.

Proof. — Let $K = \prod K_s$ be the total fraction ring of A. We have to prove that $I_\mathcal{P} K = K$, that is that for all s, $I_\mathcal{P} K_s = K_s$. Replacing X by its normalization $\widetilde{X}$, A by its integral closure in K_s, the F_i and κ_i by their composition with $\tilde{X} \longrightarrow X$, and Z by its inverse image in $\widetilde{X}$, we may assume that A is a domain, that X is normal irreductible, and what we have to prove is now that $I_\mathcal{P} \neq 0$.

Assume by contradiction that $I_\mathcal{P} = 0$. Then there are two A-valued pseudocharacters T_P and T_Q such that

$$T = T_P + T_Q, \quad \text{and } T_* \otimes k = \sum_{i \in *} \operatorname{tr} \bar\rho_i.$$

Reducing X, we may assume that X is an affinoid neighbourghood of z (note that $z \in Z$), that T_P and T_Q take values in $\mathcal{O}(X)$, that for $i \neq j$ the $\kappa_i - \kappa_j$ are invertible on X (since so they are at z), and that T_P is the generic trace of a representation of G on a finite type torsion free $\mathcal{O}(X)$-module [15], say $M(X)$. By the maximum principle, the $v(F_n)$, $n = 1, \dots, d$ are bounded on X. Hence Prop. 4.5.3 below implies that there

[15] As T_P is residually multiplicity free, the existence of such a module follows for example from Lemma 4.3.7.

is a set $I \subset \{1, \dots, d\}$ and $Z_1 \subset Z$ such that T_P is refined by the κ_n, the F_n for $n \in I$ and Z_1.

We now claim that the eigenvalues of the crystalline Frobenius on

$$(\bar{\rho}_P)_z := \oplus_{i \in P} \bar{\rho}_i$$

are the $p^{\kappa_n(z)} F_n(z)$ for $n \in I$ (in other words, we claim that we could assume that $z \in Z_1$). Indeed, by Kisin's theorem applied to the torsion free quotient of $\Lambda^k M(X)$ (apply Theorem 3.3.3 to a flatification of the latter module as in the proof of Theorem 3.4.1), $1 \leq k \leq |I| = \dim T_P$ and to the maximal ideal of A, we get denoting by I_k the first k elements in I,

$$D_{\mathrm{crys}}(\Lambda^k(\bar{\rho}_P)_z)^{\varphi = \prod_{n \in I_k} p^{\kappa_n(x)} F_n(x)} \neq 0.$$

The claim follows from this and (REG).

By definition, we thus have $R_P = I$. Similarly, since the weights of $\bar{\rho}_z$ are the $\kappa_n(z)$, $n \in I$, we have $W_P = I$. But this implies that $W_P = R_P$, a contradiction. □

Proposition 4.5.3. — *Let (X, T) be a refined family as above. We assume that X is connected, that the $\kappa_i - \kappa_j \in \mathcal{O}(X)^*$ for all $i \neq j$, and that the $v(F_n)$, $n = 1, \dots, d$ are bounded on X. If $T = T_1 + T_2$ where T_i, $i = 1, 2$ are pseudocharacters $G \to \mathcal{O}(X)$, then there is a subset I of $\{1, \dots, d\}$ and a subset Z_1 of Z such that (X, T_1) is refined by*[16] *$((\kappa_n)_{n \in I}, (F_n)_{n \in I}, Z_1)$.*

Remark 4.5.4. — As we saw in the proof of Theorem 4.5.1 we can actually enlarge Z_1 to contain all the points of Z that satisfy (REG).

Proof. — We denote by $(\bar{\rho}_1)_x$ (resp. $(\bar{\rho}_2)_x$) the semi-simple representation of trace the evaluation of T_1 (resp. T_2) at x, so that we obviously have

$$\bar{\rho}_x \simeq (\bar{\rho}_1)_x \oplus (\bar{\rho}_2)_x. \tag{58}$$

We first prove that there is an $I \subset \{1, \dots, d\}$, with $|I| = \dim T_1$, such that for all $x \in X$, the Hodge-Tate-Sen weights of $(\bar{\rho}_1)_x$ are the $\kappa_n(x)$, $n \in I$. For this we will only use property (i) of Definition 4.2.3 of a refined family. Since X is connected, and the weights everywhere distinct, it is obviously sufficient to prove it when X is replaced by any connected open subset U of an admissible covering of X. So we may assume that X is an affinoid. Since $\mathcal{O}(X)$ is noetherian, and by replacing X by a finite surjective covering if necessary, we may assume that there exists (see [**8**, Lemme 7.1]) a finite type torsion-free module $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) on $\mathcal{O}(X)$ with a continuous Galois action whose trace (defined after tensorizing by the fraction field of $\mathcal{O}(X)$) is T_1 (resp. T_2). Replacing X by a blow-up X' as in Lemma 3.4.2, we may also assume that $\mathcal{M}_1$, $\mathcal{M}_2$ are locally free, and by localizing again, that X is a connected affinoid and that

[16] The implicit ordering on I here is the natural induced by $\{1, \dots, d\}$.

$\mathcal{M}_1(X)$, $\mathcal{M}_2(X)$ are free modules. The Sen polynomial of the module $\mathcal{M}_1 \oplus \mathcal{M}_2$ is $\prod_{n=1}^d (T - \kappa_n)$. Since it is split and X is connected, it is easy to see that the Sen polynomial of $\mathcal{M}_1$ has the form $\prod_{n \in I}(T - \kappa_n)$ for some subset I of $\{1, \dots, d\}$. This proves the first assertion.

Now choose an integer C greater than $\sum_{n=1}^d \sup_X v(F_n)$ and also greater than $d^2 + 1$. Let $z \in Z_C$. By (58), there is a subset J of $\{1, \dots d\}$, with $|J| = \dim T_1$, such that the Frobenius eigenvalues of $\bar{\rho}_1(z)$ are $p^{\kappa_n(x)} F_n(x)$, $n \in J$. By admissibility of $D_{\mathrm{crys}}((\bar{\rho}_1)_z)$, we have

$$\sum_{n \in I} \kappa_n(z) = \sum_{n \in J} (v(F_n(z)) + \kappa_n(z)),$$

that is

$$\kappa_I(z) - \kappa_J(z) = \sum_{n \in J} v(F_n(z)),$$

where $\kappa_*(x) = \sum_{n \in *} \kappa_n(x)$. That implies

$$|\kappa_I(z) - \kappa_J(z)| \leq \sum_{n=1}^{d} |v(F_n(z))| \leq C,$$

so by Lemma 4.5.5 below we have $J = I$. Thus it is clear that $((\kappa_n)_{n \in I}, (F_n)_{n \in I}, Z_C)$ is a refinement of (X, T_1). □

The following lemma is a formal consequence of property (v) of refined families.

Lemma 4.5.5. — *Assume that $C \geq d^2 + 1$. If I and J are two distinct non empty subsets of $\{1, \dots, d\}$ with the same cardinality, then for all $z \in Z_C$ we have*

$$|\kappa_I(z) - \kappa_J(z)| > C.$$

Proof. — Let $n + 1$ be the greatest integer that is in I or J but not both. We assume that $n + 1 \in I$. In $n = 1$, then $\kappa_I(z) - \kappa_J(z) = \kappa_2(z) - \kappa_1(z)$ and the lemma is clear by definition of Z_C. So assume that $n \geq 2$. We have

$$\kappa_I(z) - \kappa_J(z) = \kappa_{n+1}(z) + \sum_{l=1}^{n} \epsilon_l \kappa_l(z)$$

with $\epsilon_l \in \{-1, 0, 1\}$ and $\sum_{l=1}^n \epsilon_l = -1$. By adding terms of the form $\kappa_l(z) - \kappa_l(z)$, we may write $\sum_{l=1}^n \epsilon_l \kappa_l(z)$ as $-\kappa_n(z)$ plus a sum of at most $(n+1)^2 \leq d^2$ terms of the form $\pm(\kappa_l(z) - \kappa_{l-1}(z))$, $2 \leq l \leq n$. Those terms are, in absolute value, no greater than $|\kappa_n(z) - \kappa_{n-1}(z)|$ by definition of Z_C. Thus

$$|\kappa_I(z) - \kappa_J(z)| \geq |\kappa_{n+1}(z) - \kappa_n(z)| - d^2 |\kappa_n(z) - \kappa_{n-1}(z)|.$$

By the definition of Z_C, and the fact that $C \geq d^2 + 1$, we thus have

$$|\kappa_I(z) - \kappa_J(z)| > (d^2 + 1 - d^2)|\kappa_n(z) - \kappa_{n-1}(z)| > C. \quad \square$$

CHAPTER 5

SELMER GROUPS AND A CONJECTURE OF BLOCH-KATO

We recall in this section the Galois cohomological version of the standard conjectures on the order of vanishing of arithmetic L-functions at integers. The main references are [**23**] and [**55**].

5.1. A conjecture of Bloch-Kato

5.1.1. Geometric representations. — Let E be a number field, p a prime and F a finite extension of $\mathbb{Q}_p$. Let

$$\rho : G_E \longrightarrow \mathrm{GL}_n(F)$$

be a continuous representation of the absolute Galois group G_E of E, which is *geometric* in the sense of Fontaine and Mazur (see [**55**]), that is:

- ρ is unramified outside a finite number of places of E,
- $\rho_{|G_{E_v}}$ is De Rham for each place v dividing p.

It is known that the natural Galois representation on the étale cohomology groups

$$H^i_{et}(X_{\overline{E}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} F(d),$$

where X is proper smooth over E and $d \in \mathbb{Z}$, is geometric. The Fontaine-Mazur conjecture is the statement that every irreducible geometric continuous G_E-representation ρ is a subrepresentation of such a representation on an étale cohomology group.

5.1.2. Selmer groups. — We now define the Selmer group $H^1_f(E, \rho)$ of a geometric representation ρ. This is the F-subvector space of the continuous Galois cohomology group [1] $H^1(G_E,\rho)$ that parameterizes the isomorphism classes of continuous extensions

$$0 \longrightarrow \rho \longrightarrow U \longrightarrow F \longrightarrow 0, \tag{59}$$

where F denotes the trivial $F[G_E]$-module, satisfying for each finite place v of E:

i) $\dim U^{I_v} = 1 + \dim \rho^{I_v}$ if v does not divide p,
ii) $\dim D_{\mathrm{crys}}(U_{|G_v}) = 1 + \dim D_{\mathrm{crys}}(\rho_{|G_v})$ if v divides p.

For example, such an U is unramified (resp. crystalline) at a place v whenever ρ is. Moreover, at places v dividing p, condition ii) implies $\dim D_{\mathrm{dR}}(U_{|G_v}) = 1 + \dim D_{\mathrm{dR}}(\rho_{|G_v})$ so U is De Rham since ρ is. In particular, U is geometric. As a consequence (see e.g. [**104**, Prop. B.2.7]), $H^1_f(E,\rho)$ is a finite dimensional F-vector space.

Similarly, if v is a place of E and ρ a continuous representation of G_{E_v}, we define the local Selmer group $H^1_f(E_v,\rho)$ as the subspace of $H^1(E_v,\rho)$ that parameterizes the extensions of 1 by ρ that satisfy condition i) if v is prime to p, or condition ii) if v divides p.

Remark 5.1.1. — i) The formation of $H^1_f(E,\rho)$ commutes with any finite extension of the field F of coefficients of ρ.

ii) The functors $V \mapsto V^{I_v}$ and $V \mapsto D_{\mathrm{crys}}(V)$ (on the category of continuous $F[G_{E_v}]$-modules) being left exact, both conditions i) and ii) may be viewed as the requirement that they transform the short exact sequence (59) of $F[G_E]$-modules into a short *exact* sequence of vector spaces.

iii) By Grothendieck's l-adic monodromy theorem, condition *i*) is automatic if $(U^{I_v})^{\mathrm{ss}}$ does not contain the cyclotomic character.

Example 5.1.2. — i) Assume that $\rho = \mathbb{Q}_p(1)$ is the cyclotomic character. Kummer theory (or Hilbert 90) shows that there is a canonical isomorphism

$$E^*\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p \xrightarrow{\sim} H^1(E,\mathbb{Q}_p(1)).$$

Under this identification, it is well known that [2] $\mathcal{O}_E^*\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p \xrightarrow{\sim} H^1_f(E,\mathbb{Q}_p(1))$. If we relax the hypothesis f at a finite set S of places of E, we get S-units instead of units of E.

ii) Assume that A is an abelian variety over E and take $\rho = T_p(A) \otimes \mathbb{Q}_p$. Then it known that the f condition at a place v cuts out precisely the elements of

(1) For the basic properties of continuous cohomology in this context, see e.g. [**104**, App. B].
(2) First show the local analogue with E replaced by any E_v, and conclude using the finiteness of the class number of E.

the $H^1(G_E, T_p(A))$ coming locally at v from an E_v-rational point of A (when v divides p, see [**23**]). The Kummer sequence becomes then:

$$0 \longrightarrow A(E) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow H^1_f(E, T_p(A)) \longrightarrow \mathrm{Sha}_p(A) \otimes \mathbb{Q}_p \longrightarrow 0,$$

where $\mathrm{Sha}_p(A)$ is the dual of the Tate-Shafarevich group of A. Assuming the finiteness of the Tate-Shafarevich group, $H^1_f(E, T_p(A))$ appears to be a purely Galois theoretic description for $A(E) \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

5.1.3. The general conjecture. — Let ρ be as in § 5.1 and fix embeddings $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \longrightarrow \mathbb{C}$.

It is expected that the Artin L-function $L(\rho, s)$ attached to ρ and these embeddings converges on a right half plane and admits a meromorphic (even entire when ρ is not a Tate twist of the trivial character) continuation to the whole of $\mathbb{C}$. This is known for example when ρ corresponds to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. The general conjecture is then the following.

Conjecture 5.1.3. — $\mathrm{ord}_{s=0} L(\rho, s) = \dim_F H^1_f(E, \rho^*(1)) - \dim_F(\rho^*(1))^{G_E}$.

Note that this is a conjectural equality between two terms, the one on the left being only conjecturally defined in general! There are more precise conjectures predicting the leading coefficient of $L(\rho, s)$ at 0, but we shall not deal with them in this book. In view of Examples 5.1.2, the above conjecture generalizes the Dirichlet units theorem (together with his theorem on the finiteness of the class number) and (assuming the finiteness of the Tate-Shafarevich group) the Birch and Swinnerton-Dyer conjecture.

When ρ is a cyclotomic twist of a representation with finite image, the conjecture is a theorem of Soulé [**115**]. Moreover, in the case $n = 1$ and E totally real or imaginary quadratic, the conjecture follows from Iwasawa's main conjecture for those fields, proved by Wiles and Rubin respectively. Aside from some sporadic results concerning the sign conjecture (see below), only a few cases are known when $n = 2$ and $E = \mathbb{Q}$, and then the terms in the equality are 0 or 1 (Wiles, Rubin, Gross-Zagier, Kato). Needless to say, each of those particular cases is a very deep theorem.

Remark 5.1.4. — Assume that ρ is pure of motivic weight w. Apart from the case where $w = -1$, the conjectural left hand side of the equality in Conjecture 5.1.3 can be defined explicitly without any mention of L-function.

i) ($w \leq -2$) Indeed, if $w < -2$, then $0 > 1 + w/2$ should be in the domain of convergence of the Euler product defining $L(\rho, s)$ by Weil's conjectures, thus $\mathrm{ord}_{s=0} L(\rho, s)$ should be 0 (and so should be $H^1_f(E, \rho)$). If $w = -2$ then 0 is on the boundary of the domain of convergence, and a conjecture predicts that in this case $\mathrm{ord}_{s=0} L(\rho, s)$ should be $-\dim(\rho^*(1))^{G_E}$ (this is known when ρ is automorphic, cf. [**68**]).

ii) ($w \geq 0$) Recall that we expect a functional equation

$$\Lambda(\rho, s) = \varepsilon(\rho, s)\Lambda(\rho^*(1), -s) \tag{60}$$

where $\Lambda(\rho, s)$ is the *completed* L-function, a product of $L(\rho, s)$ by a finite number of some simple explicit Γ-*factors* (see [**116**] for the recipe). Since $\rho^*(1)$ has weight $-w-2$, and by i) above, the term $\mathrm{ord}_{s=0}L(\rho, s)$ is determined when $w \geq 0$ by the order of the poles of the Γ-factors.

However, although we can predict the integer of Conjecture 5.1.3 when $w \neq -1$, it is still completely conjectural that $\dim_F H^1_f(E, \rho)$ is actually this number. When $w = -1$, e.g. as in the Birch and Swinnerton-Dyer conjecture, the situation is even much worse (and more interesting) as the integer in question is completely mysterious.

5.1.4. The sign conjecture. — Among the cases where the motivic weight of ρ is -1, of special interest are the ones where[(3)] $\Lambda(\rho) = \Lambda(\rho^*(1))$, that is where the equation (60) takes the form:

$$\Lambda(\rho, s) = \varepsilon(\rho, s)\Lambda(\rho, -s), \tag{61}$$

In this case, 0 is the "center" of the functional equation of ρ, and we have

$$\epsilon(\rho, 0) = \pm 1.$$

This number is called the *sign* of the functional equation of ρ (or shortly the sign of ρ). As the Γ-factors do not vanish on the real axis, Conjecture 5.1.3 leads to an important special case, that we will call the *sign conjecture*:

Conjecture 5.1.5. — *Assume ρ satisfies* (61). *If $\varepsilon(\rho, 0) = -1$, then*

$$H^1_f(E, \rho^*(1)) \neq 0.$$

Remark 5.1.6. — (i) The sign conjecture for $E = \mathbb{Q}$ implies the sign conjecture for any E. For if ρ is a geometric irreducible representation of G_E whose functional equation satisfies (61) with sign -1, $\tau = \mathrm{Ind}_{G_{\mathbb{Q}}}^{G_E} \rho$ is a semi-simple representation of $G_{\mathbb{Q}}$ with same sign, isomorphic Selmer group, and satisfies $\tau \simeq \tau^*(1)$ by Lemma 5.1.7. It follows that τ is a direct sum of a subrepresentation $\tau_0 \oplus \tau_0^*(1)$ (whose sign is 1) and of irreducible subrepresentations $\tau_1, \dots, \tau_l$ such that $\tau_i \simeq \tau_i^*(1)$ for $i = 1, \dots, l$. Since the product of the signs of the factors of a direct sum is the sign of that direct sum, if ρ has sign -1 there must be an i such that τ_i has sign -1. Thus the sign conjecture for $\mathbb{Q}$ asserts the existence of a non zero element in $H^1_f(\mathbb{Q}, \tau_i)$, hence in $H^1_f(\mathbb{Q}, \tau) = H^1_f(E, \rho)$.

(3) Of course, this happens for instance when (60) holds and when ρ (or some Galois conjugate) is isomorphic to $\rho^*(1)$, see Lemma 5.1.7 below.

(ii) Even if the analytic continuation at 0 of $L(\rho, s)$ is not known, it is possible to give a non conjectural meaning to the sign $\epsilon(\rho, 0)$ (which is a product of local terms), hence to the sign conjecture (see [**57**, §3]).

As an exercise, let us determine when equation (61) holds. We need a notation: for $\sigma \in \mathrm{Aut}(E)$, we denote by ρ^σ the representation (well defined up to isomorphism) $g \mapsto \rho(\tau g \tau^{-1})$ where $\tau \in G_{\mathbb{Q}}$ is an element inducing σ on E.

Lemma 5.1.7. — *We assume* (60). *Then equation* (61) *holds if there exists a* $\sigma \in \mathrm{Aut}(E)$ *such that* $\rho^*(1) \simeq \rho^\sigma$. *When E is Galois (resp. $E = \mathbb{Q}$) and ρ is irreducible (resp. semisimple), the converse holds.*

Proof. — In view of equation (60), equation (61) holds if and only if ρ and $\rho^*(1)$ have equal Λ-functions. As any $\sigma \in \mathrm{Aut}\,(E)$ induces a norm-preserving bijection on primes ideal of E, it is clear that $\Lambda(\rho, s) = \Lambda(\rho^\sigma, s)$ and the first assertion follows.

For the converse, it is enough to show that when E is Galois, if two irreducible, continuous and almost everywhere unramified, representations ρ and ρ' of G_E have the same L-function, there exists a $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ such that $\rho \simeq \rho'^\sigma$. When $E = \mathbb{Q}$ and ρ and ρ' are more generally semisimple, that is true because they have equal characteristic polynomials of Frobenii for almost all p, hence $\rho \simeq \rho'$ by Cebotarev's theorem. Now for E any number field, if ρ and ρ' are semi-simple representations of G_E having the same L-functions then this still holds for $\mathrm{Ind}_{G_{\mathbb{Q}}}^{G_E}\rho$ and $\mathrm{Ind}_{G_E}^{G_{\mathbb{Q}}}\rho'$ which hence are isomorphic. Taking the restrictions to G_E, we find that if E is Galois, we have

$$\oplus_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})}\rho^{\sigma} \simeq \oplus_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})}\rho.$$

Hence, if ρ is irreducible, then it is isomorphic to a ρ^σ. □

5.2. The quadratic imaginary case

5.2.1. Assumptions and notations. — Throughout this paper, we will assume that E is an imaginary quadratic field, and we shall denote by σ a complex conjugation in $\mathrm{Gal}(\bar{E}/\mathbb{Q})$, and by c its image in $\mathrm{Gal}(E/\mathbb{Q})$, so that $\sigma^2 = c^2 = 1$. For U any representation of G_E, we set $U^\sigma(g) = U(\sigma g \sigma)$ and we denote by $U^\perp$ of the representation

$$U^\perp := (U^\sigma)^*.$$

We shall fix a continuous geometric n-dimensional representation ρ of G_E over F, and we shall assume that

$$\rho \simeq \rho^\perp(1). \tag{62}$$

Hence ρ should satisfy Equation (61) by Lemma 5.1.7. Note that in this case

$$H^1_f(E,\rho^*(1)) \simeq H^1_f(E,\rho^\sigma) \simeq H^1_f(E,\rho).$$

Our main objectives are, assuming some widely believed (and that might well be proved soon) conjectures in the theory of automorphic forms:

(1) to prove the sign conjecture for such ρ;
(2) to give a lower bound of the Selmer groups $H^1_f(E,\rho)$ depending on the geometry of an explicit unitary eigenvariety, at an explicit point.

5.2.2. An important example. — Aside from the case $n=1$, which is already of interest, an important class of examples is provided by base change to E of classical modular forms.

Let k be an even integer, N an integer prime to p, and f a normalized cuspidal newform for $\Gamma_0(N)$. If F denotes the completion at a place dividing p of the field of coefficients of f, we shall denote by ρ_f the representation $G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(F)$ attached to f and normalized in such a way that $\rho_f^*(1) \simeq \rho_f$ and that $\det(\rho_f)$ is the cyclotomic character. (This uses that k is even: ρ_f is the twist of the usual normalization by $\mathbb{Q}_p(k/2)$.) In particular, ρ_f has weight -1. We note $\rho_{f,E}$ the restriction of ρ_f to G_E. Obviously $\rho_{f,E}$ satisfies (62). For suitable choices of E, the Selmer group of $\rho_{f,E}$ turns out not to be bigger than the Selmer group of ρ_f, as the following well known proposition shows.

Proposition 5.2.1. — *Let f be as above, and S any finite set of primes. There is an imaginary quadratic field E, split at every prime of S, such that $\rho_{f,E}$ is irreducible and*

$$H^1_f(\mathbb{Q},\rho_f) = H^1_f(E,\rho_{f,E}).$$

Proof. — Indeed, we have $H^1_f(E,\rho_{f,E}) \simeq H^1_f(\mathbb{Q},\rho_f) \oplus H^1_f(\mathbb{Q},\rho_f\otimes\chi_E)$ where χ_E is the non trivial quadratic character of $G_{\mathbb{Q}}$ with kernel G_E. By the main result of [**66**], generalizing [**121**], there is an infinite number of quadratic imaginary fields E that split at every prime of S and such that $L(\rho_f\otimes\chi_E,0)\neq 0$. For such an E, [**70**, Thm 14.2 (2)] proves that $H^1_f(\mathbb{Q},\rho_f\otimes\chi_E)=0$, hence the proposition. □

5.2.3. Upper bounds on auxiliary Selmer groups. — In [**5**] as well as in subsequent works using an automorphic method to produce elements in Selmer groups ([**112**], [**8**], this book), an important input is a result giving an *upper bound* (for instance, 0) on the dimension of auxiliary Selmer groups.

The most elementary case of such a result is the next proposition, which is of crucial importance in both proofs of chapter 8 and 9. It would become false if E were replaced by a CM field of degree greater than or equal to 2, and it is actually the only point in the proof of the sign conjecture where the fact that E is quadratic is really used.

Proposition 5.2.2. — *We have $H^1_f(E, \mathbb{Q}_p(1)) = 0$.*

Proof. — By Example 5.1.2(1), $H^1_f(E, \mathbb{Q}_p(1))$ is isomorphic to $\mathcal{O}_E^* \otimes_{\mathbb{Z}} \mathbb{Q}_p$, which is 0 as $\mathcal{O}_E^*$ is finite. □

This vanishing result turns out to be the only one necessary to the proof of the sign conjecture. However, we shall need quite a number of other vanishing results to get our second main result. The easiest ones are dealt with the following proposition.

Proposition 5.2.3. — i) *$H^1_f(E, \mathbb{Q}_p) = 0$,*

ii) *$H^1_f(E, \mathbb{Q}_p(-1)) = 0$. Moreover, for $\epsilon = \pm 1$, the subspace of $H^1(E, \mathbb{Q}_p(-1))$ parameterizing extensions U of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ such that $U^{\perp}(1) \simeq \epsilon U$ (as extensions) has dimension ≤ 1.*

Proof. — Let U be a G_p-representation which is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p$. Recall that U is crystalline (resp. Hodge-Tate) if and only if it is unramified. Indeed, the only non trivial fact is to show that "U is Hodge-Tate" implies "U is unramified", but this follows for instance from the following general result of Sen [**107**, §3.2, Corollary]: in any Hodge-Tate representation of G_p with all of its Hodge-Tate weights equal to 0, the inertia acts through a finite quotient. So $H^1_f(E, \mathbb{Q}_p) = \mathrm{Hom}(\mathrm{Gal}(E^{ur}/E), \mathbb{Q}_p)$ where E^{ur} is the maximal unramified everywhere algebraic extension of E. By class-field theory we thus have

$$H^1_f(E, \mathbb{Q}_p) = \mathrm{Cl}(\mathcal{O}_E) \otimes_{\mathbb{Z}} \mathbb{Q}_p = 0,$$

which proves (i). It is well known that part (ii) follows from results of Soulé and from the invariance of Bloch-Kato conjecture under duality. For sake of completeness, we give an argument below.

First, note that if $F/\mathbb{Q}_p$ is a finite extension, Bloch-Kato's theorem shows that $H^1_f(F, \mathbb{Q}_p(-1)) = 0$ (see e.g. formula (43) in the proof of Thm. 2.5.10). By Soulé's theorem [**115**, Thm. 1]

$$H^2(\mathcal{O}_E[1/p], \mathbb{Q}_p(2)) = 0.$$

The version of Poitou-Tate exact sequence given in [**55**, prop. 2.2.1] shows then that $H^1_f(E, \mathbb{Q}_p(-1)) = 0$ when applied to the Galois module $\mathbb{Q}_p(2)$. So we get an injection

$$H^1(E, \mathbb{Q}_p(-1)) \longrightarrow \bigoplus_{v|p} H^1(E_v, \mathbb{Q}_p(-1)),$$

which is compatible with the operation $U \mapsto U^{\sigma}$ on the domain and the exchange of v and $\bar{v}$ on the range when $p = v\bar{v}$ splits in E. We conclude as $H^1(E_v, \mathbb{Q}_p(-1))$ has dimension 1 by Tate's theorem. □

The two other vanishing results we will need are somehow deeper. Since they are expected to be proved by completely different methods (e.g. using Euler systems) than those used in this paper, it would be artificial to limit ourselves to the case where those results have actually been proved, hence we take them as assumptions as follows.

Hypothesis BK1(ρ). — $H^1_f(E, \rho(-1)) = 0$.

Hypothesis BK2(ρ). — *Every deformation $\tilde{\rho}$ of ρ over $F[\varepsilon]$ (the ring of dual numbers) that satisfies $\tilde{\rho}^{\perp}(1) \simeq \tilde{\rho}$ and whose corresponding cohomology class lies in*[4] *$H^1_f(E, \mathrm{ad}\rho)$ is trivial.*

Remark 5.2.4. — Note that both hypotheses should follow from Conjecture 5.1.3 for any ρ that is pure of weight -2 (for BK2(ρ)) or -3 (for BK1(ρ)), and they are precisely in case i) of Remark 5.1.4. Fortunately, those assumptions have already been proved in interesting cases.

This conjectural vanishing of $H^1_f(E, \mathrm{ad}\rho)$ is actually fundamental for understanding eigenvarieties. Intuitively, it can be understood as follows. Let

$$R : G_E \longrightarrow \mathrm{GL}_n(L\langle t\rangle)$$

be a continuous morphism, where $L\langle t\rangle$ is the Tate algebra over L, and assume that for each $t \in \mathbb{Z}_p$ the evaluation R_t of R at t is a geometric irreducible representation. Then the Fontaine-Mazur conjectures implies that R is conjecturally constant (up to isomorphism). Indeed, each R_t is conjecturally cut out from an E-motive. But there is only a countable number of such motives, hence of $\mathrm{tr}\,(R_t)$, so $\mathrm{tr}\,(R)$ is constant. The assertion $H^1_f(E, \mathrm{ad}\rho) = 0$ is actually a slightly stronger variant of that fact, in which we replace the Tate algebra $L\langle t\rangle$ by $L[t]/t^2$.

Proposition 5.2.5. — *BK1(ρ) holds in the following two cases:*

i) *$n = 1$ and 0 is not a Hodge-Tate weight of ρ.*

ii) *$n = 2$ and ρ is of the form $\rho_{f,E}$ (using notations of § 5.2.2) for some eigenform f of weight $k \geq 4$.*

Proof. — By the theory of CM forms, case i) follows from case ii), which in turn is a result of Kato [**70**, Thm. 14.2 (1)]. □

Proposition 5.2.6. — *BK2(ρ) holds if $n = 1$ or if $n = 2$ and ρ is of the form $\rho_{f,E}$ whenever f is not CM and satisfies one of the following conditions:*

(i) *At every prime l dividing N, f is either supercuspidal or Steinberg.*

[4] Or, in an equivalent way, such that for each finite place w, $\tilde{\rho}_{|E_w}$ is geometric (automatic condition if w is prime to p) with constant monodromy operator acting on $D_{\mathrm{pst}}(\tilde{\rho}_{|E_w})$.

(ii) *The semi-simplified reduction $\bar{\rho}_f$ of ρ_f is absolutely reducible, and is (over an algebraic closure) the sum of two characters that are distinct over $G_{\mathbb{Q}(\zeta_{p^\infty})}$.*
(iii) *For any quadratic extension $L/\mathbb{Q}$ with $L \subset \mathbb{Q}(\zeta_{p^3})$, $(\bar{\rho}_f)_{|G_L}$ is absolutely irreducible.*

Proof. — Let $\chi : G_E \longrightarrow F^*$ be any geometric character. Every deformation of χ to $F[\varepsilon]$ whose associated class lies in $H^1_f(E, \mathrm{ad}(\chi))$ is trivial by Prop. 5.2.3 i), as $\mathrm{ad}\chi$ is the trivial character. In particular, $\mathrm{BK2}(\rho)$ holds for $n = 1$.

Assume now that $\rho = \rho_{f,E}$ as in the statement. Let

$$\tilde{\rho} : G_E \longrightarrow \mathrm{GL}_2(F[\varepsilon])$$

be a lift of ρ such that $\tilde{\rho} \simeq \tilde{\rho}^*(1)$, and whose associated class in $H^1(E, \mathrm{ad}\rho)$ belongs to $H^1_f(E, \mathrm{ad}\rho)$. By the previous case, the character $\det(\tilde{\rho})$ is constant, hence equals $\det(\rho_{f,E}) = \mathbb{Q}_p(1)$. As for any 2-dimensional representation over any ring, we have $\tilde{\rho} \simeq \tilde{\rho}^* \otimes \det \tilde{\rho}$, thus we get

$$\tilde{\rho} \simeq \tilde{\rho}^*(1).$$

Together with the hypothesis $\tilde{\rho} \simeq \tilde{\rho}^\perp(1)$, we get $\tilde{\rho} \simeq \tilde{\rho}^\sigma$. That is, there is an $A \in \mathrm{GL}_2(F[\varepsilon])$ such that for all $g \in G_E$,

$$A\tilde{\rho}(g)A^{-1} = \tilde{\rho}(\sigma g \sigma^{-1}).$$

Since $\sigma^2 = \mathrm{Id}$, A^2 centralizes $\rho(G_E)$. As $\rho_{f,E}$ is absolutely irreducible we have $A[\tilde{\rho}(G_E)] = M_2(F[\varepsilon])$, so $A^2 = \lambda$ for some $\lambda \in F[\varepsilon]^*$. If $\bar{A}$ denotes the reduction of A modulo ε, we have for all $g \in G_E$

$$\bar{A}\rho(g)\bar{A}^{-1} = \rho_f(\sigma)\rho(g)\rho_f(\sigma^{-1}),$$

thus $\bar{A}^{-1}\rho_f(\sigma)$ centralizes $\rho(G_E)$. Thus we have $\bar{A} = \mu\rho_f(\sigma)$ for some $\mu \in F^*$. In particular $\lambda = A^2 \equiv \mu^2 \pmod{\varepsilon}$. Let $\tilde{\mu}$ be the square root of λ in $F[\varepsilon]^*$ lifting μ. For $g \in G_E$, set $\tilde{\rho}_f(\sigma g) = \tilde{\mu}^{-1}A\tilde{\rho}(g)$ and $\tilde{\rho}_f(g) = \tilde{\rho}(g)$: this defines a deformation

$$\tilde{\rho}_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(F[\varepsilon])$$

of ρ_f whose restriction to G_E is $\tilde{\rho}$.

Since $(\tilde{\rho}_f)_{|G_E} = \tilde{\rho}$ is geometric, so is $\tilde{\rho}_f$. But such a deformation of ρ_f is trivial by [**74**, Theorem, page 2] in the cases (ii) and (iii), and by [**122**, theorem 5.5] in case (i), hence so is its restriction $\tilde{\rho}$. □

We shall actually use assumption $\mathrm{BK1}(\rho)$ to bound the subspace

$$H^1_{f'}(E, \rho(-1)) \subset H^1(E, \rho(-1))$$

parameterizing extensions that satisfy condition i) of § 5.1.2 but not necessarily condition ii).

Proposition 5.2.7. — *Assume that $p = vv'$ splits in E, that p, p^{-2} are not eigenvalues of the crystalline Frobenius on $D_{\mathrm{crys}}(\rho_{|G_{E_v}})$, and that 0 and -1 are not Hodge-Tate weights of $\rho_{|G_{E_v}}$. Then*

$$\dim_F H^1_{f'}(E, \rho(-1)) \leq n + \dim_F H^1_f(E, \rho(-1)).$$

In particular, if BK1(ρ) holds we have $\dim_F H^1_{f'}(E, \rho(-1)) \leq n$.

Proof. — We have by definition an exact sequence

$$0 \longrightarrow H^1_f(E, \rho(-1)) \longrightarrow H^1_{f'}(E, \rho(-1)) \longrightarrow H^1_s(E_v, \rho(-1)_{|G_{E_v}}) \times H^1_s(E_{v'}, \rho(-1)_{|G_{E_{v'}}}),$$

where $H^1_s(E_w, -) := H^1(E_w, -)/H^1_f(E_w, -)$. Since $\rho^\perp \simeq \rho(1)$, the last term of the exact sequence above is isomorphic to

$$H^1_s(E_v, W) \oplus H^1_s(E_v, W^*(-1)),$$

where we have set $W = \rho(-1)_{|G_{E_v}}$. In order to conclude, it is enough to show that the dimension of this latter sum is $\leq n$.

For any de Rham p-adic representation U of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, Bloch-Kato's computation [**23**, Cor. 3.8.4] (see also the proof of Theorem 2.5.10), together with Tate's Euler chararacteristic formula for $\dim_F(H^1(\mathbb{Q}_p, U))$, imply that

$$\dim_F(H^1_s(\mathbb{Q}_p, U)) = \dim_F(U) + \dim_F(H^0(\mathbb{Q}_p, U^*(1))) - \dim_F(D_{\mathrm{DR}}(U)/\mathrm{Fil}^0(D_{\mathrm{DR}}(U))).$$

The proposition would then follow from the two following facts:

$$\dim_F(D_{\mathrm{DR}}(W)/\mathrm{Fil}^0(D_{\mathrm{DR}}(W))) + \dim_F(D_{\mathrm{DR}}(W^*(-1))/\mathrm{Fil}^0(D_{\mathrm{DR}}(W^*(-1)))) = n,$$

$$H^0(\mathbb{Q}_p, W^*(1)) = H^0(\mathbb{Q}_p, W(2)) = 0.$$

The sum in the first formula is the number of Hodge-Tate weights of W (with multiplicities) which are either < 0 or > 1. As neither 0 nor 1 is a Hodge-Tate weight of W by assumption, the equality holds. The second equality also holds, by the assumption on the eigenvalues of the crystalline Frobenius. □

CHAPTER 6

AUTOMORPHIC FORMS ON DEFINITE UNITARY GROUPS: RESULTS AND CONJECTURES

6.1. Introduction

This chapter recalls or proves all the results we shall need from the theory of representations of reductive groups and of automorphic forms.

As explained in the general introduction, the main steps of our method regarding the proof of our two main theorems are, very roughly, as follows: starting with an n-dimension Galois representation ρ such that $\varepsilon(\rho, 0) = -1$, we construct a very special, non tempered, automorphic representation π^n for a unitary groups in $m = n + 2$ variables. We deform it p-adically, in other words, we put it in an eigenvariety of the unitary group. We associate to this deformation of automorphic forms a deformation of Galois representations, or rather, a Galois pseudocharacter on the eigenvariety of the unitary group. This Galois pseudocharacter gives us the desired non trivial elements in the Selmer group of ρ.

Unfortunately, some results needed to make work two of those steps in their natural generality have not yet been published or even written down: the first step, the existence of the "very special" automorphic representation π^n, has been announced, but a written proof is only available in small dimension, namely $m \leq 3$; the third step relies on the existence and the basic properties of the Galois representations attached to (some) automorphic representations of unitary groups. Here again the desired results are only known for $m \leq 3$. Fortunately, this result is also in the process of being proved: it is one of the main goals of an ambitious project gathering many experts and participants of the GRFA seminar of the "Institut de mathématiques de Jussieu" in Paris, under the direction of Michael Harris. Their work should result in a four-volumes book (**[60]**) in the next few years that is expected to contain a construction of the Galois representations attached to automorphic forms on unitary groups in many cases, and in particular in the cases we need. An important input in this project is the recent proof by Laumon and Ngo of the so-called fundamental lemma for unitary groups.

In this chapter we formulate the two needed results as conjectures (namely conjecture REP(m) on Galois representations attached to automorphic forms on unitary groups with m variables, and conjecture AC(π) on the existence of the automorphic representation π^n constructed from the automorphic counterpart π of ρ), and we shall assume those conjectures in the proof of our main theorems in chapters 8 and 9. In view of the situation explained above, it would have been pointless to limit ourselves to the case where the needed results are already written down.

The main reason for which we are able to write down some still unwriten results and rely confidently of them is not that we are told they will be proved very soon, but because they are part of a much larger and very well corroborated set of conjectures called "the Langlands program" (and its extension by Arthur).

We believe it will be of interest to explain in greater detail how our conjectures (and much more) appear as consequences of the Langlands program, and in particular how the existence of our very special non tempered automorphic forms is enlightened as a special case of the beautiful "multiplicity formulas" of Arthur. This is the aim of the appendix to this book, that recalls the part of the Langlands and Arthur's program that we need, and where we show how our conjectures follow from theirs. This appendix may be read independently, as an introduction to Langlands and Arthur's parameterizations and multiplicity formulas. Although logically independent of it, the rest of the chapter will make frequent references to this appendix for the sake of the reader's intuition.

Although we may expect that results much more general than the modest conjectures we state to be true, and even to be proved soon, we made a great deal of effort, in this chapter and throughout this book, to keep our conjectural input to the theory of automorphic forms at the lowest possible level. One reason for doing this is obvious: the weaker the assumptions we have to assume, the stronger is our result, and the sooner it will become an unconjectural theorem. Another more serious reason is that part of our work (especially chapter 7) is also expected to be used in the book [**60**] for the construction or the proof of some properties of the Galois representations in some "limit" cases which one can not handle with a direct comparison of trace formulae. So the logical scheme would be as follows: in [**60**] should be proved "directly" for a quite "generic" set of automorphic representations the existence and properties of the associated Galois representations, which should be enough to check our conjecture Rep(m). In turn, our work on eigenvarieties should complete the picture by providing existence and properties of the Galois representations attached to the remaining (cohomological) automorphic forms. For example, our conjecture Rep(m) only requires the Galois representations for automorphic forms of regular weights. To give another example, our method (hence our conjecture Rep(m)) makes no irreducibility hypothesis on the Galois representations, [1] but instead may be used to prove many cases of irreducibility (see e.g. Theorem 7.7.1).

Let us now explain more specifically the content of the chapter.

The subsection § 6.2 deals with some general facts about unitary groups, with an emphasis on the *definite* ones and their automorphic representations. We define explicitly the unitary groups $\mathrm{U}(m)$ we will work with. We need a group that is quasi-split at every finite place (otherwise, the representation π^n can not be automorphic, as explained in the appendix - see Remark A.12.4), but that is also compact at infinity - so that we can apply the theory of eigenvarieties of [**36**]. [2] This leads to the restriction that $m \not\equiv 2 \bmod 4$.

The subsections § 6.3 to § 6.7 are local preliminaries. The short subsection § 6.3 recalls the local Langlands correspondence for GL_m, as characterized by Henniart and proved by Harris and Taylor. It will be used very frequently. The subsection § 6.4 develops the theory of refinements (sometimes called *p-stabilizations*) of unramified representations of $\mathrm{GL}_m(\mathbb{Q}_p)$ which a representation theoretic counterpart of the theory of refinements of crystalline Galois representations that we explained in chapter 2. We invite the reader to look at the introduction § 6.4 of that subsection for a more precise discussion on this concept. The subsection § 6.7 recalls two descriptions of the continuous irreducible representations of the compact group $\mathrm{U}(m)(\mathbb{R})$ and compares them.

Next come two other subsections of local preliminaries. They are both devoted to the crucial question of monodromy. [3] By "monodromy" of an admissible irreducible representation π_l of $\mathrm{U}_m(\mathbb{Q}_l)$ we mean the conjectural notion encoded in the nilpotent element that appears as part of the conjectural morphism of the Weil-Deligne group of $\mathbb{Q}_l$ to ${}^L\mathrm{U}(m)$ attached to π_l. Concretely, what we need is threefold. We need to give a non-conjectural meaning to expressions such as "π_l has no more monodromy

(1) Let us say that this is anyway a subtle point, as only the stable tempered automorphic representations should have irreducible associated Galois representations, and this property is very hard to detect in practice. This actually introduces an extra difficulty in the applications to the construction of nontrivial elements in the Selmer groups that we will explain how to circumvent. This feature was already present in [**8**], but was absent of the earliest stages of the method, like in [**5**] or later in [**112**].

(2) Note that we may not use in this context the construction of p-adic families announced recently by Urban, since the "virtual multiplicity" of our π^n might be zero.

(3) Let us say that monodromy is bound to play a crucial role in our final arguments. Indeed, it follows from the Arthur multiplicity formula that under the hypothesis $\varepsilon(\rho, 0) = 1$ (not -1) there should exist an automorphic representation π'^n for $\mathrm{U}(m)$, isomorphic to π^n at every place except one, say l with l inert in the splitting quadratic field E of $\mathrm{U}(m)$, and such that π'^n_l has the same L-parameter as π^n_l on $W_{\mathbb{Q}_l}$ but a greater monodromy. If it was possible to apply our method to π'^n, it would eventually lead to a construction of a non-trivial element in the Selmer group of ρ, element which should not exist when $L(\rho, 0) \neq 0$ according to the conjecture of Bloch-Kato. This shows that a precise control of monodromy has to play a role in our argument.

that π_l'" or "π_l has no monodromy at all". We need tools to be able to show in chapter 7 that some full irreducible components of the eigenvarieties of $\mathrm{U}(m)$ containing π^n "have no more monodromy at every place l than π_l^n has". Finally, we need to be able to translate this "control on monodromy of π_l" into a control on the action of the inertia subgroup at l on the Galois representations attached to π. The latter is a part of our conjecture $\mathrm{Rep}(m)$. The objective of § 6.5 and § 6.6 is to meet the two first needs.

In § 6.5, we deal with the monodromy of representations of $\mathrm{U}(m)(\mathbb{Q}_l)$ for l split in E, that is for $\mathrm{GL}_m(\mathbb{Q}_l)$. In this case the meaning of the monodromy is non conjectural, thanks to the local Langlands correspondence, that associates to a representation π_l a conjugacy class of nilpotent matrices $N(\pi_l) \in \mathrm{GL}_m(\mathbb{C})$; we can simply say that π_l has more monodromy than π_l' if the closure of the conjugacy class of $N(\pi_l)$ contains $N(\pi_l')$. To be able to control the variation of the monodromy in a family of such π_l, we use then the existence of some particular K-types. As we shall see, this will fullfill our second need since a general property of the eigenvarieties we will study is that the locus of points whose associated $\mathrm{U}(m)(\mathbb{Q}_l)$-representation contains a given K-type is a union of irreducible components (this actually holds for every l). Of course, the simplest example of such a K-type is the trivial representation of $\mathrm{GL}_m(\mathbb{Z}_l)$, which cuts out precisely the unramified constituent of the unramified principal series (that is, the non monodromic ones). For a general monodromy type, we use suitable K-types that have been constructed by Schneider and Zink (see § 6.5). Note that the types constructed by Bushnell and Kutzko are *a priori* of no use for our purposes because they "do not see monodromy". However, let us stress that the construction of Schneider and Zink actually relies on those types.

In § 6.6, we deal with representations of $\mathrm{U}(m)(\mathbb{Q}_l)$ for l inert or ramified in E. The group $\mathrm{U}(m)(\mathbb{Q}_l)$ is a quasi-split group, but it is not split, and the situation in this case is much less favorable. First we do not know the local Langlands correspondence for those groups, neither we know the base change to GL_m/E (from a conjectural point of view, see the final appendix). Hence there is no obvious way to define "having less monodromy than" or "having no monodromy at all" for a representation π of $\mathrm{U}(m)(\mathbb{Q}_l)$. Even worse, we were not able to come up with a plausible characterization, in terms of group theory, of those irreducible admissible representations of $\mathrm{U}(m)(\mathbb{Q}_l)$ that conjecturally have no monodromy [(4)]. Second, there is no theory of types à la Bushnell-Kutzko for $\mathrm{U}(m)(\mathbb{Q}_l)$, $m > 3$, not to speak of a theory à la Schneider-Zink. The first solution we imagined to solve those problems was to avoid them:

(4) To convince the reader that this question is not easy, let us say that for $m = 3$, there is a *supercuspidal* representation of $\mathrm{U}(3)(\mathbb{Q}_l)$, discovered by Rogawski and called π^s, whose base change has a non trivial monodromy. See § A.10 in the final appendix.

that is, to assume all our automorphic representations to be unramified [5] at inert or ramified l. An unramified representation should certainly be "non-monodromic", and unramifiedness is easy to control in deformation as explained in the GL_m-case above. But the problem is: for odd m, there is no representation of $\mathrm{U}(m)$ of the form π^n that is unramified at ramified primes. [6] So this assumption is much too restrictive. Instead, we introduce a special class of principal series representations of $\mathrm{U}(m)(\mathbb{Q}_l)$ that certainly should have no monodromy, and which will enable us to deal with a large number of ρ also when m is odd. We call those representations *Non Monodromic (Strongly Regular) Principal Series.* We show in § 6.6 that to be a non monodromic principal series is a constructible property in a family.

After these local preliminaries, we turn to global questions. In subsection § 6.8, we state our assumption $\mathrm{Rep}(m)$ on existence and simple properties of the Galois representations attached to (some) automorphic forms of $\mathrm{U}(m)$. In subsection § 6.9 we construct place by place a representation π^n of $\mathrm{U}(m)(\mathbb{A}_\mathbb{Q})$ starting from a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying some properties (recall that $m = n+2$). We then state as a conjecture $\mathrm{AC}(\pi)$ (even if as we said earlier this has been announced) that this π^n, under the assumption that $\varepsilon(\pi, 1/2) = -1$, is automorphic.

6.2. Definite unitary groups over $\mathbb{Q}$

6.2.1. Unitary groups. — Let k be a field, E/k an étale k-algebra of degree 2 with non trivial k-automorphism c, and Δ a simple central E-algebra of rank m^2 equipped with a k-algebra anti-involution $x \mapsto x^*$ of the *second kind*, *i.e.* coinciding with c on E. We can attach to this datum $(\Delta, *)$ a linear algebraic k-group G whose points on a k-algebra A are given by

$$G(A) := \{x \in (\Delta \otimes_k A)^*, xx^* = 1\}.$$

The base change $G \times_k E$ is then isomorphic to the E-group Δ^* of invertible elements of E, hence G is a twisted k-form of GL_m. Actually, as is well known, every twisted k-form of GL_m is isomorphic to such a group.

Example 6.2.1. — They are two different cases.

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[5] Recall that the notion of *unramified* representation makes sense for any quasi-split group: it means having a non-zero fixed vector by a "very special" maximal compact subgroup, in the sense of Labesse.

[6] More precisely, for odd m any discrete automorphic representation π of $\mathrm{U}(m)$ whose A-packet lies in the image of the endoscopic transfert ${}^L(\mathrm{U}(n) \times \mathrm{U}(2)) \longrightarrow {}^L\mathrm{U}(m)$ has the property that its base change π_E to GL_m/E is ramified at each prime of E ramified above $\mathbb{Q}$ (see the appendix § A.9). The reason is that for odd m the aforementionned L-morphism contains in its definition a Hecke character μ of E such that $\mu^\perp = \mu$ but which does not descend to $\mathrm{U}(1)$. Such a Hecke character is automatically ramified at the primes of E ramified over $\mathbb{Q}$ (see § 6.9.2).

i) If $E \xrightarrow{\sim} k \times k$, then $\Delta \xrightarrow{\sim} \Delta_1 \times \Delta_2$ and $* : \Delta_1 \longrightarrow \Delta_2^{\mathrm{opp}}$ is an isomorphism. In this case, the choice of $i \in \{1,2\}$ induces a k-group isomorphism $G \xrightarrow{\sim} \Delta_i^*$. In particular if $\Delta \xrightarrow{\sim} M_m(E)$, then the choice of i determines a k-isomorphism $G \xrightarrow{\sim} \mathrm{GL}_m$ which is canonical up to *inner* automorphisms.

ii) If E is a field, then we say that G (and $G(k)$) is a *unitary group* attached to E/k. When moreover $\Delta = M_m(E)$, then $*$ is necessarily the adjunction with respect to a non degenerate c-Hermitian form f on E^m, hence G is the usual unitary group attached to this form. If f is the standard anti-diagonal form

$$f(xe_i, ye_j) = c(x)y\delta_{j,m-i+1},$$

then G is quasi-split, and will be referred in the sequel as the *m-variables quasi-split unitary group attached to E/k.*

6.2.2. The definite unitary group $\mathrm{U}(m)$. — Suppose from now that $k = \mathbb{Q}$, E is a quadratic imaginary field, and assume that $\Delta = M_m(E)$ and $*$ is attached to some form f on E^m as in ii) above. Then G is a unitary group over $\mathbb{Q}$. For each place v of $\mathbb{Q}$, the local component $G \times_{\mathbb{Q}} \mathbb{Q}_v$ is then the $\mathbb{Q}_v$-group attached to the datum $(\Delta \otimes_{\mathbb{Q}} \mathbb{Q}_v, *)$, hence by Example 6.2.1:

i) If $p = xx'$ is a finite prime split in E, then $x : E \longrightarrow \mathbb{Q}_p$ induces an isomorphism $G(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_m(\mathbb{Q}_p)$,

ii) if p is inert or ramified, then $G(\mathbb{Q}_p)$ is a unitary group attached to $E_p/\mathbb{Q}_p$,

iii) each embedding $E \longrightarrow \mathbb{C}$ gives an isomorphism between $G(\mathbb{R})$ and the usual real unitary group $U(p,q)$, where (p,q) is the signature of f on $E^m \otimes_{\mathbb{Q}} \mathbb{R}$, $p+q = m$.

We say that G is *definite* if $G(\mathbb{R})$ is compact, or which is the same if $pq = 0$. We will be interested in definite unitary groups G with some prescribed local properties. Their existence can be deduced from the Hasse's principle for unitary groups over number fields for which we refer to [**40**, §2] but for subsequent computations, it may be useful to give them explicitly.

Let $N : E \longrightarrow \mathbb{Q}$, $x \mapsto xc(x)$ be the norm map, $m \geq 1$ an integer.

Definition 6.2.2. — $\mathrm{U}(m)$ is the m-variables unitary group attached to the positive definite c-hermitian form q on E^m defined by

$$q((z_1, \ldots, z_m)) = \sum_{i=1}^{m} N(z_i).$$

Proposition 6.2.3. — (i) $\mathrm{U}(m)$ *is a definite unitary group.*

(ii) *If l does not split in E, and $m \not\equiv 2 \bmod 4$, then $\mathrm{U}(m)(\mathbb{Q}_l)$ is the quasi-split m-variables unitary group attached to $E_l/\mathbb{Q}_l$.*

If $m \not\equiv 2 \bmod 4$, the group $\mathrm{U}(m)$ is the unique m-variables unitary group attached to $E/\mathbb{Q}$ that is quasi-split at every finite place and compact at infinity. If $m \equiv 2 \bmod 4$, there is no group with those properties.

Proof. — (i) is obvious and (ii) is an immediate consequence of Lemma 6.2.4 below, since $\mathrm{disc}(q) = 1$ (see [**51**, chap. VI] for the basics on hermitian forms and unitary groups). The other assertions (that we shall not use) follow from Hasse's principle ([**40**, §2]). □

In the following lemma, we write $\mathrm{disc}(q) \in \mathbb{Q}_l^*/N(E_l^*)$ for the discriminant of a non degenerate c-hermitian form q and denote by q_0 the *hyperbolic* form $q_0(x,y) = \frac{xc(y)+yc(x)}{2}$ on E_l^2. Note that $\mathrm{disc}(q_0) = -1$.

Lemma 6.2.4. — *Let q be a non degenerate c-hermitian form on E_l^m.*

(a) *If m is odd, then q is equivalent to*

$$\sum_{i=1}^{\frac{m-1}{2}} q_0(z_{2i-1}, z_{2i}) + (-1)^{\frac{m-1}{2}} \mathrm{disc}(q) N(z_m).$$

For $\lambda \in \mathbb{Q}_l^*$, $\mathrm{disc}(\lambda q) \equiv \lambda \mathrm{disc}(q)$, therefore there is a unique non-degenerate c-hermitian form up to a scalar.

(b) *If m is even, then q is equivalent to*

$$\sum_{i=1}^{\frac{m-2}{2}} q_0(z_{2i-1}, z_{2i}) + N(z_{m-1}) + (-1)^{\frac{m}{2}-1} \mathrm{disc}(q) N(z_m).$$

The index of q is $m/2$ if and only if $(-1)^{m/2}\mathrm{disc}(q) \in N(E_l^*)$.

Proof. — Recall that a quadratic form on $\mathbb{Q}_l^s$ with $s \geq 5$ always has a zero (see e.g. [**111**, Chap. IV Thm. 6]). We may view E_l^m as a $\mathbb{Q}_l$-vector space of rank $2m$ and q as a quadratic form on that space, so q has a zero when $m \geq 3$. As a consequence, q contains a hyperbolic plane and we may assume $m = 2$ by induction (or $m = 1$, but this case is obvious). Applying the previous remark to the form $q((z_1, z_2)) - N(z_3)$ on E_l^3, we get that $q(v) = 1$ for some $v \in E_l^m$, which concludes the proof. □

6.2.3. Automorphic forms and representations. — Let G be a definite unitary group. We denote by $\mathbb{A}$ the $\mathbb{Q}$-algebra of $\mathbb{Q}$-adèles and $\mathbb{A} \longrightarrow \mathbb{A}_f$ the projection to the finite adèles. We have the following two important finiteness results:

i) As $G(\mathbb{R})$ is compact, $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A}_f)$, hence for each compact open subgroup $K \subset G(\mathbb{A}_f)$, the arithmetic group $K \cap G(\mathbb{Q})$ is finite.
ii) By Borel's general result on the finiteness of the class number ([**25**]), for any K as above $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.

The space of automorphic forms of G is the representation of $G(\mathbb{A})$ by right translations on the space $A(G)$ of complex functions on $X := G(\mathbb{Q})\backslash G(\mathbb{A})$ which are smooth and $G(\mathbb{R})$-finite. The space X is compact by i) and ii). It admits a $G(\mathbb{A})$-invariant finite Radon measure, so that $A(G)$ is a pre-unitary representation.

Lemma 6.2.5. — *The representation $A(G)$ is admissible and is the direct sum of irreducible representations of $G(\mathbb{A})$:*

$$A(G) = \bigoplus_{\pi} m(\pi)\pi, \tag{63}$$

where π describes all the (isomorphism classes of) irreducible admissible representations of $G(\mathbb{A})$, and $m(\pi)$ is the (always finite) multiplicity of π in the above space.

It will be convenient to denote by $\mathrm{Irr}(\mathbb{R})$ the set (of isomorphism classes) of irreducible complex continuous (hence finite dimensional) representations of $G(\mathbb{R})$. For $W \in \mathrm{Irr}(\mathbb{R})$, we define $A(G,W)$ to be the $G(\mathbb{A}_f)$-representation by right translations on the space of smooth vector valued functions $f : G(\mathbb{A}_f) \longrightarrow W^*$ such that $f(\gamma g) = \gamma_\infty f(g)$ for all $g \in G(\mathbb{A}_f)$ and $\gamma \in G(\mathbb{Q})$.

Proof. — As $G(\mathbb{R})$ is compact the action of $G(\mathbb{R})$ on $A(G)$ is completely reducible, hence as $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$ representation we have:

$$A(G) = \bigoplus_{W \in Irr(\mathbb{R})} W \otimes (A(G) \otimes W^*)^{G(\mathbb{R})}.$$

But we check at once that the restriction map $f \mapsto f_{|1 \times G(\mathbb{A}_f)}$ induces a $G(\mathbb{A}_f)$-equivariant isomorphism

$$(A(G) \otimes W^*)^{G(\mathbb{R})} \xrightarrow{\sim} A(G,W).$$

As a consequence, ii) shows that $A(G)$ is admissible, which together with the pre-unitariness of $A(G,W)$ proves the lemma. □

Definition 6.2.6. — An irreducible representation π of $G(\mathbb{A})$ is said to be *automorphic* if $m(\pi) \neq 0$.

Let $W \in \mathrm{Irr}(\mathbb{R})$ and let us restrict it to $G(\mathbb{Q}) \hookrightarrow G(\mathbb{R})$. As is well known (see § 6.7), W comes from an algebraic representation of G, hence the choice of an embedding $\overline{\mathbb{Q}} \longrightarrow \mathbb{C}$ equips W with a $\overline{\mathbb{Q}}$-structure $W(\overline{\mathbb{Q}})$ which is $G(\overline{\mathbb{Q}})$-stable. As a consequence, the obviously defined space $A(G, W(\overline{\mathbb{Q}}))$ provides a $G(\mathbb{A}_f)$-stable $\overline{\mathbb{Q}}$-structure on $A(G,W)$.

Corollary 6.2.7. — *If $\pi = \pi_\infty \otimes \pi_f$ is an automorphic representation of G, then π_f is defined over a number field.*

6.3. The local Langlands correspondence for GL_m

Let $m \geq 1$ be any integer and p a prime. Let $F/\mathbb{Q}_p$ be a finite extension, W_F its Weil group, $I_F \subset \mathrm{W}_F$ the inertia group, and $|.|$ the absolute value of F such that the norm of a uniformizer is the reciprocal of the number of elements of the residue field. We normalize the reciprocity isomorphism of local class-field theory

$$\mathrm{rec} : F^* \longrightarrow \mathrm{W}_F^{\mathrm{ab}}$$

so that uniformizers correspond to geometric Frobenius elements. By an m-dimensional Weil-Deligne representation (r, N) of F we mean the data of a continuous homomorphism

$$r : \mathrm{W}_F \longrightarrow \mathrm{GL}_m(\mathbb{C})$$

such that $r(\mathrm{W}_F)$ consists of semi-simple elements, and of a nilpotent matrix

$$N \in M_m(\mathbb{C}),$$

satisfying $r(w)Nr(w^{-1}) = |\mathrm{rec}^{-1}(w)|N$ for all $w \in \mathrm{W}_F$.

Recall from [**62**, Thm. A] that the Langlands correspondence is known for the group $\mathrm{GL}_m(F)$, and we shall use it with the normalization given *loc. cit.* This parameterization is a bijection

$$\pi \mapsto L(\pi) = (r(\pi), N(\pi))$$

between the set $\mathrm{Irr}(\mathrm{GL}_m(F))$ of isomorphism classes of irreducible smooth complex representations π of $\mathrm{GL}_m(F)$ and the set of isomorphism classes of m-dimensional Weil-Deligne representations of F. It satisfies various properties. For example:

- When $m = 1$, $\mathrm{GL}_1(F) = F^*$, we have $N(\chi) = 0$ and $r(\chi) = \chi \circ \mathrm{rec}^{-1}$ for any smooth character $\chi : F^* \longrightarrow \mathbb{C}^*$. In general, the L-parameter of the central character of π is $\det(L(\pi))$, and for any smooth character $\chi : F^* \longrightarrow \mathbb{C}^*$, $L(\pi \otimes \chi \circ \det) = L(\pi) \otimes L(\chi)$.
- π is superscuspidal (resp. ess. square integrable) if, and only if, $L(\pi)$ is irreducible (resp. indecomposable).
- If π_i is an ess. square integrable representation of $\mathrm{GL}_{m_i}(F)$, and $\sum_i m_i = m$, then $\oplus_i L(\pi_i)$ is the L-parameter of the Langlands quotient $\bigotimes_i \pi_i$ (when it makes sense).

6.4. Refinements of unramified representations of GL_m

In this subsection, we explain some aspects of the representation theoretic counterpart [(7)] of the theory of refinements developed in Section 2. The simplest example

[(7)] Actually, the theory developed in this part is comparatively much simpler than the Galois theoretic one of Section 2, as we are reduced here to see refinements as some orderings of some Frobenius of this notion is the well known fact that any classical modular eigenform of level 1 (weight k, say) generates a two-dimensional vector space of p-old forms of level $\Gamma_0(p)$. These old forms all have the same T_l-eigenvalues for $l \neq p$, and the Atkin-Lehner U_p operator preserves this two-dimensional space with characteristic polynomial $X^2 - t_p X + p^{k-1}$.

From a representation theoretic point of view, this last computation is a purely local statement, namely the computation of the characteristic polynomial of U_p, a specific element of the Hecke-Iwahori algebra, on the space of Iwahori invariants of a given irreducible unramified smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. In what follows, we explain how this theory generalizes to $\mathrm{GL}_m(\mathbb{Q}_p)$, focusing essentially on the unramified case. In [**36**, §4.8] and [**8**, §6], we explained how to deduce them from the Bernstein presentation of the Hecke-Iwahori algebra. Here we use an alternative approach based on the Borel isomorphism and the geometrical lemma.

6.4.1. The Atkin-Lehner rings. — Let F be a finite extension of $\mathbb{Q}_p$ with uniformizer ϖ and ring of integer $\mathcal{O}_F$. We denote by G the group $\mathrm{GL}_m(F)$, B its upper Borel subgroup, N the unipotent radical of B, and T the diagonal torus of G. Let $K := \mathrm{GL}_m(\mathcal{O}_F)$, $T^0 = K \cap T$, and let I be the Iwahori subgroup of G consisting of elements of K which are upper triangular modulo ϖ.

The Hecke-Iwahori algebra is the $\mathbb{Z}[\frac{1}{p}]$-algebra $\mathcal{C}_c(I\backslash G/I, \mathbb{Z}[\frac{1}{p}])$ of bi-I-invariant and compactly supported functions on G with values in $\mathbb{Z}[\frac{1}{p}]$, for the convolution product normalized such that I has mass 1. If $g \in G$, we denote by $[IgI]$ the characteristic function of IgI. We introduce now two important subrings of $\mathcal{C}_c(I\backslash G/I, \mathbb{Z}[\frac{1}{p}])$, that we call the *Atkin-Lehner rings* following Lazarus. Let $U \subset T$ be the subgroup of diagonal elements whose entries are integral powers of ϖ, $U^- \subset U$ the submonoid whose elements have the form

$$\mathrm{diag}(\varpi^{a_1}, \varpi^{a_2}, \ldots, \varpi^{a_m}), \quad a_i \in \mathbb{Z}, \quad a_i \geq a_{i+1} \ \forall i.$$

We define $\mathcal{A}_p^- \subset \mathcal{C}_c(I\backslash G/I, \mathbb{Z})$ as the subring generated by the $[IuI]$, $u \in U^-$. Recall that for each $u \in U^-$, $[IuI]$ is invertible in $\mathcal{C}_c(I\backslash G/I, \mathbb{Z}[\frac{1}{p}])$ by [**67**, §3], hence it makes sense to define also

$$\mathcal{A}_p \subset \mathcal{C}_c(I\backslash G/I, \mathbb{Z}[\frac{1}{p}])$$

as the ring generated by the elements $[IuI]$, $u \in U^-$, and their inverses.

Proposition 6.4.1. — (i) *The subset $M := IU^-I \subset G$ is a submonoid, and the map $M \longrightarrow U$, $iui' \mapsto u$, is a well defined homomorphism.*

eigenvalues in the complex world. The relation could certainly be pushed much further, in the style of the work of M. Emerton for $\mathrm{GL}_2(\mathbb{Q}_p)$ [**52**].

(ii) *The map* $U^- \longrightarrow \mathcal{A}_p^-$, $u \mapsto [IuI]$, *extends uniquely to ring isomorphisms* $\mathbb{Z}[U^-] \xrightarrow{\sim} \mathcal{A}_p^-$ *and* $\mathbb{Z}[U] \xrightarrow{\sim} \mathcal{A}_p$.

We warn the reader that when $m > 1$, the above homomorphism does not in general send $u \in U$ to $[IuI]$, but rather on $[IaI].[IbI]^{-1}$ for any a, $b \in U^-$ such that $u = ab^{-1}$.

Proof. — By [**35**, Lemma 4.1.5], $M := \coprod_{u \in U^-} IuI \subset G$ is a disjoint union, $\forall u, u' \in U^-$, $IuIu'I = Iuu'I$, hence M is a submonoid of G, and also $[IuI].[Iu'I] = [Iuu'I]$, which proves (i) and the first part of (ii). The proposition follows then from the easy fact that $U^- \to U$ is the symmetrisation of the monoid U^-. □

Example 6.4.2. — As a consequence of Prop. 6.4.1, we will systematically view $\mathcal{A}_p$-modules as U-modules. For example, let π be a smooth representation of G, say with complex coefficients. The vector space π^I of Iwahori invariant vectors inherits a $\mathbb{C}$-linear action of $\mathcal{C}_c(I\backslash G/I, \mathbb{Z}[\frac{1}{p}])$, hence of $\mathcal{A}_p$, hence is a U-module in a natural way. It turns out that this U-module structure on π^I is related to the Jacquet-module of π via the following result of Borel-Casselman.

If V is a representation of G, we denote by V_N the Jacquet-module of V with respect to N (see e.g. § 6.6.1), that is the space of coinvariants of N, with its natural action of T.

Proposition 6.4.3. — *For any smooth complex representation* π *of* G, *the natural map*

$$\pi^I \longrightarrow (\pi_N)^{T^0} \otimes \delta_B^{-1},$$

is a $\mathbb{C}[U]$*-linear isomorphism.*

Proof. — As the $[IuI]$ are invertible in the Hecke-Iwahori algebra, we have $\pi^I = [IuI].\pi^I$ for each $u \in U^-$. The result follows then from Prop. 4.1.4 and Lemma 4.1.1 of [**35**], and from the fact that $[IuI]v = \delta_B^{-1}(u)\mathcal{P}_I(u(v))$ for each $u \in U^-$ and $v \in \pi$ by Lemma 1.5.1 of *loc. cit.* □

6.4.2. Computation of some Jacquet modules. — In order to use the previous result, we recall now the computation of the Jacquet module of some induced representations, following [**19**]. Fix $P \supset B$ a parabolic subgroup of G, L its Levi component containing T. Let $\chi : L \longrightarrow \mathbb{C}^*$ be a smooth character, viewed also as a character on P which is trivial on the unipotent radical of P. Denote by $\mathrm{Ind}_P^G(\chi)$ the unitary smooth parabolic induction of χ, that is the space of complex valued smooth functions f on G such that

$$f(pg) = \chi(p)\delta_P(g)^{1/2}f(g), \ \ \forall p \in P, \ g \in G,$$

viewed as a G representation by right translations. Here δ_P is the module character of P.

Let $\coprod_{i=1}^{r} I_i = \{1,\ldots,m\}$ be the ordered partition associated to P. If $m_i = |I_i|$, then $L = \prod_{i=1}^{s} \mathrm{GL}_{m_i}(F)$. The subgroup $W = \mathfrak{S}_m$ of permutations of $\{1,\ldots,m\}$ is a subgroup of G in the usual way ($w = (\delta_{i,w(j)})$). Let $W(P) \subset W$ be subset of elements $w \in W$ such that $w(k) < w(l)$ whenever $k < l$ and both k and l belong to the same I_i. The group W acts on the characters of T by the formula $\psi^w(t) = \psi(w^{-1}tw)$. Moreover, χ may be viewed as a character of T by restriction $T \subset P$.

Proposition 6.4.4. — *The semi-simplification of the* $\mathbb{C}[T]$*-module* $(\mathrm{Ind}_P^G \chi)_N$ *is*

$$\bigoplus_{w \in W(P)} \chi^w \delta_B^{1/2}.$$

Proof. — This is a special case of the general geometrical lemma [**19**, Lemma 2.12] (see also [**125**, Theorem 1.2]). □

6.4.3. Unramified representations. — An irreducible smooth representation of G is said to be unramified if it has a non zero vector invariant by K. The classification of unramified representations is well known and due to Satake.

Let $\chi : T \longrightarrow \mathbb{C}^*$ be a smooth character. It will be convenient for us to write χ as a product of smooth characters $\chi_i : F^* \longrightarrow \mathbb{C}^*$ such that

$$\chi((x_1,\ldots,x_m)) = \prod_{i=1}^{m} \chi_i(x_i).$$

Assume that $\chi(T^0) = 1$ and consider the induced representation $\mathrm{Ind}_B^G(\chi)$. As is easily seen, the space of its K-invariant vectors is one-dimensional hence this induced representation has a unique unramified sub-quotient $\pi(\chi)$. It turns out that:

— $\pi(\chi) \simeq \pi(\chi')$ if, and only if, $\chi = \chi^w$ for some $w \in \mathfrak{S}_m$ (see § 6.4.2 for this notation).

— each unramified representation is isomorphic to some $\pi(\chi)$.

The Langlands parameter of $\pi(\chi)$ is easy to describe. The isomorphism class of Weil-Deligne representations $L(\pi(\chi)) = (r, N)$ associated to $\pi(\chi)$ satisfies $N = 0$, $r(I_F) = 1$ (hence the name unramified). It is uniquely determined by the conjugacy class of the image of a geometric Frobenius element of W_F, namely the class of

$$\mathrm{diag}(\chi_1(\varpi),\ldots,\chi_m(\varpi)) \in \mathrm{GL}_m(\mathbb{C}).$$

Of course, this diagonal element is unique only up to permutation. We will frequently refer to this class as the semi-simple conjugacy class associated to $\pi(\chi)$.

6.4.4. Refinements. — We fix π an irreducible unramified representation of G.

Definition 6.4.5. — A refinement of π is an ordering of the eigenvalues of the semi-simple conjugacy class above associated to π. In an equivalent way, a refinement of π is a character $\chi : U \simeq T/T^0 \longrightarrow \mathbb{C}^*$ such that $\pi \simeq \pi(\chi)$, the bijection being

$$\chi \mapsto (\chi_1(\varpi), \ldots, \chi_m(\varpi)).$$

Let us chose some refinement χ of π, so that $\pi \simeq \pi(\chi)$ is an irreducible subquotient of $\mathrm{Ind}_B^G(\chi)$. By Propositions 6.4.3 and 6.4.4, we get that as a U-module

$$(\pi^I)^{ss} \hookrightarrow (\mathrm{Ind}_B^G \chi)^{I,ss} \simeq \oplus_{w \in \mathfrak{S}_m} \chi^w \delta_B^{-1/2}. \tag{64}$$

As a corollary, we have the following Proposition-Definition.

Definition 6.4.6. — If a character $\chi\delta_B^{-1/2} : U \longrightarrow \mathbb{C}^*$ occurs in π^I, or equivalently if $\chi\delta_B^{1/2}$ occurs in $\pi_N^{T_0}$, then χ is a refinement of π. We say that a refinement of π is *accessible* if it occurs this way. Equivalently, χ is an accessible refinement of π if, and only if, π occurs as a *sub*representation of $\mathrm{Ind}_B^G(\chi)$.

The equivalence in the definition above follows from Frobenius reciprocity. Note moreover that for a character of the abelian group U, it is the same to appear as a subrepresentation of π^I or as a subquotient, since π^I is finite dimensional. Any π always has at least one accessible refinement.

Remark 6.4.7. — Let π be an irreducible representation of G which is not necessarily unramified, but such that $\pi^I \neq 0$. Although we shall not use it in this book, note that it still makes sense to define a refinement of π as an unramified character $\chi : T \rightarrow \mathbb{C}^*$ such that π occurs as a subquotient of $\mathrm{Ind}_B^G(\chi)$. The above notion of accessible refinement also applies verbatim to this extended context.

For most of the representations π, all the refinements are accessible. Indeed, by [**19**][Theorem 4.2] and formula (64) we have the following positive result. Set

$$q := |\mathcal{O}_F/\varpi|.$$

Proposition 6.4.8. — *Assume that $(\chi_i/\chi_j)(\varpi) \neq q$ for all $i \neq j$. Then $\pi(\chi) = \mathrm{Ind}_B^G(\chi)$ and all the refinements of $\pi(\chi)$ are accessible.*

Example 6.4.9. —
i) If $\pi \simeq \pi(\chi)$ is tempered, then χ is known to be unitary hence Proposition 6.4.8 applies. More generally, if π is generic Proposition 6.4.8 applies.
ii) On the opposite, if π is the trivial representation then it has a unique accessible refinement, namely $\delta_B^{-1/2}$. It corresponds then to the ordering

$$(q^{\frac{m-1}{2}}, q^{\frac{m-3}{2}}, \ldots, q^{-\frac{m-1}{2}}).$$

iii) Actually, by [**19**][Rem. 4.2.2], $\pi(\chi) \simeq \mathrm{Ind}_B^G(\chi)$ if and only if the assumption of Prop. 6.4.8 is satisfied.

6.4.5. Accessible refinements of almost tempered unramified representations. — In the applications, we will need to study the accessible refinements of some π which are not tempered, which leads us to introduce the following class of unramified representations.

Let π be an unramified irreducible representation of G, and X the set of eigenvalues (with multiplicities) of the semi-simple conjugacy class attached to π, $|X| = m$. Assume that X has a partition $X = \coprod_{i=1}^r X_i$ such that:

(AT1) for each i, X_i has the form $\{x, x/q, \dots, x/q^{m_i-1}\}$ with $m_i = |X_i|$,

(AT2) the real number $|\prod_{x\in X_i} x|^{1/m_i}$ does not depend on i.

Proposition 6.4.10. — *The accessible refinements of π are the orderings $(x_1, \dots, x_m)$ on X such that there exists a bijection $\tau : \{1, \dots, m\} \to X$ with the following property: whenever $\tau(k)$ and $\tau(l)$ are in the same X_i and $x_k = qx_l$, then $k < l$.*

Proof. — Let us choose a refinement $(x_1, \dots, x_m)$ of π satisfying the condition of the statement and such that $\{x_1, \dots, x_{m_1}\} = X_1$, $\{x_{m_1+1}, \dots, x_{m_1+m_2}\} = X_2$ and so on. It exists by (AT1). Let $\chi : T \longrightarrow \mathbb{C}^*$ be the corresponding character, π is then the unramified subquotient of $\mathrm{Ind}_B^G\chi$ and we are going to identify it.

Consider the standard parabolic P of G with Levi subgroup

$$L = \mathrm{GL}_{m_1}(F) \times \mathrm{GL}_{m_2}(F) \times \cdots \times \mathrm{GL}_{m_r}(F).$$

One checks immediately that the character $\chi\delta_B^{1/2}(\delta_P)_{|B}^{-1/2}$ of T extends uniquely to a character $\psi : L \longrightarrow \mathbb{C}^*$. Explicitly, $\psi(g_1, \dots, g_r) = \prod_i \psi_i(g_i)$, where ψ_i is the unramified character of $\mathrm{GL}_{m_i}(F)$ obtained by composing the determinant $\mathrm{GL}_{m_i}(F) \to F^*$ with the character of F^* trivial on $\mathcal{O}_F^*$ and sending ϖ to the element

$$y_i := xq^{-\frac{m_i-1}{2}},$$

where x is the element of X_i appearing in (AT1). As a consequence, we have an inclusion of G-representations:

$$\mathrm{Ind}_P^G\psi \subset \mathrm{Ind}_B^G\chi.$$

Up to a twist, we may assume that the real number occurring in property (AT2) is 1. In terms of the y_i, it means that $|y_i| = 1$ for all i, *i.e.* that ψ is unitary. A theorem of Bernstein [**18**] shows then that $\mathrm{Ind}_P^G\psi$ is irreducible. As it contains obviously the K-invariant vectors of $\mathrm{Ind}_B^G\chi$, we conclude that

$$\pi \simeq \mathrm{Ind}_P^G\psi.$$

Now that we have written π as a full induced representation, the proposition is an immediate consequence of Propositions 6.4.3 and 6.4.4. □

Definition 6.4.11. — Let us say that π is *almost tempered* if X admits a partition $\{X_i\}$ satisfying $(AT1)$ and $(AT2)$, or equivalently if up to a twist π is the full parabolic induction of a unitary character.

The equivalence of the two definitions above is a consequence of the proof of Proposition 6.4.10.

Example 6.4.12. — i) If π is tempered, it is almost tempered (and the X_i's are singletons). If π is one dimensional, it is also almost tempered, for the trivial partition of X in one subset.

ii) Assume that π is the local component of a discrete automorphic representation of a unitary group (resp. of GL_m). A consequence of Arthur's conjectures (resp. of Ramanujan conjecture and Moeglin-Waldspurger's theorem [**87**]) is that π should be almost tempered. This is actually the main reason why we introduced this definition.

iii) We will need the following explicit example. Assume that π is such that X contains $q^{1/2}$ and $q^{-1/2}$ with multiplicity 1, and all of whose other elements have norm 1. Then the accessible refinements of π are exactly the orderings of X of the form
$$(\dots, q^{1/2}, \dots, q^{-1/2}, \dots),$$
that is the ones such that $q^{1/2}$ precedes $q^{-1/2}$ in the ordering.

6.5. K-types and monodromy for GL_m

We keep the notations of the preceding subsections.

6.5.1. An "ordering" relation on $\mathrm{Irr}(\mathrm{GL}_m(F))$. — Recall that the relation $\pi \sim_{I_F} \pi'$ if and only if $r(\pi)_{|I_F} \simeq r(\pi')_{|I_F}$ on $\mathrm{Irr}(\mathrm{GL}_m(F))$ is called the "*inertial equivalence*" relation. We define an order relation $\prec_{I_F}$ on each equivalence class for $\sim_{I_F}$ as follows.

Definition 6.5.1. — Let π, $\pi' \in \mathrm{Irr}(\mathrm{GL}_m(F))$, we will write $\pi \prec_{I_F} \pi'$ if $\pi \sim_{I_F} \pi'$, and if $N(\pi)$ is in the Zariski closure [8] of the set of matrices $PN(\pi')P^{-1}$ in $M_m(\mathbb{C})$ where P runs among the matrices in $\mathrm{GL}_n(\mathbb{C})$ such that $Pr(\pi')_{|I_F}P^{-1} = r(\pi)_{|I_F}$.

Remark 6.5.2. — i) In an inertial equivalence class, the minimal elements for $\prec_{I_F}$ are precisely the π with $N(\pi) = 0$, and each of them is actually a smallest element.

[8] Or in the closure for the complex topology, which amounts to the same here.

ii) Assume $m > 1$ and let 1 and St be the trivial and the Steinberg representation respectively. We have $r(1)(I_F) = r(St)(I_F) = 1$, $N(1) = 0$ and $N(St)$ has nilpotent index m, hence $1 \prec_{I_F} St$. As is well known, the $\pi \prec_{I_F} 1$ are exactly the unramified representations. Moreover, the $\pi \prec_{I_F} St$ (i.e. $\pi \sim_{I_F} 1$) can also be abstractly characterized by the property that $\pi^I \neq 0$, where I is a Iwahori subgroup, or which is the same by the property that

$$\mathrm{Hom}_K(\tau, \pi) \neq 0,$$

with $\tau = \mathrm{Ind}_I^K 1_I$ and $K = \mathrm{GL}_m(\mathcal{O}_F)$.

iii) Take $m = 2$ in the example above, the representation τ is then the direct sum of the trivial 1_H and the Steinberg St_H representation of the finite group $H := \mathrm{GL}_2(\mathbb{F}_q)$. Of course, $\pi \prec_{I_F} 1$ if and only if $\mathrm{Hom}(1_H, \pi) \neq 0$. As an exercise, the reader can check that the relation $\mathrm{Hom}_K(St_H, \pi) \neq 0$ cuts exactly the π in the trivial inertial class which are not 1-dimensional.

6.5.2. Types. — Using works of Bernstein, Zelevinski, Bushnell-Kutzko and Schneider-Zink, it turns out that Remarks 6.5.2 ii), iii) are the simplest case of a general phenomenon. We are grateful to J.-F. Dat for drawing our attention to the reference [**106**].

Proposition 6.5.3. — *Let $\pi \in \mathrm{Irr}(\mathrm{GL}_m(F))$. There exists an irreducible complex representation τ of $\mathrm{GL}_m(\mathcal{O}_F)$ such that*

i) *$\pi_{|\mathrm{GL}_m(\mathcal{O}_F)}$ contains τ with multiplicity* 1,

and for any $\pi' \in \mathrm{Irr}(\mathrm{GL}_m(F))$,

ii) $\mathrm{Hom}_{\mathrm{GL}_m(\mathcal{O}_F)}(\tau, \pi') \neq 0 \Rightarrow \pi' \prec_{I_F} \pi$,

iii) *if π' is tempered and $\pi' \prec_{I_F} \pi$, then $\mathrm{Hom}_{\mathrm{GL}_m(\mathcal{O}_F)}(\tau, \pi') \neq 0$.*

Proof. — Up to the dictionary of local Langlands correspondence, this is exactly [**106**, Prop. 6.2]. For the convenience of the reader, we explain below the relevant translation.

Fix π as in the statement, and let Ω be the (unique) Bernstein component of $\mathrm{Irr}(\mathrm{GL}_m(F))$ containing π. This component is uniquely determined by the cuspidal support of π. By the properties of the local Langlands correspondence, which is built from its restriction to the supercuspidal representations and Zelevinski's classification, this support is in turn uniquely determined by $r(\pi)_{|I_F}$: for a $\pi' \in \mathrm{Irr}(\mathrm{GL}_m(F))$, we have $\pi' \in \Omega$ if and only if π' is in the same inertial class as π.

The additional datum of the matrix $N(\pi)$ determines then the way Zelevinski realizes π as a Langlands quotient, that is the "partition" $\mathcal{P}(\pi)$ such that π lies in $Q_{\mathcal{P}(\pi)}$ in the notations of [**106**, §2]. Such a "partition" is by definition a family of Young diagrams (see below) indexed by the cuspidal support of π, the size of a diagram being the multiplicity of the associated supercuspidal. By Gerstenhaber's theorem (see Prop. 7.8.1) and by definition of the ordering on partitions *loc. cit.* §3 (which is the opposite of the dominance ordering recalled in Appendix 7.8.1), we have for a $\pi' \in \Omega$:

$$\pi' \prec_{I_F} \pi \ \Leftrightarrow\ \mathcal{P}(\pi') \succ \mathcal{P}(\pi).$$

They define then *loc. cit.* §6 an explicit representation called $\sigma_{\mathcal{P}(\pi)}(\lambda)$ of a maximal compact subgroup of $\mathrm{GL}_m(F)$, here λ is Bushnell-Kutzko's type of the Bernstein component Ω. Up to conjugation we may assume that this maximal compact subgroup is $\mathrm{GL}_m(\mathcal{O}_F)$, and we set $\tau := \sigma_{\mathcal{P}(\pi)}(\lambda)$. The proposition is then [**106**, Prop. 6.2]. □

6.6. A class of non-monodromic representations for a quasi-split group

In this subsection, we let l be a prime number, and G the group of rational points of a connected reductive quasi-split group over a field F which is a finite extension of $\mathbb{Q}_l$. We denote by S a maximal split torus in G, T the centralizer of S, which is a maximal torus in G, and $B = TN$ a Borel containing T (where N is the unipotent radical of B). We denote by W the Weyl group of S: $W = N(S)/C(S) = N(S)/T$; this groups acts on T by conjugation.

We denote by F' a finite Galois extension of F on which G splits. We denote with the same letter with a prime the set of points over F' of the algebraic group defining one of the subgroups of G defined above: hence G', its Borel $B' = T'N'$, where T' is a maximal torus of G' (which is split). We denote by W' the Weyl group of T': $W' = N(T')/T'$. We have a natural inclusion $W \subset W'$.

6.6.1. Review of normalized induction and the Jacquet functor over a base ring. — Let A be a commutative $\mathbb{Q}$-algebra that contains a square root of l. We denote by δ_B the modulus character of B which takes values in $l^{\mathbb{Z}}$ and we choose once and for all a square root $\delta_B^{1/2} : G \to A^*$ of δ_B.

We recall some terminology concerning an $A[G]$-module M: the module M is *smooth* if every $v \in M$ is fixed by some compact open subgroup U of G and it is A-admissible if for every small enough compact open subgroup U, M^U is a finite type A-module.

If V is a smooth $A[T]$-module, we denote by $\mathrm{Ind}_B^G(V)$ the normalized induction of V from B to G, that is the A-module of all locally constant functions $f : G \to V$ such that $f(bg) = b\delta_B(b)^{1/2}f(g)$ for all $b \in B$, $g \in G$. The representation $\mathrm{Ind}_B^G(V)$ is smooth and its formation commutes with every base change $A \to A'$.

If M is an $A[G]$-module, we denote by M_N the Jacquet module of M relatively to N, that is the N-coinvariant quotient $M/M(N)$, where $M(N) = \{v \in M, \int_{N_0} \pi(n)v\, dn = 0$ for some compact subgroup $N_0 \subset N\}$, seen as a representation of $T = B/N$.

Proposition 6.6.1. — (a) *If M is smooth, then so is M_N.*

(b) *The functor $M \mapsto M_N$ from the category of smooth $A[G]$-modules to the category of smooth $A[T]$-modules is exact, and commutes with $-\otimes_A M'$ for any A-module M'.*

(c) *If M is flat (resp. if A is reduced and M is torsion free) as an A-module, then so is M_N.*

(d) *If M is A-admissible and of finite type as an $A[G]$-module, then M_N is of finite type as an A-module.*

Proof. — (a) is clear. The exactness in (b) is proved exactly as in the classical case (e.g. [**35**, Proposition 3.2.3]) once noted than N_0 is a pro-l-group, hence of pro-order invertible in A. (See also [**120**, page 96])

For M' an A-module and M an $A[G]$-module, we see $M \otimes_A M'$ as an $A[G]$-module for the trivial action of G on M'. The natural map

$$M_N \otimes_A M' \longrightarrow (M \otimes_A M')_N \tag{65}$$

is an isomorphism. Indeed, using a free presentation of M' over A, the exactness of $V \mapsto V_N$ and the left exactness of tensor products, we are reduced to the case where M' is free over A, which is obvious, hence (b) is proved.

The "torsion free" part of (c) is obvious from the exactness in (b). Assume that M is flat over A and let $X \hookrightarrow Y$ be an injective morphism of A-modules. Then $M \otimes_A X \to M \otimes_A Y$ is an injection of $A[G]$-module. Hence $(M \otimes_A X)_N \hookrightarrow (M \otimes_A Y)_N$ by (b), which is $M_N \otimes_A X \hookrightarrow M_N \otimes_A Y$ by (65). Thus M_N is flat.

Let us prove (d) (a proof is sketched in [**120**, page 96]). Since M is of finite type over $A[G]$, we deduce easily from the compactness of $B\backslash G$ (see e.g the first paragraph of the proof of [**35**, Thm 3.3.1]), that M_N is finitely generated as an $A[T]$-module. Since M_N is smooth and T is abelian, there is a compact open subgroup T_0 of T such that $M_N^{T_0} = M_N$. Up to replacing T_0 by a smaller group, we see by [**35**, Prop. 1.4.4 and Thm 3.3.3] that there is a compact open subgroup U_0 with Iwahori factorization $U_0 = N_0^- T_0 M_0$ of G such that the natural map $M^{U_0} \to M_N^{T_0}$ is surjective. Since M^{U_0} is of finite type over A by A-admissibility of M, then so is $M_N^{T_0} = M_N$. □

We recall the following easy reciprocity formula:

Lemma 6.6.2. — *If M is a smooth $A[G]$-module and V a smooth $A[T]$-module, we have a canonical isomorphism*

$$\mathrm{Hom}_{A[G]}(M, \mathrm{Ind}_B^G(V)) = \mathrm{Hom}_{A[T]}(M_N, V \otimes \delta_B^{1/2}).$$

6.6.2. Non Monodromic Strongly Regular Principal Series. — In this paragraph, we keep the preceding notations but we also assume that $A = k$ is a field. We recall that a smooth character $\chi : T \to k^*$ is *regular* when there is no $w \neq 1$ in W such that $\chi^w = \chi$. We also recall the following elementary result (cf. [**97**, Proposition 1]):

Lemma 6.6.3. — *Assume $\chi : T \to k^*$ is a smooth regular character. Then:*

(a) *The representation $\mathrm{Ind}_B^G(\chi)$ has a unique irreducible subrepresentation $S(\chi)$.*
(b) *The Jacquet module $S(\chi)_N$ contains $\chi\delta_B^{1/2}$ as a T-subrepresentation.*
(c) *Any smooth G-representation M such that M_N contains $\chi\delta_B^{1/2}$ as a T-subrepresentation has a subquotient isomorphic to $S(\chi)$.*

Proof. — By the geometric lemma, the Jacquet module $\mathrm{Ind}_B^G(\chi)^N$ is semi-simple as a T-representation and is the direct sum of the distinct characters $\chi^w\delta_B^{1/2}$ for $w \in W$. Since the Jacquet functor is exact, and as $\mathrm{Ind}_B^G(\chi)$ is of finite length (use Prop. 6.6.1 (d) and [**17**, Rem. 3.12]), one and only one of the Jacquet modules of its irreducible subquotients contains $\chi\delta_B^{1/2}$. Let us call this irreducible subquotient $S(\chi)$, which makes (b) tautologic.

On the other hand, by Lemma 6.6.2, the Jacquet module of any subrepresentation of $\mathrm{Ind}_B^G(\chi)$ contains $\chi\delta_B^{1/2}$. It has an irreducible subrepresentation, hence $S(\chi)$ is the unique irreducible subrepresentation of $\mathrm{Ind}_B^G(\chi)$, which is (a). Finally if M is as in (c), we have by Lemma 6.6.2 a non-zero morphism $M \to \mathrm{Ind}_B^G(\chi)$. Its image admits $S(\chi)$ as a subrepresentation by (a), and (c) follows. □

Recall that the *base change* of a smooth character χ of T is the character χ' of T' defined as $\chi' := \chi \circ \mathrm{Nm}$, where $\mathrm{Nm} : T' \longrightarrow T$ is the Galois norm.

Definition 6.6.4. — A smooth character χ of T is said *strongly regular* if its base change χ' is regular as a character of T'.

Since $W \subset W'$, a strongly regular character χ is also regular.

We now recall some notations of [**97**], [**98**]. Let Δ be the root system of G' with respect to T'. Let $X^*(T')$ be the group of rational characters on T' and $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. If $\alpha \in \Delta$, its associated coroot $\widehat{\alpha}$ is a linear form on V. The chambers of V are the connected components of $V - \bigcup_{\alpha\in\Delta} \mathrm{Ker}\,\widehat{\alpha}$. The Borel subgroup B' determines the choice of a "positive" chamber C^+.

Let $X_*(T')$ be the group of 1-parameter subgroups of T'. There is a canonical pairing $X^*(T') \times X_*(T') \to \mathbb{Z}$. Hence each coroot $\widehat{\alpha}$ determines canonically a 1-parameter subgroup $t^\alpha : F'^* \to T'$.

If χ' is a smooth character $T' \to k^*$ we define the set $\Sigma(\chi')$ as the set of the coroots $\widehat{\alpha}$ such that $\chi' \circ t^\alpha(a) = |a| \in k^*$ for every $a \in F'^*$. When $k = \mathbb{C}$, Rodier's theory [**97**] shows that if χ' is regular, the set $\Sigma(\chi')$ determines the reducibility of $\mathrm{Ind}_{B'}^{G'}(\chi')$ (in particular, this representation has length $2^{|\Sigma(\chi')|}$).

Definition 6.6.5. — An irreducible representation of G is said to be a *non monodromic principal series* if it is isomorphic to a representation $S(\chi)$ where

(a) χ is strongly regular.
(b) For every $\widehat{\alpha} \in \Sigma(\chi')$, we have $\widehat{\alpha}(C^+) < 0$.

A more appropriate terminology may have been *non monodromic strongly regular principal series*, but we shall use the one above for short.

Remark 6.6.6. — (1) For a split group G' and a regular character χ' of T', the local Langlands correspondence has been defined by Rodier ([**98**]) for the subquotients of $\mathrm{Ind}_{B'}^{G'}(\chi')$, in a way that is compatible to the usual (that is, Henniart-Harris-Taylor's) local correspondence in the case of $\mathrm{GL}_n(F)$. The representation $L(\pi) = (r(\pi), N(\pi))$ of the Weil-Deligne group of F' corresponding to any of those subquotients π has the same $r(\pi)$ namely the composition

$$\psi_{\chi'} : W_{F'} \longrightarrow {}^LT' \longrightarrow {}^LG'$$

where the first map is the L-parameter of χ' for the torus T'. The action of the monodromy N depends on the chosen subquotient. Hypothesis (b) is equivalent to saying, by [**98**, 5.2], that $S(\chi')$ has no monodromy, that is, that $N(S(\chi')) = 0$. In other words, the L-parameter for $S(\chi')$ is just the map $\psi_{\chi'}$.

(2) In the case $G' = \mathrm{GL}_n(F')$, hypothesis (b) simply says that if T' is the diagonal torus and B' the upper diagonal Borel, and $\chi' = (\chi'_1, \dots, \chi'_n)$, then $\chi'_i = \chi'_j|\cdot|$ implies $i > j$.

(3) There should exists a base change map, sending L-packets of G to L-packets to G', and corresponding to the obvious restriction map on the L-parameters. If χ is strongly regular, it is natural to expect that the base change to G' of the L-packet of G containing the representation $S(\chi)$ contains the representation $S(\chi')$.

In the few cases where the base change has been defined, this is actually true: when $G = \mathrm{GL}_n(F)$ and F'/F is cyclic, this follows immediately from the compatibility of local base change with the local Langlands correspondence and from remark (1) above. In the more interesting case where $G = U(3)$ is the (unique) unitary group over F that splits over the quadratic extension F'/F, and $G' = \mathrm{GL}_3(F')$, this is satisfied for the base change map defined by Rogawski in [**99**].

Hence the *conjectural* L-parameter for $S(\chi)$ should be the composition $\psi_\chi : W_F \to {}^LT \longrightarrow {}^LG$. In particular, it should be non monodromic (that is, $N(S(\chi)) = 0$ with the notation of § 6.3.)

(4) Rodier's theory does not seem to have been extended to any quasi-split group, even to unitary groups. Therefore we have not felt comfortable in assuming that

for any regular χ satisfying the analog of condition (b) but for the root system of G (which might be not reduced), $S(\chi)$ should have a non monodromic L-parameter. This is however the case for the rank one group $U(3)$ by results of Keys and Rogawski (see [**99**, 12.2]).

6.6.3. The locus of non monodromic principal series is constructible. — In this paragraph, we keep the notations of § 6.6.1, and we assume moreover that A is a noetherian ring. We suppose given an A-admissible smooth $A[G]$-module which is of finite type over $A[G]$. For every $x \in X := \mathrm{Spec(A)}$, we denote by $k(x) = A_x/xA_x$ the residue field of A_x, and we set $M_x := M \otimes_A A_x$ and $\overline{M}_x := M \otimes_A k(x)$.

Let us denote by X_0 the subset of $x \in X$ such that the $k(x)[G]$-module $\overline{M}_x$ has an irreducible subquotient which is a non monodromic principal series .

Proposition 6.6.7. — (i) *X_0 is a constructible subset of X. In particular, there is a subset $U \subset X_0$ which is dense and open in $\overline{X}_0$.*

(ii) *Assume that A is reduced and that M is torsion free over A. Then $\overline{X}_0$ is a (possibly empty) union of irreducible components of X.*

Proof. — By Proposition 6.6.1, the $A[T]$-module $E := M_N$ is of finite type over A. We view E as an $A[T']$-module via the map $A[T'] \longrightarrow A[T]$ induced by the norm Nm. Let B be the image of the A-algebra $A[T']$ in $\mathrm{End}_A(E)$. It is a finite A-algebra, let $Y = \mathrm{Spec(B)}$ and $g : Y \to X$ the structural map.

A point $y \in Y$ with residue field $k(y)$ defines an A-algebra morphism $B \to k(y) = B_y/yB_y$, hence a character $\chi'_y : T' \to k(y)^*$. Let us consider the subset $Y_0 \subset Y$ of points $y \in Y$ such that the character χ'_y is regular and satisfies condition (b) of Definition 6.6.5. By definition, we have

$$Y_0 = \bigcap_{w \in W' \setminus \{1\}} \left(\bigcup_{t \in T'} D(t^w - t) \right) \cap \bigcap_{\{\widehat{\alpha} \in \widehat{\Delta} \mid \widehat{\alpha}(C) > 0\}} \left(\bigcup_{f \in F'} D(t^\alpha(f) - |f|) \right),$$

where $D(b)$ for $b \in B$ is the open subset of $Y = \mathrm{Spec}\, B$ defined by the condition $b \neq 0$. Hence Y_0 is an open subset of Y as both intersections are finite.

We claim that $X_0 = g(Y_0)$. First, by Lemma 6.6.3(c) and 6.6.1(b), observe that X_0 is also the subset of points x of X such that $E \otimes_A k(x)$ contains a character $T \to k(x)^*$ satisfying (a) and (b) of Def. 6.6.5, i.e. such that the support of the B-module $E \otimes_A k(x)$ meets Y_0. In particular, it is clear that $X_0 \subset g(Y_0)$.

Let $x \in X$. As $\widehat{A_x}$ is henselian and g is finite,

$$\widehat{B_x} := B \otimes_A \widehat{A_x} \xrightarrow{\sim} \prod_{\{y \mid g(y) = x\}} \widehat{B_y},$$

hence we can write accordingly $\widehat{E_x} := E \otimes_A \widehat{A_x} = \oplus_y E(y)$ as a direct sum of $B \otimes_A \widehat{A_x}$-modules. Moreover, by flatness of $A \to \widehat{A_x}$, $\widehat{B_x}$ identifies with its image in $\mathrm{End}_{\widehat{A_x}}(\widehat{E_x})$ hence $E(y) \neq 0$ for all $y \in g^{-1}(x)$. In particular, if $x = g(y)$ with $u \in Y_0$, then

$$E \otimes_A k(x) = \widehat{E_x}/x\widehat{E_x} \supset E(y) \otimes_{\widehat{A_x}} k(x)$$

and the latter B-module is non zero by Nakayama's lemma, hence has support $\{y\}$. This proves that $g(Y_0) \subset X_0$, hence the equality $g(Y_0) = X_0$.

In particular, by Chevalley's theorem (see e.g. [**63**, exercise II.3.19]) X_0 is constructible as Y_0 is open and g is of finite type, which proves the first part of (i). The second part of (i) is a standard consequence of being constructible (see e.g. [**63**, exercise II.3.18(b)]).

Let us prove (ii). As Y_0 is an open subset, its closure is a finite number (possibly zero) of irreducible components of Y. As $g(Y_0) = X_0$ and g is finite, we only have to check that each irreducible component of Y maps surjectively to an irreducible component of X. Note that E is torsion free over A by assumption and Proposition 6.6.1, hence so are $\mathrm{End}_A(E)$ and $B \subset \mathrm{End}_A(E)$. We conclude then by Lemma 6.6.8. □

The following lemma is a variant of [**36**, Lemma 2.6.10].

Lemma 6.6.8. — *Assume that A is a reduced notherian ring and that B is a finite, torsion free, A-algebra. Then each irreducible component of* Spec(B) *maps surjectively to an irreducible component of* Spec(A).

Proof. — We check at once that a finite type A-module M is torsion free if, and only if, it has an A-embedding $M \hookrightarrow A^n$ for some n. In particular, if M is torsion free over A, then for all $x \in \mathrm{Spec(A)}$ the A_x-module M_x is torsion free.

As the finite map $g : \mathrm{Spec(B)} \to \mathrm{Spec(A)}$ is closed, it suffices to show that the image of the generic point x of an irreducible component of Spec(B) is the generic point of an irreducible component of Spec(A). By localizing A at $g(x)$, we may assume that A is local and that $g(x)$ is a closed point. As $g^{-1}(x)$ is a discrete closed subspace of Spec(B), x is also a closed point, hence it is open as it is minimal as well, and B_x is a direct factor of B. Thus we may assume that $B = B_x$ is artinian. As $B \subset A^n$ and A is reduced, it implies easily that A is itself artinian, which concludes the proof. □

Remark 6.6.9. — (A variant) Assume that we have a finite number of quasisplit groups G_i, possibly over local fields of different characteristics, each one being equipped with a datum $(G_i, B_i, T_i, G'_i, B'_i, T'_i)$ as in the beginning of § 6.6. Then we may form $G = \prod_i G_i$, as well as B, T, G', B' and T', and all the propositions and lemmas of this § 6.6 apply verbatim to this case, as all the arguments are group theoretic. For example, in this context, a non monodromic principal series of G is a tensor product of non monodromic principal series π_i of G_i.

6.7. Representations of the compact real unitary group

Recall that the continuous, irreducible, complex representations of the compact group $\mathrm{U}(m)(\mathbb{R})$ are all finite dimensional. There are two ways to describe them: either by their highest weight or by their Langlands parameters. We give here both descriptions, as well as the relation between them.

If $\underline{k} := (k_1, \dots, k_m) \in \mathbb{Z}^m$ satisfies $k_1 \geq k_2 \geq \dots \geq k_m$, we denote by $W_{\underline{k}}$ the rational (over $\mathbb{Q}$), irreducible, algebraic representation of GL_m whose highest weight relative to the upper triangular Borel is the character [9]

$$\delta_{\underline{k}} : (z_1, \dots, z_m) \mapsto \prod_{i=1}^{m} z_i^{k_i}.$$

For any field F of characteristic 0, we get also a natural irreducible algebraic representation $W_{\underline{k}}(F) := W \otimes_{\mathbb{Q}} F$ of $\mathrm{GL}_m(F)$, and it turns out that they all have this form, for a unique $\underline{k}$.

Let us fix an embedding $E \hookrightarrow \mathbb{C}$, which allows us to see $\mathrm{U}(m)(\mathbb{R})$ as a subgroup of $\mathrm{GL}_m(\mathbb{C})$ well defined up to conjugation (see § 6.2.1). So for $\underline{k}$ as above, we can view $W_{\underline{k}}(\mathbb{C})$ as a continuous representation of $\mathrm{U}(m)(\mathbb{R})$. As is well known, the set of all $W_{\underline{k}}(\mathbb{C})$ is a system of representants of all equivalence classes of irreducible continuous representations of $\mathrm{U}(m)(\mathbb{R})$. We will say that $W_{\underline{k}}$ has *regular weight* if $k_1 > k_2 > \dots > k_m$.

On the other hand, the L-parameters of the irreducible representations of $\mathrm{U}(m)(\mathbb{R})$, are determined by their restrictions to the Weil group of $W_{\mathbb{C}} = \mathbb{C}^*$ of $\mathbb{C}$, which are, up to conjugation, all the morphisms $\phi : \mathbb{C}^* \to \mathrm{GL}_m(\mathbb{C})$ of the form

$$\phi(z) = \mathrm{diag}((z/\bar{z})^{a_1}, \dots, (z/\bar{z})^{a_m})$$

where $a_1, \dots, a_m \in \mathbb{Z} + \frac{m+1}{2}$ and $a_1 > \dots > a_m$. For the proof, see [**15**, Prop. 4.3.2]. The relation between the two descriptions is given by

$$a_i = k_i + \frac{m+1}{2} - i, \ \ i = 1 \dots m.$$

6.8. The Galois representations attached to an automorphic representation of U(m)

6.8.1. Settings and notations. — In this section, $m \geq 1$ is an integer such that $m \not\equiv 2 \pmod 4$. Let us fix a prime number p that is split in E, algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and embeddings $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$. As $G(\mathbb{R})$ is compact, for any automorphic representation π then π_∞ is algebraic and the finite part π_f is

[9] This means that the action of the diagonal torus of GL_m on the unique $\mathbb{Q}$-line stable by the upper Borel is given by the character above.

defined over $\overline{\mathbb{Q}}$ by Lemma 6.2.7, so that we may view it over $\overline{\mathbb{Q}}_p$ using $\iota_p\iota_\infty^{-1}$. Let us fix such a π.

If $l = x\bar{x}$ is a prime that splits in E, then we will denote by π_x the representation of $\mathrm{GL}_m(\mathbb{Q}_l)$ deduced from π_l and the identification $G(\mathbb{Q}_l) \xrightarrow{\sim}_x \mathrm{GL}_m(\mathbb{Q}_l)$ defined by x as in § 6.2.2.

We recalled in § 6.3 the Langlands-Harris-Taylor parameterization of complex irreducible smooth representations. This parameterization holds actually if we replace $\mathbb{C}$ everywhere there by any algebraically closed field of characteristic 0, *e.g.* $\overline{\mathbb{Q}}_p$. As a consequence, to each π_x as above viewed with $\overline{\mathbb{Q}}_p$ coefficents via $\iota_p\iota_\infty^{-1}$, is attached a unique $\overline{\mathbb{Q}}_p$-valued Weil-Deligne representation $(r(\pi_x), N(\pi_x))$. We recall also that from Grothendieck's l-adic monodromy theorem (see e.g. the Appendix 7.8.3), for any local field F over $\mathbb{Q}_l$ with $l \neq p$, there is a bijection between the isomorphism classes of continuous representations $W_F \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$, and the isomorphism classes of m-dimensional $\overline{\mathbb{Q}}_p$-valued Weil-Deligne representations of F. We shall use these bijections freely in the sequel.

We let v denote the (split) place of E above p induced by $\iota_p : E \to \overline{\mathbb{Q}}_p$, and by v_∞ the complex place of E induced by $\iota_\infty : E \to \mathbb{C}$.

Let l be a prime that does not split in E. If E is unramified at l, then $G(\mathbb{Q}_l)$ is an unramified unitary group hence it makes sense to talk about its *unramified representations*: they are the irreducible smooth representations having a nonzero vector invariant under a maximal hyperspecial subgroup. For example, for the obvious model of $\mathrm{U}(m)$ over $\mathbb{Z}$ then $G(\mathbb{Z}_l)$ is maximal hyperspecial. When l is ramified, we will also say that π_l is unramified if it has a nonzero vector invariant under a *very special* maximal compact subgroup. Following Labesse's terminology [**76**, §3.6], we say that a special maximal compact subgroup is very special if all of its constant $q_{\alpha/2}$ defined by Macdonald in [**81**][§3.1] are ≥ 1. These compact subgroups and their spherical functions have been studied in [**81**] and [**34**]. It seems to be known to experts that any reductive group over a local field admits very special maximal compact subgroups (for instance, this is claimed in [**34**, p.390–391] or in [**76**, §3.6]), but we warn the reader that there seems to be no written proof of this fact in the litterature.

6.8.2. Statement of the assumption $\mathrm{Rep}(m)$. — We now formulate a conjecture about the existence and the basic properties of the Galois representation attached to an automorphic representation of $\mathrm{U}(m)$. We expect and hope that this conjecture (and much more) will be proved in the forthcoming book [**60**] on unitary groups written under the direction of Michael Harris in Paris.

To make this hope likely, we have made a special effort throughout this book to keep the properties of those Galois representations we need under control, and in the next conjecture to formulate the weakest statement that we need.

Conjecture 6.8.1 (Rep(m)). — *Let π be an automorphic representation of* $\mathrm{U}(m)$ *such that π_∞ has regular weight. There exists a continuous, semisimple, Galois representation:*

$$\rho_\pi : G_E \longrightarrow \mathrm{GL}_m(\overline{\mathbb{Q}}_p),$$

such that the following properties are satisfied:

(P0) *if $l = xx' \neq p$ is split and π_l is unramified, then ρ is unramified above l and the characteristic polynomial of a geometric Frobenius at x is given by the Langlands conjugacy class of $\pi_x|\det|^{\frac{1-m}{2}}$.*

(P1) *If $l \neq p$ is a prime and if π_l is unramified, then ρ_π is unramified at each prime above l.*

(P2) *If $l = xx' \neq p$ is a prime that splits in E, then the nilpotent monodromy operator of the Weil-Deligne representation attached to $\rho_{\pi|\mathrm{W}_{E_x}}$ is in the closure of the conjugacy class of $N(\pi_x|\det|^{\frac{1-m}{2}})$ in $\mathrm{M_m}(\overline{\mathbb{Q}}_p)$.*

(P3) *If $l \neq p$ is a prime, x a place of E above l, and π_l is a non monodromic principal series (see Definition 6.6.5) then the Weil-Deligne representation attached to $\rho_{\pi|\mathrm{W}_{E_x}}$ has a trivial monodromy.*

(P4) *The p-adic representation $\rho_{\pi|G_{E_v}}$ is De Rham, and its Hodge Tate weights are the integers*

$$-a_1 + \frac{m-1}{2}, \dots, -a_m + \frac{m-1}{2}$$

where $a_1, \dots, a_m$ are such that the restriction to $\mathbb{C}^*$ of the L-parameter of π_∞ is $z \mapsto \mathrm{diag}((z/\bar{z})^{a_1}, \dots, (z/\bar{z})^{a_m})$ (see § 6.7).

(P5) *If π_p is unramifed, then $\rho_{\pi|G_{E_v}}$ is crystalline and the characteristic polynomial of its crystalline Frobenius is the same as the one of $\iota_p\iota_\infty^{-1}L(\pi_v|\det|^{\frac{1-m}{2}})$.*

Remark 6.8.2. — (i) By the Cebotarev density theorem, and since the primes of E which split above $\mathbb{Q}$ have density 1, the property (P_0) alone determines ρ_π up to isomorphism. Moreover, it implies that ρ_π is *conjugate selfdual* in the sense that:

$$\rho_\pi^\perp \simeq \rho_\pi(m-1).$$

(ii) The properties (P0), (P1) and (P4) imply that ρ_π is geometric.

(iii) The Langlands program and Arthur's conjectures predict that there should exist local and global base change from $\mathrm{U}(m)$ to GL_m, that a π with regular weight as in the conjecture should be tempered, and that for such a tempered π the global base change π_E should be compatible at every place with the local base change (see A.7 below). Moreover it also predicts the existence of a Galois representation ρ_π of G_E, and the Weil-Deligne representation attached to the restriction at W_{E_x} for every place x (prime to p) of E should be isomorphic to $L((\pi_E|\det|^{\frac{1-m}{2}})_x)$.

The properties (P0) to (P3) are very special cases of those predictions. This is clear for (P0), (P1), (P2); as for (P3), if $l \neq p$ is a prime such that π_l is a non monodromic principal series and x a place of E above S, then the L-parameter of the base change $(\pi_E)_x$ should be non-monodromic (see Remark 6.6.6).

Moreover, properties (P4) and (P5) are also part of the standard expectations for the Langlands correspondence at places dividing p.

(iv) The property (P3) for split primes l is a special case of (P2). This should be clear from the preceding remark.

(v) In the following chapter, the property (P2) will allow us to work with representations ρ_π that have arbitrary ramification at split primes. However, because of the weak form of (P3), we shall have to assume that the ramification, if any, is of a very special kind at non-split places, namely is a non monodromic principal series .

(vi) When $m \leq 3$, the properties (P0) and (P1) and (P3) to (P5) are known by the work of Blasius and Rogawski (cf. [**20**] and [**21**] and also [**8**, §3.3] for some details). Property (P2) is not completely known but anyway is not necessary, since (P3) is known, and all the π to which we shall need to apply Rep(m) are non monodromic principal series at every places if $m \leq 3$. Thus, in the sequel, we will allow ourselves to say that Rep(m) is known for $m \leq 3$.

(vii) Properties (P1) to (P5), except maybe (P3), are also known for any m when π admits a base change to a representation satisfying the assumptions of Harris-Taylor ([**62**], [**118**]) and which is compatible with the local base change at the split places. This includes e.g. the case of a π that is supercuspidal at two split places ([**61**, §3]). Unfortunately, the representations to which we shall apply Rep(m) will never be of this type.

Moreover, let us consider the slighty different setting where $\mathrm{U}(m)$ is replaced by a definite unitary group G such that $G(\mathbb{Q}_p) \simeq \mathrm{GL}_m(\mathbb{Q}_p)$, and that for some split $l \neq p$, $G(\mathbb{Q}_l)$ is isomorphic to the group of invertible elements of a central division algebra over $\mathbb{Q}_l$. In this case, the existence of ρ_π satisfying (P0), (P1), (P2), (P4) and (P5) is known by [**62**] and [**61**, Thm. 3.1.3].

(viii) We of course expect that in the forthcoming book [**60**] by Harris *et al.*, the representation ρ_π (or some well chosen base change of it) will be cut off in the étale cohomology of some explicit local system of the Shimura variety of some inner form of $\mathrm{U}(m)$, since π_∞ has regular weights. Hence (P0), (P1), (P4) and (P5) should follow directly from the construction and a few standard arguments (see e.g. [**8**, Prop. 3.3] for (P4) and (P5) and [**8**, Prop. 3.2] for (P1) at ramified primes).

The properties (P2) and (P3) are special cases, concerning monodromy, of the compatibility of the construction of the Galois representation to the local

Langlands correspondence that might be harder to prove. However, they only ask for an "upper bond" on the monodromy of the Galois representation, which is the "easy direction", and this should follow from an accessible (maybe already known) study of the local geometry of the special fiber of the relevant Shimura varieties.

At any rate, (P3) would follow easily if the base change (local and global, with compatibility) was known, by an argument completely similar to [**8**, Prop. 3.2]).

(ix) Note that the Langlands and Arthur's conjectures also predict some irreducibility results on the representation ρ_E (for example, if π is not endoscopic). Those results might be much harder to prove. However, a feature of our method, already present in [**8**], is that we have absolutely no need of them. Instead, we shall be able to prove, as a by-product of our work, that in many cases ρ_π is irreducible, or not too reducible. See Theorem 7.7.1.

(x) (added in 2008). Since the first version of this manuscript was made available (December 2006), important progresses have been made toward a proof of the conjecture Rep(m) by S. Morel [**88**], S.W. Shin, and all the authors of the book [**60**] (of which many chapters have been made avalailable).

6.9. Construction and automorphy of a non-tempered representation of U(m)

In this subsection we fix an integer $n \geq 1$ that is not divisible by 4, and we set $m := n+2$, so that $m \not\equiv 2 \pmod 4$ as above. For a representation π of $\mathrm{GL}_n(E_v)$, v a non split place of E (resp. of $\mathrm{GL}_n(\mathbb{A}_E)$), we note $\pi^\perp$ the representation $g \mapsto \pi^*(c(g))$, where π^* is the contragredient of π and c denotes the map on $\mathrm{GL}_n(E_v)$ (resp. $\mathrm{GL}_n(\mathbb{A}_E)$) induced by the non trivial element $c \in \mathrm{Gal}(E/\mathbb{Q})$.

6.9.1. The starting point. — We start with a cuspidal tempered [10] automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$. We make the following assumptions on π:

(i) We have $\pi^\perp \simeq \pi$.

(ii) The L-parameter of π_∞ has the form

$$z \mapsto \mathrm{diag}((z/\bar{z})^{a_1}, \dots, (z/\bar{z})^{a_n})$$

where the a_i's are distinct, $\equiv \frac{1}{2} \pmod 1$, and are different from $\pm 1/2$.

(iii) If l is a nonsplit prime, then either

[10] This temperedness should be a consequence of the cuspidality, according to the generalized Ramanujan's conjecture.

(iiia) π_l is unramified and its central character χ satisfies [11] $\chi(\varpi_l) = (-1)^n$, or

(iiib) the representation π_l is a non monodromic principal series representation $S(\eta)$, where $\eta = (\eta_1, \ldots, \eta_n)$ is a regular character of the standard maximal torus of $\mathrm{GL}_n(E_l)$, and there is no (resp. exactly one) $i \in \{1, \ldots, n\}$ if n is even (resp. if n is odd) such that $\eta_i^\perp = \eta_i$.

The aim of this section is to describe, place by place, a representation of $U(m)$, called π^n (the n stands for "non-tempered", as π^n turns out to be non tempered at every finite place) depending on π, and to state a conjecture $\mathrm{AC}(\pi)$ that π^n is automorphic if (and actually only if) $\varepsilon(\pi, 0) = -1$. The representation π^n is an endoscopic transfer of π, and the conjecture we state is a particular case of the far reaching multiplicity formula of Arthur, as will be explained in Appendix A.

Remark 6.9.1. — (i) When n is even, properties (i) and (ii) are conjecturally sufficient conditions for π to be the base change of a discrete automorphic representation of the quasisplit unitary group $\mathrm{U}(n)^*$ attached to $E/\mathbb{Q}$. When n is odd, on the contrary, a representation satisfying (ii) is not a base change from $\mathrm{U}(n)^*$, but (i) and (ii) should rather ensure that $\pi \otimes \mu$ is a base change from $\mathrm{U}(n)^*$ for any Hecke character μ as in Lemma 6.9.2(iii) (see Example 6.9.3 below).

(ii) Property (iii) is not really needed for the conjecture we are going to state, but it simplifies the exposition, allowing in particular to give a non conjectural description of π^n at non split places.

To be more precise, and conjecturally speaking, condition (iii) on π_l is the condition needed for π_l^n to be either unramified or a non monodromic principal series at l. Without this condition, there should still exist a π_l^n with suitable properties, but it could be square integrable or even supercuspidal, and it is not possible in the present state of knowledge on the representation theory of local unitary groups to construct the needed representation.

Moreover, the hypothesis that π_l^n is unramified or a non monodromic principal series is what we will need in the following sections to be able to deal with the monodromy at the nonsplit l. So it is not a big loss to assume it from now.

6.9.2. Hecke characters. — If μ is a Hecke character of E, that is a continuous morphism

$$\mu : \mathbb{A}_E^*/E^* \longrightarrow \mathbb{C}^*,$$

[11] Here ϖ_l is a uniformiser of E_l. When l is inert, $\chi(\varpi_l) = \chi(l)$ and the condition on χ is automatically satisfied, see Rem. 6.9.4. The reason for the appearence here of this condition on χ basically comes from the fact that it is not equivalent for a character of $U(1)(\mathbb{Q}_l)$ to be unramified (i.e. trivial), and to have an unramified base change. However, the two notions coincide when l is inert.

recall that $\mu^\perp$ is the Hecke character $x \mapsto \mu(c(x))^{-1}$. We say that a Hecke character μ *descends to* $U(1)$ if $\mu = \psi(x/c(x))$, for some continuous character ψ of

$$\mathrm{U}(1)(\mathbb{A}_E) = \{x \in \mathbb{A}_E^*,\ xc(x) = 1\}.$$

Obviously a character μ that descends to U(1) satisfies $\mu^\perp = \mu$. If a character satisfies $\mu^\perp = \mu$, we have $\mu_\infty(z) = (z/c(z))^a$ for all $z \in (E \otimes \mathbb{R})^*$ and some *weight* a which is either an integer or a half integer.

Lemma 6.9.2. — (i) *The subgroup of Hecke characters of E that descend to $U(1)$ is of index 2 in the group of all Hecke characters of E that satisfy $\mu^\perp = \mu$.*

(ii) *For a Hecke character μ such that $\mu^\perp = \mu$ the following are equivalent:*
- *the character μ does not descend to $U(1)$,*
- *the weight a of μ is not an integer,*
- *the restriction of μ to $\mathbb{A}_\mathbb{Q}^*$ is the order 2 character $\omega_{E/\mathbb{Q}}$ corresponding via class field theory to the extension $E/\mathbb{Q}$.*

In particular, a character that satisfies the above conditions is ramified at every ramified places of $E/\mathbb{Q}$, since so is $\omega_{E/\mathbb{Q}}$.

(iii) *There exists a Hecke character μ of E, satisfying $\mu^\perp = \mu$ with weight $1/2$ and which is ramified only at ramified places of $E/\mathbb{Q}$.*

Proof. — Both (i) and (ii) result from the following observations:
- a Hecke character descends to $U(1)$ if and only if it is trivial on $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$,
- a Hecke character μ satisfies $\mu^\perp = \mu$ if and only if it is trivial on the norm group $N(\mathbb{A}_E^*/E^*)$,
- by class field theory, $N(\mathbb{A}_E^*/E^*) = \mathrm{Ker}\ \omega_{E/\mathbb{Q}}$ is an open subgroup of index 2 in $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$.

For (iii), let S be the set of rational primes that ramify in E. For each $l \in S$, choose any finite order character $\mu_l : \mathcal{O}_{E\otimes\mathbb{Q}_l}^* \to \mathbb{C}^*$ extending $\omega_{E/\mathbb{Q},\,l}$. We fix also an isomorphism $E \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}$ for convenience, and set $\mu_\infty(z) = (z/\bar{z})^{1/2}$ for $z \in \mathbb{C}$. Assume first that the cyclic group $U = \langle u \rangle$ of units in $\mathcal{O}_E$ reduces to $\{\pm 1\}$. Then we can define μ on $\mathbb{C}^* \times \widehat{\mathcal{O}}_E^*$ to be $\mu_\infty \prod_l \mu_l$. As μ coincides with $\omega_{E/\mathbb{Q}}$ on $\mathbb{R}^* \times \widehat{\mathbb{Z}}^*$, $\mu(U) = \{1\}$. As $E^* \cap (\mathbb{C}^* \times \widehat{\mathcal{O}}_E^*) = U$, μ extends uniquely to an E^*-invariant continuous character of the open subgroup $G := E^*(\mathbb{C}^* \times \widehat{\mathcal{O}}_E^*)$ of $\mathbb{A}_E^*$. Note that G is open of finite index in $\mathbb{A}_E^*$ by the finiteness of the class number of E, hence we can extend μ to a continuous character of $\mathbb{A}_E^*/E^*$. Note that G contains $\mathbb{A}_\mathbb{Q}^*$ and that μ extends $\omega_{E/\mathbb{Q}}$ by construction, hence $\mu^\perp = \mu$, which concludes the proof.

When $|U| > 2$, then $E = \mathbb{Q}(i)$ or $\mathbb{Q}(j)$, and $S = \{l\}$ contains only one prime. In this case, note that $U \cap \mathbb{Z}_l^* = \{\pm 1\}$ hence we may first extend $\omega_{E/\mathbb{Q},\,l}$ to $U\mathbb{Z}_l^*$ by choosing $\mu_l(u) := u^{-1}$, and then extend it anyhow to a finite order character of $\mathcal{O}_{E\otimes\mathbb{Q}_l}^*$. Again, if we define μ as before, we have $\mu(U) = \{1\}$ and the same proof works. ☐

Example 6.9.3. — The central character of π is a Hecke character μ that satisfies $\mu^{\perp} = \mu$ by condition (i) of § 6.9.1, with weight $a = \sum_{i=1}^{n} a_i$, which is an integer if and only if n is even by (ii) of § 6.9.1. Hence μ descends to $U(1)$ if and only if n is even, which is also the conjectural condition for π to descend to $\mathrm{U}(n)$.

Remark 6.9.4. — Assume that l is inert in E and, in the notations of § 6.9.1, that π_l is unramified. We claim that the central character χ of π_l automatically satisfies $\chi(l) = (-1)^n$. Indeed, χ is trivial on $\mathcal{O}_E^*$ as π_l is unramified, and it satisfies $\chi^{\perp} = \chi$. By Lemma 6.9.2 and the example above, $\chi_{|\mathbb{Q}_l^*} = 1$ if, and only if, n is even, hence the claim.

Notation 6.9.5. — We choose a Hecke character μ of E as follows: μ is a character as in Lemma 6.9.2(iii) if n is odd, and $\mu = 1$ if n is even.

We are now going to construct, place by place, a representation π^n of $U(m) = U(n+2)$ whose conjectural base change to $\mathrm{GL}_m(E)$ has L-parameter

$$L(\pi)\mu \oplus |\,|^{1/2}\mu \oplus |\,|^{-1/2}\mu.$$

6.9.3. Construction of π_l^n, for l split in E. — We denote by P the upper parabolic subgroup of $\mathrm{GL}_m(\mathbb{Q}_l)$ of type $(n,2)$, whose Levi subgroup is $M = \mathrm{GL}_n(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$. For x a place of E above l, we set

$$\pi_x^n := \mathrm{Ind}_P^{\mathrm{GL}_m(\mathbb{Q}_l)}(\pi_x(\mu_x \circ \det) \otimes (\mu_x \circ \det)).$$

Here Ind is the normalized induction. Since π_x is unitary, π_x^n is irreducible (see [**18**]).

Remark 6.9.6. — Let P' be the upper parabolic of type $(n,1,1)$. Since π_x is tempered by hypothesis, Langlands' classification theorem shows that

$$\mathrm{Ind}_{P'}^{\mathrm{GL}_m(\mathbb{Q}_l)}(\pi_x(\mu_x \circ \det) \otimes |\,|^{1/2}\mu_x \otimes |\,|^{-1/2}\mu_x),$$

has a unique irreducible quotient (that is, the Langlands quotient). As we have a natural $\mathrm{GL}_m(\mathbb{Q}_p)$-equivariant surjection from the representation above to the irreducible representation π_x^n, this Langlands quotient is actually π_x^n. Thus, the L-parameter of π_x^n is $L(\pi_x)\mu_x \oplus |\,|^{1/2}\mu_x \oplus |\,|^{-1/2}\mu_x$.

Let us write $l = x\bar{x}$. By (i) of § 6.9.1, $\pi_{\bar{x}}(\mu_{\bar{x}} \circ \det)$ is dual to $\pi_x(\mu_x \circ \det)$, so π_x^n is dual to $\pi_{\bar{x}}^n$. The place x defines, up to conjugation, an identification

$$i_x : \mathrm{U}(m)(\mathbb{Q}_l) \to \mathrm{GL}_m(\mathbb{Q}_l)$$

and so does the place $\bar{x}$, in such a way that $i_{\bar{x}} \circ i_x^{-1}$ is conjugate to $g \mapsto {}^t g^{-1}$. Hence we see that $i_x^* \pi_x^n \simeq i_{\bar{x}}^* \pi_{\bar{x}}^n$, using the well known result of Zelevinski that the representation $g \mapsto \tau({}^t g^{-1})$ is the contragredient of τ for any irreducible admissible representation τ of $\mathrm{GL}_m(\mathbb{Q}_l)$.

We thus may set $\pi_l^n := i_x^*\pi_x^n$ and π_l^n does not depend on the choice of the place x above l.

6.9.4. Construction of π_l^n, for l inert or ramified in E. — We denote also by l the place of E above l. In this case $G := \mathrm{U}(m)(\mathbb{Q}_l)$ is a quasi-split unitary group, and we shall use notations compatible to those of § 6.6. We may assume that G is the unitary group defined by the following hermitian form on E_l^m:

$$f(xe_i, ye_j) = c(x)y\delta_{j,m-i+1},$$

so that the group of diagonal matrices in G,

$$T = \{\mathrm{diag}(a_1, \dots, a_m),\ a_i \in E_l^*, a_{m-i+1} = c(a_i)^{-1}, i = 1, \dots, m\},$$

is the centralizer of a maximal split torus in $\mathrm{U}(m)(\mathbb{Q}_l)$. Let B be a the upper triangular Borel. The group $G' := \mathrm{U}(m)(E_l)$ is naturally identified with $\mathrm{GL}_n(E_l)$ and T' is the standard diagonal torus. Its Weyl group W' is canonically identified with $\mathfrak{S}_{n+2}$. The action of the non trivial element c of $\mathrm{Gal}(E_l/\mathbb{Q}_l)$ on T' is

$$c(\mathrm{diag}(x_1, \dots, x_m)) = \mathrm{diag}(c(x_m)^{-1}, \dots, c(x_1)^{-1}),$$

and T is the subgroup of invariants of c in T'. There is a norm map $\mathrm{Nm} : T' \to T$,

$$x = (x_1, \dots, x_m) \mapsto xc(x) = (x_1c(x_m)^{-1}, x_2c(x_{m-1})^{-1}, \dots, x_mc(x_1)^{-1}).$$

By hypothesis (iii) of § 6.9.1, and point (iii) of the remark therein, π_l is a subquotient of the normalized induction of a character $(\eta_1, \dots, \eta_n)$ of the standard torus of GL_n, with $\eta_i^\perp = \eta_{\sigma(i)}$ for all i and some $\sigma \in \mathfrak{S}_n$. As π_l is tempered, each η_i is a unitary character.

We are going to define a character $\chi' = (\chi_1', \dots, \chi_m' = \chi_{n+2}')$ of T'. Up to reordering, the χ_i', $i = 1, \dots, m = n+2$ are the $\eta_i\mu_l$, $i = 1, \dots, n$ and $|\,|^{\pm 1/2}\mu_l$. The order is as follows:

- First we define $\chi_1' = |\,|^{-1/2}\mu_l$, $\chi_m' = |\,|^{1/2}\mu_l$.
- Next, consider the set $I \subset \{1, \dots, n\}$ of i such that $\eta_i \not\simeq \eta_i^\perp$. Clearly $|I|$ is even, say $2r$, and we may define $\chi_2', \dots, \chi_{r+1}'$ and $\chi_{m-r}', \dots, \chi_{m-1}'$ in such a way that $\chi_{m-j+1}' \simeq \chi_j'^\perp$ for $j \in \{2, m-1\}$. Finally, in case (iiib) we have $|I| = n$ if $n = 2r$ is even (in which case we are done with the definition of χ') and $|I| = n-1$ if $n = 2r+1$ is odd. In this case we have only left to define the "midpoint" character χ_{r+1}' for which we take (we have no other choice) $\eta_j\mu_l$, where $\eta_j \simeq \eta_j^\perp$ (this holds for a unique j).
- In case (iiia), the characters η_i for $i \notin I$ satisfy $\eta_i = \eta_i^\perp$, but since they are unramified, this implies $\eta_i(\varpi_l) = \pm 1$ (here ϖ_l is a uniformizer of E_l). By the assumption on the central character of π_l, the set $\{i \notin I,\ \eta_i(\varpi_l) = +1\}$ always has an even number of elements, say $2r'$. For $r+1 \leq i \leq r+r'$, we set then

$\chi'_i = \chi'_{m-i+1} = +\mu_l$ (with the obvious abuse of language), so for the remaining ones we have $\chi'_i = -\mu_l$.

Lemma 6.9.7. — *The character χ' descends to T i.e. there is a smooth character χ of T such that $\chi' = \chi \circ \mathrm{Nm}$. Moreover, χ satisfies properties (a) and (b) of Definition 6.6.5 in case (iiib). In case (iiia), χ is unramified if m is even or if l is inert in E.*

Proof. — By construction, in both cases, we have ${\chi'}^{\perp}_{m-i+1} = \chi'_i$ for all i. When $m = 2q$ is even, we define $\chi(\mathrm{diag}(a_1, \dots, a_{2q})) = \chi'_1(a_1)\dots\chi'_q(a_q)$ and it is clear that $\chi \circ \mathrm{Nm} = \chi'$.

When $m = 2q+1$, we remark that the middle character χ'_{q+1} of E_l^* actually descends to a character χ_{q+1} of $U(1)(\mathbb{Q}_l)$. Indeed

$$\chi'_{q+1}\prod_{i=2}^{q}\chi'_i{\chi'}^{\perp}_i = \det(\chi')$$

is the central character of $\pi_l\mu_l$. Since the central character of $\pi\mu$ has an integral weight (namely $\sum_{i=1}^n a_i + n/2$), it descends to $U(1)$ by Lemma 6.9.2, and so does the central character of $\pi_l\mu_l$, hence also χ'_{q+1}.

Let ψ be a smooth character of $U(1)(\mathbb{Q}_l)$ such that $\chi'_{q+1}(x) = \psi(x/c(x))$ for all $x \in E_l^*$. We define $\chi(\mathrm{diag}(a_1, \dots, a_{2q+1})) = \chi'_1(a_1)\dots\chi'_q(a_q)\psi(a_{q+1})$ and again it is obvious to see that $\chi \circ \mathrm{Nm} = \chi'$.

The other assertion is clear in case (iiib) as the η_i are unitary, as well as in case (iiia) when m is even. In the remaining case, the χ'_i are unramified for $i \neq q+1$ by choice of μ (i.e. Lemma 6.9.2 (iii)), so we only have to check that ψ is trivial. But χ'_{q+1} is trivial since it is unramified and satisfies $\chi'_{q+1}(l) = \mu_l(l)\eta_{q+1}(l) = +1$, hence the result follows from Hilbert 90. □

We now define π_l^n as the unramified subquotient of $\mathrm{Ind}_B^G\chi$ in case (iiia) and as the unique subrepresentation $S(\chi)$ of $\mathrm{Ind}_B^G\chi$ in case (iiib) (see Def. 6.6.5).

Remark 6.9.8. — The L-parameter of the conjectural base change of π_l^n to $\mathrm{GL}_m(E_l)$ should be, by Remark 6.6.6 in case (iiib) and by [**76**, §3.6] in case (iiia), the L-parameter attached to the character χ' of T', which is by construction

$$L(\pi_l)\mu_l \oplus |\,|^{1/2}\mu_l \oplus |\,|^{-1/2}\mu_l,$$

as in the split case.

6.9.5. Construction of π^s_∞. — Consider the morphism $\mathbb{C}^* \longrightarrow \mathrm{GL}_m(\mathbb{C})$ (recall that $m = n+2$)

$$\begin{aligned} z &\mapsto \mu_\infty(z)\mathrm{diag}((z/\bar{z})^{a_1}, \ldots, (z/\bar{z})^{a_n}, (z/\bar{z})^{1/2}, (z/\bar{z})^{-1/2}) \\ &= \begin{cases} \mathrm{diag}((z/\bar{z})^{a_1}, \ldots, (z/\bar{z})^{a_n}, (z/\bar{z})^{1/2}, (z/\bar{z})^{-1/2}) & \text{if } n \text{ is even} \\ \mathrm{diag}((z/\bar{z})^{a_1+1/2}, \ldots, (z/\bar{z})^{a_n+1/2}, (z/\bar{z}), 1) & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Since the a_i are half-integers, and different from $\pm 1/2$, we see by § 6.7 that this morphism is always the restriction to $\mathbb{C}^*$ of the L-parameter of a unique irreducible representation π^s_∞ of $\mathrm{U}(m)$. Here the s stands for *square integrable.* The notation π^n_∞ would be misleading since π^s_∞ is, like any irreducible representation of a compact group, finite dimensional, square integrable, hence tempered.

6.9.6. Assumption $AC(\pi)$

Conjecture 6.9.9. — *Assume that $\varepsilon(\pi, 1/2) = -1$. Then the irreducible admissible representation*

$$\pi^n := \pi^s_\infty \otimes \bigotimes_l{}' \pi^n_l$$

is automorphic.

The proof of this conjecture has recently been announced by Harris in the introduction of his preprint [**59**] (maybe under some local assumptions). Since a written proof is not yet available, we prefer to be conservative and state it as a conjecture rather than as a theorem.

Remark 6.9.10. — (i) The case $m = 3$ (that is $n = 1$) of this conjecture has been proved by Rogawski ([**101**]), using the Theta correspondence. In the case $m = 4$, the needed local computations have been published recently by Konno and Konno ([**75**]).

(ii) This conjecture is a very special case of the multiplicity formula of Arthur. Its derivation from that formula is explained in detail in the appendix. From that we shall see that the $\varepsilon(\pi, 1/2) = -1$ should also be a necessary condition for the automorphy of π^n.

(iii) Although the construction of π^n depends on the choice of the Hecke character μ (for odd n: see Notation 6.9.5), it is clear that the conjecture is independent of this choice. Indeed, if μ is changed into another character μ_1, then $\mu_1 = \mu\phi'$ where ϕ' is a Hecke character of $\mathbb{A}^*_E$ that descends to a character ϕ of $\mathrm{U}(1)$. By construction the representation π^n_1 defined using μ_1 is simply $\pi^n(\phi \circ \det)$ and it follows that the automorphy of π^n is equivalent to the automorphy of π^n_1.

Note also that the hypothesis in the conjecture is about $\varepsilon(\pi, 1/2)$, not about $\varepsilon(\pi(\mu \circ \det), 1/2)$.

CHAPTER 7

EIGENVARIETIES OF DEFINITE UNITARY GROUPS

7.1. Introduction

In this section, we introduce and study in detail the eigenvarieties of definite unitary groups and we prove the basic properties of the (sometimes conjectural) family of Galois representations that they carry. These eigenvarieties give a lot of interesting examples where all the concepts studied in this book occur, and provide also an important tool for the applications to Selmer groups in the next chapters. As a first application, we define some purely Galois theoretic global deformation rings and discuss their relations to those eigenvarieties at some specific classical points (including $R = T$ like statements). We give a second application to the construction of many irreducible Galois representations. We prove also quite a number of results of independent interests regarding the theory of eigenvarieties that we explain in details below. The organization of this section is as follows.

In the first Subsection 7.2, we give an axiomatic definition of eigenvarieties, as well as their general properties. In particular, we show that an eigenvariety is unique (up to unique isomorphism) if it exists (Prop. 7.2.8). One interest is that there are in principle many different ways two construct eigenvarieties: using coherent or Betti cohomology, a group or its inner forms (or any transfer suggested by Langland's philosophy), using Emerton's representation theoretic approach etc. Each of those constructions has its own advantages but they sometimes should lead to the same eigenvariety. The uniqueness statement alluded above will often show that they are indeed the same [1].

[1] It may be also useful to combine it with the results of [**37**].

The context is the following: [2] $E/\mathbb{Q}$ is a quadratic imaginary field and $G/\mathbb{Q}$ is a unitary group in $m \geq 1$ variables attached to $E/\mathbb{Q}$. We assume that $G(\mathbb{R})$ is the compact real unitary group (G is *definite*) and we fix a prime p such that $G(\mathbb{Q}_p) \simeq \mathrm{GL}_m(\mathbb{Q}_p)$, as well as embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \to \mathbb{C}$. An (irreducible) automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of G is automatically algebraic and has cohomology in degree 0. The finite dimensional representation π_∞ is determined by its *weight* which is a decreasing sequence of integers $\underline{k} = (k_1 \geq \cdots \geq k_m)$, and π_f is defined over $\overline{\mathbb{Q}}$ hence may be viewed over $\overline{\mathbb{Q}}_p$ via the chosen embeddings. We fix also a commutative Hecke-algebra

$$\mathcal{H} = \mathcal{A}_p \otimes \mathcal{H}_{\mathrm{ur}}$$

that contains the Atkin-Lehner algebra $\mathcal{A}_p$ of U-operators at p and a spherical part $\mathcal{H}_{\mathrm{ur}}$ outside p. We are interested in p-adically interpolating the systems of Hecke eigenvalues $\psi_\pi : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$ cut out from the π as above, and more precisely the pairs $(\psi_\pi, \underline{k})$ where $\underline{k}$ is the weight of π. Note that the systems of eigenvalues of $\mathcal{A}_p$ on the Iwahori invariants of π_p (say, if π_p is unramified) are in bijection with the *refinements* of π_p in the sense of § 6.4, so that ψ_π contains the extra datum of a choice of a refinement $\mathcal{R}$ of π_p. To keep track of this refinement, we actually denote ψ_π by $\psi_{(\pi,\mathcal{R})}$. Let us fix now a collection

$$\mathcal{Z} \subset \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^m$$

of such $(\psi_{(\pi,\mathcal{R})}, \underline{k})$. An *eigenvariety for $\mathcal{Z}$* is a 4-uple (X, ψ, ω, Z) where

(a) X is a reduced rigid analytic space over $\mathbb{Q}_p$,
(b) $\psi : \mathcal{H} \longrightarrow \mathcal{O}(X)$ is a ring homomorphism,
(c) $\omega : X \longrightarrow \mathcal{W} := \mathrm{Hom}((\mathbb{Z}_p^*)^m, \mathbb{G}_m)$ is a morphism to the *weight space* $\mathcal{W}$,
(d) $Z \subset X(\overline{\mathbb{Q}}_p)$ is a Zariski-dense subset,

such that the evaluation of ψ induces an injection

$$X(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p)$$

which itself induces a bijection [3] $Z \xrightarrow{\sim} \mathcal{Z}$. To have the uniqueness property we need of course to impose some extra conditions on (ψ, ω, Z) for which we refer to Definition 7.2.5. We show that for an eigenvariety X, the unit ball $\mathcal{O}(X)^0$ is a compact subset

(2) The choice of a quadratic imaginary field in this chapter rather than a general CM field (as well as the choice of a split p) is made mainly to simplify the exposition and also because this the only case that we shall use in the applications to Selmer groups (see § 9.5.1). All the constructions actually extend to this more general setting by combining the arguments here (or of [**36**]) and those of [**32**] (see also Yamagami's work [**124**]). Alternatively, the general *definite* case is now covered by Emerton's paper [**53**]. The main reason why we fix a split prime p is Galois theoretic: at the moment, Kisin's arguments [**73**] and the theory of trianguline representations are only written in the case where the base field is $\mathbb{Q}_p$ rather that any finite extension of $\mathbb{Q}_p$.
(3) This makes sense as $\mathcal{W}(\mathbb{Q}_p)$ naturally contains $\mathbb{Z}^m$, see § 7.2.3.

of $\mathcal{O}(X)$, which (together with (d)) is the basic property needed for the construction of Galois pseudocharacter on X.

In the second subsection § 7.3 we recall the results of one of us on the existence of eigenvarieties ([**36**]). The statement is that for any idempotent e in the Hecke-algebra $\mathcal{C}_c(G(\mathbb{A}^p), \overline{\mathbb{Q}})$ commuting with $\mathcal{H}$, there is an eigenvariety for the set $\mathcal{Z}_e$ parameterizing all the p-refined π such that $e(\pi) \neq 0$. We discuss in Example 7.3.3 which sets $\mathcal{Z}$ can be obtained this way, in representation theoretic terms (Bernstein components, type theory). In fact, those eigenvarieties *of idempotent type* have stronger properties than the general ones. As their structure plays a crucial role in our main theorem on Selmer groups, as well as in some subsequent constructions in this section, we found it necessary to review their construction in detail, as well as the theory of p-adic automorphic forms of G developed in [**36**]. This is the aim of § 7.3.2 to § 7.3.6. In fact, this part is essentially self-contained and slightly improves some results of [**36**] (*e.g.* we do not restrict to the "central part" of the weight space, or to a neat level, we release the assumption that $p \neq 2$ at some point, and we prove a stronger control theorem). We rely on the work of Buzzard on eigenvarieties [**32**]. Let us also mention here that if we had been only interested in the subset $\mathcal{Z}_{e,\mathrm{ord}} \subset \mathcal{Z}_e$ of π which are p-ordinary, the existence of X would be due to Hida (actually in a much wider context [**65**]). Moreover, there is an alternative construction of X due to Emerton in [**53**].

In a third Subsection 7.4, we show how to define some quasicoherent sheaves of admissible $G(\mathbb{A}_f^p)$-representations on an eigenvariety of idempotent type, and we prove their basic properties. As an application, we show the existence of an eigenvariety for the subset $\mathcal{Z}_{e,N}$ parameterizing the p-refined π in $\mathcal{Z}_e$ such that π_l is a non monodromic principal series (in the sense of § 6.6) for each l in a fixed finite set of places such that $G(\mathbb{Q}_l)$ is a quasisplit unitary group. We don't know if those latter eigenvarieties are of idempotent type. As a consequence of all those constructions, we introduce in Def. 7.5.2 the convenient notion of *minimal eigenvariety* containing a given p-refined automorphic representation, it is defined at the moment only when π_l is either unramified or a non monodromic principal series at each nonsplit prime l.

In the fourth part § 7.5, we explain how the existence of the expected p-adic Galois representations associated to (sufficiently many of) the π parameterized by $\mathcal{Z}$ gives rise to a continuous Galois pseudocharacter

$$T : G_E \longrightarrow \mathcal{O}(X).$$

Our results in this part are unconditional when G is attached to certain division algebras, and conditional to the assumption (Rep(m)) when G is the definite unitary group $\mathrm{U}(m)$ which is quasisplit at each finite place (so $m \neq 2 \bmod 4$). For each $x \in X(\overline{\mathbb{Q}}_p)$, we have then a canonical semisimple representations

$$\bar{\rho}_x : G_E \longrightarrow \mathrm{GL}_m(\overline{\mathbb{Q}}_p),$$

whose trace is the evaluation of T at x. The game is to understand with this weak notion of families of Galois representations (namely the mere existence of T) how to deduce from a property of the $\bar{\rho}_z$ for (a Zariski-dense subset of) $z \in Z$, a similar property for $\bar{\rho}_x$ for *any* $x \in X(\overline{\mathbb{Q}}_p)$. We are typically interested in a property concerning the restriction to a decomposition group at a finite place w of E.

At a prime w above p, we show that T is a refined family in the sense of § 4.2.2 hence we can apply to (X, T) the results of section 4. At a prime w not dividing p, it is convenient to introduced the *generic representation*

$$\rho_x^{\mathrm{gen}} : G_E \longrightarrow \mathrm{GL}_m(\overline{\mathcal{K}}_x),$$

whose trace is the composition of $T : G_E \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{K}_x := \mathrm{Frac}(\mathcal{O}_x)$ and where $\overline{\mathcal{K}}_x$ is a product of algebraic closures of each factor field of $\mathcal{K}_x$ (i.e. of the fraction fields of the germs of irreducible components of X at x). The representations $\bar{\rho}_x$ and ρ_x^{gen} have associated Weil-Deligne representations that we compare, and that we also compare with the ones of the $\bar{\rho}_z$ for $z \in Z$. For example, when X is the minimal eigenvariety containing some π of the type of Harris-Taylor, these Weil-Deligne representations are constant on X when restricted to the inertia group. The proofs rely on some lemmas on nilpotent elements in general matrix rings or GMA. Those facts are proved separately in the first subsection of an appendix § 7.8 that we devote to the general study of p-adic families of Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/F)$ when $l \neq p$ and $F/\mathbb{Q}_l$ a finite extension. In this appendix, we also recall the *dominance ordering* $\prec$ on nilpotent matrices and on Weil-Deligne representations, which is convenient to state our results.

In the next subsection § 7.6 we give an application of the techniques and results of this book to study some global Galois deformation rings, as was announced in § 2.6 of section 2. We fix a continuous absolutely irreducible Galois representation

$$\rho : G_{E,S} \longrightarrow \mathrm{GL}_m(L)$$

(L a finite extension of $\mathbb{Q}_p$, S a finite set of primes of E containing the primes dividing p) such that $\rho^{\perp} \simeq \rho(m-1)$, and which is crystalline with two-by-two distinct Hodge-Tate weights and crystalline Frobenius eigenvalues at the primes above p.

We are interested in the deformations ρ_A of ρ such that $\rho_A^{\perp} \simeq \rho_A(m-1)$, where A is a local artinian ring A with residue field L. We introduce two subfunctors of the full deformation functor: the *fine deformation functor* $X_{\rho,f}$, whose tangent space is $H^1_f(E, \mathrm{ad}(\rho))$, and the *refined deformation functor* $X_{\rho,\mathcal{F}}$, which depends on the choice of a refinement $\mathcal{F}$ of $\rho_{|E_v}$ ($p = v\bar{v}$). We show that those functors are pro-representable, we compare them when $\mathcal{F}$ is non critical, and we formulate two conjectures (C1) and (C2) concerning their structure (see Conj. 7.6.5).

For a quite general ρ, we also introduce in § 7.6.2 a definite unitary eigenvariety X and a point $z \in X$. If $\mathbb{T}$ (resp. $R_{\rho,\mathcal{F}}$) denotes the completion of the local ring of X at z (resp. the ring pro-representing $X_{\rho,\mathcal{F}}$), we show the existence of a natural map

$$R_{\rho,\mathcal{F}} \longrightarrow \mathbb{T}.$$

In this context, this arrow and the properties of eigenvarieties allow us to show that our conjectures (C1) and (C2) are actually consequences of the Bloch-Kato conjecture [(4)], which provides strong evidence for them. This leads us to conjecture that the arrow above is an isomorphism ("$R_{\rho,\mathcal{F}} = \mathbb{T}$"), and that a strong infinitesimal version of the *non critical slope forms are classical* property should hold: "eigenvarieties should be étale over the weight space (hence smooth) at non critical irreducible classical points". In turn, these last two conjectures imply (C1) and (C2).

Finally, as a simple application of the theory of refined families, we show in § 7.7 how we can construct many m-dimensional Galois representations of G_E which are unramified outside p and crystalline, irreducible, and with generic Hodge-Tate weights at the two primes of E dividing p. This application is conditional to (Rep(m)) but does not use any irreducibility assertion for the automorphic Galois representations. We rather start from the trivial representation and move in the *tame level 1* eigenvariety to find the Galois representations we are looking for.

7.2. Definition and basic properties of the Eigenvarieties

7.2.1. The setting. — Let E be a quadratic imaginary field and G be a definite unitary group in $m \geq 1$ variables attached to $E/\mathbb{Q}$, as in § 6.2, *e.g.* the group $\mathrm{U}(m)$ defined in § 6.2.2. Let us fix once and for all a rational prime p as well as field embeddings [(5)]

$$\iota_p : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p, \quad \iota_\infty : \overline{\mathbb{Q}} \longrightarrow \mathbb{C}.$$

We assume that $G(\mathbb{Q}_p) \simeq \mathrm{GL}_m(\mathbb{Q}_p)$. In particular, p splits in E and if we write $p = vv^c$ where $v : E \to \mathbb{Q}_p$ is defined by ι_p, then v induces a natural isomorphism $G(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_m(\mathbb{Q}_p)$. The embedding $E \longrightarrow \mathbb{C}$ given by ι_∞ induces an embedding $G(\mathbb{R}) \hookrightarrow_{\iota_\infty} \mathrm{GL}_m(\mathbb{C})$ well defined up to conjugation.

We fix a model of G over $\mathbb{Z}$ and a product Haar measure μ on $G(\mathbb{A}_f)$ such that $\mu(G(\widehat{\mathbb{Z}})) = 1$. We use the standard conventions for adèles: $\mathbb{A}_S$ (resp. $\mathbb{A}_f^S$) denotes the ring of finites adèles with components in (resp. outside) the set of primes S. Moreover, we denote by $\widehat{\mathbb{Z}}_S = \prod_{l \in S} \mathbb{Z}_l$ the ring of integers of $\mathbb{A}_S$.

The definition of an eigenvariety for G depends on the choice of a commutative Hecke algebra $\mathcal{H}$ that we fix once for all as follows. We fix a subset S_0 of the primes l

(4) More precisely, they are equivalent to the vanishing of $H^1_f(E, \mathrm{ad}(\rho))$.

(5) Which means that we also fix once for all an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$) of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$).

split in E such that $G(\mathbb{Q}_l) \simeq \mathrm{GL}_m(\mathbb{Q}_l)$ and $G(\mathbb{Z}_l)$ is a maximal compact subgroup [6], and set

$$\mathcal{H}_{\mathrm{ur}} := \mathcal{C}(G(\widehat{\mathbb{Z}}_{S_0})\backslash G(\mathbb{A}_{S_0})/G(\widehat{\mathbb{Z}}_{S_0}), \mathbb{Z}).$$

Recall that we defined in § 6.4.1 a subring [7] $\mathcal{A}_p \subset \mathcal{C}(I\backslash G(\mathbb{Q}_p)/I, \mathbb{Z}[1/p])$ where $I \subset G(\mathbb{Q}_p) \xrightarrow{\sim}_v \mathrm{GL}_m(\mathbb{Q}_p)$ is the standard Iwahori subgroup. We set [8]

$$\mathcal{H} := \mathcal{A}_p \otimes \mathcal{H}_{\mathrm{ur}}.$$

7.2.2. p-refined automorphic representations

Definition 7.2.1. — We say that $(\pi, \mathcal{R})$ is a p-refined automorphic representation of weight $\underline{k}$ if:

- π is an irreducible automorphic representation of G (see § 6.2.6),
- $\pi_\infty \xrightarrow{\sim}_{\iota_\infty} W_{\underline{k}}(\mathbb{C})$ (see § 6.7),
- π_p is unramified and $\mathcal{R}$ is an accessible refinement of π_p (see § 6.4.4).

Recall from § 6.4.3 that an accessible refinement of π_p is an ordering

$$\mathcal{R} = (\varphi_1, \dots, \varphi_m)$$

of the eigenvalues of Langlands'conjugacy class associated to π_p, such that π_p occurs as a subquotient of the normalised induction $\mathrm{Ind}_B^G \chi$, where χ is the unramified character of $(\mathbb{Q}_p^*)^m$ sending $(1, \dots, 1, p, 1, \dots, 1)$ to φ_i if p occurs at place i.

Definition 7.2.2. — If A is a ring, an *A-valued system of Hecke eigenvalues* is a ring homomorphism $\mathcal{H} \longrightarrow A$.

Let $(\pi, \mathcal{R})$ be as above, we can attach to it a $\overline{\mathbb{Q}}_p$-valued system of eigenvalues

$$\psi_{(\pi,\mathcal{R})} : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$$

as follows. By Definition 6.4.6, if $\chi : U \to \mathbb{C}^*$ is the character of the refinement $\mathcal{R}$ then $\chi\delta_B^{-1/2}$ occurs in π_p^I. Moreover, the restriction of the highest weight character of the algebraic representation $W_{\underline{k}}(\mathbb{Q}_p)$ (see § 6.7) to the subgroup $U \subset (\mathbb{Q}_p^*)^m$ defines a character $\delta_{\underline{k}} : U \to p^{\mathbb{Z}}$, so there is a unique ring homomorphism $\psi_p : \mathcal{A}_p \to \mathbb{C}$ such that

$$\psi_{p|U} = \chi\delta_B^{-1/2}\delta_{\underline{k}}. \tag{66}$$

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(6) In most applications, S_0 will contain almost all of the primes l as above, or at least have Dirichlet density 1. The reader may assume it from now on to fix ideas.

(7) Let us assume that $\mu(I) = 1$.

(8) We limit ourselves a bit the choice of $\mathcal{H}$ here only for notational reasons and later use. We could add for example inside $\mathcal{H}$ any commutative subring of $\mathcal{C}(K\backslash G(\mathbb{A}^{S_0 \cup \{p\}})/K, \overline{\mathbb{Q}}_p)$ for some compact open subgroup K and everything would apply verbatim.

Moreover, $\pi^{G(\widehat{\mathbb{Z}}_{S_0})}$ is one dimensional hence it defines a ring homomorphism ψ_{ur} : $\mathcal{H}_{\mathrm{ur}} \to \mathbb{C}$. By Lemma 6.2.7, the complex system of Hecke-eigenvalues $\psi_p \otimes_{\mathbb{Z}} \psi_{\mathrm{ur}}$ is actually $\overline{\mathbb{Q}}$-valued.

Definition 7.2.3. — We call $\psi_{(\pi,\mathcal{R})}$ the $\overline{\mathbb{Q}}_p$-valued system of Hecke eigenvalues associated to the p-refined automorphic representation $(\pi, \mathcal{R})$ of weight $\underline{k}$ defined by $\iota_p \iota_\infty^{-1}(\psi_p \otimes \psi_{\mathrm{ur}})$.

Remark 7.2.4. — We have $\psi_{(\pi,\mathcal{R})} = \psi_{(\pi',\mathcal{R}')}$ if, and only if, $\pi_t \simeq \pi'_t$ for each $t \in S_0 \cup \{p\}$ and $\mathcal{R} = \mathcal{R}'$.

7.2.3. Eigenvarieties as interpolations spaces of p-refined automorphic representations. — Let $\mathcal{Z}_0 \subset \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^m$ be the set of pairs $(\psi_{(\pi,\mathcal{R})}, \underline{k})$ associated to all the p-refined automorphic representations π of any weight $\underline{k}$, and let us fix $\mathcal{Z} \subset \mathcal{Z}_0$ a subset. It will be convenient to give a formal definition of what is an eigenvariety attached to $\mathcal{Z}$. We shall actually never use here the group G and the set $\mathcal{Z}$ could be replaced by any subset of $\mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^m$.

The *weight space* is the rigid analytic space over $\mathbb{Q}_p$

$$\mathcal{W} := \mathrm{Hom}_{gr-cont}(T^0, \mathbb{G}_m^{\mathrm{rig}})$$

whose points over any affinoid $\mathbb{Q}_p$-algebra A parameterize the continuous characters $T^0 = (\mathbb{Z}_p^*)^m \longrightarrow A^*$. It is isomorphic to a finite disjoint union of unit open m-dimensional balls. We view $\mathbb{Z}^m$ as embedded inside $\mathcal{W}(\mathbb{Q}_p)$, by mean of the map

$$(k_1, \ldots, k_m) \mapsto ((x_1, \ldots, x_m) \mapsto x_1^{k_1} \cdots x_m^{k_m}),$$

and we denote by $\mathbb{Z}^{m,-}$ the subset of $\mathbb{Z}^m$ consisting of strictly decreasing sequences. We will also need to fix an element of U^-. Its choice is not really important, but to fix ideas we set

$$u_0 := \mathrm{diag}(p^{m-1}, \ldots, p, 1) \in U^- \subset \mathcal{A}_p^*.$$

Let us fix $L \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$. In the definition below and in the sequel, we will always view $\mathcal{W}$, $\mathbb{G}_m$, and the affine line $\mathbb{A}^1$, as rigid analytic spaces over L even if we do not make it appear explicitly: for example, we will write $\mathcal{W}$ for $\mathcal{W} \times_{\mathbb{Q}_p} L$.

Definition 7.2.5. — An eigenvariety for $\mathcal{Z}$ is a reduced p-adic analytic space X over L equipped with:

— A ring homomorphism $\psi : \mathcal{H} \longrightarrow \mathcal{O}(X)^{\mathrm{rig}}$,

— An analytic map $\omega : X \longrightarrow \mathcal{W}$ over L,

— An accumulation and Zariski-dense subset $Z \subset X(\overline{\mathbb{Q}}_p)$,

such that the following conditions are satisfied:

(i) The map $\nu := (\omega, \psi(u_0)^{-1}) : X \longrightarrow \mathcal{W} \times \mathbb{G}_m$ is finite.

(ii) For all open affinoid $V \subset \mathcal{W} \times \mathbb{G}_m$, the natural map

$$\psi \otimes \nu^* : \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(\nu^{-1}(V))$$

is surjective.

(iii) The natural evaluation map $X(\overline{\mathbb{Q}}_p) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p)$,

$$x \mapsto \psi_x := (h \mapsto \psi(h)(x)),$$

induces a bijection $Z \xrightarrow{\sim} \mathcal{Z}$, $z \mapsto (\psi_z, \omega(z))$.

Recall from § 3.3.1 that a Zariski-dense and accumulation subset $Z \subset X$ is a subset that meets any irreducible component of X and such that for all z in Z, there is a basis of affinoid neighbourhoods U of z such that $Z \cap U$ is Zariski-dense in U.

The maps ω and ν determine each other, so in the sequel we shall use either the 4-uple $(X/L, \psi, \omega, Z)$ or $(X/L, \psi, \nu, Z)$ to denote an eigenvariety. The choice of $\psi(u_0)^{-1}$ rather than $\psi(u_0)$ in (i) is a convention (of [**44**]).

Remark 7.2.6. — i) In other words, (ψ, X) is a rigid analytic family of systems of Hecke-eigenvalues interpolating the ones in $\mathcal{Z} \xrightarrow{\sim} Z$. The Zariski-density of $\mathcal{Z}$ and (i) ensures that X is minimal with that property in some sense (see Prop. 7.2.8).

ii) We may like to think of (or define) such an eigenvariety as the "Zariski-closure" of $\mathcal{Z}$ in $\mathcal{W} \times \mathbb{G}_m \times \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \mathbb{A}^1)$. However, as this latter space is not a rigid space if $\mathcal{H}$ is not finitely generated (which will be the case in the applications), we have to be a little careful. The requirement (ii) above is a way to circumvent this problem.

It turns out that such an eigenvariety, if exists, is unique.

Lemma 7.2.7. — *Let (X, ψ, ω, Z) be an eigenvariety:*

(a) *X is an admissible increasing union of open affinoids of the form $\nu^{-1}(V)$, for $V \subset \mathcal{W} \times \mathbb{G}_m$ open affinoid. In particular, any two closed points of X lie in such an open affinoid.*

(b) *For any $x, y \in X(\overline{\mathbb{Q}}_p)$, $x = y$ if, and only if, $\psi_x = \psi_y$* and *$\omega(x) = \omega(y)$.*

Proof. — Assertion (a) follows from property (i) of eigenvarieties and the fact that $\mathcal{W} \times \mathbb{G}_m$ is an admissible increasing union of open affinoids. Part (b) is then a consequence of (a) and property (ii). □

Proposition 7.2.8. — *If $(X_1/L, \psi_1, \nu_1, Z_1)$ and $(X_2/L, \psi_2, \nu_2, Z_2)$ are two eigenvarieties for $\mathcal{Z}$, there exists a unique L-isomorphism $\zeta : X_1 \to X_2$ such that $\nu_2 \cdot \zeta = \nu_1$, and $\forall h \in \mathcal{H}$, $\psi_1(h) = \psi_2(h) \cdot \zeta \in \mathcal{O}(X_1)$.*

Proof. — Fix $(X_i, \psi_i, \omega_i, Z_i)$, $i = 1, 2$ satisfying (i) to (iii), and denote again by $\nu_i = (\omega_i \times \psi_i(u_0)^{-1}) : X_i \to \mathcal{W} \times \mathbb{G}_m$ the finite map of (i). As a consequence of Lemma 7.2.7 (b) and assumption (iii), there is a unique bijection $\zeta : Z_1 \xrightarrow{\sim} Z_2$, such that for all $z \in Z_1$, $\psi_{1_z} = \psi_{2_{\zeta(z)}}$ and $\omega_1(z) = \omega_2(\zeta(z))$. We will eventually prove that ζ extends to an isomorphism $\zeta : X_1 \xrightarrow{\sim} X_2$ as in the statement. By Lemma 7.2.7 (b), and as the X_i are reduced, such a map is actually unique if it exists.

For any admissible open $V \subset \mathcal{W} \times \mathbb{G}_m$, we set $X_{i,V} := \nu_i^{-1}(V)$, and let $\mathbb{A}_V$ denote the affine line over V. For each finite set $I \subset \mathcal{H}$ and such a V, we have a natural V-map

$$f_{i,V,I} : X_{i,V} \longrightarrow \mathbb{A}_V^I, \ \ x \mapsto (h(x))_{h \in I}$$

inducing a natural map $X_{i,V} \to \operatorname{proj\,lim}_{I \subset \mathcal{H}} \mathbb{A}_V^I$, and commuting with any base change by an open immersion $V' \subset V$. The morphism $f_{i,V,I}$ is closed by (i).

Assume V is moreover affinoid. Assumption (i) shows that there exists I_V such that for $I \supset I_V$, $f_{i,V,I}$ is a closed immersion for both i. We claim that for $I \supset I_V$, we have an inclusion $f_{1,V,I}(X_{1,V}) \subset f_{2,V,I}(X_{2,V})$. By exchanging 1 and 2 and using that both $X_{i,V}$ are reduced, it will follow that as closed subspaces of $\mathbb{A}_V^I$, for $I \supset I_V$, we have $f_{1,V,I}(X_{1,V}) = f_{2,V,I}(X_{2,V})$.

Let $x \in X_{1,V}$. If $x \in Z_1$ then $f_{1,V,I}(x) \in f_{2,V,I}(X_{2,V})$ by definition of ζ. In general, by the Zariski density of Z_1 in X_1, Lemma 7.2.7 (a) and Lemma 7.2.9 thereafter, we can find an open affinoid $V' \supset V$ such that some $z \in Z_1 \cap X_{1,V'}$ lies in the same irreducible component T of $X_{1,V'}$ as x. By the accumulation property of Z, Z is Zariski-dense in T, hence for $I' \supset I \cup I_{V'}$,

$$f_{1,V',I'}(T) \subset f_{2,V',I'}(X_{2,V'}).$$

In particular, for such an I' we have $f_{1,V,I'}(x) \in f_{2,V,I'}(X_{2,V})$ and by projecting to $\mathbb{A}_V^I$ we get that this holds also when $I' = I$, hence the claim.

We define now a V-isomorphism $\zeta_V : X_{1,V} \longrightarrow X_{2,V}$ by setting, for $I \supset I_V$,

$$\zeta_V := f_{2,V,I}^{-1} \cdot f_{1,V,I}.$$

This map does not depend on I and it obviously extends the previously defined map ζ on $Z_1 \cap X_{1,V}$. The independence of I implies that

$$\forall h \in \mathcal{H}, \ \psi_1(h) = \psi_2(h) \cdot \zeta_V \in \mathcal{O}(X_{1,V}).$$

We check at once that $\zeta_V \times_V V' = \zeta_{V'}$ for any $V' \subset V$ open affinoid, hence the ζ_V glue to a unique isomorphism $\zeta : X_1 \longrightarrow X_2$ and we are done. □

Lemma 7.2.9. — *Let X be a rigid space over $\mathbb{Q}_p$ and let $U \subset X$ be a quasi-compact admissible open of X. For any two points $x, y \in X$ which are in the same irreducible component, there is a quasi-compact admissible open $U' \subset X$ containing U, x and y, and such that x and y are in the same irreducible component of U.*

If moreover X is an admissible increasing union of open affinoids X_i, then we may choose U' to be one of the X_i.

Proof. — Let $f : \widetilde{X} \to X$ be the normalization of X. Choose x' and y' in $\widetilde{X}$ lifting respectively x and y, and in the same connected component. We may then choose a connected, quasi-compact, admissible open $V \subset \widetilde{X}$ containing x' and y'. Let $X = \bigcup_{i\in I} U_i$ be an admissible covering of X by open affinoids $U_i \subset X$. The space $\widetilde{X}$ is the admissible union of the affinoids $f^{-1}(U_i)$ (as f is finite), so we may find a finite subset $J \subset I$ such that $V \subset \cup_{i\in J} f^{-1}(U_i)$. The admissible open

$$U' := U \cup \bigcup_{i\in J} U_i$$

does the trick. The last assertion is obvious from the first one. □

Definition 7.2.10. — We say that a rigid space X over $\mathbb{Q}_p$ is nested if it has an admissible covering by some open affinoids $\{X_i, i \geq 0\}$ such that $X_i \subset X_{i+1}$ and that the natural $\mathbb{Q}_p$-linear map $\mathcal{O}(X_{i+1}) \longrightarrow \mathcal{O}(X_i)$ is compact.

Note that any such X is separated, and that any finite product of nested spaces is nested. For example, $\mathbb{A}^1$, $\mathbb{G}_m$ and $\mathcal{W}$ are easily checked to be nested, hence so is $\mathcal{W} \times \mathbb{G}_m$. However, quasicompact rigid spaces (like affinoids) are not nested.

Lemma 7.2.11. — *Assume that $X/\mathbb{Q}_p$ is nested.*

(i) *If $Y \longrightarrow X$ is a finite morphism, then Y is nested.*
(ii) *Assume that X is reduced. Then*

$$\mathcal{O}(X)^0 := \{f \in \mathcal{O}(X),\ \forall x \in X,\ |f(x)| \leq 1\}$$

is a compact subset of $\mathcal{O}(X)$.

Recall that $\mathcal{O}(X)$ is equipped the coarsest locally convex topology such that all the restriction maps $\mathcal{O}(X) \longrightarrow \mathcal{O}(U)$, $U \subset X$ an affinoid subdomain, are continuous ($\mathcal{O}(U)$ being equipped with its Banach algebra topology). It is a separated topological $\mathbb{Q}_p$-algebra.

Proof. — To show (i), it suffices to check that for each $\mathbb{Q}_p$-affinoid X, each coherent $\mathcal{O}_X$-module M, and each affinoid subdomain $U \subset X$ such that $\mathcal{O}(X) \longrightarrow \mathcal{O}(U)$ is compact, then the natural map $M(X) \longrightarrow M(U)$ is compact. Of course, $M(U)$ and $M(X)$ are equipped here with their (canonical) topology of finite module over an affinoid algebra. Let us fix an $\mathcal{O}$-epimorphism $\mathcal{O}^n \longrightarrow M$, and consider the commutative diagram of continuous $\mathbb{Q}_p$-linear maps

$$\begin{tikzcd} \mathcal{O}(X)^n \arrow[r, two heads] \arrow[d] & M(X) \arrow[d] \\ \mathcal{O}(U)^n \arrow[r, two heads] & M(U) \end{tikzcd}$$

Let $\mathcal{B} \subset M(X)$ be a bounded subset. By the open mapping theorem, there is a bounded subset $\mathcal{B}' \subset \mathcal{O}(X)^n$ whose image under the top surjection is $\mathcal{B}$. By assumption the left vertical arrow is compact hence the image of $\mathcal{B}'$ in $\mathcal{O}(U)^n$ has compact closure, hence so has the image of $\mathcal{B}$ in $M(U)$.

We show (ii) now. Let us fix $X = \cup_i X_i$ a nested covering of X, and set

$$Y_i := \overline{\mathrm{Im}(\mathcal{O}(X)^0 \to \mathcal{O}(X_i))}.$$

It is a compact subspace of $\mathcal{O}(X_i)$ as is the image of the unit ball of $\mathcal{O}(X_{i+1})$ by assumption. But the injection

$$\mathcal{O}(X)^0 \longrightarrow \prod_i \mathcal{O}(X_i)$$

has a closed image, and is a homeomorphism onto its image. We conclude as it lies in the compact subspace $\prod_i Y_i$. □

Corollary 7.2.12. — *Eigenvarieties are nested.*

For later use, let us introduce another notation. Let $(X/L, \psi, \nu, Z)$ be an eigenvariety. For $i = 1, \dots, m$, let

$$u_i = \mathrm{diag}(1, \dots, 1, p, 1, \dots, 1) \quad \in U \subset \mathcal{A}_p^*$$

where p occurs at place i.

Definition 7.2.13. — For $i \in \{1, \dots, m\}$, $F_i := \psi(u_i) \in \mathcal{O}(X)^*$. By definition and by formula (66) of § 7.2.2, they are the unique analytic functions on X such that for each $z = (\psi_{(\pi,\mathcal{R})}, \underline{k}) \in Z \xrightarrow{\sim} \mathcal{Z}$, we have

$$\iota_p \iota_\infty^{-1}(\mathcal{R} \cdot |p|^{\frac{1-m}{2}}) = (F_1(z)p^{-k_1}, \dots, F_i(z)p^{-k_i+i-1}, \dots, F_m(z)p^{-k_m+m-1}).$$

7.3. Eigenvarieties attached to an idempotent of the Hecke-algebra

7.3.1. Eigenvarieties of idempotent type. — We keep the notations above, and we fix an idempotent

$$e \in \mathcal{C}_c(G(\mathbb{A}_f^{p,S_0}), \overline{\mathbb{Q}}) \otimes 1_{\mathcal{H}_{\mathrm{ur}}} \subset \mathcal{C}_c(G(\mathbb{A}_f^p), \overline{\mathbb{Q}}).$$

Let $\mathcal{Z}_e \subset \mathcal{Z}$ be the subset of $(\psi_{(\pi,\mathcal{R})}, \underline{k})$ such that $e(\pi^p) \neq 0$. We will say that such a π is *of type* e. Note that by construction, there is a compact open subgroup of $G(\mathbb{A}_f)$

such that each π of type e has nonzero vectors invariant under this subgroup, thus the admissibility of the space of automorphic representations (Lemma 6.2.5) shows that the natural map

$$\mathcal{Z}_e \longrightarrow \mathbb{Z}^m,\ (\psi, \underline{k}) \mapsto \underline{k}, \tag{67}$$

has finite fibers. [9] We assume from now on that $\mathcal{Z}_e$ is nonempty.

We fix $L \subset \overline{\mathbb{Q}}_p$ a sufficiently large finite extension of $\mathbb{Q}_p$ such that $\iota_p(e) \in \mathcal{C}_c(G(\mathbb{A}_f^p), L)$. We will also write e instead of $\iota_p(e)$ or for $\iota_\infty(e) \in \mathcal{C}_c(G(\mathbb{A}_f^p), \mathbb{C})$ when there is no possible confusion. Recall that we defined some functions F_i in Def.7.2.13.

Theorem 7.3.1. — ([**36**, Thm. A]) *There exists a unique eigenvariety $(X/L, \psi, \nu, Z)$ for $\mathcal{Z}_e$. It has the following extra properties:*

(iv) *X is nested and equidimensionnal of dimension m. Moreover, $\nu(X)$ is a Fredholm hypersurface of $\mathcal{W} \times \mathbb{G}_m$, hence X inherits a canonical admissible covering. Precisely, X is admissibly covered by the affinoid subdomains $\Omega \subset X$ such that $\omega(\Omega) \subset \mathcal{W}$ is an open affinoid and that $\omega : \Omega \longrightarrow \omega(\Omega)$ is finite and surjective when restricted to any irreducible component of Ω.*

(v) *Let Z' be the subset of $x \in X(\overline{\mathbb{Q}}_p)$ such that*
   - (a) $\omega(x) = (k_1, \dots, k_m) \in \mathbb{Z}^{m,-}$,
   - (b) $\forall i \in \{1, \dots, m-1\},\ v(F_1(x)F_2(x)\cdots F_i(x)) < k_i - k_{i+1} + 1$,
   - (c) *if* $\varphi_i := F_i(x)p^{-k_i+i-1}$ *then* $\forall i \neq j,\ \varphi_i\varphi_j^{-1} \neq p$.

   Then $Z' \subset Z$, and Z' is an accumulation Zariski-dense subset of X.

(vi) $\psi(\mathcal{H}_{\mathrm{ur}}) \subset \mathcal{O}(X)^0$.

Proof. — The uniqueness assertion is Prop. 7.2.8. The existence of X/L satisfying (i)-(vi) is [**36**, Thm. A] (using [**32**]) when:

- $e = e_{K^p}$ for a net compact open subgroup $K = I \times K^p \subset G(\mathbb{A}_f)$,
- $\mathcal{W}$ is replaced by its open subspace of analytic characters [10],
- $p \neq 2$ for assertion (v) and the accumulation property of Z,

[9] By definition, these fibers are empty outside $\mathbb{Z}^{m,-}$. Moreover, if $d(\underline{k}) := \dim_{\mathbb{C}}(W_{\underline{k}})$ denotes the polynomial in $\underline{k}$ given by Weyl's formula, then the cardinal of the fiber of (67) above $\underline{k}$ is appoximately a constant times $d(\underline{k})$ (this constant actually depends in general on some congruence on $\underline{k}$ modulo a fixed integer). In particular, if $\mathcal{Z}_e \neq \emptyset$, this cardinal goes to infinity when $\underline{k}$ goes off the walls.

[10] Recall that a continuous character $\chi : \mathbb{Z}_p^* \to L^*$ is *analytic* if the induced map $\mathbb{Z}_p \to L$, $t \mapsto \chi(1+pt)$, is the restriction to $\mathbb{Z}_p$ of an element of the Tate algebra $L\langle t\rangle$. The analytic characters are exactly the ones that coincide over $1+p\mathbb{Z}_p$ with the character $x \mapsto x^s$ (defined by the binomial power series) where $s \in L$ is any element such that $v(s) > 1/(p-1) - 1$. A finite order character is analytic if, and only if, its order is prime to p.

- and under a stronger condition for part (b) in (v), namely that

$$v(\psi(u_0)(x)) < 1 + \mathrm{Min}_{i=1}^{m-1}(k_i - k_{i+1})$$

(note that by definition $\psi(u_0) = \prod_{i=1}^{m-1} F_1 F_2 \cdots F_i$).

We will explain in § 7.3.2 to § 7.3.6 below how to extend the construction of [**36**] to the generality above, in the spirit of [**32**]. □

Remark 7.3.2. — i) An independent construction of X has been given by M. Emerton in [**53**]. The admissible open subspace $X^{\mathrm{ord}} \subset X$ defined by $|\psi(u_0)| = 1$ was previously constructed by H. Hida in a much more general context (see [**65**]). It is actually closed and the induced map $\omega : X^{\mathrm{ord}} \longrightarrow \mathcal{W}$ is finite.

ii) Assume that $e = e_1 + e_2$ is the sum of two orthogonal idempotents. Although we will not use it in what follows, let us note that by [**37**], the eigenvariety X_i of $(e_i,\mathcal{H})$ has a natural closed embedding into X commuting with (ψ,ν). Moreover, $X = X_1 \cup X_2$ (the intersection might be non empty). Actually, we could even show that X is precisely the abstract gluing of X_1 and X_2 "over (ψ,ν)".

iii) (A variant with a *fixed weight*) Let $i_0 \in \{1,\ldots,m\}$ and $k \in \mathbb{Z}$, and consider the subset $\mathcal{Z}_{e,k_{i_0}=k} \subset \mathcal{Z}_e$ whose elements are the $(\psi_{(\pi,\mathcal{R})},\underline{k})$ such that $\underline{k}$ has its i_0^{th} term k_{i_0} equal to k. Then there exists also a unique eigenvariety X' for $\mathcal{Z}_{e,k_{i_0}=k}$ satisfying all the properties of Thm. 7.3.1, except that it is equidimensional of dimension $m-1$. This follows verbatim by the same proof (see below) if we replace everywhere the space $\mathcal{W}$ in this proof by its hypersurface $\mathcal{W}_{k_{i_0}=k} \subset \mathcal{W}$ parameterizing the characters whose i_0^{th} component is fixed and equal to $x_{i_0} \mapsto x_{i_0}^k$. In most cases, X turns out to be isomorphic to $X' \times X_1$ where X_1 is a suitable eigenvariety of U(1). As those X_1 are explicit (*e.g.* they are finite over $\mathcal{W}_1$), it is in general virtually equivalent to study X or X'.

We end this paragraph by a discussion on idempotents, so as to shed light on the kind of sets $\mathcal{Z}_e$ that we can obtain. Of course, in the applications we will mostly choose e as a tensor product of idempotents $e_l \in \mathcal{C}_c(G(\mathbb{Q}_l),\overline{\mathbb{Q}})$ ($l \neq p$) such that $e_l = 1_{G(\mathbb{Z}_l)}$ for $l \in S_0$ or l large enough.

Example 7.3.3. — (*See e.g.* [**31**, §2].)

i) Of course, the simplest class of idempotents of $\mathcal{C}_c(G(\mathbb{A}_S),\mathbb{Q})$ are the

$$e_K := \mu(K)^{-1}1_K \in \mathcal{C}(K\backslash G(\mathbb{A}_S)/K,\mathbb{Q})$$

for any compact open subgroup $K \subset G(\mathbb{A}_S)$.

ii) A little more generally, if τ is an irreducible smooth $\overline{\mathbb{Q}}$-representation of such a K (hence finite dimensional), the element

$$e_\tau \in \mathcal{C}_c(G(\mathbb{A}_S),\overline{\mathbb{Q}})$$

which is zero outside K and coincide with $\frac{\dim_{\overline{\mathbb{Q}}}\tau}{\mu(K)}\mathrm{tr}\,(\pi^*)$ on K is an idempotent. We see at once that for each smooth representation V of $G(\mathbb{A}^S)$, $e_\tau V \subset V$ is the τ-isotypic component of V.

iii) (*Special idempotents*) Let k be a field of characteristic 0, $H = \mathcal{C}_c(G(\mathbb{A}_S), k)$, $e \in H$ an idempotent, and Mod_e be the full subcategory of the category of smooth $k[G(\mathbb{A}_S)]$-representations whose objects V are generated by eV. Following the terminology of [**31**, §3], we say that e is *special* if the functor $V \mapsto eV$, $\mathrm{Mod}_e \to \mathrm{Mod}(eHe)$ is an equivalence of categories. If e is special, the induction functor

$$W \in \mathrm{Mod}(eHe) \mapsto I(W) := He \otimes_{eHe} W \ \in \mathrm{Mod}_e.$$

is a quasi-inverse of $V \mapsto eV$, hence is exact, and for any $V \in \mathrm{Mod}_e$, the natural surjection induced an isomorphism [11]

$$I(eV) \xrightarrow{\sim} V. \tag{68}$$

iv) Set $S = \{l\}$ to simplify. If $e = e_{\tau_l}$ for some K_l-type τ_l, then e is special if, and only if, τ_l is a type in the sense of Bushnell and Kutzko [**31**]. The simplest example, due to Borel, is the case where $e = e_{K_l}$ and K_l is a Iwahori subgroup of $G(\mathbb{Q}_l)$, in which case Mod_e is the unramified Bernstein component. Moreover, by [**17**, Cor. 3.9], there exist arbitrary small compact open subgroups K_l of $G(\mathbb{Q}_l)$ such that e_{K_l} is special. However, as is well known, if K_l is a maximal compact subroup then e_{K_l} is not special in general (e.g. when $G(\mathbb{Q}_l) \xrightarrow{\sim} \mathrm{GL}_m(\mathbb{Q}_l)$ for $m > 1$).

v) (Bernstein components) Set again $S = \{l\}$. By results of Bernstein (see [**17**] or [**31**, 3.12, 3.12]), if e is special then there is a finite set Σ_l of Bernstein components of $G(\mathbb{Q}_l)$ such that Mod_e is the direct sum of these components ([**31**, Prop. 3.6]). Reciprocally, for any finite set Σ_l of components we can find a special idempotent $e_{\Sigma_l} \in \mathcal{C}(G(\mathbb{Q}_l), \overline{\mathbb{Q}})$ whose associated set of components is Σ_l. This idempotent is not unique however in general, but all the equivalent ones will give rise to the same set $\mathcal{Z}_e$, hence to the same eigenvariety by virtue of the uniqueness Prop. 7.2.8.

This remark allow us in particular to say that there exist eigenvarieties for the subset $\mathcal{Z} \subset \mathcal{Z}_0$ parameterizing p-refined automorphic representations whose local components in a finite set of primes all lie in a given Bernstein component.

Remark 7.3.4. — (K-types versus general idempotents.) The aim of type theory is to show that the special idempotents e_{Σ_l} above can be chosen of the form e_τ for some explicit K_l-type τ. In our context, this extra information is helpful from a

(11) Indeed, as $eI(W) = W$ for all W, the induced map $eI(V) \to eV$ is necessarily an isomorphism, hence the kernel of the map (68) is killed by e, whence is zero as e is special property. This argument shows actually that e is special if, and only if, $\mathrm{Mod}_e(G(\mathbb{Q}_l))$ is stable by subobjects.

computational point of view. For example, if $e = e_\tau$ then we will see that the space of p-adic automorphic form of type e is

$$\mathcal{S}(V,r) = \tau \otimes_L (F(\mathcal{C}(V,r)) \otimes_L \tau^*)^{K^p}$$

which is computable in theory. In general, some e_{Σ_l} are given abstractly by images of some idempotents in the Bernstein center of $G(\mathbb{Q}_l)$ and we have very few control on them.

7.3.2. Review of the construction of the eigenvariety ([36]). — The eigenvariety X associated to e is constructed by some formal process ([**43**], [**44**], [**36**], [**32**]) from the action of the Hecke operators on the orthonormalizable Banach family of spaces of p-adic automorphic forms of G. For example, $X(\overline{\mathbb{Q}}_p)$ turns out to parameterize exactly the $\overline{\mathbb{Q}}_p$-valued systems of Hecke eigenvalues on finite slope p-adic eigenforms of type e for G.

As our main theorem relates some Selmer group to the smoothness of X at some point, and for sake of completeness, we give below an essentially self-contained overview of the construction of X and of the theory of p-adic automorphic forms alluded to above (12). Actually, we shall use also some objects occurring in this construction to define the families of admissible $G(\mathbb{A}_f^p)$-representation on X in § 7.4.1, as well as to define their non monodromic principal series locus in § 7.4.2. The construction proceeds in four steps.

7.3.3. Step I. The family of the U^--stable principal series of a Iwahori subgroup. — The theory of p-adic automorphic forms of G relies essentially on the existence and properties of the p-adic family of the U^--stable principal series of the Iwahori subgroup I of $\mathrm{GL}_m(\mathbb{Q}_p)$. We take here and below the notations of § 6.4.1 with $F = \mathbb{Q}_p$ and $\varpi = p$, except that we shall write $G(\mathbb{Q}_p) \xrightarrow{\sim}_v \mathrm{GL}_m(\mathbb{Q}_p)$ for the G *loc. cit.* which is already used here for the unitary group over $\mathbb{Q}$.

Let $\overline{N}_0$ be the subgroup of lower triangular elements of I. The product map in $G(\mathbb{Q}_p)$ induces an isomorphism

$$\overline{N}_0 \times B \xrightarrow{\sim} IB.$$

If $u \in U^-$ then $u^{-1}\overline{N}_0 u \subset \overline{N}_0$, therefore (see Proposition 6.4.1) $M^{-1}IB \subset IB$. Let

$$\chi : T^0 \longrightarrow \mathcal{O}(\mathcal{W})^*$$

denote the tautological character. If $V \subset \mathcal{W}$ is either an open affinoid or a closed point we denote by $\chi_V : T^0 \longrightarrow A(V)^*$ the induced continuous character. Fix such a V. There exists an smallest integer $r_V \geq 0$ such that for any integer $r \geq r_V$, χ_V

(12) We warn the reader that some of the various conventions that we use in this book differ from the ones used in [**36**].

restricts to an analytic $A(V)$-valued function on the subgroup of elements of T^0 with coefficients in $1+p^r\mathbb{Z}_p$. Viewing the character χ of T^0 as a character of B which is trivial on UN, it makes then sense to consider for $r \geq r_V$ the space

$$\mathcal{C}(V,r) = \left\{ \begin{array}{l} f : IB \longrightarrow A(V),\ f(xb) = \chi_V(b)f(x)\ \ \forall\, x \in IB,\ b \in B, \\ f_{|\overline{N}_0} \text{ is } r\text{-analytic.} \end{array} \right\}$$

Let us recall what r-analytic means. Let $\{n_{i,j}\}_{i>j}$ be the obvious matrix coefficients but divided by p, viewed as algebraic functions on the lower unipotent subgroup of $G(\mathbb{Q}_p)$. This collection of maps induces a homeomorphism $\overline{N}_0 \xrightarrow{\sim} \mathbb{Z}_p^{n(n-1)/2}$. A function $f : \overline{N}_0 \to A(V)$ is said to be *r-analytic* if for each $a \in \overline{N}_0$, the induced map

$$f_a : \overline{N}_0 \to A(V),\ \ n \mapsto f(an),$$

lies in the Tate algebra $A(V)\langle\{p^r n_{i,j}\}_{i>j}\rangle$. If we endow this latter algebra with the sup norm, then the norm $|f| := \sup_a |f_a|$ makes $\mathcal{C}(V,r)$ a Banach $A(V)$-module which is $A(V)$-ONable by construction. [13] It is equipped with an integral $A(V)$-linear action of M by left translations: $(m.f)(x) = f(m^{-1}x)$. If we set

$$U^{--} = \{u = (p^{a_1}, \ldots, p^{a_m}) \in U,\ a_1 > a_2 > \cdots > a_m\},$$

then an immediate computation shows that the action of any $u \in U^{--}$ on $\mathcal{C}(V,r)$ is $A(V)$-compact.

The family $\{\mathcal{C}(V,r), V, r \geq r_V\}$ of M-modules defined above satisfies some compatibilities. If $V' \subset V$ is another open affinoid or closed point, then the natural map $\mathcal{C}(V,r) \to \mathcal{C}(V',r)$ induces an M-equivariant isomorphism $\mathcal{C}(V,r)\widehat{\otimes}_{A(V)}A(V')$. Moreover, the natural inclusion $\mathcal{C}(V,r) \longrightarrow \mathcal{C}(V,r+1)$ is $A(V)[M]$-equivariant and compact. If $r > 0$ and $u \in U^{--}$, then the action of u factors through the compact inclusion $\mathcal{C}(V,r-1) \longrightarrow \mathcal{C}(V,r)$ above.

For any continuous character $\psi : T \longrightarrow L^*$ and $r \geq r_\psi := r_{\psi_{|T^0}}$, let us consider similarly the L-Banach space $i_B^{IB}(\chi,r)$ of functions $f : IB \to L^*$ whose restriction to $\overline{N}_0$ is r-analytic and which satisfy $f(xb) = \psi(b)f(x)$ for all $x \in IB$ and $b \in B$. The difference with the previous spaces is that $\psi_{|U}$ may be nontrivial. It has again an action of M by left translations. If $\psi' : T \longrightarrow L^*$ is another continuous character and $r \geq r_\psi, r_{\psi'}$, we define a natural map

$$i_B^{IB}(\psi,r) \longrightarrow i_B^{IB}(\psi'\psi,r),\ \ f \mapsto (x \mapsto \psi'(x)f(x)),$$

where for $x \in IB$, $\psi'(x) := \psi'(t)$ for $t \in T$ the unique element such that $x \in \overline{N}_0 tN$. We check at once that this map is well defined and that it induces an M-equivariant

(13) It is isometric to $A(V)\langle\{n_{i,j}\}_{i>j}\rangle^{p^{rm(m-1)/2}}$.

isomorphism [14]

$$(i_B^{IB}(\psi, r)) \otimes \psi'^{-1} \longrightarrow i_B^{IB}(\psi\psi', r). \tag{69}$$

Assume now that $V = \{\underline{k}\}$ with $\underline{k} \in \mathbb{Z}^{m,-}$, in which case $r_V = 0$. The choice of an highest-weight vector in $W_{\underline{k}}(\mathbb{Q}_p)$ with respect to B gives an M-equivariant embedding

$$W_{\underline{k}}(L)^* \longrightarrow i_B^{IB}(\delta_{\underline{k}}, 0).$$

Hence we get by (69) a canonical (up to multiplication by L^*) M-equivariant embedding [15]

$$W_{\underline{k}}(L)^* \otimes \delta_{\underline{k}} \longrightarrow \mathcal{C}(\underline{k}, 0) = i_B^{IB}(\chi_{\underline{k}}, 0).$$

Actually, the subspace of $\mathcal{C}(\underline{k}, 0)$ defined above is exactly the subspace of functions on IB which are restrictions of polynomial functions on the whole of $G(\mathbb{Q}_p)$.

7.3.4. Step II. p-adic automorphic forms. — Using the M-modules defined above as coefficient systems, we can define various Banach spaces of *p-adic automorphic forms* for G. Consider the subring $\mathcal{H}^- := \mathcal{A}_p^- \otimes \mathcal{H}_{\mathrm{ur}} \subset \mathcal{H}$. It will be convenient to introduce a functor $F : \mathrm{Mod}(L[M]) \longrightarrow \mathrm{Mod}(\mathcal{A}_p^- \otimes \mathbb{Z}[G(\mathbb{A}_f^p)])$ by

$$F(E) := \left\{ \begin{array}{l} f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow E, \\ f(g(1 \times k_p)) = k_p^{-1} f(g),\ \forall g \in G(\mathbb{A}_f),\ k_p \in I, \\ f \text{ is smooth outside } p. \end{array} \right\}$$

By *f is smooth outside p* we mean that f is invariant by right translations under some compact open subgroup of $G(\mathbb{A}_f^p)$. The group $G(\mathbb{A}_f^p)$ acts on this space in a smooth fashion by right translations, and it commutes with the natural action of $\mathcal{A}_p^-$. The direct summand $eF(E) \subset F(E)$ is then a $\mathcal{H}^-$-module in a natural way. Let $K = I \times K^p \subset G(\mathbb{A}_f)$ be a compact open subgroup which is sufficiently small so that $e = e.e_{K^p}$, and such that $K_l = G(\mathbb{Z}_l)$ for each $l \in S_0$. Let h_K be the class number $|G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K|$ (see § 6.2.3 ii)) and let us choose a decomposition

$$G(\mathbb{A}_f) = \coprod_{i=1}^{h_K} G(\mathbb{Q}) x_i K,\ \ \Gamma_i := x_i^{-1} G(\mathbb{Q}) x_i \cap K.$$

Then Γ_i is a finite group, and we may even assume by reducing K that Γ_i is trivial for each i. The map $f \mapsto (f(x_1), \ldots, f(x_{h_K}))$ induces a L-linear isomorphism

$$e_K F(E) \xrightarrow{\sim} E^{h_K}. \tag{70}$$

---

[14] Recall that by Proposition 6.4.1 (i), each character of U^- extends uniquely to a character of M trivial on I.

[15] The twists appearing there comes from the fact that we chose to extend trivially on U the induced character in the definition of $\mathcal{C}(V, r)$. This choice could have been avoided by introducing the space of p-adic characters of T rather than T^0. However, as $\mathbb{Z}^m$ is not Zariski-dense in that space, this would have introduced other nuisances...

In particular, the functor $E \mapsto eF(E)$, $\mathrm{Mod}(L[M]) \longrightarrow \mathrm{Mod}(\mathcal{H}^-)$, is an extremely well behaved functor, as a direct summand of $e_K F$. If furthermore E is equipped with a norm $|.|$ of L-vector space, then so is $F(E)$ by setting

$$|f| := \sup_{x \in G(\mathbb{A}_f)} |f(x)| = \sup_{i=1}^{h_K} |f(x_i)|.$$

The normed space $F(E)$ is isometric to E^{h_K}, therefore it inherits many of the properties of E.

Let $V \subset \mathcal{W}$ is an affinoid subdomain or a closed point, and $r \geq r_V$. We define an $\mathcal{H}^-$-module by setting

$$\mathcal{S}(V, r) := eF(\mathcal{C}(V, r)).$$

This is the space of *p-adic automorphic forms of weight in V, radius of convergence r and type e*. It has a natural structure of Banach $A(V)$-module which is a topological direct summand of an ONable Banach module [16], which is a property that Buzzard calls (Pr) in [**32**]. It is equipped with an $A(V)$-linear action of $\mathcal{H}^-$, each $h \in \mathcal{H}^-$ being bounded by 1 and each element of $U^{--} \subset \mathcal{H}^-$ being $A(V)$-compact. By formula (70), the collection of spaces $\{\mathcal{S}(V, r), V, r \geq r_V\}$ satisfies exactly the same compatibilities as $\{\mathcal{C}(V, r),\ V,\ r \geq r_V\}$.

7.3.5. Step III. Classical versus p-adic automorphic forms. — Let

$$\underline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^{m,-}.$$

We check at once using $\iota_p \iota_\infty^{-1}$ that $eF(W_{\underline{k}}(L)^*)$ is a $\mathcal{H}^-$-stable L-structure on the space $\iota_\infty(e)A(G, W_{\underline{k}}(\mathbb{C}))$ of complex automorphic forms of weight $W_{\underline{k}}(\mathbb{C})$ and type e. Moreover, we have a natural $\mathcal{H}^-$-equivariant inclusion

$$eF(W_{\underline{k}}(L)^*) \otimes \delta_{\underline{k}} \hookrightarrow \mathcal{S}(\underline{k}, 0) = eF(\underline{k}, 0), \tag{71}$$

whose image is usually referred as the subspace of *classical* p-adic automorphic forms.

In the remaining part of this paragraph § 7.3.5, we are going to prove the *control theorem*, which is a criterion ensuring that an element $f \in \mathcal{S}(\underline{k}, r)$ is classical, that is, belongs to the subspace $eF(W_{\underline{k}}(L)^*)$. A necessary condition is that f be *of finite slope* at p. Recall that an element $f \in \mathcal{S}(\underline{k}, r)$ is of finite slope if for some $u \in U^{--}$ (viewed as en element of $\mathcal{A}_p^-$):

- — the subspace $L[u].f \subset \mathcal{S}(\underline{k}, r)$ is finite dimensional,
- — and $u_{|L[u].f}$ is invertible.

---

(16) The isomorphism (70) induces an isometry $e_K F(\mathcal{C}(V, r)) \xrightarrow{\sim} \mathcal{C}(V, r)^{h_K}$ hence $e_K F(\mathcal{C}(V, r))$ is $A(V)$-ONable. We give $\mathcal{S}(V, r) \subset e_K F(\mathcal{C}(V, r))$ the subspace topology, it is closed as the image of the continuous linear projector e on $e_K F(\mathcal{C}(V, r))$.

As such a u acts compactly on $\mathcal{S}(\underline{k},r)$, and as $\mathcal{A}_p^-$ is commutative, this implies that the whole subspace $\mathcal{A}_p^-.f$ is finite dimensional over L, and that u is invertible on $\mathcal{A}_p^-.f$. As

$$\forall u \in U^{--}, \forall u' \in U^-, \exists\, k \in \mathbb{N}, \exists u'' \in U^-, \;\; u'u'' = u^k, \tag{72}$$

we see that if f is of finite slope, then *for all* $u \in U^{--}$, $L[u].f$ is finite dimensional and $u_{|L[u].f}$ is invertible. As a consequence, the finite slope elements form an L-subspace

$$\mathcal{S}(\underline{k},r)^{\mathrm{fs}} \subset \mathcal{S}(\underline{k},r),$$

over which all the elements of U^- are invertible, and $\mathcal{S}(\underline{k},r)^{\mathrm{fs}}$ extends naturally to an $\mathcal{A}_p$-module. Note that

$$eF(W_{\underline{k}}(L)^*) \otimes \delta_{\underline{k}} \subset \mathcal{S}(\underline{k},r)^{\mathrm{fs}}$$

as the $\mathcal{A}_p^-$-module structure on the finite dimensional space $\mathcal{S}(\underline{k},r)^{\mathrm{fs}}$ extends to $\mathcal{A}_p$ (and even to the full Hecke-Iwahori algebra, see § 6.4.1). Recall that for $i = 1,\dots,m$, $u_i = (1,\dots,1,p,1,\dots,1) \in U$ where p occurs at place i.

Proposition 7.3.5. — [17] *Let $f \in \mathcal{S}(\underline{k},r)^{\mathrm{fs}} \otimes_L \overline{\mathbb{Q}}_p$ be an eigenform for all $u \in U \subset \mathcal{A}_p$. For $i = 1,\dots,m$ write $u_i(f) = \lambda_i f$ for $\lambda_i \in \overline{\mathbb{Q}}_p^*$. If we have*

$$v(\lambda_1\lambda_2\cdots\lambda_i) < k_i - k_{i+1} + 1, \;\; \forall i = 1,\dots,m-1,$$

then f is classical.

More precisely, under the same condition the full generalized $\mathcal{A}_p$-eigenspace of f in $\mathcal{S}(\underline{k},r)^{\mathrm{fs}} \otimes_L \overline{\mathbb{Q}}_p$ is included in the subspace $eF(W_{\underline{k}}(L)^*) \otimes_L \overline{\mathbb{Q}}_p$ of classical forms.

This result is a variant of [**36**, Prop. 4.7.4], which is stated there with some stronger conditions on f, for instance that $u(f) = \lambda f$ with $u = (p^{m-1}, p^{m-2}, \dots, p, 1)$ and $\lambda \in \overline{\mathbb{Q}}_p{}^*$ such that

$$v(\lambda) < \mathrm{Min}_{i=1}^{m-1}(k_i - k_{i+1}) + 1. \tag{73}$$

The conditions of the statement first appear in the work of Emerton in [**53**] (he obtains them using his "Jacquet-module" functor). This is also what we would expect by a Verma module argument. These conditions are actually more meaningful than (73) above, or than the ones that the second author introduced in [**36**, §4.7, §7.5] under the name of U-non criticality, in the sense that they match perfectly with the notion of *numerically non critical refinement* that we encountered in our study of the Galois side in chapter 2 (see Remark 2.4.6 (ii) of § 2.4, as well as § 7.5.4). The proof of the proposition that we give below is a slight variant of the elementary proof originally given in [**36**, Prop. 4.7.4] which relies on a coarse but useful presentation of the M-representations $\mathcal{C}(V,r)$, given by the Plücker embedding of $\mathrm{GL}_m(\mathbb{Q}_p)$.

(17) Added in 2008.

Proof. — Choose a net compact open subgroup K as in § 7.3.4 such that $e.e_K = e_K.e = e$. As the action of e and e_K commute with $\mathcal{A}_p^-$, we may assume that $e = e_K$. Moreover, we may assume that $r = 0$ as

$$\mathcal{S}(\underline{k}, r)^{\mathrm{fs}} \subset \mathcal{S}(\underline{k}, 0), \quad \forall r \in \mathbb{N}.$$

Indeed, fix a $u \in U^{--}$. For $r \geq 1$, the operator $u : e_K F(\mathcal{C}(\underline{k}, r)) \to e_K F(\mathcal{C}(\underline{k}, r))$ factors through the (compact) restriction $e_K F(\mathcal{C}(\underline{k}, r)) \to e_K F(\mathcal{C}(\underline{k}, r-1))$, as this holds without applying the functor $e_K F$ (see § 7.3.3). As a consequence, a standard argument shows that the characteristic power series $\det(1 - Tu_{|\mathcal{S}(\underline{k},r)})$ is independent of $r \geq 0$, thus so is $\mathcal{S}(\underline{k}, r)^{\mathrm{fs}}$.

In order to define the Plücker presentation of $W_{\underline{k}}^*(L)$ and of $\mathcal{C}(\underline{k}, 0)$, we have to introduce a collection of objects which are similar to the ones introduced in § 7.3.3 (actually, they are simpler). Let us recall first a well-known construction of linear algebra. Let V be any finite dimensional $\mathbb{Q}_p$-vector space, $H = \mathrm{GL}_{\mathbb{Q}_p}(V)$, $v \in V$ a nonzero vector, $P \subset H$ the stabilizer of the line $\mathbb{Q}_p v \subset V$, and let

$$\chi : P \longrightarrow \mathbb{Q}_p^*$$

be the character of P acting on the line $\mathbb{Q}_p v$. For $n \geq 0$ an integer, the L-vector space of homogeneous polynomial functions $V \longrightarrow L$ which have degree n is isomorphic (with its natural H-action) to the L-vector space $i_P^H(\chi^n)$ of algebraic functions $f : H \longrightarrow L$ such that $f(hb) = \chi^n(b)f(h)$ for all $h \in H$ and $b \in P$ (where H acts by left translations). Indeed, the map $\varphi \mapsto (h \mapsto \varphi(h(e)))$ provides such an isomorphism. In other words, we have for $n \geq 0$ an $L[H]$-equivariant isomorphism

$$i_P^H(\chi^n) \xrightarrow{\sim} \mathrm{Symm}^n(V \otimes_{\mathbb{Q}_p} L)^*. \tag{74}$$

We go now a bit further and construct a representation which is to $i_P^H(\chi^n)$ what $\mathcal{C}(\underline{k}, 0)$ is to $W_{\underline{k}}^*(L)$. For that, we assume furthermore that we give ourselves a lattice $\mathcal{L}$ of V of the form

$$\mathcal{L} = \mathbb{Z}_p v \oplus R.$$

Let $\overline{N} \subset H$ be the subgroup fixing pointwise R and $V/R[1/p]$, it is the radical unipotent of the parabolic subgroup which is opposite to P with respect to the decomposition above. Consider also the (parahoric) compact subgroup

$$J = \{ h \in \mathrm{GL}_{\mathbb{Z}_p}(\mathcal{L}),\ h(v) \in \mathbb{Z}_p v + pR \}.$$

We have an Iwasawa decomposition

$$J = (\overline{N} \cap J) \times (P \cap J),$$

and a natural isomorphism of topological groups

$$\alpha : \overline{N} \cap J \xrightarrow{\sim} R, \quad n \mapsto \alpha(n) = (n(v) - v)/p. \tag{75}$$

Let $\mathfrak{U}^- \subset H$ (resp. $\mathfrak{U}^{--} \subset H$) be the submonoid consisting of elements u such that for some integer $a \in \mathbb{Z}$ we have $u(v) = p^a v$ and $u(R) \supset p^a R$ (resp. $u(R) \supset p^{a-1}R$). Let $\mathfrak{M} \subset H$ be the submonoid generated by J and $\mathfrak{U}^-$. It is an exercise using Iwasawa decomposition to check that

$$\mathfrak{M} = J\mathfrak{U}^- J, \ \mathfrak{M}^{-1}JP \subset JP, \ \ \forall u \in \mathfrak{U}^{--} \ \ u^{-1}\alpha^{-1}(R)u \subset \alpha^{-1}(pR). \tag{76}$$

Define $\mathcal{O}(R)$ as the Tate algebra of L-valued analytic functions over R: if $t_1, \dots, t_r$ is a $\mathbb{Z}_p$-basis of $\mathrm{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}_p)$, then it is the standard Tate algebra

$$\mathcal{O}(R) = L\langle t_1, \dots, t_r\rangle,$$

which is a Banach L-algebra equipped with its Gauss norm. By means of the isomorphism (75) above, we define the L-algebra $\mathcal{O}(\overline{N} \cap J)$ of *L-valued analytic functions on $\overline{N} \cap J$* as the functions $f : \overline{N} \cap J \to L$ such that $f \circ \alpha^{-1} \in \mathcal{O}(R)$. This being said, let us consider the space

$$\mathcal{C}(n) = \left\{ \begin{array}{l} f : JP \longrightarrow L, \ f(xb) = \chi(b)^n f(x) \ \ \forall\, x \in JP, \ b \in P, \\ f_{|\overline{N}\cap J} \in \mathcal{O}(\overline{N} \cap J). \end{array} \right\}.$$

As an L-vector space, we have canonically $\mathcal{C}(n) \xrightarrow{\sim} \mathcal{O}(\overline{N} \cap J) \underset{\alpha}{=} \mathcal{O}(R)$, which gives $\mathcal{C}(n)$ a structure of Banach space over L. By (76) it has a natural action of $\mathfrak{M}$ by left translations,[18] and the elements $u \in \mathfrak{U}^{--}$ act by compact operators. An algebraic L-valued function on H restricts to an analytic function on $\overline{N} \cap J$, so we get a natural $\mathfrak{M}$-equivariant map

$$i_P^H(\chi^n) \longrightarrow \mathcal{C}(n),$$

which is injective as the open subset $JP \subset H$ is Zariski-dense in H. For integrality reasons, it will be convenient to twist those representations by a power of the (unique) character

$$\delta : \mathfrak{M} \longrightarrow L^*$$

such that $\delta(J) = 1$ and $\delta(u) = p^a$ if $u = (p^a, u_{|R}) \in \mathfrak{U}^-$. Indeed, we claim that:

Lemma 7.3.6. — (a) *The elements of $\mathfrak{M}$ all have norm ≤ 1 on $\mathcal{C}(n) \otimes \delta^n$.*

(b) *For $m \in J\mathfrak{U}^{--}J$, $\frac{m}{p^{n+1}}$ has norm ≤ 1 on the quotient $(\mathcal{C}(n)/i_P^H(\chi)) \otimes \delta^n$.*

---

[18] That the action of J preserves the analyticity, and even $\mathbb{Z}_p\langle t_1, \dots, t_r\rangle$, can be checked by an explicit computation of the natural map

$$J \times \overline{N} \cap J \longrightarrow \overline{N} \cap J \times P \cap J,$$

$(j, n) \mapsto (a, b)$ where (a, b) is the Iwasawa decomposition of $jn \in J$. An alternative, more conceptual, argument is to note that this map is induced by a morphism of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p)$.

Proof. — We leave as an exercise to the reader to check that for $m \in J$, m has norm 1 on $\mathcal{C}(n)$. If $u = (p^a, u_{|R}) \in \mathfrak{U}^-$ and if $n \in \overline{N} \cap J$, then $u^{-1}n = u^{-1}nu.u^{-1}$ where $u^{-1}nu \in \overline{N} \cap J$ satisfies

$$\alpha(u^{-1}nu) = p^a u^{-1}(\alpha(n)). \tag{77}$$

As $\chi^n(u^{-1})\delta(u)^n = 1$, this shows (a).

To check (b) it is now enough to assume that $m \in \mathfrak{U}^{--}$. We may write $m = uu'$ with $u = (p, \mathrm{id}_R)$ and $u' \in \mathfrak{U}^-$, so we may actually assume that $m = u$. Let us identify $\mathcal{C}(n)$ with $L\langle t_1, \ldots, t_r\rangle$ as above, by means of α. By (74), $i_P^H(\chi^n)$ is its subspace of polynomials in $t_1, \ldots, t_r$ with total degree $\leq n$. But by (77), u acts on $L\langle t_1, \ldots, t_r\rangle$ by sending $f(t_1, \ldots, t_r)$ to $f(pt_1, \ldots, pt_r)$, and (b) follows. $\square$

All of this being done, we can define the Plücker presentation. Let $f_1, \ldots, f_m$ be the canonical basis of $\mathbb{Q}_p^m$. For $i = 1, \ldots, m$, we apply the above construction to:

- — the space $V_i := \Lambda^i(\mathbb{Q}_p^m)$,
- — the element $v_i := f_1 \wedge f_2 \wedge \cdots \wedge f_i$,
- — the integer $n_i = k_i - k_{i+1} \geq 0$ if $i < m$, $n_m = k_m$ otherwise,
- — the lattice $\mathcal{L}_i = \Lambda^i(\mathbb{Z}_p^m)$,
- — the decomposition $\mathcal{L}_i = \mathbb{Z}_p v_i \oplus R_i$ where R_i is the $\mathbb{Z}_p$-module generated by the elements $f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_i}$ with $j_1 < \cdots < j_i$ and $(j_1, \ldots, j_i) \neq (1, \ldots, i)$.

This gives rise to a collection of objects

$$(H_i, P_i, \chi_i, i_{P_i}^{H_i}(\chi_i^{n_i}), \overline{N}_i, J_i, \alpha_i, \mathfrak{U}_i^*, \mathfrak{M}_i, \delta_i, \mathcal{C}_i(n_i)), \ \ i = 1, \ldots, m.$$

For each i we can furthermore consider the fundamental representation

$$\Lambda^i : G(\mathbb{Q}_p) \longrightarrow H_i.$$

We check at once that $\Lambda^i(B) \subset P_i$, $\Lambda^i(\overline{N}_0) \subset \overline{N}_i$, $\Lambda^i(I) \subset J_i$, $\Lambda^i(U^*) \subset \mathfrak{U}_i^*$, and $\Lambda^i(M) \subset \mathfrak{M}_i$. We shall denote by $W_i(L)^*$ the space $i_{P_i}^{H_i}(\chi_i^{n_i})$ *viewed as a representation of* $G(\mathbb{Q}_p)$ via Λ^i. In particular,

$$W_i(L) \simeq \mathrm{Symm}^{n_i}(\Lambda^i L^m).$$

Recall that the representation $W_{\underline{k}}(L)^*$ of $G(\mathbb{Q}_p)$ is irreducible and isomorphic to the L-vector space of functions $f : G(\mathbb{Q}_p) \longrightarrow L$ such that $f(gb) = \delta_{\underline{k}}(b)f(g)$ (for left translations) where $\delta_{\underline{k}}$ is the highest weight character of $W_{\underline{k}}$. As a consequence, the map $(f_1, \ldots, f_m) \mapsto (g \mapsto \prod_{i=1}^m f_i(\Lambda^i(g)))$ induces an $L[G(\mathbb{Q}_p)]$-equivariant surjection

$$\bigotimes_{i=1}^{m} W_i(L)^* \longrightarrow W_{\underline{k}}(L)^*.$$

(Note that $x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} = x_1^{k_1-k_2}(x_1x_2)^{k_2-k_3}(x_1x_2x_3)^{k_3-k_4}\cdots(x_1x_2\cdots x_m)^{k_m}$ for all $(x_1, \ldots, x_m) \in (\mathbb{Q}_p^*)^m$.) This is the *Plücker presentation* of $W_{\underline{k}}(L)^*$.

Similarly, let us denote by $\mathcal{C}_i(n_i)'$ the representation $\mathcal{C}_i(n_i) \otimes \delta_i^{n_i}$ viewed as a representation of M via Λ^i. As for $W_{\underline{k}}(L)^*$, the natural map $(f_1, \dots, f_m) \mapsto (g \mapsto \prod_{i=1}^m f_i(\Lambda^i(g))$ induces an $L[M]$-equivariant map

$$\widehat{\otimes}_{i=1}^m \mathcal{C}_i(n_i)' \longrightarrow \mathcal{C}(\underline{k}, 0).$$

This map is continuous of norm ≤ 1 and surjective, as the natural map

$$\overline{N}_0 \longrightarrow \prod_{i=1}^m \overline{N}_i, \ \ g \mapsto (\Lambda_i(g)),$$

is easily seen to induce a norm deacreasing surjection on the Tate algebra of analytic functions (for instance as the injection above is induced by a closed immersion of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p)$). We get this way a commutative square of $L[M]$-modules:

$$\begin{array}{ccc} \otimes_{i=1}^m W_i(L)^* \otimes \delta_i^{n_i} & \longrightarrow & W_{\underline{k}}(L)^* \otimes \delta_{\underline{k}} \\ \downarrow & & \downarrow \\ \widehat{\otimes}_{i=1}^m \mathcal{C}_i(n_i)' & \longrightarrow & \mathcal{C}(\underline{k}, 0) \end{array}$$

where the horizontal arrows are surjective and the vertical ones are injective. As the functor $e_K F$ is exact, there is a similar diagram of $\mathcal{H}^-$-modules with $e_K F$ applied everywhere. Let us introduce the $L[M]$-modules

$$Q = \mathcal{C}(\underline{k}, 0)/(W_{\underline{k}}(L)^* \otimes \delta_{\underline{k}}), \ \ Q' = \left(\widehat{\otimes}_{i=1}^m \mathcal{C}_i(n_i)'\right) / \left(\otimes_{i=1}^m W_i(L)^* \otimes \delta_i^{n_i}\right).$$

We have then a natural surjection

$$e_K F(Q') \longrightarrow e_K F(Q) \longrightarrow 0, \tag{78}$$

as well as a tautological injection

$$0 \longrightarrow e_K F(Q') \longrightarrow \prod_{i=1}^m e_K F(Q'_i), \tag{79}$$

where we have set

$$Q'_i = \left(\widehat{\otimes}_{j \neq i} \mathcal{C}_j(n_j)'\right) \otimes \left(\mathcal{C}_i(n_i)'/(W_i(L)^* \otimes \delta_i^{n_i})\right).$$

Let us now prove Prop. 7.3.5. Let $w \in e_K F(Q)$ be such that $u_i(w) = \lambda_i w$ where the λ_i are as in the statement. All we have to show is that $w = 0$. By the $L[M]$-equivariant surjection (78), and as U^{--} acts compactly on $e_K F(Q)$, the generalized $\mathcal{A}_p^-$-eigenspace of w in $e_K F(Q)$ is the image of a generalized $\mathcal{A}_p^-$-eigenspace $E \subset e_K F(Q')$ for the same system of eigenvalues as w. It is enough to show that any $w' \in E$ which is an eigenvector, *i.e.* such that $u_i(w') = \lambda_i w'$ for all i, vanishes. Fix such a w'. By the $L[M]$-equivariant injection (79) it is enough to show that its image

$$w'_i \in e_K F(Q'_i)$$

vanishes for all $i = 1, \dots, m-1$ (of course, $Q'_m = 0$). Set

$$U_i = \prod_{j=1}^{i} u_i \ \in U^-.$$

We claim that the element $\frac{U_i}{p^{n_i+1}}$ has norm ≤ 1 on $e_K F(Q'_i)$. This will conclude the proof as its eigenvalue on w'_i is

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_i}{p^{n_i+1}}$$

which has norm > 1 by assumption.

Let us check the claim. By the definitions of the norm on $e_K F(-)$ (see § 7.3.4) and of Hecke operators (see e.g.[**36**, Lemme 4.5.2]), it is enough to show that each element of the form $\frac{gU_ig'}{p^{n_i+1}}$, for $g, g' \in I$, has norm 1 on Q'_i. By Lemma 7.3.6 (a), gU_ig' acts by norm ≤ 1 over $\mathcal{C}(n_j)'$ for all j. As $\Lambda^i(gU_ig') \in J_i\mathfrak{U}^{--}J_i$, assertion (b) of the same lemma conludes the proof of the claim. □

7.3.6. Step IV. Fredholm series and construction of the eigenvariety. — As any $h \in U^{--}\mathcal{H}^-$ acts compactly on the (Pr)-family of Banach modules $\{\mathcal{S}(V, r), V, r \geq r_V\}$, there is a unique power series $P_h(T) \in 1 + T\mathcal{O}(\mathcal{W})\{\{T\}\}$ such that for any $V \subset \mathcal{W}$ open affinoid or closed point and $r \geq r_V$,

$$P_h(T)_{|V} = \det(1 - Th_{|\mathcal{S}(V,r)}) \in 1 + TA(V)\{\{T\}\}.$$

A power series $P \in \mathcal{O}(\mathcal{W})\{\{T\}\}$ with $P(0) = 1$ is often called a *Fredholm* series. Set $P := P_{u_0}$, and consider the *Fredholm hypersurface* $Z(P) \subset \mathcal{W} \times \mathbb{G}_m$, that is the closed subspace defined by $P = 0$. As any Fredholm hypersurface, $Z(P)$ is canonically admissibly covered by its affinoid subdomains Ω^* such that $\mathrm{pr}_1(\Omega^*)$ is an open affinoid of $\mathcal{W}$ and that the induced map $\mathrm{pr}_1 : \Omega^* \longrightarrow \mathrm{pr}_1(\Omega^*)$ is finite. Here pr_1 is the first projection $\mathcal{W} \times \mathbb{G}_m \longrightarrow \mathcal{W}$. Let us denote by $\mathcal{C}^*$ this canonical covering. This covering is easily seen to be stable by finite intersections, by pullback over affinoid subdomains of $\mathcal{W}$, hence to satisfy the following good property:

(*) if Ω_1^*, $\Omega_2^* \in \mathcal{C}^*$ then $\Omega_1^* \cap \Omega_2^*$ is an open-closed subspace of $\Omega_1 \times_{V_1} (V_1 \cap V_2)$.

The eigenvariety X will then be constructed as a finite map $\nu : X \longrightarrow Z(P)$ as follows. Let $\Omega^* \in \mathcal{C}^*$ and $V := \mathrm{pr}_1(\Omega^*)$. There is a unique factorization $P = QR$ in $A(V)\{\{T\}\}$ where $Q \in 1 + TA(V)[T]$ has a unit leading coefficient and is such that $\Omega^* = Z(Q)$ is a closed and open subspace of $Z(P) \times_{\mathcal{W}} V$. To this factorization corresponds, for $r \geq r_V$, a unique Banach $A(V)$-module decomposition

$$\mathcal{S}(V, r) = S(\Omega^*) \oplus N(\Omega^*, r),$$

which is $\mathcal{H}^-$-stable, and such that:

— $S(\Omega^*)$ is a finite projective $A(V)$-module, which is independent of $r \geq r_V$,

— the characteristic polynomial of u_0 on $S(\Omega^*)$ is the reciprocal polynomial $Q^{\mathrm{rec}}(T)$ of $Q(T)$, and $Q^{\mathrm{rec}}(u_0)$ is invertible on $N(\Omega^*, r)$.

The local piece Ω of X is then by definition the maximal spectrum of the $A(V)$-algebra generated by the image of $\mathcal{H} = \mathcal{H}^-[u_0]^{-1}$ in $\mathrm{End}_{A(V)}(S(\Omega^*))$. It is equipped by construction with a ring homomorphism $\mathcal{H} \longrightarrow A(\Omega)$, with a finite map $\nu : \Omega \longrightarrow \Omega^*$, and with a finite $A(\Omega)$-module $S(\Omega^*)$. We check then that the Ω and the maps above glue uniquely over $\mathcal{C}^*$ to an object (X, ψ, ν) as in the statement of Proposition 7.3.1, which is easy using the property (*) mentioned above of the admissible covering $\mathcal{C}^*$. In other words, the coherent sheaves of $\mathcal{O}$-algebras $\{\widetilde{A(\Omega)}, \Omega^* \in \mathcal{C}^*\}$ glue canonically to a coherent $\mathcal{O}_{Z(P)}$-algebra, and $\nu : X \longrightarrow Z(P)$ is its relative spectrum (see [**48**, §2.2]). The space X constructed this way is actually reduced by [**37**, Prop. 3.9]. In the same way, the locally free coherent sheaves $\{\widetilde{S(\Omega^*)}, \Omega^* \in \mathcal{C}^*\}$ glue canonically to a coherent sheaf on $Z(P)$. This sheaf is a $\nu_*\mathcal{O}_X$-module in a natural way, hence has the form $\nu_*\mathcal{S}$ for as a coherent sheaf $\mathcal{S}$ on X.

***Definition** 7.3.7.* — We denote by $\mathcal{C}$ the admissible covering $\nu^{-1}\mathcal{C}^*$ of X and by $\mathcal{S}$ the coherent sheaf on X defined above. If $\Omega = \nu^{-1}(\Omega^*) \in \mathcal{C}$, then $\mathcal{S}(\Omega) = S(\Omega^*)$.

***Remark** 7.3.8.* — (*On quasicoherent and coherent sheaves on rigid spaces*) Let X be a rigid analytic space over k. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *quasicoherent* (resp. coherent) if there exists an admissible covering $\{U_i\}$ of X by affinoid subdomains such that $\mathcal{F}_{|U_i}$ is the sheaf $\widetilde{M_i}$ associated to some $\mathcal{O}(U_i)$-module M_i (resp. such that M_i is finite type over $\mathcal{O}(U_i)$) (see [**27**, §9.4.2], [**48**, §2.1]). Contrary to the case of schemes, this does not imply in general that for any affinoid subdomain U, $\mathcal{F}_{|U}$ is associated to an $\mathcal{O}(U)$-module (see Gabber's counterexample [**48**, Ex. 2.1.6]). This holds however when U lies in some U_i, when $\mathcal{F}$ is coherent ([**27**, §9.4.3]), or when $\mathcal{F}$ is globally on X a direct inductive limit of coherent $\mathcal{O}_X$-modules ([**48**, Lemma 2.1.8]).

In our applications, we will define some quasicoherent sheaves on X using the covering $\mathcal{C}$, but they will all be direct inductive limits of coherent sheaves.

7.4. The family of $G(\mathbb{A}_f^p)$-representations on an eigenvariety of idempotent type

In all this part, we keep the notations of § 7.3.1. In particular, X is the eigenvariety associated to the idempotent e given by Theorem 7.3.1, or its variant with one fixed weight as in Remark 7.3.2 (iii).

7.4.1. The family of local representations on X. — Let us fix some finite set S of primes [19] such that $S \cap (S_0 \cup \{p\}) = \varnothing$. Assume moreover that the idempotent e decomposes as a tensor product of idempotents at $l \in S$ and outside l: $e = e_S \otimes e^S$ and $e_S = \otimes_{l \in S} e_l$, $e_l^2 = e_l \in \mathcal{C}(G(\mathbb{Q}_l), \overline{\mathbb{Q}})$.

The eigenvariety X carries a natural sheaf of admissible $G(\mathbb{A}_S)$-representations that we will describe now. For $V \subset \mathcal{W}$ an open affinoid and $r \geq r_V$, we have by definition a split inclusion

$$S(V,r) = eF(\mathcal{C}(V,r)) \subset F(\mathcal{C}(V,r)), \tag{80}$$

and the latter space is a smooth $G(\mathbb{A}_S)$-module as $p \notin S$. We fix now $\Omega^* \in \mathcal{C}^*$, set $\Omega = \nu^{-1}(\Omega^*)$, $V = pr_1(\Omega^*)$, and we consider $\mathcal{S}(\Omega^*) \subset \mathcal{S}(V,r)$ as in § 7.3.6.

Definition 7.4.1. — We define $\Pi_S(\Omega)$ as the $\mathbb{Z}[G(\mathbb{A}_S)]$-submodule of $F(\mathcal{C}(V,r))$ generated by $\mathcal{S}(\Omega^*)$.

By definition, $\Pi_S(\Omega)$ is an $\mathcal{H}^- \otimes A(V)$-submodule and the natural map $\mathcal{H}^- \otimes A(V) \longrightarrow \mathrm{End}(\Pi_S(\Omega))$ factors through $A(\Omega)$ as it does on the generating subspace $\mathcal{S}(\Omega^*)$ of $\Pi_S(\Omega)$. As a consequence, $\Pi_S(\Omega)$ is an $A(\Omega)$-module in a natural way. It is independent of $r \geq r_V$ as $\mathcal{S}(\Omega^*)$ is.

Proposition 7.4.2. — *Let $\Omega \in \mathcal{C}$.*

(i) *$\Pi_S(\Omega)$ is an $A(\Omega)$-admissible smooth representation of $G(\mathbb{A}_S)$.*
(ii) *The natural inclusion $\mathcal{S}(\Omega) \longrightarrow e_S\Pi_S(\Omega)$ is an equality.*

Moreover,

(iii) *The sheaves of $\mathcal{O}$-modules $\{\widetilde{\Pi_S(\Omega)}, \Omega \in \mathcal{C}\}$ glue canonically to a quasicoherent smooth $\mathcal{O}_X[G(\mathbb{A}_S)]$-module, and* (ii) *glue to an isomorphism $\mathcal{S} \xrightarrow{\sim} e_S\Pi_S$.*
(iv) *For each compact open subgroup $J \subset G(\mathbb{A}_S)$ the subsheaf of J-invariants $\Pi_S^J \subset \Pi_S$ is a coherent $\mathcal{O}_X$-module, and $\Pi_S = \bigcup_J \Pi_S^J$.*
(v) *For all $x \in X$, $(\Pi_S)_x$ is torsion free over $\mathcal{O}_{\omega(x)}$, hence also over $\mathcal{O}_x$.*

Proof. — We check first assertion (ii). Let $Q \in A(V)[T]$ be the polynomial attached to Ω^* as in § 7.3.6, so that $\mathcal{S}(\Omega^*)$ is the Kernel of $Q^{\mathrm{rec}}(u_0)$ on $eF(\mathcal{C}(V,r))$. As $p \notin S$, $Q^{\mathrm{rec}}(u_0)\Pi_S(\Omega^*) = 0$, hence (ii) holds by definition.

We know that $\Pi_S(\Omega)$ is smooth as $F(\mathcal{C}(V,r))$ is, hence (i) follows from (ii) and Lemma 7.4.3 (i). Assertion (iii) follows easily from the properties of the admissible covering $\mathcal{C}^*$, the proof is similar to the gluing argument for the sheafs $\mathcal{S}$ and $\nu_*\mathcal{O}_X$ so

[19] We hope that there will be no possible confusion with the letter S occurring in the spaces $S(\Omega^*)$ or $S(V,r)$.

we leave the details to the reader. To prove (iv), note that for any $\mathbb{Q}$-algebra A, any A-linear representation V of J, and any A-module M, the natural map

$$V^J \otimes_A M \longrightarrow (V \otimes_A M)^J$$

is an isomorphism (argue as in Lemma 6.6.1 (b)). Part (iv) follows now from (i) and (iii).

Before showing (v), let us recall that by construction $\mathcal{O}_x$ is a $\mathcal{O}_{\omega(x)}$-subalgebra of the endomorphism ring of a finite free $\mathcal{O}_{\omega(x)}$-module, hence the total fraction ring of $\mathcal{O}_x$ identifies with $\mathcal{O}_x \otimes_{\mathcal{O}_{\omega(x)}} \mathrm{Frac}(\mathcal{O}_{\omega(x)})$. As a consequence, it suffices to check that $(\Pi_S)_x$ is torsion free over $\mathcal{O}_{\omega(x)}$. But for each Ω^*, V and Ω as above, $\Pi_S(\Omega)$ is a subpace of $F(\mathcal{C}(V,r))$, which is clearly torsion free over $A(V)$. □

***Lemma** 7.4.3.* — *(Bernstein) Let k be a field of characteristic* 0*, A a noetherian k-algebra and V a smooth $A[G(\mathbb{A}_S)]$-representation. Assume that for some decomposed idempotent $e \in \mathcal{C}_l(G(\mathbb{A}_S),k)$, eV is finite type over A and generates V as an $A[G(\mathbb{Q}_l)]$-module. Then:*

(i) *V is A-admissible,*

(ii) *if A is moreover finite dimensional over k, V is of finite length on $A[G(\mathbb{Q}_l)]$.*

Proof. — Let us show (i). By induction on $|S|$, we may assume that $S = \{l\}$. By [**17**, Prop. 3.3], and more precisely by "variante 3.3.1" and the remark following Corollary 3.4 *loc. cit.*, V is $Z(G(\mathbb{Q}_l)) \otimes_k A$-admissible where $Z(G(\mathbb{Q}_l))$ is the center of the k-valued Hecke-algebra of $G(\mathbb{Q}_l)$. As $A[G(\mathbb{Q}_l)]eV = V$, the action of $Z(G(\mathbb{Q}_l)) \otimes_k A$ on V factors through its faithful quotient $A' \subset \mathrm{End}_A(eV)$. As eV is finite type over A by assumption, and A is noetherian, so is A', hence V is A-admissible.

The second assertion follows from (i) as V is then k-admissible and of finite type over $k[G(\mathbb{A}_S)]$ (use [**17**, Cor. 3.9]). □

For sake of completeness, we end this discussion with a study of the fibers of Π_S at a point of X. We fix $x \in X$ with residue field $k(x)$, hence we get a natural system of Hecke-eigenvalues $\psi_x : \mathcal{H} \longrightarrow k(x)$. To this system of Hecke-eigenvalues corresponds a generalized $\mathcal{H}$-eigenspace $\mathcal{S}^{\psi_x} \subset \mathcal{S}(\omega(x),r) = eF(\mathcal{C}(\omega(x),r))$, for r big enough.

***Definition** 7.4.4.* — We denote by $\Pi_S^{\psi_x}$ the $\mathcal{O}_x/m_{\omega(x)}\mathcal{O}_x$-representation of $G(\mathbb{A}_S)$ generated by the (finite dimensional) subspace $\mathcal{S}^{\psi_x}$ of $F(\mathcal{C}(\omega(x),r))$. It is a finite length admissible representation of $G(\mathbb{A}_S)$ by Lemma 7.4.3 (ii).

***Definition** 7.4.5.* — Assume moreover that $x \in Z$, so $\psi_x(\mathcal{H}) \subset \overline{\mathbb{Q}}$. We denote by Π_x the $k(x)$-model[20] of the complex $G(\mathbb{A}_S)$ subrepresentation of $\mathcal{A}(G, W_{\underline{k}})$ generated

[20] Defined by ι_p and ι_∞.

by the $\mathcal{H}$-eigenspace of $e(A(G, W_{\underline{k}}))$ for the system of eigenvalues $\iota_\infty \iota_p^{-1}(\psi_x)$. It is a semisimple $k(x)$-representation.

Proposition 7.4.6. — (i) *The natural map* $(\Pi_S)_x/m_{\omega(x)}(\Pi_S)_x \longrightarrow \Pi_S^{\psi_x}$ *is surjective and induces an isomorphism* $\mathcal{S}_x/m_{\omega(x)}\mathcal{S}_x \xrightarrow{\sim} \mathcal{S}^{\psi_x}$.

(ii) *If* $x \in Z$, *then* Π_x *is a subrepresentation of* $\Pi_S^{\psi_x}$.

(iii) *If* $x \in Z$ *is numerically non critical, in the sense that it satisfies property* (v) *(b) of Thm. 7.3.1, then* $\Pi_x \xrightarrow{\sim} \Pi_S^{\psi_x}$.

Proof. — Let $\Omega^* \in \mathcal{C}^*$ containing x as above, we will argue as in the proof of Prop. 7.4.2 (ii) of which we take the notations. As $\mathcal{S}(\Omega) \subset \mathcal{S}(V, r) = eF(\mathcal{C}(V, r))$ is projective and direct summand, the natural map

$$\mathcal{S}(\Omega)/m_{\omega(x)}\mathcal{S}(\Omega) \longrightarrow eF(\mathcal{C}(\omega(x)), r),$$

is injective and the Fredholm series of u_0 on $\mathcal{S}(\Omega)/m_{\omega(x)}\mathcal{S}(\Omega)$ is the evaluation of $Q(T)$ at $\kappa(x)$. By taking the ψ_x-generalized eigenspace we get that

$$\mathcal{S}_x/m_{\omega(x)}\mathcal{S}_x \xrightarrow{\sim} \mathcal{S}^{\psi_x},$$

and (i) follows. The point (ii) follows from (71) of § 7.3.5, and (iii) from the control theorem Prop.7.3.5. □

When e_S is special (see 7.3.3 (iii)), most of our results hold in the stronger form.

Corollary 7.4.7. — *Assume that* e_S *is special.*

(i) *The natural surjection induces an isomorphism*

$$\mathcal{C}_c(G(\mathbb{A}_S), L)e_S \otimes_{e_S\mathcal{C}_c(G(\mathbb{A}_S),L)e_S} \mathcal{S} \longrightarrow \Pi_S.$$

(ii) *For all* $x \in X$, $(\Pi_S)_x$ *is flat over* $\mathcal{O}_{\omega(x)}$.

(iii) *For all* $x \in X$, *the map of Prop. 7.4.6* (i) *is an isomorphism.*

Proof. — By Prop. 7.4.2 (ii), $e_S\Pi_S = \mathcal{S}$, thus (i) follows from the discussion in Example 7.3.3. Assertion (ii) follows formally from (i), the fact that for each $x \in X$, $\mathcal{S}_x$ is finite free over $\mathcal{O}_{\omega(x)}$, and from the exactness of the functor I defined in Example 7.3.3. The map of Prop. 7.4.6 (i) induces an isomorphism after projection to e_S by *loc. cit.*, hence is an isomorphism as e_S is special, which proves (iii). □

7.4.2. The non monodromic principal series locus of X. — We keep the assumptions and notations of § 7.3.1. We fix a finite set S_N of primes l such that $G(\mathbb{Q}_l)$ is quasisplit and assume that e is a tensor product of idempotents e_l with $l \in S$ by an idempotent outside S_N. Recall that we defined some $\mathcal{O}_X[G(\mathbb{Q}_l)]$-modules Π_{S_N} in § 7.4.1.

Let $X_0 \subset X$ be the subset of points x such that for each $l \in S$, $\Pi_x \otimes_{\mathcal{O}_x} k(x)$ contains a non monodromic principal series $G(\mathbb{A}_S)$-representation in the sense of § 6.6.2 (see

Remark 6.6.9 when $|S| > 1$). Let $X_N \subset X$ be the Zariski-closure of X_0, we view it as a reduced closed subspace of X. Let also $\mathcal{Z}_{e,N} \subset \mathcal{Z}_e$ be the subset parameterizing p-refined automorphic π such that π_l is non monodromic principal series for each $l \in S$. We assume that $\mathcal{Z}_{e,N} \neq \varnothing$.

Proposition 7.4.8. — *There exists a unique eigenvariety for $\mathcal{Z}_{e,N}$, namely*

$$(X_N, \psi_{|X_N}, \nu_{|X_N}, Z \cap X_N).$$

X_N is a union of irreducible components of X, hence equidimensional of dimension m. It satisfies also properties (iv), (v) *and* (vi) *of X (see Theorem 7.3.1).*

Proof. — For any open affinoid $V \subset X$, set $V_0 = V \cap X_0$ and define $\overline{V}_0 \subset V$ to be the Zariski-closure of V_0 in V, equipped with its reduced structure. By Prop. 7.4.2 (see also Remark 7.3.8), $\Pi_{S_N|V}$ is the sheaf associated to the $A(V)$-admissible representation $\Pi_{S_N}(V)$, to which we can apply the construction of § 6.6.3. By definition, V_0 is the intersection of $\mathrm{Spec}(A(V))_0$ defined there with its subspace $V = \mathrm{Specmax}(A(V))$, and the Zariski-topology of V is by definition the topology induced from $\mathrm{Spec}(A(V))$. As $A(V)$ is a Jacobson ring, and as $\mathrm{Spec}(A(V))_0$ is constructible by Prop. 6.6.5 (i), we check easily that we also have

$$\overline{V}_0 = V \cap \overline{\mathrm{Spec}(A(V))_0}.$$

By Prop. 6.6.5(ii) and by Prop. 7.4.2 (v), we know that $\overline{V}_0$ is a (possibly empty) union of irreducible components of V.

We claim that for any two open affinoids $V, W \subset X$,

$$\overline{V}_0 \cap W = \overline{(V \cap W)_0}. \tag{81}$$

Note that $V \cap W$ is affinoid as X is separated, so by replacing W by $V \cap W$ in (81), we may assume that $W \subset V$. Moreover, the inclusion $\supset$ above is clear as $V_0 \cap W = (V \cap W)_0$, thus it only remains to prove that $\overline{V}_0 \cap W \subset \overline{W}_0$. As we know that $\overline{V}_0$ has a Zariski-dense open subset $V' \subset V_0$ by Prop. 6.6.5 (i), $V' \cap W$ is Zariski-dense in $W \cap \overline{V}_0$ by Lemma 7.4.9 (applied to $Y = \overline{V}_0$, $U = V'$, $\Omega = W \cap Y$), and we are done.

As a consequence of (81), all the $\overline{V}_0$ glue to a reduced closed subspace $T \subset X$. By construction, $X_0 \subset T$ is Zariski-dense, as it satisfies the much stronger assertion that for any open affinoid V, $V_0 = V \cap X_0$ is Zariski-dense in $\overline{V}_0 = V \cap T$. Hence $T = X_N$ and the proposition follows at once. □

Lemma 7.4.9. — *Let Y be an affinoid, $U \subset Y$ a Zariski-open subset, and $\Omega \subset Y$ an affinoid subdomain. If U is Zariski-dense in Y, then $\Omega \cap U$ is Zariski-dense in Ω.*

Proof. — Set $F := Y \backslash U$, we have to show that $F \cap \Omega$ does not contain any irreducible component of Ω. If it was the case, $F \cap \Omega$ would contain an affinoid subdomain of

Ω, hence F would contain an affinoid subdomain of Y, as well as each irreducible component of Y containing it, but this is a contradiction. □

7.5. The family of Galois representations on eigenvarieties

In this part, we explain how the existence of Galois representations attached to classical automorphic representations for G give rise to a family of Galois representations on eigenvarieties. We keep the notations and assumptions of § 7.2.1, as well as those of §6.8.1 [(21)].

7.5.1. Setting. — So as not to multiply the statements, let us assume once and for all that G is:

(a) either the group $\mathrm{U}(m)$ defined in § 6.2.2, in which case we assume that hypothesis $(\mathrm{Rep}(m))$ of § 6.8.2 holds,

(b) or a definite unitary group such that for any finite prime l, $G(\mathbb{Q}_l)$ is either quasisplit or isomorphic to the group of invertible elements of a central division algebra over $\mathbb{Q}_l$, this latter case occurring at least for one $l =: q$. Assume moreover that $G(\mathbb{Q}_p) \simeq \mathrm{GL}_m(\mathbb{Q}_p)$.

As explained in Remark 6.8.2 (vii), recall that in the second case, the obvious analog of condition $(\mathrm{Rep}(m))$ is known except for property (P3).

We assume moreover that the set S_0 defining $\mathcal{H}_{\mathrm{ur}}$ has Dirichlet density one, and we fix a decomposed compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$ as well as a finite set S of primes, such that $p \in S$ and that for each $l \notin S$ or in S_0, K_l is a maximal hyperspecial or a very special compact subgroup. We choose the decomposed idempotent e such that $ee_{K^p} = e_{K^p}$. If we are in case (b), we assume moreover that e_q vanishes on the one dimensional representations of the division algebra [(22)] $G(\mathbb{Q}_q)$. We fix also a finite set of primes $S_N \subset S$ that we assume to be empty in case (b), and define $\mathcal{Z} \subset \mathcal{Z}_e$ to be the set parameterizing p-refined automorphic representations of type e which are non monodromic principal series at primes in S_N (note that $G(\mathbb{A}_{S_N})$ is quasisplit), with one of the weights fixed if we like (see Rem. 7.3.2 (iii)), and let

$$(X, \psi, \omega, Z)$$

be the corresponding eigenvariety given by Prop. 7.4.8, which is a closed subspace of the eigenvariety X_e associated to e.

In the following important example-definition, we introduce the "minimal level" eigenvariety containing a given refined automorphic representation of G.

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(21) Of course, we take the same choice of $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$, ι_p and ι_∞ in both cases.

(22) We need to make this technical condition to ensure that automorphic representations of G admit associated Galois representations (see [**61**, Thm. 3.1.4] and § 6.8.2 (vii)).

Example 7.5.1. — *The minimal eigenvariety containing* π. Let $(\pi, \mathcal{R})$ be a p-refined automorphic representation of G.

In case (a) (resp. in case (b)), assume that π is either a non monodromic principal series or unramified (resp. is unramified) at all the finite nonsplit primes. Define S as the finite set consisting of p and of the primes l such that either π_l is ramified or $G(\mathbb{Q}_l)$ is the group of invertible elements of a division algebra (which occurs only in case (b)). Define also $S_N \subset S$ as the subset of nonsplit primes l such that π_l is a non monodromic principal series (this set is empty in case (b)). Choose $e = e_S$ such that:

(i) For $l \in S_N$, $e_l = e_{\Sigma_l}$ is a special idempotent attached to the Bernstein component attached to the inertial class Σ_l of π_l (see Example 7.3.3 (v)).

(ii) For $l = w\bar{w} \neq p \in S \backslash S_N$ such that $G(\mathbb{Q}_l) = \mathrm{GL}_m(\mathbb{Q}_l)$, $e_l = e_{\tau_l}$ where τ_l is the finite dimensional irreducible representation of $\mathrm{GL}_m(\mathcal{O}_{E_w})$ attached to π_l by Prop. 6.5.3.

(iii) For $l = w\bar{w}$ such that $G(\mathbb{Q}_l)$ is the group of invertible elements of a division algebra, $e_l = e_{\tau_l}$ where τ_l is a Bushnell-Kutzko's type for the Bernstein component of π_l. Such a τ_l exists by [**31**, Prop. 5.4].

Choose a finite extension $L/\mathbb{Q}_p$ which is sufficiently big so that $\iota_p\iota_\infty^{-1}\pi_f$ and each e_l is defined over L.

Definition 7.5.2. — Under these assumptions, the unique eigenvariety over L for $\mathcal{Z}_{e,N}$ given by Prop. 7.4.8 will be referred as *the minimal eigenvariety of G containing π* (or $(\pi, \mathcal{R})$).

Of course, in this context, if we are interested in the variant with the i_0^{th} weight fixed as in Rem. 7.3.2 (iii), we shall always choose the integer k to be the i_0^{th} weight of π_∞.

We now go back to the Galois side. Recall that $G_{E,S}$ is the Galois group of a maximal algebraic extension of E which is unramified outside the primes above S. For each regular automorphic representation π, properties (P0) and (P1) assert the existence of a unique semisimple continuous representation

$$\rho_\pi : G_{E,S} \longrightarrow \mathrm{GL}_m(\overline{\mathbb{Q}}_p),$$

such that for each prime $l = w\bar{w} \in S_0$, the trace of a geometric Frobenius at x, say

$$\mathrm{Frob}_w \in G_{E,S},$$

is the trace of the Langlands conjugacy class of $\iota_p\iota_\infty^{-1}(\pi_w.|\det|^{\frac{1-m}{2}})$. Let

$$h_w \in \mathcal{C}(K_l \backslash \mathcal{C}(G(\mathbb{Q}_l)/K_l, \mathbb{Z}) \xrightarrow{\sim}_w \mathcal{C}(\mathrm{GL}_m(\mathbb{Z}_l)\backslash \mathrm{GL}_m(\mathbb{Q}_l)/\mathrm{GL}_m(\mathbb{Z}_l), \mathbb{Z})$$

be the usual Satake element $[K_l(l, 1, \ldots, 1)K_l]$, it satisfies

$$\mathrm{tr}\,(\rho_\pi(\mathrm{Frob}_w)) = \psi_{(\pi,\mathcal{R})}(h_w).$$

Let us denote by

$$Z_{\rm reg} \subset Z \xrightarrow{\sim} \mathcal{Z}_e$$

the subset of points parameterizing the p-refined $(\mathcal{R}, \pi)$ such that π_∞ is regular, and such that the semisimple conjugacy class of π_p (see § 6.4.3) has m distinct eigenvalues. If $z \in Z_{\rm reg}$ parameterizes the regular p-refined π, we will set

$$\bar{\rho}_z := \rho_\pi.$$

The following lemma shows that $Z_{\rm reg}$ is sufficiently large.

***Lemma* 7.5.3.** — *$Z_{\rm reg}$ is a Zariski-dense subspace of X accumulating at each point of Z.*

— *Let $U \subset X$ be an open affinoid, and let $Y \subset U$ be a Zariski-dense subset such that $\omega(Y) \subset \mathbb{Z}^m$. Consider the subset $Y' \subset Y$ whose points y satisfy*

$$F_i(y)/F_j(y) \neq p^{k_i - k_j - i + j + 1} \quad \forall i \neq j$$

where $\omega(y) = (k_1, \dots, k_m)$. Then Y' is Zariski-dense in U.

Proof. — Let $(\omega_1, \dots, \omega_m) : \mathcal{W} \longrightarrow \mathbb{A}^m$ be the analytic map such that for $i = 1, \dots, m$, and $x \in X$, $\omega_i(x)$ is the derivative at 1 of the restriction to the $i^{\rm th}$ copy of $\mathbb{Z}_p^*$ of the character $\omega(x)$ (for more details about this construction, see § 7.5.4 below). For $i \neq j$, the closed subset of X defined by the equality

$$\omega_i = \omega_j + 1$$

is nowhere dense by property (iv) of the eigenvariety X (see Theorem 7.3.1). As Z is an accumulation and Zariski-dense subset of X, the first assertion is a consequence of the second one.

Let $Y' \subset Y \subset U$ be as in the second assertion. As the F_i are invertible in $\mathcal{O}(U)$, the maximum modulus principle shows that there are two integers $a, b \in \mathbb{Z}$ such that

$$p^a < |F_i(x)/F_j(x)| < p^b$$

for all $x \in U$ and $i \neq j$. In particular, $Y \backslash Y'$ is included in the closed subspace of U defined by a finite number of relations of the form $\omega_i - \omega_j = A$ with $i \neq j$ and $A \in \mathbb{Z}$. We conclude as such a subspace is nowhere dense by property (iv) of the eigenvariety X again. □

7.5.2. The family of Galois representations on X. — We adopt also from now on the notations of § 4.2.2. The first result is that the $\bar{\rho}_z$ with $z \in Z_{\rm reg}$ interpolate uniquely to a rigid analytic family of p-adic representations of $G_{E,S}$ on X. It uses only properties (P0) and (P1).

Proposition 7.5.4. — *There exists a unique continuous pseudocharacter*

$$T : G_{E,S} \longrightarrow \mathcal{O}(X)^0$$

such that for all $z \in Z$, $T_z = \mathrm{tr}\,(\bar{\rho}_{z'})$. Moreover:

(i) *$T(cg^{-1}c^{-1}) = T(g)\chi(g)^{m-1}$ for each $g \in G_{E,S}$ (see § 5.2.1),*

(ii) *for each prime $l = w\bar{w}$ in S_0, we have $T(\mathrm{Frob}_w) = \psi(h_w)$.*

In the statement above, $\chi : G_{E,S} \longrightarrow \mathbb{Z}_p^*$ is the p-adic cyclotomic character. Moreover c is the outer complex conjugation (see § 5.2.1).

Proof. — By property (vi), $\psi(\mathcal{H}_{\mathrm{ur}}) \subset \mathcal{O}(X)$ is a relatively compact subset, and by Lemma 7.5.3 Z_{reg} is Zariski-dense in X. The existence and uniqueness of T follows then from [**36**, Prop. 7.1.1]. The equalities in (i) and (ii) hold as X is reduced and as they hold on the Zariski-dense subspace Z_{reg} (see Remark 6.8.2 (i)). □

For $x \in X$, recall that $\mathcal{O}_x$ is the rigid local ring at x, $k(x)$ its residue field and $\overline{k(x)}$ an algebraic closure of $k(x)$. As $\mathcal{O}_x$ is reduced and noetherian, its total fraction ring

$$\mathcal{K}_x := \mathrm{Frac}(\mathcal{O}_x)$$

is a finite product of fields, and we will denote by $\overline{\mathcal{K}}_x$ a (finite) product of algebraic closures of each of those fields. By Taylor's theorem [**117**, Thm. 1.2], we have then two canonical representations attached to x:

(a) $\bar{\rho}_x : G_{E,S} \longrightarrow \mathrm{GL}_m(\overline{k(x)})$, which is the unique (up to isomorphism) continuous semisimple representation with trace $T_x : G \longrightarrow \mathcal{O}_x \longrightarrow k(x)$.

(b) $\rho_x^{\mathrm{gen}} : G_{E,S} \longrightarrow \mathrm{GL}_m(\overline{\mathcal{K}}_x)$, which is the unique (up to isomorphism) semisimple representation with trace $T \otimes \mathcal{K}_x : G_{E,S} \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{K}_x$.

Corollary 7.5.5. — *For each $x \in X$, and for each prime $l = w\bar{w}$ in S_0, we have $\bar{\rho}_x^{\perp} \simeq \bar{\rho}_x(m-1)$ and $\rho_x^{\mathrm{gen}\perp} \simeq \rho_x^{\mathrm{gen}}(m-1)$.*

7.5.3. Properties of T at the primes $l \neq p$ in S. — Let $l \neq p \in S$ and w a prime of E above l. We are interested in the restriction to $\mathrm{W}_{E_x} \longrightarrow G_{E,S}$ of the family T. We invite the reader to read first the Appendix 7.8 of which we will use concepts and notations.

Lemma 7.5.6. — *For each $x \in X$ and $s(x)$ a germ of irreducible component of X at x, there exists $z \in |Z_{\mathrm{reg}}|$ in the same irreducible component as x such that*

$$N_{s(x)}^{\mathrm{gen}} \sim_{I_{E_w}} \overline{N}_z.$$

Proof. — It follows from Prop. 7.8.19 (i) and (ii), and the Zariski-density of $|Z_{\mathrm{reg}}|$ in X. □

Assumption (P3) has the following consequence.

***Proposition* 7.5.7.** — *Assume that $l \in S_N$. For each $x \in X$, $\overline{N}_x = N_x^{\mathrm{gen}} = 0$.*

Proof. — By assumption (P3), $\overline{N}_z = 0$ for each $z \in |Z_{\mathrm{reg}}|$, hence we are done by Lemma 7.5.6 and Prop. 7.8.19 (iii). □

Assume now that $l = w\bar{w} \neq p$ splits in E and that $G(\mathbb{Q}_l) \simeq \mathrm{GL}_m(\mathbb{Q}_l)$. Let us fix a $\overline{\mathbb{Q}}_p$-valued d-dimensional Weil-Deligne representation (r_0, N_0) of E_w. Assume that the idempotent

$$e_l \in \mathcal{C}_c(G(\mathbb{Q}_l), \overline{\mathbb{Q}}_p) \xrightarrow{\sim}_w \mathcal{C}_c(\mathrm{GL}_m(\mathbb{Q}_l), \overline{\mathbb{Q}}_p)$$

occurring in the definition of X (see § 7.5.1) has the property that for all the irreducible smooth representations π of $\overline{\mathbb{Q}}_p[G(\mathbb{Q}_l)]$, we have

$$e(\pi) \neq 0 \Rightarrow N(\pi) \prec_{I_{\mathbb{Q}_l}} N_0. \tag{82}$$

Note that such idempotents exist by Prop. 6.5.3.

***Proposition* 7.5.8.** — *Assume that $l = w\bar{w} \neq p$ splits in E and that e_l is as above. For each $x \in X$, and each germ $s(x)$ of irreducible component at x, then*

$$\overline{N}_x \prec N_{s(x)}^{\mathrm{gen}} \prec N_0.$$

Proof. — By assumption (P2), we have $\overline{N}_z \prec N_0$ for all $z \in |Z_{\mathrm{reg}}|$. As $\prec_{I_{E_w}}$ implies $\prec$, we conclude by Lemma 7.5.6 and Prop. 7.8.19 (iii). □

***Remark* 7.5.9.** — For the sake of completeness, let us consider also the following stronger variant of condition (P2): *let $l = w\bar{w}' \neq p$ be a prime that splits in E, and (r, N) (resp. (r', N')) the $\overline{\mathbb{Q}}_p$-valued Weil-Deligne representation attached to $\pi_w|\det|^{\frac{1-m}{2}}$ (resp. $\rho_{\pi|\mathrm{W}_{E_w}}$). If $N \prec_{I_{E_w}} N_0$, then $N' \prec_{I_{E_w}} N_0$.* Under this stronger assertion, the proof of Prop. 7.5.8 shows that we even have $\overline{N}_x \prec_{I_{E_w}} N_{s(x)}^{\mathrm{gen}} \prec_{I_{E_w}} N$.

Let us give another application in a more specific situation. Let us fix $x \in X$ and assume that $\bar{\rho}_x$ is irreducible and defined over $k(x)$. Let us view T as a continuous pseudocharacter

$$T : G_{E,S} \longrightarrow \mathcal{O}_x,$$

and consider the faithful Cayley-Hamilton algebra [23]

$$S := \mathcal{O}_x[G_{E,S}]/\mathrm{Ker}\, T.$$

Then $S \simeq \mathcal{M}_d(\mathcal{O}_x)$ by Thm. 1.4.4 (i), so that T is the trace of a unique (continuous) representation

$$\rho : G_{E_S} \longrightarrow \mathrm{GL}_m(\mathcal{O}_x).$$

[23] It actually coincides with the universal Cayley-Hamilton quotient (see § 1.2.5) of $(\mathcal{O}_x[G_{E,S}], T)$ by Theorem 1.4.4 (i).

Let K be the total fraction ring of $\mathcal{O}_x$, then $\rho \otimes K$ is absolutely irreducible as $\mathcal{O}_x[G_{E,S}] \longrightarrow \mathcal{M}_m(\mathcal{O}_x)$ is surjective. In particular

$$\rho_x^{\mathrm{gen}} \simeq \rho \otimes \overline{K}.$$

By Lemma 4.3.7 and Prop. 7.8.14, ρ admits an associated $\mathcal{O}_x$-valued Weil-Deligne representation, say (r, N), $N \in M_d(\mathcal{O}_x)$.

Corollary 7.5.10. — *We keep the assumptions of Prop. 7.5.8. Assume that*

$$\overline{N}_x \sim_{I_{E_w}} N_0,$$

then N admits a Jordan normal form over $\mathcal{O}_x$ (see § 7.8.1) and $N \sim_{I_{E_w}} N_0$.

Proof. — Note that $N_1 \prec_{I_{E_w}} N_2$ and $N_1 \sim N_2$ imply $N_1 \sim_{I_{E_w}} N_2$. In particular, by Prop. 7.5.8 and the assumption, we get that for each germ of irreducible component $s(x)$ at x,

$$N_0 \sim_{I_{E_w}} \overline{N}_x \prec_{I_{E_w}} N_{s(x)}^{\mathrm{gen}} \prec N_0,$$

hence all the $\prec$ above are $\sim_{I_{E_w}}$. The corollary follows then from Lemma 7.8.9 (ii). □

7.5.4. Properties of T at the prime v. — We are interested in the restriction of the family T to

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow_v G_{E,S}$$

given by the prime v above p, as in section 4. For any representation ρ of G, we will simply say that ρ is Hodge-Tate, crystalline etc. if its restriction by the map above is.

Let $z \in Z_{\mathrm{reg}}$ parameterizing the p-refined automorphic form $(\pi, \mathcal{R})$ of weight $\underline{k} = (k_1, \ldots, k_m)$. By properties (P4) of (Rep(m)), $\bar{\rho}_z$ is Hodge-Tate, with Hodge-Tate weights the following strictly increasing sequence of integers:

$$-k_1,\ -k_2+1,\ \ldots,\ -k_m+m-1.$$

For convenience, and also in order to fit with the notations of sections 2 and 3 of this book, this shift leads us to modify a little the map ω as follows. Let

$$\log_p : \mathcal{W} \longrightarrow \mathrm{Hom}_{\mathrm{gr}}(T_0, \mathbb{A}^1).$$

be the map induced the usual p-adic logarithm $\mathbb{G}_m \to \mathbb{A}^1$ (killing p), and let us identify

$$\mathrm{Hom}_{\mathrm{gr}}(T_0, \mathbb{A}^1) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Lie}(T), \mathbb{A}^1) \xrightarrow{\sim} \mathbb{A}^m$$

via the diagonalisation $T \xrightarrow{\sim} (\mathbb{Q}_p^*)^m$. Under these identifications, $\log_p : \mathcal{W} \to \mathbb{A}^m$ associates to the character $\chi = (\chi_1, \ldots, \chi_m) \in \mathcal{W}(L)$ the element

$$\log_p(\chi) = \left(\ldots, \left(\frac{\partial \chi_i}{\partial \gamma}\right)_{\gamma=1}, \ldots\right) \in L^m.$$

In particular, the composition of the embedding $\mathbb{Z}^m \hookrightarrow \mathcal{W}$ defined in § 7.2.3 with $\log_p$ is the natural inclusion $\mathbb{Z}^m \subset \mathbb{A}^m$.

Definition 7.5.11. — The morphism $\kappa = (\kappa_1, \ldots, \kappa_m) : X \longrightarrow \mathbb{A}^m$ is the composition of the map $\log_p \cdot \omega$ by the affine change of coordinates

$$(x_1, \ldots, x_m) \mapsto (-x_1, -x_2 + 1, \ldots, -x_m + m - 1).$$

For each $z \in Z_{\mathrm{reg}}$, $\kappa_1(z), \ldots, \kappa_i(z)$ is the strictly increasing sequence of Hodge-Tate weights of $\bar{\rho}_z$.

It turns out that this is enough to imply that for each $x \in X(\overline{\mathbb{Q}}_p)$, the Sen polynomial of $\bar{\rho}_x$ is $\prod_{i=1}^m (T - \kappa_i(x))$.

Lemma 7.5.12. — *Let $T : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathcal{O}(X)$ be any m-dimensional continuous pseudocharacter on a separated rigid analytic space over $\mathbb{Q}_p$, $\kappa = (\kappa_1, \ldots, \kappa_m) : X \longrightarrow \mathbb{A}^m$ an analytic map, and $Z \subset X(\overline{\mathbb{Q}}_p)$ a Zariski-dense accumulation subset. Assume that for each $z \in Z$, the Sen polynomial of $\bar{\rho}_z$ is $\prod_{i=1}^m (T - \kappa_i(z))$.*

Then for each $x \in X(\overline{\mathbb{Q}}_p)$, the Sen polynomial of $\bar{\rho}_x$ is $\prod_{i=1}^m (T - \kappa_i(x))$. In particular, $\bar{\rho}_x$ is Hodge-Tate whenever the $\kappa_i(x)$ are distinct integers.

Proof. — By replacing X by its normalization $\tilde{X}$ and Z by its inverse image in $\tilde{X}$, we may assume that X is normal and irreducible.

Let $\Omega \subset X$ be an open affinoid. Let g, g', Y, Y' and $\mathcal{Y}$ and $\mathcal{M}_{\mathcal{Y}}$ be as in Lemma 7.8.11. For each open affinoid $V \subset \mathcal{Y}$, Sen's theory [**108**] attaches to the locally free continuous $\mathcal{O}(V)$-representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\mathcal{M}_{\mathcal{Y}}(V)$ a canonical element

$$\varphi_V \in \mathrm{End}_{\mathcal{O}(V)}(\mathcal{M}_{\mathcal{Y}}(V))_{\mathbb{C}_p},$$

whose formation commutes with any open affinoid immersion $V' \subset V$. The characteristic polynomial $P_{\varphi,V}$ of each φ_V lies in $\mathcal{O}(V)[T]$, and all of them glue to a single polynomial $P_\varphi \in \mathcal{O}(\mathcal{Y})[T]$.

Let $S \subset \Omega$ be a Zariski-dense subset. Then $g^{-1}(S)$ is Zariski-dense in Y by [**36**, Lemme 6.2.8], hence $g^{-1}(S) \cap Y'$ is Zariski-dense in $\mathcal{Y}$. Assuming that the conclusion of the lemma holds for all x in S, P_φ coincides with $\prod_{i=1}(T - \kappa_i)$ on $\mathcal{Y}^{\mathrm{red}}$, so the conclusion of the lemma holds actually on the whole of Ω (note that g and g' are surjective). In particular, by a connectedness argument it is enough to show that the conclusion of the lemma holds for each x in a single affinoid subdomain of X.

As Z is a Zariski-dense accumulation subset of X, there is an affinoid subdomain Ω of X such that Z is Zariski-dense in Ω. We claim that the conclusion of the lemma holds for each $x \in \Omega$. Indeed, this follows from the previous paragraph for that specific Ω and for $S = Z \cap \Omega$, and we are done. □

Let again $z \in Z_{\mathrm{reg}}$ parameterizing the p-refined automorphic form $(\pi, \mathcal{R})$ as above, and set

$$\mathcal{F}_z = (F_1(z)p^{\kappa_1(z)}, \ldots, F_m(z)p^{\kappa_m(z)}).$$

By Definition 7.2.13,

$$\mathcal{F}_z = \iota_p \iota_\infty^{-1}(\mathcal{R}|p|^{\frac{1-m}{2}})$$

is an accessible refinement of $\pi_p|\det|^{\frac{1-m}{2}}$. By properties (P4) and (P5) of (Rep(m)), $\bar{\rho}_z$ is a crystalline representation and $\mathcal{F}_z$ is an ordering of the eigenvalues of its crystalline Frobenius. As $z \in Z_{\text{reg}}$, all these eigenvalues are distinct hence $\mathcal{F}_z$ is also the ordered set of Frobenius eigenvalues of a unique refinement of $\bar{\rho}_x$ in the sense of § 2.4, that we will call also $\mathcal{F}_z$. If furthermore $z \in Z'$ in the sense of property (v) of X, note that $\mathcal{F}_z$ is numerically non critical (see Remark 2.4.6 (ii)).

***Proposition** 7.5.13.* — *$(X, T, \kappa_i, \{F_i\}, Z_{\text{reg}})$ is a refined family in the sense of § 4.2.3.*

Proof. — By Lemma 7.5.12 and what we just explained, $(X, T, \kappa_i, \{F_i\}, Z_{\text{reg}})$ satisfies properties (i) to (iv) of Definition 4.2.3 of a refined family. It also satisfies (*) of *loc. cit.* as for any $x \in X$ the character $\omega(x)$ of $(\mathbb{Z}_p^*)^m$ lifts $(\kappa_1, \dots, \kappa_m)$ (up to a harmless translation) by Def. 7.5.11. To prove property (v) of the definition of a refined family, we need to prove for any $z \in Z$ and any integer C that the set $(Z_{\text{reg}})_C$ accumulates at z.

Let $z \in Z$. By the second assertion of Lemma 7.5.3, it is actually enough to show that Z_C accumulates at z. By property (iv) of Thm 7.3.1, there is an open affinoid $\Omega \subset X$ such that $\kappa(\Omega)$ is an open affinoid and $\kappa_{|\Omega}$ is finite and surjective when restricted to any irreducible component of Ω. Thus $\kappa(\Omega)$ contains an open affinoid ball B of center $\kappa(z)$, and replacing Ω by $\Omega \cap \kappa^{-1}(B)$, we way even assume that $\kappa(\Omega) = B$. There is an integer N such that B contains the set $Y \subset \mathbb{Z}^m$ of m-uples $(k_1, \dots, k_m)$ with $k_1 < k_2 < \cdots < k_m$ and $(k_1, \dots, k_m) \equiv \kappa(z) \pmod{(p-1)p^N}$. For $C' \in \mathbb{R}$, let $Y_{C'} \subset Y$ be the subset of elements $(k_1, \dots, k_m)$ such that

$$\left\{ \begin{array}{l} k_2 - k_1 > C', \\ k_{i+1} - k_i > C'(k_i - k_{i-1} + 1), \ \forall i \in \{2, \dots, m-1\}. \end{array} \right. \tag{83}$$

Then by definition, Z_C contains $\kappa^{-1}(Y_{C'}) \cap Z$ for all $C' > C$. By the maximum modulus principle, we may choose $C' > C$ such that for all $i = 1, \dots, m-1$ and all $x \in \Omega$, we have $|F_1(x)F_2(x)\cdots F_i(x)| < C' - 1$. By property (v) of the eigenvariety X, $\kappa^{-1}(Y_{C'}) \subset Z_C$. From the properties of κ recalled above and by [**36**, Lemma 6.2.8], it is thus enough to prove that $Y_{C'}$ accumulates at $\kappa(z)$ in the closed ball B, which is obvious. □

7.6. The eigenvarieties at the regular, non critical, crystalline points and global refined deformation functors

In this part, we give an application of the techniques and results of this book to study some global deformation rings, as we announced in § 2.6 of section 2. This has some counterparts concerning the geometry of the minimal eigenvarieties at the classical, non critical, crystalline points. We will show that those eigenvarieties should be smooth at those points and that they are very neatly related to deformation theory. By contrast, a much more complicated situation is expected at reducible, critical points, and this will actually be the main theme of the last sections 8 and 9.

7.6.1. Some global deformation functors and a general conjecture. — Let $m \geq 1$ be an integer, $E/\mathbb{Q}$ a quadratic imaginary field, S a finite set of places of E, and

$$\rho : G_{E,S} \longrightarrow \mathrm{GL}_m(L) = \mathrm{GL}(V)$$

an absolutely irreducible continuous representation with coefficients in the finite extension $L/\mathbb{Q}_p$. We assume (see § 5.2.1) that

$$V^{\perp} \simeq V(m-1).$$

Let us assume also that the prime $p = v\bar{v}$ splits in E, that $V_p := V_{|E_v}$ is a crystalline representation whose crystalline Frobenius has m distinct eigenvalues in L^*, and that the Hodge-Tate weights of V_p are two-by-two distinct.

Choice. — *Let us choose a refinement $\mathcal{F}$ of V_p in the sense of § 2.4.*

We will introduce below a deformation functor of V depending on this choice, but let us first remind some general facts of deformation theory. Let $\mathcal{C}$ be the category of finite dimensional local $\mathbb{Q}_p$-algebras with fixed residue field L, that we introduced in § 2.3.5, H a topological group and V a finite dimensional continuous L-representation of H. Following Mazur [**84**], let

$$X_V : \mathcal{C} \longrightarrow \mathrm{Ens}$$

be the deformation functor of the H-representation V. For any $A \in \mathcal{C}$, $X_V(A)$ is by definition the set of isomorphism classes of pairs (V_A, π) where V_A is a finite free A-module equipped with a continuous A-linear action of H and $\pi : V_A \otimes_A L \xrightarrow{\sim} V$ an H-equivariant isomorphism. If V is absolutely irreducible and if the continuous cohomology group $H^1(H, \mathrm{ad}(V))$ is finite dimensional, we know from [**84**, §10] that X_V is prorepresentable by a complete local noetherian ring, with tangent space isomorphic to the cohomology group above.

Remark 7.6.1. — Note that in Mazur's theory [**84**], the residue field k of the coefficient rings is a finite field. However, everything also applies verbatim when k is a finite extension of $\mathbb{Q}_p$ (in which case it is actually even a bit simpler as k automatically lifts as a subfield of the coefficient rings). The adequate version in this setting of the *p-finiteness condition* of *loc. cit.* is the following: for any finite dimensional continuous $\mathbb{Q}_p$-representation U of H, the continuous cohomology group $H^1(H, U)$ is a finite dimensional $\mathbb{Q}_p$-vector space. By [**104**, Prop. B.2.7] and Tate's theorems, this condition holds if $H = G_{E,S}$ or the Galois group of a local field.

Let us denote by X_V and X_{V_p} the deformation functors associated respectively to the $G_{E,S}$-representation V and to the G_p-representation V_p. The choice of any embedding $E \longrightarrow \overline{\mathbb{Q}}_p$ extending v defines a natural transformation by restriction

$$X_V \longrightarrow X_{V_p},$$

that is $(V_A, \pi) \mapsto (V_{A\,|\,E_v}, \pi)$. Let us denote again by $\mathcal{F}$ the triangulation of $D_{\mathrm{rig}}(V_p)$ associated to our chosen refinement $\mathcal{F}$ by Prop. 2.4.1. Recall that we defined in § 2.3.6 a *refined deformation functor*

$$X_{V_p,\mathcal{F}} : \mathcal{C} \longrightarrow \mathrm{Set}$$

of $(V_p, \mathcal{F})$ equipped with a natural transformation $X_{V_p,\mathcal{F}} \longrightarrow X_{V_p}$. By assumption on V_p and Prop. 2.3.6 and 2.4.1, $X_{V_p,\mathcal{F}}$ is actually a subfunctor X_{V_p}.

Definition 7.6.2. — Define two subfunctors $X_{V,\mathcal{F}}$ and $X_{V,f}$ of X_V as follows. If $A \in \mathcal{C}$, say that $(V_A, \pi) \in X_{V,\mathcal{F}}(A)$ (resp. $X_{V,f}(A)$) if, and only if:

(i) $V_A^{\perp} \simeq V_A(m-1)$,

(ii) For $w \in S$ not dividing p, V_A is constant when restricted to I_{E_w}, that is

$$V_A \simeq_{I_{E_w}} V \otimes_L A.$$

(iii) $(V_{A\,|\,E_v}, \pi) \in X_{V_p,\mathcal{F}}(A)$ (resp. $V_{A\,|\,E_v}$ is crystalline).

We call $X_{V,f}$ the *fine deformation functor* of V, and $X_{V,\mathcal{F}}$ the *refined deformation functor* of V associated to $\mathcal{F}$.

Recall that the parameter of a triangulation define for each A a morphism

$$\delta = (\delta_i) : X_{V_p,\mathcal{F}}(A) \longrightarrow \mathrm{Hom}(\mathbb{Q}_p^*, A^*)^m.$$

Here Hom means *continuous group homomorphisms.* In particular, the derivative at 1 of such a morphism is an element of A^m, which gives us a morphism

$$\partial\kappa : X_{V_p,\mathcal{F}} \longrightarrow \widehat{\mathbb{G}_a^m}.$$

Denote by $\bar{\delta} : \mathbb{Q}_p^* \to (L^*)^m$ the parameter of $\mathcal{F}$ (see 2.4.1).

Proposition 7.6.3. — (i) *$X_{V,f}$ and $X_{V,\mathcal{F}}$ are prorepresentable by some local complete noetherian rings.*

(ii) *The parameter of a triangulation induces a canonical morphism*

$$\delta : X_{V,\mathcal{F}} \longrightarrow \widehat{\mathrm{Hom}(\mathbb{Q}_p^*, \mathbb{G}_m^m)}_{\bar{\delta}}.$$

(iii) *There is a canonical injective morphism $X_{V,f}(L[\varepsilon]) \longrightarrow H^1_f(E, \mathrm{ad}(V))$, whose image is the subspace of deformations $\widetilde{V}$ of V to $L[\varepsilon]$ such that $\widetilde{V}^{\perp} \simeq \widetilde{V}(m-1)$.*

Proof. — In order to prove (i), we have to check that each of the conditions (i), (ii) and (iii) in Definition 7.6.2 are *deformation conditions* in the sense of Mazur [**84**, §19, 23].

For condition (i), note that as V is absolutely irreducible, a deformation V_A is uniquely determined up to isomorphism by its trace (Serre-Carayol's theorem [**33**, Thm. 1]). It is then trivial to check conditions (1), (2) and (3) of §23 of *loc. cit.* for that deformation condition. For condition (ii), (1) is obvious, (3) follows from Prop. 7.8.5 [24], and (2) follows easily from (1) and (3) (see the proof of Prop. 2.3.9).

For condition (iii) in the refined case, it is Prop. 2.3.9. In the fine case, it follows from Ramakrishna criterion (see [**84**, §25, Prop. 1]) and from the fact that the category of crystalline representations is closed under passage to subobjects, quotients, and finite direct sums, by a result of Fontaine. That concludes the proof of part (i) of the proposition.

We already explained assertion (ii) before the statement, and assertion (iii) is now immediate. □

Let us set for short
$$H^1_{\mathcal{F}}(E, \mathrm{ad}(V)) := X_{V,\mathcal{F}}(L[\varepsilon]).$$
Recall that we defined in § 2.4.3 a notion of *non critical refinement.*

Proposition 7.6.4. — *If $\mathcal{F}$ is a non critical refinement of V_p, then $X_{V,f}$ is a subfunctor of $X_{V,\mathcal{F}}$. If moreover* $\mathrm{Hom}_{G_p}(V_p, V_p(-1)) = 0$, *then*

(i) *$X_{V,f}$ is exactly the subfunctor of $X_{V,\mathcal{F}}$ defined by the equation $\partial\kappa = 0$,*

(ii) *This inclusion induces the following exact sequence on tangent spaces:*
$$0 \longrightarrow X_{V,f}(L[\varepsilon]) \longrightarrow H^1_{\mathcal{F}}(E, \mathrm{ad}(V)) \overset{\partial\kappa(L[\varepsilon])}{\longrightarrow} L^m$$

Proof. — The first assertion follows from Prop. 2.5.8. Point (i) is Theorem 2.5.1, and (ii) is then obvious. □

We believe in the following conjectures.

Conjecture 7.6.5. — (C1) *$X_{V,f}$ is a closed point.*

(C2) *If $\mathcal{F}$ is non critical, then $\partial\kappa$ is an isomorphism. In particular, $X_{V,\mathcal{F}}$ is (formally) smooth of dimension m.*

By Prop. 7.6.3 (iii), Conjecture (C1) is actually equivalent to the conjecture $BK2(\rho)$ introduced in § 5.2.3 (see also 5.2.4). Let us record this fact in the following corollary.

[24] If (r, N) is the Weil-Deligne representation of $V_{A\,|E_w}$, note that condition (ii) is equivalent to ask that N is constant on each isotypic component of the semisimple representation $r_{|I_{E_w}}$.

Corollary 7.6.6. — *Conjecture (C1) is equivalent to the conjecture $BK2(\rho)$ (see 5.2.3). In particular, the Bloch-Kato conjecture implies (C1).*

As a consequence, (C1) is a very "safe" conjecture. In what follows, we will try to provide evidence for (C2) and we will relate it to eigenvarieties. In particular, this will shed some light on the expected structure of those eigenvarieties in some cases.

Remark 7.6.7. — Assuming that $\mathrm{Hom}_{G_p}(V_p, V_p(-1)) = 0$, Prop. 7.6.4 (ii) shows that (C2) implies (C1). As ρ is conjecturally pure, this assumption conjecturally always hold, hence *(C2) is conjecturally stronger that (C1).* As we shall see, the input of eigenvarieties will show that they are actually equivalent.

7.6.2. An automorphic special case. — We keep the assumptions on § 7.6.1. As we want to give examples providing evidence for (C2), we will focus from now on to some special cases (but still rather general, see Rem. 7.6.9) coming from the theory of automorphic forms for which everything we shall need is known. Let us fix a prime $q \neq p$ that splits in E, as well as another split prime $q' \notin \{q, p\}$ if $m \equiv 0 \bmod 4$ and such that $q' = q$ else.

Lemma 7.6.8. — *There exists a unique unitary group in m variables G attached to $E/\mathbb{Q}$ such that*

(i) *$G(\mathbb{R})$ is compact,*
(ii) *if $l \notin \{q, q'\}$, $G(\mathbb{Q}_l)$ is quasisplit,*
(iii) *if $l = q$ or q', $G(\mathbb{Q}_l)$ is the group of invertible elements of a central division algebra over $\mathbb{Q}_l$.*

Proof. — This follows from Hasse's principle (see e.g. [**40**, (2.2)]). There is no global obstruction when m is odd ([**40**, Lemme 2.1]), and a $\mathbb{Z}/2\mathbb{Z}$-obstruction when m is even. In that case, the local invariant in $\mathbb{Z}/2\mathbb{Z}$ of a division algebra is always non zero (see (2.3) of *loc. cit.*), and the one at the real place is $(-1)^{m/2}$ by [**40**, Lemme 2.2], hence the lemma. □

Let π be an automorphic representation of G such that:

(π1) π is only ramified at primes that split in E,
(π2) π_p is unramified and its Langlands conjugacy class has m distinct eigenvalues,
(π3) π_q is supercuspidal.

As $G(\mathbb{R})$ is compact, it is easy to construct automorphic representations π satisfying (π1), (π3) and such that π_p is unramified (see e.g. [**38**, Lemma 2]). Then, most of the classical points of the minimal eigenvariety of G containing π satisfy furthermore (π2), by property (v) of that eigenvariety.

Let us fix some choices of ι_∞ and ι_p as in § 7.2.1. By [**61**, Thm. 3.1.3], such a π admits a strong base change π_E to $\mathrm{GL}_m(\mathbb{A}_E)$ (it is cuspidal as π_q is supercuspidal). Moreover, this π_E satisfies the assumptions of Harris-Taylor's theorem [**62**], hence by *loc. cit.* we can attach to this π and those embeddings a Galois representation ρ with the following properties:

(ρ1) ρ has all the properties of § 7.6.1.

(ρ2) $\rho_{|E_w}$ is unramified for each nonsplit place w of E, and compatible with $\pi_E|.|^{(m-1)/2}$ at all split w (up to Frobenius semi-simplification).

(ρ3) $\rho_{|E_v}$ is crystalline and the characteristic polynomial of its crystalline Frobenius is the same as the one of $\iota_p\iota_\infty^{-1}L(\pi_p)$.

Remark 7.6.9. — If we believe in Langlands' extension of the Taniyama-Shimura-Weil and Artin conjectures, as well as the yoga of parameters (see Appendix A), any ρ as in § 7.6.1 which is unramified at nonsplit places, irreducible at q, and indecomposable at q', should occur this way.

Under assumptions (ρ3) and (π2), $\iota_p\iota_\infty^{-1}$ induces a bijection $\mathcal{R} \mapsto \mathcal{F}$ between the refinements of π_p in the sense of § 6.4.4 and the refinements of V_p in the sense of §2.4. As π_p is tempered by Harris-Taylor's theorem, all its refinements are accessible by Example 6.4.9. In particular, to any choice of any refinement $\mathcal{F}$ of V_p as in § 7.6.1 corresponds an accessible refinement of π_p and vice-versa. For some technical reasons, let us also assume that:

(ρ_4) $\mathcal{F}$ is non critical and regular (see Def. 2.5.5).

By the same argument as above, this will also be satisfied for many π's. All these assumptions being done, let us consider the minimal eigenvariety X associated to $(\pi, \mathcal{F})$ (see Example 7.5.1). Let $z \in X$ be the L-point parameterizing π equipped with its refinement $\mathcal{F}$, and set

$$\mathbb{T} := \widehat{\mathcal{O}_z}.$$

7.6.3. $R = T$ at the regular non critical crystalline points of minimal eigenvarieties.

— Assume that ρ and π are as in § 7.6.1 and § 7.6.2. Let $R_{\rho,\mathcal{F}}$ be the universal deformation ring of the refined deformation functor $X_{V,\mathcal{F}}$ given by Prop. 7.6.3 (i).

Proposition 7.6.10. — *There is a canonical commutative diagram*

$$\begin{tikzcd} R_{\rho,\mathcal{F}} \arrow[r, two heads] & \mathbb{T} \\ L[[t_1,\dots,t_m]] \arrow[u, "\partial\kappa^\sharp"] \arrow[r, equal] & \widehat{\mathcal{O}}_{\kappa(z)} \arrow[u, "\kappa^\sharp"'] \end{tikzcd} \tag{84}$$

Moreover $\mathbb{T}$ is equidimensional of dimension m and $\kappa^\sharp$ is a finite injective map.

By $\kappa^\sharp : \widehat{\mathcal{O}}_{\kappa(z)} \to \mathbb{T} = \widehat{\mathcal{O}}_z$ we mean the structural ring homomorphism on completed local rings induced from $\kappa : X \to \mathcal{W}$.

Proof. — We claim first the existence of a natural map $R_{\rho,\mathcal{F}} \to \mathbb{T}$. Let $A = \mathcal{O}_z$. As ρ is absolutely irreducible, T is the trace of a unique continuous representation

$$\rho_A : G_{E,S} \longrightarrow \mathrm{GL}_m(A),$$

hence for each cofinite length proper ideal I of A we have a canonical element $\rho_A \otimes A/I \in X_V(A/I)$ (note that $A/I = \mathbb{T}/I\mathbb{T}$). We have to show that this element falls in $X_{V,\mathcal{F}}(A/I)$, *i.e.* to check conditions (i) to (iii) in Def. 7.6.2. Condition (i) follows at once from the fact that $T^\perp = T(m-1)$ and that ρ is absolutely irreducible. Condition (ii) follows from Cor. 7.5.10, which applies by **[118]**. Finally, condition (iii) follows from Theorem 4.4.1 if we can check the assumptions of § 4.4.1 at the point z. They hold as $(X, T, \kappa, \{F_i\})$ is a refined family by Prop. 7.5.13, and as the assumptions (REG) and (NCR) of § 4.4.1 follow form (ρ4). This concludes the claim.

The existence of a commutative diagram as in the statement is moreover given by the identification of the parameter $\delta \otimes A/I$ in the statement of Theorem 4.4.1. More precisely, that theorem provides a diagram

$$\begin{tikzcd} R_{\rho,\mathcal{F}} \arrow[rr] & & \mathbb{T} \\ & \widehat{\mathcal{O}_{\overline{\delta}}} \arrow[ul, "\delta"] \arrow[ur, "\eta"'] & \end{tikzcd} \tag{85}$$

where $\overline{\delta} \in \mathrm{Hom}(\mathbb{Q}_p^*, \mathbb{G}_m)$ is the parameter of $\mathcal{F}$, and where we define the map on the right as the composite of the two other ones. The identification of the parameter δ in that Theorem shows that the composition of η with the natural map $\widehat{\mathcal{O}}_{\kappa(z)} \longrightarrow \widehat{\mathcal{O}_{\overline{\delta}}}$ obtained by differenciation at 1 is the structural map $\mathcal{O}_{\kappa(z)} \to \mathbb{T}$, and that the image of η is generated by the F_i's over $\mathcal{O}_{\kappa(z)}$.

The assertion on $\mathbb{T}$ and $\kappa^\sharp$ follow from property (iv) of the eigenvariety X, thus it only remains to check that the upper map is a surjection. By properties (ii) of eigenvarieties (see Def. 7.2.5), $\mathbb{T}$ is generated by $\mathcal{H}^{\mathrm{ur}}$ as an $\widehat{\mathcal{O}_{\overline{\delta}}}$-algebra, *i.e.* by the $T(\mathrm{Frob}_w)$'s for the primes $l = w\bar{w} \in S_0$. But each $T(\mathrm{Frob}_w)$ obviously lifts in $R_{\rho,\mathcal{F}}$ as the trace of Frob_w in the universal refined deformation, and we are done by the commutative diagram (85). □

Corollary 7.6.11. — *Assume that ρ is associated to a π as in § 7.6.2.*

(i) *Conjecture (C1) is equivalent to conjecture (C2). In particular, if the Bloch-Kato conjecture holds then (C2) holds.*

(ii) *Conjecture (C1) implies that all the maps of the diagram of Prop. 7.6.10 are isomorphisms.*

Proof. — By Harris-Taylor's theorem (especially property (P5) of 6.8.1), ρ is pure and $\mathrm{Hom}_{G_p}(V_p, V_p(-1)) = 0$, hence Prop. 7.6.4 applies. In particular, (C2) implies (C1) by Remark 7.6.7.

Assume that (C1) holds. We claim that $\partial\kappa^\sharp$ is an isomorphism. It then implies (C2) as well as assertion (ii) of the corollary, since the top arrow (resp. the right arrow) in Prop. 7.6.10 is surjective (resp. injective), so it is enough to prove the claim.

By Prop. 7.6.10, $\partial\kappa^\sharp$ is injective. As it induces an isomorphism on the residue fields L and as $R_{\rho,\mathcal{F}}$ is a complete local noetherian ring, it is enough to show that $\partial\kappa^\sharp$ induces an isomorphism on tangent spaces. Under conjecture (C1), the exact sequence of Prop. 7.6.4 (ii) shows that the tangent space of $R_{\rho,\mathcal{F}}$ has dimension $\leq m$, and it is enough to know that it has dimension exactly m in order to conclude. But by Prop. 7.6.10 $R_{\rho,\mathcal{F}}$ has Krull dimension $\geq m$, as its quotient $\mathbb{T}$ has dimension m.

Recall that the Krull dimension of a local noetherian ring is always at most the dimension of its tangent space ([**83**, Thm 13.4], with equality if and only if the ring is regular). As a consequence, the Krull dimension of $R_{\rho,\mathcal{F}}$ and the dimension of its tangent space both coincide with m (and $R_{\rho,\mathcal{F}}$ is regular of dimension m), and we are done. □

In particular, the map $R_{\rho,\mathcal{F}} \longrightarrow \mathbb{T}$ that we defined should always be an isomorphism, as it is so under (C1). Moreover, we also get that a far reaching infinitesimal version of the principle *a non critical slope form is classical* should hold for eigenvarieties.

In the following conjecture, we keep the preceding assumptions. In particular, recall that $\mathcal{F}$ is non critical.

Conjecture 7.6.12. — (R=T) *The map $R_{\rho,\mathcal{F}} \longrightarrow \mathbb{T}$ is an isomorphism.*
(CRIT) *The map $\kappa^\sharp$ is an isomorphism, i.e. the weight map κ is étale at z.*

Example 7.6.13. — Assume furthermore that $m = 1$, or that $m = 2$ and $\rho = (\rho_f)_{|G_E}$ for some classical modular eigenform f of level N which is (essentially) square integrable at all the primes dividing N. Then (C1) holds by Prop. 5.2.6, hence so do (C2), (R=T) and (CRIT).

Of course, the natural trend in the area since the work of Wiles and Taylor-Wiles is that we should first try to prove conjectures (R=T) and (CRIT) using the cohomology of Shimura varieties and the theory of automorphic forms. Note that (R=T) and (CRIT) together imply (C2) by Prop. 7.6.10. Then (C1) would follow by Cor. 7.6.11.

Corollary 7.6.14. — *Conjectures (R=T) and (CRIT) imply (C1) and (C2).*

The deepest part there is certainly to show (R=T), but we will not say more here about that conjecture (see Kisin's paper [**74**] toward a proof of (C1) in the case $m = 2$, as well as the discussion in § 5.2.3). In the remark below, we discuss instead what is known about conjecture (CRIT).

Remark 7.6.15. — (i) Assuming that $\mathcal{F}$ is furthermore numerically non critical (see Remark 2.4.6 (ii)), then by Prop. 7.4.6 (iii) - that is essentially the *small slope forms are classical* result of Prop. 7.3.5 - the refined automorphic representation $(\pi, \mathcal{R})$ does not have any infinitesimal deformation in the space of p-adic automorphic forms.

This falls short of implying that κ is étale because of a subtlety: for these general G we do not have a good control of X in terms of the spaces of p-adic automorphic forms, for instance like the pairing we have for $\mathrm{GL}_2/\mathbb{Q}$ given by the q-expansion (see *e.g.* [**9**, Prop. 1 (c)]). However, if we knew that the multiplicity one theorem holds for the automorphic representations of G which are unramified at the nonsplit primes (which is expected), then (CRIT) would follow easily [25] from Prop. 7.4.6 (iii). In particular, by results of Rogaswki we know (CRIT) in the numerically non critical case when $m \leq 3$.

(ii) Even admitting this multiplicity one result for G, it would be very interesting to have a proof of conjecture (CRIT) in general. As explained in Rem. 2.4.6 (ii), in the classical case of the eigencurve and the group $\mathrm{GL}_2/\mathbb{Q}$, the full case of (CRIT) is known and is quite deep: it follows from Coleman's theorem [**42**], including the so called *boundary case* where "$v(a_p) = k - 1$".

(iii) Let us mention that we certainly believe that conjectures (R=T) and (CRIT) also hold without the assumption that $\mathcal{F}$ is regular (but of course still non critical). However, it seems to be an interesting problem to understand the case where the eigenvalues of the crystalline Frobenius of V_p are not assumed to be two-by-two distinct any more. Indeed, there seems to be no trivial fashion to make refinements of π_p and V_p correspond without this assumption.

(iv) If we do not assume that $\mathcal{F}$ is noncritical (but, say, all the other hypotheses), the situation is actually very interesting but quite different. For example, there can be no map from $R_{\rho,\mathcal{F}}$ to $\mathbb{T}$ (compare with [**39**, Prop. 5.5 (ii)]). We postpone its study to a subsequent work.

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[25] Basically because when an A-module M is free of rank one over A, then each commutative A-subalgebra of $\mathrm{End}_A(M)$ is equal to A (hence étale over A)...

7.7. An application to irreducibility

A simple application of the results of this section, together with the generic irreducibility Theorem 4.5.1, is the existence of many n-dimensional Galois representations of G_E that are irreducible, even after restriction to a decomposition group at a place v above p. As an example, let us prove the following result:

***Theorem** 7.7.1.* — *Assume* $\mathrm{Rep}(m)$ *(actually for that matter we may release conditions (P2) and (P3)). Then for any integer C, there exists an automorphic representation π for* $\mathrm{U}(m)$ *such that the Galois representation ρ_π*

- *is unramified at each place not dividing p,*
- *is crystalline and irreducible at each of the two places dividing p,*
- *and has Hodge-Tate weights k_i, $i = 1, \ldots, m$, such that $|k_i - k_j| > C$ for every $i \neq j$.*

Proof. — Let π be the trivial representation of $\mathrm{U}(m)$. It is unramified at all the finite primes hence we may consider the minimal eigenvariety containing π as in § 7.5.1. By definition it is the eigenvariety $(X/\mathbb{Q}_p, \psi, \nu, Z)$ for the set Z of p-refined automorphic representations $(\psi_{(\pi,\mathcal{R})}, \underline{k})$ such that π is unramified at all finite places and $\mathcal{R}$ is an accessible refinement of π_p. For that reason, we may wish to call it the *unramified*, or *tame level 1*, eigenvariety for $\mathrm{U}(m)$. Note that X is not empty since it contains the point x_0 corresponding to the trivial representation π of $\mathrm{U}(m)$ with its unique refinement at p (see Ex. 6.4.9 (ii)).

By an argument similar than the one given in the proof of Prop. 7.5.13, there exists a point $x_1 \in Z$ arbitrary close to x_0 that corresponds to an automorphic representation π_1 (together with a refinement $\mathcal{R}_1$) such that:

(i) $(\pi_1)_\infty$ is regular,

(ii) the eigenvalues $\lambda_1, \ldots, \lambda_m$ of the Langlands conjugacy class of $(\pi_1)_p$ are distinct, and no quotient of two of them is equal to p,

(iii) if $I, J \subset \{1, \ldots, m\}$ are such that $|I| = |J|$, then $\prod_{i \in I} \lambda_i = \prod_{j \in J} \lambda_j \Rightarrow I = J$.

Condition (ii) ensures that $(\pi_1)_p$ is a full irreducible unramified principal series, hence that all its refinements are accessible (cf. Prop. 6.4.8).

We now use the refined family T of Galois representations on X constructed in this section. The representation $\bar{\rho}_{x_1}$ corresponding to π_1 has distinct Frobenius eigenvalues at v and distinct Hodge-Tate weights by (i). In particular, we may write $\bar{\rho}_{x_1}$ as a sum $\bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_r$ of non-isomorphic irreducible representations. As explained in § 4.4.3, there is a partition $\{1, \ldots, m\} = W_1 \coprod \ldots \coprod W_r$, such that $|W_i| = d_i$ for $i = 1, \ldots, r$, defined by setting W_i equal to the set of j's such that $\kappa_j(x_1)$ is a weight of $\bar{\rho}_j$. Moreover, as explained there, to a refinement $\mathcal{F}$ (that is, an ordering of the crystalline Frobenius eigenvalues) of $\bar{\rho}_{x_1}$ is attached a second partition $\{1, \ldots, m\} = R_1(\mathcal{F}) \coprod \ldots \coprod R_r(\mathcal{F})$

with $|R_i(\mathcal{F})| = d_i$, and all partitions of this type are attached to some refinement $\mathcal{F}$. It is a simple combinatorial task, tackled in the next lemma, to see that there is always one (and in general, many) partition (R_i) of this type which is "orthogonal" to the partition (W_i), in the sense that

$$\forall \mathcal{P} \subset \{1, \ldots, r\},\ 0 < |\mathcal{P}| < r,\ \coprod_{i \in \mathcal{P}} W_i \neq \coprod_{i \in \mathcal{P}} R_i.$$

We choose such a partition, and a refinement $\mathcal{F}$ of $\bar{\rho}_{x_1}$ defining that partition.

To $\mathcal{F}$ corresponds a refinement $\mathcal{R}_2$ of $(\pi_1)_p$, necessarily accessible. We thus may define a point x_2 of $Z \subset X$ corresponding to $(\pi_1, \mathcal{R}_2)$. Note that $\mathcal{F}$ is regular by property (iii) above of x_1. By Thm 4.5.1, and the properties of $\mathcal{R}_2$, the family of Galois representation T restricted to D_v is generically (absolutely) irreducible near x_2. Hence for any integer C, there is a point $x_3 \in Z_C$, such that $(\bar{\rho}_{x_3})_{|D_v}$ is crystalline absolutely irreducible. So is $(\bar{\rho}_{x_3})_{|D_{\bar{v}}}$, since it is the dual of the preceding representation.

It is clear that the automorphic representation π_3 corresponding to x_3 satisfies all assertions of the theorem. □

Remark 7.7.2. — The trivial representation of $\mathrm{U}(m)$ has a unique refinement, and it turns out that this refinement does not allow us to conclude using Theorem 4.5.1 that the deformation it defines of the trivial representation is generically irreducible restricted to D_v (and likely it is not for $m \geq 3$). Indeed, as explained in § 4.4.3, since the trivial representation is ordinary in the sense of *loc. cit.*, this unique refinement is characterized by a permutation σ of $\{1, \ldots, m\}$. Actually for the trivial representation we have $\sigma(1) = m, \ldots, \sigma(m) = 1$ and we see that, for $m \geq 3$, σ is not transitive, or in the language of *loc. cit.* the refinement is not *anti-ordinary* (it is not ordinary either). That is why we had to process in two steps in the proof above.

We now prove the combinatorial lemma needed in the above proof.

Lemma 7.7.3. — *For every partition $\{1, \ldots, m\} = W_1 \coprod \ldots \coprod W_r$, with $|W_i| = d_i$ for all i, there exists a partition $\{1, \ldots, m\} = R_1 \coprod \ldots \coprod R_r$, with $|R_i| = d_i$ for all i such that*

$$\forall \mathcal{P} \subset \{1, \ldots, r\},\ 0 < |\mathcal{P}| < r,\ \coprod_{i \in \mathcal{P}} W_i \neq \coprod_{i \in \mathcal{P}} R_i.$$

Proof. — Pick up an element t_i in each W_i. Choose a transitive permutation σ of $\{1, \ldots, r\}$. Put $t_{\sigma(i)}$ in R_i. Complete the construction of R_i as you like. Then for all $\mathcal{P} \subset \{1, \ldots, d\}$, $\coprod_{i \in \mathcal{P}} W_i$ contains t_i for $i \in \mathcal{P}$ and no other t_j, while $\coprod_{i \in \mathcal{P}} R_i$ contains the t_i for $i \in \sigma(\mathcal{P})$ and no other t_j. Hence if those two unions are equal, $\sigma(\mathcal{P}) = \mathcal{P}$. □

Remark 7.7.4. — The proof above may be adapted to prove the existence of partitions (R_i) satisfying further properties. For example, we shall need in a remark of chapter

8 to deal with a case where $d_1 = d_r = 1$, $W_r = \{k\}$, $W_1 = \{k+1\}$, with $1 \leq k < k+1 \leq m$; in this case we want a partition (R_i), satisfying the above properties and moreover $R_1 = \{1\}$ and $R_r = \{m\}$. It is certainly clear for the reader how the above proof has to be adapted to prove the existence of such R_i.

7.8. Appendix: p-adic families of Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ with $l \neq p$

7.8.1. Some preliminary lemmas on nilpotent matrices. — Let k be a field and $n \in M_d(k)$ a nilpotent matrix. By Jordan's normal form theorem, there exists a unique unordered partition of $\{1, \ldots, d\}$

$$\underline{t}(n) := (t_1 \geq t_2 \geq \cdots), \ \ t_i \in \mathbb{N}, \ \sum_i t_i = d,$$

such that n is conjugate in $M_d(k)$ to the direct sum of Jordan's blocks (26)

$$J_{t_1} \oplus J_{t_2} \oplus \cdots J_{t_s},$$

where s is the smallest integer such that $t_{s+1} = 0$. If $k \longrightarrow k'$ is a field embedding, then $t(n \otimes_k k') = t(n)$.

Recall that the *dominance ordering* on the set of decreasing sequences $\underline{t} = (t_1 \geq t_2 \geq \cdots)$ of integers is the partial ordering

$$\underline{t} \prec \underline{t}' \ \Leftrightarrow \ \forall i \geq 1, \ t_1 + \cdots + t_i \leq t'_1 + \cdots + t'_i.$$

We refer to [**82**, §I] for its basic properties.

Proposition 7.8.1 **(Gerstenhaber).** — *Let $n, n' \in M_d(k)$ be two nilpotent matrices. Then the following assertions are equivalent:*

(i) *n is in the Zariski-closure of the conjugacy class of n' in $M_d(\bar{k})$,*

(ii) *For all $i \geq 1$, $\mathrm{rank}\, n^i \leq \mathrm{rank}\, n'^i$,*

(iii) *$t(n) \prec t(n')$.*

Proof. — The equivalence between (i) and (ii) is [**56**, Thm. 1.7]. Assertion (ii) is equivalent to ask that for each i, $\dim(\ker(n^i)) \geq \dim(\ker(n'^i))$, which is another way to say that $t^*(n) \succ t^*(n')$. Here, $\underline{t}^*$ is the conjugate partition of $\underline{t}$, and the result follows as $\underline{t} \prec \underline{t}' \Leftrightarrow \underline{t}^* \succ \underline{t}'^*$ by [**82**, §1.11]. □

(26) For us, the Jordan block $J_d \in M_d(A)$ for any commutative ring A is the matrix of the endomorphism n of A^d defined by $n(e_1) = 0$ and $n(e_i) = e_{i-1}$ if $i > 1$.

Definition 7.8.2. — If n, n' are any two nilpotent matrices [27], we write $n \prec n'$ (resp. $n \sim n'$) if $t(n) \prec t(n')$ (resp. $t(n) = t(n')$). If $n, n' \in M_d(k)$, $n \prec n'$ if, and only if, n is in the Zariski-closure of the conjugacy class of $n' \in M_d(\bar{k})$.

Corollary 7.8.3. — *Let V be a finite dimensional k-vector space and $n \in \mathrm{End}_k(V)$ a nilpotent element. If $U \subset V$ is a $k[n]$-submodule and n' is the endomorphism induced by n on $U \oplus V/U$, then $n' \prec n$.*

Proof. — It is clear on condition (ii) of Prop. 7.8.1. □

Let us collect now some useful results about nilpotent matrices with coefficients in a ring. Let A be a commutative ring and $n \in M_d(A)$ a nilpotent matrix. We will say that n *admits a Jordan normal form over* A if n is $\mathrm{GL}_d(A)$-conjugate in $M_d(A)$ to a direct sum of Jordan blocks $J_{t_1} \oplus \cdots J_{t_s}$ for some unordered partition $(t_1 \geq t_2 \geq \cdots)$ of $\{1, \dots, d\}$ as above. Again, we see by reducing modulo any maximal ideal of m that if such a Jordan normal form exists, then the associated partition is unique. The following proposition is probably well known.

Proposition 7.8.4. — *Let A be a local ring and $n \in M_d(A)$ a nilpotent matrix. The following properties are equivalent:*

(i) *n admits a Jordan normal form over A,*
(ii) *for some faithfully flat commutative A-algebra B, the image of n in $M_d(B)$ admits a Jordan normal form over B,*
(iii) *for each integer $i \geq 1$, the submodule $n^i(A^d) \subset A^d$ is free over A and direct summand.*

Proof. — It is clear that (i) implies (ii), and also that (i) implies (iii) even if we do not assume A to be local. Note that for any element $u \in \mathrm{End}_A(A^d)$ and any faithfully flat A-algebra B, then

$$Im(u) \otimes_A B \xrightarrow{\sim} Im(u \otimes_A B),$$

and the latter is projective and direct summand in B^n as a B-module if, and only if, $\mathrm{Im}\,(u) \subset A^n$ has those properties as A-module. As a consequence, (ii) implies that $n^i(A^d)$ are projective and direct summand A-modules, hence free as A is local, hence (ii) implies (iii).

It only remains to show that (iii) implies (i), for which we argue as in the classical proof of Jordan's theorem. For $i \geq 0$ let $N_i := \mathrm{Im}\,(n^i) \in A^d$. We construct by descending induction some $A[n]$-submodules F_{i+1} and Q_i of A^d for $i = d-1, \dots, 0$, such that

- $F_d = 0$,

[27] It is not necessary to ask, in this definition, that they have the same coefficient field (or even the same size).

- F_{i+1} and Q_i are free and direct summand as A-modules,
- $n_{|Q_i}$ has a Jordan normal form over A with blocks of size $i+1$ (if any),
- $F_i = F_{i+1} \oplus Q_i$ and $n^i(F_{i+1}) \oplus n^i(Q_i) = N_i$.

Assume that F_j and Q_j are constructed for $j > i$, we have to define Q_i. Note that F_{i+1} is free and direct summand as A-module, and that $n_{|F_{i+1}}$ admits a Jordan normal form, so $n^i(F_{i+1})$ and

$$K_i := \operatorname{Ker} n \cap n^i(F_{i+1})$$

are free and direct summand as an A-module as well. In particular, K_i is a direct summand of $\operatorname{Ker} n \cap N_i$, by Lemma 7.8.7 (i) below.

As A is local, we may then find a finite free A-module $Q'_i \subset \operatorname{Ker} n \cap N_i$ which is a complement to K_i, it satisfies:

$$K_i \oplus Q'_i = \operatorname{Ker} n \cap N_i, \quad Q'_i \cap n^i(F_{i+1}) = 0. \tag{86}$$

We claim that

$$n^i(F_{i+1}) \oplus Q'_i = N_i. \tag{87}$$

Indeed, $n(N_i) = N_{i+1} = n^{i+1}(F_{i+1})$ implies that $N_i = n^i(F_{i+1}) + \operatorname{Ker} n \cap N_i$, which proves the claim by (86). Note also that

$$Q'_i \cap F_{i+1} = 0. \tag{88}$$

Indeed, the Jordan blocks of Q_j for $j > i$ have size $> i+1$, thus $\operatorname{Ker} n \cap F_{i+1} \subset n^i(F_{i+1})$. As $n(Q'_i) = 0$, we get that $Q'_i \cap F_{i+1} \subset Q'_i \cap n^i(F_{i+1}) = 0$ by (86), and we are done.

We can now conclude the proof. If $Q'_i = 0$, then we set $Q_i = 0$ and we are done. If else, we may choose $v_1, \dots, v_r$ in A^d such that $n^i(v_1), \dots, n^i(v_r)$ is an A-basis of Q'_i. Set

$$Q_i := A[n]v_1 + \cdots + A[n]v_r.$$

Note that $n^{i+1}(Q_i) = n(Q'_i) = 0$. We check at once by applying n several times that

— the $n^s(v_j)$ with $0 \leq s \leq i$ and $1 \leq j \leq r$ are an A-basis of Q_i (so in particular, $n_{|Q_i}$ admits a Jordan normal form $J_{i+1} \oplus J_{i+1} \oplus \cdots \oplus J_{i+1}$ (r times)),

— $Q_i \cap F_{i+1} = 0$ (note that $Q'_i \cap F_{i+1} = 0$ by (88)).

and we are done by (87) if we set $F_i := F_{i+1} \oplus Q_i$. □

The equivalence (i) ⇔ (ii) of Prop. 7.8.4 shows that the property of admitting a Jordan normal form (say over a local ring [28]) is invariant under faithfully flat base change. When we deal with deformation theory, the following other kind of descent is useful.

[28] In the general case, the well-behaved definition for "admitting a Jordan normal form over A" is certainly to ask that the $n^i(A^d)$ be projective and direct summand.

Proposition 7.8.5. — *Assume that $A \longrightarrow A'$ is a local homomorphism between artinian local rings inducing an isomorphism on the residue fields, and let $n \in M_d(A)$ be a nilpotent matrix. Then n admits a Jordan normal form over A if, and only if, its image in $M_d(A')$ admits a Jordan normal form over A'.*

Proof. — Assume that the image of n in $M_d(A')$ has a Jordan normal form (the other implication is obvious). For $i \geq 1$, let $N_i := Im(n^i) \subset A^d$ and $N_i' = Im((n \otimes_A A')^i) \subset A'^d$. As $A \subset A'$, we have

$$N_i \subset A'.N_i = N_i'.$$

Recall that A and A' have the same residue field $k := A/m$, and let $\bar{n} \in M_d(k)$ be the image of n. We have two natural surjections with the same image

$$N_i \otimes_A k \longrightarrow \mathrm{Im}\,(\bar{n}^i), \quad N_i' \otimes_{A'} k \xrightarrow{\sim} \mathrm{Im}\,(\bar{n}^i), \tag{89}$$

where the second map is an isomorphism as $N_i' \subset A'^d$ is a direct summand. Let $\bar{v_1}, \ldots, \bar{v_r}$ a k-basis of $\bar{n}^i(k^d)$, and $v_1, \ldots, v_r \in N_i$ some liftings of the $\bar{v_j}$. Set

$$P_i := \sum_{j=1}^{r} A v_j \subset N_i.$$

(If $\bar{n}^i = 0$ then we set $P_i = 0$). This is a (free) direct summand of A^d by Lemma 7.8.6 below. Moreover, the v_j generate N_i' over A' by (89) and Nakayama's lemma, hence $P_i A' = N_i A' = N_i'$. By Lemma 7.8.7 (ii) below, this implies that $P_i = N_i$, thus N_i is free and direct summand, and we are done by Prop. 7.8.4. □

Lemma 7.8.6. — *Let A be a local ring with residue field k. If some elements $v_1, \ldots, v_p$ in A^d have k-independent images in k^d, then they are A-independent and*

$$\bigoplus_{i=1}^{p} A v_i \subset A^d$$

is a direct summand.

Proof. — Let $M \in M_{d,p}(A)$ be the matrix defining the v_i in the canonical basis. By assumption, some $p \times p$-minor of $\bar{M} \in M_{d,p}(k)$ is nonzero, hence the same $p \times p$-minor of M is in A^*, and the v_i are A-independent. We conclude by completing the $\bar{v_i}$ in a basis of k^d and by Nakayama's lemma. □

Lemma 7.8.7. — *Let A be a commutative ring, $P \subset N \subset M$ an inclusion of A-modules such that P and M are projective, and P is a direct summand of M.*

(i) *P is a direct summand of N.*

(ii) *Assume that $A \longrightarrow B$ is an injective ring homomorphism, then $M \longrightarrow M \otimes_A B$ is injective. If $B \,.\, P = B \,.\, N$ in $M \otimes_A B$, then $P = N$.*

Proof. — We can write $M = P \oplus P'$ for some A-module $P' \subset M$. It is immediate to check that $N = P \oplus (P' \cap N)$, which proves (i). To check (ii), we may assume that M is a free A-module, and the first assertion is then obvious. Assuming that $B \,.\, P = B \,.\, N$, we have to show by (i) that $P' \cap N = 0$. But

$$(P' \cap N) \subset B \,.\, (P' \cap N) \subset B \,.\, P' \cap B \,.\, N = B \,.\, P' \cap B \,.\, P = 0,$$

which concludes the proof. □

When a nilpotent element in $M_d(A)$ (or even in a GMA over A) does not necessarily admit a Jordan normal form there are still some inequalities between the generic and residual partitions that are satisfied.

Proposition 7.8.8. — *Let A be a commutative reduced local ring whose total fraction ring is a finite product of fields*[29] $K = \prod_s K_s$, *and let k be its residue field. Let $R \subset M_d(K)$ be a standard GMA of type $(d_1, \dots, d_r)$ (see Example 1.3.4). Assume that the natural surjective map*

$$R \longrightarrow \prod_i M_{d_i}(k)$$

is a ring homomorphism[30]. *Let $n \in R$ be a nilpotent element that we write $n = (n_s) \in (M_d(K_s))$, and let $\bar{n} \in \prod_i M_{d_i}(k) \subset M_d(k)$ be its projection under the map above. Then $\bar{n} \prec n_s$ for each s.*

Proof. — Note that $\bar{n} \in M_d(k)$ is nilpotent as the map of the statement is a ring homomorphism, hence the statement makes sense.

By Prop. 7.8.1, we have to show that for each $i \geq 1$ and for each s,

$$\dim_{K_s}(n^i(K_s^d)) \geq \dim_k(\bar{n}^i(k^d)). \tag{90}$$

By replacing n by n^i we may assume that $i = 1$. Let us write $A^d = \oplus_{i=1}^r V_i$ according to the standard basis, $V_i = 0 \times A^{d_i} \times 0$. For each $i = 1, \dots, r$, let

$$\bar{n}(v_{i,1}), \dots, \bar{n}(v_{i,t_i})$$

be a k-basis of $\bar{n}(k^{d_i})$, $t_i < d_i$ (choose no v_i if $\underline{t}(\bar{n})_i = 0$). Let $w_{i,j} \in V_i$ be any lifting of $v_{i,j}$. To prove (90), it suffices to show that the elements

$$n(w_{i,j}) \in K^d, \ i = 1 \dots r, \ j = 1 \cdots t_i,$$

are K-independent. It suffices to check that for each i, if $p_i : K^d \to K^{d_i}$ denotes the canonical K-linear projection on KV_i, the elements

$$p_i(n(w_{i,j})) \in A^{d_i}, \ j = 1 \dots t_i,$$

[29] See Prop. 1.3.11 for a discussion of this assumption.

[30] It is the same to ask that the Cayley-Hamilton algebra (R, T) is residually multiplicity free, as explained in Example 1.4.2.

are A-independent. By construction these elements reduce mod m to the elements $\bar{n}(v_{i,1}), \dots, \bar{n}(v_{i,t_i}) \in k^{d_i}$ which are k-independent, hence we conclude by Lemma 7.8.6. □

Proposition 7.8.9. — *Let A be as in the statement of Prop. 7.8.8, and let $n \in M_d(A)$ be a nilpotent element.*

(i) *for each s, $n_s \succ \bar{n}$.*

(ii) *if* (i) *is an equality for each s, then n admits a Jordan normal form over A.*

Proof. — Assertion (i) follows from Prop. 7.8.8 in the special case when $R = M_d(A)$. Let us check (ii). We have to show that $N_i := n^i(A^d)$ is free and direct summand as A-module. By assumption,

$$\forall s, \ \dim_{K_s}(N_i \otimes_A K_s) = \dim_k(\bar{n}^i(k^d)) =: d_i, \tag{91}$$

and we also have a natural surjection

$$N_i \otimes_A k \longrightarrow \bar{n}^i(k^d). \tag{92}$$

By Nakayama's lemma and (92), N_i is generated over A by d_i elements, and those elements are necessarily K-independent by (91), thus N_i is free of rank d_i over A and the map in (92) is an isomorphism. We conclude by Lemma 7.8.6. □

Lemma 7.8.10. — *Let A be a noetherian commutative domain, K its fraction field. Let $n \in M_d(K)$ be a nilpotent matrix. There exists a nonzero $f \in A$ such that $n \in M_d(A_f)$ and such that for each $x \in D(f) \subset$ Spec(A), if n_x denotes the image of $n \in M_d(A_x/xA_x)$, then we have $n \sim n_x$.*

Proof. — We may assume that $n \in M_d(A)$. For $i = 1 \dots d$, let $M_i = n^i(A^d)$. As $M_i \otimes_A K$ is free and direct summand in K^d, we may assume, by replacing A by some A_f for a nonzero $f \in A$ if necessary, that all the M_i are free and direct summand in A^d. But then for each $x \in \mathrm{Spec}(A)$, $\mathrm{rk}\, n_x^i = \dim_K M_i \otimes_A K$, and we are done by Lemma 7.8.1. □

7.8.2. Preliminaries on general families of pseudocharacters. — Even in the "specific" context of the pseudocharacter T on the eigenvarieties X introduced in § 7.5.2, there is no reason to expect that T should be the trace of a representation of $G_{E,S}$ on a locally free $\mathcal{O}_X$-module of rank m, or on a torsion free $\mathcal{O}_X$-module of generic rank m, and this even locally. However, this holds for general reasons on an étale covering of the Zariski-open subspace of X consisting of the x such that $\bar{\rho}_x$ is absolutely irreducible by [**36**, Cor. 7.2.6]. Moreover, recall that in the first part of this book, we studied that question in detail locally around any point x such that $\bar{\rho}_x$ is multiplicity free.

We collect here some general facts that might be used to circumvent this problem.

Lemma 7.8.11. — *Let $T : \Gamma \longrightarrow \mathcal{O}(X)$ be any continuous m-dimensional pseudocharacter of a topological group Γ on a reduced rigid space X over $\mathbb{Q}_p$. Let $\Omega \subset X$ be an open affinoid.*

(i) *There is a normal affinoid Y, a finite dominant*[31] *map $g : Y \longrightarrow \Omega$, and a finite type torsion free $\mathcal{O}(Y)$-module $M(Y)$ of generic ranks m equipped with a continuous representation*

$$\rho_Y : \Gamma \longrightarrow \mathrm{GL}_{\mathcal{O}(Y)}(M(Y))$$

whose generic trace is T.

Moreover, ρ_Y is generically semisimple and the sum of absolutely irreducible representations. For y in a dense Zariski-open subset $Y' \subset Y$, $M(Y)_y$ is free of rank m over $\mathcal{O}_y$, and $M(Y)_y \otimes \overline{k(y)}$ is semisimple, and isomorphic to $\bar{\rho}_{g(y)}$.

(ii) *There is a blow-up $g' : \mathcal{Y} \to Y$ of a closed subset of $Y \backslash Y'$ such that the strict transform $\mathcal{M}_{\mathcal{Y}}$ of the coherent sheaf on Y associated to $M(Y)$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module of rank m. That sheaf $\mathcal{M}_{\mathcal{Y}}$ is equipped with a continuous $\mathcal{O}_{\mathcal{Y}}$-representation of Γ with trace $(g'g)^\sharp \circ T$, and for all $y \in \mathcal{Y}$, $(\mathcal{M}_{\mathcal{Y},y} \otimes \overline{k(y)})^{\mathrm{ss}}$ is isomorphic to $\bar{\rho}_{g'g(y)}$.*

Proof. — Let us prove (i). By normalizing Ω if necessary, we may assume that $\Omega = X$ is irreducible. By Taylor's theorem [**117**, Thm. 1.2], there exists a finite extension K' of $K := \mathrm{Frac}(\mathcal{O}(X))$ such that $T : \Gamma \longrightarrow K'$ is the trace of a direct sum of absolutely simple representations of $\Gamma \to \mathrm{GL}_m(K')$. If we define $\mathcal{O}(Y)$ as the normalization of X in K', the existence of a finite type, continuous, Γ-stable $\mathcal{O}(Y)$-submodule $M(Y) \in K'^m$ is [**8**, Lemme 7.1 (i), (v)].

It satisfies the "Moreover, ..." assertion by definition. By a classical result of Burnside and the generic flatness theorem, this latter fact implies that for y in a Zariski-open subset of Y, $M(Y)_y = M(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}_y$ is free of rank m over $\mathcal{O}_y$ and that $M(Y)_y \otimes k(y)$ is a semisimple $k(y)[\Gamma]$-module. In particular, for those y we have $M(Y)_y \otimes_{\mathcal{O}_y} \overline{k(y)} \simeq \bar{\rho}_{g(y)}$ as they both have the same trace, which concludes the proof of (i).

Part (ii) follows then from (i) and Lemma 3.4.2 (either reduce Y' in (i) or note that the explicit blow-up of Y used in that lemma is the blow-up of the m-th Fitting ideal of $M(Y)$, whose associated closed subset does not meet Y'.) □

7.8.3. Grothendieck's l-adic monodromy theorem in families. — From now and until the end of this section, F denotes a finite extension of $\mathbb{Q}_l$ with $l \neq p$, W_F its Weil group, $I_F \subset W_F$ its inertia group and $\varphi \in W$ a geometric Frobenius. We fix

[31] Precisely, it is surjective, and the image of any irreducible component of Y is an irreducible component of Ω.

also a nonzero continuous group homomorphism $t_p : I_F \to \mathbb{Q}_p$. The following lemma is well known.

***Lemma** 7.8.12.* — *Let B be any $\mathbb{Q}$-algebra and $\rho : \mathrm{W}_F \longrightarrow B^*$ any group homomorphism. Assume that there exists some nilpotent element $N \in B$ such that ρ coincides with $g \mapsto \exp(t_p(g)N)$ on some open subgroup of I_F. Then N is the unique element with this property. Moreover, the map*

$$r : \mathrm{W}_F \longrightarrow B^*, \ \ \varphi^n g \mapsto \rho(\varphi^n g)\exp(-t_p(g)N), \ \forall\, n \in \mathbb{Z}, \ g \in I_F,$$

is a group homomorphism, trivial on some open subgroup of W_F.

***Definition** 7.8.13.* — Let ρ be as above. If N exists, we say that (r, N) is the Weil-Deligne representation associated to ρ. By Lemma 7.8.12, it is unique and determines ρ entirely.

We are interested in the study of p-adic analytic families of representations of $\mathrm{Gal}(\overline{F}/F)$, and actually a little more generally of W_F. Let us give a version of Grothendieck's l-adic monodromy theorem adapted to this setting. We let A be an affinoid algebra over $\mathbb{Q}_p$ and B an A-algebra of finite type equipped with its canonical A-Banach algebra topology.

***Lemma** 7.8.14.* — *Let $\rho : \mathrm{W}_F \longrightarrow B^*$ be a continuous morphism, then ρ admits a Weil-Deligne representation.*

Proof. — We fix a submultiplicative norm on B and let $B^0 \subset B$ be its open unit ball. Then $\{1 + p^n B^0, n \geq 1\}$ is a basis on open neighborhoods of $1 \in B^*$ whose successive quotients are discrete and killed by p. As a consequence, the restriction ρ' of ρ to the the wild inertia subgroup of I_F (which is pro-l) has a finite image, as its Kernel contains the open subgroup ${\rho'}^{-1}(1 + pB^0)$. Let F'/F be a finite extension such that $\rho_{|I_{F'}}$ is tame and pro-p, so that it factors through a continuous morphism

$$t_p(I_{F'}) \longrightarrow B^*.$$

The derivative at $0 \in t_p(I_{F'}) \subset \mathbb{Q}_p$ of the map above gives an canonical element $N \in B$. As $\varphi N \varphi^{-1} = \lambda N$ for λ a nonzero power of l, N is nilpotent by Lemma 7.8.15, and we are done. □

***Lemma** 7.8.15.* — *Let A be a noetherian $\mathbb{Q}$-algebra and B a (non necessarily commutative) A-algebra of finite type as an A-module. If $x \in B$ is B^*-conjugate to λx for some integer $\lambda \geq 2$, then x is nilpotent.*

Proof. — Replacing x by its image in the regular representation $B \hookrightarrow \mathrm{End}_A(B)$, we may assume that $B = \mathrm{End}_A(M)$ for some finite type A-module M. When A is a field, the result is easy linear algebra. A little more generally, if $\mathrm{Supp}(M) = \{P\}$ is a closed point of

$$X := \mathrm{Spec}(A),$$

then M is of finite length, and using the B-stable filtration $\{P^n M, n \geq 1\}$ we are reduced to the previous case over the field A/P.

In the general case, we argue by noetherian induction on the closed subset $\mathrm{Supp}(M) \in X$. Let P be the generic point of an irreducible component of $\mathrm{Supp}(M)$, and let K be the kernel of the natural map $M \to M_P$, it is a B-stable submodule. By the previous case x acts nilpotently on M_P, and by notherian induction $x_{|K}$ is nilpotent since $P \notin \mathrm{Supp}(K)$, and we are done. □

7.8.4. p-adic families of W_F-representations. — Let us now fix a topological group G, a continuous homomorphism $\mathrm{W}_F \longrightarrow G$, a rigid analytic space X over $\mathbb{Q}_p$ and a continuous d-dimensional pseudocharacter

$$T : G \longrightarrow \mathcal{O}(X).$$

As already explained in § 7.5.2, we have for any $x \in X$ two canonical semisimple G-representations

$$\bar{\rho}_x \text{ and } \rho_x^{\mathrm{gen}}$$

with respective traces $T \otimes k(x)$ and $T \otimes \overline{\mathcal{K}}_x$.

As $\bar{\rho}_x$ is continuous and defined over a finite extension of $k(x)$, its restriction to W_F has an associated Weil-Deligne representation. This holds also for ρ_x^{gen}. Indeed, let us choose Ω an open affinoid neighborhood of x and apply Lemma 7.8.11 (i) to this Ω. It gives us a continuous representation $\rho_Y : G \longrightarrow \mathrm{GL}_{\mathcal{O}(Y)}(M(Y))$, which admits a Weil-Deligne representation (r_Y, N_Y) by Lemma 7.8.14. If we choose an $\mathcal{O}(\Omega)$-morphism $\mathcal{O}(Y) \to \overline{\mathcal{K}}_x$, we can compose (r_Y, N_Y) with the ring homomorphism $\mathrm{End}_{\mathcal{O}(Y)}(M(Y)) \to M_d(\overline{\mathcal{K}}_x)$ to obtain a Weil-Deligne representation for ${\rho_x^{\mathrm{gen}}}_{|\mathrm{W}_F}$, and we are done.

Definition 7.8.16. — We call respectively $(\overline{r}_x, \overline{N}_x)$ and $(r_x^{\mathrm{gen}}, N_x^{\mathrm{gen}})$ the residual and generic Weil-Deligne representation of F attached to T at x.

Let $x \in X$. As the $\mathbb{Q}_p$-algebra $\mathcal{O}_x$ is local henselian and $k(x)$ is finite over $\mathbb{Q}_p$, there is a canonical embedding $k(x) \to \mathcal{O}_x$ inducing the identity after composition with $\mathcal{O}_x \to k(x)$. In particular, we can chose an embedding

$$\iota_x : \overline{k(x)} \longrightarrow \overline{\mathcal{K}}_x,$$

and try to compare the two Weil-Deligne representations $(\overline{r}_x \otimes_{\iota_x} \overline{\mathcal{K}}_x, \overline{N}_x \otimes_{\iota_x} \overline{\mathcal{K}}_x)$ and $(r_x^{\mathrm{gen}}, N_x^{\mathrm{gen}})$.

Lemma 7.8.17. — *$r_x^{\mathrm{gen}}{}_{|I_F}$ is isomorphic to $\overline{r}_{x|I_F} \otimes_{\iota_x} \overline{\mathcal{K}}_x$. Moreover, $T_{|I_F}$ is constant on the connected component of x in X.*

Proof. — The representation $r_x^{\mathrm{gen}}{}_{|I_F}$ has a finite image by construction, hence $r_x^{\mathrm{gen}}{}_{|I_F}$ is actually a semisimple I_F-representation and its trace is $\overline{k(x)}$-valued. But this trace coincides by definition with $\iota_x(T_x)$, which proves the first part of the lemma.

Let us show the second assertion. Let $H \subset I_F$ be an open subgroup such that

$$T(gh) = T(g) \in k(x),\ \forall g \in I_F.$$

Then we just showed that the equality $T(gh) = T(g)$ holds in $\mathcal{O}_x$, which implies that it holds on each irreducible component of X containing x, and actually on the whole connected component $X(x)$ of x in X by applying the same reasoning to all the points of $X(x)$. In particular,

$$k_0 := \mathbb{Q}_p(T(I_K)) \subset \mathcal{O}(X(x))$$

is a finite dimensional $\mathbb{Q}_p$-algebra. But $\mathrm{Spec}(\mathcal{O}(X(x)))$ is connected and reduced, as so is $X(x)$, hence k_0 is a field, which concludes the proof. □

Let (r, N) be a $M_d(k)$-valued Weil-Deligne representation, with k an algebraically closed field of characteristic 0. Then the representation $r_{|I_K}$ is semisimple and commutes with N, so each of its isotypic component is preserved by N. If τ is any (k-valued) finite dimensional irreducible representation of I_F, let us denote by N_τ the induced nilpotent element acting on $\mathrm{Hom}_{I_K}(\tau, k^d)$. The following definition is a mild extension of Definition 7.8.2, and was already studied in § 6.5 when $k = \mathbb{C}$.

Definition 7.8.18. — Let (ρ_1, N_1) and (ρ_2, N_2) be two Weil-Deligne representations as above. We will write $N_1 \prec_{I_F} N_2$ (resp. $N_1 \sim_{I_F} N_2$) if for each τ, $N_{1,\tau} \prec N_{2,\tau}$ (resp. $N_{1,\tau} \sim N_{2,\tau}$).

If both (ρ_1, N_1) and (ρ_2, N_2) are $M_d(k)$-valued, $N_{1,\tau} \prec N_{2,\tau}$ if, and only if, (ρ_1, N_1) is in the Zariski-closure of the conjugacy class of (ρ_2, N_2).

Of course, $N_1 \prec_{I_F} N_2$ implies that $N_1 \prec N_2$.

For $x \in X$, let us write

$$\overline{\mathcal{K}}_x = \prod_{s(x)} \overline{\mathcal{K}}_{s(x)}$$

where $s(x)$ runs the finite set of irreducible components of $\mathrm{Spec}(\mathcal{O}_x)$, *i.e.* the germs of irreducible components of X at x. We can write in the same fashion ρ_x^{gen} and $(r_x^{\mathrm{gen}}, N_x^{\mathrm{gen}})$ as a set of $\overline{\mathcal{K}}_{s(x)}$-representations $\rho_{s(x)}^{\mathrm{gen}}$ and $(r_{s(x)}^{\mathrm{gen}}, N_{s(x)}^{\mathrm{gen}})$.

Proposition 7.8.19. — *Let $x \in X$, $s(x)$ a germ of irreducible component at x, and W the*[32] *irreducible component of x in X containing $s(x)$.*

(i) *Let $y \in W$ and $s(y)$ a germ of irreducible component of X at y belonging to W. Then $N^{\mathrm{gen}}_{s(x)} \sim_{I_F} N^{\mathrm{gen}}_{s(y)}$.*

(ii) *For each open affinoid $\Omega \subset W$, there is a Zariski-dense and Zariski-open subset $\Omega' \subset \Omega$ such that $\overline{N}_y \sim_{I_F} N^{\mathrm{gen}}_{s(x)}$ for all $y \in \Omega'$.*

(iii) $\overline{N}_x \prec_{I_F} N^{\mathrm{gen}}_{s(x)}$.

Proof. — By normalizing X if necessary, we may assume that $X = W$ is normal and irreducible. In particular, $\mathcal{O}_x$ is a domain for each x hence we will not have to specify the s any more: $r^{\mathrm{gen}}_x = r^{\mathrm{gen}}_{s(x)}$. We may also assume that X is affinoid, and it is enough to show (ii) when $\Omega = X$.

Let Y be a normal affinoid, as well as $g : Y \longrightarrow X$, $M(Y)$, ρ_Y and Y', be given by Lemma 7.8.11 (i). By replacing Y by a connected component, we may assume that Y is irreducible. Note that for each $x \in X$ and $y \in Y$ with $g(y) = x$, we have

$$\rho^{\mathrm{gen}}_y \simeq \rho^{\mathrm{gen}}_x, \quad \bar{\rho}_x \simeq \bar{\rho}_y,$$

so we may assume that $Y = X$, and that T is the trace of a continuous representation

$$\rho : G \longrightarrow \mathrm{GL}_{\mathcal{O}(X)}(M)$$

on a finite type, torsion free, and generic rank d $\mathcal{O}(X)$-module M.

Let $K = \mathrm{Frac}(\mathcal{O}(X))$. For each $y \in X$, the natural map $\mathcal{O}(X) \to \overline{\mathcal{K}}_y$ extends to an embedding $\overline{K} \to \overline{\mathcal{K}}_y$. As the $\overline{K}[G]$-module $M \otimes_{\mathcal{O}(X)} \overline{K}$ is semisimple by Lemma 7.8.11 (i), we have $\rho \otimes_K \overline{\mathcal{K}}_y \simeq \rho^{\mathrm{gen}}_y$. By Lemma 7.8.14, ρ admits a Weil-Deligne representation (r, N), so by the uniqueness of the Weil-Deligne representation we have

$$(r, N) \otimes_K \overline{\mathcal{K}}_y \xrightarrow{\sim} (r^{\mathrm{gen}}_y, N^{\mathrm{gen}}_y), \ \forall y \in X, \tag{93}$$

which proves (i).

Let us show (ii). By replacing X by a finite etale covering coming from the base, we may assume that the irreducible representations of the finite group $r(I_F)$ are all defined over some local field $k_0 \subset \mathcal{O}(X)$, hence we can write the following finite

[32] This component is defined as follows. Let Ω be any open affinoid of X containing x. We have a natural map $\mathcal{O}(\Omega)_x \to \mathcal{O}_x$ from the Zariski-local ring at x to the analytic one, which is known to be injective, and both are reduced if $\mathcal{O}(\Omega)$ is, so we get an injective morphism

$$\mathrm{Frac}(\mathcal{O}(\Omega)_x) \hookrightarrow \mathrm{Frac}(\mathcal{O}_x).$$

The image of $\mathrm{Spec}(\overline{\mathcal{K}}_{s(x)})$ in $\mathrm{Spec}(\mathcal{O}(\Omega)_x)$ is the generic point of a (unique) irreducible component W_Ω of Ω containing x. The component W alluded above is then the unique irreducible component of X containing Ω. It does not depend on the choice of Ω. Indeed, if $\Omega' \subset \Omega$ is another open affinoid containing x, then $W_{\Omega'}$ is an irreducible component of $W_\Omega \cap \Omega'$, hence is Zariski-dense in W.

decomposition of $M_{|W_F}$

$$M = \bigoplus_{\tau} \tau \otimes_{k_0} M_\tau, \quad M_\tau := \mathrm{Hom}_{k_0[r(I_F)]}(\tau, M_\tau).$$

Let us choose a nonzero $f \in \mathcal{O}(X)$ such that each $(M_\tau)_f$ is a free $\mathcal{O}(X)_f$-module, and that $M \otimes k(y) \simeq k(y)^d$ is a semisimple $k(y)[G]$-representation for each y in X_f (use Burnside's theorem). In particular,

$$M \otimes_{\mathcal{O}(X)} k(y) \simeq \bar{\rho}_y, \quad \forall y \in X_f. \tag{94}$$

Applying Lemma 7.8.10 to $N_\tau \in \mathrm{End}_{\mathcal{O}(X)_f}((M_\tau)_f)$, we may assume by changing f if necessary that

$$N_{\tau,y} \sim N_\tau \otimes K, \ \forall y \in X_f, \tag{95}$$

hence (ii) holds by (93), (94) and (95) if we take $\Omega' := X_f$.

It only remains to prove assertion (iii). We claim first that me may assume that the module M defined in the second paragraph above of the proof is free. Indeed, take $g' : \mathcal{Y} \to Y = X$ and $\mathcal{M}_{\mathcal{Y}}$ as in Lemma 7.8.11 (ii). If V an affinoid subdomain of $\mathcal{Y}$, then $V \cap Y'$ is Zariski-dense in V, so $\mathcal{M}_{\mathcal{Y}}(V)_y$ is a direct sum of absolutely irreducible G-representations for each y in a Zariski-dense subset of V. Thus $\mathcal{M}_{\mathcal{Y}}(V) \otimes_{\mathcal{O}(V)} \mathrm{Frac}(\mathcal{O}(V))$ has the same property, which proves the claim by replacing X by $\mathcal{Y}$, and then by an affinoid subdomain as $\mathcal{Y} \to X$ is surjective. In particular, the Weil-Deligne representation (r, N) is now $M_d(\mathcal{O}(X))$-valued.

As M is free over $\mathcal{O}(X)$, we have

$$(M \otimes_{\mathcal{O}(X)} \overline{k(x)})^{\mathrm{G-ss}} \simeq \bar{\rho}_x, \ \forall x \in X,$$

so Lemma 7.8.3 (i) implies that it is enough to get (iii) to check that the image N_x of N in $\mathrm{End}_{k(x)}(M \otimes k(x))$ satisfies

$$N_x \prec_{I_F} N \otimes K.$$

As this is an assertion on the action of W_F, we may decompose M (again, up to enlarging the base field if necessary) as a sum of its isotypic components M_τ as above, and the result follows then from Prop. 7.8.9 (i) applied to the Zariski-local ring A of X at x and to N acting on the free A-module $M_\tau \otimes_{\mathcal{O}(X)} A$. □

Remark 7.8.20. — Let $x \in X$. For each $s \in \mathrm{Spec}(\mathcal{O}_x)$, say with residue field $k(s)$, there exists a unique (isomorphism class of) semisimple representation

$$\rho_s : G \longrightarrow \mathrm{GL}_d(\overline{k(s)})$$

whose trace is the composite $T(s) : G \to \mathcal{O}_x \to k(s)$. The argument we gave for r_x^{gen} shows that ρ_s admits a Weil-Deligne representation (r_s, N_s). As an exercise, the reader can check using a slight variant of the proof of Prop. 7.8.19 that

$$s \in \overline{\{s'\}} \quad \Rightarrow \quad N_s \prec_{I_F} N_{s'}.$$

CHAPTER 8

THE SIGN CONJECTURE

8.1. Statement of the theorem

We use the notations of Section 5, especially of § 5.2.1: E is a quadratic imaginary field, p a prime that is split in E,

$$\rho : G_E \longrightarrow \mathrm{GL}_n(L)$$

an n-dimensional geometric semisimple representation of G_E with coefficients in a finite extension $L/\mathbb{Q}_p$, satisfying

$$\rho^{\perp} \simeq \rho(-1).$$

We fix also embeddings ι_∞ and ι_p as in §6.8.1. We denote by v and $\bar{v}$ the two places of E above p as in *loc. cit.* We make the following assumptions on ρ:

(1) The dimension n is not divisible by 4.

(2) There is a cuspidal tempered automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$, satisfying properties (i), (ii) and (iii) of § 6.9.1 and such that for every split place x of E the Weil-Deligne representation of $\iota_p\iota_\infty^{-1}\pi_x|\det|_x^{1/2}$ and the one attached to $\rho_{|E_x}$ are isomorphic up to Frobenius semi-simplification.

(3) The representation $\rho_{|E_v}$ is crystalline and the characteristic polynomial of its crystalline Frobenius is the same as the one of $\iota_p\iota_\infty^{-1}(L(\pi_v|\det|^{1/2}))$.

Note that for the sake of generality, and because irreducibility may be hard to check in applications, we do not assume that ρ is irreducible.

Example 8.1.1. — There are many known examples of such ρ, in any dimension n. Start with a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying the hypothesis (i), (ii), (iii) of § 6.9.1. Assume moreover that π is square-integrable at some finite place. [1]

[1] When $n = 2$, it is well known that this assumption is not necessary. When $n = 3$, it is actually possible to remove it, but we have to assume that p is outside a density zero set of primes depending on π. Indeed, if π satisfies (i) and (ii) of 6.9.1, $\pi\mu$ descends by Rogawski's base change to the quasi-split unitary group $\mathrm{U}(2,1)(\mathbb{Q})$ attached to E (or even to the form that is compact at infinity), there is

Then by the main result of [**62**], π is tempered and there is a Galois representation ρ satisfying (1), (2) and (3). In this case, by [**118**] we also know that ρ is irreducible. It should be possible, in a near future, to remove the square-integrability hypothesis using results of [**60**], but then the irreducibility of ρ might not be known.

Recall that we introduced previously assumptions Rep and AC (conjectures 6.8.1 and 6.9.9).

Theorem 8.1.2. — *Assume $AC(\pi)$ and* $\mathrm{Rep}(n+2)$*. Then the sign conjecture holds for ρ: namely, if $\varepsilon(\rho,0)=-1$, then* $\dim_L H^1_f(E,\rho)\geq 1$

Since hypotheses AC(ρ) for a character ρ and Rep(3) are known (see Remarks 6.8.2 (vi) and 6.9.10 (ii)), we deduce:

Corollary 8.1.3. — *If $n=1$, for a ρ as above, the sign conjecture holds.*

This result was the main result of [**8**], where it was proved by similar methods, and can also be deduced of earlier results of Rubin (see the introduction of *loc. cit.*).

For $n=2$ we can prove a result avoiding the hypotheses at nonsplit primes.

Corollary 8.1.4. — *Let f be a modular newform of even weight $k\geq 4$ and level $\Gamma_0(N)$ prime to p, and $\rho=\rho_f$ as in Example 5.2.2. Assume $AC(\pi_{f,E})$* [(2)] *and* $\mathrm{Rep}(4)$*. Then the sign conjecture holds for ρ_f, namely if $\varepsilon(\rho_f,0)=-1$, we have* $\dim_L H^1_f(\mathbb{Q},\rho_f)\geq 1$.

The reason why $k=2$ is (unfortunately) excluded will be apparent in the following proof.

Proof. — By Proposition 5.2.1, there is a quadratic imaginary field E where p and all the primes dividing N are split, and such that $\varepsilon(\rho_{f,E},0)=\varepsilon(\rho_f,0)$ and $H^1_f(\mathbb{Q},\rho_f)=H^1_f(E,\rho_{f,E})$. So the corollary follows from Theorem 8.1.2 if we verify hypotheses (1) to (3) for $\rho=\rho_{f,E}$. Assumption (1) is clear. For the automorphic representation π needed in (2) we simply take $\pi:=\pi_{f,E}$ the Langland's base change to E of the automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to f which is normalized so as to be selfdual: $\pi_f\simeq\pi_f^*$ and it has a trivial central character. It is clear that π satisfies (i) and (ii) of 6.9.1 since the L-parameter of π_∞ coincides with

$$z\mapsto \mathrm{diag}((z/\bar z)^{\frac{1-k}{2}},(z/\bar z)^{\frac{k-1}{2}})$$

---

a Galois representation that we may write $\rho\mu$ attached to $\pi\mu$ satisfying (2) by [**20**]. The temperedness of π together with the compatibility condition in (3) are then the main result of [**7**].

(2) The needed representation $\pi_{f,E}$ is actually defined in the proof. Precisely, $\pi_{f,E}$ is the base change to a quadratic imaginary field E as in Prop. 5.2.1 of the automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to f, where π_f is normalized so that $\pi_f^*\simeq\pi_f$.

on $\mathbb{C}^*$ and as $k \geq 4$ is even, and also (iii) since π is unramified at non-split places of E by construction of E. It is well known that assumption (3) holds since p does not divide N. □

Remark 8.1.5. — (i) When f is ordinary at p, this result, without its automorphic assumptions, was proved by Nekovar as a consequence of his *parity theorem* ([**89**], see also [**90**, Chap. 12]). A similar result was also proved later by Skinner-Urban in [**112**], using automorphic forms on the symplectic group GSp_4. Since the existence of Galois representations attached to such forms is known they do not assume any variant of hypothesis Rep(4) but they have stronger hypotheses on f, namely that p is an ordinary prime for f, and that $N = 1$.

(ii) It may be possible to remove the restriction $k > 2$ from this result (or, for that matter, the restriction on the weight in Thm 8.1.2) by actually deducing the $k = 2$ case from the result above and a deformation argument. We postpone this to a subsequent work.

Remark 8.1.6. — As we explained in section 5, it was not our policy in this book to assume the most general versions of Langlands and Arthur's conjectures on the discrete spectrum of unitary groups, but rather to formulate a minimal set of expected assumptions which we prove to be enough to imply the sign conjecture in a large number of cases. Indeed, the version of Theorem 8.1.2 that we state is actually the strongest that we can prove under our assumptions ($\mathrm{Rep}(n+2)$) and ($\mathrm{AC}(\pi)$). A reason for that restriction is that is not clear to us which part of those general conjectures will be proved first and in which form. This especially applies to the part concerning the Langland's parameterization for the local unitary group and the local-global compatibility of the base change from $\mathrm{U}(m)$ to GL_m at those primes (see Appendix A), of which our proof would need some properties (*e.g.* if we do not want to make assumption (iii) of § 6.9.1). Let us simply say that *at the end*, we expect that Arthur's general conjectures should imply the full case of the sign conjecture for the ρ attached to a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. We hope to go back to this extension in the future.

The following two subsections are devoted to the proof of 8.1.2.

8.2. The minimal eigenvariety X containing π^n

8.2.1. Definition of π^n and X. — From now till the end of this book, we set

$$m := n + 2.$$

Assume that $\varepsilon(\pi, 0) = -1$ and let π^n be the (endoscopic non-tempered) automorphic representation of $\mathrm{U}(m)$ given by $AC(\pi)$ (see chapter 6, § 6.9). Recall that the representation π^n depends on the choice of a Hecke character $\mu : \mathbb{A}_E^* \longrightarrow \mathbb{C}^*$ as in Definition 6.9.5. Recall that $\mu^\perp = \mu$, that $\mu = 1$ if m is even, and that μ does not descend to $U(1)$ when m is odd.

By § 6.9.1, for each prime l that does not split in E, π_l is either a non monodromic principal series or unramified, so that it makes sense to consider the *minimal eigenvariety* X containing π^n as in Example 7.5.1.[(3)] We will use in the sequel the same notations as in § 7.5. In particular, recall that S is the finite set of primes consisting of p and of the primes l such that π_l is ramified, and $L/\mathbb{Q}_p$ is a sufficiently large finite extension of $\mathbb{Q}_p$ on which π^n and X are defined. Assume also that L is large enough so that ρ is a sum of absolutely irreducible representations defined over L.

In order to associate a point of X to π^n we have to specify an accessible refinement of π_p^n. They are given by the following lemma. Recall that the place v fixes an isomorphism $U(m)(\mathbb{Q}_p) \xrightarrow{\sim}_v \mathrm{GL}_m(\mathbb{Q}_p)$.

Lemma 8.2.1. — (i) *The representation π_p^n is almost tempered (see Def. 6.4.11).*

(ii) *Its accessible refinements are the $n!\frac{(n+1)(n+2)}{2}$ orderings of the form*

$$\mu_w|.|^{-1/2}(p)(\dots,1,\dots,p^{-1},\dots),$$

where 1 precedes p^{-1}.

(iii) *The ordered set of the other eigenvalues in the dots above is any ordering of the eigenvalues of the Langlands conjugacy class $L(\pi_p|.|^{1/2})$. Each of those eigenvalues has complex norm $p^{-1/2}$, and in particular is different from $1, p^{-1}$.*

Proof. — By Remark 6.9.6, π_p^n is the unramified representations such that

$$L(\pi_p^n) = \mu_v|.|^{-1/2}(L(\pi_p|.|^{1/2}) \oplus 1 \oplus |.|),$$

and the parameter $L(\pi_p)$ is bounded as π_p is tempered by assumption. The lemma follows now from Prop. 6.4.10. □

Let us choose any such refinement $\mathcal{R}$ for the moment, which fixes an associated point $z = (\psi_{(\pi^n,\mathcal{R})}, \underline{k})$ on the eigenvariety X.

Remark 8.2.2. — Note that we did not make any assumption on the compatibility between ρ and $\pi|.|^{1/2}$ at the nonsplit primes. Actually, we can prove a version of it under the running assumptions. Indeed, as X is the minimal eigenvariety containing π^n, Prop. 7.5.7 and Prop. 7.5.8, ρ is unramified outside S, and has a trivial monodromy operator at the primes $l \in S_N$. In the same way, Lemma 7.5.12 shows that the Hodge-Tate weights of $\rho_{|E_v}$ correspond to the highest weight $\underline{k}$ of π_∞^n as in part (P4) of property (Rep(m)) of § 6.8.2. In particular, those Hodge-Tate weights are two-by-two distinct.

[(3)] The results here will apply whether we choose or not the variant with a fixed weight.

8.2.2. Normalization of the Galois representation on X. — As explained in § 7.5.2, by assumption Rep(m) we have a continuous pseudocharacter

$$T : G_{E,S} \longrightarrow \mathcal{O}(X),$$

such that $T^{\perp}(g) = \chi(g)^{m-1}T(g)$ for all $g \in G_{E,S}$. Recall that χ is the cyclotomic character. It will be convenient to twist it by a constant character as follows. The following lemma is immediate (see § 6.9.2).

Lemma 8.2.3. — *The Hecke character $\mu^{-1}|\cdot|^{\frac{m}{2}}$ has an integral weight $\delta := \frac{m}{2}$ if m is even, and $\delta := \frac{m-1}{2}$ if m is odd. In particular, it is an algebraic Hecke character.*

Enlarging a bit our base field $L/\mathbb{Q}_p$ if necessary, we may assume that $\mu^{-1}|\cdot|^{\frac{m}{2}}$ is defined over L via $\iota_p\iota_\infty^{-1}$. By class field theory, there is an associated [4] continuous character

$$\nu = \mu^{-1}|\cdot|^{\frac{m}{2}} \circ \mathrm{rec}^{-1} : \; G_{E,S} \longrightarrow L^*.$$

By Cebotarev theorem and Rem. 6.9.6, the evaluation of T at the point z is the trace of the representation

$$\rho\nu^{-1} \oplus \nu^{-1} \oplus \chi\nu^{-1}.$$

This leads us to define T', κ_i' and F_i' as follows:

- $T' := T \otimes \nu$, *i.e.* $T'(g) = T(g)\nu(g)\,\forall g \in G_{E,S}$,
- $\kappa_i' := \kappa_i - \delta$ for $i = 1, \dots, m$,
- $F_i' := F_i\iota_p\iota^{-1}(\nu_v(p))$ for $i = 1, \dots, m$.

Definition 8.2.4. — From now on, we shall use the letters T, κ_i and F_i to denote respectively T', κ_i' and F_i' above. With this choice of normalization we have

$$\bar{\rho}_z = 1 \oplus \chi \oplus \rho, \quad T^{\perp} = T(-1), \tag{96}$$

and the new $(X, T, \{\kappa_i\}, \{F_i\}, Z)$ is obviously still a refined family.

8.2.3. The faithful GMA at the point z. — Let $A := \mathcal{O}_z$ be the rigid local ring at the closed point underlying to z, and $\mathfrak{m}$ its maximal ideal, $k = k(z) = A/\mathfrak{m} \simeq L$. We will focus on the A-valued m-dimensional continuous pseudocharacter induced by T,

$$T : G_{E,S} \longrightarrow A,$$

that we denote also by T (rather than $T \otimes_{\mathcal{O}(X)} A$). Let

$$R := A[G_{E,S}]/\mathrm{Ker}\, T$$

[4] This character is the unique continuous character ν such that for each finite prime w of E prime to p,

$$\nu_{|W_{E_w}} = \iota_p\iota_\infty^{-1}(\mathrm{rec}^{-1}(\mu_w^{-1}|\cdot|^{\frac{m}{2}}))$$

where the *local* rec map is the one discussed in § 6.3.

be the faithful Cayley-Hamilton algebra associated to T. Recall that

$$\bar{\rho}_z = \rho \oplus 1 \oplus \chi.$$

As the Hodge-Tate weights of $\bar{\rho}_z$ are two-by-two distincts by Rem. 8.2.2, $\bar{\rho}_z$ is multiplicity free. For later use, let us write

$$\rho = \oplus_{j=1}^{r} \rho_j$$

where the ρ_j are pairwise non isomorphic and absolutely irreducible. As each ρ_j is defined over L by assumption on L, T is actually *residually multiplicity free* in the sense of Definition 1.4.1. By Theorem 1.4.4 and Remark 1.4.5 we get the following lemma.

Lemma 8.2.5. — *R is a GMA over A and is a finite type, torsion free, A-module.*

We will be interested in the Ext-groups between the irreducible constituents of $\bar{\rho}_z$. For this purpose we set

$$\mathcal{I} := \{\chi, 1, \rho_1, \ldots, \rho_r\},$$

which is also the set of simple R-modules by Lemma 1.2.7.

Definition 8.2.6. — If $i, j \in \mathcal{I}$ are two irreducible factors of $\bar{\rho}_z$, we set [5]

$$\mathrm{Ext}_T(i, j) := \mathrm{Ext}_{R \otimes_A k}(i, j) = \mathrm{Ext}_R(i, j).$$

It is a finite dimensional k-vector space. [6]

The following lemma follows from Prop. 1.5.10.

Lemma 8.2.7. — *The natural k-linear injection*

$$\mathrm{Ext}_T(i, j) \hookrightarrow \mathrm{Ext}_{k[G_{E,S}]}(i, j)$$

falls inside the subspace of continuous extensions of i by j as $k[G_{E,S}]$-representations.

Remark 8.2.8. — By definition, the image of the inclusion above is exactly the set of extensions of i by j that occur in some subquotients of some R-module M. As $R = A[G]/\mathrm{Ker}\, T$ is the only natural Cayley-Hamilton quotient of $(A[G], T)$ that we can consider a priori here, $\mathrm{Ext}_T(i, j)$ should be thought as *the space of extensions of i by j that we can construct from the datum of the pseudocharacter T*, hence the notation Ext_T.

We will study now the local conditions at each primes of the elements in $\mathrm{Ext}_T(i, j)$.

[5] Note that the last equality in the definition below is Remark 1.5.9.

[6] This follows for example from Theorem 1.5.5 and Lemma 8.2.5.

8.2.4. Properties at l of $\mathrm{Ext}_T(i,j)$. — Let us fix $l \neq p$ a prime, w a prime of E above l, as well as a decomposition group $G_{E_w} \longrightarrow G_{E,S}$. We begin by a general lemma.

Lemma 8.2.9. — *Let V be the semisimplification of the representation $\rho_{|G_{E_w}}$.*

(i) *Assume that l splits in E. For $d \in \mathbb{Z}$, χ^d is not a subrepresentation of V, and*

$$\mathrm{Ext}_{L[G_{E_w}]}(V, \chi^d) = \mathrm{Ext}_{L[G_{E_w}]}(\chi^d, V) = 0.$$

(ii) $\mathrm{Ext}_{L[G_{E_w}]}(\chi, 1) = 0$.

Proof. — Assume that l splits in E. We claim first that for all $d \in \mathbb{Z}$, l^d is not an eigenvalue of a Frobenius at w in $\rho_{|E_w}^{I_w}$. Indeed, as π_w is tempered, the eigenvalues of any geometric Frobenius element ϕ_w in the complex Weil-Deligne representation attached to $\pi_w|.|^{1/2}$ have norm $\sqrt{l}$. This proves already the first part of (i) by assumption (2) on ρ.

Let W be either $V(d)$, $V^*(d)$ or χ^{-1}. We need to show that $H^1(E_w, V) = 0$. As $l \neq p$, we know from Tate's theorem that

$$\dim_L H^1(E_w, W) = \dim_L H^0(E_w, W) + \dim_L H^0(E_w, W^*(1)),$$

so the case $W = \chi^{-1}$ is clear and the other ones follow from the claim and assumption (2) on ρ. □

Proposition 8.2.10. — *For each $i \neq 1$ in $\mathcal{I}$, $\mathrm{Ext}_T(1, i)$ consists of extensions which are split when restricted to I_{E_w}.*

Proof. — Let $R_w \subset R$ be the image of $A[G_{E_w}]$ in R via the natural map $A[G_{E,S}] \longrightarrow R$. It is of finite type over A as R is and as A is noetherian. Let K be the total fraction field of A, and set $R_K = R \otimes_A K$. As R is torsion free over A, $R \subset R_K$. Let us choose any datum of idempotents $e_\chi, e_1, \dots$ as well as a representation $R_K \to M_m(K)$ adapted to the chosen $\{e_i\}$ as in Theorem 1.4.4, and consider the induced representation

$$\rho_K : G_{E,S} \longrightarrow \mathrm{GL}_m(K).$$

By Lemma 4.3.9 and Prop. 1.3.12, ρ_K is semisimple and the sum of absolutely irreducible representations, so $\rho_K \otimes \overline{K} \simeq \rho_z^{\mathrm{gen}}$. In particular $\rho_{K|E_w}$ has an associated Weil-Deligne representation (r, N) with values in R_w.

The argument will be different according as l splits or not in E. As the proposition is obvious if $l \notin S$, we may assume that $l \in S$. Let us assume first that l does not split in E, which implies that $l \in S_N$. Note that for a continuous extension U of 1 by $i \neq 1 \in \mathcal{I}$ to be trivial when restricted to I_{E_w}, it is enough to check that I_{E_w} acts through a finite quotient on U, as $\mathbb{Q}$-linear representations of finte groups are semisimple. By Theorems 1.5.5 and 1.5.6 (1), it suffices to show that the image of N

in $R \subset R_K$ is trivial, because then $I_{E_w} \to R^*$ factors through a finite quotient. But as X is a minimal eigenvariety containing π^n, Prop. 7.5.7 shows that $N = 0$ when $w \in S_N$, and we are done in this case.

Let us assume now that $l = w\bar{w}$ splits in E. By Lemma 8.2.9 (i), there is nothing to prove when $i \neq \chi$, hence we concentrate from now on $\mathrm{Ext}_T(1, \chi)$. We will need to choose a specific GMA structure of R.

Lemma 8.2.11. — *Let A be a local henselian commutative noetherian ring, $\mathfrak{m}$ its maximal ideal, S an A-algebra (non necessarily commutative) which is of finite type as A-module. Let $\mathrm{Irr}(S)$ be the (finite) set of simple S-modules, or what is the same of simple $S/\mathfrak{m}S$-modules, and let $\mathcal{P} \subset \mathrm{Irr}(S)$ a subset with the following property:*

$$\forall M \in \mathcal{P},\ \ N \in \mathrm{Irr}(S)\backslash\mathcal{P},\ \mathrm{Ext}_{S/\mathfrak{m}S}(M, N) = \mathrm{Ext}_{S/\mathfrak{m}S}(N, M) = 0.$$

Then there is a unique central idempotent $e \in S$ such that for each $M \in \mathrm{Irr}(S)$, $e(M) = M$ if $M \in \mathcal{P}$, $e(M) = 0$ otherwise.

Proof. — Note that if M is a simple S module, it is monogenic over S hence of finite type over A, thus $\mathfrak{m}M = 0$ by Nakayama's lemma. It shows that $\mathrm{Irr}(S/\mathfrak{m}S) \to \mathrm{Irr}(S)$ is bijective. Moreover, $\mathfrak{m}S \subset \mathrm{rad}(S)$.

Assume first that $A = k$ is a field, hence S is any finite dimensional k-algebra. Let M be any finite type S-module, M has finite length. Define $M_{\mathcal{P}}$ (resp. $M^{\mathcal{P}}$) as the largest submodule of M all of whose simple subquotients lie in $\mathcal{P}$ (resp. in $\mathrm{Irr}(S)\backslash\mathcal{P}$). Obviously, $M_{\mathcal{P}} \cap M^{\mathcal{P}} = 0$. We claim that $M = M_{\mathcal{P}} \oplus M^{\mathcal{P}}$. Ab absurdum, as M is of finite length, we can find a submodule $M_{\mathcal{P}} \oplus M^{\mathcal{P}} \subset M' \subset M$ such that

$$Q := M'/(M_{\mathcal{P}} \oplus M^{\mathcal{P}}) \in \mathrm{Irr}(S).$$

By the Ext-assumption and an immediate induction, note that $\mathrm{Ext}_S(A, B) = \mathrm{Ext}_S(B, A) = 0$ whenever A and B are finite length S-modules such that each irreducible subquotient of A (resp. of B) lies in $\mathcal{P}$ (resp in $\mathrm{Irr}(S)\backslash\mathcal{P}$). Assume for example that $Q \notin \mathcal{P}$. The remark above shows that there is an S-submodule $M_{\mathcal{P}} \subsetneq M_0 \subset M'$ such that $M' = M_{\mathcal{P}} \oplus M_0$. But M_0 has all its subquotients in $\mathrm{Irr}(S)\backslash\mathcal{P}$, a contradicton. The case $Q \in \mathcal{P}$ is similar, which proves the claim.

We check at once that the decomposition $M = M_{\mathcal{P}} \oplus M^{\mathcal{P}}$ is stable by $\mathrm{End}_S(M)$. In particular, we can write $S = S_{\mathcal{P}} \oplus S^{\mathcal{P}}$ and we get that both $S_{\mathcal{P}}$ and $S^{\mathcal{P}}$ are two-sided ideals of S. We check now at once that the element $e \in S_{\mathcal{P}}$ given by the decomposition above of $1 = e + (e-1) \in S$ is a central idempotent with all the required properties, thus proving the assertion in the case where A is a field.

In general, we choose $e \in S/\mathfrak{m}S$ as above. As A is henselian and S finite over A, there is an idempotent $f \in S$ lifting e. By reducing mod $\mathfrak{m}$ the direct sum decomposition

$$S = fSf \oplus (1-f)Sf \oplus fS(1-f) \oplus (1-f)S(1-f),$$

and as e is central, we get that $fS(1-f) \otimes_A A/\mathfrak{m} = (1-f)Sf \otimes_A A/\mathfrak{m} = 0$, hence $fS(1-f) = (1-f)Sf = 0$ in S by Nakayama's lemma. In other words, f is central, and we are done. □

We show now that up to R^*-conjugation, R_w is bloc diagonal of type $(2, n)$ in R.

Lemma 8.2.12. — *There is a datum of idempotents $\{e_i, i \in \mathcal{I}\}$ for the generalized matrix algebra R such that $e := e_\chi + e_1$ is in the center of R_w.*

Proof. — We have $\mathrm{rad}(R) \cap R_w \subset \mathrm{rad}(R_w)$ by Lemma 1.2.7, so the set $\mathcal{I}_w$ of simple $R_w/\mathfrak{m}R_w$-modules is the set of irreducible subquotients of the $W_{|G_{E_w}}$ with $W \in \mathcal{I}$. Let us consider the subset

$$\mathcal{P} = \{\chi, 1\} \subset \mathcal{I}_w.$$

By the second assertion of Lemma 8.2.9 (i), we can apply Lemma 8.2.11 to $S = R_w$ and the set $\mathcal{P}$ above, which gives us a central idempotent $e \in R_w$.

By the first assertion of Lemma 8.2.9 (i), $T(e) = \bar{T}(e) = 2$, so if we consider now the restriction T_e of T to eRe, it is a Cayley-Hamilton pseudocharacter of dimension 2 (see Lemma 1.2.5) which is residually multiplicity free with residual representations 1 and χ. By Lemma 1.4.3, we can then write

$$e = e_\chi + e_1$$

where e_χ, $e_1 \in eRe$ lift the residual idempotents $1 = \epsilon_\chi + \epsilon_1$ (see the proof of Lemma 1.4.3). We conclude the proof by lifting then successively the remaining residual primitive idempotents in $(1-e)R(1-e)$ and arguing as in the first part of the proof of Lemma 1.4.3, or better by applying that lemma to $(1-e)R(1-e)$. □

We can now conclude the proof of Prop. 8.2.10. Let us choose a datum of idempotents $e_\chi, e_1, \dots$ as in Lemma 8.2.12 as well as a representation $R_K \to M_m(K)$ adapted to those $\{e_i\}$ as above. Note that a continuous G_{E_w}-extension of 1 by χ is trivial if and only if its monodromy operator is trivial. By Theorems 1.5.5 and 1.5.6 (1), it suffices to show that the image eN of N in $eR_we = eR_w \subset eR_Ke$ is trivial. Write $N = (N_s) \in M_d(K)$, $K = \prod_s K_s$. By the minimality of X, assumption (2), and by Prop. 7.5.8, we have for each germ s of irreducible component of X at x,

$$\overline{N}_z \prec N_s \prec \overline{N}_z,$$

hence $\overline{N}_z \sim N_s$ (recall that $\overline{N}_z$ is the monodromy operator of $\bar{\rho}_z$). We have to show that $eN = 0$. Let us write by abuse $(1-e)\overline{N}_z$ for the image of $\overline{N}_z$ in $\prod_{j=1,\dots,r} \mathrm{End}(\rho_j)$.

We have $\overline{N}_z \sim (1-e)\overline{N}_z$ as 1 and χ are unramified. But $(1-e)\overline{N}_z \prec (1-e)N_s$ by Prop. 7.8.8 applied to $(1-e)R(1-e)$ and $n = (1-e)N$, so we get

$$\forall s,\ (1-e)N_s \sim N_s \sim (1-e)\overline{N}_z \tag{97}$$

and $eN = 0$ by the first of the $\sim$'s above. □

Let us record now a fact that we will use in section 9. We assume here, and only here, that ρ is irreducible. Let $I_{\mathrm{tot}} \subset A$ be the total reducibility locus of T (see § 1.5.1 and Definition 1.5.2). Let $J \supset I_{\mathrm{tot}}$ be a proper ideal of cofinite length of A. Recall that $T \otimes A/J$ can be written uniquely as the sum of of three residually irreducible pseudocharacters

$$T \otimes A/J = T_\chi + T_1 + T_\rho$$

of respective dimension 1, 1 and n, lifting the decomposition of $T \otimes k$. Moreover, let

$$R_\rho : G_{E,S} \longrightarrow \mathrm{GL}_m(A/J)$$

be the unique (up to conjugation) continuous representation with trace T_ρ (see Def. § 1.5.3, Prop. 1.5.10).

Lemma 8.2.13. — *Assume that ρ is irreducible. The monodromy operator of $R_{\rho|E_w}$ admits a Jordan normal form over A/J.*

Proof. — We keep the notations of the proof above. As we showed, this monodromy operator is 0 if w is not split, hence we may assume that it is. As ρ is irreducible, $(1-e)R(1-e) \simeq M_n(A)$, and by Lemma 1.5.4 (ii), it suffices to show that $(1-e)N \in (1-e)R(1-e)$ admits a Jordan normal form over A. By (97),

$$\forall s,\ (1-e)N_s \sim (1-e)\overline{N}_z$$

and $(1-e)\overline{N}_z$ is the monodromy operator of $\rho_{|E_w}$ as ρ is irreducible, so the lemma follows from Prop. 7.8.9 (ii). □

8.2.5. Properties at v and $\bar{v}$ of $\mathrm{Ext}_T(1,i)$. — Let us assume from now on that the accessible refinement $\mathcal{R}$ of π_p^n has been chosed of the form

$$(1, \dots, p^{-1}).$$

There are $n!$ such refinements by Lemma 8.2.1, and the dots above are the eigenvalues of $L(\pi_p|.|^{1/2})$ chosen in some random order (they are all different from 1 and p^{-1}). We fix for $* = v, \bar{v}$ some decomposition group map $\mathrm{Gal}(\overline{E}_*/E_*) \longrightarrow G_{E,S}$.

Proposition 8.2.14. — *For all $i \neq 1$, $\mathrm{Ext}_T(1,i)$ consists of extensions which are crystalline at v and $\bar{v}$.*

Proof. — Fix $i \in \{\chi, \rho_1, \dots, \rho_r\}$. Note that a continuous $L[G_{E,S}]$-extension U of 1 by i is crystalline at $* = v$ or $\bar{v}$ if and only if

$$D_{\mathrm{crys}}(U_{|E_*})^{\varphi=1} \neq 0. \tag{98}$$

Indeed, there is an exact sequence $0 \to D_{\mathrm{crys}}(i_{|E_*}) \to D_{\mathrm{crys}}(U_{|E_*}) \to D_{\mathrm{cris}}(1_{|E_*})$ by left exactness of the functor D_{crys}. As $i_{|E_*}$ is crystalline and does not have 1 as crystalline Frobenius eigenvalue (for $i \subset \rho$, this is assumption (3) on ρ and Lemma 8.2.1), we obtain that

$$\dim_L D_{\mathrm{crys}}(U_{|E_*}) = \dim_L i + \dim_L D_{\mathrm{crys}}(U_{|E_*})^{\varphi=1},$$

hence the claim above.

For $U \in \mathrm{Ext}_T(1, i)$ we shall deduce (98) from the properties of p-adic analytic continuation of crystalline periods in weakly refined families that we obtained in section 4, especially Theorem 4.3.6. Recall from Prop. 7.5.13 that

$$(X, T, \{\kappa_i\}, \{F_i\}, Z_{\mathrm{reg}}) \tag{99}$$

is a refined family for the restriction map $\mathrm{Gal}(\overline{E}_v/E_v) \to G_{E,S}$ (beware that we use the normalization of T, κ_i, and F_i, set in Definition 8.2.4). In particular, $(X, T, \{\kappa_i\}, F_1, Z_{\mathrm{reg}})$ defines a weakly refined family (see Def. 4.2.7). At our point z, note that

$$F_1(z)p^{\kappa_1(z)} = 1$$

as the first eigenvalue of our refinement $\mathcal{R}$ is 1 by assumption. Assuming for the moment that its assumptions are satisfied at the point z, Theorem 4.3.6 shows that for any $U \in \mathrm{Ext}_T(1, i)$, formula (98) holds for $* = v$ (apply the theorem to any partition $\mathcal{P}$ containing $\{1\}$ and $\{i\}$ and to the maximal ideal $I = \mathfrak{m}$ of A). Following § 4.3, there are three assumptions to check:

(ACC) z is an accumulation point of Z_{reg}. This is the first part of Lemma 7.5.3.

(MF) T is residually multiplicity free at z. This holds for instance as $\bar{\rho}_z = 1 \oplus \chi \oplus \rho$ has distinct Hodge-Tate weights (see § 8.2.3).

(REG) The crystalline Frobenius eigenvalue 1 has multiplicity one in $D_{\mathrm{crys}}(\bar{\rho}_z)$. This follows from Lemma 8.2.1 and has been used several times already.

This concludes the proof that $\mathrm{Ext}_T(1, i)$ consists of extensions which are crystalline at v. The same arguments apply to $\bar{v}$ as follows. As $T^{\perp}(1) = T$, the refined family (99) induces formally a weakly refined family

$$(X, T, \{\kappa'_i\}, F, Z_{\mathrm{reg}}),$$

for the restriction $\mathrm{Gal}(\overline{E}_{\bar{v}}/E_{\bar{v}}) \to G_{E,S}$, where $F = F_m^{-1}$ and $\kappa'_i = -1 - \kappa'_{m-i+1}$ for each $i = 1, \dots, m$. As p^{-1} is the last eigenvalue of our refinement $\mathcal{R}$, note that

$p^{\kappa_m(z)}F_m(z) = p^{-1}$, or which is the same that

$$p^{\kappa'_1(z)}F(z) = 1.$$

The argument above applies then verbatim and shows that $\mathrm{Ext}_T(1,i)$ consists of extensions which are crystalline at $\bar{v}$, which concludes the proof. □

Recall the definition of H^1_f from § 5.1.2.

Corollary 8.2.15. — *For each $i \in \mathcal{I}$ with $i \neq 1$, $\mathrm{Ext}_T(1,i) \subset H^1_f(E,i)$.*

Proof. — It follows from Prop. 8.2.14 and Prop. 8.2.10. □

8.2.6. Symmetry properties of T. — We choose now a particular GMA datum on R using the symmetry of the pseudocharacter T (see § 1.8).

Let $\tau : \mathcal{O}(X)[G_{E,S}] \to \mathcal{O}(X)[G_{E,S}]$ be the $\mathcal{O}(X)$-linear map such that

$$\tau(g) := cg^{-1}c^{-1}\chi(g).$$

We have $\tau^2 = 1$ and $\tau(gg') = \tau(g')\tau(g)$, hence τ is an $\mathcal{O}(X)$-linear anti-involution of $\mathcal{O}(X)[G_{E_S}]$. By § 8.2.2, it satisfies

$$T \circ \tau = T.$$

As a consequence, τ induces an A-linear anti-involution on the A-algebra R and we can apply to it the results of § 1.8 (see Remark 1.8.1). The involution τ induces naturally an involution on $\mathcal{I}$ that we still denote by [7] τ, namely

$$\forall i \in \mathcal{I}, \tau(i) = i^{\perp} \otimes \chi.$$

For example, we have $\tau(1) = \chi$. By Lemma § 1.8.3, we can find a data of idempotents $\{e_i, i \in \mathcal{I}\}$ for the GMA R such that

$$\forall i \in \mathcal{I},\ \tau(e_i) = e_{\tau(i)}. \tag{100}$$

We choose now a GMA datum for R of the form $\{e_i, \psi_i, i \in \mathcal{I}\}$ with the e_i as above. It will be also convenient to fix an adapted representation

$$R \hookrightarrow M_m(K)$$

associated to this datum in the sense of Theorem 1.4.4 (ii), so that R identifies with the standard GMA of type $(1,1,d_1,\dots,d_r)$ associated to $\{A_{i,j}, i,j \in \mathcal{I}\}$ where the $A_{i,j} \subset K$ are fractional ideals. In particular, each $A_{i,j}$ is finite type over A.

Lemma 8.2.16. — *For all $i,j \in \mathcal{I}$, $A_{i,j}A_{j,i} = A_{\tau(i),\tau(j)}A_{\tau(j),\tau(i)}$.*

Proof. — We have to show that $T(e_iRe_jRe_i) = T(e_{\tau(i)}Re_{\tau(j)}Re_{\tau(i)})$, which is immediat from the fact $T \circ \tau = T$, and that $\tau(e_*) = e_{\tau(*)}$. □

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[7] This involution is denoted by σ in § 1.8.

Let $i \neq j \in \mathcal{I}$. Recall that we defined in § 1.8.5 a map

$$\perp_{i,j} : \mathrm{Ext}_{k(z)[G_{E,S}]}(i,j) \longrightarrow \mathrm{Ext}_{k(z)[G_{E,S}]}(\tau(j),\tau(i)).$$

The following lemma is a consequence of Prop. 1.8.6 and Lemma 1.8.5.

Lemma 8.2.17. — *The map $\perp_{i,j}$ induces an isomorphism*

$$\mathrm{Ext}_T(i,j) \xrightarrow{\sim} \mathrm{Ext}_T(\tau(j),\tau(i)).$$

In particular, using Prop. 8.2.14 and 8.2.10, the lemma above has the following corollary.

Corollary 8.2.18. — *For all $i \neq \chi$ in $\mathcal{I}$, $\mathrm{Ext}_T(i,\chi)$ consists of extensions which are crystalline at v and $\bar{v}$, and split when restricted to I_{E_w} for each w prime to p.*

8.3. Proof of Theorem 8.1.2

8.3.1. — Let us prove Theorem 8.1.2. By Corollary 8.2.15, it suffices to show that for some irreducible subquotient ρ_j of ρ,

$$\mathrm{Ext}_T(1,\rho_j) \neq 0.$$

For that we relate those Ext-groups to some reducibility ideal of T and we show that the associated reducibility locus is included is some explicit closed subscheme of $\mathrm{Spec}(A)$. We first draw a key consequence of the vanishing of $H^1_f(E,\chi)$ and of the work above. Recall that we fixed in § 8.2.6 a specific trace embedding $R \hookrightarrow M_m(K)$ identifying R with the standard GMA of type $(1,1,d_1,\ldots,d_r)$ associated to $\{A_{i,j}, i,j \in \mathcal{I}\}$ where the $A_{i,j} \subset K$ are fractional ideals of K (hence of finite type over A).

Lemma 8.3.1 (*Vanishing of $H^1_f(E,\chi)$*). — *We have*

$$\mathrm{Ext}_T(1,\chi) = 0 \;\text{ and }\; A_{\chi,1} = \sum_j A_{\chi,\rho_j} A_{\rho_j,1}.$$

Proof. — By Prop. 5.2.2, $H^1_f(E,\chi)$ vanishes, hence so does $\mathrm{Ext}_T(1,\chi)$ by Cor. 8.2.15. Set

$$A'_{\chi,1} = \sum_j A_{\chi,\rho_j} A_{\rho_j,1}.$$

By Theorem 1.5.5 applied to $J = \mathfrak{m}$, we get that $\mathrm{Hom}_A(A_{\chi,1}/A'_{\chi,1}, k) = 0$. But $A_{\chi,1}$ has finite type over A, hence we conclude by Nakayama's lemma. □

Let $\mathcal{P}$ be the following partition of $\mathcal{I}$: $\mathcal{P} = (\{\chi\},\{1\},\mathcal{I}\backslash\{\chi,1\})$, and $I_{\mathcal{P}} \subset A$ its reducibility locus (see Def. 1.5.2).

Lemma 8.3.2 (*Reduction to the nonvanishing of $I_{\mathcal{P}}$*). —

(i) $I_{\mathcal{P}} = A_{1,\chi}A_{\chi,1} + \sum_j A_{1,\rho_j}A_{\rho_j,1} + \sum_j A_{\chi,\rho_j}A_{\rho_j,\chi}$.

(ii) $I_{\mathcal{P}} = \sum_j A_{1,\rho_j} A_{\rho_j,1}$.
(iii) *If* $\mathrm{Ext}_T(1,\rho_j) = 0$ *for all* j, *then* $I_{\mathcal{P}} = 0$.

Proof. — Assertion (i) is Prop. 1.5.1. As $\tau(1) = \chi$, Lemma 8.2.16 shows that

$$\sum_j A_{1,\rho_j} A_{\rho_j,1} = \sum_j A_{\chi,\rho_j} A_{\rho_j,\chi}.$$

But by Lemma 8.3.1, $A_{\chi,1} = \sum_j A_{\chi,\rho_j} A_{\rho_j,1}$, hence

$$A_{\chi,1} A_{1,\chi} = \sum_j A_{\chi,\rho_j} A_{\rho_j,1} A_{1,\chi} \subset \sum_j A_{\chi,\rho_j} A_{\rho_j,\chi},$$

as $A_{\rho_j,1} A_{1,\chi} \subset A_{\rho_j,\chi}$. This proves assertion (ii).

Arguing as in the proof of Lemma 8.3.1, $\mathrm{Ext}_T(1,\rho_j) = 0$ implies that

$$A_{\rho_j,1} = \sum_{i \neq 1,\rho_j} A_{\rho_j,i} A_{i,1}.$$

Assume that this holds for each j. By applying this identity or Lemma 8.3.1 $s := 1+|\mathcal{I}|$ times, we get that for each j, $A_{1,\rho_j} A_{\rho_j,1}$ is a finite sum of terms of the form

$$A_{1,\rho_j} A_{\rho_j,i_1} A_{i_1,i_2} \cdots A_{i_s,1}, \tag{101}$$

with all the $i_k \in \mathcal{I}$, $i_1 \neq \rho_j$, $i_s \neq 1$, and $i_{k+1} \neq i_k$. As $s > |\mathcal{I}|$, for each such term there exist $k < k'$ such that $i_k = i_{k'}$, which implies that

$$A_{i_k,i_{k+1}} A_{i_{k+1},i_{k+2}} \cdots A_{i_{k'-1},i_{k'}} \subset A_{i_k,i_{k+1}} A_{i_{k+1},i_k} \subset \mathfrak{m},$$

$$A_{\rho_j,i_1} A_{i_1,i_2} \cdots A_{i_{k-1},i_k} A_{i_{k'},i_{k'+1}} \cdots A_{i_s,1} \subset A_{\rho_j,1},$$

hence that

$$A_{1,\rho_j} A_{\rho_j,i_1} A_{i_1,i_2} \cdots A_{i_s,1} \subset \mathfrak{m} A_{1,\rho_j} A_{\rho_j,1}.$$

This proves that $I_{\mathcal{P}} \subset \mathfrak{m} I_{\mathcal{P}}$, hence that $I_{\mathcal{P}} = 0$ by Nakayama's lemma. □

By Lemma 8.3.2 (iii), there only remains to show part (ii) of the following lemma. Recall that $\{\kappa_i(z)\}_{i=1\dots m}$ is the strictly increasing (see Def. 7.5.11) sequence of Hodge-Tate weights of $\bar{\rho}_z$ at v. Let $a \in \{1,\dots,m\}$ be the unique integer such that

$$\kappa_a(z) = 0.$$

Lemma 8.3.3 (*Non triviality of* $I_{\mathcal{P}}$). — (i) $(\kappa_a - \kappa_1) - (\kappa_a(z) - \kappa_1(z)) \in I_{\mathcal{P}}$.
(ii) $a \neq 1$ *and* $I_{\mathcal{P}} \neq 0$.

Proof. — As already said in the proof of Prop. 8.2.14, $(X, T, \kappa_1, F_1, Z_{\mathrm{reg}})$ is a weakly refined family for $G_{|E_v} \to G_{E,S}$, and the assumption (ASS), (MF) and (REG) of § 4.3 are satisfied. Part (i) is then Theorem 4.3.4 as

$$D_{\mathrm{crys}}(\bar{\rho}_z)^{\varphi=1} = D_{\mathrm{crys}}(1)^{\varphi=1}$$

has dimension 1 by Lemma 8.2.1 and assumption (3).

Let us show assertion (ii). By property (iv) of the eigenvariety X, the natural map $\mathcal{O}_{\kappa(z)} \longrightarrow A$ is injective, so it suffices to check that

$$(\kappa_a - \kappa_1) - (\kappa_a(z) - \kappa_1(z)) \neq 0,$$

i.e. that $a \neq 1$. If $a = 1$, $\kappa_a(z) = 0$ is the smallest Hodge-Tate weight of $\bar{\rho}_z$ at v. But this is absurd as -1 is a Hodge-Tate weight of $\bar{\rho}_z$ at v, namely the one of χ. □

8.3.2. Some remarks about the proof. — The above proof of Theorem 8.1.2 can actually be simplified in several different ways. We chose to look at the full minimal eigenvariety X containing z and its associated Galois pseudocharacter T because this is the relevant point and space for which the analysis developped here can be pushed further (and for which the Ext_T have a maximal dimension) as we will explain in the next section. All the strenght of the results proved here (especially the ones in § 8.2.1) will be used in section 9, and we found it convenient to directly include them here so as not to repeat half of the story there.

In the style of [**8**], we could have replaced X by the normalization of the germ of any irreducible curve $C \subset X$ containg z such that $Z \cap C$ is infinite and that $\kappa_a - \kappa_1$ is not constant on C. The ring A would have been a DVR which would have simplified some of the pseudocharacter theoretic arguments. Note that in the argument above, we do not really choose a stable "lattice" as in [**8**] but we rather work with the full ring theoretic image R of the family of Galois representations. This is actually convenient and it illustrates the techniques developped in the previous sections of this book. Had we worked on the germ of a smooth curve as explained above, we could have used the choice of a good lattice as in [**8**, Prop. 7.1] (as written, it requires ρ to be irreducible).

Moreover, a nice way to understand the combinatorics in (iii) of Lemma 8.3.2 is to compare it with the connected graph theorem [**6**, Thm. 1].[(8)] In our case, there would be no edge $1 \to \chi$ by Lemma 8.3.1, hence at least an edge $1 \to \rho_j$ for some j. Note that we do not claim that the pseudocharacter T used in the proof above is generically irreducible, but Lemma 8.3.3 rather says that it is not "too reducible", and this is actually enough to conclude. Actually, had we assumed that the eigenvalues of the Langland's conjugacy class of π_p are "regular", we could have chosed a refinement $\mathcal{R}$ (hence a z) leading to a generically irreducible T (even on the curve C), as follows from Rem. 7.7.4.

(8) Here is the statement: let A be a henselian DVR, $r : G \longrightarrow \mathrm{GL}_d(A)$ a generically absolutely irreductible representation, and assume that the semi-simplified residual representation $\bar{r}^{\mathrm{ss}}$ is multiplicity free. Then the oriented graph whose vertices are the irreducible constituents of $\bar{r}^{\mathrm{ss}}$, and with an edge from i to j if there is a nontrivial extension of i by j in the subquotients A-modules of A^d, is connected as an oriented graph.

As is clear from the proof, the cornerstone of the argument is the fact that

$$\mathrm{Ext}_T(1, \chi) \subset H^1_f(E, \chi),$$

(that is Lemma 8.3.1) which requires to control the deformation at *all the finite places*, from which we deduced that $\mathrm{Ext}_T(1, \chi) = 0$ using the finiteness of $\mathcal{O}_E^*$ (in terms of the graph alluded above, it is the step: "*there is no arrow* $1 \to \chi$"). This last fact fails for a general CM field E, and actually the whole argument breaks down in this generality because of that. We will discuss that issue in greater detail in Remark 9.5.1.

CHAPTER 9

THE GEOMETRY OF THE EIGENVARIETY AT SOME ARTHUR POINTS AND HIGHER RANK SELMER GROUPS

9.1. Statement of the theorem

We keep the notations of § 8.1. In particular

$$\rho : \mathrm{Gal}(\overline{E}/E) \longrightarrow \mathrm{GL}_n(L)$$

is a *modular* Galois representation attached to a cuspidal automorphic representation $\pi|.|^{1/2}$ of $\mathrm{GL}_n(\mathbb{A}_E)$ that satisfies conditions (1), (2) and (3) there. We now make the following new assumptions on ρ:

(4) $\Lambda^i\rho$ is absolutely irreducible for $i = 1, \ldots, n$.

(5) The crystalline representation $\rho_{|E_v}$ admits a regular non critical refinement (see § 2.4.3 and Def. 2.5.5).

(6) The hypotheses $BK1(\rho)$ and $BK2(\rho)$ hold (see § 5.2.3).

Remark 9.1.1. — (i) The irreducibility assumption (4) is known if $n \leq 3$. However, when $n \geq 4$ it is not always satisfied, especially in the interesting case where π is a base change from $\mathbb{Q}$ and n is even. As we shall see, the reason of this assumption here is the presence of hypothesis (MF′) in Thm. 4.4.4. As explained there, we expect that this hypothesis is actually unnecessary, but to remove it would require more work in § 4.4.4.

(ii) Recall that the regularity assumption in (5) combined with (3) means that the Langlands conjugacy class $C \in \mathrm{GL}_n(\mathbb{C})$ of the unramified representation $\pi_p|.|^{1/2}$ has distinct eigenvalues, and that those eigenvalues can be ordered as

$$(\varphi_1, \ldots, \varphi_n)$$

in such a way that for each $j = 1, \ldots, n$, $\varphi_1 \cdots \varphi_j$ is a simple eigenvalue of $\Lambda^j(C)$. If $n \leq 3$, it is equivalent to only ask that the φ_i are distinct, and if $n = 2$ (resp. $n = 1$) this is conjectured to always be the case (resp. it is obviously true).

The non critical part of the assumption of (5) means that the refinement of $\rho_{|E_v}$ associated by property (3) to the ordering above is non critical in the sense of § 2.4.3. Again, this is automatically satisfied if $n = 1$, and in most cases when $n = 2$ (see Remark 2.4.6).

(iii) As we saw in Propositions 5.2.5 and 5.2.6, the hypothesis (6) is known to hold if $n = 1$ and also in the $n = 2$ case for ρ of the form $\rho_{f,E}$ for f a modular forms of even weight with a small explicit set of exceptions.

Of course, we shall also assume that $\varepsilon(\rho, 0) = -1$, and that Rep($m$) and AC($\pi$) hold. As in § 6.9, we denote by π^n the non-tempered automorphic representation of U(m) attached to ρ by assumption AC(π), for some choice of a Hecke character μ as in Def. 6.9.5 that we fix once and for all. Recall that we defined in § 8.2.1 the minimal eigenvariety X of U(m) containing π^n. We consider here the variant where we fix one of the m weights (any one), so that X is equidimensional of dimension $m - 1 = n + 1$. Any choice of an accessible refinement $\mathcal{R}$ of π_p^n defines a point $z \in X$. By Lemma 8.2.1 and assumption (5), we may choose a refinement of the form

$$\mu_v|.|^{-1/2}(p)(1, \varphi_1, \varphi_2, \ldots, \varphi_n, p^{-1}),$$

where the φ_i are chosen as in Remark 9.1.1 to satisfy the regularity and non criticality assumption of (5). We fix once and for all such a refinement $\mathcal{R}$, hence a point

$$z \in X,$$

which is the *Arthur's* point that we refer to in the title, z is defined over L.

In all this section, we will generally follow the notations of § 8.2.1. In particular, recall that we defined in Def. 8.2.6 an L-subspace $\mathrm{Ext}_T(1, \rho) \subset \mathrm{Ext}_{G_{E,S}}(1, \rho)$ which is the space of extensions of 1 by ρ that we can construct from the Galois pseudocharacter T carried by X (see Remark 8.2.8). By Prop. 8.2.14 and 8.2.18, we know that

$$\mathrm{Ext}_T(1, \rho) \subset H^1_f(E, \rho).$$

Theorem 9.1.2. — *Assume that ρ satisfies (1) to (6), that $\varepsilon(\rho, 0) = -1$, and that AC($\pi$) and* Rep($m$) *hold. Let t be the dimension of the tangent space of X at z and h the dimension of* $\mathrm{Ext}_T(1, \rho)$, *then*

$$t \;\leq\; h\left(n + \frac{h+1}{2}\right).$$

Note that both dimensions above are taken over the residue field $k = k(z) \simeq L$. Recall that $t := \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,z}$. As $\mathcal{O}_{X,z}$ is equidimensional of dimension $n + 1$, we have

$$n + 1 \;\leq\; h\left(n + \frac{h+1}{2}\right)$$

so in particular we find again that $h \neq 0$ (i.e Theorem 8.1.2 for ρ) and get the following corollary.

Corollary 9.1.3. — *(Same assumption.) If X is not smooth at z, then*

$$\dim_L H^1_f(E,\rho) \geq \dim_L \mathrm{Ext}_T(1,\rho) \geq 2.$$

When $n = \dim(\rho) = 1$, i.e. when

$$\rho : \mathrm{Gal}(\overline{E}/E) \longrightarrow L^*$$

is any continuous character such that $\rho^\perp = \rho(-1)$, class field theory and Remark 9.1.1 imply that conditions (1) to (6) are satisfied once we assume that the two Hodge-Tate weights of $\rho_{|E_v}$ are different from 0 or -1 (because of condition (ii) on π in § 6.9.1). Moreover, by Remarks 6.8.2 (vi) and 6.9.10 (ii), assumptions Rep(3) and AC(π) are also known.

Corollary 9.1.4. — *If $n = 1$, Theorem 9.1.2 and its corollary above hold under the single assumption that* 0 *and* -1 *are not Hodge-Tate weights of the character $\rho_{|E_v}$.*

Remark 9.1.5. — As we already said, various versions of the main conjectures are known for Hecke characters from the work of Rubin [**103**]. These "main-conjectures" also imply that $\dim_L H^1_f(E,\rho) \geq 1$ when expected. However, as far as we know, they do not allow one to show that $\dim_L H^1_f(E,\rho) \geq 2$ when the L-function or rather its p-adic analogues vanish at a higher order (this phenomenon is sometimes called the possible *non semisimplicity of the Iwasawa module*). As a consequence, even the simplest case covered by the corollary above is of interest.

In all this book, we have concentrated on the Galois side part of the study, letting aside the various p-adic L-functions that should enter in the picture. We hope that once this is done, the Galois deformations studied here will shed some light on the $\leq$ part of the conjectures alluded above. From a more conjectural point of view, this remark actually applies to any ρ satisfying conditions (1) and (2).

9.2. Outline of the proof

The proof of Theorem 9.1.2 is a refinement of the proof of the sign conjecture consisting in a careful analysis of the Galois pseudocharacter

$$T : G_{E,S} \longrightarrow \mathcal{O}_{X,z}$$

at the point z. As in § 8.2.3, we let $A = \mathcal{O}_{X,z}$, $\mathfrak{m}$ the maximal ideal of A, $k = A/\mathfrak{m} \simeq L$ its residue field, and

$$R := A[G]/\mathrm{Ker}\, T$$

the faithful Cayley-Hamilton GMA associated to T at z. Recall that it is of finite type and torsion free over A (which is a reduced henselian noetherian ring). As ρ is irreducible, we have now

$$\mathcal{I} = \{\chi, \rho, 1\}.$$

As in § 8.2.6, we fix a data of idempotents (e_χ, e_ρ, e_1) for R such that $\tau(e_*) = e_{\sigma(*)}$. Note that σ fixes ρ, and exchanges 1 and χ. Last but not least, $K = \prod_s K_s$ is the total fraction ring of A, and we fix a representation

$$\rho_K : R \longrightarrow M_m(K)$$

associated to this data of idempotents, as in Theorem 1.4.4 (ii). Recall that this gives us a set of finite type A-modules $A_{i,j} \subset K$, $i, j \in \mathcal{I}$, such that $A_{i,i} = A$ for each i, $A_{i,j}A_{j,k} \subset A_{i,k}$ for each i, j, k, and $A_{i,j}A_{j,i} \subset \mathfrak{m}$ if $i \neq j$. Moreover, $R = \rho_K(R) \subset M_m(K)$ is the standard GMA of type $(1, n, 1)$ associated to these data (see Example 1.3.4), that is

$$R = \rho_K(R) = \begin{pmatrix} A & A^n_{\chi,\rho} & A_{\chi,1} \\ A^n_{\rho,\chi} & M_n(A) & A^n_{\rho,1} \\ A_{1,\chi} & A^n_{1,\rho} & A \end{pmatrix} \subset M_m(K). \tag{102}$$

Our aim will be to elucidate as much as possible the structure of the A-modules $A_{i,j}$.

One the one hand, those $A_{i,j}$ are related to the $\mathrm{Ext}_T(j, i)$ by Theorem 1.5.5. In turns, by results already proved in § 8.2.1, those $\mathrm{Ext}_T(j, i)$ are related to all the 6 possible Selmer groups occuring here, namely $H^1_f(E, *)$ where

$$* = \chi, \chi^{-1}, \rho, \rho^*, \rho\chi^{-1}, \rho^*\chi^{-1}.$$

Up to the underlying symmetries, and by $BK_1(\rho)$ when $* = \rho^*$, we will actually know all of them except the one of ρ, which is precisely the one we are interested in.

On the other hand, the $A_{i,j}$ are also related to the reducibility loci of T by Prop. 1.5.1. A remarkable fact is that we are able to compute here all the reducibility ideals of T. Precisely, we will show that all the proper reducibility loci actually coincide schematically with the closed point z. In other words, T is as irreducible as possible. The proof of this key fact will actually use all the machinery that we developed in sections 1 to 4 of the book. This will provide then the missing link between the tangent space of X at z and the $A_{i,j}$, and then with the $\mathrm{Ext}_T(1, \rho)$.

9.3. Computation of the reducibility loci of T

Let us analyse the proper reducibility loci of T (see § 1.5.1). Recall that each of them is attached to a non trivial partition $\mathcal{P}$ of $\mathcal{I} = \{\chi, 1, \rho\}$, and there are 4 such partitions. An especially interesting one is the the total reducibility ideal I_{tot}, which is attached to the finest partition $\{\{\chi\},\{1\},\{\rho\}\}$.

Lemma 9.3.1. — (i) *All these four reducibility ideals coincide with I_{tot}.*
(ii) $I_{\mathrm{tot}} = A_{1,\rho}A_{\rho,1}$.
(iii) $I_{\mathrm{tot}}K = K$.

Proof. — By Prop. 1.5.1, each proper reducibility ideal is a sum of terms of the form $A_{i,j}A_{j,i}$ with $i \neq j$, and contains $A_{*,\rho}A_{\rho,*}$ for $* = 1$ or χ. By Lemma 8.2.16, $A_{i,j}A_{j,i} = A_{\sigma(j),\sigma(i)}A_{\sigma(i),\sigma(j)}$ so

$$A_{1,\rho}A_{\rho,1} = A_{\chi,\rho}A_{\rho,\chi}. \tag{103}$$

By Lemma 8.3.1, we also have

$$A_{\chi,1} = A_{\chi,\rho}A_{\rho,1}. \tag{104}$$

But $A_{1,\chi}A_{\chi,\rho} \subset A_{1,\rho}$, so

$$A_{1,\chi}A_{\chi,1} \subset A_{1,\rho}A_{\rho,1},$$

which proves assertions (i) and (ii).

By Lemma 8.3.3 (i), the total reducibility ideal I_{tot} contains the element

$$f := \kappa_a - \kappa_1 - (\kappa_a(z) - \kappa_1(z)) \in \mathcal{O}_{\kappa(z)},$$

where a is some integer between 2 and m (note that the reducibility ideal $I_{\mathcal{P}}$ considered there is I_{tot} as ρ is irreducible). In particular, f is a nonzero element of the domain $\mathcal{O}_{\kappa(z)}$. Recall that $K = \mathrm{Frac}(A) = \prod_s K_s$. By property (iv) of the eigenvariety X, the composition of the natural maps

$$\mathcal{O}_{\kappa(z)} \longrightarrow A \longrightarrow \mathrm{Frac}(A) \longrightarrow K_s$$

is injective for each s, so $K = Kf \subset I_{\mathrm{tot}}K$, which proves (iii). □

Lemma 9.3.2. — *The representation ρ_K induces an isomorphism $R \otimes_A K \xrightarrow{\sim} M_m(K)$, and $\rho_K \otimes K_s$ is absolutely irreducible for each s.*

Proof. — This is actually a general consequence of Prop. 1.3.12 and of the fact that $IK = K$ for all irreducibility ideals I, but we argue directly. By Lemma 9.3.1 (ii) and (iii), $A_{1,\rho}K = A_{\rho,1}K = K$, and the same equality holds with 1 replaced by χ by formula (103). As $A_{1,\chi} \supset A_{1,\rho}A_{\rho,\chi}$ we get also that $A_{1,\chi}K = K$, as well as $A_{\chi,1}K = K$ by the same reasoning. This proves the first part of the lemma, of which the second part is an obvious consequence. □

Note that for the moment, we did not use assumptions (5) to (6), and only the irreducibility assumption of ρ in (4). We will now use (4) and (5) by beginning a deeper study of I_{tot}. We show first that the total reducibility locus

$$V(I_{\mathrm{tot}}) \subset \mathrm{Spec}(\mathcal{O}_z)$$

lies in the schematic fiber of the weight morphism $\kappa : X \to \mathcal{W}$ over $\kappa(z)$.

Proposition 9.3.3. — *For each integer $j \in \{1, \dots, m\}$, $\kappa_j - \kappa_j(z) \subset I_{\mathrm{tot}}$.*

To prove this proposition, we need to recall some aspects of the theory of refined deformations that we developped in § 4.4.1. By definition of the chosen refinement $\mathcal{R}$, the refinement $\mathcal{F}_z$ of $\bar{\rho}_z = 1 \oplus \rho \oplus \chi$ is

$$\mathcal{F}_z = \iota_p \iota_\infty^{-1}(1, \varphi_1, \dots, \varphi_n, p^{-1}) = (F_1(z)p^{\kappa_1(z)}, \dots, F_m(z)p^{\kappa_m(z)}).$$

This makes sense as by assumption (5) and Lemma 8.2.1, the m Frobenius eigenvalues $F_j(z)p^{\kappa_j(z)}$, $j = 1, \dots, m$, are distinct. Of course, this refinement induces also a refinement $\mathcal{F}_{z,*}$ of each $\bar{\rho}_*$: $\mathcal{F}_{z,1} = (1)$, $\mathcal{F}_{z,\chi} = (p^{-1})$ and $\mathcal{F}_{z,\rho} = \iota_p \iota_\infty^{-1}(\varphi_1, \dots, \varphi_n)$.

Recall that in this situation, we defined in § 4.4.3 a permutation

$$\sigma \in \mathfrak{S}_m$$

that encaptures how the indices i of the weights $\kappa_i(z)$ and the Frobenius eigenvalues $F_i(z)p^{\kappa_i(z)}$ are related to the decomposition $\bar{\rho}_z = 1 \oplus \rho \oplus \chi$: if

— R_* is the set of integers i such that $D_{\mathrm{crys}}(\bar{\rho}_*)^{\varphi = F_i(z)p^{\kappa_i(z)}} \neq 0$,

— and if W_* is the set of integers i such that $\kappa_i(z)$ is a Hodge-Tate weight of $\bar{\rho}_*$,

then σ is the unique permutation of $\{1, \dots, m\}$ that sends R_* onto W_*, and that is increasing on each R_*.

Lemma 9.3.4. — (i) *σ is a transitive permutation.*

(ii) *$\mathcal{F}_z$ is a critical regular refinement of $\bar{\rho}_z$. However, for each $*$, $\mathcal{F}_{z,*}$ is a non-critical regular refinement of $\bar{\rho}_*$ and R_* is a subinterval of $\{1, \dots, d\}$.*

Proof. — Let us show assertion (ii) first. As $(1, p^{-1})$ is a critical refinement of $1 \oplus \chi$, $\mathcal{F}_z$ is a critical refinement of $\bar{\rho}_z$. For the regularity property, let us fix $j \geq 1$ an integer. By Lemma 8.2.1, the eigenvalues λ of the crystalline Frobenius on $D_{\mathrm{crys}}(\Lambda^j \bar{\rho}_z)$ such that $|\iota_\infty \iota_p^{-1}(\lambda)| = \sqrt{p}^{-j+1}$ are exactly the products of $j-1$ elements of $\iota_p \iota_\infty^{-1}(\{\varphi_1, \dots, \varphi_n\})$. We conclude then by assumption (5). The non criticality of $\mathcal{F}_{z,*}$ is obvious for $* = 1, \chi$ and is assumption (5) for $* = \rho$. Moreover, the assertion on R_* is clear, namely:

$$R_1 = \{1\}, \; R_\rho = \{2, 3, \dots, m-1\}, \; R_\chi = \{m\}. \tag{105}$$

We show now (i). Let $a \in \{1, \dots, m\}$ be the unique integer such that $\kappa_a(z) = 0$. As already said, $a \geq 2$ (see Lemma 8.3.3 (ii)), and we have

$$W_1 = \{a\}, \; W_\rho = \{1, 2, \dots, a-2, a+1, \dots, m\}, \; W_\chi = \{a-1\}. \tag{106}$$

(If $a = 2$, this means that $W_\rho = \{2, 3, \ldots, m\}$.) By definition of the permutation σ, we see now that

$$\sigma(i) = \begin{cases} a, \text{ if } i = 1, \\ i - 1, \text{ if } i = 2, \ldots, a-1 \text{ and } a \neq 2, \\ i + 1, \text{ if } i = a, \ldots, m-1, \\ a - 1, \text{ if } i = m. \end{cases}$$

which is a cycle, and we are done. □

Proof. — (of Prop. 9.3.3) As one of the κ_i is constant by assumption on X, the proposition is an immediat consequence of Lemma 9.3.4 and Corollary 4.4.5, once we know that assumptions (REG), (NCR), (INT) and (MF′) of § 4.4.1 and § 4.4.4 are satisfied. But the first three ones are by Lemma 9.3.4 (ii), and (MF′) follows from assumption (4) and from the fact that 1 and χ are one dimensional. □

Corollary 9.3.5. — *I_{tot} is a cofinite lenght ideal of A.*

Proof. — It follows from Prop. 9.3.3 and from the fact that the natural map $\mathcal{O}_{\kappa(z)} \to \mathcal{O}_z = A$ is finite by property (iv) of the eigenvariety X. □

Recall from Def. 1.5.3 that for each $* \in \mathcal{I}$, there is a (unique up to isomorphism) continuous representation

$$\rho_* : G_{E,S} \longrightarrow \mathrm{GL}_{d_*}(A/I_{\mathrm{tot}})$$

lifting $\bar{\rho}_*$.

Proposition 9.3.6. — (i) *For each $* \in \{1, \chi, \rho\}$, ρ_* is crystalline at v and $\bar{v}$.*

(ii) *Moreover, the characteristic polynomial of the crystalline Frobenius on the free A/I_{tot}-module $D_{\mathrm{crys}}(\rho_{*|E_v})$ is*

$$\prod_{i \in R_*} (T - F_i p^{\kappa_i(z)}) \in (A/I_{\mathrm{tot}})[T].$$

Proof. — Note that $\mathrm{Hom}_{G_{E_v}}(\bar{\rho}_*, \bar{\rho}_*(-1)) = 0$ for each $*$. Indeed, it is clear for $* = 1, \chi$, and it holds for $* = \rho$ as $\varphi_i \neq p\varphi_j$ for each $i, j \in \{1, \ldots, n\}$ by Lemma 8.2.1. The first part of the proposition for the place v follows then from Corollary 4.4.5 (ii) (we already checked that (INT), (REG), (NCR) and (MF′) hold in the proof of Prop. 9.3.5). But $\rho_*^\perp \simeq \rho_{\tau(*)}$ for each $*$, as they share the same trace and each ρ_* is residually irreducible, so the proposition also holds for the place $\bar{v}$, which proves (i).

By Theorem 4.4.4, we know that the crystalline representation ρ_* is trianguline over A with parameters the δ_i with $i \in R_*$ such that

$$\delta_{i|\mathbb{Z}_p^*} = \chi^{-\kappa_{\sigma(i)}(z)}, \ \ \delta_i(p) = F_i p^{\kappa_i(z) - \kappa_{\sigma(i)}(z)} \in (A/I_{\mathrm{tot}})^*.$$

By Berger's Theorem 2.2.9, the characteristic polynomial of the statement can be written as the product over the $i \in R_*$ of the characteristic polynomials of φ on $D_{\mathrm{crys}}(\mathcal{R}_{A/I_{\mathrm{tot}}}(\delta_i))$, hence the statement. $\square$

We now come to the main proposition of this subsection. We will use here assumption $BK_2(\rho)$ of hypothesis (6).

Proposition 9.3.7. — *I_{tot} is the maximal ideal of A.*

Proof. — Note that the residue field $k := k(z)$ of A lifts canonically to a subfield of A by the henselian property. Let us fix a

$$\psi : A/I_{\mathrm{tot}} \longrightarrow k[\varepsilon],$$

a k-linear ring homomorphism. We claim that for each $*$,

$$\rho_{*,\psi} := \rho_* \otimes_{A/I_{\mathrm{tot}},\psi} k[\varepsilon]$$

is a trivial deformation of $\bar{\rho}_*$, which means that we have an isomorphism

$$\rho_{*,\psi} \simeq \bar{\rho}_* \otimes_k k[\varepsilon].$$

Let us assume this claim and show how to conclude. By properties (ii) and (iv) of eigenvarieties (see Def. 7.2.5), A is generated by $\mathcal{H}$ as an $\mathcal{O}_{\kappa(z)}$-algebra. As

$$\mathcal{H} = \mathcal{A}_p \otimes \mathcal{H}_{\mathrm{ur}}$$

and by assumption (2), we see that A is generated over $\mathcal{O}_{\kappa(z)}$ by the F_i's and by the $T(\mathrm{Frob}_w)$'s for the primes $l = w\bar{w} \in S_0$. Assuming that each $\rho_{*,\psi}$ is constant, we get that for any such w,

$$\psi(T(\mathrm{Frob}_w)) \in k \subset k[\varepsilon]$$

is constant. Moreover, Prop. 9.3.6 (ii) implies that for each i,

$$\psi(F_i) \in k \subset k[\varepsilon]$$

is also constant (use that the $F_i(z)p^{\kappa_i(z)}$ are two-by-two distinct by Lemma 9.3.4). Last but not least, by Prop. 9.3.3 the image of

$$\mathcal{O}_{\kappa(z)} \longrightarrow A/I_{\mathrm{tot}} \longrightarrow_{\psi} k[\varepsilon]$$

also falls into k. As A is generated over $\mathcal{O}_{\kappa(z)}$ by the F_i and the $T(\mathrm{Frob}_w)$, we get that

$$\psi(A/I_{\mathrm{tot}}) = k.$$

As this holds for all ψ, $A/I_{\mathrm{tot}} = k$ and we are done.

Let us prove the claim now. By Prop. 9.3.6, we know that $\rho_{*,\psi}$ is crystalline at v and $\bar{v}$. Moreover, $\rho_{*,\psi}$ is obviously unramified outside S. By Lemma 8.2.13 (applied to $J = \mathrm{Ker}\,\psi$) we know that for each prime w of E not dividing p, the monodromy operator of $\rho_{*,\psi|E_w}$ admits a Jordan normal form over A/I_{tot}, hence is constant, when

$* = \rho$. This trivially also holds when $* = 1$ or χ, as any continuous G_{E_w}-extension of 1 by 1 is unramified for such a w.

If $* = 1$ or χ, the finiteness of the class number of E, and more precisely Prop. 5.2.3 (i), implies then that $\rho_{*,\psi}$ is constant. If $* = \rho$, we have $\rho_{*,\psi}^{\perp} = \rho_{*,\psi}$ (see the first paragraph of the proof of Prop. 9.3.6), hence hypothesis $BK_2(\rho)$ in assumption (6) shows again that $\rho_{*,\psi}$ is constant, which completes the proof. □

Remark 9.3.8. — We could also study the proper reducibility loci of the restriction of T to G_{E_v}. For example when $\rho_{|E_v}$ is irreducible (e.g. when $n = 1$), the same proofs as above show that they all coincide and that they lie in the schematic fiber of κ above $\kappa(z)$. However, they do not necessarily coincide with the maximal ideal of A.

9.4. The structure of R and the proof of the theorem

For $i \neq j \in \mathcal{I}$, let us consider the integers

$$h_{i,j} := \dim_L \mathrm{Ext}_T(i,j).$$

We first recapitulate all that we know about those $h_{i,j}$.

Lemma 9.4.1. — *(i) $h_{1,\rho} = h_{\rho,\chi} = h$,*
(ii) $h_{1,\chi} = 0$ and $h_{\chi,1} \leq 1$,
(iii) $h_{\rho,1} = h_{\chi,\rho} \leq n$.

Proof. — The first equalities in (i) and (ii) follow from Lemma 8.2.17, which proves (i). Assertion (iii) is then a consequence of Prop. 8.2.10 and of Prop. 5.2.7 (which assumes hypothese $BK_1(\rho)$ and whose assumptions are satisfied by Lemma 8.2.1).

We already proved that $h_{1,\chi} = 0$ in Lemma 8.3.1, so it only remains to show that $h_{\chi,1} \leq 1$. That will follow from Prop. 5.2.3 (ii) if we can show that

$$\mathrm{Ext}_T(\chi, 1) \subset H^1(E, L(-1))$$

falls into an eigenspace of the endomorphism $U \mapsto U^{\perp}(-1)$ of the latter space. But this follows from Lemma 9.3.2 and Prop. 1.8.10 as τ fixes $\rho \in \mathcal{I}$. We will actually show later that $\mathrm{Ext}_T(\chi, 1)$ falls inside the part of sign $+1$. □

As $|\mathcal{I}| = 3$, recall that from Theorem 1.5.5 that for i, j and k two-by-two distinct in $\mathcal{I}$, we have an isomorphism

$$\mathrm{Hom}_L(A_{i,j}/A_{i,k}A_{k,j}, L) \xrightarrow{\sim} \mathrm{Ext}_T(j,i). \tag{107}$$

Lemma 9.4.2. — *The integer h is the minimal number of generators of $A_{\rho,1}$.*

Proof. — By (107) above and Nakayama's lemma, we have to show that

$$A_{\rho,\chi}A_{\chi,1} \subset \mathfrak{m}A_{\rho,1}.$$

But $A_{\chi,1} = A_{\chi,\rho}A_{\rho,1}$ by Lemma 8.3.1 (that is by $h_{1,\chi} = 0$ and (107)), and we conclude as $A_{\rho,\chi}A_{\chi,\rho} \subset \mathfrak{m}$. □

Lemma 9.4.3. — (i) *There are elements $f_1, \ldots, f_n \in A_{1,\rho}$ such that $A_{1,\rho} = \sum_{i=1}^n Af_i + A_{1,\chi}A_{\chi,\rho}$.*

(ii) *There is a $g \in A_{1,\chi}$ such that $A_{1,\chi} = Ag + A_{1,\rho}A_{\rho,\chi}$.*

(iii) $A_{1,\rho}A_{\rho,1} = \sum_{i=1}^n f_iA_{\rho,1} + gA_{\chi,\rho}A_{\rho,1}$.

(iv) *For some $\lambda \in K^*$, we have $A_{\chi,\rho} = \lambda A_{\rho,1}$.*

(v) $\mathfrak{m} = A_{1,\rho}A_{\rho,1}$.

Proof. — Assertions (i) and (ii) follow from Lemma 9.4.1 (ii) and (iii), formula (107) and Nakayama's lemma. By expanding $A_{1,\rho}A_{\rho,1}$ with the formulas of (i) and (ii), we get part (iii) as the missing term satisfies

$$A_{1,\rho}A_{\rho,\chi}A_{\chi,\rho}A_{\rho,1} \subset \mathfrak{m}A_{1,\rho}A_{\rho,1},$$

hence may be deleted by Nakayama's lemma.

Assertion (iv) holds as $A_{i,j}$ and $A_{\tau(j),\tau(i)}$ are A-isomorphic submodules of K by Lemma 1.8.5 (ii). Part (v) is Prop. 9.3.7 combined with Lemma 9.3.1 (ii). □

Proof. — (of Theorem 9.1.2) By computing the minimal number of generators of $\mathfrak{m}$ with formulae (v) and (iii) of Lemma 9.4.3, as well as Lemma 9.4.2, we get

$$t \leq nh + s,$$

where s is the minimal number of generators of $A_{\chi,\rho}A_{\rho,1}$. But the A-module $A_{\chi,\rho}A_{\rho,1}$ is isomorphic to $A_{\rho,1}A_{\rho,1} \subset K$ by (iv) of *loc. cit.*, so $s \leq \frac{h(h+1)}{2}$. Indeed, if $e_1, e_2, \ldots, e_h$ are generators of the A-module $A_{\rho,1}$, then the $\frac{h(h+1)}{2}$ elements:

$$e_i^2,\ i = 1, \ldots, h, \quad \text{and} \quad e_ie_j,\ 1 \leq i < j < h,$$

are generators of the A-module $A_{\rho,1}A_{\rho,1} \subset K$, and we are done. □

Let us give a simple corollary of this analysis when $H^1_f(E, \rho)$ has dimension 1 (hence $h = 1$), which is somehow the generic situation. Recall that a local noetherian ring $(A, \mathfrak{m}, k)$ is *regular* if its Krull dimension equals the dimension $\dim_k \mathfrak{m}/\mathfrak{m}^2$ of its tangent space ([**83**, §14]).

Corollary 9.4.4. — *Assume that $h = 1$. Then A is regular of dimension $n + 1$, all the inequalities of Lemma 9.4.1 are equalities, and up to a block-diagonal change of coordinates in K^{n+2}, we have*

$$R = \begin{pmatrix} A & A^n & A \\ \mathfrak{m}^n & M_n(A) & A^n \\ Ag + \mathfrak{m}^2 & \mathfrak{m}^n & A \end{pmatrix} \subset M_{n+2}(K),$$

for some $g \in \mathfrak{m} \backslash \mathfrak{m}^2$.

Proof. — The proof of Lemma 9.4.3 actually shows that

$$t \leq h_{\rho,1}\, h \,+\, h_{\chi,1}\, s.$$

If $h = 1$, the term on the right is less that $nh + h(h+1)/2 = n+1$. As $t \geq n+1$, all these inequalities are equalities, thus $h_{\chi,1} = 1$ and $h_{\rho,1} = h_{\chi,\rho} = n$. Moreover, Lemma 9.4.2 shows that $A_{\rho,1}$ is free of rank 1 over A, as well as $A_{\chi,\rho}$ by an argument similar to Lemma 9.4.3 (iv). In particular, up to a block-diagonal change of coordinates we may then assume that $A_{\rho,1} = A_{\chi,\rho} = A$, and the corollary follows at once from Lemma 9.4.3. □

9.5. Remarks, questions, and complements

9.5.1. The case of a CM field E and the sign of Galois representations. — Throughout this paper, we have made the assumption that E is a quadratic imaginary field. Actually, most of the work we have done can be extended to the case of a CM field E (say quadratic over its totally real subfield E^+, with E^+ of degree d over $\mathbb{Q}$), but the method (both for the sign conjecture and for this chapter) ultimately fails if E is not quadratic over $\mathbb{Q}$. Let us explain why.

We would work with a unitary group $\mathrm{U}(m)$ defined over E^+, which is compact at every archimedian places and quasi-split at every finite places. Such a group exists if m is odd or if $dm \not\equiv 2 \pmod 4$. Starting with a couple (π, ρ) of an automorphic cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ a Galois representation ρ of G_E satisfying the obvious analogs of the assumptions of § 8.1, there shoud exist a representation π^n of $\mathrm{U}(m)$ under the hypothesis that $\varepsilon(\rho, 0) = (-1)^d$. The results of chapter 7 would extend easily to this case. But in chapter 8 and 9, it is used in a crucial way that $\mathrm{Ext}_T(1, \chi) = 0$. This result is deduced form the fact that $H^1_f(E, \chi) = 0$, which in turn was deduced in chapter 5 from the equality $H^1_f(E, \chi) = \mathcal{O}^*_E \otimes_{\mathbb{Z}} \mathbb{Q}_p$ (Example 5.1.2) and the fact that E is quadratic imaginary. In the general CM case, we see that instead $\dim H^1_f(E, \chi) = d - 1$, from which we cannot conclude that $\mathrm{Ext}^1_T(1, \chi) = 0$. Thus we are not able to make the proof of the sign conjecture (chapter 8) work in this case and construct a non zero element of $H^1_f(E, \rho)$. This is consistent with the fact that we haven't made any hypothesis implying $L(\rho, 0) = 0$ (since $\varepsilon(\rho, 0)$ may be 1).

The discussion above is more or less the content of Remark 9.1 of [**8**]. In the context of this book, we can offer a much finer analysis of the situation in the case of a CM field E. We denote by c the non-trival element in $\mathrm{Gal}(E/E^+)$ and by σ a lifting of c in G_{E^+} satisfying $\sigma^2 = 1$. The notations U^σ and $U^\perp$ (for a representation U of G_E) are then defined as in § 5.2.1.

The operation $U \mapsto U^\perp(1)$ defines a linear involution τ on $H^1_f(E, \mathbb{Q}_p(1))$. From Prop 1.8.10, we see that there is a sign $\epsilon = \pm 1$, such that the subspace $\mathrm{Ext}^1_T(1, \chi)$ of $H^1_f(E, \mathbb{Q}_p(1))$ is in the eigenspace of eigenvalue ϵ of τ. To be more precise about ϵ we need actually the following result of independent interest:

Theorem 9.5.1. — *Assume only the hypothesis (P0) of* $\mathrm{Rep}(m)$ *(extended to the case of a CM field E). Let* π *be an automorphic representation of* $\mathrm{U}(m)(\mathbb{A}_{E^+})$ *as in* $\mathrm{Rep}(m)$ *such that the attached Galois representation* ρ_π *is absolutely irreducible. Then if* $Q \in \mathrm{GL}_m(\bar{\mathbb{Q}}_p)$ *is such that* $\rho_\pi^\perp(g) = Q\rho_\pi(g)Q^{-1}\chi(g)^{m-1}$ *for all* $g \in G_E$*, then we have* ${}^tQ = Q$.

The existence of a Q as in the statement follows from remark (i) after $\mathrm{Rep}(m)$, and it follows from the absolute irreducibility of ρ_π that ${}^tQ = \epsilon Q$ with $\epsilon = \pm 1$ (see Lemma 1.8.4). If m is odd, then it is clear that $\epsilon = 1$. We postpone the proof that this result also holds for an even m to a subsequent work [(1)].

Going back to our specific situation, we can deduce

Corollary 9.5.2. — $\mathrm{Ext}^1_T(1, \chi)$ *is a subspace of the* $+1$*-eigenspace of* τ *in* $H^1_f(E, \mathbb{Q}_p(1))$.

Proof. — By Lemma 9.3.2, the generic representation ρ_K is absolutely irreducible, hence we are in the situation of Example 1.8.7. We have in particular a collection of signs ϵ_s indexed by the irreducible components of K. By an argument already given in § 4.3.3, and the accumulation of classical points at $z \in X$, Theorem 9.5.1 shows that each of those signs is $+1$. The corollary follows then from Prop 1.8.10 (i). □

But it turns out, perhaps surprisingly, that the information given by the above corollary is empty:

Lemma 9.5.3. — *The involution* τ *is the identity of* $H^1_f(E, \mathbb{Q}_p(1))$.

Proof. — We recall the Kummer isomorphism

$$\mathrm{kum} : E^*/(E^*)^{p^n} \to H^1(E, \mathbb{Z}/p^n\mathbb{Z}(1)),$$

that sends $x \in E^*$ to the class of the cocycle $\mathrm{kum}(x)$ of G_E defined by

$$\mathrm{kum}(x)(s) = s(u)/u,$$

(1) See "The sign of the Galois representations attached to automorphic forms for unitary groups", preprint.

where $u \in \bar{E}^*$ is an element such that $u^{p^n} = x$. The conjugation by σ defines an involution of $H^1(E, \mathbb{Z}/p^n\mathbb{Z}(1))$ by sending an extension U to U^σ. In terms of cocycle, this involution sends a cocycle j to j^σ, with $j^\sigma(s) = j(\sigma s \sigma^{-1})$. Hence

$$\operatorname{kum}(x)^\sigma(s) = (\sigma s \sigma^{-1})(u)/u = \sigma(s(\sigma(u))/\sigma(u)) = \sigma(u)/s(\sigma(u)) = \operatorname{kum}(c(x))^{-1}$$

(use that $\sigma^2 = 1$ and that σ acts as the reciprocal on roots of unity).

Another natural involution on $H^1(E, \mathbb{Z}/p^n\mathbb{Z}(1))$ is $U \mapsto U^*(1)$, and it is easy to see that this involution sends $\operatorname{kum}(x)$ on $\operatorname{kum}(x^{-1})$. Finally, the involution τ on $H^1(E, \mathbb{Z}/p^n\mathbb{Z}(1))$ defined by $U \mapsto U^\perp(1)$ is the composition of the two preceding involutions, and thus sends $\operatorname{kum}(x)$ on $\operatorname{kum}(c(x))$. Taking the limit over n, tensorizing by $\mathbb{Q}_p$, and restricting to H^1_f, we see that under the Kummer isomorphism

$$\operatorname{kum} : \mathcal{O}_E^* \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^1_f(E, \mathbb{Q}_p(1)),$$

the involution τ corresponds to the conjugation c. Hence the lemma is reduced to the assertion that c acts by the identity on a finite index subgroup of $\mathcal{O}_E^*$. But this arithmetical statement is a well known consequence of Dirichlet's unit theorem, that says that $\mathcal{O}_E^*$ and $\mathcal{O}_{E+}^*$ have the same rank. □

9.5.2. When is T_A the trace of a representation over A?— Let us go back to the assumptions of § 9.1 and to Theorem 9.1.2. We keep the notations of § 9.2. If $h = 1$, then T_A is the trace of a representation $G_{E,S} \longrightarrow \mathrm{GL}_m(A)$ by Cor. 9.4.4. Another way to argue would be to say that A is regular, hence a UFD, and use Prop 1.6.1. Conversely:

Lemma 9.5.4. — *If T_A is the trace of a representation of $G_{E,S}$ over A, then we have either $h = 1$ or $h = t \geq n + 1$.*

Proof. — Since T_A is the trace of a representation we may assume that $R \subset M_m(A)$, *i.e.* that the A-modules $A_{i,j}$ are actually ideals of A (use Prop.1.6.4, Lemma 1.3.7 and Prop. 1.3.8). From Lemma 9.4.3(v) we see that either $A_{\rho,1} = A$ and $A_{1,\rho} = \mathfrak{m}$ or $A_{1,\rho} = A$ and $A_{\rho,1} = \mathfrak{m}$, and we conclude by Lemma 9.4.2. □

In the conclusion of the above lemma, the case $h = t$ seems very unlikely. However, it is not possible to exclude it by a simple GMA analysis, since the data $A_{\chi,1} = \mathfrak{m}^2$, $A_{\rho,1} = A_{\chi,\rho} = \mathfrak{m}$, $A_{1,\chi} = A_{1,\rho} = A_{\rho,\chi} = A$ define a GMA satisfying all the assertions of Lemma 9.4.1 (which is even equipped with an obvious anti-involution).

Another related intriguing question is to know whether $\operatorname{Ext}_T(\chi, 1) \neq 0$. By Cor. 9.4.4, this is the case if $h = 1$, and the example above shows that it is not formal from what we have proved.

9.5.3. Other remarks and questions. — From a philosophical point of view, a very intriguing open question is the following one.

Question. — *Should we expect that* $\mathrm{Ext}_T(1,\rho) = H^1_f(E,\rho)$*?*

On the one hand, although $\mathrm{Ext}_T(1,\rho)$ is a canonical subspace of $H^1_f(E,\rho)$, it is attached to the unitary group $\mathrm{U}(n+2)$, so its arithmetic content is somehow included in the one of the cohomology of the related unitary Shimura varieties. There is no reason a priori that all the cohomology classes in $H^1_f(E,\rho)$ be related to the cohomology of this "small" class of algebraic varieties (rather than, say, to all the algebraic varieties over E, as we might expect from the Fontaine-Mazur conjecture).

On the other hand, the trend of ideas initiated by Mazur-Wiles' proof of Iwasawa's main conjecture and by Wiles' $R = T$ philosophy rather suggests that we may have equality in our context too. This is also corroborated by our results in § 7.6.

Note that by Corollary 9.1.3, we can detect directly on the geometry of the eigenvariety X at x if $\mathrm{Ext}_T(1,\rho)$ has rank ≥ 2. It would be very interesting to find examples where it is indeed the case! As we saw, the space X is built from some rather explicit spaces of p-adic automorphic forms on the definite unitary group $\mathrm{U}(m)$, thus we hope that some numerical experiments could be made. [2] The first step is actually to find a ρ for which the Bloch-Kato conjecture predicts that $\dim_L H^1_f(E,\rho) > 1$. When $n \leq 2$, this amounts to find some modular form of even weight $k \geq 4$, whose sign is -1, and whose archimedean L-function vanishes at order ≥ 2 at $k/2$. The authors do not know any such example at the moment [3].

As explained in Remark 9.1.5, we hope that we can go further in the future and make the L-function of ρ (or say a p-adic version) enter into the picture, altough it is not clear how at the moment.

(2) Actually, for those questions as well as many others, it would be very useful to have at hand a database like the one of William Stein for classical modular forms, as well as a program computing slopes as Buzzard's one.

(3) Of course, if we could have handled the case $k = 2$ it would have sufficed to take an elliptic curve over $\mathbb{Q}$ with sign -1 and rank ≥ 2, and there are plenty of them.

APPENDIX: ARTHUR'S CONJECTURES

In this appendix,[(1)] we offer a brief and somewhat personal exposition of parts of Langlands' and Arthur's conjectural program. This exposition will allow us to check that the assumptions $\mathrm{Rep}(m)$ and $\mathrm{AC}(\pi)$ about automorphic forms on unitary groups that we have made in chapter 6 are predicted by that program. We do this for two reasons: first, this should make our assumptions more believable, and second, more importantly, putting those assumptions in the general picture of Langlands' and Arthur's program is very helpful to understand our method and how it may or may not be generalized. For a more complete overview of the conjectures, we refer the reader to [**3**], [**22**] and [**100**].

Let F be a number field and G a connected reductive group[(2)] over F. An *automorphic representation* π of G is an irreducible constituent of the right regular representation of $G(\mathbb{A}_F)$ on the Hilbert space[(3)]

$$L^2(G(F)\backslash G(\mathbb{A}_F), \omega, \mathbb{C}),$$

for some admissible character ω of $G(\mathbb{A}_F)$ which is trivial on $G(F)$. In general, this representation will have a discrete and a continuous part, which makes the previous definition rather unprecise. Recall that π is *discrete* if it occurs discretely in (that is, as a sub-representation of) the space above, such π are well defined. For example, it is known that the L^2 above are completely discrete if (and only if) G is anisotropic modulo its center (this was the case of our definite unitary groups of § 6.2). In general, Langlands' theory of Eisenstein series reduces the study of all the automorphic π to the discrete ones, and we will focus on these in this appendix. For π a unitary admissible irreducible representation of $G(\mathbb{A}_F)$ which is not a discrete automorphic representation, we set $m(\pi) = 0$.

(1) We thank Florian Herzig for his remarks concerning this appendix.

(2) The conjectures we shall describe so far have been mainly tested for classical groups—it is possible that some minor changes will be needed in the exceptional cases.

(3) Precisely, let Z be the center of G and Z_∞ be the connected component of its real points. The aforementionned space is the space of measurable complex functions f on $G(F)\backslash G(\mathbb{A}_F)$ such that the associated map $g \mapsto f(g)\omega^{-1}(g)$ is Z_∞-invariant and square-integrable on $G(F)Z_\infty\backslash G(\mathbb{A}_F)$ for a (finite) $G(\mathbb{A}_F)$-invariant measure on this latter space (which exists by a result of Borel and Harish-Chandra).

The aim of Arthur's program (an extension of Langlands' program) is to compute $m(\pi)$ for all π. This is done by a very rich set of conjectures that are not completely rigid, and that proceed in two steps: describing a natural partition of the set of all discrete automorphic representations into "packets" and understanding $m(\pi)$ for π within a given packet.

We denote by $\Pi_{\mathrm{unit}}(G, F)$ (or Π_{unit} when there is no ambiguity) the set of all isomorphism classes of irreducible unitary representations of $G(\mathbb{A}_F)$ and by $\Pi_{\mathrm{disc}}(G, F)$ the subset of discrete automorphic representations. Two interesting subsets of $\Pi_{\mathrm{disc}}(G, F)$ are $\Pi_{\mathrm{cusp}}(G, F)$, the set of cuspidal automorphic representations, and $\Pi_{\mathrm{temp}}(G, F)$, the set of tempered *discrete* automorphic representations.

Example A.0.1. — ($G = \mathrm{GL}_m$) The Ramanujan conjecture asserts that

$$\Pi_{\mathrm{cusp}}(\mathrm{GL}_m, F) = \Pi_{\mathrm{temp}}(\mathrm{GL}_m, F)$$

(this is known to be false for other reductive groups). [4] Moreover, a theorem of Moeglin-Waldspurger [**87**] shows that the full discrete spectrum of G is built from the cuspidal spectrum of the GL_d with d *dividing* m, which might be suprising. For example, if m is a prime, a discrete π is either cuspidal or one dimensional. In this context, each packet of discrete representations is actually a singleton, and each discrete representation occurs with multiplicity one (Shalika's *weak multiplicity one theorem*). All these facts are actually predicted by Arthur's philosophy, which not only predicts the $m(\pi)$ but also gives a general hint about how the discrete spectrum of a general G is constructed from the tempered one, and even from the cuspidal one of the GL_m.

A.1. Failure of strong multiplicity one and global A-packets

For two unitary irreducible representations π and π' of $G(\mathbb{A}_F)$, say $\pi \sim \pi'$ if $\pi_v \simeq \pi'_v$ for almost all primes v.

When G is GL_m, or an inner form of GL_m, it is known [5] that if π and π' are discrete automorphic then $\pi \sim \pi'$ implies that $\pi = \pi'$ as a subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \omega, \mathbb{C})$ (*strong multiplicity one*) but this statement is known to be false for some other groups, including some groups very close to GL_m like SL_m, and our unitary groups.

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(4) Actually there are trivial counterexamples, like the one dimensional representations of $G = D^*$ for a division algebra D, but there are deeper ones with G split, or like the representation π^n we are especially interested in.

(5) It is due to Miyake and Jacquet-Langlands for $m = 2$, to Jacquet, Piatietskii-Shapiro and Shalika for GL_m, to Badulescu for inner forms of GL_m which split at all archimedean places, and to Badulescu-Renard in the remaining cases.

Following Arthur we should be able to define naturally certain disjoint subsets of Π_{unit} called *global A-packets*. Every global A-packet Π should be a subset of an equivalence class for $\sim$, and each nonempty Π should contain a discrete representation. Moreover, the restriction to Π_{disc} of an equivalence class of $\sim$ should coincide with the set of discrete elements of a union of A-packets, and even of a single A-packet in many cases. [6] However, an A-packet will contain many non discrete representations in general. The motive for such an enlargement is to allow the global A-packets to be products of *local A-packets*, as we soon shall see. For GL_m and its inner forms, thus, every A-packet is a singleton. A global A-packet Π is said tempered, if every $\pi \in \Pi$ (discrete or not) is tempered.

To describe the set of A-packets we need to introduce the conjectural *Langlands group* L_F.

A.2. The Langlands groups

For K a topological group, we define $\mathrm{Rep}_m(K)$ (resp. $\mathrm{Irr}_m(K) \subset \mathrm{Rep}_m(K)$) as the set of equivalence classes of complex m-dimensional continuous (resp. moreover irreducible) representations of K whose range contains only semi-simple elements.

According to Langlands, there should exist for tannakian reasons a group L_F (called by others the Langlands group), extension of the global Weil group W_F by a compact group, with a natural bijection (the *global correspondence*) between $\mathrm{Irr}_m(L_F)$ and $\Pi_{\text{cusp}}(\mathrm{GL}_m, F)$. The collection of L-groups $\{L_F\}$ with F varying should satisfy conditions similar to the collection of global Weil groups $\{W_F\}$ (see [**116**]).

For v a place of F, we define explicitly a local Langlands group L_{F_v} as the Weil group W_{F_v} if v is archimedean, and as $W_{F_v} \times \mathrm{SU}_2(\mathbb{R})$ otherwise. In this latter case the local Langlands group is closely related to the Weil-Deligne group, in the sense that there is a simple bijection between $\mathrm{Rep}_m(L_{F_v})$ and the set of Frobenius semi-simple Weil-Deligne representations (r, N) of F_v that we recalled in §6.3. Langlands conjectured the existence of a natural bijection (the *local correspondence*) between $\mathrm{Rep}_m(L_{F_v})$ and the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_m(F_v)$. He proved that conjecture when v is archimedean, and the non-archimedean case is now also a theorem of Harris and Taylor [**62**], relying on works of many people.

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[6] The obstruction to this uniqueness property should be explained by the existence of everywhere locally equivalent A-parameters for G which are not globally equivalent (see below). Such parameters should not exist for an inner form of GL_m, but also for unitary groups (see Exp.Prop. A.11.9). We thank Toby Gee for pointing out an inconsistency in the first version of this paragraph.

It is part of the conjectures that there should exist a distinguished class of embeddings $L_{F_v} \hookrightarrow L_F$, and the global correspondence should coincide with the local correspondence after restriction to the local Langlands' groups.

A.3. Parameterization of global A-packets

We now go back to a general reductive group G. We refer to [**25**] for the definition of the L-group ${}^LG$ of G. Let us simply recall that ${}^LG$ is a semi-direct product of W_F by the dual group $\widehat{G}(\mathbb{C})$ of $G(\mathbb{C})$, that the product is direct if G is split, and that the L-group of two inner forms are canonically isomorphic.

Following Arthur, a *global A-parameter* (for G) is a continuous homomorphism

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG$$

such that

(o) $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ is holomorphic and falls into $\widehat{G}(\mathbb{C})$,

(i) for all $w \in L_F$, the image of $\psi(w)$ in the quotient W_F of ${}^LG$ is the same as the image of w by the map $L_F \to W_F$,

(ii) $\psi(w)$ is semi-simple [(7)] for all $w \in L_F$,

(iii) the image $\psi(L_F)$ is bounded in ${}^LG$ modulo the center $Z(\widehat{G}(\mathbb{C}))$ of $\widehat{G}(\mathbb{C})$,

(iv) ψ is *relevant*, that is $\psi(L_F \times \mathrm{SL}_2(\mathbb{C}))$ is not allowed to lie in a parabolic subgroup [(8)] of ${}^LG$ unless the corresponding parabolic subgroup of G is defined over F.

Note that condition (iv) is automatic if G is quasi-split since every parabolic subgroup of a quasi-split group is defined over the base field.

Remark A.3.1. — In the definition of the L-group, the Weil group W_F acts on $\widehat{G}(\mathbb{C})$ through a finite quotient $\mathrm{Gal}(E/F)$, where E is a finite Galois extension of F on which G splits. For the sake of defining global A-parameters (the same remark will apply for local A-parameters and L-parameters, see below), it would not change anything if the L-group ${}^LG$ was replaced by the *reduced L-group* of G, namely the semi-direct product of $\mathrm{Gal}(E/F)$ by $\widehat{G}(\mathbb{C})$ (being understood that in condition (i) above, each occurrence of W_F has to be replaced by $\mathrm{Gal}(E/F)$). In particular, an A-parameter for a *split* group G is simply a morphism $L_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$ satisfying conditions (o), (ii) and (iii) above—remember that (iv) is automatic.

---

(7) Recall that an element g of ${}^LG$ is semisimple if its image in each representation ${}^LG \to {}^L\mathrm{GL}_m$ has a semisimple $\mathrm{GL}_m(\mathbb{C})$ component.

(8) Recall that it is a subgroup $P \subset {}^LG$ which surjects onto W_F and which is the normalizer in ${}^LG$ of a parabolic subgroup of $\widehat{G}$, see [**25**, §3.3].

Two parameters are said *equivalent* if they are conjugate by an element of $\widehat{G}(\mathbb{C})$, up to a 1-cocycle of L_F in $Z(\widehat{G}(\mathbb{C}))$ which is locally trivial at every place (see [**100**] §2.1). In the cases we will deal with, namely GL_m and unitary groups, it turns out that every such cocycle is trivial ([**100**] §2.2). Thus in those cases the equivalence relation for parameters is just the conjugacy by an element of $\widehat{G}(\mathbb{C})$.

Definition A.3.2. — A global A-parameter ψ is said to be discrete if $C(\psi)^0 \subset Z(\widehat{G}(\mathbb{C}))$, where $C(\psi) = \{g \in \widehat{G}(\mathbb{C}), g\psi(w) = \psi(w)g \ \forall w \in L_F \times \mathrm{SL}_2(\mathbb{C})\}$ is the centralizer in $\widehat{G}(\mathbb{C})$ of the image of ψ, and $C(\psi)^0$ is its identity component.
— A global A-parameter ψ is said to be tempered [9] if its restriction to $\mathrm{SL}_2(\mathbb{C})$ is trivial.

Example A.3.3. — Assume again $G = \mathrm{GL}_m$. We see at once that a global A-parameter ψ is discrete if, and only if, the corresponding representation $L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is irreducible. In particular, there exists a divisor d of m and an irreducible tempered parameter $\psi' : L_F \to \mathrm{GL}_{\frac{m}{d}}(\mathbb{C})$ such that $\psi' = \psi \otimes [d]$, where $[d]$ denotes the unique d-dimensional holomorphic representation of $\mathrm{SL}_2(\mathbb{C})$. Moreover, using the fact that L_F (as W_F) should be an extension of an abelian group (namely $\mathbb{R}$) by a compact group, we see easily that $\mathrm{Irr}_m(L_F)$ should be in bijection with the set of global discrete tempered A-parameters of GL_m. Note that this formalism matches perfectly with Ex. A.0.1.

The first main conjecture of Arthur is the existence of a natural correspondence which associates to every global discrete A-parameter of G (up to equivalence) an A-packet of G, or the empty set.

Two A-parameters ψ and ψ' should be sent to the same (non-empty) A-packet if and only if they are equivalent. The correspondence above should send tempered A-parameters to tempered A-packets, in which case it should coincide with the former theory of Langlands. Note that this requirement, in the case $G = \mathrm{GL}_m$, is the generalized Ramanujan conjecture.

Note that some global A-parameters, even satisfying the relevance condition, may very well be sent to the empty set. We shall give an example below (see Remark A.12.4). However, this should not happen when G is quasi-split, or for a tempered A-parameter.

To understand the "naturality" of the correspondence between discrete global A-parameters and global A-packets, we need to introduce the local counterpart of global A-packets and global A-parameters. It should be stressed here that contrary to Langlands' theory of local L-packets which should apply to all the admissible irreducible

[9] There is a common abuse of language here, as strictly we should say *essentially tempered*, that is, tempered up to a twist.

representations, the introduction of Arthurs' A-packets is mainly motivated by global considerations and basically applies to local components of global automorphic representations.

A.4. Local A-packets and local A-parameters

Following Arthur, a *local A-parameter* is a continuous morphism

$$\psi_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

such that the analogues of conditions (o) to (iii) of global parameters are satisfied (of course, F has to be replaced by F_v everywhere there, and ${}^L G$ is now the L-group of G/F_v). Note that there is no relevance condition (iv) in the definition. As in the global case, a local A-parameter is said to be *tempered* if it is trivial on $\mathrm{SL}_2(\mathbb{C})$. The restriction ψ_v of a global A-parameter ψ to $L_{F_v} \times \mathrm{SL}_2(\mathbb{C})$ is obviously a local A-parameter, and ψ_v is tempered if ψ is.

Following Arthur, to every local A-parameter should correspond a finite set, possibly empty, of irreducible unitary representations of $G(F_v)$. This map will not be injective in general. Moreover, in contrast with global A-packets, local A-packets are not, in general, disjoint, and their reunion will not be the set of all unitary irreducible representations of $G(F_v)$ (but rather a subset which is pretty close to the set of local constituents of global automorphic representations).

According to Arthur, a global A-packet Π defined by an A-parameter ψ should be the set of restricted tensor products

$$\Pi = \{\pi = \otimes'_v \pi_v,\ \ \pi_v \in \Pi_v\},$$

for a set of local representations π_v belonging for each v to the local A-packet Π_v corresponding to ψ_v, and such that almost all π_v are unramified (and $G(\mathcal{O}_{F_v})$ spherical).

A tempered A-parameter should define a tempered A-packet, that is, an A-packet all of whose members are tempered.

Ultimately, local A-packets should be constructed using our understanding of the trace formula and its stabilization. One key property that an A-packet should satisfy is that a suitable non-trivial linear combination of the character-distributions of its members should be a stable distribution (that is essentially, a distribution that is invariant by conjugation by elements of $G(\bar{F}_v)$, not only of $G(F_v)$).

To say more on the correspondence between A-parameters and A-packets, we need to review the earlier notions of L-parameters and L-packets, due to Langlands.

A.5. Local L-parameters and local L-packets

Following Langlands, there should be a *partition* of the set of equivalence classes of all *admissible* irreducible representations of $G(F_v)$ into (local) L-packets. We stress that the local L-packets behave much more nicely than the local A-packets, since the former are *disjoint* and since their reunion does not miss any admissible irreducible representation.

The set of L-packets should be in bijection with the set of $\widehat{G}(\mathbb{C})$-conjugacy classes of relevant *L-parameters.* Recall that an L-parameter is a continuous morphism

$$\phi_v : L_{F_v} \longrightarrow {}^L G$$

that satisfies conditions (i) and (ii) (but not (iii)) of § A.3, it is said to be *relevant* if it satisfies moreover (iv) *loc. cit.* Moreover, an L-parameter is said to be *discrete* if its connected centralizer is central, as in Def. A.3.2.

A local A-parameter ψ_v defines a local L-parameter (that may not be relevant) by the formula

$$\phi_{\psi_v}(w) = \psi_v\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right). \tag{108}$$

Here $|.| : L_{F_v} \to \mathbb{R}^*$ is the composition of $L_{F_v} \to \mathrm{W}_{F_v}^{\mathrm{ab}} \xrightarrow{\sim}_{\mathrm{rec}^{-1}} F_v^*$ with the norm of F_v^*. The local A-packet corresponding to ψ_v should contain the local L-packet corresponding to ϕ_{ψ_v} (if relevant) and could be larger in general, but not when ψ_v is tempered.

The problem with L-packets that motivated the introduction of A-packets is that it is not always possible to construct a non-trivial linear combination of the characters of its members that is a stable distribution. This problem does not arise for tempered L-packets.

A.6. Functoriality

If G and G' are two groups as above, any admissible morphism of L-groups (that is a holomorphic group homomorphism compatible to the projection to W_F)

$${}^L G \to {}^L G'$$

induces a map from L-parameters or (local and global) A-parameters for G to similar parameters for G'. According to the conjectures described above, this should determine a map (rather, a correspondence) from the set of packets (local or global, A or L) for G to the set of packets for G'. Such a conjectural correspondence is an instance of Langlands' functorialities.

The most basic example is the case where G and G' are inner forms of each other, which is sometimes called a *Jacquet-Langlands* transfer. Then ${}^LG = {}^LG'$ and the identity map should define a correspondence between packets of G and G'. Note that even in this simplest case, this correspondence may not be a map (even for local L-packets) since a parameter relevant for G may not be relevant for G'.

Note that in defining functorialities, it is useful to work with the full L-groups, not only the reduced ones, since there are more morphisms between full L-groups.

Example A.6.1. — Let D be a quaternion division algebra over F, $G = D^*$ and $G' = \mathrm{GL}_2$. The A-parameter of the trivial, discrete, representation π of G or G' (global or local) is the discrete, non-tempered, parameter $1 \otimes [2]$ in the notations of Ex. A.3.3, and the corresponding A-packets have a single element π. Of course, the Jacquet-Langlands A-functoriality makes those trival representations correspond.

If we had tried to understand this simple transfer in the context of L-functoriality, we see that we could not have asked that the transfer of π be both discrete and compatible at all the finite places with the local Langlands correspondence. Indeed, for each finite place v such that D_v is nonsplit, this latter correspondence would match the trivial representation of D_v^* with the Steinberg representation of $\mathrm{GL}_2(F_v)$, which is infinite dimensional: that would contradict the strong approximation theorem for SL_2.

In other words, as long as we are interested in the discrete spectrum (say), and with non tempered representations, the A-functoriality is better behaved than the L-one, and it is actually made for that. This phenomenon is not just a fancy problem related to the trivial representation, but it appears in all kind of functorialities. We will give later in Ex. A.10 a deeper example due to Rogawski in the case of a base change.

A.7. Base change of parameters and packets

In this paragraph, it will be convenient to assume that F is either a global or a local field. Recall the notion of L-parameter we gave applies to the local context only, whereas the one of A-parameter does in both cases. Let E be a finite extension of F and set

$$G_E := G \times_F E.$$

The restriction of an L-parameter ϕ (resp. an A-parameter ψ) of G to L_E (resp. to $L_E \times \mathrm{SL}_2(\mathbb{C})$) defines an L-parameter ϕ_E (resp. an A-parameter ψ_E) of G_E. The map $\phi \to \phi_E$ (resp. $\psi \to \psi_E$) is called the *base change map* for parameters.[10] In general, ψ_E is tempered if ψ is, but ψ_E may be not be discrete although ψ is.

[10] This base change may be viewed as a special case of the general functoriality by considering the natural map ${}^LG \to {}^L(\mathrm{Res}_F G_E)$.

We shall be mainly interested in the case where G_E is GL_m, or an inner form of GL_m, in which case

$$^LG_E = {}^L(\mathrm{GL}_m)_E = \mathrm{GL}_m(\mathbb{C}) \times W_E.$$

Let us make this assumption until the end of § A.7.

In any of the three possible cases it is possible to attach to ϕ_E or ψ_E a single admissible irreducible representation of $\mathrm{GL}_m(E)$ (in the local case) or of $\mathrm{GL}_m(\mathbb{A}_E)$, as follows. If ϕ is a local L-parameter, ϕ_E is a local L-parameter for G_E, hence for GL_m over E (for which it is automatically relevant), and thus defines a single admissible irreducible representation of $\mathrm{GL}_m(E)$ by the local Langlands correspondence. If ψ is a local A-parameter, then so is ψ_E, and the map ϕ_{ψ_E} defined in formula (108) is an L-parameter of GL_m/E and thus defines again an admissible irreducible representation of $\mathrm{GL}_m(E)$. Finally if F is global and ψ a global A-parameter, we associate to ψ the restricted tensor product on all places w of E of the representation attached to the base change ψ_w of ψ_v (if v is the place of F below w).

To summarize, we have defined, assuming Langlands and Arthur's parameterization, a base change to GL_m (the quasisplit inner form of G_E) of a global *A-packet*[(11)] Π of G, which is a single irreducible admissible representation of $\mathrm{GL}_m(\mathbb{A}_E)$, and also for local A-packets and local L-packets for G (the result being then an admissible irreducible representation of $\mathrm{GL}_m(E)$). Note that by definition, the base change for global and local A-packets are compatible in the obvious sense.

A.8. Base change of a discrete automorphic representation

We keep our assumption that $^LG_E = {}^L(\mathrm{GL}_m)_E$. If F is a global field and π is a global discrete automorphic representation of G, it belongs to a unique global A-packet Π which has a well defined base change as we saw above. We define the *base change* of π as the base change π_E of its A-packet. If ψ is an A-parameter corresponding to Π and if ψ_E is discrete, then π_E should be a discrete automorphic representation of GL_m/E.

If F_v is a local field and π_v is an irreducible representation of $G(F_v)$, it belongs to a single L-packet Π_v. If w is a place of E above v, we may define the base change of π_v as the base change π_{E_w} of this local L-packet. Note that we can not use local A-packets to define local base change unambiguously since a representation may belong to several local A-packets that have a different base change (see Ex. A.10 below).

The inconvenience of defining local and global base change for representations using different types (A and L) of packets is that in general there is *no compatibility*

(11) Note that it is clear form the definition that even if an A-packet is defined by two different parameters, the base change representation attached to those parameters is the same.

between local and global base change for representations: it may well be the case that $(\pi_E)_w \not\simeq \pi_{E_w}$ for some $w|v$. However, if π is a global discrete automorphic representation which is *tempered*, it belongs to a tempered A-packet Π, product of tempered local A-packets Π_v that are also L-packets and contain π_v. Hence it is clear that the v-component of the global base change of Π in this case should be the base change of the local components π_v of π.

A.9. Parameters for unitary groups and Arthur's conjectural description of the discrete spectrum

In this paragraph we specialize to certain unitary groups the formalism developped above, including parameters (L or A, local or global) and the base change to GL_m. We fix a unitary group $G := \mathrm{U}(m)$ (quasi-split or not) in m variables attached to a CM extension E/F of number fields. The reduced L-group of $\mathrm{U}(m)$ is the semi-direct product

$$^L\mathrm{U}(m) = \mathrm{GL}_m(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$$

where $\mathrm{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} = \langle c \rangle$ acts on $\mathrm{GL}_m(\mathbb{C})$ by

$$cMc^{-1} := \phi_m {}^tM^{-1}\phi_m^{-1}, \quad M \in \mathrm{GL}_m(\mathbb{C}), \tag{109}$$

where,

$$\phi_m := \begin{pmatrix} & & & (-1)^{m+1} \\ & & \cdots & \\ & -1 & & \\ 1 & & & \end{pmatrix}. \tag{110}$$

Note that

$$^t\phi_m = \phi_m^{-1} = (-1)^{m+1}\phi_m, \quad c\phi_m = \phi_m c. \tag{111}$$

Note that $\mathrm{U}(m)_E$ is an inner form of GL_m, hence the theory of base change to GL_m/E explained in §A.7, A.8 applies.

An A-packet Π of G will be a tensor product of local A-packets Π_v, where Π_v will have one element when v splits in E, but more than one in general for the other places. In particular, Π may be infinite in general. We now review what Arthur's theory of parameters implies for the structure of the discrete spectrum of G (and for these packets), following Rogawski's analysis [**100**, §2.2]. If

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{U}(m)$$

is a discrete A-parameter, Rogawski shows that $\psi_E : L_E \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is a direct sum of r pairwise nonisomorphic irreducible representations ρ_j (see the proof of Lemma 2.2.1 *loc. cit.*) that satisfy [12] $\rho_j^\perp \simeq \rho_j$. He defines ψ to be *stable* if ψ_E is irreducible, and says that an A-packet Π of $\mathrm{U}(m)$ is stable if its A-parameter is. As explained in Ex. A.3.3, ψ is stable if and only if ψ_E is discrete, in which case there is a discrete automorphic representation π_E of GL_m/E which is the base change of ψ as explained in §A.8. That representation π_E is cuspidal if and only if ψ is tempered.

In general, for any (unordered) partition

$$m = m_1 + \cdots + m_r,$$

Rogawski defines an admissible map [13]

$$\xi : {}^L(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_r)) \to {}^L\mathrm{U}(m).$$

He shows then that any discrete A-parameter ψ of G can be written uniquely as

$$\xi \circ (\psi_1 \times \cdots \times \psi_r)$$

where the ψ_j are distinct stable parameters of the quasisplit group $\mathrm{U}(m_i)$ and for a unique unordered partition $m = m_1 + \cdots + m_r$ as above ([**100**, Lemma 2.2.2]). We say that ψ is *endoscopic* if it is not stable, *i.e.* if $r > 1$.

If G is quasisplit, this reduces conjecturally the study of the discrete spectrum of G to the stable parameters (compare with Ex. A.3.3), and the general case is then a matter of relevance. [14] This *structure* of the discrete spectrum of G, as well as other predictions of Arthur, have been verified by Rogawski when $m \leq 3$ [**99**].

We will refine slighty this study in § A.11 by giving some sufficient conditions on an A-parameter ψ' of GL_m/E to *descend to* $\mathrm{U}(m)$, *i.e.* ensuring that $\psi' = \psi_E$ for some discrete A-parameter ψ of $\mathrm{U}(m)$. As an exercise, [15] the reader can already check that the parameter $1 \otimes [m]$ descends to a stable nontempered parameter of $\mathrm{U}(m)$. Its associated A-packet has a single element, namely the trivial representation.

Remark A.9.1. — Let π be a discrete automorphic representation of G. If π is nontempered, the presence of a nontrivial representation of the $\mathrm{SL}_2(\mathbb{C})$ in the A-parameter of (the A-packet containing) π imposes strong restrictions on the π_v. For example, if v is

(12) If $\rho : L_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$ is a representation, we define another representation $\rho^\perp : L_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$ by setting $\rho^\perp(g, u) := {}^t\rho(dgd^{-1}, u)^{-1}$ where $d \in L_F$ is any element not belonging to L_E: see § A.11.

(13) This map is a special case of the so-called endoscopic functoriality as $\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_r)$ is not a Levi subgroup of $\mathrm{U}(m)$ when $r > 1$ (see [**100**, §1.2]). When some m_i does not have the same parity as m, ξ is actually not defined at the level of the reduced L-groups, as a character μ as in § 6.9.2 occurs in its definition, and μ is not of finite order. It is an important fact that ξ is also not canonical at all in this case, as it depends on this choice of μ. The uniqueness assertion in Rogawski's description also assumes that such a μ has been fixed once and for all.

(14) The situation is actually not as simple as it may seem in the nontempered case, as a relevant parameter may exceptionally lead to an empty packet. Moreover, the multiplicity formula is more complicated in the nonsplit case.

(15) Note that ϕ_m is antisymmetric if and only if $[m]$ is symplectic.

archimedean and $G(F_v)$ is compact, then π_v cannot be regular in the sense of § 6.8.2. In particular, the regularity assumption there should imply that π is tempered, hence that its local and global base change are compatible (see § A.8).

By contrast, there are very few conditions ensuring that π belongs to a stable A-packet. A standard sufficient condition (as in the works of Kottwitz, Clozel and Harris-Taylor) is that π_v is square-integrable at a split place v (this follows easily from Arthur's formalism), but this condition rules out a lot of very interesting stable packets.

A.10. An instructive example, following Rogawski

We give now a very instructive example of a nontempered A-packet for the group U(3) which illustrates most of the subtleties that appeared till now. It was found by Rogawski ([**99**],[**101**]), and it is probably the simplest of such examples. We stress that it should not be thought as exotic, but rather as an important intuition for the general situation. Moreover, it is exactly the kind of packet that we use in the arithmetic applications of chapters 8 and 9.

We keep here the notations of § A.9 and take $m = 3$. We are interested in the nontempered A-parameter associated to the partition $3 = 2 + 1$. These parameters have actually a nonconjectural meaning as they factor through W_F. To fix ideas we fix η an automorphic character of $\mathbb{A}_E^*$ such that $\eta^\perp = \eta$ and that η does not descend to U(1) (see § 6.9.2). By class field theory, we may view it as a continuous character of W_E. We may actually use this η to define ξ and we are interested in the parameter $\xi \circ (1 \times 1 \otimes [2])$. More explicitly, let us simply say [(16)] that there is a unique parameter ψ whose base change

$$\psi_E : W_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_3(\mathbb{C})$$

fixes the vector e_2 of the canonical basis (e_1, e_2, e_3) of $\mathbb{C}^3$, and which acts as

$$(w \times g) \mapsto \eta(w)g$$

on $\mathbb{C}e_1 \oplus \mathbb{C}e_3 = \mathbb{C}^2$.

We are going to describe completely the A-packet Π associated to ψ following Rogawski. As predicted, it is a product of local A-packets Π_v, so we are reduced to describe each of the Π_v. When v splits in E, Π_v is a singleton and coincides with its associated (non-tempered) L-packet defined in § A.5, so we concentrate on the nonsplit case.

Consider first the local quasi-split unitary group $U_3(\mathbb{Q}_l)$ attached to the quadratic extension E_v of $\mathbb{Q}_l$. Rogawski has defined a non-tempered representation $\pi^n(\eta_v)$,

(16) This suffices for the discussion here. We will say more about those parameters and their extension to L_F in a more general context in § A.12.

where $\eta_v : E_v^* \longrightarrow \mathbb{C}^*$ is the restriction of η at v. Recall that $\eta_v = \eta_v^\perp$ but note that η_v does not come by base change from $\mathrm{U}(1)(\mathbb{Q}_l)$ (see the proof of Lemma 6.9.2 (i)). The representation $\pi^n(\eta_v)$ is actually a twist of the one we constructed in § 6.9.4 as the unique subrepresentation π_l^n of a principal series of $\mathrm{U}(3)(\mathbb{Q}_l)$. This principal series has in this case two components. The other one is called $\pi^2(\eta_v)$. According to Rogawski it is square integrable.

The representation $\pi^n(\eta_v)$ forms an L-packet on its own. This L-packet is not tempered, and $\pi^n(\eta_v)$ is not stable. The L-packet containing $\pi^2(\eta_v)$ is

$$\{\pi^2(\eta_v), \pi^s(\eta_v)\},$$

where $\pi^s(\eta_v)$ is a supercuspidal representation that Rogawski constructs using global considerations involving the trace formula. Since this L-packet is tempered, it is also an A-packet. There is one A-packet containing $\pi^n(\eta_v)$, namely Π_v, and we have [17]

$$\Pi_v = \{\pi^n(\eta_v), \pi^s(\eta_v)\}.$$

In particular, $\pi^s(\eta_v)$ belongs to *two* A-packets, and actually those representations are the only ones (up to a twist) that belong to several A-packets.

The base change of the A-packet $\{\pi^n(\eta_v), \pi^s(\eta_v)\}$, and of the L-packet $\{\pi^n(\eta_v)\}$, is the irreducible admissible representation of $\mathrm{GL}_3(E_v)$ whose L-parameter is

$$\eta_v|\,|^{1/2} \oplus \eta_v|\,|^{-1/2} \oplus 1.$$

The base change of the L and A-packet $\{\pi^2(\eta_v), \pi^s(\eta_v)\}$ coincides with this latter parameter on the Weil group, but is nontrivial on the $\mathrm{SU}_2(\mathbb{R})$-factor (mixing $\eta_v|.|^{1/2}$ and $\eta_v|.|^{-1/2}$). Hence the two A-packets containing π^s have different base changes.

Assume till the end of this subsection that $G(F_v)$ is compact for each archimedean v, which is our main case of interest in this book. For v archimedean, Π_v is empty if η_v has weight $\pm 1/2$ (see § 6.9.2), a singleton otherwise: namely the one we defined in § 6.9.5 (up to a twist). This ends the description of Π.

As a consequence of all of that, we first see that Π is infinite, unless some archimedean η_v has weight $\pm 1/2$, in which case $\Pi = \varnothing$. Moreover, Rogawski computes then $m(\pi)$ for each $\pi \in \Pi$, hence $\Pi \cap \Pi_{\mathrm{disc}}(G, F)$. He shows that $m(\pi)$ is always 0 or 1. [18] Precisely, he assigns a sign $\varepsilon(\pi_v) = \pm 1$ to each $\pi_v \in \Pi_v$ as follows: $\varepsilon(\pi_v) = 1$ except when v is archimedean, or when v is a finite nonsplit place and $\pi_v = \pi^s$. The final result [**101**] is that $m(\pi) = 1$ if, and only if,

$$\prod_v \varepsilon(\pi_v) = \varepsilon(\eta, 1/2),$$

(17) Note that $\pi^n(\eta_v)$ is a non monodromic principal series , but not $\pi^s(\eta_v)$.

(18) Hence weak multiplicity one holds for the packet Π, actually Rogawski shows that it holds for the full discrete spectrum of G.

where $\varepsilon(\eta, 1/2)$ is the sign of the global functional equation of η. In particular, *one half* of Π is actually automorphic.

The formula above is a special case of Arthur's multiplicity formula. We will discuss in more details this multiplicity formula in a more general case in § A.13.

A.11. Descent from GL_m to $\mathrm{U}(m)$

In this paragraph we explain the algebraic formalism relating the parameters (L or A, local or global) of a unitary group $G = \mathrm{U}(m)$ as in § A.9 and their base change to GL_m. This formalism also applies to Galois representations instead of parameters. Our main aims are to characterize the image of the base change and to discuss uniqueness properties. Many special cases of the criteria that we give below are presumably well known to experts in Langlands' theory (see e.g. [**99**] for $m = 3$ or [**15**, §4.3, §6.2] in the real case). In the Galois theoretic case, some aspects of this study have been carried out in [**41**].

We will use systematically the notation $\mathrm{U}(m)$ for G, which frees the letter G (and G') for other notational purposes.

We consider a group G and a subgroup H of index 2. In this paragraph, we call *parameter* any morphism $\psi : G \longrightarrow {}^L\mathrm{U}(m)$ such that H is the kernel of the composition of ψ and the projection ${}^L\mathrm{U}(m) \to \mathrm{Gal}(E/F)$. We denote by c the nontrivial element of $\mathrm{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z}$.

We denote by d a fixed element of $G - H$. If $\rho : H \to \mathrm{GL}_n(\mathbb{C})$ is a morphism, define $\rho^\perp(h) = {}^t\rho(d^{-1}hd)^{-1}$. The representation $\rho^\perp$ does not depend on d up to isomorphism.

In the applications, E/F may either be an extension of global or of local fields, and G and H may be respectively L_F and L_E, their Arthur's variant $L_F \times \mathrm{SL}_2(\mathbb{C})$ and $L_E \times \mathrm{SL}_2(\mathbb{C})$, the Weil groups W_F and W_E, or the absolute Galois groups G_F and G_E. Note that in general, G is not a semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ by H.

If ψ is a parameter, we may write

$$\psi(d) = Ac.$$

where $A \in \mathrm{GL}_m(\mathbb{C})$, since the image of d in $\mathbb{Z}/2\mathbb{Z}$ is non trivial, hence equals c. For the same reason, we may write $\psi(d^{-1}) = Bc$. We thus have

$$1 = \psi(d)^{-1}\psi(d) = Bc(Ac) = B\phi_m{}^tA^{-1}\phi_m^{-1}$$

so $B = \phi_m{}^tA\phi_m^{-1}$ and

$$\psi(d^{-1}) = \phi_m{}^tA\phi_m^{-1}c. \tag{112}$$

From this we deduce, calling ρ the restriction of ψ to H:

$$\forall h \in H, \quad \rho(dhd^{-1}) = C^t\rho(h)^{-1}C^{-1}, \tag{113}$$

where $C = A\phi_m$. Indeed,

$$\begin{aligned}\psi(dhd^{-1}) &= Ac\psi(h)\phi_m{}^tA\phi_m^{-1}c \text{ by (112)}\\ &= A\phi_m{}^t\psi(h)^{-1t}\phi_m^{-1}A^{-1t}\phi_m\phi_m^{-1} \text{ by (109)}\\ &= A\phi_m{}^t\psi(h)^{-1}(A\phi_m)^{-1} \text{ using (111).}\end{aligned}$$

Note that in particular, we have

$$\rho \simeq \rho^{\perp}.$$

We also have $\rho(d^2) = \psi(d^2) = \psi(d)^2 = A\phi_m{}^tA^{-1}\phi_m^{-1}$ hence

$$\rho(d^2) = (-1)^{m+1}C^tC^{-1}. \tag{114}$$

Lemma A.11.1. — *A morphism $\rho : H \longrightarrow \mathrm{GL}_m(\mathbb{C})$ is the restriction to H of a parameter of G if and only if there exists a matrix $C \in \mathrm{GL}_m(\mathbb{C})$ that satisfies conditions* (113) *and* (114).

Proof. — We have already seen that those conditions were necessary. To prove they are sufficient, assume they are satisfied for some $C \in \mathrm{GL}_m(\mathbb{C})$ and define a map $\psi : G \to {}^L\mathrm{U}(m)$ by setting for all $h \in H$, $\psi(h) := \rho(h)$ and $\psi(hd) := \psi(h)Ac$ where $A := C\phi_m^{-1} = (-1)^{m+1}C\phi_m$. We only have to check that ψ is a group homomorphism. Let $g, g' \in G$. If $g \in H$ then it is clear by definition that $\psi(gg') = \psi(g)\psi(g')$. So suppose $g = hd$. We distinguish two cases: if $g' = h' \in H$, then we have

$$\begin{aligned}\psi(g)\psi(g') &= \psi(hd)\psi(h')\\ &= \rho(h)Ac\psi(h')\\ &= \rho(h)Ac\rho(h')c^{-1}A^{-1}Ac\\ &= \rho(h)A\phi_m{}^t\rho(h')^{-1}\phi_m^{-1}A^{-1}Ac \text{ using (109)}\\ &= \rho(h)C^t\rho(h')^{-1}C^{-1}Ac\\ &= \rho(h)\rho(dh'd^{-1})Ac \text{ using (113)}\\ &= \rho(hdh'd^{-1})Ac\\ &= \psi(hdh') = \psi(gg').\end{aligned}$$

Similarly, if $g' = h'd$, we have

$$\begin{aligned}\psi(g)\psi(g') &= \psi(hd)\psi(h'd) = \psi(hd)\psi(h')Ac\\ &= \rho(hdh'd^{-1})(Ac)^2 \text{ like in the first six lines of the above computation}\\ &= \rho(hdh'd^{-1}d^2) = \psi(gg').\end{aligned}$$ □

Remark A.11.2. — (i) The lemma above gives a criterion for a representation $\rho : H \to \mathrm{GL}_m(\mathbb{C})$ to be the restriction of a parameter of G. Note that the criterion depends on a choice of an element d in $G - H$. In each particular context, a clever choice of d may simplify the computations.

(ii) As an exercise, let us consider the special case where G is a semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ by H. This case [19] occurs for example when E/F is a CM extension of number fields, $G = G_F$ and $H = G_E$. Then we may and do choose a d such that $d^2 = 1$.

Let $\rho : H \longrightarrow \mathrm{GL}_m(\mathbb{C})$ such that $\rho^\perp \simeq \rho$. Thus there is a C such that (113) is satisfied. By applying this relation twice, and using $d^2 = 1$, we see that C^tC^{-1} centralizes $\operatorname{Im}\rho$. Assume that ρ is irreducible. Then the latter means ${}^tC = \lambda C$, from which $\lambda^2 = 1$, and we see that C is either antisymmetric, or symmetric. If m is odd, C has to be symmetric, but if m is even, it is clear that both situations may happen. On the other hand, (114) reads $1 = \rho(d^2) = (-1)^{m+1}C^tC^{-1}$ which simply says that C is symmetric if m is odd, and antisymmetric if m is even.

To summarize, an irreducible ρ such that $\rho^\perp \simeq \rho$ comes from a parameter of G always if m is odd, and 'half the time" if m is even.

Proposition A.11.3. — *Let $\rho : H \longrightarrow \mathrm{GL}_m(\mathbb{C})$ be a semisimple representation such that $\rho \simeq \rho^\perp$. We assume that*

(i) *either ρ is a sum of pairwise distinct irreducible representations,*
(ii) *or $\rho(H)$ is abelian.*

Then a parameter $\psi : G \longrightarrow {}^LU(m)$ extending ρ is unique up to conjugation by an element of $\mathrm{GL}_m(\mathbb{C})$ (if it exists).

Proof. — As ρ is semisimple, we may and do assume, up to changing ρ by a conjugate, that the algebra generated by $\rho(H)$ is stable by transposition.

Let ψ (resp. ψ') be a parameter of G whose restriction to H is ρ, and let C (resp. C') be as above the matrix such that $\psi(d) = C\phi_m^{-1}c$ (resp. $\psi'(d) = C'\phi_m^{-1}c$). The matrix C (resp. C') satisfies (113) and (114). Hence:

(a) $C^{-1}C'$ is in the centralizer of $\rho(H)$.
(b) $C^tC^{-1} = C'^tC'^{-1} = (-1)^{m+1}\rho(d^2)$.

We want to find a matrix $B \in \mathrm{GL}_m(\mathbb{C})$ in the centralizer of $\rho(H)$ such that

$$BC^tB = C'.$$

Indeed, it is clear that for such a B, $\psi' = B\psi B^{-1}$.

Assume first that we are in case (i). In this case, we may write $\rho = \rho_1 \oplus \cdots \oplus \rho_r$, with $\rho_1, \dots, \rho_r$ irreducible of dimension $d_1, \dots, d_r$, and choose a basis in which $\rho(h) =$

[19] A similar study is done in this context in the first pages of [**41**].

$\mathrm{diag}(\rho_1(h), \dots, \rho_r(h))$ for every $h \in H$. Since the ρ_i are distinct, and since C satisfies (113), it may be written as

$$C = C_0\sigma$$

where C_0 is in the standard Levi $L \subset \mathrm{GL}_m(\mathbb{C})$ of type $(d_1, \dots, d_r)$ and σ is the permutation of $\{1, \dots, r\}$ satisfying $\rho^{\perp}_{\sigma(i)} \simeq \rho_i$ (hence $\sigma^2 = 1$) seen as a permutation matrix (in $\mathrm{GL}_m(\mathbb{C})$) by blocks of type $(d_1, \dots, d_r)$. By (a), we may write

$$C' = CD$$

where D is in the centralizer of $\rho(H)$, that is of the Levi L, hence it is of the form $D = \mathrm{diag}(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r)$ where each a_i is repeated d_i times. Now we see that

$$\begin{aligned}
C^tC^{-1} &= C'^tC'^{-1} \text{ by (b)} \\
&= (C_0\sigma D)({}^tC_0^{-1}\sigma D^{-1}) \text{ using that } {}^t\sigma = \sigma^{-1} = \sigma \text{ and that } {}^tD = D \\
&= (C_0\sigma)({}^tC_0^{-1}\sigma)(\sigma D\sigma D^{-1}) \text{ using that } DC_0 = C_0D \\
&= C^tC^{-1}(\sigma D\sigma D^{-1}).
\end{aligned}$$

Hence

$$\sigma D\sigma = D,$$

and $a_i = a_{\sigma(i)}$ for all i. We thus may choose complex numbers b_i, $i = 1, \dots, r$ such that $b_ib_{\sigma(i)} = a_i$ for all i, and set $B = \mathrm{diag}(b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_r, \dots, b_r)$ where each b_i is repeated d_i times. Then $\sigma B\sigma^tB = D$, B is in the centralizer of $\rho(H)$ and

$$BC^tB = BC_0\sigma^tB = C_0\sigma\sigma B\sigma^tB = CD = C',$$

and we are done in case (i).

Assume now that we are in case (ii). Then ρ is a sum of distinct characters $\chi_1, \dots, \chi_r$, each of them with multiplicity $m_1, \dots, m_r$. So we have $m = m_1 + \dots + m_r$. We may assume that ρ acts by χ_1 on the first m_1 vectors of the basis, then by χ_2 on the next m_2 vectors, and so on. Hence $\rho(H)$ is made of diagonal matrices of the form $\mathrm{diag}(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r)$ where each a_i is repeated m_i times. In particular, $(-1)^{m+1}\rho(d^2) = \mathrm{diag}(d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_r, \dots, d_r)$ is of that form by (b), and the centralizer of $\rho(H)$ is the standard Levi L of type $(m_1, \dots, m_r)$.

Since C (resp. C') satisfies (113), it may be written as

$$C = C_0\sigma \text{ (resp. } C' = C'_0\sigma)$$

where $C_0 = \mathrm{diag}(C_1, \dots, C_r)$ (resp. $C'_0 = \mathrm{diag}(C'_1, \dots, C'_r)$) is in the Levi L and σ is the permutation of $\{1, \dots, r\}$ satisfying $\chi^{\perp}_{\sigma(i)} \simeq \chi_i$ (hence $\sigma^2 = 1$) seen as a permutation matrix (in $\mathrm{GL}_m(\mathbb{C})$) by blocks of type $(m_1, \dots, m_r)$.

Fix an $i \in \{1, \dots, r\}$. By (b) we see that

$$C_i{}^tC_{\sigma(i)}^{-1} = C'_i{}^tC'_{\sigma(i)}{}^{-1} = d_i.$$

If $\sigma(i) = i$, then $d_i \in \mathbb{C}^*$ satisfies $d_i^2 = 1$, which implies that C_i and C'_i are either both symmetric or both antisymmetric, and in any case that there exists a $B_i \in \mathrm{GL}_{m_i}(\mathbb{C})$ such that $B_i C_i {}^t B_i = C'_i$. In the other case we have $\sigma(i) = j \neq i$, and we assume that $i < j$ to fix ideas. Then $m_i = m_j$ and $C_j {}^t C_i^{-1} = C'_j {}^t C_i'^{-1}$. From that equality, it follows that if we set $B_i = C'_i C_i^{-1}$ and $B_j = \mathrm{Id}_{m_i}$, then

$$\mathrm{diag}(B_i, B_j)\mathrm{diag}(C_i, C_j)\tau {}^t\mathrm{diag}(B_i, B_j) = \mathrm{diag}(C'_i, C'_j)$$

where τ is the restriction of σ to the set $\{i, j\}$ (and is the only non-trivial permutation of that set) seen as a matrix by blocks in $\mathrm{GL}_{2m_i}(\mathbb{C})$.

Finally $B := \mathrm{diag}(B_1, \dots, B_r) \in \mathrm{GL}_m(\mathbb{C})$ is in the centralizer of $\rho(H)$ and clearly satisfies $BC^tB = C'$, and we are done. □

Example A.11.4. — Here is an example where the case (ii) of the proposition above may be used. Assume that E/F is a quadratic extension of local fields and that $\mathrm{U}(m)$ is the quasisplit unitary group in m variables attached to E/F. Let T be a maximal torus of $\mathrm{U}(m)(F)$ and $\chi : T(F) \longrightarrow \mathbb{C}^*$ an admissible character of $T(F)$. By the duality for tori this character χ defines an L-parameter

$$\phi(\chi) : W_F \longrightarrow {}^L T \subset {}^L\mathrm{U}(m).$$

In this book, the only representations of $\mathrm{U}(m)(F)$ that we consider for F nonarchimedean (actually $F = \mathbb{Q}_l$) are in L-packets of this type. Precisely, they will be either unramified, which means that χ is trivial on the maximal compact subgroup of $T(F)$, or non monodromic principal series , in which case χ is as in Def. 6.6.5.

The norm map $\mathrm{Nm} : T(E) \longrightarrow T(F)$ (see § 6.9.4) defines a *base change*

$$\chi_E := \chi \circ \mathrm{Nm} : T(E) \longrightarrow \mathbb{C}^*$$

and $\phi(\chi)_E$ is then the L-parameter of GL_m/E attached in the same way to χ_E. Case (ii) of Proposition A.11.3 shows that $\phi(\chi)$ is actually (up to conjugation) the unique L-parameter of $\mathrm{U}(m)$ whose base change is $\phi(\chi)_E$. Conversely, we may start from a principal series L-parameter $\phi_E : W_E \longrightarrow \mathrm{GL}_m(\mathbb{C})$, or which is equivalent from a character $\chi_E : T(E) \longrightarrow \mathbb{C}^*$, and ask whether it descends to an L-parameter of $\mathrm{U}(m)$ of the form $\phi(\chi)$ for some χ as above. This requires strong conditions on χ_E. In the two cases (unramified or non monodromic principal series) we are interested in, this analysis is precisely the work done in § 6.9.4: Lemma 6.9.7 shows that conditions (iiia) or (iiib) of § 6.9.1 on the (unique) representation π_l of $\mathrm{GL}_m(E)$ whose L-parameter is ϕ_E are sufficient.

This quite general uniqueness result is completed by the following more restrictive, but still very useful, existence result. We suppose given a subgroup G' of G which is not a subgroup of H. Hence $H' := G' \cap H$ has index two in G'. We choose the element d in $G' - H'$.

Proposition A.11.5. — *Let $\rho : H \to \mathrm{GL}_m(\mathbb{C})$ be a semisimple representation such that $\rho^\perp \simeq \rho$. Let ρ' be its restriction to H'. We assume that ρ' is a sum of distinct irreducible representations ρ'_i such that $\rho'_i \simeq {\rho'_i}^\perp$. Then ρ extends to a unique parameter ψ of G whenever ρ' extends to a parameter ψ' of G'.*

Moreover, if this holds, then the centralizer in $\mathrm{GL}_m(\mathbb{C})$ of the image of ψ is finite.

Proof. — Let C' be the matrix attached to the parameter ψ' of G'. Arguing as in the proof of the above proposition (case (i)) applied to ρ', we may assume that $\rho'(H')$ lies in the standard Levi L of type $(d_1, \dots, d_r)$ (here $d_i = \dim \rho'_i$), that the centralizer of $\rho'(H') = \rho(H')$ is the centralizer (and the center) $Z(L)$ of L, and that C' is in L (note that $\rho'_i \simeq {\rho'_j}^\perp$ if and only if $i = j$, so that $\sigma = \mathrm{Id}$).

Now let C be a matrix that satisfies (113) for ρ. Then $C^{-1}C'$ centralizes $\rho(H')$, hence it lies in the centralizer $Z(L)$ of L, and commutes with C'. From that we deduce that

$$C^t C^{-1} = C'^t C'^{-1}$$

and condition (114) holds for C since by hypothesis it holds for C'. Hence the existence of a parameter ψ whose restriction to H is ρ follows from Lemma A.11.1. The uniqueness follows immediately from Proposition A.11.3 (case (i)).

Finally, the centralizer $C(\psi) \subset \mathrm{GL}_m(\mathbb{C})$ of $\psi(G)$ is a subgroup of the analogous centralizer $C(\psi')$ of $\psi(G') = \psi'(G')$. This is the subgroup of the centralizer of $\psi'(H') = \rho'(H')$ that is fixed by the map $g \mapsto C^t g^{-1} C^{-1}$. Since the centralizer of $\rho'(H')$ is $Z(L)$, and since $C \in L$, this map is $g \mapsto g^{-1}$ and $C(\psi') \simeq (\mathbb{Z}/2\mathbb{Z})^r$. Hence $C(\psi)$ is finite. □

Remark A.11.6. — If ψ is a *discrete* A-parameter for ${}^L U(m)$, then Rogawski ([**100**, Lemma 2.2.1, 2.2.2]) has shown that its restriction ρ to $L_E \times \mathrm{SL}_2(\mathbb{C})$ is a sum of irreducible, pairwise non isomorphic representations ρ_i satisfying $\rho_i^\perp = \rho_i$. The above proposition, in the case $G = G'$, provides a converse to this result.

Expected Corollary A.11.7. — *Let E/F be a CM extension of number fields, and*

$$\rho : L_E \longrightarrow \mathrm{GL}_m(\mathbb{C})$$

a tempered A-parameter for GL_m/E that satisfies $\rho^\perp \simeq \rho$. Assume that there is an infinite place of F such that, for the corresponding inclusion $W_\mathbb{C} \hookrightarrow L_E$ (well defined up to conjugation in L_E), the restriction of ρ to $W_\mathbb{C}$ extends to a discrete *L-parameter of $W_\mathbb{R}$ (see Remark A.11.8 below). Then ρ extends to a discrete tempered A-parameter $L_F \longrightarrow {}^L U(m)$ of the quasisplit unitary group $\mathrm{U}(m)$.*

Indeed, this is the proposition for $G = L_F$, $H = L_E$, $G' = W_\mathbb{R}$, $H' = W_\mathbb{C}$. The hypothesis of the proposition on the restriction ρ' of ρ to H' follows immediately from

Remark A.11.8 below. The obtained parameter is discrete as its restriction to $W_{\mathbb{R}}$ is discrete.

Via the philosophy of parameters, this result shows that a sufficient condition for a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A}_E)$ that satisfies $\pi^{\perp} \simeq \pi$ to come by base change from the quasi-split unitary group in m variables attached to E/F is that there is a place v at infinity such that π_v comes by base change from a representation of the *compact* unitary group. Note that this is exactly the assumption at infinity on the representations π studied by Clozel [**40**] and Harris-Taylor [**62**].

Remark A.11.8. — Recall that following Langlands, the discrete L-parameters ϕ : $W_{\mathbb{R}} \longrightarrow {}^L\mathrm{U}(m)$ are exactly the ones whose restriction $\phi_{\mathbb{C}}$ to $\mathbb{C}^*$ is conjugate to

$$z \mapsto ((z/\bar{z})^{a_1}, \dots, (z/\bar{z})^{a_n}),$$

where the a_i are in $\frac{m+1}{2} + \mathbb{Z}$ and strictly decreasing (see e.g. [**15**, Prop. 4.3.2]). For each such sequence (a_i) there is a unique such parameter.[20] As they are discrete, they are relevant for each inner form of $\mathrm{U}(m)_{\mathbb{R}}$, and in particular for the compact one.

Expected Proposition A.11.9. — *For unitary groups* $\mathrm{U}(m)$, *two everywhere locally equivalent discrete A-parameters are actually equivalent.*

Ineed, let ψ_1 and ψ_2 be two such discrete A-parameters, and set $G = G' = L_F \times \mathrm{SL}_2(\mathbb{C})$, $H = H' = L_E \times \mathrm{SL}_2(\mathbb{C})$, and $\rho_j = \psi_{j|H}$. By Rogawski's classification, the ρ_j are semisimple and satisfy the assumption of Prop. A.11.5. By assumption, the ρ_j are also everywhere locally equivalent. By the expected "Cebotarev's theorem for Langlands' groups", the reunion of the conjugates of L_{E_w} is dense in L_E, hence a trace consideration implies that the ρ_j are actually equivalent. By Prop. A.11.5, the same thing holds then for the ψ_j.

The following corollary is an immediate consequence of the expected proposition above and of the simplest case of Arthur's mutliplicity formula.

Expected Corollary A.11.10. — *If* Π *is a stable A-packet for* $\mathrm{U}(m)$, *then for each* $\pi \in \Pi$ *we should have* $m(\pi) = 1$.

Indeed, there is a unique A-parameter ψ of $\mathrm{U}(m)$ giving rise to Π by Expected Prop. A.11.9. As Π is stable, Arthur's group $\mathbf{S}_\psi$ is trivial by [**100**, §2.2], hence Arthur's multiplicity formula ([**3**, (8.5)]) reduces to $m_\psi(\pi) = m(\pi) = 1$.

[20] This existence and uniqueness follows for example easily from Lemma A.11.1, and Prop. A.11.5, A.11.3: choose d to be the usual element $j \in W_{\mathbb{R}}$ such that $j^2 = -1$, $C = 1$ and note that $\rho(j^2) = \rho(-1) = (-1)^{m+1}$. These parameters ϕ satisfy $\phi(j) = \phi_m^{-1} c$. They are relevant because, if they lie in a parabolic subgroup P of ${}^L\mathrm{U}(m)$, then $P = \langle P_0, \phi(j)\rangle$ for some parabolic P_0 of $\mathrm{GL}_m(\mathbb{C})$ normalised by $\phi(j)$, and we see that ${}^tP_0 = P_0$, hence $P = {}^L\mathrm{U}(m)$.

A.12. Parameter and packet of the representation π^n

In this paragraph and the next one, we take for granted all the formalism of Langlands and Arthur as described above and in [**3**]. *All the lemmas, propositions and theorems we state are thus conditional on this formalism. Our aim is to study the A-packet of the representation π^n that we introduced in § 6.9.*

We use from now on the notations of § 6.9. In particular $m = n + 2$, n is not a multiple of 4, $\mathrm{U}(m)$ is definite and quasisplit at all finite places, and we fix an embedding $E \to \mathbb{C}$. Moreover, $\mu = \mu^{\perp}$ is a Hecke character of $\mathbb{A}_E/E^*$ as in Notation 6.9.5. Remember that μ is trivial if m (or n) is even, and that $\mu(z) = (z/\bar{z})^{1/2}$ for $z \in \mathbb{C}^* \subset \mathbb{A}_E^*$ if n is odd. We will see μ as a character of W_E, hence of L_E, when needed.

To the representation π of that subsection should correspond a tempered, irreducible, A-parameter

$$\rho : L_E \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

The hypothesis (i) on π there translates to [21]

$$\rho^{\perp} \simeq \rho.$$

We now define an A-parameter for $\mathrm{GL}_m(E)$ denoted by $\psi_E : L_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$, by the formula

$$\psi_E(w \times g) = \begin{pmatrix} \rho(w)\mu(w) & 0 \\ 0 & g\mu(w) \end{pmatrix}, \quad w \in L_E, \ g \in \mathrm{SL}_2(\mathbb{C}).$$

We fix an embedding $L_{\mathbb{R}} = W_{\mathbb{R}} \hookrightarrow L_{\mathbb{Q}}$, giving an embedding $W_{\mathbb{C}} = \mathbb{C}^* \hookrightarrow L_{\mathbb{Q}}$. We fix furthermore an element $j \in \mathrm{W}_{\mathbb{R}} \backslash \mathrm{W}_{\mathbb{C}}$ such that $j^2 = -1 \in \mathrm{W}_{\mathbb{C}} = \mathbb{C}^*$. Such an element j maps to the non trivial element c in $L_{\mathbb{Q}}/L_E = W_{\mathbb{R}}/W_{\mathbb{C}} = \mathrm{Gal}(E/\mathbb{Q})$. By hypothesis (ii) of § 6.9.1, we may and do assume (possibly up to changing ρ by a conjugate) that for $z \in W_{\mathbb{C}} = \mathbb{C}^*$,

$$\psi_E(z) = \mathrm{diag}((z/\bar{z})^{a_1}, \ldots, (z/\bar{z})^{a_n})$$

where the a_i are in $\frac{1}{2}\mathbb{Z} - \mathbb{Z}$, strictly decreasing, and different from $\pm 1/2$.

Expected Lemma A.12.1. — *The A-parameter ψ_E extends (uniquely up to isomorphism, that is up to conjugation) to a discrete (relevant) A-parameter $\psi : L_{\mathbb{Q}} \times$*

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(21) Strictly speaking, this uses the expected "Cebotarev theorem" for L_E, or (which is related) the weak mutliplicity one theorem for the discrete spectrum of GL_m/E.

$\mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{U}(m)$ *of the group* $\mathrm{U}(m)$. *We may choose*

$$\psi(j) = \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \phi_m^{-1} c.$$

Indeed, the restriction of ψ_E to $W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})$ is

$$\psi_E(z,g) = \mathrm{diag}((z/\bar{z})^{a_1}\mu_\infty(z), \ldots, (z/\bar{z})^{a_n}\mu_\infty(z), g\mu_\infty(z)).$$

We thus see that for

$$C := \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

we have for all z, g:

$$\psi_E(\bar{z}, g) = \psi_E(jzj^{-1}, g) = C^t\psi(z,g)^{-1}C^{-1}.$$

In other words, relation (112) holds for that C. We compute

$$\psi_E(j^2, 1) = \mathrm{diag}((-1)^{m+1}, \ldots, (-1)^{m+1}, (-1)^m, (-1)^m),$$

using that the a_i are half-integers and that $\mu_\infty(-1) = (-1)^m$. On the other hand, $C^tC^{-1} = \mathrm{diag}(1, \ldots, 1, -1, -1)$ so relation (114) holds for the restriction of ψ_E to $W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})$. This means by Lemma A.11.1, that the restriction of ψ_E to $W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})$ extends to a parameter ψ_∞ of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ that sends j to $C\phi_m^{-1}c$ (see (113)). By Proposition A.11.5, ψ_E extends to a unique, discrete, parameter ψ of $L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C})$, and we may even choose ψ such that $\psi(j) = C\phi_m^{-1}c$ by the last paragraph of the proof of that proposition.

It remains to explain why ψ is relevant. This means, since $\mathrm{U}(m)$ has no proper parabolic defined over $\mathbb{Q}$, that the only parabolic subgroup P of ${}^L\mathrm{U}(m)$ containing $\mathrm{Im}\,(\psi)$ is ${}^L\mathrm{U}(m)$ itself. Let P be such a subgroup. By definition, $P = \langle P_0, \psi(j)\rangle$ for some parabolic P_0 of $\mathrm{GL}_m(\mathbb{C})$ normalized by $\psi(j)$. We see then that $C^tP_0C^{-1} = P_0$. But P_0 contains $1_n \times \mathrm{SL}_2(\mathbb{C}) = \psi(\mathrm{SL}_2(\mathbb{C}))$, hence $C \in P_0$. As a consequence, $P_0 = {}^tP_0$, which implies that $P_0 = \mathrm{GL}_m(\mathbb{C})$, and we are done.

Remark A.12.2. — Here is another way of viewing the parameter ψ in terms of Rogawski's classification recalled in § A.9. First, using Cor. A.11.7, we see that $\rho\mu$ extends to a discrete stable tempered parameter ψ_0 of the quasisplit $\mathrm{U}(n)$. Then, using again μ to define an admissible morphism

$$\xi : {}^L(\mathrm{U}(n) \times \mathrm{U}(2)) \longrightarrow {}^L\mathrm{U}(m),$$

we have actually $\psi = \xi \circ (\psi_0 \times (1 \otimes [2]))$ (see Ex.A.3.3). In particular, ψ is nontempered and endoscopic of type $(n, 2)$. When $n = 1$, it is exactly the A-parameter that we studied in detail in § A.10.

Let us denote by [22]

$$\Pi = \prod_v{}' \, \Pi_v$$

the A-packet corresponding to ψ. Our aim is now to check that the representation π^n defined in § 6.9 belongs to Π. By definition, this amounts to checking that for each place v the representation π_v^n defined there lies in Π_v. Recall that for some reasons we have called π_∞^s the archimedean component of π^n (see § 6.9.5).

Expected Lemma A.12.3. — *The global representation π^n belongs to the global A-packet defined by ψ. Moreover, $\Pi_\infty = \{\pi_\infty^s\}$ and for each prime l, π_l^n is in the local L-packet $\Pi_{\varphi_l} \subset \Pi_l$ (see § A.5).*

Indeed, for each place v of $\mathbb{Q}$ the L-parameter $\phi_v := \phi_{\psi_v}$ associated to ψ_v satisfies for all $w \in L_E$

$$\phi_v(w) = \operatorname{diag}(\rho_v(w)\mu_v(w), |w|^{1/2}\mu_v(w), |w|^{-1/2}\mu_v(w)),$$

by definition. For any place v of $\mathbb{Q}$, the L-parameter ϕ_v defines an L-packet Π_{ϕ_v} of representations of $\mathrm{U}(m)(\mathbb{Q}_v)$, which is a subset of the A-packet Π_v.

Assume first that $v = l$ is a prime. When l splits in E, Remark 6.9.6 shows that

$$\Pi_{\varphi_l} = \{\pi_l^n\} \subset \Pi_l.$$

When l does not split, we defined in Lemma 6.9.7 a smooth character χ of the maximal torus $T(\mathbb{Q}_l)$ of $\mathrm{U}(m)(\mathbb{Q}_l)$. There were two cases. If χ satisfies conditions (a) and (b) of Def. 6.6.5, then π_l^n is the non monodromic principal series $S(\chi)$. As suggested by Rodier's work [**98**], the L-parameter of $S(\chi)$ is conjecturally the L-parameter $\phi(\chi)$ defined in Remark A.11.4. But by the same remark and by construction, the base change $\phi(\chi)_{E_l}$ is isomorphic to $(\phi_l)_{E_l}$, hence

$$\phi(\chi) \simeq \phi_l$$

by Prop. A.11.3 and we are done. In the other case χ is unramified and π_l^n is by definition a constituent of the full induced representation defined by χ having a nonzero vector invariant by a maximal hyperspecial (resp. very special in the sense of Labesse [**76**, §3.6]) compact open subgroup of $\mathrm{U}(m)(\mathbb{Q}_l)$ if l is inert (resp. ramified) in E. Thus π_l^n

[22] The notation $\Pi = \prod_v' \Pi_v$ means that Π is the subset of $\prod_v \Pi_v$ whose elements $(\pi_v)_v$ have the property that π_v is $\mathrm{U}(m)(\mathbb{Z}_l)$-spherical for almost all primes l.

conjecturally belongs again to the L-packet defined by $\phi(\chi)$ (this is a standard expectation when l is inert, and it is indicated by Labesse's work [**76**, §3.6] in general), and we conclude as above that $\phi(\chi) \simeq \phi_l$.

The end of this paragraph is now essentially devoted to the subtler case where $v = \infty$ is archimedean. Note that the L-parameter ϕ_∞ is not relevant for the compact group $\mathrm{U}(m)(\mathbb{R})$. Indeed, it is not even tempered, as its restriction to $\mathrm{W}_{\mathbb{C}}$ contains the non unitary characters $\mu_\infty | |^{\pm 1/2}$, whereas the relevant L-parameters of $\mathrm{U}(m)(\mathbb{R})$ all are (they are even discrete, and described in Remark A.11.8).

As a consequence,

$$\Pi_{\phi_\infty} = \varnothing,$$

but the A-packet Π_∞ may be larger. However, note that Π_∞ is a singleton if nonempty, since every representation of a compact real reductive group is stable (cf. [**2**]). We shall review below the description [23] of Π_∞ given in section 5 of [**3**], following [**2**].

For this, we see $\mathrm{U}(m)(\mathbb{R})$ as the unitary group for the standard diagonal positive definite hermitian form, and we consider its diagonal maximal torus $T(\mathbb{R}) = \mathrm{U}(1)(\mathbb{R})^m$. We denote by

$$L(\mathbb{R}) = U(1)(\mathbb{R})^n \times U(2)(\mathbb{R}) \subset \mathrm{U}(m)(\mathbb{R})$$

the subgroup of matrices which are diagonal by blocks of size $(1, \dots, 1, 2)$, so that $T \subset L$ and $L(\mathbb{C})$ is a Levi subgroup of $\mathrm{U}(m)(\mathbb{C}) = \mathrm{GL}_m(\mathbb{C})$. In $\widehat{G}(\mathbb{C}) \simeq \mathrm{GL}_m(C)$, $\widehat{T}(\mathbb{C})$ is the diagonal torus and $\widehat{L}(\mathbb{C})$ the standard Levi of type $(1, \dots, 1, 2)$. We thus have

$$Z(\widehat{L}(\mathbb{C})) \subset \widehat{T}(\mathbb{C}) \subset \widehat{L}(\mathbb{C}) \subset \widehat{G}(\mathbb{C}).$$

It turns out that those inclusions extend naturally to inclusions

$${}^L Z(L) \subset {}^L T \overset{\xi_{L,T}}{\hookrightarrow} {}^L L \overset{\xi_{G,L}}{\hookrightarrow} {}^L G.$$

While the first inclusion is obvious, the others two need a construction, which is recalled in [**3**]. From this construction, we shall only need the following description of the restriction of the embedding $\xi_{L,T}$ to $W_{\mathbb{C}} = \mathbb{C}^*$ (see [**3**, page 31]):

$$\xi_{L,T}(z) = \mathrm{diag}(1, \dots, 1, (z/\bar{z})^{1/2}, (z/\bar{z})^{-1/2}).$$

Similarly $\xi_{G,L}(z) \in \widehat{T}(\mathbb{C})$ is a diagonal matrix that we do not need to compute explicitly because it will cancel out in the following computations.

Now let us consider the unique L-parameter ϕ_τ for the group T such that for $z \in W_{\mathbb{C}} = \mathbb{C}^*$,

$$\phi_\tau(z) = \xi_{G,L}(z)^{-1} \mathrm{diag}((z/\bar{z})^{a_1} \mu_\infty(z), \dots, (z/\bar{z})^{a_n} \mu_\infty(z), \mu_\infty(z), \mu_\infty(z)).$$

[23] This amounts to defining, according to some rule described in [**2**] and [**3**, §5], a representation of $\mathrm{U}(m)(\mathbb{R})$. Although it is easy, of course, to specify such a representation (by giving its infinitesimal character or its highest weight), the description of that rule is rather nontrivial in general.

It is clear that ϕ_τ maps $W_{\mathbb{R}}$ to ${}^L\widehat{Z(L)} \subset {}^LT$. To such an L-parameter is attached in [**3**, page 30 second §, page 31 first §] an A-parameter for the group L, called

$$\psi_L : W_R \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LL.$$

By definition, ψ_L is ϕ_τ on $W_{\mathbb{R}}$ and sends $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ to a principal unipotent element in $\widehat{L}$. Thus it is clear that (up to conjugation by an element of $\widehat{L}(\mathbb{C})$) the A-parameter $\xi_{G,L} \circ \psi_L$ coincides with our A-parameter ψ_∞ on $W_{\mathbb{C}}$ and $\mathrm{SL}_2(\mathbb{C})$. Thus $\psi_\infty \simeq \psi_L$ by Lemma A.11.3.

Arthur's conjecture provides a description (cf. [**3**, page 33]) of the A-packet attached to $\xi_{G,L} \circ \psi_L = \psi_\infty$. He defines for that some L-packets parameterized by the set

$$\Sigma := W(L,T)\backslash W(G,T)/W_{\mathbb{R}}(G,T),$$

and for each element of this set a specific representation in the associated L-packet. Here $\Sigma = \{1\}$, and the unique L-parameter he defines is $\phi_1 := \xi_{G,L} \circ \xi_{L,T} \circ \phi_\tau$ as L is anisotropic [**3**, page30-31]. On $W_{\mathbb{C}}$, we thus have

$$\begin{aligned} \phi_1(z) &= \xi_{G,L}(z)\xi_{L,T}(z)\phi_\tau(z) \\ &= \mathrm{diag}((z/\bar{z})^{a_1}\mu_\infty(z), \dots, (z/\bar{z})^{a_n}\mu_\infty(z), (z/\bar{z})^{1/2}\mu_\infty(z), (z/\bar{z})^{-1/2}\mu_\infty(z)). \end{aligned}$$

Note that ϕ_1 is relevant since the $a_i, 1/2, -1/2$ are distinct half-integers. Actually, ϕ_1 is exactly by definition the L-parameter of π^s_∞, and its associated L-packet is a singleton. According to Arthur, we thus have $\Pi_{\psi_\infty} = \Pi_{\phi_1} = \{\pi^s_\infty\}$.

Remark A.12.4. — (i) In Lemma A.12.1, and especially in the proof that ψ is relevant, the fact that the a_i are distinct from $\pm 1/2$ is actually not needed. However, as the proof above shows, this latter assumption is necessary to ensure that Π_∞ (hence Π) is not empty. In particular, if one of the a_i is equal to $\pm 1/2$, we get an example of a parameter ψ which is relevant and whose associated A-packet is empty.

(ii) As an exercise, the reader can check that the A-parameter ψ is not relevant for an inner form of $\mathrm{U}(m)$ that is not quasi-split at every finite place.

(iii) We have $\Pi_\infty = \{\pi^s_\infty\}$ and $\Pi_l = \{\pi^n_l\}$ when l splits in E. When l does not split, Π_l (and even Π_{ϕ_l}) will have more that one element in general, but it does not seem possible at the moment to describe the full packet Π_l for a general m as Rogawski has done for $m = 3$ (see § A.10).

Remark A.12.5. — In § 6.9, our point of view for defining the representation π^n was to start from a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi^\perp \simeq \pi$ and that satifies § 6.9.1 (ii). We defined then *by hand* all the local components of the representation π^n of $\mathrm{U}(m)(\mathbb{A}_{\mathbb{Q}})$. An interest of this presentation is that it avoids assuming that $\pi\mu$ comes by base change from an automorphic representation π_0 of the quasisplit unitary group $\mathrm{U}(n)$, and more generally any appeal to a unitary group in n variables. As noticed in Remark A.12.2, such a π_0 should however always exist. More precisely, arguing as in Rem.A.11.4, conditions (iiia) and (iiib) of § 6.9.1 imply that we should be able to find such a π_0 which is either unramified or a non monodromic principal series at all nonsplit finite places. Conversely, if π_0 is such a representation which satisfies also (ii) of § 6.9.1 and which has a cuspidal base change π to GL_n/E (this is known to hold for example when π_0 is supercuspidal at two split places ([**61**, Thm.2.1.1,3.1.3]), then π satisfies our conditions. This gives a way to produce such examples (maybe using some inner forms of $\mathrm{U}(n)$ as well).

A.13. Arthur's multiplicity formula for π^n

We have checked that $\pi^n \in \Pi$ and we ask now whether $\pi^n \in \Pi_{\mathrm{disc}}(\mathrm{U}(m), \mathbb{Q})$. For that we will actually compute $m(\pi^n)$ using Arthur's multiplicity formula.

Following [**3**, page 52], let us consider the subgroup $S_\psi \subset \widehat{G}(\mathbb{C}) = \mathrm{GL}_m(\mathbb{C})$. In our situation, (24) this group is actually $Z(\widehat{G}(\mathbb{C})) \cdot C_\psi$ where C_ψ is the centraliser in $\widehat{G}(\mathbb{C})$ of the image of ψ. We set also $s_\psi = \psi(1, \mathrm{diag}(-1,-1)) \in S_\psi$ (see [**3**, page 26]) and

$$\mathbf{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G}(\mathbb{C})).$$

Expected Lemma A.13.1. — *We have*

$$S_\psi = \{\mathrm{diag}(a, \dots, a, \epsilon a, \epsilon a), \quad a \in \mathbb{C}^*, \epsilon = \pm 1\} \simeq \mathbb{C}^* \times \{\pm 1\}.$$

The character $S_\psi \to \{\pm 1\}$ sending $\mathrm{diag}(a, \dots, a, \epsilon a, \epsilon a)$ to ϵ factors through $\mathbf{S}_\psi$ and induces an isomorphism $\epsilon : \mathbf{S}_\psi \xrightarrow{\sim} \{\pm 1\}$. Moreover, $\mathbf{S}_\psi$ is generated by the image of s_ψ.

Indeed, as ρ is irreducible, the centralizer of $\psi(L_E \times \mathrm{SL}_2(\mathbb{C}))$ in $\mathrm{GL}_m(\mathbb{C})$ is

$$\{\mathrm{diag}(a, \dots, a, b, b)\} = \mathbb{C}^* \times \mathbb{C}^*.$$

Among those elements, those who commute with $\psi(j)$ are the ones of order two modulo $Z(\mathrm{GL}_m(\mathbb{C}))$, hence $S_\psi = \{\mathrm{diag}(a, \dots, a, \epsilon a, \epsilon a)\}$ and $\mathbf{S}_\psi = \pi^0(S_\psi / Z(\widehat{G})) = \{\pm 1\}$. Finally, the element s_ψ clearly generates $\mathbf{S}_\psi$.

We now introduce following Arthur ([**3**, page 54]) the representation

$$\tau : S_\psi \times L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(M_m(\mathbb{C}))$$

defined by $\tau(s, w, g) = \mathrm{Ad}(s\psi(w, g))$. As the kernel of $S_\psi \to \mathbf{S}_\psi$ is the center of $\mathrm{GL}_m(\mathbb{C})$, the action of S_ψ actually factors through $\mathbf{S}_\psi$. We note τ_E the restriction of τ to $S_\psi \times L_E \times \mathrm{SL}_2(\mathbb{C})$.

(24) Precisely, using the triviality of $\mathrm{Ker}^1(L_{\mathbb{Q}}, Z(\widehat{G}(\mathbb{C})))$ for $G = \mathrm{U}(m)$, as remarked in [**100**, §2.2].

Recall that the representation ψ_E on $\mathbb{C}^m$ is the direct sum of two representations of $L_E \times \mathrm{SL}_2(\mathbb{C})$: the representation $\rho\mu \otimes 1$ on the n-dimensional space V_1 (generated by the first n vectors of the standard basis of $\mathbb{C}^m$) and the representation $\mu \otimes [2]$ on the 2-dimensional space V_2 (generated by the last two vectors), where [2] is the standard representation of $\mathrm{SL}_2(\mathbb{C})$. Hence $M_m(\mathbb{C})$ may be written as

$$(V_1 \otimes V_1^*) \oplus (V_2 \otimes V_2^*) \oplus (V_1 \otimes V_2^*) \oplus (V_2 \otimes V_1^*),$$

all four spaces in the decomposition being stable by the adjoint action of $L_E \times \mathrm{SL}_2(\mathbb{C})$. Moreover, the adjoint action of S_ψ (that is, of $\mathbf{S}_\psi$) preserves also this decomposition and is trivial on $V_1 \otimes V_1^* \oplus V_2 \otimes V_2^*$, and given by the non-trivial character ϵ on $V_1 \otimes V_2^* \oplus V_2 \otimes V_1^*$.

Expected Lemma A.13.2. — *The spaces $V_1 \otimes V_1^*$, $V_2 \otimes V_2^*$ and $V_1 \otimes V_2^* \oplus V_2 \otimes V_1^*$ are stable by τ. The last one is isomorphic to $\epsilon \otimes \mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho \otimes [2]$.*

Indeed, the adjoint action of $\tau(j)$ on $M_m(\mathbb{C})$ is given by $M \mapsto -C^tMC^{-1}$. In particular, it stabilizes $V_1 \otimes V_1^*$ and $V_2 \otimes V_2^*$ which thus are stable by τ, and it interchanges $V_1 \otimes V_2^*$ and $V_2 \otimes V_1^*$, from which the lemma follows easily.

Arthur defines then in [**3**, (8.4)] a character

$$\epsilon_\psi : \mathbf{S}_\psi \longrightarrow \{\pm 1\}.$$

Expected Proposition A.13.3. — *The character ϵ_ψ is the trivial character if $\varepsilon(\pi, 1/2) = 1$ and the non-trivial character ϵ if $\varepsilon(\pi, 1/2) = -1$. In other words, $\epsilon_\psi(s_\psi) = \varepsilon(\pi, 1/2)$.*

Indeed, according to Arthur's recipe, to compute ϵ_ψ we have to decompose the semisimple representation τ into its irreducible components $\tau_k = \lambda_k \otimes \rho_k \otimes \nu_k$ as a representation of $\mathbf{S}_\psi \times L_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C})$. By definition, $\epsilon_\psi = \prod_{\tau_k \text{ special}} \lambda_k$ where τ_k *special* means that $\tau_k \simeq \tau_k^*$, which in our context is equivalent to $\rho_k \simeq \rho_k^*$, and that $\varepsilon(\rho_k, 1/2) = -1$.

As we already saw, we may ignore the τ_k's arising as components of either $V_1 \otimes V_1^*$ or $V_2 \otimes V_2^*$ since the corresponding λ_k are trivial. By Lemma A.13.2, the remaining τ_k's are the constituents of $\epsilon \otimes \mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho \otimes [2]$. Note that $\mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho$ is selfdual as $\rho^\perp \simeq \rho$. Let us decompose $\mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho$ as a sum of its r irreducible constituents, that we may note $\rho_1, \dots, \rho_r$ up to renumbering. Since ρ is irreducible, we have $r = 1$ or 2.

If $r = 1$, $\tau_1 = \tau_1^*$ so τ_1 is special if and only if $\varepsilon(\rho_1, 1/2) = -1$, but $\varepsilon(\rho_1, 1/2) = \varepsilon(\mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho, 1/2) = \varepsilon(\rho, 1/2) = \varepsilon(\pi, 1/2)$ and the proposition follows.

The second case $r = 2$ occurs exactly when ρ is self-conjugate, hence when ρ is selfdual since $\rho \simeq \rho^\perp$. In this case ρ extends to a representation ρ_1 of $L_\mathbb{Q}$, and we have $\mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho = \rho_1 \oplus \rho_2 = \rho_1 \oplus \rho_1 \omega_{E/\mathbb{Q}}$ and $\rho_1 \not\simeq \rho_2$. We have

$$\varepsilon(\pi, 1/2) = \varepsilon(\mathrm{Ind}_{L_E}^{L_\mathbb{Q}} \rho, 1/2) = \varepsilon(\rho_1, 1/2)\varepsilon(\rho_2, 1/2).$$

If ρ_1 and ρ_2 are selfdual, we see that there is exactly one (resp. 0 or 2) ρ_i that is special if $\varepsilon(\pi, 1/2) = -1$ (resp $+1$) and the proposition follows. If $\rho_1^* \simeq \rho_2$, then the functional equations of the L-functions of ρ_1 and ρ_2 show that $\varepsilon(\rho_1, s)\varepsilon(\rho_2, 1-s) = 1$, so $\varepsilon(\pi, 1/2) = +1$ and there are no special τ_k, which concludes this case as well.

The last ingredient in the multiplicity formula is a conjectural canonical pairing ([**3**, page 54])

$$\mathbf{S}_\psi \times \Pi \to \mathbb{R}, \text{ denoted } \langle s, \pi \rangle.$$

However, this ingredient is certainly the most difficult one in Arthur's exposition of its multiplicity formula.

Let us recall some features of this pairing in a general context, for a reductive group G/F, and a global A-packet Π with A-parameter ψ. Together with the global pairing should be defined for each place v of F a local pairing

$$\mathbf{S}_{\psi_v} \times \Pi_v \to \mathbb{R}.$$

However, this local pairing should not be canonical, but rather depends on the choice of a basis representation in the local A-packet Π_v. Still there should be a way, after a global choice ν (see below), to choose the local pairing such that the product formula holds

$$\langle s, \pi \rangle = \prod_v \langle s, \pi_v \rangle_{v,\nu}. \tag{115}$$

In the formula above, $\pi = \otimes'_v \pi_v$ is in Π, the pairings on the right hand side should be the chosen local pairings depending on the global choice ν, s in the left hand side should be any element of $\mathbf{S}_\psi$ and s in the right hand side denotes the image of s by the injective natural morphism $\mathbf{S}_\psi \hookrightarrow \mathbf{S}_{\psi_v}$. Moreover, almost all the terms in the product should be 1.

When G is a quasi-split group G^*, the global choice ν may be a nondegenerate character of the unipotent radical of a Borel subgroup of G^* defined with the help of a non trivial admissible character $F\backslash\mathbb{A}_F \longrightarrow \mathbb{C}^*$. Then for each v, the A-packet Π_v should contain one and only one representation $\pi_v^{\nu_v-\mathrm{gen}}$ that is ν_v-generic in the sense explained [25] in [**22**, 4.4]. This representation should actually belong to the L-packet Π_{ψ_v}. When that representation is chosen as the base point to define the local pairing (which are then denoted $\langle\ ,\ \rangle_{v,\nu}$) the product formula (115) should hold.

From now on, we work with our group $G = \mathrm{U}(m)/\mathbb{Q}$ defined in chapter 6. Assuming the choice of ν is made as above for its quasi-split form G^*, we may use the local pairing $\langle\ ,\ \rangle_{v,\nu}$ already chosen for G_v^* for every finite place v, since $G_v^* \simeq G_v$. Moreover, since there is only one infinite place ∞, there is a unique choice of the local pairing at infinity which makes the formula (115) true. We still denote it as $\langle\ ,\ \rangle_{\infty,\nu}$.

(25) Be careful that this representation is ν_v-generic in the usual sense only for a tempered Π_v.

The pairings above have much nicer features when restricted to the subgroup of $\mathbf{S}_\psi$ generated by the canonical element s_ψ. So we are very lucky in our case, because that subgroup is the full $\mathbf{S}_\psi$ (Lemma A.13.1). In particular, the following assertions should hold [(26)]. Below v is a place, π_v any representation of the local A-packet Π_v.

(a) There should be a sign $e(\pi_v, \nu_v) = \pm 1$ such that

$$\langle s_\psi^a, \pi_v \rangle_{v,\nu} = e(\pi_v, \nu_v)^a$$

for every $a \in \mathbb{Z}$ (or for that matter, for $a = 0, 1$).

(b) The sign $e(\pi_v, \nu_v)$ should depend on π_v only through the L-parameter of π_v, and that sign is $+1$ if this L-parameter is ϕ_{ψ_v} and if G_v is quasi-split. This should be understood in the strong sense that if two π_v's, even for two different inner forms of G_v, have the same L-parameter, then they have the same sign.

(c) The sign $e(\pi_v, \nu_v)$, hence the pairing $\langle s_\psi, \cdot\ \rangle_{v,\nu}$ is independent of ν. (Hence we may and will drop ν from the notation.) Moreover, any local pairing $\langle\ ,\ \rangle_v'$ such that $\langle s_\psi^a, \pi_v \rangle_v'$ is 1 for $a = 0, 1$ and a given representation π_v of L-parameter ϕ_{ψ_v} is actually equal to that pairing on the subgroup generated by s_ψ:

$$\langle s_\psi^a, \cdot \rangle_v' = \langle s_\psi^a, \cdot \rangle_v.$$

Hence the sign $e(\pi_v, \nu_v)$ is simply denoted $e(\pi_v)$, and sometimes even $e(\phi)$ where ϕ is the L-parameter of π_v.

Indeed, (a) follows from [**3**, Conjecture 6.1(iii)], the first assertion of (b) is clear if $v = \infty$ by the description of the pairing given [**3**, page 33], and seems implicitly assumed in the general case. Anyway, we will only use it for representations in the canonical L-packet Π_{ϕ_v} of Π_v for which it follows from [**3**, Conjecture 6.1(iv)]. The second assertion in (b) is clear since $\pi_v^{\nu_v-\mathrm{gen}}$ belongs to that L-packet and has sign $+1$ by definition. The last assertion on (b) is not explicitly written down in [**3**] but is quite natural (it holds for example for the inner forms of U(3) by Rogawski's work). The point (c) follows from (b) together with [**3**, Conjecture 6.1(iii)] since the $\pi_v^{\nu_v-\mathrm{gen}}$ belong to the same L-packet, independently of ν.

Expected Lemma A.13.4. — *The map $\mathbf{S}_\psi \to \mathbb{R}$, $s \mapsto \langle s, \pi^n \rangle$ is the non-trivial character ϵ.*

Indeed, according to the remarks above,

$$\langle s_\psi, \pi^n \rangle = \prod_v e_v((\pi^n)_v)$$

and $e_v(\pi_v^n) = 1$ for every finite place v since $\pi_v^n \in \Pi_{\phi_v}$, so we are reduce to showing that $e((\pi^n)_\infty) = e(\pi_\infty^s) = -1$.

(26) At least in all cases we will use them. We are not completely sure of their generality.

Let ϕ_s be the L-parameter of π^s_∞ and remember from the proof of Lemma A.12.3 that this is *not the same* as $\phi_\infty = \phi_{\psi_\infty}$. Note that all those L-parameters of $G(\mathbb{R}) = \mathrm{U}(m)(\mathbb{R})$, as well as the A-parameter ψ_∞, may be seen as parameters of $G^*(\mathbb{R})$ since those groups have the same L-group ${}^LG$. By (b) above we may work with the group $G^*(\mathbb{R})$, and the aim is to show that $e(\phi_s) = -1$.

Arthur, following Adams and Johnson, describes an algorithm to compute the L-parameter ϕ's of the representations belonging to Π_{ψ_∞} (we already used it for the group $G(\mathbb{R})$ in the proof of Lemma A.12.3) and to compute the local pairing. This algorithm, as well as the resulting pairing, depend on a choice of a conjugacy class of a Levi subgroup L^* of G^* whose associated L-group is the ${}^LL$ defined in the proof of Lemma A.12.3. Here we choose L^* to be quasi-split. We denote by $\langle\ ,\ \rangle_{L^*}$ the pairing described by Arthur using L^*.

The elements π_w in Π_{ψ_∞} are parameterized by the elements w of the set

$$\Sigma^* = W(L^*, T)\backslash W(G^*, T)/W_{\mathbb{R}}(G^*, T)$$

where T is a compact torus of G^* contained in L^*. Contrary to the case of the corresponding set Σ for the compact group $G(\mathbb{R})$ used in the proof of Lemma A.12.3, this set Σ^* is not a singleton, corresponding to the fact that the A-packet Π_{ψ_∞} for $G^*(\mathbb{R})$ is not a singleton. By construction, the L-parameter ϕ_w of the representation π_w and the values $\langle s, \pi_w\rangle_{L^*}$ for $s \in \mathbf{S}_\psi$ depend only on w through the Levi subgroup $L_w := wL^*w^{-1}$. This Levi subgroup is an inner form of L^* defined over $\mathbb{R}$, but is not in general conjugate to L^* in $G^*(\mathbb{R})$.

The parameter ϕ_1 for $w = 1$ is, using the fact that L^* is quasi-split,

$$\phi_1 = \phi_{\psi_\infty}$$

after [**3**, last sentence of the first paragraph page 32]. This ensures from (c) above that the pairing defined by Arthur using L^* is the canonical pairing:

$$\langle s_\psi, \cdot\rangle_{L^*} = \langle s_\psi, \cdot\rangle_v.$$

Let w be an element of Σ^* such that $L := L_w$ is the compact inner form of L^*. Then we have $\phi_w = \phi_s$. The needed computation to check that was actually done during the proof of Lemma A.12.3 since the only ingredient used there was that L was a compact Levi subgroup.

We thus are reduced to compute

$$e(\pi_s) = \langle s_\psi, \pi_w\rangle_\infty$$

for w as above.

For this we have to be a little bit more explicit. We can take for T, compatibly with the choice already done, the diagonal torus in an orthogonal basis $e_1, \dots, e_m$ (in the complex hermitian vector space (V, q) used to define $G^*(\mathbb{R})$), such that $q(e_{m-1}) =$

$q(e_m) = 1$ but $q(e_1) = -1$ (this is always possible since $m \geq 3$ and G^* is quasi-split.). We may define L as the Levi subgroup of matrices stabilizing the plane generated by e_{m-1} and e_m, and the lines generated by $e_1, \dots, e_{m-2}$: it is a compact group. And we may take for L^* the Levi subgroup of matrices stabilizing the plane generated by e_1 and e_m, and the lines generated by $e_2, \dots, e_{m-1}$: it is a quasi-split group. Now it is clear that if

$$w \in W(G^*, T) = W(G^*(\mathbb{C}), T(\mathbb{C})) \simeq \mathfrak{S}_m$$

is the transposition $(1, m-1)$, then $wL^*w^{-1} = L$. But that w is the reflexion w_β (cf. [**3**, page 33]) about the simple root β of $G^*(\mathbb{C})$ such that $\beta(\mathrm{diag}(x_1, \dots, x_m)) = x_1/x_{m-1}$. Since this root is non compact, we have by [**3**, (5.6) & (5.7)]: $e(\pi_s) = \langle s_\psi, \pi_w \rangle = \beta^\vee(s_\psi) = -1$ using that $s_\psi = \mathrm{diag}(1, \dots, 1, -1, -1)$.

Expected Theorem A.13.5. — *The multiplicity $m(\pi^n)$ of the representation π^n in the discrete spectrum of* $\mathrm{U}(m)$ *is* 1 *if* $\varepsilon(\pi, 1/2) = -1$ *and* 0 *otherwise.*

Indeed, by Expected Prop. A.11.9, ψ is the only A-parameter defining the A-packet Π of π^n, so we have $m(\pi^n) = m_\psi(\pi^n)$ according to Arthur's definitions. By [**3**, (8.5)],

$$m_\psi(\pi^n) = \frac{1}{|\mathbf{S}_\psi|} \sum_{s \in \mathbf{S}_\psi} \epsilon_\psi(s) \langle s, \pi^n \rangle = \frac{1}{2}(1 - \epsilon_\psi(s_\psi))$$

using Exp. Lemma A.13.4. The theorem then follows from Exp. Prop. A.13.3.

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