Astérisque

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Astérisque, tome 323 (2009), p. 33-40
<http://www.numdam.org/item?id=AST_2009__323__33_0>

Astérisque
323, 2009, p. 33–40

GEVREY CLASS OF THE INFINITESIMAL GENERATOR OF A DIFFEOMORPHISM

by

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Abstract. — Let F be an analytic diffeomorphism in $(\mathbb{C}^m, 0)$ tangent to the identity of order n. The infinitesimal generator of F is the formal vector field X such that $\operatorname{Exp} X = F$. In this paper we provide an elementary proof of the fact that X belongs to the Gevrey class of order $1/n$.

***Résumé* (La classe de Gevrey du générateur infinitésimal d'un difféomorphisme)**
Soit F un difféomorphisme analytique de $\mathbb{C}^m$ tangent à l'identité à l'ordre n. Le générateur infinitésimal de F est le champ de vecteurs formel X tel que $\operatorname{Exp} X = F$. Dans cet article nous donnons une preuve élémentaire du fait que X appartient à la classe Gevrey d'ordre $1/n$.

1. Introduction

For each couple of integers $m \geq 1$ and $n \geq 2$, let us denote $\hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ the module of formal vector fields of order $\geq n$ in $(\mathbb{C}^m, 0)$ and $\widehat{\mathrm{Diff}}_n(\mathbb{C}^m, 0)$ the group of formal diffeomorphisms in $(\mathbb{C}^m, 0)$ tangent to the identity of order $\geq n$, i.e, $F \in \widehat{\mathrm{Diff}}_n(\mathbb{C}^m, 0)$ if and only if $\nu(F) := \min\{\nu_0(x_i \circ F - x_i) | i = 1, \ldots, m\} - 1 \geq n$. For any $X \in \hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$, the exponential operator of X is the application $\exp X : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by the formula

$$\exp X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g)$$

2000 Mathematics Subject Classification. — 32H02, 32H50, 37F99.
Key words and phrases. — Dynamique discrète, générateur infinitesimal, série de Gevrey.

The first author was supported by CAPES, Brazil, Process: BEX3083/05-5.
The second author was supported by FPU program, Spain, Process: AP2005-3784.

where $X^0(g) = g$ and $X^{j+1}(g) = X(X^j(g))$. It is a classical result (for instance, see [5]) that the application

$$\begin{array}{rcl} \operatorname{Exp} : \hat{\mathfrak{X}}_n(\mathbb{C}^m, 0) & \to & \widehat{\operatorname{Diff}}_{n-1}(\mathbb{C}^m, 0) \\ X & \mapsto & (\exp X(x_1), \dots, \exp X(x_m)) \end{array}$$

is a bijection. The formal vector field X such that $F = \operatorname{Exp}(X)$ is called the *infinitesimal generator* of F.
Let $x = (x_1, \dots, x_m)$ and for any $s \in \mathbb{R}$ let $\mathbb{C}[[x]]_s$ denote the subset of elements of $\mathbb{C}[[x]]$ that satisfy the s-Gevrey condition, i.e.

$$f(x) = \sum_{k=0}^{\infty} f_k(x) \in \mathbb{C}[[x]]_s \quad \text{if and only if} \quad \sum_{k=0}^{\infty} \frac{f_k(x)}{k!^s} \in \mathbb{C}\{x\},$$

where $f_k(x)$ is homogeneous of degree k. Let us observe that 0-Gevrey condition means analyticity, and $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_t$ if $0 < s < t$. Let $\mathfrak{X}_n(\mathbb{C}^m, 0)_s \subseteq \hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ be the set of s-Gevrey vector fields $X = \sum_{k=1}^{m} X(x_k) \frac{\partial}{\partial x_k}$ with $X(x_k) \in \mathbb{C}[[x]]_s$ and $\operatorname{Diff}_n(\mathbb{C}^m, 0)_s = \widehat{\operatorname{Diff}}_n(\mathbb{C}^m, 0) \cap (\mathbb{C}[[x]]_s)^m$ the set of s-Gevrey diffeomorphisms tangent to the identity of order $\geq n$.
We will prove the following result

Theorem 1.1. — *For any $s \geq \frac{1}{n-1}$ the application* Exp *gives a bijection*

$$\operatorname{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_s \to \operatorname{Diff}_{n-1}(\mathbb{C}^m, 0)_s.$$

In particular, the infinitesimal generator of any tangent to the identity analytic diffeomorphism F is $\frac{1}{\nu(F)}$-Gevrey.

In general, X may be divergent for a convergent F, for instance, Szekeres [7] and Baker [2] proved that every entire holomorphic function tangent to the identity of order k in dimension 1 has a non-convergent infinitesimal generator, Ahern and Rosay [1] proved that this kind of diffeomorphisms cannot be the time-1 map of a C^{3k+3}-vector field, and finally J. Rey [6] showed that they cannot be the time-1 map of a C^{k+1}-vector field, which is the best possible bound. Thus, the map $\operatorname{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_0 \to \operatorname{Diff}_{n-1}(\mathbb{C}^m, 0)_0$ is not surjective for any couple of positive integers m, n. In addition, in dimension 1, using resummation arguments, it is proved that if an analytic diffeomorphism $f(x) = x + a_{k+1}x^{k+1} + \cdots$ with $a_{k+1} \neq 0$ has a divergent infinitesimal generator X, then X is k-summable, so X is Gevrey of order $\frac{1}{k}$, but not smaller (see [4], [3] and [5]). Therefore, the condition $s \geq \frac{1}{n-1}$ is necessary.

Acknowledgements. We would like to thank Javier Ribón for pointing out the above results on convergence and resummation in dimension 1, and giving us the idea to improve our first version of this paper. We would like to thank José Cano for fruitful conversation and computational calculations and Felipe Cano for useful conversation.

2. Technical estimations

In this paper, we take the following notations:

- $h_k(x)$ will denote the homogeneous polynomial $\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=k}} x^\alpha$.
- $H_{s,n}(x)$ the series $\sum_{q=n}^{\infty}(q+m-n)!^s h_q(x)$.
- $\frac{\partial}{\partial x}$ the differential operator $\sum_{k=1}^{m}\frac{\partial}{\partial x_k}$.

For formal series $f(x)=\sum_\alpha f_\alpha x^\alpha$ and $g(x)=\sum_\alpha g_\alpha x^\alpha$, we say that $f\preceq g$ if $|f_\alpha|\le|g_\alpha|$ for any $\alpha\in\mathbb{N}^m$. We get in this way a partial order in $\mathbb{C}[[x]]$, and also in $\hat{\mathfrak{X}}_n(\mathbb{C}^m,0)$ and $\widehat{\mathrm{Diff}}_n(\mathbb{C}^m,0)$, working on the component function. From the definition of Gevrey condition, it can be seen that $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that, for all $q\ge n$,

$$\mathrm{Coef}_q(X)\preceq(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

where $\mathrm{Coef}_q(X)$ denotes the homogeneous term of X of degree q. Thus $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that $X\preceq H_{s,n}(ax)\frac{\partial}{\partial x}$.
We need the following technical lemmas:

Lemma 2.1. — *For every $k,l\in\mathbb{N}^*$*

$$h_k\frac{\partial}{\partial x}h_l\preceq(l+m-1)\min\left\{\binom{k+m-1}{m-1},\binom{l+m-2}{m-1}\right\}h_{k+l-1}.$$

Proof. — Observe that

$$\begin{aligned}\frac{\partial}{\partial x}h_l &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}x^\alpha=\sum_{k=1}^{m}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}\alpha_k\frac{x^\alpha}{x_k}\\ &= \sum_{\substack{\beta\in\mathbb{N}^m\\|\beta|=l-1}}\sum_{k=1}^{m}(\beta_k+1)x^\beta=(l+m-1)h_{l-1}\end{aligned}$$

Now, the coefficient of x^α in the product $h_k(x)h_{l-1}(x)$ is less than or equal to the minimum between the number of monomials of h_k and the number of monomials of h_{l-1}, and the number of monomials of h_j is $\binom{j+m-1}{m-1}$, that corresponds to the number of ordered partitions of j in m parts; therefore,

$$h_k\frac{\partial}{\partial x}h_l=(l+m-1)h_kh_{l-1}\preceq(l+m-1)\binom{\min\{k,l-1\}+m-1}{m-1}h_{k+l-1}.\qquad\square$$

Lemma 2.2. — *Let $\Theta(y)=\sum_{j=n}^{\infty}\binom{m-1+j}{m-1}y^{j-n}$. Then $\Theta(y)$ converges for any $|y|<1$.*

Proof. — Since $\sum_{j=n}^{\infty} y^{m-1+j} = \frac{y^{m+n-1}}{1-y}$ converges for any $|y| < 1$ then

$$\Theta(y) = \frac{1}{(m-1)!}\frac{1}{y^n}\frac{d^{m-1}}{dy^{m-1}}\left(\frac{y^{m+n-1}}{1-y}\right)$$

converges for any $|y| < 1$. □

Lemma 2.3. — *For any $s > 0$ and integers $m \geq 1$ and $n \geq 2$, the sequence $\{b_q\}_{q \geq 2n-1}$ given by*

$$b_q = \sum_{j=n}^{\lfloor \frac{q+1}{2} \rfloor} \left(\frac{(j+m-n)!(q-j+1+m-n)!}{m!(q+m-n)!}(q-j+m)^{n-1}\right)^s \binom{j+m-1}{m-1},$$

is bounded.

Proof. — Observe that

$$\begin{aligned}\frac{(q-j+m)^{n-1}}{(q-j+2+m-n)\cdots(q-j+m)} &< \left(\frac{q-j+m}{q-j+2+m-n}\right)^{n-1}\\ &\leq \left(\frac{\frac{q-1}{2}+m}{\frac{q-1}{2}+2+m-n}\right)^{n-1} \leq \left(\frac{m+n-1}{m+1}\right)^{n-1}\end{aligned}$$

then

$$b_n \leq \left(\frac{m+n-1}{m+1}\right)^{s(n-1)} \sum_{j=n}^{\lfloor \frac{q+1}{2} \rfloor} \left(\frac{(j+m-n)!(q-j+m)!}{m!(q+m-n)!}\right)^s \binom{j+m-1}{m-1}.$$

In addition

$$\frac{m+1}{q+m-j+1} < \frac{m+2}{q+m-j+2} < \cdots < \frac{j+m-n}{q+m-n}$$

and

$$\frac{j+m-n}{q+m-n} \leq \frac{\frac{q+1}{2}+m-n}{q+m-n} \leq \max\left\{\frac{1}{2}, \frac{m}{m+n-1}\right\} = C_{m,n} < 1;$$

from lemma 2.2,

$$b_q < \left(\frac{m+n-1}{m+1}\right)^{s(n-1)} \Theta(C_{m,n}^s).$$

□

Proposition 2.4. — *Let $s \geq \frac{1}{n-1}$, $X \in \widehat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ and $a \in \mathbb{R}^+$ such that*

$$\operatorname{Coef}_q(X) \preceq (q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x}$$

for all $n \leq q \leq N$, and let us denote $A = 2m!^s \left(\frac{m+n-1}{m+1}\right)^{s(n-1)} \Theta(C_{m,n}^s)$. For every q, k with $n \leq q \leq N+k-1$,

$$\operatorname{Coef}_q(X^k) \preceq (aA)^{k-1}(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

Proof. — Since $X^k = \sum_{i=1}^{m} X^k(x_i)\frac{\partial}{\partial x_i}$, it is enough to prove the affirmation for $X^k(x_i)$, where $i \in \{1, 2, \dots, m\}$. Let us write $X = \sum_{j=n}^{\infty} X_j$, where X_j is homogeneous of degree j. We will proceed by induction on k; if $k = 1$, by hypothesis

$$X_q(x_i) \preceq (q + m - n)!^s a^q h_q(x) \quad \text{for every } n \leq q \leq N.$$

Suppose that the lemma is true for every $k \leq p$, then, since the order of X^j is greater than or equal to $(n-1)j + 1$, $\mathrm{Coef}_q(X^{p+1}) = 0$ for $n \leq q \leq (n-1)p + n - 1$ and for $(n-1)p + n \leq q \leq N + p$ we have

$$\begin{aligned}
&\mathrm{Coef}_q(X^{p+1}(x_i)) = \mathrm{Coef}_q(X(X^p(x_i))) = \mathrm{Coef}_q\Big(\sum_{j=n}^{\infty} X_j(X^p(x_i))\Big)\\
&= \sum_{j=n}^{q-(n-1)p} X_j \,\mathrm{Coef}_{q+1-j}(X^p(x_i))\\
&\preceq \sum_{j=n}^{q-(n-1)p} (j+m-n)!^s a^j h_j(x) \tfrac{\partial}{\partial x}\Big((aA)^{p-1}(q-j+1+m-n)!^s a^{q+1-j} h_{q+1-j}(x)\Big)\\
&\preceq \sum_{j=n}^{q-n+1} (j+m-n)!^s (q-j+1+m-n)!^s (q-j+m)\tbinom{\min\{j,q-j\}+m-1}{m-1} A^{p-1} a^{q+p} h_q,\\
&\preceq 2\sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q+m-j)^{n-1})^s \tbinom{j+m-1}{m-1} A^{p-1} a^{q+p} h_q.
\end{aligned}$$

Now, observe that

$$b_q m!^s (q+m-n)!^s = \sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q-j+m)^{n-1})^s \binom{j+m-1}{m-1},$$

where $\{b_q\}$ is the sequence defined in lemma 2.3; it follows that

$$\begin{aligned}
\mathrm{Coef}_q(X^{p+1}(x_i)) &\preceq 2b_q m!^s (q+m-n)!^s A^{p-1} a^{q+p} h_q\\
&\preceq (q+m-n)!^s (aA)^p a^q h_q
\end{aligned}$$

□

3. Proof of theorem 1.1.

To prove that the application $\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_s \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ is well defined for $s \geq \frac{1}{n-1}$, let $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$, $a > 0$ be such that $X \preceq H_{s,n}(ax)$, and A as in proposition 2.4.

Then by proposition 2.4 we have

$$\begin{aligned}
\mathrm{Coef}_q(\exp X(x_j)) &= \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{Coef}_q(X^k(x_j))\\
&\preceq \sum_{k=1}^{\infty} \frac{1}{k!} (aA)^{k-1} (q+m-n)!^s a^q h_q(x)
\end{aligned}$$

therefore $\operatorname{Exp}(X) \preceq \sum_{k=1}^{\infty} \frac{(aA)^{k-1}}{k!} H_{s,n}(ax)$. Now, to prove that Exp is surjective, let us consider a diffeomorphism $F(x) = (x_1 + f_1(x), \ldots, x_m + f_m(x)) \in \operatorname{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ where $f_j(x) = \sum_{q=n}^{\infty} f_{j,q}(x) \in \mathbb{C}[[x]]_s$ and $f_{j,q}(x)$ is an homogeneous polynomial of degree q. Then there exists $a > 0$ such that $f_{j,q}(x) \preceq (q+m-n)!^s a^q h_q(x)$. Observe that, making a linear change of coordinates, we can suppose that a is small enough such that $\sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1} \leq \frac{1}{2}$. If $X = \sum_{q=n}^{\infty} X_q$ is the infinitesimal generator of $F(x)$, we will show by induction on q that

$$X_q \preceq (q+m-n)!^s (2a)^q h_q(x) \frac{\partial}{\partial x}.$$

For $q = n$

$$X_n(x_j) = f_{j,n}(x) \preceq m!^s a^n h_n(x) \preceq m!^s (2a)^n h_n(x).$$

Suppose that the claim is true for any integer between n and q, it follows that

$$f_{j,q+1}(x) = \operatorname{Coef}_{q+1}\Big(\sum_{k=1}^{\infty} \frac{1}{k!} X^k(x_j)\Big) = X_{q+1}(x_j) + \sum_{k=2}^{q} \frac{1}{k!} \operatorname{Coef}_{q+1}\big(X^k(x_j)\big),$$

using proposition 2.4

$$\begin{aligned} X_{q+1}(x_j) &\preceq (q+1+m-n)!^s a^{q+1} h_{q+1}(x) \\ &\qquad + \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1}(q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x) \\ &\preceq \left(\frac{1}{2^{q+1}} + \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1}\right)(q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x) \\ &\preceq (q+1+m-n)!^s (2a)^{q+1} h_{q+1}(x), \end{aligned}$$

in other words $X \preceq H_{s,n}(2a) \frac{\partial}{\partial x}$. □

4. Case $0 \leq s < \frac{1}{n-1}$

As we indicated in the introduction, in this case, there exists $F \in \operatorname{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ such that its infinitesimal generator is not s-Gevrey , but the reciprocal is true, i.e.

Proposition 4.1. — *Let* $0 \leq s \leq \frac{1}{n-1}$, *and* $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$. *Then* $\operatorname{Exp}(X) \in \operatorname{Diff}_{n-1}(\mathbb{C}^m, 0)_s$.

Observe that the case $s = 0$ is a classical result about the existence of solution of an analytic differential equation. To prove this proposition in the case $s > 0$ we need the following lemma

Lemma 4.2. — *Let $t, r \in \mathbb{R}$ such that $0 < t < 1$ and $1 - t < r < 1$. Let $\{a_k\}$ be the sequence defined by $a_1 = a > 0$ and for $k \geq 1$, $a_{k+1} = \sup_{q \in \mathbb{N}^*} \sqrt[q+k]{\frac{(q+m)^{1-t}}{(k+1)^r}} a_k$. Then $\{a_k\}$ is increasing and convergent.*

Proof. — Taking $q \gg k$ it is clear that $\sqrt[q+k]{\frac{(q+m)^{1-t}}{(k+1)^r}} > 1$, and then $a_{k+1} > a_k$. Now, we know by Bernoulli inequality that

$$\sqrt[q+k]{\frac{q+m}{(k+1)^{\frac{r}{1-t}}}} < 1 + \frac{1}{q+k}\left(\frac{q+m}{(k+1)^{\frac{r}{1-t}}} - 1\right) < 1 + \frac{1}{(k+1)^{\frac{r}{1-t}}}$$

for $k > m$, so

$$a_{k+1} < \left(1 + \frac{1}{(k+1)^{\frac{r}{1-t}}}\right)^{1-t} a_k < \left(\prod_{j=m+1}^{k+1}\left(1 + \frac{1}{j^{\frac{r}{1-t}}}\right)\right)^{1-t} a_m,$$

and since $\frac{r}{1-t} > 1$ it follows that $\{a_k\}$ is bounded, thereby it is convergent. □

Proof of proposition 4.1. — If $s \in (0, \frac{1}{n-1})$, $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$ and $a \in \mathbb{R}^+$ such that $X \preceq H_{s,n}(ax)\frac{\partial}{\partial x}$ then for $t = s(n-1)$, $r \in (1-t, 1)$ and $\{a_k\}$ as in lemma 4.2, using the arguments of proposition 2.4 and the fact that $k^r a_k^{k+q-1} \geq (q+m)^{1-t} a_{k-1}^{k+q-1}$ for every $q \geq 2$, we can prove that

$$X^k \preceq (a_k A)^{k-1} k!^r H_{s,n}(a_k x)\frac{\partial}{\partial x},$$

where $A = 2m!^s \left(\frac{m+n-1}{m+1}\right)^{s(n-1)} \Theta(C^s_{m,n})$. Let $c = \lim_{k\to\infty} a_k$. Therefore we have

$$\mathrm{Coef}_q(\exp(X)(x_j)) = \sum_{k=1}^{\infty} \frac{1}{k!}\mathrm{Coef}_q(X^k(x_j)) \preceq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k!^{1-r}}(m+q-n)!^s c^q h_q(x)$$

Thus $\mathrm{Exp}(X) \preceq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k!^{1-r}} H_{s,n}(cx)\frac{\partial}{\partial x}$. □

References

[1] P. AHERN & J.-P. ROSAY – "Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables", *Trans. Amer. Math. Soc.* **347** (1995), p. 543–572.

[2] I. N. BAKER – "Non-embeddable functions with a fixpoint of multiplier 1", *Math. Z.* **99** (1967), p. 377–384.

[3] J. ÉCALLE – "Théorie itérative: introduction à la théorie des invariants holomorphes", *J. Math. Pures Appl. (9)* **54** (1975), p. 183–258.

[4] B. MALGRANGE – "Sommation des séries divergentes", *Exposition. Math.* **13** (1995), p. 163–222.

[5] J. MARTINET & J.-P. RAMIS – "Classification analytique des équations différentielles non linéaires résonnantes du premier ordre", *Ann. Sci. École Norm. Sup. (4)* **16** (1983), p. 571–621 (1984).

[6] J. REY – "Difféomorphismes résonnants de $(\mathbb{C}, 0)$", Ph.D. Thesis, Univ. Paul Sabatier, Toulouse, 1996.

[7] G. SZEKERES – "Regular iteration of real and complex functions", *Acta Math.* **100** (1958), p. 203–258.

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