Astérisque

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Astérisque, tome 321 (2008), p. 51-97

<http://www.numdam.org/item?id=AST_2008__321__51_0>

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GRADIENT KÄHLER RICCI SOLITONS

by

Robert L. Bryant

To Jean Pierre Bourguignon, on the occasion of his 60th birthday.

Abstract. — Some observations about the local and global generality of gradient Kähler Ricci solitons are made, including the existence of a canonically associated holomorphic volume form and vector field, the local generality of solutions with a prescribed holomorphic volume form and vector field, and the existence of Poincaré coordinates in the case that the Ricci curvature is positive and the vector field has a fixed point.

***Résumé* (Solitons gradients de Kähler-Ricci).** — Nous proposons quelques observations sur les généralités locale et globale des solitons gradients de Kähler-Ricci, y compris l'existence d'une forme de volume holomorphe et d'un champ de vecteurs canoniquement associés, la généralité locale de solutions pour une forme de volume holomorphe et un champ de vecteurs donnés, et l'existence de coordonnées de Poincaré dans le cas où la courbure de Ricci est positive et le champ de vecteurs a un point fixe.

1. Introduction and Summary

This article concerns the local and global geometry of gradient Kähler Ricci solitons, i.e., Kähler metrics g on a complex n-manifold M that admit a *Ricci potential*, i.e., a function f such that $\mathrm{Ric}(g) = \nabla^2 f$ (where ∇ denotes the Levi-Civita connection of M.

These metrics arise as limiting metrics in the study of the Ricci flow $g_t = -2\,\mathrm{Ric}(g)$ applied to Kähler metrics. Under the Ricci flow, a gradient Kähler Ricci soliton g_0

2000 Mathematics Subject Classification. — 53C55, 58G11.
Key words and phrases. — Ricci flow, solitons, normal forms.

Thanks to Duke University for its support via a research grant, to the NSF for its support via grants DMS-8905207, DMS-0103884, and DMS-0604195 and to Columbia University for its support via an Eilenberg Visiting Professorship.

evolves by flowing under the vector field ∇f, i.e.,

$$g(t) = \exp_{(-t\nabla f)}{}^{*}(g_0). \tag{1.1}$$

In particular, if the flow of ∇f is complete, then the Ricci flow with initial value g_0 exists for all time.

The reader who wants more background on these metrics might consult the references and survey articles [**3, 5, 10**]. The references [**8, 9, 6, 14**] contain further important work in the area and will be cited further below.

1.1. Basic facts. — Unless the metric g admits flat factors, the equation $\mathrm{Ric}(g) = \nabla^2 f$ determines f up to an additive constant and it does no harm to fix a choice of f for the discussion. For simplicity, it does no harm to assume that g has no (local) flat factors and so this will frequently be done. Also, the Ricci-flat case (aka the Calabi-Yau case), in which $\mathrm{Ric}(g) = 0$, is a special case that is usually treated by different methods, so it will usually be assumed that $\mathrm{Ric}(g) \neq 0$. (Indeed, most of the latter part of this article will focus on the case in which $\mathrm{Ric}(g) > 0$).

1.1.1. *The associated holomorphic vector field Z.* — One of the earliest observations [**2**] made about gradient Kähler Ricci solitons is that the vector field ∇f is the real part of a holomorphic vector field and that, moreover, $J(\nabla f)$ is a Killing field for g. In this article, I will take $Z = \frac{1}{2}\big(\nabla f - \mathrm{i}J(\nabla f)\big)$ to be the holomorphic vector field *associated* to g.

1.1.2. *The holomorphic volume form Υ.* — In the Ricci-flat case, at least when M is simply connected, it is well-known that there is a g-parallel holomorphic volume form Υ, i.e., one which satisfies the condition that $\mathrm{i}^{n^2}2^{-n}\,\Upsilon\wedge\overline{\Upsilon}$ is the real volume form determined by g and the J-orientation.

In § 2.2, I note that, for any gradient Kähler Ricci soliton g with Ricci potential f defined on a simply connected M, there is a holomorphic volume form Υ (unique up to a constant multiple of modulus 1) such that $\mathrm{i}^{n^2}2^{-n}\,\mathrm{e}^{-f}\,\Upsilon\wedge\overline{\Upsilon}$ is the real volume form determined by g and the J-orientation. Of course, Υ is not g-parallel (unless g is Ricci-flat) but satisfies $\nabla\Upsilon = \frac{1}{2}\,\partial f\otimes\Upsilon$.

This leads to a notion of *special* coordinate charts for (g, f) i.e., coordinate charts (U, z) such that the associated coordinate volume form $\mathrm{d}z = \mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n$ is the restriction of Υ to U. In such coordinate charts, several of the usual formulae simplify for gradient Kähler Ricci solitons.

1.1.3. *The Υ-divergence of Z.* — Given a vector field and and volume form, the divergence of the vector field with respect to the volume form is well defined. It turns out to be useful to consider this quantity for Z and Υ. The divergence in this case is the (necessarily holomorphic) function h that satisfies $\mathsf{L}_Z\,\Upsilon = h\,\Upsilon$.

By general principles, the scalar function h must be expressible in terms of the first and second derivatives of f. Explicit computation (Proposition 4) yields

$$2h = \mathrm{tr}_g\big(\nabla^2 f\big) + |\nabla f|^2 = R(g) + |\nabla f|^2, \tag{1.2}$$

where $R(g) = \mathrm{tr}_g\big(\mathrm{Ric}(g)\big)$ is the scalar curvature of g. In particular, h is real-valued and therefore constant. Now, the constancy of $R(g) + |\nabla f|^2$ had already been noted and utilized by Hamilton and Cao [**6**]. However, its interpretation as a holomorphic divergence seems to be new.

1.2. Generality. — An interesting question is: How many gradient Kähler Ricci solitons are there? Of course, this rather vague question can be sharpened in several ways.

The point of view adopted in this article is to start with a complex n-manifold M already endowed with a holomorphic volume form Υ and a holomorphic vector field Z and ask how many gradient Kähler solitons on M there might be (locally or globally) that have Z and Υ as their associated holomorphic data.

An obvious necessary condition is that the divergence h of Z with respect to Υ must be a real constant.

1.2.1. *Nonsingular extension.* — Away from the singularities (i.e., zeroes) of Z, this divergence condition turns out to be locally sufficient.

More precisely, I show (see Theorem 2) that if $H \subset M$ is an embedded complex hypersurface that is transverse at each of its points to Z, and g_0 and f_0 are, respectively, a real-analytic Kähler metric and function on H, then there is an open neighborhood U of H in M on which there exists a gradient Kähler Ricci soliton g with potential f whose associated holomorphic quantities are Z and Υ and such that g and f pull back to H to become g_0 and f_0. The pair (g, f) is essentially uniquely specified by these conditions. The real-analyticity of the 'initial data' g_0 and f_0 is necessary in order for an extention to exist since any gradient Kähler Ricci soliton is real-analytic anyway (see Remark 4).

Roughly speaking, this result shows that, away from singular points of Z, the local solitons g with associated holomorphic data (Z, Υ) depend on two arbitrary (real-analytic) functions of $2n-2$ variables.

1.2.2. *Singular existence.* — The existence of (local) gradient Kähler solitons in a neighborhood of a singularity p of Z is both more subtle and more interesting.

Even if the divergence of Z with respect to Υ is a real constant, it is not true in general that a gradient Kähler Ricci solition with Z and Υ as associated holomorphic data exists in a neighborhood of such a p.

I show (Proposition 6) that a necessary condition is that there exist p-centered holomorphic coordinates $z = (z^i)$ on a p-neighborhood $U \subset M$ and real numbers $h_1, \ldots, h_n$ such that, on U,

$$Z = h_1\, z^1 \frac{\partial}{\partial z^1} + \cdots + h_n\, z^n \frac{\partial}{\partial z^n}. \tag{1.3}$$

In other words, Z must be holomorphically linearizable, with real eigenvalues. [(1)]

In such a case, if $\mathsf{L}_Z \Upsilon = h\Upsilon$ where h is a constant, then $h = h_1 + \cdots + h_n$. I show (Proposition 7) that, moreover, in this case, one can always choose Z-linearizing coordinates as above so that $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$.

Thus, the possible local singular pairs (Z, Υ) that can be associated to a gradient Kähler Ricci soliton are, up to biholomorphism, parametrized by n real constants.

Using this normal form, one then observes that, by taking products of solitons of dimension 1, any set of real constants $(h_1, \ldots, h_n)$ can occur (see Remark 9). Since, for any gradient Kähler Ricci soliton g with associated holomorphic data (Z, Υ), the formula $\mathrm{Ric}(g) = \mathsf{L}_{\mathrm{Re}(Z)}\, g$ holds, it follows that if g is such a Kähler Ricci soliton defined on a neighborhood of a point p with $Z(p) = 0$, then $h_1, \ldots, h_n$ are the eigenvalues (each of even multiplicity) of $\mathrm{Ric}(g)$ with respect to g at p.

However, this does not fully answer the question of how 'general' the solitons are in a neighborhood of such a p. In fact, this very subtly depends on the numbers h_i. For example, if the $h_i \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then any gradient Kähler Ricci soliton g with associated data (Z, Υ) defined on a neighborhood of p must be invariant under the compact n-torus action generated by the closure of the flow of the imaginary part of Z. This puts severe restrictions on the possibilities for such solitons.

At the conclusion of Section § 3, I discuss the local generality problem near a singular point of Z and explain how it can best be viewed as an elliptic boundary value problem of a certain type, but do not go into any further detail. A fuller discussion of this case may perhaps be undertaken at a later date.

1.3. The positive case. — In Section § 4, I turn to an interesting special case: The case where g is complete, the Ricci curvature is positive, and the scalar curvature $R(g)$ attains its maximum at some (necessarily unique) point $p \in M$.

This case has been studied before by Cao and Hamilton [**6**], who proved that this point p is a minimum of the Ricci potential f, that f is a proper plurisubharmonic exhaustion function on M (which is therefore Stein), and that, moreover, the Killing field $J(\nabla f)$ has a periodic orbit on 'many' of its level sets.

(1) Of course, it is by no means true that every holomorphic vector field is holomorphically linearizable at each of its singular points.

For simplicity, the Ricci potential f will be be normalized so that $f(p) = 0$, so that f is positive away from p.

I show (Theorem 3) that under these assumptions there exist global Z-linearizing coordinates $z = (z^i) : M \to \mathbb{C}^n$, so that M is biholomorphic to $\mathbb{C}^n$ (which generalizes an earlier result of Chau and Tam [8]).[(2)] Moreover, as a consequence, it follows that every positive level set of f has at least n periodic orbits of $J(\nabla f)$, a considerable sharpening of Cao and Hamilton's original results.

This global coordinate system has several other applications.

For example, I show that there is a Kähler potential ϕ for g that is invariant under the flow of $J(\nabla f)$ and that this potential is unique up to an additive constant. (Which can be normalized away by requiring that $\phi(p) = 0$.)

As another application, I show how to normalize the choice of Z-linearizing holomorphic coordinates up to an ambiguity that lies in a compact subgroup of $\mathrm{U}(n)$. This makes the function $|z|$ well-defined on M, so it is available for estimates.

As an illustration of such use, I show that there are positive constants r and a_1, a_2, b_1, b_2, c_1, and c_2 such that, whenever $|z| \geq r$,

$$\begin{aligned} a_1 \log|z| &\leq f(z) \leq a_2 \log|z|\,, \\ b_1 \log|z| &\leq d(z,0) \leq b_2 \log|z|\,, \\ c_1 \left(\log|z|\right)^2 &\leq \phi(z) \leq c_2 \left(\log|z|\right)^2. \end{aligned} \tag{1.4}$$

I also give some bounds for a_1 and a_2. Perhaps these will be useful in further work.

1.4. The toric case. — This section studies the geometry of the reduced equation in the case when a gradient Käher Ricci soliton g defined on a neighborhood of $0 \in \mathbb{C}^n$ has toric symmetry, i.e., is invariant under the action of $\mathbb{T}^n$, the diagonal subgroup of $\mathrm{U}(n)$. This may seem specialized, but, for example, if the associated holomorphic vector field is Z_{h} where $\mathsf{h} = (h_1, \ldots, h_n)$ and the real numbers $h_1, \ldots, h_n$ have the 'generic' property of being linearly independent over $\mathbb{Q}$, then g has toric symmetry. Thus, metrics with toric symmetry are the rule when Z has a 'generic' singularity.

I first derive the equation satisfied by the reduced potential, which turns out to be a singular Monge-Ampére equation. (The singularities are, of course, related to the singular orbits of the $\mathbb{T}^n$-action.) I then show that, nevertheless, this singular

(2) On 27 July 2004, about 12 hours before the first version of this article was posted on the arXiv, Chau and Tam posted the first version of their article arXiv:math.DG/0407449 in which they prove, under the same hypotheses as in Theorem 3, that M is biholomorphic to $\mathbb{C}^n$. I saw their posting just before I posted this article. Their method is different and does not produce Z-linearizing coordinates, but has the advantage that it applies in the case of expanding solitons. In the second (much shortened) version of their article, posted on 2 August, 2004, they deduce their biholomorphism result from already-known results about automorphisms of complex manifolds. See [9].

equation has good regularity and its singular initial value problem is well-posed in the sense of Gèrard and Tahara [**11**].

As a consequence (Corollary 5), it follows that, for any $\mathsf{h} \in \mathbb{R}^n$, any real-analytic $\mathbb{T}^{n-1}$-invariant Kähler metric on a neighborhood of $0 \in \mathbb{C}^{n-1}$ is the restriction to $\mathbb{C}^{n-1}$ of an essentially unique toric gradient Kähler Ricci soliton on an open subset of $\mathbb{C}^n$ with associated holomorphic vector field $Z = Z_{\mathsf{h}}$ and associated holomorphic volume form $\Upsilon = \mathrm{d}z$. In particular, it follows that, in a sense made precise in that section, the toric gradient Kähler Ricci solitons on $\mathbb{C}^n$ depend on one 'arbitrary' real-analytic function of $(n-1)$ (real) variables.

Next, I show that the reduced (singular Monge-Ampère) equation is of Euler-Lagrange type, at least, away from its singular locus, and discuss some of its conservation laws via an application of Noether's Theorem. (This is in contrast to the unreduced soliton equation, which is not variational).

1.5. Acknowledgement. — This work is mostly based on notes written after a conversation with Richard Hamilton during a visit he made to Duke University in 1991. Section 4 is more recent, having been written after further conversations with Hamilton during a semester I spent at Columbia University in the spring of 2004.

It is a pleasure thank Hamilton for his interest and to thank Columbia University for its hospitality.

2. Associated Holomorphic Quantities

In this section, constructions of some holomorphic quantities associated to a gradient Kähler Ricci soliton g on a complex n-manifold M^n with Ricci potential f will be described.

2.1. Preliminaries. — In order to avoid confusion because of various different conventions in the literature, I will collect the notations, conventions, and normalizations to be used in this article.

2.1.1. *Tensors and inner products.* — Factors of 2 are sometimes troubling and confusing in Kähler geometry.

For a and b in a vector space V, I will use the conventions $a \circ b = \frac{1}{2}(a \otimes b + b \otimes a)$ and $a \wedge b = a \otimes b - b \otimes a$. In particular, $a \otimes b = a \circ b + \frac{1}{2}\, a \wedge b$.

A real-valued inner product $\langle , \rangle$ on a real vector space V can be extended to $V^{\mathbb{C}} = \mathbb{C} \otimes V$ in several different ways. A natural way is to extend it as an Hermitian form, i.e., so that

$$(2.1) \qquad \langle v_1 + \mathrm{i}v_2, w_1 + \mathrm{i}w_2 \rangle = \big(\langle v_1, w_1\rangle + \langle v_2, w_2\rangle\big) + \mathrm{i}\big(\langle v_2, w_1\rangle - \langle v_1, w_2\rangle\big)$$

and that is the convention to be adopted here.

If the real vector space V has a complex structure $J : V \to V$, then $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ where $V^{1,0}$ is the $+$i-eigenspace of J extended complex linearly to $V^{\mathbb{C}}$ while $V^{0,1}$ is the $(-\mathrm{i})$-eigenspace of J. It is common practice to identify $v \in V$ with $v^{1,0} = v - \mathrm{i}\, Jv \in V^{1,0}$, but some care must be taken with this.

For example, an inner product $\langle , \rangle$ on V is *compatible* with J if $\langle Jv, Jw\rangle = \langle v, w\rangle$ for all $v, w \in V$. Note the identity

$$\langle v^{1,0}, v^{1,0}\rangle = 2\langle v, v\rangle. \tag{2.2}$$

For any J-compatible inner product $\langle , \rangle$ on V (or equivalently, quadratic form) there is an *associated* 2-form η defined by

$$\eta(v, w) = \langle Jv, w\rangle\,. \tag{2.3}$$

2.1.2. *Coordinate expressions and the Ricci form.* — Let $z = (z^i) : U \to \mathbb{C}^n$ be a holomorphic coordinate chart on an open set $U \subset M$. The metric g restricted to U can be expressed in the form

$$g = g_{i\bar{\jmath}}\, \mathrm{d}z^i \circ \mathrm{d}\bar{z}^j \tag{2.4}$$

for some functions $g_{i\bar{\jmath}} = \overline{g_{j\bar{\imath}}}$ on U. The associated Kähler form Ω then has the coordinate expression

$$\Omega = \tfrac{\mathrm{i}}{2}\, g_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j\,. \tag{2.5}$$

Note that $g_{i\bar{\jmath}}\, \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j = g - 2\mathrm{i}\,\Omega$.

The Ricci tensor $\mathrm{Ric}(g)$ is J-compatible since g is Kähler, and hence has a coordinate expression $\mathrm{Ric}(g) = R_{j\bar{k}}\, \mathrm{d}z^j \circ \mathrm{d}\bar{z}^k$ where $R_{j\bar{k}} = \overline{R_{k\bar{\jmath}}}$. Its associated 2-form ρ is computed by the formula

$$\rho = \tfrac{\mathrm{i}}{2}\, R_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = -\mathrm{i}\, \partial\bar{\partial}\, G \tag{2.6}$$

where

$$G = \log\, \det\big(g_{i\bar{\jmath}}\big). \tag{2.7}$$

While ρ is independent of the coordinate chart used to compute it, the function G does depend on the coordinate chart.

The scalar curvature $R(g) = \mathrm{tr}_g\big(\mathrm{Ric}(g)\big)$ has the coordinate expression

$$R(g) = 2g^{i\bar{\jmath}} R_{i\bar{\jmath}} \tag{2.8}$$

and satisfies $R(g)\,\Omega^n = 2n\,\rho \wedge \Omega^{n-1}$.

2.1.3. *The gradient Kähler Ricci soliton condition.* — The following equivalent formulation of the gradient Kähler Ricci soliton condition is well-known:

Proposition 1. — *A real-valued function f on M satisfies* $\mathrm{Ric}(g) = D^2 f$ *if and only if $\rho = \mathrm{i}\,\partial\bar\partial f$ and $D^{0,2} f = 0$. This latter condition is equivalent to the condition that the g-gradient of f be the real part of a holomorphic vector field on M.* □

2.2. The associated holomorphic volume form. — In this subsection, given a gradient Kähler Ricci soliton g with Ricci potential f on a simply-connected complex n-manifold M, a holomorphic volume form on M (unique up to a complex multiple of modulus 1) will be constructed.

2.2.1. *Existence of special coordinates.* — The following result shows that there are coordinate systems in which the Ricci potential is more closely tied to the local coordinate quantities.

Proposition 2. — *If g is a gradient Kähler Ricci soliton on M with Ricci potential f, then M has an atlas of holomorphic charts (U, z) satisfying* $\log\det\bigl(g_{i\bar\jmath}\bigr) = -f$.

Proof. — To begin, let (U, z) be any local holomorphic coordinate chart on M, with quantities $g_{i\bar\jmath}$ and G defined as above.

Since f is a Ricci potential for g, i.e., $\mathrm{Ric}(g) = D^2 f$, it follows from (2.6) and Proposition 1 that

$$-\,\mathrm{i}\,\partial\bar\partial G = \mathrm{i}\,\partial\bar\partial f\,. \tag{2.9}$$

Thus, $f + G$ is pluriharmonic. Assuming further that the domain U of the coordinate system z is simply connected, there exists a holomorphic function p on U so that

$$f = -G + p + \bar p. \tag{2.10}$$

Now let w be any other local coordinate system on the same simply connected domain U in M and write

$$\Omega = \tfrac{\mathrm{i}}{2}\, h_{i\bar\jmath}\, \mathrm{d}w^i \wedge \mathrm{d}\bar w^j. \tag{2.11}$$

Then $H = \log\det\bigl(h_{i\bar\jmath}\bigr)$ is of the form

$$H = G + J + \overline{J} \tag{2.12}$$

where J is the log-determinant of the Jacobian matrix of the change of variables from z to w, i.e.,

$$\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n = e^J\, \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \cdots \wedge \mathrm{d}w^n\,. \tag{2.13}$$

It follows that every point of U has an open neighborhood V on which there exists a coordinate chart w for which $-H = f$, the Ricci potential. □

***Definition 1* (Special coordinates).** — Let g be a gradient Kähler Ricci soliton on M with Ricci potential f. A coordinate chart (U, z) for which $\log\det(g_{i\bar{j}}) = -f$ will be said to be *special* for (g, f).

***Remark 1* (The volume form in special coordinates).** — A coordinate chart (U, z) is special for (g, f) if and only if the volume form of g satisfies

$$\mathrm{dvol}_g = \frac{1}{n!}\,\Omega^n = \left(\frac{\mathrm{i}^n}{2}\right)^n \mathrm{e}^{-f}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.14}$$

***Theorem 1* (Existence of holomorphic volume forms).** — *Let M be a simply connected complex n-manifold endowed with a gradient Kähler Ricci soliton g with associated Kähler form Ω and a choice of Ricci potential f. Then there exists a holomorphic volume form Υ on M, unique up to muliplcation by a complex number of modulus* 1, *with the property that*

$$\mathrm{d}vol_g = \frac{1}{n!}\,\Omega^n = \left(\frac{\mathrm{i}^n}{2}\right)^n \mathrm{e}^{-f}\,\Upsilon \wedge \overline{\Upsilon}. \tag{2.15}$$

Proof. — For any two (g, f)-special coordinate charts z and w on the same domain U, the ratio of their corresponding holomorphic volume forms is a constant of modulus 1.

The volume forms of special coordinate systems are thus the sections of a flat connection ∇_0 on the canonical bundle of M, i.e., the bundle whose sections are the holomorphic volume forms on M. Since M is simply connected, the canonical bundle of M has a global ∇_0-flat section Υ that is unique up to a multiplicative constant.

By construction, Υ satisfies (2.15). Its uniqueness up to multiplication by a constant of modulus 1 is now evident. □

***Definition 2* (Associated holomorphic volume forms).** — Given a gradient Kähler Ricci soliton g with Ricci potential f, a holomorphic volume form Υ satisfying (2.15) will be said to be *associated* to the pair (g, f).

***Remark 2* (Scaling effects on Υ).** — Scaling a gradient Kähler Ricci soliton g by a constant produces another gradient Kähler Ricci soliton and adding a constant to f will produce another Ricci potential for g.

If Υ is associated to (g, f), then, for any real constants $\lambda > 0$ and c, the n-form $\lambda^n \mathrm{e}^c\,\Upsilon$ is associated to $(\lambda^2 g,\ f{+}2c)$.

2.3. The holomorphic flow. — Write the g-gradient of f as $Z + \bar{Z}$ where Z is of type $(1,0)$. Thus, $Z = \frac{1}{2}\big(\nabla f - \mathrm{i}\,J(\nabla f)\big)$.

2.3.1. *The infinitesimal symmetry.* — By the standard Kähler identities, Z is the unique vector field of type $(1,0)$ satisfying

$$\bar{\partial} f = -\mathrm{i}\, Z \lrcorner \Omega\,. \tag{2.16}$$

Writing $Z = X - i\,Y = X - i\,JX$, it follows that, in addition to X being the one-half the gradient of f, the vector field $Y = JX$ is Ω-Hamiltonian. Thus, the flow of Y preserves Ω.

Since Z is holomorphic by Proposition 1, the flow of Y also preserves the complex structure on M.

Hence, Y must be a Killing vector field for the metric g.

Thus, a gradient Kähler Ricci soliton that is not Ricci-flat always has a nontrivial infinitesimal symmetry.

Proposition 3. — *The singular locus of Z is a disjoint union of nonsingular complex submanifolds of M, each of which is totally geodesic in the metric g.*

Proof. — Since Z is holomorphic, its singular locus (i.e., the locus where it vanishes) is a complex subvariety of M. However, since this locus is also the zero locus of $Y = -\mathrm{Im}(Z)$, which is a Killing field for g, this locus is a submanifold that is totally geodesic with respect to g. In particular, it must be smooth and hence nonsingular as a complex subvariety. □

2.3.2. *Z in special coordinates.* — Assume (U,z) is a special local coordinate system. Since

$$\bar{\partial} G = g^{i\bar{\jmath}}\,\frac{\partial g_{i\bar{\jmath}}}{\partial \bar{z}^k}\,\mathrm{d}\bar{z}^k = -\bar{\partial} f\,, \tag{2.17}$$

the formula for Z in special coordinates is

$$Z = Z^\ell \frac{\partial}{\partial z^\ell} = -\left(2 g^{\ell\bar{k}} g^{i\bar{\jmath}} \frac{\partial g_{i\bar{\jmath}}}{\partial \bar{z}^k}\right)\frac{\partial}{\partial z^\ell}\,. \tag{2.18}$$

Thus, the equations for a gradient Kähler Ricci soliton in special coordinates are that the functions Z^ℓ defined by (2.18) be holomorphic.

In fact, the expression in (2.18) can be simplified, since the closure of Ω is equivalent to the equations

$$\frac{\partial g_{i\bar{\jmath}}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}. \tag{2.19}$$

Thus,

$$Z^\ell = -2\, g^{\ell\bar{k}} g^{i\bar{\jmath}}\, \frac{\partial g_{i\bar{\jmath}}}{\partial \bar{z}^k} = -2\, g^{i\bar{\jmath}} g^{\ell\bar{k}}\, \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j} = 2\, g^{i\bar{\jmath}} g_{i\bar{k}}\, \frac{\partial g^{\ell\bar{k}}}{\partial \bar{z}^j} = 2\,\frac{\partial g^{\ell\bar{\jmath}}}{\partial \bar{z}^j}, \tag{2.20}$$

where I have used the identity $g^{i\bar{\jmath}} g_{i\bar{k}} = \delta^{\bar{\jmath}}_{\bar{k}}$ and the identity $g_{i\bar{k}} g^{\ell\bar{k}} = \delta^\ell_i$ and its derivatives.

2.3.3. *The Υ-divergence of Z.* — Since Z is holomorphic, the Lie derivative of Υ with respect to Z must be of the form $h\,\Upsilon$ where h is a holomorphic function on M (usually called the divergence of Z with respect to Υ).

Replacing Υ by $\lambda\Upsilon$ for any $\lambda \in \mathbb{C}^*$ will not affect the definition of h, so the function h is intrinsic to the geometry of the soliton. On general principle, it must be computable in terms of the first and second covariant derivatives of f, which leads to the following interpretation of a result of Cao and Hamilton:

Proposition 4. — *The holomorphic function h is real-valued (and therefore constant). Moreover,*

$$2h = R(g) + 2|Z|^2 \tag{2.21}$$

where $R(g)$ is the scalar curvature of g and $|Z|^2$ is the squared g-norm of Z.

Proof. — In special coordinates, where $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$, the function h has the expression

$$h = \frac{\partial Z^\ell}{\partial z^\ell}. \tag{2.22}$$

Thus, by (2.20),

$$h = 2\,\frac{\partial g^{\ell\bar{\jmath}}}{\partial z^\ell\,\partial \bar{z}^j}, \tag{2.23}$$

which shows that the holomorphic function h is real-valued and therefore constant. Moreover, since $\rho = \mathrm{i}\,\partial\bar\partial f$, it follows that

$$\begin{aligned}
\left(\tfrac{\mathrm{i}}{2}\right) R_{j\bar{k}}\, \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k = \rho = \mathrm{i}\,\partial\bar\partial f &= \partial(Z \lrcorner \Omega)\\
&= \left(\tfrac{\mathrm{i}}{2}\right) \partial\left(g_{\ell\bar{k}} Z^\ell\, \mathrm{d}\bar{z}^k\right)\\
&= \left(\tfrac{\mathrm{i}}{2}\right) \left(g_{\ell\bar{k}}\frac{\partial Z^\ell}{\partial z^j} + Z^\ell \frac{\partial g_{\ell\bar{k}}}{\partial z^j}\right) \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k.
\end{aligned} \tag{2.24}$$

In particular, in view of (2.19) and (2.18),

$$\begin{aligned}
R(g) = 2g^{j\bar{k}} R_{j\bar{k}} = 2g^{j\bar{k}} \left(g_{\ell\bar{k}}\frac{\partial Z^\ell}{\partial z^j} + Z^\ell \frac{\partial g_{\ell\bar{k}}}{\partial z^j}\right) &= 2h + 2g^{j\bar{k}} Z^\ell \frac{\partial g_{\ell\bar{k}}}{\partial z^j}\\
= 2h + 2Z^\ell\, g^{j\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z^\ell} = 2h - g_{\ell\bar{k}} Z^\ell \bar{Z}^k &= 2h - 2|Z|^2,
\end{aligned} \tag{2.25}$$

as claimed. □

***Remark 3* (Interpretations).** — It was Cao and Hamilton [**6**, Lemma 4.1] who first observed that the quantity $R(g) + |\nabla f|^2$ is constant for a (steady) gradient Kähler Ricci soliton. Since $Z = \frac{1}{2}\big(\nabla f - \mathrm{i}\,J(\nabla f)\big)$, one has $2|Z|^2 = |\nabla f|^2$, so their expression is the right hand side of (2.21).

The interpretation of $R(g)+|\nabla f|^2$ as the Υ-divergence of Z seems to be new. In fact, for any gradient Ricci soliton g (not necessarily Kähler) with Ricci potential f, one has the identity

$$\mathsf{L}_{\nabla f}\big(\mathrm{e}^f\,\mathrm{d}vol_g\big) = 2h\ \mathrm{e}^f\,\mathrm{d}vol_g\,. \tag{2.26}$$

where $R(g)+|\nabla f|^2 = 2h$ is a constant. This points out the importance of the modified volume form $\mathrm{e}^f\,\mathrm{d}vol_g$ in the general case.

In a sense, this constancy can be regarded as a sort of conservation law for the Ricci flow. Note that, since $\Delta f = R(g)$, this relation is equivalent to the equation $\Delta_g(\mathrm{e}^f) = 2h\,\mathrm{e}^f$.

2.4. Examples. — The associated holomorphic objects constructed so far make it possible to simplify somewhat the usual treatment of the known explicit examples. The following examples will be useful in later discussions in this article.

***Example 1* (The one-dimensional case: Hamilton's cigar).** — Suppose that M is a Riemann surface. Then Υ is a nowhere vanishing 1-form on M and Z is a holomorphic vector field on M that satisfies $\mathrm{d}\big(\Upsilon(Z)\big) = h\,\Upsilon$, where h is a constant. There are essentially two cases to consider.

First, suppose that $h=0$. Then $\Upsilon(Z)$ is a constant, say $\Upsilon(Z)=c$.

If $c=0$, then Z is identically zero, and, from (2.20) it follows that, in special coordinates $z=(z^1)$ the real-valued function $g^{1\bar 1}$ is constant. In particular, in special coordinates $g = g_{1\bar 1}|\mathrm{d}z^1|^2$, so g is flat.

If $c\neq 0$, then Z is nowhere vanishing and, after adjusting Υ and the special coordinate system by a constant multiple, it can be assumed that $c=2$, i.e., that $\Upsilon = \mathrm{d}z^1$ and $Z = 2\,\partial/\partial z^1$. Then (2.20) implies that $g^{1\bar 1} = z^1+\bar z^1 + C$ for some constant C. By adding a constant to z^1, it can be assumed that $C=0$, so it follows that, in this coordinate system

$$g = \frac{|\mathrm{d}z^1|^2}{(z^1+\bar z^1)}. \tag{2.27}$$

Since M is supposed to be simply connected, one can take z^1 to be globally defined. Thus M is immersed into the right half-plane in $\mathbb{C}$ in such a way that g is the pullback of the conformal metric defined by (2.27). Of course, this metric is not complete, even on the entire right half-plane.

Second, assume that h is not zero. Then $\Upsilon(Z)$ is a holomorphic function on M that has nowhere vanishing differential. Write $\Upsilon(Z) = hz^1$ for some (globally defined) holomorphic immersion $z^1 : M \to \mathbb{C}$. Then, by construction, $\Upsilon = \mathrm{d}z^1$ and $Z = hz^1\,\partial/\partial z^1$. By (2.20), it follows that

$$g^{1\bar 1} = \tfrac{1}{2}(c + h\,|z^1|^2) \tag{2.28}$$

for some constant c, so $z^1(M) \subset \mathbb{C}$ must lie in the open set U in the w-plane on which $c + h|w|^2 > 0$. In fact, g must be the pullback under $z^1 : M \to U \subset \mathbb{C}$ of the metric

$$\frac{2\,|\mathrm{d}w|^2}{c + h\,|w|^2}. \tag{2.29}$$

This metric on the domain $U \subset \mathbb{C}$ is not complete unless both c and h are nonnegative and it is flat unless both c and h are positive. In this latter case, this metric is simply Hamilton's 'cigar' soliton [**12**].

Consequently, in dimension 1, the only complete gradient Kähler Ricci solitons are either flat or one of Hamilton's 'cigar' solitons (which are all homothetic to a single example).

Note that, under the Ricci flow $g_t = -2\,\mathrm{Ric}(g)$, the metric (2.29) evolves as

$$g(t) = \frac{2\,|\mathrm{d}w|^2}{\mathrm{e}^{2ht}c + h\,|w|^2} = \frac{2\,\left|\mathrm{d}(\mathrm{e}^{-ht}w)\right|^2}{c + h\,\left|\mathrm{e}^{-ht}w\right|^2} = \Phi(-t)^*(g_0) \tag{2.30}$$

where $\Phi(t)(w) = e^{ht}w$ is the flow of twice the real part of $Z = hw\,\partial/\partial w$.

***Example 2* (Products).** — By taking products of the 1-dimensional examples, one can construct a family of complete examples on $\mathbb{C}^n$: Let $h_1, \ldots, h_n$ and $c_1, \ldots, c_n$ be positive real numbers and consider the metric on $\mathbb{C}^n$ defined by

$$g = \sum_{k=1}^{n} \frac{2\,|\mathrm{d}w^k|^2}{(c_k + h_k\,|w^k|^2)}. \tag{2.31}$$

This is, of course, a gradient Kähler Ricci soliton, with associated holomorphic volume form and vector field

$$\Upsilon = \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \cdots \wedge \mathrm{d}w^n, \qquad Z = \sum_{k=1}^{n} h_k\, w^k\, \frac{\partial}{\partial w^k}. \tag{2.32}$$

The Ricci curvature is

$$\mathrm{Ric}(g) = \sum_{k=1}^{n} \frac{2c_k h_k\,|\mathrm{d}w^k|^2}{(c_k + h_k\,|w^k|^2)^2}. \tag{2.33}$$

Although these product examples are trivial generalizations of Hamilton's cigar soliton, they will be useful in observations to be made below.

Also, note that, even if the h_k are not positive, as long as the c_k are positive, the formula (2.31) defines an incomplete gradient Kähler Ricci soliton on the polycylinder defined by the inequalities $c_k + h_k|w^k|^2 > 0$.

***Example 3* (Cao's Soliton).** — One more case of an easily constructed example is the gradient Kähler Ricci soliton metric g on $\mathbb{C}^n$ that is invariant under $\mathrm{U}(n)$, discovered by H.-D. Cao [**2**]. The form of this metric can be derived as follows:

Suppose that such a metric g is given on $\mathbb{C}^n$. (One could do this analysis on any $\mathrm{U}(n)$-invariant domain in $\mathbb{C}^n$, and Cao does this, but I will not pursue this more general case further here.) The group $\mathrm{U}(n)$ must preserve the associated holomorphic volume form Υ up to a constant multiple and this implies that Υ must be a constant multiple of the standard volume form $\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$. Since Υ is only determined up to a constant multiple anyway, there is no loss of generality in assuming that $\Upsilon = \mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n$. Furthermore, the vector field Z must also be invariant under $\mathrm{U}(n)$, which implies that Z must be a multiple of the radial vector field. Since $\mathrm{d}\big(Z \lrcorner \Upsilon\big) = h\,\Upsilon$ where h is real, it follows that

$$Z = h \sum_{k=1}^{n} z^k \frac{\partial}{\partial z^k}\,. \tag{2.34}$$

Now, the condition that g be rotationally invariant with associated Kähler form closed implies that

$$g_{i\bar{j}} = a(r)\delta_{ij} + a'(r)\,\bar{z}^i\,z^j \tag{2.35}$$

for some function a of $r = |z^1|^2+\cdots+|z^n|^2$ that satisfies $ra'(r)+a(r) > 0$ and $a(r) > 0$ (when $n > 1$). Thus $G = \log\big(a(r)^{n-1}(ra'(r)+a(r))\big)$ in this coordinate system. Now, the identity $G = -f$, the equation (2.16), and the above formula for the coefficients of Ω combine to yield

$$\bar{\partial}G = \mathrm{i}\,Z \lrcorner\,\Omega = -\frac{h}{2}\big(ra'(r)+a(r)\big)\bar{\partial}r = -\frac{h}{2}\bar{\partial}\big(ra(r)\big). \tag{2.36}$$

Supposing that $n > 1$ (since the $n = 1$ case has already been treated), it follows that $G + \frac{h}{2}ra(r)$ must be constant, i.e., that

$$a(r)^{n-1}\big(ra(r)\big)'e^{(h/2)ra(r)} = a(0)^n. \tag{2.37}$$

Upon scaling Υ by a constant, it can be assumed that $a(0) = 1$, so assume this from now on. Also, one can assume that h is nonzero since, otherwise, the solution that is smooth at $r = 0$ is simply $a(r) \equiv a(0) = 1$, which gives the flat metric.

The ODE (2.37) for a is singular at $r = 0$, so the existence of a smooth solution near $r = 0$ is not immediately apparent.

Fortunately, (2.37) can be integrated by quadrature: Set $b(r) = (h/2)ra(r)$ and note that (2.37) can be written in terms of b as

$$b(r)^{n-1}e^{b(r)}\,b'(r) = (h/2)^n r^{n-1}. \tag{2.38}$$

Integrating both sides from 0 to $r > 0$ yields an equation of the form

$$(-1)^n(n-1)!\,e^{b(r)}\left(e^{-b(r)} - \sum_{k=0}^{n-1}\frac{\big(-b(r)\big)^k}{k!}\right) = \left(\frac{h}{2}\right)^n\frac{r^n}{n}. \tag{2.39}$$

Set

$$F(b) = (-1)^n(n-1)!\,e^b\left(e^{-b} - \sum_{k=0}^{n-1}\frac{(-b)^k}{k!}\right) \simeq e^b\left(\frac{b^n}{n} - \frac{b^{n+1}}{n(n+1)} + \cdots\right). \tag{2.40}$$

Now, F has a power series of the form $F(b) = \frac{1}{n}\,b^n(1 + \frac{n}{n+1}b + \cdots)$, so F can be written in the form $F(b) = \frac{1}{n}f(b)^n$ for an analytic function of the form $f(b) = b(1 + \frac{1}{n+1}b + \cdots)$. The analytic function f is easily seen to satisfy $f'(b) > 0$ for all b and to satisfy the limits

$$\lim_{b\to+\infty} f(b) = \infty \qquad\text{and}\qquad \lim_{b\to-\infty} f(b) = -\sqrt[n]{n!}\,. \tag{2.41}$$

In particular, f maps $\mathbb{R}$ diffeomorphically onto $\left(-\sqrt[n]{n!},\infty\right)$ and is smoothly invertible. Of course, $f(0) = 0$.

Since (2.39) is equivalent to $f\big(b(r)\big)^n = \left(\frac{h}{2}\,r\right)^n$, when $h > 0$ it can be solved for $r \geq 0$ by setting $b(r) = f^{-1}\left(\frac{h}{2}\,r\right)$, yielding a unique real-analytic solution with a power series of the form

$$b(r) = \frac{h}{2}\,r - \frac{h^2}{4(n+1)}\,r^2 + \cdots\ . \tag{2.42}$$

Consequently, when $h > 0$, the solution b is defined for all $r \geq 0$ and is positive and strictly increasing on the half-line $r \geq 0$. In particular, the function

$$a(r) = \frac{2}{h}\,\frac{b(r)}{r} = 1 - \frac{h}{2(n+1)}\,r + \cdots\ . \tag{2.43}$$

is a positive real-analytic solution of (2.37) that is defined on the range $0 \leq r < \infty$ and satisfies $ra'(r) + a(r) = b'(r) > 0$ on this range, so that the expression (2.35) defines a gradient Kähler Ricci soliton on $\mathbb{C}^n$.

An ODE analysis of this solution (which Cao [**2**] does) shows that when $h > 0$ the resulting metric is complete on $\mathbb{C}^n$ and has positive sectional curvature.

When $h < 0$, the solution $b(r)$ only exists for $r < -\frac{2}{h}\sqrt[n]{n!}$. It is not difficult to see that the corresponding gradient Kähler Ricci soliton on a bounded ball in $\mathbb{C}^n$ is inextendible and incomplete.

Chau and Schnürer [**7**] have shown that Cao's example is stable in a certain sense and hence is 'isolated' in an appropriately defined neighborhood in the space of Kähler-Ricci solitons on $\mathbb{C}^n$.

3. Potentials and local generality

In this section, the question of 'how many' gradient Kähler Ricci soliton metrics could give rise to specified holomorphic data (Υ, Z) on a complex manifold M will be considered. While this question is not easy to answer globally, it is not so difficult to answer locally.

Thus, throughout this section, assume that a complex n-manifold M is specified, together with a nonvanishing holomorphic volume form Υ on M and a holomorphic vector field Z on M such that $\mathrm{d}(Z \lrcorner \Upsilon) = h\,\Upsilon$ for some real constant h.

3.1. Local potentials. — Suppose that $U \subset M$ is an open subset on which there exists a function ϕ such that $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\phi$ is a positive definite $(1,1)$-form whose associated Kähler metric g is a gradient Ricci soliton with associated holomorphic data Υ and Z and Ricci potential f.

By (2.16),

$$
\begin{aligned}
2\bar\partial f &= -2\mathrm{i} Z \lrcorner \Omega = Z \lrcorner (\partial\bar\partial\phi) = -Z \lrcorner (\bar\partial\partial\phi) \\
&= -Z \lrcorner \big(\mathrm{d}(\partial\phi)\big) = -\,\mathsf{L}_Z(\partial\phi) + \mathrm{d}\big(\partial\phi(Z)\big) \\
&= \bar\partial\big(\partial\phi(Z)\big) - \big(\mathsf{L}_Z(\partial\phi) - \partial\big(\mathsf{L}_Z(\phi)\big)\big)
\end{aligned}
\tag{3.1}
$$

By decomposition into type, it follows that

$$\bar\partial\big(2f - \partial\phi(Z)\big) = 0. \tag{3.2}$$

Consequently, $F = 2f - \partial\phi(Z) = 2f - \mathrm{d}\phi(Z)$ is a holomorphic function on U.

3.2. Nonsingular extension problems. — Suppose now that $p \in U$ is not a singular point of Z. Then, by shrinking U if necessary, F can be written in the form $F = dH(Z)$ for some holomorphic function H on the p-neighborhood U. Replacing ϕ by $\phi+H+\bar H$, gives a new potential for Ω that satisfies the stronger condition

$$\partial\phi(Z) = \mathrm{d}\phi(Z) = 2f. \tag{3.3}$$

This function ϕ is unique up to the addition of the real part of a holomorphic function that is constant on the orbits of Z.

Of course, (3.3) implies that $\mathrm{d}\phi(Y) = 0$, i.e., that ϕ is invariant under the flow of Y, the imaginary part of Z.

3.2.1. *Local reduction to equations.* — In local coordinates $z = (z^i)$ for which $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$, one has $f = -G$ so ϕ satisfies the Monge-Ampère equation [3]

$$\det\left(\frac{\partial^2\phi}{\partial z^i \partial \bar z^j}\right) \mathrm{e}^{\frac{1}{2}\mathrm{d}\phi(X)} = 1 \tag{3.4}$$

as well as the equation

$$\mathrm{d}\phi(Y) = 0. \tag{3.5}$$

Conversely, if ϕ is a strictly pseudo-convex function defined on a p-neighborhood U that satisfies both (3.4) and (3.5), then the Kähler metric g whose associated Kähler

[3] It is interesting to note that this equation is not of Euler-Lagrange type, even locally, unless $Z \equiv 0$, i.e., the Ricci-flat case. Of course, in the Ricci-flat case, the variational nature of this equation is well-known.

form is $\Omega = \frac{i}{2}\,\partial\bar\partial\phi$ is a gradient Kähler Ricci soliton on U with associated holomorphic form Υ and holomorphic vector field Z.

***Remark 4* (Real-analyticity of solitons).** — Note that, because (3.4) is a real-analytic elliptic equation for the strictly pseudo-convex function ϕ, it follows by elliptic regularity that ϕ (and hence g) is real-analytic as well.

Now, (3.4) and (3.5) are two PDE for ϕ, the first of second order and the second of first order. While this is an overdetermined system, it is not difficult to show that it is involutive in Cartan's sense.

In fact, an analysis along the lines of exterior differential systems leads to the following result as a proper formulation of a 'Cauchy problem' for gradient Kähler Ricci solitons in the nonsingular case:

***Theorem 2* (Nonsingular extensions).** — *Let M^n be a complex n-manifold endowed with a holomorphic volume form Υ and a nonzero vector field Z satisfying $\mathrm{d}(Z \lrcorner \Upsilon) = h\,\Upsilon$ for some real constant h.*

Let $H^{n-1} \subset M$ be any embedded complex hypersurface that is transverse to Z, let Ω_0 be any real-analytic Kähler form on H, and let f_0 be any real-analytic function on H.

Then there is an open H-neighborhood $U \subset M$ on which there exists a gradient Kähler Ricci soliton g with associated Kähler form Ω, holomorphic volume form Υ, holomorphic vector field Z, and Ricci potential f that satisfy[4]

$$H^*\Omega = \Omega_0, \qquad \text{and} \qquad H^*f = f_0. \tag{3.6}$$

Moreover, g is locally unique in the sense that any other gradient Kähler Ricci soliton $\tilde g$ defined on an open H-neighborhood $\tilde U \subset M$ satisfying these initial conditions agrees with g on some open neighborhood of H in $U \cap \tilde U$.

Proof. — The first step in the proof will be to construct a special set of local 'flow-box' coordinate charts adapted to the hypersurface H, the holomorphic form Υ, and the holomorphic vector field Z.

To begin, note that, since, by hypothesis Z_p does not lie in $T_pH \subset T_pM$ for all $p \in H$, the $(n-1)$-form $Z \lrcorner \Upsilon$ is nonvanishing when pulled back to H.

Let $p \in H$ be fixed. Since $H^*\big(Z \lrcorner \Upsilon\big)$ does not vanish at p, there exist p-centered holomorphic coordinates $w^2, \dots, w^n$ on a p-neighborhood V in H such that $V^*\big(Z \lrcorner \Upsilon\big) = \mathrm{d}w^2 \wedge \cdots \wedge \mathrm{d}w^n$.

Since H is embedded in M, there exists an open neighborhood $U \subset M$ of $V \subset H$ with the property that $U \cap H = V$ and so that each complex integral curve $C \subset M$

[4] Notation: If $P \subset Q$ is a submanifold, and ψ is a differential form on Q, I use $P^*\phi$ to denote the pullback of ψ to P.

of Z that meets U does so in a connected open disk $U \cap C$ that intersects H in a unique point.

Consequently, there exist unique holomorphic functions $z^2, \ldots, z^n$ on U satisfying $\mathrm{d}z^2(Z) = \cdots = \mathrm{d}z^n(Z) = 0$ and $V^*(z^j) = w^j$ for $2 \leq j \leq n$. Moreover, there exists a unique function z^1 on U with the property that z^1 vanishes on $V = U \cap H$ and so that $U^*\Upsilon = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n$. Since the functions $z^1, \ldots, z^n$ have independent differentials on U, it follows that by shrinking V (and hence U) if necessary, it can be assumed that (U, z) is a p-centered holomorphic coordinate chart whose image $z(U) \subset \mathbb{C}^n$ is a polycylinder of the form $|z^i| < \rho^i$ for some $\rho^1, \ldots, \rho^n > 0$. By shrinking ρ^1 if necessary, it can be arranged that $1+h\rho^1 > 0$.

By construction, $Z = F(z)\,\partial/\partial z^1$ for some holomorphic function F defined on $z(U) \subset \mathbb{C}^n$. Thus, $U^*(Z \lrcorner \Upsilon) = F(z)\,\mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n$. Since $V^*(Z \lrcorner \Upsilon) = \mathrm{d}w^2 \wedge \cdots \wedge \mathrm{d}w^n$, it follows that $F(0, w^2, \ldots, w^n) = 1$ for $(0, w^2, \ldots, w^n) \in z(U)$. Moreover, since $\mathrm{d}\big(Z \lrcorner \Upsilon\big) = h\,\Upsilon$, it follows that $\partial F/\partial z^1 = h$. Consequently, in these coordinates $Z = (1+hz^1)\,\partial/\partial z^1$.

Now write $Z = X - \mathrm{i}Y$, where X and Y are commuting real vector fields. The integral curves of Y are transverse to the hypersurface H and there exists a real hypersurface $R \subset U$ that is the union of the integral curves of Y in U that pass through $V = U \cap H$. The vector field X is everywhere transverse to R in U.

Now let ψ_0 be a real-valued function on V such that $V^*(\Omega_0) = \frac{\mathrm{i}}{2}\,\partial\bar{\partial}\psi_0$. Such an Ω_0-potential ψ is unique up the the addition of the real part of a holomorphic function of $w^2, \ldots, w^n$. Extend ψ_0 to a function ψ_1 on R by making it constant along the integral curves of Y. Similarly, extend $V^*(f_0)$ to a function f_1 on R by making it constant along the integral curves of Y.

Finally, consider the initial value problem for a function ϕ on a neighborhood of R in U given by the real-analytic PDE

$$\det\left(\frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) \mathrm{e}^{\frac{1}{2}\mathrm{d}\phi(X)} = 1 \tag{3.7}$$

subject to the real-analytic initial conditions

$$\begin{aligned} \phi(z) &= \psi_1(z) \\ \mathsf{L}_X(\phi)(z) &= 2f_1(z) \end{aligned} \qquad \text{for all } z \in R \subset U. \tag{3.8}$$

It is easy to check that (3.7) and (3.8) constitutes a noncharacteristic Cauchy problem. Hence, by the Cauchy-Kovalewski Theorem, there exists an open neighborhood $W \subset U$ containing R on which there exists a solution ϕ to this problem.

Now, the solution ϕ produced by the Cauchy-Kovalewski Theorem is real-analytic and strictly pseudo-convex. By uniqueness in the Cauchy-Kovalewski Theorem, ϕ is the unique real-analytic solution. Since, as has already been remarked, elliptic regularity implies that any strictly pseudo-convex solution of (3.7) must be real-analytic, it follows that ϕ is the unique solution of (3.7) that satisfies (3.8).

By its very construction, the $(1,1)$-form $\Omega = \frac{i}{2}\partial\bar{\partial}\phi$ is then the Kähler form of a gradient Kähler Ricci soliton metric on $W \subset U$ that satisfies $V^*\Omega = V^*\Omega_0$, that has $W^*\Upsilon$ and W^*Z as the associated holomorphic volume form and vector field, respectively, and has $f = \frac{1}{2}\mathrm{d}\phi(X)$ as Ricci potential, which, of course, satisfies $V^*f = V^*f_0$.

Now, if one replaces ψ by $\psi + H + \bar{H}$ for some holomorphic function $H = H(w^2, \dots, w^n)$ on V, then one finds that the solution ϕ is replaced by by $\phi + H(z^2, \dots, z^n) + \overline{H(z^2, \dots, z^n)}$, so that Ω is unaffected.

The argument thus far has shown that every point $p \in H$ has an open neighborhood $U \subset M$ on which there exists a gradient Kähler-Ricci soliton g_U with the desired extension properties and associated holomorphic data. It has also shown that this extension is locally unique. Now a standard patching argument shows that there exists an open neighborhood $U \subset M$ of the entire complex hypersurface H on which such an extension exists and is unique in the sense described in the statement of the theorem. □

***Remark 5* (Local generality).** — Theorem 2 essentially says that the local gradient Kähler Ricci solitons depend on two real-analytic functions of $2n{-}2$ variables, namely the potential functions ψ_0 (which is assumed to be strictly pseudo-convex but otherwise arbitrary) and f_0 (which is arbitrary). There is, of course, some ambiguity in the choice of the holomorphic coordinates z^i, but this ambiguity turns out to depend on essentially $n{-}2$ holomorphic functions of $n{-}1$ holomorphic variables, which is negligible when compared with two arbitrary (real-analytic) functions of $2n{-}2$ real variables.

3.3. Near singular points of Z.

— The situation near a singular point of Z is considerably more delicate and interesting.

3.3.1. Linear parts and linearizability. — Recall that, at a point $p \in M$ where Z vanishes, there is a well-defined linear map $Z'_p : T_pM \to T_pM$ (often called 'the linear part of Z at p') defined by setting $Z'_p(v) = w$ if $w = [V, Z](p)$ for some (and hence any) holomorphic vector field V defined near p and satisfying $V(p) = v \in T_pM$.

In local coordinates $z = (z^i)$ centered on p, if

$$Z = Z^j(z)\,\frac{\partial}{\partial z^j}, \tag{3.9}$$

where, by assumption $Z^j(0) = 0$ for $1 \le j \le n$, then

$$Z'_p\Big(\frac{\partial}{\partial z^l}(p)\Big) = \frac{\partial Z^j}{\partial z^l}(0)\,\frac{\partial}{\partial z^j}(p). \tag{3.10}$$

The linear map $Z'_p : T_pM \to T_pM$ has a Jordan normal form and this is an important invariant of the singularity. In particular, the set of eigenvalues of Z'_p is well-defined.

Proposition 5. — *Let Z be the holomorphic vector field associated to a gradient Kähler Ricci soliton g on M. At any singular point of Z, the linear part Z'_p is diagonalizable, with all eigenvalues real.*

Proof. — If the data (Υ, Z) is associated to a gradient Kähler Ricci soliton g in a neighborhood of a singular point p of Z, then (2.24) shows that, in special coordinates centered on p, one has

$$g^{i\bar{k}}(0)\, R_{j\bar{k}}(0) = \frac{\partial Z^i}{\partial z^j}(0). \tag{3.11}$$

Because the matrices $\big(g_{i\bar{j}}(0)\big)$ and $\big(R_{i\bar{j}}(0)\big)$ are Hermitian symmetric and $\big(g_{i\bar{j}}(0)\big)$ is positive definite, one can choose the special coordinates so that $\big(g_{i\bar{j}}(0)\big)$ is a multiple of the identity matrix and $\big(R_{i\bar{j}}(0)\big)$ is diagonal. □

Definition 3. — A holomorphic vector field Z on M is said to be *linearizable* near a singular point p if there exist p-centered coordinates $w = (w^i)$ on an open p-neighborhood W and constants a^i_j such that, on W, one has

$$Z = a^i_j w^j \frac{\partial}{\partial w^i}\,. \tag{3.12}$$

The coordinates $w = (w^i)$ are said to be *linearizing* or *Poincaré* coordinates for Z near p.

Not every holomorphic vector field is linearizable near its singular points, even if the linear part at such a point has all of its eigenvalues nonzero and distinct.

***Example 4* (A nonlinearizable singular point).** — The vector field

$$Z = z^1 \frac{\partial}{\partial z^1} + \big(2z^2 + (z^1)^2\big) \frac{\partial}{\partial z^2} \tag{3.13}$$

on $\mathbb{C}^2$ is not linearizable at the origin, even though its linear part there is diagonalizable with eigenvalues 1 and 2.

This nonlinearizability is perhaps most easily seen as follows: The flow $\Phi(t)$ of the vector field Z is

$$\Phi(t)(z^1, z^2) = \big(\mathrm{e}^t\, z^1,\ \mathrm{e}^{2t}(z^2 + (z^1)^2 t)\big). \tag{3.14}$$

In particular $\Phi(t + 2\pi\mathrm{i}) \neq \Phi(t)$, which would be true if Z were holomorphically conjugate to the linear vector field

$$Z'_{(0,0)} = z^1 \frac{\partial}{\partial z^1} + 2z^2 \frac{\partial}{\partial z^2}. \tag{3.15}$$

This phenomenon, however, does not happen for singular points of holomorphic vector fields associated to a gradient Kähler Ricci soliton:

Proposition 6. — *Let Z be a nonzero holomorphic vector field on the complex n-manifold M that is associated to a gradient Kahler Ricci soliton g. Then Z is linearizable at each of its singular points. Moreover, the linear part of Z at a singular point is diagonalizable and has all its eigenvalues real.*

Proof. — Let $p \in M$ be a singular point of Z. The diagonalizability of the linear part of Z at a singular point and the reality of the corresponding eigenvalues has already been demonstrated, so all that remains is to show that Z is linearizable near p.

To do this, write $Z = X - \mathrm{i}Y$ where X and Y are, as usual, real vector fields. As has already been remarked, the vector field Y is an infinitesimal isometry of g. In particular, the flow of Y is complete in the geodesic ball $B_r(p)$ for some $r > 0$ and is a 1-parameter group of isometries of the metric g restricted to $B_r(p)$ that fixes the center p. It follows that there is a compact, connected abelian subgroup $\mathbb{T} \subset \mathrm{U}(T_pM)$ whose Lie algebra is an abelian subalgebra $\mathfrak{t} \subset \mathfrak{u}(T_pM)$ that contains $Y'_p : T_pM \to T_pM$, the linearization of Y at p and is such that the 1-parameter subgroup $\exp(tY'_p)$ is dense in $\mathbb{T}$.

Let $\Phi : \mathbb{T} \to \mathrm{Isom}\big(B_r(p), g\big)$ be the homomorphism induced by the exponential map, i.e., such that

$$\Phi(k)\big(\exp_p(v)\big) = \exp_p\big(k \cdot v\big) \tag{3.16}$$

for all $v \in B_r(0_p) \subset T_pM$. Then $\Phi(k)$ is a holomorphic isometry of g for all $k \in \mathbb{T}$.

Now let $\mathrm{d}\mu$ be Haar measure on $\mathbb{T}$ and choose any holomorphic mapping $\psi : B_r(p) \to T_pM \simeq \mathbb{C}^n$ with the property that $\psi(p) = 0$ and $\psi'(p) : T_pM \to T_{0_p}(T_pM)$ is the inverse of the exponential mapping $\exp'_p : T_{0_p}(T_pM) \to T_pM$. (It may be necessary to shrink r to do this.)

Define a holomorphic mapping $w : B_r(p) \to T_pM$ by the averaging formula

$$w(z) = \int_{\mathbb{T}} k^{-1} \cdot \psi\big(\Phi(k)z\big)\,\mathrm{d}\mu \tag{3.17}$$

for $z \in B_r(p)$. Then $w(p) = 0_p$ and, by construction, $w\big(\Phi(k)z\big) = k \cdot w(z)$ for all $z \in B_r(p)$ and all $k \in \mathbb{T}$. Moreover, also by construction, $w'(p) = \psi'(p)$. In particular, by shrinking r again, if necessary, it can be assumed that w defines a $\mathbb{T}$-equivariant holomorphic embedding of $B_r(p)$ into $T_pM \simeq \mathbb{C}^n$.

In particular, the holomorphic mapping $w : B_r(p) \to T_pM$ satisfies

$$w\big(\exp_{tY}(z)\big) = \exp(tY'_p)\big(w(z)\big), \tag{3.18}$$

for all real t. Since w is holomorphic and Y is the imaginary part of the holomorphic vector field Z, it follows that, for $z \in B_r(p)$ and t complex and of sufficiently small modulus, the identity

$$w\big(\exp_{tZ}(z)\big) = \exp(tZ'_p)\big(w(z)\big) \tag{3.19}$$

holds. In particular, the coordinate system w linearizes Z at p. □

***Remark 6* (The exponential map).** — Of course, the exponential map $\exp_p : T_pM \to M$ of g also intertwines the flow of Y'_p on T_pM with the flow of Y on M, but the exponential map is not generally holomorphic and so cannot be used to linearize Z holomorphically.

***Remark 7* (Complex vs. real flows).** — The reader may want to remember that, for a holomorphic vector field $Z = X - \mathrm{i}Y$, the two real vector fields X and Y have commuting flows and that, moreover, the identity

$$\exp_{(a+\mathrm{i}b)Z} = \exp_{2aX} \circ \exp_{2bY} \tag{3.20}$$

holds. (The factors of 2 are neglected in some references.)

Corollary 1. — *Let g be a gradient Kähler Ricci soliton on M and let Z be its associated holomorphic vector field. Let $p \in M$ be a singular point of Z and let $\lambda \in \mathbb{R}^*$ be a nonzero eigenvalue of Z'_p of multiplicity $k \geq 1$. Then there exists a k-dimensional complex submanifold $N_\lambda \subset M$ that passes through p, to which Z is everywhere tangent, and on which Y is periodic of period $4\pi/|\lambda|$.*

***Remark 8* (Nonuniqueness of the N_λ).** — The reader should be careful not to confuse the submanifolds N_λ with the images under the exponential mapping of the eigenspaces of Z'_p acting on T_pM. Indeed, the N_λ need not be unique. For example, for the linear vector field

$$Z = z^1 \frac{\partial}{\partial z^1} + 2z^2 \frac{\partial}{\partial z^2}. \tag{3.21}$$

on $\mathbb{C}^2$, each of the parabolas $z^2 - c(z^1)^2 = 0$ for $c \in \mathbb{C}$ is tangent to Z and the imaginary part of Z has period 4π on all of $\mathbb{C}^2$, so each could be regarded as N_1.

On the other hand, the line $z^1 = 0$ is the only curve that could be regarded as N_2, since this is the union of the 2π-periodic points of Y.

***Remark 9* (Existence at singular points).** — Example 2 shows that diagonalizability with real eigenvalues is sufficient for a linear vector field to be the linear part of a vector field associated to a (locally defined) gradient Kähler Ricci soliton.

3.3.2. *Prescribed eigenvalues.* — Let $\mathsf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ be a nonzero real vector and define

$$\Lambda_{\mathsf{h}} = \{\mathsf{k} \in \mathbb{Z}^n \mid \mathsf{k} \cdot \mathsf{h} = 0\} = \mathbb{Z}^n \cap \mathsf{h}^\perp \subset \mathbb{R}^n. \tag{3.22}$$

Then Λ_{h} is a free abelian group of rank $n - k$ for some $1 \le k \le n$. The number k is the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-span of the numbers $h_1, \ldots, h_n$ in $\mathbb{R}$. Let $\Lambda_{\mathsf{h}}^+ \subset \Lambda_{\mathsf{h}}$ consist of the $\mathsf{k} \in \Lambda_{\mathsf{h}}$ such that $\mathsf{k} = (k_1, \ldots, k_n)$ with each k_i nonnegative.

Consider the linear holomorphic vector field

$$Z_{\mathsf{h}} = \sum_{j=1}^{n} h_j z^j \frac{\partial}{\partial z^j} \tag{3.23}$$

on $\mathbb{C}^n$. Let $Z_{\mathsf{h}} = X_{\mathsf{h}} - \mathrm{i}Y_{\mathsf{h}}$ be the decomposition into real and imaginary parts.

The closure of the flow of Y_{h} is a connected compact abelian subgroup $\mathbb{T}_{\mathsf{h}} \subset \mathrm{U}(n)$ of dimension k. (In fact, in these coordinates, $\mathbb{T}_{\mathsf{h}}$ lies in the diagonal matrices in $\mathrm{U}(n)$.) Note that Z_{h} and (hence) X_{h} are invariant under the action of $\mathbb{T}_{\mathsf{h}}$.

3.3.3. *Normalizing volume forms.* — In addition to knowing that Z can be linearized near a singular point, it will be useful to know that this can be done in such a way that it simplifies the coordinate expression for Υ as well:

Proposition 7 **(Volume normalization at Z-singular points).** — *Set $h = h_1 + \cdots + h_n$ and let Υ be a nonvanishing holomorphic n-form defined on an open neighborhood U of the origin in $\mathbb{C}^n$ that satisfies $\mathrm{d}(Z_{\mathsf{h}} \lrcorner \Upsilon) = h\Upsilon$.*

Then there exist Z_{h}-linearizing coordinates $w = (w^i)$ near the origin in $\mathbb{C}^n$ such that, on the domain of these coordinates $\Upsilon = \mathrm{d}w^1 \wedge \cdots \wedge \mathrm{d}w^n$.

Proof. — There exists a nonvanishing holomorphic function F on U that satisfies

$$\Upsilon = F(z)\, \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \tag{3.24}$$

and the function F must be invariant under the flow of Z_{h}. In particular, it follows that F has a power series expansion of the form

$$F(z) = c_0 + \sum_{\mathsf{k} \in \Lambda_{\mathsf{h}}^+ \setminus \{0\}} c_{\mathsf{k}}\, z^{\mathsf{k}} \tag{3.25}$$

where z^{k} is the monomial $(z^1)^{k_1} \cdots (z^n)^{k_n}$ when $\mathsf{k} = (k_1, \ldots, k_n)$ and the c_{k} are constants, with $c_0 \neq 0$ (since, by hypothesis $F(0) \neq 0$).

Now, the series

$$G(z) = c_0 + \sum_{\mathsf{k} \in \Lambda_{\mathsf{h}}^+ \setminus \{0\}} \frac{c_{\mathsf{k}}}{(k_1+1)}\, z^{\mathsf{k}} \tag{3.26}$$

converges on the same polycylinder that the series (3.25) does. The resulting holomorphic function G is evidently invariant under the flow of Z_{h} and satisfies

$$G + z^1 \frac{\partial G}{\partial z^1} = F. \tag{3.27}$$

Because G satisfies (3.27), the function $w^1 = z^1 G(z)$ satisfies

$$\mathrm{d}w^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n = F(z)\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n. \tag{3.28}$$

Moreover, since G is Z_{h}-invariant, the function w^1 satisfies $\mathsf{L}_{Z_{\mathsf{h}}}\, w^1 = h_1\, w^1$.

Thus, replacing z^1 by w^1 in the coordinate chart results in a new Z_{h}-linearizing coordinate chart in which $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$. □

***Corollary 2* (Local normal form near singular points).** — *Let Z and Υ be a holomorphic vector field and volume form, respectively on a complex n-manifold M. Let $p \in M$ be a singular point of Z.*

If there exists a gradient Kähler Ricci soliton g with Ricci potential f on a neighborhood of p whose associated holomorphic vector field and volume form are Z and Υ, respectively, then there exists an $\mathsf{h} \in \mathbb{R}^n$ and a p-centered holomorphic chart $z = (z^i) : U \to \mathbb{C}^n$ such that, on U,

$$Z = h_i\, z^i \frac{\partial}{\partial z^i} \quad \text{and} \quad \Upsilon = \mathrm{d}z = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n. \tag{3.29}$$

Proof. — Apply Propositions 6 and 7. □

3.3.4. *Local solitons near a singular point.* — In view of Corollary 2, questions about the local existence and generality of gradient Kähler Ricci solitons with prescribed Z and Υ near a singular point of Z can be reduced by a holomorphic change of variables to the study of solitons on an open neighborhood of $0 \in \mathbb{C}^n$ with $Z = Z_{\mathsf{h}}$ for some $\mathsf{h} \neq 0$ and $\Upsilon = \mathrm{d}z = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$.

***Proposition 8* (Solitons with a prescribed singularity).** — *Let ϕ be a strictly pseudoconvex function defined on a $\mathbb{T}_{\mathsf{h}}$-invariant, contractible neighborhood of $0 \in \mathbb{C}^n$ that satisfies*

$$\det\left(\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right) \mathrm{e}^{\frac{1}{2}\mathrm{d}\phi(X_{\mathsf{h}})} = 1 \tag{3.30}$$

and

$$\mathrm{d}\phi(Y_{\mathsf{h}}) = 0. \tag{3.31}$$

Then $\Omega = \frac{\mathrm{i}}{2}\, \partial\bar{\partial}\phi$ is the associated Kähler form of a gradient Kähler Ricci soliton with Ricci potential $f = \frac{1}{2}\mathrm{d}\phi(X_{\mathsf{h}})$ whose associated holomorphic vector field and volume form are Z_{h} and $\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$, respectively.

Conversely, if g is a gradient Kähler Ricci soliton defined on a $\mathbb{T}_{\mathsf{h}}$-invariant, contractible neighborhood of $0 \in \mathbb{C}^n$ and f is a Ricci potential for g that satisfies $f(0) = 0$ such that the associated holomorphic vector field and volume form are Z_{h} and $\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$, respectively, then g has a Kähler potential ϕ that satisfies (3.30) *and* (3.31).

Proof. — The first part of the proposition follows by computation, so nothing more needs to be said. It remains to establish the converse statement.

Thus, consider a gradient Kähler Ricci soliton g defined on a $\mathbb{T}_{\mathsf{h}}$-invariant, contractible neighborhood U of $0 \in \mathbb{C}^n$ with Ricci potential f satisfying $f(0) = 0$ whose associated holomorphic volume form and vector field are $\Upsilon = \mathrm{d}z$ and Z_{h}, respectively.

The metric g will necessarily be invariant under $\mathbb{T}_{\mathsf{h}}$, as will its associated Kähler form Ω. Since $\Omega^n = n!\,\mathrm{i}^{n^2}\,2^{-n}\,\mathrm{e}^{-f}\,\Upsilon \wedge \overline{\Upsilon}$, it follows that f, too, must be invariant under $\mathbb{T}_{\mathsf{h}}$.

On U, there will exist some Kähler potential ϕ so that $\Omega = \frac{\mathrm{i}}{2}\partial\bar{\partial}\phi$. By averaging ϕ over $\mathbb{T}_{\mathsf{h}}$, it can be assumed that ϕ is $\mathbb{T}_{\mathsf{h}}$-invariant. By subtracting a constant, it can be assumed that $\phi(0) = 0$.

As has been already noted in § 3.1, the difference $F = 2f - \mathrm{d}\phi(Z_{\mathsf{h}})$ is a holomorphic function on U. By construction, F is also necessarily $\mathbb{T}_{\mathsf{h}}$-invariant and vanishes at 0. Since ϕ is $\mathbb{T}_{\mathsf{h}}$-invariant, it follows that $\mathrm{d}\phi(Y_{\mathsf{h}}) = 0$. Thus $F = 2f - \mathrm{d}\phi(Z_{\mathsf{h}}) = 2f - \mathrm{d}\phi(X_{\mathsf{h}})$ is real-valued and holomorphic and therefore constant. Thus, F vanishes identically, i.e., $f = \frac{1}{2}\mathrm{d}\phi(X_{\mathsf{h}})$.

Now, however, by construction, ϕ satisfies (3.30) and, since ϕ is invariant under the flow of Y_{h}, it also satisfies (3.31). □

***Remark 10* (Analyticity in the singular case).** — The equation (3.30) is a $\mathbb{T}_{\mathsf{h}}$-invariant real-analytic Monge-Ampère equation whose linearization at a strictly pseudo-convex solution ϕ is given by

$$\Delta u + 2\,\mathsf{L}_{X_{\mathsf{h}}}\,u = 0 \tag{3.32}$$

where Δ is the Laplacian with respect to the metric g associated to $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar{\partial}\phi$. Of course, this is an elliptic equation.

It follows by elliptic regularity that any gradient Kähler Ricci soliton is real-analytic, even in the neighborhood of singular points of Z.

***Example 5* (Existence with prescribed h).** — By considering Example 2, one sees that, for any h, there is a sufficiently small ball around the origin on which there is at least one strictly pseudo-convex solution ϕ to (3.30).

3.3.5. *A boundary value formulation.* — Suppose now that ϕ is a strictly pseudo-convex solution of (3.30) defined on a $\mathbb{T}_{\mathsf{h}}$-invariant bounded neighborhood $D \subset \mathbb{C}^n$

of $0 \in \mathbb{C}^n$ with smooth boundary ∂D. Let g be the corresponding gradient Kähler Ricci soliton.

Any solution u of (3.32) in D that vanishes on the boundary will also satisfy

$$0 = \int_D |\nabla u|^2 + \tfrac{1}{2} R(g)\, u^2 \, \mathrm{d}vol_g\,, \tag{3.33}$$

as follows by integration by parts using the identities $\rho = \mathsf{L}_{X_h} \Omega$ and $\mathrm{d}vol_g = \frac{1}{n!}\Omega^n$.

In particular, by shrinking D if necessary, it can be assumed that any solution u to (3.32) in D that vanishes on ∂D must vanish on D.

It then follows, by the implicit function theorem, that any $\mathbb{T}_{\mathsf{h}}$-invariant function ψ on ∂D that is sufficiently close (in the appropriate norm) to ϕ on ∂D is the boundary value of a unique pseudo-convex solution $\tilde{\phi}$ of (3.30) that is near ϕ on D. The uniqueness then implies that $\tilde{\phi}$ must also be $\mathbb{T}_{\mathsf{h}}$-invariant and so must, in particular, satisfy (3.31).

Note that the metric g does not always uniquely determine ϕ by the construction given in Proposition 8 since one can add to ϕ the real part of any $\mathbb{T}_{\mathsf{h}}$-invariant holomorphic function that vanishes at $0 \in \mathbb{C}^n$. (Depending on h, there may or may not be any nonconstant $\mathbb{T}_{\mathsf{h}}$-invariant holomorphic functions on a neighborhood of $0 \in \mathbb{C}^n$.) However, this ambiguity is relatively small.

Thus, local gradient Kähler Ricci solitons near $0 \in \mathbb{C}^n$ with prescribed holomorphic data $(Z, \Upsilon) = (Z_{\mathsf{h}}, \mathrm{d}z)$ do exist and have a 'degree of generality' that depends on the number k. The most constraints appear when k reaches its maximum value n and the least when k reaches its minimum value 1.

4. Poincaré coordinates in the positive case

Throughout this section, M will be a noncompact, simply connected complex manifold and g will be a complete gradient Kähler Ricci soliton with postive Ricci curvature. Moreover, it will be assumed that the scalar curvature $R(g)$ has at least one critical point.

4.1. First consequences. — Cao and Hamilton [**6**, Proposition 4.2] prove the following useful result:

Lemma 1. — *The scalar curvature $R(g)$ has only one critical point and it is both a local maximum and the unique critical point of f, which is a strictly convex proper function on M.*

Proof. — Since $R(g)+2|Z|^2 = 2h$ by Proposition 4, the function $R(g) \geq 0$ is bounded by the constant h and any critical point of $R(g)$ is a critical point of $|Z|^2 = \frac{1}{2}|\nabla f|^2$.

On the other hand, since $\nabla^2 f = \mathrm{Ric}(g)$, which is positive definite, the formula

$$\mathrm{d}\left(\tfrac{1}{2}|\nabla f|^2\right)(\nabla f) = \nabla^2 f(\nabla f, \nabla f) = \mathrm{Ric}(g)(\nabla f, \nabla f) \tag{4.1}$$

shows that $\frac{1}{2}|\nabla f|^2$ cannot have any critical point away from where $\nabla f = 0$. Moreover, any point p where ∇f vanishes satisfies $R(g)(p) = 2h$, which is the maximum possible value of $R(g)$.

Since $\nabla^2 f = \mathrm{Ric}(g)$ is positive definite, the function f is locally strictly convex. Since g is complete, f can have at most one critical point, i.e., point where $\nabla f = 0$, and it must be a nondegenerate minimum of f.

By hypothesis, there does exist a (unique) critical point of f; call it p. By adding a constant to f it can be assumed that $f(p) = 0$. It remains to show that f is proper, i.e., that $f^{-1}\big([a,b]\big) \subset M$ is compact for any closed interval $[a,b] \subset \mathbb{R}$.

Since $R(g) + 2|Z|^2 = 2h$ and since $R(g) > 0$, it follows that $|Z| \le \sqrt{h}$, so that Z has bounded length. In particular, writing

$$Z = X - \mathrm{i}Y = \tfrac{1}{2}\big(\nabla f - \mathrm{i}J(\nabla f)\big) \tag{4.2}$$

one has $|X|^2 = |Y|^2 = \frac{1}{2}|Z|^2 < \frac{1}{2}h$, so X and Y have bounded lengths as well. Since g is complete, their flows are globally defined on M.

Let $\gamma : \mathbb{R} \to M$ be any nonconstant integral curve of ∇f, i.e., $\gamma'(t) = \nabla f\big(\gamma(t)\big) \neq 0$ for all $t \in \mathbb{R}$. Consider the function $\phi(t) = f\big(\gamma(t)\big)$. Straightforward computation yields $\phi'(t) = \big|\nabla f\big(\gamma(t)\big)\big|^2 > 0$ and

$$\phi''(t) = 2\,\mathrm{Ric}(g)\big(\nabla f\big(\gamma(t)\big), \nabla f\big(\gamma(t)\big)\big) > 0, \tag{4.3}$$

so $\phi : \mathbb{R} \to \mathbb{R}$ is strictly convex and increasing. It follows that ϕ increases without bound along γ.

Since $\nabla^2 f$ is positive definite, the critical point p is a source singularity of the vector field ∇f. Let $U \subset M$ be the open set that consists of p and all of the points q in M whose α-limit point under ∇f is equal to p. Since f strictly increases without bound on each integral curve of ∇f, it follows that f maps each integral curve of ∇f that lies in U diffeomorphically onto $(0,\infty)$. Moreover, for each $c > 0$, the set $f^{-1}(c) \cap U$ is compact and diffeomorphic to S^{2n-1}. Indeed, $f : U \to [0,\infty)$ is proper.

Now suppose that $U \neq M$. Then, by the connectedness of M, there exists a point $q \in M \setminus U$ that is not in the interior of $M \setminus U$, i.e., a point $q \notin U$ such that there exists a sequence $q_i \in U$ that converges to q. This implies, in particular, that $f(q_i) \ge 0$ converges to $f(q) = c$. Thus, $c \ge 0$ and, for i sufficiently large, q_i must lie in $f^{-1}\big([0,c{+}1]\big) \cap U$, which has been shown to be compact and must therefore contain its limit points. Thus q lies in $f^{-1}\big([0,c{+}1]\big) \cap U$, although, by construction, $q \notin U$. Thus, $U = M$ and f is proper, as claimed. □

***Remark 11* (M is Stein).** — As Cao and Hamilton remark, since $\rho = \mathrm{i}\partial\bar{\partial} f$ is the Ricci form of g, which is positive, the proof shows that f is a strictly plurisubharmonic proper exhaustion function on M. This implies that M is Stein and, as Cao points out in [**4**, Proposition 3.2], that M is diffeomorphic to $\mathbb{R}^{2n}$.

However, as will be seen in Theorem 3, one has the stronger result that M is biholomorphic to $\mathbb{C}^n$.

The following result, also known to Cao and Hamilton, [5] gives constraints on the rate of growth of the Ricci potential.

***Lemma 2* (Growth of f).** — *Let p be the critical point of $R(g)$ and let f be the Ricci potential, normalized so that $f(p) = 0$. There exist positive constants c_1 and c_2 such that, for all $x \in M$,*

$$\sqrt{1 + \big(c_1\, d(x,p)\big)^2} - 1 \le f(x) \le c_2\, d(x,p). \tag{4.4}$$

Proof. — Since g is complete, there exists a geodesic joining p to x whose length is $d(p,x)$. Let $\alpha : \mathbb{R} \to M$ be such a unit speed geodesic with $\alpha(0) = p$ and $\alpha(s) = x$ such that $d(p,x) = s$.

Consider the function $\phi(t) = f\big(\alpha(t)\big)$. By the Chain Rule, and the fact that α has unit speed,

$$\phi'(t) = \nabla f\big(\alpha(t)\big)\cdot\alpha'(t) \le \big|\nabla f\big(\alpha(t)\big)\big| \le \sqrt{2h}. \tag{4.5}$$

Since $\phi(0) = 0$, it follows that $f(x) = f\big(\alpha(s)\big) = \phi(s) \le \sqrt{2h}\, s$. Thus, one can take $c_2 = \sqrt{2h}$.

For the other inequality, note that, again, by the Chain Rule,

$$\phi''(t) = \nabla^2 f\big(\alpha(t)\big)\big(\alpha'(t),\alpha'(t)\big) = \mathrm{Ric}(g)\big(\alpha(t)\big)\big(\alpha'(t),\alpha'(t)\big) \tag{4.6}$$

and the right hand side of this equation is positive since $\mathrm{Ric}(g)$ is positive. Moreover, if $\lambda_{\min}(g) > 0$ denotes the minimum eigenvalue of $\mathrm{Ric}(g)$, which is a positive continuous function on M, it follows that

$$\phi''(t) \ge \lambda_{\min}(g)\big(\alpha(t)\big) > 0. \tag{4.7}$$

In particular, ϕ is a convex function on $\mathbb{R}$.

Let $r_0 > 0$ be sufficiently small that it is below the injectivity radius of g at p and sufficiently small that $\lambda_{\min}(g)(y) \ge \frac{1}{2}\lambda_{\min}(g)(p)$ for all y lying within $B_{r_0}(p)$. Let $a = \frac{1}{2}\lambda_{\min}(g)(p) > 0$.

[5] H.-D. Cao, personal communication, 2 June 2004.

Then $\phi''(t) \geq a$ for $|t| \leq r_0$ while $\phi''(t) > 0$ for $|t| \geq r_0$. Because $\phi(0) = \phi'(0) = 0$, it follows that $\phi(t) \geq A(t)$ for all $t \in \mathbb{R}$ where

$$A(t) = \begin{cases} \frac{1}{2}at^2 & \text{for } |t| \leq r_0; \\ ar_0|t| - \frac{1}{2}a\,r_0{}^2 & \text{for } |t| \geq r_0. \end{cases} \tag{4.8}$$

Since there exists a positive constant c_1 such that $A(t) \geq \sqrt{1+(c_1t)^2} - 1$, the desired lower bound follows. □

***Remark 12* (An alternative growth formulation).** — Another formulation of Lemma 2 is that the function $c : M \setminus \{p\} \to \mathbb{R}$ defined by

$$c(x) = \frac{\sqrt{f(x)\big(f(x)+2\big)}}{d(x,p)} > 0 \tag{4.9}$$

is bounded above and has a positive lower bound.

The bounds of Lemma 2 can be simplified somewhat if one stays sufficiently far from p:

Corollary 3. — *For every $r > 0$, there exist positive constants c_1 and c_2 such that, for all x outside the ball of radius r, one has*

$$c_1\, d(x,p) \leq f(x) \leq c_2\, d(x,p). \tag{4.10}$$

***Remark 13* (The growth rate of f).** — For any vector $v \in TM$, one has

$$\operatorname{Ric}(g)(v,v) \leq \lambda_{\max}(g)\,|v|^2 \tag{4.11}$$

where $\lambda_{\max}(g) : M \to \mathbb{R}$ is the maximum eigenvalue function for $\operatorname{Ric}(g)$. Since g is Kähler, the eigenvalues of $\operatorname{Ric}(g)$ occur in pairs and, since $\operatorname{Ric}(g) > 0$, it follows that $\lambda_{\max}(g) \leq \frac{1}{2}R(g)$. In particular, by Proposition 4, one has the more explicit inequality

$$\operatorname{Ric}(g)(v,v) \leq \tfrac{1}{2}R(g)\,|v|^2 \leq \tfrac{1}{2}\big(2h - |\nabla f|^2\big)\,|v|^2. \tag{4.12}$$

Now let $\gamma : (0,\infty) \to M$ be the arclength parametrization of a nonconstant integral curve of ∇f, such that p is the limit of $\gamma(s)$ as $s \to 0^+$. Thus, $\big|\nabla f\big(\gamma(s)\big)\big|\,\gamma'(s) = \nabla f\big(\gamma(s)\big)$ for all $s > 0$.

Let $\phi(s) = f\big(\gamma(s)\big)$. One then computes via the Chain Rule that

$$\phi'(s) = \big|\nabla f\big(\gamma(s)\big)\big| \leq \sqrt{2h}. \tag{4.13}$$

and hence that

$$\phi''(s) = \operatorname{Ric}(g)\left(\frac{\nabla f\big(\gamma(s)\big)}{\big|\nabla f\big(\gamma(s)\big)\big|}, \frac{\nabla f\big(\gamma(s)\big)}{\big|\nabla f\big(\gamma(s)\big)\big|}\right). \tag{4.14}$$

By the positivity of $\mathrm{Ric}(g)$ and (4.12), this implies

$$0 < \phi''(s) \le \tfrac{1}{2}\left(2h - (\phi'(s))^2\right). \tag{4.15}$$

Moreover, it is clear that, as $s \to 0^+$, the quantity on the right hand side of (4.14) has $\lambda_{\min}(g)(0) > 0$ as a lower bound for its infimum limit. Thus, the infimum limit of $\phi''(s)$ as $s \to 0^+$ is positive.

From these relations, several conclusions can be drawn. The function ϕ is increasing and strictly convex up on $(0, \infty)$. On the other hand, since ϕ' is bounded above, it follows that ϕ grows at most linearly. Moreover, there must be a sequence of distances $s_k \to \infty$ such that $\phi''(s_k) \to 0$. Since, by (4.14)

$$\phi''(s_k) \ge \lambda_{\min}(g)\bigl(\gamma(s_k)\bigr), \tag{4.16}$$

it follows that $\lambda_{\min}(g)\bigl(\gamma(s_k)\bigr) \to 0$ as $k \to \infty$.

4.2. Poincaré coordinates. — Let Υ be the associated holomorphic volume form on M, normalized so that Υ has unit size at p. This determines Υ up to a complex multiple of modulus 1. Let Z be the associated holomorphic vector field.

Since Z vanishes at p, the eigenvalues of Z'_p are the eigenvalues of the Ricci tensor at p, which are real and positive, say $h_1, \ldots, h_n > 0$. Set $h = h_1 + \cdots + h_n > 0$, as usual.

***Theorem 3* (Poincaré coordinates).** — *There exists a global special coordinate system $z : M \to \mathbb{C}^n$ that linearizes Z. In particular, M is biholomorphic to $\mathbb{C}^n$.*

Proof. — By Proposition 6, there exists a small open ball U about p on which there exist p-centered holomorphic coordinates $w = (w^i) : U \to \mathbb{C}^n$ that linearize Z. By shrinking U if necessary, it can be assumed that $U = f^{-1}\bigl([0, \varepsilon)\bigr)$ for some small $\varepsilon > 0$. Note that, since the w^i linearize Z, the identity

$$w^i\bigl(\exp_{tZ}(q)\bigr) = \mathrm{e}^{h_i t}\, w^i(q) \tag{4.17}$$

holds for all $q \in U$ and all $t \in \mathbb{C}$ in the connected domain containing $0 \in \mathbb{C}$ for which $\exp_{tZ}(q)$ lies in U. In particular, this implies that

$$w^i\bigl(\exp_{2tX}(q)\bigr) = \mathrm{e}^{h_i t}\, w^i(q) \tag{4.18}$$

for all $q \in U$ and all $t \in \mathbb{R}$ in the interval containing $0 \in \mathbb{R}$ for which $\exp_{2tX}(q)$ lies in U.

Now, for $q \in M$ distinct from p, write $q = \exp_{2t'X}(q')$ for some $q' \in U$ and $t' \in \mathbb{R}$. Define

$$z^i(q) = \mathrm{e}^{h_i t'}\, w^i(q'). \tag{4.19}$$

If $\exp_{2t'X}(q') = \exp_{2t''X}(q'')$ for some $q'' \in U$ and $t'' \in \mathbb{R}$, then one sees from (4.18) that $\mathrm{e}^{h_i t''}\, w^i(q'') = \mathrm{e}^{h_i t'}\, w^i(q')$, so $z^i(q)$ is well-defined.

Since the flow of X is holomorphic and w^i is holomorphic on U, the function $z^i : M \to \mathbb{C}$ is also holomorphic. Moreover, by construction,

$$z^i\big(\exp_{2tX}(q)\big) = \mathrm{e}^{h_i t}\, z^i(q) \tag{4.20}$$

for all $q \in M$, which implies that

$$z^i\big(\exp_{tZ}(q)\big) = \mathrm{e}^{h_i t}\, z^i(q). \tag{4.21}$$

In particular, the Lie derivative of z^i by Z is $h_i z^i$.

The fact that the mapping $z = (z^i) : M \to \mathbb{C}^n$ is one-to-one and onto now follows immediately since, as was observed in the proof of Lemma 1, the gradient flow lines of $\nabla f = 2X$ all have p as α-limit point and the flow of ∇f exists for all time.

Finally, in these coordinates $\Upsilon = F(z)\,\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$ for some nonvanishing entire holomorphic function F on $\mathbb{C}^n$. However, since $\mathrm{d}(Z \lrcorner \Upsilon) = h\,\Upsilon$, it follows immediately that $\mathrm{d}F(Z) = 0$. Since all of the eigenvalues of Z'_p are positive, this is only possible if F is a constant function. By scaling one of the z^i by a constant, it can be arranged that $F \equiv 1$.

Thus, the resulting global coordinate system (M, z) is special and linearizes Z, as desired. $\square$

***Remark 14* (Previous results).** — Chau and Tam [**8**, Theorem 1.1] proved that M is biholomorphic to $\mathbb{C}^n$ under the additional hypothesis that all the eigenvalues h_i are equal. In a very recent posting to the arXiv [**8**], they prove a result that implies that M is biholomorphic to $\mathbb{C}^n$ under the hypotheses of Theorem 3. However, their result does not provide Z-linearizing coordinates, which is the main purpose of Theorem 3.

4.3. Coordinate ambiguities. — The reader may find it surprising that any local Z-linearizing coordinates z^i defined on a neighborhood of the Z-singular point p extend to global coordinates on $\mathbb{C}^n$ that are special for any gradient Käher-Ricci soliton defined on $\mathbb{C}^n$ with positive Ricci curvature whose associated holomorphic vector field is Z.

This is perhaps made less surprising by the following result:

Proposition 9. — *Let $\mathsf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ be a vector with $h_i > 0$ for $1 \le i \le n$. Consider the vector field*

$$Z_{\mathsf{h}} = h_i\, z^i \,\frac{\partial}{\partial z^i} \tag{4.22}$$

on $\mathbb{C}^n$. Then the set G_{h} of biholomorphisms $\psi : \mathbb{C}^n \to \mathbb{C}^n$ that preserve Z_{h} is a complex Lie group of dimension d_{h} where $d_{\mathsf{h}} \ge n$ is the number of vectors $\mathsf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ that satisfy $k_i \ge 0$ and $\mathsf{k} \cdot \mathsf{h} \in \{h_1, \ldots, h_n\}$.

Moreover, if $U \subset \mathbb{C}^n$ is any connected open neighborhood of $0 \in \mathbb{C}^n$, then any locally defined biholomorphism $\psi : (U, 0) \to (\mathbb{C}^n, 0)$ that preserves Z_{h} is the restriction to U of an element of G_{h}.

Proof. — Let $U \subset \mathbb{C}^n$ be an open neighborhood of 0 and let $\psi = \big(w^i(z)\big) : U \to \mathbb{C}^n$ be a local biholomorphism that preserves Z. Since Z has only one singular point, namely $0 \in \mathbb{C}^n$, it follows that $\psi(0) = 0$. Moreover, by construction, the functions w^i must satisfy $\mathrm{d}w^i(Z) = h_i w^i$. It follows that each w^i has a power series expansion about $0 \in \mathbb{C}^n$ of the form

$$w^i = \sum_{\{\mathsf{k}\geq 0 \mid \mathsf{k}\cdot\mathsf{h} = h_i\}} c^i_{\mathsf{k}}\, z^{\mathsf{k}}. \tag{4.23}$$

Since the right hand side has only a finite number of terms, this power series is a polynomial and hence globally defined on $\mathbb{C}^n$. It remains to see that it is invertible.

Consider the n-form $\mathrm{d}w = \mathrm{d}w^1 \wedge \cdots \wedge \mathrm{d}w^n$. By the above analysis $\mathrm{d}w = F(z)\,\mathrm{d}z$ for some polynomial $F(z)$. By hypothesis, ψ is a local biholomorphism, so $F(0) \neq 0$. Since $\mathsf{L}_Z\,\mathrm{d}w = (h_1 + \cdots + h_n)\,\mathrm{d}w$ by construction, it follows that $\mathrm{d}F(Z) = 0$, i.e., that F is Z-invariant. This implies that F is constant and hence nowhere vanishing.

Now, by hypothesis, ψ is locally invertible, with, say, a local inverse $\psi^{-1} : (V, 0) \to (\mathbb{C}^n, 0)$. However, by construction, ψ^{-1} preserves Z, so, by the argument given above, ψ^{-1} is also a polynomial mapping and hence extends to a global polynomial mapping $\pi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Since $\psi \circ \pi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a polynomial mapping that is the identity on some neighborhood of 0, it must be the identity everywhere on $\mathbb{C}^n$. In particular, π is the global inverse of ψ extended to $\mathbb{C}^n$, which is now revealed to be an element of G_{h}, which is what needed to be shown.

Finally, it is clear that, for any i and any choice of constants $c^i_{\mathsf{k}} \in \mathbb{C}$ for (i, k) such that $\mathsf{k} \in \mathbb{Z}^n$ satisfies $k_j \geq 0$ for $1 \leq j \leq n$ and $\mathsf{k} \cdot \mathsf{h} = h_i$, the formula (4.23) defines a polynomial w^i that satisfy $\mathsf{L}_Z\, w^i = h_i w^i$.

Moreover, for any choice of d_{h} constants $c = (c^i_{\mathsf{k}})$ where (i, k) satisfies $\mathsf{k} \in \mathbb{Z}^n$ with $k_j \geq 0$ for $1 \leq j \leq n$ and $\mathsf{k} \cdot \mathsf{h} = h_i$, the corresponding collection of functions w^i satisfies

$$\mathrm{d}w^1 \wedge \cdots \wedge \mathrm{d}w^n = F(c^i_{\mathsf{k}})\,\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n. \tag{4.24}$$

where F is a polynomial of degree n in the d parameters $c^i_{\mathsf{k}} \in \mathbb{C}$.

As long as $F(c^i_{\mathsf{k}}) \neq 0$, the polynomial mapping $\psi_c = (w^i)$ is a local (and therefore global) biholomorphism of $\mathbb{C}^n$ that preserves Z and hence lies in G_{h}. Thus, the c^i_{k} define global holomorphic coordinates on G_{h} that embed it into $\mathbb{C}^{d_{\mathsf{h}}}$ as an open set. □

***Remark 15* (The structure of G_{h}).** — If $\mu_1, \ldots, \mu_k \geq 1$ are the multiplicities of the eigenvalues $(h_1, \ldots, h_n)$, then G_{h} is the semi-direct product of a reductive subgroup isomorphic to $\mathrm{GL}(\mu_1,\mathbb{C}) \times \cdots \mathrm{GL}(\mu_k,\mathbb{C})$ with a nilpotent subgroup biholomorphic to $\mathbb{C}^\mu$ where $\mu = d_{\mathsf{h}} - {\mu_1}^2 - \cdots - {\mu_k}^2$.

When $n = 1$, one has $G_{\mathsf{h}} \simeq \mathbb{C}^* = \mathrm{GL}(1,\mathbb{C})$. When $n = 2$, one has either

1. $d_{\mathsf{h}} = 2$ if $\mathsf{h} = (h_1, h_2)$ with neither h_1/h_2 nor h_2/h_1 an integer (in which case $G_{\mathsf{h}} = \mathbb{C}^* \times \mathbb{C}^*$);
2. $d_{\mathsf{h}} = 3$ if $\mathsf{h} = (h_1, h_2)$ with either h_1/h_2 or h_2/h_1 an integer greater than 1; or
3. $d_{\mathsf{h}} = 4$ if $\mathsf{h} = (h, h)$ (in which case $G_{\mathsf{h}} = \mathrm{GL}(2,\mathbb{C})$).

When $n > 2$, there is no upper bound for d_{h} that depends only on n. For example, when $n = 3$, one has $d_{(1,1,k)} = k + 6$ for any integer $k > 1$.

4.4. Global consequences. — Throughout this section, g will be a complete gradient Kähler Ricci soliton on $\mathbb{C}^n$ with positive Ricci curvature whose associated vector field Z is given by (4.22) where $\mathsf{h} = (h_1, \ldots, h_n)$ and

$$0 < h_1 \leq h_2 \leq \cdots \leq h_n\,. \tag{4.25}$$

The compact abelian group $\mathbb{T}_{\mathsf{h}} \subset \mathrm{U}(n)$ will denote the closure of the orbit of Y, the imaginary part of Z.

The existence of global linearizing coordinates for a gradient Kähler Ricci soliton gives elementary proofs and/or improvements of several known results.

4.4.1. *Periodic orbits.* — The first result sharpens Theorem 1.1 of the article [**6**] of Cao and Hamilton.

Proposition 10 **(Periodic orbits of $J(\nabla f)$).** — *For all $c > 0$, the flow of $J(\nabla f)$ preserves the (smooth) level set $f^{-1}(c) \subset M$ and has at least n periodic orbits on $f^{-1}(c)$.*

Proof. — Since $Z = \frac{1}{2}\big(\nabla f - \mathrm{i}\, J(\nabla f)\big)$, and since $h_i > 0$ for $1 \leq i \leq n$, it follows that $J(\nabla f)$ is periodic of period $2\pi/h_i$ on the z^i-axis. Moreover, since f increases without bound as $|z^i| \to \infty$, this axis meets each level set $f^{-1}(c)$ for $c > 0$ in a circle. Thus, there are at least n distinct periodic orbits of $J(\nabla f)$ within each such level set. $\square$

4.4.2. *An invariant potential.* — As has already been seen, the metric g is invariant under $\mathbb{T}_{\mathsf{h}}$. It turns out that one can canonically choose a Kähler potential for g:

Proposition 11 **(Canonical potentials).** — *There is a unique $\mathbb{T}_{\mathsf{h}}$-invariant Kähler potential $\phi : \mathbb{C}^n \to \mathbb{R}$ satisfying $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\phi$ and $\phi(0) = 0$.*

Proof. — Since $M = \mathbb{C}^n$, there exists at least one Kähler potential ϕ for g, i.e., such that $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\phi$. Since $\mathbb{T}_{\mathsf{h}}$ is compact, by averaging ϕ over $\mathbb{T}_{\mathsf{h}}$, one can assume that ϕ is $\mathbb{T}_{\mathsf{h}}$ invariant and by adding a constant, one can assume that $\phi(0) = 0$.

If $\tilde\phi$ were also $\mathbb{T}_{\mathsf{h}}$-invariant and satisfied $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\tilde\phi$, then the difference $\tilde\phi - \phi$ would be the real part of a $\mathbb{T}_{\mathsf{h}}$-invariant holomorphic function H. In particular H would be invariant under the flow of Y and hence of Z. However, as has already been seen, the only holomorphic functions on $\mathbb{C}^n$ that are invariant under the flow of Z are the constants. Thus $\tilde{\phi} - \phi$ is constant. The normalization $\phi(0) = 0$ then guarantees the uniqueness of ϕ. □

4.4.3. *Normalized linearizing coordinates.* — The ambiguity in the linearizing coordinates for the vector field Z represented by the group G_{h} can be used to simplify the potential for g.

***Theorem 4* (Normalized coordinates).** — *Let ϕ be the unique $\mathbb{T}_{\mathsf{h}}$-invariant Kähler potential for g, normalized so that $\phi(0) = 0$. Then there exists an element $\Psi \in G_{\mathsf{h}}$, unique up to composition with an element of the compact group* $\mathrm{U}(n) \cap G_{\mathsf{h}}$*, such that*

$$\Psi^*(\phi) = |z^1|^2 + \cdots + |z^n|^2 + E_{\bar{\imath}\bar{\jmath}kl}(z)\bar{z}^i\bar{z}^j z^k z^l \tag{4.26}$$

for some real-analytic functions $E_{\bar{\imath}\bar{\jmath}kl} = E_{\bar{\jmath}\bar{\imath}kl} = E_{\bar{\imath}\bar{\jmath}lk} = \overline{E_{\bar{k}\bar{l}ij}}$ defined near $0 \in \mathbb{C}^n$.

Proof. — Let f be the Ricci potential for g, normalized so that $f(0) = 0$. Since f is $\mathbb{T}_{\mathsf{h}}$-invariant and since, by (3.2), the difference $2f - \mathrm{d}\phi(Z)$ is holomorphic and $\mathbb{T}_{\mathsf{h}}$-invariant, it follows by the same argument as above that $2f - \mathrm{d}\phi(Z)$ is constant and hence vanishes identically. Thus

$$\mathrm{d}\phi(Z) = \mathrm{d}\phi(X) = 2f. \tag{4.27}$$

Because ϕ and f are real-analytic they have convergent power series expansions near $0 \in \mathbb{C}^n$. Since $f(0) = 0$ and f has a critical point at 0, it has an expansion of the form

$$f = \tfrac{1}{2} f_{ij}\, z^i z^j + f_{i\bar{\jmath}}\, z^i \bar{z}^j + \tfrac{1}{2}\overline{f_{ij}}\, \bar{z}^i \bar{z}^j + O(|z|^3). \tag{4.28}$$

where $f_{ij} = f_{ji}$ and $f_{i\bar{\jmath}} = \overline{f_{j\bar{\imath}}}$. Because of the positivity of the h_i and the invariance of f under the flow of Y, it follows that $f_{ij} = 0$ and $(h_i - h_j) f_{i\bar{\jmath}} = 0$ for all i and j. Moreover, since f is strictly convex up at the origin, the Hermitian form $f_{i\bar{\jmath}}\, z^i \bar{z}^j$ is positive definite.

Thus, by making a linear change of variables that preserves Z (i.e., by applying a transformation in $\mathrm{GL}(n, \mathbb{C}) \cap G_{\mathsf{h}}$), it can be arranged that

$$f = \tfrac{1}{2} h_1\, |z^1|^2 + \cdots + \tfrac{1}{2} h_n\, |z^n|^2 + O(|z|^3). \tag{4.29}$$

Next, consider the part of f that is pure in z or $\bar{z}$, i.e., consider the expansion

$$f = \tfrac{1}{2} h_1\, |z^1|^2 + \cdots + \tfrac{1}{2} h_n\, |z^n|^2 + \sum_{\mathsf{k} \geq 0,\, |\mathsf{k}| \geq 3} \left(f_{\mathsf{k}}\, z^{\mathsf{k}} + \overline{f_{\mathsf{k}}}\, \bar{z}^{\mathsf{k}}\right) + f_{i\bar{\jmath}}(z) z^i \bar{z}^j. \tag{4.30}$$

where $f_{\mathsf{k}} \in \mathbb{C}$ and $f_{i\bar{\jmath}} = \overline{f_{j\bar{\imath}}}$ vanishes at $z = 0$. The invariance of f under the flow of Y implies that $f_{\mathsf{k}} = 0$ for all k, so these 'pure' terms do not appear after all.

Finally, consider the part of the remainder that is linear in the variables $\bar{z}^i$ or z^i and vanishes at $z = 0$ to order at least 3, i.e., write

$$f = \tfrac{1}{2}h_k\,|z^k|^2 + Q^i(z)\,\bar{z}^i + \overline{Q^i(z)}\,z^i + f_{\bar{\imath}\bar{\jmath}kl}(z)\bar{z}^i\bar{z}^j z^k z^l, \tag{4.31}$$

where $Q^i(z)$ is a holomorphic function of z that vanishes to order at least 2 at $z = 0$ and $f_{\bar{\imath}\bar{\jmath}kl} = f_{\bar{\jmath}\bar{\imath}kl} = f_{\bar{\imath}\bar{\jmath}lk} = \overline{f_{\bar{k}\bar{l}ij}}$.

Again, the fact that f is invariant under the flow of Y implies that Q^i must satisfy $\mathsf{L}_Z\,Q^i = h_i Q^i$, i.e., that Q^i has an expansion of the form

$$Q^i(z) = \sum_{\{\mathsf{k}\geq 0\ \mid\ \mathsf{k}\cdot\mathsf{h}=h_i\}} c^i_{\mathsf{k}} z^{\mathsf{k}} \tag{4.32}$$

with $c^i_{\mathsf{k}} = 0$ unless $|\mathsf{k}| = k_1 + \cdots + k_n > 1$. In particular, this implies that Q^i is a polynomial in z since the right hand side of (4.32) can contain only finitely many terms. Now consider the change of variables defined by

$$w^i = z^i + \frac{2}{h_i}\,Q^i(z) \tag{4.33}$$

This transformation belongs to G_{h} by definition and satisfies

$$f = \tfrac{1}{2}h_k\,|w^k|^2 + f^*_{\bar{\imath}\bar{\jmath}kl}(w)\,\bar{w}^i\bar{w}^j w^k w^l, \tag{4.34}$$

for some functions $f^*_{\bar{\imath}\bar{\jmath}kl}$ with the same symmetry and reality properties as the corresponding $f_{\bar{\imath}\bar{\jmath}kl}$.

Since $\mathsf{L}_X\,\phi = 2f$ and $\phi(0) = 0$, it follows that ϕ has a power series expansion

$$\phi = |w^k|^2 + E_{\bar{\imath}\bar{\jmath}kl}(w)\,\bar{w}^i\bar{w}^j w^k w^l, \tag{4.35}$$

as desired. The uniqueness of the transformation $\Psi = (w^i)$ up to composition with an element of $\mathrm{U}(n) \cap G_{\mathsf{h}}$ is now evident. □

4.4.4. *Totally geodesic submanifolds.* — Since the fixed locus of an isometry of g must be totally geodesic, one has the following result:

***Proposition 12* (Geodesic subspaces).** — *If h_i has multiplicity $\mu_i > 0$ and has the property that, for all k, $h_k \neq m h_i$ for any integer $m > 1$, then the μ_i-plane in $\mathbb{C}^n$ defined by $z^j = 0$ when $h_j \neq h_i$ is totally geodesic.*

More generally, if Y has a periodic point q with period $T > 0$, then the union of the T-periodic points is a nontrivial totally geodesic linear subspace of $\mathbb{C}^n$ generated by the z^i-axis lines for which h_i is an integer multiple of $4\pi/T$. □

***Remark 16* (Geodesic axes).** — The reader might wonder whether or not the hypothesis of h_i having no 'supermultiples' is necessary in order for the h_i-eigenspace of Z_{h} in $\mathbb{C}^n$ to be totally geodesic.

The answer is clearly 'yes' in general Z-linearizing coordinates: For example, if $n = 2$ and $\mathsf{h} = (1, k)$ for some integer k, then, any of the curves $z^2 = \lambda(z^1)^k$ could be taken to be the z^1-axis in Z_{h}-linearizing coordinates. They all have the same tangent space at the origin, so at most one of them could be geodesic for a given gradient Kähler Ricci soliton g defined near $0 \in \mathbb{C}^2$ with associated holomorphic vector field Z_{h}.

However, if one uses g-normalized coordinates as provided by Theorem 4, there is a canonical $\mathbb{C}^{\mu_i} \subset \mathbb{C}^n$ associated to the eigenvalue h_i of multiplicity μ_i by the equations $z^j = 0$ when $h_j \neq h_i$. It is still not clear to me whether this canonical subspace is totally geodesic unless h_i satisfies the 'no supermultiples' condition.

4.4.5. *Growth of f in linearizing coordinates.* — Now that global linearizing coordinates are available, it makes sense to ask about the growth of the metric g and its related quantities in those coordinates.

One particularly useful quantity to estimate will be the size of $|\nabla f|^2(z)$ as $|z| \to \infty$. Note that, because of (4.3), the function $|\nabla f|^2$ is strictly increasing on the nonconstant flow lines of ∇f. On the other hand, $|\nabla f|^2 = 2h - R(g)$ is bounded by $2h$. Define

$$\lambda_- = \liminf_{|z|\to\infty} |\nabla f|^2(z) > 0 \qquad \text{and} \qquad \lambda_+ = \sup_z |\nabla f|^2(z) \leq 2h. \tag{4.36}$$

Proposition 13. — *For any $r > 0$, there exist constants $a_1 > 0$, $a_2 > 0$, b_1, and b_2 such that, for all $z \in \mathbb{C}^n$ with $|z| \geq r$,*

$$a_1 \log|z| + b_1 \leq f(z) \leq a_2 \log|z| + b_2\,. \tag{4.37}$$

Explicitly, one can take

$$a_1 = \frac{1}{h_n} \inf_{|z|=r} |\nabla f(z)|^2(z) > 0 \qquad \textit{and} \qquad a_2 = \frac{\lambda_+}{h_1} \leq \frac{2h}{h_1}. \tag{4.38}$$

Proof. — Fix $r > 0$ and note that there exist constants $m_r > 0$ and $M_r > 0$ such that

$$m_r \leq f(z) \leq M_r \qquad \text{when } |z| = r. \tag{4.39}$$

Moreover, taking a_1 and a_2 as defined in (4.38) and using the fact that $|\nabla f|^2(z)$ and $|z|$ both increase along the flow lines of ∇f, one sees that

$$h_n\, a_1 \leq |\nabla f(z)|^2 \leq h_1\, a_2 \qquad \text{when } |z| \geq r. \tag{4.40}$$

Now, the flow of $\nabla f = 2\mathrm{Re}(Z)$ in Z-linearizing coordinates is

$$\exp_{t\nabla f}(z^1, \ldots, z^n) = (\mathrm{e}^{h_1 t} z^1, \ldots, \mathrm{e}^{h_n t} z^n), \tag{4.41}$$

so, since $0 < h_1 \leq \cdots \leq h_n$, it follows that

$$\mathrm{e}^{h_1 t}|z| \leq \left|\exp_{t\nabla f}(z^1, \ldots, z^n)\right| \leq \mathrm{e}^{h_n t}|z|. \tag{4.42}$$

In particular, it follows that, for $t \geq 0$.

$$t \leq \frac{1}{h_1}\left(\log\left(\left|\exp_{t\nabla f}(z^1, \ldots, z^n)\right|\right) - \log|z|\right). \tag{4.43}$$

and

$$\frac{1}{h_n}\left(\log\left(\left|\exp_{t\nabla f}(z^1,\dots,z^n)\right|\right)-\log|z|\right)\leq t. \tag{4.44}$$

On the other hand, since $\mathsf{L}_{\nabla f}\, f = |\nabla f|^2$, it follows that

$$f(z)+h_n\,a_1\,t\leq f\big(\exp_{t\nabla f}(z^1,\dots,z^n)\big)\leq f(z)+h_1\,a_2\,t \tag{4.45}$$

for all $t\geq 0$ and z satisfying $|z|=r$. Combining this with the above inequality gives, for all $t\geq 0$ and z satisfying $|z|=r$,

$$f\big(\exp_{t\nabla f}(z^1,\dots,z^n)\big)-a_2\log\left|\exp_{t\nabla f}(z^1,\dots,z^n)\right|\leq f(z)-a_2\,\log|z|. \tag{4.46}$$

Since every $w\in\mathbb{C}^n$ with $|w|\geq r$ is of the form $w=\exp_{t\nabla f}(z)$ for some $t\geq 0$ and z with $|z|=r$, it follows that

$$f(w)\leq a_2\,\log|w|+(M_r-a_2\,\log r) \tag{4.47}$$

for all $w\in\mathbb{C}^n$ with $|w|\geq r$. Thus, taking $b_2=M_r-a_2\,\log r$ verifies the claimed upper bound on f.

The lower bound follows by combining the lower bound on t with the lower bound on f:

$$m_r+a_1\left(\log\left(\left|\exp_{t\nabla f}(z^1,\dots,z^n)\right|\right)-\log|z|\right)\leq f\big(\exp_{t\nabla f}(z^1,\dots,z^n)\big), \tag{4.48}$$

which gives

$$(m_r-a_1\log r)+a_1\log|w|\leq f(w), \tag{4.49}$$

for all $w\in\mathbb{C}^n$ with $|w|\geq r$. □

Note that, as a function of r, the expression a_1 defined in (4.38) is increasing and its limit as $r\to\infty$ is λ_-/h_n.

Corollary 4. — *For any $\varepsilon>0$, there exists $r>0$ such that, for $z\in\mathbb{C}^n$ with $|z|\geq r$,*

$$\left(\frac{\lambda_-}{h_n}-\varepsilon\right)\log|z|\leq f(z)\leq\left(\frac{\lambda_+}{h_1}+\varepsilon\right)\log|z|. \tag{4.50}$$

In particular, there exist constants $b_1>0$ and $b_2>0$ such that, for all $z\in\mathbb{C}^n$ with $|z|\geq r$,

$$b_1\log|z|\leq d(z,p)\leq b_2\log|z|. \tag{4.51}$$

Proof. — The first statement follows by elementary reasoning from Proposition 13 while the second follows by combining the first with Corollary 3. □

Note that Corollary 4 implies that the ratio $f(z)/\log|z|$ is bounded above and has a positive lower bound as $|z|\to\infty$. Set

$$\mu_-=\liminf_{|z|\to\infty}\frac{f(z)}{\log|z|}\qquad\text{and}\qquad\mu_+=\limsup_{|z|\to\infty}\frac{f(z)}{\log|z|}. \tag{4.52}$$

Then Corollary 4 implies

$$\frac{\lambda_-}{h_n} \le \mu_- \le \mu_+ \le \frac{\lambda_+}{h_1}. \tag{4.53}$$

Proposition 14. — *One has the bounds $\mu_- \le 2n \le \mu_+$, in other words*

$$\liminf_{|z|\to\infty} \frac{f(z)}{\log|z|} \le 2n \le \limsup_{|z|\to\infty} \frac{f(z)}{\log|z|}. \tag{4.54}$$

Proof. — Suppose these bounds do not hold and let $R > 0$ be fixed large enough so that there exist positive constants a_1 and a_2 where either $a_2 < 2n$ or else $a_1 > 2n$ and positive constants b_1 and b_2 so that

$$a_1 \log|z| \le f(z) \le a_2 \log|z| \tag{4.55}$$

and

$$b_1 \log|z| \le d(z,0) \le b_2 \log|z| \tag{4.56}$$

hold whenever $|z| \ge R$. (Remember that, in these linearizing coordinates $p = 0$.)

Let $M > 0$ be sufficiently large that $d(z,0) \le M$ when $|z| \le R$, and consider any real number ρ that is larger than both $\log R$ and M/b_2.

Consider the g-metric ball $B_{b_1\rho}(0)$. Since $d(z,0) \le b_1\rho$ for $z \in B_{b_1\rho}(0)$, it follows that either $|z| \le R$ or $b_1 \log|z| \le b_1\rho$, i.e., $|z| \le \mathrm{e}^\rho$. Since $\mathrm{e}^\rho > R$, in either case it follows that $|z| \le \mathrm{e}^\rho$. Thus $B_{b_1\rho}(0)$ is contained in the flat metric ball $B^0_{\mathrm{e}^\rho}(0)$.

On the other hand, if $|z| \le \mathrm{e}^\rho$, then either $|z| \le R$ or else $d(z,0) \le b_2\rho$. In the former case, $d(z,0) \le M \le b_2\rho$, by construction. In either case, z lies in the g-metric ball $B_{b_2\rho}(0)$.

Thus, one has inclusions

$$B_{b_1\rho}(0) \subseteq B^0_{\mathrm{e}^\rho}(0) \subseteq B_{b_2\rho}(0). \tag{4.57}$$

Now, the volume form for g on $\mathbb{C}^n$ is

$$\mathrm{vol}_g = \mathrm{e}^{-f}\,\mathrm{vol}_0 \tag{4.58}$$

where $\mathrm{vol}_0 = \mathrm{i}^{n^2} 2^{-n} \mathrm{d}z \wedge \mathrm{d}\bar z$ is the volume form of the flat metric on $\mathbb{C}^n$.

Consequently, the g-volume of the g-metric ball $B_{b_2\rho}(0)$ is at least as large as the g-volume of the flat metric ball $B^0_{\mathrm{e}^\rho}(0)$ which is given by the integral

$$\begin{aligned}
\int_{|z|\le \mathrm{e}^\rho} \mathrm{e}^{-f}\,\mathrm{vol}_0 &= \int_{|z|\le R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \int_{|z|=R}^{|z|=\mathrm{e}^\rho} \mathrm{e}^{-f}\,\mathrm{vol}_0 \\
&\ge \int_{|z|\le R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \int_{|z|=R}^{|z|=\mathrm{e}^\rho} |z|^{-a_2}\,\mathrm{vol}_0 \\
&= \int_{|z|\le R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \mathrm{vol}(S^{2n-1}) \int_{s=R}^{s=\mathrm{e}^\rho} s^{2n-1-a_2}\,\mathrm{d}s
\end{aligned} \tag{4.59}$$

Now, if $a_2 < 2n$, then the above would imply

$$\mathrm{vol}\big(B_{b_2\rho}(0), g\big) \geq \int_{|z|\leq R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \frac{\mathrm{vol}(S^{2n-1})}{2n-a_2}\big(\mathrm{e}^{(2n-a_2)\rho} - R^{2n-a_2}\big). \tag{4.60}$$

However, because g has positive Ricci curvature, by the Bishop Comparison Theorem [**13**, Theorem 1.3] the g-volume of $B_{b_2\rho}(0)$ is bounded by a constant times ρ^{2n}. Obviously, such a bound is not compatible with (4.60) for all ρ sufficiently large. Thus, $a_2 \geq 2n$.

In the other direction, the g-volume of the g-metric ball $B_{b_1\rho}(0)$ is at most as large as the g-volume of the flat metric ball $B^0_{\mathrm{e}^\rho}(0)$, which obeys the upper bound

$$\begin{aligned}
\int_{|z|\leq \mathrm{e}^\rho} \mathrm{e}^{-f}\,\mathrm{vol}_0 &= \int_{|z|\leq R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \int_{|z|=R}^{|z|=\mathrm{e}^\rho} \mathrm{e}^{-f}\,\mathrm{vol}_0 \\
&\leq \int_{|z|\leq R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \int_{|z|=R}^{|z|=\mathrm{e}^\rho} |z|^{-a_1}\,\mathrm{vol}_0 \\
&= \int_{|z|\leq R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \mathrm{vol}(S^{2n-1}) \int_{s=R}^{s=\mathrm{e}^\rho} s^{2n-1-a_1}\,\mathrm{d}s
\end{aligned} \tag{4.61}$$

If $a_1 > 2n$, then this would imply

$$\mathrm{vol}\big(B_{b_1\rho}(0), g\big) \leq \int_{|z|\leq R} \mathrm{e}^{-f}\,\mathrm{vol}_0 + \frac{\mathrm{vol}(S^{2n-1})}{a_1-2n}\big(R^{2n-a_1} - \mathrm{e}^{(2n-a_1)\rho}\big). \tag{4.62}$$

and the right hand side is bounded as a function of ρ. Thus, $\mathrm{vol}\big(B_{b_1\rho}(0), g\big)$ would be bounded, independent of ρ, which, because g is complete and of positive Ricci curvature on the noncompact manifold $\mathbb{C}^n$, violates Theorem 4.1 of [**13**], which asserts that g must have at least linear volume growth. Thus $a_1 \leq 2n$. □

Remark 17 **(Growth of f in examples).** — In the case of Hamilton's soliton (Example 1) and, more generally Cao's soliton (Example 3), one has $h_1 = h_n$ and $\lambda_- = \lambda_+ = 2nh_1$, so equality holds in the bounds of Proposition 14.

On the other hand for the product examples (Example 2),

$$f(z) = \sum_{k=1}^{n} \log\big(1 + (h_k/c_k)|z^k|^2\big) \tag{4.63}$$

which satisfies

$$\liminf_{|z|\to\infty} \frac{f(z)}{\log|z|} = 2 \qquad \text{while} \qquad \limsup_{|z|\to\infty} \frac{f(z)}{\log|z|} = 2n. \tag{4.64}$$

In particular, note that this implies $\lambda_- \leq 2h_n < 2h$.

Remark 18 **(Growth of the potential ϕ).** — Let ϕ be the $\mathbb{T}_\mathfrak{h}$-invariant potential for g, i.e., $\Omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\phi$, and assume that ϕ is normalized so that $\phi(0) = 0$.

Since $\mathsf{L}_{\nabla f}\,\phi = f$, it follows that ϕ is determined in terms of f and that Corollary 4 implies growth bounds for ϕ as well. For example, one sees that there exist positive constants r, c_1, and c_2 so that, whenever $|z| \geq r$, one has

$$c_1 \left(\log|z|\right)^2 \leq \phi(z) \leq c_2 \left(\log|z|\right)^2. \tag{4.65}$$

It should be possible to derive C^2-bounds on ϕ (and hence on g) using the fact that ϕ satisfies an elliptic Monge-Ampére equation, but I do not see, at present, a good way to do this so as to get any useful information.

5. The toric case

In this last section, some remarks will be made about the reduction of the gradient Kähler Ricci soliton equation in the 'toric' case, which will now be defined.

Throughout this section, $\mathbb{T}^n$ will denote the maximal abelian subgroup of $\mathrm{U}(n)$ that consists of diagonal matrices. Although there is no symplectic form specified on $\mathbb{C}^n$, the mapping $\mu_n : \mathbb{C}^n \to \mathbb{R}^n$ defined by

$$\mu_n(z^1, \dots, z^n) = \left(|z^1|^2, \dots, |z^n|^2\right) \tag{5.1}$$

will sometimes be referred to as the 'momentum mapping' of $\mathbb{T}^n$.

***Definition 4* (Toric metrics).** — A $\mathbb{T}^n$-invariant Kähler metric g that is defined on a connected $\mathbb{T}^n$-invariant open neigborhood of $0 \in \mathbb{C}^n$ will be said to be *toric*.

***Remark 19* (Toric ubiquity).** — While, at first glance, the toric condition seems to be rather special, note that any gradient Kähler Ricci soliton g on a neighborhood of $0 \in \mathbb{C}^n$ that has $(Z, \Upsilon) = (Z_{\mathsf{h}}, \mathrm{d}z)$ as its associated holomorphic data is invariant under the torus $\mathbb{T}_{\mathsf{h}}$. If h is 'generic' in the sense that the real numbers $h_1, \dots, h_n$ are linearly independent over $\mathbb{Q}$, then $\mathbb{T}_{\mathsf{h}} = \mathbb{T}^n$ and hence g is toric.

Thus, in some sense, the toric case is 'generic' among complete gradient Kähler Ricci solitons with positive Ricci curvature.

5.1. Symmetry reduction in the toric case. — Assuming an n-torus symmetry allows one to reduce the number of independent variables in the gradient Kähler Ricci soliton equation (3.4).

Proposition 15. — *Let g be a toric gradient Kähler Ricci solition defined on a connected open neighborhood of $0 \in \mathbb{C}^n$ with a nonzero associated holomorphic vector field Z and holomorphic volume form Υ (defined with respect to a Ricci potential f satisfying $f(0) = 0$). Then*

1. *The vector field Z is linearized in the coordinates $z = (z^i)$, so that $Z = Z_{\mathsf{h}}$ for some nonzero $\mathsf{h} = (h_1, \dots, h_n) \in \mathbb{R}^n$;*

2. *The n-form* Υ *is* $c\,\mathrm{d}z^1 \wedge \cdots \mathrm{d}z^n$ *for some nonzero constant* c*; and*
3. g *has a unique Kähler potential satisfying* $\phi(0) = 0$ *of the form*

$$\phi(z) = u\big(|z^1|^2, |z^2|^2, \dots, |z^n|^2\big) \tag{5.2}$$

for some real-analytic function u *defined on an open neighborhood of* $0 \in \mathbb{R}^n$*. Moreover,* u *satisfies the singular real Monge-Ampère equation*

$$\det_{1\le i,j\le n}\left(r^i \frac{\partial}{\partial r^i}\left(r^j \frac{\partial u}{\partial r^j}\right)\right) \exp\left(\frac{1}{2}\sum_{j=1}^n h_j\, r^j \frac{\partial u}{\partial r^j}\right) = |c|^2\, r^1 r^2 \cdots r^n\,. \tag{5.3}$$

where

$$\prod_{j=1}^n \frac{\partial u}{\partial r^j}(0) = |c|^2 \qquad \text{and} \qquad \frac{\partial u}{\partial r^j}(0) > 0, \qquad 1 \le j \le n. \tag{5.4}$$

Conversely, for any nonzero $\mathsf{h} \in \mathbb{R}^n$ *and any nonzero complex constant* c*, if* u *is a real-analytic function defined on an open neighborhood of* $0 \in \mathbb{R}^n$ *that satisfies* (5.3) *and* (5.4)*, then the function* ϕ *defined on a* $\mathbb{T}^n$*-invariant neighborhood of* $0 \in \mathbb{C}^n$ *by* (5.2) *is the Kähler potential of a toric gradient Kähler Ricci soliton on the open neighborhood of* $0 \in \mathbb{C}^n$ *where it is strictly pseudo-convex.*

Proof. — To begin with, let me point out a fact that will be used several times in the following argument: Any $\mathbb{T}^n$-invariant holomorphic function defined on a connected open neighborhood of $0 \in \mathbb{C}^n$ is constant there. This follows, for example, by examining the effect of $\mathbb{T}^n$ on the individual terms in the power series of such a function.

Now, since g is toric, its associated holomorphic vector field Z is invariant under the action of $\mathbb{T}^n$ and hence must vanish at $0 \in \mathbb{C}^n$ and commute with each of the scaling vector fields $Z_i = z^i \frac{\partial}{\partial z^i}$. It follows easily that $Z = Z_{\mathsf{h}}$ for some $\mathsf{h} \in \mathbb{R}^n$. (For the definition of Z_{h}, see (3.23).)

Let f be the unique $\mathbb{T}^n$-invariant Ricci potential for g that satisfies $f(0) = 0$ and let Υ be a holomorphic volume form associated to g and f. Since Υ is uniquely determined up to a complex number of modulus 1, it follows that, under the action of $\mathbb{T}^n$, Υ must transform multiplicitively by a character of $\mathbb{T}^n$. It then follows easily that $\Upsilon = c\,\mathrm{d}z$ for some nonzero constant c.

Let ϕ be the unique $\mathbb{T}^n$-invariant Kähler potential for g that satisfies $\phi(0) = 0$. As has already been remarked, ϕ is real-analytic and so can be expanded as a convergent power series in the variables z^i and $\bar{z}^i$. However, $\mathbb{T}^n$-invariance evidently implies that this power series can be collected in terms of the quantities $r^i = |z^i|^2$. Thus, the existence of a function u satisfying (5.2) follows.

As argued in § 3.2, the quantity $2f - \partial\phi(Z_{\mathsf{h}})$ is a holomorphic function on a neighborhood of $0 \in \mathbb{C}^n$. By construction, it, too, is $\mathbb{T}^n$-invariant and vanishes at $0 \in \mathbb{C}^n$, which implies that it vanishes identically. Thus, $\partial\phi(Z_{\mathsf{h}}) = \mathrm{d}\phi(X_{\mathsf{h}}) = 2f$.

The rest of the argument follows by substituting the formula (5.2) into (3.4), multiplying by $r^1 \cdots r^n$, and rearranging terms, which gives (5.3).

Note that the stated positivity conditions on the first derivatives of u are needed in order that the corresponding ϕ be strictly pseudo-convex in a neighborhood of $0 \in \mathbb{C}^n$ and the relation with $|c|^2$ follows by computing the coefficient of $r^1 \cdots r^n$ in the power series expansion of the left hand side of (5.3).

The converse statement follows by computation. □

***Remark 20* (Normalizations).** — Given a solution u to (5.3) that satisfies $u(0) = 0$, one can obviously scale in the individual coordinates so as to arrange that

$$\phi = r^1 + \cdots + r^n + O(|r|^2), \tag{5.5}$$

thereby reducing to the case $|c| = 1$, so it suffices to consider this case. Note also that the resulting Kähler soliton g is already in the normalized form guaranteed by Theorem 4.

***Remark 21* (pseudo-convexity of toric potentials).** — A $\mathbb{T}^n$-invariant function ϕ of the form (5.2), i.e., $\phi = u \circ \mu_n$ for some u defined on a domain $V \subset \mathbb{R}^n$, is strictly pseudo-convex on the domain $(\mu_n)^{-1}(V) \subset \mathbb{C}^n$ if and only if the symmetric matrix

$$\left(\delta_{ij}\,\frac{\partial u}{\partial r^j} + \sqrt{r^i r^j}\,\frac{\partial^2 u}{\partial r^i \partial r^j}\right) \tag{5.6}$$

is positive definite on the part of V that lies in the orthant defined by the inequalities $r^i \geq 0$.

5.1.1. *A singular initial value problem.* — Although (5.3) is singular along the hypersurfaces $r^i = 0$ in $\mathbb{R}^n$, it turns out that the methods of Gérard and Tahara [**11**] can be used to prove an extension theorem.

Theorem 5. — *Let v be a real-analytic function on an open subset $V \subset \mathbb{R}^{n-1}$ with the property that $\psi = v \circ \mu_{n-1}$ is strictly pseudo-convex on $(\mu_{n-1})^{-1}(V) \subset \mathbb{C}^{n-1}$.*

Then there exists an open neighborhood $U \subset \mathbb{R}^n$ of $V \times \{0\}$ and a real-analytic function u on U with the properties

1. *$u(r^1, \ldots, r^{n-1}, 0) = v(r^1, \ldots, r^{n-1})$ for $(r^1, \ldots, r^{n-1}) \in V$;*
2. *u satisfies (5.3) with $|c| = 1$; and*
3. *$\phi = u \circ \mu_n$ is strictly pseudo-convex on ${\mu_n}^{-1}(U) \subset \mathbb{C}^n$.*

Moreover, u is locally unique in the sense that any for any other pair $(\tilde U, \tilde u)$ with these properties, there is an open neigborhood W of $V \times \{0\}$ contained in $U \cap \tilde U$ such that u and $\tilde u$ agree on W.

Proof. — For the sake of clarity, write $t = r^n$ and let the lower case latin indices run from 1 to $n-1$. Then after dividing both sides of (5.3) (with $|c| = 1$) by $r^1 \cdots r^{n-1}$ and the exponential factor, this equation takes the form

$$\det \begin{pmatrix} \delta_{ij}\dfrac{\partial u}{\partial r^i} + r^j \dfrac{\partial^2 u}{\partial r^i \partial r^j} & \dfrac{\partial(tu_t)}{\partial r^i} \\ r^j \dfrac{\partial(tu_t)}{\partial r^j} & (t\,\partial_t)^2 u \end{pmatrix} = t\,\mathrm{e}^{\left(-\frac{h_n}{2}\,(tu_t) - \frac{1}{2}\sum_{j=1}^{n-1} h_j\, r^j \frac{\partial u}{\partial r^j}\right)}. \tag{5.7}$$

Note the first crucial aspect of this equation, which is that the t-derivatives of u occur as either tu_t or $t(tu_t)_t = (t\partial_t)^2 u$, i.e., as the 'regular singular' versions of the t-derivatives at $t = 0$.

Expanding the left hand side of (5.7) along the last column shows that this equation can be written in the form

$$\begin{aligned} \det\left(\delta_{ij}\frac{\partial u}{\partial r^i} + r^j \frac{\partial^2 u}{\partial r^i \partial r^j}\right)\left((t\,\partial_t)^2 u\right) &= t\,\mathrm{e}^{\left(-\frac{h_n}{2}\,(tu_t) - \frac{1}{2}\sum_{j=1}^{n-1} h_j\, r^j \frac{\partial u}{\partial r^j}\right)} \\ &\quad + Q_{ij}\left(r, \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}\right) \frac{\partial(tu_t)}{\partial r^i}\,\frac{\partial(tu_t)}{\partial r^j} \end{aligned} \tag{5.8}$$

where $Q_{ij} = Q_{ji}$ are certain polynomials in the variables r^i and the first and second derivatives of u with respect to the variables r^i.

In particular, note that the right hand side of (5.8) is an entire analytic function of the variables r^i and t, the first and second derivatives of u with respect to the variables r^i, the expression tu_t and its first derivatives with respect to the r^i.

In what follows, it will be particularly important that this right hand side is also in the ideal generated by t and the quadratic expressions $\frac{\partial(tu_t)}{\partial r^i}\,\frac{\partial(tu_t)}{\partial r^j}$.

Now, set

$$u(r^1, \ldots, r^{n-1}, t) = v(r^1, \ldots, r^{n-1}) + z(r^1, \ldots, r^{n-1}, t) \tag{5.9}$$

and define

$$F_{ij}(r^1, \ldots, r^{n-1}, t) = \delta_{ij}\frac{\partial v}{\partial r^i} + r^j \frac{\partial^2 v}{\partial r^i \partial r^j} \tag{5.10}$$

Note that, by hypothesis, $\det\bigl(F_{ij}(r, 0)\bigr) \neq 0$ for $r \in V \subset \mathbb{R}^{n-1}$. In particular, the expression

$$\det\left(F_{ij}(r,t) + \delta_{ij}\frac{\partial z}{\partial r^i} + r^j \frac{\partial^2 z}{\partial r^i \partial r^j}\right), \tag{5.11}$$

which is what the coefficient of $(t\partial_t)^2 u$ on the left hand side of (5.8) becomes when one substitutes $u = v + z$ into that equation, is an analytic expression in $r \in V$, t, and the partials of z that is non-vanishing on V when one sets $t = z = 0$.

Thus, substituting $u = v + z$ into (5.8) and dividing by the determinant factor yields an equation for z of the form

$$(t\,\partial_t)^2 z = E\left(r, t, z, \frac{\partial z}{\partial r^i}, tz_t, \frac{\partial^2 z}{\partial r^i \partial r^j}, \frac{\partial(tz_t)}{\partial r^i}\right) \tag{5.12}$$

where the function E is

1. real-analytic on an open neighborhood of $V \times \{0\}$ in $V \times \mathbb{R} \times \mathbb{R}^{1+n+\frac{1}{2}n(n+1)}$ and
2. in the ideal generated by t and the products of *pairs* of the last $(n-1)$ variables (i.e., the 'slots' containing the entries $\frac{\partial(tz_t)}{\partial r^i}$).

Now, turning to Chapter 8 of Gèrard and Tahara [**11**], one sees that (5.12) is of the form to which their Theorem 8.0.3 applies. [6] Consequently, (5.12) has a unique real-analytic solution $z(r,t)$ (defined on some neighborhood of $V \times \{0\} \subset \mathbb{R}^n$) that satisfies the initial condition

$$z(r^1, \ldots, r^{n-1}, 0) = 0 \qquad \text{for } (r^1, \ldots, r^{n-1}) \in V. \tag{5.13}$$

Using this solution z to define u via (5.9), one sees that (5.7) has a correspondingly unique real-analytic solution satisfying the initial condition

$$u(r^1, \ldots, r^{n-1}, 0) = v(r^1, \ldots, r^{n-1}) \qquad \text{for } (r^1, \ldots, r^{n-1}) \in V, \tag{5.14}$$

as claimed. The existence of an open neighborhood U of $V \times \{0\}$ such that $\phi = u \circ \mu_n$ is strictly pseudo-convex on $(\mu_n)^{-1}(U) \subset \mathbb{C}^n$ is routine. □

***Corollary 5* (Singular initial value problem for toric solitons).** — *Let g' be a real-analytic toric Kähler metric on a $\mathbb{T}^{n-1}$-invariant, connected open neighborhood $V \subset \mathbb{C}^{n-1}$ of 0.*

Then, for any $\mathsf{h} \in \mathbb{R}^n$ there exists a $\mathbb{T}^n$-invariant open neighborhood $U_\mathsf{h} \subset \mathbb{C}^n$ of $V \times \{0\}$ and a toric gradient Kähler Ricci soliton g_h on U_h whose pullback to V is g', whose associated vector field is Z_h, and whose associated holomorphic volume form with respect to its $\mathbb{T}^n$-invariant Ricci potential f_h vanishing at $0 \in \mathbb{C}^n$ is $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$.

Moreover, g_h is locally unique in that any extension of g' with these properties agrees with g_h on some open neighborhood of $V \times \{0\}$.

***Remark 22* (Contrast in initial value problems).** — Note that Corollary 5 has a very different character from Theorem 2. Not only is the nature of the initial data different, but, in the case of Corollary 5, one is imposing initial conditions along a submanifold that is everywhere *tangent* to the holomorphic vector field $Z = Z_\mathsf{h}$, rather than

[6] While I do not want to state their full theorem here, I will give the gist: The two properties listed for the function E are easily seen to imply that there exists a unique formal power series solution of the form $z(r,t) = z_1(r)\,t + z_2(r)\,t^2 + \cdots$ to (5.12). The main import of the quoted Theorem 8.0.3 is that this series actually converges to an analytic solution on some open neighborhood of $V \times \{0\}$. (The need for a theorem is caused by the singularity at $t = 0$, which renders the standard method of majorants ineffective in proving the convergence of the formal series.)

everywhere *transverse.* The difference, of course, is that Corollary 5 addresses a singular initial value PDE problem that is, in many ways the analogue of the sort of ODE problem one encounters in the theory of regular singular points of ODE.

Because the generalization of the ODE concept of 'regular singular point' to the case of PDE is very delicate (cf. the book of Gèrard and Tahara), it is somewhat remarkable that this theory actually applies in this case.

5.1.2. *A Lagrangian formulation.* — While the reduced equation (5.3) is singular along the hypersurfaces $r^i = 0$, it is regular on the open simplicial cone defined by $r^i > 0$. Indeed, setting $r^i = \mathrm{e}^{\rho^i}$, the equation (5.3) with $|c|^2 = 1$ can be written in the form

$$\det_{1\leq i,j\leq n}\left(\frac{\partial^2 u}{\partial\rho^i\partial\rho^j}\right)\mathrm{e}^{\left(\frac{h_1}{2}\frac{\partial u}{\partial\rho^1}+\cdots+\frac{h_n}{2}\frac{\partial u}{\partial\rho^n}\right)} = \mathrm{e}^{\rho^1+\cdots+\rho^n}. \tag{5.15}$$

Setting $u_i = \frac{\partial u}{\partial\rho^i}$, this can be further rewritten into the form

$$\mathrm{e}^{\left(\frac{h_1}{2}u_1+\cdots+\frac{h_n}{2}u_n\right)}\,\mathrm{d}u_1\wedge\cdots\wedge\mathrm{d}u_n = \mathrm{e}^{\rho^1+\cdots+\rho^n}\,\mathrm{d}\rho^1\wedge\cdots\wedge\mathrm{d}\rho^n. \tag{5.16}$$

Thus, on $\mathbb{R}^{2n+1}$ with coordinates u, ρ^i, u_i, if one defines the contact form

$$\theta = \mathrm{d}u - u_i\,\mathrm{d}\rho^i \tag{5.17}$$

and the closed θ-primitive [(7)] n-form

$$\Psi = \mathrm{e}^{\left(\frac{h_1}{2}u_1+\cdots+\frac{h_n}{2}u_n\right)}\,\mathrm{d}u_1\wedge\cdots\wedge\mathrm{d}u_n - \mathrm{e}^{\rho^1+\cdots+\rho^n}\,\mathrm{d}\rho^1\wedge\cdots\wedge\mathrm{d}\rho^n, \tag{5.18}$$

Then the solutions of the original equation (5.3) correspond to the integral manifolds of the Monge-Ampère ideal

$$\mathcal{I} = \langle\theta, \mathrm{d}\theta, \Psi\rangle. \tag{5.19}$$

Since Ψ is closed and $\mathrm{d}\theta\wedge\Psi = 0$, the $(n{+}1)$-form $\Pi = \theta\wedge\Psi$ is closed and hence is the Poincaré-Cartan form (see [**1**]) of a contact Lagrangian for the function u. In particular, it follows by Noether's Theorem that the symmetries of the Poincaré-Cartan form give conservation laws for the reduced equation.

This is interesting because this system turns out to have a number of symmetries that are not apparent from the symmetries of the original equation.

***Remark 23* (Affine symmetries and equivalences).** — For example, consider the affine transformations on $\mathbb{R}^{2n+1}$ of the form

$$\begin{aligned} \bar u &= s\,u + a_i B^i_k\,\rho^k + c\,,\\ \bar u_i &= A^j_i\,u_j + a_i\,,\\ \bar\rho^i &= B^i_j\,\rho^j + b^i \end{aligned} \tag{5.20}$$

(7) If (M^{2n+1},θ) is a contact manifold of dimension $2n{+}1$, then an n-form Ψ on M is said to be *θ-primitive* if $\mathrm{d}\theta\wedge\Psi\equiv 0 \mod \theta$.

where A^i_j, B^i_j, $s \neq 0$, a_i, b^i, and c are real constants satisfying the n^2+2n+1 equations

$$
\begin{aligned}
A^j_i B^i_k &= s\,\delta^j_i\,,\\
\sum_i h_i A^j_i &= h_j \quad \text{for } 1 \leq j \leq n\,,\\
\sum_i B^i_j &= 1 \quad \text{for } 1 \leq j \leq n\,,\\
\mathrm{e}^{\left(\frac{h_1}{2}\,a_1+\cdots+\frac{h_n}{2}\,a_n\right)}\,\det(A) &= \mathrm{e}^{b^1+\cdots+b^n}\,\det(B)\,.
\end{aligned}
\tag{5.21}
$$

Such transformations, which constitute a Lie group of dimension n^2+1, preserve the forms θ and Υ up to constant multiples and hence preserve the system $\mathcal{I}$.

Obviously, the system depends on the vector $\mathsf{h} = (h_1, \dots, h_n)$. However, by leaving off the second of the above four conditions, one finds transformations that define equivalences between any two systems with $h = h_1 + \cdots + h_n \neq 0$ and any two systems with $h = h_1 + \cdots + h_n = 0$ but $\mathsf{h} \neq 0$. (The system corresponding to $\mathsf{h} = 0$ is, of course, the system that gives Ricci-flat toric Kähler metrics.)

***Remark 24* (Algebraic coordinates).** — The function u is, in some sense, not that important, since only the derivatives of u appear in the formula for the metric. Thus, one can actually formulate the essential part of the exterior differential system as a system on $\mathbb{R}^{2n}$.

Assuming that none of the h_i are zero, one can coordinatize the system algebraically as follows: Set $v_i = \mathrm{e}^{\frac{1}{2}h_i u_i}$. Then the form Υ can, after multiplying by a constant, be written in the form

$$
\Upsilon = \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n - \frac{h_1 \cdots h_n}{2^n}\,\mathrm{d}r^1 \wedge \cdots \wedge \mathrm{d}r^n, \tag{5.22}
$$

and the contact condition that $\mathrm{d}u - u_i\,\mathrm{d}\rho^i = 0$ can be replaced by the condition

$$
\sum_{i=1}^{n} \frac{2}{h_i}\frac{\mathrm{d}v_i}{v_i} \wedge \frac{\mathrm{d}r^i}{r^i} = 0. \tag{5.23}
$$

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