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GEOMETRY OF MODULI SPACES

by

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Dedicated to Jean Pierre Bourguignon

Abstract. — In this paper we describe some recent results on the geometry of the moduli space of Riemann surfaces. We surveyed new and classical metrics on the moduli spaces of hyperbolic Riemann surfaces and their geometric properties. We then discussed the Mumford goodness and generalized goodness of various metrics on the moduli spaces and their deformation invariance. By combining with the dual Nakano negativity of the Weil-Petersson metric we derive various consequences such that the infinitesimal rigidity, the Gauss-Bonnet theorem and the log Chern number computations.

***Résumé* (Géométrie des espaces de modules).** — Dans cet article nous décrivons certains résultats récents en géométrie de l'espace de modules des surfaces de Riemann. Nous parcourons un certain nombre de métriques classiques et nouvelles sur les les espaces de modules de surfaces de Riemann hyperboliques et leur propriétés géométriques. Ensuite nous discutons la bonté de Mumford et la bonté généralisée de différentes métriques sur l'espace de modules et leurs invariance de déformation. En combinant avec la négativité de Nakano duale de la métrique de Weil-Peterson nous en tirons différentes conséquences telles que la rigidité infinitésimale, le théorème de Gauss-Bonnet et les calculs de nombres logarithmiques de Chern.

1. Introduction

In this paper we describe our recent work on the geometry of the moduli space of Riemann surfaces $\mathcal{M}_g$. We will survey the properties of the canonical metrics especially the asymptotic behavior.

This paper is organized as follows. In the second section we will briefly recall the deformation theory of Riemann surfaces. In the third section we will recall the

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Ricci and perturbed Ricci metrics as welll as the Kähler-Einstein metric which were discussed in [**5**] and [**6**].

In the fourth section we will discuss the notion of Mumford goodness and our generalizations to the p-goodness and intrinsic goodness. We then discuss the relation of the goodness and the complex Monge-Amperé equation as well as the Kähler-Ricci flow. In the last section we will discuss the applications of these fine properties of the canonical metrics.

2. Fundamentals of Teichmüller and Moduli Spaces

We briefly recall the fundamental theory of the geometry of Teichmüller and moduli spaces of hyperbolic Riemann surfaces in this section. Most of the results can be found in [**5**], [**6**], [**7**] and [**18**].

Let $\mathcal{M}_{g,k}$ be the moduli space of Riemann surfaces of genus g with k punctures such that $2g-2+k>0$. By the uniformization theorem we know there is a unique hyperbolic metric on such a Riemann surface. To simplify the computation, through out this paper, we will assume $k=0$ and $g\geq 2$ and work on $\mathcal{M}_g$. Most of the results can be trivially generalized to $\mathcal{M}_{g,k}$.

We first recall the local geometry of $\mathcal{M}_g$. For each point $s\in\mathcal{M}_g$, let X_s be the corresponding Riemann surface. By the Kodaira-Spencer deformation theory and Hodge theory, we know

$$T_s\mathcal{M}_g\cong H^1(X_s,T_{X_s})\cong H^{0,1}(X_s,T_{X_s}).$$

It follows direct from Serre duality that

$$T_s^*\mathcal{M}_g\cong H^0(X_s,K_{X_s}^2).$$

By the Riemann-Roch theorem, we know that the complex dimension of the moduli space is $n=\dim_{\mathbb{C}}\mathcal{M}_g=3g-3$. Given a Riemann surface X of genus $g\geq 2$, we denote by λ the unique hyperbolic (Kähler-Einstein) metric on X. Let z be local holomorphic coordinate on X. We normalize λ:

$$\partial_z\partial_{\bar z}\log\lambda=\lambda. \tag{2.1}$$

Let $\mathcal{T}_g$ be the Teichmüller space. It is well known that $\mathcal{T}_g$ is a domain of holomorphy and $\mathcal{M}_g$ is a quasi-projective orbifold. There are many canonical metrics on $\mathcal{T}_g$. These are the metrics where biholomorphisms are automatically isometries and thus these metrics descent down to $\mathcal{M}_g$.

There are three complex Finsler metrics on $\mathcal{T}_g$: The Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric $\|\cdot\|_K$ and the Caratheódory metric $\|\cdot\|_C$. Each of these metrics defines a norm on the tangent space of $\mathcal{T}_g$. These metrics are non-Kähler. By the famous work of Royden we know that the Teichmüller metric coincides with the Kobayashi metric:

$$\|\cdot\|_T = \|\cdot\|_K.$$

We now describe the Kähler metrics. The first known Kähler metric is the Weil-Petersson metric ω_{WP}. Since $\mathcal{T}_g$ is a domain of holomorphy, there is a complete Kähler-Einstein metric on $\mathcal{T}_g$ due to the work of Cheng and Yau [**2**]. Since $\mathcal{M}_g$ is quasi-projective, there exist a Kähler metric on $\mathcal{M}_g$ with Poincaré growth. Furthermore, one has the Bergman metric associate to $\mathcal{T}_g$ and the Kähler metric defined by McMullen [**10**] by perturbing the Weil-Petersson metric.

In [**5**] and [**6**] we defined two new Kähler metrics: the Ricci and perturbed Ricci metrics which have very nice curvature and asymptotic properties. These metrics will be discussed in the following sections.

We now recall the construction of the Weil-Petersson metric. Let $(s_1, \cdots, s_n)$ be local holomorphic coordinates on $\mathcal{M}_g$ near a point p and let X_s be the corresponding Riemann surfaces. Let $\rho : T_s\mathcal{M}_g \to H^1(X_s, TX_s) \cong H^{0,1}(X_s, TX_s)$ be the Kodaira-Spencer map. Then the harmonic representative of $\rho\left(\frac{\partial}{\partial s_i}\right)$ is given by

$$\rho\left(\frac{\partial}{\partial s_i}\right) = \partial_{\bar z}\left(-\lambda^{-1}\partial_{s_i}\partial_{\bar z}\log\lambda\right)\frac{\partial}{\partial z}\otimes d\bar z = B_i. \tag{2.2}$$

If we let $a_i = -\lambda^{-1}\partial_{s_i}\partial_{\bar z}\log\lambda$ and let $A_i = \partial_{\bar z}a_i$, then the harmonic lift v_i of $\frac{\partial}{\partial s_i}$ is given by

$$v_i = \frac{\partial}{\partial s_i} + a_i\frac{\partial}{\partial z}. \tag{2.3}$$

The well-known Weil-Petersson metric $\omega_{WP} = \frac{\sqrt{-1}}{2}h_{i\bar j}ds_i \wedge d\bar s_j$ on $\mathcal{M}_g$ is the L^2 metric on $\mathcal{M}_g$:

$$h_{i\bar j}(s) = \int_{X_s} A_i\overline{A}_j\, dv \tag{2.4}$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\bar z$ is the volume form on X_s. It was proved by Ahlfors that the Ricci curvature of the Weil-Petersson metric is negative. The upper bound of the Ricci curvature of the Weil-Petersson metric was conjectured by Royden and was proved by Wolpert [**16**].

In our work [**5**] we defined the Ricci metric ω_τ:

$$\omega_\tau = -Ric\,(\omega_{WP}) \tag{2.5}$$

and the perturbed Ricci metric $\omega_{\widetilde{\tau}}$:

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP} \tag{2.6}$$

where C is a positive constant. These new Kähler metrics have good curvature and asymptotic properties and play important roles in out study.

Now we describe the curvature formulas of the Weil-Petersson metric. Please see [**5**] and [**6**] for details. We denote by $f_{i\bar{j}} = A_i \overline{A_j}$ where each A_i is the harmonic Beltrami differential corresponding to the local holomorphic vector field $\frac{\partial}{\partial s_i}$. It is clear that $f_{i\bar{j}}$ is a function on X. We let $\square = -\partial_z \partial_{\bar{z}}$ be the Laplace operator, let $T = (\square + 1)^{-1}$ be the Green operator and let $e_{i\bar{j}} = T(f_{i\bar{j}})$. The functions $e_{i\bar{j}}$ and $f_{i\bar{j}}$ are building blocks of these curvature formula.

Theorem 2.1. — *The curvature formula of the Weil-Petersson metric was given by*

$$R_{i\bar{j}k\bar{l}} = -\int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}})\, dv. \tag{2.7}$$

This formula was first established by Wolpert [**16**] and was generalized by Siu [**14**] and Schumacher [**13**] to higher dimensions. A short proof can be found in [**5**].

It is easy to derive information of the sign of the curvature of the Weil-Petersson metric from its curvature formula (2.7). However, the Weil-Petersson metric is incomplete and its curvature has no lower bound. Thus we need to look at its asymptotic behavior. We now recall geometric construction of the Deligne-Mumford (DM) moduli space and the degeneration of hyperbolic metrics. Please see [**5**] and [**16**] for details.

Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ and let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. It was shown in [**3**] that D is a divisor with only normal crossings. A point $y \in D$ corresponds to a stable nodal surface X_y. A point $p \in X_y$ is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v) \mid uv = 0,\ |u|, |v| < 1\} \subset \mathbb{C}^2$. Let $p_1, \cdots, p_m \in X_y$ be the nodes. X_y is stable if each connected component of $X_y \setminus \{p_1, \cdots, p_m\}$ has negative Euler characteristic.

Fix a point $y \in D$, we assume the corresponding Riemann surface X_y has m nodes. Now for any point $s \in \mathcal{M}_g$ lying in a neighborhood of y, the corresponding Riemann surface X_s can be decomposed into the thin part which is a disjoint union of m collars and the thick part where the injectivity radius with respect to the Kähler-Einstein metric is uniformly bounded from below.

There are two kinds of local holomorphic coordinate on a collar or near a node. We first recall the rs-coordinate defined by Wolpert in [**18**]. In the node case, given a nodal surface X with a node $p \in X$, we let a, b be two punctures which are glued together to form p.

Definition 2.1. — *A local coordinate chart (U, u) near a is called rs-coordinate if $u(a) = 0$ where u maps U to the punctured disc $0 < |u| < c$ with $c > 0$, and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.*

In the collar case, given a closed surface X, we assume there is a closed geodesic $\gamma \subset X$ such that its length $l = l(\gamma) < c_*$ where c_* is the collar constant.

Definition 2.2. — *A local coordinate chart (U, z) is called rs-coordinate at γ if $\gamma \subset U$ where z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$, and the Kähler-Einstein metric on X can be written as*

$$\frac{1}{2}\left(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|}\right)^2|dz|^2.$$

The existence of collar was due to Keen [**4**]. We formulate this theorem in the following:

Lemma 2.1. — *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

1. *z maps Ω to the annulus $\{\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c\}$ for $c > 0$;*
2. *the Kähler-Einstein metric on X restricted to Ω is given by*

$$\left(\frac{1}{2}u^2 r^{-2}\csc^2\tau\right)|dz|^2 \tag{2.8}$$

   where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;

3. *the geodesic γ is given by the equation $|z| = e^{-\frac{\pi^2}{l}}$;*
4. *the constant c has a lower bound such that the area of Ω is bounded from below by a universal constant.*

We call such a collar Ω a genuine collar.

Now we describe the pinching coordinate chart of $\overline{\mathcal{M}}_g$ near the divisor D [**18**]. Let X_0 be a nodal surface corresponding to a codimension m boundary point and let $p_1, \cdots, p_m$ be the nodes of X_0. Then $\widetilde{X}_0 = X_0 \setminus \{p_1, \cdots, p_m\}$ is a union of punctured Riemann surfaces. Fix rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \cdots, m$ such that all the U_i and V_i are mutually disjoint. Now pick an open set $U_0 \subset \widetilde{X}_0$ such that the intersection of each connected component of $\widetilde{X}_0$ and U_0 is a nonempty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. Now pick Beltrami differentials $\nu_{m+1}, \cdots, \nu_n$ which are supported in U_0 and span the tangent space at $\widetilde{X}_0$ of the deformation space of $\widetilde{X}_0$. Let $\Delta_\varepsilon^{n-m} \subset \mathbb{C}^{n-m}$ be the polydisc of radius ε. For $t'' = (t_{m+1}, \cdots, t_n) \in \Delta_\varepsilon^{n-m}$, let $\nu(t'') = \sum_{i=m+1}^n t_i\nu_i$. We assume $|t''| = (\sum_{i=m+1}^n |t_i|^2)^{\frac{1}{2}}$ small enough such that $|\nu(t'')| < 1$. The nodal surface $X_{0,t''}$ is obtained by solving the Beltrami equation $\bar{\partial} w = \nu(t'')\partial w$. Since $\nu(t'')$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates on $X_{0,t''}$. By the theory of Ahlfors and Bers [**1**] and Wolpert [**18**] we can assume that there are constants $\delta, c > 0$ such that when $|t''| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,t''}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$. Now we assume $t' = (t_1, \cdots, t_m)$ has small norm. We do the plumbing construction on $X_{0,t''}$ to obtain $X_t = X_{t',t''}$. For each $i = 1, \cdots, m$, we remove the discs $\{0 < |\eta_i| \le \frac{|t_i|}{c}\}$ and $\{0 < |\zeta_i| \le \frac{|t_i|}{c}\}$ from $X_{0,t''}$

and identify $\{\frac{|t_i|}{c} < |\eta_i| < c\}$ with $\{\frac{|t_i|}{c} < |\zeta_i| < c\}$ by the rule $\eta_i\zeta_i = t_i$. This defines the surface X_t. The tuple $t = (t', t'') = (t_1, \cdots, t_m, t_{m+1}, \cdots, t_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar $\{\frac{|t_i|}{c} < |\eta_i| < c\}$ the plumbing collar.

Remark 2.1. — From the estimate of Wolpert [**17**], [**18**] on the length of short geodesic, we have $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log|t_i|}$.

In [**5**] and [**6**] we derived the precise asymptotic of the Weil-Petersson metric and its curvature. This is one of the key components in the proof of its goodness. We have

Theorem 2.2. — *Let $(t, s) = (t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates near a codimension m boundary point in $\overline{\mathcal{M}}_g$. Let h be the Weil-Petersson metric. Then we have the asymptotic:*

1. $h^{i\bar{i}} = 2u_i^{-3}|t_i|^2(1 + O(u_0))$ *and* $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1 + O(u_0))$ *for* $1 \le i \le m$;
2. $h^{i\bar{j}} = O(|t_it_j|)$ *and* $h_{i\bar{j}} = O\left(\frac{u_i^3u_j^3}{|t_it_j|}\right)$, *if* $1 \le i, j \le m$ *and* $i \ne j$;
3. $h^{i\bar{j}} = O(1)$ *and* $h_{i\bar{j}} = O(1)$, *if* $m + 1 \le i, j \le n$;
4. $h^{i\bar{j}} = O(|t_i|)$ *and* $h_{i\bar{j}} = O\left(\frac{u_i^3}{|t_i|}\right)$ *if* $i \le m < j$;
5. $h^{i\bar{j}} = O(|t_j|)$ *and* $h_{i\bar{j}} = O\left(\frac{u_j^3}{|t_j|}\right)$ *if* $j \le m < i$

where $u_0 = \sum_{j=1}^m u_j + \sum_{j=m+1}^n |s_j|$.

The precise estimates of the asymptotic of the full curvature tensor of the Weil-Petersson metric, which will be used in the proof of its goodness, can be found in [**5**], [**6**] and [**7**].

3. Canonical Metrics on $\mathcal{M}_g$

Since the Weil-Petersson metric is incomplete and does not have bounded geometry, it is hard to use it to study the geometry of $\mathcal{M}_g$. In [**5**] we introduced the Ricci metric $\omega_\tau = -Ric(\omega_{WP})$ and the perturbed Ricci metric $\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$. It turns out that these new Kähler metrics have nice curvature and asymptotic properties. These new metrics are also closely related to the Kähler-Einstein metric. Especially the Ricci metric is cohomologous to the Kähler-Einstein metric as currents.

To describe the curvature formulae of the Ricci and perturbed Ricci metrics, we need to introduce several operators. We first define the operator $\xi_k : C^\infty(X_s) \to C^\infty(X_s)$ by

$$\xi_k(f) = \overline{\partial}^*(i(B_k)\partial f) = -\lambda^{-1}\partial_{\bar{z}}(A_k\partial_z f) = -A_kK_1K_0(f) \tag{3.1}$$

where K_0, K_1 are the Maass operators [**16**], [**5**].

It was proved in[**5**] that ξ_k is the commutator of the Laplace operator and the Lie derivative in the direction v_k:

$$(\Box + 1)v_k - v_k(\Box + 1) = \Box v_k - v_k \Box = \xi_k. \tag{3.2}$$

We also need the commutator of the operator v_k and $\overline{v}_l$. In [**5**] we defined the operator $Q_{k\bar{l}} : C^\infty(X_s) \to C^\infty(X_s)$ by

$$Q_{k\bar{l}}(f) = [\overline{v}_l, \xi_k](f) = \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z f_{k\bar{l}}\partial_{\bar{z}} f \tag{3.3}$$

where $P : C^\infty(X_s) \to \Gamma(\Lambda^{1,0}(T^{0,1}X_s))$ is the operator defined by $P(f) = \partial_z(\lambda^{-1}\partial_z f)$.

The terms appeared in the curvature formulae of the Ricci and perturbed Ricci metrics are formally symmetric with respect to indices. For convenience, we recall the symmetrization operator defined in [**5**].

Definition 3.1. — *Let U be any quantity which depends on indices i, k, α and $\bar{j}, \bar{l}, \bar{\beta}$. The symmetrization operator σ_1 is defined by taking the summation of all orders of the triple (i, k, α). Similarly, σ_2 is the symmetrization operator of $\bar{j}$ and $\bar{\beta}$ and $\widetilde{\sigma_1}$ is the symmetrization operator of $\bar{j}$, $\bar{l}$ and $\bar{\beta}$.*

In [**5**] we derived the curvature formulae of the new metrics. These formulae, although very complicated, are integral formulae along the fibers of the universal curve.

Theorem 3.1. — *Let $\widetilde{R}_{i\bar{j}k\bar{l}}$ and $P_{i\bar{j}k\bar{l}}$ be the curvature tensors of the Ricci and perturbed Ricci metrics respectively. In [**5**] we established the following curvature formulae of these metrics:*

$$\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}} = &- h^{\alpha\bar{\beta}} \left\{ \sigma_1\sigma_2 \int_{X_s} \left\{ (T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\bar{\beta}}) + T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}}) \right\} \, dv \right\} \\
&- h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}} \, dv \right\} \\
&+ \tau^{p\bar{q}} h^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\bar{q}}) e_{\alpha\bar{\beta}} \, dv \right\} \left\{ \widetilde{\sigma}_1 \int_{X_s} \overline{\xi}_l(e_{p\bar{j}}) e_{\gamma\bar{\delta}}) \, dv \right\} \\
&+ \tau_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}}
\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
P_{i\bar{j}k\bar{l}} = & -h^{\alpha\bar{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{T(\xi_k(e_{i\bar{j}}))\bar{\xi}_l(e_{\alpha\bar{\beta}})+T(\xi_k(e_{i\bar{j}}))\bar{\xi}_\beta(e_{\alpha\bar{l}})\right\}\,dv\right\}\\
& -h^{\alpha\bar{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\bar{\beta}}\,dv\right\}\\
& +\widetilde{\tau}^{p\bar{q}}h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\bar{\beta}}\,dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\bar{\xi}_l(e_{p\bar{j}})e_{\gamma\bar{\delta}})\,dv\right\}\\
& +\tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}}+CR_{i\bar{j}k\bar{l}}.
\end{aligned}
\tag{3.5}
$$

Unlike the case of the Weil-Petersson metric from which we can see the sign of the curvature directly, the above formulae are too complicated. On one hand we can see that these metrics are Kähler from these formulae. On the other hand, we need to look at the asymptotic of the curvature of these new metrics. In [**5**] and [**6**] we computed the asymptotic of these new metrics and their curvature:

Theorem 3.2. — *Let $u_0=\sum_{j=1}^m u_j+\sum_{j=m+1}^n|s_j|$. The Ricci metric has the asymptotic:*

1. $\tau_{i\bar{i}}=\frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *and* $\tau^{i\bar{i}}=\frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1+O(u_0))$, *if* $i\le m$;
2. $\tau_{i\bar{j}}=O\left(\frac{u_i^2u_j^2}{|t_it_j|}(u_i+u_j)\right)$ *and* $\tau^{i\bar{j}}=O(|t_it_j|)$, *if* $i,j\le m$ *and* $i\ne j$;
3. $\tau_{i\bar{j}}=O\left(\frac{u_i^2}{|t_i|}\right)$ *and* $\tau^{i\bar{j}}=O(|t_i|)$, *if* $i\le m$ *and* $j\ge m+1$;
4. $\tau_{i\bar{j}}=O(1)$, *if* $i,j\ge m+1$.

The holomorphic sectional curvature of the Ricci metric has the asymptotic:

1. $\widetilde{R}_{i\bar{i}i\bar{i}}=-\frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0))$ *if* $i\le m$;
2. $\widetilde{R}_{i\bar{i}i\bar{i}}=O(1)$ *if* $i>m$.

We also have a weak curvature estimate of the Ricci metric. Let

$$
\Lambda_i=\begin{cases}\frac{u_i}{|t_i|} & \textit{if } i\le m\\ 1 & \textit{if } i>m.\end{cases}
$$

Then

1. $\widetilde{R}_{i\bar{j}k\bar{l}}=O(1)$ *if* $i,j,k,l>m$;
2. $\widetilde{R}_{i\bar{j}k\bar{l}}=O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0)$ *if at least one of these indices* i,j,k,l *is less than or equal to* m *and they are not all equal to each other.*

The asymptotic of the perturbed Ricci metric and its curvature can be found in [**5**] and [**6**]. Also, precise estimates of the full curvature tensor of the Ricci and perturbed Ricci metrics, which will also be used in the proof of their goodness, can be found in [**7**] and [**8**].

As a simple corollary of the curvature formulae and asymptotic analysis, in [**5**] we first proved the equivalence of canonical metrics on $\mathcal{M}_g$:

Theorem 3.3. — *All the canonical metrics on the moduli space $\mathcal{M}_g$: the Teichmüller-Kobayashi metric, the Carathéodory metric, the induced Bergman metric, the asymptotic Poincaré metric, the McMullen metric, the Ricci metric, the perturbed Ricci metric and the Kähler-Einstein metric are equivalent.*

The new metrics we defined have nice curvature properties which can be used to control the Kähler-Einstein metric. In [**5**] and [**6**] we proved

Theorem 3.4. — *Let $\mathcal{M}_g$ be the moduli space of genus $g \geq 2$ Riemann surfaces. Then*

- *The Ricci and perturbed Ricci metrics are complete Kähler metrics with Poincaré growth.*
- *The Ricci and perturbed Ricci metrics as well as the Kähler-Einstein metric have bounded geometry on the Teichmüller space $\mathcal{T}_g$.*
- *The Ricci and holomorphic sectional curvatures of the perturbed Ricci metric are bounded from above and below by negative constants.*
- *All the covariant derivatives of the curvature of the Kähler-Einstein metric are bounded.*

The finer asymptotic of these metrics, their local connection forms and curvature forms will lead to the Mumford goodness which is a set of growth conditions of these metrics and their derivatives modeled on the Poincaré metric on the punctured disk. These conditions will guarantee the behavior of the Chern forms of these complete metrics.

4. Notions of Goodness

In this section we will discuss various notions of goodness. The central idea is to control the Chern forms, as currents, of singular Hermitian metrics on holomorphic vector bundles over quasi-projective varieties.

Let M be a compact complex manifold and let (E, h) be a Hermitian vector bundle over M. We denote by $(z_1, \cdots, z_n)$ the local holomorphic coordinates on M and by $(e_1, \cdots, e_m)$ the local holomorphic frame of E. Let $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$ and denote by θ and Θ the local connection and curvature forms of h. Then we have $\theta_\alpha^\gamma = \partial_i h_{\alpha\overline{\beta}} h^{\gamma\overline{\beta}} dz_i$ and $\Theta_\alpha^\gamma = R_{\alpha i\overline{j}}^\gamma dz_i \wedge d\overline{z}_j$ where $\partial_i = \frac{\partial}{\partial z_i}$ and

$$R_{\alpha i\overline{j}}^\gamma = -h^{\gamma\overline{\beta}} \left(\partial_i \partial_{\overline{j}} h_{\alpha\overline{\beta}} - h^{\tau\overline{\delta}} \partial_i h_{\alpha\overline{\delta}} \partial_{\overline{j}} h_{\tau\overline{\beta}}\right).$$

The k-th Chern form $c_k(h)$ of h is given by the coefficient of the term t^k in the polynomial $\det\left(I + \frac{\sqrt{-1}}{2\pi}\Theta\right)$. It is well known that

$$[c_k(h)] = c_k(E) \tag{4.1}$$

as cohomology classes. However, this is no long true in general when M is noncompact. One needs growth conditions on h and its derivatives. The class of noncompact manifolds we are interested in is the quasi-projective manifolds.

The first condition was given by Mumford in [**11**] which we will describe now. Let $\overline{X}^n$ be a projective manifold of complex dimension n and let $D \subset \overline{X}$ be a divisor of normal crossings. Let $X = \overline{X} \setminus D$.

We cover a neighborhood of $D \subset \overline{X}$ by finitely many polydiscs

$$\{U_\alpha = (\Delta^n, (z_1, \cdots, z_n))\}_{\alpha \in A}$$

such that $V_\alpha = U_\alpha \setminus D = (\Delta^*)^m \times \Delta^{k-m}$. Namely, $U_\alpha \cap D = \{z_1 \cdots z_m = 0\}$. We let $U = \bigcup_{\alpha \in A} U_\alpha$ and $V = \bigcup_{\alpha \in A} V_\alpha$. On each V_α we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^m \frac{1}{2|z_i|^2 (\log|z_i|)^2} dz_i \wedge d\overline{z}_i + \sum_{i=m+1}^n dz_i \wedge d\overline{z}_i\right).$$

The Mumford goodness is a growth condition on differential forms. We recall the following definitions from [**11**]:

Definition 4.1. — *Let η be a smooth local p-form defined on V_α.*

- *We say η has Poincaré growth if there is a constant $C_\alpha > 0$ depending on η such that*
$$|\eta(t_1, \cdots, t_p)|^2 \leq C_\alpha \prod_{i=1}^p \|t_i\|^2_{\omega_{P,\alpha}}$$
for any point $z \in V_\alpha$ and $t_1, \cdots, t_p \in T_z X$.
- *We say η is good if both η and $d\eta$ have Poincaré growth.*

Now let $\overline{E}$ be a holomorphic vector bundle of rank k over $\overline{X}$ and let E be the restriction of E to X. Let h be a Hermitian metric on E which may be singular along the divisor D.

Definition 4.2. — *An Hermitian metric h on E is good if for all $z \in V$, assuming $z \in V_\alpha$, and for all basis $(e_1, \cdots, e_k)$ of $\overline{E}$ over U_α we have*

- $\left|h_{\alpha\overline{\beta}}\right|, (\det h)^{-1} \leq C\left(\sum_{i=1}^m \log|z_i|\right)^{2p}$ *for some $C > 0$ and $p \geq 1$.*
- *The local 1-forms $\left(\partial h \cdot h^{-1}\right)_\alpha^\gamma$ are good on V_α. Namely the local connection and curvature forms of h have Poincaré growth.*

Remark 4.1. — It is easy to see that the definition of Poincaré growth is independent of the choice of local data.

We collect the main properties of good metrics in the following theorem which is due to Mumford. Please see [**11**] for details.

Theorem 4.1. — *Let X and E be as above. Then*

- *A form $\eta \in A^p(X)$ with Poincaré growth defines a p-current $[\eta]$ on $\overline{X}$. In fact we have*
$$\int_X |\eta \wedge \xi| < \infty$$
for any $\xi \in A^{k-p}(\overline{X})$.
- *If both $\eta \in A^p(X)$ and $\xi \in A^q(X)$ have Poincaré growth, then $\eta \wedge \xi$ has Poincaré growth.*
- *For a good form $\eta \in A^p(X)$, we have $d[\eta] = [d\eta]$.*
- *Given an Hermitian metric h on E, there is at most one extension $\overline{E}$ of E to $\overline{X}$ such that h is good.*
- *If h is a good metric on E, the Chern forms $c_i(E,h)$ are good forms. Furthermore, as currents, they represent the corresponding Chern classes $c_i(\overline{E}) \in H^{2i}(\overline{X}, \mathbb{C})$.*

The most important feature of a good metric on E is that we can compute the Chern classes of $\overline{E}$ via the Chern forms of h as currents. Namely, with the growth assumptions on the metric and its derivatives, we can integrate by part, so Chern-Weil theory still holds. However, the Mumford goodness is very strong and hard to check. Also, there are only few examples. In [**7**] we showed that the canonical metrics on the moduli space of Riemann surfaces are Mumford good.

We now give weaker notions of goodness which still have the major properties of Mumford good metrics. The definition of Mumford on Poincaré growth and good forms is quite local. We first give a global formulation of these growth conditions. Please see [**7**] for details.

We call a Kähler metric ω_P on X a Poincaré type metric if ω_P is equivalent to $\omega_{P,\alpha}$ when restricted to V_α.

Remark 4.2. — It is easy to see that

- Any two Poincaré type metrics are equivalent.
- The quasi-projective Kähler manifold (X, ω_P) is complete and has finite volume.

Our first observation is

Lemma 4.1. — *A smooth form $\eta \in A^q(X)$ has Poincaré growth if and only if $\|\eta\|_{\omega_P} \leq C$ for some constant C and a Poincaré metric on X. Namely η has L^∞ bound with respect to Poincaré metrics.*

Parallel to the Poincaré growth and good forms, we know define the p-growth and p-good forms by replacing the L^∞ norm by L^p norm.

Definition 4.3. — *Let $p \geq 1$ be a real number. A differential form $\eta \in A^q(X)$ has p-growth if*

$$\|\eta\|_{\omega_P} \in L^p(X, \omega_P).$$

The form η is p-good if both η and $d\eta$ have p-growth.

We note here that the above definition is independent of the choice of ω_P. To study the currents of p-growth forms, we need a special cut-off functions. In [**9**] we construct a desirable cut-off function:

Proposition 4.1. — *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there is a function ρ_ε such that*

1. $0 \leq \rho_\varepsilon \leq 1$.
2. *For any open neighborhood N of D in $\overline{X}$, there is $\varepsilon > 0$ such that* $\operatorname{supp}(1-\rho_\varepsilon) \subset N$.
3. *For each $\varepsilon > 0$, there is a neighborhood N of D such that $\rho_\varepsilon|_N \equiv 0$.*
4. *$\rho_{\varepsilon'} \geq \rho_\varepsilon$ for $\varepsilon' \leq \varepsilon$.*
5. *There is a constant C, independent of ε such that*
$$-C\omega_P \leq \sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \leq C\omega_P$$
and
$$|\nabla'\rho_\varepsilon| \leq C.$$
6. $$\lim_{\varepsilon\to 0} \rho_\varepsilon = 1.$$

The p-good forms have similar behavior to good forms.

Lemma 4.2. — *For $p \geq 1$, if $\eta \in A^q(X)$ has p-growth, then η defines a q-current. If η is p-good, then $d[\eta] = [d\eta]$. Furthermore, if η, η' have p and p' growth respectively, then $\eta \wedge \eta'$ has $\frac{pp'}{p+p'}$ growth.*

Now we can generalize the Mumford good metrics. Similar to Definition 4.2 we define

Definition 4.4. — *A Hermitian metric h on E is p-good if*

1. $\left|h_{\alpha\overline{\beta}}\right|, (\det h)^{-1} \leq C\left(\sum_{i=1}^m \log|z_i|\right)^{2s}$ *for some $C > 0$ and $s \geq 1$.*
2. *The local* 1*-forms $\left(\partial h \cdot h^{-1}\right)_\alpha^\gamma$ are p-good on V_α.*

We have

Theorem 4.2. — *For p large enough, if the Hermitian metric h on E is p-good, then the Chern forms of h represent the corresponding Chern classes of $\overline{E}$:*

$$[c_i(h)] = c_i(\overline{E}) \in H^{2i}(\overline{X}, \mathbb{C}).$$

The p-goodness is essentially integral conditions which is much easier to check than the Mumford goodness. Since the most important part of controlling the growth the singular metric h is to study its Chern forms, we can just take this as a definition.

Definition 4.5. — *A Hermitian metric h on E is intrinsically good if the Chern form $c_i(h)$ defines a $2i$-current and*

$$[c_i(h)] = c_i(\overline{E}).$$

It turns out that the intrinsic goodness is preserved by the continuity method and the Kähler-Ricci flow. We have the following relation:

good metrics $\Rightarrow$ p-good metrics for large p $\Rightarrow$ intrinsic good metrics

There are only few examples of Mumford good metrics. In [**11**] Mumford showed that the invariant metrics on Hermitian symmetric spaces are good. Later Wolpert [**18**] showed that the hyperbolic metric on the relative dualizing sheaf is good. In [**15**] Trapani proved that the metric on the logarithmic tangent bundle of $\overline{\mathcal{M}}_g$ is good. In the last cases, the holomorphic bundle involoved are line bundles. In [**7**] and [**8**] we prove:

Theorem 4.3. — *Let $\overline{E} = T_{\overline{\mathcal{M}}_g}(-\log D)$ be the logarithmic tangent bundle of the DM moduli space and let $E = \overline{E}\mid_{\mathcal{M})g}$. Then the metrics on E induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics are good in the sense of Mumford.*

The moduli space $\mathcal{M}_g$ together with these metrics provide very interesting examples of good geometry. It is more interesting to study the goodness of the Kähler-Einstein metric since many consequences follows.

5. The Monge-Amperé Equation and the Goodness

As we described in last section, the Chern forms of various good singular Hermitian metrics represent corresponding Chern classes. Thus it is important to study the goodness of canonical metrics on the quasi-projective manifold X such as the the Kähler-Einstein metric.

Let X be a quasi-projective manifold obtained by removing a normal crossing divisor D from a projective manifold $\overline{X}$. Let $\overline{E} = T_{\overline{X}}(-\log D)$ be the logarithmic tangent bundle and let E be the restriction of $\overline{E}$ to X. In this section we will consider Hermitian metrics on E induced from a Kähler metric on X.

Let ω_g be a Kähler metric on X. Let $U, (z_1, \cdots, z_n)$ be a chart of $\overline{X}$ such that $U \cap D = \{z_1 \cdots z_m = 0\}$. It is clear that a local frame of E is given by

$$\underline{e} = (e_1, \cdots, e_n) = \left(z_1 \frac{\partial}{\partial z_1}, \cdots, z_m \frac{\partial}{\partial z_m}, \frac{\partial}{\partial z_{m+1}}, \cdots, \frac{\partial}{\partial z_n} \right).$$

Let h be the metric on E induced by ω_g. Then under this frame we have

$$g^*_{i\bar{j}} = \begin{cases} z_i \overline{z}_j g_{i\bar{j}} & i, j \leq m \\ z_i g_{i\bar{j}} & i \leq m < j \\ \overline{z}_j g_{i\bar{j}} & j \leq m < i \\ g_{i\bar{j}} & i, j > m. \end{cases} \tag{5.1}$$

By using the above frame and the local formula of the metric h, we have

Lemma 5.1. — *The Chern forms of h and ω_g coincide. Namely,*

$$c_k(h) = c_k(g).$$

If we assume the background metric ω_g has Poincaré growth, then the induced metric h is good will imply that the metric g has bounded curvature. The converse is not true in general. But we can bound the Chern froms:

Lemma 5.2. — *If ω_g is a Kähler metric on X with bounded curvature and has Poincaré growth, then the Chern forms of the metric h on E induced by ω_g are good in the sense of Mumford.*

In the case when h is induced by the Kähler-Einstein metric on X, to ensure the Chern forms of h represent the correct Chern classes, we need control on the Kähler-Einstein metric.

The following result is a weaker version of our work. We state this version to illustrate the ideas.

Theorem 5.1. — *Let $\overline{X}$ be a projective manifold with $\dim_{\mathbb{C}} \overline{X} = n$. Let $D \subset \overline{X}$ be a divisor of normal crossings, let $X = \overline{X} \setminus D$, let $\overline{E} = T_{\overline{X}}(-\log D)$ and let $E = \overline{E}\,|_X$.*

Let ω_g be a Kähler metric on X with bounded curvature and Poincaré growth. Assume $Ric(\omega_g) + \omega_g = \partial\overline{\partial} f$ where f is a bounded smooth function. Then

- *There exist a unique Kähler-Einstein metric ω_{KE} on X with Poincaré growth.*
- *The curvature and covariant derivatives of curvature of the Kähler-Einstein metric are bounded.*
- *If ω_g is intrinsic good, then ω_{KE} is intrinsic good. Furthermore, all metrics along the paths of continuity and Kähler-Ricci flow are intrinsic good.*

Remark 5.1. — In [**8**] we will prove a stronger version of the above theorem by replacing the L^∞ bound of the Ricci potential f by L^p bound.

On the other hand, if we know the existence and properties of the Kähler-Einstein metric by other means, we can prove the above theorem by only assuming $f \in L^1(X, \omega_g)$.

Theorem 5.2. — *Let ω_g be a Kähler metric on X with Poincaré growth and bounded curvature. Assume $Ric(\omega_g)+\omega_g = \partial\overline{\partial} f$ where $f \in L^1(X, \omega_g)$ and there exist a Kähler-Einstein metric on X which is equivalent to ω_g. If ω_g is intrinsically good, then $\omega_{_{KE}}$ is also intrinsically good.*

By combining Theorem 3.3, 4.3 and 5.2 we have

Theorem 5.3. — *Let ρ be the metric on the logarithmic tangent bundle over the moduli space $\mathcal{M}_g$ induced by the Kähler-Einstein metric on $\mathcal{M}_g$. Then ρ is intrinsically good.*

The intrinsic goodness of the Kähler-Einstein metric will imply stability of the log tangent bundle and a strong Chern number inequality. As a consequence we proved in [**6**] and [**7**]

Theorem 5.4. — *The logarithmic tangent bundle $\overline{E}$ of the DM moduli space $\overline{\mathcal{M}}_g$ is stable with respect to the canonical polarization. Furthermore, we have*

$$c_1(\overline{E})^2 \leq \frac{6g-4}{3g-3} c_2(\overline{E}).$$

We now briefly describe the proof of these two theorems. Please see [**7**] and [**8**] for details.

We first deform the background ω_g along the Kähler-Ricci flow for short time such that all the covariant derivatives of ω_g are bounded. In the case, the intrinsic goodness of ω_g is also preserved.

The existence of the Kähler-Einstein metric follows from the C^k estimates of the complex Monge-Amperé equation

$$\frac{(\omega_g + \partial\overline{\partial}\varphi)^n}{\omega_g^n} = e^{\varphi + f}$$

where we use Yau's generalized maximum principle. To prove that the intrinsic goodness of ω_g is preserved along the path of continuity, if we denote by g' the Kähler-Einstein metric, we need to show that

$$c_k(g) - c_k(g')$$

is the 0-current. Let R, R', Γ, Γ' be the curvatures and connections of g and g' respectively.

We first deal with renormalized Chern character forms. For a Hermitian metric h on a holomorphic vector bundle E with curvature Θ, the k-th Chern character form is defined by

$$ch_k(h) = Tr\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)^k.$$

To simplify the notation, we drop the constant $\frac{\sqrt{-1}}{2\pi}$. As differential forms we have

$$ch_k(g) - ch_k(g') = d\left(\mathrm{Tr}\sum_{i=0}^{k-1} R^{k-1-i} \wedge (\Gamma - \Gamma') \wedge R'^i\right)$$

and

$$\Gamma'^p_{ik} - \Gamma^p_{ik} = g'^{p\bar{q}}\varphi_{;i\bar{q}k}.$$

By the C^2 and C^3 estimate we know

$$\mathrm{Tr}\left(\sum_{i=0}^{k-1} R^{k-1-i} \wedge (\Gamma - \Gamma') \wedge R'^i\right)$$

has Poincaré growth. Since both $ch_k(g)$ and $ch_k(g')$ has Poincaré growth it is easy to see $ch_k(g) - ch_k(g')$ is the 0 current.

This is proved by integration by part where we use the cut-off function as in Proposition 4.1. Finally, by the expression of $c_k(g)$ and $c_k(g')$ via $ch_k(g)$ and $ch_k(g')$ we see that $c_k(g) - c_k(g')$ is also the 0 current.

6. Rigidity and Gauss-Bonnet Theorem

In this final section we discuss the applications of the curvature and asymptotic properties of the canonical metrics on the curve moduli $\mathcal{M}_g$.

The Weil-Petersson metric has many negative curvature properties. Ahlfors showed that its Riemannian sectional curvature is negative. Later, it was proved by Wolpert that the bisectional curvature of the Weil-Petersson metric is negative. In [**12**] Schumacher showed that the curvature of the Weil-Petersson metric is strongly negative in the sense of Siu. In [**7**] we proved that the Weil-Petersson metric is dual-Nakano negative from which we will derive Nakano-type vanishing theorems.

We first recall the concept of dual Nakano negativity. Let (E^m, h) be a holomorphic vector bundle with a Hermitian metric over a complex manifold M^n. The curvature of E is given by

$$P_{i\bar{j}\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}}h_{i\bar{j}} + h^{p\bar{q}}\partial_\alpha h_{i\bar{q}}\partial_{\bar{\beta}}h_{p\bar{j}}.$$

(E, h) is Nakano semi-positive if the curvature P defines a semi-positive form on the bundle $E \otimes T_M$. Namely,

$$P_{i\bar{j}\alpha\bar{\beta}}C^{i\alpha}\overline{C^{j\beta}} \geq 0 \tag{6.1}$$

for all $m \times n$ complex matrix C. The metric h is Nakano positive if (6.1) is a strict inequality whenever $C \neq 0$. E is dual Nakano (semi) negative if the dual bundle with the induced metric (E^*, h^*) is Nakano (semi) positive.

In [**7**] we showed

Theorem 6.1. — *Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$. Then $(T_{\mathcal{M}_g}, \omega_{WP})$ is dual Nakano negative.*

Let us briefly describe the idea. Please see [**7**] for details. By the definition of the dual-Nakano negativity, we only need to show that $(T^*\mathcal{M}_g, h^*)$ is Nakano positive. Let $R_{i\bar{j}k\bar{l}}$ be the curvature of $\mathcal{M}_g$ and $P_{i\bar{j}k\bar{l}}$ be the curvature of the cotangent bundle. We first have

$$P_{m\bar{n}k\bar{l}} = -h^{i\bar{n}}h^{m\bar{j}}R_{i\bar{j}k\bar{l}}.$$

Thus if we let $a_{kj} = \sum_m h^{m\bar{j}}C^{mk}$, we have

$$P_{m\bar{n}k\bar{l}}C^{mk}\overline{C^{nl}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}a_{kj}\overline{a_{li}} = -\sum_{i,j,k,l} R_{k\bar{j}i\bar{l}}a_{kj}\overline{a_{li}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}a_{ij}\overline{a_{lk}}.$$

Recall that at $X \in \mathcal{M}_g$ we have

$$R_{i\bar{j}k\bar{l}} = -\int_X \left(e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}}\right) dv.$$

By combining the above two formulae, to prove that the WP metric is Nakano negative is equivalent to show that

$$\int_X \left(e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}}\right) a_{ij}\overline{a_{lk}}\, dv \geq 0 \tag{6.2}$$

and the left hand side of the above formula is strictly positive if $A = [a_{ij}] \neq 0$.

We now describe the proof with the assumption that the matrix $[a_{ij}]$ is invertible. The general case can be found in [**7**] which follows from the same idea.

Recall that if we let $\square = -\lambda^{-1}\partial_z\partial_{\bar{z}}$ be the Laplace operator with respect to the KE metric λ on X and let $T = (\square + 1)^{-1}$, then $e_{i\bar{j}} = T\left(f_{i\bar{j}}\right)$ where $f_{i\bar{j}} = A_i\overline{A_j}$ and A_i is the harmonic representative of the Kodaira-Spencer class of $\frac{\partial}{\partial t_i}$ where $(t_1, \cdots, t_n)$ are local coordinates on $\mathcal{M}_g$ and z is the local coordinate on X_t.

Let $B_j = \sum_{i=1}^n a_{ij}A_i$. Then the inequality (6.2) is equivalent to

$$\sum_{j,k} R(B_j, \overline{B_k}, A_k, \overline{A_j}) = \sum_{j,k}\int_X \left(T\left(B_j\overline{A_j}\right)A_k\overline{B_k} + T\left(B_j\overline{B_k}\right)A_k\overline{A_j}\right) dv \geq 0. \tag{6.3}$$

Since $\{A_k\}$ is a basis of the space $H^{0,1}(X, T_X)$ and the matrix $\{a_{ij}\}$ is an arbitrary invertible matrix, we need to show that the inequality (6.3) holds for any two bases $\{A_i\}$ and $\{B_i\}$. Of course we can choose one basis, say $\{A_i\}$, and let the other basis vary freely.

Now we prove the inequality (6.3). Let $\mu = \sum_j B_j \overline{A_j}$. Then the first term in (6.3) is

$$\sum_{j,k} \int_X T\left(B_j \overline{A_j}\right) A_k \overline{B_k}\, dv = \int_X T(\mu)\overline{\mu}\, dv \geq 0.$$

To check the second term, we let $G(z,w)$ be the Green's function of the operator T. Namely, for any function $f \in C^\infty(X)$, we have $T(f) = \int_X G(z,w) f(w)\, dv(w)$. Now we let

$$H(z,w) = \sum_j \overline{A_j}(z) B_j(w).$$

We know the second term of (6.3) is

$$\begin{aligned}\sum_{j,k} \int_X T\left(B_j \overline{B_k}\right) A_k \overline{A_j}\, dv &= \sum_{j,k} \int_X \int_X G(z,w) B_j(w) \overline{B_k}(w) A_k(z) \overline{A_j}(z)\, dv(w) dv(z)\\ &= \int_X \int_X G(z,w) H(z,w) \overline{H}(z,w) dv(w) dv(z) \geq 0\end{aligned}$$

where the last inequality follows from the fact that the Green's function G is non-negative which was proved by Wolpert in [**16**].

The asymptotic of Weil-Petersson , Ricci and perturbed Ricci metrics give us good control of the L^2 cohomology with bundle twist. In [**7**] we showed

Theorem 6.2. — *Let $\mathcal{M}_g$ be the moduli space of genus g curves and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. Then*

$$H^*_{(2)}\left((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\right) \cong H^*(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)).$$

Combining with the dual-Nakano negativity of the Weil-Petersson metric we have

Theorem 6.3. — *The Chern numbers of the log cotangent bundle $T^*_{\overline{\mathcal{M}}_g}(\log D)$ of the moduli spaces of Riemann surfaces are positive.*

More importantly, we proved that the complex structure of the moduli space is infinitesimally rigid:

Theorem 6.4. — *When $q \neq 3g-3$, the L^2 cohomology groups vanish*

$$H^{0,q}_{(2)}\left((\mathcal{M}_g, \omega_\tau), \left(T_{\overline{\mathcal{M}}_g}(-\log D), \omega_{WP}\right)\right) = 0.$$

One of the most important consequence of the curvature properties and goodness of the Ricci, perturbed Ricci and Kähler-Einstein metrics is the Gauss-Bonnet Theorem on $\mathcal{M}_g$. Together with L. Ji, we showed in [**7**]

Theorem 6.5. — *(Liu, Ji, Sun, Yau) The Gauss-Bonnet Theorem hold on the moduli space equipped with the Ricci, perturbed Ricci or Kähler-Einstein metrics:*

$$\int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{\tilde{\tau}}) = \int_{\mathcal{M}_g} c_n(\omega_{KE}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Here $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic of $\mathcal{M}_g$ and $n = 3g-3$.

The computation of the Euler characteristic of the moduli space is due to Zagier.

In the proof of the Gauss-Bonnet Theorem we used the fact that the curvature of the Ricci, perturbed Ricci and Kähler-Einstein metrics are bounded. However, the curvature of the Weil-Petersson metric is not bounded. However, as an application of the Mumford goodness of the Weil-Petersson metric and the Ricci metric we have

Theorem 6.6. — *We have*

$$\chi(T_{\overline{\mathcal{M}}_g}(-\log D)) = \int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{WP}) = \frac{B_{2g}}{4g(g-1)}$$

where $n = 3g-3$.

This theorem gave us the first log Chern number of the DM moduli space $\overline{\mathcal{M}}_g$.

Corollary 6.1. — *We have*

$$\chi(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

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