Astérisque

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Test configuration and geodesic rays

TEST CONFIGURATION AND GEODESIC RAYS

by

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Dedicated to Professor J. P. Bourguignon,
with affection, gratitude and admiration

Abstract. — This paper presents recent research findings on the connection between test configuration and geodesic ray in Kähler metric space. The purpose was to gain insight on the degeneration of Kähler metrics along geodesic rays. A result associating every smooth test configuration a $C^{1,1}$ geodesic ray is proved and exemplified with toric degenerations. Furthermore, we show that the ¥ invariant agrees with Futaki invariant, thus acts as a good substitute in general $C^{1,1}$ geodesic rays without a background test configuration. Based on the assumption of simple test configuration, we extend Donaldson's correspondence between solutions of Monge-Ampère equation and holomorphic discs. Results indicate that Chen and Tian's analysis on Monge-Ampère equation via holomoprhic discs could apply in simple test configuration.

***Résumé* (Configuration de test et rayons géodésiques).** — Cet article présente les dernières découvertes sur la connexion entre la configuration de test et les rayons géodésiques dans les espaces métriques kähleriens. Un résultat qui associe à chaque configuration de test lisse un $C^{1,1}$-rayon géodésique est démontré, et nous fournissons des exemples avec des dégénérations toriques. D'autre part, nous montrons que l'invariant ¥ s'accorde avec celui de Futaki, et forme ainsi un bon substitut dans le cas de $C^{1,1}$-rayons géodésiques généraux sans configuration de test. En nous basant sur l'hypothèse d'une configuration de test simple, nous étendons la correspondance de Donaldson entre les solution de l'équation de Monge-Ampère et les disques holomorphes. Les résultats indiquent que l'analyse de Chen et Tian sur l'équation de Monge-Ampère par le biais des disques holomorphes pourrait s'applique dans les configurations de test simples.

1. Introduction

The purpose of this paper is to explore the connection between geodesic rays in the space of Kähler metrics and test configurations in algebraic manifold [**15**]. This

is a continuation of [9] in some aspects. In [7], the first named author and E. Calabi proved that the space of Kähler potentials is a non-positive curved space in the sense of Alexanderov. As a consequence, they proved that for any given geodesic ray and any given Kähler potential outside of the given ray, there always exists a geodesic ray in the sense of metric distance (L^2 in the Kähler potentials) which initiates from the given Kähler potential and parallel to the initial geodesic ray. The initial geodesic ray, plays the role of prescribing an asymptotic direction for the new geodesic ray out of any other Kähler potential. When the initial geodesic ray is smooth and is tamed by a bounded ambient geometry, the first named author [9] proved the existence of relative $C^{1,1}$ geodesic ray from any initial Kähler potential. (These definitions can be found in Section 2.) Similarly, as remarked in [9], a test configuration should play a similar role. One would like to know if it induces a relative $C^{1,1}$ geodesic ray from any other Kähler potential in the direction of test configuration. In [3], Arezzo and Tian proved a surprising result that for a smooth test configuration with analytic (smooth) central fiber, there always exists a general fiber sufficiently closed to the central fiber, such that there exists a smooth geodesic ray initiated from that fiber metric, and be asymptotically closed to the test configuration (or approximating to some analytic metric in the central fiber). A natural question, motivated by Arezzo-Tian's work, is if there exists a relative geodesic ray from arbitrary initial Kähler metric which also reflects the same geometry (i.e., degenerations) of the underlying test configuration. In section 3, we prove

Theorem 1.1. — *Every smooth test configuration induces a relative $C^{1,1}$ geodesic ray from any Kähler potential in the given class.* [1]

Test configurations can be viewed as algebraic rays, which are geodesics in a finite dimensional subspace(with new metric) of space of Kähler metrics. The geodesic rays induced by a test configuration are the rays parallel to the algebraic ray. They automatically have bounded ambient geometry introduced by the first named author [9].

Theorem 1.2. — *For simple test configuration* [2], *if the induced geodesic ray is smooth regular* [3], *then the generalized Futaki invariant agrees with the ¥ invariant* [4].

In 1982, E. Calabi asked if there always exists an extremal Kähler metric in every Kähler class [5]. This is a very ambitious conjecture which includes his famous conjecture on Kähler Einstein metric (when the first Chern class has a definite sign) as

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[1] Following ideas of [9], the smooth assumption can be reduced to a lower bound of the Riemannian curvature of the total space.

[2] Definition 2.3.

[3] Definition 2.1, it is also equivalent to Definition 6.2 in this case.

[4] The ¥ invariant is defined by the first named author [9].

a special case. It was soon pointed out by Levine [19] that Calabi's conjecture can not hold for general Kähler class. However, it is understood among the experts that, with some modification, Calabi's conjecture might hold for general Kähler manifolds. Unfortunately, it is truely subtle and elusive to search/fromulate a correct statement regarding the existence of constant scalar curvature Kähler (cscK) metrics.

The generalized Futaki invariant or algebraic Futaki invariant is an algebraic notion which relates to the stability of projective manifolds. In the late 1990s, S. T. Yau conjectured that the existence of Kähler Einstein metrics in Fano manifolds is equivalent to some form of Stability of the underlying polarized Kähler class. Even though what stability notion to use is also part of puzzle, this is indeed a fundamental conjecture with respect to Kähler Einstein metrics. According to G. Tian [34] and Donaldson [12], this equivalence relation should be extended to include the case of the constant scalar curvature (cscK) metric in a general Kähler class. In [34] , G. Tian introduced the notion of K-Stability and in the same paper, he proved that the existence of KE metric implies weak K stability. In [13], Donaldson proved that, in algebraic manifold with discrete automorphism group, the existence of cscK metrics implies that the underlying Kähler class is Chow-Stable. In this paper, Donaldson actually formulated a new version (but equivalent) of K-Stability in terms of weights of Hilbert points. In Kähler toric varieties, the existence of cscK metrics implies that the underlying Kähler class is Semi-K stable [15]. Now it is a well-known conjecture that the existence of constant scalar curvature metrics, is equivalent to the K stability of the underlying complex polarization (the so called "Yau-Tian-Donaldson conjecture").

In [9], the first named author used the ¥ invariant to define geodesic stability. Theorem 1.2 states that geodesic stability in the algebraic manifold, is a proper generalization of K stability, at least conceptually. The first named author believes that the existence of KE metrics is equivalent to the geodesic stability introduced in [9]. Note that the geodesic stability introduced in [9] is a mild modification of a similar concept of S. K. Donaldson [12].

The Yau-Tian-Donaldson conjecture is a central problem in Kähler geometry now. Through the hard work of many mathematicians, we now know more about one direction (from existence to stability), cf. Tian [34], Donaldson [16] , Mabuchi [22], Paul-Tian [23], Phong-Sturm [24], Chen-Tian [10]... But on the direction from algebraic stability to existence, few progress has been made though. However, in toric manifolds, there has been special results of Donaldson [15] and Zhou-Zhu [37].

There is a recent intriguing work by V.Apostolov, D.Calderbank, P.Gauduchon and C.W.Tonnesen-Friedman [2]. They constructed an example which is suspected to be

algebraically K stable [5], but admits no extremal Kähler metric. Perhaps one might speculate that, the geodesic stability aforementioned is one of the possible alternatives since it appears to be stronger than K stability and it is a non algebraic notion in nature.

The converse to Theorem 1.1 is widely open. In other words, it is hard to compactify a geodesic ray. The rays induced by any test configuration is very special in many aspects. For instance, generally speaking, the foliation of a smooth geodesic ray is a family of open strips which cover the base punctured disc. However, for the smooth geodesic rays induced from a test configurations, the strips always close up as punctured disc, or we may say that, the orbits are periodic. Unfortunately, having a periodic orbit does not appear to be enough to construct a test configuration. It would be a very intriguing problem to find a sufficient condition so that we can "construct" a test configuration out of a "good" geodesic ray.

Question A. — *Is there a canonical method to construct some test configuration/algebraic ray such that it reflects the same degeneration of a geodesic ray? What is natural geometric conditions on the "good" geodesic ray?*

Our second main result is to establish the correspondence between smooth regular solutions of Homogeneous complex Monge-Ampère equation (HCMA) on simple test configurations and some family of holomorphic discs in an ambient space $\mathcal{W}$ which will be explicitly constructed. We prove, in section 5:

Theorem 1.3. — *There is a one to one correspondence between smooth regular solutions of HCMA on simple test configuration $\mathcal{M}$ and families of holomorphic discs in $\mathcal{W}$ with proper boundary condition.* [6]

Note that in the case of disc, S. K. Donaldson [14] and S. Semmes [30] established first such a correspondence between the regularity of the solution of the HCMA equation and the smoothness of the moduli space of holomorphic discs whose boundary lies in some totally real sub-manifold. The theorem above is a generalization of Donaldson's result. Following this point of view, the regularity of the solution is essentially the same as the smoothness of the moduli space of these holomorphic discs under perturbation. As in [14], we proved the openness of smooth regular solutions in Section 6.

Theorem 1.4. — *Let $\rho(t)$ be a smooth regular geodesic ray induced by a simple test configuration. Then there exists a parallel smooth regular geodesic ray for any initial point sufficiently close to $\rho(0)$ in C^∞ sense.*

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[5] Generalized K stable for extremal Kähler metrics, cf. [32].

[6] In a followup work, we expect to extend this to all smooth test configurations.

An immediate corollary is that the smooth geodesic ray constructed by Arezzo-Tian is open for small deformation of the initial Kähler potential. One may wonder what about the closeness of these solutions? Note that the first named author and Tian [**10**] studied the compactness of these holomorphic discs in the disc setting and we believe that the technique of [**10**] can be extended over here.

In Section 7, as a special case, we explore the geodesic rays induced by toric degenerations [**15**]. In particular, we found plenty of geodesic rays whose regularity is at most $C^{1,1}$ globally. We prove:

Theorem 1.5. — *The geodesic ray induced by a toric degeneration has the initial direction equal to the extremal function in the polytope representation.*

More interestingly, we can write down the geodesic ray explicitly in polytope representation. Thus, the various invariants and energies can be calculated explicitly. This should have general interest since there are very few non-trivial examples of geodesic segments or rays in the literature.

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The first named author has been lecturing on these theorems since spring of 2007. In particular, he lectured in a week long conference on geometric analysis (June 17–22, 2007) held at Luminy, France.

When we are ready to post our paper, the authors noticed Phong-Sturm's work [**27**] which overlaps with our theorem 1.1.

2. Preliminary

2.1. Geodesic rays in Kähler potential space. — Let (M, ω, J) be a compact Kähler manifold of complex dimension n. This means J is an integrable complex structure and the symplectic form ω is compatible with J. In another word, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$, and $g = \omega(\cdot, J\cdot)$ is a metric.

In local complex coordinates $z_\alpha = x_\alpha + iy_\alpha$, denote the metric $g = \omega(\cdot, J\cdot)$ by $g_{\alpha\bar\beta} dz^\alpha \otimes dz^{\bar\beta}$. Then $g_{\alpha\bar\beta}$ is the complexification of the real metric g_{ij}.

By definition, $\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$. Let

$$(1) \qquad \mathcal{H} = \{\phi \in C^\infty(M) : g_{\alpha\bar\beta} + \frac{\partial^2 \phi}{\partial z_\alpha \partial z_{\bar\beta}} > 0\}.$$

It follows from the $\partial\bar\partial$ lemma that $\mathcal{H}$ is the moduli space of all Kähler metrics in the class $[\omega]$.

$\mathcal{H}$ is an infinite dimensional manifold with formal tangent space $T\mathcal{H}_\phi = C^\infty(M)$. T. Mabuchi [21] defined a metric as the following: Let $\phi_1, \phi_2 \in T\mathcal{H}_\phi$.

$$(2) \qquad < \phi_1, \phi_2 >_{\omega_\phi} = \int_M \phi_1 \phi_2 d\mu = \int_M \phi_1 \phi_2 \frac{\omega_\phi^n}{n!} = \int_M \phi_1 \phi_2 \frac{(\omega + i\partial\bar\partial\phi)^n}{n!}.$$

This metric was also defined in S. Semmes [29] and S.K. Donaldson [12]. Under this metric, the geodesic equation for curve $\phi(t) \in \mathcal{H}$ is the following:

$$(3) \qquad \ddot\phi - g_\phi^{\alpha\bar\beta} \dot\phi_\alpha \dot\phi_{\bar\beta} = 0.$$

It is just the Euler-Lagrange equation of the energy $E(\phi(t)) = \int_0^1 \int \dot\phi^2 \frac{\omega_\phi^n}{n!} dt$. According to Semmes [29], the geodesic equation can be transferred into a Complex Monge-Ampère equation: Let $\Sigma = [0,1] \times S^1$, a Riemann surface. Now ϕ is originally defined for $t \in [0,1]$. Extend ϕ to be S^1 invariant function on Σ. Let $z = t + is$ be complex coordinate of Σ, $\{w_\alpha, 1 \le \alpha \le n\}$ be a local coordinates on M. Then the geodesic equation is transformed into

$$(4) \qquad \det \begin{pmatrix} g_{\alpha\bar\beta} + \phi_{\alpha\bar\beta} & \phi_{\alpha\bar z} \\ \phi_{z\bar\beta} & \phi_{z\bar z} \end{pmatrix} = 0.$$

In another word, it is $(\Omega + i\partial\bar\partial\phi)^{n+1} = 0$ on $M \times \Sigma$, where $\Omega = \pi^*\omega$ is the pull back of ω by the projection $\pi : M \times \Sigma \to M$.

A geodesic segment connecting two points ϕ_0 and ϕ_1 is the solution of the following Drichelet boundary value problem.

$$(5) \qquad \det \begin{pmatrix} g_{\alpha\bar\beta} + \phi_{\alpha\bar\beta} & \phi_{\alpha\bar z} \\ \phi_{z\bar\beta} & \phi_{z\bar z} \end{pmatrix} = 0 \text{ on } M \times \Sigma,$$

$$(6) \qquad \phi = \phi_0 \text{ on } M \times 0 \times S^1,$$

$$(7) \qquad \phi = \phi_1 \text{ on } M \times 1 \times S^1.$$

Definition 2.1. — Smooth regular solution: We call ϕ a smooth regular solution (sometimes smooth solution for simplicity) of the Monge-Ampère equation, if ϕ is smooth and if $g_{\alpha\bar\beta} + \phi_{\alpha\bar\beta} > 0$ hold on all fibers.

In [8], The first named author proved the existence of a $C^{1,1}$ solution to above equation. He used the continuity method to solve $\det = \epsilon f$ equation, and proved the following: For every $\epsilon > 0$, there is a unique smooth solution ϕ_ϵ with $|\partial\bar\partial\phi_\epsilon| < C$. The C only depends on the background metric and the manifold. In fact, his proof works for Monge-Ampère equation on general compact complex manifold with boundary. He also proved the uniqueness of the limit when $\epsilon \to 0$. Notice that the uniqueness is expected since $\mathcal{H}$ is negatively curved space. T. Mabuchi [21], S. Semmes [29] and Donaldson [12] showed that $\mathcal{H}$ is negatively curved in formal sense and later,

the first named author and Calabi [**7**] proved it is negatively curved in the sense of Alexanderov.

The regularity beyond $C^{1,1}$ is missing. Our example in section 7 shows a solution with no global C^3 bound. A similar setup [**14**] to the geodesic equation is concerned Monge-Ampère equation on $M \times D$ instead of $M \times (I \times S^1)$. In that setup, Donaldson showed that there exists boundary value such that there is no smooth regular solution. In this direction, a deep analytic result is [**10**]. The first named author and Tian characterized the singularity in detail by analyzing the holomorphic discs associated to a solution.

In the geodesic ray case, the equation holds on $M \times [0, \infty) \times S^1$ instead of $M \times I \times S^1$. By changing variable: $z = e^{-(t+is)}$, the strip $[0, \infty) \times S^1$ goes to a punctured disc. The equation becomes $(\Omega + i\partial\bar\partial\phi)^{n+1} = 0$ on $M \times (D-0)$. The well posed question for geodesic ray is a "starting potential", as well as prescribing an "asymptotic direction." This "asymptotic direction" is usually given by either a known geodesic ray with bounded geometry or a smooth test configuration. In [**9**], we study the existence of geodesic ray with given geodesic ray as "asymptotic direction." Part of the goal of this paper is to established the existence result with respect to test configuration and to explore the relation of geodesic rays with test configurations.

2.2. Test configuration and equivariant embedding. —

Test configuration is defined first by Donaldson [**15**]. He used test configurations to study the relation between stability of projective manifolds and the existence of extremal Kähler metrics. Test configuration is parallel to the notion "special degeneration" introduced by Tian [**34**] earlier. Both notions describe a certain degeneration of Kähler manifolds. As discussed already in [**12**], the geodesic ray represents also degeneration of Kähler metrics. Therefore, it is natural to relate these notions together.

Following Donaldson's definition,

Definition 2.2. — Let $L \to M$ be an ample line bundle over a compact complex manifold. A test configuration $\mathcal{M}$ consists of:

1. a scheme $\mathcal{M}$ with a C^*−action.
2. a C^*−equivariant line bundle $\mathcal{L} \to \mathcal{M}$.
3. a flat C^*−equivariant map $\pi : \mathcal{M} \to C$, where C^* acts on C by multiplication. Any fiber $M_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to M. The pair (L^r, M) is isomorphic to $(\mathcal{L}|_{M_t}, M_t)$ for some $r > 0$, in particular, $(L^r, M) = (L_1, M_1)$.

Test configuration is more explicit in the view of equivariant embedding [**28**]. Without loss of generality, assume $r = 1$. For large k, $\mathcal{L}^k \to \mathcal{M} \to C$ can be embedded into $\mathcal{O}(1) \to P^N \times C \to C$ equi-variantly. It means there is a C^* action on $\mathcal{O}(1) \to P^N \times C \to C$, which restricts to the C^* action of the embedded

$\mathcal{L}^k \to \mathcal{M} \to C$. In fact, the embedding of each fiber M_t is just the Kodaira embedding by the linear system $H^0(M_t, \mathcal{L}^k|_{M_t})$. Moreover, one can make the S^1 action on $\mathcal{O}(1) \to P^N \times C \to C$ unitary.

In the rest of the paper, we always treat test configurations as equi-variantly embedded with $r = 1, k = 1$. Therefore, we work on a subspace of $P^N \times C$. Also, in geodesic ray problem, there is no loss of generality to only look at truncated test configuration $\mathcal{M} \to D$.

At last, we define a special kind of test configuration.

Definition 2.3. — Simple test configuration: A test configuration $\mathcal{M} \subset P^N \times D$ is called simple if the total space is smooth ($\mathcal{M}$ is a smooth sub-manifold of $P^N \times D$) and the projection $\pi : \mathcal{M} \to D$ is submersion everywhere.

By definition, the central fiber of a simple test configuration is automatically smooth.

3. Relative $C^{1,1}$ geodesic ray from smooth test configuration

3.1. Existence. — As mentioned before, test configuration represents some degeneration of a Kähler manifold along a C^* action. Geodesic ray represents a degeneration of Kähler metrics along a punctured disc. So it is natural to relate the truncated test configuration to a geodesic ray. We have the following theorem:

Theorem 3.1. — *A smooth truncated test configuration $\mathcal{M} \to D$ induces a relative $C^{1,1}$ geodesic ray from any given initial point $p \in \mathcal{H}$.*

The existence is a direct application of the first named author's result [8]. The key ingredient of this theorem is the boundary estimate in [8]. For Homogenous complex Monge-Ampère equation, there is an extensive literature in the subject (cf. [4], [18], [35]...).

At present, we assume that the total space of the test configuration is smooth. We expect that these results can be extended to singular test configurations accordingly. For instance, in [9], the first named author took another approach to construct the geodesic ray. Using techniques in [9], the smoothness condition here can be reduced to a uniform lower bound of the Riemannian curvature of the total space.

Proof. — Consider a smooth test configuration over a disc: $(\mathcal{L} \to \mathcal{M} \to D) \hookrightarrow (\mathcal{O}(1) \to P^N \times D \to D)$. Assume the total space is smooth. i.e, $\mathcal{M} \subset P^N \times D$ is smooth. Let Ω be the Fubini-study metric on $P^N \times D$. Actually, it means the pull back of Fubini-study metric on P^N by projection: $P^N \times D \to P^N$.

Now solve the equation

$$(8) \qquad (\Omega + \sqrt{-1}\partial\bar{\partial}\psi)^{n+1} \;=\; 0 \text{ on } \mathcal{M},$$

$$(9) \qquad \psi \;=\; 0 \text{ on } \partial\mathcal{M}.$$

According to [**8**], this equation has a $C^{1,1}$ solution (it is not exactly the same situation as in [**8**], but the techniques are the same). The following shows that: This solution corresponds to a geodesic ray in the Kähler class $c_1(L)$.

The C^* action on $\mathcal{M}$ induces a biholomorphic map $i : (L_1, M_1) \times (D-0) \to (\mathcal{L}, \mathcal{M}) - M_0$. Now i maps $(e, x, z) \in (L_1, M_1) \times (D-0)$ to $z \circ (e, x, 1) \subset (\mathcal{L}, \mathcal{M})$. $z\circ$ is the C^* action of test configuration, and $(e, x, 1) \in (L_1, M_1)$. The map i pulls the equation to

$$(10) \qquad (i^*\Omega + \sqrt{-1}\partial\bar{\partial}i^*\psi)^{n+1} = 0.$$

on $M_1 \times (D-0)$, with boundary condition $i^*\psi = 0$ on $M_1 \times S^1$.

Let $\omega = \Omega|_{M_1}$, and $\pi : M_1 \times (D-0) \to M_1$ be the projection, then

Proposition 3.2. — $i^*\Omega = \pi^*\omega + \sqrt{-1}\partial\bar{\partial}\eta$ *for some smooth function* η.

Proof. — Let h be the Fubini-Study hermitian metric on $\mathcal{O}(1) \to P^N$. So $\Omega = -\sqrt{-1}\partial\bar{\partial}\log h$ and $i^*\Omega = -\sqrt{-1}\partial\bar{\partial}\log i^*h$. Note $\pi^*\omega = -\sqrt{-1}\partial\bar{\partial}\log h_1$. h_1 is the pull back of the hermitian metric on line bundle $L_1 \to M_1$ by trivial projection $\pi : (L_1, M_1) \times (D-0) \to (L_1, M_1)$. So $i^*\Omega = \pi^*\omega + \sqrt{-1}\partial\bar{\partial}\log\frac{h_1}{i^*h}$ and $\eta = \log\frac{h_1}{i^*h}$. $\square$

Proposition 3.3. — $\varphi = \eta + i^*\psi$ *is a geodesic ray.*

Proof. — We have shown $(\pi^*\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0$ on $M \times (D-0)$. It remains to show the S^1 invariance of φ. First, we check the S^1 invariance of η. By assumption, S^1 action on $\mathcal{O}(1) \to P^N \times C$ is unitary. So the h is preserved by S^1 action. This immediately implies that $\eta = \log\frac{h_1}{i^*h}$ is S^1 invariant. Now we check ψ. ψ is S^1 invariant because the boundary condition $\psi = 0$ is S^1 invariant, and the uniqueness of Monge-Ampère solution. In another word, for the unique solution, the S^1 symmetric on the boundary will force the S^1 symmetry in the interior. Now both η and ψ are S^1 invariant, so is φ. $\square$

Back to the proof of the theorem 3.1: At this moment, we have associated a relative $C^{1,1}$ geodesic ray to the test configuration. The ray starts from a fixed point p, because we solved the equation with boundary condition $\psi = 0$. However, for another arbitrary point q, one can go back to the equation 8, solve $\psi = \psi_0$ on $\partial\mathcal{M}$ and obtain the relative $C^{1,1}$ ray from q. ψ_0 is the S^1 extension of the potential difference between q and p. $\square$

In [3], Arezzo and Tian constructed an analytic geodesic ray from a test configuration when the central fiber is analytic. Such test configurations in [3] are simple test configuration (cf. Defi. 2.3). Using the openness Theorem 6.5, we know that there are smooth geodesic rays near the ray they constructed.

When the test configuration is simple (cf. Defi. 2.3), one may expect some better regularity of the induced geodesic ray. Using the correspondence in section 5, the techniques developed by the first named author and Tian in [10] would apply. We expect a similar regularity result here: For any boundary condition $\phi \in C^{k,\alpha}$, there exists nearby perturbation $\phi_\epsilon, |\phi_\epsilon - \phi|_{C^{k,\alpha}} < \epsilon$, such that the HCMA with boundary value ϕ_ϵ has a almost smooth solution [7]. When the test configuration is not simple, bad regularity may appear, maybe due to lack of the correspondence in section 5. For example, in the case of toric degenerations: The total space is smooth when the total polytope is delzant, but the central fiber is never smooth. The geodesic ray is piece wise smooth and has no global C^3 bound. The singularity set on polytope representation has real codimension 1.

Back to the question raised in the introduction: given a geodesic ray, how to construct a test configuration which represents the same degeneration? Donaldson's construction of toric degenerations [15] is very inspiring: He chose piecewise linear functions to approximate an arbitrary direction. A piecewise linear function can lead to a well defined test configuration. In principle, one might view the degenerations represented by a test configuration are dense in all possible geometrical degenerations. Donaldson's construction suggests a way to choose a good approximation, which reflects the same character of degeneration.

3.2. Special cases: geodesic line and Toric variety. — One example of geodesic ray is the geodesic line generated by a holomorphic vector field. Let M be a Kähler manifold with Kähler form ω. Let X be a holomorphic vector field such that: $X = f^{,\alpha} \frac{\partial}{\partial w^\alpha}$ for some real potential f. It is well known that $Im(X)$ is killing vector field. Let $\sigma(t)$ be the flow generated by $Re(X) = \nabla_\omega f$. Then, the 1-parameter family $\omega_{\rho(t)} = \sigma(t)^* \omega$ is a geodesic line, $t \in (-\infty, \infty)$.

Nontrivial example of geodesic rays can be explicitly constructed in toric varieties. For a toric variety, there is an associated polytope. More specifically, there is a biholomorphic map $f : M^\circ = C^n/2\pi i Z^n \to P^\circ \times T^n$. Here M° is an open dense subset of $\mathcal{M}$ where the toric action is free. P is a polytope in R^n satisfying Delzant conditions. Represent a toric-invariant Kähler metric as $\omega|_{M^\circ} = i\partial\bar{\partial}f$, then there is a map f from

$$C^n/2\pi i Z^n \longrightarrow P^\circ \times T^n, \qquad \forall (u,v) \to (x = \frac{\partial f}{\partial u}, y = v).$$

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[7] For definition of almost smooth solution, see the first named author and Tian [10].

Under this map , the Kähler form ω is translated into $dx \wedge dy$. The complex structure is translated into

$$(11) \qquad J = \begin{pmatrix} 0 & G \\ G^{-1} & 0 \end{pmatrix}.$$

where

$$(G_{ij}) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right), \qquad \text{and } g(x) + f(u) = \sum x_i \, u_i, \quad \text{at } x = \frac{\partial f}{\partial u}.$$

In another word, in the symplectic chart, the complex structure has a potential g.

This transformation is really helpful for the geodesic equation. The geodesic equation, in the polytope representation, is linear for complex structure potential $g(t)$. In other words,

$$(12) \qquad \ddot{g}(t) = 0.$$

This immediately implies the existence of smooth geodesics segment connecting any two toric metrics. It is just the linear interpolation of the two end potentials.

4. Connection between algebraic notions and geometric notions

4.1. Algebraic ray and geodesic ray. — Test configurations can be viewed as algebraic rays. The induced geodesic rays are parallel to the algebraic ray.

Definition 4.1. — Two rays $\rho_1(t)$ and $\rho_2(t)$ in the space of Kähler metrics are called parallel if $\rho_1(t) - \rho_2(t)$ is uniformly bounded.

The equality $\varphi = \eta + i^*\psi$ can be interpreted geometrically. η represents the degeneration of the metric from the algebraic C^* action. ψ is the difference between the algebraic ray and the differential geometric ray. Notice that ψ is $C^{1,1}$ bounded. We will elaborate above statement in the following:

Recall that $(L, M) = (L_1, M_1) \hookrightarrow (\mathcal{O}(1), P^N)$ is embedding. The group $GL(N + 1, C)$ acts on $(\mathcal{O}(1), P^N)$. If one looks at the dual bundle of $\mathcal{O}(1)$ (i.e. the universal bundle $\{(e, x) \in C^{N+1} \times P^N : e = \lambda x\}$), the action is simply $A(e, x) = (Ae, Ax), \forall\, A \in GL(N + 1, C)$. The natural dual map between $\mathcal{O}(1)$ and universal bundle passes the action from one to the other.

Consequently, the action acts on the Hermitian metric of $\mathcal{O}(1)$, thus on its curvature. The following lemma shows that this action preserves the positivity of the Hermitian curvature.

Lemma 4.2. — *Let $A \in GL(N + 1, C)$ and h be the Fubini-Study hermitian metric on $\mathcal{O}(1)$. Then, $-i\partial\bar{\partial} \log A^*h > 0$.*

Proof. — It suffices to prove that the action preserves the negativity of curvature on the universal bundle. Under the action A, the metric of $e = (X_0, X_1, ..., X_N) \in \mathcal{O}(-1)$ changes into $||Ae||^2$ from standard Fubini-Study metric $||e||^2$. Notice that the action $A^{-1}UA$ for $U \in U(N+1)$ is transitive on P^N and this action preserves A^*h. Thus, one only need to show the negativity at one point. Let's consider the point $p = A^{-1}(1, 0, ..., 0)^t$, and $e = (X_0, ..., X_{i-1}, 1, X_{i+1}..., X_N)$. At this point p, we have

$$(13) \qquad -\sqrt{-1}\partial\bar{\partial}\log||Ae||^2 = -\sqrt{-1}\sum_{j=1}^{n}\sum_{k,l\neq i} A_{jk}\bar{A}_{jl}dX_k \wedge d\bar{X}_l.$$

To show the positivity, it suffices to show that the null space of the matrix $A_{jk}, j \neq 1, k \neq i$ must be empty. If $v = (\alpha_0, ..., \alpha_{i-1}, \alpha_{i+1}, ...\alpha_N)$ is a null vector, then the vector Av^t must be of form $(c \neq 0, 0, 0, ..., 0)$, because of non-singularity of A. By scaling $c = 1$, A will map two vectors to $(1, 0, ..., 0)$, which is a contradiction. $\square$

As a consequence, the $GL(N+1, C)$ action induces a finite dimensional subspace $\mathcal{H}_N \subset \mathcal{H}$. Note that $\mathcal{H}_N$ consists of those metrics obtained by the $GL(N+1, C)$ action.

The space $\mathcal{H}_N$ is a symmetric space. Its dual is the unitary group $U(N+1)$. Under the natural metric of symmetric spaces, the C^* action (as a 1-parameter family of metrics) is a geodesic ray in $\mathcal{H}_N$. It is interesting to consider the limit of these algebraic rays when one raises the dimension of ambient space P^N (we can raise the power k of $\mathcal{L}^k$ and do Kodaira embedding, then pull the ray back to the class $c_1(L)$ by dividing out the scalar k). First, it is easy to derive that all the embedding induces the same geometric geodesic ray.

Lemma 4.3. — *Different embedding of a test configuration into projective spaces induce the same geodesic ray provided the rays start at the same point.*

Proof. — By different embedding, one essentially raises the power k of $\mathcal{L}^k \to \mathcal{M} \to D$ first. Then, we use sections of $H^0(\mathcal{M}, \mathcal{L}^k)$ to embed $\mathcal{L}^k \to \mathcal{M}$ into $\mathcal{O}(1) \to P^N \times D$. The Fubini-Study metric naturally induces a metric on $\mathcal{L}^k$, which has curvature in class $kc_1(\mathcal{L})$. To get a geodesic ray in the Kähler class $c_1(L)$, one takes the k-th root of the Fubini metric on $\mathcal{L}^k$ to get a Hermitian metric h_k on $\mathcal{L}$. Notice that $\log\frac{h_k}{h_n}$ is the potential difference of the background metric Ω_k and Ω_n. When we solve the Monge-Ampère equation, by uniqueness of the solution, the potential difference $\log\frac{h_k}{h_n}$ goes into the difference between the $C^{1,1}$ solutions ϕ_k and ϕ_n, such that the ray potential $\eta_k + i^*\phi_k = \eta_n + i^*\phi_n$. $\square$

As $k \to \infty$, it is expected that these algebraic rays converge to some geometric geodesic rays. This is a natural extension of the classical problem: Use Bergman

metrics to approximate a given Kähler metric. There is extensive literature on this topic, c.f. Tian [**33**], Zelditch [**36**], Lu [**20**], Phong-Sturm [**25**] and Song [**31**].

4.2. Bounded ambient geometry and test configuration. — In [**9**], the first named author introduced the notion of "bounded ambient geometry" to study geodesic rays. Briefly speaking, a geodesic ray is called to have bounded ambient geometry if the following holds: There exists a metric $\tilde{g}$ on $M \times S^1 \times [0, \infty)$ such that the ray has a $C^{1,1}$ relative potential under $\tilde{g}$, and $\tilde{g}$ has uniformly bounded curvature.

The geodesic ray induced by a smooth test configuration always has bounded ambient geometry. To see this, one restricts the metric $\Omega + idz \wedge d\bar{z}$ to the punctured part $\mathcal{M} - M_0$. Since $\Omega + idz \wedge d\bar{z}$ has bounded geometry on $\mathcal{M}$, the restriction clearly has bounded geometry. The punctured part is holomorphically identified with $M \times S^1 \times [0, \infty)$. Thus the ray has bounded ambient geometry. Actually, it is a stronger version of bounded ambient geometry since the metric $\tilde{g}$ on $M \times S^1 \times [0, \infty)$ can be compactified into a fibration over a disc. In general, this is not necessarily true.

In [**9**], it is proved that: Let $\rho(t)$ be a geodesic ray with bounded ambient geometry, then for any other potential ϕ_0, there is a unique relative $C^{1,1}$ geodesic ray starting from ϕ_0 and parallel to $\rho(t)$. Alternatively, we can use this to derive the existence of geodesic rays, based on the algebraic ray.

4.3. Futaki invariant, ¥ invariant and geodesic stability. — The classical definition of Futaki invariant is the following: Let M be a Kähler manifold with Kähler metric ω. Let X be a holomorphic vector field on M. Let h be the solution of $\Delta h = R - \underline{R}$. Futaki invariant is a linear functional: $\mathcal{F}(X) = \int_M X(h)\omega^n$. The definition is independent with the metric ω chosen in a fixed class. In particular, when $X = f^{,\alpha}\frac{\partial}{\partial w^\alpha}$, $\mathcal{F}(X) = \int_M f^{,\alpha}h_{,\alpha}\omega^n = \int_M f(\underline{R} - R)\omega^n$.

Ding and Tian [**11**] generalized the Futaki invariant to a class of singular varieties. Briefly speaking, they embed the variety into a projective space P^N, and consider the restriction of ambient holomorphic vector fields tangent to the variety on regular points. Also they consider the restriction of ambient Fubini-study metric ω and define Futaki invariant in similar fashion.

In test configuration, Donaldson's algebraic definition of Futaki invariant is: Let $\mathcal{L} \to \mathcal{M} \to D$ be a test configuration. Consider the C^* action on the central fiber $L_0 \to M_0$, and its powers $L_0^k \to M_0$. Let $d_k = \dim H_k = \dim H^0(M_0; L_0^k)$ and w_k be the weight of the C^* action on highest exterior power of H_k. Then $F(k) = w_k/kd_k$ has an expansion

$$(14) \qquad\qquad F(k) = F_0 + F_1 k^{-1} + F_2 k^{-2} + \ldots\ldots$$

The coefficient F_1 is called the Futaki invariant of the C^* action on (L_0, M_0). He proved that if the central fiber is smooth, then the algebraic Futaki invariant agrees with the classical Futaki invariant.

Using Futaki invariant, Donaldson defined stability. A pair (L, M) is K-stable if: For each test configuration for (L, M) (i.e, $(L_1, M_1) = (L, M)$), the Futaki invariant of the C^* action on (L_0, M_0) is less than or equal to zero, and the equality only occurs when the configuration is a product configuration.

This algebraic definition agrees with an early geometric definition of K-stability by Ding and Tian. In [**11**], they used a C^* action of P^N to obtain the limit of the varieties M_t, then studied the Futaki invariant of the limiting variety M_0. The spirit is similar to Donaldson's setup of test configuration.

Notice that in test configuration, the stability is to check the Futaki invariant of the central fiber. However, one would like to have some criterion that doesn't need a specific central fiber. Just as the bounded ambient geometry only concerns behavior before reaching the limit, the ¥ invariant is a nice notion parallel to Futaki invariant and doesn't need a specific central fiber.

Definition 4.4. — [**9**] For a smooth geodesic ray $\rho(t)$, ¥ invariant is defined to be

$$(15) \qquad\qquad ¥ = \lim_{t \to \infty} \frac{dE}{dt} = \lim_{t \to \infty} \int \frac{\partial \rho}{\partial t} (\underline{R} - R) \omega_\rho^n.$$

The K-engery is convex along geodesics. So $\frac{dE}{dt}$ is monotone and the limit exists (either it is positive ∞ or a finite number).

The first named author defined the notion of geodesic stability by ¥ invariant: M is weakly geodesically stable if every geodesic ray has nonnegative ¥ invariant. M is geodesically stable if every geodesic ray has positive ¥ invariant. Conceptually, this is parallel to K-stability for test configurations. However, geodesic rays represent all possible geometrical degenerations. Therefore, it would not be a total surprise if geodesic rays detect some instabilities that test configuration method can't detect.

To clarify this analogy further, we prove the following.

Theorem 4.5. — *For simple test configuration, if the induced geodesic ray is smooth regular, then ¥ invariant agrees with Futaki invariant*[8].

Proof. — By definition of simple test configuration, the central fiber is smooth. Following [**15**], the algebraic Futaki invariant is exactly the classical Futaki-invariant applying to holomorphic vector field (induced by the C^* action) in the central fiber.

Denote the associated HCMA on $\mathcal{M}$ by $(\Omega + i\partial\bar{\partial}\phi)^{n+1} = 0$, ϕ is the smooth regular solution. Let ω_c be the restriction of $\Omega + i\partial\bar{\partial}\phi$ on M_0. The S^1 action of the C^* action

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[8] It is the same up to a sign.

is a Hamiltonian action on M_0. Let f be the hamiltonian potential. In another word, $df = i_v \omega_c$, where v is the S^1 action vector field. The Futaki-invariant of the C^* action is

$$\nu = \int f(\underline{R} - R)\omega_c^n. \tag{16}$$

Now we look at $\yen = \lim_{t\to\infty} \int \frac{\partial \rho}{\partial t}(\underline{R} - R)\omega_\rho^n$.

The C^* action induces a diffeomorphism $i : \mathcal{M} - M_0 \to M \times [0, \infty) \times S^1$. Identify $M \times [0, \infty) \times S^1$ with $\mathcal{M} - M_0$ this way, then

$$\lim_{t\to\infty} i^*\omega_\rho = \omega_c, \qquad \lim_{t\to\infty} i^*R_\rho = R_{\omega_c}. \tag{17}$$

So it suffices to show,

$$\lim_{t\to\infty} i^*\frac{\partial \rho}{\partial t} = -f + \text{const.} \tag{18}$$

The assumption ϕ is smooth regular means $(\Omega + i\partial\bar{\partial}\phi)|_{M_\tau} > 0$ for all fiber $M_\tau \subset \mathcal{M}$. So it induces a smooth foliation F by holomorphic discs on $\mathcal{M}$.[9] Translate into $M \times [0, \infty) \times S^1$, i_*F is a foliation by holomorphic punctured discs. i_*F in turns induces an S^1 action on $M \times [0, \infty) \times S^1$, which is moving along the leaf of i_*F in S^1 direction. By identifying the fiber M_t with $M_{t\theta}$ ($|\theta| = 1$) trivially in $M \times [0, \infty) \times S^1$, the S^1 action is Hamiltonian action with hamiltonian $\frac{\partial \rho}{\partial t}$, under the symplectic form ω_ρ. In $M \times [0, \infty) \times S^1$, notice that the identification between M_t and $M_{t\theta}$ preserves the symplectic form since $\omega_\rho|_{M_t} = \omega_\rho|_{M_{t\theta}}$ for $|\theta| = 1$.

Translate this into the context of $\mathcal{M}$, we have: If we identify the fiber M_t with $M_{t\theta}$ in $\mathcal{M}$, via the S^1 action of the C^* action, then the S^1 action induced by foliation F (on $\mathcal{M}$) is hamiltonian action with hamiltonian $i^*\frac{\partial \rho}{\partial t}$, under symplectic form $i^*\omega_\rho$. Now we take limit towards the central fiber. Under this limit, the central fiber M_0 should be identified with itself via the S^1 rotation of the C^* action. Also, originally, the S^1 action induced by F is trivial on M_0. But, under the identification (which is distorted in M_0), the limit S^1 action should be the reverse of S^1 action of the C^* action on central fiber.

At last, we can take the limit of $i^*\frac{\partial \rho}{\partial t}$. Because the leaf vector on $M \times [0, \infty) \times S^1$ is $\frac{\partial}{\partial t} - g_\rho^{\alpha\bar{\beta}}(\frac{\partial \rho}{\partial t})_{\bar{\beta}}\frac{\partial}{\partial z^\alpha}$ and

$$\frac{\partial^2 \rho}{\partial t^2} - g_\rho^{\alpha\bar{\beta}}\left(\frac{\partial \rho}{\partial t}\right)_{\bar{\beta}}\left(\frac{\partial \rho}{\partial t}\right)_\alpha = 0 \tag{19}$$

So the $\frac{\partial \rho}{\partial t}$ is constant along leaves. Therefore, when passing into $\mathcal{M}$, the $i^*\frac{\partial \rho}{\partial t}$ is constant along leaves of F. But F is foliation of discs and well defined on the central fiber, so $i^*\frac{\partial \rho}{\partial t}$ converges smoothly as moving towards the central fiber in $\mathcal{M}$. The limit

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[9] See 5.2 for foliation induced by smooth regular solution of HCMA.

of the hamiltonian $i^* \frac{\partial \rho}{\partial t}$ is the hamiltonian of the limiting action. So $\lim_{t \to \infty} i^* \frac{\partial \rho}{\partial t} = -f + \text{const.}$ and the theorem is proved. $\qquad \square$

5. Monge-Ampère equation on Simple test configurations

Following Donaldson's idea [**14**], we want to extend the correspondence in [**14**] to the case of Monge-Ampère equation on simple test configurations.

But to explain the background and the motive, we start with a review on Donaldson's result. M is a Kähler manifold with a given Kähler form ω. We solve the equation $(\pi^* \omega + i \partial \bar{\partial} \phi)^{n+1} = 0$ on $M \times D$ with boundary condition $\phi = \phi_0$ on $M \times \partial D$. π is the natural projection to M.

Donaldson and Semmes independently constructed the following manifold $W \to M$. W is glued by local holomorphic cotangent bundle over M. There exists a lifting of M into W for every Kähler metric $\omega + i \partial \bar{\partial} \phi$. If one take the lifting of $M \times D$ into $W \times D$ by the solution $\omega + i \partial \bar{\partial} \phi$, then one will obtain a family of holomorphic discs. These discs are the lifting of the foliation induced by the degenerated form $\pi^* \omega + i \partial \bar{\partial} \phi$. Conversely, if one has the family, then it can induce a solution to Monge-Ampère equation. This correspondence is powerful. It relates the regularity of a solution of HCMA equation to the regularity of moduli space of holomorphic discs in the sense of Fredholm theory.

The construction of Donaldson and Semmes works for a product manifold like $M \times D$. However, a test configuration of real interest is not a product space. So the previous construction would not work here directly. We solve this problem by taking a new point of view on the old construction: View $W \times D$ as a global construction over $M \times D$. Then we can derive an analogy in non-product case. This viewpoint might potentially be generalized to other cases.

5.1. Construction of $\mathcal{W} \to \mathcal{M}$. — Recall a test configuration is simple (Defi. 2.3) if: The total space $\mathcal{M}$ is smooth ($\mathcal{M}$ is a smooth sub-manifold of $P^N \times D$) and the projection $\pi : \mathcal{M} \to D$ is submersion everywhere.

From the definition, any simple test configuration is a fibration over the disc. Each fiber is smooth because $\pi : \mathcal{M} \to D$ is submersion everywhere.

Let $\mathcal{M}$ be a simple test configuration. We solve $(\Omega + i \partial \bar{\partial} \phi)^{n+1} = 0$ on $\mathcal{M}$. Since $\pi : \mathcal{M} \to D$ is submersion everywhere, so $\mathcal{M}$ is locally product space. To see this explicitly in the complex coordinates: First, choose a complex coordinate $\{x_0, ..., x_n\}$ for $U \subset \mathcal{M}$. The projection $z = z(x_0, ..., x_n)$ is holomorphic and $\frac{\partial z}{\partial x_i} \neq 0$ by assumption of submersion. Now one can easily cook up a tuple $\{z, x_{i_1}, ..., x_{i_n}\}$ such that the transition between $\{z, x_{i_1}, ..., x_{i_n}\}$ and $\{x_0, ..., x_n\}$ is non-degenerate. $\{z, x_{i_1}, ..., x_{i_n}\}$ is the product holomorphic coordinate we are looking for.

In the future, such product coordinate is denoted by (z, w) with $z \in D$ and $w \in M_z$. Cover $\mathcal{M}$ with local product charts U_i. On U_i, suppose that the $\Omega = i\partial\bar{\partial}\rho_i$. Write $T^*\mathcal{M}/T^*C$ over U_α by local coordinates (z, w, q). We glue these charts together, and define the transition between (z, w, q) over U_α and (v, x, p) over U_β:

$$
\begin{aligned}
z &= v, \\
w &= w(v, x) \text{ as defined in} \mathcal{M}, \\
q_j &= p_i \frac{\partial x_i}{\partial w_j} + \frac{\partial(\rho_\beta - \rho_\alpha)}{\partial w_j}.
\end{aligned}
$$

(20)

One can verify these local charts (z, w, q) glue up to a complex manifold $\mathcal{W} \to \mathcal{M}$. Define a form Θ on each fiber of $\mathcal{W} \to D$,

$$
\Theta|_{W_t} = dq_i \wedge dw_i. \tag{21}
$$

Here Θ is well defined only on the fiber, so $\Theta|_{W_t}$ is a family of forms.

The real part of Θ is a symplectic form on W_t. So W_t is a symplectic manifold and we can talk about Lagrangian sub-manifolds of W_t.

Definition 5.1. — For a Lagrangian sub-manifold L_t, L_t is called LS-submanifold if $\Theta|_{L_t}$ is non-degenerate. L_t is called LS-graph if it is LS-submanifold and also be a graph over M_t.

By straightforward calculation, one can see: Locally, LS-graphs are of forms $\partial\phi$ for some real potential ϕ on M_t, and $\Theta|_{L_t} = \partial\bar{\partial}\phi$. Our main result in this Section is:

Theorem 5.2. — *Let $\mathcal{M}$ be a simple test configuration. There is an associated manifold $\mathcal{W} \to \mathcal{M}$ such that:*

1. *A smooth solution ϕ of $(\Omega + i\partial\bar{\partial}\phi)^{n+1} = 0, \phi = \phi_0$ on $\partial\mathcal{M}$ induces a family of holomorphic discs $G : M \times D \to \mathcal{M} \to \mathcal{W}$ factoring through the foliation on $\mathcal{M}$, such that the image of $G(\cdot, z)$ is a LS-graph in $W_z \to M_z$ for all z and $\bigcup_{z \in \partial D} G(\cdot, z)$ is a totally real sub-manifold of $\mathcal{W}$.*

2. *If a family of holomorphic discs $G : M \times D \to \mathcal{W}$ respects the projection $\mathcal{W} \to D$, i.e, $\pi \circ G : M \times D \to D$ is a projection to D. Also assume it satisfies the boundary condition $G(\cdot, z) = \Lambda_{z,\phi_0}$ for $z \in \partial D$, where Λ_{z,ϕ_0} is the lifting of M_z by metric $\Omega + i\partial\bar{\partial}\phi_0$, then the image of $G(\cdot, z)$ is a LS-submanifold in W_z for all z. Moreover, if assuming these images are LS-graphs, then the family projects to a foliation of $\mathcal{M}$ and induces a smooth solution ϕ to $(\Omega + i\partial\bar{\partial}\phi)^{n+1} = 0$ with $\phi = \phi_0$ on $\partial\mathcal{M}$.*

Following Donaldson [14], we prove this theorem by discussion from both side of this correspondence in next two subsections.

5.2. One side of the Correspondence. — Now suppose there is a smooth solution ϕ for $(\Omega + i\partial\bar{\partial}\phi)^{n+1} = 0$ on $\mathcal{M}, \phi = \phi_0$ on $\partial\mathcal{M}$, with $\Omega + i\partial\bar{\partial}\phi$ positive on M_t.

In local product coordinates (z, w) of $\mathcal{M}$, write $\Omega + i\partial\bar{\partial}\phi = i\partial\bar{\partial}f$. Since $\Omega + i\partial\bar{\partial}\phi$ has rank n, it has a 1-complex dimension kernel. Let $X = \frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w^\alpha}$ be in kernel of $i\partial\bar{\partial}f$, then

$$(22) \qquad \partial\bar{\partial}f\left(\frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w^\alpha}\right) = (\eta^\alpha f_{\alpha\bar{\beta}} + f_{z\bar{\beta}})\,dw^{\bar{\beta}} + (\eta^\alpha f_{\alpha\bar{z}} + f_{z\bar{z}})\,d\bar{z}$$
$$= 0.$$

Thus,

$$(23) \qquad\qquad\qquad \eta^\alpha = -f_{z\bar{\beta}}f^{\alpha\bar{\beta}},$$

$$(24) \qquad\qquad\qquad f_{z\bar{z}} = -\eta^\alpha f_{\alpha\bar{z}}.$$

A direct calculation shows

$$(25) \qquad [X, \bar{X}] = \left(\frac{\partial\eta^{\bar{\beta}}}{\partial z} + \eta^\alpha \frac{\partial\eta^{\bar{\beta}}}{\partial w^\alpha}\right)\frac{\partial}{\partial w^{\bar{\beta}}} - \left(\frac{\partial\eta^\alpha}{\partial\bar{z}} + \eta^{\bar{\beta}}\frac{\partial\eta^\alpha}{\partial w^{\bar{\beta}}}\right)\frac{\partial}{\partial w^\alpha}$$
$$= 0.$$

This means that the kernel distribution is holomorphically parametrized by $z \in D$. Therefore a smooth regular solution implies a foliation of $\mathcal{M}$ by holomorphic discs.

The $\mathcal{M}$ can be lifted to a graph in $\mathcal{W}$, using the form $\Omega + i\partial\bar{\partial}\phi$. On local product charts U_i, $\Omega = i\partial\bar{\partial}\rho_i$, we can lift $\mathcal{M}$ to graph $\partial(\rho_i + \phi)$ in each fiber. The lift is well defined globally due to the way we glue $\mathcal{W}$.

In [**14**], Donaldson showed in the lifting of $\mathcal{M}$, the foliation is lifted up to a family of holomorphic discs in $\mathcal{W}$. More importantly, these holomorphic discs take boundary value in a totally real sub-manifold Λ_{ϕ_0}. The same technique can be extended to our case.

Theorem 5.3. — *For a simple test configuration, the smooth solution of the HCMA equation induces a foliation of holomorphic discs on $\mathcal{M}$ which can be lifted up to a family of holomorphic discs with in $\mathcal{W}$. These discs have boundary in a totally real sub-manifold.*

Proof. — As above.   □

5.3. The other side of the correspondence. — It is reasonable to consider the reverse correspondence locally. We have the following theorem:

Theorem 5.4. — *Suppose $G : D \times U \to \mathcal{W}$ is a smooth map which respects the projection and holomorphic in D. Assume for all $\tau \in \partial D$, U is mapped to be LS-graph*

and this LS-graph family has a global potential ϕ_0. Then for each $\tau \in D$, G maps U to an immersed LS-submanifold in $\mathcal{W}$. Moreover, if assuming these LS-submanifolds are LS-graphs[10], *then this family induces a smooth solution to the Monge-Ampère equation with boundary condition $\phi = \phi_0$.*

In above theorem, U is an open set of real dimension $2n$. $G : D \times U \to \mathcal{W}$ is smooth and respects the projection. In another word, for $\pi : \mathcal{W} \to D$, $\pi \circ G$ is identity on D. G is holomorphic in D variable. For each $\tau \in \partial D$, U is mapped to be a LS graph over M_τ and this LS-graph family have a global potential ϕ_0. This just means these LS-graphs are lifting of $\mathcal{M}$ using $\Omega + i\partial\bar{\partial}\phi_0$ on the boundary.

Proof. — Consider $G^*\Theta$ on $D \times U$. Θ is well defined on fibers W_t, so $G^*\Theta$ is well defined on fibers U_t in $D \times U$. We should view $G^*\Theta$ as a family of forms on U_t. Denote real coordinates on U by q_j, write $G^*\Theta = (r_{jk} + is_{jk})dq_j \wedge dq_k$. It is straightforward to show $r_{jk} + is_{jk}$ is holomorphic function over D: Let (z, q) be coordinates on $D \times U$. Let (v, x, p) be a local coordinates in $\mathcal{W}$. The map G is $v = z, x = x(z, q), p = p(z, q)$. G is holomorphic, so $\frac{\partial x}{\partial \bar{z}} = 0, \frac{\partial p}{\partial \bar{z}} = 0$. Now $\Theta|_{W_t} = dp_i \wedge dx_i, G^*\Theta|_{U_t} = \frac{\partial p_i}{\partial q_j}\frac{\partial x_i}{\partial q_k} dq_j \wedge dq_k$, therefore $\frac{\partial}{\partial \bar{z}}(r_{jk} + is_{jk}) = \frac{\partial}{\partial \bar{z}}\frac{\partial p_i}{\partial q_j}\frac{\partial x_i}{\partial q_k} = 0$.

On the boundary $\tau \in \partial D$, G maps U to LS-graphs. But Θ is purely imaginary on LS-graphs. Thus, $G^*\Theta$ is also purely imaginary. A holomorphic function on the disc with pure imaginary value on ∂D must be constant, so $r_{ij} + is_{ij}$ must be constant on every disc in $D \times U$. This also implies the Jacobi of the map $G(\tau, \cdot) : U \to W_\tau$ is non-degenerate, since the pull back image $G^*\Theta$ is non-degenerate. It follows that the image $G(\tau, U)$ is an immersed LS-submanifold.

Now assume $G(\tau, U)$ is actually a LS-graph, i.e, the projection $\pi \circ G(\tau, \cdot)$ is diffeomorphism. Following [**10**], we find a global potential for this family of LS-graphs (modulo the local potential of the background metric).

First, consider the case when U is a very small open ball. Let D_α be a small open set in D. Without loss of generality, G maps $D_\alpha \times U$ into a single chart in $\mathcal{W}$. Since they are LS-graphs, one can solve a real potential φ_α for this family in the local product chart by $\frac{\partial \varphi_\alpha}{\partial x_i} = p_i$. φ_α is unique up to a smooth function in $z \in D$.

Choose a finite covering $D_\alpha \subset D$, and make U so small such that $D_\alpha \times U$ all fit in single charts in $\mathcal{W}$. This can be done if one fixes a finite chart covering of $\mathcal{W} \to D$ in first place and then replace U by small subset if necessary. Solve the potential φ_α respectively in each $D_\alpha \times U$, and the geometry of $\mathcal{W}$ implies $\partial(\varphi_\alpha - \rho_\alpha) = \partial(\varphi_\beta - \rho_\beta)$ on every fiber M_t of $\mathcal{M}$. So on each fiber, the difference $(\varphi_\alpha - \rho_\alpha) - (\varphi_\beta - \rho_\beta)$ must be constant. It follows that $\varphi_\alpha - \rho_\alpha$ differ with $\varphi_\beta - \rho_\beta$ by a smooth real function of

---

[10] Thanks to Song Sun, we noticed that the interior LS graphs are exact because the boundary LS graphs are exact. For definition of exact LS graphs, cf, Donaldson [**14**].

z on intersection. The fact $H^1(D, \mathcal{S}) = 0$, ($\mathcal{S}$ is the sheaf of C^∞ functions) implies one can adjust φ_α by function of z such that $\varphi_\alpha - \rho_\alpha = \varphi_\beta - \rho_\beta$. Therefore they give the global potential $\phi = \varphi_\alpha - \rho_\alpha$. ϕ is unique up to a function of z on D.

The next step is to make ϕ satisfy the boundary condition $\phi = \phi_0$. Let $X = \frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w^\alpha}$ be the tangential vector of the foliation $\pi \circ G : D \times U \to \mathcal{M}$. There exists a 1-1 form Ω' on $\mathcal{M}$ such that $i_X \Omega' = 0$ and its restriction to M_t is $i\partial\bar{\partial}\varphi_\alpha|_{M_t} = \Theta|_{L_t}$. Locally, $\Omega' = i(\frac{\partial^2 \varphi}{\partial w_\alpha w_{\bar\beta}} dw_\alpha dw_{\bar\beta} + \zeta^\alpha dw_\alpha d\bar{z} + \zeta^{\bar\beta} dw_{\bar\beta} dz + h dz d\bar{z})$, where $\zeta^\alpha = -\eta^{\bar\beta}\varphi_{\alpha\bar\beta}$ and $h = \eta^\alpha \eta^{\bar\beta}\varphi_{\alpha\bar\beta}$.

Let (v, q) be coordinates on $D \times U$, q as real coordinates. (z, w) are local coordinates on $\mathcal{M}$. We have $\eta^{\bar\beta} = \frac{\partial w^{\bar\beta}}{\partial v}$. Let ρ be local potential for background metric Ω, and $\varphi = \rho + \phi$. The disc family in $\mathcal{W}$ is holomorphic implies $\frac{\partial}{\partial \bar{v}} \frac{\partial \varphi}{\partial w_\alpha} = 0$, therefore

$$(26) \qquad\qquad 0 = \frac{\partial}{\partial \bar{v}} \frac{\partial \varphi}{\partial w_\alpha} = \frac{\partial^2 \varphi}{\partial w_\alpha \partial \bar{z}} + \frac{\partial^2 \varphi}{\partial w_\alpha \partial w_{\bar\beta}} \eta^{\bar\beta}$$

So $\zeta^\alpha = \frac{\partial^2 \varphi}{\partial w_\alpha \partial \bar{z}}$, $\Omega' = i(\partial\bar{\partial}\varphi + (h - \varphi_{z\bar{z}}) dz d\bar{z}) = i(\partial\bar{\partial}(\rho + \phi) + (h - \rho_{z\bar{z}} - \phi_{z\bar{z}}) dz d\bar{z}) = \Omega + i\partial\bar{\partial}\phi + i(h - \rho_{z\bar{z}} - \phi_{z\bar{z}}) dz d\bar{z}$.

On the other hand, Ω' is a closed form. To see this: Let $i : M_t \to \mathcal{M}$ be the embedding of fibers, then $i^* d\Omega' = d(i^*\Omega') = 0$. It suffices to show $i_X d\Omega' = 0$ since the restriction of $d\Omega'$ to the fiber is zero already. Now we show $i_X d\Omega' = L_X \Omega' - d i_X \Omega' = L_X \Omega' = 0$. Notice that Ω' is determined by $\Theta|_{L_t}$ and the condition $i_X \Omega' = 0$. If we can show $\Theta|_{L_t}$ and X are preserved by X-flow, then immediately we obtain $L_X \Omega' = 0$ by uniqueness. The fact $\Theta|_{L_t}$ is preserved follows $G^* \Theta$ is constant along leaves and the fact X is preserved follows $[X, \bar{X}] = 0$. So Ω' is closed form on $\mathcal{M}$, and $i(h - \rho_{z\bar{z}} - \phi_{z\bar{z}}) dz d\bar{z} = \Omega' - \Omega - i\partial\bar{\partial}\phi$ is closed. This implies $(h - \rho_{z\bar{z}} - \phi_{z\bar{z}})$ is just a function of z. Also, since Ω' and Ω and ϕ are globally defined, so $(h - \rho_{z\bar{z}} - \phi_{z\bar{z}}) dz d\bar{z}$ is defined globally and doesn't depend on the local representation. Therefore, the function $h - \rho_{z\bar{z}} - \phi_{z\bar{z}}$ is globally defined, since $dz d\bar{z}$ is defined on the whole disc. (Notice that the z stands for a coordinate in a local product chart, so in different product charts, $\phi_{z\bar{z}}$ is not the same though the function ϕ is the same.)

Now let $H = h - \rho_{z\bar{z}} - \phi_{z\bar{z}}$. H is defined globally on $\pi \circ G(D \times U)$, but solely depends on $z \in D$. One can solve the following equation on disc:

$$(27) \qquad\qquad\qquad \partial_{z\bar{z}} \phi' = H$$

with $\phi' = \phi_0 - \phi$ on the ∂D. Now replace ϕ by $\phi + \phi'$, then one get $\Omega' = \Omega + i\partial\bar{\partial}\phi$ and $\phi = \phi_0$ on ∂D. (Note that in different local charts, (z, w) and (v, x) in $\mathcal{M}$, where z, v project down to the same disc variable. $\partial_{z\bar{z}}\phi' = \partial_{v\bar{v}}\phi'$ since ϕ' is constant fiber-wise.) This finishes the proof of finding potential ϕ if U is sufficiently small.

Now for arbitrary U, one can always decompose it into small open balls U_i which admit potential ϕ_i. Let ρ be a local potential for the Ω on $\mathcal{M}$. Then on the leaf, we have

Lemma 5.5. — *We have $\Delta(\rho + \phi_i) = X\bar{X}(\rho + \phi_i) = 0$.*

Proof. — Let $f = \rho + \phi_i$,

$$(28) \qquad X\bar{X}f = X(\eta^{\bar{\beta}})f_{\bar{\beta}} + \partial\bar{\partial}f(X, \bar{X})$$
$$= 0. \qquad \qquad \square$$

This implies $\Delta(\phi_i - \phi_j) = 0$ on the leaf. Now with the extra condition $\phi_i = \phi_j = \phi_0$ on the ∂D, it implies $\phi_i = \phi_j$ on the intersection. The global potential is immediately obtained from this. $\qquad \square$

Remark 5.6. — The above correspondence is constructed only on simple test configurations. In these configurations, central fiber are smooth. However, we believe the techniques should work for some mild singularities in the central fiber.

Another point is that the correspondence has nothing to do with the C^* action.

6. Openness of super regular solution

In simple test configurations, we can study regularity of the solution ϕ by the associated holomorphic disc family in $\mathcal{W} \to \mathcal{M}$. [11] Donaldson's definition [**14**] of super regular discs and the linearized model could be extended to our case as well. In detail,

Definition 6.1. — In the moduli map $G : D \times U \to \mathcal{W}$, a disc $G(D, x)$ is called super regular at $z \in D$ if $d(\pi \circ G_z)_x : TU \to TM$ is isomorphism. A disc $G(D, x)$ is called super regular if it is super regular at every $z \in D$.

Definition 6.2. — A geodesic ray induced from a simple test configuration is called super regular if the disc family in $\mathcal{W}$ is super regular. [12].

For a disc $G_x = G(\cdot, x)$ in the moduli map $G : D \times U \to \mathcal{W}$, one can consider the holomorphic perturbation of G_x that satisfies the totally real boundary condition (the boundary is in the Λ_ϕ, i.e., the lifting of $M_t, t \in \partial D$ by $\Omega + i\partial\bar{\partial}\phi$). Also, we normalize the perturbation such that it preserves the projection property. In another word, $\pi \circ G : D \times U \to D$ is identity on D variable. Following Donaldson [**14**], the linearized problem is

---

[11] However, the existence so far only requires smoothness of total space.
[12] I.e.: the solution is smooth regular to the Monge-Ampère equation on the test configuration $\mathcal{M}$.

Theorem 6.3. — *In the moduli map $G : D \times U \to \mathcal{W}$ corresponding to a smooth solution ϕ, the linearized perturbation equation for a disc $G(\cdot, x)$ is*

$$(29) \qquad\qquad\qquad v \;=\; Su + A\bar{u} \quad \textbf{on } \partial D,$$

$$(30) \qquad\qquad\qquad \bar{\partial} u \;=\; 0,$$

$$(31) \qquad\qquad\qquad \bar{\partial} v \;=\; 0,$$

where S and A are maps from ∂D to complex symmetric matrices and positive hermitian matrices respectively; while u, v are C^n valued functions on D.

Proof. — The idea is the same to Donaldson [14]: Trivialize the exact sequence
$$0 \to (\pi \circ G_x)^*(T^*\mathcal{M}) \to G_x^*(T\mathcal{W}) \to (\pi \circ G_x)^*(T\mathcal{M}) \to 0. \qquad \square$$

In [14], it is showed that the problem is Fredholm and the index is $2n$. Consequently, if the disc is regular in Fredholm sense, then $G : D \times U \to \mathcal{W}$ is indeed an open set in the universal moduli space.

Regarding on the criterion of regularity for a disc, a mild modification of Donaldson's argument leads to the following:

Theorem 6.4. — *If a disc is super regular at any point $p \in \partial D$, then the disc is regular.*

Proof. — We look at the linearized model since the general case can be reduced to this simple model.

First, define $\Omega(s_1, s_2) = u_1^t v_2 - u_2^t v_1$. This is a symplectic form for $s = (u, v) \in C^{2n}$. In particular, for $s_1, s_2 \in \ker \bar{\partial}_{S,A}$, $i\Omega(s_1(\tau), s_2(\tau))$ is real and independent of τ. To see this, just notice that $i\Omega(s_1, s_2)$ is holomorphic function and on ∂D, $i\Omega(s_1, s_2) = i[u_1^t(Su_2 + A\bar{u}_2) - u_2^t(Su_1 + A\bar{u}_1)] = i(u_1^t A\bar{u}_2 - u_2^t A\bar{u}_1)$ is real.

The super regularity at $p \in \partial D$ means there are $2n$ elements $s_j = (u_j, v_j) \in \ker \bar{\partial}_{S,A}$ such that $u_j(p)$ form a R-basis for C^n. By continuity, it implies $u_j(\tau)$ form a R-basis for C^n in a neighborhood $\tau \in U_p$.

We claim $s_i(\tau)$ are generically C-linearly independent. It is equivalent to claim $\det[s_j]_{1 \leq j \leq 2n}$ has discrete zeros. Notice $\det$ is holomorphic, so the zeros are either discrete or the whole disc. Suppose it is the whole disc for contradiction. In the neighborhood U_p, assume the maximal rank of $[s_j]_{1 \leq j \leq 2n}$ for $\tau \in U_p$ is achieved at p without loss of generality, and the rank is $k < 2n$. Assume $s_1, s_2, ..., s_k$ form a basis for $span\{s_i\}$ at p, then near p, $s_{k+1} = \sum \lambda_i s_i, 1 \leq i \leq k$. λ_i is holomorphic, since it satisfies $\sum \lambda_i s_i^t s_j = s_{k+1}^t s_j, 1 \leq i, j \leq k$. In another word, it is obtained by solving the holomorphic matrix equation $\lambda[s_i^t s_j] = s_{k+1}^t s_j$. Now one finds holomorphic functions $\lambda_1, ..., \lambda_k, \lambda_{k+1} = -1, \lambda_{k+2} = 0, ..., \lambda_{2n} = 0$ near p, such that $\sum \lambda_i s_i = 0$. On the boundary ∂D near p,

$$(32) \qquad 0 = \sum \lambda_j v_j = S(\sum \lambda_j u_j) + A(\sum \lambda_j \bar{u}_j) = A(\sum \lambda_j \bar{u}_j).$$

So $\sum \lambda_j \bar{u}_j = 0$ and we also have $\sum \lambda_j u_j = 0$, so

$$(33) \qquad \sum \Im(\lambda_j) u_j = 0 = \sum \Re(\lambda_j) u_j.$$

Since u_j form R-basis near p, one has $\lambda_j = 0$ on ∂D near p, which contradicts the choice of λ_j. Therefore, the $\det[s_j]_{1 \leq j \leq 2n}$ has discrete zero.

Now suppose the $\ker \bar{\partial}_{S,A}$ has dimension strictly greater than $2n$. Then one can choose s_0 not in $span\{s_i\}, 1 \leq i \leq 2n$. Now in the $2n + 1$ dimensional vector space $span\{s_i\}$, $i\Omega$ as a skew form, must be singular. So there is a vector $s \in span\{s_0, ..., s_{2n}\}$ such that $i\Omega(s, span\{s_1, ..., s_{2n}\}) = 0$. Notice we proved $s_1, ..., s_{2n}$ form a C-basis generically, this implies $s = 0$ generically on D. Thus it implies $s = 0$, contradiction. $\qquad \square$

In particular, since the holomorphic discs associated to smooth solution ϕ are automatically super regular, above theorem proves that they are all regular and the moduli space M in the map $G : D \times M \to \mathcal{W}$ is a compact connected component of the universal moduli space. It readily implies the following theorem.

Theorem 6.5. — *Openness: If the equation $(\Omega + i\partial\bar{\partial}\phi)^{n+1} = 0, \phi = \phi_0$ on $\partial\mathcal{M}$ admits a smooth solution ϕ with $\Omega + i\partial\bar{\partial}\phi > 0$ on fibers, then for any small perturbation $\delta\phi_0 \in C^\infty(\partial\mathcal{M})$, the new boundary value problem still has smooth solution ϕ' which is close to ϕ in $C^\infty(\mathcal{M})$ and $(\Omega + i\partial\bar{\partial}\phi') > 0$ on fibers.*

Proof. — We refer the proof to [**14**], which essentially asserts that compact families of regular normalized discs are stable under small perturbations. $\qquad \square$

7. Geodesic ray from Toric degenerations

7.1. Basics of Toric degeneration. — For toric varieties, there has been extensive literature in extremal metrics. Abreu [**1**] initiated to study complex geometry on toric variety by symplectic coordinates. Afterwards, there has been much work in extremal metrics on toric variety, c.f. Donaldson [**15**], Zhou-Zhu [**37**], Gabor [**32**]. For completeness, we describe Donaldson's construction of Toric degenerations [**15**] in the following:

Let $P \subset R^n$ be a polytope associated to a toric variety M. For simplicity, let us assume P is Delzant. Let f be a rational piecewise linear function on M. One can associate it with a polytope $\hat{P} = \{(x, y) : x \in P, 0 \leq y \leq K - f\} \subset R^{n+1}$, $K = \max f$. By re-scaling $\hat{P}$, we can assume $\hat{P}$ is integral. In other words, all vertices of $\hat{P}$ are integral points.

It is a classical fact that $\hat{P}$ as above induces a toric variety $\mathcal{M}$ with a positive line bundle $\mathcal{L}$. Each integral point p in $\hat{P}$ corresponds to a section s_p of $\mathcal{L} \to \mathcal{M}$. The

correspondence is compatible with addition of integral points in $\hat{M}$ and multiplication of sections in $\mathcal{L}$. In other words, if $p_1 + p_2 = p_3 + p_4$, then $s_{p_1} s_{p_2} = s_{p_3} s_{p_4}$.

One can view $\mathcal{M}$ as a sub-variety in P^N by Kodaira embedding: $x \in \mathcal{M}, x \to [s_1(x) : s_2(x) : ... : s_i(x)...]$. i runs through the integral points of $\hat{P}$. So $\mathcal{M} \subset P^N$ is defined by homogeneous equations $F(X_i) = 0$. These equations are induced by the relations of s_i, or equivalently, by the relations of the integral points in $\hat{P}$.

There is a map $\pi : \mathcal{M} \to P^1$, defined by $\pi : x \to [s_p(x) : s_q(x)]$. $p = (t_1, ..., t_n, t_{n+1}), q = (t_1, ..., t_n, t_{n+1} + 1) \in \hat{P}$. Also, there is a natural C^* action on $\mathcal{M}$ from the torus $T^{n+1} = T^n \times C^*$. It transforms section s_p to $t^k s_p$. k is the height of p. i.e, $p = (t_1, ..., t_n, k)$. So the C^* action can be lifted to $\pi : \mathcal{M} \to P^1$ by defining $t \circ [x : y] = [x : ty]$ on P^1.

The toric degeneration is just $\mathcal{M} - \pi^{-1}([1 : 0])$. The following example shows the construction in detail.

Example. — Let $P = [0, 2] \in R$ be the base polytope. $f = \max\{0, x - 1\}$ is the piece wise linear function on P. $\hat{P} = ([0, 1] \times [0, 1]) \bigcup \{1 \le x \le 2, x + y \le 2\}$. Denote the integral points $X = (0, 0), Y = (1, 0), Z = (2, 0), U = (0, 1), V = (1, 1)$. Then the toric degenerations is the sub-variety in P^4 defined by

$$(34) \qquad\qquad XZ = Y^2, XV = UY$$

The C^* action on $\mathcal{M}$ is $t : [X : Y : Z : U : V] \to [X : Y : Z : tU : tV]$. Notice that in order to get nontrivial test configuration, we only consider the part $\mathcal{M} - \pi^{-1}([1 : 0])$. In another word, we consider the asymptotic direction when $t \to \infty$ on C^*.

The central fiber is defined by $[Y : V] = [0 : 1]$. It is the toric variety associated to the segment $y = 1, x \in [0, 1]$ and $x \in [1, 2], x + y = 2$. Geometrically, the central fiber is the union of two P^1 which intersect at one point. Notice that the ambient space $\mathcal{M}$ is smooth here, so the induced geodesic ray has ambient bounded geometry automatically.

7.2. Explicit calculation of the $C^{1,1}$ geodesic ray. — We calculate the induced geodesic ray of previous example. The idea is to first calculate the geodesic segment connecting the fiber at $[1 : 1]$ to the fiber at $[1 : e^t], t \in R \times S^1$. Then, taking the limit of these segments when $t \to \infty$, we obtain a geodesic ray.

Equipped with the natural background metric of P^4, the fiber at $w = [1 : e^t] \in P^1$ has metric potential $\frac{1}{2} \log(|X|^2 + |Y|^2 + |Z|^2 + |U|^2 + |V|^2)$. Pulling this metric to the fixed fiber M at $w = [1 : 1] \in P^1$, the potential becomes

$$(35) \qquad\qquad \frac{1}{2} \log(|X|^2 + |Y|^2 + |Z|^2 + e^{2t}|U|^2 + e^{2t}|V|^2).$$

Since the fiber M is at $[1 : 1]$, so $Y = V, X = U$. After proper normalization, the potential is

$$(36) \qquad \frac{1}{2} \log(|X|^2 + |Y|^2 + (e^{2t} + 1)^{-1}|Z|^2).$$

Now we calculate the geodesic segment connecting these two metrics.

Choose $[A, B]$ as standard P^1 coordinate on M. Thus, $X = B^2, Y = AB, Z = A^2$. Using $C^* = R \times S^1$ coordinate of P^1, $A = e^y, B = 1, y \in R \times S^1$. The Kähler potential is

$$(37) \qquad h_{0,t} = \frac{1}{2} \log(1 + e^{2y} + e^{4y}(e^{2t} + 1)^{-1}).$$

One can verify that the Legendre transform of $h_{0,t}$ maps R to $(0, 2)$ for each fixed t.

Notice that in polytope representation, the geodesic is just a straight line of convex functions. Now by straightforward calculation, one just computes the two end points associated to the two metrics in polytope representation and then take the linear interpolation. Passing to limit, one gets the $C^{1,1}$ ray in polytope representation

$$(38) \qquad u_t = u_0 + t \max(0, x - 1) \ , \ t \in [0, \infty).$$

In the standard picture of $M \times [0, \infty)$, we transform the u_t by Legendre transform and get the potential

$$(39) \qquad h_t(y) = \begin{cases} h_0(y), & \text{when } y < \frac{\log 2}{4}, \\ h_0(\frac{\log 2}{4}) + y - \frac{\log 2}{4}, & \text{when } \frac{\log 2}{4} < y < \frac{\log 2}{4} + t, \\ h_0(y - t) + t, & \text{when } \frac{\log 2}{4} + t < y. \end{cases}$$

One can verify that $h_t - h_{0,t}$ is uniformly bounded. This confirms that the geometric ray is parallel to the algebraic ray.

It is natural to extend this observation to general toric degenerations.

Theorem 7.1. — *Let $\mathcal{M}$ be a toric degeneration with extremal piece wise linear function f. Suppose the ambient polytope $\hat{P}$ is integral and the base P is delzant. Then the induced geodesic ray is $u = u_0 + tf$ in polytope representation.*

Proof. — Similar to the previous set up, we calculate the geodesic segment connecting the fiber at $[1, 1]$ to the fiber at $[1, e^t]$. Then we pass the directions of these geodesic segments to the limit as $t \to \infty$.

Under the $(C^*)^n$ coordinates of $M = M_{[1:1]}$, the projective coordinates can be represented by $[... : \exp(\sum_1^n X_i y_i) : ...]$. Let $(X_1, ..., X_n, X_{n+1}) = p$ be coordinates of those integral points in $\hat{P}$. Therefore, after proper normalization, the metric potential of the algebraic ray is:

$$(40) \qquad h_{0,t} = \frac{1}{2} \log \left(\sum_{p \in \hat{P}} \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i y_i) \right).$$

$K = \max f$, $p = (X_1, ..., X_{n+1}) \in \hat{P}$ are integral points.

Let $x = (x_1, ..., x_n) \in P$. Assume the extremal function $f = c_k x_k + d$ near x. i.e, we consider x in interior of a single definition domain of f. Under the Legendre transform of $h_{0,t}$, the pre-image $\tilde{y}$ of x satisfies

$$(41) \qquad \frac{\sum_{p \in \hat{P}} X_j \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i)}{\sum_{p \in \hat{P}} \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i)} = x_j.$$

In particular, we denote the pre-image of x at time $t = 0$ by y.

By the Legendre transform, the potential u_t in polytope representation is:

$$(42) \qquad u_t(x) = x\tilde{y} - h_{0,t}.$$

So, the limit direction is:

$$(43) \qquad \lim \frac{u_t - u_0}{t} = \lim_{t \to \infty} \frac{\sum x_k(\tilde{y}_k - y_k) - \frac{1}{2} \log \dfrac{\sum_{p \in \hat{P}} \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i)}{\sum_{p \in \hat{P}} \exp 2(\sum_1^n X_i y_i)}}{t}.$$

If we can prove $\lim \frac{\tilde{y}_k - y_k}{t} = c_k$ and $\lim \dfrac{-\frac{1}{2} \log \dfrac{\sum_{p \in \hat{P}} \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i)}{\sum_{p \in \hat{P}} \exp 2(\sum_1^n X_i y_i)}}{t} = d$, then the theorem is proved. Now, we prove that the second is an implication of the first. Assuming $\lim \frac{\tilde{y}_k - y_k}{t} = c_k$, i.e, $\tilde{y}_k - y_k = c_k t + \epsilon_k t$ where $\epsilon_k \to 0$ as $t \to \infty$. We have the following

$$(44) \qquad -Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i = \left(-K + X_{n+1} + \sum_1^n c_i X_i + d \right) t$$

$$(45) \qquad + \left(-d + \sum_1^n \epsilon_i X_i \right) t + \sum_1^n X_i y_i.$$

For integral points p in the area where $f = c_k x_k + d$, the $L(p) = -K + X_{n+1} + \sum_1^n c_i X_i + d = X_{n+1} - h(X) \leq 0$, where $h(X) = h(X_1, ..., X_n)$ is the height of $\hat{P}$ over the base point $(X_1, ..., X_n)$. i.e, $h(X) = K - f(X)$. For integral points p not in the area where $f = c_k x_k + d$, by definition of $f = \max(f_1, f_2, .., f_l)$ (f_l are linear functions), it is clear that $L(p) < -\delta$ for a fixed $\delta > 0$ Therefore,

$$(46) \qquad \sum_{p \in \hat{P}} \exp 2 \left(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i \right)$$

$$(47) = \quad \exp(-2dt) \left(\sum_{p \in A} \exp 2(-\delta_p t) \exp \left(2 \sum_1^n X_i y_i \right) + \sum_{p \in B} \exp \left(2 \sum_1^n X_i y_i \right) \right)$$

B contains integral points p in $\hat{P}$ such that their projection $(X_1, ..., X_n)$ are in the area where $f = c_k x_k + d$ and $X_{n+1} = h(X)$. A contains the rest integral points in $\hat{P}$ but not in B. The condition $\hat{P}$ is integral guarantees that B is not empty.

Now we can calculate

$$
(48) \qquad \lim_{t \to \infty} \frac{-\frac{1}{2} \log \frac{\sum_{p \in \hat{P}} \exp 2(-Kt + X_{n+1}t + \sum_1^n X_i \tilde{y}_i)}{\sum_{p \in \hat{P}} \exp 2(\sum_1^n X_i y_i)}}{t}
$$

$$
(49) \qquad = \quad d + \lim_{t \to \infty} \frac{-\frac{1}{2} \log \frac{\sum_A \exp 2(-\delta_p t) \exp(2 \sum_1^n X_i y_i) + \sum_B \exp(2 \sum_1^n X_i y_i)}{\sum_{p \in \hat{P}} \exp 2(\sum_1^n X_i y_i)}}{t}
$$

$$
(50) \qquad = \quad d.
$$

So it remains to prove $\lim_{t \to \infty} \frac{\tilde{y}_k - y_k}{t} = c_k$.

Let $y'_k = \tilde{y}_k - c_k t$, our purpose next is to prove y' is uniformly bounded for t sufficiently large. The equation 41 can be rewritten as:

$$
(51) \qquad x_k = \frac{\sum_B X_k \exp(2 \sum X_i y'_i) + \sum_A X_k \exp(2 \sum X_i y'_i) \exp(-\delta_p t)}{\sum_B \exp(2 \sum X_i y'_i) + \sum_A \exp(2 \sum X_i y'_i) \exp(-\delta_p t)}.
$$

Define a map $\phi : y' \to x$ by $x_k = \frac{\sum_B X_k \exp(2 \sum X_i y'_i)}{\sum_B \exp(2 \sum X_i y'_i)}$. Let $P' \subset P$ be the polytope where $f = c_k x_k + d$. We need the following lemma:

Lemma 7.2. — $\phi : R^n \to P'$ *is a diffeomorphism from R^n to the interior of P'*

Proof. — The lemma is a special case of a more general fact: Let $S = \{p_1, ..., p_m\}$ be a set of arbitrary points in R^n. $p_i = (X_1^i, ..., X_n^i)$. If the convex hull P spanned by S has dimension n, then the map:

$$
(52) \qquad \phi : (y_1, ..., y_n) \to (x_1, ..., x_n), x_k = \frac{\sum_S X_k^i \exp(2 \sum X_j^i y_j)}{\sum_S \exp(2 \sum X_j^i y_j)}
$$

is a diffeomorphism from R^n to the interior of P.

Notice that B projects to be a grid G on P'. G contains all the vertices of P' due to the integral condition of $\hat{P}$. P' is convex since f is convex. So P' is the convex hull spanned by G. Therefore, the above fact applies exactly. $\square$

Now, using this lemma, we can prove $\lim_{t \to \infty} \frac{\tilde{y}_k - y_k}{t} = c_k$: Choose a small closed ball $B_p \subset P'$ near $p = (x_1, ..., x_n)$. The pre-image $\phi^{-1}(B_p)$ is bounded closed set in R^n. Now consider the family of maps $\phi_t : y' \to x$ defined by equation 51. Notice that each ϕ_t is a diffeomorphism since equation 51 is just another form of equation 41, which defines the standard identification between R^n and P.

Since $\phi^{-1}(B_p)$ is bounded, it is straightforward from the equation 51 that: For any $\epsilon > 0$, there exists T such that $|\phi_t(y) - \phi(y)| < \epsilon$ for $y \in \phi^{-1}(B_p)$ and $t > T$. Thus the image $\phi_t(\phi^{-1}(B_p))$ is a ball close to B_p and contains p for t sufficiently large.

So above argument proves: For any B_p contains p and lies in interior of P', there exists $T > 0$ such that $y' = \phi_t^{-1}(p) \in \phi^{-1}(B_p)$ for $t > T$. Since $\phi^{-1}(B_p)$ is bounded,

$$
(53) \qquad \lim_{t \to \infty} \frac{\tilde{y}_k - y_k}{t} = c_k + \lim_{t \to \infty} \frac{y'_k - y_k}{t} = c_k. \qquad \square
$$

These geodesic rays show some bad regularity. In general, they behave like the following: First, they break the manifold M into several pieces. As time evolves, they will tear these pieces apart, but keep metrics on each part. The space between the teared parts has degenerated metrics and zero volume. In particular, one can verify that the 2nd derivative of these rays are piece wise smooth function on fibers. At the broken points, these 2nd derivatives have jumps, so there is no global C^3 bound for the relative geodesic ray potential.

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