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GLOBAL APPLICATIONS OF RELATIVE (φ, Γ)-MODULES I

by

Fabrizio Andreatta & Adrian Iovita

Abstract. — In this paper, given a smooth proper scheme X over a p-adic DVR and a p-power torsion étale local system $\mathbb{L}$ on it, we study a family of sheaves associated to the cohomology of local, relative (φ, Γ)-modules of $\mathbb{L}$ and their cohomology. As applications we derive descriptions of the étale cohomology groups on the geometric generic fiber of X with values in $\mathbb{L}$, as well as of their classical (φ, Γ)-modules, in terms of cohomology of the above mentioned sheaves.

Résumé **(Applications globales des (φ, Γ)-modules relatifs I).** — Étant donné un schéma propre et lisse X défini sur un anneau de valuation discrète et un système local $\mathbb{L}$, étale, de torsion sur X on étudie une famille de faisceaux associés à la cohomologie des (φ, Γ)-modules locaux relatifs de $\mathbb{L}$ et leur cohomologie. Comme application on déduit une description des groupes de cohomologie étales sur la fibre générique géométrique de X à valeurs dans $\mathbb{L}$, et de leurs (φ, Γ)-modules classiques en termes de la cohomologie des faisceaux mentionnés plus haut.

1. Introduction

Let p be a prime integer, K a finite extension of $\mathbf{Q}_p$ and V its ring of integers. In [**15**], J.-M. Fontaine introduced the notion of (φ, Γ)-modules designed to classify p-adic representations of the absolute Galois group G_V of K in terms of semi-linear data. More precisely, if T is a p-adic representation of G_V, i.e. T is a finitely generated $\mathbf{Z}_p$-module (respectively a $\mathbf{Q}_p$-vector space of finite dimension) with a continuous action of G_V, one associates to it a (φ, Γ)-module, denoted $\mathrm{D}_V(T)$. This is a finitely generated module over a local ring of dimension two $\mathbf{A}_V$ (respectively a finitely generated free module over $\mathbf{B}_V := \mathbf{A}_V \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$) endowed with a semi-linear Frobenius endomorphism φ and a commuting, continuous, semi-linear action of the group $\Gamma_V := \mathrm{Gal}(K(\mu_{p^\infty})/K)$ such that $(\mathrm{D}_V(T), \varphi)$ is étale. This construction makes

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the group whose representations we wish to study simpler with the drawback of making the coefficients more complicated. It could be seen as a weak arithmetic analogue of the Riemann-Hilbert correspondence between representations of the fundamental group of a complex manifold and vector bundles with integrable connections. The main point of this construction is that one may recover T with its G_V-action directly from $\mathrm{D}_V(T)$ and, therefore, all the invariants which can be constructed from T can be described, more or less explicitly, in terms of $\mathrm{D}_V(T)$. For example

(*) one can express in terms of $\mathrm{D}_V(T)$ the Galois cohomology groups $\mathrm{H}^i(K,T) = \mathrm{H}^i(\mathrm{G}_V,T)$ of T.

More precisely, let us choose a topological generator γ of Γ_V and consider the complex

$$\mathscr{C}^\bullet(T) : \mathrm{D}_V(T) \xrightarrow{d_0} \mathrm{D}_V(T) \oplus \mathrm{D}_V(T) \xrightarrow{d_1} \mathrm{D}_V(T)$$

where $d_0(x) = \big((1-\varphi)(x), (1-\gamma)(x)\big)$ and $d_1(a,b) = (1-\gamma)(a) - (1-\varphi)(b)$. It is proven in [**18**] that for each $i \geq 0$ there is a natural, functorial isomorphism

$$\mathrm{H}^i(\mathscr{C}^\bullet(T)) \cong \mathrm{H}^i(\mathrm{G}_V, T).$$

Moreover, for $i=1$ this isomorphism was made explicit in [**9**]: let (x,y) be a 1-cocycle for the complex $\mathscr{C}^\bullet(T)$ and choose $b \in \mathbf{A} \otimes_{\mathbf{Z}_p} T$ such that $(\varphi - 1)(b) = x$. Define the map $C_{(x,y)} \colon \mathrm{G}_V \longrightarrow \mathbf{A} \otimes_{\mathbf{Z}_p} T$ by

$$C_{(x,y)}(\sigma) = (\sigma' - 1)/(\gamma - 1)y - (\sigma - 1)b,$$

where σ' is the image of σ in Γ_V. One can prove that the image of $C_{(x,y)}$ is in fact contained in T, that $C_{(x,y)}$ is a 1-cocycle whose cohomology class $[C_{(x,y)}] \in \mathrm{H}^1(\mathrm{G}_V,T)$ only depends on the cohomology class $[(x,y)] \in \mathrm{H}^1(\mathscr{C}^\bullet(T))$. Moreover, the isomorphism $\mathrm{H}^1(\mathscr{C}^\bullet(T)) \cong \mathrm{H}^1(\mathrm{G}_V,T)$ above is then defined by $[(x,y)] \mapsto [C_{(x,y)}]$.

As a consequence of (*) we have explicit descriptions of the exponential map of Perrin-Riou (or more precisely its "inverse" (see [**15**], [**7**], [**9**]) and an explicit relationship with the "other world" of Fontaine's modules: $\mathrm{D}_{\mathrm{dR}}(T), \mathrm{D}_{\mathrm{st}}(T), \mathrm{D}_{\mathrm{cris}}(T)$ (see [**9**], [**5**]).

Despite being a very useful tool, in fact the only one which allows the general classification of integral and torsion p-adic representations of G_V, the (φ,Γ)-modules have an unpleasant limitation. Namely, $\mathrm{D}_V(T)$ could not so far be directly related to geometry when T is the G_V-representation on a p-adic étale cohomology group (over $\overline{K}$) of some smooth proper algebraic variety defined over K. Here is a relevant passage from the Introduction to [**15**]: "Il est clair que ces constructions sont des cas particuliers de constructions beaucoup plus générales. On doit pouvoir remplacer les corps que l'on considère ici par des corps des fonctions de plusieurs variables ou certaines de leurs complétions. En particulier (i) la loi de réciprocité explicite énoncée au no. 2.4 doit se généraliser et éclairer d'un jour nouveau les travaux de Kato sur ce sujet; (ii) ces constructions doivent se faisceautiser et peut être donner une approche nouvelle des théorèmes de comparaison entre les cohomologies p-adiques."

The first part of the program sketched above, i.e. the construction of relative (φ,Γ)-modules was successfully carried out in [**1**]. The main purpose of the present article is to continue Fontaine's program. In particular various relative analogues, local and global, of $(*)$ are proven.

Let us first point out that in the relative situation, over a "small"-V-algebra R (see §2) there are several variants of (φ,Γ)-module functors, denoted $\mathfrak{D}_R(-)$ (arithmetic), $\mathrm{D}_R(-)$ (geometric), $\widetilde{\mathfrak{D}}_R(-)$ (tilde-arithmetic), $\widetilde{\mathrm{D}}_R(-)$ (tilde-geometric) and their overconvergent counterparts $\mathfrak{D}_R^\dagger(-)$, $\mathrm{D}_R^\dagger(-)$, $\widetilde{\mathfrak{D}}_R^\dagger(-)$ and $\widetilde{\mathrm{D}}_R^\dagger(-)$. For simplicity of exposition let us explain our results in terms of $\mathfrak{D}_R(-)$ and $\widetilde{\mathfrak{D}}_R(-)$.

I) *Local results.* This is carried on in §3 together with the appendices §A and §B. Let R be a "small" V-algebra. Fix an algebraic closure Ω of the fraction field of R and let η be the associated geometric generic point of $\mathrm{Spec}(R)$. Denote by $\overline{R}$ the union of all normal finite extensions of R contained in Ω, which are étale R-algebras after inverting p. Let M be a finitely generated $\mathbf{Z}_p$-module with continuous action of $\mathscr{G}_R := \pi_1^{\mathrm{alg}}(\mathrm{Spm}(R_K,\eta))$ and let $\mathrm{D} := \widetilde{\mathfrak{D}}_R(M)$. Then D is a finitely generated $\widetilde{\mathbf{A}}_{\overline{R}}$-module endowed with commuting actions of a semi-linear Frobenius φ and a linear action of the group Γ_R (see §2.) As in the classical case, Γ_R is a much smaller group that $\mathscr{G}_R$. It is the semidirect product of Γ_V and of a group isomorphic to $\mathbf{Z}_p^d$ where d is the relative dimension of R over V.

Let $\mathscr{C}^\bullet(\Gamma_R,\mathrm{D})$ be the standard complex of continuous cochains computing the continuous Γ_R-cohomology of D and denote $\mathscr{T}_R^\bullet(\mathrm{D})$ the mapping cone complex of the morphism $(\varphi-1)\colon \mathscr{C}^\bullet(\Gamma_R,\mathrm{D}) \longrightarrow \mathscr{C}^\bullet(\Gamma_R,\mathrm{D})$. Then, Theorem 3.2 states that we have natural isomorphisms, functorial in R and M,

$$\mathrm{H}^i_{\mathrm{cont}}(\mathscr{G}_R,M) \cong \mathrm{H}^i\big(\mathscr{T}_R^\bullet(\mathrm{D})\big) \quad \text{for all } i\geq 0.$$

The maps are defined in §3 in an explicit way, following Colmez's description in the classical case. The input of Fontaine's construction of the classical (φ,Γ)-modules was to replace modules over perfect, non-noetherian rings with modules over smaller rings: "C'est d'ailleurs [...] que j'ai compris l'intérêt qu'il avait à ne pas remplacer $k((\pi))$ par sa clôture radicielle" Indeed, "[...]ceci permet d'introduire des techniques différentielles". Motivated by the same needs, in view of applications to comparison isomorphisms, we show in appendix §A that one can replace the module $\widetilde{\mathfrak{D}}_R(M)$ over the ring $\widetilde{\mathbf{A}}_{\overline{R}}$, which is not noetherian, with the smaller (φ,Γ_R)-module $\mathfrak{D}_R(M)\subset \widetilde{\mathfrak{D}}_R(M)$ over the noetherian, regular domain $\mathbf{A}_R$ of dimension $d+1$. We show that the natural map

$$\mathrm{H}^i_{\mathrm{cont}}(\Gamma_R,\mathfrak{D}_R(M)) \longrightarrow \mathrm{H}^i_{\mathrm{cont}}(\Gamma_R,\widetilde{\mathfrak{D}}_R(M)) \quad \text{for all } i\geq 0$$

is an isomorphism. The proof follows and slightly generalizes the Tate-Sen method in [**2**]. In particular, one has a natural isomorphism

$$\mathrm{H}^i_{\mathrm{cont}}(\mathscr{G}_R,M) \cong \mathrm{H}^i\big(\mathscr{T}_R^\bullet(\mathfrak{D}_R(M))\big) \quad \text{for all } i\geq 0,$$

where $\mathscr{T}^\bullet_R(\mathfrak{D}_R(M))$ is the mapping cone complex of the map

$$(\varphi-1)\colon \mathscr{C}^\bullet(\Gamma_R,\mathfrak{D}_R(M)) \to \mathscr{C}^\bullet(\Gamma_R,\mathfrak{D}_R(M)).$$

II) *Global results.* This is carried on in §4, §5, §6. The setting for §4 and §5 is the following. Let X be a smooth, proper, geometrically irreducible scheme of finite type over V and let $\mathbb{L}$ denote a locally constant étale sheaf of $\mathbf{Z}/p^s\mathbf{Z}$-modules (for some $s \geq 1$) on the generic fiber X_K of X. Let $\mathscr{X}$ denote the formal completion of X along its special fiber and let X_K^{rig} be the rigid analytic space attached to X_K. Fix a geometric generic point $\eta = \mathrm{Spm}(\mathbb{C}_{\mathscr{X}})$ and set $\mathbf{L}$ the fiber of $\mathbb{L}$ at η.

To each $\mathscr{U} \to \mathscr{X}$ étale such that $\mathscr{U}$ is affine, $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$, with $R_{\mathscr{U}}$ a small V-algebra and a choice of local parameters $(T_1, T_2, \ldots, T_d)$ of $R_{\mathscr{U}}$ (as in §2) we attach the relative (φ,Γ)-module $\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L}) := \widetilde{\mathfrak{D}}_{R_{\mathscr{U}}}(\mathbf{L})$. However, the association $\mathscr{U} \longrightarrow \widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L})$ is not functorial because of the dependence of $\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L})$ on the choice of the local parameters. In other words the relative (φ,Γ)-module construction does not sheafify.

Nevertheless due to I) above, the association $\mathscr{U} \longrightarrow \mathrm{H}^i\big(\mathscr{T}^\bullet_{R_{\mathscr{U}}}(\widetilde{\mathfrak{D}}_{R_{\mathscr{U}}}(\mathbf{L})\big)$ is functorial for every $i \geq 0$ and we denote by $\mathscr{H}^i(\mathbf{L})$ the sheaf on the pointed étale site $\mathscr{X}^\bullet_{\mathrm{et}}$ associated to it. In §4 we prove Theorem 4.1: there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^q(\mathscr{X}^\bullet_{\mathrm{et}}, \mathscr{H}^p(\mathbf{L})) \Longrightarrow \mathrm{H}^{p+q}(X_{K,\mathrm{et}}, \mathbb{L}).$$

We view this result as a global analogue of $(*)$: the étale cohomology of $\mathbb{L}$ is calculated in terms of local relative (φ,Γ)-modules attached to $\mathbf{L}$.

The proof of Theorem 4.1 follows a roundabout path which was forced on us by lack of enough knowledge on étale cohomology of rigid analytic spaces. More precisely, for an algebraic, possibly infinite, extension M of K contained in $\overline{K}$, Faltings defines in **[14]** a Grothendieck topology $\mathfrak{X}_M$ on X (see also §4). The local system $\mathbb{L}$ may be thought of as a sheaf on $\mathfrak{X}_M$ and it follows from **[14]**, see 4.4, that there is a natural isomorphism:

$$(**) \qquad \mathrm{H}^i\big(\mathfrak{X}_M, \mathbb{L}\big) \cong \mathrm{H}^i\big(X_{M,\mathrm{et}}, \mathbb{L}\big),$$

for all $i \geq 0$. The main tool for proving $(**)$ is the result: every point $x \in X_K$ has a neighborhood W which is $K(\pi, 1)$. Such a result, although believed to be true, is yet unproved in the rigid analytic setting. Therefore the proof of Theorem 4.1 goes as follows. Let $\mathbb{L}^{\mathrm{rig}}$ be the locally constant étale sheaf on X_K^{rig} associated to $\mathbb{L}$. We define the analogue Grothendieck topology $\widehat{\mathfrak{X}}_M$ on $\mathscr{X}$, prove that there is a spectral sequence with $E_2^{p,q} = \mathrm{H}^q\big(\mathscr{X}^\bullet_{\mathrm{et}}, \mathscr{H}^p(\mathbf{L}^{\mathrm{rig}})\big)$ abutting to $\mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}})$, then compare $\mathrm{H}^i(\mathfrak{X}_M, \mathbb{L})$ to $\mathrm{H}^i(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}})$ and in the end use Faltings' result $(**)$.

In §5 we introduce a certain family of continuous sheaves which we call *Fontaine sheaves* and which we denote by $\overline{\mathscr{O}}_{\_}$, $\mathscr{R}\big(\overline{\mathscr{O}}_{\_}\big)$, $A^+_{\mathrm{inf}}(\overline{\mathscr{O}}_{\_})$. There are algebraic and analytic variants of these: the first are sheaves on $\mathfrak{X}_M$ and the second on $\widehat{\mathfrak{X}}_M$. We would like to remark that the local sections of the Fontaine sheaves are very complicated and they are **not** relative Fontaine rings. Continuous cohomology of continuous sheaves on $\mathfrak{X}_M$ and $\widehat{\mathfrak{X}}_M$ respectively is developed in §5. As an application a geometric interpretation of $\widetilde{\mathfrak{D}}_V(W_i)$, where $W_i = \mathrm{H}^i\big(X_{\overline{K}_{\mathrm{et}}}, \mathbb{L}\big)$, for $\mathbb{L}$ an étale local system of $\mathbf{Z}/p^s\mathbf{Z}$-modules on X_K as above is given. More precisely, it is proven in §5 that there is a natural isomorphism of classical (φ,Γ)-modules:

$$\mathrm{H}^i\left(\widehat{\mathfrak{X}}_{K_\infty}, \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf}}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{K_\infty}})\right) \cong \widetilde{\mathrm{D}}_V\left(\mathrm{H}^i(X_{\overline{K},\mathrm{et}}, \mathbb{L})\right).$$

Finally, in §6 we relax our global assumptions. Now $\mathscr{X}$ denotes a formal scheme topologically of finite type over V, smooth and geometrically irreducible, not necessarily algebrizable, and X_K^{rig} denotes its rigid analytic generic fiber.

In §6 we set up the basic theory for comparison isomorphisms between the different p-adic cohomology theories in this analytic setting. Our main result is that, if $\mathbb{L}^{\mathrm{rig}}$ is a p-power torsion local system on X_K^{rig} and $\mathscr{A}^{\mathrm{Font}}$ is one of the analytic Fontaine sheaves on $\widehat{\mathfrak{X}}_M$ listed above, then the cohomology groups $\mathrm{H}^i(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{A}^{\mathrm{Font}})$ can be calculated as follows. Let us first recall that we fixed a geometric generic point $\eta = \mathrm{Spm}(\mathbb{C}_{\mathscr{X}})$, where $\mathbb{C}_{\mathscr{X}}$ is a complete, algebraically closed field which can be chosen as in 4.4. For each étale morphism $\mathscr{U} \longrightarrow \mathscr{X}$ such that $\mathscr{U}$ is affine, $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ with $R_{\mathscr{U}}$ a small V-algebra, let $\overline{R}_{\mathscr{U}}$ denote the union of all normal $R_{\mathscr{U}}$-algebras contained in $\mathbb{C}_{\mathscr{X}}$ which are finite and étale over $R_{\mathscr{U}}$ after inverting p. Write $\pi_1^{\mathrm{alg}}(\mathscr{U}_M, \eta)$ for the Galois group of $R_{\mathscr{U}} \otimes_V M \subset \overline{R}_{\mathscr{U}} \otimes_V K$. Let $\mathscr{A}^{\mathrm{Font}}(\overline{R}_{\mathscr{U}} \otimes K)$ denote the Fontaine ring constructed starting with the pair $(R_{\mathscr{U}}, \overline{R}_{\mathscr{U}})$ as in [**15**] and denote by $\mathbf{L}$, as before, the fiber of $\mathbb{L}^{\mathrm{rig}}$ at η. One can show that the association $\mathscr{U} \longrightarrow \mathrm{H}^i(\pi_1^{\mathrm{alg}}(\mathscr{U}_M, \eta), \mathbf{L} \otimes \mathscr{A}^{\mathrm{Font}}(\overline{R}_{\mathscr{U}} \otimes K))$ is functorial and denote $\mathscr{H}^i_M(\mathbb{L}^{\mathrm{rig}} \otimes \mathscr{A}^{\mathrm{Font}})$ the sheaf on $\mathscr{X}^\bullet_{\mathrm{et}}$ associated to it.

Notice that, due to the generalized Tate-Sen method of §A, if $\mathscr{A}^{\mathrm{Font}} = A^+_{\mathrm{inf}}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_K})$, the inflation defines an isomorphism:

$$\mathrm{H}^i\big(\Gamma_{R_{\mathscr{U}}}, \mathfrak{D}_{R_{\mathscr{U}}}(\mathbf{L})\big) \cong \mathrm{H}^i\big(\Gamma_{R_{\mathscr{U}}}, \widetilde{\mathfrak{D}}_{R_{\mathscr{U}}}(\mathbf{L})\big) \cong \mathrm{H}^i\big(\pi_1^{\mathrm{alg}}(\mathscr{U}, \eta), \mathbf{L} \otimes A^+_{\mathrm{inf}}(\overline{R}_{\mathscr{U}} \otimes K)\big).$$

Hence, the sheaf $\mathscr{H}^i_M\left(\mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf}}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_K})\right)$ is defined locally in terms of Γ-cohomology of relative (φ,Γ)-modules.

It is proved (Theorem 6.1) that there exists a spectral sequence

$$(***) \qquad E_2^{p,q} = \mathrm{H}^q(\mathscr{X}^\bullet_{\mathrm{et}}, \mathscr{H}^p_M(\mathbb{L}^{\mathrm{rig}} \otimes \mathscr{A}^{\mathrm{Font}})) \Longrightarrow \mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{A}^{\mathrm{Font}}).$$

At this point we would like to remark that our results in §6 are distinct from those of Faltings in [**12**], [**13**], [**14**]. Namely let us consider the following diagram of categories and functors:

$$\begin{tikzcd}
\mathbf{Sh}(\widehat{\mathfrak{X}}^\bullet_M)^{\mathbf{N}} \arrow[r, "\widehat{v}^{\mathbf{N}}_{\mathscr{X},M,*}"] \arrow[d, "\alpha"'] & \mathbf{Sh}(\mathscr{X}^\bullet_{\mathrm{et}})^{\mathbf{N}} \arrow[d, "\beta"] \\
\mathbf{Sh}(\mathscr{X}^\bullet_{\mathrm{et}}) \arrow[r, "\mathrm{H}^0(\mathscr{X}^\bullet_{\mathrm{et}}, -)"] & \mathbf{AbGr}
\end{tikzcd}$$

where if $\mathscr{C}$ is a Grothedieck topology then we denote by $\mathbf{Sh}(\mathscr{C})$ the category of sheaves of abelian groups on $\mathscr{C}$ and by $\mathbf{Sh}(\mathscr{C})^{\mathbf{N}}$ the category of continuous sheaves on $\mathscr{C}$ (see §5). We also denote $\alpha = \varprojlim \widehat{v}_{\mathscr{X},M,*}$ and $\beta = \varprojlim \mathrm{H}^0(\mathscr{X}_{\mathrm{et}}^{\bullet}, -)$.

We analyze the spectral sequence attached to the composition of functors: $H^0(\mathscr{N}_{\mathrm{et}}^{\bullet}, -) \circ \alpha$ while it appears, although very little detail is given, that Faltings considers the composition of the other two functors in the above diagram (in the algebraic setting). We believe that our point of view is appropriate for the applications to relative (φ, Γ)-modules that we have in mind.

The analysis in §6 and the spectral sequence $(***)$ have already been used in order to construct a p-adic, overconvergent, finite slope Eicher-Shimura isomorphism and to give a new, cohomological construction of p-adic families of finite slope modular forms in [**19**].

In a sequel paper ("Global applications of relative (φ, Γ)-modules, II″) we plan to first extend the constructions and results in §6 of the present paper to formal schemes over V with semi-stable special fiber and use them in order to prove comparison isomorphisms between the different p-adic cohomology theories involving Fontaine sheaves in such analytic settings. We believe that we would be able to carry on this project for spaces like: the p-adic symmetric domains, their étale covers (in the cases where good formal models exist), the p-adic period domains of Rapoport-Zink, etc.

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I. LOCAL THEORY

2. Preliminaries

2.1. The basic rings. — Let V be a complete discrete valuation ring, with perfect residue field k of characteristic p and with fraction field $K=\mathrm{Frac}(V)$ of characteristic 0. Let $\mathbf{v}$ be the valuation on V normalized so that $\mathbf{v}(p)=1$. Let $K\subset\overline{K}$ be an algebraic closure of K with Galois group $\mathrm{Gal}(\overline{K}/K)=:\mathrm{G}_V$ and denote by $\overline{V}$ the normalization of V in $\overline{K}$. Define the tower

$$K_0:=K\subset K_1=K(\zeta_p)\subset\cdots\subset K_n=K(\zeta_{p^n})\subset\cdots$$

where ζ_{p^n} is a primitive p^n-th root of unity and $\zeta_{p^{n+1}}^p=\zeta_{p^n}$ for every $n\in\mathbf{N}$. Let V_n be the normalization of V in K_n and define $K_\infty:=\cup_n K_n$. Write $\Gamma_V:=\mathrm{Gal}(K_\infty/K)$ and $\mathrm{H}_V:=\mathrm{Gal}(\overline{K}/K_\infty)$ so that $\Gamma_V=\mathrm{G}_V/\mathrm{H}_V$.

We also fix a field extension $K_\infty\subset M\subset\overline{K}$ so that $K\subset M$ is Galois with group $\mathrm{Gal}(M/K)$. We let W be the normalization of V in M. The two important cases are $M=K_\infty$ with $W=V_\infty$ and $M=\overline{K}$ with $W=\overline{V}$.

Let R be a V-algebra such that $k\subset R\otimes_V k$ is geometrically integral. Let $R^0=V\left\{T_1^{\pm1},\ldots,T_d^{\pm1}\right\}$ be p-adic completion of the polynomial algebra $V\left[T_1^{\pm1},\ldots,T_d^{\pm1}\right]$. Assume that R is obtained from R^0 iterating finitely many times the following operations:

ét) the p-adic completion of an étale extension;

loc) the p-adic completion of the localization with respect to a multiplicative system;

comp) the completion with respect to an ideal containing p.

Define

$$R_n:=R\underset{V}{\otimes}V_n\left[T_1^{\frac{1}{p^n}},T_1^{\frac{-1}{p^n}},\ldots,T_d^{\frac{1}{p^n}},T_d^{\frac{-1}{p^n}}\right],\qquad R_\infty:=\cup_n R_n.$$

Let $\overline{R}$ be the direct limit of a maximal chain of normal R_∞-algebras, which are domains and, after inverting p, are finite and étale extensions of $R_\infty\left[\frac{1}{p}\right]$.

Let $m\in\mathbf{N}$ and let S be a R_m-algebra such that S is finite as R_m-module and $R_m\subset S$ is étale after inverting p. Define S_n as the normalization of $S\otimes_{R_m}R_n$ in $S\otimes_{R_m}R_n\left[p^{-1}\right]$ for every $n\geq m$. Let $S_\infty:=\cup_{n\geq m}S_n$.

Write S_n' for the normalization of S_n in $S_n\otimes_{V_n}M$ and S_∞' for the normalization of S_∞ in $S_\infty\otimes_{V_\infty}M$. We put $S':=S_m'$. Note that $R'=R\otimes_V W$ and $R_\infty'=R_\infty\otimes_{V_\infty}W$.

Proposition 2.1. — *There exist constants $0<\varepsilon<1$ and $N=N(S)\in\mathbf{N}$, depending on S, and there exists an element p^ε of V_N of valuation ε such that $S_{n+1}^p+p^\varepsilon S_{n+1}\subset S_n+p^\varepsilon S_{n+1}$ (as subrings of S_{n+1}) and ${S_{n+1}'}^p+p^\varepsilon S_{n+1}'\subset S_n'+p^\varepsilon S_{n+1}'$ (as subrings of S_{n+1}') for every $n\geq N$.*

Proof. — The claim concerning S_{n+1} follows from [**1**, Cor. 3.7]. It follows from [**1**, Prop. 3.6] that there exists a decreasing sequence of rational numbers $\{\delta_n(S)\}$ such that $p^{\delta_n(S)}$ annihilates the trace map $\mathrm{Tr}\colon S'_n \to \mathrm{Hom}_{R'_n}(S'_n, R'_n)$. This implies that $p^{\delta_n(S)} S'_{n+1} \subset S'_n \otimes_{R'_n} R'_{n+1}$; see loc. cit. This, and the fact that the proposition holds for R' by direct check, allows to conclude; see the proof of [**1**, Cor. 3.7] for details. □

Definition 2.2. — For every R-subalgebra $S \subset \overline{R}$ as in 2.1 such that S'_∞ is an integral domain, viewed as a subring of $\overline{R}$, define

$$\mathscr{G}_S := \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/S\left[\frac{1}{p}\right]\right), \quad \Gamma_S := \mathrm{Gal}\left(S_\infty\left[\frac{1}{p}\right]/S\left[\frac{1}{p}\right]\right)$$

and

$$\mathscr{H}_S := \mathrm{Ker}\left(\mathscr{G}_S \to \Gamma_S\right).$$

Analogously, let

$$\mathrm{G}_S := \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/S'\left[\frac{1}{p}\right]\right), \quad \Gamma'_S := \mathrm{Gal}\left(S'_\infty\left[\frac{1}{p}\right]/S'\left[\frac{1}{p}\right]\right)$$

and

$$\mathrm{H}_S := \mathrm{Ker}\left(\mathrm{G}_S \to \Gamma'_S\right) = \mathrm{Gal}\left(S'_\infty\left[\frac{1}{p}\right]/S'\left[\frac{1}{p}\right]\right).$$

Since S'_∞ is an integral domain, the map $\mathscr{H}_S/H_S \to \mathrm{Gal}(M/K_\infty)$ is an isomorphism. Furthermore, Γ_S is isomorphic to the semidirect product of Γ_V and of Γ'_S. The latter is a finite index subgroup of $\Gamma'_R \cong \mathbf{Z}_p^d$. We let $\gamma_1, \ldots, \gamma_d$ be topological generators of Γ'_R.

2.2. RAE. — Following Faltings [**12**, Def. 2.1] we say that an extension $R_\infty \subset S_\infty$ is almost étale if it is finite and étale after inverting p and if, for every $n \in \mathbf{N}$, the element $p^{\frac{1}{p^n}} \mathfrak{e}_\infty$ is in the image of $S_\infty \otimes_{R_\infty} S_\infty$. Here $\mathfrak{e}_\infty \in S_\infty \otimes_{R_\infty} S_\infty[p^{-1}]$ is the canonical idempotent splitting the multiplication map $S_\infty \otimes_{R_\infty} S_\infty[p^{-1}] \to S_\infty[p^{-1}]$.

We say that such extension satisfies (RAE), *for refined almost étaleness*, if the following holds. For every $n \geq m$ let $\mathfrak{e}_n$ be the diagonal idempotent associated to the étale extension $R_n[p^{-1}] \subset S_n[p^{-1}]$. There exists $\ell \in \mathbf{N}$, independent of m, such that there exists an element $p^{\frac{\ell}{p^n}}$ of V_n of valuation $\frac{\ell}{p^n}$ and $p^{\frac{\ell}{p^n}} \mathfrak{e}_n$ lies in the image of $S_n \otimes_{R_n} S_n$.

We assume that (RAE) holds for every extension $R_\infty \subset S_\infty$ arising as in 2.1. If this holds, we say that R or equivalently $\mathrm{Spf}(R)$ is *small.*

Remark 2.3. — It is proven in [**1**, Prop. 5.10 & Thm. 5.11] that (RAE) holds if R is of Krull dimension ≤ 2 or if the composite of the extensions $V[T_1^{\pm 1}, \cdots, T_d^{\pm 1}] \to R^0 \to R$ is flat and has geometrically regular fibers. For example, this holds if R is obtained by taking the completion with respect to an ideal containing p of the localization of an étale extension of $V[T_1^{\pm 1}, \cdots, T_d^{\pm 1}]$; see [**1**, Prop. 5.12].

2.3. The rings $\widetilde{\mathbf{E}}_{S_\infty}$, $\mathbf{E}_S$, $\widetilde{\mathbf{E}}_{S'_\infty}$ and $\mathbf{E}'_S$. — Let S be as in 2.1. Define

$$\widetilde{\mathbf{E}}^+_{S_\infty} := \varprojlim \big(S_\infty/p^\varepsilon S_\infty\big), \qquad \widetilde{\mathbf{E}}^+_{S'_\infty} := \varprojlim \big(S'_\infty/p^\varepsilon S'_\infty\big)$$

where the inverse limit is taken with respect to Frobenius. Using 2.1 define the *generalized ring of norms*,

$$\mathbf{E}^+_S \subset \widetilde{\mathbf{E}}^+_{S_\infty}, \qquad \mathbf{E}'^+_S \subset \widetilde{\mathbf{E}}^+_{S'_\infty}$$

as the subring consisting of elements $(a_0,\dots,a_n,\dots)$ in $\widetilde{\mathbf{E}}^+_{S_\infty}$ (resp. in $\widetilde{\mathbf{E}}^+_{S'_\infty}$) such that a_n is in $S_n/p^\varepsilon S_n$ (resp. $S'_n/p^\varepsilon S'_n$) for every $n \geq N(S)$.

By construction $\widetilde{\mathbf{E}}^+_{S_\infty}$, $\mathbf{E}^+_S$ (resp. $\widetilde{\mathbf{E}}^+_{S'_\infty}$ and $\mathbf{E}'^+_S$) are endowed with a Frobenius homomorphism φ and a continuous action of Γ_S (resp. Γ'_S). Denote by ϵ the element $(1,\zeta_p,\dots,\zeta_{p^n},\dots) \in \mathbf{E}^+_V$ and by $\overline{\pi} := \epsilon - 1$. Put $\widetilde{\mathbf{E}}_{S'_\infty} := \widetilde{\mathbf{E}}^+_{S'_\infty}\big[\overline{\pi}^{-1}\big]$, $\widetilde{\mathbf{E}}_{S_\infty} := \widetilde{\mathbf{E}}^+_{S_\infty}\big[\overline{\pi}^{-1}\big]$, $\mathbf{E}'_S := \mathbf{E}'^+_S\big[\overline{\pi}^{-1}\big]$ and $\mathbf{E}_S := \mathbf{E}^+_S\big[\overline{\pi}^{-1}\big]$.

By abuse of notation for $\alpha \in \mathbf{Q}$, we write ${\pi_0}^\alpha$ for $a = (a_0,a_1,\dots,a_n,\dots) \in \widetilde{\mathbf{E}}^+_{V_\infty}$, if it exists, such that $\mathbf{v}(a_i) = \frac{\alpha}{p^i}$ for $i \gg 0$. For example, $\overline{\pi} = \pi_0^{\frac{p}{p-1}}$; see [**2**, Prop. 4.2(d)].

For every $i = 1,\dots,d$, let $x_i := (T_i, T_i^{\frac{1}{p}}, T_i^{\frac{1}{p^2}}, \cdots) \in \mathbf{E}^+_{R^0}$. The following hold:

1. there exists $N(S) \in \mathbf{N}$ such that the map $\mathbf{E}^+_S/\pi_0^{p^n\varepsilon}\mathbf{E}^+_S \to S_n/p^\varepsilon S_n$ and the map $\widetilde{\mathbf{E}}^+_{S_\infty}/\pi_0^{p^n\varepsilon}\widetilde{\mathbf{E}}^+_{S_\infty} \to S_\infty/p^\varepsilon S_\infty$, sending $(a_0,\dots,a_n,\dots) \mapsto a_n$, are isomorphisms for every $n \geq N(S)$ (see [**1**, Thm. 5.1]);
2. $\mathbf{E}^+_S$ is a normal ring, it is finite as $\mathbf{E}^+_R$-module and it is an étale extension of $\mathbf{E}^+_R$, after inverting $\overline{\pi}$, of degree equal to the generic degree of $R_m \subset S$ (see [**1**, Thm. 4.9 & Thm. 5.3]);
3. $\widetilde{\mathbf{E}}^+_{S_\infty}$ is normal and coincides with the $\overline{\pi}$-adic completion of the perfect closure of $\mathbf{E}^+_S$ (see [**1**, Cor. 5.4]);
4. there exists $\ell \in \mathbf{N}$ and maps $\overline{\pi}^\ell\widetilde{\mathbf{E}}^+_{S_\infty} \to \mathbf{E}^+_S \otimes_{\mathbf{E}^+_R} \widetilde{\mathbf{E}}^+_{R_\infty} \to \widetilde{\mathbf{E}}^+_{S_\infty}$ which are isomorphisms after inverting $\overline{\pi}$. In particular, $\widetilde{\mathbf{E}}^+_{S_\infty}\big[\overline{\pi}^{-1}\big] = \widetilde{\mathbf{E}}^+_{R_\infty} \otimes_{\mathbf{E}^+_R} \mathbf{E}^+_S\big[\overline{\pi}^{-1}\big]$ (see [**1**, Lem. 4.15]);
5. consider the ring

$$\lim_{\infty \leftarrow n} \widehat{S_\infty} := \left\{\big(x^{(0)}, x^{(1)},\dots,x^{(m)},\dots\big) | x^{(m)} \in \widehat{S_\infty},\ \big(x^{(m+1)}\big)^p = x^{(m)}\right\},$$

where $\widehat{S_\infty}$ is the p-adic completion of S_∞, the transition maps are defined by raising to the p-th power, the multiplicative structure is induced by the one on $\widehat{S_\infty}$ and the additive structure is defined by

$$(\dots,x^{(m)},\dots) + (\dots,y^{(m)},\dots) = \big(\dots, \lim_{n\to\infty} (x^{(m+n)} + y^{(m+n)})^{p^n},\dots\big).$$

The natural map $\lim_{\infty\leftarrow n} \widehat{S_\infty} \to \widetilde{\mathbf{E}}^+_{S_\infty}$ is a bijection (see [**1**, Lem. 4.10]).

It follows from (1), see [**1**, Cor. 4.7], that

$$\mathbf{E}^+_V \cong k_\infty[\![\pi_K]\!] \qquad \text{and} \qquad \mathbf{E}^+_{R^0} \cong \mathbf{E}^+_V\left\{x_1,\dots,x_d,\frac{1}{x_1},\dots,\frac{1}{x_d}\right\},$$

where k_∞ is the residue field of V_∞ and $\pi_K = (\ldots, \tau_n, \tau_{n+1}, \ldots)$, with $\tau_i \in V_i$ for $i \gg 0$, is a system of uniformizers satisfying $\tau_{i+1}^p \equiv \tau_i \mod p^\varepsilon$. The convergence in $x_1^{\pm 1}, \ldots, x_d^{\pm 1}$ is relative to the $\overline{\pi}$-adic topology on $\mathbf{E}_V^+$. Eventually $\mathbf{E}_R^+$ is obtained from $\mathbf{E}_{R^0}^+$ iterating the operations

ét) the $\overline{\pi}$-adic completion of an étale extension;
loc) the $\overline{\pi}$-adic completion of the localization with respect to a multiplicative system;
comp) the completion with respect to an ideal containing a power of $\overline{\pi}$.

In particular, $\{\pi_K, x_1, \ldots, x_d\}$ is an absolute p-basis of $\mathbf{E}_R^+$.

Lemma 2.4. — *Let S be as in 2.1. The following hold:*

1. *the maps $\widetilde{\mathbf{E}}_{S'_\infty}^+/\pi_0^{p^n\varepsilon}\widetilde{\mathbf{E}}_{S'_\infty}^+ \to S'_\infty/p^\varepsilon S'_\infty$ and $\mathbf{E}_S'^+/\pi_0^{p^n\varepsilon}\mathbf{E}_S'^+ \to S'_n/p^\varepsilon S'_n$, given by $(a_0, \ldots, a_n, \ldots) \mapsto a_n$, are isomorphisms for $n \geq N(S)$. In particular, $\widetilde{\mathbf{E}}_{R'_\infty}^+$ coincides with the $\overline{\pi}$-adic completion of $\widetilde{\mathbf{E}}_{R_\infty}^+ \otimes_{\widetilde{\mathbf{E}}_{V_\infty}^+} \widetilde{\mathbf{E}}_W^+$ and $\mathbf{E}_R'^+$ coincides with the $\overline{\pi}$-adic completion of $\mathbf{E}_R^+ \otimes_{\mathbf{E}_V^+} \widetilde{\mathbf{E}}_W^+$;*
2. *the extensions $\mathbf{E}_R^+ \to \widetilde{\mathbf{E}}_{R_\infty}^+$, $\mathbf{E}_R^+ \to \widetilde{\mathbf{E}}_{R'_\infty}^+$ and $\mathbf{E}_R^+ \to \mathbf{E}_R'^+$ are faithfully flat. For every finitely generated $\mathbf{E}_R^+$-module M, the base change of M via any of the above extensions is $\overline{\pi}$-adically complete and separated;*
3. *we have maps $\overline{\pi}^\ell\widetilde{\mathbf{E}}_{S'_\infty}^+ \to \mathbf{E}_S^+ \otimes_{\mathbf{E}_R^+} \widetilde{\mathbf{E}}_{R'_\infty}^+ \to \widetilde{\mathbf{E}}_{S'_\infty}^+$ and $\overline{\pi}^\ell\mathbf{E}_S'^+ \to \mathbf{E}_S^+ \otimes_{\mathbf{E}_R^+} \mathbf{E}_R'^+ \to \mathbf{E}_S'^+$. They are isomorphisms after inverting $\overline{\pi}$.*

Proof. — Statements (1) and (3) follow from 2.2 arguing as in the proofs of [**1**, Thm. 5.1] and [**1**, Lem. 4.15] respectively.

(2) By 2.3 we have $\mathbf{E}_R^+/\pi_0^{\varepsilon p^n}\mathbf{E}_R^+ \cong R_n/p^\varepsilon R_n$ and $\widetilde{\mathbf{E}}_W^+/\pi_0^{\varepsilon p^n}\widetilde{\mathbf{E}}_W^+ \cong W/p^\varepsilon W$. One deduces form (1) that $\mathbf{E}_R'^+/\pi_0^{\varepsilon p^n}\mathbf{E}_R'^+ \cong \left(R_n/p^\varepsilon R_n\right) \otimes_{V_n} W$ and that $\widetilde{\mathbf{E}}_{R'_\infty}^+/\pi_0^{\varepsilon p^n}\widetilde{\mathbf{E}}_{R'_\infty}^+ \cong \left(R_\infty/p^\varepsilon R_\infty\right) \otimes_{V_\infty} W$. By construction $R_\infty \otimes_{V_\infty} W$ is a free $R_n \otimes_{V_n} W$-module with basis $\{T_1^{\frac{\alpha_1}{p^m}} \cdots T_d^{\frac{\alpha_d}{p^m}}\}$ for $m \geq n$ and $0 \leq \alpha_i < p^{m-n}$. Hence, $\mathbf{E}_{R'}'^+$ (resp. $\widetilde{\mathbf{E}}_{R'_\infty}^+$) is the $\overline{\pi}$-adic completion $\mathbf{E}_R^+ \otimes_{\mathbf{E}_V^+} \widetilde{\mathbf{E}}_W^+$ (resp. of $\cup_n\left(\mathbf{E}_R^+ \otimes_{\mathbf{E}_V^+} \widetilde{\mathbf{E}}_W^+\right)[x_1^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^n}}]$). The $\mathbf{E}_V^+$-algebra $\widetilde{\mathbf{E}}_W^+$ is the $\overline{\pi}$-adic completion of finite, normal and generically separable extensions of the DVR $\mathbf{E}_V^+$. Those are free as $\mathbf{E}_V^+$-module. We may then apply [**1**, Lem. 8.7] to conclude. □

Given an R_∞-algebra S_∞, finite and étale over $R_\infty\left[\frac{1}{p}\right]$, there exists $m \in \mathbf{N}$ and there exists a R_m-algebra S, finite and étale over $R_m\left[\frac{1}{p}\right]$ such that S_∞, defined as in 2.1, is the normalization of $S \otimes_{R_n} R_\infty$.

Theorem 2.5. — *The functor $S_\infty \longrightarrow \mathbf{E}_S^+$ defines an equivalence of categories from the category R_∞-**AED** of R_∞-algebras which are normal domains, finite and étale over R_∞ after inverting p to the category $\mathbf{E}_R^+$-**AED** of $\mathbf{E}_R^+$-algebras, which are normal domains, finite and étale after inverting $\overline{\pi}$. In particular, this realizes $\mathscr{H}_R$ as the Galois group of $\mathbf{E}_R$.*

Proof. — See [**1**, Thm. 6.3]. □

Let $\mathbf{E}^+_{\overline{R}}$ be $\cup_{S_\infty}\mathbf{E}^+_S$ where the union is taken over all R_∞-subalgebras $S_\infty\subset\overline{R}$ such that $S_\infty\left[p^{-1}\right]$ is finite étale over $R_\infty\left[p^{-1}\right]$. Similarly, define $\mathbf{E}'^+_{\overline{R}}$ to be the union $\cup_{S_\infty}\mathbf{E}'^+_S$. Let $\widetilde{\mathbf{E}}^+_{\overline{R}}=\lim\left(\overline{R}/p^\varepsilon\overline{R}\right)$, where the inverse limit is taken with respect to Frobenius. It coincides with the $\overline{\pi}$-adic completion of $\cup_{S_\infty}\widetilde{\mathbf{E}}^+_{S_\infty}$. Denote $\mathbf{E}_{\overline{R}}:=\mathbf{E}^+_{\overline{R}}\left[\overline{\pi}^{-1}\right]$, $\mathbf{E}'_{\overline{R}}:=\mathbf{E}'^+_{\overline{R}}\left[\overline{\pi}^{-1}\right]$ and $\widetilde{\mathbf{E}}_{\overline{R}}:=\widetilde{\mathbf{E}}^+_{\overline{R}}\left[\overline{\pi}^{-1}\right]$.

Proposition 2.6. — *Let S be as in 2.2. Then,*

(a) $\widehat{\overline{R}}^{\mathscr{H}_S}=\widehat{S_\infty}$ *and* $\widehat{\overline{R}}^{\mathrm{H}_S}=\widehat{S'_\infty}$ *(here* $\widehat{\ }$ *denotes the p-adic completion);*

(b) *we have*

$$\left(\mathbf{E}^+_{\overline{R}}\right)^{\mathscr{H}_S}=\mathbf{E}^+_S,\qquad \mathbf{E}^{\mathscr{H}_S}_{\overline{R}}=\mathbf{E}_S,\qquad \left(\widetilde{\mathbf{E}}^+_{\overline{R}}\right)^{\mathscr{H}_S}=\widetilde{\mathbf{E}}^+_{S_\infty},\qquad \widetilde{\mathbf{E}}^{\mathscr{H}_S}_{\overline{R}}=\widetilde{\mathbf{E}}_{S_\infty}$$

and

$$\left(\mathbf{E}'_{\overline{R}}\right)^{\mathrm{H}_S}=\mathbf{E}'_S,\qquad \left(\widetilde{\mathbf{E}}^+_{\overline{R}}\right)^{\mathrm{H}_S}=\widetilde{\mathbf{E}}^+_{S'_\infty},\qquad \left(\widetilde{\mathbf{E}}_{\overline{R}}\right)^{\mathrm{H}_S}=\widetilde{\mathbf{E}}_{S'_\infty};$$

(c) *the maps*

$$\widetilde{\mathbf{E}}_{R_\infty}\underset{\mathbf{E}_R}{\otimes}\mathbf{E}_S\longrightarrow\widetilde{\mathbf{E}}_{S_\infty},\quad \mathbf{E}'_R\underset{\mathbf{E}_R}{\otimes}\mathbf{E}_S\longrightarrow\mathbf{E}'_S,\quad \widetilde{\mathbf{E}}_{R'_\infty}\underset{\mathbf{E}_R}{\otimes}\mathbf{E}_S\longrightarrow\widetilde{\mathbf{E}}_{S'_\infty}$$

are isomorphisms. In particular, the maps $\mathbf{E}_S\otimes_{\mathbf{E}_V}\widetilde{\mathbf{E}}_W\to\mathbf{E}'_S$ *and* $\widetilde{\mathbf{E}}_{S_\infty}\otimes_{\widetilde{\mathbf{E}}_{V_\infty}}\widetilde{\mathbf{E}}_W\to\widetilde{\mathbf{E}}_{S'_\infty}$ *are injective with dense image.*

Proof. — (a) The fact that $\widehat{\overline{R}}^{\mathscr{H}_S}=\widehat{S_\infty}$ is proven in [**1**, Lem. 6.13]. The second equality follows arguing as in loc. cit.

(b) The equalities in the first displayed formula hold due to [**1**, Prop. 6.14]. Those in the second displayed formula follow arguing as in loc. cit. In fact, $\widetilde{\mathbf{E}}^+_{\overline{R}}$ (resp. $\widetilde{\mathbf{E}}^+_{S'_\infty}$) can be written as in 2.3(5) as the limit $\lim\limits_{\infty\leftarrow n}\widehat{\overline{R}}$ (resp. $\lim\limits_{\infty\leftarrow n}\widehat{S'_\infty}$). The last two equalities in the second displayed formula follow then from (a). The fact that the inclusion $\mathbf{E}'_S\subset\left(\mathbf{E}'_{\overline{R}}\right)^{\mathrm{H}_S}$ is an equality can be checked after base change via $\mathbf{E}'^+_R\to\widetilde{\mathbf{E}}^+_{R'_\infty}$ since the latter is faithfully flat by 2.4(2). But $\mathbf{E}'_S\otimes_{\mathbf{E}'^+_R}\widetilde{\mathbf{E}}^+_{R'_\infty}\cong\mathbf{E}_{S'_\infty}$ by 2.4(3) and $\left(\mathbf{E}'^+_{\overline{R}}\right)^{\mathrm{H}_S}\otimes_{\mathbf{E}'^+_R}\widetilde{\mathbf{E}}^+_{R'_\infty}\subset\left(\widetilde{\mathbf{E}}^+_{\overline{R}}\right)^{\mathrm{H}_S}=\widetilde{\mathbf{E}}^+_{S'_\infty}$.

(c) The first equality follows from 2.3(4). The others follow from 2.4(3). In the case $S=R$ the last statement follows from 2.4(1). The general case follows from this, the equalities just proven and the fact that $\mathbf{E}_R\subset\mathbf{E}_S$ is finite étale by 2.5. □

2.4. The rings $\widetilde{\mathbf{A}}_{S_\infty}$, $\widetilde{\mathbf{A}}^\dagger_{S_\infty}$, $\mathbf{A}_S$, $\mathbf{A}_S^\dagger$, $\widetilde{\mathbf{A}}_{S'_\infty}$, $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$, $\mathbf{A}'_S$ and $\mathbf{A}'^\dagger_S$

Definition 2.7. — Define $\widetilde{\mathbf{A}}_{\overline{R}} := \mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}})$. It is endowed with the following topology, called the *weak topology*: Consider on $\widetilde{\mathbf{E}}_{\overline{R}}$ the topology having $\{\overline{\pi}^n \widetilde{\mathbf{E}}^+_{\overline{R}}\}_n$ as fundamental system of neighborhoods of 0. On the truncated Witt vectors $\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}})$ we consider the product topology via the isomorphism $\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}) \cong (\widetilde{\mathbf{E}}_{\overline{R}})^m$ given by the phantom components. Eventually, the weak topology is defined as the projective limit topology $\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}) = \lim\limits_{\infty \leftarrow m} \mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}})$.

Alternatively, let $\pi := [\varepsilon] - 1$ where $[\varepsilon]$ is the Teichmüller lift of ε. Put $\widetilde{\mathbf{A}}^+_{\overline{R}} := \mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}})^+$. For every n and $h \in \mathbf{N}$ define $U_{n,h} := p^n \widetilde{\mathbf{A}}_{\overline{R}} + \pi^h \widetilde{\mathbf{A}}^+_{\overline{R}}$. The weak topology on $\widetilde{\mathbf{A}}_{\overline{R}}$ has $\{U_{n,h}\}_{n,h\in\mathbf{N}}$ as fundamental system of neighborhoods.

Define $\mathbf{v}_{\mathbf{E}} \colon \widetilde{\mathbf{E}}_{\overline{R}} \to \mathbf{Q} \cup \{\infty\}$ by $\mathbf{v}_{\mathbf{E}}(z) = \infty$ if $z = 0$ and $\mathbf{v}_{\mathbf{E}}(z) = \frac{p}{p-1}\max\{n \in \mathbf{Q} | \overline{\pi}^{-n} z \in \widetilde{\mathbf{E}}^+_{\overline{R}}\}$. For $z = \sum_k [z_k] p^k \in \widetilde{\mathbf{A}}_{\overline{R}}$ and $N \in \mathbf{N}$ we put

$$\mathbf{v}_{\mathbf{E}}^{\leq N}(z) := \inf\{\mathbf{v}_{\mathbf{E}}(z_k) | 0 \leq k \leq N\}.$$

For every $N \in \mathbf{N}$ we have

(i) $\mathbf{v}_{\mathbf{E}}^{\leq N}(x) = +\infty \Leftrightarrow x \in p^{N+1} \widetilde{\mathbf{A}}^+_{\overline{R}}$;
(ii) $\mathbf{v}_{\mathbf{E}}^{\leq N}(xy) \geq \mathbf{v}_{\mathbf{E}}^{\leq N}(x) + \mathbf{v}_{\mathbf{E}}^{\leq N}(y)$;
(iii) $\mathbf{v}_{\mathbf{E}}^{\leq N}(x+y) \geq \min(\mathbf{v}_{\mathbf{E}}^{\leq N}(x), \mathbf{v}_{\mathbf{E}}^{\leq N}(y))$ with equality if $\mathbf{v}_{\mathbf{E}}^{\leq N}(x) \neq \mathbf{v}_{\mathbf{E}}^{\leq N}(y)$;
(iv) $\mathbf{v}_{\mathbf{E}}^{\leq N}(\overline{\pi}) = \frac{p}{p-1}$ and $\mathbf{v}_{\mathbf{E}}^{\leq N}(\overline{\pi} x) = \mathbf{v}_{\mathbf{E}}^{\leq N}(\overline{\pi}) + \mathbf{v}_{\mathbf{E}}^{\leq N}(x)$;
(v) $\mathbf{v}_{\mathbf{E}}^{\leq N}(\varphi(x)) = p\mathbf{v}_{\mathbf{E}}^{\leq N}(x)$;
(vi) $\mathbf{v}_{\mathbf{E}}^{\leq N}(\gamma(x)) = \mathbf{v}_{\mathbf{E}}^{\leq N}(x)$ for every $\gamma \in \mathscr{G}_R$.

The second claim in (iii) and property (v) follow since $\widetilde{\mathbf{E}}^+_{\overline{R}}$ is by construction the $\overline{\pi}$-adic completion of $\cup_{S_\infty} \widetilde{\mathbf{E}}^+_{S_\infty}$ and each $\widetilde{\mathbf{E}}^+_{S_\infty}$ is normal by 2.3(3). Note that the topology on $\widetilde{\mathbf{A}}_{\overline{R}}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}$ induced from the weak topology on $\widetilde{\mathbf{A}}_{\overline{R}}$ coincides with the $\mathbf{v}_{\mathbf{E}}^{\leq N}$ topology.

For every S as in 2.2 define

$$\widetilde{\mathbf{A}}_{S_\infty} := \mathbf{W}(\widetilde{\mathbf{E}}_{S_\infty}), \qquad \widetilde{\mathbf{A}}^+_{S_\infty} := \mathbf{W}(\widetilde{\mathbf{E}}^+_{S_\infty}),$$

$$\widetilde{\mathbf{A}}_{S'_\infty} := \mathbf{W}(\widetilde{\mathbf{E}}_{S'_\infty}) \qquad \text{and} \qquad \widetilde{\mathbf{A}}^+_{S'_\infty} := \mathbf{W}(\widetilde{\mathbf{E}}^+_{S'_\infty}).$$

They are subrings of $\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}})$ closed for the weak topology.

Overconvergent coefficients. — For $r \in \mathbf{Q}_{>0}$ define $\widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$ as the subring of elements $z = \sum\limits_{k=0}^{\infty} p^k [z_k]$ of $\widetilde{\mathbf{A}}_{\overline{R}}$ such that

$$\lim_{k\to\infty} r v_{\mathbf{E}}(z_k) + k = +\infty;$$

see [**2**, Prop. 4.2]. Write $\widetilde{\mathbf{A}}^{\dagger}_{\overline{R}} = \bigcup_{r\in\mathbf{Q}_{>0}} \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$. For $z = \sum_{k\in\mathbf{Z}} p^k[z_k] \in \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$, put

$$w_r(z) = \begin{cases} \infty & \text{if } z = 0; \\ \inf_{k\in\mathbf{Z}} \left(r v_{\mathbf{E}}(z_k) + k\right) & \text{otherwise.} \end{cases}$$

Thanks to [**2**, Prop. 4.2] one knows that the map $w_r\colon \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}} \to \mathbf{R}\cup\{\infty\}$ satisfies

(i) $w_r(x) = +\infty \Leftrightarrow x = 0$;
(ii) $w_r(xy) \geq w_r(x) + w_r(y)$;
(iii) $w_r(x+y) \geq \min(w_r(x), w_r(y))$;
(iv) $w_r(p) = 1$ and $w_r(px) = w_r(p) + w_r(x)$.

For every S_∞ define $\widetilde{\mathbf{A}}^{(0,r]}_{S_\infty} := \widetilde{\mathbf{A}}_{S_\infty} \cap \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$ and $\widetilde{\mathbf{A}}^{\dagger}_{S_\infty} := \widetilde{\mathbf{A}}_{S_\infty} \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$. Similarly, define $\widetilde{\mathbf{A}}^{(0,r]}_{S'_\infty} := \widetilde{\mathbf{A}}_{S'_\infty} \cap \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$ and $\widetilde{\mathbf{A}}^{\dagger}_{S'_\infty} := \widetilde{\mathbf{A}}_{S'_\infty} \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$. By [**2**, Prop. 4.2] they are w_r-adically complete and separated subrings of $\widetilde{\mathbf{A}}_{\overline{R}}$.

2.4.1. *Noetherian coefficients.* — Let S be as in §2.1. In [**1**, Appendix II] a ring $\mathbf{A}_S \subset \mathbf{W}\big(\widetilde{\mathbf{E}}_{S_\infty}\big)$ has been constructed, functorially in S, with the following properties:

(i) it is complete and separated for the weak topology. In particular, it is p-adically complete and separated.
(ii) $\mathbf{A}_S \cap \big(p\mathbf{W}\big(\widetilde{\mathbf{E}}_{\overline{R}}\big)\big) = p\mathbf{A}_S$;
(iii) $\mathbf{A}_S/p\mathbf{A}_S \cong \mathbf{E}_S$. In particular, it is noetherian and regular.
(iv) it is endowed with continuous commuting actions of Γ_S and of an operator φ lifting those defined on $\mathbf{E}_S$;
(v) $\mathbf{A}_R$ contains the Teichmüller lifts of ϵ, $x_1, \ldots, x_d$;
(vi) $\mathbf{A}_S$ is the unique finite and étale $\mathbf{A}_R$-algebra lifting the finite and étale extension $\mathbf{E}_R \subset \mathbf{E}_S$.

One also requires the existence of a subring $\mathbf{A}_S^+$ lifting $\mathbf{E}_S^+$, with suitable properties, so that $\mathbf{A}_S$ is unique. We refer to [**2**, Prop. 4.42] for details. Define $\mathbf{A}_S^{(0,r]} := \mathbf{A}_S \cap \widetilde{\mathbf{A}}^{(0,r]}_{\overline{R}}$ and $\mathbf{A}_S^{\dagger} := \mathbf{A}_S \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$. Define $\mathbf{A}'_S$ to be the closure of the image of $\mathbf{A}_S \otimes_{\mathbf{A}_V} \widetilde{\mathbf{A}}_W$ in $\widetilde{\mathbf{A}}_{\overline{R}}$ for the weak topology. Put $\mathbf{A}'^{\dagger}_S := \mathbf{A}'_S \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$.

Eventually, let $\mathbf{A}_{\overline{R}}$ (resp. $\mathbf{A}'_{\overline{R}}$) be the completion for the p-adic topology of $\cup_{S_\infty} \mathbf{A}_S$ (resp. $\cup_{S_\infty} \mathbf{A}'_S$), where the union is taken over all normal R_∞-subalgebras $S_\infty \subset \overline{R}$ such that $S_\infty\left[p^{-1}\right]$ is finite étale over $R_\infty\left[p^{-1}\right]$. Write $\mathbf{A}^{\dagger}_{\overline{R}} := \mathbf{A}_{\overline{R}} \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$ (resp. $\mathbf{A}'^{\dagger}_{\overline{R}} := \mathbf{A}'_{\overline{R}} \cap \widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$).

Proposition 2.8. — *The extensions $\widetilde{\mathbf{A}}^{\dagger}_{R_\infty} \subset \widetilde{\mathbf{A}}^{\dagger}_{S_\infty}$ and $\mathbf{A}^{\dagger}_R \subset \mathbf{A}^{\dagger}_S$ are finite and étale. Their reduction modulo p coincide with $\widetilde{\mathbf{E}}_{R_\infty} \subset \widetilde{\mathbf{E}}_{S_\infty}$ and $\mathbf{E}_R \subset \mathbf{E}_S$ respectively.*

Proof. — It is clear that $\widetilde{\mathbf{A}}^{\dagger}_{S_\infty}$ coincides with $\widetilde{\mathbf{E}}_{S_\infty}$ modulo p since it contains $\widetilde{\mathbf{A}}^{+}_{S_\infty}$ and $p\widetilde{\mathbf{A}}_{\overline{R}} \cap \widetilde{\mathbf{A}}^{\dagger}_{S_\infty} = p\widetilde{\mathbf{A}}^{\dagger}_{S_\infty}$. The fact that $\widetilde{\mathbf{A}}^{\dagger}_{R_\infty} \subset \widetilde{\mathbf{A}}^{\dagger}_{S_\infty}$ is finite and étale is proven in [**2**, Prop. 4.7]. See [**2**, Prop. 4.28] for the statements regarding $\mathbf{A}^{\dagger}_R \subset \mathbf{A}^{\dagger}_S$. □

Lemma 2.9. — *The following hold:*

(a) $\mathbf{A}_S = \mathbf{A}_{\overline{R}}^{\mathscr{H}_S}$, $\mathbf{A}'_S = \left(\mathbf{A}'_{\overline{R}}\right)^{\mathrm{H}_S}$, $\widetilde{\mathbf{A}}_{S_\infty} = \widetilde{\mathbf{A}}_{\overline{R}}^{\mathscr{H}_S}$, *and* $\widetilde{\mathbf{A}}_{S'_\infty} = \widetilde{\mathbf{A}}_{\overline{R}}^{\mathrm{H}_S}$. *The same equalities hold considering overconvergent coefficients i. e.,* $\mathbf{A}_S^{(0,r]} = \left(\mathbf{A}_{\overline{R}}^{(0,r]}\right)^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]} = \left(\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}\right)^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S'_\infty}^{(0,r]} = \left(\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}\right)^{\mathrm{H}_S}$ *and* $\mathbf{A}_S^\dagger = \left(\mathbf{A}_{\overline{R}}^\dagger\right)^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S_\infty}^\dagger = \left(\mathbf{A}_{\overline{R}}^\dagger\right)^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S'_\infty}^\dagger = \left(\widetilde{\mathbf{A}}_{\overline{R}}^\dagger\right)^{\mathrm{H}_S}$.

(b) *The natural maps* $\widetilde{\mathbf{A}}_{R_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \widetilde{\mathbf{A}}_{S_\infty}$, $\mathbf{A}'_R \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \mathbf{A}'_S$ *and* $\widetilde{\mathbf{A}}_{R'_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \widetilde{\mathbf{A}}_{S'_\infty}$ *are isomorphisms. Similarly, considering overconvergent coefficients, the maps* $\widetilde{\mathbf{A}}_{R_\infty}^\dagger \otimes_{\mathbf{A}_R^\dagger} \mathbf{A}_S^\dagger \to \widetilde{\mathbf{A}}_{S_\infty}^\dagger$, $\mathbf{A}'^\dagger_R \otimes_{\mathbf{A}_R^\dagger} \mathbf{A}_S^\dagger \to \mathbf{A}'^\dagger_S$ *and* $\widetilde{\mathbf{A}}_{R'_\infty}^\dagger \otimes_{\mathbf{A}_R^\dagger} \mathbf{A}_S^\dagger \to \widetilde{\mathbf{A}}_{S'_\infty}^\dagger$ *are isomorphisms.*

(c) *We have* $\mathbf{A}_S^\dagger/p\mathbf{A}_S^\dagger = \mathbf{A}_S/p\mathbf{A}_S = \mathbf{E}_S$, $\mathbf{A}'^\dagger_S/p\mathbf{A}'^\dagger_S = \mathbf{A}'_S/p\mathbf{A}'_S = \mathbf{E}'_S$ *and* $\widetilde{\mathbf{A}}_{S'_\infty}^\dagger/p\widetilde{\mathbf{A}}_{S'_\infty}^\dagger = \widetilde{\mathbf{A}}_{S'_\infty}/p\widetilde{\mathbf{A}}_{S'_\infty} = \widetilde{\mathbf{E}}_{S'_\infty}$.

(d) *The maps* $\widetilde{\mathbf{A}}_{S_\infty} \otimes_{\widetilde{\mathbf{A}}_{V_\infty}} \widetilde{\mathbf{A}}_W \longrightarrow \widetilde{\mathbf{A}}_{S'_\infty}$ *and* $\mathbf{A}_S \otimes_{\mathbf{A}_V} \widetilde{\mathbf{A}}_W \longrightarrow \mathbf{A}'_S$ *are injective and have dense image for the weak topology. The image of* $\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]} \otimes_{\widetilde{\mathbf{A}}_{V_\infty}^{(0,r]}} \widetilde{\mathbf{A}}_W^{(0,r]} \longrightarrow \widetilde{\mathbf{A}}_{S'_\infty}^{(0,r]}$ *is dense for the* w_r*-adic topology for every* $r \in \mathbf{Q}_{>0}$.

Proof. — (a) & (b) We have inclusions $\mathbf{A}_S \subset \mathbf{A}_{\overline{R}}^{\mathscr{H}_S}$, $\mathbf{A}'_S \subset \left(\mathbf{A}'_{\overline{R}}\right)^{\mathrm{H}_S}$, $\widetilde{\mathbf{A}}_{S_\infty} \subset \widetilde{\mathbf{A}}_{\overline{R}}^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S'_\infty} \subset \widetilde{\mathbf{A}}_{\overline{R}}^{\mathrm{H}_S}$ and maps $\widetilde{\mathbf{A}}_{R_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \widetilde{\mathbf{A}}_{S_\infty}$, $\mathbf{A}'_R \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \mathbf{A}'_S$ and $\widetilde{\mathbf{A}}_{R'_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S \longrightarrow \widetilde{\mathbf{A}}_{S'_\infty}$. The extension $\mathbf{A}_R \to \mathbf{A}_S$ is finite and étale and, hence, $\mathbf{A}_S$ is projective as $\mathbf{A}_R$-module. Since $\widetilde{\mathbf{A}}_{R_\infty}$, $\mathbf{A}'_R$ and $\widetilde{\mathbf{A}}_{R'_\infty}$ are p-adically complete and separated and p is a not a zero divisor in these rings, $\widetilde{\mathbf{A}}_{R_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S$, $\mathbf{A}'_R \otimes_{\mathbf{A}_R} \mathbf{A}_S$ and $\widetilde{\mathbf{A}}_{R'_\infty} \otimes_{\mathbf{A}_R} \mathbf{A}_S$ are p-adically complete and separated and p is a non-zero divisor. The same holds for $\mathbf{A}_S$, $\mathbf{A}_{\overline{R}}^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{S_\infty}$, $\widetilde{\mathbf{A}}_{\overline{R}}^{\mathscr{H}_S}$, $\widetilde{\mathbf{A}}_{\overline{R}}^{\mathrm{H}_S}$, $\mathbf{A}'_S$ and $\widetilde{\mathbf{A}}_{S'_\infty}$. To check that all the inclusions and all the maps above are isomorphisms it then suffices to show it modulo p. This follows from 2.6 if we prove that $\mathbf{A}'_S/p\mathbf{A}'_S = \mathbf{E}'_S$ and $\widetilde{\mathbf{A}}_{S'_\infty}/p\widetilde{\mathbf{A}}_{S'_\infty} = \widetilde{\mathbf{E}}_{S'_\infty}$. Once this is established the other statements in (a) and the first part of (b) follow.

(c) Since by construction $\mathbf{A}'^\dagger_S \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\mathbf{A}'^\dagger_S$, $\widetilde{\mathbf{A}}_{S'_\infty}^\dagger \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\widetilde{\mathbf{A}}_{S'_\infty}^\dagger$, $\mathbf{A}_S^\dagger \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\mathbf{A}_S^\dagger$ and $\mathbf{A}'_S \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\mathbf{A}'_S$, $\mathbf{A}'_{S_\infty} \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\widetilde{\mathbf{A}}_{S'_\infty}$, $\mathbf{A}_S \cap p\widetilde{\mathbf{A}}_{\overline{R}} = p\mathbf{A}_S$ the maps $\mathbf{A}_S^\dagger/p\mathbf{A}_S^\dagger \to \mathbf{A}_S/p\mathbf{A}_S \to \mathbf{E}_{\overline{R}}^{\mathscr{H}_S} = \mathbf{E}_S$, $\mathbf{A}'^\dagger_S/p\mathbf{A}'^\dagger_S \to \mathbf{A}'_S/p\mathbf{A}'_S \to \left(\mathbf{E}'_{\overline{R}}\right)^{\mathrm{H}_S} = \mathbf{E}'_S$ and $\widetilde{\mathbf{A}}_{S'_\infty}^\dagger/p\widetilde{\mathbf{A}}_{S'_\infty}^\dagger \to \widetilde{\mathbf{A}}_{S'_\infty}/p\widetilde{\mathbf{A}}_{S'_\infty} \to \widetilde{\mathbf{E}}_{\overline{R}}^{\mathscr{H}_S} = \widetilde{\mathbf{E}}_{S'_\infty}$ are injective. It follows from [**2**, §4.3(e) & Lem. 4.15] that there exists $\mathbf{A}_S^+ \subset \widetilde{\mathbf{A}}_{S_\infty}^+ \left\{\frac{p^\alpha}{\pi^\beta}\right\} \subset \widetilde{\mathbf{A}}_{S_\infty}^{(0,r]}$, for suitable $\alpha, \beta \in \mathbf{N}$ and $r \in \mathbf{Q}_{>0}$, so that $\mathbf{A}_S^+/\frac{p^\alpha}{\pi^\beta}\mathbf{A}_S^+ \cong \mathbf{E}_S^+$ and $\mathbf{A}_S^+$ is complete for the weak topology. Here, $\widetilde{\mathbf{A}}_{S_\infty}^+ \left\{\frac{p^\alpha}{\pi^\beta}\right\}$ denotes the completion of $\widetilde{\mathbf{A}}_{S_\infty}^+ \left[\frac{p^\alpha}{\pi^\beta}\right]$ with respect to the weak topology. In particular, since π-adic convergence in $\widetilde{\mathbf{A}}_{S_\infty}^+ \left\{\frac{p^\alpha}{\pi^\beta}\right\}$ implies convergence for the weak topology, $\mathbf{A}_S^+$ is π-adically complete. Note that $\mathbf{A}_S^+ \otimes_{\mathbf{A}_V^+} \widetilde{\mathbf{A}}_W^+$ and $\widetilde{\mathbf{A}}_{S_\infty}^+ \otimes_{\widetilde{\mathbf{A}}_{V_\infty}^+} \widetilde{\mathbf{A}}_W^+$

map to the π-adic completion of the image of $\widetilde{\mathbf{A}}^+_{S_\infty}\otimes_{\widetilde{\mathbf{A}}^+_{V_\infty}}\widetilde{\mathbf{A}}^+_W\left\{\frac{p^\alpha}{\pi^\beta}\right\}$ in $\widetilde{\mathbf{A}}^+_{\overline{R}}\left\{\frac{p^\alpha}{\pi^\beta}\right\}$ and that the latter is contained in $\widetilde{\mathbf{A}}^{(0,r]}_{S'_\infty}$. We conclude that $\mathbf{A}^\dagger_S/p\mathbf{A}^\dagger_S$, $\mathbf{A}'^\dagger_S/p\mathbf{A}'^\dagger_S$ and $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}/p\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$ contain the $\overline{\pi}$-adic completion of the image of $\mathbf{E}^+_S$, $\mathbf{E}^+_S\otimes_{\mathbf{E}^+_V}\widetilde{\mathbf{E}}^+_W$ and $\widetilde{\mathbf{E}}^+_{S_\infty}\otimes_{\widetilde{\mathbf{E}}^+_{V_\infty}}\widetilde{\mathbf{E}}^+_W$ respectively. Claim (c) follows then from 2.6.

(b) The fact that $\widetilde{\mathbf{A}}^\dagger_{R_\infty}\otimes_{\mathbf{A}^\dagger_R}\mathbf{A}^\dagger_S\cong\widetilde{\mathbf{A}}^\dagger_{S_\infty}$ follows from 2.8. Since $\mathbf{A}^\dagger_R\subset\mathbf{A}^\dagger_S$ is finite and étale by 2.8 there is a unique idempotent $e_{S/R}$ of $\mathbf{A}^\dagger_S\otimes_{\mathbf{A}^\dagger_R}\mathbf{A}^\dagger_S$ such that for every $x\in\mathbf{A}^\dagger_S\otimes_{\mathbf{A}^\dagger_R}\mathbf{A}^\dagger_S$ we have $m(x)=\mathrm{Tr}_{\mathbf{A}^\dagger_S/\mathbf{A}^\dagger_R}(xe_{S/R})$. Here, m is the multiplication map. Write $e_{S/R}=\sum_{i=1}^u a_i\otimes b_i$ with a_i and $b_i\in\mathbf{A}^{(0,s]}_S$ for some $s\in\mathbf{Q}_{>0}$. Then, $e_{S/R}$ is an idempotent of $\mathbf{A}'^\dagger_S\otimes_{\mathbf{A}'^\dagger_R}\mathbf{A}'^\dagger_S$ and of $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}\otimes_{\widetilde{\mathbf{A}}^\dagger_{S'_\infty}}\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$. By the first part of (b) the extensions $\mathbf{A}'_R\subset\mathbf{A}'_S$ and $\widetilde{\mathbf{A}}_{R'_\infty}\subset\widetilde{\mathbf{A}}_{S'_\infty}$ are finite and étale and $m(x)=\mathrm{Tr}_{\mathbf{A}'_S/\mathbf{A}'_R}(xe_{S/R})$ and $m(x)=\mathrm{Tr}_{\widetilde{\mathbf{A}}_{S'_\infty}/\widetilde{\mathbf{A}}_{R'_\infty}}(xe_{S/R})$. We then get that for every $x\in\mathbf{A}'^\dagger_S$ (resp. $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$) we have $x=m(x\otimes 1)=\sum_{i=1}^u\mathrm{Tr}_{\mathbf{A}'_S/\mathbf{A}'_R}(xa_i)b_i$ (resp. $x=m(x\otimes 1)=\sum_{i=1}^u\mathrm{Tr}_{\widetilde{\mathbf{A}}_{S'_\infty}/\widetilde{\mathbf{A}}_{R'_\infty}}(xa_i)b_i$). Since $\mathrm{Tr}_{\mathbf{A}'_S/\mathbf{A}'_R}$ and $\mathrm{Tr}_{\widetilde{\mathbf{A}}_{S'_\infty}/\widetilde{\mathbf{A}}_{R'_\infty}}$ send overconvergent elements to overconvergent elements, we conclude that the maps in the second part of (b) are surjective.

Since the extension $\mathbf{A}^\dagger_R\subset\mathbf{A}^\dagger_S$ is finite and étale, $\mathbf{A}^\dagger_S$ is projective as $\mathbf{A}^\dagger_R$-module. In particular, p is not a zero divisor in $\mathbf{A}'^\dagger_R\otimes_{\mathbf{A}^\dagger_R}\mathbf{A}^\dagger_S$ and in $\widetilde{\mathbf{A}}^\dagger_{R'_\infty}\otimes_{\mathbf{A}^\dagger_R}\mathbf{A}^\dagger_S$ and those rings are p-adically separated. Thus, to prove that the maps in the second part of (b) are injective, and hence are isomorphisms, it suffices to prove that they are injective modulo p. This follows from (c).

(d) Since the extensions $\mathbf{A}_V\subset\widetilde{\mathbf{A}}_{V_\infty}\subset\widetilde{\mathbf{A}}_W$ are extensions of DVR's, they are flat. Since p is not a zero divisor in $\widetilde{\mathbf{A}}_{S_\infty}$ and $\mathbf{A}_S$, it is not a zero divisor in $\widetilde{\mathbf{A}}_{S_\infty}\otimes_{\widetilde{\mathbf{A}}_{V_\infty}}\widetilde{\mathbf{A}}_W$ and $\mathbf{A}_S\otimes_{\mathbf{A}_V}\widetilde{\mathbf{A}}_W$. Thus, to check the injectivity in (d) we may reduce modulo p. The density can be checked modulo p^n for every $n\in\mathbf{N}$ and, using induction, it suffices in fact to prove it for $n=1$. Then, the first claim of (d) follows from 2.8 and 2.6(c).

We prove the second claim of (d). Suppose that $r=a/b$ with a and $b\in\mathbf{N}$ and let $A_{S_\infty,(a,b)}$ (resp. $A_{S'_\infty,(a,b)}$) denote the p-adic completion of $\widetilde{\mathbf{A}}^+_{S_\infty}\left[\frac{p^a}{[\overline{\pi}]^{\left(\frac{p-1}{p}\right)b}}\right]$ (resp. of $\widetilde{\mathbf{A}}^+_{S'_\infty}\left[\frac{p^a}{[\overline{\pi}]^{\left(\frac{p-1}{p}\right)b}}\right]$), where $[\overline{\pi}]$ is the Teichmüller lift of $\overline{\pi}$. Arguing as in the proof of [**2**, Lem. 4.15] one has

$$A_{S'_\infty,(a,b)}\subseteq\left\{x\in\widetilde{\mathbf{A}}^{(0,r]}_{S'_\infty}|w_r(x)\geq 0\right\}\subseteq A_{S'_\infty,(a,b)}\left[\frac{1}{[\overline{\pi}]}\right].$$

Since $w_r([\overline{\pi}])>0$, we conclude that $A_{S'_\infty,(a,b)}\left[\frac{1}{[\overline{\pi}]}\right]$ is dense in $\widetilde{\mathbf{A}}^{(0,r]}_{S'_\infty}$ for the w_r-adic topology. Since the $[\overline{\pi}]$-adic completion of $A_{S_\infty,(a,b)}\otimes_{\widetilde{\mathbf{A}}^+_{V_\infty}}\widetilde{\mathbf{A}}^+_W$ is contained in the

w_r-adic closure of $\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]} \otimes_{\widetilde{\mathbf{A}}_{V_\infty}^{(0,r]}} \widetilde{\mathbf{A}}_W^{(0,r]}$ and its image in $A_{S'_\infty,(a,b)}$ contains $\overline{\pi}^\ell A_{S'_\infty,(a,b)}$ by 2.4, the conclusion follows. $\square$

Corollary 2.10. — *The extensions $\widetilde{\mathbf{A}}_{R'_\infty} \subset \widetilde{\mathbf{A}}_{S'_\infty}$, $\widetilde{\mathbf{A}}^\dagger_{R'_\infty} \subset \widetilde{\mathbf{A}}^\dagger_{S'_\infty}$, $\mathbf{A}'_R \subset \mathbf{A}'_S$ and $\mathbf{A}'^\dagger_R \subset \mathbf{A}'^\dagger_S$ are finite and étale. Their reduction modulo p coincide with $\widetilde{\mathbf{E}}_{R'_\infty} \subset \widetilde{\mathbf{E}}_{S'_\infty}$ and $\mathbf{E}'_R \subset \mathbf{E}'_S$ respectively.*

2.5. (φ,Γ)-modules and Galois representations. — Let S be as in 2.2. Let $\mathrm{Rep}(\mathscr{G}_S)$ be the abelian tensor category of finitely generated $\mathbf{Z}_p$-modules endowed with a continuous linear action of $\mathscr{G}_S$.

Let $\mathbf{A}$ be one of the rings $\widetilde{\mathbf{A}}_{S_\infty}$, $\widetilde{\mathbf{A}}^\dagger_{S_\infty}$, $\mathbf{A}_S$, $\mathbf{A}^\dagger_S$, $\widetilde{\mathbf{A}}_{S'_\infty}$, $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$, $\mathbf{A}'_S$ and $\mathbf{A}'^\dagger_S$ and let Γ be respectively Γ_S or Γ'_S. Let $(\varphi,\Gamma)-\mathrm{Mod}_{\mathbf{A}}$ (resp. $(\varphi,\Gamma)-\mathrm{Mod}^{\mathrm{et}}_{\mathbf{A}}$) be the tensor category of finitely generated $\mathbf{A}$-modules D endowed with

(i) a semi-linear action of Γ;

(ii) a semi-linear homomorphism φ commuting with Γ (resp. so that $\varphi\otimes 1\colon D\otimes^{\varphi}_{\mathbf{A}}\mathbf{A}\to D$ is an isomorphism of $\mathbf{A}$-modules).

Note that if $\mathbf{A}=\mathbf{A}_S$, then $\mathbf{A}_S$ is noetherian and $(\varphi,\Gamma_S)-\mathrm{Mod}_{\mathbf{A}_S}$ (resp. $(\varphi,\Gamma_S)-\mathrm{Mod}^{\mathrm{et}}_{\mathbf{A}_S}$) is an abelian category.

For any object M in $\mathrm{Rep}(\mathscr{G}_S)$, define

$$\mathfrak{D}(M) := \left(\mathbf{A}_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathscr{H}_S}, \qquad \mathrm{D}(M) := \left(\mathbf{A}'_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathrm{H}_S}$$

$$\widetilde{\mathfrak{D}}(M) := \left(\widetilde{\mathbf{A}}_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathscr{H}_S}, \qquad \widetilde{\mathrm{D}}(M) := \left(\widetilde{\mathbf{A}}_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathrm{H}_S}.$$

Note that $\mathfrak{D}(M)$ (resp. $\widetilde{\mathfrak{D}}(M)$) is an $\mathbf{A}_S$-module (resp. $\widetilde{\mathbf{A}}_{S_\infty}$-module) endowed with a semi-linear action of Γ_S. Analogously, $\mathrm{D}(M)$ (resp. $\widetilde{\mathrm{D}}(M)$) is a $\mathbf{A}'_S$-module (resp. a $\widetilde{\mathbf{A}}_{S'_\infty}$-module) endowed with a semi-linear action of Γ'_S. Analogously, define

$$\mathfrak{D}^\dagger(M) := \left(\mathbf{A}^\dagger_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathscr{H}_S}, \qquad \mathrm{D}^\dagger(M) := \left(\mathbf{A}'^\dagger_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathrm{H}_S}$$

$$\widetilde{\mathfrak{D}}^\dagger(M) := \left(\widetilde{\mathbf{A}}^\dagger_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathscr{H}_S}, \qquad \widetilde{\mathrm{D}}^\dagger(M) := \left(\widetilde{\mathbf{A}}^\dagger_{\overline{R}} \underset{\mathbf{Z}_p}{\otimes} M\right)^{\mathrm{H}_S}.$$

Then, $\mathfrak{D}^\dagger(M)$ (resp. $\widetilde{\mathfrak{D}}^\dagger(M)$) is an $\mathbf{A}^\dagger_S$-module (resp. $\widetilde{\mathbf{A}}^\dagger_{S_\infty}$-module) endowed with a semi-linear action of Γ_S. Analogously, $\mathrm{D}^\dagger(M)$ (resp. $\widetilde{\mathrm{D}}^\dagger(M)$) is a $\mathbf{A}'^\dagger_S$-module (resp. a $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$-module) endowed with a semi-linear action of Γ'_S.

The homomorphism φ on $\widetilde{\mathbf{A}}_{\overline{R}}$ and $\widetilde{\mathbf{A}}^\dagger_{\overline{R}}$ defines a semi-linear action of φ on all these modules commuting with the action of Γ_S (resp. of Γ'_S).

Theorem 2.11. — *The functor $\mathfrak{D}$ defines an equivalence of abelian tensor categories from the category $\mathrm{Rep}(\mathscr{G}_S)$ to the category $(\varphi,\Gamma_S)-\mathrm{Mod}^{\mathrm{et}}_{\mathbf{A}_S}$. Let M be a finitely* *generated $\mathbf{Z}_p$-module endowed with a continuous action of $\mathscr{G}_S$. The inverse is defined associating to an étale (φ,Γ_S)-module D the $\mathscr{G}_R$-module $\mathscr{V}(D) := \left(\mathbf{A}_{\overline{R}} \otimes_{\mathbf{A}_S} D\right)^{\varphi=1}$.*

Proof. — See [**1**, Thm. 7.11]. □

Lemma 2.12. — *Let M be a finitely generated $\mathbf{Z}_p$-module endowed with a continuous action of $\mathscr{G}_S$. Then,*

(i) *$\mathfrak{D}(M)$ (resp. $\mathrm{D}(M)$, $\widetilde{\mathfrak{D}}(M)$, $\widetilde{\mathrm{D}}(M)$) is an étale (φ,Γ_S)-module over $\mathbf{A}_S$ (resp. $\mathbf{A}'_S$, $\widetilde{\mathbf{A}}_{S_\infty}$, $\widetilde{\mathbf{A}}_{S'_\infty}$);*

(i′) *$\mathfrak{D}^\dagger(M)$ (resp. $\mathrm{D}^\dagger(M)$, $\widetilde{\mathfrak{D}}^\dagger(M)$, $\widetilde{\mathrm{D}}^\dagger(M)$) is an étale (φ,Γ_S)-module over $\mathbf{A}_S^\dagger$ (resp. $\mathbf{A}_S'^\dagger$, $\widetilde{\mathbf{A}}_{S_\infty}^\dagger$, $\widetilde{\mathbf{A}}_{S'_\infty}^\dagger$);*

(ii) *we have*

$$\mathfrak{D}(M) = \lim_{\infty\leftarrow n} \mathfrak{D}(M/p^nM),$$
$$\mathrm{D}(M) = \lim_{\infty\leftarrow n} \mathrm{D}(M/p^nM),$$
$$\widetilde{\mathfrak{D}}(M) = \lim_{\infty\leftarrow n} \widetilde{\mathfrak{D}}(M/p^nM),$$
$$\widetilde{\mathrm{D}}(M) = \lim_{\infty\leftarrow n} \widetilde{\mathrm{D}}(M/p^nM),$$

where the limits are inverse limits with respect to $n \in \mathbf{N}$. More precisely, $\mathfrak{D}(M)/p^n\mathfrak{D}(M) \cong \mathfrak{D}(M/p^nM)$, $\mathrm{D}(M)/p^n\mathrm{D}(M) \cong \mathrm{D}(M/p^nM)$, $\widetilde{\mathfrak{D}}(M)/p^n\widetilde{\mathfrak{D}}(M) \cong \widetilde{\mathfrak{D}}(M/p^nM)$ and $\widetilde{\mathrm{D}}(M)/p^n\widetilde{\mathrm{D}}(M) \cong \widetilde{\mathrm{D}}(M/p^nM)$ for every $n \in \mathbf{N}$.

(ii′) *if M is torsion, then $\mathfrak{D}^\dagger(M) = \mathfrak{D}(M)$, $\mathrm{D}^\dagger(M) = \mathrm{D}(M)$, $\widetilde{\mathfrak{D}}^\dagger(M) = \widetilde{\mathfrak{D}}(M)$ and $\widetilde{\mathrm{D}}^\dagger(M) = \widetilde{\mathrm{D}}(M)$;*

(iii) *the natural maps*

$$\mathfrak{D}(M) \underset{\mathbf{A}_S}{\otimes} \mathbf{A}'_S \longrightarrow \mathrm{D}(M), \qquad \mathfrak{D}(M) \underset{\mathbf{A}_S}{\otimes} \mathbf{A}_{\overline{R}} \longrightarrow M \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}_{\overline{R}}$$

and

$$\mathfrak{D}(M) \underset{\mathbf{A}_S}{\otimes} \widetilde{\mathbf{A}}_{S_\infty} \longrightarrow \widetilde{\mathfrak{D}}(M), \qquad \mathfrak{D}(M) \underset{\mathbf{A}_S}{\otimes} \widetilde{\mathbf{A}}_{S'_\infty} \longrightarrow \widetilde{\mathrm{D}}(M)$$

are isomorphisms;

(iii′) *the natural maps*

$$\mathfrak{D}^\dagger(M) \underset{\mathbf{A}_S^\dagger}{\otimes} \mathbf{A}_S'^\dagger \longrightarrow \mathrm{D}^\dagger(M), \qquad \mathfrak{D}^\dagger(M) \underset{\mathbf{A}_S^\dagger}{\otimes} \mathbf{A}_{\overline{R}}^\dagger \longrightarrow M \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}_{\overline{R}}^\dagger$$

and

$$\mathfrak{D}^\dagger(M) \underset{\mathbf{A}_S^\dagger}{\otimes} \widetilde{\mathbf{A}}_{S_\infty}^\dagger \longrightarrow \widetilde{\mathfrak{D}}^\dagger(M), \qquad \mathfrak{D}^\dagger(M) \underset{\mathbf{A}_S^\dagger}{\otimes} \widetilde{\mathbf{A}}_{S'_\infty}^\dagger \longrightarrow \widetilde{\mathrm{D}}^\dagger(M)$$

are isomorphisms;

(iv) *the natural maps*

$$\mathfrak{D}^\dagger(M) \underset{\mathbf{A}_S^\dagger}{\otimes} \mathbf{A}_S \longrightarrow \mathfrak{D}(M), \qquad \mathrm{D}^\dagger(M) \underset{\mathbf{A}_S'^\dagger}{\otimes} \mathbf{A}'_S \longrightarrow \mathrm{D}(M)$$

and

$$\widetilde{\mathfrak{D}}^\dagger(M) \otimes_{\widetilde{\mathbf{A}}^\dagger_{S_\infty}} \widetilde{\mathbf{A}}_{S_\infty} \longrightarrow \widetilde{\mathfrak{D}}(M), \qquad \widetilde{\mathrm{D}}^\dagger(M) \otimes_{\widetilde{\mathbf{A}}^\dagger_{S'_\infty}} \widetilde{\mathbf{A}}_{S'_\infty} \longrightarrow \widetilde{\mathrm{D}}(M)$$

are isomorphisms.

Proof. — We refer the reader to [**1**, Thm. 7.11] and [**2**, Thm 4.35] for the proofs that $\mathfrak{D}(M)$ and $\mathfrak{D}^\dagger(M)$ are étale (φ,Γ_S)-modules and that $\mathfrak{D}(M)=\mathfrak{D}^\dagger(M)\otimes_{\mathbf{A}^\dagger_S}\mathbf{A}_S$, that $\mathfrak{D}(M)\otimes_{\mathbf{A}_S}\mathbf{A}_{\overline{R}}\cong M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}$ and $\mathfrak{D}^\dagger(M)\otimes_{\mathbf{A}^\dagger_S}\mathbf{A}^\dagger_{\overline{R}}\cong M\otimes_{\mathbf{Z}_p}\mathbf{A}^\dagger_{\overline{R}}$. Claims (i), (i′) and (iv) follow from this and the displayed isomorphisms. We prove the other statements.

Due to 2.8, 2.9 and 2.10 to prove (ii′), (iii) and (iii′) one may pass to an extension $S_\infty\subset T_\infty$ in $\overline{R}$ finite, étale and Galois after inverting p. For example, $\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathscr{H}_T} = \big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathscr{H}_S}\otimes_{\mathbf{A}_S}\mathbf{A}_T$ and $\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathrm{H}_T} = \big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathrm{H}_S}\otimes_{\mathbf{A}'_S}\mathbf{A}'_T$ by étale descent. Hence, if $\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathscr{H}_T}\otimes_{\mathbf{A}_T}\mathbf{A}'_T\to\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathrm{H}_T}$ is an isomorphism, taking the $\mathscr{H}_S$-invariants, we get the claimed isomorphism $\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathscr{H}_S}\otimes_{\mathbf{A}_S}\mathbf{A}'_S\to\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathrm{H}_S}$.

Suppose first that there exists $N\in\mathbf{N}$ such that $p^NM=0$. Then, there exists an extension $S_\infty\subset T_\infty$ such that $\mathscr{H}_T\subset\mathscr{H}_S$ acts trivially on M. Replacing S_∞ with T_∞, we may then assume that $\mathscr{H}_S$ acts trivially on M. By 2.9 we have $\mathbf{A}^\dagger_S/p^N\mathbf{A}^\dagger_S=\mathbf{A}_S/p^N\mathbf{A}_S$, $\mathbf{A}'^\dagger_S/p^N\mathbf{A}'^\dagger_S=\mathbf{A}'_S/p^N\mathbf{A}'_S$, $\widetilde{\mathbf{A}}^\dagger_{S_\infty}/p^N\widetilde{\mathbf{A}}^\dagger_{S_\infty}=\widetilde{\mathbf{A}}_{S_\infty}/p^N\widetilde{\mathbf{A}}_{S_\infty}$ and eventually $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}/p^N\widetilde{\mathbf{A}}^\dagger_{S'_\infty}=\widetilde{\mathbf{A}}_{S'_\infty}/p^N\widetilde{\mathbf{A}}_{S'_\infty}$. Furthermore, we have in this case $\mathfrak{D}(M)=M\otimes_{\mathbf{Z}_p}\mathbf{A}_S$, $\mathrm{D}(M)=M\otimes_{\mathbf{Z}_p}\mathbf{A}'_S$, $\widetilde{\mathfrak{D}}(M)=M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{S_\infty}$, $\widetilde{\mathrm{D}}(M)=M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{S'_\infty}$ and similarly for the overconvergent (φ,Γ_S)-modules. Then, the claims follow from 2.9.

Assume next that M is free of rank n. It follows from [**2**, Thm. 4.35] that there exists an extension $R_\infty\subset T_\infty$ in $\overline{R}$ finite, étale and Galois after inverting p such that $\mathfrak{D}^\dagger(M)\otimes_{\mathbf{A}^\dagger_S}\mathbf{A}^\dagger_T$ is a free $\mathbf{A}^\dagger_T$-module of rank n. As we have seen above we may and will replace S_∞ with T_∞ so that $\mathfrak{D}^\dagger(M)$ (resp. $\mathfrak{D}(M)$) is a free $\mathbf{A}^\dagger_S$-module (resp. $\mathbf{A}_S$-module). Fix a basis $\{e_1,\ldots,e_n\}$ of $\mathfrak{D}^\dagger(M)$. It is also a $\mathbf{A}_R$-basis of $\mathfrak{D}(M)$. Hence, it is a basis over $\mathbf{A}^\dagger_{\overline{R}}$ (resp. $\widetilde{\mathbf{A}}^\dagger_{\overline{R}}$, $\mathbf{A}_{\overline{R}}$, $\widetilde{\mathbf{A}}_{\overline{R}}$) of $M\otimes_{\mathbf{Z}_p}\mathbf{A}^\dagger_{\overline{R}}$ (resp. $M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}^\dagger_{\overline{R}}$, $M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}$, $M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}$). Since $\mathscr{H}_S$ and H_S act trivially on $\{e_1,\ldots,e_s\}$, we get claims (iii) and (iii′). For example, $\widetilde{\mathfrak{D}}(M)=\big(M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\big)^{\mathscr{H}_S}=\widetilde{\mathbf{A}}_{S_\infty}e_1\oplus\cdots\oplus\widetilde{\mathbf{A}}_{S_\infty}e_n=\mathfrak{D}(M)\otimes_{\mathbf{A}_S}\widetilde{\mathbf{A}}_{S_\infty}$.

We are left to prove (ii). We may assume that M is torsion free, since the claim for the torsion part is trivial. Note that $\mathfrak{D}(M)$, $\widetilde{\mathfrak{D}}(M)$, $\mathrm{D}(M)$ and $\widetilde{\mathrm{D}}(M)$ are submodules of invariants of free modules over p-adically complete and separated rings. For example, $\mathfrak{D}(M)=\big(M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}\big)^{\mathscr{H}_S}\subset M\otimes_{\mathbf{Z}_p}\mathbf{A}_{\overline{R}}$. Hence, they are themselves p-adically complete and separated. It suffices to show that for every $n\in\mathbf{N}$ the map from their quotient modulo p^n to $\mathfrak{D}\big(M/p^nM\big)$ (resp. $\widetilde{\mathfrak{D}}\big(M/p^nM\big)$, $\mathrm{D}\big(M/p^nM\big)$, $\widetilde{\mathrm{D}}\big(M/p^nM\big)$) is an isomorphism. Due to (iii) it suffices to show it for $\mathfrak{D}(M)$ and in this case it follows from the fact that $\mathfrak{D}$ is an exact functor by 2.11. □

2.6. The weak topology on the (φ,Γ)-modules. — Suppose that $M \cong \mathbf{Z}_p^n \times \prod_{i=1}^m \mathbf{Z}/p^{s_i}\mathbf{Z}$ as a $\mathbf{Z}_p$-module. Then, $M \otimes \widetilde{\mathbf{A}}_{\overline{R}}$ is isomorphic to $\widetilde{\mathbf{A}}_{\overline{R}}^n \times \prod_{i=1}^m \widetilde{\mathbf{A}}_{\overline{R}}/p^{s_i}\widetilde{\mathbf{A}}_{\overline{R}}$ as $\widetilde{\mathbf{A}}_{\overline{R}}$-module and, in particular, the product topology defines a topology on $M \otimes \widetilde{\mathbf{A}}_{\overline{R}}$. It is independent of the choice of the presentation of M as $\mathbf{Z}_p$-module and the action of $\mathscr{G}_S$ is continuous for such topology. Note that $\mathfrak{D}(M)$, $\widetilde{\mathfrak{D}}(M)$, $\mathrm{D}(M)$, $\widetilde{\mathrm{D}}(M)$, $\mathfrak{D}^\dagger(M)$, $\widetilde{\mathfrak{D}}^\dagger(M)$, $\mathrm{D}^\dagger(M)$ and $\widetilde{\mathrm{D}}^\dagger(M)$ are by construction submodules of $M \otimes \widetilde{\mathbf{A}}_{\overline{R}}$. They are then endowed with the topology induced from the one just defined on $M \otimes \widetilde{\mathbf{A}}_{\overline{R}}$. We call it the *weak topology.*

We state the following theorem relating the cohomology (continuous for the weak topology) of the various (φ,Γ)-modules introduced above.

Theorem 2.13. — *The natural maps*

$$\mathrm{H}^n(\Gamma_S,\mathfrak{D}(M)) \longrightarrow \mathrm{H}^n(\Gamma_S,\widetilde{\mathfrak{D}}(M)) \longrightarrow \mathrm{H}^n\left(\mathscr{G}_S, M \underset{\mathbf{Z}_p}{\otimes} \widetilde{\mathbf{A}}_{\overline{R}}\right),$$

$$\mathrm{H}^n(\Gamma'_S,\mathrm{D}(M)) \longrightarrow \mathrm{H}^n(\Gamma'_S,\widetilde{\mathrm{D}}(M)) \longrightarrow \mathrm{H}^n\left(\mathrm{G}_S, M \underset{\mathbf{Z}_p}{\otimes} \widetilde{\mathbf{A}}_{\overline{R}}\right),$$

$$\mathrm{H}^n(\Gamma_S,\mathfrak{D}^\dagger(M)) \longrightarrow \mathrm{H}^n(\Gamma_S,\widetilde{\mathfrak{D}}^\dagger(M)) \longrightarrow \mathrm{H}^n\left(\mathscr{G}_S, M \underset{\mathbf{Z}_p}{\otimes} \widetilde{\mathbf{A}}^\dagger_{\overline{R}}\right)$$

and

$$\mathrm{H}^n(\Gamma'_S,\mathrm{D}^\dagger(M)) \longrightarrow \mathrm{H}^n(\Gamma'_S,\widetilde{\mathrm{D}}^\dagger(M)) \longrightarrow \mathrm{H}^n\left(\mathrm{G}_S, M \underset{\mathbf{Z}_p}{\otimes} \widetilde{\mathbf{A}}^\dagger_{\overline{R}}\right)$$

are all isomorphisms.

Proof. — See A.14. □

3. Galois cohomology and (φ,Γ)-modules

In this section we show how, given a finitely generated $\mathbf{Z}_p$-module M with continuous action of $\mathscr{G}_S$, one can compute the cohomology groups $\mathrm{H}^n(\mathscr{G}_S, M)$ and $\mathrm{H}^n(\mathrm{G}_S, M)$ in terms of the associated (φ,Γ_S)-modules $\mathfrak{D}(M)$, $\widetilde{\mathfrak{D}}(M)$, $\mathfrak{D}^\dagger(M)$, $\widetilde{\mathfrak{D}}^\dagger(M)$, $\mathrm{D}(M)$, $\widetilde{\mathrm{D}}(M)$, $\mathrm{D}^\dagger(M)$ and $\widetilde{\mathrm{D}}^\dagger(M)$. We start with the following crucial:

Definition 3.1. — Let D be a continuous (φ,Γ_S)-module over $\mathbf{A}_S$ or $\widetilde{\mathbf{A}}_{S_\infty}$ or $\mathbf{A}_S^\dagger$ or $\widetilde{\mathbf{A}}_{S_\infty}^\dagger$ (resp. over $\mathbf{A}'_S$ or $\widetilde{\mathbf{A}}_{S'_\infty}$ or $\mathbf{A}_S'^\dagger$ or $\widetilde{\mathbf{A}}^\dagger_{S'_\infty}$). Define $\mathscr{C}^\bullet(\Gamma_S,\mathrm{D})$ (resp. $\mathscr{C}^\bullet(\Gamma'_S,\mathrm{D})$) to be the complex of continuous cochains with values in D.

Let $\mathscr{T}^\bullet(\mathrm{D})$ (resp. $\mathscr{T}'^\bullet(\mathrm{D})$) be the mapping cone associated to $\varphi-1\colon \mathscr{C}^\bullet(\Gamma_S,\mathrm{D}) \to \mathscr{C}^\bullet(\Gamma_S,\mathrm{D})$ (resp. to $\varphi-1\colon \mathscr{C}^\bullet(\Gamma'_S,\mathrm{D}) \to \mathscr{C}^\bullet(\Gamma'_S,\mathrm{D})$).

Theorem 3.2. — *There are isomorphisms of δ-functors from the category of* $\mathrm{Rep}(\mathscr{G}_S)$ *to the category of abelian groups:*

$$\begin{aligned}
\rho_i &: \mathrm{H}^i\big(\mathscr{T}^\bullet(\mathfrak{D}(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathscr{G}_S,_), & \rho_i' &: \mathrm{H}^i\big(\mathscr{T}'^\bullet(\mathrm{D}(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{G}_S,_),\\
\widetilde{\rho}_i &: \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathfrak{D}}(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathscr{G}_S,_), & \widetilde{\rho}_i' &: \mathrm{H}^i\big(\mathscr{T}'^\bullet(\widetilde{\mathrm{D}}(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{G}_S,_),\\
\rho_i^\dagger &: \mathrm{H}^i\big(\mathscr{T}^\bullet(\mathfrak{D}^\dagger(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathscr{G}_S,_), & \rho_i'^\dagger &: \mathrm{H}^i\big(\mathscr{T}'^\bullet(\mathrm{D}^\dagger(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{G}_S,_),\\
\widetilde{\rho}_i^\dagger &: \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathfrak{D}}^\dagger(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathscr{G}_S,_), & \widetilde{\rho}_i'^\dagger &: \mathrm{H}^i\big(\mathscr{T}'^\bullet(\widetilde{\mathrm{D}}^\dagger(_))\big) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{G}_S,_).
\end{aligned}$$

The isomorphisms ρ_i', $\widetilde{\rho}_i'$, $\rho_i'^\dagger$ and $\widetilde{\rho}_i'^\dagger$ are $\mathrm{Gal}(M/K)$*-equivariant.*

Furthermore, all the maps in the square

$$\begin{tikzcd}
\mathrm{H}^i\big(\mathscr{T}^\bullet(\mathfrak{D}^\dagger(_))\big) \arrow[r] \arrow[d] & \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathfrak{D}}^\dagger(_))\big) \arrow[d] \\
\mathrm{H}^i\big(\mathscr{T}^\bullet(\mathfrak{D}(_))\big) \arrow[r] & \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathfrak{D}}(_))\big)
\end{tikzcd}$$

induced by the natural inclusions of (φ,Γ_S)-modules $\mathfrak{D}^\dagger(W) \subset \mathfrak{D}(W) \subset \widetilde{\mathfrak{D}}(W)$ and $\mathfrak{D}^\dagger(W) \subset \widetilde{\mathfrak{D}}^\dagger(W) \subset \widetilde{\mathfrak{D}}(W)$, for $W \in \mathrm{Rep}(\mathscr{G}_S)$, are isomorphisms and they are compatible with the maps $\rho_i^\dagger$, $\widetilde{\rho}_i^\dagger$, ρ_i and $\widetilde{\rho}_i$. Similarly, all the maps in the square

$$\begin{tikzcd}
\mathrm{H}^i\big(\mathscr{T}'^\bullet(\mathrm{D}^\dagger(_))\big) \arrow[r] \arrow[d] & \mathrm{H}^i\big(\mathscr{T}'^\bullet(\widetilde{\mathrm{D}}^\dagger(_))\big) \arrow[d] \\
\mathrm{H}^i\big(\mathscr{T}'^\bullet(\mathrm{D}(_))\big) \arrow[r] & \mathrm{H}^i\big(\mathscr{T}'^\bullet(\widetilde{\mathrm{D}}(_))\big)
\end{tikzcd}$$

are isomorphisms and are compatible with the maps $\rho_i'^\dagger$, $\widetilde{\rho}_i'^\dagger$, ρ_i' and $\widetilde{\rho}_i'$

Proof. — First of all we exhibit in 3.1 the maps ρ_i and ρ_i' (with or without $\widetilde{\ }$ or $\dagger$) so that they are compatible with the displayed squares and they are compatible with the residual action of G_V (if one exists). We then prove that they are isomorphisms in 3.3. Eventually, we show that they are isomorphisms of δ-functors in 3.4. □

3.1. The maps. — Let W be a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$. Let $D(W)$ and A be (1) $\mathfrak{D}(W)$ and $\mathbf{A}$, (2) $\widetilde{\mathfrak{D}}$ and $\widetilde{\mathbf{A}}$, (3) $\mathfrak{D}^\dagger(W)$ and $\mathbf{A}^\dagger$ or (4) $\widetilde{\mathfrak{D}}^\dagger(W)$ and $\widetilde{\mathbf{A}}^\dagger$. Since in each case $D(W) \otimes_{A_S} A_{\overline{R}} \cong W \otimes_{\mathbf{Z}_p} A_{\overline{R}}$ by 2.12 and since the sequence (22) admits a right continuous splitting, we have exact sequences of $\mathscr{G}_S$-modules

$$0 \longrightarrow W \longrightarrow D(W) \underset{A_S}{\otimes} A_{\overline{R}} \xrightarrow{\varphi-1} D(W) \underset{A_S}{\otimes} A_{\overline{R}} \longrightarrow 0 \tag{1}$$

Similarly, let $D'(W)$ and A' be (1) $\mathrm{D}(W)$ and $\mathbf{A}'$, (2) $\widetilde{\mathrm{D}}$ and $\widetilde{\mathbf{A}}$, (3) $\mathrm{D}^\dagger(W)$ and $\mathbf{A}'^\dagger$ or (4) $\widetilde{\mathrm{D}}^\dagger(W)$ and $\widetilde{\mathbf{A}}^\dagger$. In each case $D'(W) \otimes_{A_S'} A_{\overline{R}}' \cong W \otimes_{\mathbf{Z}_p} A_{\overline{R}}'$ by 2.12. Thanks to B.1 we get exact sequences of $\mathscr{G}_S$-modules

$$0 \longrightarrow W \longrightarrow D'(W) \underset{A_S'}{\otimes} A_{\overline{R}}' \xrightarrow{\varphi-1} D'(W) \underset{A_S'}{\otimes} A_{\overline{R}}' \longrightarrow 0 \tag{2}$$

The maps in both exact sequences are continuous for the weak topology by 2.12 and admit a right splitting as $\mathbf{Z}_p$-modules by B.1.

Let (α,β) be an n-cochain of $\mathscr{T}^\bullet(D(W))$ i. e., in $\mathscr{C}^{n-1}(\Gamma_S,D(W))\times\mathscr{C}^n(\Gamma_S,D(W))$. Define

$$c^n_{\alpha,\beta} := \beta + (-1)^n d\big(\sigma(\alpha)\big) \in \mathscr{C}^n(\mathscr{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A_{\overline{R}}),$$

where d is the differential operator on $\mathscr{C}^n(\Gamma_S,D(W))$ and σ is the left inverse of $\varphi-1$ defined in B.1 (for each of the four possibilities for A).

Recall that the derivation on $\mathscr{T}^\bullet(D(W))$ is given by $d\big((\alpha,\beta)\big) = \big((-1)^n(\varphi-1)(\beta)+d\alpha, d\beta\big)$. Thus, (α,β) is an n-cocycle if and only if it satisfies $(-1)^n(\varphi-1)\beta+d\alpha=0$ and $d\beta=0$. In this case, $dc^n_{\alpha,\beta}=0$ and $(\varphi-1)c^n_{\alpha,\beta}=(\varphi-1)\beta+(-1)^n d(\varphi-1)\sigma(\alpha)=(\varphi-1)\beta+(-1)^n d\alpha=0$. Thus, $c^n_{\alpha,\beta}$ is an n-cocycle in $\mathscr{C}^n(\mathscr{G}_S,W)$ by (1).

Choose a different continuous left inverse σ' of $\varphi-1$. Then, $(\varphi-1)(\sigma'-\sigma)=0$ so that $\big(\sigma'-\sigma\big)(\alpha)$ lies in $\mathscr{C}^{n-1}(\mathscr{G}_S,W)$. Thus, $\beta+(-1)^n d\big(\sigma'(\alpha)\big)-\beta-(-1)^n d\big(\sigma(\alpha)\big)=(-1)^n d(\sigma'-\sigma)(\alpha)$. In particular, $c^n_{\alpha,\beta}$ depends on the choice of σ up to a continuous coboundary with values in W.

Let $(\alpha,\beta)=((-1)^{n-1}(\varphi-1)b+da, db)\in\mathscr{C}^{n-1}(\Gamma_S,D(W))\times\mathscr{C}^n(\Gamma_S,D(W))$ be an n-coboundary in $\mathscr{T}^\bullet(D(W))$. Then, $c^n_{\alpha,\beta}=db+(-1)^{2n-1}d(\sigma\circ(\varphi-1))(b)+(-1)^n d(\sigma(d(a)))$. Note that $\big(1-(\sigma\circ(\varphi-1)\big)b$ and $\sigma\big(d(a)\big)-d\big(\sigma(a)\big)$ are annihilated by $\varphi-1$. Hence, $c^n_{\alpha,\beta}$ is the image via the differential of $(1-(\sigma\circ(\varphi-1))(b)+(-1)^n\,\big(\sigma\big(d(a)\big)-d\big(\sigma(a)\big)\big)$ which lies in $\mathscr{C}^{n-1}(\mathscr{G}_S,W)$. In particular, it is a continuous coboundary.

We thus get a map

$$r_i^W : \mathrm{H}^i\big(\mathscr{T}^\bullet(D(W))\big) \longrightarrow \mathrm{H}^i(\mathscr{G}_S,W).$$

By construction it is functorial in W. In case (1) we get the map ρ_i, in case (2) we get $\widetilde{\rho}_i$, in case (3) we get the map $\rho_i^\dagger$ and in case (4) we get $\widetilde{\rho}_i^\dagger$. By construction they are compatible with the first commutative displayed square appearing in 3.2.

Analogously, using (2), one gets the claimed map $r_i'^W$. In case (1) we get the map ρ_i', in case (2) we get $\widetilde{\rho}_i'$, in case (3) we get the map $\rho_i'^\dagger$ and in case (4) we get $\widetilde{\rho}_i'^\dagger$. They are compatible with the second commutative displayed square appearing in 3.2. Furthermore, we also have actions of G_V and we claim that $r_i'^W$ is G_V-equivariant.

Indeed, let (α,β) be an n-cocycle in $\mathscr{C}^{n-1}(\Gamma'_S,D'(W))\times\mathscr{C}^n(\Gamma'_S,D'(W))$. Let $g\in\mathrm{G}_V$. Then, $g\big((\alpha,\beta)\big)=\big(g(\alpha),g(\beta)\big)$ and $c^n_{g((\alpha,\beta))}=g(\beta)+(-1)^n d\big(\sigma(g(\alpha))\big)$. On the other hand, $g\big(c^n_{\alpha,\beta}\big)=g(\beta)+(-1)^n g\big(d(\sigma(a))\big)$. Note that $g\circ d=d\circ g$ and $(\varphi-1)(\sigma\circ g-g\circ\sigma)=0$ since φ is Γ_V-equivariant. Thus, $c^n_{g((\alpha,\beta))}-g\big(c^n_{\alpha,\beta}\big)=(-1)^n d\big((\sigma\circ g-g\circ\sigma)(\alpha)\big)$ is a coboundary in $\mathscr{C}^n(\mathscr{G}_S,W)$.

Proposition 3.3. — *The maps ρ_i, $\widetilde{\rho}_i$, $\rho_i^\dagger$, $\widetilde{\rho}_i^\dagger$, ρ_i', $\widetilde{\rho}_i'$, $\rho_i'^\dagger$ and $\widetilde{\rho}_i'^\dagger$ are isomorphisms.*

Proof. — We use the notation of 3.1. Since $\mathscr{T}^\bullet(D)$ and $\mathscr{T}'^\bullet(D')$ are mapping cones, we get exact sequences

(3)
$$\mathrm{H}^{n-1}(\Gamma_S, D(W)) \xrightarrow{\hat{\delta}_n} \mathrm{H}^n(\mathscr{T}^\bullet(D(W))) \longrightarrow \mathrm{H}^n(\Gamma_S, D(W)) \xrightarrow{(-1)^n(\varphi-1)} \mathrm{H}^n(\Gamma_S, D(W))$$

and

(4)
$$\mathrm{H}^{n-1}(\Gamma'_S, D'(W)) \xrightarrow{\hat{\delta}'_n} \mathrm{H}^n(\mathscr{T}'^\bullet(D'(W))) \longrightarrow \mathrm{H}^n(\Gamma'_S, D'(W)) \xrightarrow{(-1)^n(\varphi-1)} \mathrm{H}^n(\Gamma'_S, D'(W)).$$

They are compatible with respect to the natural inclusions $\mathfrak{D}^\dagger(W) \subset \mathfrak{D}(W) \subset \widetilde{\mathfrak{D}}(W)$ and $\mathfrak{D}^\dagger(W) \subset \widetilde{\mathfrak{D}}^\dagger(W) \subset \widetilde{\mathfrak{D}}(W)$ (resp. $\mathrm{D}^\dagger(W) \subset \mathrm{D}(W) \subset \widetilde{\mathrm{D}}(W)$ and $\mathrm{D}^\dagger(W) \subset \widetilde{\mathrm{D}}^\dagger(W) \subset \widetilde{\mathrm{D}}(W)$). Thanks to A.14 we deduce that the horizontal arrows in the two displayed squares of 3.2 are isomorphisms. We are then left to prove that $\widetilde{\rho}_i$, $\widetilde{\rho}_i^\dagger$, $\widetilde{\rho}'_i$ and $\widetilde{\rho}_i'^\dagger$ are isomorphisms.

From the exactness of (1) and (2) and the existence of a continuous right splitting we get the exact sequences

(5)
$$\mathrm{H}^{n-1}(\mathscr{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A_{\overline{R}}) \xrightarrow{\delta_n} \mathrm{H}^n(\mathscr{G}_S, W) \longrightarrow \mathrm{H}^n(\mathscr{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A_{\overline{R}}) \xrightarrow{\varphi-1} \mathrm{H}^n(\mathscr{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A_{\overline{R}})$$

and

(6)
$$\mathrm{H}^{n-1}(\mathrm{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A'_{\overline{R}}) \xrightarrow{\delta'_n} \mathrm{H}^n(\mathrm{G}_S, W) \longrightarrow \mathrm{H}^n(\mathrm{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A'_{\overline{R}}) \xrightarrow{\varphi-1} \mathrm{H}^n(\mathrm{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A'_{\overline{R}})$$

Thanks to A.14 the inflation maps

$$\mathrm{Inf}_n \colon \mathrm{H}^n(\Gamma_S, D(W)) \longrightarrow \mathrm{H}^n(\mathscr{G}_S, D(W) \underset{A_S}{\otimes} A_{\overline{R}}) = \mathrm{H}^n(\mathscr{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A_{\overline{R}})$$

and

$$\mathrm{Inf}'_n \colon \mathrm{H}^n(\Gamma'_S, D'(W)) \longrightarrow \mathrm{H}^n(\mathrm{G}_S, D'(W) \underset{A'_S}{\otimes} A'_{\overline{R}}) = \mathrm{H}^n(\mathrm{G}_S, W \underset{\mathbf{Z}_p}{\otimes} A'_{\overline{R}})$$

in cases (2) and (4) of 3.1 are isomorphisms. Take a cocycle τ in $\mathscr{C}^{n-1}(\Gamma_S, D(W))$. One constructs $\delta_n\big(\mathrm{Inf}_{n-1}(\tau)\big)$ as $d\big(\sigma\big(\mathrm{Inf}_{n-1}(\tau)\big)\big)$. On the other hand, $\hat{\delta}_n(\tau) = (\tau, 0)$ in $\mathscr{C}^{n-1}(\Gamma_S, D(W)) \times \mathscr{C}^n(\Gamma_S, D(W))$ and $c^n_{\tau,0} = (-1)^n d\big(\sigma(\tau)\big)$. Thus, $\delta_n \circ (-1)^{n-1}\mathrm{Inf}_{n-1} = \rho^W_n \circ (-1)\hat{\delta}_n$. If (α,β) is an n-cocycle in $\mathscr{T}^\bullet(D(W))$, its image in $\mathrm{H}^n(\Gamma_S, D(W))$ is the class of β. The image of $c^n_{\alpha,\beta}$ in $\mathrm{H}^n(\mathscr{G}_S, W \otimes_{\mathbf{Z}_p} A_{\overline{R}})$ is the class of $\beta + (-1)^n d\big(\sigma(\alpha)\big)$ i. e., of β. We conclude that the exact sequences (3) and (5) are compatible via r^W_n and the inflation maps Inf_n and Inf_{n-1} i. e., the following diagram commutes (the rows continue on the left and on the right):

$$\begin{tikzcd}
\mathrm{H}^{n-1}(\Gamma_S, D(W)) \arrow[r, "-\hat{\delta}_n"] \arrow[d, "(-1)^{n-1}\mathrm{Inf}_{n-1}"] & \mathrm{H}^n(\mathscr{T}^\bullet(D(W))) \arrow[r] \arrow[d, "r^W_n"'] & \mathrm{H}^n(\Gamma_S, D(W)) \arrow[r, "(-1)^n(\varphi-1)"] \arrow[d, "\mathrm{Inf}_n"] & \cdots \\
\mathrm{H}^{n-1}(\mathscr{G}_S, W \otimes_{\mathbf{Z}_p} A_{\overline{R}}) \arrow[r, "\delta_n"] & \mathrm{H}^n(\mathscr{G}_S, W) \arrow[r] & \mathrm{H}^n(\mathscr{G}_S, W \otimes_{\mathbf{Z}_p} A_{\overline{R}}) \arrow[r, "\varphi-1"] & \cdots
\end{tikzcd}$$

An analogous argument implies that the exact sequences (4) and (6) are compatible via $r_n'^W$ and the inflation maps Inf_n' and Inf_{n-1}'. The proposition follows. $\square$

Proposition 3.4. — *The functors ρ_i, $\widetilde{\rho}_i$, $\rho_i^\dagger$, $\widetilde{\rho}_i^\dagger$, ρ_i', $\widetilde{\rho}_i'$, $\rho_i'^\dagger$ and $\widetilde{\rho}_i'^\dagger$ are isomorphisms of δ-functors.*

Proof. — We use the notation of 3.1. We prove the proposition for r_i. The proof for r_i' is similar. Let $0 \to W_1 \to W_2 \to W_3 \to 0$ be an exact sequence of $\mathscr{G}_S$-representations. We need to prove that one has a connecting homomorphism $\delta\colon \mathrm{H}^n\big(\mathscr{T}^\bullet(D(W_3))\big) \longrightarrow \mathrm{H}^{n+1}\big(\mathscr{T}^\bullet(D(W_1))\big)$, making $\mathrm{H}^n\big(\mathscr{T}^\bullet(D(\_))\big)$ into a δ-functor, and that the diagram

$$\begin{array}{ccc}
\mathrm{H}^n\big(\mathscr{T}^\bullet(D(W_3))\big) & \xrightarrow{\ \delta\ } & \mathrm{H}^{n+1}\big(\mathscr{T}^\bullet(D(W_1))\big) \\
{\scriptstyle r_i^{W_3}}\big\downarrow & & \big\downarrow{\scriptstyle r_i^{W_1}} \\
\mathrm{H}^i(\mathscr{G}_S, W_3) & \xrightarrow{\ \delta'\ } & \mathrm{H}^{i+1}(\mathscr{G}_S, W_1),
\end{array}$$

where δ' is the connecting homomorphisms, commutes.

We start defining δ. Let (α,β) be an n-cocycle in $\mathscr{T}^n(D(W_3)) = \mathscr{C}^{n-1}(\Gamma_S, D(W_3))\times \mathscr{C}^n(\Gamma_S, D(W_3))$. Due to A.5.2 there exist $a \in \mathscr{C}^{n-1}(\Gamma_S, D(W_2))$ and $b \in \mathscr{C}^n(\Gamma_S, D(W_2))$ lifting α and β respectively. Then, $d(a,b) = \big((-1)^n(\varphi-1)b + da, db\big)$ is an element of $\mathscr{C}^n(\Gamma_S, D(W_1)) \times \mathscr{C}^{n+1}(\Gamma_S, D(W_1))$ and it is a $n+1$-cocycle of $\mathscr{T}^\bullet(D(W_1))$. Its cohomology class is, by definition, $\delta\big((\alpha,\beta)\big)$. Note that

$$c^{n+1}_{d(a,b)} = db + (-1)^{2n+1}d\big(\sigma\circ(\varphi-1)(b)\big) + (-1)^{n+1}d\big(\sigma(da)\big).$$

On the other hand, $c^n_{(\alpha,\beta)} = \beta + (-1)^n d\big(\sigma(\alpha)\big)$. Consider $c^n_{(a,b)} = b + (-1)^n d\big(\sigma(a)\big)$ in $\mathscr{C}^n(\mathscr{G}_S, W_2\otimes_{\mathbf{Z}_p} A_{\overline{R}})$. Then, $\gamma := c^n_{(a,b)} - \sigma\big((\varphi-1)\big(c^n_{(a,b)}\big)\big)$ lies in $\mathscr{C}^n(\mathscr{G}_S, W_2)$. Furthermore, it lifts $c^n_{(\alpha,\beta)}$ since $\sigma\big((\varphi-1)c^n_{(\alpha,\beta)}\big) = 0$ because (α,β) is a cocycle. Then, the class of $\delta'\big(c^n_{(\alpha,\beta)}\big)$ is $d\gamma$. To compute it we may take the differential in $\mathscr{C}^n(\mathscr{G}_S, W_2\otimes_{\mathbf{Z}_p} A_{\overline{R}})$ i. e.,

$$dc^n_{(a,b)} - d\sigma\big((\varphi-1)\big(c^n_{(a,b)}\big)\big) = db - d\big(\sigma\circ(\varphi-1)(b)\big) + (-1)^{n+1}d\big(\sigma(da)\big).$$

Here, we used that $d\sigma\big((\varphi-1)\big(d(\sigma(a))\big)\big) = d\big(\sigma(da)\big)$. The conclusion follows. $\square$

II. GLOBAL THEORY

4. Étale cohomology and relative (φ, Γ)-modules

As in the Introduction, let X denote a smooth, geometrically irreducible and proper scheme over $\mathrm{Spec}(V)$. Fix a field extension $K \subset M \subset \overline{K}$. In this section we review a Grothendieck topology on X, introduced by Faltings in [**14**] and denoted $\mathfrak{X}_M$, and its relation to étale cohomology; see 5.11. We also define the analogue Grothendieck topology, $\widehat{\mathfrak{X}}_M$ on the formal completion $\mathscr{X}$ of X along its special fiber. In this section we study p-power torsion sheaves for these Grothendieck topologies and compare their cohomology theories. The main result of this section is the following. Let $\mathbb{L}$ be an étale local system of $\mathbf{Z}/p^n\mathbf{Z}$-modules on X_K, for some $n \geq 1$ and let $\mathbb{L}^{\mathrm{rig}}$ be the corresponding étale local system on the rigid space X_K^{rig} attached to X_K. For every pointed étale open $(\mathscr{U}, s)$ of $\mathscr{X}$ (see 4.4) such that $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ is affine and $R_{\mathscr{U}}$ is a small V-algebra (see 2.2), let $\mathbf{L}$ be the fiber of $\mathbb{L}^{\mathrm{rig}}$ at the geometric generic point of $\mathscr{U}_K^{\mathrm{rig}}$ defined by s (see 4.5). As the notation suggests it is independent of $\mathscr{U}$ and s. Let $\mathfrak{D}_{\mathscr{U}}(\mathbf{L})$, $\mathrm{D}_{\mathscr{U}}(\mathbf{L})$, $\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L})$, $\widetilde{\mathrm{D}}_{\mathscr{U}}(\mathbf{L})$ denote the respective (φ, Γ)-modules over $R_{\mathscr{U}}$. For each $i \geq 0$ the associations $(\mathscr{U}, s) \longrightarrow \mathrm{H}^i\big(\mathscr{T}^\bullet(\mathfrak{D}_{\mathscr{U}}(\mathbf{L}))\big)$, $(\mathscr{U}, s) \longrightarrow \mathrm{H}^i\big(\mathscr{T}^\bullet(\mathrm{D}_{\mathscr{U}}(\mathbf{L}))\big)$, $(\mathscr{U}, s) \longrightarrow \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L}))\big)$, $\mathscr{U} \longrightarrow \mathrm{H}^i\big(\mathscr{T}^\bullet(\widetilde{\mathrm{D}}_{\mathscr{U}}(\mathbf{L}))\big)$ are functorial and we denote by $\mathscr{H}^{i,\mathrm{ar}}(\mathbb{L})$, $\mathscr{H}^{i,\mathrm{ge}}(\mathbb{L})$, $\mathscr{H}^{i,\mathrm{t},\mathrm{ar}}(\mathbb{L})$ and $\mathscr{H}^{i,\mathrm{t},\mathrm{ge}}(\mathbb{L})$ respectively the associated sheaves on the pointed étale site $\mathscr{X}_{\mathrm{et}}^\bullet$. We have

Theorem 4.1. — *There are spectral sequences*

$$\text{i)} \quad E_2^{p,q} = \mathrm{H}^q\Big(\mathscr{X}_{\mathrm{et}}^\bullet, \mathscr{H}^{p,*,\mathrm{ge}}(\mathbb{L})\Big) \Longrightarrow \mathrm{H}^{p+q}\Big(X_{\overline{K},\mathrm{et}}, \mathbb{L}\Big).$$

$$\text{ii)} \quad E_2^{p,q} = \mathrm{H}^q\Big(\mathscr{X}_{\mathrm{et}}^\bullet, \mathscr{H}^{p,*,\mathrm{ar}}(\mathbb{L})\Big) \Longrightarrow \mathrm{H}^{p+q}\Big(X_{K,\mathrm{et}}, \mathbb{L}\Big).$$

where $*$ stands for nothing or t. *Moreover, the spectral sequence i) is compatible with the residual* G_V*-action on all of its terms.*

The proof of theorem 4.1 will take the rest of this section. In particular, see 4.5.

4.1. Some Grothendieck topologies and associated sheaves. — Following [**14**, §3, p. 214] we define the following site:

Let X be a scheme flat over V. We denote by $X_{M,\mathrm{et}}$ the small étale site of X_M and by $\mathbf{Sh}(X_{M,\mathrm{et}})$ the category of sheaves of abelian groups on $X_{M,\mathrm{et}}$.

The site $\mathfrak{X}_M$. — The objects consist of pairs (U, W) where

(i) $U \to X$ is étale;
(ii) $W \to U \otimes_V M$ is a finite étale cover.

The maps are compatible maps of pairs and the coverings of a pair (U, W) are families $\{(U_\alpha, W_\alpha)\}_\alpha$ over (U, W) such that $\amalg_\alpha U_\alpha \to U$ and $\amalg_\alpha W_\alpha \to W$ are surjective. It is easily checked that we get a Grothendieck topology in the sense of [**3**, I.0.1].

It is noetherian if X is noetherian; see [**3**, II.5.1]. Note that one has a final object, namely (X, X_M). Let $\mathbf{Sh}(\mathfrak{X}_M)$ be the category of sheaves of abelian groups in $\mathfrak{X}_M$.

Let $\mathscr{X}$ be a formal scheme, flat over $\mathrm{Spf}(V)$ and with ideal of definition generated by p. Denote by $\mathscr{X}_{\mathrm{et}}$ the small étale site on $\mathscr{X}$ and by $\mathbf{Sh}(\mathscr{X}_{\mathrm{et}})$ the category of sheaves of abelian groups on $\mathscr{X}_{\mathrm{et}}$.

The sites $\mathscr{U}^{M,\mathrm{fet}}$ and $\mathscr{U}_{M,\mathrm{fet}}$. — Let $\mathscr{U} \to \mathscr{X}$ be an étale map of formal schemes. Define $\mathscr{U}^{M,\mathrm{fet}}$ to be the category whose objects are pairs $(\mathscr{W}, L)$ where

(i) L is a finite extension of K contained in M;

(ii) $\mathscr{W} \to \mathscr{U}^{\mathrm{rig}} \otimes_K L$ is a finite étale cover of L-rigid analytic spaces; here $\mathscr{U}^{\mathrm{rig}}$ denotes the K-rigid analytic space associated to $\mathscr{U}$.

Define $\mathrm{Hom}_{\mathscr{U}^{M,\mathrm{fet}}}\big((\mathscr{W}', L'), (\mathscr{W}, L)\big)$ to be empty if $L \not\subset L'$ and to be the set of morphisms $g\colon \mathscr{W}' \to \mathscr{W} \otimes_L L'$ of L'-rigid analytic spaces if $L \subset L'$. The coverings of a pair $(\mathscr{W}, L)$ in $\mathscr{U}^{M,\mathrm{fet}}$ are families of pairs $\{(\mathscr{W}_\alpha, L_\alpha)\}_\alpha$ over $(\mathscr{W}, L)$ such that $\amalg_\alpha \mathscr{W}_\alpha \to \mathscr{W}$ is surjective. Define the fiber product of two pairs $(\mathscr{W}', L')$ and $(\mathscr{W}'', L'')$ over a pair $(\mathscr{W}, L)$ to be $(\mathscr{W}' \times_{\mathscr{W}} \mathscr{W}'', L''')$ with L''' equal to the composite of L' and L''. It is the fiber product in the category $\mathscr{U}^{M,\mathrm{fet}}$

Let $\mathscr{U}_2 \to \mathscr{U}_1$ be a map of formal schemes over $\mathscr{X}$. Assume that they are étale over $\mathscr{X}$. We then have a morphism of Grothendieck topologies $\rho_{\mathscr{U}_1,\mathscr{U}_2}\colon \mathscr{U}_1^{M,\mathrm{fet}} \to \mathscr{U}_2^{M,\mathrm{fet}}$ given on objects by $(\mathscr{W}, L) \mapsto \big(\mathscr{W} \times_{\mathscr{U}_1^{\mathrm{rig}}} \mathscr{U}_2^{\mathrm{rig}}, L\big)$. It is clear how to define such a map for morphisms and that it sends covering families to covering families.

Let $\mathscr{S}_{\mathscr{U}}$ be the system of morphisms in $\mathscr{U}^{M,\mathrm{fet}}$ of pairs $(\mathscr{W}', L') \to (\mathscr{W}, L)$ such that $g\colon \mathscr{W}' \to \mathscr{W} \otimes_L L'$ is an isomorphism. Then,

Lemma 4.2. — *The following hold:*

i) *the composite of two composable elements of $\mathscr{S}_{\mathscr{U}}$ is in $\mathscr{S}_{\mathscr{U}}$;*

ii) *given a map $\mathscr{U}_2 \to \mathscr{U}_1$ of formal schemes étale over $\mathscr{X}$, we have $\rho_{\mathscr{U}_1,\mathscr{U}_2}\big(\mathscr{S}_{\mathscr{U}_1}\big) \subset \mathscr{S}_{\mathscr{U}_2}$;*

iii) *the base change of an element of $\mathscr{S}_{\mathscr{U}}$ via a morphism in $\mathscr{U}^{M,\mathrm{fet}}$ is again an element of $\mathscr{S}_{\mathscr{U}}$;*

iv) *if $f\colon (\mathscr{W}_1, L_1) \to (\mathscr{W}, L)$ and $g\colon (\mathscr{W}_2, L_2) \to (\mathscr{W}, L)$ are morphisms lying in $\mathscr{S}_{\mathscr{U}}$ and if $h\colon (\mathscr{W}_1, L_1) \to (\mathscr{W}_2, L_2)$ is a morphism in $\mathscr{U}^{M,\mathrm{fet}}$ such that $f = g \circ h$, then h is in $\mathscr{S}_{\mathscr{U}}$.*

Proof. — Left to the reader. □

Thanks to 4.2 the category $\mathscr{U}^{M,\mathrm{fet}}$ localized with respect to $\mathscr{S}_{\mathscr{U}}$ exists and we denote it by $\mathscr{U}_{M,\mathrm{fet}}$. Note that the fiber product of two pairs over a given one exists in $\mathscr{U}_{M,\mathrm{fet}}$ and it coincides with the fiber product in $\mathscr{U}^{M,\mathrm{fet}}$. The coverings of a pair $(\mathscr{W}, L)$ in $\mathscr{U}_{M,\mathrm{fet}}$ are families of pairs $\{(\mathscr{W}_\alpha, L_\alpha)\}_\alpha$ over $(\mathscr{W}, L)$ such that $\amalg_\alpha \mathscr{W}_\alpha \to \mathscr{W}$ is surjective. By 4.2 the category $\mathscr{U}_{M,\mathrm{fet}}$ and the given families of covering define a Grothendieck topology. It is a noetherian topology if the topological space $\mathscr{X}$

is noetherian. By abuse of notation we will simply write $\mathscr{W}$ for an object $(\mathscr{W}, L)$ of $\mathscr{U}_{M,\mathrm{fet}}$.

We recall that, given pairs $(\mathscr{W}_1, L_1)$ and $(\mathscr{W}_2, L_2)$ in $\mathscr{U}_{M,\mathrm{fet}}$, one defines the set of homomorphisms

$$\mathrm{Hom}_{\mathscr{U}_{M,\mathrm{fet}}}\big((\mathscr{W}_1, L_1), (\mathscr{W}_2, L_2)\big) := \lim_{(\mathscr{W}', L')\to(\mathscr{W}_1, L_1)} \mathrm{Hom}_{\mathscr{U}^{M,\mathrm{fet}}}\big((\mathscr{W}', L'), (\mathscr{W}_2, L_2)\big),$$

where the direct limit is taken over all morphisms $(\mathscr{W}', L') \to (\mathscr{W}_1, L_1)$ in $\mathscr{S}_{\mathscr{U}}$. Equivalently, due to 4.2, it is the set of classes of morphisms $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}', L') \to (\mathscr{W}_2, L_2)$, where $(\mathscr{W}', L') \to (\mathscr{W}_1, L_1)$ is in $\mathscr{S}_{\mathscr{U}}$, and two such diagrams $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}', L') \to (\mathscr{W}_2, L_2)$ and $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}'', L'') \to (\mathscr{W}_2, L_2)$ are equivalent if and only if there is a third one $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}''', L''') \to (\mathscr{W}_2, L_2)$ mapping to the two. If $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}', L') \to (\mathscr{W}_2, L_2)$ and $(\mathscr{W}_2, L_2) \leftarrow (\mathscr{W}'', L'') \to (\mathscr{W}_3, L_3)$ are two homomorphisms, the composite $(\mathscr{W}_1, L_1) \leftarrow (\mathscr{W}''', L''') \to (\mathscr{W}_3, L_3)$ is defined by taking $(\mathscr{W}''', L''')$ to be the fiber product of $(\mathscr{W}', L')$ and $(\mathscr{W}'', L'')$ over $(\mathscr{W}_2, L_2)$.

Let $\mathscr{U}_2 \to \mathscr{U}_1$ be a map of formal schemes étale over $\mathscr{X}$. Due to 4.2, the map $\rho_{\mathscr{U}_1,\mathscr{U}_2}\colon \mathscr{U}_1^{M,\mathrm{fet}} \to \mathscr{U}_2^{M,\mathrm{fet}}$ extends to the localized categories and defines a morphism of Grothendieck topologies $\mathscr{U}_{2,M,\mathrm{fet}} \to \mathscr{U}_{1,M,\mathrm{fet}}$ which, by abuse of notation, we write $\mathscr{W} \mapsto \mathscr{W} \times_{\mathscr{U}_1^{\mathrm{rig}}} \mathscr{U}_2^{\mathrm{rig}}$.

The site $\widehat{\mathfrak{X}}_M$. — Define $\widehat{\mathfrak{X}}_M$ to be the category of pairs $(\mathscr{U}, \mathscr{W})$ where $\mathscr{U} \to \mathscr{X}$ is an étale map of formal schemes and $\mathscr{W}$ is an object of $\mathscr{U}_{M,\mathrm{fet}}$. A morphism of pairs $(\mathscr{U}', \mathscr{W}') \to (\mathscr{U}, \mathscr{W})$ is defined to be a morphism $\mathscr{U}' \to \mathscr{U}$ as schemes over $\mathscr{X}$ and a map $\mathscr{W}' \to \mathscr{W} \times_{\mathscr{U}^{\mathrm{rig}}} \mathscr{U}'^{\mathrm{rig}}$ in $\mathscr{U}'_{M,\mathrm{fet}}$. A family $\{(\mathscr{U}_\alpha, \mathscr{W}_{\alpha,\beta}, L_{\alpha,\beta})\}_\alpha \to (\mathscr{U}, \mathscr{W}, L)$ is a covering if $\{\mathscr{U}_\alpha\}_\alpha$ is an étale covering of $\mathscr{U}$ and, for every α, $\big(\mathscr{W}_{\alpha,\beta}, L_{\alpha,\beta}\big)_\beta$ is a covering of $\mathscr{W} \times_{\mathscr{U}^{\mathrm{rig}}} \mathscr{U}_\alpha^{\mathrm{rig}}$ in $\mathscr{U}_{\alpha,M,\mathrm{fet}}$.

Remark that the fiber product $(\mathscr{U}''', \mathscr{W}''')$ of two pairs $(\mathscr{U}', \mathscr{W}')$ and $(\mathscr{U}'', \mathscr{W}'')$ over a pair $(\mathscr{U}, \mathscr{W})$ exists putting $\mathscr{U}''' := \mathscr{U}' \times_{\mathscr{U}} \mathscr{U}''$ and $\mathscr{W}'''$ to be the fiber product in $\mathscr{U}'''_{M,\mathrm{fet}}$ of $\mathscr{W}' \times_{\mathscr{U}'^{\mathrm{rig}}} \mathscr{U}'''^{\mathrm{rig}}$ and $\mathscr{W}'' \times_{\mathscr{U}''^{\mathrm{rig}}} \mathscr{U}'''^{\mathrm{rig}}$ over $\mathscr{W} \times_{\mathscr{U}^{\mathrm{rig}}} \mathscr{U}'''^{\mathrm{rig}}$. The pair $(\mathscr{X}, (\mathscr{X}^{\mathrm{rig}}, K))$ is a final object in $\widehat{\mathfrak{X}}_M$. We let $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)$ be the category of sheaves of abelian groups on $\widehat{\mathfrak{X}}_M$.

We remark that in all the categories $\mathbf{Sh}(\_)$ introduced above AB3* and AB5 hold and the representable objects provide families of generators. In particular, one has enough injectives; see [**3**, Thm. II.1.6 & § II.1.8].

4.2. Morphisms of Grothendieck topologies. — One has natural functors:

(I) $u_{X,M}\colon \mathfrak{X}_M \longrightarrow (X \otimes_V M)_{\mathrm{et}}$ with $u_{X,M}(U, W) := W$;
(II.a) $v_{X,M}\colon X_{\mathrm{et}} \longrightarrow \mathfrak{X}_M$ given by $v_{X,M}(U) := \big(U, U \otimes_V M\big)$;
(II.b) $\widehat{v}_{\mathscr{X},M}\colon \mathscr{X}_{\mathrm{et}} \longrightarrow \widehat{\mathfrak{X}}_M$ given by $\widehat{v}_{\mathscr{X},M}(\mathscr{U}) := (\mathscr{U}, (\mathscr{U}^{\mathrm{rig}}, K))$.

Assume that $\mathscr{X}$ is the formal scheme associated to X i. e., that it is the formal completion of X along its special fiber. We then have:

(III) $\mu_{X,M}\colon \mathfrak{X}_M \longrightarrow \widehat{\mathfrak{X}}_M$ given by $\mu_{X,M}\big(U,W\big) := (\mathscr{U},(\mathscr{W},L))$ where $\mathscr{U}$ is the formal scheme associated to U and if the cover $W \to U\otimes_V M$ is defined over a finite extension $K\subset L$, contained in M, then $\mathscr{W}\to\mathscr{U}_L^{\mathrm{rig}}$ is the pull-back via $\mathscr{U}_L^{\mathrm{rig}}\to U_L^{\mathrm{rig}}$ of the associated finite and étale cover of rigid analytic spaces $W^{\mathrm{rig}}\to U_L^{\mathrm{rig}}$;

(IV) $\nu_X\colon X_{\mathrm{et}}\to\mathscr{X}_{\mathrm{et}}$ given by $\nu_X(U)=\mathscr{U}$ where $\mathscr{U}$ is the formal scheme associated to U.

Let $K\subset M_1\subset M_2\subset\overline{K}$ be field extensions. Define

(V.a) $\beta_{M_1,M_2}\colon \mathfrak{X}_{M_1}\to\mathfrak{X}_{M_2}$ by $\beta_{M_1,M_2}(U,W)=\big(U,W\otimes_{M_1}M_2\big)$.

(V.b) $\widehat{\beta}_{M_1,M_2}\colon \widehat{\mathfrak{X}}_{M_1}\to\widehat{\mathfrak{X}}_{M_2}$ by $\widehat{\beta}_{M_1,M_2}(\mathscr{U},\mathscr{W})=\big(\mathscr{U},\mathscr{W}\big)$.

Due to the definition of $\widehat{\mathfrak{X}}_M$, the functors $\mu_{X,M}$ and $\widehat{\beta}_{M_1,M_2}$ are well defined. More precisely, given (U,W), the image $\mu_{X,M}\big(U,W\big)$ does not depend on the subfield $L\subset M$ to which W descends. Analogously, given $\mathscr{U}\in\mathscr{X}_{\mathrm{et}}$, then $\widehat{\beta}_{M_1,M_2}$ sends the multiplicative system $\mathscr{S}_{\mathscr{U}}$, used to define $\widehat{\mathfrak{X}}_{M_1}$, to the multiplicative system $\mathscr{S}_{\mathscr{U}}$ used to define $\widehat{\mathfrak{X}}_{M_2}$.

It is clear that the above functors send covering families to covering families and commute with fiber products. In particular, they are morphisms of topologies, see [**3**, Def. II.4.5]. Given any such functor g, we let g_* and g^* be the induced morphisms of the associated category of sheaves for the given topologies; see [**3**, p. 41-42]. Note that the functors above preserve final objects and commute with finite fibred products. Therefore, the induced functor g^* on the categories of sheaves is exact by [**3**, Thm. II.4.14].

We work out an example. If $\mathscr{F}$ is a sheaf on $\widehat{\mathfrak{X}}_M$, then $\mu_{X,M,*}(\mathscr{F})$ is the sheaf on $\mathfrak{X}_M$ defined by $(U,W)\mapsto\mathscr{F}\big(\mu_{X,M}\big(U,W\big)\big)$. If $\mathscr{F}$ is a sheaf on $\mathfrak{X}_M$, then $\mu_{X,M}^*(\mathscr{F})$ is the sheaf associated to the separated presheaf defined by $(\mathscr{U},\mathscr{W})\mapsto\lim_{(U',W')}\mathscr{F}(U',W')$ where the limit is the direct limit taken over all pairs (U',W') in $\mathfrak{X}_M$ and all maps $(\mathscr{U},\mathscr{W})\to\mu_{X,M}\big(U',W'\big)$ in $\widehat{\mathfrak{X}}_M$.

Notation. — If $\mathscr{F}$ is a sheaf on X_{et} or is in $\mathbf{Sh}(X_{\mathrm{et}})^{\mathbf{N}}$ (see section 5 for the definition), we write $\mathscr{F}^{\mathrm{form}}$ for $\nu_X^*(\mathscr{F})$, respectively for $\nu_X^{*,\mathbf{N}}(\mathscr{F})$.

If $\mathbb{L}$ is a locally constant sheaf on $X_{M,\mathrm{et}}$, by abuse of notation we denote $\mathbb{L}$ its push forward $u_{X,M,*}(\mathbb{L})\in\mathbf{Sh}(\mathfrak{X}_M)$. It is a locally constant sheaf on $\mathfrak{X}_M$.

If $\mathscr{F}$ is a sheaf on $\mathfrak{X}_M$ or is in $\mathbf{Sh}(\mathfrak{X}_M)^{\mathbf{N}}$, we denote by $\mathscr{F}^{\mathrm{rig}}$ the pull-back $\mu_{X,M}^*(\mathscr{F})$. Note that if $\mathscr{F}\in\mathbf{Sh}(\mathfrak{X}_M)$ is locally constant, then $\mathscr{F}^{\mathrm{rig}}$ is also a locally constant sheaf on $\widehat{\mathfrak{X}}_M$.

4.3. Stalks [**14**, p. 214]. — Let K_x be a finite field extension of K contained in $\overline{K}$ and denote by V_x its valuation ring.

Fix a map $x\colon \mathrm{Spec}(V_x)\to X$ of V-schemes and denote by $\bar{x}\colon \mathrm{Spec}(\overline{V})\to X$ the composite of x with the natural map $\mathrm{Spec}(\overline{V})\to\mathrm{Spec}(V_x)$. Let $\mathscr{O}_{X,x}^{\mathrm{sh}}$ be the the direct limit $\lim_i R_i$ taken over all pairs $\{(R_i,f_i)\}_i$ where $\mathrm{Spec}(R_i)$ is étale over X and $f_i\colon R_i\to\overline{V}$ defines a point over $\bar{x}$. Let $\mathscr{F}$ be a sheaf in $\mathbf{Sh}\big(X_{\mathrm{et}}\big)$. The stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x is defined as $\mathscr{F}_x=\mathscr{F}\big(\mathscr{O}_{X,x}^{\mathrm{sh}}\big)$ by which we mean the direct limit $\lim_i\mathscr{F}\big(\mathrm{Spec}(R_i)\big)$.

Define $\overline{\mathcal{O}}_{X,x,M}$ as the direct limit $\lim_{i,j} R'_{i,j}$ over the pairs $\{(R'_{i,j}, R'_{i,j} \to \overline{V})\}_{i,j}$ where (1) $R'_{i,j}$ is an integral R_i-algebra and is normal as a ring, (2) $R'_{i,j} \otimes_V K$ is a finite and étale extension of $R_i \otimes_V M$, (3) the composite $R_i \otimes_V M \to R'_{i,j} \otimes_V K \to \overline{K}$ is $r \otimes \ell \mapsto f_i(r) \cdot \ell$. If $\mathscr{F}$ is a sheaf in $\mathbf{Sh}(\mathfrak{X}_M)$, we then write $\mathscr{F}_x$ or equivalently $\mathscr{F}(\overline{\mathcal{O}}_{X,x,M} \otimes_V K)$ for the direct limit $\lim_{i,j} \mathscr{F}(\mathrm{Spec}(R_i), \mathrm{Spec}(R'_{i,j} \otimes_V K))$. We call it the stalk of $\mathscr{F}$ at x.

Let $G_{x,M}$ be the Galois group of $\overline{\mathcal{O}}_{X,x,M} \otimes_V K$ over $\mathcal{O}^{\mathrm{sh}}_{X,x} \otimes_V M$. Then, $\mathscr{F}_x$ is endowed with an action of $G_{x,M}$.

Let $V \subset V_{\hat{x}}$ be a finite extension of DVRs. Let $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ be a map of V-formal schemes and let $\widehat{\overline{x}}\colon \mathrm{Spf}(\widehat{\overline{V}}) \to \mathscr{X}$ be the composite of $\hat{x}$ with $\mathrm{Spf}(\widehat{\overline{V}}) \to \mathrm{Spf}(V_x)$. Define $\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ be the direct limit $\lim_i S_i$ over all pairs $\{(S_i, g_i)\}_{i \in I}$ such that S_i is p-adically complete and separated V-algebra, $\mathrm{Spf}(S_i) \to \mathscr{X}$ is an étale map of formal schemes and $g_i\colon S_i \to \widehat{\overline{V}}$ defines a formal point over $\widehat{\overline{x}}$. If $\mathscr{F}$ is a sheaf in $\mathbf{Sh}(\mathscr{X}_{\mathrm{et}})$, the stalk $\mathscr{F}_{\hat{x}}$ of $\mathscr{F}$ at $\hat{x}$ is defined to be the direct limit $\mathscr{F}(\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}) := \lim_{i \in I} \mathscr{F}(\mathrm{Spf}(S_i))$.

Write $\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}$ for the direct limit $\lim_{i,j} S'_{i,j}$ over all triples $\{(S'_{i,j}, S'_{i,j} \to \widehat{\overline{V}}, L_{i,j})\}_{i,j}$ where (1) $L_{i,j}$ is a finite extension of K contained in M, (2) $S'_{i,j}$ is an integral extension of S_i and is normal as a ring, (3) $S'_{i,j} \otimes_V K$ is a finite and étale $S_i \otimes_V L_{i,j}$-algebra, (4) the composite $S_i \otimes_V L_{i,j} \to S'_{i,j} \otimes_V K \to \widehat{\overline{K}}$ is $a \otimes \ell \mapsto g_i(a) \cdot \ell$. Given a sheaf $\mathscr{F}$ in $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)$, denote by $\mathscr{F}_{\hat{x}}$, or equivalently $\mathscr{F}(\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M} \otimes_V K)$, the direct limit $\lim_{i,j} \mathscr{F}(\mathrm{Spf}(S_i), (\mathrm{Spm}(S'_{i,j} \otimes_V K), L_{i,j}))$. We call it the stalk of $\mathscr{F}$ at $\hat{x}$.

Denote by $G_{\hat{x},M}$ the Galois group of $\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M} \otimes_V K$ over $\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}} \cdot M$. Then, $\mathscr{F}_{\hat{x}}$ is endowed with an action of $G_{\hat{x},M}$.

Lemma 4.3. — *Let $k(x)$ (resp. $\overline{k}$) be the residue field of V_x (resp. $\overline{V}$) and denote by $x_k\colon \mathrm{Spec}(k(x)) \to X_k$ (resp. $\bar{x}_k\colon \mathrm{Spec}(\overline{k}) \to X_k$) the points induced by x or $\hat{x}$ (resp. $\bar{x}$ or $\widehat{\overline{x}}$) on the special fiber X_k of X or of $\mathscr{X}$. Then,*

i. *$\mathcal{O}^{\mathrm{sh}}_{X,x}$ coincides with the strict henselization of $\mathcal{O}_{X,x_k}$ and $\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ coincides with the strict formal henselization of $\mathcal{O}_{\mathscr{X},x_k}$.*

Assume that $\mathscr{X}$ is the formal scheme associated to X and that $\hat{x}$ is the map of formal schemes defined by x. Then,

ii. *$(\mathcal{O}^{\mathrm{sh}}_{X,x}, (p))$ and $(\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}, (p))$ are noetherian henselian pairs and the natural map $\mathcal{O}^{\mathrm{sh}}_{X,x} \to \mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ is an isomorphism after taking p-adic completions;*
iii. *the base change functor from the category of finite extensions of $\mathcal{O}^{\mathrm{sh}}_{X,x}$, étale after inverting p, to the category of finite extensions of $\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$, étale after inverting p, is an equivalence of categories;*
iv. *the maps $\mathcal{O}^{\mathrm{sh}}_{X,x}/p\mathcal{O}^{\mathrm{sh}}_{X,x} \to \mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}/p\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ and $\overline{\mathcal{O}}_{X,x,M}/p\overline{\mathcal{O}}_{X,x,M} \to \overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}/p\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}$ are isomorphisms.*
v. *Frobenius on $\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}/p\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}$ is surjective with kernel $p^{\frac{1}{p}}\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}/p\overline{\mathcal{O}}_{\mathscr{X},\hat{x},M}$.*

Proof. — (i) The strict henselization of $\mathscr{O}_{X,x_k}$ is defined as the direct limit $\lim_j T_j$ over all pairs $\{(T_j,t_j)\}_j$ where $\mathrm{Spec}(T_j)$ is étale over X and $t_j\colon T_j \to \overline{k}$ is a point over $\bar{x}_k$. In particular, we get a map $\mathscr{O}^{\mathrm{sh}}_{X,x} = \lim_i R_i \to \mathscr{O}^{\mathrm{sh}}_{X,x_k} = \lim_j T_j$ by associating to a pair $\big(R_i, f_i\colon R_i \to \overline{V}\big)$ the pair $\big(R_i, R_i \to \overline{V} \to \overline{k}\big)$. To conclude that such a map is an isomorphism it suffices to show that for any pair (T_j,t_j) there is a unique map of V-algebras $T_j \to \overline{V}$ lifting t_j and inducing the point $\bar{x}$. The base change of T_j via $\bar{x}$ defines an étale $\overline{V}$-algebra A_j and t_j induces a map of $\overline{V}$-algebras $A_j \to \overline{k}$. By étaleness of A_j the latter lifts uniquely to a map of $\overline{V}$-algebras $A_i \to \widehat{\overline{V}}$ which, since T_j is of finite type over V, factors via $\overline{V}$.

The strict formal henselization of $\mathscr{O}_{\mathscr{X},X_k}$ is defined as the direct limit $\lim_j Q_j$ over all pairs $\{Q_j,q_j\}_j$ where Q_j is a p-adically complete and separated V-algebra, $\mathrm{Spf}(Q_j) \to \mathscr{X}$ is an étale map of formal schemes and $q_j\colon Q_j \to \overline{k}$ is a point over $\bar{x}_k$. The proof that $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ is the strict formal henselization of $\mathscr{O}_{\mathscr{X},X_k}$ is similar to the first part of the proof and left to the reader.

(ii) It follows from (i) that $\mathscr{O}^{\mathrm{sh}}_{X,x}$ (resp. $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$) is a local ring with residue field $\overline{k}$ and maximal ideal $\mathfrak{m}_x$ (resp. $\mathfrak{m}_{\hat{x}}$) generated by the maximal ideal of $\mathscr{O}_{X,x_k}$. In particular, the graded rings $\mathrm{gr}_{\mathfrak{m}_x}\mathscr{O}^{\mathrm{sh}}_{X,x}$ and $\mathrm{gr}_{\mathfrak{m}_{\hat{x}}}\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ are noetherian so that $\mathscr{O}^{\mathrm{sh}}_{X,x}$ and $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ are noetherian.

We claim that $(\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}, \mathfrak{m}_{\hat{x}})$ is a henselian pair; see [**11**, §0.1]. This amounts to prove that any étale map $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}} \to B$, such that $\overline{k} = \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}/\mathfrak{m}_{\hat{x}}\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}} \to B/\mathfrak{m}_x B$ is an isomorphism, admits a section. Note that there exist i and an étale extension $S_i \to A$ such that B is obtained by base change of A via $S_i \to \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$. Via $a\colon A/\mathfrak{m}_x A \to B/\mathfrak{m}_x B \cong \overline{k}$ the pair $(\widehat{A}, a)$, where $\widehat{A}$ is the p-adic completion of A, appears in the inductive system used to define the strict formal henselization of $\mathscr{O}_{\mathscr{X},x_k}$ so that, thanks to (i), we get a natural map $A \to \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ and, hence base-changing, a map of $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$-algebras $B \to \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$. Analogously, one proves that $(\mathscr{O}^{\mathrm{sh}}_{X,x}, \mathfrak{m}_x)$ is a henselian pair.

Note that p is contained in $\mathfrak{m}_x$, so that $\big(\mathscr{O}^{\mathrm{sh}}_{X,x},(p)\big)$ and $\big(\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}},(p)\big)$ are henselian pairs. Let $\mathscr{O}^{\mathrm{sh}}_{X\otimes\mathbf{F}_p,x_k}$ be the strict henselization of the local ring of $X\otimes_V V/pV$ at x_k. By construction we have natural injective maps $\mathscr{O}^{\mathrm{sh}}_{X,x}/p\mathscr{O}^{\mathrm{sh}}_{X,x} \to \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}/p\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}} \to \mathscr{O}^{\mathrm{sh}}_{X\otimes\mathbf{F}_p,x_k}$. We claim that such maps are isomorphisms. It suffices to show that the composite is surjective. Using (i) this is equivalent to prove that the map from the strict henselization of $\mathscr{O}_{X,x_k}$ to the strict henselization of $\mathscr{O}_{X\otimes\mathbf{F}_p,x_k}$ is surjective. This amounts to show that given an étale map $f\colon \mathrm{Spec}(R) \to \mathrm{Spec}(\mathscr{O}_{X\otimes\mathbf{F}_p,x_k})$, there exists an étale map $g\colon \mathrm{Spec}(S) \to \mathrm{Spec}(\mathscr{O}_{X,x_k})$ reducing to f modulo p. By the Jacobian criterion of étaleness we have $R = \mathscr{O}_{X\otimes\mathbf{F}_p,x_k}[T_1,\dots,T_d]/(h_1,\dots,h_d)$ with $\det\left(\partial h_i/\partial T_j\right)_{i,j=1}^d$ invertible in R. Then, $S := \mathscr{O}_{X,x_k}[T_1,\dots,T_d]/(q_1,\dots,q_d)\left[\det\left(\partial q_i/\partial T_j\right)^{-1}\right]$, with q_i lifting h_i, is an étale $\mathscr{O}_{X,x_k}$-algebra and lifts R as wanted. Since p is not a zero divisor in $\mathscr{O}^{\mathrm{sh}}_{X,x}$ and in $\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$, we conclude that the graded rings $\mathrm{gr}_p\mathscr{O}^{\mathrm{sh}}_{X,x}$ and in $\mathrm{gr}_p\mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}$ are isomorphic, concluding the proof of (ii).

(iii) Let $\widehat{\mathscr{O}^{\rm sh}_{X,x}}$ (resp. $\widehat{\mathscr{O}^{\rm sh}_{\mathscr{X},x}}$) be the p-adic completion of $\mathscr{O}^{\rm sh}_{X,x}$ (resp. $\mathscr{O}^{\rm sh}_{\mathscr{X},\hat{x}}$). Thanks to [**11**, Thm. 5] one knows that the category of finite extensions of $\mathscr{O}^{\rm sh}_{X,x}$, étale after inverting p (resp. the category of finite extensions of $\mathscr{O}^{\rm sh}_{\mathscr{X},\hat{x}}$, étale after inverting p), is equivalent to the category of finite extensions of $\widehat{\mathscr{O}^{\rm sh}_{X,x}} = \widehat{\mathscr{O}^{\rm sh}_{\mathscr{X},x}}$, étale after inverting p. The claim follows from (ii).

(iv) The first claim follows from (ii). The second follows from the first and (iii).

(v) Note that $p^\alpha \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$ for every $\alpha \in \mathbf{Q}_{>0}$. It follows from [**14**, §3, Lemma 5] that Frobenius is surjective on $\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}/p^\alpha\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$ for every $0<\alpha<1$. Let $a \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$. Write $a = b^p + p^{\frac{1}{p}}c$ with b and $c \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$. Write $c = d^p + p^{1-\frac{1}{p}}e$ with $e \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$. Then, $a \equiv (b + p^{\frac{1}{p^2}}d)^p$ modulo $p\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$.

Let $a \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$ be such that $a^p \in p\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$. Then, $\frac{a^p}{p} = \left(\frac{a}{p^{\frac{1}{p}}}\right)^p$ lies in $\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$. Since the latter is a normal ring, this implies that $\frac{a}{p^{\frac{1}{p}}} \in \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$ as claimed. □

Proposition 4.4. — *The notation is as above (in (2), (3) & (5) below we also assume that $\mathscr{X}$ is the formal scheme associated to X and that $\hat{x}$ is the map of formal schemes associated to $x\colon \mathrm{Spec}(V_x) \to X$).*

1) *Suppose that X (resp. $\mathscr{X}$) is locally (topologically) of finite type over V and that every closed point of X maps to the closed point of* Spec(V). *Then, a sequence of sheaves $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ on $X_{\rm et}$ (resp. $\mathfrak{X}_M$, resp. $\mathscr{X}_{\rm et}$, resp. $\widehat{\mathfrak{X}}_M$) is exact if and only if for every point x of X (resp. $\hat{x}$ of $\mathscr{X}$) as above the induced sequence of stalks $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$ (resp. $\mathscr{F}_{\hat{x}} \to \mathscr{G}_{\hat{x}} \to \mathscr{H}_{\hat{x}}$) is exact;*
2) *let $\mathscr{F}$ be in $\mathbf{Sh}(X_{\rm et})$. Then, $\nu_X^*(\mathscr{F})_{\hat{x}} \cong \mathscr{F}_x$;*
3) *if $\mathscr{F}$ is in $\mathbf{Sh}(\mathfrak{X}_M)$, then, $\mu_{X,M}^*(\mathscr{F})_{\hat{x}} \cong \mathscr{F}_x$;*
4) *fix field extensions $K \subset M_1 \subset M_2 \subset \overline{K}$. Then, $\beta^*_{M_1,M_2}$ (resp. $\widehat{\beta}^*_{M_1,M_2}$) of a flasque sheaf is flasque. Furthermore, if $\mathscr{F}$ is in $\mathbf{Sh}(\mathfrak{X}_{M_1})$ (resp. $\mathscr{F}$ is in $\mathbf{Sh}(\widehat{\mathfrak{X}}_{M_1})$), then one has natural identifications:*
   a) *$\beta^*_{M_1,M_2}(\mathscr{F})_x = \mathscr{F}_x$ (resp. $\widehat{\beta}^*_{M_1,M_2}(\mathscr{F})_{\hat{x}} = \mathscr{F}_{\hat{x}}$);*
   b) *if $M_1 \subset M_2$ is Galois with group G, then $\mathrm{H}^0(\mathfrak{X}_{M_1},\mathscr{F}) = \mathrm{H}^0\left(\mathfrak{X}_{M_2},\beta^*_{M_1,M_2}(\mathscr{F})\right)^G$ (resp. $\mathrm{H}^0(\widehat{\mathfrak{X}}_{M_1},\mathscr{F}) = \mathrm{H}^0\left(\widehat{\mathfrak{X}}_{M_2},\widehat{\beta}^*_{M_1,M_2}(\mathscr{F})\right)^G$).*

   Assume that K_x is contained in M. Then,
5) *we have a natural isomorphisms $G_{\hat{x},M} \cong G_{x,M}$ and, if $\mathscr{F}$ is in $\mathfrak{X}_M$, the isomorphism $\mu^*_{X,M}(\mathscr{F})_{\hat{x}} \cong \mathscr{F}_x$ is compatible with the actions of $G_{\hat{x},M}$ and $G_{x,M}$*
6) *let $\mathscr{F}$ be a sheaf in $\mathfrak{X}_M$. Then, $\left(\mathrm{R}^q v_{X,M,*}(\mathscr{F})\right)_x \cong \mathrm{H}^q\left(G_{x,M},\mathscr{F}_x\right)$;*
7) *let $\mathscr{F}$ be a sheaf in $\widehat{\mathfrak{X}}_M$. Then, $\left(\mathrm{R}^q \widehat{v}_{\mathscr{X},M,*}(\mathscr{F})\right)_{\hat{x}} \cong \mathrm{H}^q\left(G_{\hat{x},M},\mathscr{F}_{\hat{x}}\right)$.*

Proof. — (1) In each case it suffices to prove that a sheaf is trivial if and only if all its stalks are.

We give a proof for a sheaf on $\mathfrak{X}_M$ and leave the other cases to the reader. Let $\mathscr{F} \in \mathbf{Sh}(\mathfrak{X}_M)$ such that for every point x of X, defined over a finite extension of K, we have $\mathscr{F}_x = 0$. Let $(U,W) \in \mathfrak{X}_M$ and let $\alpha \in \mathscr{F}(U,W)$. Then, for every $x\colon \mathrm{Spec}(V_x) \to U$ and every point $y\colon \mathrm{Spec}(K_y) \to W$ over $x \otimes_V K$, which exists since $W \to U_M$ is finite, there exist $(U_x, W_y) \in \mathfrak{X}_M$ and a map $(U_x, W_y) \to (U,W)$ such that (1) $x \otimes_V \overline{V}$ factors via U_x, (2) $y \otimes_K \overline{K}$ factors via W_y and (3) the image of α in $\mathscr{F}(U_x, W_y)$ is 0.

Due to the assumption, the set of points x (resp. y) as above are dense in U (resp. W) so that there exist points x_i and y_i such that $\amalg_i (U_{x_i}, W_{y_i}) \to (U,W)$ is a covering of (U,W) in $\mathfrak{X}_M$. Since $\mathscr{F}$ is a sheaf, the homomorphism $\mathscr{F}(U,W) \to \prod_i \mathscr{F}(U_{x_i}, W_{y_i})$ is injective. Hence, $\alpha = 0$ to start with.

(2) Since any sheaf is the direct limit of representable sheaves and direct limits commute with ν_X^* and with taking stalks, we may assume that $\mathscr{F}$ is represented by an étale X-scheme $Y \to X$. In particular, $\nu_X^*(\mathscr{F})$ is represented by the formal scheme $\mathscr{Y}$ associated to Y. Let Y_x (resp. $\mathscr{Y}_{\hat{x}}$) be the pull back of Y (resp. $\mathscr{Y}$) to $\mathrm{Spec}(\mathscr{O}_{X,x})$ (resp. $\mathrm{Spec}(\mathscr{O}_{\mathscr{X},\hat{x}})$). We then have the following diagram

$$\begin{tikzcd}
\mathscr{F}_x \arrow[d, equal] & \nu_X^*(\mathscr{F})_{\hat{x}} \arrow[d, equal] & \\
\mathrm{Hom}_{\mathscr{O}_{X,x}}\left(\mathscr{O}_{Y_x}, \mathscr{O}^{\mathrm{sh}}_{X,x}\right) \arrow[r] & \mathrm{Hom}_{\mathscr{O}_{\mathscr{X},\hat{x}}}\left(\mathscr{O}_{\mathscr{Y}_{\hat{x}}}, \mathscr{O}^{\mathrm{sh}}_{\mathscr{X},\hat{x}}\right) \arrow[r] & \mathrm{Hom}_k\left(\mathscr{O}_{Y_x} \otimes_V k, \overline{k}\right).
\end{tikzcd}$$

By 4.3(i) these maps are bijective as claimed.

(4) If $\mathscr{F}$ is in $\mathbf{Sh}(\mathfrak{X}_{M_1})$, then $\beta^*_{M_1,M_2}(\mathscr{F})$ is the sheaf in $\mathfrak{X}_{M_2}$ associated to the presheaf $\beta^{-1}_{M_1,M_2}(\mathscr{F})$ defined by $(U,W) \mapsto \lim \mathscr{F}(U',W')$ where the limit is taken over all the pairs (U',W') in $\mathfrak{X}_{M_1}$ and all the maps $(U,W) \to (U', W' \otimes_{M_1} M_2)$. This is equivalent to take the direct limit over all pairs (U,W') in $\mathfrak{X}_{M_1}$ and over all maps $(U,W) \to (U,W')$ as U_{M_1}-schemes. If $M_1 \subset M_2$ is finite, there exists an initial pair, namely (U,W) itself, viewed in $\mathfrak{X}_{M_1}$ via the finite and étale map $W \to U \otimes_V M_2 \to U \otimes_V M_1$, so that $\beta^{-1}_{M_1,M_2}(\mathscr{F})(U,W) = \mathscr{F}(U,W)$. In general, there exists a finite extension $M_1 \subset L$ contained in M_2 and a pair (U, W_L) in $\mathfrak{X}_L$ such that $W = W_L \otimes_L M_2$. Since any morphism of pairs in $\mathfrak{X}_{M_2}$ descends to a finite extension of M_1, we conclude that $\beta^{-1}_{M_1,M_2}(\mathscr{F})(U,W) = \mathscr{F}(U, W_L \otimes_L M_2)$, defined as the direct limit $\lim_{L'} \mathscr{F}(U, W_L \otimes_L L')$ taken over all finite extensions $L \subset L'$ contained in M_2, considering $(U, W_L \otimes_L L')$ in $\mathfrak{X}_{M_1}$ via the finite and étale map $W_L \otimes_L L' \to U \otimes_V L \to U \otimes_V K$. In any case, we conclude that $\beta^{-1}_{M_1,M_2}(\mathscr{F})$ is already a sheaf i. e., $\beta^{-1}_{M_1,M_2}(\mathscr{F}) = \beta^*_{M_1,M_2}(\mathscr{F})$. Furthermore, $\beta^*_{M_1,M_2}$ preserves flasque objects and satisfies (a).

For (b), recall that $\mathfrak{X}_{M_1}$ and $\mathfrak{X}_{M_2}$ have final objects so that global sections can be computed using the final objects. Since $X_{M_2} \to X_{M_1}$ is a limit of finite and étale covers with Galois group G and $\mathscr{F}$ is a sheaf on $\mathfrak{X}_{M_1}$, one has $\mathscr{F}(X, X_{M_1}) = \mathscr{F}(X, X_{M_2})^G$. Then, $\mathrm{H}^0(\mathfrak{X}_{M_1}, \mathscr{F}) = \mathscr{F}(X, X_{M_1}) = \mathscr{F}(X, X_{M_2})^G = \mathrm{H}^0\left(\mathfrak{X}_{M_2}, \beta^*_{M_1,M_2}(\mathscr{F})\right)^G$ and (b) follows.

A similar argument works for $\widehat{\beta}^*_{M_1,M_2}$. Details are left to the reader.

(3) & (5) The first claim in (5) follows from 4.3(iii). To get the second claim and (3), one argues as in (2) reducing to the case of a sheaf represented by a pair (U, W), so that $\mu_{X,M}^*(U,W) = (\mathscr{U}, \mathscr{W}, L)$, and using 4.3(iii).

(6) Consider the functor $\mathbf{Sh}(\mathfrak{X}_M) \to (G_{x,M}\text{-Modules})$, associating to a sheaf $\mathscr{F}$ its stalk $\mathscr{F}_x$. It is an exact functor. Recall that $\mathscr{F}_x = \lim_{i,j} \mathscr{F}\big(\mathrm{Spec}(R_i), \mathrm{Spec}(R'_{i,j} \otimes_V K)\big)$. Thus, the continuous Galois cohomology $\mathrm{H}^*\big(G_{x,M}, \mathscr{F}_x\big)$ is the direct limit over i and j of the Chěch cohomology of $\mathscr{F}$ relative to the covering $\big(\mathrm{Spec}(R_i), \mathrm{Spec}(R'_{i,j} \otimes_V K)\big)$. In particular, if $\mathscr{F}$ is injective, it is flasque and $\mathrm{H}^q\big(G_{x,M}, \mathscr{F}_x\big) = 0$ for $q \geq 1$.

Both $\big\{(\mathrm{R}^q v_{\mathscr{X},M,*}(\mathscr{F}))_x\big\}_q$ and $\big\{\mathrm{H}^q\big(G_{x,M}, \mathscr{F}_x\big)\big\}_q$ are δ-functors from $\mathbf{Sh}(\mathfrak{X}_M)$ to the category of abelian groups. Also $(\mathrm{R}^q v_{\mathscr{X},M,*}(\mathscr{F}))_x$ is zero for $q \geq 1$ and $\mathscr{F}$ injective. For $q = 0$ we have

$$\big(\mathrm{R}^0 v_{\mathscr{X},M,*}(\mathscr{F})\big)_x = \lim_i \mathscr{F}\big(\mathrm{Spec}(R_i), \mathrm{Spec}(R_i \otimes_V K)\big) = \mathscr{F}\big(\overline{\mathscr{O}}_{X,x,M} \otimes_V K\big)^{G_{x,M}}.$$

This proves the claim.

(7) The proof is similar to the proof of (6) and left to the reader. □

Lemma 4.5. — *Assume that $\mathscr{X}$ is the formal scheme associated to X, that X is locally of finite type over V and that every closed point of X maps to the closed point of* $\mathrm{Spec}(V)$. *We then have the following equivalences of δ-functors :*

i. $\mathrm{R}^q\big(\nu_X^* \circ v_{X,M,*}\big) = \nu_X^* \circ \mathrm{R}^q v_{X,M,*}$ *and* $\mathrm{R}^q\big(\widehat{v}_{\mathscr{X},M,*} \circ \mu_{X,M}^*\big) = \big(\mathrm{R}^q \widehat{v}_{\mathscr{X},M,*}\big) \circ \mu_{X,M}^*$;

ii. $\nu_X^* \circ \mathrm{R}^q v_{X,M,*} \xrightarrow{\sim} \big(\mathrm{R}^q \widehat{v}_{\mathscr{X},M,*}\big) \circ \mu_{X,M}^*$.

Proof. — (i) Since ν_X^* and $\mu_{X,M}^*$ are exact and $v_{X,M,*}$ and $\widehat{v}_{\mathscr{X},M,*}$ are left exact, the derived functors of $\nu_X^* \circ v_{X,M,*}$ and $\widehat{v}_{\mathscr{X},M,*} \circ \mu_{X,M}^*$ exist. By 4.4 we have

$$\nu_X^* \left(\mathrm{R}^q v_{X,M,*}(\mathscr{F})\right)_{\hat{x}} \cong \left(\mathrm{R}^q v_{X,M,*}\left(u_{X,M,*}(\mathscr{F})\right)\right)_x \cong \mathrm{H}^q\left(G_{x,M}, \mathscr{F}_x\right)$$

and

$$\mathrm{R}^q \widehat{v}_{\mathscr{X},M,*}\left(\mu_{X,M}^*(\mathscr{F})\right)_{\hat{x}} \cong \mathrm{H}^q\left(G_{\hat{x},M}, \mu_{X,M}^*(\mathscr{F})_x\right).$$

This implies that if $\mathscr{F}$ is injective, $\nu_X^*\left(\mathrm{R}^q v_{X,M,*}(\mathscr{F})\right) = 0$ and $\mathrm{R}^q \widehat{v}_{\mathscr{X},M,*}\left(\mu_{X,M}^*(\mathscr{F})\right) = 0$ for $q \geq 1$. Hence, $\mathrm{R}^q\big(\nu_X^* \circ v_{X,M,*}\big) = \nu_X^* \circ \mathrm{R}^p v_{X,M,*}$ (resp. $\mathrm{R}^q\big(\widehat{v}_{\mathscr{X},M,*} \circ \mu_{X,M}^*\big) = \mathrm{R}^q \widehat{v}_{\mathscr{X},M,*} \circ \mu_{X,M}^*$). Indeed, they are both δ-functors since ν_X^* (resp. $\mu_{X,M}^*$) is exact, they are both erasable and they coincide for $q = 0$.

(ii) We construct a map $\gamma_{\mathscr{F}} \colon \nu_X^*\left(v_{X,M,*}(\mathscr{F})\right) \longrightarrow \widehat{v}_{\mathscr{X},M,*}\left(\mu_{X,M}^*(\mathscr{F})\right)$ functorial in $\mathscr{F}$. The sheaf $\nu_X^*\left(v_{X,M,*}(\mathscr{F})\right)$ is the sheafification of the presheaf $\mathscr{F}_1$ which associates to an object $\mathscr{U}$ in $\mathscr{X}_{\mathrm{et}}$ the direct limit $\lim \mathscr{F}(U, U_K)$ taken over all $U \in X_{\mathrm{et}}$ and all maps from $\mathscr{U}$ to the formal scheme associated to U. On the other hand, the *presheaf* $\mathscr{F}_2 := \widehat{v}_{\mathscr{X},M,*}\left(\mu_{X,M}^*(\mathscr{F})\right)$ ($\mu_{X,M}^*(\mathscr{F})$ is taken as presheaf) associates to $\mathscr{U} \in \mathscr{X}_{\mathrm{et}}$ the direct limit $\lim \mathscr{F}(U, W)$ over all (U, W) in $\mathfrak{X}_M$ and all maps from $\mathscr{U}$ to the formal scheme associated to U and from $\mathscr{U}^{\mathrm{rig}}$ to the rigid analytic space defined by $W^{\mathrm{rig}} \times_{U^{\mathrm{rig}}} \mathscr{U}^{\mathrm{rig}}$. We thus get a morphism at the level of presheaves $\mathscr{F}_1 \to \mathscr{F}_2$. Passing to the associated sheaves we get the claimed map.

The map $\gamma_{\mathscr{F}}$ induces $\mathrm{R}^q\gamma_{\mathscr{F}}\colon \mathrm{R}^q\big(\nu_X^*\circ v_{X,M,*}\big)(\mathscr{F}) \longrightarrow \mathrm{R}^q\big(\widehat{v}_{\mathscr{X},M,*}\circ\mu_{X,M}^*\big)(\mathscr{F})$. Using (i), we get a natural transformation of δ-functors as claimed in (ii). We are left to prove that it is an isomorphism. This can be checked on stalks and, as explained in the proof of (i), it amounts to prove that for any sheaf $\mathscr{F}$ one has $\mathrm{H}^q\left(G_{x,M},\mathscr{F}_x\right)\cong \mathrm{H}^q\left(G_{\hat{x},M},\mu_{X,M}^*(\mathscr{F})_{\hat{x}}\right)$. The conclusion follows since $\mu_{X,M}^*(\mathscr{F})_{\hat{x}}\cong\mathscr{F}_x$ and $G_{x,M}\cong G_{\hat{x},M}$ thanks to 4.4. □

For later purposes we introduce the following variants of the topologies introduced above:

4.4. Pointed étale sites. — Les $\mathscr{X}$ be a p-adic formal scheme, formally smooth over $\mathrm{Spf}(V)$ and with $\mathscr{X}\otimes_V k$ geometrically irreducible. Let $\mathbb{K}$ be a separable closure of the field of fractions of $\mathscr{X}\otimes_V k$. Let $\mathbf{W}_{\mathbb{K}}$ be a Cohen ring for $\mathbb{K}$ i. e., a complete DVR such that $\mathbf{W}_{\mathbb{K}}/p\mathbf{W}_{\mathbb{K}}\cong\mathbb{K}$. Let $\mathbb{C}_{\mathscr{X}}$ be the p-adic completion of an algebraic closure of the fraction field of $\mathbf{W}_{\mathbb{K}}$ containing $\overline{K}$.

The site $\mathscr{X}_{\mathrm{et}}^\bullet$. — Denote by $\mathscr{X}_{\mathrm{et}}^\bullet$ the following Grothendieck topology. As a category it consists of pairs $(\mathscr{U},s)$ where $\mathscr{U}\to\mathscr{X}$ is an étale morphism of formal schemes and s is a morphism $\mathrm{Spf}\big(\mathbf{W}_{\mathbb{K}}\otimes_{\mathbf{W}(k)}V\big)\to\mathscr{U}$ of V-formal schemes inducing a geometric generic point of $\mathscr{U}_k$. A map of pairs $(\mathscr{U},s)\to(\mathscr{U}',s')$ is a map of $\mathscr{X}$-schemes $\mathscr{U}\to\mathscr{U}'$ such that the composite with s is s'. A covering $\amalg_{i\in I}(\mathscr{U}_i,s_i)\to(\mathscr{U},s)$ is defined to a map of pairs $(\mathscr{U}_i,s_i)\to(\mathscr{U},s)$ for every i such that $\amalg_i\mathscr{U}_i\to\mathscr{U}$ is an étale covering.

Fix $\hat{x}\colon \mathrm{Spf}(V_x)\to\mathscr{X}$ as in 4.3 and *choose* a homomorphism $\eta_{\hat{x}}\colon \mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}}\to \mathbf{W}_{\mathbb{K}}\otimes_{\mathbf{W}(k)}V$ inducing the geometric generic point $\mathbb{K}$ on $\mathscr{X}\otimes_V k$. Given a sheaf $\mathscr{F}$ on $\mathscr{X}_{\mathrm{et}}^\bullet$ define $\mathscr{F}_{\hat{x}}$ to be $\mathscr{F}\big(\mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}}\big)$ as in 4.3 i. e., if $\mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}}$ is the direct limit $\lim_i S_i$ as in loc. cit. and if $s_i\colon \mathrm{Spf}(\mathbf{W}_{\mathbb{K}}\otimes_{\mathbf{W}(k)}V)\to\mathrm{Spf}(S_i)$ is defined composing with $\eta_{\hat{x}}$, then $\mathscr{F}\big(\mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}}\big):=\lim_{i\in I}\mathscr{F}\big((\mathrm{Spf}(S_i),s_i)\big)$. One then proves that a sequence of sheaves on $\mathscr{X}_{\mathrm{et}}^\bullet$ is exact if and only if the associated sequence of stalks is exact for every $\hat{x}\colon\mathrm{Spf}(V_x)\to\mathscr{X}$.

The site $\widehat{\mathfrak{X}}_M^\bullet$. — Define $\widehat{\mathfrak{X}}_M^\bullet$ to be the following Grothendieck topology. Its objects are the pairs $\big((\mathscr{U},s),\mathscr{W},L\big)$ where $(\mathscr{U},\mathscr{W},L)$ is an object of $\widehat{\mathfrak{X}}_M$ and $(\mathscr{U},s)$ is an object of $\mathscr{X}_{\mathrm{et}}^\bullet$. A morphism $\big((\mathscr{U},s),\mathscr{W},L\big)\to\big((\mathscr{U}',s'),\mathscr{W}',L'\big)$ in $\widehat{\mathfrak{X}}_M^\bullet$ is a morphism $(\mathscr{U},\mathscr{W},L)\to(\mathscr{U}',\mathscr{W}',L')$ in $\widehat{\mathfrak{X}}_M^\bullet$ such that the induced map $\mathscr{U}\to\mathscr{U}'$ arises from a map $(\mathscr{U},s)\to(\mathscr{U}',s')$ in $\mathscr{X}_{\mathrm{et}}^\bullet$. A covering $\amalg_{i\in I}\big((\mathscr{U}_i,s_i),\mathscr{W}_i,L_i\big)\to\big((\mathscr{U}',s'),\mathscr{W}',L'\big)$ is the datum of morphisms $\big((\mathscr{U}_i,s_i),\mathscr{W}_i,L_i\big)\to\big((\mathscr{U}',s'),\mathscr{W}',L'\big)$ for every $i\in I$ such that $\amalg_i(\mathscr{U}_i,\mathscr{W}_i,L_i)\to(\mathscr{U}'\mathscr{W}',L')$ is a covering in $\widehat{\mathfrak{X}}_M$.

Fix $\hat{x}\colon\mathrm{Spf}(V_x)\to\mathscr{X}$ as in 4.3. Choose a map $\eta_{\hat{x}}\colon\mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}}\to\mathbf{W}_{\mathbb{K}}\otimes_{\mathbf{W}(k)}V$ inducing the geometric generic point $\mathbb{K}$ on $\mathscr{X}\otimes_V k$. Given a sheaf $\mathscr{F}$ on $\widehat{\mathfrak{X}}_M^\bullet$ put $\mathscr{F}_{\hat{x}}:=\mathscr{F}\big(\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}\otimes_V K\big)$ where, using the notation of 4.3, we write $\mathscr{F}\big(\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}\otimes_V K\big):=\lim_{i,j}\mathscr{F}\big((\mathrm{Spf}(S_i),s_i),\mathrm{Spm}(S_{i,j}'\otimes_V K),L_{i,j}\big)$. Then, a sequence of sheaves on $\widehat{\mathfrak{X}}_M^\bullet$ is exact if and only if the associated sequence of stalks is exact for every $\hat{x}\colon \mathrm{Spf}(V_x) \to \mathscr{X}$.

We have functors

(i) $a\colon \mathscr{X}_{\mathrm{et}}^{\bullet} \to \mathscr{X}_{\mathrm{et}}$ given by $a\big(\mathscr{U}, s\big) = \mathscr{U}$;

(ii) $b\colon \widehat{\mathfrak{X}}_M^{\bullet} \to \widehat{\mathfrak{X}}_M$ given by $b\big((\mathscr{U}, s), \mathscr{W}, L\big) = (\mathscr{U}, \mathscr{W}, L)$;

(iii) $\widehat{v}_{\mathscr{X},M}\colon \mathscr{X}_{\mathrm{et}}^{\bullet} \to \widehat{\mathfrak{X}}_M^{\bullet}$ given by $\widehat{v}_{\mathscr{X},M}(\mathscr{U}, s) = \big((\mathscr{U}, s), \mathscr{U}^{\mathrm{rig}}, K\big)$.

As in 4.4(7) one proves that for every point $\hat{x}\colon \mathrm{Spf}(V_x) \to \mathscr{X}$,

$$\big(\mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathscr{F})\big)_{\hat{x}} \cong \mathrm{H}^i\left(G_{\hat{x},M}, \mathscr{F}_{\hat{x}}\right), \qquad G_{\hat{x},M} := \mathrm{Gal}\big(\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M} \mathop{\otimes}_V K/\mathscr{O}_{\mathscr{X},\hat{x}}^{\mathrm{sh}} \cdot M\big).$$

Then:

Lemma 4.6. — *Assume that X is smooth and geometrically irreducible over V and let $\mathscr{X}$ be the associated p-adic formal scheme. Then: The categories $\mathscr{X}_{\mathrm{et}}^{\bullet}$ and $\widehat{\mathfrak{X}}_M^{\bullet}$ admit final objects. Furthermore, a (resp. b, resp. $\widehat{v}_{\mathscr{X},M}$) send final object to final object and a (resp. b) are surjective.*

Proof. — Since $\mathscr{X}$ is formally smooth over $\mathrm{Spf}(V)$, for every étale map $\mathscr{U} \to \mathscr{X}$ also $\mathscr{U}$ is formally smooth over V. Thus, if $\mathscr{U}$ irreducible and if we fix a geometric generic point $\overline{s}_{\mathscr{U}}\colon \mathrm{Spec}(\mathbb{K}) \to \mathscr{U}_k$, there exists a map $s_{\mathscr{U}}\colon \mathrm{Spf}\left(\mathbf{W}_{\mathbb{K}} \otimes_{\mathbf{W}(k)} V\right) \to \mathscr{X}$ lifting $\overline{s}_{\mathscr{U}}$. This proves that a, and hence b, are surjective.

We claim that $(\mathscr{X}, s_{\mathscr{X}})$ is a final object in $\mathscr{X}_{\mathrm{et}}^{\bullet}$. This implies that $\big((\mathscr{X}, s_{\mathscr{X}}), \mathscr{X}^{\mathrm{rig}}, K\big)$ is a final object in $\widehat{\mathfrak{X}}_M^{\bullet}$ and that a, b and $\widehat{v}_{\mathscr{X},M}$ preserve final objects. To prove the claim it suffices to show that given two maps s, $s'\colon \mathrm{Spf}\left(\mathbf{W}_{\mathbb{K}} \otimes_{\mathbf{W}(k)} V\right) \to \mathscr{X}$ as V-formal schemes, inducing the generic point of $\mathscr{X}_k$, there exists an automorphism ρ of $\mathbf{W}_{\mathbb{K}} \otimes_{\mathbf{W}(k)} V$ (as V-algebra) such that $s' = s \circ \rho$. Let R be the p-adic completion of the localization of $\mathscr{X}$ at its generic point. It is a DVR. Write Z for $\mathbf{W}_{\mathbb{K}} \otimes_{\mathbf{W}(k)} V$. Then s and s' induce maps f and $f'\colon R \to Z$ such that, considering f or f', the maximal ideal of Z is generated by the maximal ideal of R and the residue field of Z is a separable closure of the residue field of R. By uniqueness of étale extensions, there exists an automorphism h of Z, as V-algebra, such that $g = h \circ f$. □

Corollary 4.7. — *Let $\mathscr{F}$ be a sheaf on $\mathscr{X}_{\mathrm{et}}$ (resp. $\widehat{\mathfrak{X}}_M$). We have a natural isomorphism of δ-functors $\mathrm{H}^i\big(\mathscr{X}_{\mathrm{et}}^{\bullet}, a_*(\mathscr{F})\big) \cong \mathrm{H}^i(\mathscr{X}_{\mathrm{et}}, \mathscr{F})$ (resp. $\mathrm{H}^i\big(\widehat{\mathfrak{X}}_M^{\bullet}, b_*(\mathscr{F})\big) \cong \mathrm{H}^i(\widehat{\mathfrak{X}}_M, \mathscr{F})$).*

Proof. — We have functors

$$a_*\colon \mathbf{Sh}\big(\mathscr{X}_{\mathrm{et}}\big) \longrightarrow \mathbf{Sh}\big(\mathscr{X}_{\mathrm{et}}^{\bullet}\big), \qquad b_*\colon \mathbf{Sh}\big(\widehat{\mathfrak{X}}_M\big) \longrightarrow \mathbf{Sh}\big(\widehat{\mathfrak{X}}_M^{\bullet}\big),$$

which send flasque objects to flasque objects. Since a and b are surjective by 4.6, a_* and b_* are also exact. Since $\mathrm{H}^0\big(\mathscr{X}_{\mathrm{et}}^{\bullet}, a_*(\mathscr{F})\big) \cong \mathrm{H}^0(\mathscr{X}_{\mathrm{et}}, \mathscr{F})$ and $\mathrm{H}^0\big(\widehat{\mathfrak{X}}_M^{\bullet}, b_*(\mathscr{F})\big) \cong \mathrm{H}^0(\widehat{\mathfrak{X}}_M, \mathscr{F})$, the lemma follows. □

This allows to work with pointed sites, which are better suited for Galois cohomology computations. Indeed, let $(\mathscr{U}, s) \in \mathscr{X}_{\mathrm{et}}^{\bullet}$ with $\mathscr{U}$ connected and let $\overline{\eta}_s\colon \mathrm{Spm}(\mathbb{C}_{\mathscr{X}}) \to \mathscr{U}_K^{\mathrm{rig}}$ be the composite of s_K^{rig} and the morphism $\mathbf{W}_{\mathbb{K}} \otimes_{\mathbf{W}(k)} K \to \mathbb{C}_{\mathscr{X}}$ chosen in 4.4. It induces a map $R_{\mathscr{U}} \subset \mathbb{C}_{\mathscr{X}}$. Let $R_{\mathscr{U}} \subset \overline{R}_{\mathscr{U}}$ be the union of all finite and normal $R_{\mathscr{U}}$-subalgebras of $\mathbb{C}_{\mathscr{X}}$, which are étale after inverting p. Define $\pi_1^{\mathrm{alg}}(\mathscr{U}_M^{\mathrm{rig}},\overline{\eta}_s)$, or simply $\pi_1(\mathscr{U}_M)$, to be $\mathrm{Gal}\left(\overline{R}_{\mathscr{U}} \otimes_V K/R_{\mathscr{U}} \otimes_V M\right)$ and let $\mathrm{Rep}_{\mathrm{disc}}\left(\pi_1(\mathscr{U}_M)\right)$ be the category of abelian groups, with the discrete topology, endowed with a continuous action of $\pi_1(\mathscr{U}_M)$.

Lemma 4.8. — *The category $\mathscr{U}_{M,\mathrm{fet}}$ is equivalent, as Grothendieck topology, to the category of finite sets with continuous action of $\pi_1(\mathscr{U}_M) := \mathrm{Gal}\left(\overline{R}_{\mathscr{U}}\left[\frac{1}{p}\right]/R_{\mathscr{U}} \otimes_V M\right)$. In particular,*

1) *the functor*

$$\mathbf{Sh}(\mathscr{U}_{M,\mathrm{fet}}) \longrightarrow \mathrm{Rep}_{\mathrm{disc}}\left(\pi_1(\mathscr{U}_M)\right), \qquad \mathscr{F} \mapsto \mathscr{F}(\overline{R}_{\mathscr{U}} \underset{V}{\otimes} K),$$

with $\mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K) := \lim_{(\mathscr{U},\mathscr{W})} \mathscr{F}(\mathscr{U},\mathscr{W})$ where the direct limit is over all elements of $\mathscr{U}_{M,\mathrm{fet}}$, defines an equivalence of categories;

2) *for $\mathscr{F} \in \mathbf{Sh}(\mathscr{U}_{M,\mathrm{fet}})$ we have $\mathrm{H}^i(\mathscr{U}_{M,\mathrm{fet}},\mathscr{F}) = \mathrm{H}^i\left(\pi_1(\mathscr{U}_M),\mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)\right)$, where the latter is the derived functor of $\mathrm{Rep}_{\mathrm{disc}}\left(\pi_1(\mathscr{U}_M)\right) \ni A \mapsto A^{\pi_1(\mathscr{U}_M)}$ (the Galois invariants of A).*

Proof. — The first claim follows noting that $\mathscr{U}_{M,\mathrm{fet}}$ is the category of finite and étale covers of $R_{\mathscr{U}} \otimes_V M$. By Grothendieck's reformulation of Galois theory the latter is equivalent to the category of finite sets with continuous action of $\pi_1(\mathscr{U}_M)$.

An inverse of the functor given in (1) is given as follows. Let $G \in \mathrm{Rep}_{\mathrm{disc}}\left(\pi_1(\mathscr{U}_M)\right)$. Let $(\mathscr{U},\amalg_i(\mathscr{W}_i,L_i)) \in \widehat{\mathfrak{X}}^\bullet_M$ with $\mathscr{W}_i = \mathrm{Spm}(S_i)$ and $S_i \otimes_{L_i} M$ a domain and fix an embedding $f_i\colon S_i \otimes_{L_i} M \to \overline{R}_{\mathscr{U}} \otimes_V K$. Let $H_i := \mathrm{Gal}(\overline{R}_{\mathscr{U}} \otimes_V K/S_i \otimes_L M) \subset \pi_1(\mathscr{U}_M)$ which is independent of f_i. Then, define $\mathscr{G}\left(\mathscr{U},\amalg_i(\mathscr{W}_i,L_i)\right) = \prod_i G^{H_i}$. One verifies that $\mathscr{G}$ is a sheaf and that the two functors are the inverse one of the other.

For claims (2) we note that the cohomology groups appearing are universal δ-functors coinciding for $i=0$. □

We next show that the sites introduced above are very useful in order to compute étale cohomology:

Proposition 4.9. — *(Faltings) Assume that X is locally of finite type over V and that every closed point of X maps to the closed point of $\mathrm{Spec}(V)$. Let $\mathbb{L}$ be a finite locally constant étale sheaf on X_M annihilated by p^s. For every i the map $\mathrm{H}^i(\mathfrak{X}_M,\mathbb{L}) \longrightarrow \mathrm{H}^i(X_{M,\mathrm{et}},\mathbb{L})$, induced by pull-back along $u_{X,M}$, is an isomorphism.*

Proof. — [**14**, Rmk. p. 242] Put $G_M := \mathrm{Gal}(\overline{K}/M)$. We have a spectral sequence

$$\mathrm{H}^p\left(G_M,\mathrm{H}^q(X_{\overline{K},\mathrm{et}},\mathbb{L})\right) \Longrightarrow \mathrm{H}^{p+q}\left(X_{M,\mathrm{et}},\mathbb{L}\right)$$

and, thanks to 4.4(4.b), a spectral sequence

$$\mathrm{H}^p\left(G_M,\mathrm{H}^q(\mathfrak{X}_{\overline{K}},\mathbb{L})\right) \Longrightarrow \mathrm{H}^{p+q}\left(\mathfrak{X}_M,\mathbb{L}\right).$$

Hence, it suffices to prove the proposition for $M = \overline{K}$. Let $z_{X,\overline{K}}\colon X_{\mathrm{et}} \to X_{\overline{K},\mathrm{et}}$ be the map $U \to U \otimes_V \overline{K}$. We can factor it via the maps $v_{X,\overline{K}}\colon X_{\mathrm{et}} \to \mathfrak{X}_{\overline{K}}$ and $u_{X,\overline{K}}\colon \mathfrak{X}_{\overline{K}} \to X_{\overline{K},\mathrm{et}}$. This induces a spectral sequence $\mathrm{R}^p v_{X,\overline{K},*} \circ \mathrm{R}^q u_{X,\overline{K},*}(\mathbb{L}) \Longrightarrow \mathrm{R}^{p+q} z_{X,\overline{K},*}(\mathbb{L})$ which provides with a map $\mathrm{R}^p v_{X,\overline{K},*}\big(u_{X,\overline{K},*}(\mathbb{L})\big) \to \mathrm{R}^p z_{X,\overline{K},*}(\mathbb{L})$. As usual we write $\mathbb{L}$ for $u_{X,\overline{K},*}(\mathbb{L})$. Furthermore, since $\mathbb{L}$ is a finite locally constant étale sheaf on X_K, for every point $x\colon \mathrm{Spec}(V_x) \to X$ the stalk $\big(u_{X,\overline{K},*}(\mathbb{L})\big)_x$ is isomorphic to $\mathbb{L}_{x_K}$. One knows from [**12**, Cor. II.2.2] that Spec $\Big(\mathscr{O}^{\mathrm{sh}}_{X,x} \otimes_V \overline{K}\Big)$ is $K(\pi,1)$. This implies that the stalk $\big(\mathrm{R}^q z_{X,\overline{K},*}(\mathbb{L})\big)_x$ is $\mathrm{H}^q\big(G_{x,\overline{K}}, \mathbb{L}_x\big)$. By 4.4 also the stalk $\big(\mathrm{R}^q v_{X,\overline{K},*}(\mathbb{L})\big)_x$ coincides with $\mathrm{H}^q\big(G_{x,\overline{K}}, \mathbb{L}_x\big)$. Hence, $\mathrm{R}^q u_{X,\overline{K},*}(\mathbb{L}) \cong \mathrm{R}^q z_{X,\overline{K},*}(\mathbb{L})$. Using the spectral sequences

$$\mathrm{H}^p\big(X_{\mathrm{et}}, \mathrm{R}^q z_{X,\overline{K},*}(\mathbb{L})\big) \Longrightarrow \mathrm{H}^{p+q}\left(X_{\overline{K},\mathrm{et}}, \mathbb{L}\right)$$

and

$$\mathrm{H}^p\big(X_{\mathrm{et}}, \mathrm{R}^q v_{X,\overline{K},*}(\mathbb{L})\big) \Longrightarrow \mathrm{H}^{p+q}\left(\mathfrak{X}_{\overline{K}}, \mathbb{L}\right)$$

the proposition follows. □

4.5. Comparison between algebraic cohomology and formal cohomology

Assume that X is locally of finite type over V and that every closed point of X maps to the closed point of $\mathrm{Spec}(V)$. Let $\mathscr{X}$ be the associated formal scheme. Since ν_X^* is an exact functor, given an injective resolution $I^\bullet$ of $\mathscr{F}$, then $0 \to \nu_X^*(\mathscr{F}) \to \nu_X^*(I^\bullet)$ is exact so that given an injective resolution $J^\bullet$ of $\mathscr{F}^{\mathrm{form}} = \nu_X^*(\mathscr{F})$ we can extend the identity map on $\mathscr{F}$ to a morphism of complexes $\nu_X^*(I^\bullet) \to J^\bullet$. Since ν_X sends the final object X of X_{et} to the final object $\mathscr{X}$ of $\mathscr{X}_{\mathrm{et}}$, one has a natural map $I^\bullet(X) \to \nu_X^*(I^\bullet)(\mathscr{X})$. Then,

Definition 4.10. — One has natural maps of δ-functors

$$\rho^q_{X,\mathscr{X}}(\mathscr{F})\colon \mathrm{H}^q\big(X_{\mathrm{et}}, \mathscr{F}\big) \to \mathrm{H}^q\big(\mathscr{X}_{\mathrm{et}}, \mathscr{F}^{\mathrm{form}}\big),$$
$$\rho^q_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}(\mathscr{F})\colon \mathrm{H}^q\big(\mathfrak{X}_M, \mathscr{F}\big) \to \mathrm{H}^q\big(\widehat{\mathfrak{X}}_M, \mathscr{F}^{\mathrm{rig}}\big)$$

Note that one has spectral sequences

$$\mathrm{H}^q\big(X_{\mathrm{et}}, \mathrm{R}^p v_{X,M,*}(\mathscr{F})\big) \Longrightarrow \mathrm{H}^{p+q}(\mathfrak{X}_M, \mathscr{F}) \tag{7}$$

and

$$\mathrm{H}^q\big(\mathscr{X}_{\mathrm{et}}, \nu_X^* \mathrm{R}^p v_{\mathscr{X},M,*}(\mathscr{F})\big) = \mathrm{H}^q\big(\mathscr{X}_{\mathrm{et}}, \mathrm{R}^p \widehat{v}_{\mathscr{X},M,*}(\mathscr{F}^{\mathrm{rig}})\big) \Longrightarrow \mathrm{H}^{p+q}\big(\widehat{\mathfrak{X}}_M, \mathscr{F}^{\mathrm{rig}}\big) \tag{8}$$

where the equality on the left hand side is due to 4.5.

Proposition 4.11. — *The following hold:*

a. *If $\mathscr{F}$ in* $\mathbf{Sh}(X_{\mathrm{et}})$ *is torsion, the map $\rho^q_{X,\mathscr{X}}(\mathscr{F})$ is an isomorphism;*

b. *the spectral sequences (7) and (8) are compatible via $\rho^q_{X,\mathscr{X}}$ and $\rho^{p+q}_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}$;*

c. *if $\mathscr{F}$ is a torsion sheaf on $\mathfrak{X}_M$, the map $\rho^q_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}(\mathscr{F})$ is an isomorphism.*

Proof. — (a) Let $X_k := X \otimes_V k$ and denote by $\iota\colon X_{\mathrm{et}} \to X_{k,\mathrm{et}}$ and $\widehat{\iota}\colon \mathscr{X}_{\mathrm{et}} \to X_{k,\mathrm{et}}$ the functors induced by the closed immersions $X_k \subset X$ and $X_k \subset \mathscr{X}$ respectively. In fact, $\widehat{\iota}$ is an equivalence, since the étale sites of $\mathscr{X}$ and of X_k coincide, and $\widehat{\iota} \circ \nu_X = \iota$. For any sheaf $\mathscr{F}$ on X_{et} denote $\mathscr{F}_k := \iota^*(\mathscr{F})$ or, equivalently, $\widehat{\iota}^*\big(\mathscr{F}^{\mathrm{form}}\big)$. We then have $\mathrm{H}^q\big(X_{\mathrm{et}}, \mathscr{F}\big) \to \mathrm{H}^q\big(\mathscr{X}_{\mathrm{et}}, \mathscr{F}^{\mathrm{form}}\big) \cong \mathrm{H}^q\big(X_{k,\mathrm{et}}, \mathscr{F}_k\big)$ where the first map is $\rho_{X,\mathscr{X}}(\mathscr{F})^q$. The composite is defined by restriction and is an isomorphism if $\mathscr{F}$ is a torsion sheaf due to [**16**, Cor. 1] and the fact that X is proper over V. The conclusion follows.

(b) Left to the reader.

(c) The left hand side of the spectral sequences (7) and (8) are isomorphic by (a) since $\mathrm{R}^p v_{X,M,*}$ sends a torsion sheaf to a torsion sheaf. The conclusion follows from (b). □

Corollary 4.12. — *Let $\mathbb{L}$ be a locally constant sheaf on $\mathfrak{X}_M$ annihilated by p^s. Then, the two sides of the Leray spectral sequences*

$$\mathrm{H}^j\left(\mathscr{X}_{\mathrm{et}}, \mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathbb{L}^{\mathrm{rig}})\right) \Longrightarrow \mathrm{H}^{i+j}\left(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}}\right)$$

and

$$\mathrm{H}^j\left(X_{\mathrm{et}}, \mathrm{R}^i v_{X,M,*}(\mathbb{L})\right) \Longrightarrow \mathrm{H}^{i+j}\left(\mathfrak{X}_M, \mathbb{L}\right),$$

obtained from the morphisms of topoi $\widehat{v}_{\mathscr{X},M}\colon \widehat{\mathfrak{X}}_M \to \mathscr{X}_{\mathrm{et}}$ and $v_{X,M}\colon \mathfrak{X}_M \to X_{\mathrm{et}}$, are naturally isomorphic.

Proof. — The statements follow from 4.11. □

Proof of Theorem 4.1. — Let $\mathbb{L}$ be an étale local system of $\mathbf{Z}/p^n\mathbf{Z}$-modules on X_K, for some $n \geq 1$ and let $\mathbb{L}^{\mathrm{rig}}$ be the corresponding étale local system on the rigid space X_K^{rig} attached to X_K. Let $(\mathscr{U}, s) \in \mathscr{X}_{\mathrm{et}}^{\bullet}$ with $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ small affine; see 2.2. Put $\mathbf{L}_{\mathscr{U}} := \mathbb{L}^{\mathrm{rig}}\big(\overline{R}_{\mathscr{U}} \otimes_V K\big)$; the notation is as in 4.8. It has a continuous action of the algebraic fundamental group $\pi_1^{\mathrm{alg}}(\mathscr{U}_K^{\mathrm{rig}}, \overline{\eta}_s)$. Since $\mathbb{L}^{\mathrm{rig}}$ is finite and locally constant there exists $(\mathscr{U}, \mathscr{W}) \in \mathscr{U}_{\overline{K},\mathrm{fet}}$ such that $\mathbb{L}^{\mathrm{rig}}(\mathscr{U}, \mathscr{W})$ is trivial and then $\mathbf{L}_{\mathscr{U}} := \mathbb{L}^{\mathrm{rig}}(\mathscr{U}, \mathscr{W})$. As a $\mathbf{Z}$-module it is independent of $\mathscr{U}$ and $\mathscr{W}$ and we simply write $\mathbf{L}$ by abuse of notation. It follows from 3.2 that $\mathscr{H}_{\overline{K}}^{i,*,\mathrm{ge}}(\mathbb{L})$ is the sheaf on $\mathscr{X}_{\mathrm{et}}^{\bullet}$ associated to the following functor $(\mathscr{U}, s) \longrightarrow \mathrm{H}^i\big(\pi_1^{\mathrm{alg}}(\mathscr{U}_{\overline{K}}^{\mathrm{rig}}, \overline{\eta}_s), \mathbf{L}\big)$. Analogously, $\mathscr{H}_{\overline{K}}^{i,*,\mathrm{ar}}(\mathbb{L})$ is the sheaf on $\mathscr{X}_{\mathrm{et}}^{\bullet}$ associated to the following functor $(\mathscr{U}, s) \longrightarrow \mathrm{H}^i\big(\pi_1^{\mathrm{alg}}(\mathscr{U}_K^{\mathrm{rig}}, \overline{\eta}_s), \mathbf{L}\big)$.

We come to the proofs of (i) and (ii) of 4.1. Since they are very similar we prove only (i). Consider a point $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ of $\mathscr{X}$ and let $\hat{\overline{x}}\colon \mathrm{Spf}(\widehat{\overline{V}}) \to \mathscr{X}$ be the composite of $\hat{x}$ with $\mathrm{Spf}(\widehat{\overline{V}}) \to \mathrm{Spf}(V_{\hat{x}})$. Then, $\mathrm{G}_{\hat{x},\overline{K}}$ is the direct limit of the fundamental groups $\pi_1^{\mathrm{alg}}\big(\mathscr{U}_{\alpha,\overline{K}}^{\mathrm{rig}}, \overline{\eta}_{s_\alpha}\big)$ over a set $(\mathscr{U}_\alpha, s_\alpha)$ of affine small neighborhoods of $\hat{\overline{x}}$, which is cofinal among such étale neighborhoods and is totally ordered with respect to morphisms in $\mathscr{X}_{\mathrm{et}}^{\bullet}$. As remarked above one has a canonical identification

$(\mathbb{L}^{\mathrm{rig}})_{\hat{x}} \cong \mathbf{L}$ (as groups). We conclude that the stalk of $\mathscr{H}^{i,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L})$ at $\hat{x}$ can be described as follows:

$$\big(\mathscr{H}^{i,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L})\big)_{\hat{x}} \cong \mathrm{H}^i\big(\mathrm{G}_{\hat{x},\overline{K}}, \mathbf{L}\big).$$

On the other hand for every $(\mathscr{U}, s) \in \mathscr{X}^{\bullet}_{\mathrm{et}}$ with $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ small, the map of Grothendieck topologies $\iota\colon \mathscr{U}_{\overline{K},\mathrm{fet}} \to \widehat{\mathfrak{X}}^{\bullet}_{\overline{K}}$ (see 4.1) induces a spectral sequence

$$\mathrm{H}^i\big(\mathscr{U}_{\overline{K},\mathrm{fet}}, \mathrm{R}^j\iota_*(\mathbb{L}^{\mathrm{rig}})\big) \Longrightarrow \mathrm{R}^i\widehat{v}_{\mathscr{X},\overline{K},*}(\mathbb{L}^{\mathrm{rig}})(\mathscr{U}, s).$$

The category of sheaves on $\mathscr{U}_{\overline{K},\mathrm{fet}}$ is equivalent to the category of discrete groups with continuous action of $\pi_1^{\mathrm{alg}}\big(\mathscr{U}^{\mathrm{rig}}_{\overline{K}}, \overline{\eta}_s\big)$ by 4.8. Via this equivalence $\iota_*(\mathbb{L}^{\mathrm{rig}}) = \mathbf{L}$ as representation of $\pi_1^{\mathrm{alg}}\big(\mathscr{U}^{\mathrm{rig}}_{\overline{K}}, \overline{\eta}_s\big)$. Furthermore, $\mathrm{H}^i\big(\mathscr{U}_{\overline{K},\mathrm{fet}}, \iota_*(\mathbb{L}^{\mathrm{rig}})\big) \cong \mathrm{H}^i\big(\pi_1^{\mathrm{alg}}(\mathscr{U}^{\mathrm{rig}}_{\overline{K}}, \overline{\eta}_s), \mathbf{L}\big)$ by loc. cit. We then obtain a natural, functorial map

$$\alpha^i_{\mathscr{U}} : \mathrm{H}^i(\pi_1^{\mathrm{alg}}(\mathscr{U}^{\mathrm{rig}}_{\overline{K}}, \overline{\eta}_s), \mathbf{L}) \longrightarrow \mathrm{R}^i\widehat{v}_{\mathscr{X},\overline{K},*}(\mathbb{L}^{\mathrm{rig}})(\mathscr{U}, s),$$

which induces a morphism of sheaves on $\mathscr{X}^{\bullet}_{\mathrm{et}}$

$$\alpha^i\colon \mathscr{H}^{i,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L}) \longrightarrow \mathrm{R}^i\widehat{v}_{\mathscr{X},\overline{K},*}(\mathbb{L}^{\mathrm{rig}}).$$

For every point $\hat{x}$ of $\mathscr{X}$ the map $\alpha^i_{\hat{x}}$ induced by α^i on stalks is the canonical morphism

$$\alpha^i_{\hat{x}}\colon \big(\mathscr{H}^{i,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L})\big)_{\hat{x}} \cong \mathrm{H}^i(\mathrm{G}_{\hat{x},\overline{K}}, (\mathbb{L}^{\mathrm{rig}})_{\hat{x}}) \longrightarrow \big(\mathrm{R}^i\widehat{v}_{\mathscr{X},\overline{K},*}(\mathbb{L}^{\mathrm{rig}})\big)_{\hat{x}},$$

which by proposition 4.4(7) is an isomorphism. Therefore, α^i induces an isomorphism $\mathscr{H}^{i,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L}) \cong \mathrm{R}^i\widehat{v}_{\mathscr{X},\overline{K},*}(\mathbb{L}^{\mathrm{rig}})$ of sheaves on $\mathscr{X}^{\bullet}_{\mathrm{et}}$. Thus, the Leray spectral sequence takes the form

$$E_2^{p,q} = \mathrm{H}^q\left(\mathscr{X}^{\bullet}_{\mathrm{et}}, \mathscr{H}^{p,*,\mathrm{ge}}_{\overline{K}}(\mathbb{L})\right) \Longrightarrow \mathrm{H}^{p+q}(\widehat{\mathfrak{X}}^{\bullet}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}).$$

It follows from 4.7 that $\mathrm{H}^{p+q}(\widehat{\mathfrak{X}}^{\bullet}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}) \cong \mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}})$. Furthermore, by 4.12 we have $\mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}) \cong \mathrm{H}^{p+q}(\mathfrak{X}_{\overline{K}}, \mathbb{L})$ and, thanks to 4.4, we know that $\mathrm{H}^{p+q}(\mathfrak{X}_{\overline{K}}, \mathbb{L}) \cong \mathrm{H}^{p+q}\big(X_{\overline{K},\mathrm{et}}, \mathbb{L}\big)$. All these isomorphisms are equivariant for the residual action of G_V. This proves the claim. □

5. A geometric interpretation of classical (φ, Γ)-modules

Let the notations be as in the previous section and fix as before M an algebraic extension of K contained in $\overline{K}$. In this section we work with continuous sheaves on all our topologies (see §4). We define families of continuous sheaves denoted $\overline{\mathscr{O}}_{\mathfrak{X}_M}$, $\mathscr{R}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$, $\mathrm{A}^+_{\mathrm{inf}}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ and call them *algebraic Fontaine sheaves* on $\mathfrak{X}_M$ (respectively $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}, \mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right), \mathrm{A}^+_{\mathrm{inf}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$ called *analytic Fontaine sheaves* on $\widehat{\mathfrak{X}}_M$) and study their properties. In this section we compare the cohomology on $\mathfrak{X}_M$ of an étale local system $\mathbb{L}$ of $\mathbf{Z}/p^s\mathbf{Z}$-modules on X_K tensored by one of the algebraic Fontaine sheaves with the cohomology on $\widehat{\mathfrak{X}}_M$ of its analytic analogue. As a consequence we derive the following result.

Let us fix $M=K_\infty=K(\mu_{p^\infty})$ and consider the sheaf $\mathscr{F}_\infty:=\mathbb{L}^{\mathrm{rig}}\otimes A_{\mathrm{inf}}^+\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{K_\infty}}\big)$ on $\widehat{\mathfrak{X}}_{K\infty}$.

Theorem 5.1. — *Let $\mathbb{L}$ be an étale local system $\mathbb{L}$ of $\mathbf{Z}/p^s\mathbf{Z}$-modules on X_K. We have natural isomorphisms of classical (φ,Γ_V)-modules*

$$\mathrm{H}^i(\widehat{\mathfrak{X}}_{K_\infty},\mathscr{F}_\infty)\cong\widetilde{\mathfrak{D}}_V\Big(\mathrm{H}^i(X_{\overline{K},\mathrm{et}},\mathbb{L})\Big),$$

for all $i\geq 0$.

The proof of theorem 5.1 will take the rest of this section.

5.1. Categories of inverse systems. — We review some of the results of [**20**] which will be needed in the sequel. Let $\mathscr{A}$ be an abelian category. Denote by $\mathscr{A}^{\mathbf{N}}$ the category of inverse systems indexed by the set of natural numbers. Objects are inverse systems $\{A_n\}_n:=\cdots\to A_{n+1}\to A_n\ldots A_2\to A_1$, where the A_i's are objects of $\mathscr{A}$ and the arrows denote morphisms in $\mathscr{A}$. The morphisms in $\mathscr{A}^{\mathbf{N}}$ are commutative diagrams

$$\begin{tikzcd}\cdots \arrow[r] & A_{n+1} \arrow[r]\arrow[d] & A_n \arrow[r]\arrow[d] & \cdots \arrow[r] & A_2 \arrow[r]\arrow[d] & A_1 \arrow[d]\\ \cdots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & \cdots \arrow[r] & B_2 \arrow[r] & B_1,\end{tikzcd}$$

where the vertical arrows are morphisms in $\mathscr{A}$. Then, $\mathscr{A}^{\mathbf{N}}$ is an abelian category with kernels and cokernels taken componentwise and if $\mathscr{A}$ has enough injectives, then $\mathscr{A}^{\mathbf{N}}$ also has enough injectives; see [**20**, Prop. 1.1]. Furthermore, there is a fully faithful and exact functor $\mathscr{A}\to\mathscr{A}^{\mathbf{N}}$ sending an object A of $\mathscr{A}$ to the inverse system $\{A\}_n:=\cdots\to A\to A\cdots\to A$ with transition maps given by the identity and a morphism $f\colon A\to B$ of $\mathscr{A}$ to the map of inverse systems $\{A\}_n\to\{B\}_n$ defined by f on each component. By [**20**, Prop. 1.1] such map preserves injective objects.

Let $h\colon\mathscr{A}\to\mathscr{B}$ be a left exact functor of abelian categories. It induces a left exact functor $h^{\mathbf{N}}\colon\mathscr{A}^{\mathbf{N}}\to\mathscr{B}^{\mathbf{N}}$ which, by abuse of notation and if no confusion is possible, we denote again by h. If $\mathscr{A}$ has enough injectives, then also $\mathscr{A}^{\mathbf{N}}$ does and the injective objects of $\mathscr{A}^{\mathbf{N}}$ are of the form (I_n,d_n) where $I_n\in\mathscr{A}$ is injective and d_n is a split surjection; see [**20**, Prop. 1.1]. One can derive the functor $h^{\mathbf{N}}$. It is proven in [**20**, Prop. 1.2] that $\mathrm{R}^i\big(h^{\mathbf{N}}\big)=\big(\mathrm{R}^ih\big)^{\mathbf{N}}$.

If inverse limits over $\mathbf{N}$ exist in $\mathscr{B}$, define the left exact functor $\varprojlim h\colon\mathscr{A}^{\mathbf{N}}\to\mathscr{B}$ as the composite of $h^{\mathbf{N}}$ and the inverse limit functor $\varprojlim\colon\mathscr{B}^{\mathbf{N}}\to\mathscr{B}$. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. For every $A=\{A_n\}_n\in\mathscr{A}^{\mathbf{N}}$ one then has a spectral sequence

$$\varprojlim{}^{(p)}\mathrm{R}^qh(A_n)\Longrightarrow\mathrm{R}^{p+q}\big(\varprojlim h(A)\big),$$

where $\varprojlim^{(p)}$ is the p-th derived functor of $\varprojlim$ in $\mathscr{B}$. If in $\mathscr{B}$ infinite products exist and are exact functors, then $\varprojlim^{(p)}=0$ for $p\geq 2$ and the above spectral sequence reduces to the simpler exact sequence

$$0 \longrightarrow \varprojlim^{(1)} \mathrm{R}^{i-1}h(A_n) \longrightarrow \mathrm{R}^i(\varprojlim h)(A) \to \varprojlim \mathrm{R}^i h(A_n) \longrightarrow 0; \tag{9}$$

see [**20**, Prop. 1.6]. In particular, if A is injective, then $\varprojlim^{(p)} \mathrm{R}^q h(A_n) = 0$ for $q \geq 1$ and for $q = 0$ and $p \geq 1$ by the structure of the injective objects in $\mathscr{A}^{\mathbf{N}}$. In a particular, injective objects of $\mathscr{A}^{\mathbf{N}}$ are acyclic for $\mathrm{R}^{p+q} \varprojlim h$.

Note that via the map $\mathscr{A} \to \mathscr{A}^{\mathbf{N}}$ given above, if $A \in \mathscr{A}$ then $\mathrm{R}^i h^{\mathbf{N}}(\{A\}_n) = \{\mathrm{R}^i h(A)\}_n$ and $\mathrm{R}^i \varprojlim h(\{A\}_n) = \mathrm{R}^i h(A)$.

5.2. Example. — [**20**, §2] Let G be a profinite group. Let $\mathscr{A}$ be the category of discrete modules with continuous action of G and let $\mathscr{B}$ be the category of abelian groups. For every j let $\mathrm{H}^j(G, \_) : \mathscr{A}^{\mathbf{N}} \to \mathscr{B}$ be the j-th derived functor of $\varprojlim \mathrm{H}^0(G, \_)$ on $\mathscr{A}^{\mathbf{N}}$. By loc. cit. for every inverse system $T = \{T_n\}_n \in \mathscr{A}^{\mathbf{N}}$ we have an exact sequence

$$0 \longrightarrow \varprojlim^{(1)} \mathrm{H}^{j-1}(G, T_n) \longrightarrow \mathrm{H}^j(G, T) \longrightarrow \varprojlim \mathrm{H}^j(G, T_n) \longrightarrow 0. \tag{10}$$

Moreover given $\{(N_n, d_n)\}_n \in \mathscr{B}^{\mathbf{N}}$, one computes $\varprojlim^{(1)} N_n$ as the cokernel of the map

$$\prod_n (\mathrm{Id} - d_n) \colon \prod_n N_n \longrightarrow \prod_n N_n. \tag{11}$$

For later use we remark the following. Assume that each N_n is a module over a ring C and that $d_n \colon N_{n+1} \to N_n$ is a homomorphism of C-modules. Suppose that for every n there exists an element $c_n \in C$ annihilating the cokernel of d_n. One then proves by induction on $m \in \mathbf{N}$ that the cokernel of $\prod_n (\mathrm{Id} - d_n) \colon \prod_{n \leq m} N_n \to \prod_{n \leq m} N_n$ is annihilated by $c_1 \cdots c_m$. In particular, if C is a complete local domain and $c_n = c^{\frac{1}{p^n}} \in C$ for every n for some c in the maximal ideal of C so that the product $\prod_m c_m$ converges to $c^{\frac{p}{p-1}}$ in C, then $c^{\frac{p}{p-1}}$ annihilates $\varprojlim^{(1)} N_n$.

For every $\{(T_n, d_n)\} \in \mathscr{A}^{\mathbf{N}}$ one defines $\mathrm{H}^j_{\mathrm{cont}}\left(G, \lim_{\infty \leftarrow n} T_n\right)$ as the continuous cohomology defined by continuous cochains modulo continuous coboundaries with values in $\lim_{\infty \leftarrow n} T_n$ endowed with the inverse limit topology considering on each T_n the discrete topology. As explained in [**20**, Pf. of Thm. 2.2] there exists a canonical complex $D^\bullet(G, T_n)$ whose G-invariants define the continuous cochains $C^\bullet(G, T_n)$ of G with values in T_n and such that each $D^i(G, T_n)$ is G-acyclic. This resolution is functorial so that we get a resolution

$$(T_n, d_n) \subset (D^1(G, T_n), d_n^1) \to (D^2(G, T_n), d_n^2) \to \cdots .$$

The continuous cohomology $\mathrm{H}^i_{\mathrm{cont}}(G, \lim_{\infty \leftarrow n} T_n)$ is obtained by applying $\lim_{\infty \leftarrow n} \mathrm{H}^0(G, \_)$ to this resolution and taking homology. Due to (10), since $D^i(G, T_n)$ is G-acyclic, we have

$$\mathrm{H}^i_{\mathrm{cont}}\big(G, \lim_{\leftarrow}(D^j(G,T_n),d_n^j)\big) = \begin{cases} 0 & \text{if } i \geq 2 \\ \overset{(1)}{\lim\limits_{\leftarrow}} C^j(G,T_n) & \text{for } i=1 \\ \lim\limits_{\infty\leftarrow n} C^j(G,T_n) & \text{if } i=0 \end{cases} \tag{12}$$

In particular, if the system $\{T_n\}_n$ is Mittag-Leffler, then $(D^j(G,T_n),d_n^j)$ is acyclic for every j and we obtain

$$\mathrm{H}^i_{\mathrm{cont}}\big(G, \lim_{\infty\leftarrow n} T_n\big) \xrightarrow{\sim} \mathrm{H}^i(G,T).$$

Next, assume as before that there exists a complete local domain C such that T_n is a C-module and d_n is a homomorphism of C-modules. Suppose also that there is c in the maximal ideal of C such that $c^{\frac{1}{p^n}} \in C$ and $c^{\frac{1}{p^n}}$ annihilates the cokernel of d_n. Then, $c^{\frac{1}{p^n}}$ annihilates also the cokernel of $C^i(G,T_{n+1}) \to C^i(G,T_n)$ so that $c^{\frac{p}{p-1}}$ annihilates $\mathrm{H}^1_{\mathrm{cont}}\big(G,(D^j(G,T_n),d_n^1)\big)$. This implies that if we invert $c^{\frac{p}{p-1}}$ we have an isomorphism

$$\mathrm{H}^i_{\mathrm{cont}}\big(G, \lim_{\infty\leftarrow n} T_n\big)\big[c^{-\frac{p}{p-1}}\big] \xrightarrow{\sim} \mathrm{H}^i(G,T)\big[c^{-\frac{p}{p-1}}\big]. \tag{13}$$

5.3. Fontaine sheaves on $\mathfrak{X}_M$ and $\widehat{\mathfrak{X}}_M$. — We now come to the definition of a family of sheaves on $\mathfrak{X}_M$ and $\widehat{\mathfrak{X}}_M$ which will play a crucial role in the sequel. See 5.11.

Definition 5.2. — [**14**, p. 219-221] The notation is as in 4.1. Let $\overline{\mathscr{O}}_{\mathfrak{X}_M}$ be the sheaf of rings on $\mathfrak{X}_M$ defined requiring that for every object (U,W) in $\mathfrak{X}_M$ the ring $\overline{\mathscr{O}}_{\mathfrak{X}_M}(U,W)$ consists of the normalization of $\Gamma\big(U,\mathscr{O}_U\big)$ in $\Gamma\big(W,\mathscr{O}_W\big)$.

Denote by $\mathscr{R}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ the sheaf of rings in $\mathbf{Sh}(\mathfrak{X}_M)^{\mathbf{N}}$ given by the inverse system $\big\{\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big\}$, where the transition maps are given by Frobenius.

For every $s\in\mathbf{N}$ define the sheaf of rings $A^+_{\mathrm{inf,s}}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ in $\mathbf{Sh}(\mathfrak{X}_M)^{\mathbf{N}}$ as the inverse system $\big\{\mathbf{W}_s\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)\big\}$. Here, $\mathbf{W}_s\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ is the sheaf $\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)^s$ with ring operations defined by Witt polynomials and the transition maps in the inverse system are defined by Frobenius. Define $A^+_{\mathrm{inf}}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ to be the inverse system of sheaves $\big\{\mathbf{W}_n\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)\big\}_n$ where the transition maps are defined as the composite of the projection $\mathbf{W}_{n+1}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)\to\mathbf{W}_n\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$ and Frobenius on $\mathbf{W}_n\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)$.

Similarly, $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$ is the sheaf of rings on $\widehat{\mathfrak{X}}_M$ associating to an object $(\mathscr{U},\mathscr{W},L)$ in $\widehat{\mathfrak{X}}_M$ the ring $\overline{\mathscr{O}}(\mathscr{U},\mathscr{W})$ defined as the normalization of $\Gamma\big(\mathscr{U},\mathscr{O}_{\mathscr{U}}\big)$ in $\Gamma\big(\mathscr{W},\mathscr{O}_{\mathscr{W}}\big)\otimes_L M$.

Let $\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$ be the sheaf of rings in $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)^{\mathbf{N}}$ given by $\left\{\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\right\}$, where the inverse system is taken using Frobenius as transition map.

For $s\in\mathbf{N}$ define the sheaf of rings $A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$ in $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)^{\mathbf{N}}$ as the inverse system $\left\{\mathbf{W}_s\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\right\}$ with transition maps given by Frobenius. Eventually, let $A^+_{\mathrm{inf}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$ in $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)^{\mathbf{N}}$ be the sheaf $\left\{\mathbf{W}_n(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M})\right\}$ where the transition maps are defined as the composite

$$\mathbf{W}_{n+1}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}) \longrightarrow \mathbf{W}_n(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}) \longrightarrow \mathbf{W}_n(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M});$$

here, the first map is the natural projection and the second is Frobenius.

We denote by φ the Frobenius operator acting on the sheaves, or inverse systems of sheaves, introduced above. One can define analogous sheaves for the pointed sites $\mathfrak{X}^\bullet_M$ and $\widehat{\mathfrak{X}}^\bullet_M$; we leave the details to the reader.

Remark 5.3. — Note that if $X = V$ and $M = \overline{K}$, one has $\mathrm{H}^0_{\mathrm{cont}}\left((V,\overline{K}),\mathscr{R}(\overline{\mathscr{O}}_{\overline{K}})\right) = \widetilde{\mathbf{E}}^+_{\overline{V}}$, $\mathrm{H}^0_{\mathrm{cont}}\left((V,\overline{K}),A^+_{\mathrm{inf,s}}(\overline{\mathscr{O}}_{\overline{K}})\right) = \mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}})$ and $\mathrm{H}^0_{\mathrm{cont}}\left((V,\overline{K}),A^+_{\mathrm{inf}}(\overline{\mathscr{O}}_{\overline{K}})\right) = \widetilde{\mathbf{A}}^+_{\overline{V}}$.

For later use, we recall that we denote by π the element $[\varepsilon]-1$ of $\widetilde{\mathbf{A}}^+_{\overline{V}}$ where ε is the element $(1,\zeta_p,\zeta_{p^2},\cdots) \in \mathbf{E}^+_{\overline{V}}$ and $[\varepsilon]$ is its Teichmüller lift.

Notation. — If $\mathscr{F}$ is in $\mathbf{Sh}(\mathfrak{X}_M)^{\mathbf{N}}$ (resp. $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)^{\mathbf{N}}$) write $\mathrm{H}^i_{\mathrm{cont}}(\mathfrak{X}_M,\mathscr{F})$ (respectively $\mathrm{H}^i_{\mathrm{cont}}(\widehat{\mathfrak{X}}_M,\mathscr{F})$) for the i-th derived functor of $\varprojlim \mathrm{H}^0(\mathfrak{X}_M,\_\_)$ (resp. $\varprojlim \mathrm{H}^0(\widehat{\mathfrak{X}}_M,\_\_)$) applied to $\mathscr{F}$. Note that if $\mathscr{F} = \{\mathscr{G}\}_n$ with $\mathscr{G} \in \mathbf{Sh}(\mathfrak{X}_M)$ (resp. in $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)$), then $\mathrm{H}^i_{\mathrm{cont}}(\mathfrak{X}_M,\mathscr{F}) = \mathrm{H}^i(\mathfrak{X}_M,\mathscr{G})$ (resp. $\mathrm{H}^i_{\mathrm{cont}}(\widehat{\mathfrak{X}}_M,\mathscr{F}) = \mathrm{H}^i(\widehat{\mathfrak{X}}_M,\mathscr{G})$).

One proves as in 4.7 that if $\mathscr{F} \in \mathbf{Sh}(\mathscr{X}_{\mathrm{et}})^{\mathbf{N}}$ (resp. $\mathbf{Sh}(\widehat{\mathfrak{X}}_M)^{\mathbf{N}}$), we have a natural isomorphism of δ-functors $\mathrm{H}^i_{\mathrm{cont}}\left(\mathscr{X}^\bullet_{\mathrm{et}},a^{\mathbf{N}}_*(\mathscr{F})\right) \cong \mathrm{H}^i_{\mathrm{cont}}(\mathscr{X}_{\mathrm{et}},\mathscr{F})$ (resp. $\mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}^\bullet_M,b^{\mathbf{N}}_*(\mathscr{F})\right) \cong \mathrm{H}^i_{\mathrm{cont}}(\widehat{\mathfrak{X}}_M,\mathscr{F})$).

From now on we assume that X is locally of finite type over V and that every closed point of X maps to the closed point of $\mathrm{Spec}(V)$. We let $\mathscr{X}$ be the formal scheme associated to X.

Lemma 5.4. — *One has* $A^+_{\mathrm{inf},*}(\overline{\mathscr{O}}_{\mathfrak{X}_M})^{\mathrm{rig}} \xrightarrow{\sim} A^+_{\mathrm{inf},*}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M})$ *where* $* = s \in \mathbf{N}$ *or* $* = \varnothing$.

Proof. — Consider a pair (U,W) in $\mathfrak{X}_M$, with W defined over some finite extension $K \subset L$ contained in M. Recall from section 4 that $\mu_{X,M}(U,W) := (\mathscr{U},\mathscr{W},L)$. We have a natural map $\overline{\mathscr{O}}_{\mathfrak{X}_M}(U,W) \to \mu_{X,M,*}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)(U,W)$ i. e., a map from the normalization of $\Gamma(U,\mathscr{O}_U)$ in $\Gamma(W,\mathscr{O}_W)\otimes_L M$ to the normalization of $\Gamma(\mathscr{U},\mathscr{O}_{\mathscr{U}})$ in $\Gamma(\mathscr{W},\mathscr{O}_{\mathscr{W}})\otimes_L M$. This induces a natural morphism $\overline{\mathscr{O}}_{\mathfrak{X}_M} \to \mu_{X,M,*}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$ and, hence, a morphism $\mu^*_{X,M}\left(\overline{\mathscr{O}}_{\mathfrak{X}_M}\right) \longrightarrow \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$, coming from adjunction of $\mu_{X,M,*}$ and $\mu^*_{X,M}$. We then get a homomorphism

$$\mu^{*,\mathbf{N}}_{X,M}\left(A^+_{\mathrm{inf},*}(\overline{\mathscr{O}}_{\mathfrak{X}_M})\right) \to A^+_{\mathrm{inf},*}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}).$$

We claim that these maps are isomorphisms. It suffices to prove it componentwise and by devissage it is enough to show that $\mu^*_{X,M}\left(\overline{\mathscr{O}}_{\mathfrak{X}_M}/p\overline{\mathscr{O}}_{\mathfrak{X}_M}\right) \longrightarrow \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$ is an isomorphism. Due to 4.4(3) this amounts to prove that, for every point x of X as in 4.3, the natural map $\overline{\mathscr{O}}_{X,x,M}/p\overline{\mathscr{O}}_{X,x,M} \to \overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}/p\overline{\mathscr{O}}_{\mathscr{X},\hat{x},M}$ is an isomorphism. This follows from 4.3(iv). □

Lemma 5.5. — *We have the following equivalences of δ-functors :*

i. $\mathrm{R}^q\big(\nu_X^{*,\mathbf{N}} \circ v_{X,M,*}^{\mathbf{N}}\big) = \nu_X^{*,\mathbf{N}} \circ \mathrm{R}^p v_{X,M,*}^{\mathbf{N}}$ *and* $\mathrm{R}^q\big(\widehat{v}_{\mathscr{X},M,*}^{\mathbf{N}} \circ \mu_{X,M}^{*,\mathbf{N}}\big) = \big(\mathrm{R}^q\widehat{v}_{\mathscr{X},M,*}^{\mathbf{N}}\big) \circ \mu_{X,M}^{*,\mathbf{N}}$;

ii. $\nu_X^{*,\mathbf{N}} \circ \mathrm{R}^q v_{X,M,*}^{\mathbf{N}} \xrightarrow{\sim} \big(\mathrm{R}^q\widehat{v}_{\mathscr{X},M,*}^{\mathbf{N}}\big) \circ \mu_{X,M}^{*,\mathbf{N}}$.

Proof. — The result follows for lemma 4.5 and 5.1. □

5.4. Comparison between algebraic and formal cohomology of continuous sheaves. — Since $\nu_X^{*,\mathbf{N}}$ is an exact functor, as in section 4.5, given an injective resolution $I^\bullet$ of a continuous sheaf $\mathscr{F}$, then $0 \to \nu_X^{*,\mathbf{N}}(\mathscr{F}) \to \nu_X^{*,\mathbf{N}}(I^\bullet)$ is exact so that given an injective resolution $J^\bullet$ of $\mathscr{F}^{\mathrm{form}} = \nu_X^{*,\mathbf{N}}(\mathscr{F})$ we can extend the identity map on $\mathscr{F}$ to a morphism of complexes $\nu_X^{*,\mathbf{N}}(I^\bullet) \to J^\bullet$. Since ν_X sends the final object X of X_{et} to the final object $\mathscr{X}$ of $\mathscr{X}_{\mathrm{et}}$, one has a natural map $I^\bullet(X) \to \nu_X^{*,\mathbf{N}}(I^\bullet)(\mathscr{X})$. Then,

Definition 5.6. — One has natural maps of δ-functors

$$\rho_{X,\mathscr{X}}^{\mathrm{cont},q}(\mathscr{F})\colon \mathrm{H}^q_{\mathrm{cont}}\big(X_{\mathrm{et}}, \mathscr{F}\big) \to \mathrm{H}^q_{\mathrm{cont}}\big(\mathscr{X}_{\mathrm{et}}, \mathscr{F}^{\mathrm{form}}\big),$$

and

$$\rho_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}^{\mathrm{cont},q}(\mathscr{F})\colon \mathrm{H}^q_{\mathrm{cont}}\big(\mathfrak{X}_M, \mathscr{F}\big) \to \mathrm{H}^q_{\mathrm{cont}}\big(\widehat{\mathfrak{X}}_M, \mathscr{F}^{\mathrm{rig}}\big).$$

Note that one has spectral sequences

(14)
$$\mathrm{H}^q_{\mathrm{cont}}\big(X_{\mathrm{et}}, \mathrm{R}^p v_{X,M,*}^{\mathbf{N}}(\mathscr{F})\big) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{cont}}(\mathfrak{X}_M, \mathscr{F}),$$

and

(15)
$$\mathrm{H}^q_{\mathrm{cont}}\big(\mathscr{X}_{\mathrm{et}}, \nu_X^*\mathrm{R}^p v_{\mathscr{X},M,*}^{\mathbf{N}}(\mathscr{F})\big) = \mathrm{H}^q_{\mathrm{cont}}\big(\mathscr{X}_{\mathrm{et}}, \mathrm{R}^p\widehat{v}_{\mathscr{X},M,*}^{\mathbf{N}}(\mathscr{F}^{\mathrm{rig}})\big) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{cont}}\big(\widehat{\mathfrak{X}}_M, \mathscr{F}^{\mathrm{rig}}\big),$$

where the equality on the left hand side is due to 5.5.

Proposition 5.7. — *The following hold:*

a. *If $\mathscr{F}$ is a torsion sheaf on $\mathbf{Sh}(X_{\mathrm{et}})^{\mathbf{N}}$, then $\rho_{X,\mathscr{X}}^{\mathrm{cont},q}(\mathscr{F})$ is an isomorphism;*

b. *the spectral sequences (14) and (15) are compatible via $\rho_{X,\mathscr{X}}^{\mathrm{cont},q}$ and $\rho_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}^{\mathrm{cont},p+q}$;*

c. *if $\mathscr{F}$ is a torsion sheaf in $\mathbf{Sh}(\mathfrak{X}_M)^{\mathbf{N}}$, the map $\rho_{\mathfrak{X}_M,\widehat{\mathfrak{X}}_M}^{\mathrm{cont},q}(\mathscr{F})$ is an isomorphism.*

Proof. — (a) follows from 4.11 (a) and the exact sequence (9) noting that the inverse limit of a torsion inverse system of sheaves is itself torsion; (b) is left to the reader; (c) is proven similarly to 4.11 (c). □

Corollary 5.8. — *Let $\mathbb{L}$ be a locally constant sheaf on $\mathfrak{X}_M$ annihilated by p^s. Then, the two sides of the Leray spectral sequences*

$$\mathrm{H}^j_{\mathrm{cont}}\Big(\mathscr{X}_{\mathrm{et}}, \mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}^{\mathbf{N}}\Big(\mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf,s}}\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\Big)\Big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{cont}}\Big(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf,s}}\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\Big)$$

and

$$\mathrm{H}^j_{\mathrm{cont}}\Big(X_{\mathrm{et}}, \mathrm{R}^i v_{X,M,*}^{\mathbf{N}}\Big(\mathbb{L} \otimes A^+_{\mathrm{inf,s}}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)\Big)\Big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{cont}}\Big(\mathfrak{X}_M, \mathbb{L} \otimes A^+_{\mathrm{inf,s}}\big(\overline{\mathscr{O}}_{\mathfrak{X}_M}\big)\Big)$$

are isomorphic.

Proof. — The statements follow from 5.7 and 5.4. □

Proposition 5.9. — (Faltings) *Let $\mathbb{L}$ be a finite locally constant étale sheaf on $X_{\overline{K}}$ annihilated by p^s. For every i the kernel and the cokernel of the induced map of $\mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}})$-modules*

$$\mathrm{H}^i\left(\mathfrak{X}_{\overline{K}}, \mathbb{L}\right) \otimes \mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}}) \longrightarrow \mathrm{H}^i_{\mathrm{cont}}\left(\mathfrak{X}_{\overline{K}}, \mathbb{L} \otimes A^+_{\mathrm{inf,s}}(\overline{\mathscr{O}}_{\mathfrak{X}_{\overline{K}}})\right)$$

are annihilated by the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$.

Proof. — By devissage one reduces to the case $s = 1$. The statement follows then from [**14**, §3, Thm. 3.8]. □

Proposition 5.10. — *We have a commutative square*

$$\begin{array}{ccc}
\mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}}) & \longrightarrow & \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf,s}}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}})\right) \\
\| \downarrow & & \downarrow \\
\varprojlim \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{W}_s(\overline{V}/p\overline{V}) & \longrightarrow & \varprojlim \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right),
\end{array}$$

where the inverse limits are taken with respect to Frobenius. The kernel and the cokernel of any two maps in the square are annihilated by the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$. Furthermore, each map

$$\mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{W}_s(\overline{V}/p\overline{V}) \longrightarrow \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right), \tag{16}$$

appearing in the inverse limits in the displayed square, has kernel and cokernel annihilated by the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$.

Proof. — We first of all construct the maps in the square. The top horizontal map is defined by the natural map $\mathbb{L}^{\mathrm{rig}} \to \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$. Similarly, the lower horizontal arrow is induced by the map $\mathbb{L}^{\mathrm{rig}} \to \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$. Note that $\mathrm{H}^0_{\mathrm{cont}}\big(\widehat{\mathfrak{X}}_{\overline{K}}, (\mathscr{F}_n)_n\big)$ is the composite of the functors $\lim\limits_{\infty \leftarrow n} \mathrm{H}^0\big(\widehat{\mathfrak{X}}_{\overline{K}}, \mathscr{F}_n\big)_n$. This gives a spectral sequence in which the derived functors $\varprojlim^{(i)}$ of $\varprojlim$ on the category of abelian groups appear. Since $\varprojlim^{(i)} = 0$ for $i \geq 2$, see [**20**, §1], we get an exact sequence

$$\begin{aligned}
0 \to \varprojlim{}^{(1)} \mathrm{H}^{i-1}_{\mathbf{N}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right)\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \longrightarrow& \\
\longrightarrow \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \longrightarrow& \\
\longrightarrow \varprojlim \mathrm{H}^i_{\mathbf{N}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\mathbf{W}_s\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \longrightarrow 0.&
\end{aligned}$$

This provides the right vertical map in the square. Clearly the square commutes. The fact that the top horizontal arrow has kernel and cokernel annihilated by the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$ follows by 5.8 and 5.9. The equality on the left hand side follows since $\mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\right)$ is a finite group being isomorphic to $\mathrm{H}^i\left(X_{\overline{K},\mathrm{et}},\mathbb{L}\right)$ by 4.12 and 4.4.

To conclude the proof, it suffices to show that the kernel and cokernel of (16) are annihilated by the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$. We may reduce to the case $s=1$ i. e., to prove that the map

$$f_j\colon \mathrm{H}^j\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\right)\otimes\widetilde{\mathbf{E}}^+_{\overline{V}}\longrightarrow \mathrm{H}^j_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\otimes\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right)$$

has kernel and cokernel annihilated by any any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$. For any integer $m\geq 1$ let $\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)^{\geq m}$ be the inverse system $\left\{\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right\}$ where the transition maps are the identity in degree $\geq m$ and are Frobenius in degree $<m$. Let $\beta_m\colon\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\longrightarrow\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)^{\geq m}$ be the map of inverse systems whose n-th component is $\varphi^{n-m}\colon\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\to\overline{\mathscr{O}}_{\overline{K}}/p\overline{\mathscr{O}}_{\overline{K}}$ for $n>m$ and is the identity for $n<m$. We claim that β_m is surjective. It suffices to check it componentwise and, for each component, to check surjectivity of $\varphi^n\colon\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\to\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}$ on stalks. This follows from 4.3(v). Consider $\pi_0^{p^m}\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$ with $\pi_0:=(p,p^{\frac{1}{p}},p^{\frac{1}{p^2}},\cdots)$. Then, $\pi_0^{p^m}\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$ is the inverse system $\{p^{\frac{1}{p^{n-m}}}\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\}_n$ with transition map given by Frobenius. We claim that $\mathrm{Ker}(\beta_n)=\pi_0^{p^m}\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$. This also can be checked componentwise, for each component it can be checked on stalks and it follows from 4.3(v). Note that

$$\mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_M,\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)^{\geq m}\right)\cong\mathrm{H}^i\left(\widehat{\mathfrak{X}}_M,\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right).$$

Indeed, by [**20**, Prop. 1.1] an injective resolution of $(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}})^{\geq m}$ is given by an injective resolution of each component of this inverse system which is constant in degree $n\geq m$. Take the long exact sequence of the groups $\mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\_\right)$ associated to the short exact sequence

$$0\longrightarrow\mathbb{L}^{\mathrm{rig}}\otimes\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\xrightarrow{1\otimes\pi_0^{p^m}}\mathbb{L}^{\mathrm{rig}}\otimes\mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\xrightarrow{1\otimes\beta_m}\left(\mathbb{L}^{\mathrm{rig}}\otimes\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)^{\geq m}\longrightarrow 0.$$

We get the exact sequence

$$\begin{aligned}\mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \xrightarrow{\pi_0^{p^m}} \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \longrightarrow \\ \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right) \xrightarrow{\delta_i} \mathrm{H}^{i+1}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right) \\ \xrightarrow{\pi_0^{p^m}} \mathrm{H}^{i+1}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{R}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right)\end{aligned}$$

which we will compare with the exact sequence

$$\begin{aligned}\mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \widetilde{\mathbf{E}}^+_{\overline{V}} \xrightarrow{\pi_0^{p^m}} \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \widetilde{\mathbf{E}}^+_{\overline{V}} \longrightarrow \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes (\overline{V}/p\overline{V}) \xrightarrow{0} \\ \longrightarrow \mathrm{H}^{i+1}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \widetilde{\mathbf{E}}^+_{\overline{V}} \xrightarrow{\pi_0^{p^m}} \mathrm{H}^{i+1}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \widetilde{\mathbf{E}}^+_{\overline{V}}\end{aligned}$$

via the map $g_i\colon \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \overline{V}/p\overline{V} \longrightarrow \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)$, defined by (16), and via f_j for $j = i$ or $j = i+1$.

Set $\delta_{-1} = 0$ and let us denote for the rest of the proof $\mathscr{F} := \mathbb{L}^{\mathrm{rig}} \otimes \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}$, $\mathscr{G} := \mathbb{L}^{\mathrm{rig}} \otimes \mathscr{R}(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}})$ and $\mathbf{E} := \widetilde{\mathbf{E}}^+_{\overline{V}}$. Fix $m \geq 1$ and $i \geq 0$ and consider the (not necessarily commutative) diagram

$$\begin{tikzcd} \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \overline{V}/p\overline{V} \arrow[r, "0"] \arrow[d, "g_i"] & \mathrm{H}^{i+1}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{E} \arrow[r, "\pi_0^{p^m}"] \arrow[d, "f_{i+1}"] & \mathrm{H}^{i+1}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{E} \arrow[d, "f_{i+1}"] \\ \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathscr{F}\right) \arrow[r, "\delta_i"] & \mathrm{H}^{i+1}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathscr{G}\right) \arrow[r, "\pi_0^{p^m}"] & \mathrm{H}^{i+1}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathscr{G}\right). \end{tikzcd}$$

Let us remark that the right square of the diagram is commutative and that the rows are exact. We claim that the image of δ_i is annihilated by every element of the maximal ideal of $\mathbf{E}$, i.e. that δ_i is "almost zero". For every $\epsilon \in \mathbf{Q}$ with $\epsilon > 0$ let us denote by π_0^ϵ any element r of $\mathbf{E}$ such that $v_{\mathbf{E}}(r) = \epsilon$. Let us fix any such ϵ and let $x \in \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathscr{F}\right)$. Denote by $y = \delta_i(x) \in \mathrm{Ker}(\pi_0^{p^m})$. As the cokernel of f_{i+1} is annihilated by any element of the maximal ideal of $\mathbf{E}$, $\pi_0^{\epsilon/2} y = f_{i+1}(t)$ for some $t \in \mathrm{H}^{i+1}\left(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}}\right) \otimes \mathbf{E}$ and therefore $0 = \pi_0^{p^m}(\pi_0^{\epsilon/2} y) = \pi_0^{p^m} f_{i+1}(t) = f_{i+1}(\pi_0^{p^m} t)$. As the kernel of f_{i+1} is also annihilated by every element of the maximal ideal of $\mathbf{E}$ we have $0 = \pi_0^{\epsilon/2}(\pi_0^{p^m} t) = \pi_0^{p^m}(\pi_0^{\epsilon/2} t)$ and because multiplication by $\pi_0^{p^m}$ is injective on the top row of the diagram, we deduce $\pi_0^{\epsilon/2} t = 0$. Thus $\pi_0^\epsilon \delta_i(x) = \pi_0^{\epsilon/2}(f_{i+1}(t)) = f_{i+1}(\pi_0^{\epsilon/2} t) = 0$, which proves the claim.

Now we consider the diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\right)\otimes \mathbf{E} \arrow[r, "\pi_0^{p^m}"] \arrow[d, "\overline{f}_i"] & \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\right)\otimes \mathbf{E} \arrow[r] \arrow[d, "f_i"] & \mathrm{H}^i\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\right)\otimes \overline{V}/p\overline{V} \arrow[r] \arrow[d, "g_i"] & 0 \\
0 \arrow[r] & \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathscr{G}\right)/M_{i-1} \arrow[r, "\pi_0^{p^m}"] & \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathscr{G}\right) \arrow[r] & \mathrm{H}^i_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathscr{F}\right) \arrow[r, "\delta_i"] & M_i
\end{tikzcd}$$

where for every $i\geq 0$ we denoted by M_i the image of δ_i in $\mathrm{H}^{i+1}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathscr{G}\right)$ and $\overline{f}_i$ is the composition of f_i with the natural projection. It is clear that the diagram is commutaive and the rows are exact. Moreover, the snake lemma and the fact that $\delta_i\circ g_i=0$ give the following exact sequence of $\mathbf{E}$-modules.

$$\mathrm{Ker}(f_i)\to\mathrm{Ker}(g_i)\to\mathrm{Coker}(\overline{f}_i)\to\mathrm{Coker}(f_i)\to\mathrm{Coker}(g_i)\to M_i.$$

As $\mathrm{Coker}(\overline{f}_i)$ is a quotient of $\mathrm{Coker}(f_i)$, we deduce that the modules $\mathrm{Ker}(f_i)$, $\mathrm{Coker}(\overline{f}_i)$, $\mathrm{Coker}(f_i)$ and M_i are annihilated by every element of the maximal ideal of $\mathbf{E}$, and therefore the same holds for $\mathrm{Ker}(g_i)$ and $\mathrm{Coker}(g_i)$. This finishes the proof of Proposition 5.10. □

Theorem 5.11. — *Let $\mathbb{L}$ be a locally constant étale sheaf on X_M annihilated by p^s. We have a first quadrant spectral sequence:*

$$\mathrm{H}^j\left(\mathscr{X}_{\mathrm{et}},\mathrm{R}^i\widehat{v}^{\mathrm{cont}}_{\mathscr{X},M,*}\left(\mathbb{L}^{\mathrm{rig}}\otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)\right)\right)\Longrightarrow \mathrm{H}^{i+j}_{\mathrm{cont}}\left(\mathfrak{X}_M,\mathbb{L}\otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\mathfrak{X}_M}\right)\right).$$

If $M=\overline{K}$, there is a map of $\mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}})$-modules

$$\mathrm{H}^n(X_{\overline{K},\mathrm{et}},\mathbb{L})\otimes\mathbf{W}_s(\widetilde{\mathbf{E}}^+_{\overline{V}})\longrightarrow \mathrm{H}^n_{\mathrm{cont}}\left(\mathfrak{X}_{\overline{K}},\mathbb{L}\otimes A^+_{\mathrm{inf,s}}(\overline{\mathscr{O}}_{\mathfrak{X}_{\overline{K}}})\right),$$

which is an isomorphism after inverting π.

Proof. — The first spectral sequence abuts to $\mathrm{H}^{i+j}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_M,\mathbb{L}^{\mathrm{rig}}\otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)\right)$. The first statement follows then from 5.8. The second one is the content of 5.9. □

5.5. Proof of theorem 5.1. — The groups $\mathrm{H}^n_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{K_\infty},\mathscr{F}_\infty\right)\left[\pi^{-1}\right]$ are modules for the ring $\mathbf{W}\left(\widetilde{\mathbf{E}}_{\overline{V}}\right)^{H_V}=\mathbf{W}\left(\widetilde{\mathbf{E}}_{V_\infty}\right)$ and have residual action of Γ_V and φ. By 4.4 the functor $\beta^{*,\mathbf{N}}_{K_\infty,\overline{K}}\colon \mathbf{Sh}(\widehat{\mathfrak{X}}_{K_\infty})^{\mathbf{N}}\to\mathbf{Sh}(\widehat{\mathfrak{X}}_{\overline{K}})^{\mathbf{N}}$ is exact, sends flasque objects to flasque objects and $\mathrm{H}^0_{\mathrm{cont}}(\widehat{\mathfrak{X}}_{K_\infty},\mathscr{F})$ is equal to $\varprojlim\mathrm{H}^0\left(H_V,\mathrm{H}^0_{\mathbf{N}}(\widehat{\mathfrak{X}}_{\overline{K}},\beta^{*,\mathbf{N}}_{K_\infty,\overline{K}}(\mathscr{F}))\right)$ for every $\mathscr{F}$ in $\mathbf{Sh}(\widehat{\mathfrak{X}}_{K_\infty})^{\mathbf{N}}$. Here, $\mathrm{H}^0_{\mathbf{N}}(\widehat{\mathfrak{X}}_{\overline{K}},\_)$ is the functor from $\mathbf{Sh}(\widehat{\mathfrak{X}}_{\overline{K}})^{\mathbf{N}}$ to the category of inverse systems of H_V-modules mapping $\{\mathscr{G}_n\}_n\mapsto\{\mathrm{H}^0(\widehat{\mathfrak{X}}_{\overline{K}},\mathscr{G}_n)\}_n$. We then get a spectral sequence

$$\mathrm{H}^j\left(H_V,\mathrm{H}^i_{\mathbf{N}}\left(\widehat{\mathfrak{X}}_{\overline{K}},\mathbb{L}^{\mathrm{rig}}\otimes A^+_{\mathrm{inf,s}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\right)\right)\right)\Longrightarrow\mathrm{H}^{i+j}_{\mathrm{cont}}\left(\widehat{\mathfrak{X}}_{K_\infty},\mathscr{F}_\infty\right).\tag{17}$$

Here, $\mathrm{H}^j\left(H_V,\_\right)$ stands for the j-th derived functor of $\varprojlim\mathrm{H}^0(H_V,\_)$ on the category of inverse systems of H_V-modules.

Put $M := \mathrm{H}^i_{\mathbf{N}}\Big(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes A^+_{\mathrm{inf},s}\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\big)\Big)$. Then, M is the inverse system $\{M_n\}_n$ with $M_n := \mathrm{H}^i\Big(\widehat{\mathfrak{X}}_{\overline{K}}, \mathbb{L}^{\mathrm{rig}} \otimes \mathbf{W}_s\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_{\overline{K}}}\big)\Big)$ and transition maps $d_n \colon M_{n+1} \to M_n$ given by Frobenius. By 5.10 each d_n has cokernel annihilated the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$. Let $C^\bullet(H_V, M_n)$ be the complex of continuous cochains with values in M_n. For every $i \in \mathbf{N}$ the transition maps in $\{C^i(H_V, M_n)\}_n$ are given by Frobenius and their cokernels are also annihilated the Teichmüller lift of any element in the maximal ideal of $\widetilde{\mathbf{E}}^+_{\overline{V}}$. We deduce from (13) and the following discussion that we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{cont}}\big(H_V, \lim_{\infty\leftarrow n} M_n\big)\big[\pi^{-1}\big] \xrightarrow{\sim} \mathrm{H}^i(H_V, M)\big[\pi^{-1}\big],$$

where $\mathrm{H}^i_{\mathrm{cont}}\big(H_V, \lim_{\infty\leftarrow n} M_n\big)$ is continuous cohomology. Eventually, we conclude from 5.11 that

$$\mathrm{H}^j(H_V, M)\big[\pi^{-1}\big] \cong \mathrm{H}^j\left(H_V, \mathrm{H}^i(X_{\overline{K},\mathrm{et}}, \mathbb{L}) \otimes \mathbf{W}\big(\widetilde{\mathbf{E}}_{\overline{V}}\big)\right).$$

By A.5 the latter is zero for $j \geq 1$ and is equal to the invariants under H_V for $j = 0$. In particular, the spectral sequence (17) degenerates if we invert π. Since $\mathbb{L}$ is defined on X_K the isomorphism one gets is compatible with respect to the residual action of Γ_V and the action of Frobenius. The H_V-invariants of $\mathrm{H}^n\big(X_{\overline{K},\mathrm{et}}, \mathbb{L}\big) \otimes \mathbf{W}\big(\widetilde{\mathbf{E}}_{\overline{V}}\big)$ coincide by definition with $\widetilde{\mathfrak{D}}_V\big(\mathrm{H}^n(X_{\overline{K},\mathrm{et}}, \mathbb{L})\big)$.

6. The cohomology of Fontaine sheaves

In this section $\mathscr{X}$ denotes a formal scheme topologically of finite type, smooth and geometrically irreducible over V and let X_K^{rig} be its generic fiber. Let X_K^{rig} be the Grothendieck topology defined by étale and quasi-compact maps. We refer to [**21**, §3.1 & 3.2] for generalities about étale morphisms of rigid analytic spaces. We study the cohomology on $\widehat{\mathfrak{X}}_M$ of continuous sheaves satisfying certain assumptions (see 6.10). For example, it follows from 6.16 that these sheaves $\mathscr{F}$ can be taken to be of the following form:

1) If $\mathbb{L}$ is a p-power torsion étale local system on X_K^{rig} we set $\mathscr{F} := \mathbb{L} \otimes A^+_{\mathrm{inf}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right)$.
2) If $\mathbb{L}$ is an étale sheaf on X_K^{rig} such that $\mathbb{L} = \lim\limits_{\leftarrow} \mathbb{L}_n$, with each $\mathbb{L}_n$ a locally constant $\mathbf{Z}/p^n\mathbf{Z}$-module and we set $\mathscr{F} := \mathbb{L}\hat{\otimes}\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$.

Then the cohomology groups $\mathrm{H}^i\big(\widehat{\mathfrak{X}}_M, \mathscr{F}\big)\big[\pi^{-1}\big]$ can be calculated as follows (here π is $\big[\epsilon\big] - 1 \in A^+_{\mathrm{inf}}(\overline{V})$ if $\mathscr{F}$ is of the first type and π is p if $\mathscr{F}$ is of the second).

Let us fix a geometric generic point $\eta = \mathrm{Spm}(\mathbf{C}_{\mathscr{X}})$ as in §5 and for each small formal scheme $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ (see 6.9) with a map $\mathscr{U} \longrightarrow \mathscr{X}$ which is étale, define $\overline{R}_{\mathscr{U}}$ to be the union of all finite, normal $R_{\mathscr{U}}$-algebras contained in $\mathbf{C}_{\mathscr{X}}$, which are étale after inverting p. Denote by $\mathscr{F}(\overline{R}_{\mathscr{U}} \otimes K)$ the inductive limit of the sections $\mathscr{F}(\mathscr{U}, \mathscr{W})$, where $\mathscr{W}$ runs over all objects of $\mathscr{U}_K^{\mathrm{fet}}$. Then $\mathscr{F}(\overline{R} \otimes_V K)$ is a continuous representation of $\pi_1^{\rm alg}(\mathscr{U}_K,\eta)$. Moreover (see 6.20) $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\overline{R}_{\mathscr{U}}\otimes_V K)\big[p^{-1}\big]\cong\widehat{\overline{R}}_{\mathscr{U}}\big[p^{-1}\big]$ and $A_{\rm inf}^+\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\big(\overline{R}_{\mathscr{U}}\otimes_V K\big)\big[\pi^{-1}\big]$ is isomorphic to the relative Fontaine ring $A_{\rm inf}^+$ (in which π was inverted) constructed using the pair $(R_{\mathscr{U}},\overline{R}_{\mathscr{U}})$. For any such $\mathscr{U}=\mathrm{Spf}(R_{\mathscr{U}})$, the association $\mathscr{U}\longrightarrow \mathrm{H}^i\big(\pi_1^{\rm alg}(\mathscr{U}_K,\eta),\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)\big)\big[\pi^{-1}\big]$ is functorial and we denote by $\mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})$ the sheaf on $\mathscr{X}_{\rm et}$ associated to it. Then the main result of this section is:

Theorem 6.1. — *Assume that the above assumption holds. Then, there exists a spectral sequence*

$$E_2^{p,q}=\mathrm{H}^q\big(\mathscr{X}^\bullet_{\rm et},\mathscr{H}^p_{\mathrm{Gal}_M}(\mathscr{F})\big)\Longrightarrow \mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_M,\mathscr{F}).$$

As mentioned in the Introduction, theorem 6.1 is the main technical tool needed to prove comparison isomorphisms relating different p-adic cohomology theories on $X_K^{\rm rig}$. The proof of theorem 6.1 will take the rest of the section.

6.1. Remarks on various Grothendieck topologies. — Denote by $\mathscr{X}_{\rm Zar}$ the Zariski topology on $\mathscr{X}$.

The site $\widehat{\mathfrak{X}}_{M,\rm Zar}$. — Let the underlying category of $\widehat{\mathfrak{X}}_{M,\rm Zar}$ be the full subcategory of the category of $\widehat{\mathfrak{X}}_M$ defined in 4.1 whose objects are pairs $(\mathscr{U},\mathscr{W})$ with $(\mathscr{U},\mathscr{W})\in\widehat{\mathfrak{X}}_M$ and $\mathscr{U}\to\mathscr{X}$ is a Zariski open formal subscheme. We define a family of maps in $\widehat{\mathfrak{X}}_{M,\rm Zar}$ to be a covering family if it is a covering family when considered in $\widehat{\mathfrak{X}}_M$. We let

$$\iota\colon \widehat{\mathfrak{X}}_{M,\rm Zar}\longrightarrow\widehat{\mathfrak{X}}_M$$

be the natural functor. We also denote by

$$\widehat{v}_{\mathscr{X},M}\colon \mathscr{X}_{\rm Zar}\longrightarrow\widehat{\mathfrak{X}}_{M,\rm Zar}$$

the map of Grothendieck topologies given by $\widehat{v}_{\mathscr{X},M}(\mathscr{U}):=(\mathscr{U},(\mathscr{U}^{\rm rig},K))$. Since ι sends covering families to covering families, it is clear that $\iota_*\colon \mathbf{Sh}\big(\widehat{\mathfrak{X}}_M\big)\to\mathbf{Sh}\big(\widehat{\mathfrak{X}}_{M,\rm Zar}\big)$ and $\iota_*^{\mathbf{N}}\colon\mathbf{Sh}\big(\widehat{\mathfrak{X}}_M\big)^{\mathbf{N}}\to\mathbf{Sh}\big(\widehat{\mathfrak{X}}_{M,\rm Zar}\big)^{\mathbf{N}}$ send flasque objects to flasque objects.

Stalks. — Let $\hat{x}\colon\mathrm{Spf}(V_{\hat{x}})\to\mathscr{X}$ be a closed immersion of formal schemes with $V\subset V_{\hat{x}}(\subset\overline{K})$ a finite extension of discrete valuation rings. Let $\mathscr{O}_{\mathscr{X},\hat{x}}$ be the local ring of $\mathscr{O}_{\mathscr{X}}$ at $\hat{x}$. Define $\overline{\mathscr{O}}^{\rm Zar}_{\mathscr{X},\hat{x},M}$ to be the limit $\lim_{i,j}S_{i,j}$ over all quadruples $(R_i,S_{i,j},S_{i,j}\to\widehat{\overline{V}},L_{i,j})$ where (1) $\mathrm{Spf}(R_i)\subset\mathscr{X}$ is a Zariski open neighborhood of $\hat{x}$, (2) $L_{i,j}$ is a finite extension of K contained in M, (3) $R_i\subset S_{i,j}$ is an integral extension with $S_{i,j}$ normal, (4) $S_{i,j}\otimes_V K$ is a finite and étale $R_i\otimes_V L_{i,j}$-algebra, (5) the composite $R_i\otimes_V L_{i,j}\to S_{i,j}\otimes_V K\to\widehat{\overline{K}}$ is $a\otimes\ell\mapsto\hat{x}^*(a)\cdot\ell$. If $\mathscr{F}$ is a sheaf on $\widehat{\mathfrak{X}}_{M,\rm Zar}$, define the stalk of $\mathscr{F}$ at $\hat{x}$ to be

$$\mathscr{F}_{\hat{x}}=\mathscr{F}\big(\overline{\mathscr{O}}^{\rm Zar}_{\mathscr{X},\hat{x},M}\big):=\lim_{i,j}\mathscr{F}\big(\mathrm{Spf}(R_i),(\mathrm{Spm}(S_{i,j}\mathop{\otimes}_V K),L_{i,j})\big).$$

A sequence of sheaves on $\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}$ is exact if and only if the induced sequence of stalks is exact for every closed immersion $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ as above. As in 4.4 one proves that $(\mathrm{R}^q\widehat{v}_{\mathscr{X},M,*}(\mathscr{F}))_{\hat{x}} \cong \mathrm{H}^q\big(G_{\hat{x},M}, \mathscr{F}_{\hat{x}}\big)$ where $G_{\hat{x},M} := \mathrm{Gal}\big(\overline{\mathscr{O}}^{\mathrm{Zar}}_{\mathscr{X},\hat{x},M}/\mathscr{O}_{\mathscr{X},\hat{x}} \otimes_V K\big)$.

6.1.1. *The site* $\mathscr{U}_{M,\mathrm{fet}}$. — Let $\mathscr{U} \subset \mathscr{X}$ be a Zariski open formal subscheme or an object of $\mathscr{X}^\bullet_{\mathrm{et}}$. Let $\mathscr{U}_{M,\mathrm{fet}}$ be the Grothendieck topology $\mathscr{U}_{M,\mathrm{fet}}$ introduced in 4.1. It is a full subcategory of $\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}$ (resp. $\widehat{\mathfrak{X}}_M$). If $\mathscr{U}' \to \mathscr{U}$ is a morphism in $\mathscr{X}_{\mathrm{Zar}}$ (resp. $\mathscr{X}^\bullet_{\mathrm{et}}$), we have a map of Grothendieck topologies

$$\rho_{\mathscr{U},\mathscr{U}'}\colon \mathscr{U}_{M,\mathrm{fet}} \longrightarrow \mathscr{U}'_{M,\mathrm{fet}}$$

letting $\rho_{\mathscr{U},\mathscr{U}'}\big(\mathscr{U},\mathscr{W}\big)$ be the pair $(\mathscr{U}',\mathscr{W}')$ where $\mathscr{W}' := \mathscr{W} \times_{\mathscr{U}^{\mathrm{rig}}} \mathscr{U}'^{\mathrm{rig}}$; see 4.1.

Assume that $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ is affine. By 4.6 we have an inclusion $R_{\mathscr{U}} \subset \mathbb{C}_{\mathscr{X}}$ (this way we work with $\mathscr{X}^\bullet_{\mathrm{et}}$ instead of $\mathscr{X}_{\mathrm{et}}$). Let $R_{\mathscr{U}} \subset \overline{R}_{\mathscr{U}}$ be the union of all finite and normal $R_{\mathscr{U}}$-subalgebras of $\mathbb{C}_{\mathscr{X}}$, which are étale after inverting p. If $\mathscr{U} = \amalg_i \mathscr{U}_i$, with $\mathscr{U}_i$ of the type above for every i, define $\overline{R}_{\mathscr{U}} := \prod_i \overline{R}_{\mathscr{U}_i}$.

Define $\pi_1(\mathscr{U}_M)$ to be $\mathrm{Gal}\big(\overline{R}_{\mathscr{U}} \otimes_V K/R_{\mathscr{U}} \otimes_V M\big)$ and let $\mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)$ be the category of abelian groups, with the discrete topology, endowed with a continuous action of $\pi_1(\mathscr{U}_M)$. We have proved in 4.8 that the functor $\mathscr{F} \mapsto \mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)$ defines an equivalence of categories from the category $\mathbf{Sh}\,(\mathscr{U}_{M,\mathrm{fet}})$ to the category $\mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)$. Taking continuous sheaves we get:

Lemma 6.2. — 1) *The functor*

$$\mathbf{Sh}\,(\mathscr{U}_{M,\mathrm{fet}})^{\mathbf{N}} \longrightarrow \mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)^{\mathbf{N}}, \qquad \{\mathscr{F}_n\} \to \{\mathscr{F}_n(\overline{R}_{\mathscr{U}} \underset{V}{\otimes} K)\}$$

is an equivalence of categories;

2) *for every* $\mathscr{F} \in \mathbf{Sh}\,(\mathscr{U}_{M,\mathrm{fet}})^{\mathbf{N}}$ *we have*

$$\mathrm{H}^i_{\mathrm{cont}}\big(\mathscr{U}_{M,\mathrm{fet}}, \mathscr{F}\big) = \mathrm{H}^i\big(\pi_1(\mathscr{U}_M), \mathscr{F}(\overline{R}_{\mathscr{U}} \underset{V}{\otimes} K)\big),$$

where the latter is the i-th derived functor of $\mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)^{\mathbf{N}} \to \mathrm{AbGr}$ *given by* $\{A_n\} \mapsto \lim\limits_{\infty \leftarrow n} A_n^{\pi_1(\mathscr{U}_M)}$.

Definition 6.3. — Let $\mathscr{F}$ be in $\mathbf{Sh}\big(\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}\big)$ (or in $\mathbf{Sh}(\widehat{\mathfrak{X}}^\bullet_M)$, or in $\mathbf{Sh}\big(\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}\big)^{\mathbf{N}}$, or in $\mathbf{Sh}\big(\widehat{\mathfrak{X}}^\bullet_M\big)^{\mathbf{N}}$). We define $\mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)$ as the image of $\mathscr{F}$ in $\mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)$ (or in $\mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)^{\mathbf{N}}$) of $\mathscr{F}$ via the pull-back maps $\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}) \to \mathbf{Sh}(\mathscr{U}_{M,\mathrm{fet}}) \cong \mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)$ (respectively via the pull-back $\mathbf{Sh}(\widehat{\mathfrak{X}}^\bullet_M) \to \mathrm{Rep}_{\mathrm{disc}}\big(\pi_1(\mathscr{U}_M)\big)$, etc.).

Convention 6.4. — From now on we simply write $\widehat{\mathfrak{X}}_{M,*}$ for $\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}$ or $\widehat{\mathfrak{X}}^\bullet_M$ and $\mathscr{X}_*$ for $\mathscr{X}_{\mathrm{Zar}}$ or, respectively, $\mathscr{X}^\bullet_{\mathrm{et}}$.

6.2. The sheaf $\mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})$. — Let $\mathscr{F}\in\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})$. Let $\mathscr{U}'\to\mathscr{U}$ be a map in $\mathscr{X}_*$ with $\mathscr{U}'$ and $\mathscr{U}$ affine. We then get an induced map

$$\mathrm{H}^i\big(\pi_1(\mathscr{U}_M),\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)\big)\longrightarrow \mathrm{H}^i\big(\pi_1(\mathscr{U}'_M),\mathscr{F}(\overline{R}_{\mathscr{U}'}\otimes_V K)\big).$$

In particular, $\mathscr{U}\to\mathrm{H}^i\big(\pi_1(\mathscr{U}_M),\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)\big)$ is a contravariant functor on the category of affine objects of $\mathscr{X}_*$.

Definition 6.5. — Define $\mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})$ to be the sheaf on $\mathscr{X}_*$ associated to the contravariant functor given by $\mathscr{U}\to\mathrm{H}^i\big(\pi_1(\mathscr{U}_M),\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)\big)$ for $\mathscr{U}$ affine.

6.2.1. *The standard resolution.* — Let $\mathscr{G}$ be a presheaf on $\widehat{\mathfrak{X}}_{M,*}$. For $i\in\mathbf{N}$ and for $\mathscr{U}=\mathrm{Spf}(R_{\mathscr{U}})$ an affine object of $\mathscr{X}_*$, define

$$E^i(\mathscr{G})_{\mathscr{U}}:=\mathrm{Hom}_{\mathbf{Z}}\Big(\mathbf{Z}\left[\pi_1(\mathscr{U}_M)^{i+1}\right],\mathscr{G}(\overline{R}_{\mathscr{U}}\otimes_V K)\Big).$$

It is endowed with an action of $\pi_1(\mathscr{U}_M)$ defined as follows. For every $\gamma,g_0,\dots,g_i\in\pi_1(\mathscr{U}_M)$ and every $f\in E^i(\mathscr{G})_{\mathscr{U}}$ put $\gamma\cdot f(g_0,\dots,g_i)=\gamma^{-1}\big(f(\gamma g_0,\dots,\gamma g_i)\big)$. Denote by $C^i(\mathscr{G})_{\mathscr{U}}\subset E^i(\mathscr{G})_{\mathscr{U}}$ the subgroup of invariants for the action of $\pi_1(\mathscr{U}_M)$. Consider the map

$$d_i\colon \mathbf{Z}\left[\pi_1(\mathscr{U}_M)^{i+1}\right]\to\mathbf{Z}\left[\pi_1(\mathscr{U}_M)^{i}\right],$$
$$(g_0,\dots,g_i)\mapsto\sum_{j=0}^{i}(-1)^j(g_0,\dots,g_{j-1},g_{j+1},\dots,g_i)$$

for $i\geq 1$ and given by $g_0\mapsto 1$ for $i=0$. We then get an exact sequence of $\pi_1(\mathscr{U}_M)$-modules

$$\cdots\longrightarrow\mathbf{Z}\left[\pi_1(\mathscr{U}_M)^2\right]\longrightarrow\mathbf{Z}\left[\pi_1(\mathscr{U}_M)^1\right]\longrightarrow\mathbf{Z}\longrightarrow 0.$$

Taking $\mathrm{Hom}_{\mathbf{Z}}\big(\_,\mathscr{G}(\overline{R}_{\mathscr{U}}\otimes_V K)\big)$ we get an exact sequence of $\pi_1(\mathscr{U}_M)$-modules

$$0\longrightarrow\mathscr{G}(\overline{R}_{\mathscr{U}}\otimes_V K)\longrightarrow E^0(\mathscr{G})_{\mathscr{U}}\longrightarrow E^1(\mathscr{G})_{\mathscr{U}}\longrightarrow\cdots\tag{18}$$

which provides a resolution of $\mathscr{G}(\overline{R}_{\mathscr{U}}\otimes_V K)$ by acyclic $\pi_1(\mathscr{U}_M)$-modules. Using 4.8 we define the sheaf $\mathscr{W}\mapsto E^i(\mathscr{G})(\mathscr{U},\mathscr{W})$ on the category $\mathscr{U}_{M,\mathrm{fet}}$ associated to $E^i(\mathscr{G})_{\mathscr{U}}$. Furthermore, $(\mathscr{U},\mathscr{W})\mapsto E^i(\mathscr{G})(\mathscr{U},\mathscr{W})$ is a contravariant functor defined on the subcategory of $\widehat{\mathfrak{X}}_{M,*}$ of pairs $(\mathscr{U},\mathscr{W})$ with $\mathscr{U}$ affine.

Definition 6.6. — Let $\mathscr{F}\in\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})$. For every $i\in\mathbf{N}$ define $\mathfrak{E}^i(\mathscr{F})$ to be the sheaf on $\widehat{\mathfrak{X}}_{M,*}$ associated to the contravariant functor $(\mathscr{U},\mathscr{W})\to E^i(\mathscr{F})(\mathscr{U},\mathscr{W})$ for $\mathscr{U}$ affine. Define $\mathscr{C}^i(\mathscr{F})$ to be the sheaf on $\mathscr{X}_*$ associated to the contravariant functor associating to an affine $\mathscr{U}$ the continuous i-th cochains of $\pi_1(\mathscr{U}_M)$ with values in $\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)$ i. e., $\mathscr{C}^i(\mathscr{F})(\mathscr{U})=E^i(\mathscr{F})_{\mathscr{U}}^{\pi_1(\mathscr{U}_M)}$.

Proposition 6.7. — *The following hold:*

i) *the differentials* d_i *of* 6.2.1 *define an exact sequence of sheaves on* $\widehat{\mathfrak{X}}_{M,*}$

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathfrak{E}^0(\mathscr{F}) \longrightarrow \mathfrak{E}^1(\mathscr{F}) \longrightarrow \mathfrak{E}^2(\mathscr{F}) \longrightarrow \cdots;$$

ii) *for every* $j \geq 1$ *and every* i *one has* $\mathrm{R}^j\widehat{v}_{\mathscr{X},M,*}\mathfrak{E}^i(\mathscr{F}) = 0$*;*

iii) *for every* i *one has* $\widehat{v}_{\mathscr{X},M,*}\mathfrak{E}^i(\mathscr{F}) = \mathscr{C}^i(\mathscr{F})$*.*

Proof. — (i) let $(\mathscr{U},(\mathscr{W},L)) \in \widehat{\mathfrak{X}}_{M,*}$ with $\mathscr{U}$ affine. Suppose that $\mathscr{W} = \mathrm{Spm}(S)$ with $S\otimes_L M$ an integral domain. Write $\mathrm{Gal}_M(\mathscr{W}) := \mathrm{Gal}\big(\overline{R}_{\mathscr{U}}\otimes_V K/S\otimes_L M\big)$. Then, $E^i(\mathscr{F})(\mathscr{U},\mathscr{W})$ is $E^i(\mathscr{F})_{\mathscr{U}}^{\mathrm{Gal}_M(\mathscr{W})}$. In particular, using (18), it follows that the kernel of $E^0(\mathscr{F})(\mathscr{U},\mathscr{W}) \to E^1(\mathscr{F})(\mathscr{U},\mathscr{W})$ is $\mathscr{F}\big(\overline{R}_{\mathscr{U}}\otimes_V K\big)^{\mathrm{Gal}_M(\mathscr{W})}$. This coincides with $\mathscr{F}(\mathscr{U},\mathscr{W})$ since $\mathscr{F}$ is a sheaf thanks to 6.2. In particular, the kernel of $\mathfrak{E}^0(\mathscr{F}) \to \mathfrak{E}^1(\mathscr{F})$ is $\mathscr{F}$. To check the exactness of the sequence in (i) it is enough to pass to the stalks. Given $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ as in 6.1 or 4.4, the stalk $\mathfrak{E}^i(\mathscr{F})_{\hat{x}}$ is the direct limit $\lim E^i(\mathscr{F})(\mathscr{U},\mathscr{W})$ over all $(\mathscr{U},\mathscr{W})$ with $\mathscr{U}$ an affine neighborhood of $\hat{x}$ and $\mathscr{W} = \mathrm{Spm}(S)$ with $S\otimes_L M \subset \overline{R}_{\mathscr{U}}\otimes_V K$. Hence, $\mathfrak{E}^i(\mathscr{F})_{\hat{x}} = \lim \mathfrak{E}^i(\mathscr{F})_{\mathscr{U}}$ where the limit is now taken over all affine open neighborhoods $\mathscr{U}$ of $\hat{x}$. Since for any such (18) is exact, we conclude that the stalk at $\hat{x}$ of the sequence in (i) is exact as well.

(ii) The claim can be checked on stalks. As explained in 6.1 or 4.4, given $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ as before, one has $\big(\mathrm{R}^q\widehat{v}_{\mathscr{X},M,*}(\mathfrak{E}^i(\mathscr{F}))\big)_{\hat{x}} \cong \mathrm{H}^q\big(G_{\hat{x},M},\mathfrak{E}^i(\mathscr{F})_{\hat{x}}\big)$. But $\mathfrak{E}^i(\mathscr{F})_{\hat{x}}$ coincides with the direct limit $\lim \mathfrak{E}^i(\mathscr{F})_{\mathscr{U}}$ taken over all affine neighborhoods $\mathscr{U}$ of $\hat{x}$. Hence,

$$\mathfrak{E}^i(\mathscr{F})_{\hat{x}} = \lim_{\hat{x}\in\mathscr{U}} \mathfrak{E}^i(\mathscr{F})_{\mathscr{U}} = \lim_{\hat{x}\in\mathscr{U}} \mathrm{Hom}\Big(\mathbf{Z}\left[\pi_1(\mathscr{U}_M)^{i+1}\right], \mathscr{F}(\overline{R}_{\mathscr{U}}\underset{V}{\otimes}K)\Big) =$$

$$= \mathrm{Hom}\Big(\varinjlim \mathbf{Z}\left[\pi_1(\mathscr{U}_M)^{i+1}\right], \varinjlim \mathscr{F}(\overline{R}_{\mathscr{U}}\underset{V}{\otimes}K)\Big) = \mathrm{Hom}\Big(\mathbf{Z}\left[(G_{\hat{x}})^{i+1}\right], \mathscr{F}_{\hat{x}}\Big),$$

where $G_{\hat{x}}$ is $G_{\hat{x},M}^{\mathrm{Zar}}$ or $G_{\hat{x},M}$ depending whether $\widehat{\mathfrak{X}}_{M,*}$ is $\widehat{\mathfrak{X}}_{M,\mathrm{Zar}}$ or $\widehat{\mathfrak{X}}_M^{\bullet}$. In particular, $\big(\mathrm{R}^q\widehat{v}_{\mathscr{X},M,*}(\mathfrak{E}^i(\mathscr{F}))\big)_{\hat{x}} = 0$ if $q\geq 1$. Claim (ii) follows.

(iii) For every affine open $\mathscr{U} \subset \mathscr{X}$ there exists a map from the group of i-th cochains $C^i\big(\pi_1(\mathscr{U}_M),\mathscr{F}(\overline{R}_{\mathscr{U}}\otimes_V K)\big) = \big(E^i_{\mathscr{U}}\big)^{\pi_1(\mathscr{U}_M)}$ to $\widehat{v}_{\mathscr{X},M,*}\mathfrak{E}^i(\mathscr{F})(\mathscr{U})$. This provides a natural map $\mathscr{C}^i(\mathscr{F}) \to \widehat{v}_{\mathscr{X},M,*}\mathfrak{E}^i(\mathscr{F})$. On the other hand, it follows from the discussion above that $\big(\widehat{v}_{\mathscr{X},M,*}(\mathfrak{E}^i(\mathscr{F}))\big)_{\hat{x}}$ is equal to the group of i-th cochains $C^i\big(G^{\mathrm{Zar}}_{\hat{x},M},\mathscr{F}_{\hat{x}}\big)$ i. e., the stalk of $\mathscr{C}^i(\mathscr{F})$. The claim follows. □

Corollary 6.8. — *If* $\mathscr{F} \in \mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})$*, then* $\mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathscr{F}) \cong \mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})$ *functorially in* $\mathscr{F}$*.*

6.3. The sheaf $\mathscr{H}^i_{\mathrm{Gal}_M,\mathrm{cont}}$. — We wish to prove an analogue of 6.8 in the case of a continuous sheaf $\mathscr{F} = \{\mathscr{F}_n\}_n \in \mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})^{\mathbf{N}}$. We need some assumptions.

Definition 6.9. — Consider a small object $\mathscr{U}$ of $\mathscr{X}_*$. Define $R_{\mathscr{U},M,\infty}$ to be the normalization of $R_{\mathscr{U},\infty}$ in the subring of $\overline{R}_{\mathscr{U}}\otimes_V K$ generated by M and $R_{\mathscr{U},\infty}$, where $R_{\mathscr{U},\infty}$

is defined as in 2.1. Denote by $\Gamma_{\mathscr{U},M}$ the group $\mathrm{Gal}\big(R_{\mathscr{U},M,\infty}\otimes_V K/R_{\mathscr{U}}\otimes_V M\big)$. Let $\mathscr{H}_{\mathscr{U},M}$ be the kernel of the map $\pi_1\big(\mathscr{U}_M\big)\to\Gamma_{\mathscr{U},M}$. Let us remark that the definitions of $R_{\mathscr{U},\infty}, R_{\mathscr{U},M,\infty}, \mathscr{H}_{\mathscr{U},M}, \Gamma_{\mathscr{U},M}$ depend on a choice of local parameters of $R_{\mathscr{U}}$ and so are not canonical.

6.3.1. *The site* $\widehat{\mathfrak{U}}_{M,*}(\infty)$. — Let $\mathscr{U}$ be a small object of $\mathscr{X}_*$. For every map $\mathscr{U}'\to\mathscr{U}$ with $\mathscr{U}':=\mathrm{Spf}(R_{\mathscr{U}'})$ affine and $R_{\mathscr{U}'}\otimes_V K$ an integral domain, we let $\mathscr{H}_{\mathscr{U}',M}$ be the kernel of $\pi_1(\mathscr{U}'_M)\to\Gamma_{\mathscr{U},M}$. Note that such a map is surjective.

Let $\widehat{\mathfrak{U}}_{M,*}(\infty)$ be the following full subcategory of $\widehat{\mathfrak{U}}_{M,*}$. Let $(\mathscr{U}',\mathscr{W}')\in\widehat{\mathfrak{U}}_{M,*}$ and assume that $\mathscr{U}':=\amalg_i\mathscr{U}'_i$ with $\mathscr{U}'_i$ connected. Then, $\mathscr{W}'$ lies in $\mathscr{U}'_{M,\mathrm{fet}}$ which, via the equivalence of 6.2, is equivalent to the category of finite sets with continuous action of $\pi_1(\mathscr{U}'_M)=\prod_i\pi_1(\mathscr{U}'_{i,M})$. We then say that $(\mathscr{U}',\mathscr{W}')$ lies in $\widehat{\mathfrak{U}}_{M,*}(\infty)$ if and only if $\mathscr{W}'$ lies in the subcategory of finite sets with continuous action of $\prod_i\Gamma_{\mathscr{U}}$ (viewed as a quotient of $\pi_1(\mathscr{U}'_M)$). We then have natural maps of Grothendieck topologies $\mathscr{U}_*\xrightarrow{\alpha}\widehat{\mathfrak{U}}_{M,*}(\infty)\xrightarrow{\beta}\widehat{\mathfrak{U}}_{M,*}$ giving rise to maps on the category of sheaves

$$\mathbf{Sh}(\widehat{\mathfrak{U}}_{M,*})\xrightarrow{\beta_*}\mathbf{Sh}(\widehat{\mathfrak{U}}_{M,*}(\infty))\xrightarrow{\alpha_*}\mathbf{Sh}(\mathscr{U}_*)$$

whose composite is $\widehat{v}_{\mathscr{U},M,*}$. As in 6.1 or 4.4 one has a notion of stalks in $\mathbf{Sh}(\widehat{\mathfrak{U}}_{M,*}(\infty))$. For $\hat{x}\colon\mathrm{Spf}(V_{\hat{x}})\to\mathscr{X}$ a point as in 4.3, let $H_{\hat{x},M}$ be the kernel of the map $G_{\hat{x},M}\to\Gamma_{\mathscr{U},M}$. If $\mathscr{F}\in\mathbf{Sh}(\widehat{\mathfrak{U}}_M)$ and $\mathscr{F}_{\hat{x}}$ is its stalk, one proves as in 4.4 that

$$\mathrm{R}^q\beta_*(\mathscr{F})_{\hat{x}}\cong\mathrm{H}^q\big(H_{\hat{x},M},\mathscr{F}_{\hat{x}}\big).$$

Caveat: The site $\widehat{\mathfrak{U}}_M(\infty)$ depends on the choice of an extension $R_{\mathscr{U}}\subset R_{\mathscr{U},\infty}$. In particular, if $\{\mathscr{U}_i\}_i$ is a covering of $\mathscr{X}$ by small objects, the sites $\widehat{\mathfrak{U}}_{i,M}(\infty)$ do not necessarily glue so that the site $\widehat{\mathfrak{X}}_M(\infty)$ is not defined in general.

Assumption 6.10. — We suppose that

i) $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ is a sheaf of $A^+_{\mathrm{inf}}(V_\infty)$-modules (resp. of $\{V_\infty/p^nV_\infty\}_n$-modules) on $\widehat{\mathfrak{X}}_{M,*}$;

ii) $\mathscr{X}$ admits
   a) a covering $\mathscr{S}:=\{\mathscr{W}_i\}_i$ in $\mathscr{X}_*$ by small objects $\mathscr{W}_i:=\mathrm{Spf}\big(R_{\mathscr{W}_i}\big)$,
   b) a choice $R_{\mathscr{W}_i}\subset R_{\mathscr{W}_i,\infty}$ as in 2.1,
   c) for every i a basis $\mathscr{T}_i:=\{\mathscr{U}_{i,j}\}_j$ of $\mathscr{W}_i$ by small objects such that, putting $R_{\mathscr{U}_{i,j},\infty}$ to be the normalization of $R_{\mathscr{U}_{i,j}}\otimes_{R_{\mathscr{W}_i}}R_{\mathscr{W}_i,\infty}$, condition (RAE) holds for $R_{\mathscr{U}_{i,j},\infty}$.

Furthermore, for every i, j and $n\in\mathbf{N}$, putting $\mathscr{U}:=\mathscr{U}_{i,j}$, the following hold:

iii) the cokernel of $\mathscr{F}_{n+1}(\overline{R}_{\mathscr{U}}\otimes_V K)\to\mathscr{F}_n(\overline{R}_{\mathscr{U}}\otimes_V K)$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(\overline{V}/p\overline{V}\big)$ (resp. $\overline{V}$);

iv) for every $q\geq 1$ the group $\mathrm{H}^q\left(\mathscr{H}_{\mathscr{U},M},\mathscr{F}_n\big(\overline{R}_{\mathscr{U}}\otimes_V K\big)\right)$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(\overline{V}/p\overline{V}\big)$ (resp. $\overline{V}$);

v) the cokernel of the transition maps $\mathscr{F}_{n+1}(R_{\mathscr{U},M,\infty}\otimes_V K)\to\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞);

vi) for every covering $\mathscr{Z} \to \mathscr{U}$ by small obiects in $\mathscr{X}_*$ and every $q \geq 1$ the Chech cohomology group $\mathrm{H}^q\big(\mathscr{Z} \to \mathscr{U}, \mathscr{F}_n(R_{\mathscr{Z},\infty} \otimes_V K)\big)$ is annihilated by any element of the maximal ideal of $\mathbf{W}(V_\infty/pV_\infty)$ (resp. V_∞).

We write π for the element $[\epsilon]-1$ in $\widetilde{\mathbf{A}}^+_{V_\infty}$ if $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ is a sheaf of $\widetilde{\mathbf{A}}^+_{V_\infty}$-modules. Instead, we put $\pi = p$ if $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ is a sheaf of $\{V_\infty/p^nV_\infty\}_n$-modules. It follows from (iii) and 5.2 that we have an isomorphism

$$\mathrm{H}^i\big(\pi_1(\mathscr{U}_M), \mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)\big)\big[\pi^{-1}\big] \cong \mathrm{H}^i_{\mathrm{cont}}\big(\pi_1(\mathscr{U}_M), \lim_{\infty\leftarrow n} \mathscr{F}_n(\overline{R}_{\mathscr{U}} \otimes_V K)\big)\big[\pi^{-1}\big].$$

If $\mathscr{U}' \to \mathscr{U}$ is a map in $\mathscr{X}_*$ with $\mathscr{U}'$ and $\mathscr{U}$ small objects in $\mathscr{T}_i$, we then get an induced map

$$\mathrm{H}^i\big(\pi_1(\mathscr{U}_M), \mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)\big)\big[\pi^{-1}\big] \longrightarrow \mathrm{H}^i\big(\pi_1(\mathscr{U}'_M), \mathscr{F}(\overline{R}_{\mathscr{U}'} \otimes_V K)\big)\big[\pi^{-1}\big].$$

As in 6.5, we define

Definition 6.11. — Assume that $\mathscr{F}$ satisfies the assumption above. Let $\mathscr{H}^i_{\mathrm{Gal}_M,\mathrm{cont}}(\mathscr{F})$ be the sheaf on $\mathscr{X}_*$ associated to the contravariant functor sending an object $\mathscr{U}$ of $\mathscr{X}_*$, with $\mathscr{U} \in \cup_i \mathscr{T}_i$, to $\mathrm{H}^i\big(\pi_1(\mathscr{U}_M), \mathscr{F}(\overline{R}_{\mathscr{U}} \otimes_V K)\big)\big[\pi^{-1}\big]$.

We want to prove the following:

Theorem 6.12. — *Let $\mathscr{F} \in \mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})^{\mathbf{N}}$ be such that the conditions of 6.10 are fulfilled. Then, $\mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathscr{F})\big[\pi^{-1}\big] \cong \mathscr{H}^i_{\mathrm{Gal}_M,\mathrm{cont}}(\mathscr{F})$. The isomorphism is functorial in $\mathscr{F}$.*

Proof. — It suffices to prove that for every small object $\mathscr{W}_i \in \mathscr{S}$, we have an isomorphism $\mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathscr{F})\big[\pi^{-1}\big]|_{\mathscr{W}_i} \cong \mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})|_{\mathscr{W}_i}$ functorially in $\mathscr{W}_i$ and $\mathscr{F}$. We construct the isomorphism and leave it to the reader to check the functoriality in $\mathscr{W}_i$ and $\mathscr{F}$. □

We may and will, *till the end of this section*, assume that $\mathscr{X} = \mathscr{W}_i$ *is small.* We put $\mathscr{T} := \mathscr{T}_i$ and we write Γ for $\Gamma_{\mathscr{W}_i,M}$. Consider the maps on the category of sheaves

$$\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})^{\mathbf{N}} \xrightarrow{\beta_*^{\mathbf{N}}} \mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*}(\infty))^{\mathbf{N}} \xrightarrow{\lim_{\leftarrow} \alpha_*} \mathbf{Sh}(\mathscr{X}_*),$$

introduced in 6.3.1. The composite is $\varprojlim \widehat{v}_{\mathscr{X},M,*}$. Since α_* and β_* are left exact and β_* sends injective to injective, we have a spectral sequence

$$\text{(19)} \qquad \mathrm{R}^p \varprojlim \alpha_*\big(\mathrm{R}^q\beta_*^{\mathbf{N}}(\mathscr{F})\big) \Longrightarrow \mathrm{R}^{p+q}\big(\varprojlim \widehat{v}_{\mathscr{X},M,*}\big)(\mathscr{F}).$$

Lemma 6.13. — *For every $q \geq 1$ the group $\mathrm{R}^q\beta_*(\mathscr{F})$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞).*

Proof. — Since $\mathrm{R}^q\beta_*^{\mathbf{N}} = \big(\mathrm{R}^q\beta_*\big)^{\mathbf{N}}$ as remarked in 5.1, it suffices to prove that for every $n \in \mathbf{N}$ and every $q \geq 1$ the sheaf $\mathrm{R}^q\beta_*(\mathscr{F}_n)$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞). It suffices to prove the vanishing on stalks. But for $\hat{x}\colon \mathrm{Spf}(V_{\hat{x}}) \to \mathscr{X}$ a point as in 4.3, we have $\mathrm{R}^q\beta_*(\mathscr{F}_n)_{\hat{x}} \cong \mathrm{H}^q\big(H_{\hat{x},M}, \mathscr{F}_{\hat{x}}\big)$ as explained in 6.3.1. The latter coincides with the direct limit $\lim \mathrm{H}^q\left(\mathscr{H}_{\mathscr{U},M},\mathscr{F}_n\big(\overline{R}_{\mathscr{U}}\otimes_V K\big)\right)$ taken over the small objects belonging to a basis of $\mathscr{X}_*$ containing $\hat{x}$. The claim then follows from 6.10(i) & (iv). □

Using 6.13 and (19) we conclude that

$$\mathrm{R}^p \lim_{\leftarrow} \alpha_*\big(\beta_*^{\mathbf{N}}(\mathscr{F})\big)\left[\pi^{-1}\right] \cong \mathrm{R}^p\big(\lim_{\leftarrow}\widehat{v}_{\mathscr{X},M,*}\big)(\mathscr{F})\left[\pi^{-1}\right].$$

We are left to compute $\mathrm{R}^p \lim\limits_{\leftarrow} \alpha_*$. For this we use the analogue of 6.2.1 on $\widehat{\mathfrak{X}}_{M,*}(\infty)$.

Given $\mathscr{U}$ in $\mathscr{T}$, write $R_{\mathscr{U},M,\infty}$ as the union $\cup_n R_{\mathscr{U},M,n}$ of finite $R_{\mathscr{U}}$-algebras such that $R_{\mathscr{U}}\otimes_V K\subset \mathrm{R}_{\mathscr{U},M,n}\otimes_V K$ is finite and étale. Then, for every covering $\mathscr{U}'\to\mathscr{U}$ with $\mathscr{U}'\in\mathscr{T}$, we have $R_{\mathscr{U}',M,\infty}\otimes_V K\cong \cup_n R_{\mathscr{U}'}\otimes_{R_{\mathscr{U}}} R_{\mathscr{U},M,n}\otimes_V K$ by construction. Let $\mathscr{U}''\to\mathscr{U}'\times_{\mathscr{U}}\mathscr{U}'$ be a covering with $\mathscr{U}''$ in $\mathscr{T}$. Then, we also have $R_{\mathscr{U}'',M,\infty}\otimes_V K\cong \cup_n R_{\mathscr{U}''}\otimes_{R_{\mathscr{U}}} R_{\mathscr{U},M,n}\otimes_V K$. Since $\mathscr{F}_n$ is a sheaf, we conclude that the sequence

$$0\longrightarrow \mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)\longrightarrow \mathscr{F}_n(R_{\mathscr{U}',M,\infty}\otimes_V K)\longrightarrow \mathscr{F}_n(R_{\mathscr{U}'',M,\infty}\otimes_V K)$$

is exact i. e., $\mathscr{U}\to\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)$ satisfies the sheaf property with respect to coverings $\mathscr{U}'\to\mathscr{U}$ with $\mathscr{U}'$ and $\mathscr{U}$ small and lying in $\mathscr{T}$. Then, the following makes sense:

Definition 6.14. — For every small object $\mathscr{U}\to\mathscr{X}$ lying in $\mathscr{T}$ and every i, $n\in\mathbf{N}$ define $E^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$ to be $\mathrm{Hom}_{\mathbf{Z}}\left(\mathbf{Z}\left[\Gamma^{i+1}\right],\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)\right)$. Define $\mathfrak{E}^i(\Gamma,\mathscr{F}_n)$ to be the sheaf on $\widehat{\mathfrak{X}}_{M,*}(\infty)$ characterized by the property that, for every small object $\mathscr{U}\in\mathscr{T}$, its restriction to $\mathscr{U}_{M,\mathrm{fet}}$ (see 4.1) is $E^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$ as representation of $\Gamma_{\mathscr{U}}$. Let $\mathfrak{E}^i(\Gamma,\mathscr{F}):=\left\{\mathfrak{E}^i(\Gamma,\mathscr{F}_n)\right\}_n$.

Let $C^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}\subset E^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$ be the subgroup of invariants for the action of Γ i. e., the group of i-th cochains of Γ with values in $\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)$. Denote by $\mathscr{C}^i(\Gamma,\mathscr{F}_n)$ the unique sheaf on $\mathscr{X}_*$ whose value for every small object $\mathscr{U}$ is $C^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$. Eventually, let $\mathscr{C}^i(\Gamma,\mathscr{F}):=\left\{\mathscr{C}^i(\Gamma,\mathscr{F}_n)\right\}_n$.

Proposition 6.15. — *Assume that $\mathscr{F}$ satisfies 6.10. Then:*

i) *we have an exact sequence in* $\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*}(\infty))^{\mathbf{N}}$

$$0\longrightarrow \beta_*^{\mathbf{N}}(\mathscr{F})\longrightarrow \mathfrak{E}^0(\Gamma,\mathscr{F})\longrightarrow \mathfrak{E}^1(\Gamma,\mathscr{F})\longrightarrow\cdots$$

ii) $\mathrm{R}^q\lim\limits_{\leftarrow}\alpha_*\left(\mathfrak{E}^i(\Gamma,\mathscr{F})\right)\left[\pi^{-1}\right]=0$ *for every* $q\geq 1$ *and every* i;

iii) $\lim\limits_{\leftarrow}\alpha_*\left(\mathfrak{E}^i(\Gamma,\mathscr{F})\right)\left[\pi^{-1}\right]$ *is the sheaf associated to the contravariant functor sending a small object* $\mathscr{U}$ *to* $\lim\limits_{\infty\leftarrow n}\mathscr{C}^i\left(\Gamma,\mathscr{F}_n\big(R_{\mathscr{U},M,\infty}\otimes_V K\big)\right)\left[\pi^{-1}\right]$.

In particular, $\mathrm{R}^q\lim\limits_{\leftarrow}\alpha_*(\beta_*^{\mathbf{N}}(\mathscr{F}))\left[\pi^{-1}\right]$ *is the q-th cohomology of the complex*

$$\lim_{\infty\leftarrow n}\mathscr{C}^0(\Gamma,\mathscr{F}_n)\left[\pi^{-1}\right]\longrightarrow\lim_{\infty\leftarrow n}\mathscr{C}^1(\Gamma,\mathscr{F}_n)\left[\pi^{-1}\right]\longrightarrow\lim_{\infty\leftarrow n}\mathscr{C}^2(\Gamma,\mathscr{F}_n)\left[\pi^{-1}\right]\longrightarrow\cdots$$

proving 6.12.

Proof. — Claim (i) can be checked componentwise and then it follows as in the proof of 6.7(i).

(ii)–(iii) We use the spectral sequence

$$\lim^{(p)}\big(\mathrm{R}^q\alpha_*(\mathfrak{E}^i(\Gamma,\mathscr{F}_n))\big) \Longrightarrow \mathrm{R}^{p+q}\lim_{\leftarrow}\alpha_*(\mathfrak{E}^i(\Gamma,\mathscr{F}))$$

given in 5.1. Since each $\mathscr{F}_n$ is a sheaf we have $\mathfrak{E}^i(\Gamma,\mathscr{F}_n)(R_{\mathscr{U},M,\infty}\otimes_V K) = E^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$. Hence, $\mathrm{H}^q\big(\Gamma,\mathfrak{E}^i(\Gamma,\mathscr{F}_n)(R_{\mathscr{U},M,\infty}\otimes_V K)\big)$ is 0 for every $q\geq 1$ and it coincides with the cochains $\mathscr{C}^i(\Gamma,\mathscr{F}_n)(R_{\mathscr{U},M,\infty}\otimes_V K) = C^i(\Gamma,\mathscr{F}_n)_{\mathscr{U}}$ for $q=0$.

Arguing as in 6.7(ii) we conclude that $\mathrm{R}^q\alpha_*\big(\beta_*(\mathfrak{E}^i(\Gamma,\mathscr{F}_n))\big)=0$ for $q\geq 1$. We are left to compute $\lim^{(p)}\alpha_*\big(\mathfrak{E}^i(\Gamma,\mathscr{F}_n)\big)=\lim^{(p)}\mathscr{C}^i(\Gamma,\mathscr{F}_n)$.

Due to 6.10(vi), for every small object $\mathscr{U}\in\mathscr{T}$ and every n the Chech cohomology group $\mathrm{H}^q\left(\mathscr{X}\to\mathscr{U},\mathscr{F}_n(R_{\mathscr{X},\infty}\otimes_V K)\right)$, relative to every covering $\mathscr{X}\to\mathscr{U}$ by small objects lying in $\mathscr{T}$, is annihilated by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞). But we have

(20)

$$\mathscr{C}^i(\Gamma,\mathscr{F}_n(R_{\_,\infty}\otimes K)) = \lim_{m\to\infty}\mathscr{C}^i(\Gamma/p^m\Gamma,\mathscr{F}_n(R_{\_,\infty}\otimes K)) = \lim_{m\to\infty}\Big(\prod_{\Gamma/p^m\Gamma}\mathscr{F}_n(R_{\_,\infty}\otimes K)\Big).$$

As both inductive limit and finite products are exact functors we deduce that the Chech cohomology group $\mathrm{H}^q\left(\mathscr{X}\to\mathscr{U},\mathscr{C}^i\big(\Gamma,\mathscr{F}_n(R_{\mathscr{X},\infty}\otimes_V K)\big)\right)$ relative to every covering $\mathscr{X}\to\mathscr{U}$ with $\mathscr{X}$ and $\mathscr{U}\in\mathscr{T}$ is annihilated by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞). Hence, the restriction of $\mathscr{C}^i\big(\Gamma,\mathscr{F}_n(R_{\_,\infty}\otimes_V K)\big)$ to $\mathscr{U}$ is flasque, see [**3**, II.4.2], up to multiplication by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞). In particular, $\mathrm{H}^q\big(\mathscr{U},\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)$ is almost zero for every $q\geq 1$; see [**3**, II.4.4]. Due to 6.10(v) the projective system $\{\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)\}_n$ is almost Mittag-Lefler and using once again (20) we also have that the projective system $\big\{\mathscr{C}^i\big(\Gamma,\mathscr{F}_n\big)\big\}_n$ is almost Mittag-Leffler. Hence, $\lim^{(1)}\mathscr{C}^i(\Gamma,\mathscr{F}_n)$ is almost zero.

By [**20**, Lemma 3.12] the sheaf $\lim^{(q)}\mathscr{C}^i(\Gamma,\mathscr{F}_n)$ is the sheaf associated to the presheaf $\mathscr{U}\mapsto\mathrm{H}^q\big(\mathscr{U},\big(\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)_n\big)$. We have, for each $q\geq 1$, exact sequences

$$0\to\lim^{(1)}\mathrm{H}^{q-1}\big(\mathscr{U},\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)\longrightarrow\mathrm{H}^q\big(\mathscr{U},\big(\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)_n\big)\longrightarrow\lim_{\infty\leftarrow n}\mathrm{H}^q\big(\mathscr{U},\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)\to 0.$$

For $q\geq 2$, using the fact proved above that $\mathrm{H}^s\big(\mathscr{U},\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)$ is almost zero for $s\geq 1$, we deduce that $\mathrm{H}^q\big(\mathscr{U},\big(\mathscr{C}^i(\Gamma,\mathscr{F}_n)\big)_n\big)$ is almost zero. We conclude that $\lim^{(q)}\mathscr{C}^i(\Gamma,\mathscr{F}_n)$ is annihilated by every element of the maximal ideal $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞) for $q\geq 2$.

Thus,

$$R^p\lim_{\leftarrow}\alpha_*\big(\mathfrak{E}^j(\Gamma,\mathscr{F})\big)\big[\pi^{-1}\big]=0$$

for $p\geq 1$ and all $j\geq 0$, and

$$\lim_{\leftarrow}\alpha_*\mathfrak{E}^\bullet(\Gamma,\mathscr{F})\big[\pi^{-1}\big]\cong\lim_{\infty\leftarrow n}\mathscr{C}^\bullet(\Gamma,\mathscr{F}_n)\big[\pi^{-1}\big].$$

The conclusion follows. □

Theorem 6.16. — *Let $\mathscr{X}$ be formally smooth, topologically of finite type and geometrically irreducible over V. Let $\mathbb{L}=(\mathbb{L}_n)_n$ be a projective system of sheaves such that $\mathbb{L}_n\cong\mathbb{L}_{n+1}/p^n\mathbb{L}_{n+1}$ for every n. Let $\mathscr{F}\in\mathbf{Sh}(\mathscr{X}_*)^{\mathbf{N}}$ be a sheaf of one of the following types:*

A) *$\mathscr{F}$ is $\mathbb{L}^{\mathrm{rig}}\otimes A^+_{\mathrm{inf}}\left(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\right):=\left(\mathbb{L}_n\otimes\mathbf{W}_n\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\right)_n$;*

B) *$\mathscr{F}:=\left(\mathbb{L}_n\otimes\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p^n\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)\right)_n$ where $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p^{n+1}\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\to\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p^n\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$ is the natural projection for each $n\in\mathbf{N}$.*

Then, the assumptions in 6.10 hold.

Remark 6.17. — Assumption (ii) of 6.10 holds. Indeed, $\mathscr{X}$ is formally smooth and topologically of finite type over V. In particular, it is Zariski locally the p-adic completion of a smooth scheme over $\operatorname{Spec}(V)$. Thus, $\mathscr{X}$ admits a basis by affine subschemes satisfying (RAE) due to 2.3.

In case (A), assume further that $\mathbb{L}$ is a p-power torsion i. e., annihilated by p^s for some s. Then, one can compute the sheaf $\mathscr{H}^i_{\mathrm{Gal}_M,\mathrm{cont}}(\mathscr{F})$ introduced in 6.11 via relative (φ,Γ)-modules. Indeed, assume that $\mathscr{U}=\operatorname{Spf}(R_{\mathscr{U}})$ is small and that (RAE) holds for $R_{\mathscr{U},\infty}$.

For $M=K$ we have $\pi_1(\mathscr{U}_M)=\mathscr{G}_{R_{\mathscr{U}}}$ and by A.14 the inflation

$$\mathrm{H}^i(\Gamma_{R_{\mathscr{U}}},\mathfrak{D}(\mathbf{L}))\longrightarrow\mathrm{H}^i\big(\pi_1(\mathscr{U}_K),\mathbf{L}\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{R_{\mathscr{U}}}\big)\cong\mathrm{H}^i\big(\pi_1(\mathscr{U}_K),\mathscr{F}(\overline{R}_{\mathscr{U}}\underset{V}{\otimes}K)\big)\left[\pi^{-1}\right]$$

is an isomorphism.

Analogously, for $M=\overline{K}$ we have $\pi_1(\mathscr{U}_M)=\mathrm{G}_{R_{\mathscr{U}}}$ so that

$$\mathrm{H}^i(\Gamma'_{R_{\mathscr{U}}},\mathrm{D}_{\overline{K}}(\mathbf{L}))\xrightarrow{\sim}\mathrm{H}^i\big(\pi_1(\mathscr{U}_{\overline{K}}),\mathbf{L}\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}_{\mathscr{U}}}\big)\cong\mathrm{H}^i\big(\pi_1(\mathscr{U}_{\overline{K}}),\mathscr{F}(\overline{R}_{\mathscr{U}}\underset{V}{\otimes}K)\big)\left[\pi^{-1}\right].$$

Here, $\mathrm{D}_{\overline{K}}(\mathbf{L})$ is the (φ,Γ'_R)-module associated to $\mathbf{L}$ and the field $\overline{K}$ as in §2.

For $M=K_\infty$, the group $\pi_1(\mathscr{U}_M)$ is the subgroup of $\mathscr{G}_{R_{\mathscr{U}}}$ generated by $\mathrm{G}_{R_{\mathscr{U}}}$ and H_V. In this case

$$\mathrm{H}^i(\Gamma'_{R_{\mathscr{U}}},\mathrm{D}_{K_\infty}(\mathbf{L}))\xrightarrow{\sim}\mathrm{H}^i\big(\pi_1(\mathscr{U}_{K_\infty}),\mathbf{L}\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}_{\mathscr{U}}}\big)\cong\mathrm{H}^i\big(\pi_1(\mathscr{U}_{K_\infty}),\mathscr{F}(\overline{R}_{\mathscr{U}}\underset{V}{\otimes}K)\big)\left[\pi^{-1}\right]$$

where $\mathrm{D}_{K_\infty}(\mathbf{L})$ is the (φ,Γ'_R)-module defined in §2 using the field K_∞.

6.4. Proof of Theorem 6.16. — We start with some preliminary results.

Lemma 6.18. — *Let R be as in 2.1. Let $S_\infty\subset T_\infty$ be integral extensions of R_∞ such that $S_\infty\otimes_V K=T_\infty\otimes_V K$ and $R_\infty\subset S_\infty$ is almost étale (see 2.2). Then, the cokernel of $S_\infty\subset T_\infty$ is annihilated by any element of the maximal ideal of V_∞.*

Proof. — Let $\mathfrak{e}_\infty$ be the canonical idempotent of the étale extension $R_\infty\left[p^{-1}\right]\subset S_\infty\left[p^{-1}\right]=T_\infty\left[p^{-1}\right]$. Since $R_\infty\subset S_\infty$ is almost étale, for every $\alpha\in\mathbf{Z}\left[p^{-1}\right]_{>0}$ we may write $p^\alpha\mathfrak{e}_\infty$ as a finite sum $\sum_i a_i\otimes b_i$ with a_i and b_i in S_∞. Let $m\colon S_\infty\left[p^{-1}\right]\otimes_{R_\infty}S_\infty\left[p^{-1}\right]\to S_\infty\left[p^{-1}\right]$ be the multiplication map and let $\mathrm{Tr}\colon S_\infty\left[p^{-1}\right]\to R_\infty\left[p^{-1}\right]$ be the trace map. Then, $\mathfrak{e}_\infty$ is characterized by the property that $m(x\otimes y) = (\mathrm{Tr}\otimes \mathrm{Id})\big((x\otimes y)\cdot \mathfrak{e}_\infty\big)$. In particular, for every $x\in T_\infty$ we have $p^\alpha x = m\big(p^\alpha x\otimes 1\big) = \sum_i \mathrm{Tr}(a_i x) b_i$. But $\mathrm{Tr}(a_i x)\in R_\infty$ since x and a_i are integral over R_∞ and R_∞ is integrally closed. Hence, $p^\alpha x = \sum_i \mathrm{Tr}(a_i x) b_i$ lies in S_∞ as claimed. □

Lemma 6.19. — *Let $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ be an affine small object of $\mathscr{X}_*$ and let $\mathscr{U}'\to\mathscr{U}$ be a covering with $\mathscr{U}'$ affine. Then, $R_{\mathscr{U}',M,\infty}\cong R_{\mathscr{U},M,\infty}\otimes_{R_{\mathscr{U}}} R_{\mathscr{U}'}$.*

Proof. — Write the composite of M and K_∞ (in $\overline{K}$) as the union $\cup_n M_n$ where $M_0 = K\subset\cdots\subset M_n\subset\cdots$ and $K\subset M_n$ is a finite extension for every n. Let W_n be the ring of integers of M_n and let $\mathbf{F}_n$ be its residue filed. Let $T_1,\ldots,T_d\in R_{\mathscr{U}}$ be parameters as in 2.1. Since $R_{\mathscr{U}}\otimes_V k$ is a smooth k-algebra, then $R_{\mathscr{U}}\big[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big]\otimes_V \mathbf{F}_n$ is a smooth k-algebra. Hence, $R_{\mathscr{U}}\otimes_V W_n\big[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big]$ is a regular ring modulo the maximal ideal of W_n and, hence, it is a regular ring itself. In particular, it is normal. This implies that $R_{\mathscr{U},M,\infty}\cong \cup_n R_{\mathscr{U}}\otimes_V W_n\big[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big]$.

Since $\mathscr{U}'\to\mathscr{U}$ is formally étale, then $R_{\mathscr{U}'}\otimes_V k$ is a smooth k-algebra. Reasoning as above we conclude that $R_{\mathscr{U}',M,\infty}\cong\cup_n R_{\mathscr{U}'}\otimes_V W_n\big[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big]$. The lemma follows. □

Let $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$ be an affine small object of $\mathscr{X}_*$. Let A be the union of some collection of almost étale, integral $R_{\mathscr{U},M,\infty}$-subalgebras of $\overline{R}_{\mathscr{U}}$. Write

$$\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)(A\underset{V}{\otimes}K) := \lim\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)(\mathscr{U},\mathscr{W},L)$$

where the direct limit is taken over all $(\mathscr{U},(\mathscr{W},L))\in\widehat{\mathfrak{X}}_M$ with $\mathscr{W} = \mathrm{Spm}(S_{\mathscr{W}})$ such that $S_{\mathscr{W}}\otimes_L M\subset A\otimes_V K$.

Proposition 6.20. — *Assume that $R_{\mathscr{U}}$ is small over V. Then, the natural map*

$$A/pA \longrightarrow \big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)(A\underset{V}{\otimes}K)$$

has kernel and cokernel annihilated by any element of the maximal ideal of V_∞.

Proof. — The presheaf $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}$ is separated i. e., if $(\mathscr{U}',\mathscr{W}',L')\to(\mathscr{U},\mathscr{W},L)$ is a covering map, the natural map

$$\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U},\mathscr{W},L)/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U},\mathscr{W},L)\longrightarrow\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U}',\mathscr{W}',L')/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U}',\mathscr{W}',L')$$

is injective. This implies that we have an injective map

$$A/pA = \overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(A\underset{V}{\otimes}K)/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(A\underset{V}{\otimes}K)\hookrightarrow\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)(A\underset{V}{\otimes}K).$$

We also get that the sheaf associated to the presheaf associating to a triple $(\mathscr{U},\mathscr{W},L)$ the ring $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U},\mathscr{W},L)/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U},\mathscr{W},L)$ is defined by taking $\big(\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}\big)(\mathscr{U},\mathscr{W},L)$ to be the direct limit, over all coverings $(\mathscr{U}',\mathscr{W}',L')$ of $(\mathscr{U},\mathscr{W},L)$ with $\mathscr{U}'$ affine, of the elements b in the group $\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U}',\mathscr{W}',L')/p\overline{\mathscr{O}}_{\widehat{\mathfrak{X}}_M}(\mathscr{U}',\mathscr{W}',L')$ such that the image of b in $\overline{\mathcal{O}}_{\widehat{\mathfrak{X}}_M}(\mathcal{U}'',\mathcal{W}'',L'')/p\overline{\mathcal{O}}_{\widehat{\mathfrak{X}}_M}(\mathcal{U}'',\mathcal{W}'',L'')$ is 0 where $(\mathcal{U}'',\mathcal{W}'',L'')$ is the fiber product of $(\mathcal{U}',\mathcal{W}',L')$ with itself over $(\mathcal{U},\mathcal{W},L)$. Hence,

$$\big(\overline{\mathcal{O}}_{\widehat{\mathfrak{X}}_M}/p\overline{\mathcal{O}}_{\widehat{\mathfrak{X}}_M}\big)(A\underset{V}{\otimes}K)=\lim_{S,T}\mathrm{Ker}_{S,T}$$

where the notation is as follows. The direct limit is taken over all normal $R_{\mathcal{U},M,\infty}$-subalgebras S of A, finite and étale after inverting p over $R_{\mathcal{U},M,\infty}$, all covers $\mathcal{U}'\to\mathcal{U}$ and all normal extensions $R_{\mathcal{U}',M,\infty}\otimes_{R_\mathcal{U}}S\to T$, finite, étale and Galois after inverting p. Eventually, we put $\mathcal{U}'':=\mathrm{Spf}(R_{\mathcal{U}''})$ to be the fiber product of $\mathcal{U}'$ with itself over $\mathcal{U}$ i. e., $R_{\mathcal{U}''}:=\widehat{R_{\mathcal{U}'}\otimes_{R_\mathcal{U}}R_{\mathcal{U}'}}$. We let

$$\mathrm{Ker}_{S,T}:=\mathrm{Ker}\Big(\,T/pT\rightrightarrows\widetilde{T}_S/p\widetilde{T}_S\,\Big),$$

where $\widetilde{T}_S$ is the normalization of the base change to $R_{\mathcal{U}''}$ of $T\otimes_{(R_{\mathcal{U}',M,\infty}\otimes_{R_{\mathcal{U},M,\infty}}S)}T$. For every S and T as above, write $G_{S,T}:=\mathrm{Gal}\big(T\otimes_V K/S\otimes_{R_{\mathcal{U},M,\infty}}R_{\mathcal{U}',M,\infty}\otimes_V K\big)$. Then, $\widetilde{T}_S$ is the product $\prod_{g\in G_{S,T}}\widetilde{T\otimes_{R_{\mathcal{U}'}}R_{\mathcal{U}''}}$ where tilde stands for the normalization (of $T\otimes_{R_{\mathcal{U}'}}R_{\mathcal{U}''}$) and we view $R_{\mathcal{U}''}$ as $R_{\mathcal{U}'}$-algebra choosing the left action. Hence,

$$\mathrm{Ker}_{S,T}=\mathrm{Ker}\left(T/pT\rightrightarrows\prod\nolimits_{g\in G_{S,T}}\frac{\widetilde{T\otimes_{R_{\mathcal{U}'}}R_{\mathcal{U}''}}}{p\widetilde{T\otimes_{R_{\mathcal{U}'}}R_{\mathcal{U}''}}}\right).$$

The two maps in the display are $a\mapsto(a,\dots,a)$ and $a\mapsto\big(g(a)\big)_{g\in G_{S,T}}$.

For the rest of this proof we make the following notations: if B is a $R_{\mathcal{U},M,\infty}$-algebra we denote by $B':=B\otimes_{R_{\mathcal{U},M,\infty}}R_{\mathcal{U}',M,\infty}=B\otimes_{R_\mathcal{U}}R_{\mathcal{U}'}$, by $B'':=B\otimes_{R_{\mathcal{U}',M,\infty}}R_{\mathcal{U}'',M,\infty}=B\otimes_{R_{\mathcal{U}'}}R_{\mathcal{U}''}$ (the second equalities above follow form 6.19) and by $\widetilde{B}$ the normalization of B in $B\big[p^{-1}\big]$. We then get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S/pS & \longrightarrow & S'/pS' & \rightrightarrows & S''/pS'' \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ 0 & \longrightarrow & \mathrm{Ker}_{S,T} & \longrightarrow & T/pT & \rightrightarrows & \widetilde{T}_S/p\widetilde{T}_S=\prod_{g\in G_{S,T}}\big(\widetilde{T}''/p\widetilde{T}''\big). \end{array}\tag{21}$$

The top row is exact by étale descent and the bottom row is exact by construction. We claim that the kernel and cokernel of the map $S/pS\longrightarrow\mathrm{Ker}_{S,T}$ are annihilated by any element of the maximal ideal of V_∞. To do this we analyze the maps α and β.

Analysis of $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Ker}(\beta)$. — Note that he extension $R_{\mathcal{U},M,\infty}\subset S$ is integral and almost étale by 2.3. Hence, the extensions $R_{\mathcal{U}',M,\infty}\subset S'$ and $R_{\mathcal{U}'',M,\infty}\subset S''$ are integral and almost étale as well. Since the extension $R_\mathcal{U}\to R_{\mathcal{U}'}$ (resp. $R_\mathcal{U}\to R_{\mathcal{U}''}$) is faithfully flat, the rings S' and S'' have no non-trivial p-torsion. In particular, S' (resp. S'') injects into its normalization $\widetilde{S}'$ (resp. $\widetilde{S}''$) which is $T^{G_{S,T}}$. Thanks to Lemma 6.18 the cokernel of $S'\to\widetilde{S}'=T^{G_{S,T}}$ (resp. $S''\to\widetilde{S}''$) is annihilated by any element of the maximal ideal of V_∞.

Consider the following. If $0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$ is an exact sequence of abelian groups then the kernel of the induced map $B/pB \to C/pC$ is the image in B/pB of the group of p-torsion elements of D. In particular if B, C, D are V_∞-modules and D is annihilated by an element $a \in V_\infty$ then $\mathrm{Ker}(B/pB \to C/pC)$ is also annihilated by a. It follows from this obvious fact that the kernel of the map $S'/pS' \to \widetilde{S}'/p\widetilde{S}'$ and the kernel of the map $S''/pS'' \to \widetilde{S}''/p\widetilde{S}''$ are annihilated by any element of the maximal ideal of V_∞.

The map $\widetilde{S}'/p\widetilde{S}' \to T/pT$ (resp. $\widetilde{S}''/p\widetilde{S}'' \to \widetilde{T}_S/p\widetilde{T}_S$) is injective since $\widetilde{S}' \to T$ (resp. $\widetilde{S}'' \to \widetilde{T}_S$) is an integral extension of normal rings. Hence, the kernel of α and the kernel of β are annihilated by any element of the maximal ideal of V_∞.

Analysis of the image of $\mathrm{Coker}(S/pS \longrightarrow \mathrm{Ker}_{S,T})$ *in* $\mathrm{Coker}(\alpha)$. — Define Z as $Z := \mathrm{Coker}\big(S'/pS' \to (T/pT)^{\mathrm{G}_{S,T}}\big) \subset \mathrm{Coker}(\alpha)$. Since $\mathrm{Ker}_{S,T}$ is $\mathrm{G}_{S,T}$-invariant (by definition), the image of $\mathrm{Coker}(S/pS \longrightarrow \mathrm{Ker}_{S,T})$ in $\mathrm{Coker}(\alpha)$ is contained in Z. Put $Y := \mathrm{Coker}\big(S'/pS' \to \widetilde{S}'/p\widetilde{S}'\big)$. Let us remark that we have an exact sequence of groups:

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow \mathrm{Coker}\big(\widetilde{S}'/p\widetilde{S}' \longrightarrow (T/pT)^{\mathrm{G}_{S,T}}\big) \longrightarrow 0.$$

We know that Y is annihilated by any element of the maximal ideal of V_∞, so let us examine the last term of the sequence. This is the same as $\mathrm{Coker}\big(T^{\mathrm{G}_{S,T}}/pT^{\mathrm{G}_{S,T}} \longrightarrow (T/pT)^{\mathrm{G}_{S,T}}\big)$. Consider the exact sequence

$$0 \longrightarrow T^{G_{S,T}}/pT^{G_{S,T}} \longrightarrow \big(T/pT\big)^{G_{S,T}} \longrightarrow \mathrm{H}^1\big(G_{S,T}, T\big).$$

Since $R_{\mathscr{U}',M,\infty} \to T$ is almost étale, the group $\mathrm{H}^1\big(G_{S,T}, T\big)$ is annihilated by any element of the maximal ideal of V_∞; see [**12**, Thm. I.2.4(ii)]. Hence, the cokernel of $T^{G_{S,T}}/pT^{G_{S,T}} \longrightarrow \big(T/pT\big)^{G_{S,T}}$ is annihilated by any element of the maximal ideal of V_∞. We deduce that the same is true for the module Z above.

Now using the snake lemma applied to the commutative diagram (21), we get that the kernel and cokernel of the map $S/pS \longrightarrow \mathrm{Ker}_{S,T}$ are annihilated by any element of the maximal ideal of V_∞ as claimed.

This concludes the proof in the case that A is the union of almost étale, integral and normal $R_{\mathscr{U},M,\infty}$-subalgebras of $\overline{R}_{\mathscr{U}}$. In the general case, assume that Q is an almost étale, integral $R_{\mathscr{U},M,\infty}$-subalgebra of A and let S be its normalization. Then, the cokernel of $Q \to S$ annihilated by any element of the maximal ideal of V_∞ by Lemma 6.18. The same then applies to the kernel and the cokernel of $Q/pQ \to S/pS$. The conclusion follows. □

6.4.1. *End of proof of 6.16.* — Assumption (i) clearly holds. We let $\{\mathscr{W}_i\}_i = \mathscr{S}$ be a covering of $\mathscr{X}$ and let $\mathscr{T}_i := \{\mathscr{U}_{ij}\}_j$ be a basis of $\mathscr{W}_i$ as in 6.10(ii). Let $\mathscr{U} \in \mathscr{T}_i$ for some i.

(iii) The group $\mathbb{L}_n(\overline{R}_{\mathscr{U}} \otimes_V K)$ is constant on the connected components of $\mathscr{U}$ and does not depend on $\mathscr{U}$ itself. It then suffices to verify assumption (iii) for $\mathbb{L}_n$ the constant sheaf i. e, $\mathbb{L}_n = \mathbf{Z}/p^s\mathbf{Z}$ for some s in case (A) or $\mathbb{L}_n = (\mathbf{Z}/p^n\mathbf{Z})$ in case (B). In this case (iii) follows from 6.20 with $A = \overline{R}_{\mathscr{U}}$.

(iv) Due to 6.20 it suffices to prove that $\mathrm{H}^q\big(\mathscr{H}_{\mathscr{U}},\mathbb{L}_n(\overline{R}_{\mathscr{U}}\otimes_V K)\otimes\mathbf{W}_n(\overline{R}_{\mathscr{U}}/p\overline{R}_{\mathscr{U}})\big)$ (resp. $\mathrm{H}^q\big(\mathscr{H}_{\mathscr{U}},\mathbb{L}_n(\overline{R}_{\mathscr{U}}\otimes_V K)\otimes\overline{R}_{\mathscr{U}}/p^n\overline{R}_{\mathscr{U}}\big)$ is annihilated by any power of the maximal ideal of $\mathbf{W}_n\big(V_\infty/pV_\infty\big)$ (resp. V_∞) for every $q\geq 1$. In both cases, one reduces by devissage to the case $n=1$. The claim then follows from A.5 and A.3.

(v) Given $n\in\mathbf{N}$, let $R_{\mathscr{U}}\otimes_V K\subset B$ be a finite and étale extension such that $\mathbb{L}_{n+1}$ and $\mathbb{L}_n$ are constant on the étale site of B_M. We may assume that B is defined over a finite extension $K\subset L$ contained in M and that $R_{\mathscr{U}}\otimes_V M\subset B\otimes_L M$ is a Galois extension of integral domains. Define $A_{\mathscr{U}}$ as the normalization of $R_{\mathscr{U}}$ in the subring of $\overline{R}_{\mathscr{U}}\otimes_V M$ generated by $R_{\mathscr{U},M,\infty}$ and B. Then, assumption (v), with $A_{\mathscr{U}}$ in place of $R_{\mathscr{U},M,\infty}$, holds due to 6.20 since we may reduce to the case where $\mathbb{L}$ is trivial.

Let D_n be the kernel of $\mathscr{F}_{n+1}(A_{\mathscr{U}}\otimes_V K)\to\mathscr{F}_n(A_{\mathscr{U}}\otimes_V K)$. Let M_n be the kernel of $L_{n+1}(B)\to L_n(B)$. It is an $\mathbf{F}_p$-vector space. In case (B), the sequence $0\longrightarrow A_{\mathscr{U}}/pA_{\mathscr{U}}\longrightarrow A_{\mathscr{U}}/p^{n+1}A_{\mathscr{U}}\longrightarrow A_{\mathscr{U}}/p^nA_{\mathscr{U}}\to 0$ is exact since $A_{\mathscr{U}}$ is normal. Tensor it with $L_{n+1}(B)$ and put $E_n:=M_n\otimes\big(A_{\mathscr{U}}/pA_{\mathscr{U}}\big)$. Since $L_n(B)=L_{n+1}(B)/p^nL_{n+1}(B)$, the sequence $0\longrightarrow M_n\otimes A_{\mathscr{U}}/pA_{\mathscr{U}}\longrightarrow L_{n+1}(B)\otimes A_{\mathscr{U}}/p^{n+1}A_{\mathscr{U}}\longrightarrow L_n(B)\otimes A_{\mathscr{U}}/p^nA_{\mathscr{U}}\to 0$ is exact. Thanks to 6.20 we get that the natural map $E_n\to D_n$ has kernel and cokernel annihilated by any element of the maximal ideal of V_∞/pV_∞.

In case (B) consider the exact sequence $0\longrightarrow A_{\mathscr{U}}/p^{\frac{1}{p}}A_{\mathscr{U}}\longrightarrow\mathbf{W}_{n+1}\big(A_{\mathscr{U}}/pA_{\mathscr{U}}\big)\longrightarrow\mathbf{W}_n\big(A_{\mathscr{U}}/pA_{\mathscr{U}}\big)$ where the last map is the natural projection composed with Frobenius. Tensoring it with $L_{n+1}(B)$ we get the exact sequence $0\longrightarrow M_n\otimes A_{\mathscr{U}}/p^{\frac{1}{p}}A_{\mathscr{U}}\longrightarrow L_{n+1}(B)\otimes\mathbf{W}_{n+1}\big(A_{\mathscr{U}}/pA_{\mathscr{U}}\big)\longrightarrow L_n(B)\otimes\mathbf{W}_n\big(A_{\mathscr{U}}/pA_{\mathscr{U}}\big)$. Put $F_n:=M_n\otimes A_{\mathscr{U}}/p^{\frac{1}{p}}A_{\mathscr{U}}$. Also in this case the natural map $F_n\to D_n$ has kernel and cokernel annihilated by any element of the maximal ideal of V_∞/pV_∞. It follows from A.5 and A.3 that $\mathrm{H}^q(\mathscr{H}_{\mathscr{U}},E_n)$ and $\mathrm{H}^q(\mathscr{H}_{\mathscr{U}},F_n)$ are annihilated by any element of the maximal ideal of V_∞/pV_∞. Thus, the same applies to $\mathrm{H}^q(\mathscr{H}_{\mathscr{U}},D_n)$ and, hence, to the cokernel of the map from $\mathscr{F}_{n+1}(A_{\mathscr{U}}\otimes_V K)^{\mathscr{H}_{\mathscr{U}}}=\mathscr{F}_{n+1}(R_{\mathscr{U},M,\infty}\otimes_V K)$ to $\mathscr{F}_n(A_{\mathscr{U}}\otimes_V K)^{\mathscr{H}_{\mathscr{U}}}=\mathscr{F}_n(R_{\mathscr{U},M,\infty}\otimes_V K)$. This concludes the proof of (v).

(vi) For every covering $\mathscr{X}\to\mathscr{U}$ in $\mathscr{X}_*$ with $\mathscr{X}\in\mathscr{T}_i$ define $\mathrm{H}^q_{\mathscr{F}_n}(\mathscr{X}\to\mathscr{U})$ as the Chech cohomology group

$$\mathrm{H}^q_{\mathscr{F}_n}(\mathscr{X}\to\mathscr{U}):=\mathrm{H}^q\big(\mathscr{X}\to\mathscr{U},\mathbb{L}_n(B)\otimes\mathbf{W}_n(A_{\mathscr{U}}\underset{R_{\mathscr{U}}}{\otimes}R_{\mathscr{X}}/pR_{\mathscr{X}})\big)$$

respectively

$$\mathrm{H}^q_{\mathscr{F}_n}(\mathscr{X}\to\mathscr{U}):=\mathrm{H}^q\big(\mathscr{X}\to\mathscr{U},\mathbb{L}_n(B)\otimes(A_{\mathscr{U}}\underset{R_{\mathscr{U}}}{\otimes}R_{\mathscr{X}}/p^nR_{\mathscr{X}})\big).$$

See the proof of (v) for the notation. For every $q\geq 1$ the group $\mathrm{H}^q_{\mathscr{F}_n}(\mathscr{X}\to\mathscr{U})$ is 0 since the sheaves considered are quasi-coherent.

Due to 6.20 we conclude that assumption (vi) holds using $A_{\mathscr{U}}\otimes_{R_{\mathscr{U}}}R_{\mathscr{X}}\otimes_V K$ instead of $R_{\mathscr{X},M,\infty}$. Let G be the Galois group of $A_{\mathscr{U}}\otimes_V K$ over $R_{\mathscr{U},M,\infty}\otimes_V K$. Using the spectral sequence

$$\mathrm{H}^p\big(G,\mathrm{H}^q\big(\mathscr{X}\to\mathscr{U},\mathscr{F}_n(A_{\mathscr{U}}\underset{R_{\mathscr{U}}}{\otimes}(R_{\mathscr{X}}\underset{V}{\otimes}K)\big)\Longrightarrow\mathrm{H}^{p+q}\big(\mathscr{X}\to\mathscr{U},\mathscr{F}_n(R_{\mathscr{X},\infty}\underset{V}{\otimes}K)\big)$$

we deduce that the group $\mathrm{H}^p\big(\mathscr{X} \to \mathscr{U}, \mathscr{F}_n(R_{\mathscr{X},\infty} \otimes_V K)\big)$ is isomorphic to the group $\mathrm{H}^p\big(G, \mathrm{H}^0\big(\mathscr{X} \to \mathscr{U}, \mathscr{F}_n(A_{\mathscr{U}} \otimes_{R_{\mathscr{U}}} (R_{\mathscr{X}} \otimes_V K)\big)\big)$ i. e., $\mathrm{H}^p\big(G, \mathscr{F}_n(A_{\mathscr{U}} \otimes_V K)\big)$. Let C be the kernel of the surjective map $\mathscr{H}_{\mathscr{U}} \to G$. Consider the spectral sequence

$$\mathrm{H}^p\big(G, \mathrm{H}^q(C, \mathbb{L}_n(B) \otimes \overline{R}_{\mathscr{U}}/p\overline{R}_{\mathscr{U}}\big) \Longrightarrow \mathrm{H}^{p+q}\big(\mathscr{H}_U, \mathbb{L}_n(B) \otimes \overline{R}_{\mathscr{U}}/p\overline{R}_{\mathscr{U}}\big).$$

Note that $\mathrm{H}^q(C, \mathbb{L}_n(B) \otimes \overline{R}_{\mathscr{U}}/p\overline{R}_{\mathscr{U}}\big)$ and $\mathrm{H}^q\big(\mathscr{H}_{\mathscr{U}}, \mathbb{L}_n(B) \otimes \overline{R}_{\mathscr{U}}/p\overline{R}_{\mathscr{U}}\big)$ are annihilated by multiplication by any element of the maximal ideal of $\mathbf{W}\big(V_\infty/pV_\infty\big)$ (resp. V_∞) for $q \geq 1$ due to A.5 and A.3. Hence, the same must hold for $\mathrm{H}^q\big(G, \mathbb{L}_n(B) \otimes A_{\mathscr{U}}/pA_{\mathscr{U}}\big)$. By devissage and 6.20 one concludes that the same must hold for $\mathrm{H}^q\big(G, \mathscr{F}_n(A_{\mathscr{U}} \otimes_V K)\big)$. Thus, (vi) holds. □

6.5. Proof of theorem 6.1. — By theorem 6.12 if $\mathscr{F}$ is a sheaf of $\mathbf{Sh}(\widehat{\mathfrak{X}}_{M,*})^{\mathbf{N}}$ such that the assumptions 6.10 are satisfied then $\mathrm{R}^i\widehat{v}_{\mathscr{X},M,*}(\mathscr{F})\big[\pi^{-1}\big] \cong \mathscr{H}^i_{\mathrm{Gal}_M}(\mathscr{F})$. Using this isomorphism the Leray spectral sequence for the composition of functors $\mathrm{H}^0(\mathscr{X}_*, -) \circ \widehat{v}_{\mathscr{X},M,*}$ becomes

$$E_2^{p,q} = \mathrm{H}^q\big(\mathscr{X}_*, \mathscr{H}^p_{\mathrm{Gal}_M}(\mathscr{F})\big) \Longrightarrow \mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_{M,*}, \mathscr{F}).$$

In particular, we obtain a spectral sequence for $* = \bullet$. Now theorem 6.1 follows as the functors $\mathrm{H}^i(\widehat{\mathfrak{X}}^\bullet_M, -)$ and $\mathrm{H}^i(\widehat{\mathfrak{X}}_M, -)$ are canonically isomorphic; see 4.7.

Appendix A
Galois cohomology via the Tate-Sen method

The goal of this section is to prove Proposition A.5 stating that, if M is a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$, then the groups $\mathrm{H}^i(\mathscr{H}_S, \mathfrak{D}(M) \otimes_{\mathbf{A}_S} \mathbf{A}_{\overline{R}})$, $\mathrm{H}^i(\mathscr{H}_S, \widetilde{\mathfrak{D}}(M) \otimes_{\widetilde{\mathbf{A}}_S} \widetilde{\mathbf{A}}_{\overline{R}})$, $\mathrm{H}^i(\mathrm{H}_S, D(M) \otimes_{\mathbf{A}'_S} \mathbf{A}'_{\overline{R}})$ and $\mathrm{H}^i(\mathrm{H}_S, \widetilde{\mathrm{D}}(M) \otimes_{\widetilde{\mathbf{A}}_{S'_\infty}} \widetilde{\mathbf{A}}_{\overline{R}})$ are trivial for $i \geq 1$. This is the key tool to compute the Galois cohomology of M in terms of the associated (φ, Γ_S)-modules.

To treat all the cases above, we follow the axiomatic approach started by Colmez in [**10**, §3.2 & 3.3] and developed in [**2**, §2].

A.1. The axioms. — Let $\mathscr{G}$ be a profinite group and let $\widetilde{\Lambda}$ be $\mathbf{Z}_p$-algebra which is an integral domain and is endowed with a map $v\colon \widetilde{\Lambda} \to \mathbf{R} \cup \{+\infty\}$ such that:

(i) $v(x) = +\infty \Leftrightarrow x = 0$;
(ii) $v(xy) \geq v(x) + v(y)$;
(iii) $v(x + y) \geq \min(v(x), v(y))$;
(iv) $v(p) > 0$ and $v(px) = v(p) + v(x)$.

We endow $\widetilde{\Lambda}$ with the (separated) topology induced by v. We assume that $\widetilde{\Lambda}$ is complete for this topology and that it is endowed with a continuous action of $\mathscr{G}$ such that $v(g(x)) = v(x)$ for $x \in \widetilde{\Lambda}$ and for $g \in \mathscr{G}$.

Let $\mathscr{H}$ a closed normal subgroup of $\mathscr{G}$ such that $\Gamma = \mathscr{G}/\mathscr{H}$ is endowed with a continuous homomorphism $\chi\colon \Gamma \to \mathbf{Z}_p^\times$ with open image, with kernel isomorphic to

$\mathbf{Z}_p^d$ and such that $\gamma g \gamma^{-1} = g^{\chi(\gamma)}$ for every $g \in \operatorname{Ker}(\chi)$ and every $\gamma \in \Gamma$. Let $\gamma_0 \in \Gamma$ be such that $\operatorname{Im}(\chi) = \mathbf{Z}_p \chi(\gamma_0) \oplus F$ with F a finite group. Assume that there exist $\gamma_1, \dots, \gamma_d \in \operatorname{Aut}(\widetilde{\Lambda})$ such that $\operatorname{Ker}(\chi)$ is an open subgroup of $\mathbf{Z}_p\gamma_1 \oplus \cdots \oplus \mathbf{Z}_p\gamma_d$ and let $m_0 \in \mathbf{N}$ be such that $p^{m_0} \oplus_{i=1}^d \mathbf{Z}_p\gamma_i \subset \operatorname{Ker}(\chi)$. Let $\mathrm{G} \subset \mathscr{G}$ be a closed normal subgroup, put $\mathrm{H} := \mathrm{G} \cap \mathscr{H}$ and assume that $\Gamma' := \mathrm{G}/\mathrm{H} \xrightarrow{\sim} \operatorname{Ker}(\chi)$.

Assume that for every open normal subgroup $\mathscr{H}' \subset \mathscr{H}$ there exists an integer $m_{0,\mathscr{H}'} \geq m_0$ such that for every $i \in \{0, \dots, d\}$ one has

(a) a lifting $\gamma_i^{p^{m_{0,\mathscr{H}'}}} \in \mathscr{G}/\mathscr{H}'$ centralizing $\mathscr{H}/\mathscr{H}'$;

(b) an increasing sequence $\left(\Lambda_{m,\mathscr{H}'}^{(i)}\right)_{m \geq m_{0,\mathscr{H}'}}$ of *closed* subrings of $\widetilde{\Lambda}^{\mathscr{H}'}$;

(c) maps $\left(\tau_{m,\mathscr{H}'}^{(i)} : \widetilde{\Lambda}^{\mathscr{H}'} \to \Lambda_{m,\mathscr{H}'}^{(i)}\right)_{m \geq m_{0,\mathscr{H}'}}$

and the following axioms à la *Tate-Sen* hold:

(TS1) there exists $c_1 \in \mathbf{R}_{>0}$ such that for every open normal subgroups $H_1 \subseteq H_2$ of $\mathscr{H}$ (resp. of H), there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $v(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$;

(TS2) there exists $c_{2,\mathscr{H}'} \in \mathbf{R}_{>0}$ such that for every i and $j \in \{0, \dots, d\}$ and every $m \geq m_{0,\mathscr{H}'}$:

(a) $\tau_{m,\mathscr{H}'}^{(i)}$ is $\Lambda_{m,\mathscr{H}'}^{(i)}$-linear and $\tau_{m,\mathscr{H}'}^{(i)}(x) = x$ if $x \in \Lambda_{m,\mathscr{H}'}^{(i)}$;

(b) one has $v\left(\tau_{m,\mathscr{H}'}^{(i)}(x)\right) \geq v(x) - c_{2,\mathscr{H}'}$ and $\lim_{m \to \infty} \tau_{m,\mathscr{H}'}^{(i)}(x) = x$ for every $x \in \widetilde{\Lambda}^{\mathscr{H}'}$;

(c) $\tau_{m,\mathscr{H}'}^{(i)}$ commutes with $\tau_{m',\mathscr{H}'}^{(j)}$;

(d) the ring $\Lambda_{m,\mathscr{H}'}^{(i)}$ is stable under $\mathscr{G}/\mathscr{H}'$ and $\tau_{m,\mathscr{H}'}^{(i)}$ commutes with the action of $\mathscr{G}/\mathscr{H}'$ for $i \in \{1, \dots, d\}$;

(d′) the ring $\Lambda_{m,\mathscr{H}'}^{(0)}$ is stable under $\gamma_0^{p^{m_{0,\mathscr{H}'}}}$ and $\tau_{m,\mathscr{H}'}^{(0)}$ commutes with $\gamma_0^{p^{m_{0,\mathscr{H}'}}}$ and $\tau_{m,\mathscr{H}'}^{(0)} \circ \tau_{m,\mathscr{H}'}^{(1)} \circ \cdots \circ \tau_{m,\mathscr{H}'}^{(d)}$ commutes with the action of $\mathscr{G}/\mathscr{H}'$;

(e) we have $\Lambda_{m,\mathscr{H}'}^{(i)} \subset \Lambda_{m,\mathscr{H}''}^{(i)}$, as subrings of $\widetilde{\Lambda}$, for every open normal subgroup $\mathscr{H}'' \subset \mathscr{H}'$ and the following diagram commutes

$$\begin{array}{ccc}
\widetilde{\Lambda}^{\mathscr{H}'} & \xrightarrow{\tau_{m,\mathscr{H}'}^{(i)}} & \Lambda_{m,\mathscr{H}'}^{(i)} \\
\downarrow & & \downarrow \\
\widetilde{\Lambda}^{\mathscr{H}''} & \xrightarrow{\tau_{m,\mathscr{H}''}^{(i)}} & \Lambda_{m,\mathscr{H}''}^{(i)}
\end{array}$$

(TS3) let $X_{m,\mathscr{H}'}^{(i)} = \left(1 - \tau_{m,\mathscr{H}'}^{(i)}\right)\left(\widetilde{\Lambda}^{\mathscr{H}'}\right)$. Then,

(a) there exists $c_{3,\mathscr{H}'} \in \mathbf{R}_{>0}$ such that for every $m \geq m_{0,\mathscr{H}'}$ and every $i \in \{0, \dots, d\}$, the map $1 - \gamma_i^{p^m}$ is invertible on $X_{m,\mathscr{H}'}^{(i)}$ and for every $x \in X_{m,\mathscr{H}'}^{(i)}$, one has $v\left(\left(1 - \gamma_i^{p^m}\right)^{-1}(x)\right) \geq v(x) - c_{3,\mathscr{H}'}$.

(b) There exists $c_{4,\mathscr{H}'} \in \mathbf{R}_{>0}$ such that for every $m \geq m_{0,\mathscr{H}'}$ and every $i \in \{1,\ldots,d\}$ and every $x \in \Lambda^{(i)}_{m,\mathscr{H}'}$, one has $v\left(\left(\gamma_i^{p^m}-1\right)(x)\right) \geq v(x) + c_{4,\mathscr{H}'}$.

(TS4) Let $\mathrm{H}' \subset \mathrm{H}$ be an open normal subgroup. Assume that there exists an integer $m_{0,\mathrm{H}'} \geq m_0$ and that for every $i \in \{1,\ldots,d\}$ one has a lifting $\gamma_i^{p^{m_{0,\mathrm{H}'}}} \in \mathrm{G}/\mathrm{H}'$ centralizing H/H' and an increasing sequence $\left(\Lambda^{(i)}_{m,\mathrm{H}'}\right)_{m\geq m_{0,\mathrm{H}'}}$ of *closed* subrings of $\widetilde{\Lambda}^{\mathrm{H}'}$ stable under G/H' and maps $\left(t^{(i)}_{m,\mathrm{H}'}\colon \widetilde{\Lambda}^{\mathrm{H}'} \to \Lambda^{(i)}_{m,\mathrm{H}'}\right)_{m\geq m_{0,\mathscr{H}'}}$ such that the analogues of (TS2) and (TS3) hold.

We followed closely the formalism of [**2**, §2] with the differences that we added (TS2)(e) and (TS4).

A.2. Notation. — Let W be a free $\widetilde{\Lambda}$-module of finite rank a i. e., $W := \widetilde{\Lambda}^a$. We consider it as a topological module with respect to the (separated) topology defined as the product topology considering on $\widetilde{\Lambda}$ the v-adic topology. Note that such topology is independent of the choice of $\widetilde{\Lambda}$-basis of W. For every positive $n \in \mathbf{Q}$ write $\widetilde{\Lambda}_{\geq n}$ for the subgroup of $\widetilde{\Lambda}$ consisting of elements x such that $v(x) \geq n$. They are a fundamental system of neighborhoods for the topology on $\widetilde{\Lambda}$ for $n \to \infty$. Let $W_{\geq n}$ be the image of $\widetilde{\Lambda}^a_{\geq n}$ in W; they form a fundamental system of neighborhoods for the given topology on W. Assume that W is endowed with a continuous action of $\mathscr{G}$. We consider continuous cohomology of a closed subgroup H' of $\mathscr{G}$ with values in W. If $f \in C^r\big(H', W\big)$ is a continuous cochain, with $r \geq 0$ and with the profinite topology on H', write $v(f) := \min\{n \in \mathbf{N} | (\forall g_1,\ldots,g_r \in H') f(g_1,\ldots,g_r) \in W_{\geq n}\}$. We write $\partial\colon C^r\big(H',W\big) \to C^{r+1}\big(H',W\big)$ for the boundary map.

Lemma A.1. — [**23**, §3.2] *Let H_0 be an open subgroup of H (resp. of $\mathscr{H}$) and let f be an r-cochain of H_0 with values in W for $r \geq 1$.*

(1) *Assume that there exists an open normal subgroup $H_1 \subset H_0$ such that f factors via an r-cochain of H_0/H_1. Then, there exists an $(r-1)$-cochain h of H_0/H_1 with values in W such that $v\big(f - \partial h\big) > v(\partial f) - c_1$ and $v(h) > v(f) - c_1$.*

(2) *There exists a sequence of open normal subgroups $H_n \subset H_0$ and continuous cochains $f_n \in C^r\big(H_0/H_n, W\big)$ for $n \in \mathbf{N}$ such that $f \equiv f_n$ modulo $W_{\geq n}$ for $n \to \infty$.*

Proof. — We work out the case of $H_0 \subset H$. For $H_0 \subset \mathscr{H}$ the argument is analogous and the details are left to the reader.

(1) Let $\alpha \in \widetilde{\Lambda}^{H_1}$ be an element satisfying (TS1). Define the $(r-1)$-cochain $\alpha \cup f$ of H_0/H_1 with values in W by

$$\big(\alpha \cup f\big)\big(g_1,\ldots,g_{r-1}\big) := (-1)^r \sum_{t\in H_0/H_1} g_1\cdots g_{r-1}t(\alpha)\cdot f\big(g_1,\ldots,g_{r-1},t\big).$$

One computes that

$$\begin{aligned}\partial(\alpha\cup f)(g_1,\ldots,g_r) &= g_1\big((\alpha\cup f)(g_2,\ldots,g_r)\big)+\sum_{j=1}^{r-1}(-1)^j(\alpha\cup f)(\ldots,g_jg_{j+1},\ldots)+\\ &+(-1)^r(\alpha\cup f)(g_1,\ldots,g_{r-1})=\\ &=(-1)^r\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot g_1 f\big(g_2,\ldots,g_r,t\big)+\\ &+(-1)^r\sum_{j=1}^{r-1}\sum_{t\in H_0/H_1}(-1)^j g_1\cdots g_r t(\alpha)\cdot f\big(g_1,\ldots,g_jg_{j+1},g_r,t\big)+\\ &+\sum_{t\in H_0/H_1} g_1\cdots g_{r-1}t(\alpha)\cdot f\big(g_1,\ldots,g_{r-1},t\big)\end{aligned}$$

and

$$\begin{aligned}\big(\alpha\cup\partial f\big)(g_1,\ldots,g_r) &= (-1)^{r+1}\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot\partial f\big(g_1,\ldots,g_r,t\big)=\\ &=(-1)^{r+1}\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot g_1 f(g_2,\ldots,g_r,t)+\\ &+(-1)^{r+1}\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot\sum_{j=1}^{r-1}(-1)^j f\big(g_1,\ldots,g_jg_{j+1},g_r,t\big)+\\ &-\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot f\big(g_1,\ldots,g_r t\big)+\\ &+\sum_{t\in H_0/H_1} g_1\cdots g_r t(\alpha)\cdot f(g_1,\ldots,g_r)\end{aligned}$$

Since $\sum_{t\in H_0/H_1} t(\alpha)=1$, we have $\big(\alpha\cup\partial f\big)=f-\partial\big(\alpha\cup f\big)$. Put $h=\alpha\cup f$. Then, $v(h)>v(f)-c_1$ and $v(f-\partial h)=v\big(\alpha\cup\partial f\big)\geq v(\alpha)+v(\partial f)$.

(2) Since f is continuous there exists an open normal subgroup H_n such that the composite $\bar f_n\colon H_0^r\to W\to W/W_{\geq n}$ factors via $\big(H_0/H_n\big)^r$. Let f_n be the composite of $\bar f_n$ with a splitting $W/W_{\geq n}\to W$ (as sets). Then, f_n is a continuous cochain and $v(f-f_n)\geq n$. □

Proposition A.2. — [**23**, Prop. 10] *We have* $\mathrm{H}^r\big(\mathrm{H},W\big)=0$ *for* $r\geq 1$ *and* $\mathrm{H}^r\big(\mathscr{H},W\big)=0$ *for* $r\geq 1$. *In particular,* $\mathrm{H}^r\big(G,W\big)=\mathrm{H}^r\big(\Gamma',W^H\big)$ *and* $\mathrm{H}^r\big(\mathscr{G},W\big)=\mathrm{H}^r\big(\Gamma,W^{\mathscr{H}}\big)$.

Proof. — The last statement follows from the first one and from the spectral sequences $\mathrm{H}^j(\Gamma,\mathrm{H}^j(\mathscr{H},\_))\implies\mathrm{H}^{i+j}(\mathscr{G},\_)$ and $\mathrm{H}^j(\Gamma',\mathrm{H}^j(\mathrm{H},\_))\implies\mathrm{H}^{i+j}(\mathrm{G},\_)$. Let $H_0:=H$ or $\mathscr{H}$. Let f be an r-th cochain of H_0, for $r\geq 1$, with values in W. Let $\{H_n,f_n\}_n$ be as in A.1(2) and, for each n, write h_n for the continuous $(r-1)$-cochain satisfying A.1(1) i. e., $v\big(f_n-\partial h_n\big)>v(\partial f_n)-c_1$ and $v(h_n)>v(f_n)-c_1$.

Then, $\{h_n\}$ is Cauchy so that it converges to a continuous $(r-1)$-cochain h. Furthermore, $v(f_n - \partial h_n) \geq n - c_1$ for every n so that $\partial h_n \to f$ for $n \to \infty$. We conclude that $f = \partial h$ as claimed. $\square$

Let $\widetilde{\Lambda}_{\geq n}$ be the subset of $\widetilde{\Lambda}$ consisting of elements b such that $v(b) \geq n$. Then, $\widetilde{\Lambda}_{\geq 0}$ is a ring and $\widetilde{\Lambda}_{\geq n}$ is an ideal for every $n \geq 0$ due to the properties of v. We write $\overline{\Lambda}_n$ for the quotient $\widetilde{\Lambda}_{\geq 0}/\widetilde{\Lambda}_{\geq n}$. Assume that the following strengthening of (TS1) holds:

(TS1′) for every $c \in \mathbf{R}_{>0}$ and for every open normal subgroups $H_1 \subseteq H_2$ of $\mathscr{H}$ (resp. of H), there exists $\alpha \in \widetilde{\Lambda}_{\geq 0}^{H_1}$ such that $v\left(\sum_{\tau \in H_2/H_1} \tau(\alpha)\right) \leq c$.

One then has the following variant of A.2:

Proposition A.3. — *Let W be a free $\overline{\Lambda}_n$-module of finite rank a endowed with a continuous action of $\mathscr{G}$. Then, for every $c \in \mathbf{R}_{>0}$ and every integer $r \geq 1$ there exists an element $\gamma_c \in \widetilde{\Lambda}_{\geq 0}^{\mathscr{H}}$ of valuation $v(\gamma_c) < c$ such that $\gamma_c \cdot \mathrm{H}^r\big(\mathrm{H}, W\big) = 0$ and $\gamma_c \cdot \mathrm{H}^r\big(\mathscr{H}, W\big) = 0$.*

A.3. Decompletion. — The notation is as in A.2. Write $\mathfrak{D}(W) := W^{\mathscr{H}}$ and $\mathrm{D}(W) := W^{\mathrm{H}}$. They are closed subgroups of W endowed with the topology induced from W.

It is proven in [**2**, Cor. 2.3] that (TS1) implies that there exists an open normal subgroup $\mathscr{H}_W \subset \mathscr{H}$ and a $\widetilde{\Lambda}$-basis $e_1, \ldots, e_a$ of W such that $W^{\mathscr{H}_W} \cong \widetilde{\Lambda}^{\mathscr{H}_W} e_1 \oplus \cdots \oplus \widetilde{\Lambda}^{\mathscr{H}_W} e_a$. For every $i = 0, \ldots, d$ and every $m \geq m_W = m_{0,\mathscr{H}_W}$ define the map $\tau^{(i)}_{m,\mathscr{H}_W} : W^{\mathscr{H}_W} \to W^{\mathscr{H}_W}$ by $\sum_{i=1}^a \beta_i e_i \mapsto \sum_{i=1}^a \tau^{(i)}_{m,\mathscr{H}_W}(\beta_i) e_i$. Due to (TS2) such map is independent of $\mathscr{H}_W$ and the basis $e_1, \ldots, e_a$ and it descends to a map on $\mathfrak{D}(W) = W^{\mathscr{H}}$. Due to (TS2)(b) it is continuous for the topology on $W^{\mathscr{H}_W}$ induced from W. We then drop the index $\mathscr{H}_W$ and we write simply

$$\tau^{(i)}_m : \mathfrak{D}(W) \longrightarrow \mathfrak{D}(W) \quad \text{for } i = 0, \ldots, d, \quad m \geq m_W.$$

Using (TS4) and repeating the construction above, we get similarly continuous maps

$$t^{(i)}_m : \mathrm{D}(W) \longrightarrow \mathrm{D}(W) \quad \text{for } i = 1, \ldots, d, \quad m \geq m_W.$$

For every $m \geq m_W$ due to (TS2) we have a decomposition

$$\mathfrak{D}(W) := \mathfrak{D}_m(W) \oplus \mathfrak{D}^{(0)}_m(W) \oplus \cdots \oplus \mathfrak{D}^{(d)}_m(W),$$

where $\mathfrak{D}^{(d)}_m(W) := \big(1 - \tau^{(d)}_m\big)\big(\mathfrak{D}(W)\big)$, $\mathfrak{D}^{(d-1)}_m(W) := \big(1 - \tau^{(d-1)}_m\big)\left(\mathfrak{D}(W)_{\tau^{(d)}_m = 1}\right)$, $\ldots$, $\mathfrak{D}^{(0)}_m(W) := \big(1 - \tau^{(0)}_m\big)\left(\mathfrak{D}(W)_{\tau^{(d)}_m = 1, \ldots, \tau^{(1)}_m = 1}\right)$ and $\mathfrak{D}_m(W) = \mathfrak{D}(W)_{\tau^{(d)}_m = 1, \ldots, \tau^{(0)}_m = 1}$. They are closed Γ-submodules of $\mathfrak{D}(W)$. We endow them with the induced topology. By (TS2) the decomposition above is an isomorphism of topological Γ-modules. Similarly, we have an isomorphism of topological Γ'-modules

$$\mathrm{D}(W) := \mathrm{D}_m(W) \oplus \mathrm{D}^{(1)}_m(W) \oplus \cdots \oplus \mathrm{D}^{(d)}_m(W),$$

where $\mathrm{D}_m^{(d)}(W) := \bigl(1-t_m^{(d)}\bigr)(\mathrm{D}(W)), \ldots, \mathrm{D}_m^{(1)}(W) := \bigl(1-t_m^{(1)}\bigr)\left(\mathrm{D}(W)_{t_m^{(d)}=1,\ldots,t_m^{(2)}=1}\right)$ and $\mathrm{D}_m(W)=\mathrm{D}(W)_{t_m^{(d)}=1,\ldots,t_m^{(1)}=1}$ are closed Γ'-submodules of $\mathrm{D}(W)$.

Proposition A.4. — *There exists an integer $N\geq m_W$ such that for every $n\geq N$, if $\gamma_i^{p^n}\in\Gamma$ the map $\gamma_i^{p^n}-1$ is bijective with continuous inverse on $\mathfrak{D}_m^{(i)}(W)$ for $i=0,\ldots,d$ (resp. $\mathrm{D}_m^{(i)}(W)$ for $i=1,\ldots,d$).*

Then, the maps of continuous cohomology groups $\mathrm{H}^j\bigl(\Gamma,\mathfrak{D}_m(W)\bigr)\to\mathrm{H}^j\bigl(\Gamma,\mathfrak{D}(W)\bigr)$ and $\mathrm{H}^j\bigl(\Gamma',\mathrm{D}_m(W)\bigr)\to\mathrm{H}^j\bigl(\Gamma',\mathrm{D}(W)\bigr)$ are isomorphisms.

Proof. — We deduce from the first statement that $\mathrm{H}^j\left(\mathbf{Z}_p\gamma_i^{p^n},\mathfrak{D}_m^{(i)}(W)\right)=0$ and that $\mathrm{H}^j\left(\mathbf{Z}_p\gamma_i^{p^n},\mathrm{D}_m^{(i)}(W)\right)=0$ for every $j\geq 0$. We get from the Hochschild–Serre spectral sequence that $\mathrm{H}^j\left(\Gamma,\mathfrak{D}_m^{(i)}(W)\right)=0$ and $\mathrm{H}^j\left(\Gamma',\mathrm{D}_m^{(i)}(W)\right)=0$ for $i\geq 1$. For $i=0$ the first statement implies that $\gamma_0^{p^n}-1$ is bijective with continuous inverse on the group $\mathrm{H}^j(\Gamma',\mathfrak{D}_m^{(0)}(W))$. By Hochschild-Serre $\mathrm{H}^j\left(\Gamma,\mathfrak{D}_m^{(i)}(W)\right)=0$ for $i=0$ as well. The second statement follows.

Since $\gamma_i^{p^t}-1=(\gamma_i^{p^s}-1)\bigl(\sum_{j=0}^{p^{t-s}-1}\gamma_i^{p^sj}\bigr)$ for $t\geq s$, if $\gamma_i^{p^t}-1$ is bijective with continuous inverse on $\mathfrak{D}_m^{(i)}(W)$ (resp. $\mathrm{D}_m^{(i)}(W)$) also $\gamma_i^{p^s}-1$ is. Hence, it suffices to prove that $\gamma_i^{p^m}-1$ is invertible with continuous inverse.

We prove the statement for $\mathfrak{D}_m^{(i)}(W)$. The proof for $\mathrm{D}_m^{(i)}(W)$ is similar and the details are left to the reader. Write $W^{\mathscr{H}_W}\cong\widetilde{\Lambda}^{\mathscr{H}_W}e_1\oplus\cdots\oplus\widetilde{\Lambda}^{\mathscr{H}_W}e_a$ as in A.3 and write $W_m^{\mathscr{H}_W,(i)}:=\bigl(1-\tau_m^{(i)}\bigr)\left(W^{\mathscr{H}_W}_{\tau_m^{(i-1)}=1,\ldots,\tau_m^{(1)}=1}\right)$. Due to the assumptions in A.1 we have a lifting $\gamma^{p^{m_{0,\mathscr{H}_W}}}\in\mathscr{G}/\mathscr{H}_W$ commuting with the elements of $\mathscr{H}/\mathscr{H}_W$. Since $\mathfrak{D}_m^{(i)}(W)=\bigl(W_m^{\mathscr{H}_W,(i)}\bigr)^{\mathscr{H}}$ by (TS2) it then suffices to prove that $\gamma_i^{p^m}-1$ is invertible with continuous inverse on $W_m^{\mathscr{H}_W,(i)}$.

Extend v on $\widetilde{\Lambda}e_1\oplus\cdots\oplus\widetilde{\Lambda}e_a$ by $v(\sum_{j=1}^a z_je_j):=\inf\{v(z_j)|j=1,\ldots,a\}$. It defines the weak topology on $\widetilde{\Lambda}e_1\oplus\cdots\oplus\widetilde{\Lambda}e_a$. Since the action of $\mathscr{G}/\mathscr{H}_W$ on $W^{\mathscr{H}_W}$ is continuous, there exists an integer $N\geq m_{0,\mathscr{H}_W}$ such that $\gamma_i^{p^N}$ acts trivially on $W^{\mathscr{H}_W}/W^{\mathscr{H}_W}_{\geq c_{3,\mathscr{H}_W}+1}\cong\oplus_{j=1}^a\widetilde{\Lambda}^{\mathscr{H}_W}/\widetilde{\Lambda}^{\mathscr{H}_W}_{\geq c_{3,\mathscr{H}_W}+1}e_j$. Take $m\geq N$. Following [**8**, Prop. II.6.4] define

$$f_i\colon W_m^{\mathscr{H}_W,(i)}\longrightarrow W_m^{\mathscr{H}_W,(i)},\quad f_i\bigl(\sum_{j=1}^a z_je_j\bigr):=\sum_{j=1}^a\bigl(1-\gamma_i^{p^m}\bigr)^{-1}(z_j)e_j.$$

It is well defined, continuous, bijective and with continuous inverse (for the weak topology) due to (TS3). Then, $z-f_i\bigl((1-\gamma_i^{p^m})(z)\bigr)=-f_i\left(\sum_{j=1}^a\gamma_i^{p^m}(z_j)\bigl(1-\gamma_i^{p^m}\bigr)(e_j)\right)$. Write

$$g_{i,z}\colon W_m^{\mathscr{H}_W,(i)}\longrightarrow W_m^{\mathscr{H}_W,(i)},\quad g_{i,z}(y):=y-f_i\left(\bigl(1-\gamma_i^{p^m}\bigr)(y)-z\right).$$

Then, $v\bigl(f_i(z)\bigr)\geq v(z)-c_{3,\mathscr{H}_W}$ by (TS3) and $v\bigl(g_{i,0}(y)\bigr)\geq v(y)+\inf\{v\bigl((1-\gamma_i^{p^m})(e_i)\bigr)|i=1,\ldots,a\}-c_{3,\mathscr{H}_W}\geq v(y)+1$. Hence, $v\bigl(g_{i,z}(y_1)-g_{i,z}(y_2)\bigr)=$

$v(g_{i,0}(y_1-y_2)) \geq v(y_1-y_2)+1$. This implies that $g_{i,z}$ is a contracting operator for the v-adic topology so that there exists a unique fixed point y_z. Since f_i is bijective, we get that y_z is the only solution of $\left(1-\gamma_i^{p^m}\right)(y)=z$. We deduce that $1-\gamma_i^{p^m}$ is bijective on $W_m^{\mathscr{H}_W,(i)}$. Furthermore, since y_z is the limit of the sequence $g_{i,z}^n(z)$ and $g_{i,z}(z)-z=f_i(\gamma_i^{p^m}(z))$, we have $v(y_z-z)\geq v\big(g_{i,z}(z)-z\big)\geq v\big(\gamma_i^{p^m}(z)\big)-c_{3,\mathscr{H}_W}$. Hence, $\left(1-\gamma_i^{p^m}\right)^{-1}$ is continuous on $W_m^{\mathscr{H}_W,(i)}$. □

We are ready to apply the considerations above in the cases of interest to us. Let S be as in 2.2. Let M be a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$. Let $M \cong \mathbf{Z}_p^a \oplus_{i=1}^b \mathbf{Z}_p/p^{c_i}\mathbf{Z}_p$. For $A=\widehat{\overline{R}}\otimes_V K$, $\mathbf{A}_{\overline{R}}$, $\mathbf{A}_{\overline{R}}^\dagger$, $\widetilde{\mathbf{A}}_{\overline{R}}$ or $\widetilde{\mathbf{A}}_{\overline{R}}^\dagger$, then $M\otimes_{\mathbf{Z}_p} A = A^a\oplus_{i=1}^b\big(A/p^{c_i}A\big)$. We consider $M\otimes_{\mathbf{Z}_p}A$ as topological module for the product topology considering on A the topology induced from the p-adic topology on $\widehat{\overline{R}}\otimes_V K$ or form the weak topology on $\mathbf{A}_{\overline{R}}$ and considering on each $A/p^{c_i}A$ the quotient topology.

Proposition A.5. — *We have:*

1) *the ring* $\widetilde{\Lambda}:=\widehat{\overline{R}}\otimes_V K$ *with* $v(b):=\min\left\{\alpha\in\mathbf{Q}\,|\,\frac{b}{p^\alpha}\in\widehat{\overline{R}}\right\}$ *satisfies (TS1). Furthermore, the following holds*
   (TS1′) *for every* $c\in\mathbf{R}_{>0}$ *and every open normal subgroups* $H_1\subsetneq H_2$ *of* $\mathscr{H}$ *(resp. of* H*), there exists* $\alpha\in\widehat{\overline{R}}^{H_1}$ *such that* $\sum\limits_{\tau\in H_2/H_1}\tau(\alpha)$ *is an element of* $\overline{V}$ *of valuation* $\leq c$*;*
2) *for every* $r\in\mathbf{Q}_{>0}$ *the ring* $\widetilde{\Lambda}:=\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ *with* $v=w_r$ *satisfies (TS1);*
3) *for every* $N\in\mathbf{N}$ *the ring* $\widetilde{\Lambda}:=\widetilde{\mathbf{A}}_{\overline{R}}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}$ *with* $v=\mathbf{v}_{\widetilde{\mathbf{E}}}^{\leq N}$ *satisfies (TS1);*
4) $\mathrm{H}^i\big(\mathscr{H}_S, M\otimes_{\mathbf{Z}_p}(\widehat{\overline{R}}\otimes_V K)\big)=0$ *for every* $i\geq 1$*;*
5) (a) $\mathrm{H}^i\big(\mathscr{H}_S, M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\big)=0$, (b) $\mathrm{H}^i\Big(\mathscr{H}_S, M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^\dagger\Big)=0$ *for every* $i\geq 1$*;*
6) (a) $\mathrm{H}^i\big(\mathrm{H}_S, M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\big)=0$, (b) $\mathrm{H}^i\Big(\mathrm{H}_S, M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^\dagger\Big)=0$ *for every* $i\geq 1$.

Proof. — For open normal subgroups $\mathscr{H}_1\subset\mathscr{H}_2$ of $\mathscr{H}$ claim (1) follows from [**2**, Prop 3.4 & Rmk. 3.5] and claim (2) follows from [**2**, Prop. 4.4].

Let $\mathrm{H}_1\subset\mathrm{H}_2$ be normal subgroups of H. They correspond to extensions $R'_\infty\subset {S'_\infty}^2\subset {S'_\infty}^1$ which are finite and Galois over $R'_\infty\left[p^{-1}\right]$ of degree d_1 and d_2 respectively. In particular, there exists an extension $V_\infty\subset {V_\infty}^\natural$, finite and Galois after inverting p, such that they arise by taking the normalization of the base change of extensions of $R_\infty\otimes_{V_\infty}{V_\infty}^\natural$ finite and Galois after inverting p of degree d_1 and d_2 respectively. This is equivalent to require that there exist open normal subgroups of $\mathscr{H}_1\subset\mathscr{H}_2$ of $\mathscr{H}$ such that $\mathscr{H}_1\cap\mathrm{H}=\mathrm{H}_1$, $\mathscr{H}_2\cap\mathrm{H}=\mathrm{H}_2$ and $\mathrm{H}_1/\mathrm{H}_2\cong\mathscr{H}_1/\mathscr{H}_2$. Then, (TS1) (resp. (TS1′)) for $\mathscr{H}_1\subset\mathscr{H}_2$ implies (TS1) (resp. (TS1′)) for $\mathrm{H}_1\subset\mathrm{H}_2$. Hence, (1) and (2) follow.

(3) Let $H_1\subset H_2$ be open normal subgroups of $\mathscr{H}$ (resp. H) and let α_r be an element of $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ satisfying (TS1). If we write $\alpha_r:=\sum_k p^k[z_k]$ with $z_k\in\widetilde{\mathbf{E}}_{\overline{R}}$, since $w_r(\alpha_r)=$

$\inf\{r\mathbf{v}_{\mathbf{E}}(z_k)+k\} \geq -c_1$, we get that $\mathbf{v}_{\mathbf{E}}(z_k) \geq \frac{-c_1-k}{r}$. Hence, for every $N \in \mathbf{N}$ we have $\mathbf{v}_{\mathbf{E}}^{\leq N}(\alpha_r) \geq \frac{-c_1-N}{r}$. In particular, the claim follows.

(4) It follows from A.2.

(5)–(6) Since $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger} = \widetilde{\mathbf{A}}_{\overline{R}}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}$, claims (5) and (6) follow from A.2 for M a $\mathscr{G}_S$-representation which is free as $\mathbf{Z}_p/p^{N+1}\mathbf{Z}_p$-module. In particular, (5) and (6) hold for torsion representations.

We may then assume that M is torsion free. Let $A := \widetilde{\mathbf{A}}_{\overline{R}}$ or $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$ and let $H = \mathscr{H}_S$ or H_S. Let f be an i-th cocycle of H with values in $M \otimes_{\mathbf{Z}_p} A$, continuous for the weak topology. If $A = \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$, then $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger} = \cup_r \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ and $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ is open in $\widetilde{\mathbf{A}}_{\overline{R}}$ for the weak topology since it contains $\widetilde{\mathbf{A}}_{\overline{R}}^{+}$. In particular, since H is compact in this case f takes values in $M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ for some r.

Since f is continuous, for every $n \in \mathbf{N}$ there exists an open normal subgroup H_n of H such that the composite $\bar{f}_n \colon H^i \to M \otimes_{\mathbf{Z}_p} A \to M \otimes_{\mathbf{Z}_p} \big(A/(U_{n+1,[\frac{c_1+n}{r}]+n} \cap A)\big)$ factors via $\big(H_0/H_n\big)^i$; see 2.4 for the notation $U_{n,h}$. Here, for $u \in \mathbf{Q}$ we write $[u]$ for the smallest positive integer bigger or equal to u. Let f_n be the composite of $\bar{f}_n$ with a splitting $M \otimes_{\mathbf{Z}_p} \big(A/(U_{n+1,[\frac{c_1+n}{r}]+n} \cap A)\big) \longrightarrow M \otimes_{\mathbf{Z}_p} A$ (as sets). Then, f_n is a continuous i-cochain and we also have $f_n \equiv f$, if viewed as cochains with values in $M \otimes_{\mathbf{Z}_p} \big(A/(U_{n+1,[\frac{c_1+n}{r}]+n} \cap A)\big)$. For every n let $h_n := \alpha_r \cup f_n$ be the continuous $(i-1)$-cochain defined as in the proof of A.1(1). The computations in loc. cit. show that $f_n - \partial h_n \equiv \alpha \cup \partial f_n \equiv 0$ and $h_{n+1} \equiv h_n$ in $M \otimes_{\mathbf{Z}_p} \big(A/(U_{n+1,n} \cap A)\big)$. Then, $\{h_n\}$ is Cauchy for the weak topology and $\{\partial h_n\}_n$ converges to f for the weak topology. In particular, h_n converges to a continuous $(i-1)$-cochain h with values in $M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}$ and $\partial h = f$.

If $A = \widetilde{\mathbf{A}}_{\overline{R}}$, this concludes the proof. If $A = \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$, since $p^n \widetilde{\mathbf{A}}_{\overline{R}} \cap \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]} = p^n \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ and since $w_r(p) = 1$ and $w_r(\pi) \geq \frac{pr}{p-1}$ by [**2**, Prop. 4.2(d)], we conclude that $\{h_n\}$ is Cauchy for the w_r-adic topology as well. Since $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ is complete and separated for the w_r-adic topology by [**2**, Prop. 4.2(c)], we conclude that h in fact takes values in $M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$. The conclusion follows. □

A.4. Sen's theory for $\widehat{\overline{R}}[p^{-1}]$. — Before passing to the (φ, Γ)-modules, we first show that our theory applies in the case of $\widehat{\overline{R}}[p^{-1}]$-representations. These results are due to Sen [**22**], in the classical case of a DVR with perfect residue field, and are due to [**6**] for a DVR with imperfect residue field. The key point is of course to show that A.4 applies. This follows essentially from results proven in [**2**]. We review some of the basic definitions and properties from loc. cit.

Let S be a R-algebra as in 2.1. Fix $m_{0,S} \in \mathbf{N}$ such that $p^{m_{0,S}} \bigoplus_{i=1}^{d} \mathbf{Z}_p\gamma_i \subseteq \Gamma_S$. Then, for every $m \geq m_{0,S}$, the ring $S_{m+1}[p^{-1}]$ is a free $S_m[p^{-1}]$-module of rank p^{d+1} (resp. $S_{m+1} \cdot W[p^{-1}]$ is a free $S_m \cdot W[p^{-1}]$-module of rank p^d). For every $i \in \{1, \dots, d\}$

and every $n \in \mathbf{N}$, define

$$S^{(i)}_{n,*} = S\big[T_1^{\frac{1}{p^n}},\ldots,T_{i-1}^{\frac{1}{p^n}},T_{i+1}^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big].V_n$$

and

$$S'^{(i)}_{n,*} = S'\big[T_1^{\frac{1}{p^n}},\ldots,T_{i-1}^{\frac{1}{p^n}},T_{i+1}^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big].$$

For $i=0$, one puts $S^{(i)}_{n,*} = S\big[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\big]$. Eventually, let $S^{(i)}_{\infty,*} = \bigcup_{n\in\mathbf{N}} S^{(i)}_{n,*}$ and $S'^{(i)}_{\infty,*} = \bigcup_{n\in\mathbf{N}} S'^{(i)}_{n,*}$. For every $i\in\{0,\ldots,d\}$ and $m\in\mathbf{N}$, one defines

$$\widehat{S}^{(i)}_{\infty,m,K} = \begin{cases} \Big(\widehat{S^{(0)}_{\infty,*}\cdot V_m}\Big)[p^{-1}] & \text{if } i=0,\\ \Big(\widehat{S^{(i)}_{\infty,*}\cdot S_m}\Big)[p^{-1}] & \text{if } i\in\{1,\ldots,d\},\end{cases}$$

where the hat stands for p-adic completion. Similarly, for $i\in\{1,\ldots,d\}$ and $m\in\mathbf{N}$, put

$$\widehat{S}'^{(i)}_{\infty,m,K} = \Big(\widehat{S'^{(i)}_{\infty,*}.S_m}\Big)[p^{-1}].$$

Note that $\widehat{S}^{(i)}_{\infty,m,K}\subset \widehat{S}_\infty[p^{-1}]$ for every $i\in\{0,\ldots,d\}$ and $m\in\mathbf{N}$ and that $\widehat{S}'^{(i)}_{\infty,m,K}\subset \widehat{S'_\infty}[p^{-1}]$. For $n\geq m\geq m_0$ and $x\in S_n[p^{-1}]$, one puts

$$\tau^{(i)}_m(x) = \begin{cases} \frac{1}{p^{n-m}}\mathrm{Tr}_{S_n/S^{(0)}_{n,*}\cdot V_m}(x) & \text{if } i=0,\\ \frac{1}{p^{n-m}}\mathrm{Tr}_{S_n/S^{(i)}_{n,*}\cdot S_m}(x) & \text{if } i\in\{1,\ldots,d\}.\end{cases}$$

For $n\geq m\geq m_0$ and $x\in S'_n[p^{-1}]$ and every $i=1,\ldots,d$ define

$$t^{(i)}_m(x) = \frac{1}{p^{n-m}}\mathrm{Tr}_{S'_n/S'^{(i)}_{n,*}.S_m}(x).$$

Such maps do not depend on n for $n\gg 0$ so that they are defined on $S_\infty[p^{-1}]$ (resp. $S'_\infty[p^{-1}]$).

Proposition A.6. — *For every $i=0,\ldots,d$ and every $m\geq m_0$ the map $\tau^{(i)}_m$ is continuous for the p-adic topology so that it extends to a unique $\widehat{S}^{(i)}_{\infty,m,K}$-linear map*

$$\tau^{(i)}_m : \widehat{S_\infty}[p^{-1}] \longrightarrow \widehat{S}^{(i)}_{\infty,m,K}.$$

Analogously, for every $i=1,\ldots,d$ and every $m\geq m_0$ the map $t^{(i)}_m$ is continuous for the p-adic topology so that it extends to a unique $\widehat{S}'^{(i)}_{\infty,m,K}$-linear map

$$t^{(i)}_m : \widehat{S'_\infty}[p^{-1}] \longrightarrow \widehat{S}'^{(i)}_{\infty,m,K}.$$

Proof. — The claim for $\tau^{(i)}_m$ follows from [**2**, Lem. 3.8]. Since $t^{(i)}_m$ is obtained from $\tau^{(i)}_m$ by base-change from V_∞ to W, the claim for $t^{(i)}_m$ follows as well. □

Note that $\big(\widehat{\overline{R}}[p^{-1}]\big)^{\mathscr{H}_S} = \widehat{S_\infty}[p^{-1}]$ and $\big(\widehat{\overline{R}}[p^{-1}]\big)^{\mathrm{H}_S} = \widehat{S'_\infty}[p^{-1}]$ due to 2.6. Furthermore,

Proposition A.7. — *The rings $\widehat{S}^{(i)}_{\infty,m,K}$ and the applications $\tau^{(i)}_m$ satisfy (TS2) and (TS3). The rings $\widehat{S}'^{(i)}_{\infty,m,K}$ with the applications $t^{(i)}_m$ satisfy (TS4).*

Proof. — The fact that (TS2) and (TS3) hold is proven in [**2**, Prop. 3.9] and in [**2**, Prop. 3.11]. Axiom (TS4) follows from this since $t^{(i)}_m$ is obtained from $\tau^{(i)}_m$ by base-change from $\widehat{V_\infty}$ to $\widehat{W}$ and taking p-adic completions. □

Lemma A.8. — *We have $\bigcup_m\left(\bigcap_i \widehat{S}^{(i)}_{\infty,m,K}\right) = S_\infty[p^{-1}]$. Analogously, we also have $\bigcup_m\left(\bigcap_i \widehat{S}'^{(i)}_{\infty,m,K}\right) = \bigcup_m \widehat{S'_m}[p^{-1}]$.*

Proof. — The first claim follows from [**2**, Lem. 3.12]. The second is proven as in loc. cit. □

Let M be a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$ and let $Q := M\otimes_{\mathbf{Z}_p}\widehat{\overline{R}}[p^{-1}]$. Due to A.2 we know that the natural maps

$$\mathrm{H}^n\left(\Gamma_S, Q^{\mathscr{H}_S}\right) \longrightarrow \mathrm{H}^n\left(\mathscr{G}_S, Q\right) \quad\text{and}\quad \mathrm{H}^n\left(\Gamma'_S, Q^{\mathrm{H}_S}\right) \longrightarrow \mathrm{H}^n\left(\mathrm{G}_S, Q\right)$$

are isomorphisms. Furthermore,

Theorem A.9. — *There exists a finitely generated, projective $S_\infty[p^{-1}]$-submodule $N\subset Q^{\mathscr{H}_S}$, stable under Γ_S, such that $N\otimes_{S_\infty}\widehat{S_\infty}\cong Q^{\mathscr{H}_S}$ and the natural map*

$$\mathrm{H}^n(\Gamma_S, N) \longrightarrow \mathrm{H}^n\left(\Gamma_S, Q^{\mathscr{H}_S}\right)$$

is an isomorphism. Furthermore, if $N' := N\otimes_{S_\infty}\left(\bigcup_m \widehat{S'_m}\right)$, then $N'\otimes_{\bigcup_m\widehat{S'_m}}\widehat{S'_\infty}\cong Q^{\mathrm{H}_S}$ and the natural map

$$\mathrm{H}^n\left(\Gamma'_S, N'\right) \longrightarrow \mathrm{H}^n\left(\Gamma'_S, Q^{\mathrm{H}_S}\right)$$

is an isomorphism.

Proof. — Put N to be the base change of $\mathfrak{D}_m(Q)$, as defined in A.3, via the natural map $\bigcap_i \widehat{S}^{(i)}_{\infty,m,K}\to S_\infty[p^{-1}]$. Similarly, put N' to be the base change of $\mathrm{D}_m(Q)$ via $\bigcap_i \widehat{S}^{('i)}_{\infty,m,K}\to\bigcup_m\widehat{S'_m}[p^{-1}]$.

Due to [**2**, Thm. 3.1] there exists an $\widehat{\overline{R}}[p^{-1}]$-basis $e_1,\dots,e_a$ of $M\otimes_{\mathbf{Z}_p}\widehat{\overline{R}}[p^{-1}]$ stable under an open subgroup $\mathscr{H}_Q$ of $\mathscr{H}_S$, normal in $\mathscr{G}_S$. Let $S_\infty[p^{-1}]\subset T_\infty[p^{-1}]$ be the corresponding Galois extension. Then $\mathfrak{D}_m(Q)$ (resp. $\mathrm{D}_m(Q)$) is by construction the set of $\mathscr{H}_S/\mathscr{H}_Q$-invariants (resp. $\mathrm{H}_S/\mathrm{H}_Q$-invariants) of the free $\bigcap_i\widehat{T}^{(i)}_{\infty,m,K}$-module (resp. $\bigcap_i\widehat{T}'^{(i)}_{\infty,m,K}$-module) with basis $e_1,\dots,e_a$. By [**1**, Cor. 3.11] we have $\widehat{T_\infty}[p^{-1}]\cong\widehat{S_\infty}[p^{-1}]\otimes_{S_\infty}T_\infty$ so that the extension $\widehat{S_\infty}[p^{-1}]\subset\widehat{T_\infty}[p^{-1}]$ is finite, étale and Galois with group $\mathscr{H}_S/\mathscr{H}_Q$. Similarly, one proves that $\widehat{T'_\infty}[p^{-1}]\cong\widehat{S'_\infty}[p^{-1}]\otimes_{S_\infty}T_\infty$ so that the extension $\widehat{S'_\infty}[p^{-1}]\subset\widehat{T'_\infty}[p^{-1}]$ is also finite, étale and Galois with group $\mathrm{H}_S/\mathrm{H}_Q$. Then, the claims that $N' = N\otimes_{S_\infty}\left(\bigcup_m\widehat{S'_m}\right)$ and that N and N' satisfy the requirements of the theorem follow as in the proof of A.4 and étale descent. □

A.5. Sen's theory for $\widetilde{\mathbf{A}}_{\overline{R}}$ and $\widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$. — We recall some facts proven in [**2**, §4] needed in order to prove that (TS2) and (TS3) hold also for the rings $\widetilde{\mathbf{A}}_{\overline{R}}$ and $\widetilde{\mathbf{A}}^{\dagger}_{\overline{R}}$. Let S be a R-algebra as in 2.1. For every $i=0,\ldots,d$ let

$$\mathbf{A}_S^{(i)}(\infty) := \cup_n \mathbf{A}_S\left[[x_0]^{\frac{1}{p^n}},\ldots,[x_{i-1}]^{\frac{1}{p^n}},[x_{i+1}]^{\frac{1}{p^n}},\ldots,[x_d]^{\frac{1}{p^n}}\right]$$

and let $\overline{\mathbf{A}_S^{(i)}(\infty)}$ be the closure of $\mathbf{A}_S^{(i)}(\infty)$ in $\widetilde{\mathbf{A}}_{S_\infty}$ for the weak topology. Here, we write x_0 for the element ϵ and we write $[x_i]$ for the Teichmüller lift of x_i. By construction it is stable under $\mathscr{G}_R$ if $i\neq 0$ and it is stable under $\gamma_0^{p^s}$, for $s\gg 0$, if $i=0$. Then,

Proposition A.10. — *For every $m\geq 0$ and every $i=0,\ldots,d$ there exists a homomorphism*

$$\tau_m^{(i)}=\tau_{S,m}^{(i)}\colon \widetilde{\mathbf{A}}_{S_\infty}\longrightarrow \overline{\mathbf{A}_S^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right],$$

called the generalized trace à la Tate, such that

(i) *it is $\overline{\mathbf{A}_S^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right]$-linear and it is the identity on $\overline{\mathbf{A}_S^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right]$;*

(ii) *it is continuous for the weak topology;*

(iii) *it commutes with the action of* $\mathrm{Gal}(S_\infty/R)$ *if $i\neq 0$ and it commutes with the action of $\gamma_0^{\mathbf{Z}_p}$ if $i=0$. Furthermore, $\tau_m^{(i)}\circ\tau_n^{(j)}=\tau_n^{(j)}\circ\tau_m^{(i)}$ for m, $n\in\mathbf{N}$ and i, $j\in\{0,\ldots,d\}$ and $\tau_m^{(0)}\circ\tau_m^{(1)}\circ\cdots\circ\tau_m^{(d)}$ commutes with the action of $\mathscr{G}_R$;*

(iv) *for every $n\in\mathbf{Z}$ such that $m+n\geq 0$ we have $\varphi^n\circ\tau_{m+n}^{(i)}=\tau_m^{(i)}\circ\varphi^n$;*

(v) *it is compatible for varying S i. e., given a map of R-algebras $S\longrightarrow T$ as in 2.1 we have that $\tau_{m,T}^{(i)}$ restricted to $\widetilde{\mathbf{A}}_{S_\infty}$ coincides with $\tau_{m,S}^{(i)}$;*

(vi) *there exists $r_S\in\mathbf{Q}_{>0}$ such that (TS2) and (TS3) hold for every $0<r<r_S$ with $\widetilde{\Lambda}:=\mathbf{A}_{\overline{R}}^{(0,r]}$ and $v=w_r$, taking $\widetilde{\Lambda}_m^{(i)}$ for $m\geq 0$ to be the closure of $\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]}\cap\mathbf{A}_S^{(i)}(\infty)\left[[x_i]^{\frac{1}{p^m}}\right]$ in $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ and taking $\tau_{S,m}^{(i)}$ for $m\geq 0$ to be the restriction of the maps defined in (i);*

(vii) *for every $N\in\mathbf{N}$ (TS2) and (TS3) hold for $\widetilde{\Lambda}:=\mathbf{A}_{\overline{R}}/p^{N+1}\mathbf{A}_{\overline{R}}$ and $v=\mathbf{v}_{\mathbf{E}}^{\leq N}$, taking $\widetilde{\Lambda}_m^{(i)}$ for $m\geq 0$ to be $\overline{\mathbf{A}_S^{(i)}(\infty)}/p^{N+1}\overline{\mathbf{A}_S^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right]$ and taking $\tau_{S,m}^{(i)}$ for $m\geq 0$ to be the reduction modulo p^{N+1} of the maps defined in (i);*

(viii) *there exists $m_S\in\mathbf{N}$ such that for $m\geq m_S$ the map $\gamma_i^{p^m}-1$ is an isomorphism on $\left(1-\tau_m^{(i)}\right)\left(\widetilde{\mathbf{A}}_{S_\infty}\right)$ with continuous inverse (for the weak topology).*

Proof. — Claims (i)-(v) follow from [**2**, Prop. 4.11]. The verification of (TS2) (resp. of (TS3)) in (vi) follows from [**2**, Prop. 4.19] (resp. [**2**, Prop. 4.26 & Prop. 4.28]). The fact that (TS2) holds in (vii) follows from (ii) and the fact that the weak topology on $\mathbf{A}_{\overline{R}}/p^{N+1}\mathbf{A}_{\overline{R}}$ is the $\mathbf{v}_{\mathbf{E}}^{\leq N}$-adic topology.

Since $\left(1-\tau_m^{(i)}\right)\left(\widetilde{\mathbf{A}}_{S_\infty}\right)$ is p-adically complete and separated, the fact that $\gamma_i^{p^m}-1$ is bijective can be verified modulo p and follows from [**2**, Prop. 4.26 & Prop. 4.28]. In particular, $(\gamma_i^{p^m}-1)^{-1}$ is bijective on $p^{N+1}\left(1-\tau_m^{(i)}\right)\left(\widetilde{\mathbf{A}}_{S_\infty}\right)$ and, consequently, on $\big(1-\tau_m^{(i)}\big)\big(\widetilde{\mathbf{A}}_{S_\infty}/p^{N+1}\widetilde{\mathbf{A}}_{S_\infty}\big)$ for every $N\in\mathbf{N}$. Note that for every $h\in\mathbf{N}$ the group $\pi^h\widetilde{\mathbf{A}}_{S_\infty}^+$ is contained in the subgroup of elements $x\in\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]}$ such that $w_r(x)\geq hw_r(\pi)$. By (TS3) for $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$ there exist constants $c_{3,S}$ and $c_{4,S}$ such that for every element z in $\pi^h(1-\tau_m^{(i)})\widetilde{\mathbf{A}}_{S_\infty}^+$ (resp. $\overline{\mathbf{A}_S^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right]\cap\pi^h\widetilde{\mathbf{A}}_{S_\infty}^+$) one has

$$r\mathbf{v}_{\mathbf{E}}(z_k)^{\leq N}\big((1-\gamma_i^{p^m})^{-1}(z)\big)\geq w_r\big((1-\gamma_i^{p^m})^{-1}(z)\big)-N\geq w_r(z)-c_{3,S}-N$$

and, respectively,

$$r\mathbf{v}_{\mathbf{E}}(z_k)^{\leq N}\big((1-\gamma_i^{p^m})(z)\big)\geq w_r\big((1-\gamma_i^{p^m})(z)\big)-N\geq w_r(z)+c_{4,S}-N.$$

Since $w_r(z)\geq r\mathbf{v}_{\widetilde{\mathbf{E}}}^{\leq N}(z)$, we conclude that $\mathbf{v}_{\mathbf{E}}(z_k)^{\leq N}\big((1-\gamma_i^{p^m})^{-1}(z)\big)\geq\mathbf{v}_{\widetilde{\mathbf{E}}}^{\leq N}(z)-\frac{c_{3,S}+N}{r}$ and $\mathbf{v}_{\mathbf{E}}(z_k)^{\leq N}\big((1-\gamma_i^{p^m})(z)\big)\geq\mathbf{v}_{\widetilde{\mathbf{E}}}^{\leq N}(z)+\frac{c_{4,S}-N}{r}$. Hence, (vii) and (viii) follow. □

Similarly, given a R-algebra S as in 2.1, for every $i=1,\ldots,d$ let $\mathbf{A}_S'^{(i)}(\infty):=\cup_n\mathbf{A}_S'\left[[x_1]^{\frac{1}{p^n}},\ldots,[x_{i-1}]^{\frac{1}{p^n}},[x_{i+1}]^{\frac{1}{p^n}},\ldots,[x_d]^{\frac{1}{p^n}}\right]$ and let $\overline{\mathbf{A}_S'^{(i)}(\infty)}$ be the closure of $\mathbf{A}_S'^{(i)}(\infty)$ in $\widetilde{\mathbf{A}}_{S_\infty'}$ for the weak topology. Note that since $i\geq 1$ the ring $\overline{\mathbf{A}_S^{(i)}(\infty)}$ contains the closure for the weak topology of $\cup_n\mathbf{A}_V\left[[x_0]^{\frac{1}{p^n}}\right]$ which is $\widetilde{\mathbf{A}}_{V_\infty}$ by [**2**, Cor. 4.10]. Then, $\overline{\mathbf{A}_S^{(i)}(\infty)}\otimes_{\widetilde{\mathbf{A}}_{V_\infty}}\widetilde{\mathbf{A}}_W$ maps to $\overline{\mathbf{A}_S'^{(i)}(\infty)}$ and the image is dense for the weak topology. Recall that $\widetilde{\mathbf{A}}_{S_\infty}\otimes_{\widetilde{\mathbf{A}}_{V_\infty}}\widetilde{\mathbf{A}}_W$ injects and is dense in $\widetilde{\mathbf{A}}_{S_\infty'}$ and $\mathbf{A}_S\otimes_{\mathbf{A}_V}\widetilde{\mathbf{A}}_W$ injects and is dense in $\mathbf{A}_S'$ by 2.9. Hence, for every $i=1,\ldots,d$ we may base-change $\tau_{S,m}^{(i)}$ via $\otimes_{\widetilde{\mathbf{A}}_{V_\infty}}\widetilde{\mathbf{A}}_W$ and complete with respect to the weak topology. We obtain a map

$$t_m^{(i)}=t_{S,m}^{(i)}\colon\widetilde{\mathbf{A}}_{S_\infty'}\longrightarrow\overline{\mathbf{A}_S'^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right].$$

Proposition A.11. — *The analogues of the statements (i)-(viii) of A.10 hold for $\widetilde{\mathbf{A}}_{S_\infty'}$, the rings $\overline{\mathbf{A}_S'^{(i)}(\infty)}\left[[x_i]^{\frac{1}{p^m}}\right]$ and the maps $t_m^{(i)}$.*

Proof. — The proposition follows from A.10, from the construction of $t_m^{(i)}$ and density arguments. For (vi) note that $\widetilde{\mathbf{A}}_{S_\infty}^{(0,r]}\otimes_{\widetilde{\mathbf{A}}_{V_\infty}^{(0,r]}}\widetilde{\mathbf{A}}_W^{(0,r]}$ maps to $\widetilde{\mathbf{A}}_{S_\infty'}{}^{(0,r]}$ and has dense image for the w_r-adic topology by 2.9(d). □

Lemma A.12. — *We have $\widetilde{\mathbf{A}}_{S_\infty}^{\tau_0^{(0)}=1,\ldots,\tau_0^{(d)}=1}=\mathbf{A}_S$ and $\big(\widetilde{\mathbf{A}}_{S_\infty'}\big)^{t_0^{(1)}=1,\ldots,t_0^{(d)}=1}=\mathbf{A}_S'$.*

Proof. — By [**2**, Cor. 4.10] the monomials $\{[x_0]^{\frac{\alpha_1}{p^n}}\cdots,[x_d]^{\frac{\alpha_d}{p^n}}\}_{0\leq\alpha_i<p^n}$ form an $\mathbf{A}_S$-basis of $\varphi^{-n}(\mathbf{A}_S)=\mathbf{A}_S\left[[x_0]^{\frac{1}{p^n}},\ldots,[x_d]^{\frac{1}{p^n}}\right]$ and $\cup_n\varphi^{-n}(\mathbf{A}_S)$ is dense in $\widetilde{\mathbf{A}}_{S_\infty}$ for the weak topology. In particular, $\cup_n\varphi^{-n}(\mathbf{A}_S)^{\tau_0^{(0)}=1,\ldots,\tau_0^{(d)}=1}$ is dense in $\widetilde{\mathbf{A}}_{S_\infty}^{\tau_0^{(1)}=1,\ldots,\tau_0^{(d)}=1}$ and $\cup_n\varphi^{-n}(\mathbf{A}_S\otimes_{\mathbf{A}_V}\widetilde{\mathbf{A}}_W)^{t_0^{(0)}=1,\ldots,t_0^{(d)}=1}$ is dense in $\big(\widetilde{\mathbf{A}}_{S_\infty'}\big)^{t_0^{(1)}=1,\ldots,t_0^{(d)}=1}$ respectively.

From the fact that $\tau_0^{(i)}\big([x_i]^{\frac{\alpha}{p^n}}\big) = 0$ for $0 < \alpha < p^n$, we get that $\cap_{i=0}^d \mathbf{A}_S^{(i)}(\infty) = \mathbf{A}_S$ is dense in $\widetilde{\mathbf{A}}_{S_\infty}^{\tau_0^{(0)}=1,\ldots,\tau_0^{(d)}=1}$ and $\mathbf{A}_S \otimes_{\mathbf{A}_V} \widetilde{\mathbf{A}}_W$ is dense in $\big(\widetilde{\mathbf{A}}_{S'_\infty}\big)^{t_0^{(1)}=1,\ldots,t_0^{(d)}=1}$ respectively. The conclusion follows from 2.9. □

A.5.1. *The operators $\tau_m^{(i)}$ and $t_m^{(i)}$ on (φ,Γ)-modules.* — Let S be as in 2.2. Let M be a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$. If $M \cong \mathbf{Z}_p^a \oplus_{i=1}^p \mathbf{Z}_p/p^{c_i}\mathbf{Z}_p$, then $M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}} \cong \widetilde{\mathbf{A}}_{\overline{R}}^a \oplus_{i=1}^b \widetilde{\mathbf{A}}_{\overline{R}}/p^{c_i}\widetilde{\mathbf{A}}_{\overline{R}}$ and on the latter we have the product topology considering on $\widetilde{\mathbf{A}}_{\overline{R}}$ the weak topology. Recall that we have defined $\widetilde{\mathfrak{D}}(M) = \big(M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}\big)^{\mathscr{H}_S}$ and $\widetilde{\mathrm{D}}(M) := \big(M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}\big)^{\mathrm{H}_S}$. We then define the weak topology on $\widetilde{\mathfrak{D}}(M)$ and $\widetilde{\mathrm{D}}(M)$ to be the topology induced from the inclusions $\widetilde{\mathfrak{D}}(M) \subset \widetilde{\mathrm{D}}(M) \subset M \otimes_{\mathbf{Z}_p} \widetilde{\mathbf{A}}_{\overline{R}}$.

Assume first that M is killed by p^{N+1}. Since $\widetilde{\Lambda} := \widetilde{\mathbf{A}}_{\overline{R}}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}$ satisfies (TS1)-(TS4) due to A.10 & A.11(vii), we may apply A.3 and define the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) on $\widetilde{\mathfrak{D}}(M)$ and on $\widetilde{\mathrm{D}}(M)$ and we get decompositions

$$\widetilde{\mathfrak{D}}(M) := \widetilde{\mathfrak{D}}_m(M) \oplus \widetilde{\mathfrak{D}}_m^{(0)}(M) \oplus \cdots \oplus \widetilde{\mathfrak{D}}_m^{(d)}(M)$$

and

$$\widetilde{\mathrm{D}}(M) := \widetilde{\mathrm{D}}_m(M) \oplus \widetilde{\mathrm{D}}_m^{(1)}(M) \oplus \cdots \oplus \widetilde{\mathrm{D}}_m^{(d)}(M).$$

By devissage we get the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) on $\widetilde{\mathfrak{D}}(M)$ and on $\widetilde{\mathrm{D}}(M)$ and the decomposition above for any torsion $\mathscr{G}_S$-representation M.

If M is torsion free, $\widetilde{\mathfrak{D}}(M) := \lim\limits_{\infty\leftarrow n} \widetilde{\mathfrak{D}}(M/p^nM)$ and $\widetilde{\mathrm{D}}(M) := \lim\limits_{\infty\leftarrow n} \widetilde{\mathrm{D}}(M/p^nM)$ by 2.12. Using the construction for the torsion case and passing to the limit, we get the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) on $\widetilde{\mathfrak{D}}(M)$ and on $\widetilde{\mathrm{D}}(M)$ and the decomposition above.

Proposition A.13. — *Let S be as in 2.2 and let M be a $\mathbf{Z}_p$-representation of $\mathscr{G}_S$. Then,*

1) *$\widetilde{\mathfrak{D}}_0(M) = \mathfrak{D}(M)$ and $\widetilde{\mathrm{D}}_0(M) = \mathrm{D}(M)$;*
2) *the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) on $\widetilde{\mathfrak{D}}(M)$ (resp. $\widetilde{\mathrm{D}}(M)$) are continuous for the weak topology;*
3) *the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) preserve $\widetilde{\mathfrak{D}}^\dagger(M)$ (resp. $\widetilde{\mathrm{D}}^\dagger(M)$). In particular, we have*
$$\widetilde{\mathfrak{D}}^\dagger(M) := \widetilde{\mathfrak{D}}_m^\dagger(M) \oplus \widetilde{\mathfrak{D}}_m^{\dagger,(0)}(M) \oplus \cdots \oplus \widetilde{\mathfrak{D}}_m^{\dagger,(d)}(M)$$
and
$$\widetilde{\mathrm{D}}^\dagger(M) := \widetilde{\mathrm{D}}_m^\dagger(M) \oplus \widetilde{\mathrm{D}}_m^{\dagger,(1)}(M) \oplus \cdots \oplus \widetilde{\mathrm{D}}_m^{\dagger,(d)}(M),$$
where the modules on the right hand side are defined as in A.3;
4) *if $\gamma_i^{p^n} \in \Gamma_S$ (resp. Γ'_S) then $\gamma_i^{p^n} - 1$ is bijective on $\widetilde{\mathfrak{D}}_m^{(i)}(M)$ (resp. $\widetilde{\mathrm{D}}_m^{(i)}(M)$) with continuous inverse (for the weak topology);*
5) *if $\gamma_i^{p^n} \in \Gamma_S$ (resp. Γ'_S) then $\gamma_i^{p^n} - 1$ is bijective on $\widetilde{\mathfrak{D}}_m^{\dagger,(i)}(M)$ (resp. $\widetilde{\mathrm{D}}_m^{\dagger,(i)}(M)$) with continuous inverse (for the weak topology);*
6) *$\widetilde{\mathfrak{D}}_m^\dagger(M) = \widetilde{\mathfrak{D}}_m(M) \cap \widetilde{\mathfrak{D}}^\dagger(M)$, $\widetilde{\mathfrak{D}}_m^{\dagger,(i)}(M) = \widetilde{\mathfrak{D}}_m^{(i)}(M) \cap \widetilde{\mathfrak{D}}^\dagger(M)$, $\widetilde{\mathrm{D}}_m^{\dagger,(i)}(M) = \widetilde{\mathrm{D}}_m^{(i)}(M) \cap \widetilde{\mathrm{D}}^\dagger(M)$ and $\widetilde{\mathrm{D}}_m^\dagger(M) = \widetilde{\mathrm{D}}_m(M) \cap \widetilde{\mathrm{D}}^\dagger(M)$. In particular, $\widetilde{\mathfrak{D}}_0^\dagger(M) = \mathfrak{D}^\dagger(M)$ and $\widetilde{\mathrm{D}}_0^\dagger(M) = \mathrm{D}^\dagger(M)$.*

Proof. — Since $\gamma_i^{p^t}-1=(\gamma_i^{p^s}-1)\big(\sum_{j=0}^{p^{t-s}-1}\gamma_i^{p^sj}\big)$ for $t\geq s$, it suffices to prove the bijectivity and the existence of a continuous inverse in (4) and (5) for $n\gg 0$. Assuming (3), we have $\widetilde{\mathfrak{D}}_m^{\dagger,(i)}(M)=\widetilde{\mathfrak{D}}_m^{(i)}(M)\cap\widetilde{\mathfrak{D}}^\dagger(M)$ and $\widetilde{\mathrm{D}}_m^{\dagger,(i)}(M)=\widetilde{\mathrm{D}}_m^{(i)}(M)\cap\widetilde{\mathrm{D}}^\dagger(M)$. Then, (4) and the bijectivity in (5) imply the existence of a continuous inverse in (5). Claim (6) follows from the others. For every m and $n\in\mathbf{N}$ the maps

$$\varphi^n\otimes 1\colon \widetilde{\mathfrak{D}}(M)\overset{\varphi^n}{\underset{\mathbf{A}_S}{\otimes}}\mathbf{A}_S\overset{\sim}{\longrightarrow}\widetilde{\mathfrak{D}}(M),\qquad \varphi^n\otimes 1\colon \widetilde{\mathfrak{D}}^\dagger(M)\overset{\varphi^n}{\underset{\mathbf{A}_S^\dagger}{\otimes}}\mathbf{A}_S^\dagger\overset{\sim}{\longrightarrow}\widetilde{\mathfrak{D}}^\dagger(M)$$

and

$$\varphi^n\otimes 1\colon \widetilde{\mathrm{D}}(M)\overset{\varphi^n}{\underset{\mathbf{A}'_S}{\otimes}}\mathbf{A}'_S\overset{\sim}{\longrightarrow}\widetilde{\mathrm{D}}(M),\qquad \varphi^n\otimes 1\colon \widetilde{\mathrm{D}}^\dagger(M)\overset{\varphi^n}{\underset{\mathbf{A}_S'^\dagger}{\otimes}}\mathbf{A}_S'^\dagger\overset{\sim}{\longrightarrow}\widetilde{\mathrm{D}}^\dagger(M)$$

are isomorphisms by 2.12(i). It follows from A.10 and A.11(iv) that $\big(\varphi^n\otimes 1\big)\circ\tau_{m+n}^{(i)}=\tau_m^{(i)}\circ\big(\varphi^n\otimes 1\big)$ and $\big(\varphi^n\otimes 1\big)\circ\tau_{m+n}^{(i)}=t_m^{(i)}\circ\big(\varphi^n\otimes 1\big)$ and that $\varphi^n\otimes 1$ defines an isomorphism from $\widetilde{\mathfrak{D}}_{m+n}(M)\otimes_{\mathbf{A}_S}^{\varphi^n}\mathbf{A}_S$ (respectively from $\widetilde{\mathfrak{D}}_{m+n}^{(i)}(M)\otimes_{\mathbf{A}_S}^{\varphi^n}\mathbf{A}_S$, respectively from $\widetilde{\mathrm{D}}_{m+n}(M)\otimes_{\mathbf{A}'_S}^{\varphi^n}\mathbf{A}'_S$, respectively from $\widetilde{\mathrm{D}}_{m+n}^{(i)}(M)\otimes_{\mathbf{A}_S}^{\varphi^n}\mathbf{A}_S$) to $\widetilde{\mathfrak{D}}_m(M)$ (respectively $\widetilde{\mathfrak{D}}_m^{(i)}(M)$, $\widetilde{\mathrm{D}}_m(M)$, $\widetilde{\mathrm{D}}_m^{(i)}(M)$). Hence, it suffices to prove claims (2), (4) and (5) for $m\gg 0$ to deduce it for every $m\in\mathbf{N}$.

Since $\widetilde{\mathfrak{D}}(M):=\lim\limits_{\infty\leftarrow n}\widetilde{\mathfrak{D}}(M/p^nM)$ and $\widetilde{\mathrm{D}}(M):=\lim\limits_{\infty\leftarrow n}\widetilde{\mathrm{D}}(M/p^nM)$ by 2.12 and the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) are constructed on each $\widetilde{\mathfrak{D}}(M/p^nM)$ (resp. $\widetilde{\mathrm{D}}(M/p^nM)$) passing to the limit, to prove (1), (2) and (4) one may assume that M is a torsion representation. By devissage one may also assume that M is a free $\mathbf{Z}/p^{N+1}\mathbf{Z}$-module for some $N\in\mathbf{N}$. Note that $\tau_m^{(i)}$ and $t_m^{(i)}$ commute with the Galois action and are compatible with extensions $S_\infty\subset T_\infty$ and $S'_\infty\subset T'_\infty$ by A.10 and A.11. Due to 2.8, 2.9 and 2.10 and étale descent, it then suffices to prove (1), (2) and (4) passing to an extension $S_\infty\subset T_\infty$ in $\overline{R}$ finite, étale and Galois after inverting p i. e., for $\big(M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\big)^{\mathscr{H}_T}$ instead of $\widetilde{\mathfrak{D}}(M)$ and $\big(M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\big)^{\mathrm{H}_T}$ instead of $\widetilde{\mathrm{D}}(M)$. We may then assume that $\mathscr{H}_T$, and hence H_T, act trivially on M. Claim (1) follows then from A.12. Claim (2) follows from A.10(ii) and A.11(ii). Claim (4) for $m\gg 0$ follows from A.4 since $\widetilde{\Lambda}:=\widetilde{\mathbf{A}}_{\overline{R}}/p^{N+1}\widetilde{\mathbf{A}}_{\overline{R}}$ satisfies (TS1)-(TS4). This concludes the proof of (1), (2) and (4).

If M is a torsion representation, then (3) and the bijectivity in (5) follow from 2.12(ii′). Assume that M is free of rank n. Thanks to 2.8, 2.9 and 2.10 and étale descent we may pass to an extension $S_\infty\subset T_\infty$ in $\overline{R}$ finite, étale and Galois after inverting p. By [**2**, Thm. 4.35] there exists such an extension $S_\infty\subset T_\infty$ so that $\mathfrak{D}^\dagger(M)\otimes_{\mathbf{A}_S^\dagger}\mathbf{A}_T^\dagger$ is a free $\mathbf{A}_T^\dagger$-module of rank n. Fix a basis $\{e_1,\dots,e_n\}$ and choose $r\in\mathbf{Q}_{>0}$ such that these elements lie in $M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$. By 2.12(iii′) we have $M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^\dagger=\widetilde{\mathbf{A}}_{\overline{R}}^\dagger e_1\oplus\cdots\widetilde{\mathbf{A}}_{\overline{R}}^\dagger e_n$. For every $s<\min\{r,r_T\}$, see A.10(vi) & A.11, let $W_s := M\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^{(0,s]}$. Define $\widetilde{\mathfrak{D}}^{(0,s]}(M) := W_s^{\mathscr{H}_T}$ and $\widetilde{\mathrm{D}}^{(0,s]}(M) := W_s^{\mathrm{H}_T}$. Then, $\widetilde{\mathfrak{D}}^{\dagger}(M) = \cup_s\widetilde{\mathfrak{D}}^{(0,s]}(M)$ and $\widetilde{\mathrm{D}}^{\dagger}(M) = \cup_s\widetilde{\mathrm{D}}^{(0,s]}(M)$.

Note that $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,s]}$ satisfies (TS1)-(TS4) by A.10(vi) & A.11. Hence, the operators $\tau_m^{(i)}$ (resp. $t_m^{(i)}$) preserve $\widetilde{\mathfrak{D}}^{(0,s]}(M)$ (resp. $\widetilde{\mathrm{D}}^{(0,s]}(M)$) and we further have decompositions

$$\widetilde{\mathfrak{D}}^{(0,s]}(M) := \widetilde{\mathfrak{D}}_m^{(0,s]}(M)\oplus\widetilde{\mathfrak{D}}_m^{(0,s],(0)}(M)\oplus\cdots\oplus\widetilde{\mathfrak{D}}_m^{(0,s],(d)}(M)$$

and

$$\widetilde{\mathrm{D}}^{(0,s]}(M) := \widetilde{\mathrm{D}}_m^{(0,s]}(M)\oplus\widetilde{\mathrm{D}}_m^{(0,s],(1)}(M)\oplus\cdots\oplus\widetilde{\mathrm{D}}_m^{(0,s],(d)}(M)$$

by A.3. This proves (3) in the overconvergent case. It follows from A.4 that if $\gamma_i^{p^n}\in\Gamma_S$ (resp. Γ'_S) then $\gamma_i^{p^n}-1$ is bijective on $\widetilde{\mathfrak{D}}_m^{(0,s],(i)}(M)$ (resp. $\widetilde{\mathrm{D}}_m^{(0,s],(i)}(M)$ for $m\gg 0$. We conclude that the bijectivity in (5) holds. Claim (5) follows. □

We deduce from A.4, A.5 and A.13 the following theorem which summarizes the results proven so far:

Theorem A.14. — *The natural maps*

$$\mathrm{H}^n\big(\Gamma_S,\mathfrak{D}(M)\big)\longrightarrow\mathrm{H}^n\big(\Gamma_S,\widetilde{\mathfrak{D}}(M)\big)\longrightarrow\mathrm{H}^n\left(\mathscr{G}_S, M\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}}\right),$$

$$\mathrm{H}^n\big(\Gamma'_S,\mathrm{D}(M)\big)\longrightarrow\mathrm{H}^n\big(\Gamma'_S,\widetilde{\mathrm{D}}(M)\big)\longrightarrow\mathrm{H}^n\left(\mathrm{G}_S, M\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}}\right),$$

$$\mathrm{H}^n\big(\Gamma_S,\mathfrak{D}^{\dagger}(M)\big)\longrightarrow\mathrm{H}^n\big(\Gamma_S,\widetilde{\mathfrak{D}}^{\dagger}(M)\big)\longrightarrow\mathrm{H}^n\left(\mathscr{G}_S, M\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}\right)$$

and

$$\mathrm{H}^n\big(\Gamma'_S,\mathrm{D}^{\dagger}(M)\big)\longrightarrow\mathrm{H}^n\big(\Gamma'_S,\widetilde{\mathrm{D}}^{\dagger}(M)\big)\longrightarrow\mathrm{H}^n\left(\mathrm{G}_S, M\underset{\mathbf{Z}_p}{\otimes}\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}\right)$$

are all isomorphisms.

A.5.2. *The structure of δ-functors.* — The cohomology groups appearing in A.14 are δ-functors i. e., given an exact sequence $0\to W_1\to W_2\to W_3\to 0$ of $\mathscr{G}_S$-representations we get an associated long exact sequence of cohomology groups, functorial with respect to morphisms of short exact sequences.

For the cohomology groups on the right hand side it suffices to construct a left inverse (as sets) of the surjection $W_2\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}\to W_3\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}$ which is continuous for the weak topology and has the property that the image of $W_3\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$ is contained in $W_2\otimes_{\mathbf{Z}_p}\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$. We consider two cases.

The first case is that W_3 is a free $\mathbf{Z}_p$-module. Any left inverse $W_2\to W_3$ as $\mathbf{Z}_p$-modules induces by extension of scalars the claimed inverse.

The second case is that W_3 is a torsion $\mathbf{Z}_p$-module. By devissage one can suppose that W_3 is a free $\mathbf{Z}/p^n\mathbf{Z}$-module. The splitting is defined lifting a basis of W_3 to W_2 and constructing a left inverse ζ_n to the projection $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}\to\widetilde{\mathbf{A}}_{\overline{R}}/p^n\widetilde{\mathbf{A}}_{\overline{R}}$.

Since $\widetilde{\mathbf{A}}_{\overline{R}}/p^n\widetilde{\mathbf{A}}_{\overline{R}} = \mathbf{W}_n(\widetilde{\mathbf{E}}_{\overline{R}})$ we define such inverse sending $(a_0,\ldots,a_n) \mapsto (a_0,\ldots,a_n,0,\ldots)$.

For the cohomology groups of (φ,Γ)-modules we argue as follows. Due to A.13 we have continuous right inverses of the inclusions $\mathfrak{D}^*(W_i) \subset \widetilde{\mathfrak{D}}^*(W_i)$ and $\mathrm{D}^*(W_i) \subset \widetilde{\mathrm{D}}^*(W_i)$, where $*$ stands for nothing or $\dagger$. Such inverses are compatible with the morphisms of (φ,Γ)-modules induced by the map $W_2 \to W_3$. Thus, it suffices to construct a left inverse to the map $\widetilde{\mathrm{D}}(W_2) \to \widetilde{\mathrm{D}}(W_3)$ sending $\widetilde{\mathrm{D}}^\dagger(W_3)$ to $\widetilde{\mathrm{D}}^\dagger(W_2)$ (resp. to the map $\widetilde{\mathfrak{D}}(W_2) \to \widetilde{\mathfrak{D}}(W_3)$ sending $\widetilde{\mathfrak{D}}^\dagger(W_3)$ to $\widetilde{\mathfrak{D}}^\dagger(W_2)$). As before we distinguish two cases.

The first is when W_3 is a free $\mathbf{Z}_p$-module. Due to [2, Thm. 4.35] there exists an extension $S \subset T$, finite and étale after inverting p, so that $\mathfrak{D}^\dagger(W_3)\otimes_{\mathbf{A}_S^\dagger}\mathbf{A}_T^\dagger$ is a free $\mathbf{A}_T^\dagger$-module. Since $\mathbf{A}_S^\dagger \subset \mathbf{A}_T^\dagger$ is finite and étale by 2.8 we deduce that $\mathfrak{D}^\dagger(W_3)$ is a projective $\mathbf{A}_S^\dagger$-module so that we can find a continuous left inverse to the surjection $\mathfrak{D}^\dagger(W_2) \to \mathfrak{D}^\dagger(W_3)$ as $\mathbf{A}_S^\dagger$-modules. Thanks to 2.12 we conclude the construction in this case simply extending scalars.

The second case is when W_3 is a torsion $\mathbf{Z}_p$-module. By devissage we may assume that W_3 is a free $\mathbf{Z}/p^n\mathbf{Z}$-module. Let $S \subset T$ be an extension such that $\mathcal{G}_T$ acts trivially on W_3. Then, $\mathfrak{D}(W_3)\otimes_{\mathbf{A}_S}\mathbf{A}_T = W_3\otimes\mathbf{A}_T$ is a free $\mathbf{A}_T/p^n\mathbf{A}_T$-module. Due to 2.9 and 2.12 the various (φ,Γ)-modules associated to W_i as $\mathcal{G}_T$-representations, for $i=2$ and 3, are defined by the corresponding (φ,Γ)-modules as $\mathcal{G}_S$-representations extending scalars via the finite and étale extension $\mathbf{A}_S^\dagger \subset \mathbf{A}_T^\dagger$. A splitting as $\mathbf{A}_S^\dagger$-modules to the inclusion $\mathbf{A}_S^\dagger \subset \mathbf{A}_T^\dagger$ produces at the level of (φ,Γ)-modules a left inverse $\zeta_{T/S}$ to the process of extending scalars. To conclude it suffices to construct the inverse considering $\mathcal{G}_T$-representations composing then with $\zeta_{T/S}$. One gets the seeked for map lifting a basis of W_3 to W_2 and using the left inverse to the projection $\widetilde{\mathbf{A}}_{\overline{R}}^\dagger \to \widetilde{\mathbf{A}}_{\overline{R}}/p^n\widetilde{\mathbf{A}}_{\overline{R}}$ constructed above. This map has the required properties since $\zeta_n\colon \widetilde{\mathbf{A}}_{\overline{R}}/p^n\widetilde{\mathbf{A}}_{\overline{R}} \to \widetilde{\mathbf{A}}_{\overline{R}}$ sends $\widetilde{\mathbf{A}}_{T_\infty}/p^n\widetilde{\mathbf{A}}_{T_\infty}$ to $\widetilde{\mathbf{A}}_{T_\infty}^\dagger$ and $\widetilde{\mathbf{A}}_{T_\infty'}/p^n\widetilde{\mathbf{A}}_{T_\infty'}$ to $\widetilde{\mathbf{A}}_{T_\infty'}^\dagger$.

Appendix B
Artin-Schreier theory

The aim of this section is to prove the following:

Proposition B.1. — *The map $\varphi-1$ on $\widetilde{\mathbf{A}}_{\overline{R}}$, $\widetilde{\mathbf{A}}_{\overline{R}}^\dagger$, $\mathbf{A}_{\overline{R}}$, $\mathbf{A}_{\overline{R}}^\dagger$, $\mathbf{A}_{\overline{R}}'$ and $\mathbf{A}_{\overline{R}}'^\dagger$ is surjective and its kernel is $\mathbf{Z}_p$. Furthermore, the exact sequence*

$$0 \longrightarrow \mathbf{Z}_p \longrightarrow \widetilde{\mathbf{A}}_{\overline{R}} \xrightarrow{\varphi-1} \widetilde{\mathbf{A}}_{\overline{R}} \longrightarrow 0 \tag{22}$$

admits a continuous right splitting $\sigma\colon \widetilde{\mathbf{A}}_{\overline{R}} \to \widetilde{\mathbf{A}}_{\overline{R}}$ (as $\mathbf{Z}_p$-modules) so that so that $\sigma(\mathbf{A}_{\overline{R}}) \subset \mathbf{A}_{\overline{R}}$, $\sigma(\widetilde{\mathbf{A}}_{\overline{R}}^\dagger) \subset \widetilde{\mathbf{A}}_{\overline{R}}^\dagger$, $\sigma(\mathbf{A}_{\overline{R}}^\dagger) \subset \mathbf{A}_{\overline{R}}^\dagger$, $\sigma(\mathbf{A}_{\overline{R}}') \subset \mathbf{A}_{\overline{R}}'$ and $\sigma(\mathbf{A}_{\overline{R}}'^\dagger) \subset \mathbf{A}_{\overline{R}}'^\dagger$.

Proof. — Note that by [**2**, Prop. 4.2] we have $\varphi(\mathbf{A}_{\overline{R}}^{(0,r]}) \subset \mathbf{A}_{\overline{R}}^{(0,r/p]}$ and $\varphi(\widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}) \subset \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r/p]}$ so that $(\varphi-1)(\mathbf{A}_{\overline{R}}^{\dagger}) \subset \mathbf{A}_{\overline{R}}^{\dagger}$ and $(\varphi-1)(\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}) \subset \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$. We know from 2.4 and 2.9 that p is a regular element of $\mathbf{A}_{\overline{R}}$, $\mathbf{A}'_{\overline{R}}$ and $\widetilde{\mathbf{A}}_{\overline{R}}$ and that $\mathbf{A}_{\overline{R}}/p\mathbf{A}_{\overline{R}} = \mathbf{E}_{\overline{R}}$, $\widetilde{\mathbf{A}}_{\overline{R}}/p\widetilde{\mathbf{A}}_{\overline{R}} = \widetilde{\mathbf{E}}_{\overline{R}}$ and that, thanks to 2.6, the image of $\mathbf{E}_{\overline{R}} \otimes_{\mathbf{E}_V} \widetilde{\mathbf{E}}_W \to \mathbf{A}'_{\overline{R}}/p\mathbf{A}'_{\overline{R}}$ is dense for the $\overline{\pi}$-adic topology. In particular, to prove that $\varphi-1$ on $\widetilde{\mathbf{A}}_{\overline{R}}$, $\mathbf{A}_{\overline{R}}$ and $\mathbf{A}'_{\overline{R}}$ is surjective and its kernel is $\mathbf{Z}_p$, it suffices to prove that the kernel of $\varphi-1$ on $\widetilde{\mathbf{E}}_{\overline{R}}$ is $\mathbf{F}_p$ and that $\varphi-1$ is surjective on $\mathbf{E}_{\overline{R}}$, on $\widetilde{\mathbf{E}}_{\overline{R}}$ and on the completion of $\mathbf{E}_{\overline{R}} \otimes_{\mathbf{E}_V} \widetilde{\mathbf{E}}_W$ for the $\overline{\pi}$-adic topology. Since $\widetilde{\mathbf{E}}_{\overline{R}}$ is an integral domain by 2.3(5), the kernel of $\varphi-1$ is $\mathbf{F}_p$. The other claim follows from B.2.

Since $\mathbf{A}_{\overline{R}}^{\dagger} = \mathbf{A}_{\overline{R}} \cap \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$, to conclude that $\varphi-1$ is surjective on $\mathbf{A}_{\overline{R}}^{\dagger}$ and on $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$, it suffices to prove that for every $z \in \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$ the solutions $y \in \widetilde{\mathbf{A}}_{\overline{R}}$ of $(\varphi-1)(y) = z$ lie in $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$. Since any such solutions differ by an element of $\mathbf{Z}_p$ and the latter is contained in $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$, it suffices to show that $\varphi-1$ is surjective on $\widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$. Let $z \in \widetilde{\mathbf{A}}_{\overline{R}}^{\dagger}$ and choose $r \in \mathbf{Q}_{>0}$ so that $z \in \widetilde{\mathbf{A}}_{\overline{R}}^{(0,r]}$. Write $z = \sum_k [z_k]p^k$ with $z_k \in \widetilde{\mathbf{E}}_{\overline{R}}$. Then, putting $c = \min\{-1, w_r(z)\}$, we have $r\mathbf{v}_{\mathbf{E}}(z_k) + k \geq c$ for every $k \in \mathbf{N}$ i. e., $\mathbf{v}_{\mathbf{E}}(z_k) \geq \frac{c-k}{r}$. By B.2 there exists $y_k \in \widetilde{\mathbf{E}}_{\overline{R}}$ such that $(\varphi-1)(y_k) = z_k$ and $0 \leq \mathbf{v}_{\mathbf{E}}(y_k) \leq \mathbf{v}_{\mathbf{E}}(z_k)$ if $\mathbf{v}_{\mathbf{E}}(z_k) \geq 0$ or $\mathbf{v}_{\mathbf{E}}(y_k) = \frac{\mathbf{v}_{\mathbf{E}}(z_k)}{p}$ if $\mathbf{v}_{\mathbf{E}}(z_k) < 0$. In any case, $\mathbf{v}_{\mathbf{E}}(y_k) \geq \frac{c-k}{pr}$. Hence, $y := \sum_k p^k[y_k]$ lies in $\widetilde{\mathbf{A}}_{\overline{R}}^{(0,pr]}$ and $(\varphi-1)(y) = z$. □

Lemma B.2. — *The map $\varphi-1$ is surjective on $\mathbf{E}_{\overline{R}}$, $\mathbf{E}_{\overline{R}}^{+}$, $\widetilde{\mathbf{E}}_{\overline{R}}$ and $\widetilde{\mathbf{E}}_{\overline{R}}^{+}$. Furthermore, given a and $b \in \widetilde{\mathbf{E}}_{\overline{R}}$ such that $a^p - a = b$ we have $0 \leq \mathbf{v}_{\mathbf{E}}(a) \leq \mathbf{v}_{\mathbf{E}}(b)$ if $\mathbf{v}_{\mathbf{E}}(b) \geq 0$; and $\mathbf{v}_{\mathbf{E}}(a) = \mathbf{v}_{\mathbf{E}}(b)/p$ if $\mathbf{v}_{\mathbf{E}}(b) < 0$.*

Proof. — Recall that $\mathbf{E}_{\overline{R}} := \cup_{S_\infty} \mathbf{E}_S$ (resp. $\mathbf{E}_{\overline{R}}^{+} := \cup_{S_\infty} \mathbf{E}_S^{+}$) and the union is taken over a maximal chain of finite normal extensions of $\mathbf{E}_R$ (resp. $\mathbf{E}_R^{+}$), étale after inverting $\overline{\pi}$. Then, $\mathrm{H}^1_{\mathrm{et}}(\mathbf{E}_{\overline{R}}, \mathbf{Z}/p\mathbf{Z}) = 0$ and $\mathrm{H}^1_{\mathrm{et}}(\mathbf{E}_{\overline{R}}^{+}, \mathbf{Z}/p\mathbf{Z}) = 0$, $\mathrm{H}^1_{\mathrm{et}}(\varphi^{-\infty}(\mathbf{E}_{\overline{R}}), \mathbf{Z}/p\mathbf{Z}) = 0$ and $\mathrm{H}^1_{\mathrm{et}}(\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+}), \mathbf{Z}/p\mathbf{Z}) = 0$. By Artin-Schreier theory $\mathbf{E}_{\overline{R}}/(\varphi-1)\mathbf{E}_{\overline{R}}$ injects in $\mathrm{H}^1_{\mathrm{et}}(\mathbf{E}_{\overline{R}}, \mathbf{Z}/p\mathbf{Z})$ and, hence, it is zero. Analogously, $\mathbf{E}_{\overline{R}}^{+}/(\varphi-1)\mathbf{E}_{\overline{R}}^{+} = 0$. This implies that $\varphi^{-\infty}(\mathbf{E}_{\overline{R}})/(\varphi-1)(\varphi^{-\infty}(\mathbf{E}_{\overline{R}})) = 0$ and $\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})/(\varphi-1)(\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})) = 0$. By 2.3 the ring $\widetilde{\mathbf{E}}_{\overline{R}}^{+}$ is the $\overline{\pi}$-adic completion of $\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})$ and $\widetilde{\mathbf{E}}_{\overline{R}} = \widetilde{\mathbf{E}}_{\overline{R}}^{+}[\overline{\pi}^{-1}]$. In particular, $\widetilde{\mathbf{E}}_{\overline{R}} = \varphi^{-\infty}(\mathbf{E}_{\overline{R}}) + \overline{\pi}\widetilde{\mathbf{E}}_{\overline{R}}^{+}$ and we are left to prove that given a power series $b = \sum_{n=1}^{+\infty} b_n \overline{\pi}^n$ with $\{b_n\}_n$ in $\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})$, we can solve the equation $(\varphi-1)(a) = b$. It suffices to find $\{a_n\}_n$ in $\varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})$ such that $\overline{\pi}^{(p-1)n}a_n^p - a_n = b_n$. Indeed, if we put $a := \sum_{n=1}^{\infty} a_n \overline{\pi}^n$, then $(\varphi-1)(a) = b$. Given $b_n \in \varphi^{-\infty}(\mathbf{E}_{\overline{R}}^{+})$ there exists S_∞ and m such that $b_n \in \varphi^{-m}(\mathbf{E}_S^{+})$. But $\mathbf{E}_S^{+}$ and $\varphi^{-m}(\mathbf{E}_S^{+})$ are $\overline{\pi}$-adically complete and separated, the equation $\overline{\pi}^{(p-1)n}X^p - X = b_n$ in the variable X has 1 as derivative and admits b_n as solution modulo $\overline{\pi}$. By Hensel's lemma it admits a unique solution a_n in $\varphi^{-m}(\mathbf{E}_S^+)$. The first part of the lemma follows.

Assume that $a^p - a = b$. Then, the properties of $\mathbf{v}_{\mathbf{E}}^{\leq 1}$ recalled in 2.4 imply that if $\mathbf{v}_{\mathbf{E}}(a) < 0$ we have $\mathbf{v}_{\mathbf{E}}(a^p) = p\mathbf{v}_{\mathbf{E}}(a) < \mathbf{v}_{\mathbf{E}}(a)$ and $\mathbf{v}_{\mathbf{E}}(b) = \mathbf{v}_{\mathbf{E}}(a^p - a) = p\mathbf{v}_{\mathbf{E}}(a)$. On the other hand, if $\mathbf{v}_{\mathbf{E}}(a) > 0$ we have $\mathbf{v}_{\mathbf{E}}(a^p) = p\mathbf{v}_{\mathbf{E}}(a) > \mathbf{v}_{\mathbf{E}}(a)$ and $\mathbf{v}_{\mathbf{E}}(b) = \mathbf{v}_{\mathbf{E}}(a^p - a) = \mathbf{v}_{\mathbf{E}}(a)$. If $\mathbf{v}_{\mathbf{E}}(a) = 0$, then $\mathbf{v}_{\mathbf{E}}(a^p - a) \geq 0$ and $\mathbf{v}_{\mathbf{E}}(b) \geq \mathbf{v}_{\mathbf{E}}(a)$. The second claim follows. □

Lemma B.3. — *For every m and $n \in \mathbf{N}$ we have $(\varphi-1)\big([\overline{\pi}]^n\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)+p^m\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)\big) = [\overline{\pi}]^n\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)+p^m\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)$ where $[\overline{\pi}]$ is the Teichmüller lift of $\overline{\pi}$. In particular, the map $\varphi - 1\colon \widetilde{\mathbf{A}}_{\overline{R}} \to \widetilde{\mathbf{A}}_{\overline{R}}$ is open for the weak topology.*

Proof. — By construction $\{[\overline{\pi}]^n\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+) + p^m\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)\}_{m,n}$ is a fundamental system of neighborhoods for the weak topology on $\widetilde{\mathbf{A}}_{\overline{R}}$. Since $\varphi - 1$ is linear, the first claim implies the second. Since $(\varphi - 1)(p^m a) = p^m(\varphi - 1)(a)$ for every $a \in \mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}})$, since $\varphi-1$ is surjective on $\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)$ by B.2 and since $\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+) = \mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)/p^m\mathbf{W}(\widetilde{\mathbf{E}}_{\overline{R}}^+)$, it is enough to prove that for every n we have $(\varphi-1)\big([\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)\big) = [\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)$. Indeed, $(\varphi - 1)\big([\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)\big) \subset \big([\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)\big)$ remarking that $(\varphi - 1)([\overline{\pi}]^n a) = [\overline{\pi}]^{pn}a^p - [\overline{\pi}]^n a = [\overline{\pi}]^n\big([\overline{\pi}]^{(p-1)n}a^p - a\big)$. On the other hand, $[\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+) \subset (\varphi - 1)\big([\overline{\pi}]^n\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)\big)$ since for every $b \in \mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)$ the equation $[\overline{\pi}]^{(p-1)n}X^p - X = b$ admits a solution modulo p (cf. proof of B.2) and, hence, in $\mathbf{W}_m(\widetilde{\mathbf{E}}_{\overline{R}}^+)$ by Hensel's lemma. The lemma follows. □

Lemma B.4. — *There exists a left inverse ρ as $\mathbf{Z}_p$-modules of the inclusion $\iota\colon \mathbf{Z}_p \to \widetilde{\mathbf{A}}_{\overline{R}}$ of (22), which is continuous for the weak topology.*

Proof. — Let R^* be the p-adic completion of the localization of R at the generic point of $R\otimes_V k$. We then have a map $\widetilde{\mathbf{A}}_{\overline{R}} \to \widetilde{\mathbf{A}}_{\overline{R^*}}$, which is continuous for the weak topology, so that it suffices to construct ρ for R^*. We may then assume that $R = R^*$ is a complete discrete valuation ring with residue field L. In particular, $\mathbf{E}_R$ is a discrete valuation field with valuation ring $\mathbf{E}_R^+$ and $\mathbf{A}_R$ is a complete discrete valuation ring with uniformizer p and residue field $\mathbf{E}_R$.

Recall that $\mathbf{E}_{\overline{R}}$ is the union $\cup_S \mathbf{E}_S$ over all finite normal extensions $R \subset S \subset \overline{R}$, étale after inverting p. Let $R \subset S$ be any such. Since R is a complete discrete valuation ring, also S is a complete discrete valuation ring. Then, $\mathbf{E}_S^+$ is a complete discrete valuation ring. For $S \subset T \subset \overline{R}$ finite normal extensions, étale after inverting p of degree $n_{S,T}$, we get that $\mathbf{E}_T^+$ is a finite and torsion free as $\mathbf{E}_S^+$-module, of rank $n_{S,T}$; see 2.3. By loc. cit. the choice of a $\mathbf{E}_S^+$-basis of $\mathbf{E}_T^+$ determines a $\varphi^{-m}(\mathbf{E}_S^+)$-basis of $\varphi^{-m}(\mathbf{E}_T^+)$ for every $m \in \mathbf{N}$ and maps

$$\overline{\pi}^{\frac{\ell_{S,T}}{p^m}}\widetilde{\mathbf{E}}_{T_\infty}^+ \to \varphi^{-m}(\mathbf{E}_T^+) \underset{\varphi^{-m}(\mathbf{E}_S^+)}{\otimes} \widetilde{\mathbf{E}}_{S_\infty}^+ \cong \big(\widetilde{\mathbf{E}}_{S_\infty}^+\big)^{n_{S,T}} \to \widetilde{\mathbf{E}}_{T_\infty}^+,$$

where $\ell_{S,T}$ is a constant depending on $S \subset T$. We thus get an isomorphism $\widetilde{\mathbf{E}}_{S_\infty}^{n_{S,T}} \to \widetilde{\mathbf{E}}_{T_\infty}$ as topological groups (for the $\overline{\pi}$-adic topology). Since $\mathbf{E}_S^+$ is integrally closed in $\mathbf{E}_T^+$, we may assume that the given $\mathbf{E}_S^+$-basis of $\mathbf{E}_T^+$ contains 1. Suppose furthermore that $\frac{\ell_{S,S'}}{p^m} < 1$. We then get a splitting of the inclusion $\widetilde{\mathbf{E}}_{S_\infty} \subset \widetilde{\mathbf{E}}_{T_\infty}$ as $\widetilde{\mathbf{E}}_{S_\infty}$-modules such that $\overline{\pi}\widetilde{\mathbf{E}}_{T_\infty}^+$ is mapped to $\widetilde{\mathbf{E}}_{S_\infty}^+$. Consider the set $\mathscr{F}$ of pairs (A, t) where A is a normal sub-$\widetilde{\mathbf{E}}_{R_\infty}$-algebra of $\cup_S \widetilde{\mathbf{E}}_{S_\infty}$ and $t\colon A \to \widetilde{\mathbf{E}}_{R_\infty}$ is a splitting of the inclusion $\widetilde{\mathbf{E}}_{R_\infty} \subset A$ as $\widetilde{\mathbf{E}}_{R_\infty}$-modules such that $t\left(A \cap \left(\overline{\pi} \cdot \cup_S \widetilde{\mathbf{E}}_{S_\infty}^+\right)\right) \subset \widetilde{\mathbf{E}}_{R_\infty}^+$. It is an ordered set in which every chain has a maximal element. Zorn's lemma implies that $\mathscr{F}$ has a maximal element which, by the discussion above, must coincide with $\cup_S \widetilde{\mathbf{E}}_{S_\infty}$. We conclude that there exists a left inverse ξ as $\widetilde{\mathbf{E}}_{R_\infty}$-modules of the inclusion $\widetilde{\mathbf{E}}_{R_\infty} \subset \cup_S \widetilde{\mathbf{E}}_{S_\infty}$ such that $\overline{\pi}\widetilde{\mathbf{E}}_{S_\infty}^+$ is mapped to $\widetilde{\mathbf{E}}_{R_\infty}^+$ for every S. Since $\widetilde{\mathbf{E}}_{\overline{R}}$ (resp. $\widetilde{\mathbf{E}}_{\overline{R}}^+$) is the $\overline{\pi}$-adic completion of $\cup_S \widetilde{\mathbf{E}}_{S_\infty}$ (resp. $\cup_S \widetilde{\mathbf{E}}_{S_\infty}^+$) and since $\widetilde{\mathbf{E}}_{R_\infty}$ (resp. $\widetilde{\mathbf{E}}_{R_\infty}^+$) is $\overline{\pi}$-adically complete and separated, ξ extends to a left inverse ζ as $\widetilde{\mathbf{E}}_{R_\infty}$-modules of the inclusion $\widetilde{\mathbf{E}}_{R_\infty} \subset \widetilde{\mathbf{E}}_{\overline{R}}$ mapping $\overline{\pi}\widetilde{\mathbf{E}}_{\overline{R}}^+$ to $\widetilde{\mathbf{E}}_{R_\infty}^+$. In particular, ζ is continuous for the $\overline{\pi}$-adic topology.

On the other hand, recall from 2.3 that $\mathbf{E}_R^+ = L \otimes_k k_\infty[\![\pi_K]\!]$ and that $\widetilde{\mathbf{E}}_{R_\infty}^+$ is the completion of $\cup_n \mathbf{E}_R^+ \left[\pi_K^{\frac{1}{p^n}}, x_1^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^n}}\right]$ for the topology defined by the fundamental system of neighborhoods $\left\{\pi_K^m \left(\cup_n L \otimes_k \mathbf{E}_V^+[\pi_K^{\frac{1}{p^n}}] \left[x_1^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^n}}\right]\right)\right\}_m$. Define

$$\delta\colon \cup_n L \mathop{\otimes}_k k_\infty(\!(\pi_K)\!) \left[\pi_K^{\frac{1}{p^n}}, x_1^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^n}}\right] \to L \mathop{\otimes}_k k_\infty$$

as the L-linear map sending $\pi_K^{i_0} x_1^{i_1} \cdots x_d^{i_d}$ to 0 for every $(i_0, i_1, \ldots, i_d) \in \mathbf{Q}^{d+1}$ such that $(i_1, \ldots, i_d)$ is not equal to 0 in $(\mathbf{Q}/\mathbf{Z})^d$. It is well defined since $\{\pi_K, x_1, \ldots, x_d\}$ is an absolute p-basis of $\mathbf{E}_R^+$. Furthermore, δ is continuous for the π_K-topology and, hence, it extends to a continuous left inverse ν as L-modules of the inclusion $L \otimes_k k_\infty \subset \widetilde{\mathbf{E}}_{R_\infty}$ considering the $\overline{\pi}$-adic topology on $\widetilde{\mathbf{E}}_{R_\infty}$ and the discrete topology on L. Finally, choose a left splitting τ as $\mathbf{F}_p$-vector spaces of $\mathbf{F}_p \subset L \otimes_k k_\infty$.

Let $\delta\colon \widetilde{\mathbf{A}}_{\overline{R}} \to \mathbf{Z}_p$ be the map sending a Witt vector $(a_0, \ldots, a_n, \ldots)$ of $\widetilde{\mathbf{A}}_{\overline{R}} = \mathbf{W}\big(\widetilde{\mathbf{E}}_{\overline{R}}\big)$ to $\big(\tau \circ \nu \circ \zeta(a_0), \ldots, \tau \circ \nu \circ \zeta(a_n), \ldots\big)$. It is a left inverse of the inclusion $\mathbf{Z}_p \subset \widetilde{\mathbf{A}}_{\overline{R}}$ and it is continuous for the weak topology on $\widetilde{\mathbf{A}}_{\overline{R}}$ and on $\mathbf{Z}_p$. Note that the topology induced on $\mathbf{Z}_p$ from the weak topology on $\widetilde{\mathbf{A}}_{\overline{R}}$ is the p-adic topology. The lemma follows. □

End of the proof of Proposition B.1. — With the notations of B.4, let $e := \iota \circ \rho\colon \widetilde{\mathbf{A}}_{\overline{R}} \to \widetilde{\mathbf{A}}_{\overline{R}}$. It is a continuous homomorphism of $\mathbf{Z}_p$-modules and $e^2 = e$. Thus, if $M := \mathrm{Ker}(e) = \mathrm{Im}(e-1)$, we have that M is closed in $\widetilde{\mathbf{A}}_{\overline{R}}$ and $\widetilde{\mathbf{A}}_{\overline{R}} = \mathbf{Z}_p \oplus M$. Then, $(\varphi - 1)|_M\colon M \to \widetilde{\mathbf{A}}_{\overline{R}}$ is bijective. It is open thanks to B.3. Hence, its inverse is a continuous homomorphism of $\mathbf{Z}_p$-modules. We let σ be the composite of $\big((\varphi-1)|_M\big)^{-1}$ and the inclusion $M \subset \widetilde{\mathbf{A}}_{\overline{R}}$. It satisfies the requirements of B.1. □

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