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Takuro Mochizuki

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T. Mochizuki
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan.
E-mail : takuro@math.kyoto-u.ac.jp

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To my parents

KOBAYASHI-HITCHIN CORRESPONDENCE FOR TAME HARMONIC BUNDLES AND AN APPLICATION

Takuro Mochizuki

Abstract. — We establish the correspondence between tame harmonic bundles and μ_L-polystable parabolic Higgs bundles with trivial characteristic numbers. We also show the Bogomolov-Gieseker type inequality for μ_L-stable parabolic Higgs bundles.

Then we show that any local system on a smooth quasiprojective variety can be deformed to a variation of polarized Hodge structure. As a consequence, we can conclude that some kind of discrete groups cannot be a split quotient of the fundamental group of a smooth quasiprojective variety.

***Résumé* (La correspondance de Kobayashi-Hitchin pour les fibrés harmoniques modérés et une application)**

Nous établissons la correspondance de Kobayashi-Hitchin entre les fibrés harmoniques modérés et fibrés de Higgs paraboliques μ_L-polystables dont les deux premiers nombres de Chern sont nuls. Ensuite, nous montrons que tout système local sur une variété quasi-projective lisse peut être déformé vers une variation de structure de Hodge polarisée. En conséquence, nous pouvons conclure que certains groupes discrets ne peuvent pas apparaître comme quotient scindé d'un groupe fondamental d'une variété quasi-projective lisse.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Background

1.1.1. Kobayashi-Hitchin correspondence. — We briefly recall some aspects of the so-called Kobayashi-Hitchin correspondence. (See the introduction of [**38**] for more detail.) In 1960's, M. S. Narasimhan and C. S. Seshadri proved the correspondence between irreducible flat unitary bundles and stable vector bundles with degree 0, on a compact Riemann surface ([**47**]). Clearly, it was desired to extend their result to the higher dimensional case and the non-flat case.

In early 1980's, S. Kobayashi introduced the Hermitian-Einstein condition for holomorphic bundles on Kahler manifolds ([**30**], [**31**]). He and M. Lübke ([**37**]) proved that the existence of Hermitian-Einstein metric implies the polystability of the underlying holomorphic bundle. S. K. Donaldson pioneered the way for the inverse problem ([**12**] and [**13**]). He attributed the problem to Kobayashi and N. Hitchin. The definitive result was given by K. Uhlenbeck, S. T. Yau and Donaldson ([**64**] and [**14**]). We also remark that V. Mehta and A. Ramanathan ([**40**]) proved the correspondence in the case where the Chern class is trivial, i.e., the correspondence of flat unitary bundles and stable vector bundles with trivial Chern classes.

On the other hand, it was quite fruitful to consider the correspondences for vector bundles with some additional structures like Higgs fields, which was initiated by Hitchin ([**22**]). He studied the Higgs bundles on a compact Riemann surface and the moduli spaces. His work has influenced various fields of mathematics. It involves a lot of subjects and ideas, and one of his results is the correspondence of the stability and the existence of Hermitian-Einstein metrics for Higgs bundles on a compact Riemann surface.

1.1.2. A part of C. Simpson's work. — C. Simpson studied the Higgs bundles over higher dimensional complex manifolds, influenced by the work of Hitchin, but motivated by his own subject: Variation of Polarized Hodge Structure. He made great innovations in various areas of algebraic geometry. Here, we recall just a part of his huge work.

Let X be a smooth irreducible projective variety over the complex number field, and E be an algebraic vector bundle on X. Let (E, θ) be a Higgs bundle, i.e., θ is a holomorphic section of $\mathrm{End}(E) \otimes \Omega_X^{1,0}$ satisfying $\theta^2 = 0$. The "stability" and the "Hermitian-Einstein metric" are naturally defined for Higgs bundles, and Simpson proved that there exists a Hermitian-Einstein metric of (E, θ) if and only if (E, θ) is polystable. In the special case where the Chern class of the vector bundle is trivial, the Hermitian-Einstein metric gives the pluri-harmonic metric. Together with the result of K. Corlette who is also a great progenitor of the study of harmonic bundles ([**6**]), Simpson obtained the Trinity on a smooth projective variety:

(1)	**Algebraic Geometry** polystable Higgs bundle (trivial Chern class)	$\leftrightarrow$	**Differential Geometry** harmonic bundle	$\leftrightarrow$	**Topology** semisimple local system

If (E, θ) is a stable Higgs bundle, then $(E, \alpha\cdot\theta)$ is also a stable Higgs bundle. Hence we obtain the family of stable Higgs bundles $\{(E, \alpha\cdot\theta) \mid \alpha \in \boldsymbol{C}^*\}$. Correspondingly, we obtain the family of flat bundles $\{L_\alpha \mid \alpha \in \boldsymbol{C}^*\}$. Simpson showed that we obtain the variation of polarized Hodge structure as a limit $\lim_{\alpha\to 0} L_\alpha$. In particular, it can be concluded that any flat bundle can be deformed to a variation of polarized Hodge structure. As one of the applications, he obtained the following remarkable result ([**55**]):

***Theorem 1.1* (Simpson)**. — *Let Γ be a rigid discrete subgroup of a real algebraic group which is not of Hodge type. Then Γ cannot be a split quotient of the fundamental group of a smooth irreducible projective variety.*

There are classical known results on the rigidity of subgroups of Lie groups. The examples of rigid discrete subgroups can be found in 4.7.1–4.7.4 in the 53 page of [**55**]. The classification of real algebraic group of Hodge type was done by Simpson. The examples of real algebraic group which is not of Hodge type can be found in the 50 page of [**55**]. As a corollary, he obtained the following.

Corollary 1.2. — *$SL(n, \mathbb{Z})$ $(n \geq 3)$ cannot be a split quotient of the fundamental group of a smooth irreducible projective variety.*

1.2. Main Purpose

1.2.1. Kobayashi-Hitchin correspondence for parabolic Higgs bundles

It is an important and challenging problem to generalize the correspondence (1) to the quasiprojective case from the projective case. As for the correspondence of harmonic bundles and semisimple local systems, an excellent result was obtained by J. Jost and K. Zuo [**29**], which says there exists a tame pluri-harmonic metric on any semisimple local system over a quasiprojective variety. The metric is called the Corlette-Jost-Zuo metric.

In this paper, we restrict ourselves to the correspondence between Higgs bundles and harmonic bundles on a quasiprojective variety Y. More precisely, we should consider not Higgs bundles on Y but *parabolic* Higgs bundles on (X, D), where (X, D) is a pair of a smooth irreducible projective variety and a normal crossing divisor such that $Y = X - D$. Such a generalization has been studied by several people. In the non-Higgs case, J. Li [**35**] and B. Steer-A. Wren [**62**] established the correspondence. In the Higgs case, Simpson established the correspondence in the one dimensional case [**52**], and O. Biquard established it in the case where D is smooth [**5**].

Remark 1.3. — Their results also include the correspondence in the case where the characteristic numbers are non-trivial.

For applications, however, it is desired that the correspondence for parabolic Higgs bundles should be given in the case where D is not necessarily smooth, which we would like to discuss in this paper.

We explain our result more precisely. Let X be a smooth irreducible projective variety over the complex number field provided an ample line bundle L. Let D be a simple normal crossing divisor of X. The main purpose of this paper is to establish the correspondence between tame harmonic bundles and μ_L-parabolic Higgs bundles whose characteristic numbers vanish. (See Chapter 3 for the meaning of the words.)

***Theorem 1.4* (Proposition 5.1–5.3, and Theorem 9.4)**. — *Let $\big(\boldsymbol{E}_*, \theta\big)$ be a regular filtered Higgs bundle on (X, D), and we put $E := \boldsymbol{E}_{|X-D}$. It is μ_L-polystable with trivial characteristic numbers, if and only if there exists a pluri-harmonic metric h of (E, θ) on $X - D$ which is adapted to the parabolic structure. Such a metric is unique up to an obvious ambiguity.*

Remark 1.5. — Regular Higgs bundles and parabolic Higgs bundles are equivalent. See Chapter 3.

Remark 1.6. — More precisely on the existence result, we can show the existence of the adapted pluri-harmonic metric for μ_L-stable reflexive saturated regular filtered Higgs sheaf on (X, D) with trivial characteristic numbers. (See Sections 3.1–3.2 for the definition.) Then, due to our previous result in [**44**], it is a regular filtered Higgs bundle on (X, D), in fact.

We are mainly interested in the μ_L-stable parabolic Higgs bundles whose characteristic numbers vanish. But we also obtain the following theorem on more general μ_L-stable parabolic Higgs bundles.

***Theorem 1.7* (Theorem 6.5)**. — *Let X be a smooth irreducible projective variety of an arbitrary dimension, and D be a simple normal crossing divisor. Let L be an ample* *line bundle on X. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-stable regular filtered Higgs bundle in codimension two on (X, D). Then the following inequality holds:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) - \frac{\int_X \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2 \operatorname{rank} E} \leq 0.$$

Such an inequality is called Bogomolov-Gieseker inequality.

1.2.2. Strategy for the proof of Bogomolov-Gieseker inequality. — We would like to explain our strategy for the proof of the main theorems. First we describe an outline for Bogomolov-Gieseker inequality (Theorem 1.7), which is much easier. We have only to consider the case $\dim X = 2$. Essentially, it consists of the following two parts.

(1) The correspondence in the graded semisimple case :
We establish the Kobayashi-Hitchin correspondence for *graded semisimple* parabolic Higgs bundles. In particular, we obtain the Bogomolov-Gieseker inequality in this case.

(2) Perturbation of the parabolic structure and taking the limit :
Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a given $\boldsymbol{c}$-parabolic μ_L-stable Higgs bundle, which is not necessarily graded semisimple. For any small positive number ϵ, we take a perturbation $\boldsymbol{F}^{(\epsilon)}$ of $\boldsymbol{F}$ such that $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is a graded semisimple μ_L-stable parabolic Higgs bundle. Then the Bogomolov-Gieseker inequality holds for $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$. By taking a limit for $\epsilon \longrightarrow 0$, we obtain the Bogomolov-Gieseker inequality for the given $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

Let us describe for more detail.

(1) In [**55**], Simpson constructed a Hermitian-Einstein metric for Higgs bundle by the following process:

(i) : Take an appropriate initial metric.
(ii) : Deform it along the heat equation.
(iii) : Take a limit, and then we obtain the Hermitian-Einstein metric.

If the base space is compact, the steps (ii) and (iii) are the main issues, and the step (i) is trivial. Actually, Simpson also discussed the case where the base Kahler manifold is non-compact, and he showed the existence of a Hermitian-Einstein metric if we can take an initial metric whose curvatures satisfy some finiteness condition. (See Section 2.2 for more precise statements.) So, for a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ on (X, D), where X is a smooth projective surface and D is a simple normal crossing divisor, ideally, we would like to take an initial metric of $E := {}_{\boldsymbol{c}}E_{|X-D}$ adapted to the parabolic structure. But, it is rather difficult, and the author is not sure whether such a good metric can always be taken for any parabolic Higgs bundles. It seems one of the main obstacles to establish the Kobayashi-Hitchin correspondence for parabolic Higgs bundles.

However, we can easily take such a good initial metric, if we assume the vanishing of the nilpotent part of the residues of the Higgs field on the graduation of the parabolic filtration. Such a parabolic Higgs bundle will be called *graded semisimple* in this paper. We first establish the correspondence in this easy case. (Proposition 6.1).
(2) Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X, D), where $\dim X = 2$. We take a perturbation of $\boldsymbol{F}^{(\epsilon)}$ as in Section 3.3. In particular, $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is a μ_L-stable graded semisimple $\boldsymbol{c}$-parabolic Higgs bundle, and the following holds:

$$\text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}),$$

$$\left| \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}) - \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) \right| \leq C \cdot \epsilon.$$

Then we obtain the Bogomolov-Gieseker inequality for $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ by using the Hermitian-Einstein metric obtained in (1). By taking the limit $\epsilon \to 0$, we obtain the desired inequality for the given $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

1.2.3. Strategy for the proof of Kobayashi-Hitchin correspondence. — Let X be a smooth projective surface, and D be a simple normal crossing divisor. Let L be an ample line bundle on X, and ω be the Kahler form representing $c_1(L)$. Roughly speaking, the correspondence on (X, D) as in Theorem 1.4 can be divided into the following two parts:

- For a given tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$, we obtain the μ_L-polystable parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ with the trivial characteristic numbers.
- On the converse, we obtain a pluri-harmonic metric of $(E, \overline{\partial}_E, \theta)$ on $X - D$ for such $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

As for the first issue, most problem can be reduced to the one dimensional case, which was established by Simpson [**52**]. However, we have to show the vanishing of the characteristic numbers, for which our study of the asymptotic behaviour of tame harmonic bundles ([**44**]) is useful.

As for the second issue, we use the perturbation method, again. Namely, let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). Take a perturbation $\boldsymbol{F}^{(\epsilon)}$ of the filtration $\boldsymbol{F}$ for a small positive number ϵ. We also take metrics appropriate ω_ϵ of $X - D$ such that $\lim_{\epsilon \to 0} \omega_\epsilon = \omega$, and then we obtain Hermitian-Einstein metrics h_ϵ for the Higgs bundle $(E, \overline{\partial}_E, \theta)$ on $X - D$ with respect to ω_ϵ, which is adapted to the parabolic structure $\boldsymbol{F}^{(\epsilon)}$. Ideally, we would like to consider the limit $\lim_{\epsilon \to 0} h_\epsilon$, and we expect that the limit gives the Hermitian-Einstein metric h for $(E, \overline{\partial}_E, \theta)$ with respect to ω, which is adapted to the given filtration $\boldsymbol{F}$. Perhaps, it may be correct, but it does not seem easy to show, in general.

We restrict ourselves to the simpler case where the characteristic numbers of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ are trivial. Under this assumption, we show such a convergence. More precisely, we show that there is a subsequence $\{\epsilon_i\}$ such that $\big\{(E,\overline{\partial}_E,\theta,h_{\epsilon_i}\big\}$ converges to a harmonic bundle $(E',\overline{\partial}_{E'},\theta',h')$ on $X-D$, and we show that the given $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)$ is isomorphic to the parabolic Higgs bundles obtained from $(E',\overline{\partial}_{E'},\theta',h')$.

Remark 1.8. — We obtained a similar correspondence for λ-connections in [**46**]. Although the essential ideas are same, we need some additional argument in the case of λ-connections.

1.3. Additional Results

1.3.1. Torus action and the deformation of a G-flat bundle. — Once Theorem 1.4 is established, we can use some of the arguments for the applications given in the projective case. For example, we can deform any flat bundle to the one which comes from a variation of polarized Hodge structure. We follow the well known framework given by Simpson with a minor modification. We briefly recall it, and we will mention the problem that we have to care about in the process.

Let X be a smooth irreducible projective variety, and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S}D_i$. Let x be a point of $X-D$. Let Γ denote the fundamental group $\pi_1(X-D,x)$. Any representation of Γ can be deformed to a semisimple representation, and hence we start with a semisimple one.

Let (E,∇) be a flat bundle over $X-D$ such that the induced representation $\rho:\Gamma\longrightarrow \mathrm{GL}(E_{|x})$ is semisimple. Recall we can take a Corlette-Jost-Zuo metric of (E,∇), as mentioned in Subsection 1.2.1. Hence we obtain a tame pure imaginary harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $X-D$, and the induced μ_L-polystable $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)$ on (X,D), where $\boldsymbol{c}$ denotes any element of $\boldsymbol{R}^S$. We have the canonical decomposition $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)=\bigoplus_i({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,\theta_i)^{\oplus m_i}$, where each $({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,\theta_i)$ is μ_L-stable.

Let us consider the family of $\boldsymbol{c}$-parabolic Higgs bundles $\big({}_{\boldsymbol{c}}E,\boldsymbol{F},t\cdot\theta\big)$ for $t\in\boldsymbol{C}^*$, which are μ_L-polystable. Due to the standard Langton's trick [**33**], we have the semistable $\boldsymbol{c}$-parabolic Higgs sheaves $({}_{\boldsymbol{c}}\widetilde{E}_i,\widetilde{\boldsymbol{F}}_i,\widetilde{\theta}_i)$ which are limits of $\big({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,t\cdot\theta_i\big)$ in $t\to 0$. On the other hand, we can take a pluri-harmonic metric h_t of the Higgs bundle $(E,\overline{\partial}_E,t\cdot\theta)$ on $X-D$ for each t, which is adapted to the parabolic structure. (Theorem 1.4). Then we obtain the family of flat bundles $(E,\mathbb{D}^1_t)$, and the associated family of the representations $\big\{\rho_t:\Gamma\longrightarrow\mathrm{GL}(E_{|x})\,\big|\,t\in\boldsymbol{C}^*\big\}$. Since $(E,\overline{\partial}_E,t\cdot\theta,h_t)$ is tame pure imaginary in the case $t\in\boldsymbol{R}_{>0}$, the representations ρ_t are semisimple. The family $\{\rho_t\,|\,t\in\boldsymbol{C}^*\}$ should be continuous with respect to t, and the limit $\lim_{t\to 0}\rho_t$ should exist, ideally. We formulate the continuity of ρ_t with respect to t and the convergence of ρ_t in $t\to 0$, as follows. Let V be a $\boldsymbol{C}$-vector space such that $\mathrm{rank}(V)=\mathrm{rank}(E)$. Let h_V denote the metric of V, and let $U(h_V)$ denote the unitary group for h_V. We put $R(\Gamma,V):=Hom(\Gamma,\mathrm{GL}(V))$. By the conjugate, $U(h_V)$ acts on the space

$R(\Gamma, V)$. Let $M(\Gamma, V, h_V)$ denote the usual quotient space. Let $\pi_{\mathrm{GL}(V)} : R(\Gamma, V) \longrightarrow M(\Gamma, V, h_V)$ denote the projection.

By taking any isometry $(E_{|x}, h_{t|x}) \simeq (V, h_V)$, we obtain the representation $\rho'_t : \Gamma \longrightarrow \mathrm{GL}(V)$. We put $\mathcal{P}(t) := \pi_{\mathrm{GL}(V)}(\rho'_t)$, and we obtain the map $\mathcal{P} : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$. It is well defined. Then, we obtain the following partial result.

Proposition 1.9 **(Theorem 10.1, Lemma 10.2, Proposition 10.3)**

1. *The induced map $\mathcal{P}$ is continuous.*
2. $\mathcal{P}\bigl(\{0 < t \leq 1\}\bigr)$ *is relatively compact in* $M(\Gamma, V, h_V)$.
3. *If each* $\bigl({}_{\boldsymbol{c}}\widetilde{E}_i, \widetilde{\boldsymbol{F}}_i, \widetilde{\theta}_i\bigr)$ *is stable, then the limit* $\lim_{t\to 0} \mathcal{P}(t)$ *exists, and the limit flat bundle underlies the variation of polarized Hodge structure. As a result, we can deform any flat bundle to a variation of polarized Hodge structure.*

We would like to mention the point which we will care about. For simplicity, we assume $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ is μ_L-stable, and $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}, t\cdot\theta\bigr)$ converges to the μ_L-stable parabolic Higgs bundle $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$. Let $\{t_i\}$ denote a sequence converging to 0. By taking an appropriate subsequence, we may assume that the sequence $\{(E, \overline{\partial}_E, h_{t_i}, t_i \cdot \theta_i)\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, h', \theta')$ weakly in L_2^p locally over $X - D$, which is due to Uhlenbeck's compactness theorem and the estimate for the Higgs fields. Then we obtain the induced parabolic Higgs bundle $({}_{\boldsymbol{c}}E', \boldsymbol{F}', \theta')$. We would like to show that $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$ and $({}_{\boldsymbol{c}}E', \boldsymbol{F}', \theta')$ are isomorphic. Once we have known the existence of a non-trivial map $G : {}_{\boldsymbol{c}}E' \longrightarrow {}_{\boldsymbol{c}}\widetilde{E}$ which is compatible with the parabolic structure and the Higgs field, it is isomorphic due to the stability of $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$. Hence the existence of such G is the main issue for this argument. We remark that the problem is rather obvious if D is empty.

Remark 1.10. — Even if $({}_{\boldsymbol{c}}\widetilde{E}_i, \widetilde{\boldsymbol{F}}_i, \widetilde{\theta}_i)$ are not μ_L-stable, the conclusion in the third claim of Proposition 1.9 should be true. In fact, Simpson gave a detailed argument to show it, in the case where D is empty ([**56**], [**57**]). More strongly, he obtained the homeomorphism of the coarse moduli spaces of semistable flat bundles and semistable Higgs bundles.

In this paper, we do not discuss the moduli spaces, and hence we omit to discuss the general case. Instead, we use an elementary inductive argument on the rank of local systems, which is sufficient to obtain a deformation to a variation of polarized Hodge structure. However, it would be desirable to arrive at the thorough understanding as Simpson's work, in future.

Remark 1.11. — For an application, we have to care about the relation between the deformation and the monodromy groups. We will discuss only a rough relation in Section 10.2. More precise relation will be studied elsewhere.

Once we can deform any local system on a smooth quasiprojective variety to a variation of polarized Hodge structure, preserving some compatibility with the monodromy group, we obtain the following corollary. It is a natural generalization of Theorem 1.1.

Corollary 1.12. — *Let* Γ *be a rigid discrete subgroup of a real algebraic group which is not of Hodge type. Then* Γ *cannot be a split quotient of the fundamental groups of any smooth irreducible quasiprojective variety.*

Remark 1.13. — Such a deformation of flat bundles on a quasiprojective variety was also discussed in [**28**] in a different way.

1.3.2. Tame pure imaginary pluri-harmonic reduction (Appendix). — Let G be a linear algebraic group defined over $\boldsymbol{C}$ or $\boldsymbol{R}$. We will discuss a characterization of reductive representations $\pi_1(X-D,x) \longrightarrow G$ via the existence of tame pure imaginary pluri-harmonic reduction. Here a representation is called reductive, if the Zariski closure of the image is reductive. Such a kind of characterization was given by Jost and Zuo ([**29**]) directly for G, although their definition of reductivity looks different from ours. It is our purpose to explain that the problem can be reduced to the case $G = GL(n)$ by Tannakian consideration. Some results are used in Chapter 10.

1.4. Outline

Chapter 2 is an elementary preparation for the discussion in the later chapters. The reader can skip this chapter. Chapter 3 is preparation about parabolic Higgs bundles. We discuss the perturbation of a given filtration in Section 3.3, which is one of the keys in this paper.

In Chapter 4, an ordinary metric for parabolic Higgs bundle is given. We follow the construction in [**35**] and [**36**]. Our purpose is to establish the relation between the parabolic characteristic numbers and some integrals, in the case of graded semisimple parabolic Higgs bundles.

In Chapter 5, we show the fundamental properties of the parabolic Higgs bundles obtained from tame harmonic bundles. Namely, we show the μ_L-stability and the vanishing of the characteristic numbers. In Chapter 6, we show the preliminary Kobayashi-Hitchin correspondence for graded semisimple parabolic Higgs bundles. Bogomolov-Gieseker inequality can be obtained as an easy corollary of this preliminary correspondence and the perturbation argument of the parabolic structure.

In Chapter 7, we construct a frame around the origin for a tame harmonic bundle on a punctured disc. It is a technical preparation to discuss the convergence of a sequence of tame harmonic bundles. Such a convergence is shown in Chapter 8. We also give a preparation for the existence theorem of pluri-harmonic metric, which is completed in Chapter 9.

Once the Kobayashi-Hitchin correspondence for tame harmonic bundles is established, we can apply Simpson's argument of the torus action, and we can obtain some topological consequence of quasiprojective varieties. It is explained in Chapter 10. Chapter 10.2.3 is regarded as an appendix, in which we recall something related to pluri-harmonic metrics of G-flat bundles.

1.5. Acknowledgement

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CHAPTER 2

PRELIMINARY

This chapter is a preparation for the later discussions. We will often use the notation given in Sections 2.1–2.2, especially.

2.1. Notation and Words

We use the notation $\mathbb{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ to denote the set of integers, rational numbers, real numbers and complex numbers, respectively. For a real number a, we put $\boldsymbol{R}_{>a} := \{x \in \boldsymbol{R} \,|\, x > a\}$. We use the notation $\mathbb{Z}_{>a}$, $\mathbb{Z}_{\geq a}$, $\boldsymbol{Q}_{>a}$, etc. in a similar meaning.

For real numbers a, b, we put as follows:

$$[a,b] := \{x \in \boldsymbol{R} \,|\, a \leq x \leq b\} \quad [a,b[:= \{x \in \boldsymbol{R} \,|\, a \leq x < b\}$$
$$]a,b] := \{x \in \boldsymbol{R} \,|\, a < x \leq b\} \quad]a,b[:= \{x \in \boldsymbol{R} \,|\, a < x < b\}$$

The notation $\delta_{i,j}$ will be Kronecker's delta, i.e., $\delta_{i,j} = 1$ $(i = j)$ and $\delta_{i,j} = 0$ $(i \neq j)$.

A normal crossing divisor D of a complex manifold X will be called *simple*, if each irreducible component is non-singular. Let $D = \bigcup_{i \in S} D_i$ be the irreducible decomposition. For elements $\boldsymbol{a} \in \boldsymbol{R}^S$, a_i will denote the i-th component of $\boldsymbol{a}$ $(i \in S)$. The notation ${}_{\boldsymbol{a}}E$ is often used to denote a vector bundle on X, and we often put $E := {}_{\boldsymbol{a}}E_{|X-D}$.

Let Y be a manifold, E be a vector bundle on Y, and $\{f_i\}$ be a sequence of sections of E. We say $\{f_i\}$ converges to f weakly in L_l^p locally on Y, if the restriction $\{f_{i\,|\,K}\}$ converges to $f_{|K}$ weakly in $L_l^p(K)$ for any compact subset $K \subset Y$.

Let $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \theta^{(i)})\big\}$ be a sequence of Higgs bundles on Y. We say that the sequence $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \theta^{(i)})\big\}$ converges to $(E^{(\infty)}, \overline{\partial}^{(\infty)}, \theta^{(\infty)})$ weakly in L_2^p (resp. in C^1) locally on Y, if there exist locally L_2^p-isomorphisms (resp. C^1-isomorphisms) $\Phi^{(i)} : E^{(i)} \longrightarrow E^{(\infty)}$ on Y such that the sequences $\{\Phi^{(i)}(\overline{\partial}^{(i)})\}$ and $\{\Phi^{(i)}(\theta^{(i)})\}$ weakly converge to $\overline{\partial}^{(\infty)}$ and $\theta^{(\infty)}$ respectively in L_1^p (resp. C^0) locally on Y.

Let E be a vector bundle on Y with a hermitian metric h. For an operator $F \in \mathrm{End}(E) \otimes \Omega_Y^{p,q}$, we use the notation $F_h^{\dagger} \in \mathrm{End}(E) \otimes \Omega_Y^{q,p}$ to denote the adjoint of F with respect to h. The notation $F^{\dagger}$ is often used, if there are no risk of confusion.

Let (S_i, φ_i) $(i = 1, 2, \ldots, \infty)$ be a pair of discrete subsets $S_i \subset \boldsymbol{R}$ and functions $\varphi_i : S_i \longrightarrow \mathbb{Z}_{>0}$. We say that $\big\{(S_i, \varphi_i) \,\big|\, i = 1, 2, \ldots\big\}$ converges to $(S_\infty, \varphi_\infty)$, if there exists i_0 for any $\epsilon > 0$ such that (i) any $b \in S_i$ $(i > i_0)$ is contained in $]a - \epsilon, a + \epsilon[$ for some $a \in S_\infty$, (ii) $\sum_{b \in S_i, |a-b|<\epsilon} \varphi_i(b) = \varphi_\infty(a)$ is satisfied.

2.2. Review of some Results of Simpson on Kobayashi-Hitchin Correspondence

2.2.1. Analytic stability and Hermitian-Einstein metric. — We recall some results in [**51**]. Let Y be an n-dimensional connected complex manifold which is not necessarily compact. Let ω be a Kahler form of Y. The adjoint for the multiplication of ω is denoted by Λ_ω, or simply by Λ if there are no confusion. The Laplacian for ω is denoted by Δ_ω.

Condition 2.1

1. The volume of Y with respect to ω is finite.
2. There exists an exhaustion function ϕ on Y such that $0 \leq \sqrt{-1}\partial\overline{\partial}\phi \leq C \cdot \omega$ for some positive constant C.
3. There exists an increasing function $\boldsymbol{R}_{\geq 0} \longrightarrow \boldsymbol{R}_{\geq 0}$ such that $a(0) = 0$ and $a(x) = x$ for $x \geq 1$, and the following holds:
   - Let f be a positive bounded function on Y such that $\Delta_\omega f \leq B$ for some positive number B. Then $\sup_Y |f| \leq C(B) \cdot a\Big(\int_Y f\Big)$ for some positive constant $C(B)$ depending on B. Moreover $\Delta_\omega f \leq 0$ implies $\Delta_\omega f = 0$.

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on Y. Let h be a hermitian metric of E. Then we have the $(1,0)$-operator ∂_E determined by $\overline{\partial} h(u, v) = h\big(\overline{\partial}_E u, v\big) + h\big(u, \partial_E v\big)$. We also have the adjoint $\theta^{\dagger}$. If we emphasize the dependence on h, we use the notation $\partial_{E,h}$ and $\theta_h^{\dagger}$. We obtain the connections $D_h := \overline{\partial}_E + \partial_E$ and $\mathbb{D}^1 := D_h + \theta + \theta^{\dagger}$. The curvatures of D_h and $\mathbb{D}^1$ are denoted by $R(h)$ and $F(h)$ respectively. When we emphasize the dependence on $\overline{\partial}_E$, they are denoted by $R(\overline{\partial}_E, h)$ and $F(\overline{\partial}_E, h)$. We also use $R(E, h)$ and $F(E, h)$, if we emphasize the bundle.

Condition 2.2. — $F(h)$ is bounded with respect to h and ω.

When Condition 2.2 is satisfied, we put as follows:

$$\deg_\omega(E, h) := \frac{\sqrt{-1}}{2\pi} \int_Y \mathrm{tr}\big(F(h)\big) \cdot \omega^{n-1} = \frac{\sqrt{-1}}{2\pi} \int_Y \mathrm{tr}\,\Lambda\big(F(h)\big) \cdot \frac{\omega^n}{n}$$

Note $\mathrm{tr}\, F(h) = \mathrm{tr}\, R(h)$. Recall that a subsheaf $V \subset E$ is called saturated if the quotient E/V is torsion-free. For any saturated Higgs subsheaf $V \subset E$, there is a

Zariski closed subset Z of codimension two such that $V_{|Y-Z}$ gives a subbundle of $E_{|Y-Z}$, on which the metric h_V of $V_{|Y-Z}$ is induced. Let π_V denote the orthogonal projection of $E_{|Y-Z}$ onto $V_{|Y-Z}$. Let tr_V denote the trace for endomorphisms of V.

***Proposition 2.3* ([51] Lemma 3.2).** — *When the conditions* 2.1 *and* 2.2 *are satisfied, the integral*

$$\deg_\omega(V,h_V) := \frac{\sqrt{-1}}{2\pi}\int_Y \mathrm{tr}_V\big(F(h_V)\big)\cdot\omega^{n-1}$$

is well defined, and it takes the value in $\boldsymbol{R}\cup\{-\infty\}$. *The Chern-Weil formula holds as follows, for some positive number* C*:*

$$\deg_\omega(V,h_V) = \frac{\sqrt{-1}}{2\pi}\int_Y \mathrm{tr}\Big(\pi_V\circ\Lambda_\omega F(h)\Big)\cdot\frac{\omega^n}{n} - C\int_Y \big|D''\pi_V\big|_h^2\cdot\mathrm{dvol}_\omega\,.$$

Here we put $D''=\overline{\partial}_E+\theta$. *In particular, if the value* $\deg_\omega(V,h_V)$ *is finite,* $\overline{\partial}_E(\pi_V)$ *and* $[\theta,\pi_V]$ *are* L^2.

For any $V\subset E$, we put $\mu_\omega(V,h_V) := \deg_\omega(V,h_V)/\operatorname{rank} V$.

***Definition 2.4* ([51]).** — A metrized Higgs bundle $(E,\overline{\partial}_E,\theta,h)$ is called analytic stable, if the inequalities $\mu_\omega(V,h_V) < \mu_\omega(E,h)$ hold for any non-trivial Higgs saturated subsheaves $(V,\theta_V)\subsetneq(E,\theta)$.

The following important theorem is crucial for our argument.

***Proposition 2.5* (Simpson).** — *Let* (Y,ω) *be a Kahler manifold satisfying Condition* 2.1, *and let* $(E,\overline{\partial}_E,\theta,h_0)$ *be a metrized Higgs bundle satisfying Condition* 2.2. *If it is analytic stable, then there exists a hermitian metric* $h=h_0\cdot s$ *satisfying the following conditions:*

- h *and* h_0 *are mutually bounded.*
- $\det(h)=\det(h_0)$. *In particular, we have* $\mathrm{tr}\,F(h)=\mathrm{tr}\,F(h_0)$.
- $D''(s)$ *is* L^2 *with respect to* h_0 *and* ω.
- *It satisfies the Hermitian-Einstein condition* $\Lambda_\omega F(h)^\perp=0$, *where* $F(h)^\perp$ *denotes the trace free part of* $F(h)$.
- *The following equalities hold:*

$$\int_Y \mathrm{tr}\Big(F(h)^2\Big)\cdot\omega^{n-2} = \int_Y \mathrm{tr}\Big(F(h_0)^2\Big)\cdot\omega^{n-2}, \tag{2}$$

$$\int_Y \mathrm{tr}\Big(F(h)^{\perp\,2}\Big)\cdot\omega^{n-2} = \int_Y \mathrm{tr}\Big(F(h_0)^{\perp\,2}\Big)\cdot\omega^{n-2}. \tag{3}$$

Proof. — Condition 2.2 implies $\Lambda_\omega F(h)$ is bounded. Applying Theorem 1 in [**51**], we obtain the hermitian metric h satisfying the first four conditions. Due to Proposition 3.5 in [**51**], we obtain the inequality $\int_Y \mathrm{tr}\big(F(h)^2\big)\cdot\omega^{n-2}\leq\int_Y \mathrm{tr}\big(F(h_0)^2\big)\cdot\omega^{n-2}$. Since we have assumed the boundedness of $F(h_0)$, we also obtain $\int_Y \mathrm{tr}\big(F(h)^2\big)\cdot\omega^{n-2}\geq$

$\int_Y \operatorname{tr}\big(F(h_0)^2\big)\cdot\omega^{n-2}$ due to Lemma 7.4 in [**51**], as mentioned in the remark just before the lemma. Therefore, we obtain (2). Since we have $\operatorname{tr} F(h_0) = \operatorname{tr} F(h)$, we also obtain (3). □

2.2.2. Uniqueness. — The following proposition can be proved by the methods in [**51**].

Proposition 2.6. — *Let (Y,ω) be a Kahler manifold satisfying Condition 2.1, and $(E,\overline{\partial}_E,\theta)$ be a Higgs bundle on Y. Let h_i $(i=1,2)$ be hermitian metrics of E such that $\Lambda_\omega F(h_i)=0$. We assume that h_1 and h_2 are mutually bounded. Then the following holds:*

- *We have the decomposition of Higgs bundles $(E,\theta)=\bigoplus(E_a,\theta_a)$ which is orthogonal with respect to both of h_i.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then there exist positive numbers b_a such that $h_{1,a}=b_a\cdot h_{2,a}$.*

Proof. — We take the endomorphism s_1 determined by $h_2=h_1\cdot s_1$. Then we have the following inequality due to Lemma 3.1 (d) in [**51**] on $X-D$:

$$\Delta_\omega \log\operatorname{tr}\big(s_1\big)\leq \big|\Lambda_\omega F(h_1)\big|+\big|\Lambda_\omega F(h_2)\big|=0.$$

Here we have used $\Lambda_\omega F(h_i)=0$. Then we obtain $\Delta_\omega\operatorname{tr}\big(s_1\big)\leq 0$. Since the function $\operatorname{tr}(s_1)$ is bounded on Y, we obtain the harmonicity $\Delta_\omega\operatorname{tr}(s_1)=0$ due to Condition 2.1.

We put $D''=\overline{\partial}+\theta$ and $D':=\partial_{E,h_1}+\theta^\dagger_{h_1}$, where $\theta^\dagger_{h_1}$ denotes the adjoint of θ with respect to the metric h_1. Then we also have the following equality:

$$0=F\big(h_2\big)-F\big(h_1\big)=D''\big(s_1^{-1}D's_1\big)=-s_1^{-1}D''s_1\cdot s_1^{-1}\cdot D's_1+s_1^{-1}D''D's_1.$$

Hence we obtain $D''D's_1=D''s_1\cdot s_1^{-1}\cdot D's_1$. As a result, we obtain the following equality:

$$\int\big|s_1^{-1/2}D''s_1\big|^2_{h_1}\operatorname{dvol}_\omega=-\sqrt{-1}\int\Lambda_\omega\operatorname{tr}\big(D''D's_1\big)\operatorname{dvol}_\omega=-\int\Delta_\omega\operatorname{tr}(s_1)\operatorname{dvol}_\omega=0.$$

Hence we obtain $D''s_1=0$, i.e., $\overline{\partial}s_1=\big[\theta,s_1\big]=0$. Since s_1 is self-adjoint with respect to h_1, we obtain the flatness $\big(\overline{\partial}+\partial_{E,h_1}\big)s_1=0$. Hence we obtain the decomposition $E=\bigoplus_{a\in S}E_a$ such that $s_a=\bigoplus b_a\cdot\operatorname{id}_{E_a}$ for some positive constants b_a. Let π_{E_a} denote the orthogonal projection onto E_a. Then we have $\overline{\partial}\pi_{E_a}=0$. Hence the decomposition $E=\bigoplus_{a\in S}E_a$ is holomorphic. It is also compatible with the Higgs field. Hence we obtain the decomposition as the Higgs bundles. Then the claim of Proposition 2.6 is clear. □

Remark 2.7. — We have only to impose $\Lambda_\omega F(h_1)=\Lambda_\omega F(h_2)$ instead of $\Lambda_\omega F(h_i)=0$, which can be shown by a minor refinement of the argument.

2.2.3. The one dimensional case. — In the one dimensional case, Simpson established the Kobayashi-Hitchin correspondence for parabolic Higgs bundle. Here we recall only the special case. (See Chapter 3 for some definitions.)

***Proposition 2.8* (Simpson).** — *Let X be a smooth irreducible projective curve, and D be a simple divisor of X. Let $(\boldsymbol{E}_*, \theta)$ be a filtered regular Higgs bundle on (X, D). We put $E = {}_{\boldsymbol{c}}E_{|X-D}$. The following conditions are equivalent:*

- *$(\boldsymbol{E}_*, \theta)$ is poly-stable with* $\text{par-deg}(\boldsymbol{E}_*) = 0$.
- *There exists a harmonic metric h of (E, θ), which is adapted to the parabolic structure of $\boldsymbol{E}_*$.*

Moreover, such a metric is unique up to obvious ambiguity. Namely, let h_i $(i = 1, 2)$ be two harmonic metrics. Then we have the decomposition of Higgs bundles $(E, \theta) = \bigoplus (E_a, \theta_a)$ satisfying the following:

- *The decomposition is orthogonal with respect to both of h_i.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then there exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.*

Proof. — See [**52**]. We give only a remark on the uniqueness. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X - D$, and h_i $(i = 1, 2)$ be harmonic metrics on it. Assume that the induced prolongments ${}_{\boldsymbol{c}}E(h_i)$ are isomorphic. (See Section 3.5 for prolongment.) Recall the norm estimate for tame harmonic bundles in the one dimensional case ([**52**]), which says that the harmonic metrics are determined up to boundedness by the parabolic filtration and the weight filtration. Hence we obtain the mutually boundedness of h_1 and h_2. Then the uniqueness follows from Proposition 2.6. □

2.3. Weitzenböck Formula

Let (Y, ω) be a Kahler manifold. Let h be a Hermitian-Einstein metric for a Higgs bundle $(E, \overline{\partial}_E, \theta)$ on Y. More strongly, we assume $\Lambda_\omega F(h) = 0$. The following lemma is a minor modification of Weitzenböck formula for harmonic bundles by Simpson ([**52**]).

Lemma 2.9. — *Let s be any holomorphic section of E such that $\theta s = 0$. Then we have $\Delta_\omega \log |s|_h^2 \leq 0$, where Δ_ω denotes the Laplacian for ω.*

Proof. — We have $\partial\overline{\partial}|s|_h^2 = \partial\big(s, \partial_E s\big) = (\partial_E s, \partial_E s) + (s, \overline{\partial}_E \partial_E s) = (\partial_E s, \partial_E s) + (s, R(h)s)$. Then we obtain the following:

$$\partial\overline{\partial} \log |s|_h^2 = \frac{\partial\overline{\partial}|s|^2}{|s|^2} - \frac{\partial|s|^2 \cdot \overline{\partial}|s|^2}{|s|^4} = \frac{(s, R(h)s)}{|s|^2} + \frac{(\partial_E s, \partial_E s)}{|s|^2} - \frac{\partial|s|^2 \cdot \overline{\partial}|s|^2}{|s|^4}.$$

We have $R(h) = -(\theta^\dagger\theta + \theta\theta^\dagger) + F(h)^{(1,1)}$, where $F(h)^{(1,1)}$ denotes the $(1,1)$-part of $F(h)$. Hence we have the following:

$$(4) \quad \Lambda_\omega\big(s, R(h)s\big) = \Lambda_\omega\Big(s,\ (-\theta\theta^\dagger - \theta^\dagger\theta)s\Big) + \Lambda_\omega\big(s, F(h)^{(1,1)}s\big)$$
$$= -\Lambda_\omega\big(\theta^\dagger s, \theta^\dagger s\big) - \Lambda_\omega\big(\theta s, \theta s\big) + \Lambda_\omega\big(s, F(h)^{(1,1)}s\big) = -\Lambda_\omega(\theta^\dagger s,\ \theta^\dagger s).$$

Here we have used $\Lambda_\omega F(h) = \Lambda_\omega F(h)^{(1,1)} = 0$. Therefore we obtain the following:

$$-\sqrt{-1}\Lambda_\omega\big(s, R(h)s\big) = \sqrt{-1}\Lambda_\omega(\theta^\dagger s, \theta^\dagger s) = -\big|\theta^\dagger s\big|_h^2.$$

On the other hand, we also have the following:

$$-\sqrt{-1}\Lambda_\omega\left(\frac{(\partial s, \partial s)}{|s|^2} - \frac{\partial|s|^2\overline{\partial}|s|^2}{|s|^4}\right) \leq 0.$$

Hence we obtain $\Delta_\omega \log|s|^2 \leq 0$. □

2.4. A Priori Estimate of Higgs Fields

2.4.1. On a disc. — We put $X(T) := \big\{z \in \boldsymbol{C} \,\big|\, |z| < T\big\}$ for any positive number T. In the case $T = 1$, $X(1)$ is denoted by X. We will use the usual Euclidean metric $g = dz \cdot d\bar{z}$ and the induced measure dvol_g. The corresponding Kahler form ω is given by $\sqrt{-1}dz \wedge d\bar{z}/2$. Let Δ'' denote the Laplacian $-\sqrt{-1}\Lambda_\omega\partial\overline{\partial} = -2\partial_z\overline{\partial}_z$. By the standard theory of Dirichlet problem, there exists a constant C' such that the following holds:

- We have the solution ψ of the equation $\Delta''\psi = \kappa$ such that $|\psi(P)| \leq C' \cdot \|\kappa\|_{L^2}$ for any L^2-function κ and for any $P \in X$.

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X with a hermitian metric h. We have the expression $\theta = f \cdot dz$. We would like to estimate of the norm $\big|f\big|_h$ by the eigenvalues of g and the L^2-norm $\big\|F(h)\big\|_{L^2} := \int_X |F(h)|_{h,g}^2 \cdot \mathrm{dvol}_g$.

Proposition 2.10. — *Let t be any positive number such that $t < 1$. There exist constants C and C' such that the following inequality holds on $X(t)$:*

$$|f|_h^2 \leq C \cdot e^{10C'||F(h)||_{L^2}}.$$

The constant C' is as above. The constant C depends only on t, the rank of E and the eigenvalues of f.

Proof. — Let us begin with the following lemma, which is just a minor modification of the fundamental inequality in the theory of harmonic bundles.

Lemma 2.11. — *We have the inequality:*

$$\Delta'' \log|f|_h^2 \leq -\frac{\big|[f, f^\dagger]\big|_h^2}{|f|_h^2} + 5|F(h)|_{h,g}.$$

Proof. — By a general formula, we have the following inequality:

$$-\sqrt{-1}\Lambda_\omega \partial\overline{\partial}\log|f|_h^2 \leq -\sqrt{-1}\Lambda_\omega \frac{\big(f,\,[R(h),f]\big)_h}{|f|_h^2}.$$

We obtain the desired inequality from $R(h) = F(h) - [\theta, \theta^\dagger] = F(h) - [f, f^\dagger]\cdot dz\cdot d\bar{z}$. □

Let us take a function A satisfying $\Delta'' A = 5|F(h)|_h$ and $|A| \leq 5C'||F(h)||_{L^2}$. Then we obtain the following:

$$\Delta''\big(\log|f|_h^2 - A\big) = \Delta''\log\big(|f|_h^2\cdot e^{-A}\big) \leq -\frac{\big|[f,f^\dagger]\big|_h^2}{|f|_h^2}.$$

For any $Q \in X$, let $\alpha_1(Q),\ldots,\alpha_{\mathrm{rank}(E)}(Q)$ denote the eigenvalues of $f_{|Q}$. We put $\nu(Q) := \sum_{i=1}^{\mathrm{rank}(E)}|\alpha_i(Q)|^2$ and $\mu(Q) := |f_{|Q}|_h^2 - \nu(Q)$. It can be elementarily shown that there exists a constant C_1 which depends only on the rank of E, such that $C_1\cdot\mu^2 \leq \big|[f,f^\dagger]\big|_h^2$. Hence, the following inequality holds:

$$\Delta''\log\big(e^{-A}\cdot|f|_h^2\big) \leq -C_1\cdot\frac{\mu^2}{|f|_h^2}.$$

We also have a constant C_2 which depends only on the eigenvalues of f, such that $\nu \leq C_2$ holds.

Let T be a number such that $0 < T < 1$, and $\phi_T : X(T) \longrightarrow \boldsymbol{R}$ is given by the following:

$$\phi_T(z) = \frac{4T^2}{(T^2 - |z|^2)^2}.$$

Then we have $\Delta''\log\phi_T = -\phi_T$ and $\phi_T \geq 2$. In particular, we have $\nu \leq C_2\cdot\phi_T/2$. The following lemma is clear.

Lemma 2.12. — *Either one of $|f_{|Q}|_h^2 \leq C_2\cdot\phi_T(Q)$ or $|f_{|Q}|_h^2 \leq 2\mu(Q)$ holds for any $Q \in X$.*

We take a constant $\widehat{C}_3 > 0$ satisfying $\widehat{C}_3 > C_2$ and $\widehat{C}_3 > 4\cdot C_1^{-1}$, and we put $C_3 := \widehat{C}_3\cdot e^{5C'\|F(h)\|_{L^2}}$. We put $S_T := \big\{P \in X(T)\,\big|\,\big(e^{-A}\cdot|f|^2\big)(P) > C_3\cdot\phi_T(P)\big\}$. For any point $P \in S_T$, we have $|f(P)|_h^2 > C_3\cdot e^{A(P)}\cdot\phi_T(P) > C_2\cdot\phi_T(P)$. Due to Lemma 2.12, we obtain the following:

$$\Delta''\log\big(e^{-A}\cdot|f|_h^2\big)(P) \leq -\frac{C_1}{4}\cdot|f(P)|_h^2 \leq -\frac{1}{C_3}\big(e^{-A}\cdot|f|_h^2\big)(P).$$

On the other hand, we have the following:

$$\Delta''\log(C_3\cdot\phi_T) = -\frac{1}{C_3}(C_3\cdot\phi_T).$$

Moreover, it is easy to see $\partial S_T \cap \{|z| = T\} = \varnothing$. Hence, we obtain $S_T = \varnothing$ by a standard argument. (See [**1**], [**52**] or the proof of Proposition 7.2 in [**44**].) Namely, we obtain the inequality $e^{-A}|f|_h^2 \leq \widehat{C}_3\cdot e^{5C'\|F(h)\|_{L^2}}\cdot\phi_T$ on $X(T)$. Taking a limit for

$T \to 1$, we obtain $|f|_h^2 \leq e^{10C'||F(h)||_{L^2}} \cdot \widehat{C}_3 \cdot (1-|z|^2)^{-1}$ on X. Then the claim of Proposition 2.10 follows. $\square$

2.4.2. A Priori Estimate on a Multi-disc. — For a positive number T, we put $Y(T) := \{(z_1, \ldots, z_n) \,\big|\, |z_i| < T\}$. Let g denote the metric $\sum dz_i \cdot d\bar{z}_i$ of $Y(T)$. Let ω be a Kahler form on $Y(T)$ such that there exists a constant $C > 0$ such that $C^{-1} \cdot \omega \leq g \leq C \cdot \omega$. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle with a hermitian metric h, which is Hermitian-Einstein with respect to ω. For simplicity, we restrict ourselves to the case $\Lambda_\omega F(h) = 0$. We assume $||F(h)||_{L^2} < \infty$, where $\|F(h)\|_{L^2}$ denotes the L^2-norm of $F(h)$ with respect to ω and h. We have the expression $\theta = \sum f_i \cdot dz_i$ for holomorphic sections $f_i \in \mathrm{End}(E)$ on $Y(T)$.

Lemma 2.13. — *Take $0 < T_1 < T$. There exist some constants C_1 and C_2 such that the following inequality holds for any $P \in Y(T_1)$:*

$$\log |f_i|^2(P) \leq C_1 \cdot \big\|F(h)\big\|_{L^2} + C_2.$$

The constants C_1 and C_2 are good in the sense that they depend only on T, T_1, $\mathrm{rank}\, E$, the eigenvalues of f_i $(i = 1, 2, \ldots, n)$ and the constant C.

Proof. — We take a positive number T_2 such that $T_1 < T_2 < T$. The induced Higgs field and the metric of $\mathrm{End}(E)$ are denoted by $\widetilde{\theta}$ and $\widetilde{h}$. Then the metric $\widetilde{h}$ is a Hermitian-Einstein metric of $\big(\mathrm{End}(E), \widetilde{\theta}\big)$ such that $\Lambda_\omega F(\widetilde{h}) = 0$. Because of $\widetilde{\theta}(f_i) = 0$, we have the subharmonicity $\Delta_\omega \log |f_i|_{\widetilde{h}}^2 \leq 0$ due to Lemma 2.9. We use Theorem 9.20 in [**18**]. Note that $\Delta_\omega u = -\sqrt{-1}\Lambda_\omega \partial\overline{\partial} u$ is expressed as $-\sum a^{i,j}\partial_{x_i}\partial_{x_j} u$, where we use the real coordinate given by $z_i = x_i + \sqrt{-1}x_{n+i}$. (In terms of Chapter 9 of [**18**], we consider the case $b^i = c = 0$.) The matrix $\mathcal{A} = (a_{i,j})$ is symmetric and positive definite, and the eigenvalues are bounded uniformly, due to the condition $C^{-1} \cdot \omega \leq g \leq C \cdot \omega$. Hence, we obtain the following inequality for $P \in Y(T_1)$:

$$\log |f_i|^2(P) \leq C_3 \cdot \int_{Y(T_2)} \log^+ |f_i|^2 \cdot \mathrm{dvol}_g\,.$$

Here we put $\log^+(y) := \max\{0, \log y\}$, and C_3 denotes a good constant.

The $(1,1)$-part of $F(h)$ is expressed as $\sum F_{i,j} \cdot dz_i \cdot d\bar{z}_j$. Due to Proposition 2.10, there exist good constants C_j $(j = 4, 5)$ such that the following inequality holds for any point $(z_1, \ldots, z_n) \in Y(T_2)$:

$$\log |f_1|^2(z_1, \ldots, z_n) \leq C_4 \cdot \left(\int_{|w_1|<T} |F_{1,1}(w_1, z_2, \ldots, z_n)|^2 \cdot \sqrt{-1} dw_1 \wedge d\overline{w}_1 \right)^{1/2} + C_5.$$

Then the claim of Lemma 2.13 follows. $\square$

2.5. Norm Estimate for Tame Harmonic Bundle in Two Dimensional Case

2.5.1. Norm estimate. — We recall some results in [**44**]. We use bold symbols like $\boldsymbol{a}$ to denote a tuple, and a_i denotes the i-th component of $\boldsymbol{a}$. We say $\boldsymbol{a} \leq \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^2$ if $a_i \leq b_i$. We put $X := \{(z_1, z_2) \in \boldsymbol{C}^2 \,\big|\, |z_i| < 1\}$, $D_i := \{z_i = 0\}$ and $D := D_1 \cup D_2$. Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$. For each $\boldsymbol{c} = (c_1, c_2) \in \boldsymbol{R}^2$, we obtain the locally free sheaf ${}_{\boldsymbol{c}}E$ on X with parabolic structure ${}^iF$ $(i = 1, 2)$, as in Section 3.5. We also obtain the Higgs field θ of ${}_{\boldsymbol{c}}E_*$. The residue of θ induces the endomorphism $\mathrm{Gr}^F \mathrm{Res}_i(\theta) \in \mathrm{End}({}^i\,\mathrm{Gr}^F(E_{|D_i}))$ whose eigenvalues are constant on D_i. Thus, the nilpotent part $\mathcal{N}_i$ of $\mathrm{Gr}^F \mathrm{Res}_i(\theta)$ is well defined. It is shown that the conjugacy classes of $\mathcal{N}_{i\,|\,P}$ are independent of $P \in D_i$. Let ${}^{\underline{1}}W$ denote the weight filtration of $\mathcal{N}_1$ on ${}^1\,\mathrm{Gr}^F(E_{|D_1})$.

We have two filtrations ${}^iF$ $(i = 1, 2)$ on ${}_{\boldsymbol{c}}E_{|O}$. We put ${}^{\underline{2}}\,\mathrm{Gr}^F_{\boldsymbol{a}} := {}^2\,\mathrm{Gr}^F_{a_2}\,{}^1\,\mathrm{Gr}^F_{a_1}({}_{\boldsymbol{c}}E_{|O})$. The maps $\mathcal{N}_i$ induce the endomorphisms of ${}^{\underline{2}}\,\mathrm{Gr}^F_{\boldsymbol{a}}$ which are denoted by ${}^{\underline{2}}\mathcal{N}_i$. Let ${}^{\underline{2}}W$ denote the weight filtration of ${}^{\underline{2}}\mathcal{N}_1 + {}^{\underline{2}}\mathcal{N}_2$. We also have the filtration induced by ${}^{\underline{1}}W$, which is denoted by the same notation. We can take a decomposition ${}_{\boldsymbol{c}}E = \bigoplus_{(\boldsymbol{a},\boldsymbol{k}) \in \boldsymbol{R}^2 \times \mathbb{Z}^2} U_{(\boldsymbol{a},\boldsymbol{k})}$ satisfying the following conditions:

- ${}^iF_b({}_{\boldsymbol{c}}E_{|D_i}) = \bigoplus_{a_i \leq b} U_{\boldsymbol{a},\boldsymbol{k}\,|\,D_i}$ and ${}^1F_{b_1}({}_{\boldsymbol{c}}E_{|O}) \cap {}^2F_{b_2}({}_{\boldsymbol{c}}E_{|O}) = \bigoplus_{\boldsymbol{a} \leq \boldsymbol{b}} U_{\boldsymbol{a},\boldsymbol{k}|O}$
- We have ${}^{\underline{1}}W_k\big({}^1\,\mathrm{Gr}^F_b({}_{\boldsymbol{c}}E_{|D_1})\big) = \bigoplus_{a_1 = b, k_1 \leq k} U_{\boldsymbol{a},\boldsymbol{k}\,|\,D_1}$ under the isomorphism ${}^1\,\mathrm{Gr}^F_b({}_{\boldsymbol{c}}E_{|D_1}) \simeq \bigoplus_{a_1 = b} U_{\boldsymbol{a},\boldsymbol{k}\,|\,D_1}$.
- We have ${}^{\underline{1}}W_{k_1} \cap {}^{\underline{2}}W_{k_2}\big({}^{\underline{2}}\,\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|O})\big) = \bigoplus_{\boldsymbol{l} \leq \boldsymbol{k}} U_{\boldsymbol{a},\boldsymbol{l}}$ under the isomorphism ${}^{\underline{2}}\,\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|O}) \simeq \bigoplus_{\boldsymbol{k}} U_{\boldsymbol{a},\boldsymbol{l}}$.

We take a holomorphic frame $\boldsymbol{v} = (v_1, \ldots, v_r)$ which is compatible with the decomposition, i.e., for each v_i we have $(\boldsymbol{a}(v_i), \boldsymbol{k}(v_i)) \in \boldsymbol{R}^2 \times \mathbb{Z}^2$ such that $v_i \in U_{\boldsymbol{a}(v_i),\boldsymbol{k}(v_i)}$. Let $\widehat{h}_1$ be a hermitian metric of E given as follows:

$$\widehat{h}_1(v_i, v_j) = \delta_{i,j} \cdot |z_1|^{-2a_1(v_i)} |z_2|^{-2a_2(v_i)} \big(-\log|z_1|\big)^{k_1(v_i)} \big(-\log|z_2|\big)^{k_2(v_i) - k_1(v_i)}$$

We put $Z := \big\{(z_1, z_2) \,\big|\, |z_1| < |z_2|\big\}$.

Lemma 2.14. — *h and $\widehat{h}_1$ are mutually bounded on Z.*

2.5.2. Some estimate for related metrics. — We put $\widetilde{X} := \big\{(\zeta_1, \zeta_2) \,\big|\, |\zeta_i| < 1\big\}$, $\widetilde{D}_i := \{\zeta_i = 0\}$ and $\widetilde{D} := \widetilde{D}_1 \cup \widetilde{D}_2$. Let $\pi : \widetilde{X} - \widetilde{D} \longrightarrow X - D$ denote the map given by $\pi(\zeta_1, \zeta_2) = (\zeta_1\zeta_2, \zeta_2)$. Then, we have $\pi^{-1}(Z) = \widetilde{X} - \widetilde{D}$. Hence Lemma 2.14 is reworded as π^*h and $\pi^*\widehat{h}_1$ are mutually bounded.

We give a preparation for later use. We put $\widetilde{E} := \pi^*E$. For $\boldsymbol{a} = (a_1, a_2) \in \boldsymbol{R}^2$, we put $\widetilde{\boldsymbol{a}} := (a_1, a_1 + a_2)$. Then, we put $\pi^*U_{\boldsymbol{a},\boldsymbol{k}} =: \widetilde{U}_{\widetilde{\boldsymbol{a}},\boldsymbol{k}}$. We put $\widetilde{\boldsymbol{v}} := \pi^*\boldsymbol{v}$. We put $\widetilde{a}_1(\widetilde{v}_i) := a_1(v_i)$, $\widetilde{a}_2(\widetilde{v}_i) = a_1(v_i) + a_2(v_i)$, $k_j(\widetilde{v}_i) := k_j(v_i)$. Then, $\widetilde{v}_i$ is a section of $\widetilde{U}_{\widetilde{\boldsymbol{a}}(\widetilde{v}_i),\boldsymbol{k}(\widetilde{v}_i)}$.

Let χ be a non-negative valued function on $\boldsymbol{R}$ such that $\chi(t) = 1$ $(t \leq 1/2)$ and $\chi(t) = 0$ $(t \geq 2/3)$. Let $\rho(\zeta) : \boldsymbol{C}^* \longrightarrow \boldsymbol{R}$ be the function given by $\rho(\zeta) = -\chi(|\zeta|) \cdot \log|\zeta|^2$. Then, we will use the following metrics later (Section 5.2)

$$h_0(\widetilde{v}_i, \widetilde{v}_j) := \delta_{i,j} \cdot \prod_k |\zeta_k|^{-2a_k(v_i)}$$

$$h_1(\widetilde{v}_i, \widetilde{v}_j) := h_0(\widetilde{v}_i, \widetilde{v}_j) \cdot \big(1 + \rho(\zeta_1) + \rho(\zeta_2)\big)^{k_1(\widetilde{v}_i)} \cdot \big(1 + \rho(\zeta_2)\big)^{k_2(\widetilde{v}_i) - k_1(\widetilde{v}_i)}$$

Then, h_1 and $\pi^* h$ are mutually bounded. The curvature $R(h_0)$ is 0. Let $\widetilde{\omega}$ denote the Poincaré metric of $\widetilde{X} - \widetilde{D}$:

$$\widetilde{\omega} = \sum_{i=1,2} \frac{d\zeta_i \cdot d\overline{\zeta}_i}{|\zeta_i|^2 (-\log|\zeta_i|^2)^2}$$

Lemma 2.15. — *$R(h_1)$ and $\partial_{h_1} - \partial_{h_0}$ are bounded with respect to $(\widetilde{\omega}, h_i)$ $(i = 0, 1)$.*

Proof. — $\partial \log\big(1 + \rho(\zeta_2)\big)$, $\partial\overline{\partial} \log\big(1 + \rho(\zeta_2)\big)$, $\partial \log(1 + \rho(\zeta_1) + \rho(\zeta_2))$ and $\partial\overline{\partial} \log(1 + \rho(\zeta_1) + \rho(\zeta_2))$ are bounded with respect to $\widetilde{\omega}$. Then, the boundedness of $R(h_1)$ and $\partial_{h_1} - \partial_{h_0}$ follow. □

2.6. Preliminary from Elementary Calculus

Take $\epsilon > 0$ and $N > 1$. In this section, we use the following volume form $\mathrm{dvol}_{\epsilon,N}$ of a punctured disc Δ^*:

$$\mathrm{dvol}_{\epsilon,N} := \big(\epsilon^{N+2} \cdot |z|^{2\epsilon} + |z|^2\big)^{-1} \frac{\sqrt{-1} dz \wedge d\bar{z}}{|z|^2}$$

Let f be a function on a punctured disc Δ^* such that $\|f\|_{L^2}^2 := \int_{\Delta^*} |f|^2 \cdot \mathrm{dvol}_{\epsilon,N} < \infty$. We use the polar coordinate $z = r \cdot e^{\sqrt{-1}\theta}$. For the decomposition $f = \sum f_n(r) \cdot e^{\sqrt{-1}n\theta}$, we have $\|f\|_{L^2}^2 = 2\pi \sum_n \|f_n\|_{L^2}^2$, where $\|f_n\|_{L^2}^2$ are given as follows:

$$\|f_n\|_{L^2}^2 := \int_0^1 |f_n(\rho)|^2 \cdot \big(\epsilon^{N+2} \rho^{2\epsilon} + \rho^2\big)^{-1} \frac{d\rho}{\rho}.$$

Proposition 2.16. — *Let f be as above. Then we have a function v satisfying the following:*

$$\overline{\partial}\partial v = f \cdot \frac{d\bar{z} \wedge dz}{|z|^2}, \quad \big|v(z)\big| \leq C \cdot \Big(|z|^{\epsilon} \epsilon^{(N-1)/2} + |z|^{1/2}\Big) \cdot \|f\|_{L^2}.$$

The constant C can be independent of ϵ, N and f.

Proof. — We use the argument of S. Zucker in [**66**]. First let us consider the equation $\overline{\partial} u = f \cdot d\bar{z}/\bar{z}$. For the decomposition $u = \sum u_n(\rho) \cdot e^{\sqrt{-1}n\theta}$, it is equivalent to the following equations:

$$\frac{1}{2}\left(r \frac{\partial}{\partial r} u_n - n \cdot u_n\right) = f_n, \quad (n \in \mathbb{Z}).$$

We put as follows:

$$u_n := \begin{cases} 2r^n \int_0^r \rho^{-n-1} f_n(\rho)\cdot d\rho & (n \leq 0), \\ 2r^n \int_A^r \rho^{-n-1} f_n(\rho)\cdot d\rho & (n > 0). \end{cases}$$

Then $u = \sum u_n \cdot e^{\sqrt{-1}n\theta}$ satisfies the equation $\overline{\partial} u = f \cdot d\bar{z}/\bar{z}$.

Lemma 2.17. — *There exists $C_1 > 0$ such that*

$$|u_n(r)| \leq C_1 \cdot \|f_n\|_{L^2} \cdot \left(\frac{\epsilon^{(N+2)/2}\cdot r^{\epsilon}}{|2\epsilon - 2n|^{1/2}} + \frac{r^{1/2}}{(1+|n|)^{1/2}} \right).$$

The constant C_1 is independent of n, ϵ, N and f.

Proof. — In the case $n \leq 0$, we have the following:

$$(5) \quad |u_n(r)| \leq 2r^n \left(\int_0^r |f_n(\rho)|^2 (\epsilon^{N+2}\rho^{2\epsilon} + \rho^2)^{-1} \frac{d\rho}{\rho} \right)^{1/2} \times \left(\int_0^r \rho^{-2n-1}(\epsilon^{N+2}\rho^{2\epsilon} + \rho^2)\cdot d\rho \right)^{1/2}$$

We have the following:

$$\int_0^r \rho^{-2n-1}(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)d\rho = \frac{\epsilon^{N+2}\cdot r^{2\epsilon-2n}}{2\epsilon - 2n} + \frac{r^{-2n+2}}{-2n+2}.$$

Hence we obtain the following:

$$|u_n(r)| \leq 2\|f_n\|_{L^2} \cdot \left(\frac{\epsilon^{(N+2)/2}\cdot r^{\epsilon}}{|2\epsilon - 2n|^{1/2}} + \frac{r}{|2-2n|^{1/2}} \right).$$

In the case $n > 0$, we also have the following:

$$|u_n(r)| \leq 2r^n \cdot \|f_n\|_{L^2} \left| \int_A^r \rho^{-2n-1}(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)d\rho \right|^{1/2}.$$

We have the following:

$$\left| \int_A^r \rho^{-2n-1}\epsilon^{N+2}\cdot\rho^{2\epsilon}\cdot d\rho \right| \leq \frac{\epsilon^{N+2}}{|-2n+2\epsilon|} r^{-2n+2\epsilon}.$$

We also have the following:

$$\int_A^r \rho^{-2n+1} d\rho = \begin{cases} \log r - \log A & (n=1) \\ (-2n+2)^{-1}(r^{-2n+2} - A^{-2n+2}) & (n \geq 2) \end{cases}$$

Therefore we obtain the following:

$$|u_n(r)| \leq C \cdot \|f_n\|_{L^2} \left(\frac{\epsilon^{(N+2)/2}\cdot r^{\epsilon}}{|2\epsilon - 2n|^{1/2}} + \frac{r^{1/2}}{(1+|n|)^{1/2}} \right)$$

Thus we are done. □

Then let us consider the equation $\partial v = u \cdot dz/z$. For the decomposition $v = \sum v_n \cdot e^{\sqrt{-1}n\theta}$, it is equivalent to the following equations:

$$\frac{1}{2}\left(r\frac{\partial v_n}{\partial r} + n \cdot v_n\right) = u_n, \quad (n \in \mathbb{Z}).$$

We put as follows:

$$v_n(r) := \begin{cases} 2r^{-n} \cdot \int_0^r \rho^{n-1} u_n(\rho) \cdot d\rho & (n \geq 0) \\ \\ 2r^{-n} \cdot \int_A^r \rho^{n-1} u_n(\rho) \cdot d\rho & (n < 0). \end{cases}$$

Then we have $\partial v = u \cdot dz/z$ for $v := \sum v_n \cdot e^{\sqrt{-1}n\theta}$. From Lemma 2.17, we obtain the following in the case $n > 0$:

$$\begin{aligned} (6) \quad |v_n(r)| &\leq 2r^{-n} \int_0^r \rho^{n-1}\left(\frac{\epsilon^{(N+1)/2} \cdot \rho^\epsilon}{|2\epsilon - 2n|^{1/2}} + \frac{\rho^{1/2}}{(1+|n|)^{1/2}}\right) d\rho \cdot \|f_n\|_{L^2} \\ &\leq C_2 \cdot \|f_n\|_{L^2} \cdot \left(\frac{\epsilon^{(N+2)/2}}{|2\epsilon - 2n|^{1/2}} \frac{r^\epsilon}{|n+\epsilon|} + \frac{1}{(1+|n|)^{1/2}} \frac{r^{1/2}}{n+1/2}\right). \end{aligned}$$

We have a similar estimate in the case $n < 0$. Hence we obtain the following:

$$|v(z)| \leq \sum_n |v_n(r)| \leq C_4 \cdot (\epsilon^{(N-1)/2} r^\epsilon + r^{1/2}) \cdot \|f\|_{L^2}.$$

Thus the proof of Proposition 2.16 is finished. □

2.7. Reflexive Sheaf

We recall some general facts about reflexive sheaves. See [**21**] and [**41**] for some more properties of reflexive sheaves. Let X be a complex manifold. Recall that a coherent $\mathcal{O}_X$-module $\mathcal{E}$ is called reflexive, if $\mathcal{E}$ is isomorphic to the double dual $\mathcal{E}^{\vee\vee} := \mathcal{H}om\big(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X\big)$ of $\mathcal{E}$. Recall we can take a resolution *locally* on X (Lemma 3.1 of [**41**]):

$$(7) \qquad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{V}_0 \longrightarrow \mathcal{V}_1 \longrightarrow 0$$

Here $\mathcal{V}_0$ is locally free and $\mathcal{V}_1$ is torsion-free. The following Hartogs type theorem is well known.

Lemma 2.18. — *Let Z be a closed subset of X whose codimension is larger than 2. Let f be a section of a reflexive sheaf $\mathcal{E}$ on $X \setminus Z$. Then f is naturally extended to the section of $\mathcal{E}$ over X.*

Proof. — We have only to check the claim locally. Let us take a resolution (7), and then f induces the section of $\widetilde{f}$ of $\mathcal{V}_0$ on $X - Z$. Due to the Hartogs' theorem, $\widetilde{f}$ can be extended to the section on X. Since it is mapped to 0 in $\mathcal{V}_1$, we obtain the section of $\mathcal{E}$ on X. □

The converse is also true.

Lemma 2.19. — *Let $\mathcal{F}$ be a torsion-free coherent sheaf on X such that any section f of $\mathcal{F}$ on $U - Z$ is extended to the section on U, where U denotes an open subset and Z denotes a closed subset with* $\operatorname{codim} Z \geq 2$. *Then $\mathcal{F}$ is reflexive.*

Proof. — We have the inclusion $\iota : \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$, which is isomorphic outside of the subset $Z_0 \subset X$ with $\operatorname{codim}(Z_0) \geq 2$. Then, we obtain the surjectivity of ι from the given property of $\mathcal{F}$, and thus ι is isomorphic. □

Lemma 2.20. — *If $\mathcal{E}$ is reflexive, $\mathcal{E} \otimes \mathcal{O}_D$ is torsion-free for a divisor D.*

Proof. — Take a resolution as in (7). Because of $\operatorname{Tor}^1(\mathcal{V}_1, \mathcal{O}_D) = 0$, we obtain the injection $\mathcal{E} \otimes \mathcal{O}_D \longrightarrow \mathcal{V}_0 \otimes \mathcal{O}_D$, and hence $\mathcal{E} \otimes \mathcal{O}_D$ is torsion-free. □

Lemma 2.21. — *If $\mathcal{E}$ is a reflexive sheaf, $\mathcal{H}om(\mathcal{F}, \mathcal{E})$ is also reflexive for any coherent sheaf $\mathcal{F}$.*

Proof. — Let us check the condition in Lemma 2.19. Let U be a small open subset, on which we have a resolution $\mathcal{V}_{-1} \xrightarrow{a} \mathcal{O}_U^{\oplus r} \xrightarrow{b} \mathcal{F} \longrightarrow 0$ on U. Let f be a homomorphism $\mathcal{F} \longrightarrow \mathcal{E}$ on $U \setminus Z$, where $\operatorname{codim} Z \geq 2$. The morphism $\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{E}$ is naturally induced on $U \setminus Z$, which is naturally extended to the morphism $\varphi : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{E}$ on U by the Hartogs property. Since $\varphi \circ a$ is 0, φ induces the extension of f. □

2.8. Moduli Spaces of Representations

Let Γ be a finitely presented group, and V be a finite dimensional vector space over $\boldsymbol{C}$. For $a, f \in \mathrm{GL}(V)$, we put $\mathrm{ad}(a)(f) := a \circ f \circ a^{-1}$. The space of homomorphisms $R(\Gamma, V) := Hom(\Gamma, \mathrm{GL}(V))$ is naturally an affine variety over $\boldsymbol{C}$. We regard it as a Hausdorff topological space with the usual topology, not the Zariski topology. We have the natural action of $\mathrm{GL}(V)$ on $R(\Gamma, V)$ given by ad. Let h_V be a hermitian metric of V, and let $U(h_V)$ denote the unitary group of V with respect to h_V. The usual quotient space $R(\Gamma, V)/U(h_V)$ is denoted by $M(\Gamma, V, h_V)$. Let $\pi_{\mathrm{GL}(V)}$ denote the projection $R(\Gamma, V) \longrightarrow M(\Gamma, V, h_V)$.

More generally, we consider the moduli space of representations to a complex reductive subgroup G of $\mathrm{GL}(V)$. We put $R(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$, which we regard as a Hausdorff topological space with the usual topology. It is the closed subspace of $R(\Gamma, V)$.

Let K be a maximal compact subgroup of G. Assume that the hermitian metric h_V of V is K-invariant. We put $N_G\big(h_V\big) := \big\{u \in U(h_V) \,\big|\, \mathrm{ad}(u)(G) = G\big\}$ which is compact. We have the natural adjoint action of $N_G\big(h_V\big)$ on G, which induces the action on $R(\Gamma, G)$. The usual quotient space is denoted by $M(\Gamma, G, h_V)$. Let π_G denote the projection $R(\Gamma, G) \longrightarrow M(\Gamma, G, h_V)$. We have the naturally defined map $\Phi : M(\Gamma, G, h_V) \longrightarrow M(\Gamma, V, h_V)$. The map Φ is clearly proper in the sense that the inverse image of any compact subset via Φ is also compact.

A representation $\rho \in R(\Gamma, G)$ is called Zariski dense, if the image of ρ is Zariski dense in G. Let $\mathcal{U}$ be the subset of $R(\Gamma, G)$, which consists of Zariski dense representations. Then the restriction of Φ to $\mathcal{U}$ is injective.

Let ρ and ρ' be elements of $R(\Gamma, G)$. We say that ρ and ρ' are isomorphic in G, if there is an element $g \in G$ such that $\mathrm{ad}(g) \circ \rho = \rho'$. We say ρ' is a deformation of ρ in G, if there is a continuous family of representations $\rho_t : [0, 1] \times \Gamma \longrightarrow G$ such that $\rho_0 = \rho$ and $\rho_1 = \rho'$. We say ρ' is a deformation of ρ in G modulo $N_G(h_V)$, if there is an element $u \in N_G(h_V)$ such that ρ can be deformed to $\mathrm{ad}(u) \circ \rho'$ in G. The two notions are different if $N_G(h_V)$ is not connected, in general. We also remark that ρ can be deformed to ρ' in G modulo $N_G(h_V)$, if and only if $\pi_G(\rho)$ and $\pi_G(\rho')$ are contained in the same connected component of $M(\Gamma, G, h_V)$.

We recall some deformation invariance from [**55**]. A representation $\rho \in R(\Gamma, G)$ is called rigid, if the orbit $G \cdot \rho$ is open in $R(\Gamma, G)$.

Lemma 2.22. — *Let $\rho \in R(\Gamma, G)$ be a rigid and Zariski dense representation. Then any deformation ρ' of ρ in G is isomorphic to ρ in G.*

Proof. — If ρ is Zariski dense, then $G \cdot \rho$ is closed in $R(\Gamma, G)$. Hence it is a connected component. □

CHAPTER 3

PARABOLIC HIGGS BUNDLE AND REGULAR FILTERED HIGGS BUNDLE

We recall the notion of parabolic structure, and then we give some detail about the characteristic numbers for parabolic sheaves. In Section 3.3, a perturbation of the filtration is given, which will be useful in our later argument.

3.1. Parabolic Higgs Bundle

3.1.1. $\boldsymbol{c}$-Parabolic Higgs sheaf. — Let us recall the notion of parabolic structure and the Chern characteristic numbers of parabolic bundles following [**35**], [**39**], [**51**], [**52**], [**62**] and [**65**]. Our convention is slightly different from theirs.

Let X be a connected complex manifold and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S}D_i$. Let $\boldsymbol{c}=(c_i\,\big|\,i\in S)$ be an element of $\boldsymbol{R}^S$. Let $\mathcal{E}$ be a torsion-free coherent $\mathcal{O}_X$-module. Let us consider a collection of the increasing filtrations ${}^i\mathcal{F}$ $(i\in S)$ indexed by $]c_i-1,c_i]$ such that (i) ${}^i\mathcal{F}_a(\mathcal{E})\supset\mathcal{E}(-D_i)$ for any $a\in]c_i-1,c_i]$, (ii) ${}^i\mathcal{F}_a(\mathcal{E})=\bigcap_{a<b}{}^i\mathcal{F}_b(\mathcal{E})$. We put ${}^i\operatorname{Gr}^{\mathcal{F}}_a\mathcal{E}:={}^i\mathcal{F}_a(\mathcal{E})/{}^i\mathcal{F}_{<a}(\mathcal{E})$. We assume that the sets $\big\{a\,\big|\,{}^i\operatorname{Gr}^{\mathcal{F}}_a\mathcal{E}\neq 0\big\}$ are finite for any i. Such tuples of filtrations are called the $\boldsymbol{c}$-parabolic structure of $\mathcal{E}$ at D, and the tuple $\big(\mathcal{E},\{{}^i\mathcal{F}\,|\,i\in S\}\big)$ is called a $\boldsymbol{c}$-parabolic sheaf on (X,D). We will sometimes omit to denote $\boldsymbol{c}$. We say $\big(\mathcal{E},\{{}^i\mathcal{F}\,|\,i\in S\}\big)$ is reflexive, if $\mathcal{E}$ is reflexive. (See [**21**] and [**41**] for reflexive sheaves. See also Section 2.7.)

Definition 3.1. — For a reflexive $\boldsymbol{c}$-parabolic sheaf $\big(\mathcal{E},\{{}^i\mathcal{F}\,|\,i\in S\}\big)$, we say that the parabolic structure is saturated, if $\mathcal{E}/{}^i\mathcal{F}_a$ are torsion-free $\mathcal{O}_{D_i}$-modules for any i and a.

We remark that each ${}^i\mathcal{F}_a$ are also reflexive. To see it, let us see the inclusion ${}^i\mathcal{F}_a\longrightarrow{}^i\mathcal{F}_a^{\vee\vee}$. Since $\mathcal{E}$ is reflexive, the inclusion ${}^i\mathcal{F}_a\longrightarrow\mathcal{E}$ is extended to the injection ${}^i\mathcal{F}_a^{\vee\vee}\longrightarrow\mathcal{E}$. (See the proof of Lemma 2.21.) Hence we obtain the inclusion ${}^i\mathcal{F}_a^{\vee\vee}/{}^i\mathcal{F}_a\longrightarrow\mathcal{E}/{}^i\mathcal{F}_a$. The codimension of the support of ${}^i\mathcal{F}_a^{\vee\vee}/{}^i\mathcal{F}_a$ is larger than 2, and $\mathcal{E}/{}^i\mathcal{F}_a$ is torsion-free as an $\mathcal{O}_{D_i}$-module. Hence we obtain ${}^i\mathcal{F}_a^{\vee\vee}/{}^i\mathcal{F}_a=0$

We will use the notation $\mathcal{E}_*$ instead of $\big(\mathcal{E},\{{}^i\mathcal{F}\}\big)$ for simplicity. When we emphasize $\boldsymbol{c}$, we will often use the notation ${}_{\boldsymbol{c}}\mathcal{E}$ and ${}_{\boldsymbol{c}}\mathcal{E}_*$ instead of $\mathcal{E}$ and $\mathcal{E}_*$. In the case $\boldsymbol{c}=(0,\dots,0)$, the notation ${}^\diamond\mathcal{E}_*$ is used. We will also use the following notation.

(8) $\quad \mathcal{P}ar\big(\mathcal{E}_*,i\big):=\big\{a\,\big|\,{}^i\mathrm{Gr}^{\mathcal{F}}_a(\mathcal{E})\neq 0\big\},\quad \mathcal{P}ar'\big(\mathcal{E}_*,i\big):=\mathcal{P}ar\big(\mathcal{E}_*,i\big)\cup\{c_i,c_i-1\},$

(9) $\mathrm{gap}(\mathcal{E}_*,i):=\min\big\{|a-b|\,\big|\,a,b\in\mathcal{P}ar'(\mathcal{E}_*,i),\ a\neq b\big\},\quad \mathrm{gap}(\mathcal{E}_*):=\min_{i\in S}\mathrm{gap}\big(\mathcal{E}_*,i\big).$

Let us recall a Higgs field ([**65**]) of a $\boldsymbol{c}$-parabolic sheaf on (X,D). A holomorphic homomorphism $\theta:\mathcal{E}\longrightarrow\mathcal{E}\otimes\Omega^{1,0}_X(\log D)$ is called a Higgs field of $\mathcal{E}_*$, if the following holds:

- The naturally defined composite $\theta^2=\theta\wedge\theta:\mathcal{E}\longrightarrow\mathcal{E}\otimes\Omega^{2,0}_X(\log D)$ vanishes.
- $\theta\big({}^i\mathcal{F}_a\big)\subset{}^i\mathcal{F}_a\otimes\Omega^{1,0}_X(\log D)$

Such a tuple $(\mathcal{E}_*,\theta)$ is called a $\boldsymbol{c}$-parabolic Higgs sheaf on (X,D).

A $\boldsymbol{c}$-parabolic Higgs sheaf $(\mathcal{E}_*,\theta)$ on (X,D) is called reflexive and saturated, if the underlying $\boldsymbol{c}$-parabolic sheaf is reflexive and saturated. A morphism between $\boldsymbol{c}$-parabolic Higgs sheaves is defined to be a morphism of the underlying sheaf which is compatible with the parabolic structures and the Higgs fields.

Lemma 3.2. — *Let $(\mathcal{E}_*,\theta)$ be any $\boldsymbol{c}$-parabolic Higgs sheaf on (X,D). Then there exists the reflexive saturated parabolic Higgs sheaf $(\mathcal{E}'_*,\theta')$, such that we have the morphism $(\mathcal{E}_*,\theta)\longrightarrow(\mathcal{E}'_*,\theta')$ which is isomorphic in codimension one, i.e. isomorphic outside of the subset with codimension two. Such $(\mathcal{E}'_*,\theta')$ is unique up to the canonical isomorphism.*

Proof. — Let $\mathcal{E}'$ denote the double dual of $\mathcal{E}$. We have the canonical morphism $\mathcal{E}\longrightarrow\mathcal{E}'$ which is isomorphic outside of the subset Z of codimension two. Let ${}^i\mathcal{F}^1_a$ denote the subsheaf of $\mathcal{E}'$ which consists of the sections f of $\mathcal{E}'$ such that $f_{|X-Z}\in{}^i\mathcal{F}_a$. Such a subsheaf is coherent ([**60**]). We have $\mathcal{E}'(-D_i)\subset{}^i\mathcal{F}^1_a$ for any $a\in]c_i-1,c_i]$. We have the natural surjection $\pi_{i,a}:\mathcal{E}'\longrightarrow\mathcal{E}'/{}^i\mathcal{F}^1_a$, and the target is the $\mathcal{O}_{D_i}$-module. Let $T_{i,a}$ denote the torsion part of $\mathcal{E}'/{}^i\mathcal{F}^1_a$ as an $\mathcal{O}_{D_i}$-module, and we put ${}^i\mathcal{F}'_a:=\pi^{-1}_{i,a}(T_{i,a})$. Then, it is easy to see that $\big\{{}^i\mathcal{F}'\,\big|\,i\in S\big\}$ gives the saturated $\boldsymbol{c}$-parabolic structure of $\mathcal{E}'$. The Higgs field θ naturally induces the morphism $\mathcal{E}\longrightarrow\mathcal{E}'\otimes\Omega^{1,0}_X(\log D)$. Due to the reflexivity of $\mathcal{E}'$, we obtain $\theta':\mathcal{E}'\longrightarrow\mathcal{E}'\otimes\Omega^{1,0}_X(\log D)$ satisfying $\theta^2=0$. It is easy to check $\theta({}^i\mathcal{F}'_a)\subset{}^i\mathcal{F}'_a\otimes\Omega^{1,0}_X(\log D)$. The uniqueness is clear. □

For a $\boldsymbol{c}$-parabolic Higgs sheaves $(\mathcal{E}_{i\,*},\theta_i)$ $(i=1,2)$ on (X,D), we obtain the sheaf of the morphisms $\mathcal{H}om\big((\mathcal{E}_{1\,*},\theta_1),(\mathcal{E}_{2\,*},\theta_2)\big)$.

Lemma 3.3. — *If $(\mathcal{E}_{2\,*},\theta_2)$ is reflexive and saturated, $\mathcal{H}om\big((\mathcal{E}_{1\,*},\theta_1),(\mathcal{E}_{2\,*},\theta_2)\big)$ is reflexive.*

Proof. — We have only to check the condition in Lemma 2.19. Let f be a section of $\mathcal{H}om\big((\mathcal{E}_{1\,*},\theta_1),(\mathcal{E}_{2\,*},\theta_2)\big)$ on $U\setminus Z$, where U denotes an open subset and Z denotes a closed subset with $\operatorname{codim}(Z) \geq 2$. Since $\mathcal{E}_2$ is reflexive, it is extended to the homomorphism $\widetilde{f} : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ on U, which is compatible with θ_i. We have the induced map $\varphi : {}^i\mathcal{F}(\mathcal{E}_1) \longrightarrow \mathcal{E}_2/{}^i\mathcal{F}(\mathcal{E}_2)$. The codimension of the support of $\operatorname{Im}(\varphi)$ is larger than 2, and $\mathcal{E}_2/{}^i\mathcal{F}(\mathcal{E}_2)$ is a torsion-free $\mathcal{O}_{D_i}$-module. Hence, we obtain $\operatorname{Im}(\varphi) = 0$, i.e., $\widetilde{f}$ preserves the filtration. □

Assume X is projective. Let Y be a sufficiently ample and generic hypersurface of X. We put $D_Y := D \cap Y$, which is assumed to be a simple normal crossing divisor of Y. Let $(\mathcal{E}_{i\,*|Y}, \theta_{iY})$ denote the induced parabolic Higgs sheaf on (Y, D_Y) by $(\mathcal{E}_{i\,*}, \theta_i)$. If $\mathcal{E}_{i\,*}$ is reflexive and saturated, so is $\mathcal{E}_{i\,*|Y}$. (See Corollary 3.1.1 of [**41**].)

Lemma 3.4. — *Assume* $\dim X \geq 2$ *and that* $\mathcal{E}_{2\,*}$ *is saturated and reflexive. For any morphism* $f : (\mathcal{E}_{1\,*|Y}, \theta_{1\,Y}) \longrightarrow (\mathcal{E}_{2\,*|Y}, \theta_{2\,Y})$, *we have* $F : (\mathcal{E}_{1\,*}, \theta_1) \longrightarrow (\mathcal{E}_{2\,*}, \theta_2)$ *which induces* f.

Proof. — Let $\theta_{i|Y} : \mathcal{E}_{i*|Y} \longrightarrow \mathcal{E}_{i*|Y} \otimes \Omega_X^{1,0}(\log D)_{|Y}$ denote the restriction of θ_i to Y. We have the induced morphism $G : f \circ \theta_{1|Y} - \theta_{2|Y} \circ f : \mathcal{E}_{1*|Y} \longrightarrow \mathcal{E}_{2*|Y} \otimes \Omega_X^{1,0}(\log D)_{|Y}$. Because of $f \circ \theta_{1\,Y} - \theta_{2\,Y} \circ f = 0$ in $\mathcal{H}om\big(\mathcal{E}_{1\,*|Y}, \mathcal{E}_{2\,*|Y}\big) \otimes \Omega_Y^{1,0}(\log D_Y)$, G induces the map $\mathcal{E}_{1\,*|Y} \longrightarrow \mathcal{E}_{2\,*|Y} \otimes \mathcal{O}(-Y)_{|Y}$. We regard it as the section of $\mathcal{J} := \mathcal{H}om\big(\mathcal{E}_{1\,*}, \mathcal{E}_{2\,*}\big) \otimes \mathcal{O}(-Y)_{|Y}$. Since $\mathcal{G} := \mathcal{H}om\big(\mathcal{E}_{1\,*}, \mathcal{E}_{2\,*}\big)$ is reflexive, we have $H^i\big(X, \mathcal{G} \otimes \mathcal{O}(-Y)\big) = 0$ $(i = 0, 1)$, if Y is sufficiently ample. (See the proof of Proposition 3.2 in [**41**].) Hence, we have $H^0(Y, \mathcal{J}) = 0$, i.e., $G = 0$. Then, the claim of the lemma follows from Generalized Enriques Severi Lemma (Proposition 3.2 in [**41**]) and Lemma 3.3. □

Remark 3.5. — We also have the parallel notion of $\boldsymbol{c}$-parabolic sheaves on smooth varieties with simple normal crossing divisors over a field k.

Remark 3.6. — Sometimes, it will be convenient to consider filtrations ${}^i\mathcal{F}$ such that $S({}^i\mathcal{F}) = \big\{a \in \boldsymbol{R} \,\big|\, {}^i\operatorname{Gr}_a^{\mathcal{F}}(\mathcal{E}) \neq 0\big\}$ is not contained in an interval $]c_i - 1, c_i]$ for some c_i. In that case, we will call $\{{}^i\mathcal{F} \mid i \in S\}$ a generalized parabolic structure. Higgs field is also defined as in the standard case, i.e., a holomorphic map $\theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X^{1,0}(\log D)$ such that $\theta^2 = 0$ and $\theta\big({}^i\mathcal{F}_a\big) \subset {}^i\mathcal{F}_a \otimes \Omega_X^{1,0}(\log D)$.

3.1.2. The parabolic first Chern class and the degree.

— For a $\boldsymbol{c}$-parabolic sheaf $\mathcal{E}_*$ on (X, D), we put as follows:

$$\operatorname{wt}(\mathcal{E}_*, i) := \sum_{a \in]c_i - 1, c_i]} a \cdot \operatorname{rank}_{D_i} {}^i\operatorname{Gr}_a^{\mathcal{F}}(\mathcal{E}).$$

Here $\operatorname{rank}_{D_i} {}^i\operatorname{Gr}_a^{\mathcal{F}}(\mathcal{E})$ denotes the rank as an $\mathcal{O}_{D_i}$-module. In the following, we will often denote it by $\operatorname{rank} {}^i\operatorname{Gr}_a^{\mathcal{F}}(\mathcal{E})$, if there are no risk of confusion. The parabolic first Chern class of $\mathcal{E}_*$ is defined as follows:

$$\operatorname{par-c}_1(\mathcal{E}_*) := c_1(\mathcal{E}) - \sum_{i \in S} \operatorname{wt}(\mathcal{E}_*, i) \cdot [D_i] \in H^2(X, \boldsymbol{R}).$$

Here $[D_i]$ denotes the cohomology class given by D_i. If X is an n-dimensional compact Kahler manifold with a Kahler form ω, we put as follows:

$$\text{par-deg}_\omega(\mathcal{E}_*) := \int_X \text{par-c}_1(\mathcal{E}_*) \cdot \omega^{n-1}, \quad \mu_\omega(\mathcal{E}_*) := \frac{\text{par-deg}_\omega(\mathcal{E}_*)}{\operatorname{rank}\mathcal{E}}.$$

If ω is the first Chern class of an ample line bundle L, we also use the notation $\text{par-deg}_L(\mathcal{E}_*)$ and $\mu_L(\mathcal{E}_*)$.

Lemma 3.7. — *Let $\mathcal{E}_*^{(i)}$ $(i = 1, 2)$ be $\boldsymbol{c}$-parabolic sheaves on (X, D), and let $f : \mathcal{E}_*^{(1)} \longrightarrow \mathcal{E}_*^{(2)}$ be a morphism which is generically isomorphic. Then, we have $\mu(\mathcal{E}_*^{(1)}) \leq \mu(\mathcal{E}_*^{(2)})$. If the equality occurs, f is isomorphic in codimension one.*

Proof. — By considering the restriction to a generic complete intersection curve, we have only to discuss the case $\dim X = 1$. Let P be any point of D. We put $F_a^{(i)} := \operatorname{Im}\bigl({}^P\mathcal{F}_a(\mathcal{E}^{(i)})_{|P} \longrightarrow \mathcal{E}_{|P}^{(i)}\bigr)$ for $a \in]c(P)-1, c(P)]$, which give the filtration $F^{(i)}$ of $\mathcal{E}_{|P}^{(i)}$. We have the induced map $f_{|P} : \mathcal{E}_{|P}^{(1)} \longrightarrow \mathcal{E}_{|P}^{(2)}$ which preserves the filtrations. We put $I := \operatorname{Im}(f_{|P})$, $K := \operatorname{Ker}(f_{|P})$ and $C := \operatorname{Cok}(f_{|P})$. Let $F(K)$ (resp. $F^{(1)}(I)$) denote the induced filtration on K (resp. I) by $F^{(1)}$. Let $F(C)$ (resp. $F^{(2)}(I)$) denote the induced filtration on C (resp. I) by $F^{(2)}$. We put as follows:

$$w(K) := \sum a \cdot \operatorname{Gr}_a^F(K), \quad w^{(i)}(I) := \sum a \cdot \operatorname{Gr}_a^{F^{(i)}}(I), \quad w(C) := \sum a \cdot \operatorname{Gr}_a^F(C)$$

Then, we have $-w^{(1)}(I) \leq -w^{(2)}(I)$ and $-w(K) < -w(C) + r_0$, where $r_0 = \operatorname{rank} K = \operatorname{rank} C$. It is easy to obtain the claims of the lemma from these relations. □

Remark 3.8. — For the parabolic first Chern class on algebraic varieties, we have only to replace the cohomology group and the integral by the Chow group and the degree of the 0-cycles.

3.1.3. μ_L-Stability. — Let X be a smooth projective variety with an ample line bundle L over a field k, and D be a simple normal crossing divisor of X. The μ_L-stability of $\boldsymbol{c}$-parabolic Higgs sheaves is defined as usual. Namely, a $\boldsymbol{c}$-parabolic Higgs sheaf $(\mathcal{E}_*, \theta)$ is called μ_L-stable, if the inequality $\text{par-deg}_L(\mathcal{E}'_*) < \text{par-deg}_L(\mathcal{E}_*)$ holds for any saturated non-trivial subsheaf $\mathcal{E}' \subsetneq \mathcal{E}$ such that $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega^{1,0}(\log D)$. (Recall a subsheaf $\mathcal{E}' \subset \mathcal{E}$ is called saturated, if $\mathcal{E}/\mathcal{E}'$ is torsion-free.) Here the parabolic structure of $\mathcal{E}'_*$ is the naturally induced one from the parabolic structure of $\mathcal{E}_*$. Similarly, μ_L-semistability and μ_L-polystability are also defined in a standard manner.

Let $\bigl(\mathcal{E}_*^{(i)}, \theta^{(i)}\bigr)$ $(i = 1, 2)$ be μ_L-semistable $\boldsymbol{c}$-parabolic Higgs sheaves such that $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. Let $f : (\mathcal{E}_*^{(1)}, \theta^{(1)}) \longrightarrow (\mathcal{E}_*^{(2)}, \theta^{(2)})$ be a non-trivial morphism. Let $(\mathcal{K}_*, \theta_{\mathcal{K}})$ denote the kernel of f with the naturally induced parabolic structure and the Higgs field. Let $\mathcal{I}$ denote the image of f, and $\widetilde{\mathcal{I}}$ denote the saturated subsheaf of $\mathcal{E}^{(2)}$ generated by $\mathcal{I}$. The parabolic structures of $\mathcal{E}_*^{(1)}$ and $\mathcal{E}_*^{(2)}$ induce the parabolic structures of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$, respectively. We denote the induced parabolic sheaves by $(\mathcal{I}_*, \theta_{\mathcal{I}})$ and $(\widetilde{\mathcal{I}}_*, \theta_{\widetilde{\mathcal{I}}})$.

Lemma 3.9. — *$(\mathcal{K}_*, \theta_{\mathcal{K}})$, $(\mathcal{I}_*, \theta_{\mathcal{I}})$ and $(\widetilde{\mathcal{I}}_*, \theta_{\widetilde{\mathcal{I}}})$ are also μ_L-semistable such that $\mu_L(\mathcal{K}_*) = \mu_L(\mathcal{I}_*) = \mu_L(\widetilde{\mathcal{I}}_*) = \mu_L(\mathcal{E}_*^{(i)})$. Moreover, $\mathcal{I}_*$ and $\widetilde{\mathcal{I}}_*$ are isomorphic in codimension one.*

Proof. — Using Lemma 3.7 and μ_L-semistability of $(\mathcal{E}_*^{(i)}, \theta^{(i)})$, we have $\mu(\mathcal{E}_*^{(1)}) \leq \mu(\mathcal{I}_*) \leq \mu(\widetilde{\mathcal{I}}_*) \leq \mu(\mathcal{E}_*^{(2)})$. Since the equalities hold, the claim of the lemma follows. □

Lemma 3.10. — *Let $\big(\mathcal{E}_*^{(i)}, \theta^{(i)}\big)$ $(i = 1, 2)$ be μ_L-semistable reflexive saturated parabolic Higgs sheaves such that $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. Assume either one of the following:*

1. *One of $(\mathcal{E}_*^{(i)}, \theta^{(i)})$ is μ_L-stable, and $\operatorname{rank}(\mathcal{E}^{(1)}) = \operatorname{rank}(\mathcal{E}^{(2)})$ holds.*
2. *Both of $(\mathcal{E}_*^{(i)}, \theta^{(i)})$ are μ_L-stable.*

If there is a non-trivial map $f : (\mathcal{E}_*^{(1)}, \theta^{(1)}) \longrightarrow (\mathcal{E}_*^{(2)}, \theta^{(2)})$, then f is isomorphic.

Proof. — If $(\mathcal{E}_*^{(1)}, \theta^{(1)})$ is μ_L-stable, the kernel of f is trivial due to Lemma 3.9. If $(\mathcal{E}_*^{(2)}, \theta^{(2)})$ is μ_L-stable, the image of f and $\mathcal{E}^{(2)}$ are same at the generic point of X. Thus, we obtain that f is generically isomorphic in any case. Then, we obtain that f is isomorphic in codimension one, due to Lemma 3.7. Since both of $\mathcal{E}_*^{(i)}$ are reflexive and saturated, we obtain that f is isomorphic. □

Corollary 3.11. — *Let $(\mathcal{E}_*, \theta)$ be a μ_L-polystable reflexive saturated Higgs sheaf. Then we have the unique decomposition:*

$$(\mathcal{E}_*, \theta) = \bigoplus_j (\mathcal{E}_*^{(j)}, \theta^{(j)}) \otimes \boldsymbol{C}^{m(j)}.$$

Here, $(\mathcal{E}_*^{(j)}, \theta^{(j)})$ are μ_L-stable with $\mu_L(\mathcal{E}_*^{(j)}) = \mu(\mathcal{E}_*)$, and they are mutually non-isomorphic. It is called the canonical decomposition in the rest of the paper.

3.1.4. $\boldsymbol{c}$-Parabolic Higgs bundle in codimension k. — We will often use the notation ${}_{\boldsymbol{c}}E$ instead of $\mathcal{E}$. We put as follows, for each $i \in S$:

$${}^iF_a\big({}_{\boldsymbol{c}}E_{|D_i}\big) := \operatorname{Im}\Big({}^i\mathcal{F}_a({}_{\boldsymbol{c}}E)_{|D_i} \longrightarrow {}_{\boldsymbol{c}}E_{|D_i}\Big).$$

The tuple $\big({}^i\mathcal{F} \,\big|\, i \in S\big)$ can clearly be reconstructed from the tuple of the filtrations $\boldsymbol{F} := \big({}^iF \,\big|\, i \in S\big)$. Hence we will often consider $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}\big)$ instead of $\big({}_{\boldsymbol{c}}E, \{{}^i\mathcal{F} \,|\, i \in S\}\big)$, when ${}_{\boldsymbol{c}}E$ is locally free. We put $D_I := \bigcap_{i \in I} D_i$ for any subset $I \subset S$, on which we have the induced filtrations ${}^I\boldsymbol{F} := \big({}^iF_{|D_I} \,\big|\, i \in I\big)$ of ${}_{\boldsymbol{c}}E_{|D_I}$.

Definition 3.12. — Let ${}_{\boldsymbol{c}}E_* = ({}_{\boldsymbol{c}}E, \boldsymbol{F})$ be a $\boldsymbol{c}$-parabolic sheaf such that ${}_{\boldsymbol{c}}E$ is locally free. If the following conditions are satisfied, ${}_{\boldsymbol{c}}E_*$ is called a $\boldsymbol{c}$-parabolic bundle.

- Each ${}^iF$ of ${}_{\boldsymbol{c}}E_{|D_i}$ is the filtration in the category of vector bundles on D_i. Namely, ${}^i\operatorname{Gr}_a^F({}_{\boldsymbol{c}}E_{|D_i}) = {}^iF_a/{}^iF_{<a}$ are locally free $\mathcal{O}_{D_i}$-modules.

– The tuple of the filtrations $\boldsymbol{F}$ is compatible in the sense of Definition 4.37 in [**44**]. (In this case, the decompositions are trivial.) Namely, for any subset $I \subset S$ we have a decomposition $\bigoplus_{\boldsymbol{a}\in \boldsymbol{R}^I} U_{\boldsymbol{a}} = {}_{\boldsymbol{c}}E_{|D_I}$ locally on D_I, such that $\bigcap_{i\in I} {}^iF_{a_i|D_I} = \bigoplus_{\boldsymbol{b}\leq \boldsymbol{a}} U_{\boldsymbol{b}}$.

We remark that the second condition is trivial in the case $\dim X = 2$.

Remark 3.13. — Our compatibility condition of the parabolic filtrations are same as the “locally abelian” condition given in [**27**]. (See Corollary 4.48 of [**44**], for example.)

The notion of $\boldsymbol{c}$-parabolic bundle is too restrictive in the case $\dim X > 2$. Hence we will also use the following notion in the case $k = 2$.

Definition 3.14. — Let ${}_{\boldsymbol{c}}E_*$ be a $\boldsymbol{c}$-parabolic sheaf on (X, D). It is called a $\boldsymbol{c}$-parabolic bundle in codimension k, if the following condition is satisfied:

– There is a Zariski closed subset $Z \subset D$ with $\operatorname{codim}_X(Z) > k$ such that the restriction of ${}_{\boldsymbol{c}}E_*$ to $(X - Z, D - Z)$ is a $\boldsymbol{c}$-parabolic bundle.

It is easy to observe that a reflexive saturated $\boldsymbol{c}$-parabolic Higgs sheaf is a $\boldsymbol{c}$-parabolic Higgs bundle in codimension two.

3.1.5. The characteristic number for $\boldsymbol{c}$-parabolic bundle in codimension two. — For any $\boldsymbol{c}$-parabolic bundle ${}_{\boldsymbol{c}}E_*$ in codimension two, the parabolic second Chern character $\text{par-ch}_2({}_{\boldsymbol{c}}E_*) \in H^4(X, \boldsymbol{R})$ is defined as follows:

$$\text{par-ch}_2({}_{\boldsymbol{c}}E_*) := \operatorname{ch}_2({}_{\boldsymbol{c}}E) - \sum_{\substack{i\in S\\ a\in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)}} a \cdot \iota_{i\,*}\Big(c_1\big({}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E)\big)\Big) + \frac{1}{2}\sum_{\substack{i\in S\\ a\in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)}} a^2 \cdot \operatorname{rank}\big({}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E)\big)\cdot [D_i]^2 + \frac{1}{2}\sum_{\substack{(i,j)\in S^2\\ i\neq j}} \sum_{\substack{P\in \operatorname{Irr}(D_i\cap D_j)\\ (a_i,a_j)\in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, P)}} a_i\cdot a_j \cdot \operatorname{rank} {}^P\operatorname{Gr}^F_{(a_i,a_j)}({}_{\boldsymbol{c}}E)\cdot [P]. \tag{10}$$

Let us explain some of the notation:

– $\operatorname{ch}_2({}_{\boldsymbol{c}}E)$ denotes the second Chern character of ${}_{\boldsymbol{c}}E$.

– ι_i denotes the closed immersion $D_i \longrightarrow X$, and $\iota_{i\,*} : H^2(D_i) \longrightarrow H^4(X)$ denotes the associated Gysin map.

– $\operatorname{Irr}(D_i \cap D_j)$ denotes the set of the irreducible components of $D_i \cap D_j$.

– Let P be an element of $\operatorname{Irr}(D_i \cap D_j)$. The generic point of the component is also denoted by P. We put ${}^PF_{(a,b)} := {}^iF_{a\,|\,P} \cap {}^jF_{b\,|\,P}$ and ${}^P\operatorname{Gr}^F_{\boldsymbol{a}} := {}^PF_{\boldsymbol{a}} \big/ \sum_{\boldsymbol{a}'\lneq \boldsymbol{a}} {}^PF_{\boldsymbol{a}'}$. Then $\operatorname{rank} {}^P\operatorname{Gr}^F_{\boldsymbol{a}}$ denotes the rank of ${}^P\operatorname{Gr}^F_{\boldsymbol{a}}$ as an $\mathcal{O}_P$-module.

– We put $\mathcal{P}ar({}_{\boldsymbol{c}}E_*, P) := \big\{\boldsymbol{a} \,\big|\, {}^P\operatorname{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E) \neq 0\big\}$.

– $[D_i] \in H^2(X, \boldsymbol{R})$ and $[P] \in H^4(X, \boldsymbol{R})$ denote the cohomology classes given by D_i and P respectively.

If X is an n-dimensional compact Kahler manifold with a Kahler form ω, we put as follows:

$$\text{par-ch}_{2,\omega}({}_{\boldsymbol{c}}E_*) := \text{par-ch}_2({}_{\boldsymbol{c}}E_*) \cdot \omega^{n-2}, \quad \text{par-c}^2_{1,\omega}({}_{\boldsymbol{c}}E_*) := \text{par-c}_1({}_{\boldsymbol{c}}E_*)^2 \cdot \omega^{n-2}.$$

If ω is the first Chern class of an ample line bundle L, we use the notation $\text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*)$ and $\text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*)$. In the case $\dim X = 2$, we have the obvious equalities $\text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*) = \text{par-c}^2_1({}_{\boldsymbol{c}}E_*)$ and $\text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \text{par-ch}_2({}_{\boldsymbol{c}}E_*)$.

Definition 3.15. — Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a μ_L-polystable reflexive saturated $\boldsymbol{c}$-parabolic Higgs sheaf on (X, D). We say that $({}_{\boldsymbol{c}}E_*, \theta)$ has trivial characteristic numbers, if any stable component $({}_{\boldsymbol{c}}E'_*, \theta')$ of $({}_{\boldsymbol{c}}E_*, \theta)$ satisfies $\text{par-deg}_L({}_{\boldsymbol{c}}E'_*) = \int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E'_*) = 0$

3.2. Filtered Sheaf

3.2.1. Definitions. — We recall the notion of filtered sheaf by following [**52**]. Let X be a complex manifold, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. For $\boldsymbol{a} \in \boldsymbol{R}^S$, a_i denotes the i-th component of $\boldsymbol{a}$ for $i \in S$. A filtered sheaf on (X, D) is defined to be a tuple $\boldsymbol{E}_* = \big(\boldsymbol{E}, \{{}_{\boldsymbol{c}}E \,|\, \boldsymbol{c} \in \boldsymbol{R}^S\}\big)$ as follows:

– $\boldsymbol{E}$ is a quasi coherent $\mathcal{O}_X$-module. We put $E := \boldsymbol{E}_{|X-D}$.
– ${}_{\boldsymbol{c}}E$ are coherent $\mathcal{O}_X$-submodules of $\boldsymbol{E}$ for any $\boldsymbol{c} \in \boldsymbol{R}^S$ such that ${}_{\boldsymbol{c}}E_{|X-D} = E$.
– In the case $\boldsymbol{a} \leq \boldsymbol{b}$, we have ${}_{\boldsymbol{a}}E \subset {}_{\boldsymbol{b}}E$, where $\boldsymbol{a} \leq \boldsymbol{b}$ means $a_i \leq b_i$ for all $i \in S$. We also have $\bigcup_{\boldsymbol{a} \in \boldsymbol{R}^S} {}_{\boldsymbol{a}}E = \boldsymbol{E}$ and ${}_{\boldsymbol{a}}E = \bigcap_{\boldsymbol{a} < \boldsymbol{b}} {}_{\boldsymbol{b}}E$.
– We have ${}_{\boldsymbol{a}'}E = {}_{\boldsymbol{a}}E \otimes \mathcal{O}_X(-\sum n_j \cdot D_j)$ as submodules of $\boldsymbol{E}$, where $\boldsymbol{a}' = \boldsymbol{a} - (n_j \,|\, j \in S)$ for some integers n_j.
– For each $\boldsymbol{c} \in \boldsymbol{R}^S$, the filtration ${}^i\mathcal{F}$ of ${}_{\boldsymbol{c}}E$ indexed by $]c_i - 1, c_i]$ is given as follows:

$$^i\mathcal{F}_d({}_{\boldsymbol{c}}E) := \bigcup_{\substack{a_i \leq d \\ \boldsymbol{a} \leq \boldsymbol{c}}} {}_{\boldsymbol{a}}E.$$

Then the tuple $\big({}_{\boldsymbol{c}}E, \{{}^i\mathcal{F} \,|\, i \in S\}\big)$ is a $\boldsymbol{c}$-parabolic sheaf, i.e., the sets $\big\{a \in \,]c_i - 1, c_i] \,\big|\, {}^i\mathrm{Gr}^{\mathcal{F}}_a({}_{\boldsymbol{c}}E) \neq 0\big\}$ are finite.

Remark 3.16. — By definition, we obtain the $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}E_*$ obtained from filtered sheaf $\boldsymbol{E}_*$ for any $\boldsymbol{c} \in \boldsymbol{R}^S$, which is called the $\boldsymbol{c}$-truncation of $\boldsymbol{E}_*$. On the other hand, a filtered sheaf $\boldsymbol{E}_*$ can be reconstructed from any $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}E_*$. So we can identify them.

Definition 3.17. — A filtered sheaf $\boldsymbol{E}_*$ is called reflexive and saturated, if any $\boldsymbol{c}$-truncations are reflexive and saturated.

A filtered sheaf $\boldsymbol{E}_*$ is called a filtered bundle in codimension k, if any $\boldsymbol{c}$-truncations are $\boldsymbol{c}$-parabolic bundle in codimension k.

Remark 3.18. — In the definition, "any $\boldsymbol{c}$" can be replaced with "some $\boldsymbol{c}$".

A Higgs field of $\boldsymbol{E}_*$ is defined to be a holomorphic homomorphism $\theta : \boldsymbol{E} \longrightarrow \boldsymbol{E} \otimes \Omega^{1,0}(\log D)$ satisfying $\theta({}_{\boldsymbol{c}}E) \subset {}_{\boldsymbol{c}}E \otimes \Omega^{1,0}_X(\log D)$.

Let $\boldsymbol{E}^{(i)}_*$ $(i = 1, 2)$ be a filtered bundle on (X, D). We put as follows:

$$\widetilde{\boldsymbol{E}} := Hom(\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}), \qquad {}_{\boldsymbol{a}}\widetilde{E} := \{f \in \widetilde{\boldsymbol{E}} \,|\, f({}_{\boldsymbol{c}}E^{(1)}) \subset {}_{\boldsymbol{c}+\boldsymbol{a}}E^{(2)},\ \forall \boldsymbol{c}\}.$$

$$\widehat{\boldsymbol{E}} := \boldsymbol{E}^{(1)} \otimes \boldsymbol{E}^{(2)}, \qquad {}_{\boldsymbol{a}}\widehat{E} := \sum_{\boldsymbol{a}_1+\boldsymbol{a}_2\leq \boldsymbol{a}} {}_{\boldsymbol{a}_1}E^{(1)} \otimes {}_{\boldsymbol{a}_2}E^{(2)}.$$

Then $(\widetilde{\boldsymbol{E}}, \{{}_{\boldsymbol{a}}\widetilde{E}\})$ and $(\widehat{\boldsymbol{E}}, \{{}_{\boldsymbol{a}}\widehat{E}\})$ are also filtered bundles. They are denoted by $Hom(\boldsymbol{E}^{(1)}_*, \boldsymbol{E}^{(2)}_*)$ and $\boldsymbol{E}^{(1)}_* \otimes \boldsymbol{E}^{(2)}_*$.

Let $(\boldsymbol{E}_*, \theta)$ be a regular filtered Higgs bundle. Let a and b be non-negative integers. Applying the above construction, we obtain the parabolic structures and the Higgs fields on $T^{a,b}(\boldsymbol{E}) := Hom(\boldsymbol{E}^{\otimes a}, \boldsymbol{E}^{\otimes b})$. We denote it by $(T^{a,b}\boldsymbol{E}_*, \theta)$.

3.2.2. The characteristic numbers of filtered bundles in codimension two

Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $\boldsymbol{E}_*$ be a filtered bundle in codimension two on (X, D).

Lemma 3.19. — *For any* $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, *we have* $\text{par-c}_1({}_{\boldsymbol{c}}E_*) = \text{par-c}_1({}_{\boldsymbol{c}'}E_*)$ *in* $H^2(X, \boldsymbol{R})$.

Proof. — The j-th components of $\boldsymbol{c}$ and $\boldsymbol{c}'$ are denoted by c_j and c'_j for any $j \in S$. Take an element $i \in S$. We have only to consider the case $c_j = c'_j$ $(j \neq i)$. We may also assume $c'_i \in \mathcal{P}ar(\boldsymbol{E}_*, i)$ and $c_i < c'_i$. Moreover it can be assumed that c_i is sufficiently close to c'_i. Then we have the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \longrightarrow {}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}'}E \longrightarrow {}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E_{|D_i}) \longrightarrow 0.$$

We put $c := c'_i - 1$. Then we have the following:

$$(11) \quad {}^i\mathrm{Gr}^F_c({}_{\boldsymbol{c}}E) \otimes \mathcal{O}(D_i) \simeq {}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E), \qquad {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E) \simeq {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}'}E),\ (c < a < c'_i).$$

Therefore we have $\mathrm{wt}({}_{\boldsymbol{c}}E_*, i) = \mathrm{wt}({}_{\boldsymbol{c}'}E_*, i) - \mathrm{rank}\, {}^i\mathrm{Gr}^F_c({}_{\boldsymbol{c}}E)$. On the other hand, we have $c_1({}_{\boldsymbol{c}'}E) = c_1({}_{\boldsymbol{c}}E) + c_1(\iota_*{}^i\mathrm{Gr}^F_{c'}({}_{\boldsymbol{c}'}E))$. There is a closed subset $W \subsetneq D_i$ such that ${}^i\mathrm{Gr}^F_{c'}({}_{\boldsymbol{c}'}E)_{|D_i - W}$ is isomorphic to a direct sum of $\mathcal{O}_{D_i - W}$. We remark that $H^2(X, \boldsymbol{R}) \simeq H^2(X \setminus W, \boldsymbol{R})$, because the codimension of W in X is larger than two. Then it is easy to check $c_1(\iota_*{}^i\mathrm{Gr}^F_{c'}({}_{\boldsymbol{c}'}E)) = \mathrm{rank}\, {}^i\mathrm{Gr}^F_c({}_{\boldsymbol{c}}E) \cdot [D_i]$. Then the claim of the lemma immediately follows. □

Corollary 3.20. — *For any* $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, *we have the following:*

$$\text{par-deg}_L({}_{\boldsymbol{c}}E_*) = \text{par-deg}_L({}_{\boldsymbol{c}'}E_*), \qquad \int_X \text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*) = \int_X \text{par-c}^2_{1,L}({}_{\boldsymbol{c}'}E_*).$$

In particular, the characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) := \text{par-deg}_L({}_{\boldsymbol{c}}E_*)$ *and* $\int_X \text{par-c}^2_{1,L}(\boldsymbol{E}_*) := \int_X \text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*)$ *are well defined.*

Remark 3.21. — The μ_L-stability of a regular filtered Higgs bundle is defined, which is equivalent to the stability of any $\boldsymbol{c}$-truncation. Due to Corollary 3.20, it is independent of a choice of $\boldsymbol{c}$.

Proposition 3.22. — *For any* $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, *we have the following:*

$$\int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}'}E_*).$$

In particular, $\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) := \int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*)$ *is well defined.*

Proof. — We have only to consider the case $\dim X = 2$. We use the following lemma.

Lemma 3.23. — *Let* Y *be a smooth projective surface, and* D *be a smooth divisor of* Y. *Let* $\mathcal{F}$ *be an* $\mathcal{O}_D$*-coherent module. Then we have the following:*

$$\int_X \text{ch}_2(\iota_*\mathcal{F}) = \deg_D \mathcal{F} - \frac{1}{2}\operatorname{rank}_D(\mathcal{F}) \cdot (D, D).$$

Proof. — By considering the blow up of $D \times \{0\}$ in $Y \times \boldsymbol{C}$ as in [**17**], we can reduce the problem in the case Y is a projective space bundle over D. We can also reduce the problem to the case $\mathcal{F}$ is a locally free sheaf on D. Then, in particular, we may assume that there is a locally free sheaf $\widetilde{\mathcal{F}}$ such that $\widetilde{\mathcal{F}}_{|D} = \mathcal{F}$. In the case, we have the K-theoretic equality $\iota_*\mathcal{F} = \widetilde{\mathcal{F}} \cdot \big(\mathcal{O} - \mathcal{O}(-D)\big)$. Therefore we have the following:

$$\text{ch}(\iota_*\mathcal{F}) = \text{ch}(\widetilde{\mathcal{F}}) \cdot \big(D - D^2/2\big) = \operatorname{rank}\widetilde{\mathcal{F}} \cdot D + \left(-\frac{1}{2}\operatorname{rank}\widetilde{\mathcal{F}} \cdot D^2 + c_1(\widetilde{\mathcal{F}}) \cdot D\right).$$

Then the claim of the lemma is clear. □

Let us return to the proof of Lemma 3.22. We use the notation in the proof of Lemma 3.19. We have the following equalities:

$$\begin{aligned}(12) \quad \int_X \text{ch}_2({}_{\boldsymbol{c}'}E) &= \int_X \text{ch}_2({}_{\boldsymbol{c}}E) + \deg_{D_i}({}^i\operatorname{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)) - \frac{1}{2}\operatorname{rank}{}^i\operatorname{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E) \cdot D_i^2 \\ &= \int_X \text{ch}_2({}_{\boldsymbol{c}}E) + \deg_{D_i}({}^i\operatorname{Gr}^F_{c}({}_{\boldsymbol{c}}E)) + \frac{1}{2}\operatorname{rank}{}^i\operatorname{Gr}^F_{c}({}_{\boldsymbol{c}}E) \cdot D_i^2.\end{aligned}$$

Here we have used (11). We also have the following:

$$c'_i \cdot \deg_{D_i}({}^i\operatorname{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)) = (c+1) \cdot \Big(\deg_{D_i}({}^i\operatorname{Gr}^F_{c}({}_{\boldsymbol{c}}E)) + \operatorname{rank}{}^i\operatorname{Gr}^F_{c}({}_{\boldsymbol{c}}E) \cdot D_i^2\Big).$$

We remark the isomorphism ${}^{P}\mathrm{Gr}^{F}_{(c_i',a)}({}_{\boldsymbol{c}'}E) \simeq {}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E)$ and the following exact sequence:

$$0 \longrightarrow {}^{j}\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}}E) \longrightarrow {}^{j}\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}'}E) \longrightarrow \bigoplus_{P\in D_i\cap D_j} {}^{P}\mathrm{Gr}^{F}_{(c_i',a)}({}_{\boldsymbol{c}'}E) \longrightarrow 0.$$

Hence we obtain the following equality:

$$a\cdot \deg_{D_j}\big({}^{j}\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}'}E)\big) = a\cdot \deg_{D_j}\big({}^{j}\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}}E)\big) + a\cdot \sum_{P\in D_i\cap D_j} \operatorname{rank} {}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E).$$

We have the following equalities:

(13)

$$\frac{1}{2}c_i'^{2}\cdot \operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c_i'}({}_{\boldsymbol{c}'}E)\cdot D_i^2 = \frac{1}{2}c^2 \operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2 + \left(c+\frac{1}{2}\right)\cdot \operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c_i'}({}_{\boldsymbol{c}'}E)\cdot D_i^2.$$

(14) $c_i'\cdot a\cdot \operatorname{rank}{}^{P}\mathrm{Gr}^{F}_{(c_i',a)}\big({}_{\boldsymbol{c}'}E\big) = c\cdot a\cdot \operatorname{rank}{}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E) + a\cdot \operatorname{rank}{}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E).$

Then we obtain the following:

(15)

$$\int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}'}E_*) - \int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \deg_{D_i}({}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)) + \frac{1}{2}\operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2$$
$$-\deg_{D_i}({}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)) - (c+1)\operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)D_i^2 - \sum_{j\neq i}\sum_{P\in D_i\cap D_j}\sum_{a} a\cdot \operatorname{rank}{}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E)$$
$$+\left(c+\frac{1}{2}\right)\operatorname{rank}{}^{i}\mathrm{Gr}^{F}_{c}({}_{\boldsymbol{c}}E)D_i^2 + \sum_{j\neq i}\sum_{P\in D_i\cap D_j}\sum_{a} a\cdot \operatorname{rank}{}^{P}\mathrm{Gr}^{F}_{(c,a)}({}_{\boldsymbol{c}}E) = 0.$$

Thus we are done. □

Definition 3.24. — Let $(\boldsymbol{E}_*,\theta)$ be a μ_L-polystable reflexive saturated regular filtered Higgs sheaf on (X,D). We say that $(\boldsymbol{E}_*,\theta)$ has trivial characteristic numbers, if any stable component $(\boldsymbol{E}'_*,\theta')$ of $(\boldsymbol{E}_*,\theta)$ satisfies $\text{par-deg}(\boldsymbol{E}'_*) = \int_X \text{par-ch}_2(\boldsymbol{E}'_*) = 0$.

3.3. Perturbation of Parabolic Structure

Let X be a smooth projective *surface* over $\boldsymbol{C}$ with an ample line bundle L, and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S} D_i$. (Remark that each D_i is smooth by definition of *simple* normal crossing divisor. See Section 2.1.) Let $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle over (X,D). Due to the projectivity of D_i, the eigenvalues of $\mathrm{Res}_i(\theta)\in \mathrm{End}\big({}_{\boldsymbol{c}}E_{|D_i}\big)$ are constant. Hence we obtain the generalized eigen decomposition with respect to $\mathrm{Res}_i(\theta)$:

$${}^{i}\mathrm{Gr}^{F}_{a}\big({}_{\boldsymbol{c}}E_{|D_i}\big) = \bigoplus_{\alpha\in\boldsymbol{C}} {}^{i}\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}\big({}_{\boldsymbol{c}}E_{|D_i}\big).$$

Let $\mathcal{N}_i$ denote the nilpotent part of the induced endomorphism $\mathrm{Gr}^F \mathrm{Res}_i(\theta)$ on ${}^{i}\mathrm{Gr}^{F}_{a}({}_{\boldsymbol{c}}E_{|D_i})$.

Definition 3.25. — The $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ is called graded semisimple, if $\mathcal{N}_i$ are 0 for any $i \in S$.

For simplicity, we assume $c_i \notin \mathcal{P}ar\big({}_{\boldsymbol{c}}E_*, i\big)$ for any i, where $\boldsymbol{c} = (c_i \,|\, i \in S)$.

Proposition 3.26. — *Let ϵ be any positive number satisfying $\epsilon \cdot 100\,\mathrm{rank}(E) \leq \mathrm{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F})$. There exists a $\mathbf{c}$-parabolic structure $\boldsymbol{F}^{(\epsilon)} = \big({}^iF^{(\epsilon)} \,\big|\, i \in S\big)$ such that the following holds:*

- $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)})$ *is a graded semisimple* $\mathbf{c}$*-parabolic Higgs bundle.*
- *We have* $\mathrm{wt}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, i) = \mathrm{wt}({}_{\boldsymbol{c}}E, \boldsymbol{F}, i)$. *(See Subsection* 3.1.2 *for* wt.*) In particular, we have* $\text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F})$.
- *There is a constant C, which is independent of ϵ, such that the following holds:*
$$\left|\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) - \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F})\right| \leq C \cdot \epsilon,$$
- $\mathrm{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \epsilon$.

Such $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is called an ϵ-perturbation of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

Proof. — To take a refinement of the filtration ${}^iF$, we see the weight filtration induced on ${}^i\,\mathrm{Gr}^F$. Let η be a generic point of D_i. We have the weight filtration W_η of the nilpotent map $\mathcal{N}_{i,\eta}$ on ${}^i\,\mathrm{Gr}^F\big({}_{\boldsymbol{c}}E_{|D_i}\big)_{|\eta}$, which is indexed by $\mathbb{Z}$. We recall the following general lemma.

Lemma 3.27. — *Let C be a smooth irreducible projective curve over $\boldsymbol{C}$. The generic point of C is denoted by η, and let $K(\eta)$ denote the corresponding field. Let V be an algebraic vector bundle on C. The fiber of V over η is denoted by $V_{|\eta}$, which is the $K(\eta)$-vector space.*

If we are given a $K(\eta)$-vector subspace $V'_\eta \subset V_{|\eta}$, then there exists the unique vector subbundle V' of V, whose fiber over η is V'_η.

Proof. — We put $t := \mathrm{rank}\, V$ and $s := \mathrm{rank}\, V'_\eta$. Let $G(t,s)$ denote the Grassmann variety of the s-dimensional subspaces of $\boldsymbol{C}^t$. Let Q be any closed point of C. We take a local frame $u_1, \dots, u_t$ of V on a Zariski neighbourhood of Q. Let $A(Q)$ denote the local ring at Q in C. The fraction field of $A(Q)$ is naturally isomorphic to $K(\eta)$. By using the frame $u_1, \dots, u_t$, we identify $V \otimes A(Q)$ and $A(Q)^{\oplus t}$. The $K(\eta)$-subspace V'_η of $A(Q)^{\oplus t} \otimes K(\eta) = K(\eta)^{\oplus t}$ gives the morphism $\varphi : \mathrm{Spec}\, K(\eta) \longrightarrow G(t,s)$ over $\mathrm{Spec}(\boldsymbol{C})$. Since A is a discrete valuation ring and $G(t,s)$ is proper, the morphism φ is uniquely extended to $\overline{\varphi} : \mathrm{Spec}(A) \longrightarrow G(t,s)$ by the valuative criterion for properness. (See Theorem 4.7 in [**20**], for example.) It gives the extension of V'_η around Q. □

By using the lemma, we can extend W_η to the filtration W of ${}^i\,\mathrm{Gr}^F\big({}_{\boldsymbol{c}}E_{|D_i}\big)$ in the category of vector bundles on D_i due to the smoothness of D_i and $\dim D_i = 1$. By our construction, $\mathcal{N}_i(W_k) \subset W_{k-2}$ and $\dim \mathrm{Gr}^W_k = \dim \mathrm{Gr}^W_{-k}$. The endomorphism

$\mathrm{Res}_i(\theta)$ preserves the filtration W on ${}^i\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$, and the nilpotent part of the induced endomorphisms on $\mathrm{Gr}^W\,{}^i\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ are trivial.

Let us take the refinement of the filtration ${}^iF$. For any $a \in]c_i - 1, c_i]$, we have the surjection $\pi_a : {}^iF_a({}_{\boldsymbol{c}}E_{|D_i}) \longrightarrow {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})$. We put ${}^i\widetilde{F}_{a,k} := \pi_a^{-1}(W_k)$. We use the lexicographic order on $]c_i - 1, c_i] \times \mathbb{Z}$. Thus we obtain the increasing filtration ${}^i\widetilde{F}$ indexed by $]c_i - 1, c_i] \times \mathbb{Z}$. The set $\widetilde{S}_i := \big\{(a,k) \in]c_i - 1, c_i] \times \mathbb{Z} \,\big|\, {}^i\mathrm{Gr}^{\widetilde{F}}_{(a,k)} \neq 0\big\}$ is finite.

Let $\varphi_i : \widetilde{S}_i \longrightarrow]c_i - 1, c_i]$ be the increasing map given by $\varphi_i(a,k) := a + k\epsilon$. We put as follows:

$$ {}^iF^{(\epsilon)}_b = \bigcup_{\varphi_i(a,k)\leq b} {}^i\widetilde{F}_{(a,k)} $$

Thus we obtain the $\boldsymbol{c}$-parabolic structure $\boldsymbol{F}^{(\epsilon)} = \big({}^iF^{(\epsilon)} \,\big|\, i \in S\big)$.

Let P be any point of D_i. Take a holomorphic coordinate neighbourhood (U_P, z_1, z_2) around P such that $U_P \cap D_i = \{z_1 = 0\}$. Then we have the expression $\theta = f_1(z_1,z_2)\cdot dz_1/z_1 + f_2(z_1,z_2)\cdot dz_2$. Then, $f_j(0,z_2)$ $(j = 1,2)$ preserve the filtration ${}^iF^{(\epsilon)}$. Therefore, it is easy to see that $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta\big)$ is $\boldsymbol{c}$-parabolic Higgs bundle on (X,D). By our construction, it has the desired property. Thus the proof of Proposition 3.26 is finished. □

The following proposition is standard.

Proposition 3.28. — *Assume that $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta\big)$ is μ_L-stable. If ϵ is sufficiently small, then the ϵ-perturbation $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta\big)$ is also μ_L-stable.*

Proof. — Let ${}_{\boldsymbol{c}}\widehat{E} \subset {}_{\boldsymbol{c}}E$ be a saturated subsheaf such that $\theta\big({}_{\boldsymbol{c}}\widehat{E}\big) \subset {}_{\boldsymbol{c}}\widehat{E} \otimes \Omega^{1,0}(\log D)$. Let $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{F}}^{(\epsilon)}$ be the tuples of the filtrations of ${}_{\boldsymbol{c}}\widehat{E}$ induced by $\boldsymbol{F}$ and $\boldsymbol{F}^{(\epsilon)}$ respectively. There is a constant C, which is independent of choices of ${}_{\boldsymbol{c}}\widehat{E}$ and small $\epsilon > 0$, such that $\big|\mu_L({}_{\boldsymbol{c}}\widehat{E}, \widehat{\boldsymbol{F}}) - \mu_L({}_{\boldsymbol{c}}\widehat{E}, \widehat{\boldsymbol{F}}^{(\epsilon)})\big| \leq C\cdot\epsilon$. Therefore, we have only to show the existence of a positive number η satisfying the inequalities $\mu_L({}_{\boldsymbol{c}}\widehat{E}, \boldsymbol{F}) + \eta < \mu_L({}_{\boldsymbol{c}}E, \boldsymbol{F})$, for any saturated Higgs subsheaf $0 \neq {}_{\boldsymbol{c}}\widehat{E} \subsetneq {}_{\boldsymbol{c}}E$ under the μ_L-stability of $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta\big)$. It is standard, so we give only a brief outline. Due to a lemma of A. Grothendieck (see Lemma 2.5 in [**19**]) we know the boundedness of the family $\mathcal{G}(A)$ of saturated Higgs subsheaves ${}_{\boldsymbol{c}}\widehat{E} \subsetneq {}_{\boldsymbol{c}}E$ such that $\deg_L({}_{\boldsymbol{c}}\widehat{E}) \geq -A$ for any fixed number A.

Let us consider the case where A is sufficiently large. Then $\mu_L({}_{\boldsymbol{c}}\widehat{E}_*)$ is sufficiently small for any ${}_{\boldsymbol{c}}\widehat{E} \notin \mathcal{G}(A)$. On the other hand, since the family $\mathcal{G}(A)$ is bounded, the function μ_L on $\mathcal{G}(A)$ have the maximum, which is strictly smaller than $\mu_L({}_{\boldsymbol{c}}E_*)$ due to the μ_L-stability. Thus we are done. □

3.4. Mehta-Ramanathan Type Theorem

3.4.1. Statement. — We discuss the Mehta-Ramanathan type theorem for parabolic Higgs sheaves. Let X be an n-dimensional smooth irreducible projective variety over $\boldsymbol{C}$ with an ample line bundle L. For simplicity, we assume the characteristic number of k is 0. Let D be a simple normal crossing divisor of X.

Proposition 3.29. — *Let (V_*, θ) be a parabolic Higgs sheaf over (X, D). It is μ_L-(semi)stable, if and only if $(V_*, \theta)_{|Y}$ is μ_L-(semi)stable, where Y denotes a complete intersection of sufficiently ample generic hypersurfaces.*

We closely follow the arguments of V. Mehta, A. Ramanathan ([**41**], [**40**]) and Simpson ([**55**]). See the papers for more detail.

3.4.2. $\mathcal{W}$-operator. — In the following, let k denote a field of characteristic 0. Let $\mathcal{X}$ be a smooth projective variety over k, with an ample line bundle L. Let $\mathcal{D}$ be a simple normal crossing divisor of $\mathcal{X}$. Let $\mathcal{W}$ be a vector bundle on $\mathcal{X}$. A $\mathcal{W}$-valued operator of a parabolic sheaf V_* on $(\mathcal{X}, \mathcal{D})$ is defined to be a morphism $\eta : V_* \longrightarrow V_* \otimes \mathcal{W}$. A $\mathcal{W}$-subobject of (V_*, η) is a saturated subsheaf $F \subset V$ such that $\eta(F) \subset F \otimes \mathcal{W}$. We endow F with the induced parabolic structure. A parabolic sheaf with a $\mathcal{W}$-valued operator (V_*, η) is defined to be μ_L-semistable if and only if $\mu_L(F_*) \leq \mu_L(V_*)$ holds for any $\mathcal{W}$-subobject $F_* \subset V_*$. The μ_L-stability is also defined similarly.

In general, we have the $\mathcal{W}$-subobjects $F_* \subset V_*$ with the properties: (i) $\mu_L(G_*) \leq \mu_L(F_*)$ for any $\mathcal{W}$-subobject G_* of (V_*, η), (ii) if $\mu_L(G_*) = \mu_L(F_*)$, we have $\operatorname{rank}(G) \leq \operatorname{rank}(F)$. Such F_* is uniquely determined, which can be shown by using an argument similar to the last part of the proof of Proposition 3.28. It is called the β-$\mathcal{W}$-subobject of (V_*, η). By a similar argument, we also obtain the Harder-Narasimhan filtration.

3.4.3. Weil's Lemma. — In general, for a given projective variety $\mathcal{X}$ with a normal crossing divisor $\mathcal{D} = \bigcup_{j \in S} \mathcal{D}_j$, a pair of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a tuple $\boldsymbol{a} = (a_j \mid j \in S) \in \boldsymbol{R}^S$ is called a parabolic line bundle on $(\mathcal{X}, \mathcal{D})$. We can regard them as the $\boldsymbol{a}$-parabolic sheaf on $(\mathcal{X}, \mathcal{D})$ in an obvious manner. Let $\operatorname{Pic}(\mathcal{X}, \mathcal{D})$ denote the set of parabolic line bundles on $(\mathcal{X}, \mathcal{D})$.

Let us return to the setting in Subsection 3.4.1. For simplicity, we assume $H^i(X, L^m) = 0$ for any $m \geq 1$ and $i > 0$. We put $S_m := H^0(X, L^m)$ for $m \in \mathbb{Z}_{\geq 1}$. For $\boldsymbol{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}_{\geq 1}^{n-1}$, we put $S_{\boldsymbol{m}} := \prod_{i=1}^{t} S_{m_i}$. Let $Z_{\boldsymbol{m}}$ denote the correspondence variety, i.e., $Z_{\boldsymbol{m}} = \big\{(x, s_1, \ldots, s_{n-1}) \in X \times S_{\boldsymbol{m}}, \bigm| s_i(x) = 0,\ 1 \leq i \leq n-1\big\}$. The natural morphisms $Z_{\boldsymbol{m}} \longrightarrow S_{\boldsymbol{m}}$ and $Z_{\boldsymbol{m}} \longrightarrow X$ are denoted by $q_{\boldsymbol{m}}$ and $p_{\boldsymbol{m}}$, respectively. We put $Z_{\boldsymbol{m}}^D := Z_{\boldsymbol{m}} \times_X D$ and $Z_{\boldsymbol{m}}^{D_j} := Z_{\boldsymbol{m}} \times_X D_j$. Recall that $Z_{\boldsymbol{m}}^{D_j}$ are irreducible, because $Z_{\boldsymbol{m}}^{D_j}$ is a vector bundle over D_j. Let $K_{\boldsymbol{m}}$ denote the function field of $S_{\boldsymbol{m}}$. We put $Y_{\boldsymbol{m}} := Z_{\boldsymbol{m}} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$, $Y_{\boldsymbol{m}}^{D_j} := Z_{\boldsymbol{m}}^{D_j} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$ and $Y_{\boldsymbol{m}}^D := Z_{\boldsymbol{m}}^D \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$.

The irreducible decomposition of $Z^D_{\boldsymbol{m}} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$ is given by $\bigcup_j Z^{D_j}_{\boldsymbol{m}} \times_{S_{\boldsymbol{m}}} K_{\boldsymbol{m}}$. Recall the following result of Mehta and Ramanathan, by whom such a type of lemma is called Weil's Lemma.

Lemma 3.30. — *Assume $n \geq 2$. For $\boldsymbol{m} = (m_1, \dots, m_{n-1})$ with each $m_i \geq 3$, the natural map* $\mathrm{Pic}(X, D) \longrightarrow \mathrm{Pic}(Y_{\boldsymbol{m}}, Y^D_{\boldsymbol{m}})$ *is bijective.*

Proof. — Since we have the natural correspondence between the irreducible components of D and $Y^D_{\boldsymbol{m}}$, the claim is obviously reduced to Proposition 2.1 of [**41**]. □

3.4.4. A family of degenerating curves. — As in [**41**], we fix a sequence of integers $(\alpha_1, \dots, \alpha_{n-1})$ with $\alpha_i \geq 2$. We put $\alpha := \prod \alpha_i$. For a positive integer m, let (m) denote $(\alpha_1^m, \dots, \alpha_{n-1}^m)$. Let V_* be a coherent parabolic sheaf on (X, D). For each m, we can take an open subset $U_m \subset S_{(m)}$ such that (i) $q^{-1}_{(m)}(s)$ are smooth $(s \in U_m)$, (ii) $q^{-1}_{(m)}(s)$ intersects with the smooth part of D transversally, (iii) V_* is a parabolic bundle on an appropriate neighbourhood of each $q^{-1}_{(m)}(s) \subset X$. In the following, we will shrink U_m, if necessary. In Section 5 of [**41**], Mehta and Ramanathan constructed a family of degenerating curves. Take integers $l > m > 0$. Let A be a discrete valuation ring over k with the quotient field K. Then there exists a curve C over $\mathrm{Spec}\, A$ with a morphism $\varphi : C \longrightarrow X \times \mathrm{Spec}\, A$ over $\mathrm{Spec}\, A$ with the properties: (i) C is smooth, (ii) the generic fiber C_K gives a sufficiently general K-valued point in U_l, (iii) the special fiber C_k is reduced with smooth irreducible components C^i_k $(i = 1, \dots, \alpha^{l-m})$ which are sufficiently general k-valued points in U_m. We use the notation D_C to denote $C \times_X D$. We also use the notation $D_{j,C}$, D_{j,C_K} and D_{j,C^i_k} in similar meanings. Then, we obtain the parabolic bundle $\varphi^*(V_*)$ on (C, D_C), which is denoted by $V_{*|C}$. The restriction to C_K and C^i_k are denoted similarly. Let W_* be a parabolic subsheaf of $V_{*\,|C_K}$. Recall that W can be extended to the subsheaf $\widetilde{W} \subset V_{|C}$, flat over $\mathrm{Spec}\, A$ with the properties: (i) $\widetilde{W}$ is a vector bundle over C, (ii) $\widetilde{W}_{|C^i_k} \longrightarrow V_{|C^i_k}$ are injective. (See Section 4 of [**41**].) In particular, we have $\deg_L(\det(\widetilde{W}_{|C_K})) = \sum \deg_L(\det(\widetilde{W}_{|C^i_k}))$. We have the induced parabolic structure of $\widetilde{W}_{|C^i_k}$ as the subsheaf of $V_{*|C^i_k}$, for which we have $\mathrm{wt}(W_{l*}, D_{j,C_K}) \geq \mathrm{wt}(\widetilde{W}_{|C^i_k *}, D_{j,C^i_k})$ for each D_j. Therefore, we obtain $\mu_L(\widetilde{W}_{*|C_K}) \leq \sum_i \mu_L(\widetilde{W}_{|C^i_k,*})$. If the equality occurs, we have $\mathrm{wt}(W_{l*}, D_{j,C_K}) = \mathrm{wt}(\widetilde{W}_{|C^i_k *}, D_{j,C^i_k})$ for any i and j, and $\widetilde{W}_*$ with the induced parabolic structure is the parabolic bundle.

3.4.5. The arguments of Mehta and Ramanathan. — Let $\mathcal{W}$ be a vector bundle on X. Let (V_*, η) be a parabolic sheaf with a $\mathcal{W}$-operator on (X, D).

Lemma 3.31. — *(V_*, η) is μ_L-semistable, if and only if there exists a positive integer m_0 such that $(V_*, \eta)_{|Y_{(m)}}$ is also μ_L-semistable for any $m \geq m_0$.*

Proof. — We have only to show the "only if" part. We reproduce the argument in [**41**]. First, assume $(V_*,\eta)_{|Y_{(m)}}$ is μ_L-semistable for some m, and we show that $(V_*,\eta)_{|Y_{(l)}}$ is μ_L-semistable for any $l>m$. We take a family of degenerating curves C as in Subsection 3.4.4. We have the β-$\mathcal{W}$-subobject $W_{l,*}\subset V_{*|C_K}$. We extend it to $\widetilde{W}\subset V_{|C}$. Note that it is naturally the $\mathcal{W}$-subobject. Since we have $\mu_L(W_{l\,*})\leq\sum_i\mu_L(\widetilde{W}_{|C_k^i\,*})$ and $\mu_L(V_{*|C_K})=\sum_i\mu_L(V_{*|C_k^i})$, we obtain $\mu_L(W_{l\,*})\leq\mu_L(V_{*|C_K})$. Thus, we obtain the semistability of $V_{*|Y_{(l)}}$.

We will show that V_* is not semistable if $V_{*|Y_{(m)}}$ are not semistable for any m. By shrinking U_m appropriately, we may have $\mathcal{W}$-subobjects $W_{m\,*}$ of $p_{(m)}^*V_{*|q_{(m)}^{-1}U_m}$ such that $W_{m\,*|\,q_{(m)}^{-1}(s)}$ is the β-$\mathcal{W}$-subobject of $(V_*,\eta)_{|q_{(m)}^{-1}(s)}$ for any $s\in U_m$. The restriction $W_{m\,*|Y_{(m)}}$ is the β-$\mathcal{W}$-subobject of $(V_*,\eta)_{|Y_{(m)}}$. We have the parabolic line bundle $\mathcal{L}_{m\,*}\in\operatorname{Pic}(X,D)$ corresponding to $\det(W_{m,*})_{|Y_{(m)}}\in\operatorname{Pic}(Y_{(m)},Y_{(m)}^D)$.

We put $\beta_m:=\mu_L(W_{m,*|Y_{(m)}})$. For $l>m$, we obtain $\beta_l\leq\alpha^{l-m}\cdot\beta_m$ by using a family of degenerating curves. Since we have $\beta_m=\alpha^m\cdot\mu_L(\mathcal{L}_{m\,*})/\operatorname{rank}(W_m)$, we obtain $\mu_L(\mathcal{L}_{l\,*})/\operatorname{rank}W_l\leq\mu_L(\mathcal{L}_{m\,*})/\operatorname{rank}W_m$. On the other hand, we have $\beta_m\geq\alpha^m\mu_L(V_*)$, and the sequence $\{\mu_L(\mathcal{L}_{m\,*})\}$ is bounded. Since $\{\operatorname{wt}(\mathcal{L}_m,D_j)\}$ is finite, we may take a subsequence $Q\subset\{m\}$ such that $\deg_L(\mathcal{L}_m)$, $\operatorname{wt}(\mathcal{L}_m,D_j)$ and $\operatorname{rank}(W_m)$ are independent of the choice of $m\in Q$.

Let us show that $\mathcal{L}_m$ $(m\in Q)$ are isomorphic. Take $l>m$ in Q. We take a family of degenerating curves as above. We extend $W_{l|C_K}$ to $\widetilde{W}$ on C. From $\beta_l=\alpha^{l-m}\beta_m$, $\beta_l=\mu_L(W_{l\,*})\leq\sum\mu_L(\widetilde{W}_{|C_k^i\,*})$ and $\mu_L(\widetilde{W}_{|C_k^i\,*})\leq\beta_m$, we obtain $\mu_L(\widetilde{W}_{|C_k^i\,*})=\beta_m$, and thus $\widetilde{W}_{|C_k^i\,*}$ are β-$\mathcal{W}$-subobjects of $V_{*|C_k^i}$. In particular, $\mu_L(\det(\widetilde{W}_{|C_k^i\,*}))=\mu_L(\mathcal{L}_{l\,*|C_k^i})$. We also obtain $\mu_L(W_{l\,*})=\sum\mu_L(\widetilde{W}_{|C_k^i\,*})$, and hence $\operatorname{wt}(\widetilde{W}_{|C_k^i\,*},D_{j,C_k^i})=\operatorname{wt}(W_{l\,*},D_{j,C_K})=\operatorname{wt}(\mathcal{L}_{l\,*},D_j)$. Hence we obtain $\deg_L\bigl(\det(\widetilde{W}_{|C_k^i})\bigr)=\deg_L(\mathcal{L}_{l|C_k^i})$, and thus $\mathcal{L}_{l|C}\simeq\det(\widetilde{W})$. Since the parabolic weights are also same, we have $\det(\widetilde{W})_*\simeq\mathcal{L}_{l\,*|C}$. Since C_k^i are sufficiently general in U_m, we obtain $\mathcal{L}_{l\,*|Y_{(m)}}\simeq\mathcal{L}_{m\,*|Y_{(m)}}$, and hence $\mathcal{L}_{l\,*}$ and $\mathcal{L}_{m\,*}$ are isomorphic. Now, let $\mathcal{L}_*$ denote $\mathcal{L}_{l\,*}$ $(l\in Q)$.

Let us show the existence of a $\mathcal{W}$-subsheaf $\widetilde{W}$ of V, such that $\widetilde{W}_{|q_{(m)}^{-1}(s)}=W_{m|q_{(m)}^{-1}(s)}$ for a sufficiently large m. Such $\widetilde{W}$ will contradict with the semistability of (V_*,η). Let U be an open subset of X on which V is a vector bundle. We may assume that $\operatorname{codim}(X-U)\geq 2$. We put $r=\operatorname{rank}(W_m)$ for $m\in Q$. Let G denote the bundle of Grassmann varieties on U, whose fiber over $q\in U$ consists of the subspaces of $V_{|q}$ with rank r. We have the natural embedding of G into the projectivization of $\bigwedge^r V_{|U}$. Let $\Sigma\subset\bigwedge^r V_{|U}$ denote the cone over G.

Let F denote the double dual of $\bigwedge^r V$. We have the naturally induced saturated parabolic structure of F. Let $\mathcal{H}om(\mathcal{L}_*,F_*)$ denote the sheaf of homomorphisms from $\mathcal{L}_*$ to F_*, which is reflexive. We put $H:=H^0\bigl(X,\mathcal{H}om(\mathcal{L}_*,F_*)\bigr)$. For any $\phi\in H$, we put $\Sigma(\phi) := \{x \in U \,|\, \phi(x) \in \Sigma\}$. Since $\{\Sigma(\phi)\,|\,\phi \in H\}$ is bounded family, we have $q_{(m)}^{-1}(s) \not\subset \Sigma(\phi)$ for a sufficiently large m and $s \in U_m$, unless $\Sigma(\phi) \neq U$. On the other hand, there exists a non-trivial morphism $\phi \in H$ such that $q_{(m)}^{-1}(s) \subset \Sigma(\phi)$ for such m and s, due to the above consideration and General Enriques-Severi Lemma (Proposition 3.2 [**41**]). Hence, we obtain $\Sigma(\phi) = U$ for such ϕ. The image of ϕ naturally induces the saturated subsheaf $\widetilde{W} \subset V$. If m is sufficiently large, we also obtain $\eta(\widetilde{W}) \subset \widetilde{W} \otimes \mathcal{W}$. To see it, we recall the boundedness of the family $\mathcal{S}$ of the saturated subsheaves F of V such that $\deg(F) \geq C$, for some fixed C (Lemma 2.5 in [**19**]). So we can take a large m such that $\eta(F) \subset F \otimes \mathcal{W}$ ($F \in \mathcal{S}$) if and only if $\eta(F_{|q^{-1}(s)}) \subset F_{|q^{-1}(s)} \otimes \mathcal{W}$ for a sufficiently general $s \in U_m$. Thus we are done. □

Lemma 3.32. — *(V_*, η) is μ_L-stable, if and only if there exists a positive integer m_0 such that $(V_*, \eta)_{|Y_{(m)}}$ is also μ_L-stable for any $m \geq m_0$.*

Proof. — We reproduce the argument in [**40**]. First, let us see $(V_*, \eta)_{|q_{(m)}^{-1}(s)}$ is simple for sufficiently large m if (V_*, η) is μ_L-stable. To show it, we have only to consider the case V_* is reflexive and saturated. Let $\mathcal{H}om((V_*, \eta), (V_*, \eta))$ be the sheaf of endomorphisms of V which preserves the parabolic structure and commutes with η. Then, it is easy to check $\mathcal{H}om((V_*, \eta), (V_*, \eta))$ is reflexive by using Lemma 2.19, and hence the claim is shown by applying General Enriques-Severi Lemma.

Let us recall the notion of *socle* of semistable objects, which is the direct sum of stable subobjects (See [**40**] for more precise. Recall we have assumed the characteristic of k is 0.) Assume that $(V_*, \eta)_{|Y_{(m)}}$ is stable for some m. Then, it can be shown that $(V_*, \eta)_{|Y_{(l)}}$ is also stable for any $l > m$ by using a family of degenerating curves and the socle of $(V_*, \eta)_{|Y_{(l)}}$, instead of β-$\mathcal{W}$-subobjects. So we assume that $(V_*, \eta)_{|Y_{(m)}}$ is not stable for any m, and we will show that (V_*, η) is not μ_L-stable.

Let N be sufficiently large. By shrinking U_m appropriately for $m \geq N$, we may assume (i) $(V_*, \eta)_{|q_{(m)}^{-1}(s)}$ is simple and semistable for any $s \in U_m$, (ii) the socle of $(V_*, \eta)_{|Y_{(m)}}$ is extended to $W_{m\,*} \subset p_{(m)}^* V_{*|q_{(m)}^{-1}(U_m)}$, (iii) $W_{m\,*|q_{(m)}^{-1}(s)}$ is the socle of $(V_*, \eta)_{|q_{(m)}^{-1}(s)}$ for any $s \in U_m$. Since $(V_*, \eta)_{|q_{(m)}^{-1}(s)}$ are simple, $W_m \neq p_{(m)}^* V_{|q_{(m)}^{-1}(U_m)}$. We have the parabolic line bundle $\mathcal{L}_{m\,*}$ on (X, D) corresponding to $\det(W_{m\,*|Y_{(m)}})$ on $(Y_{(m)}, Y_{(m)}^D)$. We have $\mu_L(\mathcal{L}_{m,*}) = \operatorname{rank}(W_m) \cdot \mu_L(V_*)$. Hence, we can take a subsequence $Q \subset \{m\}$ such that $\operatorname{rank} W_m$, $\operatorname{wt}(\mathcal{L}_{m\,*}, D_i)$ and $\deg(\mathcal{L}_m)$ are independent of $m \in Q$. We put $r := \operatorname{rank} W_m$ for $m \in Q$.

Let G_m denote the bundle of Grassmann varieties on $q_{(m)}^{-1}(U_m)$, whose fiber over $Q \in q_{(m)}^{-1}(U_m)$ consists of the subspace of $p_{(m)}^*(V)_{|Q}$ with rank r. We have the natural embedding of G_m into the projectivization of $p_{(m)}^*(\bigwedge^r V)_{|q_{(m)}^{-1}(U_m)}$. Let $\widehat{G}_m$ denote the cone over G_m.

Take $m_0 \in Q$, and let E denote the set of $\mathcal{L}_* \in \operatorname{Pic}(X, D)$ with $\mu_L(\mathcal{L}_*) = r \cdot \mu_L(V_*)$ such that there exists $\phi : \mathcal{L}_{*|Y_{(m_0)}} \longrightarrow \bigwedge^r V_{*|Y_{(m_0)}}$ with $\phi(\mathcal{L}_{|Y_{(m_0)}}) \subset \widehat{G}_{m_0}$ and

$\eta(\operatorname{Im}\phi) \subset \operatorname{Im}\phi \times W$. By the same argument as the proof of Lemmas 2.7–2.8 of [**40**], it can be shown that E is finite.

Let us show that $\mathcal{L}_l \in E$ for any $l \in Q$ with $l > m_0$. Let C be a family of degenerating curves. We extend $W_{l|C_K}$ to $\widetilde{W} \subset V_*$. We have the inequalities $\mu_L(W_{l\,*}) \leq \sum \mu_L(\widetilde{W}_{|C^i_k\,*})$, $\mu_L(\widetilde{W}_{|C^i_k\,*}) \leq \alpha^m \mu_L(V_*)$ and the equality $\mu_L(W_{l\,*}) = \alpha^l \mu_L(V_*)$. Thus, the inequalities are actually equalities. Hence, we have $\operatorname{wt}(\det(\widetilde{W})_{|C^i_k\,*}, D_{j,C^i_k}) = \operatorname{wt}(\mathcal{L}_{l\,*}, D_j)$ and $\mu_L(\det(\widetilde{W})_{|C^i_k,*}) = \mu_L(\mathcal{L}_{l\,*|C^i_k})$. Therefore, we obtain $\mathcal{L}_{l\,*|C} \simeq \det(\widetilde{W})_*$. In particular, $\mathcal{L}_{l\,*|C^i_k} \simeq \det(\widetilde{W}_{|C^i_k})_*$. Since C^i_k are sufficiently general, we obtain $\mathcal{L}_{l\,*} \in E$.

Then, we can take a subsequence $Q' \subset Q$ such that $\mathcal{L}_{m\,*}$ are isomorphic ($m \in Q'$). The rest of the argument is same as the last part of the proof of Lemma 3.31. □

3.4.6. End of Proof of Proposition 3.29. — We have only to show the "only if" part. We reproduce the argument in [**55**]. Assume the μ_L-stability of (V_*, θ). Let $Y = Y_1 \cap \cdots \cap Y_t$ be a generic complete intersection, where $\deg_L(Y_i)$ are appropriately large numbers. We put $Y^{(i)} := Y_1 \cap \cdots \cap Y_i$ and $Y^{(0)} := X$. We also put $D^{(i)} := D \cap Y^{(i)}$ and $D^{(0)} = D$. We put $C_1 := \prod_{i=1}^t \big(\deg_L(Y_i)/\int_X c_1(L)^n\big)$. We put $\mathcal{W}^{(i)} := \Omega_{Y^{(i)}}(\log D^{(i)})_{|Y}$. Let $\theta_Y^{(i)}$ denote the induced $\mathcal{W}^{(i)}$-operation of $V_{*|Y}$. We may assume that $(V_{*|Y}, \theta_Y^{(0)})$ is μ_L-stable due to Lemma 3.32. By applying the Mehta-Ramanathan type theorem to the Harder-Narasimhan filtration of V_*, we may have a constant B such that (i) it is independent of the choice of Y_i and a sufficiently large $\deg_L(Y_i)$, (ii) $\text{par-deg}_L(F_*) \leq B \cdot C_1$ for any $F_* \subset V_{*|Y}$. We show that $(V_*, \theta^{(i)})$ are μ_L-stable by an induction.

Assume that the claim holds for $i-1$. Let F_* be a $\mathcal{W}^{(i)}$-object of $V_{*|Y}$ such that $\mu_L(F_*) \geq \mu_L(V_{*|Y}) = \mu_L(V_*) \cdot C_1$, and we will derive the contradiction. We put $G := V/F$, which is provided with the induced parabolic structure. Then, we have the induced map $\theta : F_* \longrightarrow G_*(-Y_i)$. Let H denote the kernel. Let N denote the saturated subsheaf of $G(-Y_i)$ generated by F/H, provided with the induced parabolic structure. We have $\mu\big((F/H)_*\big) \leq \mu(N_*)$. Let $J \subset E_*(-Y_i)$ denote the pull back of N via $E(-Y_i) \longrightarrow G(-Y_i)$ with the induced parabolic structure. We obtain the following:

$$
\begin{aligned}
(16)\quad B \cdot C_1 &\geq \text{par-deg}_L(J(Y_i)_*) \geq \text{par-deg}_L(F_*) + \text{par-deg}_L\big(N_*(Y_i)\big) \\
&\geq 2\,\text{par-deg}_L(F_*) - \text{par-deg}_L(H_*) + \operatorname{rank}(F/H) \cdot \deg_L(\mathcal{O}(Y_i)_{|Y}) \\
&\geq \big(2\operatorname{rank}(F) \cdot \mu(V_*) - B\big) \cdot C_1 + \operatorname{rank}(F/H) \cdot \deg_L(\mathcal{O}(Y_i)_{|Y})
\end{aligned}
$$

If $\deg_L(Y_i)$ is sufficiently large, $\deg_L(\mathcal{O}(Y_i)_{|Y})$ is much larger than C_1. Hence $\operatorname{rank}(F/H)$ must be 0, and hence F is actually a $\mathcal{W}^{(i-1)}$-subobject, which contradicts with the μ_L-semistability of $(V_{*|Y}, \theta^{(i-1)})$. Thus the induction can proceed.

3.5. Adapted Metric and the Associated Parabolic Flat Higgs Bundle

We recall a 'typical' example of filtered sheaf. Let E be a holomorphic vector bundle on $X-D$. If we are given a hermitian metric h of E, we obtain the $\mathcal{O}_X$-module ${}_{\boldsymbol{c}}E(h)$ for any $\boldsymbol{c}\in\boldsymbol{R}^S$, as is explained in the following. Let us take hermitian metrics h_i of $\mathcal{O}(D_i)$. Let $\sigma_i:\mathcal{O}\longrightarrow\mathcal{O}(D_i)$ denote the canonical section. We denote the norm of σ_i with respect to h_i by $|\sigma_i|_{h_i}$. For any open set $U\subset X$, we put as follows:

$$\Gamma\big(U,{}_{\boldsymbol{c}}E(h)\big):=\Big\{f\in\Gamma(U\setminus D,E)\,\Big|\,|f|_h=O\Big(\prod|\sigma_i|_{h_i}^{-c_i-\epsilon}\Big)\ \forall\epsilon>0\Big\}.$$

Thus we obtain the $\mathcal{O}_X$-module ${}_{\boldsymbol{c}}E(h)$. We also put $\boldsymbol{E}(h):=\bigcup_{\boldsymbol{c}}{}_{\boldsymbol{c}}E(h)$.

Remark 3.33. — In general, ${}_{\boldsymbol{c}}E(h)$ are not coherent.

Definition 3.34. — Let $\widetilde{\boldsymbol{E}}_*$ be a filtered vector bundle. We put $E:=\widetilde{E}=\widetilde{\boldsymbol{E}}_{|X-D}$. A hermitian metric h of E is called adapted to the parabolic structure of $\widetilde{\boldsymbol{E}}_*$, if the isomorphism $E\simeq\widetilde{E}$ is extended to the isomorphisms ${}_{\boldsymbol{c}}E(h)\simeq{}_{\boldsymbol{c}}\widetilde{E}$ for any $\boldsymbol{c}\in\boldsymbol{R}^S$.

The following result is proved in [**44**].

Proposition 3.35. — *Let $(E,\overline{\partial}_E,\theta,h)$ be a tame harmonic bundle on $X-D$. Then, we obtain the* **c***-parabolic Higgs bundle $({}_{\boldsymbol{c}}E(h)_*,\theta)$ on (X,D) by the above construction.*

Proof. — By Theorems 8.58 and 8.59 in [**44**] (the $\lambda=0$ case), ${}_{\boldsymbol{c}}E(h)_*$ with the induced filtrations is a $\boldsymbol{c}$-parabolic bundle. By Corollary 8.89 in [**44**], θ is regular. □

3.6. Convergence

We give the definition of convergence of a sequence of parabolic Higgs bundles. Although we need such a notion only in the case where the base complex manifold is a curve, the definition is given generally. Let X be a complex manifold, and $D=\bigcup_{j\in S}D_j$ be a simple normal crossing divisor of X. Let p be a number which is sufficiently larger than $\dim X$. Let b be any positive integer.

Definition 3.36. — Let $\big(E^{(i)},\overline{\partial}^{(i)},\boldsymbol{F}^{(i)},\theta^{(i)}\big)$ $(i=1,2,\dots)$ be a sequence of $\boldsymbol{c}$-parabolic Higgs bundles on (X,D). We say that the sequence $\big\{(E^{(i)},\overline{\partial}^{(i)},\boldsymbol{F}^{(i)},\theta^{(i)})\big\}$ weakly converges to $\big(E^{(\infty)},\overline{\partial}^{(\infty)},\boldsymbol{F}^{(\infty)},\theta^{(\infty)}\big)$ in L_b^p on X, if there exist locally L_b^p-isomorphisms $\Phi^{(i)}:E^{(i)}\longrightarrow E^{(\infty)}$ on X satisfying the following conditions:

- The sequence $\{\Phi^{(i)}(\overline{\partial}^{(i)})-\overline{\partial}^{(\infty)}\}$ converges to 0 weakly in L_{b-1}^p locally on X.
- The sequence $\{\Phi^{(i)}(\theta^{(i)})-\theta^{(\infty)}\}$ converges to 0 weakly in L_{b-1}^p locally on X, as sections of $\mathrm{End}(E^{(\infty)})\otimes\Omega^{1,0}(\log D)$.
- For simplicity, we assume that $\Phi^{(i)}$ are C^∞ around D.

– The sequence $\{\Phi^{(i)}\big({}^jF^{(i)}\big)\}$ converges to ${}^jF^{(\infty)}$ in an obvious sense. More precisely, for any $\delta > 0$, $j \in S$ and $a \in]c_j - 1, c_j]$, there exists m_0 such that $\operatorname{rank}{}^jF^{(\infty)}_a = \operatorname{rank}{}^jF^{(i)}_{a+\delta}$ and that ${}^jF^{(\infty)}_a$ and $\Phi^{(i)}\big({}^jF^{(i)}_{a+\delta}\big)$ are sufficiently close in the Grassmann varieties, for any $i > m_0$.

Lemma 3.37. — *Let X be a smooth projective variety, and D be a simple normal crossing divisor of X. Assume that a sequence of $\boldsymbol{c}$-parabolic Higgs bundles $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)})\big\}$ on (X, D) converges to $(E^{(\infty)}, \overline{\partial}^{(\infty)}, \boldsymbol{F}^{(\infty)}, \theta^{(\infty)})$ weakly in L^p_b on X. Assume that there exist non-zero holomorphic sections $s^{(i)}$ of $(E^{(i)}, \overline{\partial}^{(i)})$ such that $\theta^{(i)}(s^{(i)}) = 0$ and that $s^{(i)}_{|P} \in {}^jF_0\big(E^{(i)}_{|P}\big)$ for any $P \in D_j$ and $j \in S$.*

Then there exists a non-zero holomorphic section $s^{(\infty)}$ of $\big(E^{(\infty)}, \overline{\partial}^{(\infty)}\big)$ such that $\theta^{(\infty)}(s^{(\infty)}) = 0$ and that $s^{(\infty)}_{|P} \in {}^jF_0\big(E^{(\infty)}_{|P}\big)$ for any $P \in D_j$ and $j \in S$.

Proof. — Let us take a C^∞-metric $\widetilde{h}$ of $E^{(\infty)}$ on X. We put $t^{(i)} := \Phi^{(i)}(s^{(i)})$. Since p is large, we remark that $\Phi^{(i)}$ are C^0. Hence we have $\max_{P\in X} |t^{(i)}(P)|_{\widetilde{h}}$. We may assume $\max_{P\in X} |t^{(i)}(P)|_{\widetilde{h}} = 1$.

We have $\Phi^{(i)}(\overline{\partial}^{(i)}) = \overline{\partial}^{(\infty)} + a_i$, and hence $\overline{\partial}^{(\infty)} t^{(i)} = -a_i(t^{(i)})$. Due to $|t^{(i)}| \leq 1$ and $a_i \longrightarrow 0$ weakly in L^p_{b-1}, the L^p_b-norm of $t^{(i)}$ are bounded. Hence we can take an appropriate subsequence $\{t^{(i)} \mid i \in I\}$ which weakly converges to $s^{(\infty)}$ in L^p_b on X. In particular, $\{t^{(i)}\}$ converges to a section $s^{(\infty)}$ in C^0. Due to $\max_P |s^{(\infty)}(P)|_{\widetilde{h}} = 1$, the section $s^{(\infty)}$ is non-trivial. We also have $\overline{\partial}^{(\infty)} s^{(\infty)} = 0$ in L^p_{b-1}, and hence $s^{(\infty)}$ is a non-trivial holomorphic section of $(E^{(\infty)}, \overline{\partial}^{(\infty)})$. It is easy to see that $s^{(\infty)}$ has the desired property. □

Corollary 3.38. — *Let (X, D) be as in Lemma 3.37. Assume that a sequence of $\boldsymbol{c}$-parabolic Higgs bundles $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)})\big\}$ on (X, D) weakly converges to both $(E, \overline{\partial}_E, \boldsymbol{F}, \theta)$ and $(E', \overline{\partial}_{E'}, \boldsymbol{F}', \theta')$ in L^p_b on X. Then there exists a non-trivial holomorphic map $f : (E, \overline{\partial}_E) \longrightarrow (E', \overline{\partial}_{E'})$ on X which is compatible with the parabolic structures and the Higgs fields.*

CHAPTER 4

AN ORDINARY METRIC FOR A PARABOLIC HIGGS BUNDLE

In this chapter, we would like to explain about an ordinary metric for parabolic Higgs bundles, which is a metric adapted to the parabolic structure. Such a metric has been standard in the study of parabolic bundles (for example, see [**4**], [**36**] and [**35**]). It is our purpose to see that it gives a rather good metric when the parabolic Higgs bundle is *graded semisimple*. (If it is not graded semisimple, we need more complicated metric as discussed in [**5**] and [**52**].) After giving estimates around the intersection and the smooth part of the divisor in Sections 4.1 and 4.2, we see some properties of an ordinary metric in Section 4.3.

4.1. Around the Intersection $D_i \cap D_j$

4.1.1. Construction of a metric. — We put $X := \{(z_1, z_2) \in \boldsymbol{C}^2 \,\big|\, |z_i| < 1\}$, $D_i := \{z_i = 0\}$ and $D = D_1 \cup D_2$. Take a positive number ϵ, and let ω_ϵ denote the following metric, for some positive number N:

$$\sum \bigl(\epsilon^{N+2} \cdot |z_i|^{2\epsilon} + |z_i|^2\bigr) \cdot \frac{dz_i \cdot d\bar{z}_i}{|z_i|^2}.$$

Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). We put $E := {}_{\boldsymbol{c}}E_{|X-D}$. We take a positive number ϵ such that $10\epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E_*)$. We have the description:

$$\theta = f_1 \cdot \frac{dz_1}{z_1} + f_2 \cdot \frac{dz_2}{z_2}, \quad f_i \in End({}_{\boldsymbol{c}}E).$$

We have $\mathrm{Res}_i(\theta) = f_{i\,|\,D_i}$.

Assumption 4.1

- The eigenvalues of $\mathrm{Res}_i(\theta)$ are constant. The sets of the eigenvalues of $\mathrm{Res}_i(\theta)$ are denoted by S_i.
- We have the decomposition:

$${}_{\boldsymbol{c}}E = \bigoplus_{\boldsymbol{\alpha} \in S_1 \times S_2} {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}} \quad \text{such that} \quad f_i({}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}) \subset {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}.$$

There are some positive constants C and η such that any eigenvalue β of $f_{i\,|\,E_{\boldsymbol{\alpha}}}$ satisfies $|\beta-\alpha_i|\leq C\cdot|z_i|^{\eta}$ for $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$.

Remark 4.2. — The first condition is satisfied, when we are given a projective surface X' with a simple normal crossing divisor D' and a $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}'}E'_*,\theta')$ on (X',D'), such that $(X,D)\subset(X',D')$ and $({}_{\boldsymbol{c}}E_*,\theta)=({}_{\boldsymbol{c}'}E'_*,\theta')_{|X}$. The second condition is also satisfied, if we replace X with a smaller open subset around the origin $O=(0,0)$.

In the following, we replace X with a smaller open subset containing O without mentioning, if it is necessary. Let us take a holomorphic decomposition ${}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}=\bigoplus_{\boldsymbol{a}\in\boldsymbol{R}^2}U_{\boldsymbol{\alpha},\boldsymbol{a}}$ satisfying the following conditions, where b_i denotes the i-th component of $\boldsymbol{b}$:

$$\bigoplus_{\boldsymbol{b}\leq\boldsymbol{a}}U_{\boldsymbol{\alpha},\boldsymbol{b}\,|\,O}={}^1F_{a_1\,|\,O}\cap{}^2F_{a_2\,|\,O}\cap{}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}\,|\,O},\qquad \bigoplus_{b_i\leq a}U_{\boldsymbol{\alpha},\boldsymbol{b}\,|\,D_i}={}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}\,|\,D_i}\cap{}^iF_a.$$

We take a holomorphic frame $\boldsymbol{v}=(v_1,\dots,v_r)$ compatible with the decomposition, i.e., we have $(\boldsymbol{a}(v_j),\boldsymbol{\alpha}(v_j))\in\boldsymbol{R}^2\times\boldsymbol{C}^2$ for each v_j such that $v_j\in U_{\boldsymbol{\alpha}(v_j),\boldsymbol{a}(v_j)}$. Let h_0' be the hermitian metric of ${}_{\boldsymbol{c}}E$ for which $\boldsymbol{v}$ is orthonormal. Let h_0 be the hermitian metric of E such that $h_0(v_i,v_j)=h_0'(v_i,v_j)\cdot|z_1|^{-2a_1(v_i)}\cdot|z_2|^{-2a_2(v_i)}$, where $a_j(v_i)$ denotes the j-th component of $\boldsymbol{a}(v_i)$. We put as follows:

$$A=A_1+A_2,\quad A_i=\bigoplus\Big(-a_i\frac{dz_i}{z_i}\Big)\cdot\mathrm{id}_{U_{\boldsymbol{\alpha},\boldsymbol{a}}}.$$

Then, we have $\partial_{h_0}=\partial_{h_0'}+A$. We also have $R(h_0)=R(h_0')=0$.

4.1.2. Estimate of $F(h_0)$ in the graded semisimple case

Proposition 4.3. — *If $({}_{\boldsymbol{c}}E_*,\theta)$ is graded semisimple in the sense of Definition 3.25, then $F(h_0)$ is bounded with respect to ω_ϵ and h_0.*

Proof. — Since we have $F(h_0)=R(h_0)+[\theta,\theta^\dagger]+\partial_{h_0}\theta+\overline{\partial}\theta^\dagger$, we have only to estimate $[\theta,\theta^\dagger]$, $\partial_{h_0}\theta$ and $\overline{\partial}\theta^\dagger$. We have the natural decompositions $f_i=\bigoplus f_{i\,\boldsymbol{\alpha}}$ for $i=1,2$, where $f_{i\,\boldsymbol{\alpha}}\in\mathrm{End}({}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}})$. Since the decomposition of $E=\bigoplus E_{\boldsymbol{\alpha}}$ is orthogonal with respect to h_0, the adjoint $f_i^\dagger$ of f_i with respect to h_0 preserves the decomposition. Hence we have the decomposition $f_i^\dagger=\bigoplus f_{i\,\boldsymbol{\alpha}}^\dagger$, and $f_{i\,\boldsymbol{\alpha}}^\dagger$ is the adjoint of $f_{i\,\boldsymbol{\alpha}}$ with respect to $h_{0|U_{\boldsymbol{a},\boldsymbol{\alpha}}}$.

Let us show that $\big[\theta,\theta^\dagger\big]$ is bounded with respect to h_0 and ω_ϵ. We put $N_i:=f_i-\bigoplus_{\boldsymbol{\alpha}}\alpha_i\cdot\mathrm{id}_{{}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}}$ for $i=1,2$, and then we have $\big[f_i,f_j^\dagger\big]=\bigoplus_{\boldsymbol{\alpha}}\big[N_i,N_j^\dagger\big]$. Since $({}_{\boldsymbol{c}}E_*,\theta)$ is graded semisimple, we have $N_{1\,|\,D_1}\big({}^1F_a\big)\subset{}^1F_{<a}$. We also have $N_{1\,|\,D_2}\big({}^2F_a\big)\subset{}^2F_a$. Hence, we obtain $\big|N_1\big|_{h_0}\leq C\cdot|z_1|^{2\epsilon}$ for some positive constant C. Similarly we can obtain the estimate $\big|N_2\big|_{h_0}\leq C\cdot|z_2|^{2\epsilon}$. Thus we obtain the boundedness of $[\theta,\theta_{h_0}^\dagger]$ with respect to h_0 and ω_ϵ.

Let us see the estimate of $\partial_{h_0}\theta$. We have the following, where α_1 denotes the first component of $\boldsymbol{\alpha}$:

$$\partial_{h_0}\left(f_1\cdot\frac{dz_1}{z_1}\right)=\partial_{h_0}\left(\sum_{\boldsymbol{\alpha}}\alpha_1\cdot\mathrm{id}_{E_{\boldsymbol{\alpha}}}\cdot\frac{dz_1}{z_1}\right)+\partial_{h_0'}\left(N_1\frac{dz_1}{z_1}\right)+\left[A_2,\ N_1\frac{dz_1}{z_1}\right].$$

The first term is 0. We put $\Omega := dz_1\wedge dz_2/z_1\cdot z_2$. Let us see the second term $\partial_{h_0'}N_1\cdot dz_1/z_1 =: G_0\cdot\Omega$. Then, G_0 is a C^∞-section of $\mathrm{End}(E)$ satisfying $G_{0\,|\,D_1}({}^1F_a)\subset {}^1F_{<a}$ and $G_{0\,|\,D_2}=0$. Let us see the third term $[A_2,N_1]\cdot dz_2/z_2 =: G_1\cdot\Omega$. Then, G_1 is a C^∞-section of $\mathrm{End}(E)$ such that $G_{1\,|\,D_i}\big({}^iF_a\big)\subset {}^iF_{<a}$. Hence, the second and the third terms are bounded. Thus we obtain the boundedness of $\partial_{h_0}\theta$. Since $\overline{\partial}\theta^\dagger_{h_0}$ is adjoint of $\partial_{h_0}\theta$ with respect to h_0, it is also bounded. Thus the proof of Proposition 4.3 is finished. □

4.2. Around a Smooth Point of the Divisor

4.2.1. Setting. — Let Y be a complex curve, and L be a line bundle on Y. Let $\mathcal{U}$ be a neighbourhood of Y in L. The projection $L\longrightarrow Y$ induces $\pi:\mathcal{U}\longrightarrow Y$. Let σ denote the canonical section of π^*L. Let $|\cdot|$ be a hermitian metric of π^*L. Thus, we obtain the function $\big|\sigma\big|:\mathcal{U}\longrightarrow \boldsymbol{R}$. Let J_0 denote the complex structure of $\mathcal{U}$ as the open subset of L, and let J be any other integrable complex structure such that $J-J_0=O(|\sigma|)$. We regard $\mathcal{U}$ as a complex manifold via the complex structure J. The $(0,1)$-operator $\overline{\partial}$ is induced by J.

Let $(E,\overline{\partial}_E)$ be a holomorphic vector bundle on $\mathcal{U}$. We put $E_Y := E_{|Y}$, and let F be a filtration of E_Y in the category of holomorphic vector bundles indexed by $\boldsymbol{R}$. For later use, we also consider the case where F is not necessarily a c-parabolic filtration for any $c\in\boldsymbol{R}$, i.e., $S(F)=\big\{a\mid \mathrm{Gr}^F_a(E)\neq 0\big\}$ is not contained in any interval $]c-1,c]$ of the length 1. Thus $E_*=(E,F)$ may be a parabolic bundle in a slightly generalized sense (Remark 3.6). But, if F is not a c-parabolic filtration, we will assume (i) $J=J_0$ and (ii) the decomposition $E=\bigoplus E_u$ (see Subsection 4.2.2) is given holomorphically. In the case F is a c-parabolic filtration, we have the number $\mathrm{gap}(F) := \mathrm{gap}(E,F)$ as in Subsection 3.1.1. Otherwise, we put $\mathrm{gap}(F) := \max\{|a-b|\neq 0\mid a,b\in S(F)\}$. Let ϵ be a positive number such that $10\epsilon<\mathrm{gap}(F)$. Let ω be a Kahler form of $\mathcal{U}$. Take a small positive number C and a large real number N. Then, we put $\omega_\epsilon := \omega + C\cdot\epsilon^N\sqrt{-1}\partial\overline{\partial}|\sigma|^{2\epsilon}$, which gives a Kahler form of $\mathcal{U}\setminus Y$.

Let θ be a Higgs field of E_* in the sense of Remark 3.6. We put $f := \mathrm{Res}(\theta)\in End(E_Y)$.

Assumption 4.4. — The eigenvalues of f are assumed to be constant on Y. (See Remark 4.2.)

4.2.2. Construction of a metric. — We construct a hermitian metric of $E_{|\mathcal{U}-Y}$ adapted to the filtration, by following [**35**] and [**36**] essentially. (See also [**4**].) We have the generalized eigen decomposition $E_Y = \bigoplus_{\alpha\in \boldsymbol{C}} \mathrm{Gr}^{\mathbb{E}}_{\alpha}(E_Y)$ with respect to f. We also have the generalized eigen decomposition $\mathrm{Gr}^F_a(E_Y) = \bigoplus_{\alpha} \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_Y)$ of $\mathrm{Gr}^F_a(E_Y)$ with respect to $\mathrm{Gr}^F(f)$. Then we put $\widehat{E}_{Y,u} := \mathrm{Gr}^{F,\mathbb{E}}_u(E_Y)$ for $u \in \boldsymbol{R}\times\boldsymbol{C}$, and $\widehat{E}_Y := \bigoplus \widehat{E}_{Y,u}$.

Let h_0' be a C^∞-metric of E on $\mathcal{U}$. The holomorphic structure of E and the metric h_0' induces the unitary connection ∇_0 of E on $\mathcal{U}$. We put $h_Y := h'_{0|Y}$. We assume that the decomposition $E_Y = \bigoplus \mathrm{Gr}^{\mathbb{E}}_{\alpha}(E_Y)$ is orthogonal with respect to h_Y. The holomorphic structure of E_Y and the metric h_Y induce the unitary connection ∇_{E_Y} of E_Y. Thus the connection $\nabla_{\pi^*E_Y}$ is induced on π^*E_Y. Then, we can take a C^∞-isometry $\Phi : \pi^*E_Y \longrightarrow E$ such that $\nabla_0\circ\Phi - \Phi\circ\pi^*\nabla_{E_Y} = O(|\sigma|)$ with respect to ω, as in [**35**]. To see it, we take any isometry Φ' such that $\Phi'_{|Y}$ is the identity. We identify E and π^*E via Φ' for a while. Let $\mathfrak{u}(E)$ be the bundle of anti-hermitian endomorphisms of E. We have the section $A = \nabla_0 - \nabla_{\pi^*E_Y}$ of $\mathfrak{u}(E)\otimes\Omega^1_{\mathcal{U}}$. We can take a C^∞-section B of $\mathfrak{u}(E)$ such that $B = O(|\sigma|)$ and $\nabla_{\pi^*E_Y}B - A = O(|\sigma|)$, which can be easily checked by using the partition of unity on Y. Then we obtain $g^{-1}\circ\nabla_{\pi^*E_Y}g - \nabla_0 = O(|\sigma|)$ for $g = \exp(B)$, which implies the existence of an appropriate isometry Φ. We identify E and π^*E_Y via such a Φ as C^∞-bundles.

The metric h_Y induces the orthogonal decomposition $\mathrm{Gr}^{\mathbb{E}}_{\alpha}(E_Y) = \bigoplus_{a\in\boldsymbol{R}} \mathcal{G}_{(a,\alpha)}$ such that $\bigoplus_{a\le b}\mathcal{G}_{(a,\alpha)} = F_b\,\mathrm{Gr}^{\mathbb{E}}_{\alpha}(E)$. We have the natural C^∞-isomorphism $\mathcal{G}_u \simeq \widehat{E}_{Y,u}$, and thus $E_Y \simeq \widehat{E}_Y$. We identify them as C^∞-bundles via the isomorphism. Let $h_{Y,u}$ denote the restriction of h_Y to $\mathcal{G}_u$ for $u\in\boldsymbol{R}\times\boldsymbol{C}$. We put $E_u := \pi^*\mathcal{G}_u$, and thus $E = \bigoplus E_u$ and $h_0' = \pi^*h_Y = \bigoplus \pi^*h_{Y,u}$. We put as follows:

$$h_0 := \bigoplus \pi^* h_{Y,(a,\alpha)}\cdot|\sigma|^{-2a}. \tag{17}$$

4.2.3. Estimate of $R(h_0)$. — We put $\Gamma := \bigoplus a\cdot\mathrm{id}_{E_{a,\alpha}}$.

Lemma 4.5. — *$R(h_0,\overline{\partial}_E)$ is bounded with respect to ω_ϵ and h_0. More strongly, we have the following estimate, with respect to h_0 and ω_ϵ:*

$$R(h_0,\overline{\partial}_E) = \bigoplus_{u\in\boldsymbol{R}\times\boldsymbol{C}} \pi^*R(h_{Y,u},\overline{\partial}_{\widehat{E}_{Y,u}}) + \Gamma\cdot\overline{\partial}\partial\log|\sigma|^{-2} + O\big(|\sigma|^{\epsilon}\big). \tag{18}$$

Proof. — Let $\overline{\partial}_1$ denote the $(0,1)$-part of $\pi^*\nabla_{\widehat{E}_Y}$. Let T denote the $(0,1)$-part of $\nabla_0 - \pi^*\nabla_{E_Y}$. We put $S = \overline{\partial}_{E_Y} - \overline{\partial}_{\widehat{E}_Y}$. We put $Q = T + \pi^*S$. Then, we have $\overline{\partial}_E = \overline{\partial}_1 + Q$. We have $S(F_a)\subset F_{<a}\otimes\Omega^{0,1}_Y$, and $T_{|Y} = 0$ in $\big(\mathrm{End}(E)\otimes\Omega^1_{\mathcal{U}}\big)_{|Y}$. Hence, we have $Q = O(|\sigma|^{4\epsilon})$. The operator ∂_{1,h_0} is determined by the condition $\overline{\partial}h_0(u,v) = h_0(\overline{\partial}_1 u, v) + h_0(u,\partial_{1,h_0}v)$ for smooth sections u and v of E. Similarly, we obtain the operator $\partial_{1,h_0'}$.

Let $Q^{\dagger}_{h_0}$ denote the adjoint of Q with respect to h_0, and then $\partial_{E,h_0} = \partial_{1,h_0} - Q^{\dagger}_{h_0}$. Hence we obtain $R(\overline{\partial}_E, h_0) = \big[\overline{\partial}_1, \partial_{1,h_0}\big] - \overline{\partial}_1 Q^{\dagger}_{h_0} + \partial_{1,h_0} Q - \big[Q, Q^{\dagger}_{h_0}\big]$. Since Q and $Q^{\dagger}_{h_0}$ are $O(|\sigma|^{4\epsilon})$ with respect to ω_ϵ and h_0, so is $[Q, Q^{\dagger}_{h_0}]$. We have $\partial_{1,h_0} Q = \partial_{1,h_0'} Q + \partial \log |\sigma|^{-2} [\Gamma, Q]$. Since Q is sufficiently small, the second term is $O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. Since $T_{|Y}$ is 0 in $\big(\mathrm{End}(E) \otimes \Omega^1_{\mathcal{U}}\big)_{|Y}$, we have $\partial_{1,h_0'} T = O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. Since $\big(\partial_{1,h_0'} S\big)_{|Y}(F_a) \subset \big(F_{<a} \otimes \Omega^{1,0}(\log Y) \otimes \Omega^{0,1}\big)_{|Y}$, we have $\partial_{1,h_0'} S = O(|\sigma|^{2\epsilon})$ with respect to h_0 and ω_ϵ. Thus, $\partial_{1,h_0} Q$ and the adjoint $\overline{\partial}_1 Q^{\dagger}_{h_0}$ are also $O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. We have $\big[\overline{\partial}_1, \partial_{1,h_0}\big] = \big[\overline{\partial}_1, \partial_{1,h_0'}\big] + \Gamma \cdot \overline{\partial}\partial \log |\sigma|^{-2}$. Since we have $\overline{\partial}_1 + \partial_{1,h_0'} = \nabla_{\pi^* \widehat{E}_Y}$ by our construction, we obtain $\big[\overline{\partial}_1, \partial_{1,h_0'}\big] = \pi^* R(h_Y, \overline{\partial}_{\widehat{E}_Y}) + [\overline{\partial}_1, \overline{\partial}_1] + [\partial_{1,h_0'}, \partial_{1,h_0'}] = \pi^* R(h_Y, \overline{\partial}_{\widehat{E}_Y}) + O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. Thus Lemma 4.5 is proved. □

Corollary 4.6. — *We have the following estimate with respect to* ω_ϵ*:*

$$\mathrm{tr}\, R(h_0, \overline{\partial}_E) = \sum_{(a,\alpha)} \pi^* \,\mathrm{tr}\, R(h_{Y,(a,\alpha)}, \overline{\partial}_{E_{Y,(a,\alpha)}}) + \sum a \cdot \mathrm{rank}\, \mathrm{Gr}^F_a(E) \cdot \overline{\partial}\partial \log |\sigma|^{-2} + O(1)$$

4.2.4. Estimate of $F(h_0)$ in the graded semisimple case. — In this subsection, we assume that the filtration F (Subsection 4.2.1) is a c-parabolic filtration for some $c \in \boldsymbol{R}$.

Proposition 4.7. — *If (E_*, θ) is graded semisimple, $F(h_0)$ is bounded with respect to ω_ϵ and h_0.*

Proof. — We put $\rho_0 := \bigoplus \alpha \cdot \mathrm{id}_{E_{(a,\alpha)}}$ and $\overline{\rho}_0 := \bigoplus \overline{\alpha} \cdot \mathrm{id}_{E_{(a,\alpha)}}$. Let P be any point of Y. Let (U, z_1, z_2) be a holomorphic coordinate neighbourhood of $(\mathcal{U}, J)$ around P such that $U \cap Y = \{z_1 = 0\}$. We are given the Higgs field:

$$\theta = f_1 \cdot \frac{dz_1}{z_1} + f_2 \cdot dz_2.$$

Since $f_{2|Y}$ preserves the filtration F, f_2 is bounded with respect to h_0. It is easy to see $[\rho_0, f_2]_{|Y} = 0$. Hence $[\rho_0, f_2]$ is $O(|\sigma|^{2\epsilon})$ with respect to h_0. We put $f_1' = f_1 - \rho_0$. Due to the graded semisimplicity of (E_*, θ), we have $f'_{1|Y}\big(F_a\big) \subset F_{<a}$. Hence f_1' is $O(|\sigma|^{2\epsilon})$ with respect to h_0. Then it is easy to check the boundedness of $[\theta, \theta^{\dagger}]$ with respect to ω_ϵ and h_0, by a direct calculation.

We have the following:

$$\partial_{E,h_0}(f_1) \cdot \frac{dz_1}{z_1} = \partial_{1,h_0'}(f_1') \cdot \frac{dz_1}{z_1} + \big[\Gamma, f_1'\big] \cdot \partial \log |\sigma|^2 \cdot \frac{dz_1}{z_1} - \big[Q^{\dagger}_{h_0}, f_1\big] \cdot \frac{dz_1}{z_1}$$

Then, $\partial_{1,h_0'} f_1' = A \cdot dz_2 \cdot dz_1/z_1$ is C^∞-$(2,0)$-form of $\mathrm{End}(E)$, and $A_{|Y}(F_a) \subset F_{<a}$. Hence the first term is $O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. Similarly, the same estimate holds for the second term. Since $Q^{\dagger}_{h_0} = O(|\sigma|^{2\epsilon})$, the third term is $O(|\sigma|^{\epsilon})$.

We have $\partial_{E,h_0} f_2 \cdot dz_2 = \partial_{1,h_0'} f_2 \cdot dz_2 + [\Gamma, f_2] \cdot \partial \log |\sigma|^2 \cdot dz_2 - [Q^{\dagger}_{h_0}, f_2] \cdot dz_2$. Since the first term is a C^∞-2-form of $\mathrm{End}(E)$, it is $O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0. The same estimate holds for the second term because of $[\Gamma, f_2](F_a) \subset F_{<a}$. Since $Q^{\dagger}_{h_0}$ is $O(|\sigma|^{2\epsilon})$, the third term is $O(|\sigma|^{\epsilon})$ with respect to ω_ϵ and h_0. Then Proposition 4.7 is proved. □

4.2.5. Preliminary for the calculation of the integral. — Let $\widehat{h}_Y = \bigoplus \widehat{h}_{Y,u}$ be a hermitian metric of E_Y for which $\bigoplus \widehat{E}_{Y,u}$ is orthogonal. We put $\widehat{h} := \pi^* \widehat{h}_Y$. We put $A := \partial_{E,h_0} - \partial_{E,\widehat{h}}$.

Lemma 4.8. — *We have the following estimates with respect to ω_ϵ:*

$$\operatorname{tr} A = \sum_a a \cdot \operatorname{rank} \operatorname{Gr}^F_a(E) \cdot \partial \log |\sigma|^{-2} + O(1) \tag{19}$$

$$\operatorname{tr}\big(A \cdot R(h_0)\big) = \sum_{a,\alpha} \pi^* \operatorname{tr} R(\widehat{E}_{Y,a,\alpha}, h_{Y,a,\alpha}) \cdot a \cdot \partial \log |\sigma|^{-2} \tag{20}$$

$$+ \sum_{a,\alpha} \operatorname{rank} \widehat{E}_{Y,a,\alpha} \cdot a^2 \cdot \overline{\partial}\partial \log |\sigma|^{-2} \partial \log |\sigma|^{-2} + \operatorname{tr}\Big(Q^{\dagger}_{\widehat{h}} \cdot \big[\Gamma \cdot \partial \log |\sigma|^{-2},\, Q\big]\Big) + O(1)$$

$$\operatorname{tr}\big(A \cdot R(\widehat{h})\big) = \sum_{a,\alpha} \pi^* \operatorname{tr} R(\widehat{E}_{Y,a,\alpha}, \widehat{h}_{Y,a,\alpha}) \cdot a \cdot \partial \log |\sigma|^{-2} \tag{21}$$

$$- \operatorname{tr}\big(\Gamma \cdot \partial \log |\sigma|^{-2} [Q, Q^{\dagger}_{\widehat{h}}]\big) + O(1)$$

Proof. — We have $\partial_{E,h_0} = \partial_{1,h_0'} - Q^{\dagger}_{h_0} + \Gamma \cdot \partial \log |\sigma|^{-2}$ and $\partial_{E,\widehat{h}} = \partial_{1,\widehat{h}} - Q^{\dagger}_{\widehat{h}}$. We put $P = \partial_{1,h_0'} - \partial_{1,\widehat{h}}$, which is a C^∞-section of $\bigoplus \operatorname{End}(E_u) \otimes \Omega^{1,0}$. Thus, we have $A = P + Q^{\dagger}_{\widehat{h}} - Q^{\dagger}_{h_0} + \Gamma \cdot \partial \log |\sigma|^{-2}$. Since $Q^{\dagger}_{\widehat{h}}$ and $Q^{\dagger}_{h_0}$ are bounded with respect to $(\omega_\epsilon, \widehat{h})$, we obtain (19).

Let us show (20). Since $P + Q^{\dagger}_{h_0}$ is bounded with respect to h_0 and ω_ϵ, we have the boundedness of $\operatorname{tr}\big((P + Q^{\dagger}_{h_0}) \cdot R(h_0)\big)$ with respect to ω_ϵ. From (18), we obtain the following:

$$\operatorname{tr}\big(\Gamma \cdot \partial \log |\sigma|^{-2} \cdot R(h_0)\big) = \sum_{a,\alpha} \pi^* \operatorname{tr} R(\widehat{E}_{Y,a,\alpha}, h_{Y,a,\alpha}) \cdot a \cdot \partial \log |\sigma|^{-2} \tag{22}$$

$$+ \sum_{a,\alpha} \operatorname{rank} \widehat{E}_{Y,a,\alpha} \cdot a^2 \cdot \overline{\partial}\partial \log |\sigma|^{-2} \cdot \partial \log |\sigma|^{-2} + O(1).$$

Let us see $\operatorname{tr}\big(Q^{\dagger}_{\widehat{h}} \cdot R(h_0)\big)$. We decompose it as follows:

$$\operatorname{tr}\big(Q^{\dagger}_{\widehat{h}} \cdot [\overline{\partial}_1, \partial_{1,h_0}]\big) - \operatorname{tr}\big(Q^{\dagger}_{\widehat{h}} \cdot \overline{\partial}_1 Q^{\dagger}_{h_0}\big) + \operatorname{tr}\big(Q^{\dagger}_{\widehat{h}} \cdot \partial_{1,h_0} Q\big) - \operatorname{tr}\big(Q^{\dagger}_{\widehat{h}} \cdot [Q, Q^{\dagger}_{h_0}]\big) \tag{23}$$

Since $[\overline{\partial}_1, \partial_{1,h_0}]$ is bounded with respect to $(\omega_\epsilon, \widehat{h})$, we obtain the boundedness of the first term. Recall $Q^{\dagger}_{h_0} = (\pi^* S)^{\dagger}_{h_0} + T^{\dagger}_{h_0}$. Because of $T_{|Y} = 0$ in $(\operatorname{End}(E) \otimes \Omega^{0,1})_{|Y}$ and $\partial_{1,h_0} T = \partial_{1,h_0'} T + \big[\Gamma \cdot \partial \log |\sigma|^{-2}, T\big]$, we have $\partial_{1,h_0} T_{|Y} = 0$ in $\big(\operatorname{End}(E) \otimes \Omega^{1,0}(\log Y) \otimes \Omega^{0,1}\big)_{|Y}$. Because of $\overline{\partial}_1 T^{\dagger}_{h_0} = \big(\partial_{1,h_0} T\big)^{\dagger}_{h_0}$, it is easy to obtain $\overline{\partial}_1 T^{\dagger}_{h_0} = O(|\sigma|^{2\epsilon})$ with respect to $(\widehat{h}, \omega_\epsilon)$. We also have $T^{\dagger}_{h_0} = O(|\sigma|^{2\epsilon})$ with respect to $(\widehat{h}, \omega_\epsilon)$. Since $\pi^* S$ is a section of $\bigoplus_{a>a'} Hom(E_{a,\alpha}, E_{a',\alpha'}) \otimes \Omega^{0,1}$, we have $\pi^* S^{\dagger}_{h_0} = O(|\sigma|^{2\epsilon})$ with respect to $(\widehat{h}, \omega_\epsilon)$. Hence, $Q^{\dagger}_{h_0}$ and $[Q^{\dagger}_{h_0}, Q]$ are $O(|\sigma|^{2\epsilon})$ with respect to $(\omega_\epsilon, \widehat{h})$. Therefore, the fourth term in (23) is bounded. Because of $\overline{\partial}_1 \pi^* S^{\dagger}_{h_0} = \big(\partial_{1,h'_0} \pi^* S\big)^{\dagger}_{h_0} + \big([\Gamma \cdot \partial \log|\sigma|^{-2}, \pi^* S]\big)^{\dagger}_{h_0}$, it is easy to obtain $\overline{\partial}_1 \pi^* S^{\dagger}_{h_0} = O(|\sigma|^{2\epsilon})$ with respect to $(\omega_\epsilon, \widehat{h})$. Together with the estimate of $\overline{\partial}_1 T^{\dagger}_{h_0}$ above, we obtain the boundedness of $\overline{\partial}_1 Q^{\dagger}_{h_0}$ with respect to $(\omega_\epsilon, \widehat{h})$. Hence, we obtain the boundedness of the second term in (23). We have $\partial_{1,h_0} Q = \partial_{1,h'_0} Q + [\Gamma \cdot \partial \log|\sigma|^{-2}, Q]$, and $\partial_{1,h'_0} Q$ is bounded with respect to $(\omega_\epsilon, \widehat{h})$. Therefore, the third term is $O(1) + \operatorname{tr}\big(Q^{\dagger}_{h_0} [\Gamma \cdot \partial \log|\sigma|^{-2}, Q]\big)$. Thus we obtain (20).

Let us show (21). Since P, $Q^{\dagger}_{\widehat{h}}$ and $Q^{\dagger}_{h_0}$ are bounded with respect to $(\omega_\epsilon, \widehat{h})$, we have $\operatorname{tr}\big((P + Q^{\dagger}_{\widehat{h}} - Q^{\dagger}_{h_0}) R(\widehat{h})\big) = O(1)$ with respect to ω_ϵ. We have $R(\widehat{h}) = [\overline{\partial}_1, \partial_{1,\widehat{h}}] - \overline{\partial}_1 Q^{\dagger}_{\widehat{h}} + \partial_{1,\widehat{h}} Q - [Q^{\dagger}_{\widehat{h}}, Q]$. Because of $\partial_{1,\widehat{h}} T = O(|\sigma|^{2\epsilon})$ with respect to $(\omega_\epsilon, \widehat{h})$ and $\partial_{1,\widehat{h}} \pi^* S \in \bigoplus_{a>a'} \operatorname{Hom}(E_{a,\alpha}, E_{a',\alpha'}) \otimes \Omega^2$, we have $\operatorname{tr}\big(\Gamma \cdot \partial \log|\sigma|^{-2} \cdot \partial_{1,\widehat{h}} Q\big) = O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ. By a similar reason, $\operatorname{tr}\big(\Gamma \cdot \partial \log|\sigma|^{-2} \overline{\partial}_1 Q^{\dagger}_{\widehat{h}}\big) = O(|\sigma|^{2\epsilon})$. Since we have $[\overline{\partial}_1, \partial_{1,\widehat{h}}] = \pi^* R(\widehat{E}, \widehat{h}_Y) + O(|\sigma|^{2\epsilon})$ with respect to $(\widehat{h}, \omega_\epsilon)$, we obtain (21). □

Corollary 4.9. — *We have the following estimates with respect to ω_ϵ:*

(24)
$$\operatorname{tr}\big(A \cdot R(h_0) + A \cdot R(\widehat{h})\big) = \sum_u \pi^* \big(\operatorname{tr} R(\widehat{E}_{Y,u}, h_{Y,u}) + \operatorname{tr} R(\widehat{E}_{Y,u}, \widehat{h}_{Y,u})\big) \cdot a \cdot \partial \log|\sigma|^{-2} + \sum_u a^2 \cdot \operatorname{rank} \widehat{E}_{Y,u} \cdot \partial \log|\sigma|^{-2} \cdot \partial\overline{\partial} \log|\sigma|^{-2} + O(1)$$

Here, $u = (a, \alpha)$.

4.2.6. Estimate of a related metric. — For later use (Section 5.2), we consider a related metric in the case where one more filtration W is given on $\operatorname{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E)$ indexed by $\mathbb{Z}$. The argument and the calculation are essentially contained in those of Section 3.A in [**5**]. Since our purpose is more restricted, the construction of the metric can be more rough.

We put $\widetilde{E}_{u,k} := \operatorname{Gr}^W_k \operatorname{Gr}^{F,\mathbb{E}}_u (E_Y)$ for $(u,k) \in (\boldsymbol{R} \times \boldsymbol{C}) \times \mathbb{Z}$ and $\widetilde{E}_Y := \bigoplus \widetilde{E}_{u,k}$. We put $\widetilde{F}_{(a,k)} \operatorname{Gr}^{\mathbb{E}}_\alpha(E) := \pi_a^{-1}(W_k)$, where π_a denotes the projection $F_a \operatorname{Gr}^{\mathbb{E}}_\alpha(E) \longrightarrow \operatorname{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E)$. The metric h_Y induces the orthogonal decomposition $\operatorname{Gr}^{\mathbb{E}}_\alpha(E) = \bigoplus_{(a,k) \in \boldsymbol{R} \times \boldsymbol{C}} \mathcal{G}_{a,\alpha,k}$ such that $\widetilde{F}_{(b,l)} \operatorname{Gr}^{\mathbb{E}}_\alpha(E) = \bigoplus_{(a,k) \leq (b,l)} \mathcal{G}_{a,\alpha,k}$. We have the natural C^∞-isomorphism $\mathcal{G}_{u,k} \simeq \operatorname{Gr}^W_k \operatorname{Gr}^{F,\mathbb{E}}_u (E_Y)$ for $(u,k) \in (\boldsymbol{R} \times \boldsymbol{C}) \times \mathbb{Z}$. Thus, we obtain the C^∞-identification of E_Y and $\widetilde{E}_Y$. Let $h_{Y,u,k}$ denote the restriction of h_Y to $\mathcal{G}_{u,k}$.

Via the identification $\Phi : \pi^* E_Y \simeq E$, we obtain the C^∞-decomposition $E = \bigoplus E_{a,\alpha,k}$. Then, we put as follows:

$$h_1 := \bigoplus_{a,\alpha,k} \pi^* h_{Y,a,\alpha,k} \cdot |\sigma|^{-2a} \cdot \big(-\log|\sigma|^2\big)^k.$$

There exist some constants C and N such that $C^{-1}\cdot h_0\cdot(-\log|\sigma|)^{-N}\leq h_1\leq C\cdot h_0\cdot(-\log|\sigma|)^{N}$.

For appropriate constants C_1, we put $\widetilde{\omega}:=\omega+C_1\cdot\partial\overline{\partial}\log(-\log|\sigma|^2)$, which gives the Poincaré like metric on $\mathcal{U}\setminus Y$.

Lemma 4.10. — *$R(h_1)$ is bounded with respect to $\widetilde{\omega}$ and h_i $(i=0,1)$. The difference $\partial_{E,h_1}-\partial_{E,h_0}$ is bounded with respect to $\widetilde{\omega}$ and h_0.*

Proof. — Under the identification $E_Y=\widetilde{E}_Y$, we put $\widetilde{S}=\overline{\partial}_{E_Y}-\overline{\partial}_{\widetilde{E}_Y}$. We put $S':=\widetilde{S}-S$. As before, we have $\overline{\partial}_E=\overline{\partial}_2+\widetilde{Q}$ and $\widetilde{Q}=T+\pi^*\widetilde{S}$. We also have $\overline{\partial}_1=\overline{\partial}_2+\pi^*S'$. Because of $T_{|Y}=0$ in $\big(\mathrm{End}(E)\otimes\Omega^1_{\mathcal{U}}\big)_{|Y}$, T and $T^{\dagger}_{h_1}$ are $O(|\sigma|^{2\epsilon})$ with respect to $(h_i,\widetilde{\omega})$ $(i=0,1)$. Because of $\widetilde{S}(\widetilde{F}_{(a,k)})\subset\widetilde{F}_{<(a,k)}\otimes\Omega^{0,1}_Y$, $\widetilde{S}$ and $\widetilde{S}^{\dagger}_{h_1}$ are $O\big((-\log|\sigma|)^{-1/2}\big)$ with respect to $(h_1,\widetilde{\omega})$. We also obtain $\widetilde{S}=O(1)$ and $\widetilde{S}^{\dagger}_{h_1}=O\big((-\log|\sigma|)^{-1/2}\big)$ with respect to $(h_0,\widetilde{\omega})$. In particular, $\widetilde{Q}$ and $\widetilde{Q}^{\dagger}_{h_1}$ are bounded with respect to $(h_i,\widetilde{\omega})$ $(i=0,1)$.

We put $\mathcal{K}:=\bigoplus k/2\cdot\mathrm{id}_{E_{u,k}}$. Then, we obtain the following:

$$
\begin{aligned}
(25)\quad \partial_{E,h_1}=\partial_{2,h_1}-\widetilde{Q}^{\dagger}_{h_1}&=\partial_{2,h_0}+\mathcal{K}\cdot\partial\log(-\log|\sigma|^2)-\widetilde{Q}^{\dagger}_{h_1}\\
&=\partial_{1,h_0}+(\pi^*S')^{\dagger}_{h_0}+\mathcal{K}\cdot\partial\log(-\log|\sigma|^2)-\widetilde{Q}^{\dagger}_{h_1}\\
&=\partial_{E,h_0}+Q^{\dagger}_{h_0}+\big(\pi^*S'\big)^{\dagger}_{h_0}+\mathcal{K}\cdot\partial\log(-\log|\sigma|^2)-\widetilde{Q}^{\dagger}_{h_1}.
\end{aligned}
$$

It is easy to see that π^*S' and $(\pi^*S')^{\dagger}_{h_0}$ are bounded with respect to h_0. Thus, we obtain the boundedness of $\partial_{E,h_1}-\partial_{E,h_0}$ with respect to $(\widetilde{\omega},h_0)$.

We decompose $R(h_1)$ as follows:

$$
(26)\qquad R(h_1)=\big[\overline{\partial}_2,\partial_{2,h_1}\big]+\partial_{2,h_1}\widetilde{Q}-\overline{\partial}_2\widetilde{Q}^{\dagger}_{h_1}-\big[\widetilde{Q},\widetilde{Q}^{\dagger}_{h_1}\big]
$$

We decompose the second term as follows:

$$
\begin{aligned}
(27)\quad \big[\partial_{2,h_1},\widetilde{Q}\big]&=\big[\mathcal{K}\cdot\partial\log(-\log|\sigma|^2),\widetilde{Q}\big]\\
&\quad+\big[\partial_{2,h_0'}+\Gamma\cdot\partial\log|\sigma|^{-2},T\big]+\big[\partial_{2,h_0'}+\Gamma\cdot\partial\log|\sigma|^{-2},\widetilde{S}\big]
\end{aligned}
$$

Since $\partial\log(-\log|\sigma|^2)$ is bounded with respect to $\widetilde{\omega}$, we have the boundedness of $\mathcal{K}\cdot\partial\log(-\log|\sigma|^2)$ with respect to $(\widetilde{\omega},h_i)$ $(i=0,1)$. Hence, the first term in (27) is bounded. The adjoint with respect to h_1 also satisfies the same estimate.

We have $T=O(|\sigma|^{3\epsilon})$ with respect to $(\widetilde{\omega},h_i)$ $(i=0,1)$ and $[\partial_{2,h_0'},T]_{|Y}=0$ in $\big(\mathrm{End}(E)\otimes\Omega^{1,0}(\log D)\otimes\Omega^{0,1}\big)_{|Y}$. Hence $\big[\Gamma\cdot\partial\log|\sigma|^2,T\big]$ and $[\partial_{2,h_0'},T]$ are $O(|\sigma|^{3\epsilon})$ with respect to $(\widetilde{\omega},h_i)$ $(i=0,1)$. Their adjoints with respect to h_1 are also $O(|\sigma|^{2\epsilon})$ with respect to $(\widetilde{\omega},h_i)$. Therefore, we obtain the boundedness of the second term in (27) and the adjoint.

Let $\widetilde{S}=A\cdot d\overline{z}_1+B\cdot d\overline{z}_2$ be the expression for a local coordinate (U,z_1,z_2) such that $z_1^{-1}(0)=Y\cap U$. Then, we have $A_{|Y}=0$ and $B_{|Y}(\widetilde{F}_{(a,k)})\subset\widetilde{F}_{<(a,k)}$. We have $[\Gamma,B]_{|Y}(F_a)\subset F_{<a}$. Thus $[\Gamma\cdot\partial\log|\sigma|^{-2},\widetilde{S}]$, and the adjoint with respect to h_1 are

$O(|\sigma|^{2\epsilon})$ with respect to $(\widetilde{\omega}, h_i)$ $(i = 0, 1)$. We have $\big[\partial_{2,h_0'},\, A \cdot d\bar{z}_1\big]_{|Y} = 0$ in $\big(\mathrm{End}(E) \otimes \Omega^{1,0}(\log Y) \otimes \Omega^{0,1}\big)_{|Y}$. For the expression $\big[\partial_{2,h_0'}, Bd\bar{z}_2\big] = \big(C_1 \cdot dz_1/z_1 + C_2 dz_2\big) \cdot d\bar{z}_2$, we have $C_{1|Y} = 0$ and $C_{2|Y}(\widetilde{F}_{(a,k)}) \subset \widetilde{F}_{<(a,k)}$. Hence, $\big[\partial_{2,h_0'}, \widetilde{S}\big]$ and the adjoint with respect to h_1 are bounded with respect to both of $(\widetilde{\omega}, h_i)$ $(i = 0, 1)$. Therefore, we obtain the boundedness of the third term in (27) and the adjoint. Thus we obtain the boundedness of the second and third terms in (26).

We have $\big[\overline{\partial}_2, \partial_{2,h_1}\big] = \big[\overline{\partial}_2, \partial_{2,h_0'}\big] + \overline{\partial}\partial \log |\sigma|^{-2} \cdot \Gamma + \overline{\partial}\partial \log(-\log|\sigma|^2) \cdot \mathcal{K}$ which is bounded with respect to $(\widetilde{\omega}, h_i)$ $(i = 0, 1)$. Thus we obtain the boundedness of $R(h_1)$. □

4.3. Global Ordinary Metric

4.3.1. Decomposition and metric of a base space. — Let X be a smooth projective complex surface, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We also assume that D is ample. Let L be an ample line bundle on X, and ω be a Kahler form which represents $c_1(L)$. For any point $P \in D_i \cap D_j$, we take a holomorphic coordinate (U_P, z_i, z_j) around P such that $U_P \cap D_k = \{z_k = 0\}$ $(k = i, j)$ and $U_P \simeq \Delta^2$ by the coordinate. Let us take a hermitian metric g_i of $\mathcal{O}(D_i)$ and the canonical section $\mathcal{O} \longrightarrow \mathcal{O}(D_i)$ is denoted by σ_i. We may assume $|\sigma_k|_{g_k}^2 = |z_k|^2$ $(k = i, j)$ on U_P for $P \in D_i \cap D_j$.

Let us take a hermitian metric g of the tangent bundle TX such that $g = dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$ on U_P. It is not necessarily same as ω. The metric g induces the exponential map $\exp : TX \longrightarrow X$. Let $N_{D_i}X$ denote the normal bundle of D_i in X. We can take a sufficiently small neighbourhood U_i' of D_i in $N_{D_i}X$ such that the restriction of $\exp_{|U_i'}$ gives the diffeomorphism of U_i' and the neighbourhood U_i of D_i in X. We may assume $U_i \cap U_j = \coprod_{P \in D_i \cap D_j} U_P$.

Let p_i denote the diffeomorphism $\exp_{|U_i'} : U_i' \longrightarrow U_i$. Let π_i denote the natural projection $U_i' \longrightarrow D_i$. Via the diffeomorphism p_i, we also have the C^∞-map $U_i \longrightarrow D_i$, which is also denoted by π_i. On U_P, π_i is same as the natural projection $(z_i, z_j) \longmapsto z_j$. Via p_i, we have two complex structures J_{U_i} and $J_{U_i'}$ on U_i. Due to our choice of the hermitian metric g, p_i preserves the holomorphic structure (i.e., $J_{U_i'} - J_{U_i} = 0$) on U_P. The derivative of p_i gives the isomorphism of the complex bundles $T(N_{D_i}(X))_{|D_i} \simeq TD_i \oplus N_{D_i}X \simeq TX_{|D_i}$ on D_i. Hence we have $J_{U_i} - J_{U_i'} = O(|\sigma|)$.

Let ϵ be any number such that $0 < \epsilon < 1/2$. Let us fix a real number N, which is sufficiently large, say $N > 10$. We put as follows, for some positive number $C > 0$:

$$\omega_\epsilon := \omega + \sum_i C \cdot \epsilon^N \cdot \sqrt{-1}\partial\overline{\partial}|\sigma_i|_{g_i}^{2\epsilon}.$$

Proposition 4.11. — *If C is sufficiently small, then ω_ϵ are Kahler metrics of $X - D$ for any $0 < \epsilon < 1/2$.*

Proof. — We put $\phi_i := |\sigma_i|^2_{g_i}$. We have $\sqrt{-1}\cdot\partial\overline{\partial}\phi_i^{\epsilon} = \sqrt{-1}\cdot\epsilon^2\cdot\phi_i^{\epsilon}\cdot\partial\log\phi_i\cdot\overline{\partial}\log\phi_i + \sqrt{-1}\cdot\epsilon\cdot\phi_i^{\epsilon}\cdot\partial\overline{\partial}\log\phi_i$. Hence the claim of Proposition 4.11 immediately follows from the next lemma. □

Lemma 4.12. — *We put $f_t(\epsilon) := \epsilon^l\cdot t^{2\epsilon}$ for $0<\epsilon\leq 1/2$ and for $l\geq 1$. The following inequality holds:*

$$f_t(\epsilon)\leq\left(\frac{l}{-\log t^2}\right)^l\cdot e^{-l}\qquad (0<t<e^{-l}) \tag{28}$$

$$f_t(\epsilon)\leq\left(\frac{1}{2}\right)^l\cdot t\qquad (t\geq e^{-l}) \tag{29}$$

Proof. — We have $f_t'(\epsilon)=\epsilon^{l-1}t^{2\epsilon}\cdot\big(l+\epsilon\log t^2\big)$. If $t<e^{-l}$, we have $\epsilon_0 := l\times(-\log t^2)^{-1}<1/2$ and $f_t'(\epsilon_0)=0$. Hence f_t takes the maximum at $\epsilon=\epsilon_0$, and we obtain (28). If $t\geq e^{-1}$, we have $f_t'(\epsilon)>0$ for any $0<\epsilon<1/2$, and thus $f_t(\epsilon)$ takes the maximum at $\epsilon=1/2$. Thus we obtain (29). □

The Kahler forms ω_ϵ behave well around any point of D in the following sense, which is clear from the construction.

Lemma 4.13. — *Let P be any point of $D_i\cap D_j$. Then there exist positive constants C_i $(i=1,2)$ such that the following holds on U_P, for any $0<\epsilon<1/2$:*

$$C_1\cdot\omega_\epsilon\leq\sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dz_i\cdot d\bar{z}_i}{|z_i|^{2-2\epsilon}}+\frac{dz_j\cdot d\bar{z}_j}{|z_j|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dz_i\cdot d\bar{z}_i+dz_j\cdot d\bar{z}_j\big)\leq C_2\cdot\omega_\epsilon.$$

Let Q be any point of D_i°, and (U,w_1,w_2) be a holomorphic coordinate around Q such that $U\cap D_i=\{w_1=0\}$. Then there exist positive constants C_i $(i=1,2)$ such that the following holds for any $0<\epsilon<1/2$ on U:

$$C_1\cdot\omega_\epsilon\leq\sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dw_1\cdot d\bar{w}_1}{|w_1|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dw_1\cdot d\bar{w}_1+dw_2\cdot d\bar{w}_2\big)\leq C_2\cdot\omega_\epsilon.$$

***Lemma 4.14* (Simpson [51], Li [35])**. — *Let us consider the case $\epsilon=1/m$ for some positive integer m. Then the metric ω_ϵ satisfies Condition* 2.1.

Proof. — We use the argument of Simpson in [**51**]. The first condition is easy to check. Since we have assumed that D is ample, we can take a C^∞-metric $|\cdot|$ of $\mathcal{O}(D)$ with the non-negative curvature. We put $\phi := -\log|\sigma|$, where σ denote the canonical section. Then $\sqrt{-1}\partial\overline{\partial}\phi$ is a non-negative C^∞-2-form, and it is easy to check that the second condition is satisfied.

To check the condition 3, we give the following remark. Let P be a point of $D_i\cap D_j$. For simplicity, let us consider the case $(i,j)=(1,2)$. We put $V_P := \big\{(\zeta_1,\zeta_2)\,\big|\,|\zeta_i|<1\big\}$. Let us take the ramified covering $\varphi: V_P\longrightarrow U_P$ given by $(\zeta_1,\zeta_2)\longmapsto(\zeta_1^m,\zeta_2^m)$. Then it is easy to check that $\widetilde{\omega}=\varphi^{-1}\omega_\epsilon$ naturally gives the C^∞-Kahler form on V_P. If f is a bounded positive function on $U_P\setminus D$ satisfying $\Delta_{\omega_\epsilon}(f)\leq B$ for some constant

B, we obtain $\Delta_{\widetilde{\omega}}\bigl(\varphi^* f\bigr) \leq B$ on $V_P - \varphi^{-1}(D \cap U_P)$. Since $\widetilde{\omega}$ is C^∞ on V_P, we may apply the argument of Proposition 2.2 in [**51**]. Hence $\Delta_{\widetilde{\omega}}\bigl(\varphi^* f\bigr) \leq B$ holds weakly on V_P. Then we can apply the arguments of Proposition 2.1 in [**51**], and we obtain an appropriate estimate for the sup norm of f. By a similar argument, we obtain such an estimate around any smooth points of D. Thus we are done. □

4.3.2. A construction of an ordinary metric of the bundle.

— Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). In the following, we shrink the open sets U_i without mentioning, if it is necessary. We put $D_i^\circ := D_i \setminus \bigcup_{j \neq i} D_j$.

On each D_i, we have the generalized eigen decomposition with respect to $\mathrm{Res}_i(\theta)$:

$$ {}_{\boldsymbol{c}}E_{|D_i} = \bigoplus_{\alpha}{}^i \,\mathrm{Gr}^{\mathbb{E}}_{\alpha}({}_{\boldsymbol{c}}E_{|D_i}) \tag{30} $$

For each point $P \in D_i \cap D_j$, we may assume that there is a decomposition ${}_{\boldsymbol{c}}E_{|U_P} = \bigoplus {}^P U_{\boldsymbol{a},\boldsymbol{\alpha}}$ as in Section 4.1. Let ${}^P\boldsymbol{v}$ be a holomorphic frame compatible with the decomposition. We take a C^∞-hermitian metric $\widehat{h}_0$ of ${}_{\boldsymbol{c}}E$ such that ${}^P\boldsymbol{v}$ is an orthonormal frame on U_P and that the decomposition (30) is orthogonal. We have the induced unitary connections $\nabla_{0,i}$ and $\nabla_{{}_{\boldsymbol{c}}E_{|D_i}}$ on ${}_{\boldsymbol{c}}E_{|U_i}$ and ${}_{\boldsymbol{c}}E_{|D_i}$, respectively. Then, we can take a C^∞-isomorphism ${}^i\Phi : \pi_i^*\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr) \simeq {}_{\boldsymbol{c}}E$ on U_i such that (i) the restriction of ${}^i\Phi$ to D_i is the identity, (ii) the restriction of ${}^i\Phi$ to U_P is given by the frames ${}^P\boldsymbol{v}$ and $\pi_i^*\bigl({}^P\boldsymbol{v}_{|U_P \cap D_i}\bigr)$, (iii) $\nabla_{0,i} \circ {}^i\Phi - {}^i\Phi \circ \pi^*\nabla_{{}_{\boldsymbol{c}}E_{|D_i}} = O(|\sigma_i|_{g_i})$. ([**35**]. See also the explanation in Subsection 4.2.2.) We also obtain the orthogonal decompositions $\mathrm{Gr}^{\mathbb{E}}_{\alpha}\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr) = \bigoplus_{a \in \boldsymbol{R}} {}^i\mathcal{G}_{(a,\alpha)}$ with respect to $\widehat{h}_0$ such that ${}^iF_b\,\mathrm{Gr}^{\mathbb{E}}_{\alpha}\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr) = \bigoplus_{a \leq b} \mathcal{G}_{(a,\alpha)}$. They induce the C^∞-decompositions ${}_{\boldsymbol{c}}E_{|U_i} = \bigoplus {}^i_{\boldsymbol{c}}E_{(a,\alpha)}$.

We can take a hermitian metric h_0 of E on $X - D$, which is as in Subsection 4.1.1 on U_P, and as in Subsection 4.2.2 on $U_i \setminus \bigcup U_P$. More precisely, we take a hermitian metric h_{D_i} of ${}_{\boldsymbol{c}}E_{|D_i^\circ}$ such that (i) the decomposition ${}_{\boldsymbol{c}}E_{|D_i^\circ} = \bigoplus {}^i\mathcal{G}_{u|D_i^\circ}$ is orthogonal, (ii) $h_{D_i}({}^Pv_k, {}^Pv_l) = \delta_{k,l} \cdot |z_j|^{-2a_j({}^Pv_k)}$ for each $P \in D_i \cap D_j$ $(j \neq i)$. Let $h_{D_i,u}$ denote the restriction of h_{D_i} to ${}^i\mathcal{G}_{u|D_i^\circ}$. Then, h_0 is given by (17) on $U_i \setminus D$. We have $h_0({}^Pv_k, {}^Pv_l) = \delta_{k,l} \cdot |z_i|^{-2a_i({}^Pv_k)} \cdot |z_j|^{-2a_j({}^Pv_k)}$ on $U_P \setminus D$ for $P \in D_i \cap D_j$. Thus, we obtain the metric of E on $\bigcup_i U_i \setminus D$. We extend it to the metric of E on $X - D$. Such a metric h_0 is called an ordinary metric, in this paper. The following lemma immediately follows from Proposition 4.3 and Proposition 4.7.

Lemma 4.15. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, then $F(h_0)$ is bounded with respect to h_0 and ω_ϵ.*

4.3.3. Calculation of the integrals

Lemma 4.16

$$ \Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \int_{X-D} \bigl(\mathrm{tr}\, R(h_0)\bigr)^2 = \int_X \mathrm{par\text{-}c}_1^2({}_{\boldsymbol{c}}E_*). $$

Proof. — We have $\big(\operatorname{tr} R(h_0)\big)^2 = \big(\operatorname{tr} R(\widehat{h}_0)\big)^2 + \operatorname{tr} R(\widehat{h}_0)\cdot\overline{\partial}\operatorname{tr} A + \operatorname{tr} R(h_0)\cdot\overline{\partial}\operatorname{tr} A$. We have the following equality:

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X-D}\big(\operatorname{tr} R(\widehat{h}_0)\big)^2 = \int_X c_1({}_{\boldsymbol{c}}E)^2.$$

Due to (19), we obtain the following:

(31)
$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X-D}\operatorname{tr} R(\widehat{h}_0)\cdot\overline{\partial}\operatorname{tr} A = \sum_i\frac{\sqrt{-1}}{2\pi}\int_{D_i}\operatorname{tr} R(\widehat{h}_{0|D_i},{}_{\boldsymbol{c}}E_{|D_i})\cdot\big(-\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\big)$$
$$= \sum_i -\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\cdot\deg_{D_i}({}_{\boldsymbol{c}}E_{|D_i}) = -\sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\int_X c_1({}_{\boldsymbol{c}}E)\cdot[D_i]$$

We put $\widehat{E}_{D_i,u} := {}^i\operatorname{Gr}_u^{F,\mathbb{E}}(E_{|D_i})$, which is naturally isomorphic to ${}^i\mathcal{G}_u$ as C^∞-bundles. Hence the metric $h_{D_i,u}$ on $\widehat{E}_{D_i,u}$ is induced (Subsection 4.3.2). Then, we obtain the following, using Corollary 4.6:

(32)
$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X-D}\operatorname{tr} R(h_0)\cdot\overline{\partial}\operatorname{tr} A = -\sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\sum_u\frac{\sqrt{-1}}{2\pi}\int_{D_i}\operatorname{tr} R(h_{D_i,u},\widehat{E}_{D_i,u})$$
$$+\sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)^2\cdot\frac{\sqrt{-1}}{2\pi}\int_{D_i}\overline{\partial}\partial\log|\sigma_i|^2$$

We have the naturally induced parabolic structure of ${}_{\boldsymbol{c}}E_{|D_i}$ at $D_i\cap\bigcup_{j\neq i}D_j$. Then we have the following equality:

(33)
$$\sum_u\frac{\sqrt{-1}}{2\pi}\int_{D_i}\operatorname{tr} R(h_{D_i,u},\widehat{E}_{D_i,u}) = \operatorname{par-deg}_{D_i}({}_{\boldsymbol{c}}E_{|D_i\,*})$$
$$= \deg_{D_i}({}_{\boldsymbol{c}}E_{|D_i}) - \sum_{j\neq i}\operatorname{wt}({}_{\boldsymbol{c}}E_*,j)\cdot\int_X[D_i]\cdot[D_j].$$

We also have $\frac{\sqrt{-1}}{2\pi}\int_{D_i}\overline{\partial}\partial\log|\sigma_i|^2 = \int_X[D_i]^2$. Thus we obtain the following:

(34)
$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X-D}\operatorname{tr} R(h_0)\cdot\overline{\partial}\operatorname{tr} A = -\sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\int_X c_1({}_{\boldsymbol{c}}E)\cdot[D_i]$$
$$+\sum_i\sum_{j\neq i}\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\cdot\operatorname{wt}({}_{\boldsymbol{c}}E_*,j)\int_X[D_i]\cdot[D_j] + \sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)^2\cdot\int_X[D_i]^2$$
$$= -\sum_i\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\int_X c_1({}_{\boldsymbol{c}}E)\cdot[D_i] + \sum_i\sum_j\operatorname{wt}({}_{\boldsymbol{c}}E_*,i)\cdot\operatorname{wt}({}_{\boldsymbol{c}}E_*,j)\int_X[D_i]\cdot[D_j].$$

Then the claim of the lemma follows. □

Corollary 4.17

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \big(\operatorname{tr} F(h_0)\big)^2 = \int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*).$$

Proof. — It follows from $\big(\operatorname{tr} F(h_0)\big)^2 = \big(\operatorname{tr} R(h_0)\big)^2$ and the previous lemma. □

Proposition 4.18. — *If $(\boldsymbol{E}_*, \theta)$ is graded semisimple, the following equality holds:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(F(h_0)^2\Big) = 2\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*).$$

Proof. — We have only to show the following two equalities:

$$\int_{X-D} \operatorname{tr}\Big(F(h_0)^2\Big) = \int_{X-D} \operatorname{tr}\Big(R(h_0)^2\Big). \tag{35}$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(R(h_0)^2\Big) = 2\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*). \tag{36}$$

Let us show (35). By a direct calculation or the classical Chern-Simons theory, we obtain the following equality:

$$\begin{aligned}\operatorname{tr}\big(F(h_0)^2\big) &= \operatorname{tr}\big(R(h_0)^2\big) + 2\operatorname{tr}\Big(\big(\partial_{h_0}\theta + \overline{\partial}\theta^\dagger_{h_0}\big)\cdot R(h_0)\Big) \\ &\quad + d\Big(\operatorname{tr}\big((\theta+\theta^\dagger_{h_0})\cdot(\partial_{h_0}\theta + \overline{\partial}\theta^\dagger_{h_0})\big) + (2/3)\cdot \operatorname{tr}\big((\theta+\theta^\dagger_{h_0})^3\big)\Big).\end{aligned} \tag{37}$$

Since $R(h_0)$, $\partial_{h_0}\theta$ and $\overline{\partial}\theta^\dagger_{h_0}$ are a $(1,1)$-form, a $(2,0)$-form and a $(0,2)$-form respectively, we obtain the vanishing of the second term in the right hand side. It is easy to obtain $\operatorname{tr}\big((\theta+\theta^\dagger_{h_0})^3\big)=0$ from $\theta^2 = \theta^{\dagger\,2}_{h_0} = 0$.

We put $Y_i(\delta) := \Big\{x \in X \,\Big|\, |\sigma_i(x)| = \min_j |\sigma_j(x)| = \delta\Big\}$ and $Y(\delta) := \bigcup_i Y_i(\delta)$. From the estimate in Sections 4.1–4.2, $\operatorname{tr}\big(\theta\cdot\overline{\partial}\theta^\dagger_{h_0}\big)$ and $\operatorname{tr}\big(\theta^\dagger_{h_0}\cdot\partial_{h_0}\theta\big)$ are bounded with respect to $\omega_{\epsilon'}$ for some $0<\epsilon'<\epsilon$. Hence, we obtain the following convergence:

$$\lim_{\delta\to 0}\int_{Y(\delta)} \operatorname{tr}(\theta\cdot\overline{\partial}\theta^\dagger_{h_0}) = \lim_{\delta\to 0}\int_{Y(\delta)} \operatorname{tr}(\theta^\dagger_{h_0}\cdot\partial_{h_0}\theta) = 0$$

Then, we obtain the formula (35):

Let us see (36). We put $A := \partial_{h_0} - \partial_{\widehat{h}_0}$. Then we have $\operatorname{tr}\big(R(h_0)^2\big) = \operatorname{tr}\big(R(\widehat{h}_0)^2\big) + d\operatorname{tr}\big(A\cdot R(h_0) + A\cdot R(\widehat{h}_0)\big)$. The contribution of the first term is as follows:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\big(R(\widehat{h}_0)^2\big) = 2\operatorname{ch}_2({}_{\boldsymbol{c}}E).$$

As for the second term, we obtain the following from Corollary 4.9:

(38)

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2\int_{X-D} d\,\mathrm{tr}\Big(A\cdot R(h_0)+A\cdot R(\widehat{h}_0)\Big)=-\sum_{i,a,\alpha}a\cdot\deg_{D_i}\big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})\big)$$
$$-\sum_{i,a,\alpha}a\cdot\text{par-deg}_{D_i}\big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})_*\big)+\sum_{i,a,\alpha}a^2\,\mathrm{rank}\,{}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)\int_X[D_i]^2.$$

Here ${}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*$ is the parabolic bundle on $\big(D_i, D_i\cap\bigcup_{j\neq i}D_j\big)$ with the canonically induced parabolic structure. We have the following:

$$\sum_{\alpha}\text{par-deg}_{D_i}\Big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*\Big)=\text{par-deg}_{D_i}\Big({}^i\,\mathrm{Gr}^{F}_{a}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*\Big)\tag{39}$$
$$=\deg_{D_i}\big({}^i\,\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E_{|D_i}\big)\big)-\sum_{\substack{j\neq i,\\ P\in D_i\cap D_j}}\ \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar({}_{\boldsymbol{c}}E,P)\\ a_i=a}}a_j\cdot\mathrm{rank}\big({}^P\,\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|O})\big).$$

Then (36) immediately follows. □

4.3.4. The degree of subsheaves. — Let V be a saturated coherent $\mathcal{O}_{X-D}$-submodule of E. Let π_V denote the orthogonal projection of E onto V with respect to h_0, which is defined outside the Zariski closed subset of codimension two. Let h_V be the metric of V induced by h_0. The following lemmas are the special case of the results of J. Li [**35**].

Lemma 4.19. — *$\overline{\partial}\pi_V$ is L^2 with respect to h_0 and ω_ϵ if and only if there exists a coherent subsheaf ${}_{\boldsymbol{c}}V\subset{}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D}=V$.*

Lemma 4.20. — *$\deg_{\omega_\epsilon}(V,h_V)=\text{par-deg}_\omega({}_{\boldsymbol{c}}V_*)$ holds.*

Proof. — We give just an outline of a proof of Lemma 4.20. By considering the exterior product of E and V, we may assume $\mathrm{rank}\,V=1$. We may assume that L is very ample. Let C be a smooth divisor of X with $\mathcal{O}(C)\simeq L$ such that (i) ${}_{\boldsymbol{c}}V$ is locally free on a neighbourhood of C, (ii) C intersects with the smooth part of D transversally, (iii) ${}^iF$ is a filtration in the category of the vector bundles on D_i around $C\cap D_i$. We can take a smooth $(1,1)$-form τ whose support is contained in a sufficiently small neighbourhood of C, such that τ and ω represents the same cohomology class. Then we have $\int\mathrm{tr}\,R(h_V)\cdot\omega=\int\mathrm{tr}\,R(h_V)\cdot\tau$. It can be checked $\frac{\sqrt{-1}}{2\pi}\int\mathrm{tr}\,R(h_V)\cdot\tau=\text{par-deg}_\omega({}_{\boldsymbol{c}}V_*)$ by an elementary argument. □

CHAPTER 5

PARABOLIC HIGGS BUNDLE ASSOCIATED TO TAME HARMONIC BUNDLE

In this chapter, we show the fundamental property of the parabolic Higgs bundles associated to tame harmonic bundles, such as μ_L-polystability and the vanishing of characteristic numbers. We also see the uniqueness of the adapted pluri-harmonic metric. These results give the half of Theorem 1.4.

5.1. Polystability and Uniqueness

Let X be a smooth irreducible projective variety over $\boldsymbol{C}$, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let L be any ample line bundle of X.

Proposition 5.1. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$, and let $({}_{\boldsymbol{c}}E_*, \theta)$ denote the associated $\boldsymbol{c}$-parabolic Higgs bundle for any $\boldsymbol{c} \in \boldsymbol{R}^S$. (See Section 3.5.)*

- *$({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-polystable, and $\text{par-deg}_L({}_{\boldsymbol{c}}E_*) = 0$.*
- *Let $({}_{\boldsymbol{c}}E_*, \theta) = \bigoplus_i ({}_{\boldsymbol{c}}E_{i*}, \theta_i) \otimes \boldsymbol{C}^{p(i)}$ be the canonical decomposition (Corollary 3.11). Then we have the orthogonal decomposition $h = \bigoplus_i h_i \otimes g_i$. Here h_i are pluri-harmonic metrics for $(E_i, \overline{\partial}_{E_i}, \theta_i)$, and g_i are hermitian metrics of $\boldsymbol{C}^{p(i)}$.*

Proof. — The equality $\text{par-deg}_L({}_{\boldsymbol{c}}E_*) = 0$ can be easily reduced to the curve case (Proposition 2.8). It also follows from the curve case that $({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-semistable.

Let us show $({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-polystable. Let $({}_{\boldsymbol{c}}V_*, \theta_V)$ be a non-trivial saturated Higgs subsheaf of $({}_{\boldsymbol{c}}E_*, \theta)$ such that $\mu_L({}_{\boldsymbol{c}}V_*) = \mu_L({}_{\boldsymbol{c}}E_*) = 0$ and $\text{rank}(V) < \text{rank}(E)$. Recall that we have the closed subset $Z \subset X$ such that ${}_{\boldsymbol{c}}V_{|X-Z}$ is the subbundle of ${}_{\boldsymbol{c}}E_{|X-Z}$. The codimension of Z is larger than 2. We have the orthogonal projection $\pi_V : E \longrightarrow V$ on the open set $X - (Z \cup D)$. Let $C \subset X$ be any smooth curve such that (i) C intersects with the smooth part of D transversally, (ii) $C \cap Z = \varnothing$. Let θ_C denote the induced Higgs field of $E_{|C\setminus D}$. Due to the result in the curve case, we obtain that $\pi_{V|C}$ is holomorphic and that θ_C and $\pi_{V|C}$ commute. Then, we obtain that $\pi_{V|X-(D\cup Z)}$ is holomorphic and that $[\pi_V,\theta]=0$. Since the codimension of Z is larger than two, π_V naturally gives the holomorphic map $E\longrightarrow E$ on $X-D$, which is also denoted by π_V. It is easy to see $\pi_V^2=\pi_V$, and that the restriction of π_V to V is the identity. Hence we obtain the decomposition $E=V\oplus V'$, where we put $V'=\operatorname{Ker}\pi_V$. We can conclude that V and V' are vector subbundles of E, and the decomposition is orthogonal with respect to the metric h. Since we have $[\pi_V,\theta]=0$, the decomposition is also compatible with the Higgs field. Hence we obtain the decomposition of $(E,\overline{\partial}_E,\theta,h)$ into $(V,\overline{\partial}_V,\theta_V,h_V)\oplus(V',\overline{\partial}_{V'},\theta_{V'},h_{V'})$ as harmonic bundles. Then it is easy that $({}_{\boldsymbol{c}}E_*,\theta)$ is also decomposed into $({}_{\boldsymbol{c}}V_*,\theta_V)\oplus({}_{\boldsymbol{c}}V'_*,\theta_{V'})$. Since both of $({}_{\boldsymbol{c}}V_*,\theta_V)$ and $({}_{\boldsymbol{c}}V'_*,\theta_{V'})$ are obtained from tame harmonic bundles, they are μ_L-semistable. And we have $\operatorname{rank}(V)<\operatorname{rank}(E)$ and $\operatorname{rank}(V')<\operatorname{rank}(E)$. Hence the μ_L-polystability of $({}_{\boldsymbol{c}}E,\theta)$ can be shown by an easy induction on the rank.

From the argument above, the second claim is also clear. □

Proposition 5.2. — *Let $({}_{\boldsymbol{c}}E_*,\theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X,D). We put $E:={}_{\boldsymbol{c}}E_{|X-D}$. Assume that we have pluri-harmonic metrics h_i of $(E,\overline{\partial}_E,\theta)$ $(i=1,2)$, which are adapted to the parabolic structures. Then we have the decomposition of Higgs bundles $(E,\theta)=\bigoplus_a(E_a,\theta_a)$ satisfying the following conditions:*

- *The decomposition is orthogonal with respect to both of h_i. The restrictions of h_i to E_a are denoted by $h_{i,a}$.*
- *There exist positive numbers b_a such that $h_{1,a}=b_a\cdot h_{2,a}$.*

We remark that the decomposition $(E,\theta)=\bigoplus(E_a,\theta_a)$ induces the decomposition of the $\boldsymbol{c}$-parabolic Higgs bundles $\big({}_{\boldsymbol{c}}E_*,\theta\big)=\bigoplus({}_{\boldsymbol{c}}E_{a*},\theta_a)$.

Proof. — Recall the norm estimate for tame harmonic bundles ([**44**]) which says that the harmonic metrics are determined up to boundedness by the parabolic filtration and the weight filtration. Hence we obtain the mutually boundedness of h_1 and h_2. Then the uniqueness follows from Proposition 2.6. (The Kahler metric of $X-D$ is given by the restriction of a Kahler metric of X. It satisfies Condition 2.1, according to [**51**].) □

5.2. Vanishing of Characteristic Numbers

Proposition 5.3. — *Let $(E,\overline{\partial}_E,\theta,h)$ be a tame harmonic bundle on $X-D$, and $({}_{\boldsymbol{c}}E_*,\theta)$ be the induced $\boldsymbol{c}$-parabolic Higgs bundle. Then we have the vanishing of the following characteristic numbers:*

$$\int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*)=0,\qquad \int_X \text{par-c}_{1,L}^2({}_{\boldsymbol{c}}E_*)=0.$$

Proof. — We may and will assume $\dim X=2$. Let h_0 be an ordinary metric for the parabolic Higgs bundle $({}_{\boldsymbol{c}}E_*,\theta)$. We have only to show $\int\operatorname{tr}\big(R(h_0)^2\big)=\int\operatorname{tr}\big(R(h_0)\big)^2=0$.

Let $I(D)$ denote the set of the intersection points of D. Let $\pi : \widetilde{X} \longrightarrow X$ be a blow up at $I(D)$. We put $\widetilde{D} := \pi^{-1}(D)$. Let $\widetilde{D}_i$ denote the proper transform of D_i, and let $\widetilde{D}_P$ denote the exceptional curve $\pi^{-1}(P)$. We put $\widetilde{S} := S \cup I(D)$. Then, we have $\widetilde{D} = \bigcup_{i \in \widetilde{S}} \widetilde{D}_i$. We take neighbourhoods $\widetilde{U}_i$ of $\widetilde{D}_i$ with retractions $\widetilde{\pi}_i : \widetilde{U}_i \longrightarrow \widetilde{D}_i$ for $i \in \widetilde{S}$, as in Subsection 4.3.1.

We put $\widetilde{E} := \pi^{-1}({}_{\boldsymbol{c}}E)$. On $\widetilde{E}_{|\widetilde{D}_i}$ $(i \in S)$, we have the naturally induced filtration ${}^iF$. For any intersection point $P \in D_i \cap D_j$, we have the isomorphism $\widetilde{E}_{|\widetilde{D}_P} \simeq {}_{\boldsymbol{c}}E_{|P} \otimes \mathcal{O}_{\widetilde{D}_P}$. We have the filtrations ${}^iF$ and ${}^jF$ on ${}_{\boldsymbol{c}}E_{|P}$. We take a decomposition ${}_{\boldsymbol{c}}E_{|P} = \bigoplus U_{\boldsymbol{a}}$ such that ${}^1F_{a_1} \cap {}^2F_{a_2} = \bigoplus_{\boldsymbol{b} \leq \boldsymbol{a}} U_{\boldsymbol{b}}$. Then, we put ${}^PF_b({}_{\boldsymbol{c}}E_{|P}) := \bigoplus_{b_1+b_2 \leq b} U_{\boldsymbol{b}}$, which gives the filtration of ${}_{\boldsymbol{c}}E_{|P}$. The induced filtration on $\widetilde{E}_{|\widetilde{D}_P}$ is also denoted by ${}^PF$. The tuple of filtrations $\big({}^iF \,\big|\, i \in \widetilde{S}\big)$ is denoted by $\boldsymbol{F}$.

We put $\widetilde{\theta} := \pi^{-1}\theta$. Then $(\widetilde{E}, \boldsymbol{F}, \widetilde{\theta})$ is a generalized parabolic Higgs bundle in the sense of Remark 3.6. The residue $\operatorname{Res}_i \widetilde{\theta}$ preserves the filtration ${}^iF$. On each $i \in \widetilde{S}$, the residue $\operatorname{Res}_i \widetilde{\theta}$ induces the endomorphism of ${}^i\operatorname{Gr}^F_a(\widetilde{E})$. The eigenvalues are constant, and hence the nilpotent part $\mathcal{N}_i$ is well defined. The conjugacy classes of $\mathcal{N}_{i\,|\,P}$ are independent of the choice of $P \in \widetilde{D}_i$ ([**44**]). Thus, we obtain the weight filtration ${}^iW$ on ${}^i\operatorname{Gr}^F_a(\widetilde{E})$. We put $\widetilde{F}_{(a,k)} := \pi_a^{-1}({}^iW_k)$, where π_a denotes the projection ${}^iF_a \longrightarrow {}^i\operatorname{Gr}^F_a(\widetilde{E})$.

Let $\widetilde{P}_i$ denote the intersection point of $\widetilde{D}_i$ and $\widetilde{D}_P$ for $i \in S$ and $P \in I(D)$. Around $\widetilde{P}_i$, we have the holomorphic frame ${}^{P\,i}\widetilde{\boldsymbol{v}}$, as in Subsection 2.5.2. Namely, we take a holomorphic frame ${}^{P\,i}\boldsymbol{v}$ around P as in Subsection 2.5.1, (D_i plays the role of D_1, there) and we put ${}^{P\,i}\widetilde{\boldsymbol{v}} := \pi^{-1}({}^{P,i}\boldsymbol{v})$ around $\widetilde{P}_i$. We take a hermitian metric $\widehat{h}_1$ of $\widetilde{E}$ such that ${}^{P,i}\widetilde{\boldsymbol{v}}$ around $\widetilde{P}_i$ are orthonormal with respect to $\widehat{h}_1$. By using it, we take C^∞-isomorphisms $\widetilde{\Phi}_i : \widetilde{\pi}_i^* \widetilde{E}_{|\widetilde{D}_i} \simeq \widetilde{E}_{|\widetilde{U}_i}$ on $\widetilde{U}_i$ $(i \in \widetilde{S})$ as in Subsection 4.3.2. Then, we can take a hermitian metric h_1 which is as in Subsection 2.5.2 for the frame ${}^{P\,i}\widetilde{\boldsymbol{v}}$ around $\widetilde{P}_i$, and as in Subsection 4.2.6 around $\widetilde{D}_i$.

Lemma 5.4. — *We have* $\int \operatorname{tr}\big(R(h_0)^2\big) = \int \operatorname{tr}\big(R(h_1)^2\big)$ *and* $\int \operatorname{tr}\big(R(h_0)\big)^2 = \int \operatorname{tr}\big(R(h_1)\big)^2$.

Proof. — Let $\widetilde{\omega}$ denote a Poincaré like metric on $\widetilde{X} - \widetilde{D}$. Let $\widetilde{h}_0$ be an ordinary metric for $(\widetilde{E}, \boldsymbol{F}, \widetilde{\theta})$ as constructed in Subsection 4.3.2. Then, $\pi^* h_0$ and $\widetilde{h}_0$ are mutually bounded. Both of $\pi^* R(h_0)$ and $R(\widetilde{h}_0)$ are bounded with respect to $\widetilde{h}_0$ and $\widetilde{\omega}$.

Let us see that $A_0 = \partial_{\widetilde{E}, \pi^{-1}h_0} - \partial_{\widetilde{E}, \widetilde{h}_0}$ is bounded with respect to $\widetilde{\omega}$ and $\widetilde{h}_0$. Let us recall the description of A_0 around $\widetilde{P}_i$. We take a holomorphic coordinate neighbourhood (U, z_1, z_2) such that $\{z_1 \cdot z_2 = 0\} = U \cap (\widetilde{D}_i \cup \widetilde{D}_P)$. We put $D'_j := \{z_j = 0\}$. We have two holomorphic decomposition $\widetilde{E}_{|U} = \bigoplus U_{\boldsymbol{a}} = \bigoplus \widetilde{U}_{\boldsymbol{a}}$ such that ${}^jF_b = \bigoplus_{a_j \leq b} U_{\boldsymbol{a}|D'_j} = \bigoplus_{a_j \leq b} \widetilde{U}_{\boldsymbol{a}|D'_j}$, where a_j denotes the j-th component of $\boldsymbol{a}$. We put $\Gamma_j = \bigoplus a_j \cdot \mathrm{id}_{U_{\boldsymbol{a}}}$ and $\widetilde{\Gamma}_j = \bigoplus a_j \cdot \mathrm{id}_{\widetilde{U}_{\boldsymbol{a}}}$. We have $\partial_{\widetilde{E}, \pi^{-1}(h_0)} = \mathcal{D}_1 - \sum \Gamma_j \cdot dz_j / z_j$ and $\partial_{\widetilde{E}, \widetilde{h}_0} = \mathcal{D}_2 - \sum \widetilde{\Gamma}_j \cdot dz_j / z_j$, where $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) is the $(1,0)$-operator of $\widetilde{E}_{|U}$, preserving the decomposition $\widetilde{E}_{|U} = \bigoplus U_{\boldsymbol{a}}$ (resp. $\widetilde{E}_U = \bigoplus \widetilde{U}_{\boldsymbol{a}}$). Then, we have $A_0 = \sum(\widetilde{\Gamma}_j - \Gamma_j)dz_j/z_j + (\mathcal{D}_1 - \mathcal{D}_2)$. Because of $(\widetilde{\Gamma}_j - \Gamma_j)_{|D'_j}{}^jF_a \subset {}^jF_{<a}$ and $(\mathcal{D}_1 - \mathcal{D}_2)_{|D'_j}{}^jF_a \subset {}^jF_a \otimes \Omega^{1,0}$, we obtain the boundedness of A_0 with respect to $\pi^{-1}h_0$ and $\widetilde{\omega}$, around $\widetilde{P}_i$.

Let us recall the description of A_0 around $Q \in \widetilde{D}_i$ $(i \in \widetilde{S})$. Let (U, z_1, z_2) be a holomorphic coordinate around Q such that $z_1^{-1}(0) = U \cap \widetilde{D}_i$. We have two C^∞-decomposition $\widetilde{E}_{|U} = \bigoplus E_a = \bigoplus \widetilde{E}_a$ such that ${}^iF_b = \bigoplus_{a \leq b} E_{a|\widetilde{D}_i} = \bigoplus_{a \leq b} \widetilde{E}_{a|\widetilde{D}_i}$. We put $\Gamma := \bigoplus a \cdot \mathrm{id}_{E_a}$ and $\widetilde{\Gamma} := \bigoplus a \cdot \mathrm{id}_{\widetilde{E}_a}$. We have a description $\partial_{\pi^{-1}h_0} = \partial_{1,\pi^{-1}h'_0} - \Gamma dz_1/z_1 + O(1)$, where $O(1)$ denotes the bounded one form with respect to h_0 and $\widetilde{\omega}$, and $\partial_{1,\pi^{-1}h'_0}$ is operator on $\widetilde{E}_{|U}$ (not on $\widetilde{E}_{U \setminus \widetilde{D}_i}$) such that $\partial_{1,\pi^{-1}h'_0|\widetilde{D}_i}$ preserves the filtration ${}^iF$. (See the proof of Lemma 4.5.) Similarly, we have $\partial_{\widetilde{h}_0} = \partial_{1,\widetilde{h}'_0} - \widetilde{\Gamma} dz_1/z_1 + O(1)$. For the expression $\partial_{1,\pi^{-1}h'_0} - \partial_{1,\widetilde{h}'_0} = B_1 \cdot dz_1/z_1 + B_2 \cdot dz_2$, we have $B_{1|\widetilde{D}_i} = 0$ and $B_{2|\widetilde{D}_i}({}^iF_a) \subset {}^iF_a$. We also have $(\widetilde{\Gamma} - \Gamma)_{|\widetilde{D}_i}{}^iF_a \subset {}^iF_{<a}$. Thus, we obtain the boundedness of $\partial_{\pi^{-1}h_0} - \partial_{\widetilde{h}_0}$. Now, it is easy to obtain $\int \mathrm{tr}\big(R(h_0)^2\big) = \int \mathrm{tr}\big(R(\widetilde{h}_0)^2\big)$ and $\int \mathrm{tr}\big(R(h_0)\big)^2 = \int \mathrm{tr}\big(R(\widetilde{h}_0)\big)^2$.

Due to the lemmas 2.15, 4.5 and 4.10, $R(\widetilde{h}_0)$, $R(h_1)$ and $A_0 := \partial_{h_1} - \partial_{\widetilde{h}_0}$ are bounded with respect to $(\widetilde{h}_0, \widetilde{\omega})$. Hence, $\mathrm{tr}\big(A_0\big)$, $\mathrm{tr}\big(R(\widetilde{h}_0)\big)$, $\mathrm{tr}\big(R(\widetilde{h}_0) \cdot A_0\big)$, $\mathrm{tr}\big(R(h_1)\big)$ and $\mathrm{tr}\big(R(h_1) \cdot A_0\big)$ are bounded with respect to $\widetilde{\omega}$. Then, it is easy to show $\int \mathrm{tr}\big(R(\widetilde{h}_0)^2\big) = \int \mathrm{tr}\big(R(h_1)^2\big)$ and $\int \mathrm{tr}\big(R(\widetilde{h}_0)\big)^2 = \int \mathrm{tr}\big(R(h_1)\big)^2$. □

Due to the norm estimate (Lemma 2.14), $\widetilde{h} := \pi^* h$ and h_1 are mutually bounded. We also have that $R(\widetilde{h})$ is bounded with respect to $\widetilde{h}$ and $\widetilde{\omega}$. Let s denote the self-adjoint endomorphism of $\pi^{-1}(E)$ with respect to $\widetilde{h}$ and h_1, determined by $\widetilde{h} = h_1 \cdot s$. We have $\partial_{\widetilde{h}} - \partial_{h_1} = s^{-1}\partial_{h_1} s$ and $\overline{\partial}\big(s^{-1}\partial_{h_1} s\big) = R(\widetilde{h}) - R(h_1)$, which is bounded with respect to h_1 and $\widetilde{\omega}$.

Let us show the following equality for any test function χ on $\widetilde{X} - \widetilde{D}$:

$$(40) \quad \int \big(s^{-1}\partial_{h_1}(\chi \cdot s),\, \partial_{h_1}(\chi \cdot s)\big)_{h_1} \cdot \widetilde{\omega} = \int \big(\chi \cdot \overline{\partial}(s^{-1}\partial_{h_1} s),\, \chi \cdot s\big) \cdot \widetilde{\omega} + \int \partial\chi \cdot \overline{\partial}\chi \cdot \mathrm{tr}(s) \cdot \widetilde{\omega}.$$

We have the following:

$$(41) \quad \int \big(s^{-1}\partial_{h_1}(\chi \cdot s),\, \partial_{h_1}(\chi \cdot s)\big)_{h_1} = \int \Big(\overline{\partial}\Big(s^{-1} \cdot \partial_{h_1}(\chi \cdot s)\Big),\, \chi \cdot s\Big)_{h_1}$$
$$= \int \big(\overline{\partial}\partial\chi,\ \chi \cdot s\big)_{h_1} + \int \Big(\chi \cdot \overline{\partial}\big(s^{-1}\partial_{h_1} s_0\big),\, \chi \cdot s\Big)_{h_1} + \int \big(\overline{\partial}\chi \cdot s^{-1}\partial_{h_1} s,\ \chi \cdot s\big)_{h_1}.$$

Moreover, we have the following:

$$(42) \quad \big(\overline{\partial}\partial\chi, \chi \cdot s\big)_{h_1} + \big(\overline{\partial}\chi \wedge s^{-1}\partial_{h_1} s,\ \chi \cdot s\big)_{h_1} = \mathrm{tr}\big(\overline{\partial}\partial\chi \cdot \chi \cdot s\big) + \mathrm{tr}\big(\overline{\partial}\chi \cdot \partial_{h_1} s \cdot \chi\big)$$
$$= -\partial\Big(\mathrm{tr}\big(\overline{\partial}\chi \cdot \chi \cdot s\big)\Big) - \mathrm{tr}\big(\overline{\partial}\chi\partial\chi \cdot s\big).$$

Thus we obtain (40).

Lemma 5.5. — *$s^{-1}\partial_{h_1}s$ is L^2 with respect to $\widetilde{\omega}$ and $\widetilde{h}$.*

Proof. — Let ρ be a non-negative valued function on $\boldsymbol{R}$ satisfying $\rho(t)=1$ for $t\leq 1/2$ and $\rho(t)=0$ for $t\geq 2/3$. Take hermitian metrics g_i of the line bundles $\mathcal{O}(\widetilde{D}_i)$ $(i\in\widetilde{S})$. Let σ_i denote the canonical section of $\mathcal{O}(\widetilde{D}_i)$, and $|\sigma_i|$ denote the norm function of σ_i with respect to g_i. We may assume $|\sigma_i|<1$. We put $\chi_N:=\prod_{i\in\widetilde{S}}\rho\bigl(-N^{-1}\log|\sigma_i|^2\bigr)$. Then, $\partial\chi_N$ is bounded with respect to $\widetilde{\omega}$, independently of N. By using (40), we obtain $\int\bigl|s^{-1}\partial_{h_1}(\chi_N s)\bigr|^2_{h_1}\,\mathrm{dvol}_{\widetilde{\omega}}<C$ for some constant C, and thus we obtain the claim of the lemma. $\square$

We put $A_1:=s^{-1}\partial_{h_1}s$, which is L^2 with respect to $\widetilde{\omega}$ and h_1. We have $R(h_1)=R(\widetilde{h})-\overline{\partial}A_1$. Since we have $\operatorname{tr}R(\widetilde{h})=\operatorname{tr}F(\widetilde{h})=0$, we have $\operatorname{tr}\bigl(R(h_1)\bigr)^2=-d\bigl(\operatorname{tr}R(h_1)\cdot\operatorname{tr}A_1\bigr)$. Since $R(h_1)$ is bounded with respect to $\widetilde{\omega}$ and h_1, we obtain that $\operatorname{tr}R(h_1)\cdot\operatorname{tr}A_1$ is L^2 with respect to $\widetilde{\omega}$. We also know that $d\bigl(\operatorname{tr}R(h_1)\cdot\operatorname{tr}(A_1)\bigr)$ is integrable. Then we obtain the vanishing, due to Lemma 5.2 in [**51**]:

$$\int\bigl(\operatorname{tr}R(h_1)\bigr)^2=\int d\bigl(\operatorname{tr}R(h_1)\cdot\operatorname{tr}A\bigr)=0.$$

(Note that $\widetilde{\omega}$ satisfies the condition of the lemma.) Thus, we obtain $\int\text{par-c}_1({}_{\boldsymbol{c}}E_*)^2=(\frac{\sqrt{-1}}{2\pi})^2\int\bigl(\operatorname{tr}R(h_0)\bigr)^2=(\frac{\sqrt{-1}}{2\pi})^2\int\bigl(\operatorname{tr}R(h_1)\bigr)^2=0$.

Because of $R(\widetilde{h})=-[\theta,\theta^\dagger_{\widetilde{h}}]$ and $\theta^2=0$, we easily obtain $\operatorname{tr}\bigl(R(\widetilde{h})^2\bigr)=0$. Thus we obtain the following:

$$\operatorname{tr}\bigl(R(h_1)^2\bigr)+\overline{\partial}\Bigl(\operatorname{tr}\bigl(A_1\cdot R(h_1)\bigr)+\operatorname{tr}\bigl(A_1\cdot R(\widetilde{h})\bigr)\Bigr)=0$$

From the boundedness of $R(h_1)$ and $R(\widetilde{h})$ with respect to $\widetilde{\omega}$ and h_1, we obtain that $\operatorname{tr}(A_1\cdot R(h_1))$ and $\operatorname{tr}(A_1\cdot R(\widetilde{h}))$ are L^2 with respect to $\widetilde{\omega}$. Thus we obtain the vanishing, by using Lemma 5.2 in [**51**] again:

$$\int\overline{\partial}\Bigl(\operatorname{tr}\bigl(A_1\cdot R(h_1)\bigr)+\operatorname{tr}\bigl(A_1\cdot R(\widetilde{h})\bigr)\Bigr)=0.$$

Thus, we obtain $\int_X\text{par-ch}_2({}_{\boldsymbol{c}}E_*)=(\frac{\sqrt{-1}}{2\pi})^2\int\operatorname{tr}\bigl(R(h_0)^2\bigr)=(\frac{\sqrt{-1}}{2\pi})^2\int\operatorname{tr}\bigl(R(h_1)^2\bigr)=0$. $\square$

CHAPTER 6

PRELIMINARY CORRESPONDENCE AND BOGOMOLOV-GIESEKER INEQUALITY

In this chapter, we show the existence of the adapted pluri-harmonic metric for *graded semisimple* μ_L-stable parabolic Higgs bundles on a surface (Proposition 6.1). We will use it together with the perturbation of the parabolic structure (Section 3.3) to derive more interesting results. One of the immediate consequences is Bogomolov-Gieseker inequality (Theorem 6.5).

6.1. Graded Semisimple Parabolic Higgs Bundles on Surface

We show an existence of Hermitian-Einstein metric for μ_L-stable parabolic Higgs bundle on a surface under the graded semisimplicity assumption, which makes the problem much easier. Later, we will discuss such existence theorem for parabolic Higgs bundle with trivial characteristic numbers in the case where the graded semisimplicity is not assumed.

Proposition 6.1. — *Let X be a smooth irreducible projective complex surface with an ample line bundle L, and D be a simple normal crossing divisor. Let ω be a Kahler form of X, which represents $c_1(L)$. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D), which is μ_L-stable and* graded semisimple. *Let us take a positive number ϵ satisfying the following:*

- $10\epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E_*)$, *and* $\epsilon = m^{-1}$ *for some positive integer* m.

We take a Kahler form ω_ϵ of $X - D$, as in Subsection 4.3.1. *We put $E = {}_{\boldsymbol{c}}E_{|X-D}$, and the restriction of θ to $X - D$ is denoted by the same notation. Then there exists a hermitian metric h of E satisfying the following conditions:*

- *Hermitian-Einstein condition $\Lambda_{\omega_\epsilon} F(h) = a \cdot \mathrm{id}_E$ for some constant a determined by the following equation:*

$$a \cdot \frac{\sqrt{-1}}{2\pi} \frac{\operatorname{rank} E}{2} \int_{X-D} \omega_\epsilon^2 = a \cdot \frac{\sqrt{-1}}{2\pi} \frac{\operatorname{rank}(E)}{2} \int_X \omega^2 = \text{par-deg}_\omega({}_{\boldsymbol{c}}E_*). \tag{43}$$

– *h is adapted to the parabolic structure of ${}_{\boldsymbol{c}}E_*$.*
– $\deg_{\omega_\epsilon}(E,h) = \text{par-deg}_\omega({}_{\boldsymbol{c}}E_*)$.
– *We have the following equalities:*

$$\int_X 2\,\text{par-ch}_2({}_{\boldsymbol{c}}E_*) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\bigl(F(h)^2\bigr),$$

$$\int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\bigl(F(h)\bigr)^2.$$

Proof. — Let us take an ordinary metric h_0 for the parabolic bundle $({}_{\boldsymbol{c}}E_*,\theta)$ as in Section 4.3. Note we have $\Lambda_{\omega_\epsilon}\operatorname{tr}R(h_0) = \Lambda_{\omega_\epsilon}\operatorname{tr}F(h_0)$. We put $\gamma_i := \operatorname{wt}({}_{\boldsymbol{c}}E_*,i)$.

Let us see the induced metric $\det(h_0)$ of $\det(E)$. Due to our construction, $\det(h_0)$ is of the form $\tau\cdot|z_i|^{-2\gamma_i}\cdot|z_j|^{-2\gamma_j}$ around $P\in D_i\cap D_j$, where τ denotes a positive C^∞-metric of $\det\bigl({}_{\boldsymbol{c}}E\bigr)_{|U_P}$. If P is a smooth point of D_i. then the metric $\det(h_0)$ is of the form $\tau\cdot|\sigma_i|_{g_i}^{-2\gamma_i}$, where τ and γ_i are as above. Therefore, $\operatorname{tr}R(h_0) = R\bigl(\det(h_0)\bigr)$ is C^∞ on X. If a is determined by (43), we have $\int_{X-D}\bigl(\operatorname{tr}\Lambda_{\omega_\epsilon}F(h_0) - \operatorname{rank}(E)\cdot a\bigr)\cdot\omega_\epsilon^2 = 0$. Recall $\epsilon = m^{-1}$ for some positive integer m. Then the following lemma can be shown by a consideration of orbifolds.

Lemma 6.2. — *We can take a bounded C^∞-function g on $X-D$ satisfying the conditions* (i) $\Delta_{\omega_\epsilon}g = \sqrt{-1}\Lambda_{\omega_\epsilon}\operatorname{tr}\bigl(F(h_0)\bigr) - \sqrt{-1}\operatorname{rank}(E)\cdot a$, *where a is determined by the equation* (43), (ii) *∂g, $\overline{\partial}g$, and $\partial\overline{\partial}g$ are bounded with respect to ω_ϵ.*

We put $g' := g/\operatorname{rank}E$ and $h_{in} := h_0\cdot\exp(-g')$. We remark that the adjoints θ for h_0 and h_{in} are same. We also remark that $\partial_{h_{in}} - \partial_{h_0}$ and $R(h_{in}) - R(h_0)$ are just multiplications $-\partial g'\cdot\mathrm{id}_E$ and $\partial\overline{\partial}g'\cdot\mathrm{id}_E$ respectively, which are bounded with respect to ω_ϵ.

Lemma 6.3. — *The metric h_{in} satisfies the following conditions:*

– *h_{in} is adapted to the parabolic structure of ${}_{\boldsymbol{c}}E_*$.*
– *$F(h_{in})$ is bounded with respect to h_{in} and ω_ϵ.*
– *Let V be any saturated coherent subsheaf of E, and let π_V denote the orthogonal projection of E onto V. Then $\overline{\partial}\pi_V$ is L^2 with respect to h_{in} and ω_ϵ, if and only if there exists a saturated coherent subsheaf ${}_{\boldsymbol{c}}V$ of ${}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D} = V$. Moreover we have* $\text{par-deg}_\omega({}_{\boldsymbol{c}}V_*) = \deg_{\omega_\epsilon}(V,h_{in,V})$, *where $h_{in,V}$ denotes the metric of V induced by h_{in}.*
– $\operatorname{tr}\Lambda_{\omega_\epsilon}F(h_{in}) = \operatorname{rank}(E)\cdot a$ *for the constant a determined by the equation* (43).
– *The following equalities hold:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\bigl(F(h_{in})^2\bigr) = \int_X 2\,\text{par-ch}_2({}_{\boldsymbol{c}}E_*),$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\bigl(F(h_{in})\bigr)^2 = \int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*).$$

Due to the third condition, (E, h_{in}, θ) is analytic stable with respect to ω_ϵ, if and only if $({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-stable.

Proof. — Since g' is bounded and since h_0 is adapted to the parabolic structure, h_{in} is also adapted to the parabolic structure. We have $F(h_{in}) = F(h_0) + \partial\overline{\partial}g' \cdot \mathrm{id}_E$. Hence the boundedness of $F(h_{in})$ with respect to ω_ϵ and h_0 follows from those of $F(h_0)$ and $\partial\overline{\partial}g'$.

For any saturated subsheaf $V \subset E$, the orthogonal decomposition $\pi_V^{h_0}$ and $\pi_V^{h_{in}}$ are same. Hence $\overline{\partial}\pi_V^{h_{in}}$ is L^2, if and only if there exists a coherent subsheaf ${}_{\boldsymbol{c}}V \subset {}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D} = V$, by Lemma 4.19. Let $h_{0,V}$ and $h_{in,V}$ denote the metrics of V induced by h_0 and h_{in}. We have $\operatorname{tr} F(h_{in,V}) = \operatorname{tr} F(h_{0,V}) + \operatorname{rank}(V) \cdot \partial\overline{\partial}g'$. Then we obtain $\deg_{\omega_\epsilon}(V, h_{0,V}) = \deg_{\omega_\epsilon}(V, h_{in,V})$ from the boundedness of $\partial\overline{\partial}g'$ and $\partial g'$ with respect to ω_ϵ. Therefore the third condition is satisfied. The fourth condition is satisfied by our construction. The fifth condition is also checked by using the boundedness of $F(h_{in})$, $F(h_0)$, $\overline{\partial}\partial g'$ and $\overline{\partial}g'$. □

Now Proposition 6.1 follows from Lemma 6.3 and Proposition 2.5. □

6.2. Bogomolov-Gieseker Inequality

We have an immediate and standard corollary of Proposition 6.1, as in [**51**].

Corollary 6.4. — *Let X be a smooth irreducible projective surface with an ample line bundle L, and let D be a simple normal crossing divisor of X. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a μ_L-stable $\boldsymbol{c}$-parabolic graded semisimple Higgs bundle on (X, D). Then we have the following inequality:*

$$\int_X \operatorname{par-ch}_2({}_{\boldsymbol{c}}E_*) - \frac{\int_X \operatorname{par-c}_1^2({}_{\boldsymbol{c}}E_*)}{2 \operatorname{rank} E} \leq 0.$$

Proof. — Let h be the metric of E as in Proposition 6.1. Then we have the following:

$$\int_X \operatorname{par-ch}_2({}_{\boldsymbol{c}}E_*) - \frac{\int_X \operatorname{par-c}_1^2({}_{\boldsymbol{c}}E_*)}{2 \operatorname{rank} E} = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(F(h)^{\perp\, 2}\Big).$$

Then the claim follows from $\operatorname{tr}\Big(F(h)^{\perp\, 2}\Big) \geq 0$. (See the pages 878–879 in [**51**].) □

By using the perturbation of the parabolic structure, we can remove the assumption of graded semisimplicity. We can also remove the assumption $\dim X = 2$ by using Mehta-Ramanathan type theorem.

***Theorem 6.5* (Bogomolov-Gieseker inequality)**. — *Let X be a smooth irreducible projective variety of an arbitrary dimension with an ample line bundle L, and let D be* *a simple normal crossing divisor. Let* $(\boldsymbol{E}_*,\theta)$ *be a* μ_L*-stable regular Higgs bundle in codimension two on* (X,D)*. Then the following inequality holds:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) - \frac{\int_X \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2\operatorname{rank} E} \leq 0.$$

(See Subsection 3.1.5 *for the characteristic numbers.)*

Proof. — Due to the Mehta-Ramanathan type theorem (Proposition 3.29), the problem can be reduced to the case where X is a surface. Take a real number $c_i \notin \mathcal{P}ar(\boldsymbol{E}_*, i)$ for each i, and let us consider the $\boldsymbol{c}$-truncation $({}_{\boldsymbol{c}}E_*,\theta)$. Let $\boldsymbol{F}$ denote the induced $\boldsymbol{c}$-parabolic structure of ${}_{\boldsymbol{c}}E$. Let ϵ be any sufficiently small positive number, and let us take an ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ of $\boldsymbol{F}$ as in Section 3.3. Since $({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\epsilon)},\theta)$ is μ_L-stable and graded semisimple, we obtain the following inequality due to Corollary 6.4:

$$\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\epsilon)}) - \frac{\int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\epsilon)})}{2\operatorname{rank} E} \leq 0.$$

By taking the limit in $\epsilon \to 0$, we obtain the desired inequality. □

Corollary 6.6. — *Let X be a smooth irreducible projective surface with an ample line bundle L, and let D be a simple normal crossing divisor. Let* $(\boldsymbol{E}_*,\theta)$ *be a* μ_L*-stable parabolic Higgs bundle on* (X,D)*. Assume* $\int_X \text{par-ch}_2(\boldsymbol{E}_*) = \text{par-deg}_L(\boldsymbol{E}_*) = 0$*. Then we have* $\text{par-c}_1(\boldsymbol{E}_*) = 0$.

Proof. — $\text{par-deg}_L(\boldsymbol{E}_*) = 0$ implies $\int_X \text{par-c}_1(\boldsymbol{E}_*)\cdot c_1(L) = 0$. Due to the Hodge index theorem, it implies $-\int \text{par-c}_1^2(\boldsymbol{E}_*) \geq 0$, and if the equality holds then $\text{par-c}_1(\boldsymbol{E}_*) = 0$. On the other hand, we have the following inequality, due to Theorem 6.5:

$$-\frac{\int_X \text{par-c}_1^2(\boldsymbol{E}_*)}{2\operatorname{rank} E} \leq -\int_X \text{par-ch}_2(\boldsymbol{E}_*) = 0.$$

Thus the claim follows. □

CHAPTER 7

CONSTRUCTION OF A FRAME

We put $X(T) := \{z \in \boldsymbol{C} \,|\, |z| < T\}$ and $X^*(T) := X(T) - \{O\}$, where O denotes the origin. In the case $T = 1$, we omit to denote T. Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on X^*. Recall that the coefficients $a_j(z)$ of $P(z,t) := \det(t - f_0(z)) = \sum a_j(z) \cdot t^j$ are holomorphic on X, where $f_0 \in \mathrm{End}(E)$ is given by $\theta = f_0 \cdot dz/z$. The set of the solutions of the polynomial $P(0,t)$ is denoted by S_0.

Assumption 7.1. — We assume the following:

1. We have the decomposition $E = \bigoplus_{\alpha \in S_0} E_\alpha$, such that $f_0(E_\alpha) \subset E_\alpha$. In particular, we have the decomposition $f_0 = \bigoplus f_{0\,\alpha}$.
2. There exist some positive numbers $T_0 < 1$, C_0 and ϵ_0 such that $|\beta - \alpha| < C_0 \cdot |z(Q)|^{\epsilon_0}$ holds for any eigenvalue β of $f_{\alpha\,|\,Q}$ $(Q \in X^*(T_0))$.
3. We put $\xi := \sum_{\alpha \in S_0} \mathrm{rank}(E_\alpha) \cdot |\alpha|^2$. We assume $\xi < K_0$ for a given constant K_0.

Remark 7.2. — The conditions 1 and 2 are always satisfied, if we replace X by a smaller open set. Moreover, it is controlled by the behaviour of the eigenvalues of f_0.

We obtain the parabolic Higgs bundle $\left({}_aE_*, \theta\right)$ for $a \in \boldsymbol{R}$ from $(E, \overline{\partial}_E, h)$, where ${}_aE$ is as in Section 3.5 ([**52**]). In the case $a = 0$, we use the notation ${}^\diamond E$. Thus we have the parabolic filtration F of ${}_aE_{|O}$ and the sets $\mathcal{P}ar({}_aE) := \left\{b \,\middle|\, \mathrm{Gr}^F_b({}_aE_{|O}) \neq 0\right\}$. For any $b \in \mathcal{P}ar({}_aE)$, we put $\mathfrak{m}(b) := \dim \mathrm{Gr}^F_b({}_aE_{|O})$. Recall $\det({}_aE) \simeq {}_{\widetilde{a}}\det(E)$, where $\widetilde{a}$ is given as follows:

$$\widetilde{a} := \sum_{b \in \mathcal{P}ar({}_aE)} \mathfrak{m}(b) \cdot b.$$

Let U_0 be a finite subset of $]a-1, a[$, and let η_0 be a sufficiently small positive numbers such that $U_0 \subset]a - 1 + 10 \cdot \eta_0, a - 10 \cdot \eta_0[$ and $|b - c| > 10 \cdot \eta_0$ for any distinct elements $b, c \in U_0$. We make an additional assumption.

Assumption 7.3. — For any $c \in \mathcal{P}ar({}_aE)$, there exists $b \in U_0$ such that $|c - b| < \eta_0$.

We put $\mathcal{P}(b) := \{c \in \mathcal{P}ar({}_aE) \,\big|\, |c - b| < \eta_0\}$. We obtain the decomposition $\mathcal{P}ar({}_aE) = \coprod_{b \in U_0} \mathcal{P}(b)$. We put $\overline{b} := \max \mathcal{P}(b)$.

In the following of this chapter, we say that a constant C is good, if it depends only on T_0, C_0, ϵ_0, K_0, η_0 and $r := \operatorname{rank}(E)$. We say a constant $C(B)$ is good if it depends also on additional data B.

Proposition 7.4. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on X^* satisfying the assumptions* 7.1 *and* 7.3.

- *There exist holomorphic sections $F_1, \dots, F_r$ of ${}_aE$ on $X(\gamma_0)$ with the numbers $b_1, \dots, b_r \in U_0$, such that $|F_i|_h \leq C_{10} \cdot |z|^{-\overline{b}_i} \cdot (-\log|z|^2)^N$ holds. Here γ_0, C_{10} and N are good constants. We have $\#\{b_i = b\} = \#\mathcal{P}(b)$.*
- *$C_{11}^{-1} \cdot |z|^{-\widetilde{a}} \leq \big|\bigwedge_{i=1}^r F_i\big|_h \leq C_{11} \cdot |z|^{-\widetilde{a}}$ holds for a good constant C_{11}. In particular, $F_1, \dots, F_r$ give the frame of ${}_aE$.*
- *On any compact subset $H \subset X^*(\gamma_0)$, we have $\|F_{i|H}\|_{L_1^p, h} \leq C_{12}(H, p)$, where p is an arbitrarily large number.*

We will prove the proposition in the rest of this chapter.

7.1. A Priori Estimate of Higgs Field on a Punctured Disc

Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on X^* as in Proposition 7.4. We know that the curvature $R(h)$ of $\overline{\partial}_E + \partial_E$ is bounded with respect to h and the Poincaré metric $\widetilde{g} = |z|^{-2}(-\log|z|)^{-2} dz \cdot d\overline{z}$ on $X^*(T)$ for $T < 1$. ([**52**]. See also [**44**]). We would like to show that the estimate is uniform, when we vary the set S_0 boundedly.

Proposition 7.5. — *$\big|R(h)\big|_{h, \widetilde{g}} \leq K_{10}$ holds on $X^*(T_1)$ for some good constants T_1 and K_{10}.*

Proof. — In the following argument, K_i, ϵ_i and T_i will denote good constants, and Δ denotes the Laplacian $-\partial_z \partial_{\overline{z}}$ (up to the positive constant). Let $\mathcal{L}$ be a line bundle $\mathcal{O}_{X^*} \cdot e$ with the Higgs field $\theta_{\mathcal{L}}$ and the metric $h_{\mathcal{L}}$ given by $\theta_{\mathcal{L}}(e) = e \cdot \beta \cdot dz/z$ ($\beta \in \boldsymbol{C}$) and $h_{\mathcal{L}}(e, e) = 1$. Since we have only to consider $(E, \overline{\partial}_E, \theta, h) \otimes (\mathcal{L}, \theta_{\mathcal{L}}, h_{\mathcal{L}})$, we may and will assume $0 < K_1 < \xi < K_2$.

By an elementary argument, we can take a decomposition $S_0 = \coprod_{i=1}^{k_0} S_i^{(1)}$ with the following property:

- $|\alpha_j - \alpha_k| \leq 1$ for any $\alpha_j, \alpha_k \in S_i^{(1)}$.
- $|\alpha_j - \alpha_k| > \operatorname{rank}(E)^{-1}$ for $\alpha_j \in S_i^{(1)}$ and $\alpha_k \in S_0 - S_i^{(1)}$.

We put $\mathcal{S}(1) := \{1, \dots, k_0\} \subset \mathbb{Z}_{>0}$. Inductively on n, we take a subset $\mathcal{S}(n) \subset \mathbb{Z}_{>0}^n$ and a decomposition $S_0 = \coprod_{I \in \mathcal{S}(n)} S_I^{(n)}$ as follows. Assume $\mathcal{S}(n)$ and $S_I^{(n)}$ ($I \in \mathcal{S}(n)$) are already given. We can take a decomposition $S_I^{(n)} = \coprod_{i=1}^{k(I)} S_{I,i}^{(n+1)}$ with the following property:

- $|\alpha_j - \alpha_k| \leq (n+1)^{-1}$ for $\alpha_j, \alpha_k \in S_{I,i}^{(n+1)}$,
- $|\alpha_j - \alpha_k| > (n+1)^{-1} \cdot \operatorname{rank}(E)^{-1}$ for $\alpha_j \in S_{I,i}^{(n+1)}$ and $\alpha_k \in S_I^{(n)} - S_{I,i}^{(n+1)}$.

Then we put $\mathcal{S}(n+1) := \big\{(I,i) \,\big|\, I \in \mathcal{S}(n),\ i = 1, \ldots, k(I)\big\}$ and $S_{(I,i)}^{(n+1)} := S_{I,i}^{(n+1)}$, where $(I,i) \in \mathbb{Z}_{>0}^{n+1}$ denotes the element naturally determined by I and i.

We have the lexicographic order on $\mathbb{Z}_{>0}^n$, which induces the order on $\mathcal{S}(n)$. Take a total order $\leq_1$ on S_0, which satisfies the following condition for any n:

- Let $\alpha \in S_I^{(n)}$ and $\beta \in S_J^{(n)}$. If $I < J$ in $\mathcal{S}(n)$, we have $\alpha \leq_1 \beta$.

We put $F_\alpha E := \bigoplus_{\beta \leq_1 \alpha} E_\beta$ and $F_{<\alpha} E := \bigoplus_{\beta <_1 \alpha} E_\beta$. Let E'_α denote the orthogonal complement of $F_{<\alpha}(E)$ in $F_\alpha(E)$. We put $\rho := \bigoplus_{\alpha \in S_0} \alpha \cdot \mathrm{id}_{E_\alpha}$ and $\rho' := \bigoplus_{\alpha \in S_0} \alpha \cdot \mathrm{id}_{E'_\alpha}$. We have $|\rho'|_h^2 = \xi$. The following lemma is shown in the proof of Simpson's Main estimate. (See [**52**] and the proof of Proposition 7.2 of [**44**].)

Lemma 7.6. — *$|f_0 - \rho'|_h \leq K_{11} \cdot \big(-\log|z|\big)^{-1}$ holds on $X^*(T_1)$.*

For $J \in \mathcal{S}(n)$, we put $E_J^{(n)} := \bigoplus_{\alpha \in S_J^{(n)}} E_\alpha$ and $E_J'^{(n)} := \bigoplus_{\alpha \in S_J^{(n)}} E'_\alpha$. We have the natural decomposition $\mathrm{End}(E) = \bigoplus_{J_1, J_2 \in \mathcal{S}(n)} \mathcal{H}om(E_{J_1}'^{(n)}, E_{J_2}'^{(n)})$. For $I \in \mathcal{S}(n-1)$ and $A \in \mathrm{End}(E)$, let $A_{n,I,i,j}$ denote the $\mathcal{H}om(E_{I,i}'^{(n)}, E_{I,j}'^{(n)})$-component.

Lemma 7.7. — *We have $\big|\big[\rho'^{\dagger}, f_0\big]_{n,I,i,j}\big|_h \leq K_{30} \cdot (-\log|z|)^{-2}$ for $i \neq j$ on $X^*(T_2)$.*

Proof. — We put $\kappa := \bigoplus_{i=1}^{k(I)} i \cdot \mathrm{id}_{E_{I,i}^{(n)}}$ and $\kappa' := \bigoplus_{i=1}^{k(I)} i \cdot \mathrm{id}_{E_{I,i}'^{(n)}}$. We also put $q := \kappa - \kappa' \in \mathcal{T} := \bigoplus_{J_1 > J_2} \mathcal{H}om(E_{J_1}'^{(n)}, E_{J_2}'^{(n)})$. First, we give some estimate of q.

Let $\varphi : X^* \longrightarrow X^*$ denote the map given by $\varphi(z) = z^n$. We remark that $\varphi^*(E, \overline{\partial}_E, \theta, h)$ satisfies Assumption 7.1 independently of n, if we replace C_0 with a larger good constant. We put $\widetilde{h} := \varphi^* h$. We put $\widetilde{f}_0 := n \cdot \varphi^{-1}(f_0)$, i.e., $\varphi^{-1}\theta = \widetilde{f}_0 \cdot dz/z$. Let $\widetilde{f}_0^{\dagger}$ denote the adjoint of $\widetilde{f}_0$ with respect to $\widetilde{h}$. We also put $\widetilde{\rho}' := n \cdot \varphi^{-1}(\rho')$.

Let $F_{\widetilde{f}_0}$ denote the endomorphism of $\varphi^{-1}\mathcal{T}$ induced by the adjoint of $\widetilde{f}_0$, i.e., $F_{\widetilde{f}_0}(x) = [\widetilde{f}_0, x]$. Let $\pi_{\mathcal{T}}$ denote the orthogonal projection of $\varphi^{-1}\,\mathrm{End}(E)$ onto $\varphi^{-1}\mathcal{T}$. The composite of the adjoint of $\widetilde{f}_0^{\dagger}$ and $\pi_{\mathcal{T}}$ induces the endomorphism $G_{\widetilde{f}_0^{\dagger}}$ of $\varphi^{-1}\mathcal{T}$.

Lemma 7.8(1). — *$F_{\widetilde{f}_0}$ and $G_{\widetilde{f}_0^{\dagger}}$ are invertible on $X^*(T_3)$ and the norms of their inverses are dominated by a good constant.*

Proof. — Let H denote the endomorphism of $\varphi^{-1}\mathcal{T}$ induced by the adjoint of $\widetilde{\rho}'$, and we put $H_1 := G_{\widetilde{f}_0^{\dagger}} - H$. For any $\alpha \in S_I^{(n)}$ and $\beta \in S_J^{(n)}$ $(I \neq J)$, we have $n \cdot |\alpha - \beta| > \operatorname{rank}(E)^{-1}$. Hence the norm of H^{-1} is dominated by a good constant. From $|\widetilde{f}_0 - \widetilde{\rho}'|_{\widetilde{h}} \leq K_{31} \cdot (-\log|z|)^{-1}$, the norm of H_1 is dominated by a sufficiently

(0)The communication with the referee clarified a confusion, for which the author is obliged.

small good constant on $X^*(T_3)$. We put $H_2 := H^{-1} \circ H_1$. Then, $(1+H_2)$ is invertible, and the norm of the inverse is dominated by a good constant on $X^*(T_3)$. Then, the claim for $G^{-1}_{f_0^\dagger} = H_1^{-1} \circ (1+H_2)^{-1}$ can be easily checked. The claim for $F^{-1}_{f_0}$ can be checked similarly. □

We put $\widetilde{\kappa} := \varphi^{-1}\kappa$, $\widetilde{\kappa}' := \varphi^{-1}\kappa'$ and $\widetilde{q} := \varphi^{-1}q$. We have $0 = [\widetilde{f}_0, \widetilde{\kappa}] = [\widetilde{f}_0 - \widetilde{\rho}', \widetilde{\kappa}'] + [\widetilde{f}_0, \widetilde{q}]$. Due to Lemma 7.6 and Lemma 7.8, we obtain the estimate $|\widetilde{q}|_{\widetilde{h}} \leq K_{32}(-\log|z|)^{-1}$ on $X^*(T_3)$. From $[\widetilde{\kappa}, \widetilde{f}_0^\dagger] = [\widetilde{\kappa} - \widetilde{\kappa}', \widetilde{f}_0^\dagger] + [\widetilde{\kappa}', \widetilde{f}_0^\dagger]$, we obtain $\big|[\widetilde{\kappa}, \widetilde{f}_0^\dagger]\big|^2_{\widetilde{h}} \geq \big|\pi_S\big([\widetilde{\kappa} - \widetilde{\kappa}', \widetilde{f}_0^\dagger]\big)\big|^2_{\widetilde{h}} = \big|G_{\widetilde{f}_0^\dagger}(\widetilde{q})\big|^2_{\widetilde{h}}$. Hence, we obtain $|\widetilde{q}|^2_{\widetilde{h}} \leq K_{33}\big|[\widetilde{\kappa}, \widetilde{f}_0^\dagger]\big|_{\widetilde{h}}$ on $X^*(T_3)$. Due to $[\widetilde{\kappa}, \widetilde{f}_0] = 0$, we obtain the following:

$$\Delta \log |\widetilde{\kappa}|^2_{\widetilde{h}} \leq -\frac{\big|[\widetilde{f}_0^\dagger, \widetilde{\kappa}]\big|^2_{\widetilde{h}}}{|z|^2 \cdot |\widetilde{\kappa}|^2_{\widetilde{h}}} \leq -K_{35}\frac{|\widetilde{q}|_{\widetilde{h}}}{|z|^2}$$

We put $\xi' := \sum_{i=1}^{k(I)} i^2 = |\widetilde{\kappa}'|^2_{\widetilde{h}}$ and $k := \log\big(\xi'^{-1}|\widetilde{\kappa}|^2_{\widetilde{h}}\big)$. Because of $k \leq \xi'^{-1}|\widetilde{q}|^2_{\widetilde{h}}$, we obtain $\Delta k \leq -K_{36} \cdot |z|^{-2} \cdot k$. By an argument in [**52**] (see also the proof of Proposition 7.2 of [**44**]), we obtain $k \leq K_{37} \cdot |z|^{\epsilon_{38}}$. Then, we can derive $|\widetilde{q}|_{\widetilde{h}} \leq K_{39} \cdot |z|^{\epsilon_{38}}$ on $X^*(T_{40})$. Hence we obtain $|q|_h \leq K_{39} \cdot |z|^{\epsilon_{38}/n}$ on $X^*(T_{40}^n)$.

Let us finish the proof of Lemma 7.7. First we show the estimate on $X^*(T_{40}^n)$. We have $0 = [\kappa, f_0] = [\kappa', f_0] + [q, \rho'] + [q, f_0 - \rho']$. We have the following on $X^*(T_{40}^n)$:

$$\big|[q, f_0 - \rho']\big|_h \leq \frac{K_{41}|z|^{\epsilon_{38}/n}}{-\log|z|} \leq \frac{K_{42}|z|^{\epsilon_{38}/n}}{n}$$

Recall we have $|\alpha - \beta| \leq (n-1)^{-1}$ for $\alpha \in S_{I,i}^{(n-1)}$ and $\beta \in S_{I,j}^{(n-1)}$. Hence we have $\big|[q, \rho']_{n,I,i,j}\big|_h \leq K_{43} \cdot |z|^{\epsilon_{38}/n} \cdot n^{-1}$. Therefore, we obtain $\big|[\kappa', f_0]_{n,I,i,j}\big|_h \leq K_{44} \cdot |z|^{\epsilon_{38}/n} \cdot n^{-1}$, which implies $\big|(f_0)_{n,I,i,j}\big|_h \leq K'_{44} \cdot |z|^{\epsilon_{38}/n} \cdot n^{-1}$ $(i \neq j)$. Then, we obtain the estimate on $X^*(T_{40}^n)$:

$$\big|[\rho'^{\dagger}, f_0]_{n,I,i,j}\big|_h \leq K_{45} \cdot |z|^{\epsilon_{38}/n} \cdot n^{-2} \leq K_{46} \cdot \big(-\log|z|\big)^{-2}.$$

On the other hand, $[\rho'^{\dagger}, f_0]_{n,I,i,j}$ is dominated by $K_{47} \cdot (-\log|z|)^{-1} \cdot n^{-1}$ on $X^*(T_1)$, which is obtained by the estimate of $f_0 - \rho'$ (Lemma 7.6) and our choice of $S_{I,k}^{(n)}$ $(k = i, j)$. Outside of $X^*(T_{40}^n)$, we have $K_{47} \cdot (-\log|z|)^{-1} \cdot n^{-1} \leq K_{48} \cdot (-\log|z|)^{-2}$. Thus we are done. □

Let us finish the proof of Proposition 7.5. We have the following:

$$R(h) = -[\theta, \theta^\dagger] = -\Big([\rho', (f_0 - \rho')^\dagger] + [f_0 - \rho', \rho'^{\dagger}] + [(f_0 - \rho'), (f_0 - \rho')^\dagger]\Big) \cdot \frac{dz \cdot d\bar{z}}{|z|^2}$$

The second term is estimated by Lemma 7.7. The first term is adjoint of the second term. The estimate of the third term follows from Lemma 7.6. Thus we are done. □

7.2. Construction of Local Holomorphic Frames

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle of rank r on X^* as in Proposition 7.4. We will construct the desired holomorphic sections in Proposition 7.4. By considering the tensor product of the line bundle with the metric $|z|^{-c}$, we have only to discuss the case $a = 0$. We use the metrics g and $\widetilde{g}$ of X^* given as follows:

$$g := dz \cdot d\overline{z}, \quad \widetilde{g} := \frac{dz \cdot d\overline{z}}{|z|^2 \cdot \big(-\log|z|\big)^2}$$

By considering a pull back via the map $\phi_\gamma : X^* \longrightarrow X^*$ given by $\phi_\gamma(z) = \gamma \cdot z$, we may assume the following, due to Proposition 7.5.

Assumption 7.9. — The norm of $R(h)$ with respect to h and g is dominated as follows:

$$\big|R(h)\big|_{h,g} \leq C_1 \cdot \frac{1}{|z|^2 \cdot \big(-\log|z| + 1\big)^2}.$$

The constant C_1 is good.

Lemma 7.10. — *There exists a C^1-orthonormal frame $\boldsymbol{v}$ of E, for which $\overline{\partial}_E$ is represented as follows:*

$$\overline{\partial}_E \boldsymbol{v} = \boldsymbol{v} \cdot \left(-\frac{\Gamma}{2} + A\right) \cdot \frac{d\overline{z}}{\overline{z}}$$

Here Γ is a constant diagonal matrix whose (i,i)-th components α_i satisfy $0 \leq \alpha_r \leq \cdots \leq \alpha_1 < 1$, and A is a matrix-valued continuous function such that $|A| \leq C_2 \cdot (-\log|z| + 1)^{-1}$. The constant C_2 is good.

Proof. — [2] Let (r, θ) be the polar coordinate of X^*. Let ∇ denote the unitary connection $\overline{\partial}_E + \partial_{E,h}$. Take an orthonormal frame of $E_{|\partial X(\gamma)}$ for some $0 < \gamma < 1$. Extend it to the orthonormal frame $\boldsymbol{e}$ of $E_{|X^*(\gamma)}$ by using the parallel transport along each ray towards the origin. Then the connection form of ∇ with respect to $\boldsymbol{e}$ is of the form $A'(r,\theta) \cdot d\theta$, and the curvature form is given by $dA'(r,\theta) \wedge d\theta$. By Assumption 7.9, we obtain $\partial A'(r,\theta)/\partial r = O\Big((\log r)^{-2} \cdot dr/r\Big)$. Hence, $A'(r,\theta)$ converges to a function $A_0(\theta)$ for $r \to 0$, and $A'(r,\theta) - A_0(\theta) = O((-\log r)^{-1})$. We can take a gauge transform $g(\theta)$ for which $A_0(\theta)$ is transformed to $\Gamma \cdot d\theta$ for some Γ as in the claim of the lemma. □

7.2.1. Preliminary for a construction.

— We recall some results on the solvability of the $\overline{\partial}$-equation. For any real numbers b and M, we put $h(b,M) := h \cdot |z|^{2b} \cdot (-\log|z|)^M$. Let $A^{0,1}_{b,M}(E)$ denote the space of sections of $E \otimes \Omega^{0,1}$, which are L^2 with respect to $h(b,M)$ and $\widetilde{g}$. Let $A^{0,0}_{b,M}(E)$ denote the space of sections f of E such that f and $\overline{\partial} f$ are L^2 with respect to $h(b,M)$ and $\widetilde{g}$. The norm and the hermitian pairing

[2] The author thanks the referee who explained this simple proof.

of $A^{p,q}_{b,M}(E)$ are denoted by $\|\cdot\|_{b,M}$ and $\langle\cdot,\cdot\rangle_{b,M}$. On the other hand, $|\cdot|_{b,M}$ denote the norm at fibers. In the following argument, B_i will denote good constants.

We use some arguments of [**44**] based on the ideas in [**2**] and [**8**]. (But we change the signature here.) Recall the result in Section 2.8.6[(3)] of [**44**]. We take a sufficiently large good constant $N > 1$, which depends only on C_1 in Assumption 7.9. Let $\overline{\partial}_E^*$ denote the adjoint of $\overline{\partial}_E$ with respect to $\widetilde{g}$ and $h(b,N)$. Let $A^{0,1}_c(E)$ denote the space of the C^∞-sections of $E\otimes\Omega^{0,1}$ whose support is compact. Then, the following inequality holds for any $\rho\in A^{0,1}_c(E)$:

$$\big\|\overline{\partial}_E^*\rho\big\|_{b,N}\geq\|\rho\|_{b,N}$$

Lemma 7.11 **([44])**. — *For any $f_1\in A^{0,1}_{b,N}(E)$, we have $f_2\in A^{0,0}_{b,N}$ satisfying $\overline{\partial}f_2=f_1$ and $\|f_2\|_{b,N}\leq B_1\cdot\|f_1\|_{b,N}$.*

Proof. — Let $\widetilde{A}^{0,1}(E)$ denote the space of sections ρ of $E\otimes\Omega^{0,1}$ such that $\|\rho\|^2_{b,N}+\|\overline{\partial}_E^*\rho\|^2_{b,N}<\infty$. It is the L^2-space, and we have the continuous inclusion $\widetilde{A}^{0,1}(E)\longrightarrow A^{0,1}(E)$. Since $A^{0,1}_c(E)$ is dense in $\widetilde{A}^{0,1}(E)$ due to the completeness of $(X^*,\widetilde{g})$, we have $\|\rho\|_{b,N}\leq\|\overline{\partial}_E^*\rho\|_{b,N}$ for any $\rho\in\widetilde{A}^{0,1}(E)$. Hence, $\widetilde{A}^{0,1}(E)$ can be the Hilbert space with the Hermitian pairing $(\rho_1,\rho_2)\longmapsto\langle\overline{\partial}_E^*\rho_1,\overline{\partial}_E^*\rho_2\rangle_{b,N}$.

We have $\langle f_1,\rho\rangle_{b,N}\leq\|f_1\|_{b,N}\cdot\|\rho\|_{b,N}\leq\|f_1\|_{b,N}\cdot\|\overline{\partial}_E^*\rho\|_{b,N}$ for any $\rho\in\widetilde{A}^{0,1}(E)$. Due to Riesz representation theorem, there exists $f_3\in\widetilde{A}^{0,1}(E)$ such that $\|\overline{\partial}_E^*f_3\|_{b,N}\leq\|f_1\|_{b,N}$ and $\langle f_1,\rho\rangle_{b,N}=\langle\overline{\partial}_E^*f_3,\overline{\partial}_E^*\rho\rangle_{b,N}$ for any $\rho\in\widetilde{A}^{0,1}(E)$. We put $f_2=\overline{\partial}_E^*f_3$ which has the desired property. □

On the other hand, if f is a holomorphic section of E, we have the subharmonicity $\Delta\log|f|_{b,-N}\leq 0$ by using the argument in Section 2.8.7 of [**44**]. Hence, if we have $\|f\|_{b,N}<\infty$, the following holds around the origin O:

$$\begin{aligned}(44)\quad \log|f(z)|^2_{b,-N}&\leq\frac{4}{\pi|z|^2}\int_{|w-z|\leq|z|/2}\log|f(w)|^2_{b,-N}\cdot\mathrm{dvol}_g\\ &\leq\log\left(\frac{4}{\pi|z|^2}\int_{|w-z|\leq|z|/2}|f(w)|^2_{b,-N}\cdot\mathrm{dvol}_g\right)\leq\log\bigl(B_2\cdot\|f\|^2_{b,N}\bigr)\end{aligned}$$

Here, we have used $|f(w)|^2_{b,-N}\cdot(-\log|w|)^2\leq|f(w)|^2_{b,N}$. Hence, we obtain the following lemma.

Lemma 7.12. — *For a holomorphic section f of E such that $\|f\|_{b,N}<\infty$, we have $|f|_h\leq B_2\|f\|_{b,N}\cdot|z|^{-b}\cdot(-\log|z|)^{N/2}$.*

We give one more elementary remark.

(3) The inequality (2.30) *loc. cit.* should be $\langle\overline{\partial}_E\eta,\overline{\partial}_E\eta\rangle_{\boldsymbol{a},N}+\langle\overline{\partial}_E^*\eta,\overline{\partial}_E^*\eta\rangle_{\boldsymbol{a},N}\geq\|\eta\|^2_{\boldsymbol{a},N}$.

Lemma 7.13. — *Let f be a holomorphic section of ${}_{b}E$ on $X(\gamma')$. Then the maximum principle holds for $H(z) := |f(z)|_h^2 \cdot |z|^{2b} \cdot (-\log|z|)^{-N}$ on $X(\gamma'')$ for $\gamma'' < \gamma'$, i.e., $\sup_{X^*(\gamma'')} H(z) = \max_{\partial X(\gamma'')} H(z)$.*

Proof. — We put $H_\epsilon(z) := |f(z)|_h^2 \cdot |z|^{2b+\epsilon} \cdot (-\log|z|)^{-N}$ for any $\epsilon > 0$. We have $\Delta \log H_\epsilon \leq 0$ on $X^*(\gamma')$ and $\lim_{z\to 0} \log H_\epsilon(z) = -\infty$. Therefore, the maximum principle holds for $\log H_\epsilon$ on $X(\gamma'')$. Then it is easy to derive the maximum principle for H. □

7.2.2. Construction. — Take $0 < \eta \leq \eta_0$. Let Γ and $\boldsymbol{v} = (v_1, \ldots, v_r)$ be as in Lemma 7.10. We put $S(\Gamma) := \{\alpha_1, \ldots, \alpha_r\}$. Let T_A denote the section of $\mathrm{End}(E) \otimes \Omega^{0,1}$ determined by $\boldsymbol{v}$ and $A \cdot d\overline{z}/\overline{z}$, i.e., $T_A(\boldsymbol{v}) = \boldsymbol{v} \cdot A \cdot d\overline{z}/\overline{z}$. We put $\overline{\partial}_0 := \overline{\partial} - T_A$. We put $f_i := |z|^{\alpha_i} \cdot v_i$. Then we have $\overline{\partial}_0 f_i = 0$ and $|f_i|_h = |z|^{\alpha_i}$. In particular, we have $f_i \in A^{0,0}_{-\alpha_i+\eta,N}(E)$. Take $g_i \in A^{0,0}_{-\alpha_i+\eta,N}$ satisfying $\overline{\partial} g_i = T_A(f_i)$ and $\big\|g_i\big\|_{-\alpha_i+\eta,N} \leq B_1 \cdot \big\|T_A(f_i)\big\|_{-\alpha_i+\eta,N}$ as in Lemma 7.11. We put $F_i := f_i - g_i$. Then we have $\overline{\partial} F_i = 0$, $F_i \in A^{0,0}_{-\alpha_i+\eta,N}(E)$, and the following estimate:

$$\big\|F_i\big\|_{-\alpha_i+\eta,N} \leq \big\|f_i\big\|_{-\alpha_i+\eta,N} + B_1 \cdot \big\|T_A(f_i)\big\|_{-\alpha_i+\eta,N}.$$

We have the following:

$$\overline{\partial}_0 g_i = -T_A(g_i) + T_A(f_i). \tag{45}$$

Hence we obtain $g_i \in L_1^2(H)$ for any compact subset $H \subset X^*$, and the L_1^2-norm is dominated by $||T_A(f_i)||_{-\alpha_i+\eta,N}$ multiplied by some constant depending only on H. Hence for some number $p > 2$ and some good constant $C'(H,p)$, we have the following:

$$\big\|g_i\big\|_{L^p(H)} \leq C'(H,p) \cdot \big\|T_A(f_i)\big\|_{-\alpha_i+\eta,N}$$

Due to (45), we have the following, for some good constant $C''(H,p)$:

$$\big\|g_i\big\|_{L_1^p(H)} \leq C''(H,p) \cdot \Big(\big\|T_A(f_i)\big\|_{-\alpha_i+\eta,N} + \sup_H \big|T_A(f_i)\big|_{h,\widetilde{g}}\Big). \tag{46}$$

By a standard boot strapping argument, p can be arbitrarily large.

We put $\widetilde{\alpha} := \mathrm{tr}(\Gamma)$ and $\widetilde{0} := \sum_{b \in \mathcal{P}ar(^\diamond E)} \mathfrak{m}(b) \cdot b$. Since we have $\mathrm{tr}(R(h)) = \mathrm{tr}(F(h)) = 0$, the induced metric $\det(h)$ of $\det(E)$ is flat. Hence we have a holomorphic section s of ${}_{\widetilde{0}}\det(E) = \det(^\diamond E)$ such that $|s|_h = |z|^{-\widetilde{0}}$ and $\partial_{\det(E)} s = s \cdot (-\widetilde{0}) \cdot dz/z$. It is easy to see $n = \widetilde{\alpha} + \widetilde{0}$ is an integer by considering the limit of the monodromy of $\det(E)$ around the origin. We put $\widetilde{s} := z^n \cdot s$, which gives the section of ${}_{-\widetilde{\alpha}}\det(E)$.

Remark 7.14. — We will show that $-\widetilde{\alpha} = \widetilde{0}$, i.e. $s = \widetilde{s}$ later (Lemma 7.17).

Let us consider the function $\widetilde{F}$ determined by $\widetilde{F} \cdot \widetilde{s} = F_1 \wedge \cdots \wedge F_r$. We put $H_0 := \{z \,|\, 3^{-1} \leq |z| \leq 2 \cdot 3^{-1}\}$.

Lemma 7.15. — *There exists a small good constant B_{15} with the following property:*

– *Assume the following inequalities hold:*

$$\sup_{H_0} |A|_{\widetilde{g}} < B_{15}, \quad \|T_A \cdot f_i\|_{-\alpha_i+\eta,N} < B_{15}, \quad (i = 1, \dots, r). \tag{47}$$

Then, there exist $z_0 \in \{z \in \boldsymbol{C} \mid |z| = 1\}$ and a good constant $0 < B_{16} < 1/2$ such that $\widetilde{F}(H_0) \subset \{z \in \boldsymbol{C} \mid |z - z_0| < B_{16}\}$.

Proof. — From (46) and (47), we obtain $|F_1 \wedge \cdots \wedge F_r - f_1 \wedge \cdots \wedge f_r| < 4^{-1}$ holds on H_0, if B_{15} is sufficiently small. Since $v_1, \dots, v_r$ are orthonormal and f_i are given as $|z|^{\alpha_i} \cdot v_i$, we have $f_1 \wedge \cdots \wedge f_r = \exp(\sqrt{-1}\kappa) \cdot \widetilde{s}$ for some real valued functions κ. If B_{15} is sufficiently small, κ is a sum of a constant κ_0 and a function κ_1 satisfying $\sup_{H_0} |\kappa_1(z)| < 100^{-1}$ because of (47). Then the claim of the lemma follows. □

For any number $0 < \gamma < 1$, let us consider the map $\phi_\gamma : X^* \longrightarrow X^*$ given by $z \longmapsto \gamma \cdot z$. We put $\big(E(\gamma), \overline{\partial}_{E(\gamma)}, \theta(\gamma), h(\gamma)\big) := \phi_\gamma^*(E, \overline{\partial}_E, \theta, h)$. It is easy to check Assumption 7.9 for $\big(E(\gamma), \overline{\partial}_{E(\gamma)}, h(\gamma)\big)$. We have the orthonormal frame $\phi_\gamma^* \boldsymbol{v}$ of $E(\gamma)$ for which we have the following:

$$\overline{\partial}_{E(\gamma)}(\phi_\gamma^* \boldsymbol{v}) = \phi_\gamma^* \boldsymbol{v} \cdot \left(-\frac{1}{2}\Gamma + \phi_\gamma^* A\right) \cdot \frac{d\overline{z}}{\overline{z}}.$$

Note we have the following:

$$|\phi_\gamma^* A|_{h(\gamma),\widetilde{g}} \leq C_3 \cdot \frac{-\log|z| + 1}{-\log|z| - \log|\gamma| + 1} \cdot \big(-\log|z| + 1\big)^{-1}. \tag{48}$$

Hence $\phi_\gamma^* \boldsymbol{v}$ satisfies the claim of Lemma 7.10. We put $f_i^{(\gamma)} := |z|^{\alpha_i} \cdot \phi_\gamma^* v_i$. We construct the sections $g_i^{(\gamma)}$ and $F_i^{(\gamma)}$ as above. We also take $\widetilde{s}^{(\gamma)}$ and $s^{(\gamma)}$.

Lemma 7.16. — *For $\eta > 0$, there exists $\gamma_1 = \gamma_1(\eta) > 0$ such that the assumptions of Lemma 7.15 are satisfied for $\big(E(\gamma_1), \overline{\partial}_{E(\gamma_1)}, h(\gamma_1)\big)$ and $\phi_{\gamma_1}^* \boldsymbol{v}$.*

Proof. — If γ is sufficiently small, then we may assume $\sup_{H_0} |\phi_\gamma^* A|_{\widetilde{g}} \leq B_{15}$ due to (48). We also have the following:

$$\begin{aligned} &\int \big|T_{\phi_\gamma^* A} \cdot f_i^{(\gamma)}\big|^2_{h(\gamma),\widetilde{g}} \cdot |z|^{-2\alpha_i+2\eta} \cdot \big(-\log|z|\big)^N \cdot \mathrm{dvol}_{\widetilde{g}} \\ &\qquad \leq B_{18} \cdot \int \left| \frac{-\log|z| + 1}{-\log|z| - \log\gamma + 1} \right|^2 \cdot |z|^{2\eta} \cdot \big(-\log|z|\big)^N \cdot \mathrm{dvol}_{\widetilde{g}}. \end{aligned} \tag{49}$$

Since the right hand side converges to 0 in $\gamma \longrightarrow 0$, we can take γ_1 such that the inequality $\|T_{\phi_{\gamma_1}^* A} f_i^{(\gamma_1)}\|_{-\alpha_i+\eta,N} < B_{15}$ holds. □

Now we have the holomorphic sections $F_1^{(\gamma_1(\eta))}, \dots, F_r^{(\gamma_1(\eta))}$ of ${}^\diamond E(\gamma_1(\eta))$, satisfying $|F_i^{(\gamma_1(\eta))}|_{h(\gamma_1(\eta))} \leq C(\eta) \cdot |z|^{\alpha_i - \eta}\big(-\log|z|\big)^N$. We put $a_i(\eta) := \max\big\{b \in \mathcal{P}ar({}^\diamond E) \mid b \leq -\alpha_i + \eta\big\}$, and then $F_i^{(\gamma_1(\eta))}$ are sections of ${}_{a_i(\eta)}E(\gamma_1(\eta))$.

Lemma 7.17. — *We have $\mathcal{P}ar({}^{\diamond}E) = S(\Gamma)$ which preserves the multiplicity. Hence, we have $-\widetilde{\alpha} = \widetilde{0}$.*

Proof. — If η is sufficiently small, we have $a_i(\eta) \le -\alpha_i$ and hence $\sum a_i(\eta) \le -\widetilde{\alpha}$. We put $\gamma_2 := \gamma_1(\eta)$. Hence we obtain $\big|\bigwedge_{i=1}^r F_i^{(\gamma_2)}\big|_{h(\gamma_2)} = O(|z|^{\widetilde{\alpha}})$, which implies $\widetilde{F}$ is holomorphic on X, where $\widetilde{F}$ is given by $\bigwedge_{i=1}^r F_i^{(\gamma_2)} = \widetilde{F}\cdot\widetilde{s}^{(\gamma_2)}$. Due to Lemma 7.15 and the maximum principle, we obtain $B_{20}^{-1} \le |\widetilde{F}(z)| \le B_{20}$ for $z \in X(2/3)$. Hence, we obtain $\sum a_i(\eta) = -\widetilde{\alpha}$.

We put $S(b) := \{i \mid -\alpha_i = b\}$ for $b \in \mathcal{P}ar\big({}^{\diamond}E(\gamma_2)\big)$. For $i \in S(b)$, we have $F_i^{(\gamma_2)} \in {}_bE(\gamma_2)$, which induces $\overline{F}_i^{(\gamma_2)} \in \mathrm{Gr}_b^F(E(\gamma_2))$. From $B_{20}^{-1} \le |\widetilde{F}(z)| \le B_{20}$ for $z \in X(2/3)$, we have the lower estimate $|\bigwedge_{i\in S(b)} F_i^{(\gamma_2)}|_{h(\gamma_2)} \ge C_\delta |z|^{-|S(b)|\cdot b+\delta}$ for any $\delta > 0$. Hence we obtain the linearly independence of $\overline{F}_i^{(\gamma_2)}$ $(i \in S(b))$. Then, it is easy to show that $F_1^{(\gamma_2)},\dots,F_r^{(\gamma_2)}$ give the frame of ${}^{\diamond}E(\gamma_2)$ compatible with the parabolic structure, whose parabolic degrees are $-\alpha_1,\dots,-\alpha_r$, respectively. □

Now let us fix $\eta = \eta_0$. We put $\gamma_3 := \gamma_1(\eta_0)$. We have the holomorphic sections $F_i^{(\gamma_3)}$ of ${}^{\diamond}E(\gamma_3)$ on X satisfying $|F_i^{(\gamma_3)}|_{h(\gamma_3)} \le B_{30}\cdot|z|^{\alpha_i(\eta_0)-\eta_0}$. Since we have $s^{(\gamma_3)} = \widetilde{s}^{(\gamma_3)}$, the function $\widetilde{F}$ determined by $F_1^{(\gamma_3)}\wedge\cdots\wedge F_i^{(\gamma_3)} = \widetilde{F}\cdot\widetilde{s}^{(\gamma_3)}$ is holomorphic on X. Thus, we have $B_{31}^{-1} \le |\widetilde{F}(z)| \le B_{31}$ for $z \in X(2/3)$ due to the maximum principle and Lemma 7.15.

The holomorphic sections $F_i^{(\gamma_3)}$ of ${}^{\diamond}E(\gamma_3)$ on X naturally give the holomorphic sections $\widehat{F}_i$ of ${}^{\diamond}E$ on $X(\gamma_3)$. We take $\gamma_0 < \gamma_3$ appropriately, and we put $F_i := \widehat{F}_{i\,|\,X(\gamma_0)}$. It is clear that they satisfy the second and third claims of Proposition 7.4.

For each $a_i(\eta_0)$, we have the number $b_i \in U_0$ such that $a_i(\eta_0) \in \mathcal{P}(b_i)$. We obtain $F_i \in {}_{\overline{b}_i}E$. Then, the first claim of Proposition 7.4 follows from Lemma 7.13 and the third claim. □

CHAPTER 8

SOME CONVERGENCE RESULTS

8.1. Convergence of a Sequence of Tame Harmonic Bundles

Let X be a smooth projective variety of an arbitrary dimension over $\boldsymbol{C}$, and D be a simple normal crossing divisor of X. Let $(E_m, \overline{\partial}_m, \theta_m, h_m)$ $(m = 1, 2, \dots,)$ be a sequence of tame harmonic bundles of rank r on $X - D$. We have the associated parabolic Higgs bundles $({}_{\boldsymbol{c}}E_{m*}, \theta_m)$ on (X, D).

Theorem 8.1. — *Assume that the sequence of the sections* $\{\det(t - \theta_m)\}$ *of* $\mathrm{Sym}^{\cdot}\,\Omega_X^{1,0}(\log D)[t]$ *are convergent. Then the following claims hold:*

- *There exists a subsequence* $\big\{(E_m, \overline{\partial}_m, \theta_m, h_m)\,\big|\,m \in I\big\}$ *which converges to a tame harmonic bundle* $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ *on* $X - D$*, weakly in* L_2^p *locally on* $X - D$*, in the sense of Section* 2.1*. Here* p *denotes an arbitrarily large number.*
- *If we are given a parabolic Higgs sheaf* $({}_{\boldsymbol{c}}E_*, \theta)$ *such that* $\big\{({}_{\boldsymbol{c}}E_{m*}, \theta_m)_{|C}\big\}$ *converges to* $({}_{\boldsymbol{c}}E_*, \theta)_{|C}$ *for any generic curve* C*. Then we have a non-trivial holomorphic morphism* $f : ({}_{\boldsymbol{c}}E_*, \theta) \longrightarrow ({}_{\boldsymbol{c}}E_{\infty*}, \theta_\infty)$*.*

  If $({}_{\boldsymbol{c}}E_*, \theta)$ *is a* μ_L*-stable reflexive saturated parabolic Higgs sheaf,* f *is isomorphic. (See Lemma* 3.10*.)*

Proof. — The first claim is well known. We recall only an outline. The sequence of sections $\{\det(t - \theta_m)\}$ of $\mathrm{Sym}^{\cdot}\,\Omega_X^{1,0}[t]$ converges to $\det(t - \theta)$. Hence we obtain the estimate of the norms of θ_m locally on $X - D$ (See Lemma 2.13, for example). We also obtain the estimate of the curvatures $R(h_m)$ because of the relation $R(h_m) + [\theta_m, \theta_m^\dagger] = 0$. Therefore, we obtain the local convergence result like the first claim. (See **[55]** in the page 26–28, for example.) Thus we obtain the harmonic bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$.

Let us show the second claim in Subsection 8.1.3 after some preparation.

8.1.1. On a punctured disc. — Let us explain the setting in this subsection. Let $X(\gamma)$ and $X^*(\gamma)$ denote the disc $\{z \in \boldsymbol{C} \,|\, |z| < \gamma\}$ and the punctured disc $X(\gamma) - \{0\}$. In the case $\gamma = 1$, we use the notation X and X^*. We put $D := \{0\}$. Let

$(E_m, \overline{\partial}_m, \theta_m, h_m)$ $(m = 1, 2, \ldots, \infty)$ be a sequence of tame harmonic bundles of rank r on a punctured disc X^*. We have the associated parabolic Higgs bundles $({}_cE_{m*}, \theta_m)$ on (X, D) for $c \in \boldsymbol{R}$. Assume the following:

- $\{(E_m, \overline{\partial}_m, \theta_m, h_m) \,|\, m < \infty\}$ converges to $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ in C^1 locally on Δ^* via the isometries $\Phi_m : (E_m, h_m) \longrightarrow (E_\infty, h_\infty)$.
- Assumption 7.1 is satisfied for any m. The constants are independent of the choice of m.
- There exists a finite subset $U_0 \subset]c-1, c[$ and a function $\overline{\mathfrak{m}} : U_0 \longrightarrow \mathbb{Z}_{>0}$ such that $\{(\mathcal{P}ar({}_cE_m), \mathfrak{m}) \,|\, m < \infty\}$ converges to $(U_0, \overline{\mathfrak{m}})$ in the sense of Section 2.1. We put $u := \sum_{b \in U_0} \overline{\mathfrak{m}}(b) \cdot b$.

Lemma 8.2. — *We have holomorphic isomorphisms $\Psi_{m'} : {}_cE_{m'} \longrightarrow {}_cE_\infty$ on $X(\gamma)$ for some $\gamma < 1$ and some subsequence $\{m'\} \subset \{m\}$, with the following properties:*

- *$\Psi_m - \Phi_m \longrightarrow 0$ weakly in L_1^p locally on $X^*(\gamma)$.*
- *$\Psi_m(\theta_m) - \theta_\infty \to 0$ as holomorphic sections of $\mathrm{End}({}_cE_\infty) \otimes \Omega^{1,0}(\log D)$ on $X(\gamma)$.*
- *Let $F^{(m)}({}_cE_m)$ denote the parabolic filtrations of ${}_cE_{m|D}$ induced by h_m. Then the sequence of the filtrations $\{\Psi_m(F^{(m)}({}_cE_{m|D}))\}$ converges to $F^{(\infty)}({}_cE_\infty)_{|D}$ in the sense of Definition 3.36.*

Proof. — After going to a subsequence, we may assume that Assumption 7.3 is satisfied for $(E_m, \overline{\partial}_m, \theta_m, h_m)$ $(m < \infty)$ with some $\eta_0 > 0$. We take holomorphic sections $F_1^{(m)}, \ldots, F_r^{(m)}$ of ${}_cE_m$ on $X(\gamma)$ with $b_1^{(m)}, \ldots, b_r^{(m)} \in U_0$ as in Proposition 7.4, with some $\gamma < 1$. We may assume that $b_i^{(m)}$ are independent of m, which are denoted by b_i. There exists a subsequence $\{m'\}$ such that $\{\Phi_{m'}(F_i^{(m')})\}$ are convergent weakly in L_1^p locally on $X(\gamma)^*$. The limits are denoted by $F_i^{(\infty)}$. They are holomorphic with respect to $\overline{\partial}_\infty$. We replace $\{m\}$ with the subsequence $\{m'\}$, and we assume that the above convergence holds from the beginning.

For each $b \in U_0$, we put $\overline{b}(m) := \max\{a \in \mathcal{P}ar({}_cE_m) \,|\, |a - b| < \eta_0\}$. Then, we have $|F_i^{(m)}|_{h_m} < C \cdot |z|^{-\overline{b}_i(m)} \cdot (-\log|z|)^N$, where the constants C and N are independent of m. Since we have $\overline{b}_i(m) \to b_i$ for $m \to \infty$, we obtain $|F_i^{(\infty)}|_{h_\infty} < C \cdot |z|^{-b_i} \cdot (-\log|z|)^N$, and hence $F_i^{(\infty)} \in {}_{b_i}E_\infty$.

We put $\widetilde{c}(m) := \sum_{b \in \mathcal{P}ar({}_cE_m)} b \cdot \mathfrak{m}(b)$. The sequence $\{\widetilde{c}(m)\}$ converges to u. We have $C_1^{-1} \cdot |z|^{-\widetilde{c}(m)} \leq \big|\bigwedge_{i=1}^r F_i^{(m)}\big|_{h_m} \leq C_1 \cdot |z|^{-\widetilde{c}(m)}$, and hence $C_1^{-1} \cdot |z|^{-u} \leq \big|\bigwedge_{i=1}^r F_i^{(\infty)}\big|_{h_\infty} \leq C_1 \cdot |z|^{-u}$. We put $S_b := \{i \,|\, b_i = b\}$. For $i \in S_b$, we have $F_i^{(\infty)} \in {}_bE_\infty$, which induces $\overline{F}_i^{(\infty)} \in \mathrm{Gr}_b^F(E_{\infty|D})$. We have the lower estimate $\big|\bigwedge_{i \in S_b} F_i^{(\infty)}\big|_{h_\infty} \geq C_\delta \cdot |z|^{-|S_b| \cdot b + \delta}$ for any $\delta > 0$, from which we obtain the linearly independence of $\overline{F}_i^{(\infty)}$ $(i \in S_b)$ in $\mathrm{Gr}_b^F(E_{\infty|D})$. Then, it can be shown that the sections $F_1^{(\infty)}, \ldots, F_r^{(\infty)}$ give a holomorphic frame of ${}_cE_\infty$, which is compatible with the parabolic structure, and $b_i^{(\infty)}$ are the degrees of $F_i^{(\infty)}$ with respect to the parabolic structure.

We construct the holomorphic map $\Psi_m : {}_cE_m \longrightarrow {}_cE_\infty$ on $X(\gamma)$ by the correspondence $\Psi_m(F_i^{(m)}) = F_i^{(\infty)}$. The first and third claims of the lemma are satisfied by our construction. Let K be any compact subset of $X^*(\gamma)$. Since $\Psi_m - \Phi_m$ converges to 0 in L_1^p on K, we have the C^0-endomorphisms G_m of $E_{\infty|K}$ for any sufficiently large m such that (i) $\Psi_{m|K} = G_m \circ \Phi_{m|K}$, and (ii) $G_m \to \mathrm{id}_{E_{\infty|K}}$ in C^0 for $m \to \infty$. Then, $\Psi_m(\theta_m)_{|K} = G_m \circ \Phi_m(\theta_m)_{|K} \circ G_m^{-1}$ converges to $\theta_{\infty|K}$ in C^0 on K. Hence, we also have the convergence of $\Psi_m(\theta_m) - \theta_\infty$ to 0 in C^0 on any compact subset of $X^*(\gamma)$. The Higgs fields θ_m and the holomorphic frames $F_1^{(m)}, \dots, F_r^{(m)}$ determine the matrix valued holomorphic $\Omega^{1,0}(\log D)$-forms Θ_m. Similarly, we obtain Θ_∞. Due to the above argument, we have the local convergence of Θ_m to Θ_∞ on $X^*(\gamma)$. Since they are holomorphic, we obtain the convergence on $X(\gamma)$. Thus the second claim also holds. □

8.1.2. On a curve. — Let us explain the setting in this subsection. Let C be a smooth projective curve with a finite subset $D_C \subset C$. Let $(E_m, \overline{\partial}_m, h_m, \theta_m)$ $(m = 1, 2, \dots, \infty)$ be a sequence of harmonic bundles of rank r on $C - D_C$. We have the associated Higgs bundles $({}_{\boldsymbol{c}}E_{m\,*}, \theta_m)$, where $\boldsymbol{c} = (c(P) \mid P \in D) \in \boldsymbol{R}^D$. We assume the following:

- The sequence $\{(E_m, \overline{\partial}_m, h_m, \theta_m)\}$ converges to $(E_\infty, \overline{\partial}_\infty, h_\infty, \theta_\infty)$ in C^1 locally on $C - D_C$ via isometries $\Phi_m : (E_m, h_m) \longrightarrow (E_\infty, h_\infty)$.
- For each i, a finite subset $U(P) \subset]c(P) - 1, c(P)[$ and a function $\mathfrak{m} : U(P) \longrightarrow \mathbb{Z}_{>0}$ are given, and $\{(\mathcal{P}ar(E_m, P), \mathfrak{m}) \mid m < \infty\}$ converges to $(U(P), \mathfrak{m})$.

By the first condition, the sequence $\det(t - \theta_m) \in \mathrm{Sym}^{\cdot}\, \Omega_C^{1,0}(\log D_C)$ converges to $\det(t - \theta_\infty)$. Around each point $P \in D_C$, we can take a coordinate neighbourhood V_P such that Assumption 7.1 is satisfied on V_P for any $m < \infty$, and that the constants are independent of m.

Lemma 8.3. — *$\{({}_{\boldsymbol{c}}E_{m'}, \boldsymbol{F}^{(m')}, \theta_{m'}) \mid m' \in I\}$ converges to $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}^{(\infty)}, \theta_\infty)$ for an appropriate subsequence $I \subset \{m\}$ in the sense of Definition* 3.36.

Proof. — We would like to replace $\Phi_{m'}$ with $\Psi_{m'} : {}_{\boldsymbol{c}}E_{m'} \longrightarrow {}_{\boldsymbol{c}}E_\infty$ for an appropriate subsequence $\{m'\} \subset \{m\}$. By shrinking V_P appropriately, we take the holomorphic maps ${}^P\Psi_{m'} : {}_{c(P)}E_{m'} \longrightarrow {}_{c(P)}E_\infty$ on V_P for some subsequence $\{m'\} \subset \{m\}$ for each point $P \in D_C$, as in Lemma 8.2. We replace $\{m\}$ with $\{m'\}$.

Let $\chi_P : C \longrightarrow [0, 1]$ denote a C^∞-function which is constantly 1 around P, and constantly 0 on $C - V_P$. Let $\Psi_m : E_m \longrightarrow E_\infty$ be the L_1^p-map given as follows:

$$\Psi_m := \sum_P \chi_P \cdot {}^P\Psi_m + \Big(1 - \sum_P \chi_P\Big) \cdot \Phi_m. \tag{50}$$

If m is sufficiently large, then Ψ_m are isomorphisms. We have the following:

$$(51)\quad \Psi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Psi_m = \sum \overline{\partial}\chi_P \cdot ({}^P\Psi_m - \Phi_m) + \Big(1 - \sum \chi_P\Big) \cdot \Big(\Phi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Phi_m\Big).$$

Hence the sequence $\big\{\Psi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Psi_m\big\}$ converges to 0 weakly in L^p on C. By construction, the sequence of the parabolic filtrations of ${}_{\boldsymbol{c}}E_{m\,*}$ converges that of ${}_{\boldsymbol{c}}E_{\infty\,*}$. We also have the convergence of $\Psi_m(\theta_m) - \theta_\infty$ to 0 weakly in L^p on C. Hence we obtain the convergence of $\big\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}^{(m)}, \theta_m) \,\big|\, m < \infty\big\}$ to $\big({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}^{(\infty)}, \theta_\infty\big)$ weakly in L_1^p on C. □

8.1.3. The end of Proof of Theorem 8.1. — Let us return to the setting for Theorem 8.1. Let $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ be a harmonic bundle obtained as a limit. We obtain the parabolic Higgs bundle $({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty)$. We would like to show the existence of a non-trivial holomorphic homomorphism $\big({}_{\boldsymbol{c}}E_*, \theta\big) \longrightarrow \big({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty\big)$. Due to Lemma 3.4, we have only to show the existence of a non-trivial map $f_C : \big({}_{\boldsymbol{c}}E_*, \theta\big)_{|C} \longrightarrow \big({}_{\boldsymbol{c}}E_{\infty\,*}, \theta_\infty\big)_{|C}$ for some sufficiently ample generic curve $C \subset X$. We may and will assume that $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E, i)$.

We have the convergence of the sequence $\big\{\big({}_{\boldsymbol{c}}E_{m*}, \theta_m\big)_{|C} \,\big|\, m\big\}$ to $\big({}_{\boldsymbol{c}}E_*, \theta\big)_{|C}$ on C. In particular, we have the convergence $\big\{(\mathcal{P}ar({}_{\boldsymbol{c}}E_{m|C}, P), \mathfrak{m}) \,\big|\, m < \infty\big\}$ to $(\mathcal{P}ar({}_{\boldsymbol{c}}E_{|C}, P), \mathfrak{m})$ for any $P \in C \cap D$. The sequence $\big\{(E_m, \overline{\partial}_m, \theta_m, h_m)_{|C\setminus D}\big\}$ is convergent to $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)_{|C\setminus D}$ in C^1 locally on $C \setminus D$. After going to a subsequence, we obtain the convergence of $\big\{\big({}_{\boldsymbol{c}}E_{m*},, \theta_m\big)_{|C} \,\big|\, m\big\}$ to $\big({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty\big)_{|C}$ weakly in L_1^p on C, due to Lemma 8.3. Thus we obtain the existence of the desired non-trivial map f_C due to Corollary 3.38. Thus the proof of Theorem 8.1 is finished. □

8.2. Preparation for the Proof of Theorem 9.1

Let C be a smooth projective curve over $\boldsymbol{C}$ with a simple effective divisor D. Let $\{({}_{\boldsymbol{c}}E_{m*}, \theta_m)\}$ be a sequence of stable parabolic Higgs bundles on (C, D) with par-deg$({}_{\boldsymbol{c}}E_{m*}) = 0$, which converges to a stable Higgs bundle $({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty)$. We take pluri-harmonic metrics $h_0^{(m)}$ of $(E_m, \overline{\partial}_{E_m}, \theta_m)$ adapted to the parabolic structure ($m = 1, 2, \ldots, \infty$) (Proposition 2.8), where $E_m := {}_{\boldsymbol{c}}E_{m|C-D}$. We put $\mathcal{D}_m := \overline{\partial}_{E_m} + \theta_m$ and $\mathcal{D}_m^\star := \partial_{E_m, h_0^{(m)}} + \theta^\dagger_{m, h_0^{(m)}}$ ($m = 1, 2, \ldots, \infty$).

Take a sequence of small positive numbers $\{\epsilon_m\}$. For each $P \in D$, let (V_P, z) be a holomorphic coordinate around P such that $z(P) = 0$. Let N be a large positive number, for example $N > 10$. Let g_m be Kahler metrics of $C - D$ with the following form on V_P for each $P \in D$:

$$\big(\epsilon_m^{N+2}|z|^{2\epsilon_m} + |z|^2\big)\frac{dz \cdot d\overline{z}}{|z|^2}.$$

We assume that $\{g_m\}$ converges to a smooth Kahler metric g_0 of C in the C^∞-sense locally on $C-D$.

In the following argument, $\|\rho\|_{h,g}$ will denote the L^2-norm of a section ρ of $E_m \otimes \Omega^{p,q}_{C-D}$ or $\mathrm{End}(E_m)\otimes\Omega^{p,q}_{C-D}$, with respect to a metric g of $C-D$ and a metric h of E_m. On the other hand, $|\rho|_{h,g}$ will denote the norm at fibers.

Proposition 8.4. — *Let $h^{(m)}$ $(m<\infty)$ be hermitian metrics of E_m with the following properties:*

1. *Let $s^{(m)}$ be determined by $h^{(m)} = h_0^{(m)}\cdot s^{(m)}$. Then* (i) *$s^{(m)}$ is bounded with respect to $h_0^{(m)}$,* (ii) *$\det s^{(m)} = 1$,* (iii) *$\big\|\mathcal{D}_m s^{(m)}\big\|_{h_0^{(m)},g_m} < \infty$. (The estimates may depend on m.)*
2. *We have $\|F(h^{(m)})\|_{h^{(m)},g_m} < \infty$ and $\lim_{m\to\infty}\|F(h^{(m)})\|_{h^{(m)},g_m} = 0$.*
3. *There exists a tame harmonic bundle $(E',\overline{\partial}_{E'},\theta',h')$ such that the sequence $\big\{(E_m,\overline{\partial}_{E_m},\theta_m,h^{(m)})\big\}$ converges to $(E',\overline{\partial}_{E'},\theta',h')$ in C^1 locally on $C-D$.*

Then, after going to a subsequence, $\{({}_{\boldsymbol{c}}E_{m*},\theta_m)\}$ converges to $({}_{\boldsymbol{c}}E'_*,\theta')$ weakly in L_1^p on C.

Proof. — We may and will assume that $\{(E_m,\overline{\partial}_{E_m},\theta_m,h_0^{(m)})\}$ converges to $(E_\infty,\overline{\partial}_{E_\infty},\theta_\infty,h_\infty)$ via the isometries $\Phi_m : (E_m,h_0^{(m)}) \longrightarrow (E_\infty,h_\infty)$, due to Theorem 8.1. First, let us show that $s^{(m)}$ are bounded independently of m.

8.2.1. Uniform boundedness of $s^{(m)}$.

— For any point $P\in C-D$, let $SE(s^{(m)})(P)$ denote the maximal eigenvalue of $s^{(m)}_{|P}$. There exists a constant $0<C_1<1$ such that $C_1\cdot|s^{(m)}_{|P}|_{h_0^{(m)}} \le SE(s^{(m)})(P) \le |s^{(m)}_{|P}|_{h_0^{(m)}}$. Because of $\det s^{(m)}_{|P} = 1$, we have $SE(s^{(m)})(P)\ge 1$ for any P.

Let us take $b_m>0$ satisfying $2\le b_m\cdot\sup_P SE(s^{(m)})(P)\le 3$. We put $\widetilde{s}^{(m)} = b_m\cdot s^{(m)}$ and $\widetilde{h}^{(m)} := h_0^{(m)}\cdot\widetilde{s}^{(m)}$. Then $\widetilde{s}^{(m)}$ are self-adjoint and uniformly bounded with respect to both of $h_0^{(m)}$ and $\widetilde{h}^{(m)}$. We remark $F(\widetilde{h}^{(m)}) = F(h^{(m)})$. We also remark that $h^{(m)}$ and $\widetilde{h}^{(m)}$ induce the same metric of $\mathrm{End}(E_m)$.

Recall the following equality (Lemma 3.1 of [**51**]):

$$\Delta_{g_0,h_0^{(m)}}\widetilde{s}^{(m)} = \widetilde{s}^{(m)}\sqrt{-1}\Lambda_{g_0}F(\widetilde{h}^{(m)}) + \sqrt{-1}\Lambda_{g_0}\mathcal{D}_m\widetilde{s}^{(m)}\big(\widetilde{s}^{(m)}\big)^{-1}\mathcal{D}^\star_m\widetilde{s}^{(m)}. \tag{52}$$

Because of $\|\mathcal{D}_m s^{(m)}\|_{h_0^{(m)},g_m} = \|\mathcal{D}_m s^{(m)}\|_{h_0^{(m)},g_0} < \infty$ and the boundedness of $\widetilde{s}^{(m)}$, we have $\int\Delta_{g_0}\operatorname{tr}\widetilde{s}^{(m)}\cdot\mathrm{dvol}_{g_0} = 0$. Hence, we obtain the following inequality from (52) and the uniform boundedness of $\widetilde{s}^{(m)}$ with respect to $\widetilde{h}^{(m)}$:

$$(53)\quad \int \big|\mathcal{D}_m \widetilde{s}^{(m)} \cdot (\widetilde{s}^{(m)})^{-1/2}\big|^2_{g_0, h_0^{(m)}} \,\mathrm{dvol}_{g_0} \leq A_1 \int \big|\mathrm{tr}\big(\widetilde{s}^{(m)} \cdot \Lambda_{g_0} F(\widetilde{h}^{(m)})\big)\big| \cdot \mathrm{dvol}_{g_0}$$
$$\leq A_2 \cdot \int \big|\Lambda_{g_0} F(\widetilde{h}^{(m)})\big|_{\widetilde{h}^{(m)}} \cdot \mathrm{dvol}_{g_0} = A_2 \cdot \int \big|\Lambda_{g_m} F(\widetilde{h}^{(m)})\big|_{\widetilde{h}^{(m)}} \cdot \mathrm{dvol}_{g_m}$$
$$\leq A_3 \cdot \big\|F(\widetilde{h}^{(m)})\big\|_{\widetilde{h}^{(m)}, g_m}.$$

Here, A_i denote the constants which are independent of m, and we have used the inequality $\big|\mathrm{tr}\big(\widetilde{s}^{(m)} \cdot \Lambda_{g_0} F(\widetilde{h}^{(m)})\big)\big| \leq |\widetilde{s}^{(m)}|_{\widetilde{h}^{(m)}} \cdot |\Lambda_{g_0} F(\widetilde{h}^{(m)})|_{\widetilde{h}^{(m)}}$. In particular, we obtain the following inequality for some constant A_4:

$$(54)\qquad \big\|\mathcal{D}_m \widetilde{s}^{(m)}\big\|^2_{h_0^{(m)}, g_0} \leq A_4 \cdot \big\|F(\widetilde{h}^{(m)})\big\|_{\widetilde{h}^{(m)}, g_m}$$

We put $\widetilde{t}^{(m)} := \Phi_m(\widetilde{s}^{(m)}) \in \mathrm{End}(E_\infty)$.

Lemma 8.5. — *After going to an appropriate subsequence, $\{\widetilde{t}^{(m)}\}$ converges to a positive constant multiplication weakly in L_1^2 locally on $C - D$.*

Proof. — $\{\widetilde{t}^{(m)}\}$ is L_1^2-bounded on any compact subset of $C - D$ due to (54). By going to an appropriate subsequence, it is weakly L_1^2-convergent locally on $C - D$. Let $\widetilde{t}^{(\infty)}$ denote the weak limit. We obtain $\mathcal{D}_\infty \widetilde{t}^{(\infty)} = 0$ from (54). By construction, $\widetilde{t}^{(\infty)}$ is also bounded with respect to $h_0^{(\infty)}$. Therefore $\widetilde{t}^{(\infty)}$ gives an automorphism of $({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty)$. Due to the stability of $({}_{\boldsymbol{c}}E_{\infty *}, \theta_\infty)$, $\widetilde{t}^{(\infty)}$ is a constant multiplication.

We would like to show $\widetilde{t}^{(\infty)} \neq 0$. Let us take any point $Q_m \in C - D$ satisfying the following:

$$SE(s^{(m)})(Q_m) \geq \frac{9}{10} \cdot \sup_{P \in C - D} SE(s^{(m)})(P).$$

Then we have $\log \mathrm{tr}\, \widetilde{t}^{(m)}(Q_m) \geq \log(9/5)$. By taking an appropriate subsequence, we may assume that the sequence $\{Q_m\}$ converges to a point Q_∞. We have two cases (i) $Q_\infty \in D$ (ii) $Q_\infty \notin D$. We discuss only the case (i). The other case is similar and easier.

We have $\mathrm{tr}\, \widetilde{s}^{(m)} = \mathrm{tr}\, \widetilde{t}^{(m)}$, which we do not distinguish in the following. We use the coordinate neighbourhood (U, z) such that $z(Q_\infty) = 0$. For any point $P \in U$, we put $\Delta(P, T) := \{Q \in U \,|\, |z(P) - z(Q)| < T\}$. Let $g = dz \cdot d\bar{z}$ denote the standard metric of U. We have the following inequality on $U - \{Q_\infty\}$ (Lemma 3.1 of [**51**]):

$$\Delta_g \log \mathrm{tr}\, \widetilde{s}^{(m)} \leq \big|\Lambda_g F(\widetilde{h}^{(m)})\big|_{\widetilde{h}^{(m)}}.$$

Let $B^{(m)}$ be the endomorphism of E_m determined as follows:

$$F(\widetilde{h}^{(m)}) = F(h^{(m)}) = B^{(m)} \cdot \frac{dz \cdot d\bar{z}}{|z|^2},$$

Then we have the following estimate:

$$\int \big|B^{(m)}\big|^2_{\widetilde{h}^{(m)}} \big(\epsilon_m^{N+1} |z|^{2\epsilon_m} + |z|^2\big)^{-1} \frac{\mathrm{dvol}_g}{|z|^2} \leq A \int \big|F(\widetilde{h}^{(m)})\big|^2_{\widetilde{h}^{(m)}, g_m} \cdot \mathrm{dvol}_{g_m}.$$

Here A denotes a constant independent of m. Due to Lemma 2.17, there exist $v^{(m)}$ such that the following inequalities hold for some positive constant A':

$$\overline{\partial}\partial v^{(m)} = \big|B^{(m)}\big|_{\widetilde{h}^{(m)}} \frac{dz\cdot d\overline{z}}{|z|^2}, \qquad \big|v^{(m)}(z)\big| \leq A'\cdot \big\|F(\widetilde{h}^{(m)})\big\|_{\widetilde{h}^{(m)},g_m}$$

Then we have $\Delta_g\big(\log\operatorname{tr}\widetilde{t}^{(m)} - v^{(m)}\big) \leq 0$ on $U-\{Q_\infty\}$. Since $s^{(m)}$ and $(s^{(m)})^{-1}$ are bounded on $C-D$, $\log\operatorname{tr} s^{(m)}$ is bounded on $C-D$. Hence, $\Delta_g\big(\log\operatorname{tr}\widetilde{t}^{(m)} - v^{(m)}\big)\leq 0$ holds on U as distributions. (See Lemma 2.2 of [**52**], for example.) Therefore, we obtain the following:

$$\log\operatorname{tr}\widetilde{t}^{(m)}(Q_m) - v^{(m)}(Q_m) \leq A''\cdot\int_{\Delta(Q_m,1/2)}\Big(\log\operatorname{tr}\widetilde{t}^{(m)} - v^{(m)}\Big)\cdot \mathrm{dvol}_g\,.$$

Here A'' denotes a positive constant independent of m. Then we obtain the following inequalities, for some positive constants C_i $(i=1,2)$ which are independent of m:

$$\log(9/5)\leq \log\operatorname{tr}\widetilde{t}^{(m)}(Q_m)\leq C_1\cdot\int_{\Delta(Q_m,1/2)}\log\operatorname{tr}\widetilde{t}^{(m)}\cdot\mathrm{dvol}_g + C_2.$$

Recall that $\log\operatorname{tr}\widetilde{t}^{(m)}$ are uniformly bounded from above. Therefore there exists a positive constant C_3 such that the following holds for any sufficiently large m:

$$\int_{\Delta(Q_m,1/2)} -\min(0,\log\operatorname{tr}\widetilde{t}^{(m)})\cdot\mathrm{dvol}_g\leq C_3.$$

Due to Fatou's lemma, we obtain the following:

$$\int_{\Delta(Q_\infty,1/2)} -\min\big(0,\log\operatorname{tr}\widetilde{t}^{(\infty)}\big)\cdot\mathrm{dvol}_g\leq C_3.$$

It means $\widetilde{t}^{(\infty)}$ is not constantly 0 on $\Delta(Q_\infty,1/2)$. In all, we can conclude that $\widetilde{t}^{(\infty)}$ is a positive constant multiplication. Thus the proof of Lemma 8.5 is finished. □

Let $\big\{\widetilde{t}^{(m')}\big\}$ be a subsequence as in Lemma 8.5. It is almost everywhere convergent to some constant multiplication. Then we obtain the convergence of $\big\{\det\widetilde{t}^{(m')} = b_{m'}^{\operatorname{rank}E}\cdot \mathrm{id}_{\det(E)}\big\}$ to a positive constant multiplication, i.e., $\{b_{m'}\}$ is convergent to a positive constant. It means the uniform boundedness of $\{s^{(m')}\}$ with respect to $h_0^{(m')}$.

8.2.2. Construction of maps. — By assumption, we are given C^1-isometries $\Phi'_m:(E_m,h_m)\longrightarrow(E',h')$ for which $\big\{(E_m,\overline{\partial}_{E_m},\theta_m)\big\}$ converges to $(E',\overline{\partial}_{E'},\theta')$. By modifying them, we would like to construct the maps $\Psi'_m: {}_{\boldsymbol{c}}E_m\longrightarrow {}_{\boldsymbol{c}}E'$ for which a subsequence of $\big\{({}_{\boldsymbol{c}}E_{m*},\theta_m)\big\}$ converges to $({}_{\boldsymbol{c}}E'_*,\theta')$. The argument is essentially same as that in Subsections 8.1.1–8.1.2.

We put $V_P^* := V_P - \{P\}$. We will shrink V_P in the following argument if it is necessary. We may assume that Assumption 7.1 is satisfied on V_P for any $m<\infty$, and that the constants are independent of m. We have the convergence $\big\{(\mathcal{P}ar({}_{\boldsymbol{c}}E_{m*},P),\mathfrak{m})\big\}$ to $\big(\mathcal{P}ar({}_{\boldsymbol{c}}E_{\infty*},P),\mathfrak{m}\big)$. Take $\eta>0$, and we may assume that Assumption 7.3 is satisfied on V_P for any $m < \infty$, after going to a subsequence. By applying Proposition 7.4 to harmonic bundles $(E_m, \overline{\partial}_m, \theta_m, h_0^{(m)})_{|V_P^*}$, we obtain holomorphic sections $F_1^{(m)}, \dots, F_r^{(m)}$ of ${}_{\boldsymbol{c}}E_m$ on V_P with numbers $b_1^{(m)}, \dots, b_r^{(m)}$ as in Proposition 7.4. We may assume $b_i^{(m)}$ are independent of the choice of m, which are denoted by b_i. For $b \in \mathcal{P}ar({}_{\boldsymbol{c}}E_\infty, P)$, we put $\overline{b}(m) := \max\{a \in \mathcal{P}ar({}_{\boldsymbol{c}}E_m) \,|\, |a-b| < \eta_0\}$. We put $\widetilde{c}(m) := \sum_{a \in \mathcal{P}ar({}_{\boldsymbol{c}}E_m, P)} a \cdot \mathfrak{m}(a)$. Because of the uniform boundedness of $s^{(m)}$, we obtain $|F_i^{(m)}|_{h^{(m)}} \leq C \cdot |z|^{-\overline{b}_i(m)}(-\log|z|)^N$ and $C_1 \cdot |z|^{-\widetilde{c}(m)} \leq \bigl|\bigwedge_{i=1}^r F_i^{(m)}\bigr|_{h^{(m)}} \leq C_2 \cdot |z|^{-\widetilde{c}(m)}$, where the constants are independent of m. After going to a subsequence, we may assume that $\bigl\{\Phi'_m(F_i^{(m')})\bigr\}$ are convergent weakly in L_1^p locally on V_P^*. The limits are denoted by F'_i, which are holomorphic with respect to $\overline{\partial}_{E'}$. We have $|F'_i|_{h'} \leq C \cdot |z|^{-b_i}(-\log|z|)^N$ and $C_1 \cdot |z|^{-\widetilde{c}} \leq \bigl|\bigwedge_{i=1}^r F'_i\bigr|_{h^{(m)}} \leq C_2 \cdot |z|^{-\widetilde{c}}$, where $\widetilde{c} := \sum_{b \in \mathcal{P}ar({}_{\boldsymbol{c}}E_\infty, P)} \mathfrak{m}(b) \cdot b$. By the same argument as the proof of Lemma 8.2, we obtain that $F'_1, \dots, F'_r$ gives a frame of ${}_{\boldsymbol{c}}E'$ around P which is compatible with the parabolic structure. (In particular, we obtain $\mathcal{P}ar({}_{\boldsymbol{c}}E', P) = \mathcal{P}ar({}_{\boldsymbol{c}}E_\infty, P)$).

We obtain the holomorphic morphism ${}^P\Psi'_m : {}_{\boldsymbol{c}}E_{m|V_P} \longrightarrow {}_{\boldsymbol{c}}E'_{|V_P}$ by the correspondence ${}^P\Psi'_m(F_i^{(m)}) = F'_i$. By our construction, (i) ${}^P\Psi'_m - \Phi'_{m|V_P^*}$ converges to 0 weakly in L_1^p locally on V_P^*, (ii) ${}^P\Psi'_m(\theta_m) - \theta'$ converges to 0 on V_P as holomorphic sections of $\mathrm{End}({}_{\boldsymbol{c}}E') \otimes \Omega^{1,0}(\log P)$ (see the last part of the proof of Lemma 8.2), (iii) the parabolic filtrations of ${}_{\boldsymbol{c}}E_{m|P}$ converges to the parabolic filtration of ${}_{\boldsymbol{c}}E'_{|P}$ via ${}^P\Psi'_m$. Then, we construct Ψ'_m similarly to (50), which gives the convergence of $\bigl\{({}_{\boldsymbol{c}}E_{m*}, \theta_m)\bigr\}$ to $({}_{\boldsymbol{c}}E'_*, \theta')$. □

CHAPTER 9

EXISTENCE OF ADAPTED PLURI-HARMONIC METRIC

9.1. The Surface Case

Let X be a smooth irreducible projective surface over $\boldsymbol{C}$, and D be a simple normal crossing divisor of X. Let L be an ample line bundle, and ω be a Kahler form representing $c_1(L)$.

Theorem 9.1. — *Let $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta\big)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). Assume that the characteristic numbers vanish:*

$$\operatorname{par-deg}_L({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \int_X \operatorname{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}) = 0.$$

Then there exists a pluri-harmonic metric h of $(E, \theta) = ({}_{\boldsymbol{c}}E, \theta)_{|X-D}$ which is adapted to the parabolic structure.

Proof. — We may and will assume $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F}, i)$. We take a sequence $\{\overline{\epsilon}_m\}$ converging to 0, such that $\overline{\epsilon}_m = N_m^{-1}$ for some integers N_m and that $\overline{\epsilon}_m < \operatorname{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F})/100 \operatorname{rank}(E)$. We take the perturbation of parabolic structures $\boldsymbol{F}^{(\overline{\epsilon}_m)}$ as in Section 3.3. We put $\epsilon_m = \overline{\epsilon}_m/100$, and we take the Kahler metrics ω_{ϵ_m} of $X - D$ as in Subsection 4.3.1. For simplicity of the notation, we denote them by $\boldsymbol{F}^{(m)}$ and $\omega^{(m)}$, respectively. We may assume that $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(m)}\big)$ are μ_L-stable.

Due to Corollary 6.6, we have already known $\operatorname{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \operatorname{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(m)}) = 0$. Thus, we can take a pluri-harmonic metric $h_{\det E}$ of $\det(E)$ adapted to the parabolic structure. Due to Proposition 6.1, we have the Hermitian-Einstein metric $h^{(m)}$ of $(E, \overline{\partial}_E, \theta)$ with respect to $\omega^{(m)}$ such that $\Lambda_{\omega^{(m)}} F(h^{(m)}) = \operatorname{tr} F(h^{(m)}) = 0$ and $\det(h^{(m)}) = h_{\det E}$, which is adapted to the parabolic structure $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(m)})$. We remark that the sequence of the L^2-norms $\|F(h^{(m)})\|_{h^{(m)},\omega^{(m)}}$ of $F(h^{(m)})$ with respect to $h^{(m)}$ and $\omega^{(m)}$ converges to 0 in $m \to \infty$, because of the relation $\|F(h^{(m)})\|^2_{h^{(m)},\omega^{(m)}} = C \cdot \operatorname{par-ch}_{2,L}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(m)})$ for some non-zero constant C. We will show the local convergence of the sequence $\big\{(E, \overline{\partial}_E, \theta, h^{(m)})\big\}$ on $X - D$.

9.1.1. Local convergence. — In the following argument, B_i will denote positive constants which are independent of m. We use the notation $\|\rho\|_{h',\omega'}$ to denote the L^2-norm of a section ρ of $E'\otimes\Omega^{i,j}$ or $\mathrm{End}(E')\otimes\Omega^{i,j}$, where h' and ω' denote metrics of a vector bundle E' and a base space. On the other hand, $|\rho|_{h',\omega'}$ denotes the norms at fibers.

Let P be any point of $X-D$. We take a holomorphic coordinate (U,z_1,z_2) around P such that $z_i(P)=0$ and that $\omega_{|P}=\sum dz_i\cdot d\bar{z}_i$ on the tangent space at P. We have the expression $\theta=\sum f_i\cdot dz_i$.

Let η be a positive number. If m is sufficiently large, we have $\|F(h^{(m)})\|_{\omega^{(m)},h^{(m)}}\leq\eta$. Due to Lemma 2.13, there exists a constant B_1, such that $B_1^{-1}\cdot|f_i|_{h^{(m)}}\leq\eta$. Take a large number $B_2>B_1$, and we put $w_i:=B_2\cdot z_i$, $\widetilde{Y}(T):=\big\{(w_1,w_2)\,\big|\,\sum|w_i|^2\leq T\big\}$, $\widetilde{g}:=\sum dw_i\cdot d\overline{w}_i$ and $\widetilde{\omega}^{(m)}:=B_2^2\cdot\omega^{(m)}$. Then, we obtain the following:

$$\big\|R(h^{(m)})_{|\widetilde{Y}(1)}\big\|_{h^{(m)},\widetilde{g}}\leq\big\|F(h^{(m)})_{|\widetilde{Y}(1)}\big\|_{h^{(m)},\widetilde{g}}+\big\|[\theta,\theta^{\dagger}_{h^{(m)}}]_{|\widetilde{Y}(1)}\big\|_{h^{(m)},\widetilde{g}}\leq B_3\cdot\eta$$

Let d^* denote the formal adjoint of the exterior derivative d on $\widetilde{Y}(1)$ with respect to $\widetilde{g}$. If η is sufficiently small, we can apply Uhlenbeck's theorem ([**63**]). Namely, we can take an orthonormal frame $\boldsymbol{v}_m$ of $\big(E,h^{(m)}\big)_{|\widetilde{Y}(1)}$ such that the connection form A_m of $\overline{\partial}_E+\partial_{E,h^{(m)}}$ with respect to $\boldsymbol{v}_m$ satisfies the conditions:

(i) : $d^*A_m=0$,

(ii) : $\|A_m\|_{L_1^p,\widetilde{g}}\leq C(p)\cdot\|dA_m+A_m\wedge A_m\|_{L^p,\widetilde{g}}$ $(p\geq 2)$, where $C(p)$ denotes the constant depending only on p.

By our choice of B_2, we also have the following:

(iii) : Let $\Pi^{(m)}$ denote the orthogonal projection of Ω^2 onto the self-dual part with respect to $\widetilde{\omega}_m$. Then, $\big|\Pi^{(m)}\big(dA_m+A_m\wedge A_m\big)\big|_{\widetilde{\omega}^{(m)}}\leq B_4\eta$ because of $\Lambda_{\widetilde{\omega}}R(h^{(m)})=\Lambda_{\widetilde{\omega}}[\theta,\theta^{\dagger}_{h^{(m)}}]$.

From (i) and (iii), we have $\big|\big(d^*+\Pi^{(m)}\circ d\big)(A_m)+\Pi^{(m)}(A_m\wedge A_m)\big|_{\widetilde{g}}\leq B_5$. If B_2 and m are sufficiently large, $\widetilde{\omega}^{(m)}$ and $\widetilde{g}$ are sufficiently close. Recall that $d^*+\Pi\circ d$ is elliptic, where Π denotes the orthogonal projection of Ω^2 onto the self-dual part with respect to $\widetilde{g}$. Using the boot strapping argument of Donaldson for Corollary 23 in [**13**], we obtain that the L_1^p-norm of A_m on $\widetilde{Y}(T)$ $(T<1)$ is dominated by a constant B_6. Let Θ_m be determined by $\theta(\boldsymbol{v}_m)=\boldsymbol{v}_m\cdot\Theta_m$. The sup norm of Θ_m with respect to $\widetilde{g}$ is small, due to our choice of B_2. We also obtain the L_1^p-bound of Θ_m because of $\overline{\partial}\Theta_m+[A_m^{0,1},\Theta_m]=0$, where $A_m^{0,1}$ denotes the $(0,1)$-part of A_m.

Lemma 9.2. — *After going to a subsequence, $\big\{\big(E,\overline{\partial}_E,h^{(m)},\theta\big)\,\big|\,m\in I\big\}$ converges to a tame harmonic bundle $(E_\infty,\overline{\partial}_\infty,h_\infty,\theta_\infty)$ weakly in L_2^p locally on $X-D$.*

Proof. — Due to the above arguments, we can take a locally finite covering $\big\{(U_\alpha,z_1^{(\alpha)},z_2^{(\alpha)})\,\big|\,\alpha\in\Gamma\big\}$ of $X-D$ and the numbers $\big\{m(\alpha)\,\big|\,\alpha\in\Gamma\big\}$ with the following property:

– Each U_α is relatively compact in $X - D$.
– For any $m \geq m(\alpha)$, we have orthonormal frames $\boldsymbol{v}_{\alpha,m}$ of $(E, h^{(m)})$ on U_α such that the L_1^p-norms of $A_{\alpha,m}$ are sufficiently small with respect to the metrics $\sum dz_j^{(\alpha)} \cdot d\bar{z}_j^{(\alpha)}$ independently of m, where $A_{\alpha,m}$ denote the connection forms of $(\partial_{E,h^{(m)}} + \overline{\partial}_E)$ with respect to $\boldsymbol{v}_{\alpha,m}$.
– Let $\Theta_{\alpha,m}$ be the matrix valued $(1,0)$-forms given by $\theta \cdot \boldsymbol{v}_{\alpha,m} = \boldsymbol{v}_{\alpha,m} \cdot \Theta_{\alpha,m}$. Then the L_1^p-norms of $\Theta_{\alpha,m}$ are sufficiently small with respect to $\sum dz_j^{(\alpha)} \cdot d\bar{z}_j^{(\alpha)}$, independently of m.

Let $g_{\beta,\alpha,m}$ be the unitary transformation on $U_\alpha \cap U_\beta$ determined by $\boldsymbol{v}_{\alpha,m} = \boldsymbol{v}_{\beta,m} \cdot g_{\beta,\alpha,m}$. Once α and β are fixed, the L_2^p-norms of $g_{\beta,\alpha,m}$ are bounded independently of m. By a standard argument, we can take a subsequence $I \subset \{m\}$ such that the sequences $\{A_{\alpha,m} \mid m \in I\}$, $\{\Theta_{\alpha,m} \mid m \in I\}$ are weakly L_1^p-convergent for each α, and that the sequence $\{g_{\alpha,\beta,m} \mid m \in I\}$ is weakly L_2^p-convergent for each (α, β). Then, we obtain the limit Higgs bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty)$ with the metric h_∞ on $X - D$. From the convergence $\|F(h^{(m)})\|_{L^2, h^{(m)}, \omega^{(m)}} \to 0$, we obtain $\|F(h_\infty)\|_{L^2, h_\infty, \omega} = 0$, and hence $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ is a harmonic bundle. By using the argument of Uhlenbeck [**63**], we obtain locally L_2^p-isometries $\Phi_m : (E, h^{(m)}) \longrightarrow (E_\infty, h_\infty)$, via which $\{(E, \overline{\partial}_E, \theta, h^{(m)})\}$ converges to $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ weakly in L_2^p locally on $X - D$. Since we have $\det(t - \theta) = \det(t - \theta_\infty)$ by construction, the tameness of $(E_\infty, \overline{\partial}_{E_\infty}, h_\infty, \theta_\infty)$ follows. Thus, Lemma 9.2 is proved. □

We obtain the associated parabolic Higgs bundle $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)$. We would like to show that it is isomorphic to the given parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$. For that purpose, we have only to show the existence of a non-trivial morphism $f : ({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta) \longrightarrow ({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)$, because of the μ_L-stability of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ and the μ_L-polystability of $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)$. Moreover, we have only to show the existence of a non-trivial map $f_C : ({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)_{|C} \longrightarrow ({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)_{|C}$ for a sufficiently ample generic curve $C \subset X$, due to Lemma 3.4. So we show that such f_C exists for almost all C, in the next subsections.

9.1.2. Selection of a curve. — Let L^N be sufficiently ample. We put $\mathcal{V} := H^0(X, L^N)$. For any $s \in \mathcal{V}$, we put $X_s := s^{-1}(0)$. Recall Mehta-Ramanathan type theorem (Proposition 3.29), and let $\mathcal{U}$ denote the Zariski open subset of $\mathcal{V}$ which consists of the points s with the properties: (i) X_s is smooth, and $X_s \cap D$ is a simple normal crossing divisor, (ii) $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)_{|X_s}$ is μ_L-stable.

We will use the notation $X_s^* := X_s \setminus D$ and $D_s := X_s \cap D$. We have the metric $\omega_s^{(m)}$ of X_s^*, induced by $\omega^{(m)}$. The induced volume form of X_s^* is denoted by $\mathrm{dvol}_s^{(m)}$. We put $({}_{\boldsymbol{c}}E_s, \boldsymbol{F}_s^{(m)}, \theta_s) := ({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(m)}, \theta)_{|X_s}$. We have the metric $h_s^{(m)} := h^{(m)}_{|X_s^*}$ of $E_s := E_{|X_s^*}$. Since there exists m_0 such that $({}_{\boldsymbol{c}}E_s, \boldsymbol{F}_s^{(m)}, \theta_s)$ is stable for any point $s \in \mathcal{U}$ and for any $m \geq m_0$, we have the harmonic metric $h_{s,0}^{(m)}$ of (E_s, θ_s) adapted to the parabolic structure $\boldsymbol{F}_s^{(m)}$ with $\det h_{s,0}^{(m)} = h_{\det E\,|\,X_s^*}$ (Proposition 2.8). Let $u_s^{(m)}$ be the endomorphism of $E_{|X_s^*}$ determined by $h_s^{(m)} = h_{s,0}^{(m)} \cdot u_s^{(m)}$. Then, $u_s^{(m)}$ is bounded, and it satisfies $\det u_s^{(m)} = 1$. We put $\mathcal{D}_s := (\overline{\partial}_E + \theta)_{|X_s}$.

Lemma 9.3. — *For almost every $s \in \mathcal{U}$, the following holds:*

– *We have the following convergence in $m \to \infty$:*

$$\big\| F(h_s^{(m)}) \big\|_{h_s^{(m)},\omega_s^{(m)}} \longrightarrow 0. \tag{55}$$

– *For each m, we have the finiteness:*

$$\big\| \mathcal{D}_s u_s^{(m)} \big\|_{h_{s,0}^{(m)},\omega_s^{(m)}} < \infty. \tag{56}$$

Let $\widetilde{\mathcal{U}}$ denote the set of s for which both of (55) *and* (56) *hold.*

Proof. — Let us discuss the condition (55). Let us fix $s_1 \in \mathcal{U}$. We take generic $s_i \in \mathcal{U}$ $(i = 2, 3)$, i.e., X_{s_1} is transversal with X_{s_i} $(i = 2, 3)$ and $X_{s_1} \cap X_{s_2} \cap X_{s_3} = \varnothing$. Take open subsets $W_i^{(j)}$ $(j = 1, 2,\ i = 2, 3)$ such that (i) $X_{s_1} \cap X_{s_i} \subset W_i^{(1)} \subset W_i^{(2)}$, (ii) $W_i^{(1)}$ is relatively compact in $W_i^{(2)}$. Take an open neighbourhood U_1 of s_1, which is relatively compact in $\mathcal{U}$, such that X_s is transversal with X_{s_i} $(i = 2, 3)$ and $X_s \cap X_{s_i} \subset W_i^{(1)}$ for any $s \in U_1$.

Take $T > 0$, and we put $\mathcal{B} := \big\{z \in \boldsymbol{C} \,\big|\, |z| \le T\big\}$. Let q_i denote the projection of $X \times U_1 \times \mathcal{B}$ onto the i-th component. We put $\mathcal{Z}_2 := \big\{(x, s, t) \in X \times U_1 \times \mathbb{P}^1 \,\big|\, (ts_2 + (1-t)s)(x) = 0\big\}$. The fiber over $s \in U_1$ via $q_{2\,|\,\mathcal{Z}_2}$ is the closed region of the Lefschetz pencil of s and s_2.

We fix any Kahler forms ω_{U_1} and $\omega_{\mathcal{B}}$ of U_1 and $\mathcal{B}$. The induced volume forms are denoted by dvol_{U_1} and $\mathrm{dvol}_{\mathcal{B}}$. Then we have the following convergence in $m \to \infty$:

$$\int_{\mathcal{Z}_2} q_1^*\Big(\big|F(h^{(m)})\big|^2_{h^{(m)},\omega^{(m)}} \cdot \mathrm{dvol}_{\omega^{(m)}} \Big) \cdot \mathrm{dvol}_{U_1} \longrightarrow 0.$$

We put $\mathcal{Z}_2' := \mathcal{Z}_2 \setminus q_1^{-1}(W_2^{(2)})$. Then the following convergence is obtained, in particular:

$$\int_{\mathcal{Z}_2'} q_1^*\Big(\big|F(h^{(m)})\big|^2_{h^{(m)},\omega^{(m)}} \cdot \mathrm{dvol}_{\omega^{(m)}} \Big) \cdot \mathrm{dvol}_{U_1} \longrightarrow 0. \tag{57}$$

Let $\psi : \mathcal{Z}_2 \longrightarrow U_1 \times \mathcal{B}$ denote the projection. For $(s, t) \in U_1 \times \mathcal{B}$, we put $X_{(s,t)} := \psi^{-1}(s, t) = \big(ts_2 + (1-t)s\big)^{-1}(0) = X_{ts_2+(1-t)s}$. On $X_{(s,t)}$, we have the induced Kahler form $\omega_{(s,t)}^{(m)}$, the induced volume forms $\mathrm{dvol}_{(s,t)}^{(m)}$ and the hermitian metric $h_{(s,t)}^{(m)} := h^{(m)}_{|X_{(s,t)}}$. The family $\{\mathrm{dvol}_{(s,t)}^{(m)} \mid (s, t) \in U_1 \times \mathcal{B}\}$ gives the C^∞-relative volume form

$\mathrm{dvol}^{(m)}_{\mathcal{Z}_2'/U_1\times\mathcal{B}}$ of $\mathcal{Z}_2' \longrightarrow U_1 \times \mathcal{B}$. There exists a constant A such that the following holds on $\mathcal{Z}_2'$:

$$(58)\quad A\cdot q_1^*\Big(\big|F(h^{(m)})\big|^2_{h^{(m)},\omega^{(m)}}\,\mathrm{dvol}_{\omega^{(m)}}\Big)\,\mathrm{dvol}_{U_1} \geq \big|F(h^{(m)}_{(s,t)})\big|^2_{h^{(m)}_{(s,t)},\omega^{(m)}_{(s,t)}}\,\mathrm{dvol}^{(m)}_{\mathcal{Z}_2'/U_1\times\mathcal{B}}\cdot\mathrm{dvol}_{\mathcal{B}}\,\mathrm{dvol}_{U_1}$$

Therefore, we obtain the following convergence for almost every $(s,t)\in U_1\times\mathcal{B}$, from (57):

$$(59)\qquad \int_{X^*_{(s,t)}\setminus W_2^{(2)}} \big|F(h^{(m)}_{(s,t)})\big|^2_{h^{(m)}_{(s,t)},\omega^{(m)}_{(s,t)}}\,\mathrm{dvol}^{(m)}_{(s,t)} \longrightarrow 0.$$

Let $\mathcal{S}$ denote the set of the points $(s,t)\in U_1\times\mathcal{B}$ such that the above convergence (59) does not hold. The measure of $\mathcal{S}$ is 0 with respect to $\mathrm{dvol}_{U_1}\times\mathrm{dvol}_{\mathcal{B}}$.

Let $\mathcal{J}: U_1\times\mathcal{B}\longrightarrow\mathcal{V}$ denote the map given by $(s,t)\longmapsto ts_2+(1-t)s$. We have the open subset $\mathcal{J}^{-1}(U_1)\subset U_1\times\mathcal{B}$ and the measure of $\mathcal{S}\cap\mathcal{J}^{-1}(U_1)$ is 0 with respect to $\mathrm{dvol}_{U_1}\cdot\mathrm{dvol}_{\mathcal{B}}$. We have $\mathcal{S}\cap\mathcal{J}^{-1}(U_1)=\mathcal{J}^{-1}\big(\mathcal{J}(\mathcal{S})\cap U_1\big)$, and hence the measure of $\mathcal{T}(\mathcal{S})\cap U_1$ is 0 with respect to ω_{U_1}. Namely, we have the following convergence for almost every $s\in U_1$:

$$\int_{X_s^*\setminus W_2^{(2)}} \big|F(h_s^{(m)})\big|^2_{h_s^{(m)},\omega_s^{(m)}}\cdot\mathrm{dvol}_s^{(m)} \longrightarrow 0.$$

Similarly, we can show the following convergence for almost every $s\in U_1$:

$$\int_{X_s^*\setminus W_3^{(2)}} \big|F(h_s^{(m)})\big|^2_{h_s^{(m)},\omega_s^{(m)}}\cdot\mathrm{dvol}_s^{(m)} \longrightarrow 0$$

Then, we obtain that the condition (55) holds for almost all $s\in\mathcal{U}$.

The condition (56) can be discussed similarly. We give only an outline. Let $h^{(m)}_{in}$ be an initial metric which was used for the construction of $h^{(m)}$. (See the proof of Proposition 6.1.) We remark that $h^{(m)}_{in}$ and $h^{(m)}$ are mutually bounded. Let $t^{(m)}$ be determined by $h^{(m)}=h^{(m)}_{in}\cdot t^{(m)}$. Then, we have $\big\|\mathcal{D}t^{(m)}\big\|_{\omega^{(m)},h^{(m)}}<\infty$ due to Proposition 2.5. We put $h^{(m)}_{s,in}:=h^{(m)}_{in|X_s^*}$ and $t_s^{(m)}:=t^{(m)}_{|X_s^*}$ for $s\in\mathcal{U}$. By an above argument, we obtain $\big\|\mathcal{D}_s t_s^{(m)}\big\|_{\omega_s^{(m)},h^{(m)}_{s,in}}<\infty$ for almost all $s\in\mathcal{U}$. On the other hand, let $\widetilde{t}_s^{(m)}$ be determined by $h^{(m)}_{s,0}=h^{(m)}_{s,in}\cdot\widetilde{t}_s^{(m)}$. We can use $h^{(m)}_{s,in}$ as the initial metric for the construction of $h^{(m)}_{s,0}$. Hence, we have $\big\|\mathcal{D}_s\widetilde{t}_s^{(m)}\big\|_{\omega_s^{(m)},h^{(m)}_{s,in}}<\infty$. Since we have $u_s^{(m)}=\widetilde{t}_s^{(m)\,-1}\cdot t_s^{(m)}$, the condition (56) is satisfied for almost $s\in\mathcal{U}$. Thus, the proof of Lemma 9.3 is finished. □

9.1.3. End of the proof of Theorem 9.1. — Let us finish the proof of Theorem 9.1. Take $s\in\widetilde{\mathcal{U}}$, and we put $C=X_s$. We have the convergence of $\big\{(E,\overline{\partial}_E,\theta,h^{(m)})\big\}$ to $(E_\infty,\overline{\partial}_\infty,\theta_\infty,h_\infty)$ weakly in L_2^p locally on $X-D$ via isometries $\Phi_m:(E,h^{(m)})\longrightarrow(E_\infty,h_\infty)$. The restriction of Φ_m to $C\setminus D$ induce the

C^1-convergence of $\{(E,\overline{\partial},\theta,h^{(m)})_{|C\setminus D}\}$ to $(E_\infty,\overline{\partial}_\infty,\theta_\infty,h_\infty)_{|C\setminus D}$. By using Proposition 8.4, we obtain the convergence of $\{({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(m')},\theta)_{|C}\}$ to $({}_{\boldsymbol{c}}E_\infty,\boldsymbol{F}_\infty,\theta_\infty)_{|C}$ weakly in L_1^p on C for some subsequence. We also have the convergence of $\{({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(m)},\theta)_{|C}\}$ to $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)_{|C}$. Due to Corollary 3.38, we obtain the desired non-trivial map $f_C : ({}_{\boldsymbol{c}}E_\infty,\boldsymbol{F}_\infty,\theta_\infty)_{|C} \longrightarrow ({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)_{|C}$. Thus we are done. □

9.2. The Higher Dimensional Case

Now the main existence theorem is given.

Theorem 9.4. — *Let X be an irreducible projective variety over $\boldsymbol{C}$ with an ample line bundle L. Let $D=\bigcup_i D_i$ be a simple normal crossing divisor of X. Let $(\boldsymbol{E}_*,\theta)$ be a μ_L-stable regular filtered Higgs bundle with* $\text{par-deg}_L(\boldsymbol{E}_*)=\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*)=0$. *We put $E:=\boldsymbol{E}_{|X-D}$. Then there exists a pluri-harmonic metric h of $(E,\overline{\partial}_E,\theta)$, which is adapted to the parabolic structure. Such a metric is unique up to constant multiplication.*

Proof. — We may assume that D is ample. We can also assume that L is sufficiently ample as in Proposition 3.29. The uniqueness follows from the more general result (Proposition 5.2). We use an induction on $n=\dim X$. We have already known the existence for $n=2$ (Theorem 9.1).

Let $(\boldsymbol{E}_*,\theta)$ be a regular filtered Higgs bundle on (X,D). Assume that it is stable with $\text{par-deg}_L(\boldsymbol{E}_*)=\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*)=0$. For any element $s\in\mathbb{P}:=\mathbb{P}(H^0(X,L)^\vee)$ determines the hypersurface $Y_s=\{x\in X \mid s(x)=0\}$. The subset $\mathcal{X}_L\subset X\times\mathbb{P}$ is given by $\mathcal{X}_L:=\{(x,s)\mid x\in Y_s\}$. Let $\mathcal{U}$ be a Zariski open subset of $\mathbb{P}$ which consists of $s\in\mathbb{P}$ such that $(\boldsymbol{E}_*,\theta)_{|Y_s}$ is μ_L-stable. Since L is assumed to be sufficiently ample, $\mathcal{U}$ is not empty (Proposition 3.29). The image of the naturally defined map $\mathcal{X}_L\times_{\mathbb{P}}\mathcal{U}\longrightarrow X$ is Zariski open in X. The complement is denoted by W which consists of, at most, finite points of X due to the ampleness of L.

Let s be any element of $\mathcal{U}$. We have a pluri-harmonic metric h_s of $(E,\theta)_{|Y_s}$, which is adapted to the induced parabolic structure, due to the hypothesis of the induction.

Let s_i $(i=1,2)$ be elements of $\mathcal{U}$ such that Y_{s_1} and Y_{s_2} are transversal and that $Y_{s_1,s_2}:=Y_{s_1}\cap Y_{s_2}$ is transversal to D. We remark that $\dim Y_{s_1}\cap Y_{s_2}\geq 1$. We may also assume that $({}_{\boldsymbol{c}}E,\theta)_{|Y_{s_1,s_2}}$ is μ_L-stable (Proposition 3.29). Hence $h_{s_1\,|\,Y_{s_1,s_2}}$ and $h_{s_2\,|\,Y_{s_1,s_2}}$ are same up to constant multiplication. Then, we obtain the metric h of $E_{|X-(D\cup W)}$ such that $h_{|Y_s}=h_s$.

Let P be any point of $X-(D\cup W)$. We can take a coordinate neighbourhood $(U_P,z_1,\dots,z_n)$ around P such that (i) each hypersurface $\{z_i=a\}$ of U_P is a part of some Y_s, (ii) $U_P\subset X-(D\cup W)$. In the following, we will shrink U_P without mentioning. Since the restriction of h to $\{z_i=a\}$ is pluri-harmonic, we obtain the boundedness of θ and $\theta^\dagger$ with respect to h around P. (See Proposition 2.10, for example.)

For any $Q \in U_P$, let us take a path γ connecting P and Q, which is contained in some Y_s. Then, the parallel transport $\Pi_{P,Q} : E_{|P} \longrightarrow E_{|Q}$ is induced from the flat connection associated to the harmonic bundle $(E, \overline{\partial}_E, \theta)_{|Y_s}$ with $h_{|Y_s}$. The map $\Pi_{P,Q}$ is independent of the choice of γ and Y_s. From the frame of $E_{|P}$, we obtain the frame $\boldsymbol{v} = (v_1, \ldots, v_r)$ of $E_{|U_P}$. The trivialization gives the structure of flat bundle to $E_{|U_P}$. For the distinction, we use the notation (V, ∇) to denote the obtained flat bundle. The restriction of h, θ and $\theta^\dagger$ to U_P are denoted by the same notation. By the flat structure, we can regard the metric h as the map $\varphi_h : U_P \longrightarrow \mathrm{GL}(n)/U(n)$, and $\theta + \theta^\dagger$ can be regarded as the differential of the map. Let $d_{\mathrm{GL}(n)/U_n}$ denote the invariant distance of $\mathrm{GL}(n)/U_n$. Due to the boundedness of $\theta + \theta^\dagger$ with respect to h, there exists a constant C such that $d_{\mathrm{GL}(n)/U(n)}\big(\varphi_h(\gamma(0)), \varphi_h(\gamma(1))\big)$ is less than C times the length of γ for any path γ contained in some Y_s. In particular, h is a continuous metric of V.

Let H be the hermitian-matrices valued function whose (i,j)-th component is $h(v_i, v_j)$. Let $\Theta = (\Theta_{i,j})$ and $\Theta^\dagger = (\Theta^\dagger_{i,j})$ be determined by $\theta v_i = \sum \Theta_{j,i} \cdot v_j$ and $\theta^\dagger v_i = \sum \Theta^\dagger_{j,i} \cdot v_j$. We have $d\overline{H} = \overline{H}(\Theta + \Theta^\dagger)/2$ and $\overline{\partial}\Theta + [\Theta^\dagger, \Theta] = 0$ for the point-wise partial derivatives, which can be shown by considering the restriction of $(E, \overline{\partial}_E, h, \theta)$ to hyperplanes $\{z_i = a\}$. The equality holds as distributions, which follows from Fubini's theorem and the boundedness of $\overline{H}$, Θ and $\Theta^\dagger$. In particular, H and Θ are locally L^p_1, and hence $\Theta^\dagger$ is also locally L^p_1. By a standard boot strapping argument, we obtain that H, Θ and $\Theta^\dagger$ are C^∞ functions. In other words, h is a C^∞-metric of V, and $\theta^\dagger$ is a C^∞-section of $\mathrm{End}(V) \otimes \Omega^{0,1}$. We also obtain that the C^∞-structure of E and V are same because of $\overline{\partial}_E = d''_V - \theta^\dagger$, where d''_V denotes the $(0,1)$-part of ∇. Thus, we obtain that h is a C^∞-metric of $E_{|X-(D\cup W)}$. The pluri-harmonicity of h is easily obtained.

Let P be any point of W. We take a holomorphic coordinate neighbourhood $(U_P, z_1, \ldots, z_n)$ around P such that $z_i(P) = 0$ for any i and $U_P \simeq \{(z_1, \ldots, z_n) \,|\, |z_i| < 1\}$ via the coordinate. We assume $U_P \cap W = \{P\}$, and we put $U_P^* := U_P - \{P\}$. Let π_i denote the projection of U_P onto $Z := \{(w_1, \ldots, w_{n-1}) \,|\, |w_j| < 1\}$ by forgetting the i-th component. The origin of Z is denoted by O. We have the expression $\theta_{|U_P} = \sum_{i=1}^n f_i \cdot dz_i$. Since the eigenvalues of f_i are bounded on U_P, there exists a constant $C > 0$ such that $|f_{i\,|\,\pi_i^{-1}(Q)}|_h \leq C$ for any $Q \in Z$ such that $Q \neq O$ and for any i. By the continuity, we obtain $|f_i|_h \leq C$ on U_P^*. Hence $\theta + \theta^\dagger$ is bounded on U_P^*.

We have the flat bundle $V := E_{|U_P^*}$ with $\nabla := \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$. It is naturally extended to the flat bundle $(\widetilde{V}, \widetilde{\nabla})$ on U_P, and we can take a flat trivialization $\boldsymbol{v}$ of $\widetilde{V}$. Let H, Θ and $\Theta^\dagger$ are given on U_P^* as above. They are bounded. We have the relation $d\overline{H} = \overline{H} \cdot (\Theta + \Theta^\dagger)/2$ and $\overline{\partial}\Theta + [\Theta^\dagger, \Theta] = 0$ on U_P^*. The equality holds as distributions on U_P, which follows from Fubini's theorem and the boundedness of $\overline{H}$, Θ and $\Theta^\dagger$. By using an elliptic regularity argument, it can be shown that H, Θ and $\Theta^\dagger$ are C^∞. Let d''_V denote the $(0,1)$-part of the flat connection of $\widetilde{V}$. We have

$(\widetilde{V}, d''_V - \theta^\dagger)_{|U_P^*} \simeq (E, \overline{\partial}_E)_{|U_P^*}$ which is extended to the isomorphism $(\widetilde{V}, d''_V - \theta^\dagger) \simeq (E, \overline{\partial}_E)_{|U_P}$. Namely, $h_{|U_P^*}$ is naturally extended to the C^∞-metric of $E_{|U_P}$. Thus we obtain the tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$.

Let $\boldsymbol{c}$ be any element of $\boldsymbol{R}^S$. We obtain the parabolic Higgs bundle $({}_{\boldsymbol{c}}E(h)_*, \theta)$ on (X, D). (See Section 3.5 for the prolongment.)

Lemma 9.5. — *There exists a closed subset $W' \subset D$ with the following properties:*

- *The codimension of W' in X is larger than 2.*
- *The identity of E is extended to the holomorphic isomorphism ${}_{\boldsymbol{c}}E_{|X-W'} \longrightarrow {}_{\boldsymbol{c}}E(h)_{|X-W'}$.*

Proof. — Let P be any general point of the smooth part of D. We can take a holomorphic coordinate neighbourhood $(U_P, z_1, \dots, z_n)$ around P such that (i) U_P is isomorphic to $\{(z_1, \dots, z_n) \,|\, |z_i| < 1\}$ via the coordinate, (ii) $z_1^{-1}(0) = D \cap U$, (iii) each $\pi_1^{-1}(Q)$ $(Q \in Z)$ is a part of Y_s $(s \in \mathcal{U})$, where π_1 denotes the projection of U_P onto $Z := \{(z_2, \dots, z_n) \,|\, |z_i| < 1\}$. Let f be a holomorphic section of ${}_{\boldsymbol{c}}E$ on U_P. By the construction of the metric h, each restriction $f_{|\pi_1^{-1}(Q)}$ $(Q \in Z)$ gives the local section of ${}_{\boldsymbol{c}}(E_{|\pi_1^{-1}(Q)})(h)$. By using Corollary 2.53 in [**44**], we can show that f gives the section of ${}_{\boldsymbol{c}}E(h)$ on U_P. Thus, the identity of E on $U_P \setminus D$ is naturally extended to the morphism $\varphi : {}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}}E(h)$ around P. It is also easy to check the surjectivity of the specialization $\varphi_{|P}$ at P. Since both of ${}_{\boldsymbol{c}}E$ and ${}_{\boldsymbol{c}}E(h)$ are locally free, φ is isomorphic around P. □

Since both of ${}_{\boldsymbol{c}}E$ and ${}_{\boldsymbol{c}}E(h)$ are locally free, they are isomorphic. In particular, we can conclude that h is adapted to the parabolic structure. □

CHAPTER 10

TORUS ACTION AND THE DEFORMATION OF REPRESENTATIONS

We see that any flat bundle on a smooth irreducible quasiprojective variety can be deformed to a Variation of Polarized Hodge Structure. We can derive a result on the fundamental group.

We owe the essential ideas in this chapter to Simpson [**55**]. In fact, our purpose is to show a natural generalization of his results on smooth projective varieties. We will use his ideas without mentioning his name. This section is included for a rather expository purpose.

10.1. Torus Action on the Moduli Space of Representations

10.1.1. Notation. — We begin with a general remark. Let V and V' be vector spaces over $\boldsymbol{C}$, and $\Phi : V \longrightarrow V'$ be a linear isomorphism. Let Γ be any group, and $\rho : \Gamma \longrightarrow \mathrm{GL}(V)$ be a homomorphism. Then Φ and ρ induce the homomorphism $\Gamma \longrightarrow \mathrm{GL}(V')$, which is denoted by $\Phi_*(\rho)$. We also use the notation in Subsection 2.8.

10.1.2. Continuity. — Let X be a smooth irreducible projective variety with a polarization L, and D be a normal crossing divisor. Let x be a point of $X - D$. We put $\Gamma := \pi_1(X - D, x)$. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-polystable regular filtered Higgs bundle on (X, D) with trivial characteristic numbers. We put $E := \boldsymbol{E}_{|X-D}$. Since $(\boldsymbol{E}_*, t \cdot \theta)$ are also μ_L-polystable, we have a pluri-harmonic metric h_t for $(E, \overline{\partial}_E, t \cdot \theta)$ on $X - D$ adapted to the parabolic structure, due to Theorem 9.4. Therefore, we obtain the family of the representations $\rho'_t : \Gamma \longrightarrow \mathrm{GL}(E_{|x})$ $(t \in \boldsymbol{C}^*)$. We remark that ρ'_t are independent of the choice of pluri-harmonic metrics h_t.

Let V be a $\boldsymbol{C}$-vector space whose rank is same as $\operatorname{rank} E$. Let h_V be a hermitian vector space of V. For any $t \in \boldsymbol{C}^*$, we take isometries $\Phi_t : (E_{|x}, h_{t\,|\,x}) \longrightarrow (V, h_V)$, and then we obtain the family of representations $\rho_t := \Phi_{t\,*}(\rho'_t) \in R(\Gamma, \mathrm{GL}(V))$. We remark that $\pi_{\mathrm{GL}(V)}(\rho_t)$ are independent of choices of Φ_t. Thus we obtain the map $\mathcal{P} : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$ by $\mathcal{P}(t) = \pi_{\mathrm{GL}(V)}(\rho_t)$.

Theorem 10.1. — *The induced map $\mathcal{P}$ is continuous.*

Proof. — We may and will assume that $(\boldsymbol{E}_*,\theta)$ is μ_L-stable for the proof. Let $\{t_i\in \boldsymbol{C}^*\,|\,i\in\mathbb{Z}_{>0}\}$ be a sequence converging to t_0. We have only to take a subsequence $\{t_i\,|\,i\in S\}$ and a sequence of isometries $\bigl\{\Psi_i:\bigl(E_{|x},h_{t_i\,|\,x}\bigr)\longrightarrow\bigl(E_{|x},h_{t_0\,|\,x}\bigr)\,\big|\,i\in S\bigr\}$ such that $\bigl\{\Psi_{i\,*}(\rho_{t_i})\,\big|\,i\in S\bigr\}$ converges to ρ_{t_0}. Since the sections $\det(T-t_i\cdot\theta)$ of $\operatorname{Sym}^{\cdot}\Omega^{1,0}[T]$ converges to $\det(T-t_0\cdot\theta)$, we may apply Theorem 8.1. Hence there exists a subsequence $\bigl\{t_i\,\big|\,i\in S'\bigr\}$ such that $\bigl\{\bigl(E,\overline{\partial}_E,h_{t_i},t_i\cdot\theta_i\bigr)\,\big|\,i\in S'\bigr\}$ converges to a tame harmonic bundle $(E',\overline{\partial}_{E'},h',\theta')$ in L_2^p locally on $X-D$ via some isometries $\Phi_i:(E,h_{t_i})\longrightarrow(E',h')$ $(i\in S')$. It is easy to see that the representations $\Phi_{i\,|\,x\,*}(\rho_{t_i})$ converges to ρ' in $R(\Gamma,E'_{|x},h'_{|x})$, where ρ' is associated to the flat connection $\overline{\partial}_{E'}+\partial_{E'}+\theta'+\theta'^{\,\dagger}$.

We also have the non-trivial holomorphic map $f:{}_{\boldsymbol{c}}E'\longrightarrow{}_{\boldsymbol{c}}E$ which is compatible with the parabolic structure and the Higgs fields due to Theorem 8.1. Since $({}_{\boldsymbol{c}}E'_*,\theta')$ is μ_L-polystable and $({}_{\boldsymbol{c}}E_*,t_0\cdot\theta)$ is μ_L-stable, the map f is isomorphic. Then we have $f_{|x\,*}(\rho')=\rho_{t_0}$. By replacing f appropriately, we may assume $f:E'\longrightarrow E$ is isometric with respect to h' and h_{t_0}. Hence $\Psi_i:=\bigl(f\circ\Phi_i\bigr)_{|x}$ gives the desired isometries. Thus Theorem 10.1 is proved. □

10.1.3. Limit

Lemma 10.2. — *$\mathcal{P}\bigl(\{t\in\boldsymbol{C}^*\,|\,|t|<1\}\bigr)$ is relatively compact in $M(\Gamma,V,h_V)$.*

Proof. — The sequence of sections $\det(T-t\cdot\theta)$ of $\operatorname{Sym}^{\cdot}\Omega^{1,0}[T]$ clearly converges to $T^{\operatorname{rank}E}$ when $t\to0$. Hence we may apply the first claim of Theorem 8.1, and we obtain a subsequence $\{t_i\}$ converging to 0 such that $\bigl\{(E,\overline{\partial}_E,t_i\cdot\theta,h_{t_i})\bigr\}$ converges to a tame harmonic bundle $(E',\overline{\partial}_{E'},\theta',h')$ weakly in L_2^p locally on $X-D$. Then we easily obtain the convergence of the sequence $\bigl\{\pi_{\mathrm{GL}(V)}(\rho_{t_i})\bigr\}$ in $M(\Gamma,V,h_V)$. □

Ideally, the sequence $\{\mathcal{P}(t)\}$ should converge in $t\to0$, and the limit should come from a Variation of Polarized Hodge Structure. We discuss only a partial but useful result about it.

Let us recall relative Higgs sheaves. In the following, we put $\boldsymbol{C}_t:=\operatorname{Spec}\boldsymbol{C}[t]$ and $\boldsymbol{C}_t^*:=\operatorname{Spec}\boldsymbol{C}[t,t^{-1}]$. For a smooth morphism $Y_1\longrightarrow Y_2$, the sheaf of relative holomorphic $(1,0)$-forms are denoted by $\Omega^{1,0}_{Y_1/Y_2}$. We put $\mathfrak{X}:=X\times\boldsymbol{C}_t$ and $\mathfrak{X}^*:=X\times\boldsymbol{C}_t^*$. Similarly, $\mathfrak{D}:=D\times\boldsymbol{C}_t$ and $\mathfrak{D}^*:=D\times\boldsymbol{C}_t^*$. We put ${}_{\boldsymbol{c}}\widetilde{E}_*:={}_{\boldsymbol{c}}E_*\otimes\mathcal{O}_{\boldsymbol{C}_t^*}$ which is $\boldsymbol{c}$-parabolic bundle on $(\mathfrak{X}^*,\mathfrak{D}^*)$. Then, $t\cdot\theta$ gives the relative Higgs field $\widetilde{\theta}$, which is a homomorphism ${}_{\boldsymbol{c}}\widetilde{E}_*\longrightarrow{}_{\boldsymbol{c}}\widetilde{E}_*\otimes\Omega^{1,0}_{\mathfrak{X}^*/\boldsymbol{C}_t^*}(\log\mathfrak{D}^*)$ such that $\widetilde{\theta}^2=0$. Using the standard argument of S. Langton [**33**], we obtain the $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}\widetilde{E}'_*$ and relative Higgs field $\widetilde{\theta}':{}_{\boldsymbol{c}}\widetilde{E}'_*\longrightarrow{}_{\boldsymbol{c}}\widetilde{E}'_*\otimes\Omega^{1,0}_{\mathfrak{X}/\boldsymbol{C}_t}$ satisfying the following (see [**65**]):

– ${}_{\boldsymbol{c}}\widetilde{E}'_*$ is flat over $\boldsymbol{C}_t$, and the restriction to $\mathfrak{X}^*$ is ${}_{\boldsymbol{c}}\widetilde{E}_*$.
– The restriction of $\widetilde{\theta}'$ to $\mathfrak{X}^*$ is $\widetilde{\theta}$.
– $({}_{\boldsymbol{c}}\widehat{E}'_*,\widehat{\theta}'):=\bigl({}_{\boldsymbol{c}}\widetilde{E}'_*,\widetilde{\theta}'\bigr)_{|X\times\{0\}}$ is μ_L-semistable.

Let $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ denote the reflexive saturated regular filtered Higgs sheaf associated to $({}_{\boldsymbol{c}}\widehat{E}', \widehat{\theta}')$. (See Lemma 3.2.) We put $\widehat{E} := {}_{\boldsymbol{c}}\widehat{E}_{|X-D}$.

Proposition 10.3. — *Assume that* $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ *is* μ_L*-stable.*

- $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ *is a Hodge bundle, i.e.,* $({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta}) \simeq ({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ *for any* $\alpha \in \boldsymbol{C}^*$.
- *We have a pluri-harmonic metric* $\widehat{h}$ *of a Hodge bundle* $(\widehat{E}, \widehat{\theta})$ *on* $X-D$*, which is adapted to the parabolic structure. It induces the Variation of Polarized Hodge Structure. Thus we obtain the corresponding representation* $\widehat{\rho} : \pi_1(X-D, x) \longrightarrow \mathrm{GL}(\widehat{E}_{|x})$ *which underlies a Variation of Polarized Hodge Structure.*
- *Take any isometry* $G : (\widehat{E}_{|x}, \widehat{h}_{|x}) \simeq (V, h_V)$*. Then the sequence* $\{\pi_{\mathrm{GL}(V)}(\rho_t)\}$ *converges to* $\pi_{\mathrm{GL}(V)}\big(G_*(\widehat{\rho})\big)$ *in* $M(\Gamma, V, h_V)$ *for* $t \to 0$.
- *In particular, the map* $\pi_{\mathrm{GL}(V)}(\rho_t) : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$ *is continuously extended to the map of* $\boldsymbol{C}$ *to* $M(\Gamma, V, h_V)$.

Proof. — The argument is essentially due to Simpson [**55**]. The fourth claim follows from the third one. Let $\{t_i \,|\, i \in \mathbb{Z}_{>0}\}$ be a sequence converging to 0. Due to Theorem 8.1, there exists a subsequence $\{t_i \,|\, i \in S\}$ such that the sequence $\big\{(E, \overline{\partial}_E, h_{t_i}, t_i\cdot\theta) \,\big|\, i \in S\big\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, h', \theta')$ weakly in L_2^p locally on $X-D$, via isometries $\Phi_i : (E, h_{t_i}) \longrightarrow (E', h')$. Let $\rho' : \pi_1(X-D, x) \longrightarrow \mathrm{GL}(E'_{|x})$ denote the representation associated to the flat connection $\overline{\partial}_{E'} + \partial_{E'} + \theta' + \theta'^{\dagger}$. Then we have the convergence of $\big\{\Phi_{i|x\,*}(\rho_{t_i}) \,\big|\, i \in S''\big\}$ to ρ' in $M(\Gamma, \widehat{E}_{|x}, \widehat{h}_{|x})$. Due to Theorem 8.1, we also have a non-trivial morphism $f : {}_{\boldsymbol{c}}\widehat{E}' \longrightarrow {}_{\boldsymbol{c}}E'$ which is compatible with the parabolic structures and the Higgs fields. It induces the morphism ${}_{\boldsymbol{c}}\widehat{E} \longrightarrow {}_{\boldsymbol{c}}E'$ compatible with the parabolic structures and the Higgs fields. Then it must be isomorphic due to μ_L-polystability of $({}_{\boldsymbol{c}}E'_*, \theta')$ and μ_L-stability of $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$. In particular, $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ is a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle. The metric $\widehat{h}$ of $\widehat{E}$ is given by h' and f. Thus the third claim is obtained.

Let us consider the morphism $\phi_\alpha : \boldsymbol{C}_t \longrightarrow \boldsymbol{C}_t$ given by $t \longmapsto \alpha\cdot t$. We have the natural isomorphism $\phi_\alpha^*\big({}_{\boldsymbol{c}}\widetilde{E}_*, \widetilde{\theta}\big) \simeq \big({}_{\boldsymbol{c}}\widetilde{E}_*, \alpha\cdot\widetilde{\theta}\big)$ which can be extended to the morphism $\phi_\alpha^*({}_{\boldsymbol{c}}\widetilde{E}'_*, \widetilde{\theta}') \longrightarrow ({}_{\boldsymbol{c}}\widetilde{E}'_*, \alpha\cdot\widetilde{\theta}')$ such that the specialization $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta}) \longrightarrow ({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta})$ at $t=0$ is not trivial. Since $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ and $({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta})$ are μ_L-stable, the map is isomorphic. Hence $({}_{\boldsymbol{c}}\widehat{E}, \widehat{\theta})$ is a Hodge bundle. Thus the first is proved.

Since $(\widehat{E}, \overline{\partial}_{\widehat{E}}, \widehat{\theta})$ is a Hodge bundle, we have the action κ of $S^1 = \{t \in \boldsymbol{C} \,|\, |t| = 1\}$ on $\widehat{E}$ such that $\kappa(t) : \big(\widehat{E}, \overline{\partial}_{\widehat{E}}, \widehat{\theta}\big) \simeq \big(\widehat{E}, \overline{\partial}_{\widehat{E}}, t\cdot\widehat{\theta}\big)$ for any $t \in S^1$. The metric $\kappa(t)_*\widehat{h}$ is determined by $\kappa(t)_*\widehat{h}(u,v) = \widehat{h}\big(\kappa(t)(u), \kappa(t)(v)\big)$, which is also the pluri-harmonic metric of $(\widehat{E}, \overline{\partial}_{\widehat{E}}, t\cdot\theta)$. Since $(\boldsymbol{E}_*, t\cdot\widehat{\theta})$ is μ_L-stable, the pluri-harmonic metric is unique up to a positive constant multiplication. Hence we obtain the map $\nu : S^1 \longrightarrow \boldsymbol{R}_{>0}$ such that $\kappa(t)_*\widehat{h} = \nu(t)\cdot\widehat{h}$. Let $\widehat{E} = \bigoplus \widehat{E}_w$ be the weight decomposition. For $v_i \in \widehat{E}_{w_i}$ $(w_1 \neq w_2)$, we have $\nu(t)\cdot\widehat{h}(v_1, v_2) = \kappa(t)_*\widehat{h}(v_1, v_2) = t^{w_1-w_2}\widehat{h}\big(v_1, v_2\big)$. Hence, we obtain $\widehat{h}(v_1, v_2) = 0$ and $\nu(t) = 1$. Namely, $\widehat{h}$ is S^1-invariant, which means

$\big(\widehat{E},\overline{\partial}_{\widehat{E}},\widehat{\theta},\widehat{h}\big)$ gives a Variation of Polarized Hodge Structure. Thus the second claim is proved. □

Lemma 10.4. — *Assume $({}_{\boldsymbol{c}}\widehat{E}_*,\widehat{\theta})$ is not μ_L-stable. Let ρ_0 be an element of $R(\Gamma,V)$ such that $\pi_{\mathrm{GL}(V)}(\rho_0)$ is the limit of a subsequence $\big\{\pi_{\mathrm{GL}(V)}(\rho_{t_i})\big\}$ for $t_i\to 0$. Then ρ_0 is not simple.*

Proof. — Let $\{t_i\}$ be a sequence converging to 0 such that $\big\{(E,\overline{\partial}_E,t_i\cdot\theta,h_{t_i})\big\}$ converges to a tame harmonic bundle $(E',\overline{\partial}_{E'},\theta',h')$ in L_2^p locally on $X-D$. We may assume that ρ_0 is the associated representation to $(E',\overline{\partial}_{E'},\theta',h')$. We have a non-trivial map $f:{}_{\boldsymbol{c}}E'\longrightarrow{}_{\boldsymbol{c}}\widehat{E}$ compatible with the parabolic structures and the Higgs fields. If ρ_0 is simple, then $({}_{\boldsymbol{c}}E'_*,\theta')$ is μ_L-stable, and it can be shown that the map f has to be isomorphic. But it contradicts with the assumption that $({}_{\boldsymbol{c}}\widehat{E}_*,\widehat{\theta})$ is not μ_L-stable. □

10.1.4. Deformation to a Variation of Polarized Hodge Structure. — Let Y be a smooth irreducible quasiprojective variety over $\boldsymbol{C}$ with a base point x. We may assume $Y=X-D$, where X and D denote a smooth projective variety and its simple normal crossing divisor, respectively. A representation $\rho:\pi_1(Y,x)\longrightarrow\mathrm{GL}(V)$ induces a flat bundle (E,∇). We say that ρ comes from a Variation of Polarized Hodge Structure, if (E,∇) underlies a Variation of Polarized Hodge Structure. For simplicity of the notation, we put $\Gamma:=\pi_1(Y,x)$.

Theorem 10.5. — *Let $\rho\in R(\Gamma,V)$ be a representation. Then it can be deformed to a representation $\rho'\in R(\Gamma,V)$ which comes from a Variation of Polarized Hodge Structure on Y.*

Proof. — We essentially follow the argument of Theorem 3 in [**55**]. Any representation $\rho\in R(\Gamma,V)$ can be deformed to a semisimple representation $\rho'\in R(\Gamma,V)$. Therefore we may assume that ρ is semisimple from the beginning. Let (E,∇) be the corresponding semisimple flat bundle on $X-D$. We can take a Corlette-Jost-Zuo metric h of (E,∇), and hence we obtain the tame pure imaginary harmonic bundle $(E,\overline{\partial}_E,\theta,h)$. Let $(\boldsymbol{E}_*,\theta)$ denote the associated regular filtered Higgs bundle on (X,D). We have the canonical decomposition (Corollary 3.11):

$$(\boldsymbol{E}_*,\theta)=\bigoplus_{j\in\Lambda}(\boldsymbol{E}_{i*},\theta_i)\otimes\boldsymbol{C}^{m(j)}.$$

We put $r(\rho):=\sum_{j\in\Lambda}m(j)$. Note that $r(\rho)\leq\operatorname{rank}E$, and we have $r(\rho)=\operatorname{rank}E$ if and only if $(\boldsymbol{E}_*,\theta)$ is a direct sum of Higgs bundles of rank one. We use a descending induction on $r(\rho)$.

We obtain the family of regular filtered Higgs bundles $\big\{(\boldsymbol{E}_*,t\cdot\theta)\,\big|\,t\in\boldsymbol{C}^*\big\}$ $(t\in\boldsymbol{C}^*)$. In particular, we have the associated deformation of representations $\{\rho_t\in R(\Gamma,V)\,|\,t\in\boldsymbol{R}_{>0}\}$ as in Subsection 10.1.2. We may assume $\rho_1=\rho$. We have the induced map $\mathcal{P}:]0,1]\longrightarrow M(\Gamma,V,h_V)$ given by $\mathcal{P}(t):=\pi_{\mathrm{GL}(V)}(\rho_t)$, which is continuous due to

Theorem 10.1. The image is relatively compact due to Lemma 10.2. We take a representation $\rho_0 \in R(\Gamma, V)$ such that $\pi_{\mathrm{GL}(V)}(\rho_0)$ is the limit of a subsequence of $\{\pi_{\mathrm{GL}(V)}(\rho_t) \,\big|\, t \in]0,1]\}$. We may assume that it comes from a tame harmonic bundle as in the proof of Lemma 10.2.

The case 1. Let $(\boldsymbol{E}_*, \theta) = \bigoplus (\boldsymbol{E}_{i\,*}, \theta_i)^{\oplus m_i}$ be the canonical decomposition. Assume that each family $\{(\boldsymbol{E}_{i\,*}, t\cdot\theta_i) \,|\, t \in \boldsymbol{C}^*\}$ converges to the μ_L-stable regular filtered Higgs sheaf. Then ρ_0 comes from a Variation of Polarized Hodge Structure due to Proposition 10.3.

We remark that the rank one Higgs bundle is always stable. Hence the case $r(\rho) = \operatorname{rank} E$ is done, in particular.

The case 2. Assume that one of the families $\{(\boldsymbol{E}_*, t\cdot\theta_i) \,|\, t \in \boldsymbol{C}^*\}$ converges to the semistable parabolic Higgs sheaf, which is not μ_L-stable. Then we have $r(\rho) < r(\rho_0)$ due to Lemma 10.4. Hence the induction can proceed. □

10.2. Monodromy Group

We discuss the monodromy group for the Higgs bundles or flat bundles, by following the ideas in [**55**].

10.2.1. The Higgs monodromy group.

— Let X be a smooth irreducible projective variety with an ample line bundle L, and D be a simple normal crossing divisor. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-polystable regular filtered Higgs bundle on (X, D) with trivial characteristic numbers. For any non-negative integers a and b, we have the regular filtered Higgs bundles $(T^{a,b}\boldsymbol{E}_*, \theta)$. (See Subsection 3.2.1 for the explanation.) Since we have a pluri-harmonic metric h of $(E, \overline{\partial}_E, \theta)$ adapted to the parabolic structure, the regular filtered Higgs bundles $T^{a,b}(\boldsymbol{E}_*, \theta)$ are also μ_L-polystable. In particular, we have the canonical decompositions of them. We recall the definition of the Higgs monodromy group given in [**55**]. Let x be a point of $X - D$.

Definition 10.6. — The Higgs monodromy group $M(\boldsymbol{E}_*, \theta, x)$ of μ_L-polystable Higgs bundle $(\boldsymbol{E}_*, \theta)$ is the subgroup of $\mathrm{GL}(E_{|x})$ defined as follows: An element $g \in \mathrm{GL}(E_{|x})$ is contained in $M(\boldsymbol{E}_*, \theta, x)$, if and only if $T^{a,b}g$ preserves the subspace $F_{|x} \subset T^{a,b}E_{|x}$ for any stable component $(\boldsymbol{F}_*, \theta_F) \subset T^{a,b}(\boldsymbol{E}_*, \theta)$.

Remark 10.7. — Although such a Higgs monodromy group should be defined for semistable parabolic Higgs bundles as in [**55**], we do not need it in this paper.

We have an obvious lemma.

Lemma 10.8. — *We have $M(\boldsymbol{E}_*, \theta, x) = M(\boldsymbol{E}_*, t\cdot\theta, x)$ for any $t \in \boldsymbol{C}^*$, i.e., the Higgs monodromy group is invariant under the torus action.*

Let us take a pluri-harmonic metric h of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ on $X - D$, which is adapted to the parabolic structure. Then we obtain the flat connection $\mathbb{D}^1 = \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$. Then we obtain the monodromy group $M(E, \mathbb{D}^1, x) \subset \mathrm{GL}(E_{|x})$ of the flat connection. (See Subsection A.1.4.)

Lemma 10.9. — *We have $M(E, \mathbb{D}^1, x) \subset M(\boldsymbol{E}_*, \theta, x)$. For a tame pure imaginary harmonic bundle, we have $M(E, \mathbb{D}^1, x) = M(\boldsymbol{E}_*, \theta, x)$.*

Proof. — A stable component $(\boldsymbol{F}_*, \theta_F) \subset (\boldsymbol{E}_*, \theta)$ induces the flat subbundle of $F \subset T^{a,b}(E, \mathbb{D}^1)$. If $g \in M(E, \mathbb{D}^1, x)$, we have $T^{a,b} g(F_{|x}) \subset F_{|x}$. Hence, $M(E, \mathbb{D}^1, x) \subset M(\boldsymbol{E}_*, \theta, x)$. In the pure imaginary case, a flat subbundle $F \subset T^{a,b}(E, \mathbb{D}^1)$ induces $(\boldsymbol{F}_*, \theta_F) \subset (\boldsymbol{E}_*, \theta)$. Therefore, we obtain $M(E, \mathbb{D}^1, x) = M(\boldsymbol{E}_*, \theta, x)$. □

10.2.2. The deformation and the monodromy group. — For simplicity of the description, we put $\Gamma := \pi_1(X - D, x)$. Let (E, ∇) be a semisimple flat bundle over $X - D$. We have a Corlette-Jost-Zuo metric h of (E, ∇), and thus we obtain a tame pure imaginary harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$. The associated regular filtered Higgs bundle is denoted by $(\boldsymbol{E}_*, \theta)$, which is μ_L-polystable with trivial characteristic numbers.

As in Subsection 10.1.2, we have the pluri-harmonic metrics h_t for any $(E, \overline{\partial}_E, t \cdot \theta)$ $(t \in \boldsymbol{C}^*)$. Hence we obtain the flat connections $\mathbb{D}^1_t$ of E, and the representations $\rho_t : \Gamma \longrightarrow \mathrm{GL}(E_{|x})$. We also obtain the monodromy group $M(E, \mathbb{D}^1_t) \subset \mathrm{GL}(E_{|x})$.

Lemma 10.10. — *We have $M(E, \mathbb{D}^1_t) \subset M(E, \mathbb{D}^1_1)$ for $t \in \boldsymbol{C} - \{0\}$, and $M(E, \mathbb{D}^1_t) = M(E, \mathbb{D}^1_1)$ for $t \in \boldsymbol{R} - \{0\}$.*

Proof. — It follows from Lemma 10.8 and Lemma 10.9. □

We put $G_0 := M(E, \mathbb{D}^1_t, x)$ for $t \in \boldsymbol{R}_{>0}$ which is independent of the choice of t. Let $U(E, h_t, x)$ denote the unitary group for the metrized space $(E_{|x}, h_{t\,|\,x})$. Due to Lemma A.16, G_0 is reductive, and the intersection $K_{0,t} := G_0 \cap U(E, h_t, x)$ is a compact real form of G_0.

We put $V := E_{|x}$ and $h_V := h_{1\,|\,x}$. We denote G_0 and $K_{0,1}$ by G and K respectively, when we regard it as the subgroup of $\mathrm{GL}(V)$. Then we can take an isometry $\nu_t : (E_{|x}, h_{t\,|\,x}) \simeq (V, h_V)$ such that $\nu_t(G_0) = G$ and $\nu_t(K_{0\,t}) = K$ for each t. Such a map is unique up to the adjoint of $N_G(h_V)$. Thus we obtain the family of representations $\widetilde{\rho}_t := \nu_{t\,*}(\rho_t) \in R(\Gamma, G)$ $(t \in \boldsymbol{R}_{>0})$.

Lemma 10.11. — *The induced map $\pi_G(\widetilde{\rho}_t) : \boldsymbol{R}_{>0} \longrightarrow M(\Gamma, G, h_V)$ is continuous.*

Proof. — Let M' denote the subset of $M(\Gamma, G, h_V)$ which consists of the Zariski dense representations. The natural morphism $M' \longrightarrow M(\Gamma, V, h_V)$ is injective, and the image of $\pi_G(\widetilde{\rho}_t)$ is contained in M'. Hence the claim of the lemma follows from Theorem 10.1 and the properness of $M(\Gamma, G, h_V) \longrightarrow M(\Gamma, V, h_V)$. □

Lemma 10.12. — *The image* $\pi_G(\widetilde{\rho}_t)\big(]0,1]\big)$ *is relatively compact in* $M(\Gamma, G, h_V)$.

Proof. — It follows from Lemma 10.2 and the properness of the map $M(\Gamma, G, h_V) \longrightarrow M(\Gamma, V, h_V)$. □

10.2.3. Non-existence result about fundamental groups. — Let Y be a smooth irreducible quasiprojective variety. We put $\Gamma := \pi_1(Y, x)$. Let V be a finite dimensional $\boldsymbol{C}$-vector space. Let G be a reductive subgroup of $\mathrm{GL}(V)$. We see the convergence of $\pi_G(\widetilde{\rho}_t)$ $(t \to 0)$ in a simple case.

Lemma 10.13. — *Let* ρ *be an element of* $R(\Gamma, G)$. *We assume that there exists a subgroup* Γ_0 *such that* $\rho_{|\Gamma_0} : \Gamma_0 \longrightarrow G$ *is Zariski dense and rigid. Then we can take a deformation* $\rho' \in R(\Gamma, G)$ *of* ρ *which comes from a Variation of Polarized Hodge Structure on* Y.

Proof. — We take a tame pure imaginary pluri-harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ whose associated representation gives ρ, and we take the deformation $\pi_G(\widetilde{\rho}_t)$. Let us take $\rho_0 \in R(\Gamma, G)$ such that some sequence $\{\pi_G(\widetilde{\rho}_{t_i})\}$ converges to $\pi_G(\rho_0)$. We remark that $\rho_{0\,|\,\Gamma_0} : \Gamma_0 \longrightarrow G$ is also Zariski dense and rigid (Lemma 2.22). If ρ_0 comes from a Variation of Polarized Hodge Structure, we are done. If ρ_0 does not come from a Variation of Polarized Hodge Structure, we deform ρ_0 as above, again. The process will stop in the finite steps by Theorem 10.5. □

The following lemma is a straightforward generalization of Lemma 4.4 in [**55**]. (See also Lemma A.16, where we will see the argument of Lemma 4.4 can be generalized in our situation.)

Lemma 10.14. — *Let* $\rho : \Gamma \longrightarrow G$ *be a Zariski dense homomorphism. If* ρ *comes from a Variation of Polarized Hodge Structure, then the real Zariski closure* W *of* ρ *is a real form of* G, *and* W *is a group of Hodge type in the sense of Simpson. (See the page* 46 *in* [**55**].*)*

The following lemma is essentially same as Corollary 4.6 in [**55**].

Proposition 10.15. — *Let* G *be a complex reductive algebraic group, and* W *be a real form of* G. *Let* $\rho : \Gamma \longrightarrow G$ *be a representation such that* $\operatorname{Im}\rho \subset W$. *Assume that there exists a subgroup* $\Gamma_0 \subset \Gamma$ *such that* $\rho_{|\Gamma_0}$ *is rigid and Zariski dense in* G. *Then* W *is a group of Hodge type, in the sense of Simpson.*

Proof. — We reproduce the argument of Simpson. Since $\rho(\Gamma_0)$ is Zariski dense in G, W is also the real Zariski closure of $\rho(\Gamma_0)$. We take a deformation ρ' of ρ, which comes from a Variation of Polarized Hodge Structure as in Lemma 10.13. Then there exists an element $u \in N(G, U)$ such that $\mathrm{ad}(u) \circ \rho_{|\Gamma_0} \simeq \rho'_{|\Gamma_0}$ due to Lemma 2.22. Let W' denote the real Zariski closure of $\rho'(\Gamma_0)$, which is also the real Zariski closure of

ρ'. It is a group of Hodge type (Lemma 10.14). Since W and W' are isomorphic, we are done. □

Corollary 10.16. — *Let Γ_0 be a rigid discrete subgroup of a real algebraic group, which is not of Hodge type. Then Γ_0 cannot be a split quotient of the fundamental groups of any smooth irreducible quasiprojective variety.*

Proof. — It follows from Lemma 10.14 and Proposition 10.15. (See the pages 52–54 of [**55**]). □

APPENDIX

G-HARMONIC BUNDLE

A.1. G-Principal Bundles with Flat Structure or Holomorphic Structure

We recall the Tannakian consideration about harmonic bundles given in [**55**] by Simpson.

A.1.1. A characterization of algebraic subgroup of GL. — We recall some facts on algebraic groups. (See also I. Proposition 3.1 in [**11**], for example.) Let V be a vector space over a field k of characteristic 0. We put $T^{a,b}V := Hom(V^{\otimes a}, V^{\otimes b})$. Let G be an algebraic subgroup of $\mathrm{GL}(V)$, defined over k. We have the induced G-action on $T^{a,b}V$. Let $\mathcal{S}(V,a,b)$ denote the set of G-subspaces of $T^{a,b}V$, and we put $\mathcal{S}(V) = \coprod_{a,b} \mathcal{S}(V,a,b)$.

Let g be an element of $\mathrm{GL}(V)$. We have the induced element $T^{a,b}(g) \in \mathrm{GL}\big(T^{a,b}V\big)$. Then, it is known that $g \in \mathrm{GL}(V)$ is contained in G, if and only if $T^{a,b}(g)W \subset W$ holds for any $(W,a,b) \in \mathcal{S}(V)$. Suppose G is reductive. Then there is an element v of $T^{a,b}(V)$ for some (a,b) such that g is contained in G if and only if $g \cdot v = v$ holds.

We easily obtain a similar characterization of Lie subalgebras of $\mathfrak{gl}(V)$ corresponding to algebraic subgroups of $\mathrm{GL}(V)$.

A.1.2. A characterization of connections of principal G-bundle. — Let k denote the complex number field $\boldsymbol{C}$ or the real number field $\boldsymbol{R}$. Let G be an algebraic group over k. Let P_G be a G-principal bundle on a manifold X in the C^{∞}-category. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be a representation defined over k, such that the induced morphism $d\kappa : \mathfrak{g} \longrightarrow End(V)$ is injective. We put $E := P_G \times_G V$. We have $T^{a,b}E := Hom(E^{\otimes a}, E^{\otimes b}) \simeq P_G \times_G T^{a,b}V$. We have the subbundle $E_U = P_G \times_G U$ of $T^{a,b}E$ for each $U \in \mathcal{S}(V,a,b)$. A connection ∇ on E induces the connection $T^{a,b}\nabla$ on $T^{a,b}E$. Let $\mathcal{A}_G(E)$ be the set of the connections ∇ of E such that the induced connections $T^{a,b}\nabla$ preserve the subbundle E_U for any $(U,a,b) \in \mathcal{S}(V)$.

Let $\mathcal{A}(P_G)$ denote the set of the connections of P_G. If we are given a connection of P_G, the connection ∇ of E is naturally induced. It is clear that the connection

$T^{a,b}\nabla$ preserves $E_U \subset T^{a,b}E$ for any $(U,a,b) \in \mathcal{S}(V)$. Hence we have the map $\varphi : \mathcal{A}(P_G) \longrightarrow \mathcal{A}_G(E)$.

Lemma A.1. — *The map φ is bijective.*

Proof. — Since $d\kappa$ is injective, the map φ is injective. Let us take a connection $\nabla \in \mathcal{A}_G(E)$ and a connection ∇_0 which comes from a connection of P_G. Then $f = \nabla - \nabla_0$ is a section of $\mathrm{End}(E) \otimes \Omega^1$. Since $T^{a,b}f$ preserves E_U for any (a,b) and $U \subset \mathcal{S}(V,a,b)$, f comes from a section of $\mathrm{ad}(P_G) \otimes \Omega^1 \subset \mathrm{End}(E) \otimes \Omega^1$. □

A.1.3. K-Reduction of holomorphic G-principal bundle and the induced connection. — Let G be a linear reductive group defined over $\boldsymbol{C}$. Let P_G be a *holomorphic* G-principal bundle on X. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be a representation defined over $\boldsymbol{C}$, such that $d\kappa : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is injective. We put $E := P_G \times_G V$. Let K be a compact real form of G. Let $P_K \subset P_G$ be a K-reduction in the C^∞-category, i.e., $P_K \times_K G \simeq P_G$. Then the connection of P_K is automatically induced. We have the canonical G-decomposition for each (a,b):

$$T^{a,b}V = \bigoplus_{\rho \in \mathrm{Irrep}(G)} V_\rho^{(a,b)}. \tag{60}$$

Here $\mathrm{Irrep}(G)$ denotes the set of the equivalence classes of irreducible representations of G. Each $V_\rho^{(a,b)}$ is isomorphic to the tensor product of the irreducible representation ρ and the trivial representation $\boldsymbol{C}^{m(a,b,\rho)}$. The decomposition (60) is same as the canonical K-decomposition. Take a K-invariant hermitian metric h of V. It induces the hermitian metric $T^{a,b}h$ of $T^{a,b}V$, for which the decomposition (60) is orthogonal. The restriction of $T^{a,b}h$ to $V_\rho^{(a,b)}$ is isomorphic to a tensor product of a K-invariant hermitian metric on ρ and a hermitian metric on $\boldsymbol{C}^{m(a,b,\rho)}$. The metric h induces the hermitian metric of E, which is also denoted by h. From the holomorphic structure $\overline{\partial}_E$ and the metric h, we obtain the unitary connection $\nabla = \partial_E + \overline{\partial}_E$. The induced connection $T^{a,b}\nabla$ on $T^{a,b}E$ is the unitary connection determined by $T^{a,b}h$ and the holomorphic structure of $T^{a,b}E$. Then it is easy to see that $T^{a,b}\nabla$ preserves E_U for any $U \in \mathcal{S}(a,b,V)$. Hence the connection ∇ comes from P_G. Since ∇ also preserves the unitary structure, we can conclude that ∇ comes from the connection of P_K.

A.1.4. The monodromy group. — We recall the monodromy group of flat bundles ([**55**]). Let X be a connected complex manifold with a base point x. The monodromy group of a flat bundle (E,∇) at x is defined to be the Zariski closure of the induced representation $\pi_1(X,x) \longrightarrow \mathrm{GL}(E_{|x})$. It is denoted by $M(E,\nabla,x)$. Let us recall the case of principal bundles. Let G be a linear algebraic group over $\boldsymbol{R}$ or $\boldsymbol{C}$, and P_G be a G-principal bundle on X with a flat connection in the C^∞-category. Take a point $\widetilde{x} \in P_{G|x}$. Then we obtain the representation $\rho : \pi_1(X,x) \longrightarrow G$. Then the monodromy group $M(P_G,\widetilde{x}) \subset G$ is defined to be the Zariski closure of the image of ρ. We obtain the canonical reduction of principal bundles $P_{M(P_G,\widetilde{x})} \subset P_G$. The monodromy groups of flat vector bundles and flat principal bundles are related as follows. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be an injective representation. Then we have the flat bundle $E = P_G \times_G V = P_{M(P_G,\widetilde{x})} \times_{M(P_G,\widetilde{x})} V$. Via the identification $V = E_{|x}$ given by $\widetilde{x}$, we are given the inclusion $M(P_G, \widetilde{x}) \subset \mathrm{GL}(E_{|x})$. Clearly $M(P_G, \widetilde{x})$ is same as $M(E, \nabla, x)$ and it is independent of the choice of $\widetilde{x}$. Hence we can reduce the problems of the monodromy groups of flat principal G-bundles to those for flat vector bundles.

For a flat bundle (E, ∇), let $T^{a,b}E$ denote the flat bundle $Hom(E^{\otimes a}, E^{\otimes b})$ provided the canonically induced flat connection. Let $\mathcal{S}(E, a, b)$ denote the set of flat subbundles U of $T^{a,b}E$, and we put $\mathcal{S}(E) := \coprod_{(a,b)} \mathcal{S}(E, a, b)$. Let g be an element of $\mathrm{GL}(E_{|x})$. Then g is contained in $M(E, \nabla, x)$ if and only if $T^{a,b}g$ preserves U_x for any $(U, a, b) \in \mathcal{S}(E)$. If $M(E, \nabla, x)$ is reductive, we can find some (a, b) and $v \in T^{a,b}E_{|x}$ such that $g \in M(E, \nabla, x)$ if and only if $g \cdot v = v$. Hence there exists a flat subbundle $W \subset T^{a,b}E$ such that $g \in M(E, \nabla, x)$ if and only if $T^{a,b}g_{|W} = \mathrm{id}_W$.

A.2. Definitions

A.2.1. A G-principal Higgs bundle and a pluri-harmonic reduction. — Let G be a linear reductive group defined over $\boldsymbol{C}$, and K be a compact real form. Let X be a complex manifold and P_G be a holomorphic G-principal bundle on X. Let $\mathrm{ad}(P_G)$ be the adjoint bundle of P_G, i.e., $\mathrm{ad}(P_G) = P_G \times_G \mathfrak{g}$. Recall that a Higgs field of P_G is defined to be a holomorphic section θ of $\mathrm{ad}(P_G) \otimes \Omega^{1,0}$ such that $\theta^2 = 0$.

Let $P_K \subset P_G$ be a K-reduction of P_G in C^∞-category, then we have the natural connection ∇ of P_K, as is seen in Subsection A.1.3. We also have the adjoint $\theta^\dagger$ of θ, which is a C^∞-section of $\mathrm{ad}(P_G) \otimes \Omega^{0,1}$. Then we obtain the connection $\mathbb{D}^1 := \nabla + \theta + \theta^\dagger$ of the principal bundle P_G.

Definition A.2. — If $\mathbb{D}^1$ is flat, then the reduction $P_K \subset P_G$ is called pluri-harmonic, and the tuple $(P_K \subset P_G, \theta)$ is called a G-harmonic bundle.

Let V be a $\boldsymbol{C}$-vector space. A representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ is called immersive if $d\kappa$ is injective, in this paper. Take an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric h_V. From a G-principal Higgs bundle (P_G, θ) with a K-reduction $P_K \subset P_G$, we obtain the Higgs bundle $(E, \overline{\partial}_E, \theta)$ with the hermitian metric h.

Lemma A.3. — *Let (P_G, θ) be a G-principal Higgs bundle, and $P_K \subset P_G$ be a K-reduction. The following conditions are equivalent.*

1. *The reduction $P_K \subset P_G$ is pluri-harmonic.*
2. *For any representation $G \longrightarrow \mathrm{GL}(V)$ and any K-invariant hermitian metric of $\boldsymbol{C}$-vector space V, the induced Higgs bundle with the hermitian metric is a harmonic bundle.*

3. *There exist an immersive representation $G \longrightarrow \mathrm{GL}(V)$ and a K-invariant hermitian metric of $\boldsymbol{C}$-vector space V, such that the induced Higgs bundle with the hermitian metric is a harmonic bundle.*

Proof. — If $G \longrightarrow \mathrm{GL}(V)$ is immersive, then a connection of P_G is flat if and only if the induced connection on $P_G \times_G V$ is flat. Therefore the desired equivalence is clear. □

A.2.2. A flat G-bundle and a pluri-harmonic reduction. — Let G be a linear reductive group over $\boldsymbol{R}$ or $\boldsymbol{C}$, and let (P_G, ∇) be a flat G-bundle over a complex manifold X. If a K-reduction $P_K \subset P_G$ is given, we obtain the connection ∇_0 of P_K and the self-adjoint section $\varphi \in \mathrm{ad}(P_G) \otimes \Omega^1$ such that $\nabla = \nabla_0 + \varphi$ ([**7**]), which can be shown by a Tannakian consideration as in Subsection A.1.3, for example. Let $\nabla_0 = \nabla_0' + \nabla_0''$ and $\varphi = \theta + \theta^\dagger$ be the decomposition into the $(1,0)$-part and the $(0,1)$-part. The connection ∇_0 induces the connection on $\mathrm{ad}(P_G)$, which is also denoted by $\nabla_0 = \nabla_0' + \nabla_0''$. From ∇_0'' and the complex structure of X, the $(0,1)$-operator of $\mathrm{ad}(P_G) \otimes \Omega^{1,0}$ is induced, which is also denoted by ∇_0''.

Definition A.4. — A reduction $P_K \subset P_G$ is called pluri-harmonic, if $\theta^2 = 0$ and $\nabla_0''(\theta) = 0$ hold.

Let V be a vector space over $\boldsymbol{C}$. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be a representation, which induces the flat bundle (E, ∇_E). We take a K-invariant metric h_V, which induces the metric h_E of E. We obtain the decomposition $\nabla_E = \overline{\partial}_E + \partial_E + \theta_E + \theta_E^\dagger$ as in Section 21.4.3 of [**44**]. They are induced by ∇_0'', ∇_0', θ and $\theta^\dagger$, respectively. Thus, if $P_K \subset P_G$ is pluri-harmonic, we have $\theta_E^2 = \overline{\partial}_E \theta_E = 0$. Recall that they imply $\overline{\partial}_E^2 = 0$. Hence, (E, ∇_E, h) is a harmonic bundle. On the contrary, if κ is immersive and (E, ∇_E, h) is a harmonic bundle, we obtain the vanishings $\theta^2 = \nabla_0''\theta = 0$. Hence, $P_K \subset P_G$ is pluri-harmonic. Therefore, we obtain the following lemma.

Lemma A.5. — *The following conditions are equivalent.*

1. *The reduction $P_K \subset P_G$ is pluri-harmonic, in the sense of Definition* A.4.
2. *For any representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of a vector space V over $\boldsymbol{C}$, the induced flat bundle with the hermitian metric is a harmonic bundle.*
3. *There exist an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric of a vector space V over $\boldsymbol{C}$, such that the induced flat bundle with the hermitian metric is a harmonic bundle.*

Let $\pi : \widetilde{X} \longrightarrow X$ denote a universal covering. Take base points $x \in X$ and $x_1 \in \widetilde{X}$ such that $\pi(x_1) = x$. Once we pick a point $\widetilde{x} \in P_{G|x}$, the homomorphism $\pi_1(X, x) \longrightarrow G$ is given. If a K-reduction $P_K \subset P_G$ is given, we obtain a $\pi_1(X,x)$-equivariant map $F : \widetilde{X} \longrightarrow G/K$, where the $\pi_1(X,x)$-action on G/K is given by the homomorphism

$\pi_1(X,x) \longrightarrow G$. If $P_K \subset P_G$ is pluri-harmonic, then F is pluri-harmonic ([**67**]) in the sense that any restriction of F to holomorphic curve is harmonic.

A.2.3. A tame pure imaginary G-harmonic bundle. — Let G be a linear reductive group over $\boldsymbol{C}$. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$, and let W denote the Weyl group. We have the natural real structure $\mathfrak{h}_{\boldsymbol{R}} \subset \mathfrak{h}$. Hence we have the subspace $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}} \subset \mathfrak{h}$. We have the W-invariant metric of $\mathfrak{h}$, which induces the distance d of $\mathfrak{h}/W$. Let $B(\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}},\epsilon)$ denote the set of the points x of $\mathfrak{h}/W$ such that there exists a point $y \in \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}/W$ satisfying $d(x,y) < \epsilon$.

Let $(P_K \subset P_G, \theta)$ be a G-harmonic bundle on Δ^*. We have the expression $\theta = f \cdot dz/z$, where f is a holomorphic section of $\mathrm{ad}(P_G)$ on Δ^*. It induces the continuous map $[f] : \Delta^* \longrightarrow \mathfrak{h}/W$.

Definition A.6

- A G-harmonic bundle $(P_K \subset P_G, \theta)$ is called tame, if $[f]$ is bounded.
- A tame G-harmonic bundle $(P_K \subset P_G, \theta)$ is called pure imaginary, if for any $\epsilon > 0$ there exists a positive number r such that $\big[f(z)\big] \in B\big(\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}},\epsilon\big)$ for any $|z| < r$.

Lemma A.7. — *Let $(P_K \subset P_G, \theta)$ be a harmonic bundle on Δ^*. The following conditions are equivalent.*

1. *It is tame (pure imaginary).*
2. *For any $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*
3. *For some immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and some K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*

Proof. — The implications $1 \Longrightarrow 2 \Longrightarrow 3$ are clear. The implication $3 \Longrightarrow 1$ follows from the injectivity of $d\kappa : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. □

Let X be a smooth projective variety, and D be a normal crossing divisor.

Definition A.8. — A harmonic G-bundle $(P_K \subset P_G, \theta)$ on $X - D$ is called tame (pure imaginary), if the restriction $(P_K \subset P_G, \theta)_{|C\setminus D}$ is tame (pure imaginary) for any curve $C \subset X$ which is transversal with D.

Remark A.9. — Tameness and pure imaginary property are defined for principal G-Higgs bundles.

Remark A.10. — Tameness and pure imaginary property are preserved by pull back. We also remark the curve test for usual tame harmonic bundles.

Let us consider the case where G is a linear reductive group defined over $\boldsymbol{R}$, with a maximal compact group K. We have the complexification $G_{\boldsymbol{C}}$ with a maximal compact group $K_{\boldsymbol{C}}$ such that $K = K_{\boldsymbol{C}} \cap G$.

Definition A.11. — Let (P_G, ∇) be a flat bundle. A pluri-harmonic reduction $(P_K \subset P_G, \nabla)$ is called a tame pure imaginary, if the induced reduction $(P_{K_{\boldsymbol{C}}} \subset P_{G_{\boldsymbol{C}}}, \nabla)$ is a tame pure imaginary.

Lemma A.12. — *Let $(P_K \subset P_G, \theta)$ be a harmonic bundle on $X - D$. The following conditions are equivalent.*

1. *It is tame (pure imaginary).*
2. *For any $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*
3. *There exist an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric of V such that the induced harmonic bundle is tame (pure imaginary).*

A.3. Semisimplicity and Pluri-Harmonic Reduction

A.3.1. Preliminary. — Let X be a smooth irreducible quasiprojective variety with a base point x. We put $\Gamma := \pi_1(X, x)$ for simplicity of the notation. Recall the existence and the uniqueness of tame pure imaginary pluri-harmonic metric ([**29**], [**45**]), which is called the Corlette-Jost-Zuo metric. Let (E, ∇) be a semisimple flat bundle, and let $\rho : \Gamma \longrightarrow \mathrm{GL}(E_{|x})$ denote the corresponding representation. We have the canonical decomposition of $E_{|x}$:

$$E_{|x} = \bigoplus_{\chi \in \mathrm{Irrep}(\Gamma)} E_{|x,\chi}.$$

Here $\mathrm{Irrep}(\Gamma)$ denotes the set of irreducible representations, and $E_{|x,\chi}$ denotes a Γ-subspace of $E_{|x}$ isomorphic to $\chi^{\oplus m(\chi)}$. Correspondingly, we have the canonical decomposition of the flat bundle (E, ∇):

$$(E, \nabla) = \bigoplus_{\chi \in \mathrm{Irrep}(\Gamma)} E_\chi.$$

The flat bundle E_χ is isomorphic to a tensor product of a trivial bundle $\boldsymbol{C}^{m(\chi)}$ and a flat bundle L_χ whose monodromy is given by χ.

Lemma A.13

- *There exists a Corlette-Jost -Zuo metric h_χ of L_χ, which is unique up to positive constant multiplication.*
- *Under the isomorphism $(E, \nabla) \simeq \bigoplus_\chi L_\chi \otimes \boldsymbol{C}^{m(\chi)}$, any Corlette-Jost-Zuo metric of (V, ∇) is of the following form:*

$$\bigoplus_\chi h_\chi \otimes g_\chi.$$

Here g_χ denote any hermitian metrics of $\boldsymbol{C}^{m(\chi)}$. In other words, the ambiguity of the Corlette-Jost-Zuo metrics is a choice of hermitian metrics g_χ of $\boldsymbol{C}^{m(\chi)}$, once we fix h_χ.

– *The decomposition of flat connection* $\nabla = \partial + \overline{\partial} + \theta + \theta^{\dagger}$ *is independent of a choice of* g_{χ}.

Proof. — The first claim is proved in [**29**]. (See also [**45**].) The second claim easily follows from the proof of the uniqueness result in [**45**]. (See the argument of Proposition 2.6). The third claim follows from the second claim. □

We also have the following lemma (see [**50**] or [**45**])

Lemma A.14. — *If there exists a Corlette-Jost-Zuo metric on a flat bundle* (E, ∇), *then the flat bundle is semisimple.*

We have the involution $\chi \longmapsto \overline{\chi}$ on $\mathrm{Irrep}(\Gamma)$ such that $\chi \otimes_{\boldsymbol{R}} \boldsymbol{C} = \chi \oplus \overline{\chi}$. If $\overline{\chi} = \chi$, we have the real structure of L_{χ}. If $\overline{\chi} \neq \chi$, we have the canonical real structure of $L_{\chi} \otimes \boldsymbol{C} = L_{\chi} \oplus L_{\overline{\chi}}$.

Let us consider the case where a semisimple flat bundle (E, ∇) has the flat real structure $E_{\boldsymbol{R}}$ such that $E = E_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Let $\iota : E \longrightarrow E$ denote the conjugate with respect to $E_{\boldsymbol{R}}$. Then (E, ∇) is isomorphic to the following:

$$\bigoplus_{\overline{\chi}=\chi} L_{\chi} \otimes \boldsymbol{C}^{m(\chi)} \oplus \bigoplus_{\overline{\chi}\neq\chi} \big(L_{\chi} \oplus L_{\overline{\chi}}\big) \otimes \boldsymbol{C}^{m(\chi)}.$$

The real structure of (E, ∇) is induced from the real structures of L_{χ} $(\overline{\chi} = \chi)$ and $L_{\chi} \otimes \boldsymbol{C}$ $(\overline{\chi} \neq \chi)$. For a hermitian metric h of E, the hermitian metric $\iota^{*}h$ is given by $\iota^{*}h(u, v) = \overline{h(\iota(u), \iota(v))}$. Then the following lemma is clear.

Lemma A.15. — *When* (E, ∇) *has a real structure, there exists a Corlette-Jost-Zuo metric of* (E, ∇) *which is invariant under the conjugation. The ambiguity of the metric is a choice of the metrics of the vector spaces* $\boldsymbol{C}^{m(\chi)}$.

A.3.2. Pluri-harmonic reduction of the principal bundle associated with the monodromy group.

— Let $G_0 \subset \mathrm{GL}(E_{|x})$ denote the monodromy group $M(E, \nabla, x)$. We obtain the principal G_0-bundle P_{G_0} with the flat connection. If the flat bundle (E, ∇) is semisimple, we have a Corlette-Jost-Zuo metric h of (E, ∇). Let $U = U(E_{|x}, h_{|x})$ denote the unitary group of the metrized vector space $(E_{|x}, h_{|x})$, and we put $K_0 := G_0 \cap U$.

Lemma A.16. — G_0 *is reductive, and* K_0 *is a compact real form of* G_0.

Proof. — The argument was given by Simpson (Lemma 4.4 in [**55**]) for a different purpose. We reproduce it here with a minor change for our purpose. We have the canonical decomposition $T^{a,b}(E) = \bigoplus_{\chi \in \mathrm{Irrep}(\Gamma)} L_{\chi} \otimes \boldsymbol{C}^{m(a,b,\chi)}$. The decomposition is orthogonal with respect to the induced Corlette-Jost-Zuo metric $T^{a,b}(h)$. Namely, $T^{a,b}(h)$ is of the form $\bigoplus_{\chi \in \mathrm{Irrep}(\Gamma)} h_{\chi} \otimes h(a, b, \chi)$, where h_{χ} denotes a Corlette-Jost-Zuo metric of L_{χ}, and $h(a, b, \chi)$ denotes hermitian metric of $\boldsymbol{C}^{m(a,b,\chi)}$.

For any $f \in \mathrm{End}(E_{|x})$, let $f^{\dagger}$ denote the adjoint of f with respect to $h_{|x}$. For any $g \in G_0$, we have the unique expression $g = u \cdot \exp(y)$, where $u \in U$ and $y = y^{\dagger}$. The decomposition is compatible with tensor products and g-invariant orthogonal decompositions. It follows that $T^{a,b}u$ and $T^{a,b}y$ preserves the components $L_{\chi|x} \otimes \boldsymbol{C}^{m(a,b,\chi)}$. Namely, we have the decomposition $T^{a,b}g = (\bigoplus T^{a,b}g)_{\chi}$, $T^{a,b}u = (\bigoplus T^{a,b}u)_{\chi}$ and $T^{a,b}y = (\bigoplus T^{a,b}y)_{\chi}$.

Let κ be an isometric automorphism of $\big(\boldsymbol{C}^{m(a,b,\rho)}, h(a,b,\chi)\big)$. Then, $(T^{a,b}g)_{\chi}$ and $\mathrm{id}_{L_{\chi|x}} \otimes \kappa$ are commutative. Hence, $(T^{a,b}u)_{\chi}$ and $\mathrm{id}_{L_{\chi|x}} \otimes \kappa$ are commutative, and thus $(T^{a,b}u)_{\chi}$ is induced by the automorphism of $L_{\chi|x}$. Similarly, $(T^{a,b}y)_{\chi}$ is induced by the endomorphism of $L_{\chi|x}$. Hence, $L_{\chi|x} \otimes H_{\chi}$ is preserved by $(T^{a,b}u)_{\chi}$ and $(T^{a,b}y)_{\chi}$ for any subspace $H_{\chi} \subset \boldsymbol{C}^{m(a,b,\chi)}$. Since any G_0-invariant subspace of $T^{a,b}E_{|x}$ is of the form $\bigoplus L_{\chi|x} \otimes H_{\chi}$, we obtain $u \in G_0 \cap U = K_0$ and $y \in \mathfrak{g}_0 \subset End(E_{|x})$, where $\mathfrak{g}_0$ denotes the Lie subalgebra of $\mathrm{End}(E_{|x})$ corresponding to G_0.

Let $\tau : \mathrm{GL}(E_{|x}) \longrightarrow \mathrm{GL}(E_{|x})$ be the anti-holomorphic involution such that $\tau(g) = (g^{\dagger})^{-1}$. We obtain that $\tau(g) = u \cdot \exp\big(-y\big)$ is contained in G_0. Namely, τ gives the real structure of G_0. Since we have the decomposition $g = u \cdot \exp(y)$ for any $g \in G_0$, K_0 intersects with any connected components of G_0. Let G_0^0 denote the connected component of G_0 containing the unit element. It is easy to see that $K_0 \cap G_0^0$ is maximal compact in G_0^0, and hence K_0 is maximal compact of G_0. Since $K_0 \cap G_0^0$ is the fixed point set of $\tau_{|G_0^0}$, we obtain that K_0^0 is a compact real form of G_0^0. Thus K_0 is a compact real form of G_0. Since K_0 is maximal compact, G_0 is reductive. □

Let us consider the case where (E, ∇) has the real structure. We have the real parts $E_{\boldsymbol{R}\,|\,x} \subset E_{|x}$ and $G_{0\boldsymbol{R}} := G_0 \cap \mathrm{GL}(E_{\boldsymbol{R}\,|\,x})$. We take a Corlette-Jost-Zuo metric of h which is invariant under the conjugation ι. We put $K_{0\boldsymbol{R}} = G_{0\boldsymbol{R}} \cap K_0 = G_{0\boldsymbol{R}} \cap U$. The map ι induces the real endomorphism of $\mathrm{End}(E_{|x})$ given by $\iota(f) = \iota \circ f \circ \iota$.

Lemma A.17. — *$K_{0\boldsymbol{R}}$ is maximal compact in $G_{0\boldsymbol{R}}$.*

Proof. — We use the notation in the proof of Lemma A.16. Since $h_{|x}$ is invariant under the conjugation ι, U is stable under ι, and τ and ι are commutative. Let g be an element of $G_{0\boldsymbol{R}}$. We have the decomposition $g = u \cdot \exp(y)$ as in the proof of Lemma A.16, where u denotes an element of K_0 and y denotes an element of $\mathfrak{g}_0$ such that $y^{\dagger} = y$. Since $\iota(g) = g$, we have $\iota(u) \cdot \exp\big(\iota(y)\big) = u \cdot \exp(y)$. Since we have $\iota(u) \in \iota(U) = U$ and $(\iota(y))^{\dagger} = \iota(y^{\dagger}) = -\iota(y)$, we obtain $\iota(u) = u$ and $\iota(y) = y$. Namely $u \in K_{0\boldsymbol{R}}$ and $y \in \mathfrak{g}_{0\boldsymbol{R}}$. Then we can show $K_{0\boldsymbol{R}}$ is maximal compact in $G_{0\boldsymbol{R}}$, by an argument similar to the proof of Lemma A.16. □

Proposition A.18. — *Assume that (E, ∇) is semisimple. Then there exists the unique tame pure imaginary pluri-harmonic reduction $P_{K_0} \subset P_{G_0}$. Assume (E, ∇) has the flat real structure, moreover. Then, it is induced from the pluri-harmonic reduction of $P_{G_{0\boldsymbol{R}}}$.*

Proof. — Let h be a Corlette-Jost-Zuo metric of (E, ∇). For any point $z \in X$, let $M(E, \nabla, z)$ denote the monodromy group at z, and $U(E_{|z}, h_{|z})$ denote the unitary group of $E_{|z}$ with the metric $h_{|z}$. Then the intersection $M(E, \nabla, z) \cap U(E_{|z}, h_{|z})$ is a maximal compact subgroup of $M(E, \nabla, z)$, due to Lemma A.16. Hence they give the reduction $P_{K_0} \subset P_{G_0}$, which is pluri-harmonic. By using a similar argument and Lemma A.17, we obtain the compatibility with the real structure, if (E, ∇) has the flat real structure. The uniqueness of the pluri-harmonic reduction follows from the uniqueness result in Lemma A.13. Hence we are done. □

A.3.3. Characterization of the existence of pluri-harmonic reduction. — Let G be a linear reductive algebraic group over $\boldsymbol{C}$ or $\boldsymbol{R}$. Let $\widetilde{X}$ be a universal covering of X. The following corollary immediately follows from Proposition A.18.

Corollary A.19. — *Let P_G be a flat G-principal bundle over X. Assume that the image of the induced representation $\Gamma \longrightarrow G$ is Zariski dense in G. Then there exists the unique tame pure imaginary pluri-harmonic reduction of P_G. Correspondingly, we obtain the Γ-equivariant pluri-harmonic map $\widetilde{X} \longrightarrow G/K$.*

Proposition A.20. — *Let P_G be a flat G-bundle on X. The monodromy group G_0 is reductive if and only if there exists a tame pure imaginary pluri-harmonic reduction $P_K \subset P_G$. If such a reduction exists, the decomposition $\nabla = \nabla_K + (\theta + \theta^\dagger)$ does not depend on a choice of a pluri-harmonic reduction $P_K \subset P_G$, and there is the corresponding Γ-equivariant pluri-harmonic map $\widetilde{X} \longrightarrow G/K$.*

Proof. — If a pluri-harmonic reduction exists, the monodromy group is reductive due to Lemma A.7 and Lemma A.16. Assume G_0 is reductive. Let K_0 be a maximal compact group of G_0. Then we have the unique tame pure imaginary pluri-harmonic reduction $P_{K_0} \subset P_{G_0}$. We take K such as $K \cap G_0 = K_0$. Then the pluri-harmonic reduction $P_K \subset P_G$ is induced, and thus the first claim is proved. The second claim is clear. □

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