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FOUR DIMENSIONAL GALOIS REPRESENTATIONS

by

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Abstract. — We construct four dimensional irreducible mixed l-adic representations of the absolute Galois group of $\mathbb{Q}$, which are attached to irreducible cuspidal automorphic representations Π of the symplectic group of similitudes GSp(4), whose archimedean component Π_∞ belongs to the discrete series, and discuss some of the properties of these l-adic representations.

***Résumé* (Représentations galoisiennes de dimension quatre).** — Nous construisons et étudions certaines représentations l-adiques mixtes irréductibles de dimension quatre du groupe de Galois absolu de $\mathbb{Q}$, attachées à des representations automorphes cuspidales irréductibles Π du groupe de similitudes symplectiques GSp(4), dont la composante archimédienne Π_∞ appartient à la série discrète. Nous présentons également quelques propriétés de ces représentations l-adiques.

Introduction

It is well known how to associate two dimensional λ-adic representations to irreducible cuspidal automorphic representations of the group $\mathrm{Gl}(2,\mathbb{A})$, whose archimedian component is a discrete series representation, for the ring $\mathbb{A}$ of rational adeles. In the case Gl(2) the condition at the archimedian place leads to the study of classical holomorphic cuspforms of weight $k \geqslant 2$. Already for the symplectic group of similitudes $\mathrm{GSp}(4,\mathbb{A})$ the corresponding situation is not understood as well. In this paper we derive analogous results for the group GSp(4) by constructing corresponding four dimensional Galois representations. Furthermore we discuss various properties of these representations. Proofs are based on certain fundamental assertions, in particular from spectral theory, which itself will not be discussed in this paper. Some of them are available only in preprint form. For the convenience of the reader they will here be formulated as hypotheses, in order to make the paper self contained.

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That for an irreducible cuspidal automorphic representation Π of $\mathrm{GSp}(4,\mathbb{A})$ the representation Π_∞ at the archimedian place belongs to the discrete series, does not necessarily lead to the study of holomorphic cuspforms. The reason is, that the discrete series representations of $\mathrm{GSp}(4,\mathbb{R})$ are parameterized by L-packets. Each L-packet contains two classes of irreducible representations. One of them is a member of the holomorphic discrete series and does not have a Whittaker model, whereas the other is nonholomorphic but has a Whittaker model. The packets itself are parameterized, up to a character twist, by what we call their weight. The weight is described by a pair of integers (k_1, k_2) such that $k_1 \geqslant k_2 \geqslant 3$. An irreducible, cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$, whose archimedian component is holomorphic of weight (k_1, k_2), corresponds to classical vector valued holomorphic Siegel modular forms $f(\Omega)$ on the Siegel upper half space H of genus two $f : H \to V_\rho$ with the transformation property

$$f((A\Omega + B)(C\Omega + D)^{-1}) = \rho(C\Omega + D) \cdot f(\Omega), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the representation $\rho = \mathrm{Symm}^{k_1-k_2} \otimes \det^{k_2}$ of $\mathrm{Gl}(2,\mathbb{C})$ on V_ρ, where M is in a congruence subgroup of the Siegel modular group. In the case $k = k_1 = k_2$ we obtain classical Siegel modular forms of weight k. The lowest K_∞-type of the holomorphic discrete series representation is characterized by its highest weight vector, which is defined by the weight (k_1, k_2). In the Whittaker case the corresponding K_∞-type has highest weight $(k_1, 2 - k_2)$.

Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of rational adeles. In the following let $\Pi = \Pi_\infty \Pi_f$ be an irreducible cuspidal automorphic representation of the group $\mathrm{GSp}(4,\mathbb{A})$, whose component Π_∞ belongs to the discrete series lying in a L-packet of weight (k_1, k_2). We abbreviate this by saying, that Π has weight (k_1, k_2). The ramified places of Π are the archimedian place and the finite places, where Π is not spherical. The first result is

Theorem I. — *Suppose Π is a unitary cuspidal irreducible automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ for which Π_∞ belongs to the discrete series of weight (k_1, k_2). Let S denote the set of ramified places of the representation Π. Put $w = k_1 + k_2 - 3$. Then there exists a number field E, such that for primes $p \notin S$ the local L-factor*

$$L_p(p^{-s}) = L_p(\Pi_p, s - w/2), \quad L_p(X)^{-1} \in E[X]$$

of the degree 4 spinor L-series (for the 'algebraic' normalization involving the shift by $-w/2$ as above) has coefficients in E, and such that for any prime number l and any extension λ of l to E there exists a four dimensional semisimple Galois representation

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \longrightarrow \mathrm{Gl}(4, \overline{E_\lambda}),$$

which is unramified outside $S \cup \{l\}$, so that for $p \notin S \cup \{l\}$ the following holds

$$L_p(\Pi_p, s - w/2) = \det(1 - \rho_{\Pi,\lambda}(\mathrm{Frob}_p) p^{-s})^{-1}.$$

The eigenvalues of $\rho_{\Pi,\lambda}(\mathrm{Frob}_p)$ for $p \neq l, p \notin S$ are algebraic integers. The representation $\rho_{\Pi,\lambda}$ arises from a λ-adic representation, if E is chosen large enough. The so defined λ-adic representation $\rho_{\Pi,\lambda}$ is mixed. If Π is not a CAP representation (for this notation see [**S**]*) the representation $\rho_{\Pi,\lambda}$ is pure of weight w, i.e. for all isomorphisms $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ the image of the eigenvalues of $\rho_{\Pi,\lambda}(\mathrm{Frob}_p)$ has absolute value $p^{w/2}$ for $p \neq l$ and $p \notin S$.*

1. Remark. — Frob_p is the geometric Frobenius. The chosen normalization of the λ-adic representation $\rho_{\Pi,\Lambda}$ is of cohomological nature. In the most relevant cases the formulas above first arise from a right action of the Galois group on certain cohomology groups. In order to obtain a representation in the usual sense one has to consider the transposed, which then defines a representation in the usual sense. Since characteristic polynomials do not change under transposition this does not matter. Nevertheless it is the dual of the representation so obtained, which corresponds to what usually appears in the literature on elliptic modular forms. The dual representation $\rho^{\vee}_{\Pi,\lambda}$ is

$$\rho^{\vee}_{\Pi,\lambda} \cong \rho_{\Pi,\lambda} \otimes \chi^{-1}, \quad \chi = \omega_\Pi \cdot \mu_l^{-w},$$

where μ_l is the cyclotomic character $\mu_l(\mathrm{Frob}_p) = p^{-1}$ and $\omega_\Pi(\mathrm{Frob}_p) = \omega_\Pi(p)$, where ω_Π is the central character of Π. This is a consequence of the Tchebotarev density theorem, since the formulas $L_p(\Pi_p, s - w/2) = \det(1 - \rho_{\Pi,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}$ and $\Pi \cong \Pi^\vee \otimes \omega_\Pi$ imply that the two semisimple representations $\rho^{\vee}_{\Pi,\lambda}$ and $\rho_{\Pi,\lambda} \otimes \chi^{-1}$ (with $\chi = \omega_\Pi \cdot \mu_l^{-w}$) have the same character. Similarly, the class of the representation $\rho_{\Pi,\lambda}$ only depends on the weak equivalence class of Π. Two irreducible automorphic representations Π_1, Π_2 are called weakly equivalent, if they are isomorphic locally $\Pi_{1,v} \cong \Pi_{2,v}$ at almost all places v.

Theorem II. — *The representations $\rho_{\Pi,\lambda}$ constructed in theorem I are never reducible of the form $\rho_{\Pi,\lambda} \cong \rho_0 \oplus \rho_0$, for a two-dimensional λ-adic representations ρ_0 of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *They contain a one dimensional invariant subspace if and only if Π is a CAP-representation (of Saito-Kurokawa type* [**P**]*).*

Suppose E is chosen large enough, so that the representation $\rho_{\Pi,\lambda}$ is defined over E_λ. Then $\rho_{\Pi,\lambda}$ can be viewed as representations of dimension $4 \cdot [E_\lambda : \mathbb{Q}_l]$ over $\mathbb{Q}_l$. But in fact, by the way they will be constructed, these $\mathbb{Q}_l$-vector spaces then turn out to be Hodge-Tate modules of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ using [**CF**] theorem 6.2. Moreover if we exclude certain exceptional cases — for the notion of a weak endoscopic lift see the definition further below — we have

Theorem III. — *Suppose the cuspidal representation Π is neither CAP nor a weak endoscopic lift and weakly equivalent to a multiplicity one representation. Then the* *representations* $\rho_{\Pi,\lambda}$ *define Hodge-Tate modules over* $\mathbb{Q}_l$ *with four different Hodge types*

$$(k_1+k_2-3,0),\ (k_1-1,k_2-2),\ (k_2-2,k_1-1),\ (0,k_1+k_2-3)$$

each of which occurs with the same $\mathbb{Q}_l$*-dimension* $[E_\lambda:\mathbb{Q}_l]$.

Theorem III is deduced from proposition 1.5.

2. Remark. — A weaker version of theorem I was obtained in [**T**]. As in [**T**], p. 291ff we use the fact, that the representation Π contributes to the (interior) cohomology of a suitably defined projective limit M of Siegel modular threefolds with respect to a coefficient system $\mathcal{V}_\mu(\overline{\mathbb{Q}}_l)$, which only depends on the weight (k_1,k_2) of Π. Our result is also deduced from the study of the etale cohomology groups. That the etale cohomology of M defines mixed Galois representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, is seen by discussing three cases separately: the case of CAP-representations, the case of weak endoscopic lifts and the remaining case. In the last case the representations $\rho_{\Pi,\lambda}$ defined above do naturally occur in the third cohomology of M. This fails to hold in the case, where Π is a CAP-representation of Saito-Kurokawa type. It also fails to hold in the case, where Π is a weak endoscopic lift considered below. Both these exceptional cases are interesting for various reasons.

Definition. — A unitary irreducible cuspidal representation Π of $\mathrm{GSp}(4,\mathbb{A})$ is called a *weak endoscopic lift*, if there exist two unitary irreducible cuspidal automorphic forms π_1,π_2 of $\mathrm{Gl}(2,\mathbb{A})$ with central characters $\omega_{\pi_1}=\omega_{\pi_2}$, such that

$$L_v(\Pi,s)=L_v(\pi_1,s)L_v(\pi_2,s)$$

holds for almost all places. Here $L_v(\Pi,s)$ denotes the local L-factor of the degree 4 spinor L-series.

Let Π be a weak endoscopic lift attached to π_1,π_2. Then under the hypothesis A formulated in the next paragraph we get $\omega_{\pi_i}=\omega_\Pi$. Furthermore, if we consider representations Π for which Π_∞ belongs to the discrete series, $\pi_{\infty,i}$ will belong to the discrete series of weight r_i, such that $r_1>r_2\geqslant 2$ holds for a suitable ordering. Conversely for π_1,π_2 with archimedian components as above $\sigma=(\pi_1,\pi_2)$ lifts to a global nontrivial L-packet, defined as the weak equivalence class of unitary cuspidal irreducible automorphic representations Π of $\mathrm{GSp}(4,\mathbb{A})$, whose components at infinity belong to the discrete series of weight (k_1,k_2) such that the L-identities (1) from above hold at almost all places. The integers k_i and r_i are related by the formulas $r_1=k_1+k_2-2$ and $r_2=k_1-k_2+2$.

Hypothesis A. — Let F be a totally real number field. Let $\sigma=(\pi_1,\pi_2)$ be a pair of irreducible cuspidal representations of $\mathrm{Gl}(2,\mathbb{A}_F)$ with a common central character

$\omega_{\pi_1} = \omega_{\pi_2}$. Let Π be a unitary cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_F)$, which is supposed to be a weak endoscopic lift attached to $\sigma = (\pi_1, \pi_2)$. Then

(1) Π is not CAP.

(2) The central character of Π is $\omega_\Pi = \omega_{\pi_1} = \omega_{\pi_2}$.

(3) If Π_v belongs to the discrete series of weight k_1, k_2 for an archimedian place v of F, then there exist integers r_1, r_2 such that $r_1 > r_2 \geqslant 2$ (given by $r_1 = k_1 + k_2 - 2$, $r_2 = k_1 - k_2 + 2$), so that $(\pi_i)_v$ belongs to the holomorphic discrete series of weight r_i. The converse also holds.

(4) For the finitely many places v of F, for which Π_v belongs to the discrete series, Π_v is contained in a local L-packet $\Pi_v \in \{\Pi_v^+(\sigma_v), \Pi_v^-(\sigma_v)\}$ consisting of two classes of irreducible admissible representations $\Pi_v^\pm(\sigma_v)$ of $\mathrm{GSp}(4, F_v)$, which only depend on $\sigma_v = (\pi_{1,v}, \pi_{2,v})$. At the remaining places v of F, where Π_v does not belong to the discrete series, $\Pi_v \cong \Pi_v^+(\sigma_v)$ is uniquely determined by σ_v.

(5) The representations $\Pi_v^+(\sigma_v)$ have Whittaker models. The representations $\Pi_v^-(\sigma_v)$ do not have Whittaker models. For discrete series representations σ_v at the archimedian places as in (3) the representation $\Pi_v^-(\sigma_v)$ is in the holomorphic discrete series and $\Pi_v^+(\sigma_v)$ is in the nonholomorphic discrete series.

(6) For any representation $\Pi' = \otimes_v \Pi'_v$ of $\mathrm{GSp}(4, \mathbb{A}_F)$ with local components $\Pi'_v \cong \Pi_v^\pm(\sigma_v)$ as in (4) the multiplicity $m(\Pi')$ of Π' in the cuspidal (or the discrete) spectrum is one or zero, depending on whether the number of places of F, where Π'_v is in the discrete series without admitting a Whittaker model, is even or odd.

(7) For $F = \mathbb{Q}$ and Π_∞ in the discrete series the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module

$$W_{\Pi_f} = \mathrm{Hom}_{\mathrm{GSp}(4,\mathbb{A}_f)}(H^3(M, \mathcal{V}_\mu(\overline{\mathbb{Q}}_l)), \Pi_f),$$

defined as in section 2, is a two dimensional $\overline{\mathbb{Q}}_l$-representation of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It is isomorphic to either the two dimensional Galois $\overline{\mathbb{Q}}_l$-representations ρ_{π_1} or $\rho_{\pi_2} \otimes_{\overline{\mathbb{Q}}_l} \mu_l^{\otimes(2-k_2)}$ (attached to the cuspidal representations π_1 resp. π_2) depending on whether Π_∞ is holomorphic or not.

A proof of this hypothesis is contained in the preprints [**W**] and [**W1**], where these statements are deduced from the global trace formula and where also a further description of the local L-packets is given. A completely different proof of A(6) was also obtained by Roberts.

The Abel-Jacobi map. — The particular interesting case $k_1 = k_3 = 3$ is relevant for the description of the 'interior' part (image of the cohomology with compact supports) or equivalently the cuspidal part $H_P^3(M, \overline{\mathbb{Q}}_l)$ of the etale cohomology groups $H_{\text{ét}}^3(M, \overline{\mathbb{Q}}_l)$. The corresponding coefficient system $\mathcal{V}_\mu(\overline{\mathbb{Q}}_l)$ is the constant sheaf $\overline{\mathbb{Q}}_l$. Instead of M we now consider a single Siegel threefold for some fixed level, also denoted M in this subsection. We choose a projective smooth model M obtained by a toroidal compactification of M. Then the cuspidal cohomology $H_P^3(M, \mathbb{C})$ embeds as a subspace into the third cohomology $H^3(\mathrm{M},\mathbb{C})$ of this projective model. For the projective smooth model M consider the Griffiths intermediate Jacobian

$$T_2(\mathrm{M}) = F^3(\mathrm{M})\backslash H^3(\mathrm{M},\mathbb{C})/H^3(\mathrm{M},\mathbb{Z}) \cong (H^{(1,2)}(\mathrm{M},\mathbb{C}) \oplus H^{(0,3)}(\mathrm{M},\mathbb{C}))/\operatorname{im}(H^3(\mathrm{M},\mathbb{Z})).$$

Its abelian variety part $A_2(\mathrm{M}) \subset T_2(\mathrm{M})$ ([**G**],p. 18) is defined as the image of $X_{\mathbb{C}}$ in $T_2(\mathrm{M})$, where $X_{\mathbb{C}} = X \otimes_{\mathbb{Q}} \mathbb{C}$ for $X = (H^{(2,1)}(\mathrm{M},\mathbb{C}) \oplus H^{(1,2)}(\mathrm{M},\mathbb{C})) \cap H^3(\mathrm{M},\mathbb{Q})$. It contains as an abelian subvariety the image $I_2^0(\mathrm{M})$ of the cycles classes in $\mathcal{A}_2(\mathrm{M})$ under the Abel-Jacobi map. The Chow group $\mathcal{C}_2(\mathrm{M})$ of cycles of codimension two modulo rational equivalence contains subgroups $\mathcal{C}_2(\mathrm{M}) \supset \mathcal{H}_2(\mathrm{M}) \supset \mathcal{A}_2(\mathrm{M}) \supset \mathcal{I}_2(\mathrm{M}) \supset \mathcal{K}_2(\mathrm{M})$, where $\mathcal{H}_2(\mathrm{M})$ the subgroup of cycles homologically equivalent to zero, *i.e.* the kernel of the natural map $\mathcal{C}_2(\mathrm{M}) \to H^4(\mathrm{M},\mathbb{C})$ or $H^4_{\text{ét}}(\mathrm{M},\overline{\mathbb{Q}}_l)$. $\mathcal{A}_2(\mathrm{M})$ is the subgroup of cycles, which are algebraically equivalent to zero, $\mathcal{I}_2(\mathrm{M})$ is the subgroup of cycles, which are incident equivalent to zero, and $\mathcal{K}_2(\mathrm{M})$ is the kernel of the Abel-Jacobi map. The Abel-Jacobi map $AJ : \mathcal{H}_2(\mathrm{M}) \to T_2(\mathrm{M})$ is a naturally defined homomorphism. The image $I_2^0(\mathrm{M}) = AJ(\mathcal{A}_2(\mathrm{M})) \subset T_2(\mathrm{M})$ defines an abelian subvariety of $A_2(\mathrm{M})$. By [**G**] prop. 3.5 the kernel $\mathcal{K}_2(\mathrm{M})$ has finite index in $\mathcal{I}_2(\mathrm{M})$.

Let Π be a cuspidal automorphic representation, which is not CAP. Assume Π is a weak endoscopic lift of a pair of classical holomorphic cuspidal new forms of weight $r_1 = 4$ and $r_2 = 2$ respectively. Let $\sigma = (\pi_1,\pi_2)$ be the corresponding automorphic representations. π_2 is holomorphic of weight 2. Hence there is an abelian variety $A[\pi_2]$ in the Jacobi variety of a modular curve, which is attached to the orbit $(\pi_2)^\tau, \tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ (and will be considered only up to isogeny in the following). In the weak equivalence class of Π there exists a representation, whose archimedian component has a Whittaker model. See hyp. A(6). So assume Π itself has this property. If the level is chosen suitably small, Π contributes to the cohomology of M. In other words the cuspidal Π_f-isotypic subspace $H^3_P(M,\mathbb{C})(\Pi_f)$ of $H^{(2,1)}(M,\mathbb{C}) \oplus H^{(1,2)}(M,\mathbb{C}))$ is nontrivial. Then let $H^3_P(M,\mathbb{C})[\Pi]$ denote the subspace, which is generated by the $H^3_P(M,\mathbb{C})(\Pi'_f)$ for all weak endoscopic lifts Π', that are weak lifts attached to some $\sigma^\tau = (\pi_1^\tau,\pi_2^\tau), \tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and whose archimedian component Π'_∞ has a Whittaker model. By cuspidality there exists a natural projection map

$$p_\Pi : H^3(M,\mathbb{C}) \longrightarrow H^3_P(M,\mathbb{C}) \longrightarrow H^3_P(M,\mathbb{C})[\Pi],$$

which is compatible with the action of the Hecke correspondences. Again by cuspidality of Π we can identify $H^3_P(M)[\Pi]$ with a subspace of $H^3_c(M,\mathbb{C})$, which as a subspace is mapped isomorphically to $H^3_P(M,\mathbb{C})$. Consider the exact sequence $H^2(\partial\mathrm{M},\mathbb{C}) \to H^3_c(M,\mathbb{C}) \to H^3(\mathrm{M},\mathbb{C})$. For the standard toroidal compactifications M of M, defined by Igusa, the boundary $\partial\mathrm{M}$ is a union of elliptic surfaces fibered over modular curves. This structure of the boundary implies, that the image of the cohomology of $H^2(\partial M,\mathbb{C})$ in $H^3_c(M,\mathbb{C})$ is of Eisenstein type. Since Π is not CAP, the natural map from the Π_f-isotypic component of $H^3_c(M,\mathbb{C})$ to $H^3(\mathrm{M},\mathbb{C})$ therefore is injective. We get the following commutative diagram

$$\begin{array}{ccc} H^3_c(M,\mathbb{C}) & \longrightarrow & H^3(\mathrm{M},\mathbb{C}) \\ \uparrow & \searrow & \downarrow \\ H^3_P(M,\mathbb{C})[\Pi] & \hookrightarrow & H^3(M,\mathbb{C}) \end{array}$$

So $H^3_P(M,\mathbb{C})[\Pi]$ can be naturally viewed as a direct summand of $H^3(\mathrm{M},\mathbb{C})$. We claim, that furthermore the subspace $H^3_P(M,\mathbb{C})[\Pi]$ of $H^3_P(M,\mathbb{C})$ is defined over $\mathbb{Q}$ — with respect to the $\mathbb{Q}$-structure defined by the Betti cohomology. Hence $H^3_P(M,\mathbb{C})[\Pi]$ contributes to $X_{\mathbb{C}}$ and naturally maps to the abelian variety $A_2(\mathrm{M})$

$$H^3_P(M,\mathbb{C})[\Pi] \longrightarrow A_2(\mathrm{M}) \subset T^3(\mathrm{M},\mathbb{C}).$$

Since the Betti $\mathbb{Q}$-structure defines an action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on $H^3_P(M,\mathbb{C})$, which commutes with the action of $\mathrm{GSp}(4,\mathbb{A}_f)$, for the proof it is enough to show that the subspace $H^3_P(M,\mathbb{C})[\Pi]$ is stable under the action of $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Since $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ permutes the $\mathrm{GSp}(4,\mathbb{A}_f)$-isotypic components, this amounts to show that Π_f^τ contributes to $H^3_P(M,\mathbb{C})[\Pi]$, if Π_f contributes. Since $\Pi_f \cong \prod_{v\neq\infty} \Pi_v$, we get $(\Pi_f)^\tau \cong \prod_{v\neq\infty}(\Pi_v)^\tau$ for certain irreducible admissible complex representations of $\mathrm{GSp}(4,\mathbb{Q}_v)$. An inspection at the unramified places immediately implies, that Π^τ is again a weak endoscopic lift, now attached to the automorphic representation (π_1^τ, π_2^τ). It only remains to control $(\Pi^\tau)_\infty$. By the multiplicity formula (6) of hypothesis A a weak endoscopic lift $(\Pi^\tau)_\infty$ is determined by $(\Pi^\tau)_f$, more precisely by the number of its nonarchimedian local representations which do not have a local Whittaker model. Let T be a finite set of nonarchimedian places and $\mathbb{A}_T = \prod_{v\in T}\mathbb{Q}_v$. Then the following statements are equivalent: (1) The restriction of of Π_f to $\mathrm{GSp}(4,\mathbb{A}_T)$ admits a nontrivial Whittaker functional. (2) $\Pi_T = \prod_{v\in T}\Pi_v$ admits a nontrivial Whittaker functional for $\mathrm{GSp}(4,\mathbb{A}_T)$. (3) All $\Pi_v, v\in T$ have a local Whittaker model. Furthermore the restriction of Π_f to $\mathrm{GSp}(4,\mathbb{A}_T)$ admits a nontrivial Whittaker functional l iff the restriction of $(\Pi_T)^\tau$ to $\mathrm{GSp}(4,\mathbb{A}_T)$ admits a nontrivial Whittaker functional $\widetilde{l}$. Simply put $l(v) = \widetilde{l}(\tau(v))$. So this implies, that $(\Pi^\tau)_v$ has a Whittaker model iff Π_v has a Whittaker model for all nonarchimedian places v, hence also for the archimedian place. Therefore $\Pi_-(\sigma_v)^\tau \cong \Pi_-(\sigma_v^\tau)$ and $H^3_P(M,\mathbb{C})[\Pi]$ is defined over $\mathbb{Q}$. This implies, that weak endoscopic lifts Π gives rise to certain nontrivial abelian varieties

$$A[\Pi] \subset A_2(\mathrm{M}),$$

which appear in the middle intermediate Jacobian $T_2(\mathrm{M})$ as image of $H^3_P(M,\mathbb{C})[\Pi]$. By hyp. A(7) the λ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^1(A[\Pi],\overline{\mathbb{Q}}_l)(-1)$ can be expressed completely in terms of the λ-adic representations $\rho_{\pi_2,\lambda}\otimes_{\overline{\mathbb{Q}}_l}\mu_l^{-1}$, so that $A[\Pi]$ is isogenous by the Tate conjecture to a product of the abelian variety $A[\pi_2]$ attached to the orbit $(\pi_2)^\tau, \tau\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. So the weak endoscopic lift Π considered above relates $A[\pi_2]$ to the abelian variety $A_2(\mathrm{M})$ attached to the Siegel modular threefold M. $A[\pi_2]$ is a factor of $A_2(\mathrm{M})$, up to isogeny. By varying π_4 we can embed $A[\pi_2]$ in different ways (up to isogeny).

It is now tempting to ask for an analog of Heegner points. These generalized Heegner points could arise as images of suitable linear combinations of curves on M under the Abel-Jacobi homomorphism and the projections defined above (up to isogeny)

$$\mathcal{A}_2(\mathrm{M}) \longrightarrow A_2(\mathrm{M}) \longrightarrow A[\Pi] \longrightarrow A[\pi_2].$$

Particular interesting are curves contained in Hilbert-Blumenthal subvarieties (Humbert surfaces). One might ask, whether this construction is able to produce a new class of points in $A[\pi_2]$, especially when $A[\pi_2]$ is an elliptic curve defined over $\mathbb{Q}$.

CAP-representations. — If the representation Π of $\mathrm{GSp}(4,\mathbb{A})$ considered in theorem I is not a weak endoscopic lift but a CAP-representation, then - again for primes outside some finite set S including the ramified places - $L^S(\Pi,s) = L^S(\pi_1,s)L^S(\pi_2,s)$ holds for a pair of automorphic representations π_1, π_2 of $\mathrm{Gl}(2,\mathbb{A})$. π_1 is still cuspidal and at the archimedean place still belongs to the discrete series. However π_2 now is of Eisenstein type of weight $k_2 = 2$. Then, up to a character twist, the central character of Π can assumed to be trivial, and

$$L^S(\Pi,s) = L^S(\chi, s-1/2)L^S(\pi_1,s)L^S(\chi,s+1/2)$$

holds for a quadratic character χ, *i.e.* $\chi^2 = 1$ ([**P**],[**S**]). Since the central character of Π is a square, $k_1 + k_2$ is even because $\omega_{\Pi_\infty}(-1) = (-1)^{k_1+k_2}$. Therefore the semisimple representation $\rho_{\Pi,\lambda}$ comes with a monodromy filtration whose graded pieces are of weight $w+1, w, w-1$

$$\rho_{\Pi,\lambda} = \chi\mu_l^{(2-k_1-k_2)/2} \oplus \rho_{\pi_1,\lambda} \oplus \chi\mu_l^{(4-k_1-k_2)/2}.$$

This can also be understood in terms of pairings (as in appendix D).

Therefore in both these special cases the statement of the main theorem immediately reduces to the corresponding statement for $\mathrm{Gl}(2,\mathbb{A})$. If $\rho_{\pi_i,\lambda}$ are the corresponding two-dimensional λ-adic Galois representations (for the 'algebraic' normalization), the representation $\rho_{\Pi,\lambda}$ of theorem I in these cases is just formally defined as the direct sum

$$\rho_{\Pi,\lambda} = \rho_{\pi_1,\lambda} \oplus (\rho_{\pi_2,\lambda} \otimes_{\overline{\mathbb{Q}}_l} \mu_l^{\otimes(2-k_2)}).$$

However, and this should be emphasized, by hyp. A(7) the so defined representation is not the one, that naturally occurs in the cohomology of Siegel modular threefolds. For instance, if Π is a weak endoscopic lift, only the two dimensional representation $\rho_{\pi_1,\lambda}$ occurs in the Π_f-isotypic component of the third cohomology of the Siegel threefold if Π_∞ is in the holomorphic discrete series; respectively only the other two dimensional summand $\rho_{\pi_2,\lambda} \otimes_{\overline{\mathbb{Q}}_l} \mu_l^{\otimes(2-k_2)}$ occurs, if Π_∞ has a Whittaker model.

3. Remark. — Existence of a number field E, as stated in theorem I, is clear for the CAP-cases and the cases of weak endoscopic lifts. Otherwise, its existence follows from the finite dimensionality of the Betti-cohomology of Siegel modular threefolds for fixed levels.

This being said we may assume for the proof of theorem I, that Π is not a lift of CAP-type or a weak endoscopic lift. Under this additional assumption - which will be maintained for simplicity starting from section 4 - the Galois representation constructed in theorem I appear in the cohomology of M as such, and they will be shown to be pure λ-adic representations of weight w, *i.e.* the Ramanujan conjecture holds for all unramified places of Π as explained in section 1.

The Zariski closure G. — Let G denote the Zariski closure of the image $\rho_{\Pi,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ of the absolute Galois group in $\mathrm{Gl}(4,k)$, which is an algebraic group defined over the algebraic closure k of E_λ. Let G^0 be its connected component, and $\pi_0(G)$ the group of components of G for the Zariski topology. Then by the analysis of [**T**] the possible cases for G and $\pi_0(G)$ are included in a list of eleven cases. What makes the first three cases look interesting is, that here the group $\pi_0(G)$ of connected components can be more complicated. Using also results of appendix B, they are

– (later case 1a or 1b) G^0 is a k-torus of dimension 2, and $\pi_0(G)$ contains as subgroup of index at most 2 a finite subgroup of $\mathrm{PGl}(2,k)$, or

– (later case 1a) G^0 is a k-torus of dimension 1, and $\pi_0(G)$ contains a cyclic normal subgroup, whose quotient Δ is a finite subgroup of $\mathrm{PGl}(2,k)$, or

– (case 2) G is a subgroup of the normalizer $N(T)$ in $\mathrm{GSp}(4,k)$ of a maximal torus $T \subset \mathrm{GSp}(4,k)$ and $\pi_0(G)$ is contained in the Weyl group D_8 of $\mathrm{GSp}(4,k)$, or

– (later case 3) $\pi_0(G)$ is a finite subgroup of $\mathrm{PGl}(2,k)$ and $G^0 = \mathrm{Gl}(2,k)$.

In the cases 1 and 2 there exists a subgroup of index two in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, such that the restriction of $\rho_{\Pi,\lambda}$ to this subgroup becomes reducible. This follows from appendix B and C, the case 1 being obvious. In the situation of Theorem IV the same also holds for case 3, since then the representation $s = \rho \times t$ considered in lemma B.2 is odd. Therefore appendix D, remark 2 implies that $\rho : \widetilde{N} \to \mathrm{Gl}(2,k)$ is even, since the standard representation t is odd. This rules out the case $\pi_0(G) \cong A_4, S_4, A_5$ (see appendix A for further details).

In the remaining eight cases of [**T**], that were not listed above, the group $\pi_0(G)$ is a subgroup of the group O of [**T**]. Hence $\pi_0(G)$ is either trivial or of order two. Furthermore G^0 is either a subgroup of $\mathrm{GSp}(4,k)$, or it is $G^0 = \mathrm{GO}(4,k)^0$ (case 10 of [**T**]). In the situation of Theorem IV below, $G^0 = \mathrm{GO}(4,k)$ cannot occur, so that: Either $\rho_{\Pi,\lambda}$ is irreducible with closure $G = \mathrm{GSp}(4,k)$ where $\rho_{\Pi,\lambda}$ induces the four dimensional standard representation of G, or $G = \mathrm{Gl}(2,k)$ where $\rho_{\Pi,\lambda}$ induces the third symmetric power of the two dimensional standard representation, or G^0 is a proper subgroup of $\mathrm{GSp}(4,k)$ such that the restriction of the representation $\rho_{\Pi,\lambda}$ becomes reducible on a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of index at most two.

Theorem IV. — *Suppose* Π *(as in theorem I) is weakly equivalent to a multiplicity one representation. Then the representation* $\rho_{\Pi,\lambda}$ *preserves a nondegenerate symplectic* $\overline{\mathbb{Q}}_l$*-bilinear form* $\langle .,. \rangle$, *such that the Galois group acts with the multiplier* $\omega_\Pi \mu_l^{-w}$

$$\langle \rho_{\Pi,\lambda}(g)v, \rho_{\Pi,\lambda}(g)w \rangle = \omega_\Pi(g)\mu_l^{-w}(g) \cdot \langle v, w \rangle, \quad g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

where μ_l *is the cyclotomic character.*

Concerning the conditions of theorem III and IV on the existence of a multiplicity 1 representation in the weak equivalence class let me remark, that a globally generic irreducible representation Π of $\mathrm{GSp}(4,\mathbb{A})$ has multiplicity one ([**V**] p. 506 and [**Sha**], [**S2**]). Of course one expects the following: If Π is a cuspidal irreducible automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$, which is not CAP and for which Π_∞ belongs to the discrete series, then Π is weakly equivalent to a globally generic representation Π_{gen}, whose archimedian component is the nonholomorphic discrete series representation of the local L-packet of Π_∞. In fact we are able to deduce this from recent results of U.Weselmann on the twisted topological formula, but this will be considered elsewhere.

To outline the organization of this paper, we first remark that the theorems I-IV are derived from hypothesis A above and hypothesis B (see section 1). The first sections 1 and 2 give some overview on facts, that follow from the study of the cohomology of Siegel threefolds. In section 1 we review results on cohomology, which can be obtained from the stabilization of the topological trace formula, and give a proof of theorem III. Section 2 gives an overview of results of Taylor. The results of these two sections allow to deduce theorem I except for some special cases. In fact these — so called critical — cases are first defined in a group theoretic way in terms of G as the cases 1 and 3 specified above. As shown in appendix B these cases can be understood in terms of the underlying automorphic representation Π: Π is D-critical. This notion is discussed in section 3. It is of relevance for the cases 1, 2 and 3. The reformulation from group theory is obtained in appendix B. In fact, in these cases the structure of the finite group $\pi_0(G)$ is of great influence, and D-critical representations arise naturally in this context. Some exceptional cases where D is large, which a priori could arise from the classification of balanced representations (appendix A) can later be excluded. In section 4 and in appendix C it is shown, that all relevant D-critical representations Π contain — after restriction from $\mathrm{GSp}(4,\mathbb{A})$ to $\mathrm{Sp}(4,\mathbb{A})$ — a theta lift from an automorphic representation π of $\mathrm{Gl}(2,\mathbb{A}_K)$ where K is a quadratic algebra over $\mathbb{Q}$ with involution σ. In sections 5-8 properties of such theta lifts are studied. In section 9 a pole number $n_K(\Pi)$ is defined. Using $n_K(\Pi)$ it is shown in section 10 and appendix C, that $\sigma(\pi) \cong \pi \otimes \chi$ holds for some character χ if D is large. The remaining sections 11 and 12 provide proofs for the theorems I and II. It is easy to see that Theorem I holds except for some exceptional D-critical cases where D is large. In section 11 these are excluded by analyzing the property

$\sigma(\pi) \cong \pi \otimes \chi$ of the associated representation π at the archimedian place. The idea for the proof of theorem II given in section 12 is similar, and amounts to exclude 1-critical representations. Appendix D is concerned with pairings and contains a proof of theorem IV. I am grateful to E. Urban for a discussion, which made me aware of an error in an earlier version of section D.

1. Multiplicity Results and Cohomology

In the following we fix an irreducible cuspidal admissible representation $\Pi = \Pi_\infty \Pi_f$ of $\mathrm{GSp}(4, \mathbb{A})$, whose infinite component Π_∞ belongs to the discrete series of weight (k_1, k_2). We do not assume Π to be unitary in this section. Instead we use a different normalization, where

$$\Pi \cong \Pi_0 \otimes \|\cdot\|^{-\frac{c}{2}}, \quad c = k_1 + k_2 - 6 = w - 3$$

and Π_0 is unitary. See also [**T**] section 1. Character twisting is understood with respect to the similitude character of the group $\mathrm{GSp}(4)$. The central characters satisfy $\omega_\Pi = \omega_0 \cdot \|\cdot\|^{-c}$. This normalization will be used in this section, the next section and in the appendices B and D. It arises naturally in the study of pairings between certain cohomology groups. In the remaining sections we deal with L-series, where we consider the unitary representation Π_0 (and where Π_0 is then often called Π for simplicity of notation).

Let $\Pi = \Pi_0 \otimes \|\cdot\|^{-c/2}$ be as above for unitary Π_0. Let $m(\Pi)$ denote its cuspidal multiplicity. In [**T**], section 1 Taylor defines a finite dimensional $\overline{\mathbb{Q}}_l$-representation $W^\bullet_{\Pi_f}$ of the absolute Galois group of $\mathbb{Q}$, such that $W^\bullet_{\Pi_f}$ arises from an λ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by extension of scalars. It is obtained from the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Π_f-isotypic component in the interior cohomology of Siegel modular threefolds (image of the cohomology of compact supports in the cohomology)

$$H^\bullet_{!}(M, \mathcal{V}_\mu(\overline{\mathbb{Q}}_l)) \cong \bigoplus_{\Pi_f} W^\bullet_{\Pi_f} \otimes_{\overline{\mathbb{Q}}_l} \Pi_f$$

for a coefficient system $\mathcal{V}_\mu$ depending on the weight (k_1, k_2). We tacitly assume the choice of an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, which allows to identify the complex representation Π_f with a representation over the field $\overline{\mathbb{Q}}_l$. See also [**T**] p. 296.

The Shimura variety M for $\mathrm{GSp}(4)$, that we consider, can be identified with the projective limit of the moduli spaces of principal polarized abelian varieties of genus $g = 2$ with symplectic level structures. It is defined over $\mathbb{Q}$. Its complex analytic points are the cosets $M = \mathrm{GSp}(4, \mathbb{Q}) \backslash (X \times \mathrm{GSp}(4, \mathbb{A}_f))$ where $X = H \cup -H$ is the union of the upper and lower Siegel halfspace of genus two; in fact $X = \mathrm{GSp}(4, \mathbb{R})/\mathrm{Stab}(iE)$ for $iE \in H$. Complex conjugation acts on $H^3(M, \mathcal{V}_\mu(\mathbb{C})) = H^3_B(M, \mathcal{V}_\mu) \otimes_{\mathbb{Q}} \mathbb{C}$ on the coefficients by sending $\eta = \eta_0 \otimes z$ to $\overline{\eta} = \eta_0 \otimes \overline{z}$ for $\eta_0 \in H^3_B(M, \mathcal{V}_\mu)$ and $z \in \mathbb{C}$. Frobenius $F_\infty \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at infinity (*i.e.* complex conjugation) acts on the $\mathbb{C}$-valued points of M and induces an antiholomorphic automorphism such that

$F_\infty(\mathrm{GSp}(4,\mathbb{Q})\cdot[Z,g_f]) = \mathrm{GSp}(4,\mathbb{Q})\cdot[\overline{Z},g_f]$ for representatives $Z \in X$ and $g_f \in \mathrm{GSp}(4,\mathbb{A}_f)$ ([**MS**], p. 309). Here $\overline{Z}$ is the complex conjugate of the period matrix $Z \in H \cup -H$.

The coefficient systems $\mathcal{V}_\mu$ are defined as follows. Up to Tate twists they are obtained from the decomposition of tensor products of the direct image sheaf $R^1p_*\overline{\mathbb{Q}}_l$, where $p : A \longrightarrow M$ is the universal principally polarized abelian variety of genus two over M. Notice that the level can be assumed to be larger than 3. The index μ indicates the choice of a rational finite dimension irreducible representation μ of $\mathrm{GSp}(4,\mathbb{Q})$ (warning: our μ is dual to the one used in [**T**]). The coefficient system admits a natural pairing

$$\lambda_\mu : \mathcal{V}_\mu \otimes_{\overline{\mathbb{Q}}_l} \mathcal{V}_\mu \longrightarrow \overline{\mathbb{Q}}_l(-c)$$

where $c = (k_1-3)+(k_2-3)$. To define it we use the Lieberman trick. This reduces to consider the following three cases.

(a) For $k_1-3=1, k_2-3=0$ we put $\mathcal{V}_\mu = Rp_*(\overline{\mathbb{Q}}_l)$. This is a smooth $\overline{\mathbb{Q}}_l$-sheaf which is pure of weight $c=1$. The pairing λ_μ is the odd Weil pairing. The corresponding rational representation μ of $\mathrm{GSp}(4,\mathbb{Q})$ is the *dual* of the four dimensional standard representation of $\mathrm{GSp}(4,\mathbb{Q})$. (This convention is reasonable, since we consider direct images in cohomology and not in 'homology').

(b) For $k_1-3=1, k_2-3=1$ we put $\mathcal{V}_\mu = R^2p_*(\overline{\mathbb{Q}}_l)^{\mathrm{prim}}$ (primitive part). Again this is a smooth $\overline{\mathbb{Q}}_l$-sheaf, which is now pure of weight $c=2$. It carries a natural pairing λ_μ defined via the polarization, and this pairing is even.

(c) For $\mu = \nu^{-1}$ the inverse of the character ν of similitudes of $\mathrm{GSp}(4,\mathbb{Q})$ the sheaf $\mathcal{V}_\mu$ is the $\overline{\mathbb{Q}}_l(-1)$ and has weight 2. The corresponding Galois action is given by $\mu_l^{-1}(\mathrm{Frob}_p) = p$, where μ_l is the cyclotomic character.

All other irreducible representations $\mathcal{V}_\mu$ are obtained as constituents $\mu \hookrightarrow (R^1p_*\overline{\mathbb{Q}}_l)^{\otimes i} \otimes (\nu^{-1})^{\otimes j}$ by decomposing the tensor product copying the way in which μ is obtained from the decomposition of the corresponding rational representations of $\mathrm{GSp}(4,\mathbb{Q})$. This easily gives $c = c(\mu) = i+2j$ in general, and the induced pairings λ_μ on $\mathcal{V}_\mu$ are always of parity $(k_1-3)+(k_2-3) \equiv k_1+k_2 \mod 2$. The trivial representation μ corresponds to the constant etale $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{V}_\mu = \overline{\mathbb{Q}}_l$.

$H^3(M,\mathcal{V}_\mu(\mathbb{C}))$ contains the cuspidal cohomology $H^3_P(M,\mathcal{V}_\mu(\mathbb{C}))$, which has a pure Hodge structure and decomposes into Hodge types $H^{p,q}_P(M,\mathcal{V}_\mu(\mathbb{C}))$. These are permuted by the $\mathbb{C}$-antilinear map $\eta \mapsto \overline{\eta}$ and the $\mathbb{C}$-linear map F^*_∞. They are preserved by the $\mathbb{C}$-antilinear map $\eta \mapsto F^*_\infty(\overline{\eta}) = \overline{F^*_\infty(\eta)}$, whose square is the identity. There is a decomposition into $\mathrm{GSp}(4,\mathbb{A}_f)$-isotypic components

$$H^3_P(M,\mathcal{V}_\mu(\mathbb{C})) = \bigoplus_{\Pi_f} V_{\Pi_f}, \quad V_{\Pi_f} = W^{\mathbb{C}}_{\Pi_f} \otimes_{\mathbb{C}} \Pi_f .$$

The $\mathrm{GSp}(4,\mathbb{A}_f)$-action is completely decomposable on the cuspidal part of the cohomology. By the comparison isomorphism with etale cohomology we obtain corresponding spaces W_{Π_f} over $\overline{\mathbb{Q}}_l$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. The spaces $W^{\mathbb{C}}_{\Pi_f} = \bigoplus W^{p,q}_{\Pi_f}$

decompose into Hodge types, such that $F_\infty^* : W_{\Pi_f}^{p,q} \to W_{\Pi_f}^{q,p}$. Similar for complex conjugation, which sends $W_{\Pi_f}^{p,q}$ to $W_{\overline{\Pi}_f}^{q,p}$. Put $V_{\Pi_f}^{p,q} = W_{\Pi_f}^{p,q} \otimes_{\mathbb{C}} \Pi_f$.

On the cuspidal part $H_P^3(M,\mathcal{V}_\mu)$ of the third cohomology group the cup product defines a map

$$H_P^3(M,\mathcal{V}_\mu) \times H_P^3(M,\mathcal{V}_\mu) \longrightarrow H_P^6(M,\mathcal{V}_\mu \otimes_{\overline{\mathbb{Q}}_l} \mathcal{V}_\mu).$$

The map $\lambda_\mu : \mathcal{V}_\mu \otimes_{\overline{\mathbb{Q}}_l} \mathcal{V}_\mu \to \overline{\mathbb{Q}}_l(-c)$ therefore induces a pairing

$$H_P^3(M,\mathcal{V}_\mu) \times H_P^3(M,\mathcal{V}_\mu) \longrightarrow H_c^6(M,\overline{\mathbb{Q}}(-c)) \xrightarrow{\ \mathrm{tr}\ } \overline{\mathbb{Q}}(-3-c) = \overline{\mathbb{Q}}_l(-w)$$

with $c = (k_1-3)+(k_2-3)$, which is written $\mathrm{tr}(\eta\cup\eta')$ for $\eta,\eta' \in H_P^3(M,\mathcal{V}_\mu)$. The Galois representation on the cuspidal cohomology group $H_P^3(M,\mathcal{V}_\mu)$ is pure of weight $w = k_1+k_2-3$. This follows from the general Weil conjectures and the fact, that the cuspidal cohomology in degree coincides with the interior cohomology. Milne p. 172, remark 1.18 determines the parity of the cup product pairing as

$$\mathrm{tr}(\eta\cup\eta') = -(-1)^{k_1+k_2}\cdot \mathrm{tr}(\eta'\cup\eta).$$

The groups $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathrm{GSp}(4,\mathbb{A}_f)$ act on $H_P^3(M,\mathcal{V}_\mu)$. The cup product pairing is equivariant in the sense, that

$$\mathrm{tr}((\sigma\times g)\cdot\eta \cup (\sigma\times g)\cdot\eta') = \mu_l(\sigma)^{-c-3}\|g\|^{-c}\cdot \mathrm{tr}(\eta\cup\eta')$$

holds for $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $g\in\mathrm{GSp}(4,\mathbb{A}_f)$. See [**T**], page 295-296.

Therefore the restriction of the cup product pairing to $(W_{\Pi_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi_f)\times(W_{\Pi'_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi'_f)$ is zero unless $(W_{\Pi_f}\mu_l^{c+3})\otimes_{\overline{\mathbb{Q}}_l}(\Pi_f\|\cdot\|^c) \cong (W_{\Pi'_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi'_f)^\vee$. Equivalently $(W_{\Pi_f})^\vee \cong W_{\Pi'_f}\otimes\mu_l^{c+3}$ and $\Pi'_f \cong (\Pi_f)^\vee\otimes\|\cdot\|^{-c}$ must hold. This is equivalent to $\Pi'_f \cong \Pi_f\otimes\omega_{\Pi_f}^{-1}\|\cdot\|^{-c}$ by the next lemma 1.1. Hence $W_{\Pi'_f} \cong W_{\Pi_f\otimes_{\overline{\mathbb{Q}}_l}\omega_{\Pi_f}^{-1}\|\cdot\|^{-c}} \cong W_{\Pi_f}\otimes\omega_{\Pi_f}^{-1}\mu_l^{-c}$. So it follows $(W_{\Pi_f})^\vee \cong W_{\Pi_f}\otimes\mu_l^3\omega_{\Pi_f}^{-1}$. In particular the one dimensional representation $\mu_l^3\omega_{\Pi_f}^{-1}$ has weight $-2w$, since W_{Π_f} is pure of weight $w=c+3$. Hence $\omega_{\Pi_f}^{-1} = \mu_l^c\omega_0^{-1}$ holds for a Dirichlet character ω_0 of finite order. Furthermore $W_{\Pi'_f}\cong W_{\Pi_f}\otimes\omega_0^{-1}$ and

$$\Pi'_f \cong \Pi_f\otimes\omega_0^{-1}.$$

Since μ_l corresponds to $\|\cdot\|$ (idele class norm of the similitude character) we therefore write $\Pi = \Pi_0\otimes\|\cdot\|^{-c/2}$ and

$$\omega_\Pi = \|\cdot\|^{-c}\omega_0.$$

Π_0 is a unitary representation with unitary central character ω_0. We have $\overline{\Pi}_f\otimes\omega_0 \cong \Pi_f$ and $\Pi'_f\cong\overline{\Pi}_f$, where complex conjugation acts by the chosen isomorphism $\overline{\mathbb{Q}}_l\cong\mathbb{C}$. Notice $\overline{\Pi}_f = \overline{\Pi}_{0,f}\otimes\|\cdot\|^{-c/2} \cong \Pi_{0,f}^\vee\otimes\|\cdot\|^{-c/2} \cong \overline{\Pi}_{0,f}\otimes\omega_0^{-1}\|\cdot\|^{-c/2} = \Pi'_f$ by the next lemma. (One might prefer to normalize the coefficient systems $\mathcal{V}_\mu$ by a Tate twist, so that $c=0$ and $w=3$ always holds. But this is possible only for even c. So the normalization chosen above is the most natural one).

1.1. Lemma. — *Any irreducible automorphic representation* Π *of* $\mathrm{GSp}(4,\mathbb{A})$ *is isomorphic to its dual, twisted by the central character:* $\Pi \cong \Pi^\vee \otimes \omega_\Pi$.

Proof. — In fact this is a local statement. Consider the trace $\chi_{\Pi^\vee}(g) = \chi_\Pi(g^{-1})$. Since g is a symplectic similitude with similitude factor $\lambda(g)$, we have $g^{-1} = \lambda(g)^{-1}\cdot J^{-1}g'J$ with

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Hence it is enough to show $\chi_\Pi(g) = \chi_\Pi(g')$ for the transposed matrix g' of g. We may assume $g \in \mathrm{GSp}(4,\mathbb{Q}_v)$. It is then enough to prove the trace identity for regular semisimple elements g by a density argument, using well known properties of the trace distribution. Obviously, g and g' are stably conjugate, since g can be diagonalized $g = \gamma a \gamma^{-1}, a' = a$ for some similitudes a,γ defined over the algebraic closure of $\mathbb{Q}_v$. The analysis of instability for the two types of unstable tori in $\mathrm{GSp}(4,\mathbb{Q}_v)$, given in [**W**], then implies, that g' and g are conjugate already over $\mathbb{Q}_v$. We leave this easy verification as an exercise for the reader. □

The cup product pairing is nondegenerate. For $\eta,\eta' \in W_{\Pi_f}\otimes\Pi_f$ we have $\eta'\cup\omega_0^{-1} \in W_{\Pi'_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi'_f$, where ω_0^{-1} denotes a section of the one dimensional ω_0^{-1}-eigenspace of $H^0(M,\overline{\mathbb{Q}}_l)$. With this notation $\eta,\eta' \mapsto \mathrm{tr}(\eta\cup\eta'\cup\omega_0^{-1})$ defines a nondegenerate pairing on the space $W_{\Pi_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi_f$, which is equivariant with respect to the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the $\mathrm{GSp}(4,\mathbb{A}_f)$-action

$$(W_{\Pi_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi_f)\otimes_{\overline{\mathbb{Q}}_l}(W_{\Pi_f}\otimes_{\overline{\mathbb{Q}}_l}\Pi_f) \longrightarrow \omega_0\mu_l^{-3-c}\otimes_{\overline{\mathbb{Q}}_l}\omega_0\|\cdot\|^{-c} = \omega_\Pi\mu_l^{-3}\otimes_{\overline{\mathbb{Q}}_l}\omega_\Pi.$$

The map θ_∞. — The map $\sigma_\infty(\eta) = \overline{\eta}\cup\omega_0$ is $\mathbb{C}$-antilinear on de Rham cohomology, so that $\sigma_\infty(V^{p,q}_{\Pi_f}) = V^{q,p}_{\Pi_f}$. The map F^*_∞ is $\mathbb{C}$-linear so that $F^*_\infty(V^{p,q}_{\Pi_f}) = V^{q,p}_{\Pi_f}$. Therefore

$$\theta_\infty(\eta) = \overline{F^*_\infty(\eta)}\cup\omega_0 = \sigma_\infty\circ F^*_\infty$$

is $\mathbb{C}$-antilinear and preserves $V^{p,q}_{\Pi_f}$. Furthermore $\theta^2_\infty = \omega_0(-1)\sigma^2_\infty = \omega_0(-1)\cdot\mathrm{id}$, since $F^*_\infty\circ\sigma_\infty = \omega_0(-1)\cdot\sigma_\infty\circ F^*_\infty$. For this notice $F^*_\infty(\omega_0) = \omega_0(-1)\cdot\omega_0$ and $\omega_{0,\infty}(-1) = \omega_{\Pi_\infty}(-1) = (-1)^{k_1+k_2} = (-1)^c$ (see appendix D). For the $\mathrm{GSp}(4,\mathbb{A}_f)$ action we get $\theta_\infty\circ\Pi_f(g_f) = \omega_0(g_f)^{-1}\Pi_f(g_f)\circ\theta_\infty$.

Dimension formulas. — In the following hypothesis B we collect all facts needed from spectral theory, which allow us to express the action of the Frobenius on cohomology in terms of Hecke operators.

Hypothesis B. — Let $K = \prod_p K_p$ be a compact open subgroup of $\mathrm{GSp}(4,\mathbb{A}_f)$, where all K_p are principal congruence subgroups of $\mathrm{GSp}(4,\mathbb{Z}_p)$. Then, for a finite set S of places, including the archimedean place, $K_p = \mathrm{GSp}(4,\mathbb{Z}_p)$ holds for all primes $p \notin S$. Let $\Pi = \otimes_v\Pi_v$ be an irreducible, cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$, which is not CAP. Suppose S contains the set of places, for which Π is not unramified. Assume the archimedean component Π_∞ belongs to the discrete series. For the corresponding coefficient system $\mathcal{V}_\mu$ on M let $H^i(M,\mathcal{V}_\mu)(\Pi_f)$ denote the generalized Π_f-eigenspace with respect to the natural action of $\mathrm{GSp}(4,\mathbb{A}_f)$ on $H^i(M,\mathcal{V}_\mu)$. Then

(1) $H^i(M,\mathcal{V}_\mu)(\Pi_f) = H^i_P(M,\mathcal{V}_\mu)(\Pi_f)$ for all i.

(2) $H^i(M,\mathcal{V}_\mu)(\Pi_f) = 0$ for $i \neq 3$ both for the complex and the etale cohomology.

For $p \notin S$ the Shimura variety $M_K = M/K$ has a model over $\mathrm{Spec}(\mathbb{Z}_p)$ with good reduction modulo p, the moduli space of principal polarized abelian varieties with level K-structure. Its special fiber $M_K(\overline{\mathbb{F}}_p)$ over a $\overline{\mathbb{F}}_p$-valued geometric point of $\mathrm{Spec}(\mathbb{Z}_p)$ allows to define corresponding generalized eigenspaces $H^i_{P,\text{ét}}(M_K(\overline{\mathbb{F}}_p),\mathcal{V}_\mu)(\Pi_f)$ of $\mathcal{H}(K)$ for the etale l-adic cohomology groups for any prime l different from p. The subspaces in $H^i(M,\mathcal{V}_\mu)(\Pi_f) = 0$ of the K-invariant vectors can be identified with the generalized $(\Pi_f)^K$-isotypic submodules $H^i(M_K,\mathcal{V}_\mu)(\Pi_f) \subset H^i(M_K,\mathcal{V}_\mu)$ with respect to the action of the Hecke algebra $\mathcal{H}(K) = C_c^\infty(K\backslash \mathrm{GSp}(4,\mathbb{A}_f)/K)$, so that

(3) There exists an isomorphism $H^i_P(M_K(\overline{\mathbb{F}}_p),\mathcal{V}_\mu)(\Pi_f) \cong H^i_P(M_K,\mathcal{V}_\mu)(\Pi_f)$, which respects the action of the Frobenius Frob_p.

For $f = \prod_{v\neq\infty} f_v$ in $C_c^\infty(K\backslash \mathrm{GSp}(4,\mathbb{A}_f)/K)$ and a prime p not in S where $f = f^p f_p$ and $f_p = 1_{K_p}$ consider the supertraces on $H^\bullet_P(M_K,\mathcal{V}_\mu)(\Pi_f)$ respectively $H^\bullet_{P,\text{ét}}(M_K(\overline{\mathbb{F}}_p),\mathcal{V}_\mu)(\Pi_f)$ and let $T(n,f^p,\Pi_f,\mu)$ denote the difference

$$4\cdot \mathrm{Trace}\Big(f^p\cdot \mathrm{Frob}_p^n; H^\bullet_{P,\text{ét}}(M_K(\overline{\mathbb{F}}_p),\mathcal{V}_\mu)(\Pi_f)\Big) - \mathrm{Trace}\Big(f^p\cdot h_p^{(n)}; H^\bullet_P(M_K,\mathcal{V}_\mu)(\Pi_f)\Big).$$

for nonnegative integers n. Here $h_p^{(n)} \in C_c^\infty(K_p\backslash \mathrm{GSp}(4,\mathbb{Q}_p)/K_p)$ is a certain spherical Hecke operator (see Kottwitz [**K**], th. 2.1.3). Then the following holds:

(4) There exists a suitable function $f^M = \prod_v f_v^M$ on $M(A) \cong \mathrm{Gl}(2,\mathbb{A}) \times \mathrm{Gl}(2,\mathbb{A})/\mathbb{A}^*$, where f_∞^M and f_v^M for $v\neq p$ depend on f^p, K, μ and f_p^M also depends on n, such that for n sufficiently large $T(n,f^p,\Pi_f,\mu)$ is equal to the stable elliptic trace $ST_e^*(h_\infty^M h_p^M f_\pi^M)$

$$T(n,f^p,\Pi_f,\mu) = ST_e^*(h_\infty^M h_p^M f_\pi^M)$$

(using notation of [**K2**],p. 189) and $T(n,f^p,\Pi_f,\mu)$ vanishes unless Π is a weak endoscopic lift.

A proof of the statements (1), (2), (3) of hypothesis B is given in [**W**]. (2) this should be a general phenomenon for Shimura varieties. Statement (4) is deduced in [**W**] from [**K2**] and the topological trace of Goresky-MacPherson [**GMP**], [**GMP2**]. [**Lau**] gives a formula for $T(n,f^p,\Pi_f,\mu)$, which holds without the assumption, that Π is not CAP. Nevertheless it seems not completely obvious how to pass from [**Lau**], th. 23.3 and prop. 20.2 to statement (4). (For a comparison we remark that our f^M differs by a sign from the one in [**Lau**], since normalization of the transfer factor Δ_∞ in *loc. cit.* differs from the one used in [**KS**] or [**W**]).

We now apply hyp. B(2) to obtain dimension formulas. Assume that Π is not CAP nor a weak endoscopic lift. By the assumptions then $W^i_{\Pi_f} = 0$ vanishes except for the middle cohomology degree $i = 3$. We therefore write $W_{\Pi_f} = W^3_{\Pi_f}$. An irreducible cuspidal representation Π contributes nontrivially $W_{\Pi_f} \neq 0$ iff Π_∞ belongs to the discrete series of weight (k_1, k_2). Let Π^+_∞ and Π^-_∞ denote the discrete series representations of weight (k_1, k_2) of Whittaker type respectively of holomorphic type. Abbreviate $m^+(\Pi_f) = m(\Pi^+_\infty \Pi_f)$ and $m^-(\Pi_f) = m(\Pi^-_\infty \Pi_f)$. Then the dimension of W_{Π_f} is

$$\dim_{\overline{\mathbb{Q}}_l}(W_{\Pi_f}) = 2 \cdot m^+(\Pi_f) + 2 \cdot m^-(\Pi_f).$$

The factor 2 is due to the two Hodge types $(k_1 + k_2 - 3, 0)$ and $(0, k_1 + k_2 - 3)$ resp. $(k_1 - 1, k_2 - 2)$ and $(k_2 - 2, k_1 - 1)$, to which Π_∞ contributes. In particular $\dim_{\overline{\mathbb{Q}}_l}(W_{\Pi_f})$ is even.

Ramanujan's conjecture. — Still assume, that Π is neither CAP nor a weak endoscopic lift. Fix a nonarchimedian place v different from l, where Π_v is unramified. Then by hyp. B(4) the trace identity $T(n, f^p, \Pi_f, \mu) = 0$ holds for all sufficiently large integers n. Since by definition $T(n, f^p, \Pi_f, \mu)$ is a finite sum $\sum_\nu c_\nu z_\nu^n$ with $c_\nu \in \mathbb{Z}, z_\nu \in \mathbb{C}$ for all n, $T(n, f^p, \Pi_f, \mu) = 0$ follows for all n. Summing over $n \geqslant 0$ then implies, that the local L-factors at v satisfy the identity

$$(*) \qquad \prod_{\Pi'_f} \det(1 - \mathrm{Frob}_v \,|W_{\Pi'_f} \cdot p_v^{-s})^{-4} = \prod_{\Pi'} L_v(\Pi'_v, s - 3/2)^{2 \cdot m(\Pi')}.$$

The products are over all isomorphism classes of irreducible cuspidal automorphic representations Π'_f of $\mathrm{GSp}(4, \mathbb{A}_f)$ respectively Π' of $\mathrm{GSp}(4, \mathbb{A})$, which are unramified at v and for which Π' is isomorphic to Π at all places different from v and ∞.

By the Weil conjectures and the cuspidality assumption the Euler factors on the left side of the formula above all have weight $3 + c$. Hence the unitary representation $\Pi_v \otimes \|\cdot\|_v^{c/2}$ satisfies Ramanujan's conjecture at the unramified place v. By varying v (and by varying l if necessary) Ramanujan's conjecture therefore holds for all unramified nonarchimedian places of Π. *A refinement*: In general it is expected, that weakly equivalent cuspidal irreducible representations Π and Π' of $G(\mathbb{A})$ for a reductive connected group G over $\mathbb{Q}$ are locally isomorphic at all nonarchimedian places v, where Π and Π' are unramified. This in fact would follow from a good theory of L-series for such representations. For the group GSp(4) a good theory for the spinor L-series exists ([**PS**]). This implies

1.2. Lemma. — *Suppose Π and Π' are weakly equivalent irreducible cuspidal automorphic representations of* $\mathrm{GSp}(4, \mathbb{A})$. *Assume $\Pi_\infty \cong \Pi'_\infty$ or assume that Π_∞, Π'_∞ are discrete series representations of the same weight (k_1, k_2). Then Π_v and Π'_v are isomorphic for all places v, where both Π and Π' are both unramified, and the Ramanujan conjecture holds for these places.*

Proof. — That Ramanujan's conjecture holds here more precisely means, that it holds at the unramified places of the unitary representations $\Pi\otimes\|\cdot\|^{c/2}$ and $\Pi'\otimes\|\cdot\|^{c/2}$. For simplicity of notation assume $c=0$ for the proof. From the global functional equation of the L-series of Π and Π' ([**P**],[**PS**]) we obtain $\prod_v \gamma(\Pi,s)=\prod_v \gamma(\Pi',s)$. This is a finite product over all places v, where the representations are not isomorphic. From our assumption $\Pi_\infty\cong\Pi'_\infty$ at the archimedian place we obtain $\gamma_\infty(\Pi,s)=\gamma_\infty(\Pi',s)$; for the slightly more general case considered here this also holds, at least up to a constant, by the remark below. Therefore, being rational functions in p_v^{-s}, we get $\gamma_v(\Pi,s)=c_v\cdot\gamma_v(\Pi',s)$ for certain constants c_v for all v. For unramified Π_v,Π'_v the ε-factors are well known and easy to compute. This immediately implies

$$L_v(\Pi^\vee,1-s)/L_v(\Pi,s)=L_v(\Pi'^\vee,1-s)/L_v(\Pi',s).$$

By comparing both sides $L_v(\Pi,s)=L_v(\Pi',s)$ follows. A cancellation of Euler factors is not possible, since Ramanujan's conjecture holds. The equality of L-factors for the unramified representations in turn implies $\Pi_v\cong\Pi'_v$.

Concerning the archimedian place: Suppose Π_∞ and Π'_∞ are in the same discrete series L-packet. Then by hyp. A(6) one finds two auxiliary weakly isomorphic representations constructed as weak endoscopic lifts, with the same central character and these given representations at infinite in the discrete series L-packet $\Pi^+_\infty,\Pi^-_\infty$ of fixed weight (k_1,k_2). Then a similar argument as above, applied for these auxiliary representations, gives $\gamma_\infty(\Pi^+_\infty,s)=c_\infty\cdot\gamma_\infty(\Pi^-_\infty,s)$ for some nonvanishing constant c_∞. This completes the proof. □

If we insert the information from lemma 1.2 into the formula $(*)$, we obtain an improved formula, since now we know that to the product on the left side of formula $(*)$ only the single representation $\Pi'_f=\Pi_f$ can contribute. On the right side of formula $(*)$ only the archimedian component of $\Pi'=\Pi'_\infty\Pi'_f$ may vary in $\Pi'_\infty\in\{\Pi^\pm_\infty(\sigma_\infty)\}$. We thus obtain

1.3. Corollary. — *Suppose that* Π *is an irreducible cuspidal automorphic representation of* $\mathrm{GSp}(4,\mathbb{A})$. *Suppose* Π *is neither CAP nor a weak endoscopic lift. If* Π_∞ *is in the archimedian local discrete series L-packet* $\Pi^+_\infty,\Pi^-_\infty$ *of weight* (k_1,k_2), *and if* v *is some unramified place of* Π *different from* l, *then the following holds*

$$L_v(\Pi_v,s-3/2)^{2\cdot m^+(\Pi_f)+2\cdot m^-(\Pi_f)}=\det(1-\mathrm{Frob}_v\,|W_{\Pi_f}\cdot p_v^{-s})^{-4}.$$

Some additional remark. — The Tchebotarev density theorem for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations implies that the semisimplification of Galois representations W_{Π_f} attached to weakly isomorphic irreducible cuspidal representations are isomorphic. In other words, the semisimplification $W^{\mathrm{ss}}_{\Pi_f}$ of the Galois representation only depends on the weak equivalence class of Π. Let us draw some consequences from theorem I and corollary 1.3. Together they imply that $W^{\mathrm{ss}}_{\Pi_f}=n\cdot\rho_{\Pi,\lambda}$ has to be an isotypic multiple of the four dimensional representation $\rho_{\Pi,\lambda}$ defined in theorem I. In fact by corollary 1.3 it has to be an isotypic multiple of some λ-adic representation ρ. If ρ where different from $\rho_{\Pi,\lambda}$, then $\rho_{\Pi,\lambda}$ were a multiple of ρ. On the other hand the dimension of $\rho_{\Pi,\lambda}$ is four and $\rho_{\Pi,\lambda}$ is not reducible of form $\rho_{\Pi,\lambda} \cong 2 \cdot \rho_0$ for some two dimensional representation ρ_0 by theorem II. Therefore $\rho = \rho_{\Pi,\lambda}$ holds. Therefore the dimension formula of cor. 1.3 implies

$$L_v(\Pi_v, s - 3/2)^{(m^+(\Pi_f)+m^-(\Pi_f))/2} = \det(1 - \rho_{\Pi,\lambda}(\mathrm{Frob}_v) \cdot p_v^{-s})^{-n}.$$

If we replace Π by its unitary character twist $\Pi_0 = \Pi \otimes \|\cdot\|^{c/2}$, the left side becomes a power of the L-factor $L_p((\Pi_0)_p, s - w/2)$ defined in theorem I. A comparison with theorem I gives

1.4. Lemma. — *Suppose Π is an irreducible cuspidal automorphic representation, which is neither a CAP-representation nor a weak endoscopic lift such that Π_∞ belongs to the discrete series. Then $m^+(\Pi_f) + m^-(\Pi_f) = 2n$ is an even integer.*

We remark that the corresponding assertion becomes false, if Π is CAP or a weak endoscopic lift. In the second case $m^+(\Pi_f) + m^-(\Pi_f) = 1$ holds by hypothesis A. For the first case see [**P**].

***1.5. Proposition* (Stability).** — *Suppose Π is an irreducible cuspidal automorphic representation, so that Π_∞ belongs to the discrete series. Suppose Π is neither a CAP-representation nor a weak endoscopic lift. Suppose Π is weakly equivalent to a multiplicity one representation Π' (for instance a globally generic representation). Then*

$$m^+(\Pi'_f) = m^-(\Pi'_f)$$

holds for all Π', which are weakly equivalent to Π.

Proof. — Replace Π' by Π. By multiplicity 1 for Π either $m^+(\Pi_f)$ or $m^-(\Pi_f) = 1$. Therefore $m^{\pm 1}(\Pi_f) > 0$ by the last lemma 1.4. The Galois representation on the underlying $\mathbb{Q}_l$-vector space is Hodge-Tate by [**CF**] theorem 6.2(ii). The theorem of Sen therefore implies the existence of four different eigenvalues each with multiplicity $m^+(\Pi_f)$, $m^+(\Pi_f)$, $m^-(\Pi_f)$, $m^-(\Pi_f)$ respectively, as in [**T**] at the end of section 1, p. 296, Theorem I and corollary 1.3 relate the Galois representation with the L-series and Tchebotarev implies, that Galois substitutions with four different eigenvalues have eigenspaces of equal dimension $2m^+(\Pi_f) + 2m^-(\Pi_f)$. But since the multiplicities are equal, we obtain from the theorem of Sen $m^+(\Pi_f) = m^+(\Pi_f) = m^-(\Pi_f) = m^-(\Pi_f)$. This proves $m^+(\Pi_f) = m^-(\Pi_f)$. Since the Galois representation depends only on the weak isomorphism class of Π, this arguments extends to any Π' in the weak equivalence class of Π. □

2. A review of Taylor's Results

We introduce some notation following [**T**], section 1 and 2. Fix an irreducible representation Π such that Π_∞ belongs to the discrete series of weight (k_1, k_2). Assume the central characters of Π normalized to be nonunitary as in the last section. Assume Π is neither CAP nor a weak endoscopic lift. Let S be a finite set of places of $\mathbb{Q}$ containing all archimedian places, and all places where the representations Π is ramified. Consider the semisimplification $W^{ss}_{\Pi_f}$ of the Galois representation W_{Π_f}. This notation is consistent with [**T**], top of p. 297, since $\Pi^\vee \cong \Pi \otimes \omega_\Pi^{-1}$ by lemma 1.1.

The representation W attached to Π. — Put $k = \overline{\mathbb{Q}}_l$. Define W to be the finite dimensional k-vector space $W = W^{ss}_\Pi \oplus W^{ss}_\Pi \oplus W^{ss}_\Pi \oplus W^{ss}_\Pi$, The absolute Galois group acts on W. Let G denote the Zariski closure of the image of the Galois group. This is a reductive group G defined over k. The obvious embedding defines a faithful algebraic representation $s : G \hookrightarrow \mathrm{Gl}(W)$. For ease of notation we also write G for its group of k-valued points. Then we have a homomorphism

$$r : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow G.$$

Let G^0 be the connected component of G in the Zariski topology. Let T be a maximal split torus in G^0. Let $R \in X^*(T)$ denote the roots of T of the representation s.

Properties of Π ([**T**], *lemma 1 and corollary 1*)

(a) There exists an algebraic homomorphism $n : G \to k^*$, $n|G^0 \neq 1$, which is nontrivial on G^0.

(b) For all $g \in G$ the matrix $s(g) \in \mathrm{Gl}(W)$ has at most 4 eigenvalues, which come in at most two pairs $\alpha, n(g)/\alpha$.

(c) *Root condition*: For all roots $\lambda \in R$ we have $\lambda^2 \neq n$ as characters on T.

As explained in the last section we have from the stabilization of the trace formula:

(d) The irreducible unitary cuspidal automorphic representation $\Pi_0 = \Pi \otimes \|\cdot\|^{c/2}$ satisfies the Ramanujan conjecture for all unramified nonarchimedian places (Π is not CAP by assumption).

(e) By cor. 1.3 the k-dimension of W is $4m$ and

$$L_v(\Pi, s - 3/2)^m = \det(1 - s(\mathrm{Frob}_v)p_v^{-s})^{-1}$$

for all unramified nonarchimedian places v. (Π is not CAP nor a weak endoscopic lift). We may furthermore assume $n \circ r = \omega_\Pi \mu_l^{-3}$.

A list of possible cases ([**T**], *p. 298*). — We list the different possibilities for (G, s). This list was proven in [**T**] under the hypotheses (a), (b), (c) together with a weaker version of (e). There are 11 possible cases for the group G^0 and the representations $s|G^0$. In the cases 4–11 of this list the index G/G^0 is at most 2, since the quotient group G/G^0 embeds into the group O in these cases 4–11 ([**T**], page 299 bottom). The group O is tabulated in the right column of the list, and is either trivial or cyclic of order two in these cases. (See also the discussion on the bottom of page 301 of [**T**] and the lemma below. The critical cases occur in Proposition 1, part 5 and 6 of [**T**]).

The cases 1 and 3 of Taylor's list will be called the *critical cases*.

First critical case (case 1). — $G^0 = \mathbb{G}_m^r$ for $r = 1, 2$ and $s|G^0 = (\chi \oplus n\chi^{-1})^{2m}$ and $O = \mathbb{Z}/2\mathbb{Z}$. The center $Z(G^0) = G^0$ is connected.

Second critical case (case 3). — $G^0 = \mathrm{Gl}(2)$ and $s|G^0 = (st)^{2m}$ and $O = \{1\}$, where st denotes the 2-dimensional standard representation. The center $Z(G^0) = \mathbb{G}_m$ is connected.

The additional conditions (d) and (e) simplify Taylor's list. The proof of the main theorem reduces to the two critical cases

2.1. Lemma. — *In all the cases of Taylor's list except in the* critical *cases W is a isotypic multiple of a 4-dimensional representation ρ_Π and the main theorem holds.*

Proof. — It is enough to show, that except in the critical cases W is an isotypical copy of a 4-dimensional representation of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We prove slightly more: Property (e) further simplifies Taylor's list, since each eigenvalue of $s(g)$, $g \in G^0$ occurs with multiplicity divisible by m where $\dim(W) = 4m$. In particular, the coefficient e, f which appear in the second column of this list of [**T**] must satisfy: $e = f = m$ in case 2, 4, 5 and 9 and $f = 0$ in case 8. Here it is supposed that $e, f > 0$ in case 2, $f > 0$ in case 4 and 5, $e, f > 0$ in case 9. This implies, that in all the cases 4–11 of this list, the representation s of G^0 on W is a m-fold direct sum of a four dimensional representation $\widetilde{s} : G \to \mathrm{Gl}(4, k)$. The image $\widetilde{s}(G^0)$ is contained in $\mathrm{GSp}(4, k)$ in the cases 2, 4, 5, 6, 7, 8, 9, 11 and in $\mathrm{GO}(4, k)$ in case 10. Since the group O is independent from e, and $N \subset G^0$ in cases 4–11 and contained in the scalars for case 2 — see [**T**], p. 299 — we conclude that also W decomposes as a representation of the full group G into a m-fold isotypic direct sum of a 4-dimensional representation ρ_Π with similar restrictions of the image. This completes the proof of the lemma. □

Notations. — We need further information on the group $\pi_0(G)$ of connected components in these cases. Let $\overline{G} \subset G$ be the kernel of the natural map $G \to O$, where $O = \{g \in \mathrm{Out}(G^0) \mid s \circ g|G^0 \cong s|G^0\}$. Then

$$G/\overline{G} \hookrightarrow O.$$

Let N be the centralizer of G^0 in $\overline{G}$. Then $N \cap G^0 = Z(G^0)$ and $\overline{G} = (N \times G^0)/Z(G^0)$ and we have exact sequences

$$\begin{array}{ccccc} & & (G^0)_{\mathrm{ad}} & & \\ & & \uparrow\!\!\uparrow & & \\ G^0 & \hookrightarrow & \overline{G} & \twoheadrightarrow & N/Z(G^0) \\ \uparrow & & \uparrow & & \\ Z(G^0) & \hookrightarrow & N & & \end{array}$$

for the normal subgroups G^0 and N respectively. Since $\overline{G}/G^0$ imbeds into the finite group $\pi_0(G)$ of connected components, the group $N/Z(G^0)$ is a finite group. In particular $Z(G^0) = N^0$, if the center of G^0 is connected.

For the proof of theorem I it remains to understand, what happens in the critical cases 1 and 3 of the table in [**T**], p. 298. From now on consider the cases one and three of Taylor's list. In the two critical cases the group $Z(G^0)$ is connected and therefore $Z(G^0) = N^0$, which is a torus over k of rank r for $r = 1$ or $r = 2$. Hence

$$\pi_0(\overline{G}) \cong N/Z(G^0) \cong \pi_0(N),$$

which is a subgroup of $\pi_0(G)$ of index at most 2 in the two critical cases

$$0 \longrightarrow \pi_0(N) \longrightarrow \pi_0(G) \longrightarrow G/\overline{G} \longrightarrow 0.$$

The group $G/\overline{G}$ may be nontrivial only in the first critical case. We later distinguish the two subcases 1a and 1b, where $G/\overline{G}$ is trivial or nontrivial $G/\overline{G} = \mathbb{Z}/2\mathbb{Z}$.

The finite group $\widetilde{N}$. — N and $\overline{G}$ can be obtained as pushouts. Choose an integer n which annihilates $H^2(\pi_0(N), k^*)$, e.g. which annihilates the order of $\pi_0(N)$. (n should not be confused with the character n introduced earlier.) There exists a *finite* group $\widetilde{N}$ and a central extension

$$0 \longrightarrow (\mu_n)^r \longrightarrow \widetilde{N} \longrightarrow \pi_0(N) \longrightarrow 0,$$

such that $N = (\widetilde{N} \times N^0)/(\mu_n)^r$ and $\overline{G} = (\widetilde{N} \times G^0)/(\mu_n)^r$, where $(\mu_n)^r$ is the group of n-torsion points in the torus $N^0 \cong Z(G^0)$. The restriction of the representation s of G to the finite subgroup $\widetilde{N}$ defines a faithful representation

$$\rho : \widetilde{N} \longrightarrow \mathrm{Gl}(W).$$

A detailed analysis of the representation ρ in the critical cases is given in appendix B. It shows, that either ρ or its restriction to a subgroup of index two is a balanced representation, and these balanced representations are classified in appendix A. This leads to the proposition 3.1 of the next section.

3. D-critical automorphic representations

In the following sections we analyze the analytic behaviour of certain L-series. For this it is most convenient to assume the underlying cuspidal irreducible automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A})$ to be *unitary*. As explained already, this unitary normalization differs from the algebraic normalization used in the last sections by the twist $\Pi_{\mathrm{alg}} = \Pi_{\mathrm{unit}} \|\cdot\|^{-c/2}$ for $c = w - 3$. From now on we assume, that Π is unitary (except for appendix B and D where the unitary representation will be called Π_0).

Let χ denote a Dirichlet character of finite order. The L-series $\zeta(\Pi, \chi, s)$ studied below does not change, if we replace Π by a character twist. Therefore we often switch between unitary representations Π and Π', which only differ by a twist with an idele character. Sometimes the character twist Π' of Π need not even be automorphic.

Outside a finite set S of bad places including the archimedean place, the representation $\Pi = \otimes_v \Pi_v$ has unramified local representations Π_v and is completely characterized by its Satake parameters $(\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v)$. These four complex parameters uniquely determined by Π_v up to a reparameterization under the Weyl group, such that $\nu_v \mu_v = \widetilde{\nu}_v \widetilde{\mu}_v$. There are two L-series attached to Π. One of them is the degree four spinor L-series

$$L^S(\Pi \otimes \chi, s) = \prod_{v \notin S} L_v(\Pi \otimes \chi, s),$$

which appeared in theorem I. The other one is the degree 5 standard L-series

$$\zeta^S(\Pi, \chi, s) = \prod_{v \notin S} \zeta_v(\Pi, \chi, s).$$

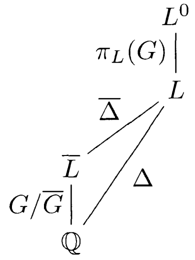
The local L-factors $\prod_j (1 - c_{v,j} \chi_v(p_v) p_v^{-s})^{-1}$ in the first case are obtained from the constants $c_{v,j}$, $j = 1, \ldots, 4$, which are the Satake parameters $\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v$ of Π_v. The second degree 5 L-series is obtained from choosing the constants $c_{v,j}$ in $\{1, \frac{\nu_v}{\widetilde{\mu}_v}, \frac{\widetilde{\mu}_v}{\nu_v}, \frac{\nu_v}{\widetilde{\nu}_v}, \frac{\widetilde{\nu}_v}{\nu_v}\}$.

In section 2 the proof of theorem I has been reduced to the study of two critical cases. In these critical cases, the properties (a)-(e) of the automorphic representation Π (listed in the last section) have to be reformulated in terms of the Satake parameters of Π. By elementary group theory this is done in the appendices A and B. To summarize the result we introduce further notation:

In appendix B we define a cyclic normal subgroup $\pi_L(G) \subset \pi_0(\overline{G})$ of $\pi_0(G)$, which is trivial except in case 1a. Let Δ be the quotient group

$$0 \longrightarrow \pi_L(G) \longrightarrow \pi_0(G) \longrightarrow \Delta \longrightarrow 0$$

and let L be the finite extension of $\mathbb{Q}$ with Galois group Δ defined by the surjective homomorphism $r : \mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \to \pi_0(G) = G/G^0 \to \Delta$. Let $\overline{L}$ be the field attached to the quadratic character with values in $G/\overline{G}$. Hence $\overline{L} = \mathbb{Q}$ if $G = \overline{G}$, or $\overline{L}$ is a quadratic extension of $\mathbb{Q}$ contained in L. Let $\overline{\Delta}$ denote the Galois group $\mathrm{Gal}(L/\overline{L})$. This gives field extensions



There exists a finite set of roots of unity ζ_v and a set T of places of Dirichlet density zero containing S, such that the Satake parameters of Π_v for $v \notin T$ have the shape $(\nu_v, \nu_v \zeta_v^{-1}, \mu_v, \mu_v \zeta_v)$ with $\omega_{\Pi,v}(p_v) = \nu_v \mu_v$. Furthermore $\zeta_v = 1$ holds if and only if v

splits in L. The logarithmic local zeta factor for $v \notin T$ has the following asymptotic expansion

$$\log \zeta_v(\Pi_v, \chi_v, s) = w_v \cdot \chi(p_v) p_v^{-s} + O(p_v^{-2s})$$

with real *weights* $w_v = \zeta_v + \zeta_v^{-1} + {}^*\mathrm{Ad}_v$ and real numbers $-1 \leqslant {}^*\mathrm{Ad}_v \leqslant 3$ defined by

$${}^*\mathrm{Ad}_v = 1 + \frac{\nu_v}{\mu_v \zeta_v} + \frac{\mu_v \zeta_v}{\nu_v}.$$

In fact, should theorem I not hold for Π, lemma B.2 and lemma B.3 of the appendix B imply: The roots of unity are $\zeta_v = \pm 1$ and the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is either an elementary abelian 2-group of order $D \geqslant 4$, or it is dihedral with normal elementary abelian subgroup $\mathrm{Gal}(L/\overline{L})$ of order $\overline{D} \geqslant 4$. More precisely

3.1. Proposition. — *If under the assumptions of theorem I the assertion of theorem would not hold for* Π, *then* Π *is* D*-critical either of CM type with* $D \geqslant 8$ *or nondegenerate of two-abelian type with* $D \geqslant 4$ *in the following sense*

3.2. Definition. — A unitary irreducible cuspidal representation Π of $\mathrm{GSp}(4, \mathbb{A})$ is called *D-critical*, if it is neither CAP nor a weak endoscopic lift and if the following holds:

(i) There exists a Galois extension $L/\mathbb{Q}$ with Galois group of degree $D = [L : \mathbb{Q}]$, a finite set S of $\mathbb{Q}$-places containing the ramified and archimedian places of Π and L, a set T of $\mathbb{Q}$-places containing S of Dirichlet density zero, such that the following holds: For all $v \notin T$ the representation Π_v has Satake parameters

$$(\nu_v, \varepsilon_v \nu_v, \mu_v, \varepsilon_v \mu_v)$$

with $\varepsilon_v = \pm 1$ and $\varepsilon_v = 1$, if and only if v splits in $L/\mathbb{Q}$.

(ii) For $v \notin T$ we have $\omega_\Pi(p_v) = \nu_v \mu_v$ for the central character ω_Π of Π.

(iii) The Ramanujan conjecture holds for all $v \notin S$, hence $|\nu_v| = |\mu_v| = 1$ holds for all $v \notin T$.

A D-critical representation Π will be called *nondegenerate*, if $(\nu_v/\mu_v)^{2D} \neq 1$ for all $v \notin T$. It is said to be of *abelian* resp. *two-abelian type*, if the Galois group $\Delta = \mathrm{Gal}(L/\mathbb{Q})$ is an abelian group resp. an elementary abelian 2-group. It is said to be of *CM type*, if Δ contains an elementary abelian 2-group $\overline{\Delta}$ as normal subgroup of index 2 and order $\overline{D} \geqslant 4$, whose fixed field is a distinguished quadratic extension field $\overline{L}$ of $\mathbb{Q}$ in L such that for $v \notin T$ the Satake parameters are determined by a pair of Grossencharacters as in Lemma B.5 and B.6 of appendix B.

3.3. Remark. — For a D-critical representation Π define $\mathrm{Ad}_v = 1 + \frac{\mu_v}{\nu_v} + \frac{\nu_v}{\mu_v}$. Condition (iii) and the discussion preceding prop 3.1 imply

(iv) The numbers $\mathrm{Ad}_v, v \notin T$ are real and satisfy $-1 \leqslant \mathrm{Ad}_v \leqslant 3$ and the weights of the asymptotic expansion of the logarithmic zeta function $\log \zeta_v(\Pi_v, \chi_v, s)$ are

$$w_v = 1 + \varepsilon_v \cdot (\mathrm{Ad}_v + 1).$$

Therefore property (iii) implies for $s \to 1^+$ the asymptotic behaviour

$$\log \zeta^S(\Pi, \chi, s) \sim \log L^S(\chi, s) + \sum_v \varepsilon_v \cdot (\mathrm{Ad}_v + 1) \cdot \chi(p_v) p_v^{-s},$$

for all unitary characters χ of $\mathbb{A}^*/\mathbb{Q}^*$.

4. Theta lifts

From now on assume Π is D-critical as in proposition 3.1 of the last section. We claim that this implies

4.1. Proposition. — *Suppose* Π *is* D*-critical of abelian or of CM type. Then the restriction of* Π *to* $\mathrm{Sp}(4, \mathbb{A})$ *contains a theta lift in the sense of theorem 4.2.*

This proposition is the quintessence of the following theorem 4.2 and the next three lemmas, which summarize results from appendix A and B.

Let $T \in \mathrm{Symm}^2(\mathbb{Q}^2), \det(T) \neq 0$ belong to a nondegenerate non vanishing Fourier coefficient of Π as in [**KRS**], p. 531. Such T always exists, and T defines a binary quadratic space V_T over $\mathbb{Q}$ with character $\chi_T = (., \Delta(V_T))$, where $\Delta(V) = (-1)^{\dim(V)/2} \det(V)$ is the discriminant of the quadratic space V of even dimension over $\mathbb{Q}$. Its value in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ only depends on the isomorphism class of the quadratic space. Note $\Delta(V \oplus V') = \Delta(V)\Delta(V')$.

4.2. Theorem **([S], [KRS], th. 7.1)**. — *Suppose* Π *is unitary and cuspidal. If* $\zeta^S(\Pi, \chi_0, s)$ *has a pole at* $s = 1$ *(for some sufficiently large finite set* S *and some unitary character* χ_0*) then*

(1) $\zeta^S(\Pi, \chi_0, s)$ *has a simple pole at* $s = 1$.

(2) *The restriction of* Π *to* $\mathrm{Sp}(4, \mathbb{A})$ *contains a theta lift from some automorphic representation of the orthogonal group* $\mathrm{O}(V, \mathbb{A})$*, where* V *is a four dimensional quadratic space* $V_T \oplus V_{T'}$. V_T *is as above, and* $V_{T'}$ *is the binary quadratic space with quadratic character* $\chi_0 \chi_T$.

Let K be the rank two commutative algebra over $\mathbb{Q}$, which is attached to the quadratic character $\chi_K = \chi_0$ by class field theory. Note, that χ_K is the quadratic character attached to the discriminant $\Delta(V)$ of $V = V_T \oplus V_{T'}$. K is a quadratic field extension of $\mathbb{Q}$ unless χ_K is the trivial character. Then $K = k^2$.

4.3. Remark **([W1], section 3)**. — In [**S**] the important assertion 4.2.1 is formulated only for $\chi_0 = \chi_T$. In general it can be deduced from [**S**], th. 2.4. This also implies, that a unitary character χ_0, for which $\zeta^S(\Pi, \chi_0, s)$ has a pole at $s = 1$, must be quadratic $\chi_0^2 = 1$ (see also the introduction of [**KRS**]).

4.4. Lemma. — *Suppose Π is D-critical of abelian type. Then for $s \to 1^+$ we have the asymptotic behaviour*

$$\log\Big(\prod_{\chi\in\widehat{\Delta}}\zeta(\Pi,\chi,s)\Big) \sim D\cdot \sum_{v\ L\text{-split}} (\mathrm{Ad}_v+2)\cdot p_v^{-s},$$

where the product is over all D characters χ of the Galois group Δ of $L/\mathbb{Q}$. Hence $\zeta(\Pi,\chi_0,s)$ has a pole at $s=1$ for at least one of the characters $\chi_0\in\widehat{\Delta}$, whereas $\prod_{\chi\in\widehat{\Delta}}\zeta(\Pi,\chi,s)$ has a pole at $s=1$ of order at most 5.

Proof. — The first statement is an immediate consequence of the asymptotic formula for the logarithmic zeta function, stated at the end of the last section. The primes $p_v, v\in T$ which are completely split in L, are called L-split. They have Dirichlet density $1/D$. Since $1\leqslant \mathrm{Ad}_v+2\leqslant 5$ for these primes, we get $\log\zeta(s)\prec \log(\prod_\chi \zeta(\Pi,\chi,s))\prec 5\cdot\log\zeta(s)$, where $\prec$ means $\leqslant$ up to some function growing like $o(\log\zeta(s))$ in the limit $s\to 1^+$. Hence $\zeta(\Pi,\chi,s)$ has a pole for at least one character χ of Δ. □

4.5. Lemma. — *Let Π be D-critical of CM type. Then $D=8$ and $\mathrm{Gal}(L/\mathbb{Q})$ is either the dihedral group D_8 or elementary abelian of order eight.*

Proof. — Consider the quadratic extension field $\overline{L}$ defined by the nontrivial quadratic character $\chi_Q=\chi_{\overline{L}/\mathbb{Q}}$, related to the nontrivial quadratic character $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to G/\overline{G}$. Then $\mathrm{Gal}(L/\overline{L})$ has order $\overline{D}\geqslant 4$ with $D=2\overline{D}\geqslant 8$.

Then similar to lemma 4.4 in the limit $s\to 1^+$ we have

$$\log(\zeta(\Pi,\chi_Q,s)\zeta(\Pi,1,s)) \sim 2\cdot \sum_{v\ \overline{L}\text{-split}} (1+\varepsilon_v\cdot(\mathrm{Ad}_v+1))\cdot p_v^{-s}.$$

By lemma B.5 and B.6 of appendix B we can replace the terms $(1+\varepsilon_v\cdot(\mathrm{Ad}_v+1))$ in the sum by the weights $w_v=3$ in the case A where v is L-split, and by $w_v=-1$ in the case B where v is $\overline{L}$-split but not L-split. These cases have Dirichlet density $1/2\overline{D}$ and Dirichlet density $(\overline{D}-1)/2\overline{D}$ respectively. This asymptotically gives

$$2\cdot\Big(-\frac{\overline{D}-1}{2\overline{D}}+3\cdot\frac{1}{2\overline{D}}\Big)\cdot\log\zeta(s)=\Big(\frac{4}{\overline{D}}-1\Big)\cdot\log\zeta(s)$$

on the right side. However, as the logarithm of a meromorphic function, the number $(4/\overline{D})-1$ has to be an integer. Since $\overline{D}\geqslant 4$ this forces $\overline{D}=4$, hence implies $D=8$ and $\mathrm{Gal}(L/\mathbb{Q})\cong\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^3$ or D_8. See the example in appendix B after lemma B.3. Finally, the first group $\langle N,S\mid N^4=S^2=1, NS=SN\rangle$ is easily excluded. The character χ of this group defined by $\chi(S)=1$ and $\chi(N)=i$ takes imaginary values except on the subgroup with elements $1,N^2,S,SN^2$. This is the elementary abelian 2-group $\mathrm{Gal}(L/\overline{L})$ contained in $\mathrm{Gal}(L/\mathbb{Q})$. These elements belong to the cases A,B,B,B with statistical weights $w_1=3, w_{N^2}=w_S=w_{SN^2}=-1$ and

Dirichlet densities 1/8 in the sense of lemma B.5 of appendix B. Therefore the real part of $\log\zeta(\Pi,\chi,s)$ behaves as

$$(3/8 - 1/8\chi_N(N^2) - 1/8\chi_N(S) - 1/8\chi_N(SN^2))\cdot\log\zeta(s) = \frac{1}{2}\cdot\log\zeta(s)$$

for $s\to 1^+$. This is impossible, since $\zeta^S(\Pi,\chi,s)$ is meromorphic at $s=1$. □

4.6. Lemma. — *Suppose* Π *is D-critical of CM type with group* $\mathrm{Gal}(L/\mathbb{Q}) = D_8$ *or* $(\mathbb{Z}/2\mathbb{Z})^3$. *Then there exists an abelian character* χ *of* $\mathrm{Gal}(L/\mathbb{Q})$, *such that* $\zeta(\Pi,\chi,s)$ *has a pole at* $s=1$.

Proof. — If the Galois group is abelian, this is covered by lemma 4.4. We can therefore assume $\mathrm{Gal}(L/\mathbb{Q}) = D_8$. The group $\Delta = D_8$ has four different abelian characters $1,\chi_Q,\chi_P,\chi_R$. Their common kernel in D_8 is the group commutator group $\{1,N^2\}$ of order two. The notation is as in the remark after lemma B.3 of appendix B. The nontrivial element N^2 in the commutator group belongs to case B, whereas the element 1 belongs to case A, in the sense of lemma B.5. The corresponding statistical weights are $w_{N^2} = -1$ and $w_1 = 3$. They occur with Dirichlet density 1/8, hence

$$\log\Big(\prod_{\chi\in\widehat{\Delta}}\zeta(\Pi,\chi,s)\Big) \sim 4\cdot(3/8-1/8)\cdot\log\zeta(s).$$

Therefore the function $\zeta(\Pi,\chi,s)$ has a pole at $s=1$ at least for one character $\chi\in\widehat{\Delta}$. □

Further information on the analytic behaviour of these zeta functions will be needed later. This information can not entirely be obtained by the method used above. But as shown above Π contains a theta lift. On the other hand the primes of $\mathbb{Q}$ have a certain decomposition behaviour in the field extension $L/\mathbb{Q}$. The two facts together impose strong conditions on the possible behaviour of $\zeta(\Pi,\chi,s)$ at the point $s=1$ in the D-critical cases of CM type. For details we refer to appendix C.

5. The orthogonal group of similitudes GSO(V)

For a nondegenerate quadratic space of dimension four over a local or global field k of characteristic zero let $\mathrm{GO}(V)$ be the orthogonal group of similitudes and let $\mathrm{GSO}(V)$ be its subgroup of proper similitudes. $\mathrm{GSO}(V)$ is geometrically connected in the Zariski topology and of index two in $\mathrm{GO}(V)$. The kernel of the similitude character is the special orthogonal group $\mathrm{SO}(V)$. $\mathrm{SO}(V)$ is of index two in the orthogonal group $\mathrm{O}(V)$ and $\mathrm{GO}(V)$ is generated by $\mathrm{GSO}(V)$ and $\mathrm{O}(V)$.

$$0\longrightarrow \mathrm{SO}(V)(k)\longrightarrow \mathrm{GSO}(V)(k)\longrightarrow M(k)\longrightarrow 0,\quad (k^*)^2\subset M(k)\subset k^*.$$

Let K is a field extension of k of degree two, or $K=k\oplus k$. Let σ be the canonical nontrivial involution of the algebra K/k. The norm form defines a nondegenerate binary quadratic form over k with discriminant Δ_K, such that $K=k[T]/(T^2-\Delta_K)$.

Consider a central simple algebra D_K of rank four over K, such that $D_K = D\otimes_k K$ holds for some central simple algebra D of rank four over k. Then the involution σ of $K \hookrightarrow D_K$ extends to a σ-linear involution $d \mapsto d^*$ of D_K, such that $(d_1d_2)^* = d_2^*d_1^*$ and $(d_1+d_2)^* = d_1^*+d_2^*$ holds, such that it commutes with a fixed K-linear standard involution $z \mapsto \overline{z}$ of D_K. The symmetric elements $V = \{d \in D_K \mid d = d^*\}$ define a four dimensional k subspace of D_K. The restriction of the reduced norm $N(z) = z\overline{z}$ to V has values in k and defines a nondegenerate four dimensional quadratic space over k with discriminant $\Delta(V) = \Delta_K \bmod (k^*)^2$. The group D_K^* acts on V by $d \mapsto gdg^*$ as proper orthogonal similitudes with similitude factor $\mathrm{Norm}_{K/k}(N(g))$. The induced homomorphism of algebraic k-groups $\mathrm{Res}_{K/k}(D_K^*) \to \mathrm{GSO}(V)$ is surjective. Its kernel is the subgroup A of all elements in the center $K^x = \mathrm{Res}_{K/k}(K^*)$ of $\mathrm{Res}_{K/k}(D_K^*)$, whose K/k-norm is trivial. $\mathrm{GO}(V)$ is the group generated by $\mathrm{GSO}(V)$ and the reflection ε defined by $\varepsilon(d) = (\overline{d})^* = \sigma(d)$. Conjugation by ε acts on $\mathrm{GSO}(V)$ via $g \mapsto \sigma(g)$.

5.1. Lemma. — *Suppose k is a local or global field of characteristic zero. Suppose V is a nondegenerate quadratic space over k of dimension four and discriminant $\Delta(V)$. Let K be the algebra $k[T]/(T^2-\Delta(V))$. There exists a central simple algebra $D_K = D\otimes_k K$ of rank 4 over K extended from a central simple algebra D of rank 4 over k, such that* $\mathrm{GSO}(V)$ *is isomorphic to the quotient group defined by the exact sequence*

$$1 \longrightarrow A \longrightarrow \mathrm{Res}_{K/k}(D_K^*) \longrightarrow \mathrm{GSO}(V) \longrightarrow 1,$$

where A is the norm 1 subgroup of the center K^x of $\mathrm{Res}_{K/k}(D_K^*)$. $\mathrm{GO}(V)$ *is isomorphic to the semidirect product* $\mathrm{GSO}(V).(\mathbb{Z}/2\mathbb{Z})$, *with the action on the normal subgroup* $\mathrm{GSO}(V)$ *defined by σ.*

Fix a k-group $\mathrm{GSO}(V)$ isomorphic to $\mathrm{Res}_{K/k}(D_K^*)/A$ for some choice of K and D_K. For algebraic extension fields F of k the natural map $D_K^*(F) \to \mathrm{GSO}(V)(F)$ need not be surjective. However

$$0 \longrightarrow A(F) \longrightarrow D_K^*(F) \longrightarrow \mathrm{GSO}(V)(F) \longrightarrow H^1(F,A) \longrightarrow 0$$

is exact. Moreover $K^x/A \cong \mathbb{G}_m$ for $K^x = \mathrm{Res}_{K/k}(\mathbb{G}_m)$ under the K/k-norm, hence

$$0 \longrightarrow A(F) \longrightarrow K^x(F) \longrightarrow F^* \longrightarrow H^1(F,A) \longrightarrow 0$$

is exact. Since K^x embeds into the center of D_K^*, we obtain as in [**HST**], p. 380

5.2. Lemma. — *For* $\mathrm{GSO}(V) \cong \mathrm{Res}_{K/k}(D_K^*)/A$ *as above there exists an isomorphism*

$$\mathrm{GSO}(V)(F) \cong \big(D_K^*(F)\times F^*\big)/K^x(F)$$

of the groups of F-valued points, where the quotients are with respect to the inverse of the natural central inclusion $K^*(F) \to D_K^*(F)$ respectively the norm map $\mathrm{Norm}_{K/k}$: $K^x(F) \to F^*$. *Furthermore the projection* $p : \mathrm{GSO}(V)(F) \to F^*/K^x(F)$ *is surjective* *with kernel* $D_K^*(F)/A(F)$. *The similitude morphism* ν *is induced by* $D_K^*(F) \times F^* \ni (g,t) \mapsto \mathrm{Norm}_{K/k}(N(g)) \cdot t^2$. *In the local field case one has an exact sequence*

$$0 \longrightarrow D_K^0(F)/A(F) \longrightarrow \mathrm{SO}(V)(F) \xrightarrow{\ p\ } F^*/K^x(F) \longrightarrow 0,$$

where $D_K^0(F) \subset D_K^*(F)$ *is the subgroup of all elements* g *for which* $\mathrm{Norm}_{K/k}(N(g)) = 1$. *(Of course elements of* $F^*/K^x(F)$ *usually cannot be lifted to elements that centralize* $D_K^0(F)/A(F)$*).*

Proof. — The first statements are clear. Notice $(F^*)^2 \subset \mathrm{Norm}_{K/k}(N(D_K^*(F)))$ in the local field case. This is obvious for $F = \mathbb{C}$. For $F = \mathbb{R}$ this holds, since every positive real number is a norm. It is true in the nonarchimedian case, since then N is surjective by Eichler's theorem. Hence for $t \in F^*$ there exists $g \in D_K^*(F)$, such that $\mathrm{Norm}_{K/k}(N(g))t^2 = 1$. Hence $(g,t) \in \mathrm{SO}(V,F)$ is a preimage of the class of $t \in F^*$ under the projection map $p(g,t) = t$. The kernel of this map is $D_K^0(F)/A(F)$. □

5.3. Corollary. — *For a local field* k *of characteristic zero irreducible admissible representations of* $\mathrm{GSO}(V)(k)$ *correspond to pairs* $(\pi^\vee, \omega)$, *where* $\pi^\vee$ *is an irreducible admissible representation of* $D_K^*(k)$ *with central character* $\omega_{\pi^\vee}$ *and* ω *is a character of* k^*, *such that* $\omega_{\pi^\vee} = \omega \circ \mathrm{Norm}_{K/k}$ *of* $\pi^\vee$. *There exist at most two nonisomorphic extensions to an irreducible representation* $(\pi^\vee, \omega, \delta)$ *of* $\mathrm{GO}(V,k)$. *This extension* $(\pi^\vee)^+$ *is unique (*$\delta = +$*) iff* $\sigma(\pi^\vee, \omega) \cong (\sigma(\pi^\vee), \omega)$ *is not isomorphic to* $(\pi^\vee, \omega)$ *or equivalently iff the induced representation is irreducible or equivalently iff* $(\pi^\vee)^+$ *is isomorphic to its twist by the nontrivial character of* $\mathrm{GO}(V)(k)/\mathrm{GSO}(V,k)$. *Otherwise there are two extensions* $(\pi^\vee)^+, (\pi^\vee)^-$ *(hence* $\delta = \pm$*). This case occurs if and only if the restriction of the irreducible representation* $(\pi^\vee)^+$ *to* $\mathrm{GSO}(V)(k)$ *remains irreducible. Each of the extensions in this case is obtained from the other as twist with the nontrivial character of* $\mathrm{GO}(V)(k)/\mathrm{GSO}(V,k)$.

As in the last corollary assume k to be local. Then admissible representations $\pi^\vee$ of $D_K^*(k)$ can be related to irreducible admissible representations π of $\mathrm{Gl}(2,K)$ via the Jaquet-Langlands correspondence. Therefore irreducible admissible representations of $\mathrm{GSO}(V)(k)$ as described in corollary 5.3 may be uniquely characterized by the corresponding admissible irreducible representation (π, ω) of the group $\mathrm{Gl}(2,K) \times k^*$. Notice that $\omega_\pi = \omega \circ \mathrm{Norm}_{K/k}$ holds since $\omega_\pi = \omega_{\pi^\vee}$.

To twist a representation of $\mathrm{GSO}(V)(k)$ by a one dimensional character χ composed with the similitude homomorphism amounts to replace $(\pi^\vee, \omega)$ by $(\pi^\vee, \omega) \otimes \chi = (\pi^\vee \otimes (\chi \circ \mathrm{Norm}_{K/k}), \omega\chi^2)$ respectively to replace (π, ω) by $(\pi, \omega) \otimes \chi = (\pi \otimes (\chi \circ \mathrm{Norm}_{K/k}), \omega\chi^2)$. Since $\sigma(\pi, \omega) = (\sigma(\pi), \omega)$ and since $\chi \circ \mathrm{Norm}_{K/k}$ is invariant under σ, the notion of twist makes also sense for irreducible admissible representations of the group $\mathrm{GO}(V)(k)$.

5.4. Corollary. — *For local fields k any irreducible component of the restriction of an irreducible admissible representation $(\pi^\vee,\omega)$ of* $\mathrm{GSO}(V)(k)$ *to its subgroup* $\mathrm{SO}(V)(k)$ *determines the representation of* $\mathrm{GSO}(V)(k)$ *up to a twist as defined above.*

Proof. — By lemma 5.2 the group $\mathrm{GSO}(V)(k)$ is generated by the subgroups $\mathrm{SO}(V)(k)$ and $D_K^*(k)$ and there is an isomorphism $\mathrm{GSO}(V)(k)/\mathrm{SO}(V)/k) \cong D_K^*(k)/D_K^0(k)$. The composed norms $\mathrm{Norm}_{K/k}\circ N$ define an injection $D_K^*(k)/D_K^0(k)$ into k^* with an image of finite index in k^*. Any character on the image can be extended to a character χ on k^*. If H is a normal subgroup of G with abelian quotient, then two irreducible representations of G with a common constituent for H differ by a twist with a character of the quotient group G/H ([**V**], p. 480). Suppose $(\pi_i^\vee,\omega_i)$ are irreducible representations of $\mathrm{GSO}(V)(k)$, such that the restrictions to $\mathrm{SO}(V)(k)$ have a common constituent. This implies $\pi_1^\vee|D_K^0(k)$ and $\pi_2^\vee|D_K^0(k)$, hence $\pi_2^\vee\cong\pi_1^\vee\otimes\widetilde{\chi}$ for a character $\widetilde{\chi}$ of $D_K^*(k)/D_K^0(k)$ and moreover $(\pi_1^\vee\otimes\widetilde{\chi},\omega_1)\cong(\pi_2^\vee,\omega_2)$ as representations of $D_K^*(k)\times k^*$. The first assertion implies $\widetilde{\chi}=\chi\circ\mathrm{Norm}_{K/k}$ for some character χ of k^*. The first and the second assertion combined imply the existence of an automorphism A such that $\pi_1^\vee(g)\chi(\mathrm{Norm}_{K/k}(N(g)))\omega_2(t) = A\pi_1^\vee(g)\omega_1(t)A^{-1}$ holds for all $g\in D_K^*(k)$ and all $t\in k^*$. By Schur's lemma A is a scalar, hence can be omitted. For $t\in k^*$ there exists $g\in\mathrm{SO}(V)(k)$ with $\mathrm{Norm}_{K/k}(N(g))=t^2$. This gives $\chi^2(t)\omega_2(t)=\omega_1(t)$ and proves the corollary. □

We remark that restrictions of representations of $D_K^*(k)$ to $D_K^0(k)$ are multiplicity free. Suppose D_K is split. Then $\pi^\vee=\pi$. Hence by [**LL**], p. 737, an irreducible constituent $\widetilde{\pi}$ of the restriction of an admissible representation $(\pi^\vee,\omega)$ of $\mathrm{GSO}(V)(k)$ to $\mathrm{SO}(V)(k)$ uniquely determines $(\pi^\vee,\omega)$ up to a twist $(\pi,\omega)\otimes\chi$, where $\chi^2=1$. If D_K is not split, then $K=k^2$. Again an irreducible constituent $\widetilde{\pi}$ of the restriction of an admissible representation $(\pi^\vee,\omega)$ of $\mathrm{GSO}(V)(k)$ to $\mathrm{SO}(V)(k)$ uniquely determines $(\pi^\vee,\omega)$ up to a twist $(\pi,\omega)\otimes\chi$, where $\chi^2=1$. Now this follows from [**HPS**], lemma 7.2 and page 92 in the nonarchimedian case. For the archimedian case this follows from [**HPS**], page 95.

5.5. The theta correspondence. — The generalized theta correspondence is a correspondence between irreducible automorphic representations $(\pi^\vee,\omega,\delta)$ of $\mathrm{GO}(V,\mathbb{A})$ and irreducible automorphic representation $\Pi'=\theta(\pi^\vee,\omega,\delta)$ of $\mathrm{GSp}(4,\mathbb{A})$. It is defined for local and for global representations and we refer to Π' as a 'theta lift'. Our conventions will be those of [**V**]. In particular, Π' and $(\pi^\vee,\omega,\delta)$ have the same central character in the sense that

$$\omega=\omega_{\Pi'}.$$

To avoid confusion for later references we notice a slight difference to the conventions of [**HST**], p. 387. If Π' and $(\pi^\vee,\omega,\delta)$ are in correspondence in our sense, Π' is in correspondence with the contragredient of $(\pi^\vee,\omega,\delta)$ in [**HST**]. For a comparison with [**KRS**], where the Howe theta correspondence is considered in its proper sense — on the level of the groups $\mathrm{Sp}(4,\mathbb{A})$ and $\mathrm{O}(V,\mathbb{A})$ — we have

5.6. Lemma **([V], p. 480)**. — *Suppose k is a nonarchimedian local field as above. For irreducible admissible representations Π and $(\pi^\vee,\omega,\delta)$ of $\mathrm{GSp}(4,k)$ and $\mathrm{GO}(V)(k)$, which correspond under the generalized theta correspondence, also their twists by a character are in correspondence the generalized theta correspondence. The restrictions of these representations to $\mathrm{Sp}(4,k)$ respectively $\mathrm{O}(V,k)$ contain constituents, which are in correspondence for the $(\mathrm{Sp}(4),\mathrm{O}(V))$-Howe correspondence. Conversely, if an irreducible $\mathrm{Sp}(4,k)$-constituent $\widetilde{\Pi}$ of Π and an irreducible $\mathrm{O}(V)(k)$-constituent $\widetilde{\pi}$ of $(\pi^\vee,\omega,\delta)$ are in correspondence for the Howe theta correspondence, then there exists a quadratic character χ, such that $\Pi\otimes\chi$ and $(\pi^\vee,\omega,\delta)$ are in correspondence for the generalized theta correspondence.*

Our interest in these correspondences is the following: For an irreducible automorphic representation Π of $\mathrm{GSp}(4,\mathbb{A})$ as in proposition 4.1 or theorem 4.2, the restriction of Π to $\mathrm{Sp}(4,\mathbb{A})$ contains an irreducible constituent $\widetilde{\Pi}$ for which there exists an irreducible automorphic representation $\widetilde{\pi}$ of $\mathrm{O}(V,\mathbb{A})$ which is in correspondence with $\widetilde{\Pi}$ under the Howe theta correspondence. $\widetilde{\pi}$ can be extended to an irreducible automorphic representation $(\pi^\vee,\omega,\delta)$ of $\mathrm{GO}(V,\mathbb{A})$, such that $\omega=\omega_{\widetilde{\Pi}}$ holds. By lemma 5.2 the group $\mathrm{GO}(V,\mathbb{A})$ is obtained from $\mathrm{O}(V)(\mathbb{A})$ as a pushout from the subgroup $D_K^0(\mathbb{A})/A(\mathbb{A})$ via the inclusion $D_K^0(\mathbb{A})/A(\mathbb{A})\hookrightarrow D_K^*(\mathbb{A})/A(\mathbb{A})$. So we have to extend the automorphic representation $\widetilde{\pi}^\vee$ of $D_K^0(\mathbb{A})$ to an automorphic representation $\pi^\vee$ of $D_K(\mathbb{A})$, such that $\widetilde{\pi}^\vee$ is a constituent under $D_K^0(\mathbb{A})$. For this first extend $\widetilde{\pi}^\vee$ to $(D_K^0(\mathbb{A})\cdot K^*(\mathbb{A}))/A(\mathbb{A})$, so that $\widetilde{\pi}^\vee$ acts by $\omega_{\widetilde{\Pi}}\circ\mathrm{Norm}_{K/k}$ on the center. For the remaining extension use induction and choose an irreducible constituent $\pi^\vee$. Moreover $\pi_v^\vee$ should be chosen unramified, whenever this is possible. By the ambiguities of the choices in the last step $\pi^\vee$ is uniquely determined only up to a character twist. More precisely, $(\pi^\vee,\omega,\delta)$ is in fact unique up to a character twist which replaces $(\pi^\vee,\omega,\delta)$ by $(\pi^\vee,\omega,\delta)\otimes\chi$ by corollary 5.4. Since we insisted on fixing ω moreover $\chi^2=1$. Then by lemma 5.6 an irreducible representation Π' of $\mathrm{GSp}(4,\mathbb{A})$ exists, such that all Π'_v and $(\pi^\vee,\omega,\delta)_v$ are in correspondence for the global generalized theta correspondence. Therefore, again by lemma 5.6

$$\Pi'=\Pi\otimes\chi$$

holds for some quadratic character χ of $\mathbb{A}^*$ (a priori not necessarily automorphic). Π' need not be automorphic, since the character χ need not be automorphic. For our purposes we may ignore this, since we only use the fact that this implies, that

$$\zeta^S(\Pi,\chi',s)=\zeta^S(\Pi',\chi',s)$$

holds for all idele class characters χ' of $\mathbb{A}^*/\mathbb{Q}^*$. Here S is a sufficiently large finite set S of exceptional places outside which χ_v,χ'_v,Π_v are all unramified. That χ_v is unramified at almost all places may be achieved from the construction above.

6. The spherical lift $\Pi'(\pi, \omega)$

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ with central character ω_Π. Let S be a finite set of places, such that Π_v is unramified and nonarchimedean for $v \notin S$. Let $\Pi^S = \otimes_{v \notin S} \Pi_v$ be the restricted tensor product over constituents $v \notin S$.

6.1. Restriction to $\mathrm{Sp}(4)$. — The restriction of Π to $\mathrm{Sp}(4, \mathbb{A})$ may not be irreducible. Let $\widetilde{\Pi}$ be an irreducible constituent. Similarly let $\widetilde{\Pi}^S$ be a spherical irreducible constituent of the restriction of Π^S. If $\widetilde{\Pi}$ is suitably chosen these choices are compatible. With this notation consider now a pair of irreducible cuspidal representations $\Pi_i, i = 1, 2$ of $\mathrm{GSp}(4, \mathbb{A})$, both unramified outside S. Then the following statements are easily seen to be equivalent:

(1) $\widetilde{\Pi}_1^S \cong \widetilde{\Pi}_2^S$

(2) $\zeta^S(\Pi_1, \tau, s) = \zeta^S(\Pi_2, \tau, s)$ holds for some unramified character τ^S.

(3) $\Pi_1^S \cong \Pi_2^S \otimes \chi^S$ holds for some unramified character χ^S of $(\mathbb{A}^S)^*$ (which not be assumed to be automorphic).

This implies that the assignment $\Pi^S \mapsto (\widetilde{\Pi}^S, \omega_\Pi^S)$ now determines Π^S up to an unramified quadratic character χ^S (as in (3)), since the central character is fixed.

6.2. The unramified theta correspondence. — Choose K and D_K, so that $\mathrm{GSO}(V)$ is described as in the last section. Let S be a finite set of places containing the archimedean place, so that $K/\mathbb{Q}$ and D_K are unramified outside S. For $v \notin S$ the generalized theta correspondence matches unramified representations $(\pi_v^\vee, \omega_v, \delta_v)$ of $\mathrm{GO}(V, \mathbb{Q}_v)$ with unramified representations Π'_v of $\mathrm{GSp}(4, \mathbb{Q}_v)$ ([**V**],p. 482). Notice, for $(\pi_v^\vee, \omega_v, \delta_v) = (\pi_v^\vee, \omega_v)^+$ unramified $(\pi_v^\vee, \omega_v, -\delta_v) = (\pi_v^\vee, \omega_v)^-$ is never unramified (if it exists), and the restriction to $\mathrm{GSO}(V, \mathbb{Q}_v)$ contains an unramified constituent. So, by abuse of notation, the irreducible unramified admissible representations of $\mathrm{GO}(V)(\mathbb{Q}_v)$ are determined by the unramified irreducible representations $(\pi_v^\vee, \omega_v)$ of $(D_K^*(\mathbb{Q}_v) \times (\mathbb{A}_v)^*))/A(\mathbb{Q}_v)$. Since D_K splits for $v \notin S$ furthermore $(\pi_v^\vee, \omega_v) = (\pi_v, \omega_v)$.

6.3. Satake parameters. — Hence the irreducible unramified admissible representations of $\mathrm{GO}(V)(\mathbb{A}^S)$ are uniquely characterized by the corresponding pairs of unramified admissible representations (π^S, ω^S), where π^S is an irreducible representation of $\mathrm{Gl}(2, \mathbb{A}_K^S)$ with central character $\omega_{\pi,v} = \omega_v \circ \mathrm{Norm}_K$ and ω^S is a character of $(\mathbb{A}^S)^*$. So the generalized theta correspondence in the unramified case relates the unramified representations

$$(\pi^S, \omega^S) \longleftrightarrow (\pi^{\vee,S}, \omega^S)^+ \longleftrightarrow (\Pi')^S$$

of the groups $\mathrm{Gl}(2, K_v) \times {K_v}^*$ and $\mathrm{GSp}(4, \mathbb{Q}_v)$. In terms of Satake parameters, the relationship between the spherical local representation π_v, ω_v of $\mathrm{Gl}(2, K_v) \times {K_v}^*$ and the spherical local representation $\Pi'_v = \Pi'(\pi_v, \omega_v)$ of $\mathrm{GSp}(4, \mathbb{Q}_v)$ is described as follows.

6.4. Lemma **(nonarchimedian unramified place).** — *Suppose $v \notin S$. Let $\Pi'_v = \Pi'_v(\pi_v, \omega_v)$ be a theta lift of some unramified admissible irreducible representation (π_v, ω_v). Then Π'_v is an unramified admissible irreducible representation of* $\mathrm{GSp}(4, \mathbb{Q}_v)$.

Suppose v is K-split*: Let w, w' be the two extensions of v.*

– If $\pi_v = \pi_w \times \pi_{w'}$ has Satake parameters α_v, β_v and α'_v, β'_v with $\alpha_v\beta_v = \alpha'_v\beta'_v = \omega(p_v)$, then Π'_v has Satake parameters $\alpha_v, \alpha'_v, \beta_v, \beta'_v$ with $\alpha_v\beta_v = \alpha'_v\beta'_v = \omega(p_v)$.

Suppose v is K-inert*:*

– If π_v is given by the Satake parameters α_v, β_v with $\alpha_v\beta_v = \omega(p_v^2)$, then Π'_v has Satake parameters $\alpha_v^{1/2}, \alpha_v^{1/2}\chi_K(p_v), \beta_v^{1/2}, \beta_v^{1/2}\chi_K(p_v)$ with $\alpha_v^{1/2}\beta_v^{1/2} = \omega(p_v)$.

Proof. — See [**HST**], lemma 10 and 11 or [**V**], p. 494ff. □

6.5. Corollary. — *The spinor L-series of the irreducible representation $(\Pi')^S = \Pi'(\pi^S, \omega^S)$, matching with the irreducible unramified representation π^S of* $\mathrm{Gl}(2, \mathbb{A}_K^S)$ *under the generalized theta correspondence as above, satisfies*

$$L^S(\Pi', s) = L^S(\pi, s).$$

The typical lift. — Let K and K' be two different quadratic extension fields of $\mathbb{Q}$ with composite field $L = K \cdot K'$ and quadratic idele class character $\chi_{L/K} = \chi_{K'/\mathbb{Q}} \circ \mathrm{Norm}_{K/\mathbb{Q}}$ of K. Let σ be the involution of $K/\mathbb{Q}$ and π be some irreducible cuspidal (generic) unitary automorphic representation of $\mathrm{Gl}(2, \mathbb{A}_K)$, so that π^S satisfies the Ramanujan conjecture at almost all places.

$$\begin{array}{ccc} & L & \\ \diagup & & \diagdown \\ K' & & K, \pi \\ \diagdown & & \diagup \\ & \mathbb{Q} & \end{array}$$

Suppose

$$\sigma(\pi) \ncong \pi \quad \text{but} \quad \sigma(\pi) \cong \pi \otimes \chi_{L/K}.$$

Then for a unitary character ω of $\mathbb{A}^*/\mathbb{Q}^*$ the theta lift $\Pi' = \Pi'(\pi, \omega)$ is nontrivial and cuspidal ([**V**], p. 507). Such Π' cannot be CAP. Either Π' is a weak endoscopic lift. Or Π' is D-critical of two-abelian type with $D = 4$, as an immediate consequence from the last lemma and definition 3.2. A partial converse for this is proposition 10.3.

6.6. Character twists. — Suppose Π is an irreducible automorphic cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$ with central character $\omega = \omega_\Pi$. Let S be a finite set of places chosen as in 6.2 so that Π^S is unramified outside S. Suppose an irreducible spherical automorphic constituents $\widetilde{\Pi}$ of the restriction of Π to $\mathrm{Sp}(4, \mathbb{A})$ is a theta lift in the sense of [**KRS**]. Then by the construction at the end of section 5, there exists an irreducible automorphic representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$ with central character

$\omega_\pi = \omega \circ \mathrm{Norm}_{K/\mathbb{Q}}$ and an irreducible representation Π' of $\mathrm{GSp}(4,\mathbb{A})$ with central character ω, such that

$$(\Pi')^S \cong \Pi^S \otimes \chi^S, \quad (\chi^S)^2 = 1.$$

χ^S is unramified but need not be automorphic.

So temporarily consider unramified irreducible representations Π^S of $\mathrm{GSp}(4,\mathbb{A}^S)$ etc., which are not necessarily automorphic! Then we have canonical bijections between the following sets:

– Equivalence classes of unramified irreducible representations Π^S of $\mathrm{GSp}(4,\mathbb{A}^S)$, where $\Pi_1^S \simeq \Pi_2^S$ iff $\Pi_1^S \cong \Pi_2^S \otimes \chi^S$ for some unramified character χ^S of $(\mathbb{A}^S)^*$ (not necessarily automorphic).

– Equivalence classes of unramified irreducible representations π^S, ω^S of $\mathrm{Gl}(2,\mathbb{A}_K^S)\times(\mathbb{A}^S)^*$ with central character $\omega_{\pi^S} = \omega^S \circ \mathrm{Norm}_{K/\mathbb{Q}}$, with equivalence $(\pi_1^S,\omega_1^S) \simeq (\pi_2^S,\omega_2^S)$ iff $\pi_1^S \cong \pi_2^S \otimes (\chi^S \circ \mathrm{Norm}_{K/\mathbb{Q}})$ and $\omega_1^S = \omega_2^S(\chi^S)^2$ holds for some unramified character χ^S of $(\mathbb{A}^S)^*$ (not necessarily automorphic).

– Equivalence classes of unramified irreducible representations π^S of $\mathrm{Gl}(2,\mathbb{A}_K^S)$ with trivial central character ω_{π^S}, where $\pi_1^S \simeq \pi_2^S$ iff $\pi_1^S \cong \pi_2^S \otimes \widetilde{\chi}^S$ for some unramified quadratic character $\widetilde{\chi}^S = \chi^S \circ \mathrm{Norm}_{K/\mathbb{Q}}$, where χ^S is a character of $(\mathbb{A}^S)^*$ (not necessarily automorphic).

– Isomorphism classes of unramified irreducible representations π^S of

$$\mathrm{Gl}(2,\mathbb{A}_K^S)^0/A(\mathbb{A}^S)$$

with trivial central character. Here $\mathrm{Gl}(2,\mathbb{A}_K^S)^0$ denotes the subgroup of elements $g \in \mathrm{Gl}(2,\mathbb{A}_K^S)$ with $\mathrm{Norm}_{K/\mathbb{Q}}(\det(g)) = 1$.

– Isomorphism classes of spherical irreducible representations π^S of

$$\mathrm{Gl}(2,\mathbb{A}_K^S)^0/A(\mathbb{A}^S).$$

The first bijection is induced by the generalized theta correspondence (lemma 6.4, corollary 6.5). Since every unramified character ω is a square of an unramified character, ω can be normalized to be trivial. Hence the second bijection follows. For the third bijection notice that character twists by the $\widetilde{\chi}$ disappear under restriction from $\mathrm{Gl}(2,\mathbb{A}_K^S)$ to $\mathrm{Gl}(2,\mathbb{A}_K^S)^0$. Since $\mathbb{A}_K^* \cdot \mathrm{Gl}(2,\mathbb{A}_K^S)^0$ is normal in $\mathrm{Gl}(2,\mathbb{A}_K^S)$ with two abelian quotient, conversely the restriction to $\mathrm{Gl}(2,\mathbb{A}_K^S)^0$ determines π^S up to a character twist. Since π^S has trivial central character by assumption, this character twist must be of the form $\pi \otimes \widetilde{\chi}$ for a character $\widetilde{\chi} = \chi \circ \mathrm{Norm}_{K/Q}$. This is shown as in the proof of corollary 5.4. The next bijection follows by definition. For the last bijection notice, that the center of $\mathrm{Gl}(2,\mathbb{A}_K^S)^0/A(\mathbb{A}^S)$ is finite. Hence an unramified representation is trivial on the center.

Let Π^S be spherical with central character $\omega^S = \omega_{\Pi^S}$. Let $\widetilde{\Pi}^S$ be a spherical constituent of the restriction of Π^S from $\mathrm{GSp}(4,\mathbb{A})$ to $\mathrm{Sp}(4,\mathbb{A})$. $\widetilde{\Pi}^S$ is associated to an unramified representation $\widetilde{\pi}^S$ of $\mathrm{O}(V,\mathbb{A}^S)$ by the Howe correspondence. The restriction of $\widetilde{\pi}^S$ to $\mathrm{SO}(V)(\mathbb{A}^S)$ contains an unramified constituent. It is unique up to the automorphism σ. By lemma 5.4 it can be extended to the representation $(\pi^\vee, \omega)^S$ of the group $\mathrm{GSO}(V)(\mathbb{A}^S)$. This produces a representation π^S of $\mathrm{Gl}(2, \mathbb{A}^S)$, unique up to a twist by a character of the form $\widetilde{\chi}S = (\chi')^S \circ \mathrm{Norm}_{K/k}$ and up to replacing π^S by $\sigma(\pi^S)$. (π^S, ω) determines an associated generalized theta lift $(\Pi')^S$, hence determines Π^S up to quadratic character twist as explained at the end of section 5.

We remark that the partial L-series

$$\zeta^S(\Pi, \chi, s), \quad L_K^S(\pi \times \pi^* \times \chi, s), \quad L_K^S(\sigma(\pi) \times \pi^* \times \chi, s)$$

only depend on $\widetilde{\Pi}^S$. They do not change, if we replace π^S by $\sigma(\pi^S)$ or by $\pi^S \otimes ((\chi')^S \circ \mathrm{Norm}_{K/k})$.

6.7. Corollary. — *For $v \notin S$ in the situation above the following statement are equivalent:*

(1) Π_v *is unitary, and the Ramanujan conjecture holds for* Π_v
(2) Π'_v *is unitary, and the Ramanujan conjecture holds for* Π'_v
(3) π_v *and* ω_v *are unitary, and the Ramanujan conjecture holds for* π_v.

We remark that in particular this conditions exclude one dimensional representations. The next proposition is concerned with the L-series attached to the automorphic representation π^S of $\mathrm{Gl}(2, \mathbb{A}_K^S)$. Observe that a replacement

$$\pi^S \longmapsto \pi^S \otimes ((\chi')^S \circ \mathrm{Norm}_K)$$

coming from the equivalence relations defined above, has no effect on both the L-series $L_K^S\big(\pi \times \pi^* \otimes (\chi \circ \mathrm{Norm}_K), s\big)$ and $L_K^S\big(\sigma(\pi) \times \pi^* \otimes (\chi \circ \mathrm{Norm}_K), s\big)$. Hence they are uniquely determined by the representation Π^S.

6.8. Proposition. — *Suppose Π is an unitary irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$. Suppose Ramanujan's conjecture holds outside a finite set of places S. Assume S is chosen large enough to contain the ramified places of K and π. Also assume, that the restriction of Π to $\mathrm{Sp}(4, \mathbb{A})$ contains a theta lift. Then there exists an algebra $K/\mathbb{Q}$ of degree two with corresponding idele class character χ_K of $\mathbb{A}^*/\mathbb{Q}^*$ and an irreducible automorphic representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$ as in 6.4, such that the order at $s = 1$ of the following meromorphic functions coincides*

(1) $\zeta^S\big(\Pi, \chi, s\big) \cdot \zeta^S\big(\Pi, \chi\chi_K, s\big)$
(2) $L_K^S\big(\chi \circ \mathrm{Norm}_K, s\big) \cdot L_K^S\big(\sigma(\pi) \times \pi^* \otimes (\chi \circ \mathrm{Norm}_K), s\big)$.

Proof. — We may replace $\zeta^S(\Pi, \chi, s)$ by $\zeta^S(\Pi', \chi, s)$. Suppose $v \notin S$. By assumption Ramanujan's conjecture holds for $v \notin S$. To compute the order at $s = 1$ we can ignore all places $v \notin S$ where v is K-inert. Since they have Dirichlet K-density zero we can ignore them in (2). Similarly the influence of the K-inert places $v \notin S$ on the asymptotic at $s = 1$ is eliminated by character χ_K. So we can restrict ourselves to the K-split places outside S. Suppose the Satake parameters of Π_v are $\alpha_v, \alpha'_v, \beta_v, \beta'_v$

(notation of lemma 6.4). Then $\zeta_v(\Pi', \chi, s)$ and $\zeta_v(\Pi', \chi\chi_K, s)$ are equal, and each of them is a product of five Euler factors attached to the parameters

$$\left\{\chi_v, \chi_v\frac{\alpha'_v}{\alpha_v}, \chi_v\frac{\alpha'_v}{\beta_v}, \chi_v\frac{\beta'_v}{\alpha_v}, \chi_v\frac{\beta'_v}{\beta_v}\right\}.$$

$\alpha_v, \beta_v, \alpha'_v, \beta'_v$ are the Satake parameters of $\pi_v = \pi_w \times \pi_{w'}$ by lemma 6.4. Replacing α_v, β_v with α'_v, β'_v and vice versa switches between $\sigma(\pi_v)$ and π_v. The local Euler-factor of the L-series of 6.8(2) are attached to ten parameters

$$\left\{\chi_v, \chi_v\frac{\alpha'_v}{\alpha_v}, \chi_v\frac{\alpha'_v}{\beta_v}, \chi_v\frac{\beta'_v}{\alpha_v}, \chi_v\frac{\beta'_v}{\beta_v}\right\} \cup \left\{\chi_v, \chi_v\frac{\beta_v}{\beta'_v}, \chi_v\frac{\alpha_v}{\beta'_v}, \chi_v\frac{\beta_v}{\alpha'_v}, \chi_v\frac{\alpha_v}{\alpha'_v}\right\}.$$

The two subsets correspond to the two extensions w and w' of v. The identity $\alpha_v\beta_v = \omega_v = \alpha'_v\beta'_v$ implies, that the two sets are equal. This completes the proof. □

7. The adjoint L-series of π

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$, for which some constituent of the restriction to $\mathrm{Sp}(4, \mathbb{A})$ is a theta lift. To Π, or more precisely to some character twist Π' of Π, we associated an irreducible automorphic representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$. Its partial adjoint L-series

$$\zeta_K^S(\mathrm{Ad}(\pi), \chi, s)$$

is uniquely determined by the unramified twist equivalence class of Π^S. Here S is assumed to be sufficiently large in particular containing the ramified places of $K/\mathbb{Q}$. This adjoint L-series has the following property

7.1. Proposition. — *For an irreducible automorphic representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$ and an idele class character χ of $\mathbb{A}_K/K^*$ and a finite set S of places, such that π and χ are unramified outside S the following holds*

$$L_K^S(\pi^* \times (\pi \otimes \chi), s) = \zeta_K^S(\mathrm{Ad}(\pi), \chi, s) \cdot L_K^S(\chi, s).$$

For irreducible unitary automorphic representations π_1 and π_2 of $\mathrm{Gl}(2, \mathbb{A}_K)$ let S be a set of places, such that π_1, π_2 are unramified outside S. Suppose each representation π_1^S, π_2^S is either cuspidal or fully induced from a pair of unitary characters (such induced representations are irreducible) or in case $K = \mathbb{Q}^2$ a combination of both (tempered). Then

$$L_K^S(\pi_1 \times \pi_2, s)$$

does not vanish at $s = 1$. Furthermore $\zeta_K^S(\mathrm{Ad}(\pi), 1, 1) \neq 0$. Assume K is a number field. Further suppose one of the representations π_i is cuspidal or the central characters $\omega_{\pi_1} = \omega_{\pi_2}$ coincide. Then $L_K^S(\pi_1^* \otimes \pi_2, s)$ has a pole at $s = 1$ if and only if $\pi_1 \cong \pi_2$. This pole is simple if and only if $\pi_1 \cong \pi_2$ is cuspidal. Otherwise it is of order 2 or 4.

Proof. — These statements are well known. It is due to Jaquet-Shalika in the case, where either π_1 or π_2 is cuspidal. On the other hand suppose both $\pi_i \cong \mathrm{Ind}(\chi_i, \chi_i')$ are induced for characters χ_i, χ_i' of $\mathbb{A}_K^*/K^*$ (Eisenstein case). If the characters χ_i, χ_i' are unitary outside S they are unitary characters, since the kernel of the idele norm is a compact subgroup of $\mathbb{A}_K^*/K^*$. A pole of $L_K^S(\pi_1^* \times \pi_2, s)$ at $s = 1$ forces $\chi_1 = \chi_2$ (up to permutations of χ_i, χ_i'). Then $\omega_{\pi_1} = \omega_{\pi_2}$ implies $\chi_1' = \chi_2'$ or $\pi_1 \cong \pi_2$. The converse is obvious. □

7.2. Remark. — Proposition 7.1 will later be applied for an irreducible automorphic representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$ constructed from some D-critical Π. This representation π need not be cuspidal. If it is cuspidal it is also unitary, since its central character is obtained from the central character ω_Π or ω_Π^S by pullback with the norm. The central character ω_Π is unitary by our temporary assumptions. If π is Eisenstein, it is a constituent of some $\mathrm{Ind}(\chi, \chi')$. Here χ, χ' are idele class characters for K. We later apply 7.1 in the situation of corollary 6.7 where π^S satisfies the Ramanujan conjecture. Hence χ^S, χ'^S and also χ, χ' must be unitary. This implies $\pi \cong \mathrm{Ind}(\chi, \chi')$. Therefore π is unitary, since this induced representations are irreducible and unitary. Hence the assumption of proposition 7.1, π has to be unitary, is satisfied.

7.3. Corollary. — *In the situation of proposition 6.8 the following holds*

(1) $\zeta^S(\Pi, 1, s)$ *does not vanish at* $s = 1$.

(2) $\zeta^S(\Pi, 1, s)$ *has a pole at* $s = 1$ *if and only if* $\sigma(\pi) \cong \pi$, *when* K *is a number field.*

Proof. — $\zeta^S(\Pi, \chi_K, s)$ has a simple poles at $s = 1$ (prop. 4.1 and th. 4.2). $L_K^S(\sigma(\pi) \times \pi^*, s)$ does not vanish at $s = 1$ (prop. 7.1 and remark 7.2). $L_K(1, s)$ has a pole at $s = 1$. So prop. 6.8 implies assertion (1). Suppose $\zeta^S(\Pi, 1, s)$ has a pole at $s = 1$. Then the first L-series of prop. 6.8 has a pole of order $\geqslant 2$ at $s = 1$, since $\zeta^S(\Pi, \chi_K, s)$ has a pole at $s = 1$ (th. 4.2). Then proposition 6.8 implies, that $L_K(1, s) \cdot L_K^S(\sigma(\pi) \times \pi^*, s)$ has a pole of order two at $s = 1$. Therefore $L_K^S(\sigma(\pi) \times \pi^*, s)$ has a simple pole at $s = 1$, since K is a number field. Prop. 7.1 and remark 7.2 then imply $\sigma(\pi) \cong \pi$, since π and $\sigma(\pi)$ have the same central character. The converse is similar. □

8. Theta lifts in the D-critical cases

Let Π be a D-critical representation of $\mathrm{GSp}(4, \mathbb{A})$. Let K be a rank two algebra over $\mathbb{Q}$, and let π be an irreducible automorphic representation of $\mathrm{Gl}(2, \mathbb{A})$. Let $\Pi' = \Pi'(\pi, \omega)$ be an irreducible automorphic representation on $\mathrm{GSp}(4, \mathbb{A})$ related to π via a theta lift as in the last sections. Let S be a finite set of places containing the archimedian places and the places where K, Π, Π' and π are ramified. Let T be a set of density zero containing S, for which the Satake parameters of Π_v have the form

$\nu_v, \varepsilon_v\nu_v, \mu_v, \varepsilon_v\mu_v$ of Π_v for $v \notin T$. Suppose $\Pi'^S \cong \Pi^S \otimes \chi^S$, for some unramified character χ^S of $\mathbb{A}^S$ (not necessarily automorphic). Under these assumptions the relation between π and Π can be specified in terms of the

8.1. Satake Parameters. — Definition 3.2(i) and (ii) and lemma 6.4 determine the Satake parameters of π as follows. For K*-split* places $v \notin T$ the Satake parameters of $\pi_v = \pi_w \times \pi_{w'}$ are

(a) $(t_v\nu_v, t_v\mu_v)$ for π_w and $(t_v\nu_v, t_v\mu_v)$ for $\pi_{w'}$, if v splits completely in $K \cdot L$.
(b) $(t_v\nu_v, t_v\mu_v)$ for π_w and $(-t_v\nu_v, -t_v\mu_v)$ for $\pi_{w'}$, if v splits in K but not in L.
For K*-inert cases* places $v \notin T$ the Satake parameters of π_v are
(c) $(t_v\nu_v^2, t_v\mu_v^2)$, if v does not split in K.

The adjoint L-series $\zeta_K^S(\mathrm{Ad}(\pi), \chi, s)$ does not change under character twists of π. Hence it is uniquely determined by Π. To simplify the formulas it is of no harm to assume $\chi^S = 1$. Under this simplifying assumption $\Pi = \Pi'$ the parameters t_v becomes 1.

8.2. Lemma. — *Let π be an automorphic representation of $\mathrm{Gl}(2, \mathbb{A}_K)$ attached to the D-critical automorphic representation Π as above. Then for $v \notin T$ the local L-factor of $\zeta_K^S(\mathrm{Ad}(\pi), 1, s)$ is*

$$\zeta_w(\mathrm{Ad}(\pi_w), 1, s)^{-1} = \Big(1 - \frac{\nu_v}{\mu_v} p_v^{-s}\Big)(1 - p_v^{-s})\Big(1 - \frac{\mu_v}{\nu_v} p_v^{-s}\Big),$$

if v splits in K and both extensions w, w' of the place v we have the same local factor. If v is inert in $K/\mathbb{Q}$ the local L-factor is

$$\zeta_v(\mathrm{Ad}(\pi_v), 1, s)^{-1} = \Big(1 - \frac{\nu_v^2}{\mu_v^2} p_v^{-2s}\Big)(1 - p_v^{-2s})\Big(1 - \frac{\mu_v^2}{\nu_v^2} p_v^{-2s}\Big).$$

Proof. — Obvious. □

8.3. Corollary. — *For K, Π and π as in the last lemma let ψ be an idele class character of $\mathbb{A}_K^*/K^*$. Then asymptotically for $s \to 1^+$*

$$\log \zeta^S(\mathrm{Ad}(\pi), \psi, s) \sim \sum_{v\ K\text{-split}} \mathrm{Ad}_v \cdot (\psi_v + \psi'_v) \cdot p_v^{-s},$$

where the sum is over all places $v \notin T$ of $\mathbb{Q}$ which are split in K. ψ_v, ψ'_v denote the values $\psi_w(\pi_w)$ for the two extensions w of the split place v to K. For $\psi = \chi \circ \mathrm{Norm}_K$ this is $\psi_v + \psi'_v = 2 \cdot \chi_v(p_v)$.

9. The pole order $n_K(\Pi)$

Let Π be a D-critical, unitary cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$ with notations as in the last section. In particular let K be a quadratic $\mathbb{Q}$-algebra, let π be the irreducible automorphic representation of $\mathrm{Gl}(2, \mathbb{A}_K)$ attached to Π. Let $L/\mathbb{Q}$ be the

Galois extension with Galois group Δ of order D attached to Π. If K is a field let KL denote the field generated by K and L. Otherwise KL denotes L.

9.1. Lemma. — *For D-critical representations Π and characters χ of Δ, the orders at $s=1$ of the meromorphic functions*

$$f_1(s) = \zeta^S(\Pi,\chi,s)\cdot\zeta^S(\Pi,\chi\chi_K,s)\cdot\zeta_K^S(\mathrm{Ad}(\pi),\chi\circ\mathrm{Norm}_K,s)$$
$$f_2(s) = L_K^S\big(\pi\times(\pi^*\otimes\chi\circ\mathrm{Norm}_K),s\big)\cdot L_K^S\big(\sigma(\pi)\times(\pi^*\otimes\chi\circ\mathrm{Norm}_K),s\big)$$

coincide. This common order is an integer $n_K(\Pi)$, which does not depend on the choice of the character χ of Δ and satisfies $1\leqslant n_K(\Pi)\leqslant 16/[KL:\mathbb{Q}]$. In particular $[KL:\mathbb{Q}]\leqslant 16$ and $n_K(\Pi)=1$ for $8<[KL:\mathbb{Q}]$. Furthermore the asymptotic behaviour at $s\to 1^+$ is

$$\log f_i(s)\sim\sum_{v\ K\text{-split},\ L\text{-split}} 4\cdot(\mathrm{Ad}_v+1)p_v^{-s}.$$

Proof. — The first assertion concerning the orders of $f_1(s), f_2(s)$ follows from Proposition 6.8 and the first part of 7.1. That the pole order is independent of $\chi\in\widehat{\Delta}$ follows once we have shown the last assertion of lemma 9.1, since the right side of the asymptotic formula is obviously independent of χ. The asymptotic formula is shown as follows: For $\chi,\chi\chi_K$ with $\chi\in\widehat{\Delta}$ we use the asymptotic formula $\log\zeta^S(\Pi,\chi,s)\sim\sum_v w_v\cdot\chi(p_v)p_v^{-s}$ obtained in remark 3.3. The weights were $w_v=1+\varepsilon_v\cdot(\mathrm{Ad}_v+1)$ with $\varepsilon_v=\pm1$, where $\varepsilon=1$ if and only if v splits in L. Hence the asymptotic behaviour of *log*$(\zeta^S(\Pi,\chi,s)\zeta^S(\Pi,\chi\chi_K,s))$ is

$$\sim\sum_{v\ K\text{-split, not } L\text{-split}}(-2\cdot\mathrm{Ad}_v)\cdot\chi_v(p_v)p_v^{-s}+2\sum_{v\ K\text{-split}, L\text{-split}}(\mathrm{Ad}_v+2)p_v^{-s}.$$

On the other hand corollary 8.3 implies

$$\log\zeta_K^S(\mathrm{Ad}(\pi),\chi\circ\mathrm{Norm}_K,s)\sim\sum_{vK\text{-split}}2\cdot\mathrm{Ad}_v\cdot\chi_v(p_v)p_v^{-s}$$

for $\psi=\chi\circ\mathrm{Norm}_K$ and $\chi\in\widehat{\Delta}$. In both cases the sum is over a subset of the places of $\mathbb{Q}$ omitting a set of $\mathbb{Q}$-density zero. Both formulas combined give the asymptotic of $\log f_1(s)$. It remains to estimate $n_K(\Pi)$. The right side of the asymptotic formula is a sum over set of $\mathbb{Q}$-primes of Dirichlet-density $[KL:\mathbb{Q}]^{-1}$. By property (iv) of remark 3.3 of a D-critical representation we have $0\leqslant\mathrm{Ad}_v+1\leqslant 4$. Hence

$$n_K(\Pi)\leqslant\frac{4(\mathrm{Ad}_v+1)}{[KL:\mathbb{Q}]}\leqslant\frac{16}{[KL:\mathbb{Q}]}.$$

This gives the upper estimate for $n_K(\Pi)$. On the other hand put $\chi=1$. Then there is at least one pole of $f_1(s)$ coming from $\zeta^S(\Pi,\chi_K,s)$, since $\zeta_K^S(\mathrm{Ad}(\pi),1,1)\neq0$ (prop. 7.1 and remark 7.2) and since $\zeta^S(\Pi,1,1)\neq0$ by cor. 7.3.1. Hence the pole order $n_K(\Pi)$ is strictly positive. □

9.2. Corollary. — *In the situation of 9.1 assume* Δ *to be an abelian group. Then* $\mathrm{ord}_{s=1} \prod_{\chi\in\widehat{\Delta}} L_K^S(\pi\times(\pi^*\otimes\chi\circ\mathrm{Norm}_K),s)\cdot L_K^S(\sigma(\pi)\times(\pi^*\otimes\chi\circ\mathrm{Norm}_K),s) = D\cdot n_K(\Pi)$.

9.3. Corollary. — *In the situation of 9.1 assume* $KL = L$ *and assume to* Δ *be abelian. Then*

(1) $\mathrm{ord}_{s=1} \prod_{\chi\in\widehat{\Delta}} \zeta_K^S(\mathrm{Ad}(\pi),\chi\circ\mathrm{Norm}_K,s) = \frac{D}{2}\cdot n_K(\Pi) - 2.$
(2) $\mathrm{ord}_{s=1} \prod_{\chi\in\widehat{\Delta}} \zeta^S(\Pi,\chi,s)\zeta^S(\Pi,\chi\chi_K,s) = \frac{D}{2}\cdot n_K(\Pi) + 2.$

So the order of $\mathrm{ord}_{s=1} \prod_{\chi\in\widehat{\Delta}} \zeta^S(\Pi,\chi,s)$ at $s=1$ is $\frac{D}{4}n_K(\Pi)+1$ and there exist at least $\frac{D}{4}n_K(\Pi)+1$ characters χ of Δ for which $\zeta^S(\Pi,\chi,s)$ has a pole at $s=1$.

Proof. — By lemma 9.1 and cor. 9.2 the sum of (1) and (2) is $D\cdot n_K(\Pi)$. So it suffices to show formula (2). Since $KL = L$ implies $\chi_K \in \widehat{\Delta}$ lemma 4.4 gives

$$\log \prod_{\chi\in\widehat{\Delta}} \zeta^S(\Pi,\chi,s)\zeta^S(\Pi,\chi\chi_K,s) \sim 2\cdot D\cdot \sum_{v\ L\text{-split}} (\mathrm{Ad}_v+2)\cdot p_v^{-s} \sim: \kappa\cdot\log\zeta^S(s).$$

Lemma 9.1 and cor. 9.2 imply $D\cdot\sum_{v\ L\text{-split}} 4\cdot(\mathrm{Ad}_v+1)p_v^{-s} \sim D\cdot n_K(\Pi)\cdot\log\zeta^S(s)$. Elimination of the Ad_v-terms in these two equations gives

$$(2\cdot\kappa - D\cdot n_K(\Pi))\cdot\log\zeta^S(s) \sim (8D-4D)\cdot\sum_{v\ L\text{-split}} p_v^{-s} = 4\cdot\log\zeta^S(s).$$

This implies the second claim $\kappa = \frac{D}{2}n_K(\Pi)+2$. The assertion on the minimal number of poles is an immediate consequence, since poles of $\zeta^S(\Pi,\chi,s)$ at $s=1$ are at most simple poles (theorem 4.2). □

9.4. Corollary. — *In the situation of 9.1 for* $D\geqslant 4$ *and abelian* Δ *one can always choose* K *as a field.*

Proof. — For $K=\mathbb{Q}^2$ the assumptions of corollary 9.3 hold. Since $\frac{D}{4}n_K(\Pi)+1\geqslant 2$, there exists a nontrivial character $\chi_{K'}$ of Δ for which $\zeta^S(\Pi,\chi_{K'},s)$ has a pole at $s=1$. So replace K by K'. □

9.5. Corollary. — *In the situation of 9.1 for* $D\geqslant 4$, Δ *abelian and* K *a field the pole order* $n_K(\Pi)$ *is* $1,2$ *or* 4.

Proof. — $n_K(\pi)$ is the order of $f_1(s)$ by lemma 9.1. Choose $\chi=1$, then $\zeta^S(\Pi,1,1)\neq 0$ (cor. 7.3.1) and $\zeta^S(\Pi,\chi_K,s)$ has a simple pole at $s=1$. Hence $n_K(\pi)-1$ is the order of $\zeta_K^S(\mathrm{Ad}(\pi),s)$ at $s=1$. This order is 0,1 or 3 depending on the CM-type of π (cuspidal or Eisenstein). This follows from remark 7.2 and proposition 7.1. □

9.6. Remark. — If $\chi\in\widehat{\Delta}$ is not a quadratic character, then in particular $\chi\neq 1,\chi_K$. Therefore remark 4.3 and lemma 9.1 imply $n_K(\Pi) = \mathrm{ord}_{s=1}\zeta_K^S(\mathrm{Ad}(\pi),\chi\circ\mathrm{Norm}_K,s)$. By assumption $\chi\circ\mathrm{Norm}_K$ is a nontrivial torsion character of $\mathbb{A}_K^*/K^*$. So if K is a field and Π is D-critical and nondegenerate, then π cannot be of CM-type and one can easily show that $\mathrm{ord}_{s=1}\zeta_K^S(\mathrm{Ad}(\pi),\chi\circ\mathrm{Norm}_K,s)=0$ (see lemma 8.2). Therefore

$n_K(\Pi) = 0$ in contradiction to lemma 9.1. This argument shows, that $\widehat{\Delta}$ must be an elementary abelian 2-group if K is a field and Π is D-critical and nondegenerate.

10. Nondegenerate D-critical representations of abelian type

10.1. Proposition. — *Suppose* Π *is a* D*-critical irreducible automorphic representation, which is* nondegenerate *of* abelian type. *Let* K *the corresponding* $\mathbb{Q}$*-algebra* K *of degree two with involution* σ. *Let* π *be the irreducible automorphic representation of* $\mathrm{Gl}(2, \mathbb{A}_K)$ *attached to* Π. *Then* π *cannot be of CM-type*

(1) $\pi \cong \pi \otimes \chi$ *for a character* χ *of* $\mathbb{A}_K^*/K^*$ *implies* $\chi = 1$, *and for* $D \geqslant 4$ *the following holds*

(2) $\sigma(\pi) \not\cong \pi$.

(3) *Suppose* $D \geqslant 8$, *or* $D = 4$ *and* K *is a field not contained in* L, *or suppose* $[LK : K] \geqslant 3$, *then* $\sigma(\pi) \not\cong \pi \otimes \chi$ *holds for all characters* χ *of* A_K^*/K^* *of finite order.*

(4) *If* $D = 4$ *and* K *is a field contained in* L, *then* $\sigma(\pi) \cong \pi \otimes (\chi \circ \mathrm{Norm}_K)$ *holds for the two characters* $\chi \neq 1, \chi_K$ *of the abelian group* $\Delta = \mathrm{Gal}(L/\mathbb{Q})$.

We remind the reader, that π was constructed in the sections 4 an 5. For $D \geqslant 4$ we also recall, that K could be chosen to be a field for abelian Δ (cor. 9.4).

Proof

(1) $\pi \cong \pi \otimes \chi$ implies $\chi^2 = 1$. It is enough to show $\chi_w = 1$ outside a set of K-density zero. Hence discard places w which are K-inert or lie above places v in the exceptional set T of definition 3.2. Then $\pi \cong \pi \otimes \chi$ either implies $\chi_w = 1$ or $\chi_w = \mu_v/\nu_v = \nu_v/\mu_v$ by 8.1. Hence $\chi_w = 1$, since Π is nondegenerate by assumption which means $(\mu_v/\nu_v)^2 \neq 1$ for $v \notin T$ (def. 3.2).

(2) For $D \geqslant 4$ the set of $\mathbb{Q}$-places $v \notin T$, split in K but not in L, has positive Dirichlet density. For such a place v choose an extension w to K. By 8.1 the representation π_w has Satake parameters $t_v\nu_v, t_v\mu_v$ and $\sigma(\pi)_w$ has Satake parameters $-t_v\nu_v, -t_v\mu_v$. Hence $\pi \cong \sigma(\pi)$ is impossible, since Π is nondegenerate.

(3) Consider the same K-places w as in the proof of (2). By assumption this set now has K-density $\geqslant 3/4$ (or $\geqslant 2/3$). For w and $w' = \sigma(w)$ extending the $\mathbb{Q}$-place v the assumption $\sigma(\pi) \cong \pi \otimes \chi$ and 8.1 together imply either $(-\nu_v, -\mu_v) = (\nu_v\chi_w, \mu_v\chi_w)$ or $(-\mu_v, -\nu_v) = (\nu_v\chi_w, \mu_v\chi_w)$. Hence $\chi_w = -1$ or $\chi_w = -\mu_v/\nu_v = -\nu_v/\mu_v = \chi_w^{-1}$. Thus $\chi_w^2 = 1$. This holds for a set of K-density $> 1/2$, hence χ is quadratic $\chi^2 = 1$. Since Π is nondegenerate $\chi_w = -1$ holds for a set of K-density $> 1/2$, which is impossible for a quadratic character.

(4) By assumption $\mathbb{Q} \subset K \subset L$ and $[L : K] = 2$ and $[K : \mathbb{Q}] = 2$. The quadratic idele class character $\chi_{L/K}$ of $\mathbb{A}_K^*/K^*$ has the form $\chi_{L/K} = \chi \circ \mathrm{Norm}_{K/\mathbb{Q}}$, where χ is one of the two quadratic characters of $\mathrm{Gal}(L/\mathbb{Q})$ different from $\chi \neq 1, \chi_K$, independent from whether $L/\mathbb{Q}$ is cyclic of order four or an abelian 2-group. For all K-places w

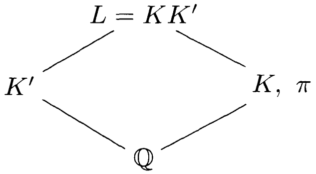
over K-split places $v \notin T$ we get $\sigma(\pi)_w \cong \pi_w \otimes (\chi_v \circ \mathrm{Norm}_K)$ by formula 8.1 (a) and (b). Since this holds for a set of K-places of K-density 1, and since π^S is unitary and satisfies the Ramanujan conjecture (see def. 3.2 and cor. 6.7), this implies that $L^S_K(\pi^* \times \sigma(\pi) \otimes (\chi^{-1} \circ \mathrm{Norm}_K), s)$ has a pole at $s = 1$. The central characters coincide, since $\chi \circ \mathrm{Norm}_K = \chi_{L/K}$ is a quadratic character. Therefore $\sigma(\pi) \cong \pi \otimes \chi \circ \mathrm{Norm}_K$ holds by proposition 7.1 and remark 7.2.

□

10.2. Corollary. — *Suppose* Π *is* D*-critical, nondegenerate of abelian type and* $D \geqslant 4$. *Then* $\zeta^S(\Pi, 1, s)$ *does not have a pole at* $s = 1$. *In particular* K *must be a field.*

Proof. — We can assume K to be a field by 9.4. Then existence of a pole of $\zeta^S(\Pi, \chi, s)$ at $s = 1$ for $\chi = 1$ is equivalent to $\sigma(\pi) \cong \pi$ by 7.3.2. This contradicts 10.1.2, since $D \geqslant 4$. □

10.3. Proposition. — *Suppose* Π *is* D*-critical, nondegenerate of abelian type with* $D \geqslant 4$. *Then* K *is a quadratic extension field of* $\mathbb{Q}$ *and a subfield of* L. L *is a abelian noncyclic extension of* $\mathbb{Q}$ *of degree* $D = [L : \mathbb{Q}] = 4$



Furthermore

$$(1) \qquad \sigma(\pi) \ncong \pi$$

$$(2) \qquad \sigma(\pi) \cong \pi \otimes \chi_{L/K}.$$

$\zeta^S(\Pi, \chi, s)$ *has a pole at* $s = 1$ *for* $n_K(\Pi) + 1$ *characters* $\chi \in \widehat{\Delta}$. *The pole number* $n_K(\Pi)$ *is 1 for cuspidal* π, *or* $n_K(\Pi)$ *is 1 or 2 for Eisenstein* π. *For* $\chi = 1$ *there is no pole at* $s = 1$. *The four L-series* $\zeta^S(\Pi, \chi, s), \chi \in \widehat{\Delta}$ *are nonzero at* $s = 1$.

Proof. — By cor. 10.2 K must be a field. By remark 9.6 Δ must be a elementary abelian 2-group. For the first assertion we have to exclude case (3) of prop. 10.1. For $D \geqslant 4$ there exists a character χ of Δ, such that $\psi = \chi \circ \mathrm{Norm}_K$ is nontrivial and quadratic. For this use that Δ is an elementary abelian 2-group. Notice $\chi \circ \mathrm{Norm}_K$ is a quadratic character of $\mathbb{A}^*_K/K^*$, hence π and $\pi \otimes \chi \circ \mathrm{Norm}_K$ we have the same central characters. This we need for prop. 7.1, if π is not cuspidal. By cor. 10.2 K is a number field. Therefore prop. 7.1 can be applied. By prop 7.1 and remark 7.2 and prop. 10.1.1 and prop. 10.1.3 the order of $L^S\big(\pi \times \pi^* \otimes (\chi \circ \mathrm{Norm}_K)\big) \cdot L^S\big(\sigma(\pi) \times \pi^* \otimes (\chi \circ \mathrm{Norm}_K)\big)$ at $s = 1$ is zero. This order is $n_K(\Pi)$ and this contradicts the lower bound $n_K(\Pi) \geqslant 1$ of prop. 9.1. Hence case (3) of prop. 10.1 is now excluded. Therefore $[LK : K] \leqslant 2$. Since $D \geqslant 4$, this implies $D = 4$ and K must be a subfield of L.

Lemma 9.3 for $D = 4$ implies that the number of characters $\chi \in \widehat{\Delta}$, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$, is $n_K(\Pi)+1$. The trivial character does not give a pole (cor. 10.2). Hence $n_K(\Pi) \leqslant 3$ since $D = 4$. The order of $L_K^S(\pi\times\pi^*, s)\cdot L_K^S(\sigma(\pi)\times\pi^*, s)$ at $s = 1$ is $n_K(\Pi)$ by definition (lemma 9.1). The order of $L_K^S(\sigma(\pi) \times \pi^*, s)$ at $s = 1$ is trivial (prop 7.1, remark 7.2 and prop 10.1.2). Therefore $n_K(\Pi)$ is the order of $L_K^S(\pi \times \pi^*, s)$ at $s = 1$. This number is 1 for cuspidal π, or 2 or 4 in the case where π is an Eisenstein representation (prop 7.1). The order of $\zeta^S(\Pi, \chi, s)$ at $s = 1$ is at most 1 (th. 4.2) and $\zeta^S(\Pi, 1, 1) \neq 0, \infty$ (cor. 7.3.1 and cor. 10.2). Since $\mathrm{ord}_{s=1} \prod_{\chi\in\widehat{\Delta}} \zeta^S(\pi, \chi, s) = \frac{D}{4} \cdot n_K(\Pi) + 1 \geqslant 2$ (by cor. 9.3 and $D = 4, n_K(\Pi) = 1, 2$) the number of $\chi \in \Delta$ for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$ is therefore equal to $n_K(\Pi) + 1$. Furthermore $\zeta^S(\Pi, \chi, 1) \neq 0$ holds for all four characters $\chi \in \widehat{\Delta}$. □

11. Proof of Theorem I

Let Π be a cuspidal irreducible unitary automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ whose archimedian component Π_∞ belongs to the discrete series of weight (k_1, k_2). For the proof of theorem I it can be assumed that Π is neither a CAP-representation nor a weak endoscopic lift, since both cases were already considered. Then Ramanujan's conjecture holds for all unramified Π_v as shown in section 1. By lemma 2.2 theorem I holds except in two critical cases. These are studied in appendix B. By the results of appendix B either theorem I holds or the representation Π is D-critical (prop. 3.1 and def. 3.2). More precisely, Π is either D-critical and nondegenerate of two-abelian type with $D \geqslant 4$ or D-critical of CM-type with $D \geqslant 8$. By proposition 4.1 this implies, that the restriction of Π to $\mathrm{Sp}(4, \mathbb{A})$ contains an irreducible constituent $\widetilde{\Pi}$, which is a theta lift attached to some irreducible representation $(\widetilde{\pi})^+$ of $\mathrm{O}(V, \mathbb{A})$, where V is a nondegenerate four dimensional quadratic space over $\mathbb{Q}$. Attached to V is a quadratic $\mathbb{Q}$-algebra K, a simple central algebra D over $\mathbb{Q}$ and the K-algebra $D_K = D \otimes_{\mathbb{Q}} K$.

As explained in section 5 we can find a finite set of places S, an irreducible unramified representation Π'^S of $\mathrm{GSp}(4, \mathbb{A}^S)$, a quadratic character χ^S of $(\mathbb{A}^S)^*$ such that $\Pi'^S \cong \Pi^S \otimes \chi^S$ so that Π'^S is a theta lift of $(\pi^S, \omega^S, \delta^S)$. Although Π' need not be automorphic, this allows to compute the L-series $\zeta^S(\Pi, \tau, s)$. The L-series do not depend on character twists of Π. So it is enough to know the Satake parameters of Π'_v at the unramified places, in order to compute $\zeta^S(\Pi, \tau, s)$. In fact $\Pi'^S = \prod_{v\notin S} \Pi'_v$ is described in lemma 6.4 up to some twist. This describes $\zeta^S(\Pi, \tau, s)$ in terms of the automorphic representation $(\pi^S)^\vee = \pi^S$. Ramanujan's conjecture also holds for π^S (cor. 6.7). This allows to study the analytic properties of the degree five L-series $\zeta^S(\Pi, \tau, s)$ attached to Π in terms of π. The behaviour at $s = 1$ depends only on a set of places of density one. That Π is D-critical, implies strong restrictions. See prop. 6.8, cor. 7.3, cor. 8.3 and the complete section 9, in particular lemma 9.1. By these results it was shown in proposition 10.3, that for the nondegenerate D-critical representation Π of two-abelian type the algebra K is a quadratic field extension K of $\mathbb{Q}$ which is contained in the number filed L (see def. 3.2) of absolute degree $D = [L : \mathbb{Q}] = 4$, such that conjugation by the nontrivial substitution $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ satisfies the properties

$$\sigma(\pi) \ncong \pi, \quad \sigma(\pi) \cong \pi \otimes \chi_{L/K}.$$

A corresponding statement holds in the D-critical cases of CM-type by appendix C. The proof in this case is similar but more involved. Again K is a subfield of L. Now $\mathrm{Gal}(L/\mathbb{Q})$ is either dihedral or elementary abelian of order $D = 8$. (lemma 4.5 and appendix C). Again

$$\sigma(\pi) \ncong \pi, \quad \sigma(\pi) \cong \pi \otimes \chi_{F/K}$$

for some character $\chi_{F/K}$ attached to a quadratic extension $K \subset F$ in L. Hence in all the relevant D-critical cases we obtain $\sigma(\pi) \cong \pi \otimes \chi$ for some character χ of $\mathbb{A}_K^*/K^*$.

Now consider the archimedian place. Then π_∞ is associated to $\pi_\infty^\vee$, which occurs in a nontrivial theta correspondence $\Pi'_\infty \leftrightarrow (\pi_\infty^\vee, \omega_\infty, \delta_\infty)$. Since $\sigma(\pi) \cong \pi \otimes \chi$ implies $\sigma(\pi^\vee) \cong \pi^\vee \otimes \chi$, therefore

$$\sigma(\pi^\vee) \cong \pi^\vee \otimes \chi.$$

Since $\Pi'_\infty \cong \Pi_\infty \otimes \chi_\infty$ belongs to the discrete series, we can ask whether the theta lift allows to match discrete series Π'_∞ on $\mathrm{GSp}(4,\mathbb{R})$ with representations $(\pi_\infty^\vee, \omega_\infty, \delta_\infty)$, for which

$$\sigma_\infty(\pi_\infty^\vee) \cong \pi_\infty^\vee \otimes \chi_\infty$$

holds. The crucial fact is, that this is impossible by the following lemmas 11.1 and 11.2. For $K_\infty = \mathbb{R}^2$ the theta correspondence is sufficiently known, at least on the level of the dual pair $\mathrm{Sp}(4,\mathbb{R}), \mathrm{O}(V,\mathbb{R})$. This suffices for our purposes, since discrete series on $\mathrm{GSp}(4,\mathbb{R})$ correspond to discrete series on $\mathrm{Sp}(4,\mathbb{R})$ except for some splitting into subrepresentations under restrictions. So the case $K_\infty = \mathbb{R}^2$ is excluded by

11.1. Lemma. — *Suppose $K_\infty = \mathbb{R}^2$. Let $\widetilde{\Pi}_\infty$ be an irreducible representation of* $\mathrm{Sp}(4,\mathbb{R})$ *in the discrete series, which is the local theta lift of an irreducible representation $(\widetilde{\pi}_\infty, \delta_\infty)$ of* $\mathrm{O}(V,\mathbb{R})$, *where $\widetilde{\pi}_\infty$ is an irreducible representation of* $\mathrm{SO}(V,\mathbb{R})$. *Then $\widetilde{\pi}_\infty$ is in the discrete series of* $\mathrm{SO}(V,\mathbb{R})$ *and $\sigma_{\widetilde{\infty}}(\pi_\infty) \ncong \widetilde{\pi}_\infty \otimes \chi_\infty$ holds for all twist by characters χ_∞.*

Proof. — Since K_∞ splits, the connected component $\mathrm{SO}(V,\mathbb{R})^0$ of $\mathrm{SO}(V,\mathbb{R})$ in the analytic topology is either $\mathrm{SO}(2,2) \cong \mathrm{Sl}(2,\mathbb{R})^2/\pm$ or $\mathrm{SO}(4) \cong (H^1)^2/\pm$, where H^1 is the group of Hamilton quaternions of norm one. In these cases, the statement of the lemma can be found in [**Pr**] and [**KV**]. In fact checking these cases is tedious. Therefore we at least include some detailed references:

For the first case see [**Pr**], theorem 3.6.1 and 3.3.1, where it is shown, that $\widetilde{\pi}_\infty$ has to be in the discrete series and is not $\sigma_\infty = \varepsilon_\infty$ invariant. The ε_∞-invariant discrete series are listed in [**Pr**], (2.5.35-2.5.38). See also [**Pr**] 2.5.3. Since V is split the orthogonal group $\mathrm{O}(V)$ is isomorphic to $\mathrm{O}(2,2)$ in the sense of [**Pr**], (2.1.4). By [**Pr**] th. 3.6.1 (case 3.3.1) a discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$ ([**Pr**] (2.4.42)) corresponds to discrete series representations of $\mathrm{Sl}(2,\mathbb{R})^2/\pm$ (the connected component of $\mathrm{O}(2,2)(\mathbb{R})$ of index 4) after restriction. Only those discrete series representations appear in the image, which are not conjugation invariant under the twisted action of the nontrivial element $\varepsilon_\infty \in \mathrm{O}(2,2)(\mathbb{R})/\mathrm{SO}(2,2)(\mathbb{R})$. Note that $\mathrm{diag}(1,1,1,-1)$ is a representative in $\mathrm{O}(2,2)(\mathbb{R})$ for ε_∞, and that conjugation by this representative is easily studied on K_∞ types for $\mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$ embedded into $\mathrm{O}(2,2)(\mathbb{R})$ as in [**Pr**], (2.1.4). It replaces the type (m,n) by $(m,-n)$. This easily allows to determine the ε-invariant discrete series representations of $\mathrm{SO}(2,2)$. In the notation of [**Pr**], (2.5.3) the ε_∞-conjugation invariant discrete series representations of $\mathrm{SO}(2,2)(\mathbb{R})$ are $\pi_{m+1,0}, \pi_{0,n-1}$, which extend to the irreducible representations $\pi^p_{m+1,0}, \pi^p_{m+1,0}$ $(p=0,1)$ of $\mathrm{O}(2,2)(\mathbb{R})$. Exactly these representations are not in the image of a discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$, under the Howe lift. We like to understand this in terms of the representation π on $\mathrm{Gl}(2,\mathbb{R}) \times \mathrm{Gl}(2,\mathbb{R})$. It defines a representation of the quotient group $GSO_{2,2}(\mathbb{R})$. Its restriction to $SO_{2,2}(\mathbb{R})$ has to be described in the notations of [**Pr**]. We can identify $\mathrm{SO}(2,2)$ and $SO_{2,2}$ using notation of [**Pr**]. Then $\mathrm{SO}(2,2)(\mathbb{R})$ corresponds to the group $SO_{2,2}(\mathbb{R})$ defined by all pairs $(g_1,g_2) \in \mathrm{Gl}(2,\mathbb{R})$ with $\det(g_1) = \det(g_2) = \pm 1$ modulo $(g_1,g_2) \sim (-g_1,-g_2)$ embedded in $\mathrm{Gl}(4,\mathbb{R})$ by

$$(g_1,g_2) \longmapsto \mathrm{diag}(g_1, \det(g_1)(g_1')^{-1}) \cdot \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}, \quad g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using this isomorphism our representative for ε_∞ in $\mathrm{O}(2,2)(\mathbb{R})$ corresponds to the matrix

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in $O_{2,2}(\mathbb{R})$. Obviously conjugation by ε of the image of (g_1,g_2) flips the variables $g_1,g_2 \mapsto g_2,g_1$. Thus the representation $\pi = \pi_1 \times \pi_2$ of $SO_{2,2}(\mathbb{R})$ with $\pi_1 \cong \pi_2 \otimes \chi_\infty$ is invariant under conjugation by ε up to isomorphism. This gives a contradiction to what was explained above. The case, where the space V splits is therefore understood.

The definite case was studied in [**KV**]. For the convenience of the reader, we again give a sketch of what happens: If $\widetilde{\pi}_\infty$ comes from a pair $\pi^\vee_{1,\infty} \times \pi^\vee_{2,\infty}$ of representations $\pi^\vee_{i,\infty}$ of H^1 of dimension $\dim(\pi^\vee_{i,\infty}) = d_i$, then $\pi_{1,\infty} \times \pi_{2,\infty}$ belongs to the discrete series of $\mathrm{Sl}(2,\mathbb{R})^2$ of weight $r_i = d_i + 1$ for $\pi_{i,\infty}$ respectively (analog of Jaquet-Langlands lift). Therefore we may assume $r_1 \geqslant r_2 \geqslant 2$ and furthermore $r_1 \equiv r_2(2)$, since the central characters coincide on the diagonal embedded subgroup $\{\pm 1\} \subset \mathrm{Sl}(2,\mathbb{R})^2$. The theta lift $\widetilde{\Pi}_\infty$ of $\pi^\vee_\infty$ is a holomorphic/antiholomorphic discrete series or a limit of the holomorphic/antiholomorphic discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ of weight

$k_1 \geqslant k_2$, where

$$k_1 = \frac{1}{2}(r_1 + r_2), \quad k_2 = \frac{1}{2}(r_1 - r_2) + 2.$$

Notice $k_1 \geqslant k_2 \geqslant 2$. $\widetilde{\Pi}_\infty$ belongs to the discrete series, if and only if $k_2 \geqslant 3$ respectively $r_1 > r_2$. In fact this amounts to or at least implies the condition $\sigma_\infty(\widetilde{\pi}_\infty) \not\cong \widetilde{\pi}_\infty \otimes \chi_\infty$, since σ_∞ permutes the weights r_1, r_2. For these statements see [**KV**], p. 26 and p. 28(8.2). However notation in [**KV**] is different, since these authors use a different way to parameterize representations of $\mathrm{SO}(4,\mathbb{R})$. □

For the proof of theorem I the case $K_\infty \cong \mathbb{C}$ remains. Although the theta correspondence seems not been worked out completely for the pair $(\mathrm{GSp}(4,\mathbb{R}), \mathrm{GO}(3,1))$, enough information is provided by [**HST**]. It allows to complete the proof of theorem I by the next lemma. Again $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ contradicts the fact, that a restriction of π_∞ lifts to a constituent $\widetilde{\Pi}_\infty$ of the discrete series representation Π_∞. $\widetilde{\Pi}_\infty$ can be extended to a theta lift Π'_∞. Since $\widetilde{\Pi}_\infty$ belongs to the discrete series, also Π'_∞ belongs to the discrete series. So the proof of theorem I is completed by the next lemma.

11.2. Lemma. — *Suppose $K_\infty = \mathbb{C}$. Let Π'_∞ be an irreducible representation of* $\mathrm{GSp}(4,\mathbb{R})$ *contained in the discrete series. Assume that Π'_∞ is a nontrivial theta lift of an irreducible representation $(\pi_\infty, \omega_\infty, \delta_\infty)$ of* $\mathrm{GO}(3,1)$ *or* $\mathrm{GO}(1,3)$. *Then the irreducible representation π_∞ of* $\mathrm{Gl}(2,\mathbb{C})$ *can not satisfy $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ for a character χ_∞.*

Proof. — Suppose $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ holds for a character χ_∞. Note $\omega_{\pi_\infty} = \omega_{\sigma_\infty(\pi_\infty)}$, since π_∞ and $\sigma_\infty(\pi_\infty)$ have the same central character $\omega_\infty \circ \mathrm{Norm}_{\mathbb{C}/\mathbb{R}}$. Therefore $\chi_\infty^2 = 1$, since $\omega_{\sigma_\infty(\pi_\infty)} = \omega_{\pi_\infty \otimes \chi_\infty} = \omega_{\pi_\infty}\chi_\infty^2$. But $\chi_\infty^2 = 1$ implies $\chi_\infty = 1$, since the character $\chi_\infty(z)$ is of the form $|z|^s(\frac{z}{|z|})^n$ for some $n \in \mathbb{Z}$. Therefore

$$\sigma_\infty(\pi_\infty) \cong \pi_\infty.$$

By a character twist with a character $\tau_\infty \circ \mathrm{Norm}_{\mathbb{C}/\mathbb{R}}$ we may reduce to the case, where the central character of π_∞ is trivial. Then [**JL**], lemma 6.1 implies $\pi_\infty \cong \mathrm{Ind}(\mu_\infty, \mu_\infty^{-1})$, where $\sigma_\infty(\mu_\infty) = \mu_\infty^{-1}$

$$\mu_\infty(z) = (z/|z|)^n, \quad n \in \mathbb{Z}$$

or $\sigma_\infty(\mu_\infty) = \mu_\infty$ where

$$\mu_\infty(z) = |z|^s, \quad s \notin \mathbb{Z}.$$

The first case - with $n \neq 0$ - was considered in [**HST**] lemma 12. According to *loc. cit.* Π'_∞ must then have a $K_\infty = \mathrm{U}(2)$-type of highest weight $(n+1, 1)$ or $(n+1, 0)$ or $(n+1, 2)$ and infinitesimal Harish-Chandra parameter $(n, 0; *)$. According to [**Pr**] 2.4.42 this can be no discrete series representation, since the infinitesimal Harish-Chandra parameter $(n, 0)$ of the restriction of Π'_∞ to $\mathrm{Sp}(4,\mathbb{R})$ is not regular. The infinitesimal character $\gamma'_{n,0}$ (see [**Pr**] page 30) is excluded in [**Pr**] 2.4.42. This shows, that in the first case we do not obtain Π'_∞ in the discrete series. For $n = 0$ in the first case or also in the second case $\mu_\infty(z) = |z|^s$, the irreducible representation π_∞ is an induced representation, which contains the trivial representation of the connected maximal compact subgroup $\mathrm{SO}(3) = \mathrm{U}(2)/\mathrm{U}(1)$ of $\mathrm{Gl}(2,\mathbb{C})/\mathrm{U}(1)$. There are four possibilities to extend this to a representation of the maximal compact subgroup $\mathrm{O}(3) \times \mathrm{O}(1)$ of $\mathrm{GO}(3,1)$. These are denoted $(0,+,+),(0,+,-),(0,-,+),(0,-,-)$ in [**HST**], page 395. Therefore the computation of [**HST**], page 395 implies, that Π'_∞ has a $\mathrm{U}(2)$-type of weight $(1,1),(1,0),\varnothing,\varnothing$ (in the last two cases, the theta correspondence is trivial). These are Howe minimal $K_\infty = \mathrm{U}(2)$-types arising from the theta correspondence for the pair $(\mathrm{O}(V)(\mathbb{R}),\mathrm{Sp}(4,\mathbb{R})$. We show, that they can not occur in a discrete series representation $\widetilde{\Pi}'_\infty$ of $\mathrm{Sp}(4,\mathbb{R}) \subset \mathrm{GSp}(4,\mathbb{R})$. A discrete series representation is tempered and has real infinitesimal character. Therefore [**Pr**] 2.3.20 and 2.3.23 can be applied with $\gamma = \lambda$, with the notations of *loc. cit.* For the two K_∞-types $(1,1),(1,0)$ considered we have $\|\pi'_{m,n}\|_\lambda = m-1 = 0$ according to [**Pr**] (2.1.22). Therefore the infinitesimal character has norm $\|\lambda\| = \|\gamma\| = \|\pi'_{m,n}\|_\lambda = 0$ by [**Pr**] (2.3.21). Thus this K_∞-type must be a lowest K_∞-type of $\widetilde{\Pi}'_\infty$ in the sense of Vogan. But discrete series representation $\widetilde{\Pi}'_\infty$ of $\mathrm{Sp}(4,\mathbb{R})$ have a unique lowest K_∞-type. According to [**Pr**] (2.4.44-47) the types $\pi'_{1,1},\pi'_{1,0}$ do not occur. □

12. Proof of theorem II

Suppose Π is unitary and satisfies the conditions of theorem I. Suppose the representation $\rho_{\Pi,\lambda}$ constructed in theorem I is reducible of the form $\rho_{\Pi,\lambda} = 2\cdot\rho_0$. Let $V = V_{\rho_0}$ be the representation space of the two dimensional representation ρ_0. Then ρ_0 is an E-rational λ-adic representation. Therefore $\det(\rho_0)$ is automorphic [**He**], hence attached to an automorphic character ω_0 of $\mathbb{A}^*/\mathbb{Q}^*$. Furthermore the group G defined in section 1 satisfies $G \subset \mathrm{Gl}(V_{\rho_0}) \cong \mathrm{Gl}(2,k)$.

Since $G \subset \mathrm{Gl}(2)$, only the cases 1, 2, 3, 5, 8 of Taylor's list [**T**], p. 298 are possible. We leave it for the reader to check that this reduces us to the cases 1 or 3, since in case 2,5,8 the representation $\rho_{\Pi,\lambda}$ can not be of the form $2\cdot\rho_0$. See also section 2 and appendix B for the following notations.

Case 1. — In this case $\overline{G} \subset G$ has index at most two. $\overline{G} = G^0 \cdot N$, where N is the centralizer of G^0 in $\overline{G}$. We can compute N inside $\mathrm{Gl}(V_{\rho_0})$. G^0 is a torus. Since G^0 is a torus and acts on V_{ρ_0} by the characters $\chi_1 \neq n\chi_1^{-1}$ in the notation of [**T**], p. 298, N and G^0 are contained in a common maximal torus of $\mathrm{Gl}(V_{\rho_0})$. In particular $\overline{G}$ is abelian. Hence the restriction of ρ_0 to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{L})$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (of index at most two) is abelian. So the representation $\rho_{0,\lambda}$ is either abelian, a sum of two characters, or induced from a character χ_3 of $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{L})$

$$\rho_0 \cong \chi_1 \oplus \chi_2 \quad \text{or} \quad \rho_0 = \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}},\overline{L})}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\chi_3).$$

Again [**He**] shows, that χ_1, χ_2 are attached to characters on $\mathbb{A}^*/\mathbb{Q}^*$ respectively χ_3 is attached to a Grossencharacter of $\mathbb{A}_{\overline{L}}^*/\overline{L}^*$. Hence in both cases ρ_0 is attached to an irreducible automorphic representation π on $\mathrm{Gl}(2,\mathbb{A})$. We normalize π by a twist with the idele norm character so that

$$L_v(\pi, s - w/2) = L_v(\rho_0, s)$$

holds for almost all unramified v. Then Π is a weak lift, which is impossible by looking at the archimedian place. But we can give a direct argument as follows:

Suppose Π satisfies Ramanujan's conjecture for all unramified places. (If Π does not satisfy Ramanujan's conjecture, then Π is a CAP representation. For CAP representations see the discussion at the end of this section). Since $L^S(\Pi, s) = L^S(\pi, s)^2$ the representation π also satisfies the Ramanujan conjecture for $v \notin S$. Hence by remark 7.2 and prop. 7.1 the L-series $\zeta^S(\mathrm{Ad}(\pi), 1, s)$ does not vanish at $s = 1$. $L^S(\Pi, s) = L^S(\pi, s)^2$ for a suitably finite set S implies

$$L^S(\omega_\Pi\omega_\pi^{-1}, s)\zeta^S(\Pi, \omega_\Pi\omega_\pi^{-1}, s) = \zeta^S(s)^2 L^S(\pi \otimes \pi^\vee, s) = \zeta^S(s)^3\zeta^S(\mathrm{Ad}(\pi), 1, s).$$

This gives a pole for $\zeta^S(\Pi, \omega_\Pi\omega_\pi^{-1}, s)$ of order $\geqslant 3$ for $\omega_\Pi \neq \omega_\pi$ at $s = 1$, and at least a pole of order 2 if $\omega_\Pi = \omega_\pi$. This contradicts theorem 4.2. Hence case 1 is excluded.

Case 3. — In this case $G = \mathrm{Gl}(2, k)$ and $\pi_0(G) = 1$. By the assumption $\rho_{\Pi,\lambda} = 2 \cdot \rho_0$ the Satake parameters of $\Pi_v, v \notin S$ are highly restricted. Suppose $\alpha_v, \beta_v, \alpha_v\beta_v = \omega_\pi(p_v)$ are the eigenvalues of Frob_v for ρ_0, up a twist by the factor $p_v^{-w/2}$. Then the Satake parameters of Π_v are either $(\alpha_v, \alpha_v, \beta_v, \beta_v) \sim (\alpha_v, \beta_v, \beta_v, \alpha_v)$ or $(\alpha_v, \beta_v, \alpha_v, \beta_v)$. Since $G = \mathrm{Gl}(2, k)$ we can assume $\alpha_v \neq \pm\beta_v$ for all v in a set of primes of Dirichlet density 1. This excludes $(\alpha_v, \beta_v, \alpha_v, \beta_v)$ for v in this set of density 1, since otherwise $\alpha_v^2 = \omega_\Pi(p_v) = \beta_v^2$ and therefore $\alpha_v = \pm\beta_v$. Hence we are in the first case and $\alpha_v\beta_v = \omega_{\Pi,v}(p_v)$. In fact we have found a set T of places of density zero, outside of which Π_v satisfies condition (i) and (ii) and (iii) of definition 3.2. Suppose Π is neither CAP nor a weak endoscopic lift. Then Ramanujan's conjecture holds for Π at almost all places. We claim Π is D-critical with $D = 1$ respectively $L = \mathbb{Q}$. Put $L = \mathbb{Q}$ and $\nu_v = \alpha_v, \mu_v = \beta_v$ and $\zeta_v = 1$. That $(\alpha_v, \alpha_v, \beta_v, \beta_v)$ are the Satake parameters for Π_v for $v \notin T$ is assertion (i) of def. 3.2. Assertion (ii) of def. 3.2 is the identity $\alpha_v\beta_v = \omega_{\Pi_v}(p_v)$ for $v \notin T$ shown above.

Lemma 4.4 applied for this 1-critical representation implies that $\zeta^S(\Pi, 1, s)$ has a pole at $s = 1$. More precisely $\log \zeta^S(\Pi, 1, s) \sim \sum_v(\mathrm{Ad}_v + 2) \cdot p_v^{-s}$ at $s \to 1^+$. So the restriction of Π to $\mathrm{Sp}(4, \mathbb{A})$ contains a theta lift, with associated algebra $K = \mathbb{Q}^2$. Hence there is a unitary representation π of $\mathrm{Gl}(2, \mathbb{A}_K) = \mathrm{Gl}(2, \mathbb{A})^2$ associated to Π as in section 4. In fact, Π must be a weak lift by lemma 6.4. Alternatively, since a pole of $\zeta^S(\Pi, 1, s)$ is simple (theorem 4.2) we get $\sum_v(\mathrm{Ad}_v + 1) \cdot p_v^s \sim 0$ at $s \to 1^+$. Therefore $n_K(\Pi) = 0$ by the asymptotic formula of lemma 9.1. But this also contradicts the assertion $n_K(\Pi) \geqslant 1$ of lemma 9.1.

So it remains to discuss, whether Π can be CAP or a weak endoscopic lift. If it were, then $\rho_{\Pi,\lambda} \cong \rho_1 \oplus \rho_2$, where ρ_i are attached to irreducible automorphic representations π_i of $\mathrm{Gl}(2, \mathbb{A})$. If Π is a weak endoscopic lift, then $\pi_{\infty,i}$ belong to the discrete series of weight r_i with $r_1 > r_2$. By a theorem of Ribet ρ_1, ρ_2 are irreducible. Hence $\rho_1 \cong \rho_0 \cong \rho_2$. This contradicts $r_1 \neq r_2$. So Π must be a CAP representation.

The CAP representation case. — Then $\rho = \rho_1 \oplus \rho_2$ and one of the representation ρ_1, ρ_2 is reducible. Then $\rho_{\Pi,\lambda} = 2 \cdot \rho_0$ implies that ρ_0 is reducible, $\rho_0 \cong \chi_1 \oplus \chi_2$ for Dirichlet characters χ_1, χ_2 of $\mathbb{A}^*/\mathbb{Q}^*$, not necessarily unitary.

This implies that Π is a CAP representation associated to the Borel subgroup B of $\mathrm{GSp}(4)$. These were classified in [**P**] and [**S**]. The description given in [**S**] of such CAP representations in terms of binary theta lifts implies, that Π_∞ can be a limit of discrete series at the archimedian place but cannot belong to the discrete series itself. We have now shown that $\rho_\Pi = 2 \cdot \rho_0$ cannot occur. □

Finally we consider what it means, that $\rho_{\Pi,\lambda}$ contains a one dimensional subrepresentation ρ_1. Still Π is unitary satisfying the assumptions of theorem I. To prove the last assertion of theorem II we must show, that Π is a CAP representation. It can not be a weak endoscopic lift $\rho_{\Pi,\lambda} = \rho_{\pi_1} \oplus \rho_{\pi_2}$, since the two dimensional representations ρ_{π_i} for the classical elliptic holomorphic cusp forms are known to be irreducible as shown by Ribet. So if Π is not CAP the Galois representations $\rho_{\Pi,\lambda}$ is a subrepresentation of the Galois-representation W_{Π_λ} constructed from the third cohomology of the Shimura variety. So we are in the situation of section 2. Put $\omega = \omega_\Pi \mu_l^{-w}$. Then by duality $\rho_{\Pi,\lambda}^\vee \otimes \omega \cong \rho_{\Pi,\lambda}$ another one dimensional character $\rho_2 = \rho_1^\vee \otimes \omega$ occurs in $\rho_{\Pi,\lambda}$. As $\rho_{\Pi,\lambda}$ is a subrepresentation of W_{Π_λ} we have $\rho_2 \neq \rho_1$, since otherwise $(\rho_1)^2 = \omega$. This would imply $\lambda^2 = n$, since ω is induced by n, and would thus contradict the root condition property (c) of Π formulated in section 2 ([**T**], lemma 1 and cor. 1). Hence there exists a decomposition $\rho_{\Pi,\lambda} = \rho_1 \oplus \rho_2 \oplus \rho_3$ with one dimensional subrepresentations $\dim_{\overline{\mathbb{Q}}_l}(\rho_1) = \dim_{\overline{\mathbb{Q}}_l}(\rho_2) = 1$. Now apply theorem III. So $\rho_{\Pi,\lambda}$ can be viewed as a $\mathbb{Q}_l$-representation of dimension $4 \cdot [E_\lambda : \mathbb{Q}_l]$ and this representation is of Hodge-Tate type. Hence the characters ρ_1, ρ_2 are locally algebraic and have the form $\rho_i = \chi_i \mu_l^{n_i}$ for some characters χ_i of $\mathbb{A}^*/\mathbb{Q}^*$ of finite order ($i = 1, 2$). If Π is not CAP the representation $\rho_{\Pi,\lambda}$ is pure of weight w (the Ramanujan conjecture holds). Therefore $n_1 = n_2 = -w/2$ and w must be even. Then $n = \omega_\Pi \mu_l^{-w}$ for some character ω_Π of finite order. Hence for a suitable power $\rho_1^{2\nu} = \rho_2^{2\nu} = n^\nu$ (for an integer $\nu > 0$). This is a contradiction to the root condition property (c) already mentioned (it is $\lambda^2 \neq n$ and explained in section 2). The proof of theorem II is complete. □

Appendix A
Balanced representations

In this section $\widetilde{N}$ is a finite group, and k is an algebraically closed field of characteristic zero. A group of type $(\mathbb{Z}/2\mathbb{Z})^r$ for some r is called elementary two-abelian.

Definition. — A representation

$$\rho : \widetilde{N} \longrightarrow \mathrm{Gl}(V)$$

on a finite dimensional k vector space V is said to have "only two eigenvalues" if for all $g \in \widetilde{N}$ the matrix $\rho(g)$ has at most two different eigenvalues $\zeta_1(g), \zeta_2(g)$. Let their multiplicities be $a_1(g), a_2(g)$ with the convention $\zeta_1(g) = \zeta_2(g)$, if only one eigenvalue occurs. Also put $\zeta(g) = \zeta_1(g)/\zeta_2(g)$ (well defined in k^* up to inverse).

Let ρ be a representation with only two eigenvalues. For the kernel $Z(\rho)$ of the associated projective representation

$$\overline{\rho} : \widetilde{N} \longrightarrow \mathrm{PGl}(V)$$

we have $\rho|Z(\rho) = \chi \cdot \mathrm{id}$, for some character of χ of $Z(\rho)$. Let $D(\rho)$ be the cardinality of $Q(\rho) = \widetilde{N}/Z(\rho)$. Consider

$$0 \longrightarrow Z(\rho) \longrightarrow \widetilde{N} \longrightarrow Q(\rho) \longrightarrow 0$$
$$D(\rho) = \#Q(\rho),$$

For fixed ρ we write Z, Q, D instead of $Z(\rho), Q(\rho), D(\rho)$. Then $g \in Z(\rho)$ iff $\zeta(\rho, g) = 1$ and the order of root of unity $\zeta(\rho, g)$ is the order of the image of g in $Q(\rho)$.

Decompose $\rho \cong \bigoplus_j \rho_j$ into irreducible representations ρ_j. If ρ has only two eigenvalues, then also the representations ρ_j. The possible irreducible representations ρ_j with only two eigenvalues were classified in [**T2**], lemma 9 and corollary 1. There are four possible types of irreducible representations ρ_j

(1) $\dim(\rho_j) = 1$.

(2) $\dim(\rho_j) = 2$ and either ρ_j is dihedral (including D_4, *i.e.* induced from a character of a subgroup of index two) or the image $Q_j = Q(\rho_j)$ of $\widetilde{N}$ in the associated two dimensional projective group $\mathrm{PGl}(2,k)$ is A_4 or S_4 or A_5.

(3) $\dim(\rho_j) = d_j \geqslant 4$. In this case there is a subgroup N_j of $\widetilde{N}$ of index $\geqslant 16$. The quotient $Q_j = \widetilde{N}/N_j$ is an elementary abelian two group, N_j acts by scalars under ρ_j, and the elements of $\widetilde{N} \setminus N_j$ have trace zero and eigenvalues of the form $\zeta_1(n) = -\zeta_2(n)$. Thus $\zeta(\rho_j, n) = 1$ resp. $\zeta(\rho_j, n) = -1$ if $n \in N_j$ resp. $n \notin N_j$.

Definition. — A representation ρ with only two eigenvalues is called *balanced* if $d = \dim(\rho)$ is even and $a_1(g) = a_2(g) = d/2$ holds for all $g \neq 1$ in $Q(\rho)$. Formally put $a_1(g) = a_2(g)$ for $g = 1$ in $Q(\rho)$.

Example. — Any two dimensional representation with at most two eigenvalues is balanced.

A.1. Lemma. — *Suppose ρ is balanced. If ρ is not isotypic multiple of a two dimensional representation, then the associated group $Q(\rho)$ is elementary two-abelian of order $D(\rho) \geqslant 4$. For $D = 4$ the representation ρ is a multiple of $\chi \otimes (1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1\chi_2)$ for characters χ, χ_1, χ_2 with $\chi_1^2 = \chi_2^2 = 1$.*

Proof. — A consequence of claim 7 and 8 below. □

Claim 1. — *Let $\rho : \widetilde{N} \to \mathrm{Gl}(V)$ be balanced of dimension d. Let Z be the kernel of $\overline{\rho}$, then $\rho|Z = \chi \cdot \mathrm{id}$ for some character of χ of Z. Fix (Z, χ) for the moment. Let ρ' be a finite dimensional irreducible representation of $\widetilde{N}$ of dimension d' such that $\rho'|Z = \chi \cdot \mathrm{id}$. Then there exists an integer $1/c(Q)$ depending only on $Q = \widetilde{N}/Z$, so that the multiplicity $m(\rho, \rho')$ for ρ' in ρ is either zero or*

$$m(\rho, \rho') = c(Q)dd'.$$

For a decomposition $\rho \cong \oplus m(\rho, \rho_i)\rho_i$ into irreducible, nonisomorphic representations ρ_i of dimensions d_i

$$\textstyle\sum_i d_i^2 = 1/c(Q).$$

Proof. — Put $D = D(\rho)$. The multiplicity $m(\rho, \rho') = D^{-1} \sum_{g \in Q} \chi_\rho(g)\overline{\chi}_{\rho'}(g)$ is zero unless ρ' appears in ρ. Then ρ' satisfies $\chi_{\rho'}(g) = a_1'(g)\zeta_1(g) + a_2'(g)\zeta_2(g)$, $a_1'(g) + a_2'(g) = d'$. The integer $m(\rho, \rho')$ can be expressed in the form

$$D^{-1} \cdot \sum_{g \in Q} \big(a_1(g) + a_2(g)\big)\big(a_1(g)' + a_2'(g)\big) + a_1(g)a_2'(g)\big(\zeta(g) - 1\big) + a_2(g)a_1'(g)\big(\zeta(g)^{-1} - 1\big)\big).$$

In this formula replace $\zeta = \zeta(g)$ or ζ^{-1} by the $\mathbb{Q}$-projection $[\mathbb{Q}(\zeta) : \mathbb{Q}]^{-1}\mathrm{trace}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta)$. This $\mathbb{Q}$-projection only depends on the order $m = \mathrm{ord}(g)$ of g and defines a weakly multiplicative function $s(m)$ on $\mathbb{N}$. For prime powers p^n

$$s(p^n) = 1,\ -1/(p-1),\ 0 \quad \text{for} \quad n = 0,\ 1,\ n \geqslant 2.$$

In particular, the numbers $\alpha_i = (1 - s(i))/2$ satisfy $\alpha_1 = 0$, $\alpha_2 = 1$ and $0 \leqslant \alpha_i \leqslant 3/4 < 1$ for $i \geqslant 3$. With these notations $a_1(g) = a_2(g) = d/2$ and $a_1'(g) + a_2'(g) = d'$ implies $m(\rho, \rho') = dd' \cdot c(Q)$, where $c(Q) := 1 - d(Q)$ and $d(Q) := \frac{1}{2D} \sum_{g \in Q}\big(1 - s(\mathrm{ord}(g))\big)$. Obviously $c(Q) > 0$, since not all $m(\rho, \rho')$ are zero.

Let $n_i(Q)$ denote the numbers of elements in Q of order i. Then

$$D \cdot d(Q) = \sum n_i(Q)\alpha_i = n_2 + \frac{3}{4}n_3 + \frac{1}{2}n_4 + \cdots.$$

Decompose $\rho \cong \bigoplus_{t=1}^{t} m(\rho, \rho_i)\rho_i$ into irreducible non isomorphic constituents ρ_i. Then $m(\rho, \rho_i)c(Q)^{-1} = dd_i$ and $\sum_i m(\rho, \rho_i)d_i = d$ imply $\sum_{i=1}^{t} d_i^2 = 1/c(Q)$. Hence $c(Q)^{-1}$ is an integer. Claim 1 is proved. □

Claim 2. — *Let Q be a finite group of order D. Define $c(Q)$ and $d(Q)$ as above. Let $n_i(Q)$ denote the numbers of elements in Q of order i. Then*

(a) *$D/2 \geqslant n_2(Q)$ implies $c(Q) > 1/8$.*

(b) $3D/4 \geqslant n_2(Q) > D/2$ *implies* $c(Q) > 1/16$ *and* $c(Q) > 1/8$, *if* Q *is a* 2-*Sylow group.*

(c) $n_2(Q) > 3D/4$ *implies* $Q \cong (\mathbb{Z}/2\mathbb{Z})^r$ *and* $c(Q) = \frac{1}{D}$.

Proof of (a)-(c). — The first statement of (c) is $Q \cong (\mathbb{Z}/2\mathbb{Z})^r$. It follows from elementary group theory and the knowledge $n_2(Q) > 3D/2$ of the elements of order two. Fix $g \in Q, g^2 = 1$. Any $h \in Q$, for which $h^2 = (gh)^2 = 1$ holds, is in the centralizer $Z_Q(g)$ of g. By the assumption on $n_2(Q)$ the number of such elements h is $> D/2$. Therefore $Z_Q(g) = Q$. Thus $g \in Z(Q)$. The centre has more than $3D/4$ elements and $Q = Z(Q)$.

For the statements (a), (b) and (c) concerning the value $c(Q)$ use $c(Q) = 1 - d(Q)$ and $D \cdot d(Q) = \sum n_i(Q)\alpha_i$ with $\alpha_i \leqslant 3/4$ for $i \geqslant 3$. Hence

$$d(Q) \leqslant D^{-1}\Big(n_2(Q) + \frac{3}{4}(D - 1 - n_2(Q))\Big) \leqslant \frac{n_2(Q)}{4D} + \frac{3}{4}(1 - D^{-1}).$$

If $n_2(Q) \leqslant D/2$, therefore $d(Q) < 7/8$ and $c(Q) > 1/8$. If $n_2(Q) \leqslant 3D/4$, therefore $d(Q) \leqslant 3/16 + 3/4 - 3/4D < 15/16$ and $c(Q) > 1/16$. If $n_2(Q) > D/2$ and Q is a 2-Sylow group, then $n_i(Q) \neq 0$ unless i is a power of 2. Hence

$$d(Q) = D^{-1}\Big(n_2(Q) + \sum_{i>2} \frac{1}{2} n_i(Q)\Big) = (2D)^{-1}\big(n_2(Q) + D - 1\big) < \frac{n_2(Q)}{2D} + \frac{1}{2}.$$

For $n_2(Q) \leqslant 3D/4$ this implies $d(Q) < 7/8$ and $c(Q) > 1/8$. Then Q is elementary two-abelian, if $n_2(Q) > 3D/4$. Hence $n_i(Q) = 0$ for $i > 2$ and $d(Q) = (D-1)/D$ resp. $c(Q) = 1/D$. Claim 2 follows. □

Claim 3. — *Suppose* ρ *is balanced. Suppose* $\chi_1 \oplus \chi_2 \oplus \chi_3 \hookrightarrow \rho$ *contains three non-isomorphic one dimensional representations* $\rho_i = \chi_i$, $i = 1, 2, 3$. *Then all quotients* χ_i/χ_j *are quadratic characters.*

Proof. — By a twist assume $\chi_1 = 1$. Suppose χ_2 were a character of order $N \geqslant 2$ and χ_3 of order $M \geqslant 2$. Since ρ has only two eigenvalues, $\chi_2(g) \neq 1$ and $\chi_3(g) \neq 1$ implies $\chi_2(g) = \chi_3(g)$. For simplicity reduce to $Q = (\mathbb{Z}/NM\mathbb{Z})^2$; then χ_3/χ_2 is trivial on at least $1 + (N-1)(M-1)$ elements, hence on $> NM/3$ elements of this group of order NM resp. $> NM/2$ elements if $N \neq 2$. For $N \neq 2$ this forces $\chi_2(g) = \chi_3(g)$ for all g contradicting the assumption $\chi_2 \neq \chi_3$. Hence $N = 2$, and still this forces χ_3/χ_2 to be quadratic. But then χ_2 is quadratic ($N = 2$) and also χ_3 is quadratic. This completes the proof. □

Claim 4. — *Suppose* ρ *is balanced. Suppose* $\rho_1 \oplus \rho_2 \hookrightarrow \rho$, *where* ρ_1 *irreducible of dimension* 2 *and* ρ_2 *is a one dimensional character. Then* $Q = Q(\rho_1)$ *is an elementary two-abelian group.*

Proof. — Assume $\rho_2 = 1$ by a twist. Then one eigenvalue is always $\zeta_1(g) = 1$. The representation ρ_1 is automatically balanced. Put $Q = Q(\rho_1)$ and $Z = Z(\rho_1) = \mathrm{Kern}(\overline{\rho}_1)$. Suppose $\rho_1|Z = \chi$. By assumption $\sum_{g\in\tilde{N}} \overline{\chi}_{\rho_1}(g)$ vanishes. Furthermore $\chi_{\rho_1}(g) = 2\chi(g)$ for $g \in Z$ and $\chi_{\rho_1}(g) = 1 + \zeta(g)$ for $g \notin Z$, since 1 is always an eigenvalue. Hence

$$\#Z \cdot \sum_{g\neq 1\in Q} (1 + \overline{\zeta}(g)) + 2 \cdot \sum_{g\in Z} \overline{\chi}(g) = 0.$$

For $\chi = 1$ the left side would be > 0. Hence $\chi \neq 1$. Therefore the sum of the $1 + \overline{\zeta}(g)$ extended over all $g \in Q$ including $g = 1$ is 2. Hence

$$\frac{1}{2D} \sum_{g\in Q} (1 - \overline{\zeta}(g)) = 1 - D^{-1}.$$

In other words $c(Q) = 1/D$. The claim now follows from

Claim 5. — *Let Q be a finite group Q of order D. Then $c(Q) = 1/D$ iff Q is elementary two-abelian.*

We have $c(D) = 1 - (D)^{-1}(n_2 + \sum_{i\geqslant 3} \alpha_i n_i)$ with $0 \leqslant \alpha_i < 1$. Thus $1/D = c(Q)$ is equivalent to $\sum_{i\geqslant 3} \alpha_i n_i = \sum_{i\geqslant 3} n_i$ or $n_i = 0, i \geqslant 3$. This is equivalent to the fact, that Q is elementary two-abelian.

Claim 6. — *Suppose ρ is balanced and $Q = Q(\rho)$. If Q is elementary two-abelian, then $\zeta(g) = -1$ if $g \neq 1$ in $Q(\rho)$ and $\zeta(g) = 1$ if $g = 1$ in Q.*

Claim 7. — *Suppose ρ is balanced with $Q = Q(\rho)$. Suppose $c(Q) > 1/8$. Then ρ is an isotypic multiple $\rho = a \cdot \rho_1$ of a two dimensional representation ρ_1, or Q is an elementary abelian two-group of order $D = D(\rho) = 4$. In the second case ρ is an isotypic multiple of $\chi \otimes (1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1\chi_2)$.*

Proof. — Decompose $\rho = \oplus m(\rho, \rho_i)\rho_i$ into irreducible representations ρ_i of dimension d_i. Then $c(Q) > 1/8$ or $c(Q)^{-1} \leqslant 7$ implies $\sum_i d_i^2 \leqslant 7$ by claim 1. Hence there is at most one i with $d_i = 2$, all others satisfy $d_j = 1, j \neq i$.

Hence either ρ is an isotypic multiple of an irreducible two dimensional representation. Otherwise there are two alternatives: Either ρ is a direct sum of characters (all $d_i = 1$ involving at least three, but at most seven different characters), each with the same multiplicity. Then all characters are quadratic and Q is elementary two-abelian (claim 3). Then $D = c(Q)^{-1} < 8$ is one of the two-powers $\sum_i d_i^2 = D = 1,\ 2,\ 4$. Hence $D = 4$ and the character ρ has to be of the form indicated. On the other hand there is the case, where there is one irreducible two dimensional constituent ρ_1 with multiplicity $2a$, and there are (up two 3 distinct, at least one) characters χ_i each with multiplicity a (claim 1) $\rho = 2a \cdot \rho_1 \oplus \bigoplus_{i=1}^t a \cdot \chi_i$ (with $1 \leqslant t \leqslant 3$). By a character twist we may assume $\chi_1 = 1$. Let us show that this second case cannot occur.

Consider the group $Q_1 = Q(\rho_1)$ of the balanced two dimensional representation ρ_1. By claim 4 the group Q_1 is elementary two-abelian.

Let $Q(\rho) \to Q(\rho_1)$ be the natural surjection and U its kernel. We claim that U is trivial. Therefore $Q(\rho) \cong Q(\rho_1)$, which is elementary two-abelian. Hence $c(Q) = 1/D$ is a 2-power (by claim 5) and the assumption $c(Q) > 1/8$ implies $D \leqslant 4$. Since $D = \sum d_i^2 > 4$ this is a contradiction, which excludes this case. Triviality of U: Fix $\overline{g} \in U$ and choose a representative $g \in \widetilde{N}$. It remains to show $\overline{g} = 1$. We obtain $4a$ times the eigenvalue $\chi(g)$ on $2a \cdot \rho_1$, and at most $3a$ eigenvalues on the sum of character spaces including 1. Therefore $\chi(g) = \chi_i(g) = 1$ for all $i = 1, \dots, t$, since ρ is balanced. But this implies $\rho(g) = 1$, hence $\overline{g} = 1$ in $Q(\rho)$. So U is trivial. □

Claim 8. — *Suppose ρ is balanced with $Q = Q(\rho)$. Suppose $c(Q) \leqslant 1/8$. Then Q is elementary two-abelian and $D = D(\rho) \geqslant 8$.*

Proof. — $n_2(Q) > D/2$ holds by our assumption $c(Q) \leqslant 1/8$ (claim 2a). If $n_2(Q) > 3D/4$, then Q is elementary two-abelian (claim 2c) of order $D \geqslant 8$ (claim 5) and we are done. In the remaining case $c(Q) > 1/16$ (claim 2b). Furthermore Q cannot be a 2-Sylow group (claim 2b). In fact, this leads to a contradiction:

On one hand $\sum_i d_i^2 < 16$ (claim 1), hence in particular $d_i < 4$. This implies $d_i = 2$ or $d_i = 1$. All irreducible constituents of $\rho = \bigoplus_i m_i \rho_i$ therefore have dimension $d_i \leqslant 2$. Since Q is not a 2-Sylow group, at least one representation ρ_i has dimension $d_i = 2$.

The natural map $Q(\rho) \to \prod_i Q(\rho_i)$ has an abelian kernel K. K is contained in the center of $Q(\rho)$.

Suppose first, that one of the ρ_i is one dimensional. Then all $Q(\rho_i)$ are elementary two-abelian (claim 4). Thus $Q(\rho)$ is a central extension of a two-abelian group by an abelian group K. Therefore $Q \cong Q' \times K'$, where Q' is a 2-Sylow group and K' is odd abelian. Then $n_2(Q) > D/2$ implies $K' = 0$ and $Q = Q'$. A contradiction, since Q was shown not to be a 2-Sylow group. Therefore all ρ_i must be 2-dimensional. At most three of them occur, each with equal multiplicity, which we can assume to be one. Then claim 1 gives $m(\rho, \rho_i) c(Q)^{-1} = dd_i$, hence $c(Q)^{-1} = dd_i = 2d = 2 \cdot 2t$, where $t = 2, 3$. $t = 1$ is excluded by the assumption $c(Q) \leqslant 1/8$.

Each of the maps $Q(\rho) \to Q(\rho_i)$ is surjective. Therefore $n_2(Q(\rho)) > D(\rho)/2$ implies $n_2(Q(\rho_i)) > D(\rho_i)/2$. The list in [**T2**] therefore excludes the case $Q(\rho_i) = A_4, S_4, A_5$. There only remain two dimensional irreducible representations ρ_i. In particular, $\widetilde{N}$ is not abelian. Therefore the $Q(\rho_i)$ are dihedral groups $D_{2n_i} = \langle S, N | S^2 = N^{n_i} = 1, SNS = N^{-1} \rangle$, since $Q(\rho_i)$ is not cyclic (otherwise $\widetilde{N}$ is a central extension of an cyclic group by a cyclic group, hence abelian.) Hence ρ is the sum of two (or three) irreducible 2-dimensional representation of dihedral type.

Let $U_i \subset Q$ be the normal subgroup of index two belonging to the subgroup of index 2 in the group $Q(\rho_i)$ attached to ρ_i. Then the restriction of ρ to U is still balanced, and contains two different one dimensional characters. By claim 4 the groups U_i are therefore elementary two-abelian. U_i is the extension of an elementary two-abelian group by a central abelian group Z_i. Furthermore Q is an extension of a 2-group by the central abelian group Z_i. As above this implies, that $Q \cong Q' \times K'$. Here Q' is a 2-Sylow group of Q and K' is abelian of odd order. Then, as above, we show $K' = 0$ and $Q = Q'$ is a 2-Sylow group. This again gives a contradiction. □

Appendix B
The Cases 1 and 3

We freely use notations from [**T**] reviewed in section 2. In particular we use the definitions of the groups G, G^0, $\overline{G}$, the centralizer N of G^0 and the finite group $\widetilde{N}$ attached to Galois representation of Π. G is the Zariski closure of the image of the Galois group and the character n. G contains a subgroup of finite index $\overline{G}$. In this appendix we discuss the cases 1 and 3. So $\overline{G}$ can be written in terms of the connected component G^0 as a pushout $\overline{G} = (\widetilde{N} \times G^0)/(\mu_n)^r$. We want to understand the underlying representation of G in terms of G^0 and $\widetilde{N}$. The set X of semisimple conjugacy classes $\mathrm{PGl}(2,k)_{\mathrm{ss}}/\sim$ can be identified with the set of elements x in k^*, up to replacing elements x by their inverse x^{-1}. Let Ω be the subset of X represented by roots of unity x of order diving $2|\pi_0(N)|$.

The third case of [**T**]. — Here the situation is $G = \overline{G}$ and $G^0 = \mathrm{Gl}(2,k)$. The representation s of G, when restricted to $G^0 = \mathrm{Gl}(2,k)$, becomes an $2m$-fold copy of the two dimensional standard representation t of $\mathrm{Gl}(2,k)$ ([**T**], p. 301 bottom) and s is a tensor product of t with a finite dimensional representation ρ of the finite group $\widetilde{N}$

$$s \cong \rho \otimes t, \quad \rho(zg) = z^{-1}\rho(g) \text{ for } z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in \mu_n.$$

Since $G^0 = \mathrm{Gl}(2,k)$ the relation between the L-functions of Π and the Galois representation (property (e) of section 2) imply by Tchebotarev, that for any $g \in \widetilde{N}$ the matrix $\rho(g)$ can have at most two different eigenvalues $\zeta_1(g), \zeta_2(g)$, with multiplicities $a_1(g), a_2(g)$, such that $a_1(g) + a_2(g) = 2m$. If there is only one eigenvalue put $\zeta_1(g) = \zeta_2(g)$. Then the ratio

$$\zeta(g) = \zeta_1(g)/\zeta_2(g) \in \Omega$$

is well defined up to inverse and depends only on the image of g in the quotient group $\pi_0(N) = N/Z(G^0) \cong \widetilde{N}/\mu_n$. Therefore ζ can be extended via $\overline{G} \to \widetilde{N}/\mu_n$ to a function on the pushout $G = \overline{G}$ with values in $\Omega \subset X$. We now define a second map $G \to X$. Any g in $G = (\widetilde{N} \times \mathrm{Gl}(2,k))/\mu_n$ is conjugate to some element with representative

$$\widetilde{n}(g) \times \begin{pmatrix} u(g) & * \\ 0 & v(g) \end{pmatrix} \in \widetilde{N} \times \mathrm{Gl}(2,k).$$

The ratio $u(g)/v(g)$ is well defined in X and only depends on the conjugacy class of $g \in G$. In fact it is the image class in $G/N \cong \mathrm{PGl}(2,k)$. Let $\Omega \subset X$ be the finite set defined above. Let $G_\Omega \subset G$ be the subset of all elements g in G, whose image under this map $G \to X$ is in Ω. The set $T = T_\Omega$ of primes v with the Frobenius elements Frob_v in G_Ω has Dirichlet density zero ([**T**], lemma 2). Enlarge T by a finite set of places so that it in particular contains the archimedian and ramified places of Π.

B.1. Lemma. — *Suppose the representation* Π *satisfies the assumptions (a) - (e) formulated in section 2 and suppose we are in case three of Taylor's list. Then there exists a set T of places v of Dirichlet $\mathbb{Q}$-density 0 for which the Satake parameters of* Π_v *are of the form*

$$\Pi_v \sim (\nu_v, \nu_v\zeta_v^{-1}, \mu_v, \mu_v\zeta_v), \qquad \nu_v\mu_v = \widetilde{\nu}_v\widetilde{\mu}_v = n(\mathrm{Frob}_v)$$

so that ζ_v is in Ω and so that ν_v/μ_v is not in Ω (both viewed in the set $X = k^*$ modulo inverse).

Proof. — For $v \in T = T_\Omega$ as above property (e) implies that the set of Satake parameters of Π_v is $\{v(g)\zeta_1(g), v(g)\zeta_2(g), u(g)\zeta_1(g), u(g)\zeta_2(g)\}$. By abuse of notation $g = r(\mathrm{Frob}_v)$ or also $g = \mathrm{Frob}_v$ is the Frobenius class. $\zeta_i(g)$ are roots of unity as defined above. For $u(g), v(g)$ suitably chosen we can assume $\zeta_1(g) = 1$. Then $\zeta(g) = \zeta_2(g)^{-1}$. Since the Weyl group acts transitively on the Satake parameters the first can be chosen to be $\nu_v = v(g)$. The set of Satake parameters $\{v(g), v(g)\zeta^{-1}(g), u(g), u(g)\zeta^{-1}(g)\}$ usually does not determine the Weyl group orbit of the tuple $(\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v)$ of Satake parameters. In our special case it does, since $v \notin T$ and $\mu_v = n(\mathrm{Frob}_v)/\nu_v$ implies $\mu_v \neq v(g)\zeta^{-1}(g)$. Notice that otherwise $\zeta^{-1}(g)v(g)^2 = n(\mathrm{Frob}_v) = \zeta^{-1}(g)u(g)^2$ — following from $\nu_v\mu_v = \widetilde{\nu}_v\widetilde{\mu}_v = n(\mathrm{Frob}_v)$ — contradicts $u(g)/v(g) \notin \Omega$. Since the Weyl group contains an element fixing ν_v, μ_v, which permutes the other Satake parameters $\widetilde{\nu}_v$ and $\widetilde{\mu}_v$, we can therefore assume $\widetilde{\nu}_v = v(g)\zeta^{-1}(g)$. This proves the lemma. □

B.2. Lemma. — *Suppose* Π *were a counterexample for theorem I as in lemma B.1. Then* Π *is a nondegenerate D-critical representation of two-abelian type with $D \geqslant 4$ and the Galois group* $\mathrm{Gal}(L:\mathbb{Q}) \cong \pi_0(G)$.

Proof. — For $T = T_\Omega$ the Frobenii classes 'generate' the finite group $\pi_0(N) = \pi_0(G)$. From lemma B.1 we get $(\mu_v/\nu_v)^{2D} \neq 1$ and $(\widetilde{\mu}_v/\widetilde{\nu}_v)^{2D} \neq 1$ for $v \notin T$. Here $D = |\pi_0(G)|$. For $v \notin T$ all Satake parameters are different from each other, unless $\zeta(\mathrm{Frob}_v) = 1$ or equivalently $r(\mathrm{Frob}_v) \in G^0$. Furthermore $\zeta(\mathrm{Frob}_v) \in \Omega$ is characterized as the unique quotient of Satake parameters contained in Ω. The relation between Galois representations and L-series therefore implies, that the representation $s = \rho \otimes t$ defines a balanced representation ρ of the finite subgroup $\widetilde{N}$ of N with only two eigenvalues. The kernel $Z(\rho)$ of the associated projective representation $\overline{\rho}$ of $\widetilde{N}$ (see appendix A) is characterized by $g \in Z(\rho)$ iff $\zeta(g) = 1$ iff $g \in Z(G^0) \cap \widetilde{N} = \mu_n$.

By the classification of balanced representations obtained in lemma A of appendix A, ρ is either a multiple of a 2-dimensional representation or $Q(\rho) = \widetilde{N}/Z(\rho)$ is an elementary two-abelian group of order $D \geqslant 4$. In the first case the representations s is a multiple of a 4-dimensional representation and the main theorem holds for Π, and $\pi_0(G) = Q(\rho)$ must be a finite subgroup of $\mathrm{PGl}(2,k)$. In the second case the representation Π is D-critical. This follows from lemma B.1 since $\widetilde{N}/Z(\rho) = \widetilde{N}/\mu_n = \pi_0(G)$ is elementary two-abelian. Hence $\zeta(\mathrm{Frob}_v) = \pm 1$ for $v \notin T$. Notice, the order of $\zeta(g)$ is the order of the image of $g \in \widetilde{N}$ in the quotient group $Q(\rho)$, which is elementary two-abelian. In particular $\zeta(\mathrm{Frob}_v) = 1$ holds iff $\mathrm{Frob}_v \in G^0$. The corresponding number field L has therefore as its Galois group $\mathrm{Gal}(L/\mathbb{Q})$ the group $\pi_0(G) = G/G^0$. Lemma B.2 is proven. Of course lemma B.2 holds in case 1 whenever $\pi_0(G)$ is a elementary abelian 2-group.

The first case of [**T**]. — In this case we have distinguish the two cases $G = \overline{G}$ and $[G : \overline{G}] = 2$ also called cases 1a and 1b. Now the connected component is a torus $G^0 = \mathbb{G}_m^r$ of rank $r = 1, 2$ over k, and is contained in N. Hence $N = \overline{G}$, and $\overline{G}$ is obtained from the finite subgroup $\widetilde{N}$ and the torus G^0 by the pushout

$$\overline{G} = (\widetilde{N} \times G^0)/\mu_n^r.$$

G^0-eigenspaces. — The restriction $\overline{s}$ of the representation s of G to $\overline{G}$ decomposes

$$\overline{s} = \rho \oplus \rho'.$$

Here ρ and ρ' are representations of $\overline{G}$, whose restriction to G^0 are characters χ resp. χ' of the torus G^0, such that $\chi\chi' = n$. This is clear for the subtorus G^0 and this extends to $\overline{G}$, since $\overline{G} = N$ is the centralizer of this subtorus (see Taylor's list and [**T**], p. 302). So ρ is the representation of $\overline{G}$ on the χ-eigenspace of G^0 and ρ' is the representation of $\overline{G}$ on the χ'-eigenspace of G^0. The root condition $\lambda^2 \neq b$ (property (c) in section 2) implies $\chi^j \neq (\chi')^j$ for all powers $j \geqslant 1$, since otherwise $\chi^{2j} = n^j$ in contradicts this root property of the Galois representation of Π. In particular the character χ'/χ is not of finite order.

In the present case the finite subgroup $\widetilde{N}$ of $N = \overline{G}$ is a normal subgroup with quotient $N/\widetilde{N} \cong G^0/G^0[n]$. The characters χ^n, χ'^n of the torus G^0 define a faithful embedding of $G^0 \cong G^0/G^0[n]$ into $\mathbb{G}_m^2$. The two projection maps define composite maps $\overline{G} = N \to N/\widetilde{N} \to \mathbb{G}_m$, which give the two characters χ^n, χ'^n of $\overline{G}$. The two characters χ^n, χ'^n of the torus G^0 therefore extend to characters of $N = \overline{G}$ and are called

$$\Phi, \Psi : \overline{G} \longrightarrow k^*.$$

In particular, $\tau(g) = (\chi'(g)/\chi(g))^n$ extends to a character of $\overline{G} = N$, which is not of finite order.

Claim. — *The representations ρ and ρ' of $\widetilde{N}$ are balanced with at most two eigenvalues.*

Proof. — With the notations above put $n = |\pi_0(G)|$ and let $\Omega \subset X$ be the subset represented by roots of unity of order $2n$. The set of places for which either $\tau(r(\mathrm{Frob}_v))$ or $n(r(\mathrm{Frob}_v))$ is in Ω has Dirichlet density zero. Let T be a suitable set of Dirichlet density zero containing these places.

Elements in $\pi_0(N)$ have a lift g in $\overline{G}$, so that $g = r(\mathrm{Frob}_v)$ for some $v \notin T$. For $g = r(\mathrm{Frob}_v)$ let $\zeta_i = \zeta_i(g)$ denote the eigenvalues for the representation ρ, and similarly let ζ_j' denote the eigenvalues for ρ'. The order of $\pi_0(N)$ is n, hence $\zeta_1^{2n} = \zeta_2^{2n} = \cdots = \lambda$ and $\zeta_1'^{2n} = \zeta_2'^{2n} = \cdots = \lambda'$ for some $\lambda \neq \lambda' \in k^*$. Notice, that g^{2n} is contained in the torus G^0 so that $\chi(g^{2n})$ and $\chi'(g^{2n})$ are defined. Now $\tau(g) \notin \Omega$ or $\tau(g^{2n}) \neq 1$, hence $\chi'(g^{2n})^n \neq \chi'(g^{2n})^n$ and therefore also $\lambda^n \neq \lambda'^n$ if g is suitably chosen.

The set of eigenvalues $\{\zeta_1, \ldots, \zeta_1', \ldots\}$ has cardinality at most 4 and in fact is the set of Satake parameters $\{\zeta_1, \ldots, \zeta_1', \ldots\} = \{\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v\}$ with $\nu_v\mu_v = \widetilde{\nu}_v\widetilde{\mu}_v = n(\mathrm{Frob}_v)$. This follows from section 2 property (e), which relates the Galois representation and the L-series of Π. Without restriction of generality we can assume $\nu_v = \zeta_1$. We claim, that among the Satake parameters there exists at most two different eigenvalues ζ_i' (with prime index). Otherwise the Satake parameters would be $\zeta_1, \zeta_1', \zeta_2', \zeta_3'$, and the Satake parameter relations $\zeta_1\zeta_2' = \zeta_1'\zeta_3'$ raised to the $2n$-th power would contradict the root condition $\lambda \neq \lambda'$. But this implies that ρ' has at most two eigenvalues. The same holds now for ρ, by reversing the roles of ρ and ρ'.

Next we claim that the Satake parameters must be of the form $(\zeta_i, \zeta_j, \zeta_a', \zeta_b')$ with $\zeta_i\zeta_a' = \zeta_j\zeta_b'$, since otherwise there are relations of the form $\zeta_i\zeta_j = \zeta_k'\zeta_l'$. Raising to the n-th resp. $2n$-th power would contradict the root condition $\lambda \neq \lambda'$; notice $(\zeta_i\zeta_j)^n = \lambda$. This being said, put $\zeta(g) = \zeta_1(g)/\zeta_2(g)$ or $\zeta(g) = 1$ if there is only one eigenvalue; this number is an n-th root of unity uniquely defined up to inverse and the Satake parameters of Π_v are

$$(\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v) = (\nu_v, \nu_v\zeta(\mathrm{Frob}_v)^{-1}, \mu_v, \mu_v\zeta(\mathrm{Frob}_v)), \quad \nu_v\mu_v = \omega_\Pi(p_v).$$

Here of course we used the freedom to normalize the Satake parameter subject to a reparameterization under the Weyl group. Furthermore $\zeta(g)$ defined for the eigenvalues of the representation ρ, and $\zeta'(g)$ defined for the eigenvalues of the representation ρ' coincide

$$\zeta(\mathrm{Frob}_v) = \zeta'(\mathrm{Frob}_v)$$

up to inverse. Hence section 2, property (e) implies that the representations ρ and ρ' are balanced. Either $\zeta(g) = \zeta'(g) = 1$ holds or all four Satake parameters are pairwise different and each of them occurs with multiplicity m. This proves the last claim. □

Remark. — For $g \in \overline{G}$ the values $\zeta(g), \zeta'(g)$ (up to inverse) depend only on the image of g in the finite quotient group $\overline{G}/G^0 = \pi_0(N)$. By a density argument the above information for Frobenius elements therefore implies

$$\zeta(g) = \zeta'(g) \quad \text{for all} \quad g \in \widetilde{N}.$$

Let $\overline{\rho}, \overline{\rho}'$ be the projective representations of $\widetilde{N}$ associated to ρ, ρ' with the kernels $Z(\rho)$ and $Z(\rho')$. Put $Q(\rho) = \widetilde{N}/Z(\rho)$ and similar for ρ'. Then $\zeta(g) = 1$ is equivalent with $g \in Z(\rho)$ or $g \in Z(\rho')$. We have an injective homomorphism $\widetilde{N}/Z(\rho) \to Q(\rho) \times Q(\rho')$ and the composition with each projection is injective and surjective. Therefore $Q(\rho) \cong Q(\rho')$. Let $\pi_L(G)$ be the image of $Z(\rho)$ in $\pi_0(N) = \pi_0(\overline{G})$. In fact this group is a normal subgroup of $\pi_0(G)$. If the torus G^0 is two-dimensional, then $\pi_L(G)$ is trivial. If G^0 is the one dimensional torus, then $\pi_L(G)$ is a finite (hence cyclic) subgroup of this k-torus. We get an exact sequence

$$0 \longrightarrow \pi_L(G) \longrightarrow \pi_0(G) \longrightarrow \Delta \longrightarrow 0.$$

Let L be the field attached to the kernel of the surjective group homomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \Delta.$$

If ρ and ρ' are both m-fold multiples of a two dimensional representation, then s and r is an m-fold multiple of a four dimensional induced representation. This follows from $[G : \overline{G}] \leqslant 2$. Hence theorem I holds for Π, so that Δ is a finite subgroup of $\mathrm{PGl}(2, k)$. If ρ and ρ' are not m-fold multiples of two dimensional representations, the main result on balanced representations implies, that the isomorphic groups $Q(\rho) \cong Q(\rho')$ are elementary two-abelian of order $D(\rho) \geqslant 4$. In particular $\zeta(g) \in \{\pm 1\}$ and we claim that the representation Π is a D-critical automorphic representation with $D = [L : \mathbb{Q}]$. Obviously $D \geqslant D(\rho) \geqslant 4$. From the assertions already made this claim follows in the case 1a, when $G = \overline{G}$. For $\overline{G} \neq G$ this follows from the next

B.3. Lemma. — *Suppose* Π *were a counterexample to theorem I and suppose* Π *belongs to case one of Taylor's list. Then* Π *is a* D*-critical automorphic representation whose underlying Galois group* $\mathrm{Gal}(L/\mathbb{Q})$

$$\pi_0(G) \twoheadrightarrow \Delta = \mathrm{Gal}(L/\mathbb{Q})$$

is a 2*-group of order* $D \geqslant 4$*, which is isomorphic to* $\pi_0(G)$ *divided by a normal cyclic subgroup. For* $G = \overline{G}$ *in case 1a the group* $\mathrm{Gal}(L/\mathbb{Q})$ *is elementary two-abelian and* Π *is nondegenerate. In case 1b the group* $\mathrm{Gal}(L/\mathbb{Q})$ *is a metabelian group isomorphic to* $\pi_0(G)$ *with an elementary two-abelian normal subgroup isomorphic to* $\pi_0(\overline{G})$ *of order* $\overline{D} \geqslant 4$ *and quotient group* $\pi_0(G)/\pi_0(\overline{G}) = \mathbb{Z}/2\mathbb{Z}$.

Example. — The case where $\pi_0(G)$ has 8 elements, hence $\pi_0(\overline{G}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, turns out to be most relevant (lemma 4.5). Hence we further specify the structure of $\pi_0(G)$ in this case. Since the group is of order 8, it is either abelian of type $(\mathbb{Z}/2\mathbb{Z})^3$ or $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or isomorphic to the dihedral group of order eight generated by N, S with $S^2 = 1, N^4 = 1, N^S = N^{-1}$. (since the quaternion group does not contain a normal subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, Huppert th. 14.10). The case $(\mathbb{Z}/2\mathbb{Z})^3$ arises iff $g^2 = 1$ for all $g \in \pi_0(G)$. Furthermore: The dihedral group D_8 of order eight has the following normal subgroups: The center $\langle N^2 \rangle$ of order two, the subgroup $R = \langle N \rangle \cong \mathbb{Z}/4\mathbb{Z}$, and the two subgroups $P = \{1, N^2, SN, SN^3\}$ and

$Q = \{1, N^2, S, SN^2\}$, which are both isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Using the isomorphism induced by $S \mapsto SN$, which changes the presentation, we can assume without restriction of generality that the subgroup $Q \subset D_8$ defined above can be identified with the given group $Q = \pi_0(\overline{G}) \subset \pi_0(G)$. There are four additional subgroups of order two generated by S, SN^2, SN and SN^3, which are not normal in Δ. The conjugacy classes are $\{1\}, \{N^2\}, \{N, N^3\}, \{S, SN^2\}$ and $\{SN, SN^3\}$. The commutator group $[D_8, D_8]$ is cyclic of order two and generated by N^2. Hence D_8 has four different abelian characters $1, \chi_P, \chi_Q$ and $\chi_R = \chi_P\chi_Q$, where χ_P and χ_Q have kernels P and Q.

Proof of lemma B.3. — The lemma has already been shown in case 1a. Therefore assume $G \neq \overline{G}$. Then $G = \langle \overline{G}, \sigma\rangle$ with $\sigma^2 \in \overline{G}$ where σ acts nontrivially on G^0, which is a k-torus of dimension one or two. In fact the dimension of $T = G^0$ has to be two, since otherwise σ acts by the inversion on T. Since σ also interchanges the two characters χ and χ' this would imply $n = \chi\chi' = 1$ contrary to the assumption (a) of section 2: $n|G^0 \neq 1$.

Since then the dimension of the torus G^0 is two and since $\chi \oplus \chi'$ is faithful on G^0, the morphism $(\chi, \chi') : \mathbb{G}_m^2 \cong G^0 \to \mathbb{G}_m^2$ is an isomorphism. So we can identify $G^0 = \mathbb{G}_m^2$ in such a way, that $\chi(t_1, t_2) = t_1, \chi'(t_1, t_2) = t_2$ for $(t_1, t_2) \in (k^*)^2$ are the two projection maps. Since $\chi' \neq \chi$ ($\dim(G^0) = 2$) and $\chi \neq \chi^{-1}$ (the assumption $n \neq 1$), a similar argument implies that σ interchanges the two characters. Therefore $\sigma(t_1, t_2)\sigma^{-1} = (t_2, t_1)$.

Recall that the restriction $\overline{s} \cong \rho \oplus \rho'$ of s to $\overline{G}$ decomposes. The restrictions of ρ resp. ρ' to G^0 are the characters χ resp. χ' of G^0. Put $\overline{s}^\sigma(g) = \overline{s}(\sigma g \sigma^{-1})$, then $\overline{s}^\sigma \cong \overline{s}$ by assumption. This implies $\rho^\sigma \cong \rho', \rho'^\sigma \cong \rho$, since $\chi^\sigma = \chi', \chi'^\sigma = \chi$. But σ acts nontrivially on G^0. Therefore $\overline{s} \cong \rho \oplus \rho^\sigma$. The representation ρ of $\overline{G}$ on V_ρ was shown to be balanced. Therefore and without restriction of generality we may assume for the proof of lemma B.3 from now on

Assumption. — ρ not to be an isotypic multiple of a 2-dimensional representation. The group $Q(\rho)$ attached to the balanced representation ρ of the group $\widetilde{N}_\rho$ is elementary two-abelian of order $D = D(\rho) \geqslant 4$.

Consider the Zariski closure $\overline{G}_\rho$ of $\overline{G}$ in $\mathrm{Gl}(V_\rho)$. Then $(\overline{G}_\rho)^0 \cong \mathbb{G}_m$ and $\overline{G}_\rho \cong (\mathbb{G}_m \times \widetilde{N}_\rho)/\mu_n$ and the group $Q(\rho)$ is isomorphic to $\pi_0(\overline{G}_\rho)$. Namely $\overline{\rho}(g) = 1$ implies $\rho(g) \in k^* \cdot \mathrm{id}_{V_\rho}$, hence $g \in (\mathbb{G}_m \cap \widetilde{N}_\rho)(k) = \mu_n$. Now n can be chosen to be $n = 2$, since $Q(\rho)$ is elementary two-abelian. Thus

$$\overline{G}_\rho = (\mathbb{G}_m \times \widetilde{N}_\rho)/\mu_2$$

$$0 \longrightarrow \mu_2 \longrightarrow \widetilde{N}_\rho \longrightarrow Q(\rho) \longrightarrow 0.$$

The central extension $\widetilde{N}_\rho$ defines a cohomology class in $H^2(Q, \mu_2)$ for the group $Q = Q(\rho)$. Let $a(g_1, g_2)$ be a representing 2-cocycle (for the inhomogeneous bar resolution), normalized such that $a(0,0)=1$. Then $q(g)=a(g,g)\in\mu_2$ is a function $q:Q\to\mu_2$ independent of the choice of cocycle. $q(g)=1$ holds iff $\widetilde{g}^2=1$ for a (any) lift $\widetilde{g}\in\widetilde{N}_\rho$ of $g\in Q$. Notice $\widetilde{g}_1\widetilde{g}_2=a(g_1,g_2)\widetilde{(g_1g_2)}$. Furthermore $q(g_1g_2)=q(g_1)q(g_2)[g_1,g_2]$ defines a quadratic form q on Q with associated bimultiplicative form $[g_1,g_2]=\widetilde{g}_1^{-1}\widetilde{g}_2\widetilde{g}_1\widetilde{g}_2^{-1}$. It is bimultiplicative since it has values in the central subgroup μ_n of $\widetilde{N}$. We have a similar situation for $\rho'=\rho^\sigma$. We identify $Q(\rho)$ and $Q'=Q(\rho')$ via σ. The corresponding function q' on Q attached to the extension class of $\widetilde{N}_{\rho'}$ in $H^2(Q,\mu_2)$ is $q'(g)=q(\sigma g\sigma^{-1})$.

Recall $Q(\rho)\cong Q(\rho')\cong\pi_0(\overline{G})$. Using the cocycles defining $\widetilde{N}_\rho$ and $\widetilde{N}_{\rho'}$ we find a finite σ-stable subgroup $\widetilde{N}$, such that

$$\overline{G}\cong(\mathbb{G}_m^2\times\widetilde{N})/\mu_2^2$$
$$0\longrightarrow\mu_2\times\mu_2\longrightarrow\widetilde{N}\longrightarrow Q\longrightarrow 0.$$

The group $Q=Q(\rho)$ was shown to elementary two-abelian and $Q(\rho')$ is identified with $Q(\rho)$ by the action of σ. We also view q,q' as functions on $\overline{G}$, which factorize over the quotient $\pi_0(\overline{G})$.

The previous decomposition of $\overline{G}$ can be extended to G. We find a subgroup H, such that

$$G\cong(\mathbb{G}_m^2\times H)/\mu_2^2$$
$$0\longrightarrow\widetilde{N}\longrightarrow H\longrightarrow\mathbb{Z}/2\mathbb{Z}\longrightarrow 0.$$

For this find a representative $\sigma\in G\setminus\overline{G}$, such that $\sigma^2\in\widetilde{N}$ and put $H=\langle\widetilde{N},\sigma\rangle$. For any choice of σ we have $\sigma^2=t\cdot g$ with $t\in G^0,g\in\widetilde{N}$. By definition $\sigma tg\sigma^{-1}=tg$, hence $t^\sigma/t=g(g^\sigma)^{-1}\in G^0\cap\widetilde{N}=\mu_2^2$. If $t=(a,b)\in(k^*)^2$, then $t^\sigma/t=(b/a,a/b)$, hence $b=\pm a$. So we can replace σ by $\widetilde{\sigma}=\sigma(a^{-1},1)\in\sigma\cdot G^0$ in such a way, that $\widetilde{\sigma}^2=(1,1)$ or $(1,-1)$ is contained in $\widetilde{N}$. μ_2^2 is a normal subgroup of H and $H/\mu_2^2\cong\pi_0(G)$. The proof of lemma B.3 is now completed by the following two lemmas. □

B.4. Lemma. — *Assume $\sigma\notin\overline{G}$. Then $s(\sigma)$ has either two different eigenvalues $t,-t$ or four different eigenvalues $t,-t,it,-it$ depending on whether $\sigma^2\in G^0$ holds or not. Each eigenvalue occurs with equal multiplicity.*

Proof. — For any choice of $\sigma\in G$, $\sigma\notin\overline{G}$ we have

(1) $(\sigma\cdot(t_1,t_2))^2=\sigma^2\cdot(t,t)$ for $t=t_1t_2\in k^*$ and the element $(t,t)\in G^0$ acts on W by the scalar $t\cdot\mathrm{id}_W$

(2) σ^4 always acts on W by a scalar. Namely $\sigma^4\in(G^0)^\sigma=\{(t,t)\mid t\in k^*\}$. Hence the eigenvalues of $s(\sigma)$ are of the form $\pm t,\pm it$ for some $t\in k^*$.

On the other hand $\sigma^2\in G^0$ implies $\sigma\in(G^0)^\sigma$. Therefore $s(\sigma^2)$ acts by a scalar on W iff $\sigma^2\in G^0$. Let $a,b,c,d\in\mathbb{N}$ denote the multiplicities of the four possible eigenvalues of $s(\sigma)$. $s(\sigma)$ permutes V_ρ and $V_{\rho'}$. Therefore $Tr_W(s(\sigma))=a-b+i\cdot c-i\cdot d=0$. This implies $a=b$ and $c=d$. Hence $s(\sigma^2)$ acts by a scalar on W, either if $a = b = 0$ or $c = d = 0$; and this is the case where $\sigma^2 \in G^0$. Hence $s(\sigma)$ has two different eigenvalues $t, -t$ with equal multiplicity.

In the remaining case $\sigma^2 \notin G^0$ both $a = b, c = d$ are nonnegative. So there are four different eigenvalues of type $\pm t, \pm it$. In this case $s(\sigma^2)$ has the eigenvalues t^2 and $-t^2$ with the multiplicities $2a$ and $2c$. It remains to show $a = c$. The image of $\sigma^2 \in \overline{G}$ in $\pi_0(\overline{G}) = \overline{G}/G^0$ is nonzero. In terms of the function $\zeta(g) = \zeta'(g), g \in \overline{G}$ introduced earlier, this means $\zeta(\sigma^2) \neq 1$. Therefore σ^2 has two different eigenvalues on both subrepresentations V_ρ and $V_{\rho'}$. The representations ρ, ρ' are balanced representations of $\overline{G}$. Hence each of the two different eigenvalues of σ^2 on V_ρ has equal multiplicities $\dim(V_\rho)/2$ and similar for ρ'. We already know, that there are only two eigenvalues $t^2, -t^2$ of σ^2. Hence $a = c$ follows from $\dim(V_\rho) = \dim(V_{\rho'})$. Lemma B.4 is proved. □

The Satake parameters for $\mathrm{Frob}_v \notin \overline{G}$. — It remains to determine the Satake parameters of Π_v in the case 1b to complete the proof lemma B.3.

B.5. Lemma. — *Suppose the assumptions are those of lemma B.2 and suppose we are in case 1b. Then* Π *is D-critical, where* $\mathrm{Gal}(L/\mathbb{Q}) = \pi_0(G)$ *is a metabelian extension of a* $\mathbb{Z}/2\mathbb{Z}$ *by an elementary two-abelian group* $Q(\rho) = \pi_0(\overline{G})$ *of order* $\geqslant 4$. *There exists a set* T *of Dirichlet density 0, such that for* $v \notin T$ *the local representation* Π_v *has Satake parameters*

$$(\nu_v, \zeta_v^{-1}\nu_v, \mu_v, \zeta_v\mu_v), \quad \nu_v\mu_v = n(\mathrm{Frob}_v),$$

with $\zeta_v = \zeta(\mathrm{Frob}_v) = \pm 1$ *and* $\zeta_v = 1$ *iff* $\mathrm{Frob}_v \in G^0$.

Proof. — *First the cases C,D and E*, where $\mathrm{Frob}_v \notin \overline{G}$.

There is the *case C*, where $\mathrm{Frob}_v^2 \notin G^0$. By lemma B.4 and property (e) of Π

$$\Pi_v \sim (t_v, -t_v, it_v, -it_v) \sim (t_v, \pm it_v, \mp it_v, -t_v)$$

up to replacement of i by $-i$. Thus $\nu_v/\mu_v = \pm i$. The case $(t_v, \pm it_v, -t_v, \mp it_v)$ is impossible, since it contradicts $\nu_v\mu_v = \widetilde{\nu}_v\widetilde{\mu}_v$. Furthermore $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, -1, -1/i, -i, -1)$ and therefore $\log\zeta_v(\Pi_v, 1, s) = -p_v^{-s} + O(p_v^{-2s})$.

In the next *case D* $\mathrm{Frob}_v^2 \in G^0$ and by lemma B.4 and property (e) of Π

$$\Pi_v \sim (t_v, -t_v, -t_v, t_v) \sim (t_v, t_v, -t_v, -t_v) \sim (-t_v, -t_v, t_v, t_v).$$

Hence $\nu_v/\mu_v = -1$ and $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, 1, -1, -1, 1)$. Thus $\log\zeta_v(\Pi_v, 1, s) = p_v^{-s} + O(p_v^{-2s})$.

Now the *case E*, where $\mathrm{Frob}_v^2 \in G^0$ and

$$\Pi_v \sim (t_v, -t_v, t_v, -t_v).$$

Then $\nu_v/\mu_v = 1$ and $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, -1, -1, -1, -1)$, hence $\log\zeta_v(\Pi_v, 1, s) = -3p_v^{-s} + O(p_v^{-2s})$.

Finally the cases *case A and B*, where $\mathrm{Frob}_v \in \overline{G}$ and either $\mathrm{Frob}_v \in G^0$ or $\mathrm{Frob}_v \notin G^0$. Then according to the discussion of case 1a, the representation Π_v has Satake parameters

$$(\nu_v, \nu_v\zeta_v^{-1}, \mu_v, \mu_v\zeta_v),$$

where $\zeta_v = \zeta(\mathrm{Frob}_v) \in \{\pm 1\}$ and $= 1$ iff $\mathrm{Frob}_v \in G^0$. This proves lemma B.5. □

The above arguments show, that Π is D-critical, but not nondegenerate degenerate, in case 1b. A substitute for the previous arguments will be necessary and finally is provided by a further analysis of the cases A and B where $\mathrm{Frob}_v \in \overline{G}$. In fact we use $\overline{G} = (\mathbb{G}_m^2 \times \widetilde{N})/\mu_2^2$ to write $r(\mathrm{Frob}_v)$ in the form $(t_v, t'_v) \times g_v$ mod μ_2^2 with $(t_v, t'_v) \in (k^*)^2$ and $g_v \in \widetilde{N}$. Let $\alpha_v, \zeta_v\alpha_v$ resp. $\alpha'_v, \zeta_v\alpha'_v$ be the eigenvalues of $\rho(g_v)$ resp. $\rho'(g_v)$ under the balanced representations ρ and ρ' of $\overline{G}$. Since $\widetilde{N}$ is a finite group, and every element has order dividing 4, these eigenvalues are contained in $\pm 1, \pm i$. Obviously $\alpha_v^2 = q(\mathrm{Frob}_v)$ and ${\alpha'_v}^2 = q'(\mathrm{Frob}_v)$, where

$$q, q' : \pi_0(\overline{G}) \longrightarrow \mu_2$$

are the quadratic forms on $\pi_0(\overline{G})$ extended to $\overline{G}$, that were defined earlier related to the extension $0 \to \mu_2^2 \to \widetilde{N} \to \pi_0(\overline{G}) \to 0$. In particular

$$\rho(\mathrm{Frob}_v) \sim t_v \begin{pmatrix} \alpha_v E & * \\ 0 & \alpha_v\zeta_v E \end{pmatrix}$$
$$\rho'(\mathrm{Frob}_v) \sim t'_v \begin{pmatrix} \alpha'_v E & * \\ 0 & \alpha'_v\zeta_v E \end{pmatrix}.$$

Without restriction of generality, by changing α'_v to $\alpha'_v\zeta_v$, the Satake parameters of Π_v therefore are

$$(t_v\alpha_v, t_v\alpha_v\zeta_v, t'_v\alpha'_v, t'_v\alpha'_v\zeta_v)$$

with $n(\mathrm{Frob}_v) = \nu_v\mu_v = (t_v\alpha_v)(t'_v\alpha'_v)$. Hence

$$t_v\alpha_v/t'_v\alpha'_v = (t_v\alpha_v)^2/n(\mathrm{Frob}_v) = t_v^2 q(\mathrm{Frob}_v)/n(\mathrm{Frob}_v) = q\frac{\Psi}{n}(\mathrm{Frob}_v)$$
$$t_v\alpha_v/t'_v\alpha'_v = n(\mathrm{Frob}_v)(t'_v\alpha'_v)^{-2} = {t'_v}^{-2}q'(\mathrm{Frob}_v)n(\mathrm{Frob}_v) = q'\frac{n}{\Phi}(\mathrm{Frob}_v).$$

Hence $q'q = q'/q$ is the quadratic character $(q'/q)(\mathrm{Frob}_v) = (t_v)^2(t'_v)^2/n^2(\mathrm{Frob}_v)$

$$qq' = \Psi\Phi/n^2$$

of $\overline{G}$, where

$$\Psi, \Phi : \overline{G} \longrightarrow k^*$$

are the characters $(t, t') \times g$ mod $\mu_2^2 \mapsto t^2$ resp. $(t')^2$ of $\overline{G}$ defined earlier. Recall that $\widetilde{N}$ is a central extension of μ_2 in our present situation. Also define $\Psi' : \overline{G} \to k^*$ by $\Psi' = n^2/\Psi$. Then

$$\Psi' = q'q\Phi.$$

The quadratic character $q'q$ is trivial on G^0 and may be viewed as a character of $\pi_0(\overline{G})$. Let L be the fixed field of the inverse image of G^0 in $\mathrm{Gal}(\overline{\mathbb{Q}}:\mathbb{Q})$ and let $\overline{L}$ be the fixed field of the inverse image of $\overline{G}$ in $\mathrm{Gal}(\overline{\mathbb{Q}}:\mathbb{Q})$.

Asymptotics. — We now determine the *weights* w_v in the asymptotic formula

$$\log\zeta_v(\Pi_v,\chi_v,s) = w_v\cdot\chi_v(p_v)p_v^{-s} + O(p_v^{-2s}),$$

where $\zeta_v(\Pi_v,\chi_v,s)$ denotes the local factor of the degree 5 standard L-series of the automorphic representation Π at a nonarchimedian unramified place $v\notin T_\Omega$ and $w_v = 1+\varepsilon_v(\mathrm{Ad}_v+1)$, $\mathrm{Ad}_v = \frac{\mu_v}{\nu_v}+\frac{\nu_v}{\mu_v}+1$, where $(\nu_v,\varepsilon_v\nu_v,\mu_v,\varepsilon_v\mu_v)$ with $\varepsilon\in\{\pm 1\}$ are the Satake parameters of Π_v.

The weights w_v. — For the five possible cases A,B,C,D,E we get

A,B: Here $\mathrm{Frob}_v\in\overline{G}$ and $w_v=(1+2\varepsilon_v)+\varepsilon_v\cdot q(\mathrm{Frob}_v)\cdot(\frac{\Psi}{n}(\mathrm{Frob}_v)+\frac{\Psi'}{n}(\mathrm{Frob}_v))$.

C,D,E: Here $\mathrm{Frob}_v\notin\overline{G}$ with $\nu_v/\mu_v=\pm i,\ -1,\ 1$ respectively. The weights are $-1,1,-3$ respectively.

That the weights are $w_v=-1,\ 1,\ -3$ in the cases C,D,E follows from the proof of lemma 7. In *case A* resp. *case B* (v splits in L or not) we find

$$\mathrm{Ad}_v = 1+\frac{t_v\alpha_v}{t'_v\alpha'_v}+\frac{t'_v\alpha'_v}{t_v\alpha_v},$$

or with the notations above

$$\mathrm{Ad}_v = 1+q(\mathrm{Frob}_v)\cdot\Big(\frac{\Psi}{n}(\mathrm{Frob}_v)+\frac{\Psi'}{n}(\mathrm{Frob}_v)\Big).$$

In the next lemma Ψ and Ψ' are shown to be nontrivial Grossencharacters of the quadratic extension field $\overline{L}$ of $\mathbb{Q}$. For this reason — 'in a statistical sense' — the weight factors in the cases A and B will behave for the further analysis of $\zeta^S(\Pi,\chi,s)$ in appendix C exactly as if they were the constant weight factors $w_v=3,-1$ for any additional character χ of finite order, since Ψ/n and Ψ'/n themselves are characters which are not of finite order and since both q and ε - viewed in their dependence on Frob_v - are constant functions on the cosets $\mathrm{Gal}(\overline{\mathbb{Q}}:\overline{L})/\mathrm{Gal}(\overline{\mathbb{Q}}:L)\cong\pi_0(\overline{G})$. Namely

B.6. Lemma. — *Ψ/n and Ψ'/n are inverse algebraic Grossencharacters of the quadratic extension field $\overline{L}$ of $\mathbb{Q}$. Both Ψ/n and Ψ'/n are characters, which are not of finite order.*

Proof. — Let E be some number field. Then a λ-adic representation ρ is called E-rational, if the representation is unramified at almost all places so that $trace\rho(\mathrm{Frob}_v)^i\in E$ holds for all $i\in\mathbb{N}$. The following properties are obvious: If ρ is E-rational, then its restriction to a subgroup of finite index is again E-rational. If ρ and ρ' are E-rational so is $\rho\otimes\rho'$. If $\rho=\chi\otimes\rho_0$ is E-rational and if χ is a character and ρ_0 a representation with finite image of order N, then χ is $E(\zeta_N)$-rational if ρ is E-rational and vice versa.

The representation r of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on W (see section 2) arises from a semisimple λ-adic representation, which is E-rational. For a number field $E = E_\Pi$ as in [**T**], p. 297 the characteristic polynomial $P_v(X)^m$ of $r(\mathrm{Frob}_v)$ is a polynomial in $E[X]$. This follows from $L_v(\Pi_v, s) = P_v(p^{-s})^{-1}$ (property (e)).

The restriction $\rho \oplus \rho'$ of s to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{L})$ decomposes and is E-rational. $\rho \otimes \rho'$ is of the form $n \otimes \rho_0$. Since $\overline{G} \cong (\mathbb{G}_m \times \widetilde{N})/\mu_2^2$, the representation ρ_0 has finite image and n is the character defined by Π (see section 2, property (a) or (e)). By our assumptions on Π the character n is E-rational. Hence, if E is suitably enlarged by a root of unity, $\rho \otimes \rho'$ is E-rational.

The restriction $\rho \oplus \rho'$ of s to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{L})$ decomposes and is E-rational, as well as the tensor square of this restriction. It contains two copies of the representation $\rho \otimes \rho'$ as direct summands. Hence $(\rho \otimes \rho) \oplus (\rho' \otimes \rho')$ defines a λ-adic semisimple abelian E-rational representation of the absolute Galois group of $\overline{L}$. By [**He**], p. 113 this representation must be locally algebraic. Since $\rho \otimes \rho = \Psi \otimes (\oplus_i \chi_i)$ for finitely many quadratic characters of the elementary two-abelian finite group $\widetilde{N}/\mu_2$, Ψ is locally algebraic and then also Ψ/n. This proves the first part of the claim.

Assumption (a) together with assumption (c) of section 2 for Π show, when applied for the characters χ and χ' of the torus $G^0 \subset G$, that the Grossencharacters Ψ/n and Ψ'/n are of infinite order. This proves lemma B.6. □

Appendix C

Poles at $s = 1$ in the CM case

Suppose Π is D-critical of CM type. By appendix B and lemma 4.5 and 4.6 the corresponding field extension $L/\mathbb{Q}$ is a Galois extension of degree $D = 8$ with Galois group either D_8 or $(\mathbb{Z}/2\mathbb{Z})^3$ with a distinguished subfield $\overline{L}$, and Π is a theta lift associated to some $\pi = \pi_K$ for $\mathrm{Gl}(2, \mathbb{A}_K)$, where $K/\mathbb{Q}$ is an algebra of degree 2 over $\mathbb{Q}$. The main result of this appendix is the

Proposition. — *Suppose* Π *is D-critical of CM type. Then any choice of K, for which* Π *is a theta lift from* $\mathrm{Gl}(2, \mathbb{A}_K)$, *is a field and this field is contained in L and is different from $\overline{L}$. These fields K_i correspond to the unitary characters $\chi = \chi_{K_i}$, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$. For any such K has three quadratic extension fields $\overline{L}K, F, F'$ in L. The automorphic representation $\pi = \pi_K$ on* $\mathrm{Gl}(2, \mathbb{A}_K)$ *associated to* Π *satisfies*

(1) $\pi \cong \pi \otimes \chi_{K\overline{L}/K} \not\cong \sigma(\pi)$

(2) $\sigma(\pi) \cong \pi \otimes \chi$ *for* $\chi = \chi_{F/K}$ *and* $\chi = \chi_{F'/K}$.

Proof. — This is a summary of lemma C.3-C.5 and C.3'-C.5'. □

Let Δ be the Galois group of $L/\mathbb{Q}$, a group of order D. Let $X_1, \ldots$ be the conjugacy classes of this group. Fix an abelian character χ of Δ. Then $\chi_i = \chi(g)$ for $g \in X_i$ is well defined.

C.1. Lemma. — *There are real numbers $w(X_i)$, such that for $s \to 1^+$*

$$\log \zeta^S(\Pi, \chi, s) \sim \Big(\frac{1}{D}\sum_i \#(X_i)\cdot w(X_i)\cdot \chi_i\Big)\cdot \log\zeta(s).$$

Proof. — Simply put

$$w(X_i) = \lim_{s\to 1^+} D\#(X_i)^{-1} \sum_{v\in T, \mathrm{Frob}_v \in X_i} (1+\varepsilon_v(\mathrm{Ad}_v + 1))p_v^{-s}/\log\zeta(s).$$

Let us show, that these limits exist. It was already shown in appendix B that "statistical" weights exist for the images of Frobenius elements $g = \mathrm{Frob}_v \in \Delta$ (depending on whether Frob_v belongs to the cases A,B,C,D,E and they were $3, -1, -1, 1, -3$ respectively). If all $\mathrm{Frob}_v, v \in T$ for $g = \mathrm{Frob}_v \in X_i$ belong to one and the same case A-E, this implies existence of weights $w(X_i)$ attached to the conjugacy class X_i in the sense above. Whether Frob_v belongs to case A-C can be completely characterized in terms of group theoretical properties of the image of Frob_v in Δ. Hence only the cases D and E might cause trouble.

For abelian Δ the existence of $w(X_i)$ in this case is an immediate consequence of the Tchebotarev density theorem, although there are several classes which might be a mixture of case D and E. Simply vary $\chi \in \widehat{\Delta}$. Since the conjugacy classes of an abelian group are separated by abelian characters we get $w(X_i) = \frac{1}{D}\sum_\chi \overline{\chi}(X_i)\cdot n(\chi)$, where

$$\log\zeta(\Pi,\chi,s) \sim n(\chi)\cdot\log\zeta(s), \quad n(\chi)\in\mathbb{Z},$$

and $n(\chi)$ is the order of the meromorphic functions $\zeta(\Pi,\chi,s)$ at $s = 1$. In the dihedral case $\Delta = D_8$, although the group is nonabelian, fortunately there is only one conjugacy class, which contains Frobenius elements with mixed cases D or E. Using the notations of the remark after lemma B.3 in appendix B, this conjugacy class is the class of the two elements SN and SN^2. All other classes have a well defined weight. Therefore the existence of the weight $w = w(\{SN, SN^3\})$ for the remaining class follows as in the abelian case. □

For the conjugacy classes of type A,B,C the weights are $w(X_i) = 3, -1, -1$ respectively. For a class which is a mixture of case E and D, we only know that the weight $w = w(X_i)$ satisfies $-3 \leqslant w \leqslant 1$.

The CM case $\Delta = D_8$. — For notations see the example in appendix B.

Classes X_i	1	$\{N^2\}$	$\{S, SN^2\}$	$\{N, N^3\}$	$\{SN, SN^3\}$
Cases	A	B	B	C	$D+E$
Weights $w(X_i)$	3	-1	-1	-1	w
Cardinality $\#(X_i)$	1	1	2	2	2
$\chi = 1$	1	1	1	1	1
$\chi = \chi_Q$	1	1	1	-1	-1
$\chi = \chi_P$	1	1	-1	-1	1
$\chi = \chi_R$	1	1	-1	1	-1

C.2. Lemma. — *The order* $n(\chi)$ *of* $\zeta^S(\Pi, \chi, s)$ *at* $s = 1$ *for the characters* $\chi = 1, \chi_Q, \chi_P, \chi_R$ *is* $0, 0, 1, 0$ *respectively. Up to a set of density zero the class* $\{SN, SN^3\}$ *has type* D *and* $w = 1$.

Proof. — For characters χ of Δ we have $8 \cdot n(\chi) = 3 - \chi(N^2) - 2 \cdot \chi(S) - 2 \cdot \chi(N) + 2 \cdot w \cdot \chi(SN)$, by definition. This implies $-n(1) = n(\chi_Q) = n(\chi_R) = \frac{1}{4}(1 - w)$ and $n(\chi_P) = \frac{1}{4}(w+3)$. Since $-3 \leqslant w \leqslant 1$ and since the $n(\chi)$ are integers, it follows $w = 1$ or $w = -3$. Suppose $w = -3$. Then $\zeta^S(\Pi, 1, s)$ would have a zero at $s = 1$, which is impossible by cor. 7.3.1. Hence $w = 1$. □

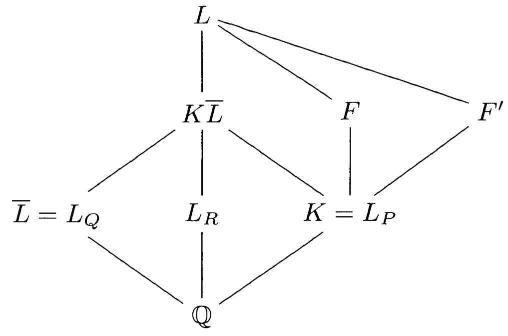
By the last lemma $\zeta^S(\Pi, \chi_P, s)$ has a pole at $s = 1$ and Π is associated to some automorphic representation $\pi = \pi_K$ of $\mathrm{Gl}(2, \mathbb{A}_K)$. K is the quadratic extension field of $\mathbb{Q}$ defined by the character $\chi_K = \chi_P$ (theorem 4.2), hence a subfield of L.

C.3. Lemma. — *Suppose* χ *is a unitary character. If* $\zeta^S(\Pi, \chi, s)$ *has a pole at* $s = 1$ *then* $\chi = \chi_P$.

Proof. — By remark 4.3 we know $\chi^2 = 1$. Let L_χ be the corresponding quadratic extension field of $\mathbb{Q}$. By lemma C.2 we may suppose $\chi \neq 1, \chi_P, \chi_Q, \chi_P\chi_Q$. Then L and L_χ are linear disjoint fields and $\mathrm{Gal}(LL_\chi/\mathbb{Q}) = \mathrm{Gal}(L/\mathbb{Q}) \times \mathrm{Gal}(L_\chi/\mathbb{Q})$. Since Π^S satisfies the Ramanujan conjecture, lemma C.3 is an immediate consequence of the Tchebotarev density theorem and the fact, that up to a set of primes of density zero the Frobenius elements mapping to the conjugacy classes of $\mathrm{Gal}(L/\mathbb{Q}) = D_8$ in the table above belong to the cases A, B, B, C, D respectively (lemma C.2). □

C.4. Corollary. — K *and* π *are unique* $\chi_K = \chi_P$. □

Let σ be the nontrivial automorphism of $K/\mathbb{Q}$. Consider the field extensions



where $K = L_P$ is the fixed field of the subgroup $P = \{1, N^2, SN, SN^3\}$, and where L_Q is the field $\overline{L}$ attached to the subgroup Q. The composite $K\overline{L}$ is the fixed field of N^2. Let F be the fixed field of the group element $SN \in D_8$, which is a quadratic extension of K. Conjugation by σ fixes K, hence induces a permutation of the quadratic extension fields $K\overline{L}, F, F'$ of K in L. It fixes $K\overline{L}$ and permutes the other two F and $\sigma(F) = F'$. Since SN and SN^3 are conjugate, F' is the fixed field of the element $SN^3 \in D_8$. We obtain thus quadratic characters $\chi_{F/K}, \chi_{F'/K}, \chi_{K\overline{L}/K}$ of $\mathbb{A}_K^*/K^*$ where $\chi_{K\overline{L}/K} = \chi_Q \circ \mathrm{Norm}_{K/\mathbb{Q}}$.

C.5. Lemma. — *The representation π of $\mathrm{Gl}(2, \mathbb{A}_K)$ associated to Π has the following CM-properties:*

(1) $\pi \cong \pi \otimes \chi_{K\overline{L}/K}$.
(2) $\sigma(\pi) \ncong \pi \otimes \chi_{K\overline{L}/K}$.
(3) $\sigma(\pi) \cong \pi \otimes \chi$ *for* $\chi = \chi_{F/K}$ *and* $\chi = \chi_{F'/K}$.

Proof. — (1) follows from (3). Prop. 6.8 for $\chi = \chi_Q$ with $\chi\chi_K = \chi_Q\chi_P = \chi_R$ gives by lemma C.2

$$\mathrm{ord}_{s=1}\, L_K^S(\chi_Q \circ \mathrm{Norm}_K) L_K^S(\sigma(\pi) \times \pi^* \otimes (\chi_Q \circ \mathrm{Norm}_K), s) = 0.$$

Hence $L_K^S(\sigma(\pi) \times (\pi^* \otimes (\chi_Q \circ \mathrm{Norm}_K), s)$ has no pole at $s = 1$. By prop. 7.1 this implies $\sigma(\pi) \ncong \pi \otimes (\chi_Q \circ \mathrm{Norm}_K)$. This proves (2). Since the Ramanujan conjecture holds for π for the proof of (3) it suffices to show, that for a set of places w of K-density 1 the representations π_w and $\pi_w \otimes \chi_{F/K,w}$ are locally isomorphic. The local isomorphisms imply $L^S(\pi^* \times (\pi \otimes \chi_{F/K}), s) \sim \zeta_K^S(s)\zeta^S(\mathrm{Ad}(\pi), 1, s)$ at $s = 1$. $\zeta_K^S(s)$ has a pole at $s = 1$ and $\zeta^S(\mathrm{Ad}(\pi), 1, 1) \neq 0$. Hence the left side has a pole at $s = 1$, which implies (3) by prop 7.1. So it remains to show the local isomorphisms. The set of places w of K, which lie over $K = L_P$-split primes p_v of $\mathbb{Q}$ with $v \notin T$, has K-density one. For such v there are two extension w, w' to K. By 8.1 we have $\pi_w \times \pi_{w'} \sim (t_v\nu_v, t_v\mu_v) \times (t_v\widetilde{\nu}_v, t_v\widetilde{\mu}_v)$ for certain t_v, if $\Pi_v \sim (\nu_v, \widetilde{\nu}_v, \mu_v, \widetilde{\mu}_v)$. We get the following list

Frob_v	1	N^2	SN	SN^3
Cases	A	B	D	D
Π_v	$(\nu_v, \nu_v, \mu_v, \mu_v)$	$(\nu_v, -\nu_v, \mu_v, -\mu_v)$	$***$	$(\nu_v, -\nu_v, -\nu_v, \nu_v)$
$\chi_{F/K}$	1	-1	1	-1
$\chi_{F'/K}$	1	-1	-1	1

The corresponding representations $\sigma(\pi_w \times \pi_{w'}) = \pi_{w'} \times \pi_w$ of $\mathrm{Gl}(2, K_v) = \mathrm{Gl}(2, \mathbb{Q}_v) \times \mathrm{Gl}(2, \mathbb{Q}_v)$ have Satake parameters

$$\begin{array}{llll} \pi_w, \pi_{w'} & t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v); & t_v(\nu_v, \mu_v), -t_v(\nu_v, \mu_v); & t_v(\nu_v, -\nu_v), t_v(-\nu_v, \nu_v) \\ \pi_{w'}, \pi_w & t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v); & -t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v); & t_v(-\nu_v, \nu_v), t_v(\nu_v, -\nu_v) \end{array}$$

In these lists the entry for Π_v and $\pi_v = \pi_w \times \pi_{w'}$ for the two conjugate elements SN and SN^3 are the same. Hence there is only one entry. By inspection in the four possible cases we see $\sigma(\pi_v) \cong \pi_v \otimes \chi_{F/K,v}$. □

The degenerate abelian CM case $\Delta = (\mathbb{Z}/2\mathbb{Z})^3$. — Let $Q = \overline{\Delta}$ be the subgroup of $\Delta = \mathrm{Gal}(L/\mathbb{Q})$, whose fixed field is the distinguished field $\overline{L}$. Suppose Q is generated by N, S with $N^2 = S^2 = 1$ and suppose $\Delta = \langle Q, M\rangle$ with $M^2 = 1$. Let χ_Q be the nontrivial quadratic character of Δ, which is trivial on the subgroup Q. Let χ_N, χ_S be the characters defined by $\chi_N(N) = -1, \chi_N(M) = \chi_N(S) = 1$ and $\chi_S(S) = -1, \chi_S(N) = \chi_S(M) = 1$. The weights for $\mathrm{Frob}_v = 1, N, S, NS, M, MN, MS, MNS$ are $3, -1, -1, -1, w_1, w_2, w_3, w_4$ with the corresponding cases A, B, B, B, $D+E$, $D+E$, $D+E$, $D+E$. As in the proof of lemma C.5 one then shows for $m(1) = 0$, $m(\chi_N) = 1$, $m(\chi_S) = 1$, $m(\chi_N\chi_S) = 1$ for

$$m(\chi) = \mathrm{ord}_{s=1}\, \zeta^S(\Pi, \chi, s)\zeta^S(\Pi, \chi\chi_Q, s).$$

Let $m = \mathrm{ord}_{s=1}\, \zeta^S(\Pi, 1, s)$ and x, y, z be the orders of $\mathrm{ord}_{s=1}\, \zeta^S(\Pi, \chi, s)$ for $\chi = \chi_N, \chi_S, \chi_N\chi_S$. Then $(w_1 + w_2 + w_3 + w_4)/8 = m$ and $(w_1 - w_2 + w_3 - w_4)/8 + 1/2 = x$ and $(w_1 + w_2 - w_3 - w_4)/8 + 1/2 = y$ and $(w_1 - w_2 - w_3 + w_4)/8 + 1/2 = z$. Then $-3 \leqslant w_i \leqslant 1$ implies $m \in \{-1, 0\}$ and $x, y, z \in \{0, 1\}$. Suppose $m = -1$. Proposition 6.8 gives $\mathrm{ord}_{s=1}\, \zeta^S(\Pi, 1, s)\zeta^S(\Pi, \chi_K, s) \geqslant 1$. The order of $\zeta^S(\Pi, \chi_K, s)$ is at most 1. This is a contradiction. Hence $m = 0$. Solving the inequalities for the w_i above, gives $0 \leqslant x + y + z \leqslant 2$ and $-2 \leqslant -x + y - z \leqslant 0$ and $-2 \leqslant x - y + z \leqslant 0$ and $-2 \leqslant -x - y + z \leqslant 0$. Therefore either $x = 1, y = 1, z = 0$ (without restriction of generality) or $x = y = z = 0$. In the first case $w_1 = 1, w_2 = 1, w_3 = 1, w_4 = -3$ and in the second we get $w_1 = -3, w_2 = 1, w_3 = 1, w_4 = 1$. Then for all $\mathbb{Q}$-places outside a set of density zero the possible types are A, B, B, B, D, D, D, E resp. A, B, B, B, E, D, D, D. This proves

C.3'. Lemma. — *There are exactly three characters $\chi = \chi_1, \chi_2, \chi_3$ of Δ for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$. For all other characters χ (of Δ) the order of $\zeta^S(\Pi, \chi, s)$ at $s = 1$ is zero. Furthermore*

$$\widehat{\Delta} = \{1, \chi_Q, \chi_1, \chi_1\chi_Q, \chi_2, \chi_2\chi_Q, \chi_3, \chi_3\chi_Q\}.$$

Since K is a subfield of L and the group Δ is abelian, cor. 9.3 and lemma C.3' imply $6 = 2 \cdot \mathrm{ord}_{s=1} \prod_{\chi \in \widehat{\Delta}} \zeta^S(\Pi, \chi, s) = n_K(\Pi) \cdot D/2 + 2 = 4 \cdot n_K(\Pi) + 2$. Hence the number $n_K(\Pi)$ defined in lemma 9.1 is $n_K(\Pi) = 1$. Cor.9.3 thus implies

$$\mathrm{ord}_{s=1} \prod_{\chi \in \widehat{\Delta}} \zeta_K^S(\mathrm{Ad}(\pi), \chi \circ \mathrm{Norm}_K, s) = n_K(\Pi) \cdot D/2 - 2 = 2.$$

By prop. 7.1 therefore there are two cases two cases where there are poles of $\zeta^S(\mathrm{Ad}(\pi), \chi \circ \mathrm{Norm}_K, s)$, and they are of the form $\chi \in \{\chi_4, \chi_4\chi_K\}$ for some $\chi_4 \in \widehat{\Delta}$. Lemma 9.1 and $n_K(\Pi) = 1$ and the nonvanishing result of lemma C.3' imply for all $\chi \in \widehat{\Delta}$, that precisely one of the three functions $\zeta^S(\Pi, \chi, s)$ or $\zeta^s(\pi, \chi\chi_K, s)$ or $\zeta^S(\mathrm{Ad}(\pi_K), \chi \circ \mathrm{Norm}_K, s)$ has a pole at $s = 1$. This implies $\{\chi_i, \chi_i\chi_K\} \cap \{\chi_4, \chi_4\chi_K\} = \varnothing$ for $i = 1, 2, 3$. Thus χ_i for $i = 1, \ldots 4$ are coset representatives with respect to $\{1, \chi_K\}$. Furthermore this implies $\chi_i\chi_K \notin \{\chi_1, \chi_2, \chi_3\}$. Since χ_K can be any of the characters χ_1, χ_2, χ_3, these properties imply

C.4'. Lemma. — $\chi_4 = \chi_1\chi_2\chi_3 = \chi_Q$.

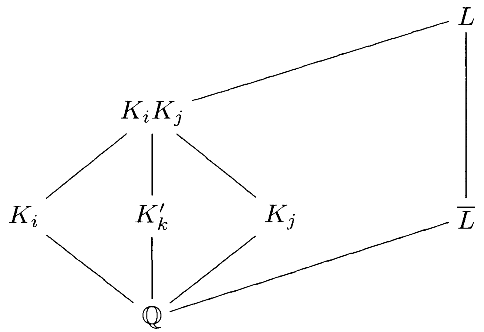
Proof. — For $1 \leqslant i, j \leqslant 3$ we can not have $\chi_i\chi_j = \chi_Q$, since otherwise $\chi_i = \chi_j\chi_Q$ contradicts lemma C.3'. The product is not in $\{\chi_1, \chi_2, \chi_3\}$ either, as stated above. Therefore $\chi_1\chi_2 = \chi_3\chi_Q$ by lemma C.3'. More symmetrically $\chi_1\chi_2\chi_3 = \chi_Q$. Now fix any $i \in \{1, 2, 3\}$. Then $\widehat{\Delta} = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_1\chi_i, \chi_2\chi_i, \chi_3\chi_i, \chi_4\chi_i\}$. Since $\chi_k\chi_i = \chi_j\chi_Q$ or 1, lemma C.3' implies $\chi_Q \in \{\chi_4, \chi_4\chi_i\}$. Since this holds for every choice of $i = 1, 2, 3$, this forces $\chi_4 = \chi_Q$. □

Prop. 7.1 and lemma C.4' imply, that π has complex multiplication $\pi \cong \pi \otimes (\chi \circ \mathrm{Norm}_K)$ for $\chi \in \widehat{\Delta}$ iff χ is in the subgroup $\{1, \chi_K, \chi_Q, \chi_Q\chi_K\}$ of $\widehat{\Delta}$. Since $n_K(\Pi) = 1$ lemma 9.1 implies, that $\sigma(\pi_K) \cong \pi_K \otimes (\chi \circ \mathrm{Norm}_K)$ holds precisely for the χ in the complementary coset in $\widehat{\Delta}$. Since $\chi_Q = \chi_{\overline{L}}$ therefore

C.5'. Lemma. — *For any choice $K = K_1, K_2, K_3$ let σ denote the nontrivial involution of $K/\mathbb{Q}$. The automorphic representation $\pi = \pi_K$ of $\mathrm{Gl}(2, \mathbb{A}_K)$ associated to Π by the theta lift has the following CM-properties*

(1) $\pi \cong \pi \otimes (\chi \circ \mathrm{Norm}_K)$ *for the characters χ in* $\langle \chi_{\overline{L}}, \chi_K \rangle = \{1, \chi_{\overline{L}}, \chi_K, \chi_{\overline{L}}\chi_K\}$.

(2) $\sigma(\pi) \cong \pi \otimes (\chi \circ \mathrm{Norm}_K) \not\cong \pi$ *for* $\chi \in \widehat{\Delta} - \langle \chi_{\overline{L}}, \chi_K \rangle$.



There are six quadratic subfields $K_1, K_2, K_3, K'_1, K'_2, K'_3$ with $\{ijk\} = \{1,2,3\}$ plus the distinguished quadratic subfield $\overline{L}$.

Appendix D
Pairings

Let G be a group, let π be an irreducible representation of G on the vector space V_π over a field k of characteristic zero. Let $\omega : G \to k^*$ be a one dimensional character of G. Let (V,ρ) be an isotypic multiple of V_π. In fact, we want that

(1) Schur's lemma holds, and

(2) that there is a natural notion of dual representation $(V,\rho)^\vee$, so that there exists a canonical isomorphism $(V,\rho) \to ((V,\rho)^\vee)^\vee$ of representations.

For instance: G is a compact group, and the representations are continuous and finite dimensional; or G is the group of F-valued points of a reductive group over F, where F is a local nonarchimedian field, and the representations are finitely generated and admissible. Also consider $(\mathcal{G}, K)$-modules for real Lie groups, or finite dimensional algebraic representations of reductive groups over an algebraically closed field of characteristic zero. So assume that we are in one of these situations.

The Parity. — Suppose, there exists an isomorphism $\psi : \rho \cong \rho^\vee \otimes \omega$ with underlying map $\psi : V \to V^\vee$ such that $\psi(\rho(g)v) = \omega(g)\rho^\vee(g)(\psi(v))$. Then we get another isomorphism by dualizing $\psi^\vee : (V^\vee)^\vee \to V^\vee$. By our assumption we can identify $(V^\vee)^\vee$ and V, so we view $\psi^\vee$ as a map from V to $V^\vee$ by abuse of notation. Obviously, $\psi^\vee$ again satisfies $\psi^\vee(\rho(g)v) = \omega(g)\rho^\vee(g)(\psi^\vee(v))$. ψ is called ε-symmetric, if

$$\psi^\vee = \varepsilon \cdot \psi$$

holds for some constant $\varepsilon = \varepsilon(G,\rho,\omega,\psi)$ in k. Since $(\psi^\vee)^\vee = \psi$ the number $\varepsilon(G,\rho,\omega)$ is either 1 or -1. It is called the parity of (G,ρ,ω).

In the special case, where (V,ρ) is an irreducible representation with $\rho \cong \rho^\vee \otimes \omega$, the isomorphism ψ is unique up to a constant (Schur's lemma). In particular, ψ is

ε-symmetric for a unique parity ε. This parity does not depend on the choice of ψ, but only depends on G, ρ, ω. This defines $\varepsilon(G, \rho, \omega)) = \varepsilon(G, V, \omega)$. To determine $\varepsilon(G, \rho, \omega)$ amounts to decide, whether there exists a k-bilinear nontrivial ε-symmetric pairing

$$\langle ., . \rangle : V \times V \longrightarrow k, \quad \langle v, v' \rangle = \varepsilon \cdot \langle v', v \rangle$$

such that $\langle gv, gv' \rangle = \omega(g) \cdot \langle v, v' \rangle$ holds for all $g \in G$, and all v, v' in V. Of course, $\langle v, v' \rangle = \psi(v)(v')$.

Remark 1. — $\varepsilon(G, \rho, \omega)$ does not change under twisting by a one dimensional character χ of G, $\varepsilon(G, \rho, \omega) = \varepsilon(G, \rho \otimes \chi, \omega\chi^2)$. If G is compact and $k = \mathbb{C}$, (V, ρ) is an irreducible continuous finite dimensional representation of G with character χ_ρ. Let dg be a Haar measure of volume one, then $\int_G \omega^{-1}(g)\chi_\rho(g^2)dg$ is a real number. It is zero, unless $\rho \cong \rho^\vee \otimes \omega$. If $\rho \cong \rho^\vee \otimes \omega$, this number is $\varepsilon(G, \rho, \omega)$.

Remark 2. — Suppose $G = G_1 \times G_2$ and $(V, \rho) = (V_1, \rho_1) \otimes_k (V_2, \rho_2)$, with a given ε-symmetric isomorphism $\psi : \rho \cong \rho^\vee \otimes \omega$. A character ω of G can be viewed as a character of G_1, G_2 in the obvious way. We assume that ρ_2 is irreducible and that ρ_1 is a finite isotypic multiple of an irreducible representation. Then the ε_1-symmetric G_1-equivariant homomorphisms ψ_1 in $\mathrm{Hom}_{G_1,\varepsilon_1}(\rho_1, \rho_1^\vee \otimes \omega)$ can be identified with the $\varepsilon = \varepsilon_1\varepsilon_2$-symmetric G-equivariant homomorphisms ψ in $\mathrm{Hom}_{G,\varepsilon}(\rho, \rho^\vee \otimes \omega)$ by the isomorphism $\psi_1 \mapsto \psi = \psi_1 \otimes \psi_2$, where $\psi_2 : \rho_2 \cong \rho_2^\vee \otimes \omega$ is the (up to a scalar) unique isomorphism for G_2. This follows from Schur's lemma by a dimension count. Hence

$$\varepsilon(G, \rho, \omega, \psi) = \varepsilon(G_1, \rho_1, \omega, \psi_1) \cdot \varepsilon(G_2, \rho_2, \omega).$$

Suppose (V_π, π) is an irreducible representation of G and suppose ψ is an ε-symmetric isomorphism as above. Suppose K is a subgroup of G, such that $\pi = \oplus_{\pi'} m(\pi')\pi'$ as a representation of K with finite multiplicities $m(\pi') > 0$ and irreducible representations π' of K. Suppose π' is such that $(\pi')^\vee \otimes \omega \cong \pi'$. Then $\varepsilon(G, \pi, \omega, \psi) = \varepsilon(K, \rho, \omega, \psi)$ for $\rho = m(\pi')\pi'$, where ψ, ω are obtained by restriction from G to K. In particular for $m(\pi') = 1$

$$\varepsilon(G, \pi, \omega) = \varepsilon(K, \pi', \omega).$$

The reducible case. — If on the other hand $(\pi')^\vee \otimes \omega \cong \pi''$ such that $\pi' \not\cong \pi''$, then $V_{\pi'} \oplus V_{\pi''}$ can be endowed both with a nondegenerate symplectic (or alternatively symmetric) nondegenerate pairing, which is K-equivariant with multiplier ω. Hence in this case no information on $\varepsilon(G, \pi, \omega)$ is obtained. This also made us restrict to the case of isotypic representations (V, ρ) at the beginning.

***Remark 3* (Schur multiplier)**. — Suppose k/k_0 is a Galois extension with Galois group Γ and G, V, π, ω, k are given as above so that (V, π) is irreducible over k. We say π has no CM, if $\pi \otimes \eta \cong \pi$ for a character η implies $\eta = 1$. Choose a basis of V. This defines a k_0-vector space structure and defines $\pi^\gamma, \gamma \in \Gamma$ by $\pi^\gamma(g)(v) = \gamma(\pi(g)\gamma^{-1}(v))$. Suppose $A_\gamma : \pi \cong \pi^\gamma \otimes \omega_\gamma$ holds for endomorphisms $A_\gamma : V \to V$ and certain characters

$\omega_\gamma, \gamma \in \Gamma$, *i.e.* $A_\gamma \pi(g) A_\gamma^{-1} = \omega_\gamma(g)\pi^\gamma(g)$. For simplicity assume the cocycle condition $\omega_{\tau\gamma} = (\omega_\gamma)^\tau \omega_\tau$ (which is automatic if π has no CM). Then $\tau(A_\gamma)A_\tau = A_{\tau\gamma} \cdot \lambda_{\tau,\gamma}$ for certain $\lambda_{\tau,\gamma} \in k^*$. Changing the basis amounts to $A_\tau \mapsto \tau(B)A_\tau B^{-1}$. The A_τ are uniquely defined up to constants. Therefore the coboundary $\lambda_{\tau,\gamma}$ defines a cohomology class in the Brauer group $H^2(\Gamma, k^*)$. This class is trivial iff the A_τ can be chosen to define a 1-cocycle with values in $\mathrm{Gl}(V)$. Then $A_\tau = \tau(B)B^{-1}$ for some $B \in \mathrm{Gl}(V)$ by Hilbert 90 (this is true if V is finite dimensional, but the argument carries over to the case of admissible representations). Then $\omega_\gamma(g)\pi^\gamma(g) = \pi(g)$ for all $\gamma \in \Gamma$, for the choice of basis determined by B. In other words, as for the usual Schur multiplier, π can be defined over the fixed field k_0 but only in a 'twisted sense'. The converse is also true. Now suppose, that in addition an isomorphism $\psi : V^\vee \otimes \omega \cong V$ of the underlying representation is given, and assume the coboundary condition $\omega_\gamma^{-2} = \omega^\gamma/\omega$ for all $\gamma \in \Gamma$. If π has no CM, this is a consequence of $(\pi^\vee)^\gamma = (\pi^\gamma)^\vee$ and $\pi^\vee \otimes \omega \cong \pi$ and $\pi^\gamma \otimes \omega_\gamma \cong \pi$. Under the assumptions above the class of $\lambda_{\tau,\gamma}$ is an element of the two-torsion subgroup of the Brauer group $Br(k/k_0)$. Notice, if ψ is defined over k_0, then $\psi^\gamma = \psi$ implies that $\omega(g)\pi(g)^\bullet = \pi(g)$, where $X^\bullet = \psi^{-1}(X^\vee)^{-1}\psi$. Note $(XY)^\bullet = X^\bullet Y^\bullet$ and $\gamma\bullet = \bullet\gamma$ and $c^\bullet = c^{-1}$ for constants $c \in k^*$. We get $A_\gamma^\bullet \pi(g) = (\omega/\omega_\gamma\omega^\gamma)(g)\pi^\gamma(g)A_\gamma^\bullet$ from $A_\gamma\pi(g) = \omega_\gamma(g)\pi^\gamma(g)A_\gamma$. Hence by Schur's lemma $A_\gamma^\bullet = c_\gamma \cdot A_\gamma$ for some constants $c_\gamma \in k^*$. Applying $(.)^\bullet$ to the defining equation for the cocycle $\lambda_{\tau,\gamma}$ we conclude $c_\gamma^\tau c_\tau \tau(A_\gamma)A_\tau = A_{\tau\gamma} \cdot \lambda_{\tau,\gamma}^{-1} c_{\tau\gamma}$. Therefore $c_\gamma^\tau c_\tau = \lambda_{\tau,\gamma}^{-2} c_{\tau\gamma}$. Hence $\lambda_{\tau,\gamma}^2$ is a coboundary.

The special case $k = \mathbb{C}$, $k_0 = \mathbb{R}$. — Suppose

$$\overline{\pi} \cong \pi^\vee \cong \pi \otimes \omega^{-1}$$

is irreducible and ω is a unitary character. Since $\overline{\pi} \cong \pi^\vee$ there exists a nondegenerate invariant hermitian symmetric form. The case we have in mind is where the representation π is unitary, so that this invariant hermitian form on V is positive definite. In this case choose the basis to be an orthonormal basis of V with respect to this hermitian form. Then we get $\overline{\pi}(g) = \pi^\vee(g) = \pi(g^{-1})'$ (This is true in the finite dimensional case and carries over in our situations). Put $\omega_\gamma = \omega$ for the complex conjugation. Then the maps $A = A_\gamma$ (γ complex conjugation) and ψ defined above coincide up to a constant $\psi = const. \cdot A$. The cocycle condition gives $\overline{A}A = \lambda \cdot E$. Hence $\overline{A}A = A\overline{A}$ and therefore $\lambda \in \mathbb{R}$. Hence for A suitably chosen we can assume $\lambda_\pi := \lambda \in \{\pm 1\}$. In fact this sign 'is' the cohomology class in $H^2(\Gamma, \mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$. Put $\theta(v) = \gamma(A(v))$ for the chosen complex conjugation γ on V. Then θ is $\mathbb{C}$-antilinear. We get $\theta \circ \pi(g) = \overline{\omega}(g) \cdot \pi(g) \circ \theta$ and $\theta^2 = \overline{A} \cdot A = \lambda_\pi$

$$\theta^2 = \lambda_\pi \cdot \mathrm{id}\,.$$

***Remark 4* (antilinear maps).** — Suppose V, W are complex vector spaces with nondegenerate hermitian symmetric forms. Let $\phi : V \to W$ be a γ-linear map, where $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then define the adjoint $\phi^* : W \to V$ by $\langle \phi(v), w\rangle_W = \overline{\langle v, \phi^*(w)\rangle_V}$, if

γ is complex conjugation. Then ϕ^* is γ-linear and $(\phi_1 \circ \phi_2)^* = \phi_2^* \circ \phi_1^*$, $(\phi^*)^* = \phi$. If ϕ is a γ-linear isomorphism, then also ϕ^{-1}. Hence $\phi \mapsto (\phi^*)^{-1}$ defines an involution of the group of all γ-linear automorphisms of V. Suppose V, W are complex vector spaces with representations π_V, π_W of G. Suppose $\langle .,. \rangle_W$ a nondegenerate hermitian-symmetric bilinear form on V, which is G invariant $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$. Suppose the same for W. Suppose ω is a character of G. Consider ω-equivariant γ-linear homomorphisms $\theta : V \to W$ for which $\theta \circ \pi_V(g) = \gamma(\omega(g)) \cdot \pi_W(g) \circ \theta$. We then say θ is a (ω, γ)-homomorphism. The composition $\theta' \circ \theta$ of a (ω, γ)-homomorphism and a (ω', γ')-homomorphism is a $(\gamma^{-1}(\omega')\omega, \gamma'\gamma)$-homomorphism. The transpose of a (ω, γ)-homomorphism is a $(\gamma(\omega)^{-1}, \gamma)$-homomorphism. Furthermore, if θ is invertible, then $(\theta^*)^{-1}$ is again a (ω, γ)-homomorphisms and θ^{-1} is a $(\gamma(\omega)^{-1}, \gamma)$-homomorphism. Suppose $V = W$ and suppose γ is the complex conjugation. Suppose θ is an $\mathbb{C}$-antilinear automorphism $\theta : V \to V$, so that

$$\theta \circ \pi(g) = \overline{\omega}(g) \cdot \pi(g) \circ \theta.$$

Assume $\theta^* = \varepsilon \cdot \theta$ for some $\varepsilon \in \{\pm 1\}$. If the representation π of G on V is irreducible, then Schur's lemma implies that the $\mathbb{C}$-vector space of (ω, γ)-endomorphisms of V is one dimensional. Hence $\theta^* = \varepsilon \cdot \theta$ for $\varepsilon \in \{\pm 1\}$ holds automatically. Define $[v, w] = \langle \theta(v), w \rangle$. This is a $\mathbb{C}$-bilinear form such that $[\pi(g)v, \pi(g)w] = \omega(g)[v, w]$ and such that $[w, v] = \varepsilon \cdot [v, w]$. If π is irreducible, then $\theta^{-1} = \lambda^{-1} \cdot \theta$ and $\lambda \in \mathbb{C}^*$ and $(\theta^*)^{-1} = \rho \cdot \theta$ and $\rho \in \mathbb{C}^*$ and $1 = \rho\lambda\varepsilon$. If $\langle .,. \rangle$ is definite, then $\rho \cdot \mathrm{id}_V = \theta^*\theta$ is positive definite and $\rho \in \mathbb{R}^*_{>0}$ and also $\lambda \in \mathbb{R}^*$. Replacing θ by a suitably multiple we get a well defined $\lambda = \lambda_\pi \in \{\pm 1\}$ such that $\rho = 1$ and $\theta^2 = \lambda_\pi \cdot \mathrm{id}_V$. Furthermore $\lambda_\pi = \varepsilon = \varepsilon(G, \pi, \omega)$. Hence

$$\theta^2 = \varepsilon(G, \pi, \omega) \cdot \mathrm{id}\,.$$

***Remark 5* (Whittaker models).** — Suppose G is a quasisplit reductive group with Borel group B over a local nonarchimedian field F. Suppose ψ is a generic unitary character of its unipotent radical N. Suppose $g \mapsto g^x$ is an involutive automorphism of (G, B) which maps ψ to its inverse $\overline{\psi}$. Let W_ψ be the induced space $\mathrm{Ind}_{N(F)}^{G(F)}(\psi)$ with the left action of $G(F)$ by the right translations $R_\psi(g), g \in G(F)$. Let (π, V) be an admissible irreducible representation of $G(F)$ such that $\pi^x \cong \overline{\pi}$, where $\pi^x(g) = \pi(g^x)$ is defined as a representation of $G(F)$ on V. Also the complex conjugate $\overline{\pi}$ (for a suitably choice of basis respecting admissibility) is defined as a representation of $G(F)$ on V. Furthermore suppose that π has a unique Whittaker model for ψ. So we can identify π with its π-isotypic component in W_ψ.

Claim. — *In this situation there exists a nontrivial antilinear endomorphism $\theta_x : V \to V$ of the representation space V of π such that $\theta_x^2 = \mathrm{id}_V$ and $\theta_x \circ \pi = \pi^x \circ \theta_x$ holds. The map θ_x is unique up to a constant.*

Proof. — Let $A : W_\psi \to W_{\overline{\psi}}$ be defined by $(Af)(g) = f(g^x)$ and $B : W_{\overline{\psi}} \to W_\psi$ by $(Bf)(g) = \overline{f}(g)$. Then $\theta_x = B \circ A : W_\psi \to W_\psi$ is an antilinear isomorphism, satisfies $\theta_x^2 = \mathrm{id}$, such that $\theta_x \circ R_\psi^x(g) = R_\psi(g) \circ \theta_x$. This is clear, since $A \circ R_\psi(g^x) = R_{\overline{\psi}}(g) \circ A$ and since $B \circ R_{\overline{\psi}}(g) = R_\psi(g) \circ B$. But then also $\theta_x \circ R_\psi(g) = R_\psi^x(g) \circ \theta_x$ by the involutive property. Hence θ_x maps the π-isotypic subspace bijectively to the π^x-isotypic subspace. □

Modifications. — Suppose $s \in G(F)$, so that $ss^x \in Z_G(F)$ is in the center of $G(F)$. Then $\iota_s(g) = sg^x s^{-1}$ is an involutive automorphism of G. Furthermore $\theta_s = \pi(s)\theta_x$ is $\mathbb{C}$-antilinear and satisfies $\theta_s \circ \pi(g) = \pi(sg^x s^{-1}) \circ \theta_s$. Then $(\theta_s)^2 = \omega_\pi(ss^x) \cdot \mathrm{id}_V$ by the last claim. In certain relevant cases s can be chosen so that $\pi(sg^x s^{-1}) = \overline{\omega}(g)\pi(g)$ holds. If this is the case, then $\varepsilon(G, \pi, \omega) = \omega_\pi(ss^x)$ by remark 4.

1. Example. — Let G be the group of F-valued points of a reductive group over a nonarchimedian local field, and let K be a suitable compact open subgroup. Assume that $\pi \cong \pi^\vee \otimes \omega$ holds, and that π contains the trivial representation of K with multiplicity one. If ω is trivial on K, then $\varepsilon(G, \pi, \omega) = 1$. Hence for unramified representations ω, π we get $\varepsilon(G, \pi, \omega) = 1$.

2. Example. — For $G = \mathrm{Gl}(N)$ and $g \in \mathrm{Gl}(N, F)$ define $g^* = (g')^{-1}$. Let κ be the matrix in $\mathrm{Gl}(N, F)$ with entry 1 in the antidiagonal and zero else. Then $g^x = \kappa g^* \kappa$ is an involutive automorphism of $\mathrm{Gl}(N, F)$. There exists a Borel group $B \subset \mathrm{Gl}(N, F)$ preserved by this involution and a generic unitary character ψ of its unipotent radical such that $\psi(n^x) = \overline{\psi}(n)$. Suppose π is irreducible, unitary and nondegenerate with ψ-Whittaker model. [**Sha**] th. 3.1. Then by [**Sha**], p. 185 we know $\pi^x \cong \pi^\vee$, hence for unitary π also $\pi^x \cong \overline{\pi}$, and we are in a situation as in remark 5. Let θ_x be as in remark 5.

The case $\mathrm{Gl}(2, F)$. — Then $\overline{\pi} \cong \pi^\vee \cong \overline{\omega}_\pi \otimes \pi$ holds. Put $\theta_s = \pi(s)\theta_x$ for $s = J$, J as below. Then $\theta_s \circ \pi(g) = (\omega_\pi^{-1} \otimes \pi)(g) \circ \theta_s$ since $Jg^x J^{-1} = \det(g)^{-1} g$. Furthermore $(\theta_s)^2 = \omega_\pi(-1) \cdot \mathrm{id}_V$ since $ss^x = s^2 = -E$. Therefore $\varepsilon(\mathrm{Gl}(2, F), \pi, \omega_\pi) = \omega_\pi(-1)$ by remark 4. It is not difficult to see that this assertion remains true also for nonunitary irreducible representations π.

3. Example. — For $G = \mathrm{GSp}(2n)$ consider

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

of rank $N = 2n$. Then $g \in \mathrm{GSp}(2n, F)$ is equivalent to $Jg^* J^{-1} = \lambda(g)^{-1} \cdot g$ for some multiplier $\lambda(g) \in F^*$. Furthermore $\iota(g) = \lambda(g)^{-1} sgs$ is an involutive automorphism of $\mathrm{GSp}(2n, F)$ such that $\pi^\iota \cong \omega_\pi^{-1} \otimes \pi$ holds if

$$s = \mathrm{diag}(1, -1, 1, -1, \ldots, \pm 1, -1, 1, -1, \ldots, \mp 1), \quad \lambda(s) = -1.$$

There exists a Borel group $B \subset \mathrm{GSp}(2n,F)$ and a generic character ψ of its unipotent radical, so that $\psi(\iota(n)) = \overline{\psi}(n)$. Suppose (V,Π) is an irreducible admissible unitary representation of $\mathrm{GSp}(2n,F)$ for a local nonarchimedian field. Suppose Π is generic, so admits a Whittaker model for ψ. Then $\overline{\Pi} \cong \Pi^\vee$, hence $\overline{\Pi} \cong \Pi^\iota$ (for $n=2$ by lemma 1.1). So we can apply remark 5 for the involutive automorphism $g^x = \iota(g)$. This gives a $\mathbb{C}$-antilinear automorphism θ_x. Put $\theta_s = \Pi(s)\theta_x$. Then the modification θ_s satisfies $(\theta_s)^2 = \omega_\Pi(s\iota(s)) = \omega_\Pi(-1)\cdot \mathrm{id}_V$ and $\theta_s \circ \Pi(g) = \Pi(s\iota(g)s^{-1})\circ\theta_s = (\omega_\Pi^{-1}\otimes\Pi)(g)\circ\theta_s$. Hence by remark 4

$$\varepsilon(\mathrm{GSp}(4,F),\Pi,\omega_\Pi) = \omega_\Pi(-1).$$

***4. Example* (Generalized Whittaker models).** — Suppose $G = \mathrm{GSp}(4,F)$. Let $P = MN$ be the Siegel parabolic as in [**PS**], p. 507. Suppose Π is a unitary irreducible admissible representation of $\mathrm{GSp}(4,F)$ for a local nonarchimedian field F, which has a nontrivial generalized Whittaker functional $\nu\otimes\psi_T$ attached to a nondegenerate symmetric matrix $T = T' \in M_{2,2}(F)$. According to [**PS**] theorem 1.1 this functional is unique, if it exists. We briefly recall part of the definition.

The matrix T defines a character ψ_T of $N(F) \to \mathbb{C}^*$. The stabilizer of this character in $M(F)$ is a semidirect product $K^*\cdot(\mathbb{Z}/2\mathbb{Z})$, where $K/\mathbb{F}$ is a quadratic algebra determined by the discriminant of T. K^* is embedded in $M(F)\cong \mathrm{Gl}(2,F)$ by the regular representation. The similitude factor $\lambda(y)$ of $y\in K^*$ with respect to this embedding is $\mathrm{Norm}_{K/F}(y)$. The group $\mathbb{Z}/2\mathbb{Z}$ induces the involution σ of $K/\mathbb{Q}$, so we view the generator σ of this group as an element $\sigma\in M(F)\subset \mathrm{GSp}(4,F)$ in this way. In fact

$$\sigma = \mathrm{diag}(1,-1,1,-1) \quad \text{respectively} \quad \sigma = \mathrm{diag}\Big(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Big)$$

in the first respectively second case of [**PS**]. Then $\sigma\in M(F)$ has multiplier $\lambda(\sigma)=1$. Finally ν is a character of K^*. Consider the automorphism $g^\sigma = \sigma(g\lambda(g)^{-1})\sigma$ on $\mathrm{GSp}(4,F)$. Therefore $g\mapsto g^\sigma$ is an involutive automorphism of $\mathrm{GSp}(4,\mathbb{F})$. If ν is a unitary character then $(\nu\otimes\psi_T)(p^\sigma) = (\overline{\nu}\otimes\psi_T)(p)$ for all p in the semidirect product $N(F).K^*$. In fact, for $y\in K^*\subset M(F)$ we get $y^\sigma = \sigma(y)\,\mathrm{Norm}_{K/F}(y)^{-1}$. Since $\sigma(y)y$ is in the center of $\mathrm{GSp}(4,F)$ we necessarily have $\nu(\sigma(y)y) = \omega_{\Pi_v}(\sigma(y)y)$ and $\nu(y^\sigma) = \nu(\sigma(y)\,\mathrm{Norm}_{K/F}^{-1}(y)) = \nu(y)^{-1}$. Composed with additional conjugation by $\mathrm{diag}(E,-E)$ we get the new involution $g\mapsto\widetilde{\iota}(g)$, which is similar to the involution ι considered above. $\widetilde{\iota}$ has the property $(\nu\otimes\psi_T)(\widetilde{\iota}(p)) = \overline{\nu\otimes\psi_T}(p)$ for all p in the semidirect product $N(F).K^*$.

This being said we are in a situation, which is similar to the one of remark 5. (V,Π) is an irreducible, admissible unitary representation of $\mathrm{GSp}(4,F)$ for a nonarchimedian local field F, which has a generalized Whittaker model for $\nu\otimes\psi_T$. The involutive automorphism $\widetilde{\iota}$ of $\mathrm{GSp}(4,F)$ preserves $K^*\cdot N$ and maps $\nu\otimes\psi_T$ to its complex conjugate. Furthermore $(V,\Pi^{\widetilde{\iota}})\cong(V,\Pi^\vee)\cong(V,\overline{\Pi})$ by lemma 1.1. So the argument of remark 5 carries over. Hence there exists an $\mathbb{C}$-antilinear automorphism θ of V such that $\theta^2 = \mathrm{id}_V$ and such that $\theta \circ \Pi(g) = \Pi(\widetilde{\iota}(g)) \circ \theta$ holds, and θ is unique up to a constant.

Modifications. — For $s^{-1} = \mathrm{diag}(E, -E)\sigma$ we have $s\widetilde{\iota}(s) = -E$ and $s\widetilde{\iota}(g)s^{-1} = \lambda(g)^{-1}g$. Then $\theta_s = \Pi(s)\theta$ is antilinear, such that $(\theta_s)^2 = \Pi(s\widetilde{\iota}(s)) = \omega_\Pi(-1) \cdot \mathrm{id}_V$ and $\theta_s \circ \Pi(g) = \Pi(s\widetilde{\iota}(g)s^{-1}) \circ \theta_s = (\omega_\Pi^{-1} \otimes \Pi)(g) \circ \theta_s$. By remark 4 therefore

$$\varepsilon(\mathrm{GSp}(4, F), \Pi, \omega_\Pi) = \omega_\Pi(-1).$$

5. Example. — Suppose Π_∞ is an irreducible representation of the group $\mathrm{GSp}(4, \mathbb{R})$, which is in the discrete series of weight (k_1, k_2). Then

$$\varepsilon(\mathrm{GSp}(4, \mathbb{R}), \Pi_\infty, \omega_{\Pi_\infty}) = \omega_{\Pi_\infty}(-1) = (-1)^{k_1+k_2}.$$

Proof. — We restrict to the minimal K_∞-type. K_∞ is the semidirect product of $\mathrm{U}(2)$ and ι_∞, where conjugation by ι_∞ induces complex conjugation on the unitary group $\mathrm{U}(2)$. The minimal K_∞-type τ occurs with multiplicity one and is induced from the irreducible representation τ_{k_1,k_2} of $\mathrm{U}(2)$ of highest weight (k_1, k_2), where $k_1 \geqslant k_2 \geqslant 3$ in the holomorphic case. Similar for $\tau_{k_1,2-k_2}$ in the Whittaker case. As a representation of $\mathrm{U}(2)$ it decomposes into the two nonisomorphic representations $\tau_{k_1,k_2} \oplus \tau_{k_1,k_2}^\vee$ resp. $\tau_{k_1,2-k_2} \oplus \tau_{k_1,2-k_2}^\vee$. Since $k_1 \neq -k_2$ resp. $k_1 \neq k_2 - 2$, the two constituents $\sigma, \sigma^{\iota_\infty}$ of τ restricted to $\mathrm{U}(2)$ are not isomorphic. Therefore the space of $\mathrm{U}(2)$-homomorphisms $\tau \to \tau^\vee \otimes \omega_{\Pi_\infty}$ is two dimensional: Let $\langle .,. \rangle$ be a form define by such a homomorphism. Write $V_\tau = V_\sigma \oplus V_\sigma$, such that $\iota_\infty \in K_\infty$ acts by $\iota_\infty(v_1, v_2) = (v_2, v_1)$ and $\mathrm{U}(2)$ acts by $\sigma \oplus \sigma^{\iota_\infty}$. Choose a $\mathbb{C}$-bilinear form (v_1, v_2) on V_σ, such that $(v_1, v_2) = (v_2, v_1) = (\sigma(k)v_1, \sigma(k^{\iota_\infty})v_2)$. Then $\langle (v_1, v_2), (w_1, w_2) \rangle = \alpha \cdot (v_1, w_2) + \beta \cdot (w_1, v_2)$. Since $\iota_\infty \in K_\infty$ acts by $\iota_\infty(v_1, v_2) = (v_2, v_1)$, $\omega_{\Pi_\infty}(\iota_\infty) = \omega_{\Pi_\infty}(-1)$ implies $\alpha = \beta \cdot \omega_{\Pi_\infty}$. Therefore the parity of $\langle .,. \rangle$ is $\alpha/\beta = \omega_{\Pi_\infty}(-1)$. Finally $\omega_\infty(-1) = \omega_{\tau_{k_1,k_2}}(-1) = (-1)^{k_1+k_2}$. □

6. Example. — Consider representations $I_P(\sigma)$ induced from a standard parabolic subgroup $P(F) = M(F)N(F)$ of the group $G(F)$ of F-valued points of a connected reductive group G over a local nonarchimedian field F of characteristic zero for an irreducible admissible representation (σ, V_σ) of the Levi $M(F)$ group. For $f \in I_P(\sigma)$ and $h \in I_P(\sigma^\vee)$ the integral $\int_{P(F)\smallsetminus G(F)} \langle f(g), h(g) \rangle d\dot{g}$ is well defined. This defines a pairing and induces a canonical isomorphism $I_P(\sigma^\vee) = I_P(\sigma)^\vee$. For a character ω of $G(F)$ let ω also denote its restriction to $M(F)$. So $I_P(\sigma)^\vee \otimes \omega = I_P(\sigma^\vee) \otimes \omega = I_P(\sigma^\vee \otimes \omega)$, in a sense, holds canonically. For an element $\widetilde{w}$ in the Weyl group, which stabilizes M and maps P to the opposite parabolic $w(P) = \overline{P}$, choose a representative $w \in G(F)$. Assume

$$\sigma^\vee \otimes \omega \cong w(\sigma), \quad w(w(\sigma)) = \sigma,$$

where $(w(\sigma), V_\sigma)$ is defined by $w(\sigma)(m) = \sigma(wmw^{-1})$. This defines a nondegenerate $G(F)$-equivariant pairing $(.,.) : I_P(\sigma) \times I_P(w(\sigma)) \to \omega$ from the induced isomorphism $I_P(\sigma^\vee \otimes \omega) \cong I_P(w(\sigma))$. So to give a $G(F)$-equivariant map $\psi_I : I_P(\sigma) \to I_P(\sigma)^\vee \otimes \omega$

is the same as to give an $G(F)$-equivariant map $A(\sigma, w) : I_P(\sigma) \to I_P(w(\sigma))$, where the pairing $[.,.]$ corresponding to ψ_I is $[f,h] = (f, A(\sigma,w)(h))$

$$\begin{array}{ccc} I_P(\sigma) & \xrightarrow{\psi_I} & I_P(\sigma)^\vee \otimes \omega \\ & {\scriptstyle A(\sigma,w)}\searrow & \downarrow \cong \\ & & I_P(w(\sigma)) \end{array}$$

Suppose $\sigma = \sigma^0 \otimes \chi$ for a tempered representation σ^0 of $M(F)$ and an unramified character χ of $M(F)$. The unramified characters of $M(F)$ define a complex manifold $X(\sigma_0)$. If χ varies, the representation spaces $I_P(\sigma)$ can be considered to be independent from χ (via the Iwasawa decomposition). If χ varies over a subset $Y(\sigma_0) \subset X(\sigma_0)$ of characters for which $\chi(w(m)) = \chi(m)^{-1}$ holds, also the pairing $(.,.)$ resp. the fixed isomorphism $I_P(\sigma)^\vee \otimes \omega \cong I_P(\sigma(w))$ can be chosen to be independent from χ.

If $|\chi| > 0$ is in the positive cone in the sense of [**BW**], XI prop. 2.6, then $I_P(\sigma)$ has a unique irreducible quotient $\pi = J_P(\sigma)$, the Langlands quotient. It is the image of the intertwiner $j : I_P(\sigma) \to I_{\overline{P}}(\sigma)$ defined by $j(f)(g) = \int_{\overline{N}(F)} f(\overline{n}g)d\overline{n}$. This integral is absolutely convergent and also uniformly in g on compacta for $|\chi| > 0$. j is a nonzero map, whose image is the unique irreducible submodule of $I_{\overline{P}}(\sigma)$. See [**BW**], XI prop. 2.6. Since $f(g) \in I_{\overline{P}}(\sigma)$ iff $F(g) = f(w^{-1}g) \in I_P(w(\sigma))$ by our assumptions on w, the properties of j are inherited by the operator $A(\sigma,w) : I_P(\sigma) \to I_P(w(\sigma))$ defined to be $j(f)(w^{-1}g)$

$$A(\sigma,w)(f)(g) = \int_{N(F)} f(w^{-1}ng)dg.$$

Viewed as a function in χ it is a holomorphic operator in the domain $|\chi| > 0$. Furthermore the corresponding ψ_I induces an isomorphism $\psi : \pi \to \pi^\vee \otimes \omega$ on the Langlands quotient

$$\begin{array}{ccccc} I_P(\sigma) & = & I_P(\sigma) & \twoheadrightarrow & \pi \\ {\scriptstyle A(\sigma,w)}\downarrow & & & & \\ I_P(w(\sigma)) & & \downarrow \psi_I & & \cong \vdots\, \psi \\ \cong \downarrow & & & & \\ I_P(\sigma)^\vee \otimes \omega & = & I_P(\sigma)^\vee \otimes \omega & \hookleftarrow & \pi^\vee \otimes \omega \end{array}$$

Suppose $\psi_{I_\chi} : I_\chi \to I_\chi^\vee \otimes \omega$ defines an analytic family of $\pm$-selfdual intertwining operators for $I_\chi = I_P(\sigma^0 \otimes \chi)$. Hence $\psi_{I_\chi}^\vee = \varepsilon(\chi) \cdot \psi_{I_\chi}$ and $\varepsilon(\chi)$ is an analytic function in χ, hence constant $\varepsilon(\chi) = \varepsilon$. Such families exist, if for $\chi > 0$ in general position the representation I_χ is irreducible and if $X(\sigma_0) = Y(\sigma_0)$. Since there are $\pm$-self dual isomorphisms $I_\chi^\vee \otimes \omega \cong I_\chi$ for χ in general position, the existence of a family of $\pm$-selfdual intertwining operators follows by meromorphic continuation on the parameter space. Returning to the Langlands quotient π, for fixed $|\chi|>0$ the pairing on π induced by the map $\psi=\psi_\chi$ is $(.,A(\sigma,w).)$. Hence the parity satisfies $\varepsilon(G(F),\pi,\omega)=\varepsilon(G(F),I_P(\sigma),\omega,\psi_I)$. Since varying χ does not change the parity of ψ_{I_χ}, the computation of this sign can be reduced to the tempered case by meromorphic continuation to the unitary line $|\chi|=1$. This reduces the computation of ε to the case of a unitary character χ respectively a tempered representation σ.

Remark. — The situation above applies for the group $\mathrm{GSp}(2n,F)$, where $\omega=\omega_\Pi\circ\lambda$ and where ω_Π is the central character of the induced representation $\Pi=I_P(\sigma)$. Furthermore, because -1 is in the center of $G(F)$ and $M(F)$, for unramified χ the central characters satisfy $\omega_{I_P(\sigma)}(-1)=\omega_\sigma(-1)=\omega_{\sigma\otimes\chi}(-1)=\omega_{I_P(\sigma\otimes\chi)}(-1)$, since $\chi(-1)=1$. Hence this argument shows, that $\varepsilon(\mathrm{GSp}(4,F),\Pi,\omega_\Pi)=\omega_\Pi(-1)$ holds for all irreducible admissible representations if it holds for all tempered irreducible representations.

Notice $wgw^{-1}=g^*\lambda(g)$ for $g\in\mathrm{GSp}(2n,F)$ by definition, where $g^*=(g')^{-1}$. For $G=\mathrm{GSp}(2n)$ consider standard parabolic group P, whose elements g are of the form

$$\begin{pmatrix} A & 0 & B & * \\ * & a & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & d \end{pmatrix},$$

where A,B,C,D define an element in $h\in\mathrm{GSp}(2n_0,F)$ and a is a block diagonal matrix in $\prod_{i\geqslant1}\mathrm{Gl}(n_i,F)$ such that $\sum_{i\geqslant0}n_i=n$. In the case $n_0=0$ formally define $\mathrm{GSp}(0,F)=F^*$, and then $g=\mathrm{diag}(a,d)$ with $d=a^*h$ and $h\in\mathrm{GSp}(0,F)$. Elements in the Levi group are given by $m=(a,h)$ so that $(a,h)^*=(a^*,h^*)$ and $\lambda(a,h)=\lambda(h)$, where λ is the similitude homomorphism of $\mathrm{GSp}(2n,F)$ or $\mathrm{GSp}(2n_0,F)$. The matrix $w=J$ of example 3 represents the longest element in the Weyl group and satisfies $wPw^{-1}=\overline{P}$, $wMw^{-1}=M$. Furthermore $\chi(w(m))=\chi(m)^{-1}$ holds for characters χ of $M(F)$. An irreducible representation σ of $M(F)$ has the form $\sigma=\pi\boxtimes\tau$, for an irreducible representation τ of $\mathrm{GSp}(2n_0,F)$ and an irreducible representation π of $\prod_{i\geqslant1}\mathrm{Gl}(n_i,F)$. The central character of σ is $\omega_\sigma=\omega_\pi\omega_\tau$ for $n_0\neq0$ resp. $\omega_\sigma=\omega_\pi\omega_\tau^2$ for $n_0=0$. The central character ω_σ of σ restricted to the center of $\mathrm{GSp}(2n,F)$ coincides with the central character $\omega_{\Pi(\sigma)}$ of the induced representation $\Pi=I(\sigma)$ restricted to $M(F)$. Furthermore $wgw^{-1}=g^*\lambda(g)$ implies, that the condition $\sigma(w(\sigma))\cong\sigma^\vee\otimes(\omega_\Pi\circ\lambda)$ is equivalent to the condition

$$\sigma(m^*)\cong\sigma^\vee(m)$$

or $\tau^\vee(h)\cong\tau(h^*)$ and $\pi(a^*)\cong\pi^\vee(a)$. The latter conditions should always hold. For $n\leqslant2$ this follows from lemma 1.1.

7. Example. — Let F be a local nonarchimedian field of characteristic zero. Let $G=\mathrm{GSp}(2n,F)$ and $P=MN$ be a standard parabolic subgroup as in the last example.

Suppose σ is an irreducible unitary generic discrete series representation of $M(F)$ such that

$$\sigma^* \cong \sigma^\vee \cong \overline{\sigma}.$$

This is always true for $n = 2$ as explained above. The second condition always holds for unitary σ. Then there exists an antilinear isomorphism θ of the representation space of σ such that $\theta \circ \sigma = \sigma^* \circ \theta$ and $\theta^2 = \mathrm{id}$ hold. E.g. for $n = 2$ set $\theta = \sigma(\kappa)\theta_x$ as in example 2 (use $\kappa\kappa^x = 1$ and remark 5).

Let $\Pi = I(\sigma) = \mathrm{Ind}_{P(F)}^{\mathrm{GSp}(4,F)}(\sigma)$ be the unitary induced representation of $\mathrm{GSp}(4, F)$ on the representation space $I(\sigma)$. Put $\omega = \omega_\Pi \circ \lambda$. The intertwining operators $A(\sigma, w) : I(\sigma) \to I(\widetilde{w}(\sigma))$ for $\widetilde{w}$ in the Weyl group with representatives $w \in \mathrm{GSp}(4, F)$ are defined, by meromorphic continuation, from the integral

$$(A(\sigma, w)f)(g) = \int_{N_w(F)} f(w^{-1}ng)dn$$

as in [**Sh**]. Suppose this is well-defined and nontrivial. See [**Sh**] cor. 7.6 for a criterion for cuspidal σ. If w is in the center with image $\widetilde{w} = 1$ in the Weyl group, then $A(\sigma, w) = \omega_\Pi(w) \cdot \mathrm{id}$. In particular $A(\sigma, -E) = \omega_\Pi(-1) \cdot \mathrm{id}$. Furthermore $\Pi(\sigma \otimes \omega) = \Pi(\sigma) \otimes \omega$ holds for the restrictions of the unitary characters ω of $\mathrm{GSp}(4, F)$ to $M(F)$. Then $\Pi(\sigma)$ and $\Pi(\sigma \otimes \chi)$ have the same representation space. With this identification $A(\sigma \otimes \omega, w) = A(\sigma, w)$. This is clear from the above integral formula, since there exists $n = n^- mk$ with $\lambda(m) = \lambda(k) = 1$ for $n \in N_w(F), m \in M(F), k \in K$ for the similitude character λ of $\mathrm{GSp}(4)$.

By functoriality the $\mathbb{C}$-antilinear $M(F)$-equivariant map $\theta : \sigma \to \sigma^*$ induces a $\mathbb{C}$-antilinear map $I(\theta) : I(\sigma) \to I(\sigma^*)$. Now put $w = J$ and $\Theta = A(\sigma^*, J) \circ I(\theta)$. Then Θ is $\mathbb{C}$-antilinear so that

$$\Theta : I(\sigma) \xrightarrow{I(\theta)} I(\sigma^*) \xrightarrow{A(\sigma^*, J)} I(J(\sigma^*)).$$

As explained in example 6, $w = J$ represents the longest element of W such that $w(\sigma^*) = \omega_\sigma^{-1} \otimes \sigma = \omega_\Pi^{-1} \otimes \sigma$ (as identities). Therefore $I(J(\sigma^*)) = I(\sigma \otimes \omega_\Pi^{-1}) = I(\sigma) \otimes \omega_\Pi^{-1}$. Hence Θ can be viewed as an endomorphism of the representation space $I(\sigma)$. It commutes with the group action up to a character, and defines a (ω_Π, γ)-linear endomorphism

$$\Theta : I(\sigma) \longrightarrow I(\sigma).$$

From the integral formula for the intertwiner and $\theta\sigma^*\theta = \sigma$ we get $I(\theta)A(\sigma^*, J)I(\theta) = A(\sigma, J)$. Since $I(\theta^2) = I(\mathrm{id}) = \mathrm{id}$, we get for $\Theta^2 = \lambda_{\Pi(\sigma)} \cdot \mathrm{id}$ the expression

$$\Theta^2 = A(\sigma^*, J)I(\theta)A(\sigma^*, J)I(\theta) = A(\sigma^*, J)I(\theta^2)A(\sigma, J) = A(\sigma^*, J)A(\sigma, J).$$

Now $A(\sigma, J) = A(\sigma \otimes \omega_\Pi^{-1}, J) = A(J(\sigma^*), J)$. Since $J^2 = -id$, hence

$$\Theta^2 = A(\sigma^*, J)A(J(\sigma^*), J) = \omega_\Pi(-1) \cdot A(\sigma^*, J)A(J(\sigma^*), J^{-1}).$$

But

$$A(\sigma^*, J)A(J(\sigma^*), J^{-1}) = \gamma(0, \sigma^*, r_J, \overline{\psi})^{-1}\gamma(0, (\sigma^*)^\vee, r_J, \psi)^{-1} \cdot \mathrm{id}$$

is the Plancherel measure in the sense of Shahidi [**Sh**], p. 274. Now use $\gamma(0,\sigma,r_J,\overline{\psi}) = \gamma(0,\sigma^\vee,\widetilde{r}_J,\overline{\psi}) = \overline{\gamma(0,\sigma,\widetilde{r}_J,\psi)}$ and $\gamma(0,\sigma^\vee,r_J,\psi) = \gamma(0,\sigma,\widetilde{r}_J,\psi)$ ([**Sh**](7.8.1) for $s = 0$). Here $r \mapsto \widetilde{r}$ is involutive and related to the functional equation [**Sh**] prop. 7.8. Therefore

$$A(\sigma^*, J)A(J(\sigma^*), J^{-1}) = |\gamma(0,\sigma^*,\widetilde{r}_J,\psi)|^{-2} \cdot \mathrm{id}\,.$$

This positive real number is another expression for the Plancherel measure. So $\Theta^2 = \omega_\Pi(-1)\cdot \mathrm{id}$ holds after a suitable normalization. In other words

$$\varepsilon(\mathrm{GSp}(2n,F),\Pi_v,\omega_\Pi) = \omega_\Pi(-1).$$

This follows by remark 4, since $\Pi = I(\sigma)$ is unitary.

8. Example. — Suppose $k = \overline{\mathbb{Q}}_l$. Let $\rho_{\Pi_0,\lambda}$ be the four dimensional semisimple representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to a unitary irreducible automorphic representation Π_0 (as in theorem I). The identity $\Pi_0 \cong \Pi_0^\vee \otimes \omega_{\Pi_0}$ and the identity $L_p(\Pi_{0,p}, s-w/2) = \det(1-\rho_{\Pi,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}$ imply

$$\rho_{\Pi,\lambda}^\vee \otimes (\omega_{\Pi_0}\cdot \mu_l^{-w}) \cong \rho_{\Pi,\lambda}.$$

Let us discuss, whether $\rho_{\Pi,\lambda}$ admits a nondegenerate equivariant symplectic pairing with multiplier $\omega = \omega_{\Pi_0}\mu_l^{-w}$. Decompose $\rho_{\Pi,\lambda} = \oplus m_i\rho_i$ into irreducible subrepresentations. Then either $\rho_i^\vee \otimes \omega \cong \rho_j$ with $\rho_j \not\cong \rho_i$. (For instance, if ρ_i is one dimensional. This follows from property (c) of Π formulated in section 2). The representation spanned by these subrepresentations ρ_i admits a symplectic pairing — for trivial reasons — as in remark 2 above. Then there is the case of two dimensional subrepresentations ρ_i, such that $\rho_i^\vee \otimes \omega \cong \rho_i$. These subrepresentations obviously admit a nondegenerate symplectic form with the multiplier ω. This leaves us with the remaining nontrivial case, where $\rho_{\Pi,\lambda}$ itself is irreducible.

9. Example. — Consider the special case of the group GSp(4) over $\mathbb{Q}$. Let $\mathcal{V}_\mu$ be a coefficient system for a Siegel modular threefold attached to the discrete series of weight (k_1,k_2) of $\mathrm{GSp}(4,\mathbb{R})$. On the cuspidal part of the third cohomology group $H^3_P(M,\mathcal{V}_\mu)$ with $k = \overline{\mathbb{Q}}_l$ we already have defined the modified cup-pairing $\eta,\eta' \mapsto \mathrm{tr}(\eta\cup\eta'\cup\omega_0^{-1})$. It is a nondegenerate equivariant pairing $(W_{\Pi_f}\otimes\Pi_f)\otimes(W_{\Pi_f}\otimes\Pi_f) \to \omega_\Pi\mu_l^{-3}\otimes\omega_\Pi$ of parity $-(-1)^{k_1+k_2}$. Notice $\Pi = \Pi_0 \otimes \|\cdot\|^{-\frac{c}{2}}$ for unitary Π_0. Hence the modified cup-pairing defines an ε-symmetric isomorphism

$$\psi : (W_{\Pi_f}\otimes\Pi_f) \longrightarrow (W_{\Pi_f}\otimes\Pi_f)^\vee \otimes (\omega_\Pi\mu_l^{-3}\otimes\omega_{\Pi_f})$$

of parity $-(-1)^{k_1+k_2}$. As explained in example 8 we can reduce to the following case

Assumption. — Suppose W_{Π_f} is an isotypic multiple of an irreducible representation of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and Π is neither CAP nor a weak endoscopic lift.

Since Π_f is irreducible and $\Pi_f \cong \Pi_f^\vee \otimes \omega_{\Pi_f}$ is essentially unique, its parity is well defined. Hence by remark 2 the modified cup-pairing is induced from an ε-symmetric isomorphism

$$\psi : W_{\Pi_f} \longrightarrow (W_{\Pi_f})^\vee \otimes (\omega_\Pi \mu_l^{-3})$$

such that $\varepsilon(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), W_{\Pi_f}, \omega_\Pi \mu_l^{-3}, \psi) \cdot \varepsilon(G(\mathbb{A}_f), \Pi_f, \omega_{\Pi_f}) = -(-1)^{k_1+k_2}$ holds. Therefore remark 2 and example 5 imply

D.1. Lemma. — *The representation of the Galois group* $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q})$ *on* W_{Π_f} *preserves a nondegenerate bilinear form with multiplier* $\omega_\Pi \mu_l^{-3}$ *and parity* $-\varepsilon(\mathrm{GSp}(4,\mathbb{A}), \Pi, \omega_\Pi)$. *The following statements are equivalent:*

(1) $\varepsilon(\mathrm{GSp}(4,\mathbb{A}), \Pi, \omega_\Pi) = 1$

(2) $\varepsilon(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), W_{\Pi_f}, \omega_{\Pi_f} \mu_l^{-3}, \psi) = -1$.

In the rest of this section we proof assertion (1) and also compute each local parity $\varepsilon(\mathrm{GSp}(\mathbb{Q}_v), \Pi_v, \omega_{\Pi_v})$. *In fact we show* $\varepsilon(\mathrm{GSp}(\mathbb{Q}_v), \Pi_v, \omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$. *Hence assertion (1) follows from the product formula* $\prod_v \omega_\Pi(-1) = 1$. *Before that we first also give a global argument.*

We have $V_{\Pi_f} \otimes \cong W_{\Pi_f} \otimes_C \Pi_f$. Fix a nontrivial antilinear (ω_0, γ)-automorphism θ_f of $\Pi_{0,f}$. Suitably normalized it satisfies $\theta_f^2 = \varepsilon(\mathrm{GSp}(4,\mathbb{A}_f), \Pi_f, \omega_{\Pi_f}) \in \{\pm 1\}$ by remark 1. There is a notion of tensor product for γ-linear homomorphisms $\phi : V \to W$, $\phi' : V' \to W'$ such that $(\phi \otimes_\mathbb{C} \phi')(v \otimes_\mathbb{C} v') = \phi(v) \otimes_\mathbb{C} \phi'(v')$. This is a well-defined γ-linear homomorphism from $V \otimes_\mathbb{C} V'$ to $W \otimes_\mathbb{C} W'$. By Schur's lemma we obtain an isomorphism between the $\mathbb{C}$-vector space of (ω_0, γ)-endomorphisms θ_∞ of V_{Π_f} and the γ-linear endomorphisms θ_Π of W_{Π_f}, such that $\theta_\infty = \theta_\Pi \otimes \theta_f$. This is not a ring homomorphism unless $\lambda_{\Pi_f} = 1$, since $\theta_\infty^2 = \theta_\Pi^2 \cdot \varepsilon_{\Pi_f}$. The induced morphism $\theta_\Pi : W_{\Pi_f} \to W_{\Pi_f}$ is $\mathbb{C}$-antilinear and satisfies $\theta_\Pi^2 = \lambda_{\Pi_f} \cdot \mathrm{id}$. We apply this for $\theta_\infty(\eta) = \overline{F_\infty^*(\eta)} \cup \omega_0$. This is a (ω_0, γ)-endomorphisms of V_{Π_f}. It is a $\mathbb{C}$-antilinear homomorphism, which maps $V_{\Pi_f}^{p,q}$ bijectively to itself such that $\theta_\infty^2 = \omega_{\Pi_\infty}(-1)$. Hence it induces a $\mathbb{C}$-antilinear automorphisms θ_Π of $W_{\Pi_f}^{p,q}$ so that $\theta_\Pi^2 = \varepsilon(\mathrm{GSp}(4,\mathbb{A}_f), \Pi_f, \omega_{\Pi_f}) \omega_{\Pi_\infty}(-1) \cdot \mathrm{id}$. Therefore

D.2. Lemma. — *The statements of lemma D.1 are equivalent to any of the following two assertions:*

(3) $\varepsilon(\mathrm{GSp}(4,\mathbb{A}_f), \Pi_f, \omega_{\Pi_f}) \omega_{\Pi_\infty}(-1) = 1$.

(4) $\theta_\Pi^2 = 1$ *for the antilinear operator* $\theta_\Pi : W_{\Pi_f}^{pq} \to W_{\Pi_f}^{pq}$.

Proof. — The equivalence of (3) and (4) has been shown above. For the equivalence of (1) and (2) and (3) is clear by example 5. □

Proof of theorem IV. — Since any antilinear nontrivial endomorphism of $\mathbb{C}$ has positive square, we get $\theta_\Pi^2 = 1$ if $\dim(W_{\Pi_f}^{pq}) = 1$ holds for at least one choice of q, p. So if $m(\Pi_\infty \Pi_f) = 1$ has multiplicity 1 for one choice of Π_∞ in the L-packet, then property

(4) does hold. So all the equivalent properties (1)-(4) hold. Of course property (2) implies theorem IV. □

D.3. Lemma. — *Let* Π_v *be an irreducible admissible representations* $\mathrm{GSp}(4, F_v)$ *for a local nonarchimedian field* F_v *of characteristic zero. Then the parity for the central character* ω_{Π_v} *is determined by*

$$\varepsilon(\mathrm{GSp}(4,F), \Pi_v, \omega_{\Pi_v}) = \omega_{\Pi_v}(-1).$$

Proof. — Assume Π_v is cuspidal. Then Π_v can be globally embedded into a cuspidal irreducible automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_F)$. For such Π there exists $T = T' \in M_{2,2}(F)$, so that $\det(T) \neq 0$ and a $\phi \in \Pi$, such that $\int_{N(F)\setminus N(\mathbb{A}_F)} \phi(ng)\overline{\psi}_T(n)dn \neq 0$. See [**S**]. Here $P = MN$ is the Siegel parabolic of upper triangular block matrices. Attached to T is the quadratic algebra K/F defined by the discriminant of T. $K^x(\mathbb{A}_F) = \mathbb{A}_K^*$ stabilizes the character ψ_T. Since Π is cuspidal the integrals $\int_{K^*\setminus\mathbb{A}_K^*} \int_{N(F)\setminus N(\mathbb{A}_F)} \phi(yng)(\nu\otimes\psi_T)(yn)dy^*dn < \infty$ are well defined by a well known Sobolev type argument. Furthermore, for suitable choice of ϕ, there exists a ν for which one of these integrals is not zero. ν can be chosen to be unitary, since then $\nu(\sigma(y)y) = \omega_{\Pi_v}(\sigma(y)y)$. This defines a nontrivial global generalized Whittaker model of Π in the sense of [**PS**]. It induces a local nontrivial generalized Whittaker model of Π_v. Hence $\varepsilon(\Pi_v, \omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$ by example 4.

Since for $\mathrm{GSp}(4, F_v)$ an irreducible representation Π_v in the discrete series is generic, if it is not cuspidal, the claim follows from example 3 above.

If Π_v is tempered, but not in the discrete series, then Π_v is unitary induced from a discrete series representation σ_v of the Levi subgroup of a proper parabolic subgroup. Since the Levi subgroups are build from groups $\mathrm{Gl}(1)$ and $\mathrm{Gl}(2)$, such representations σ_v are necessarily generic. For $\mathrm{GSp}(4, F_v)$ properly induced representations of unitary discrete series representations are irreducible (hence again generic [**Sh**] and covered by example 3) except for the case, where the parabolic is the Klingen parabolic subgroup Q. Then these induced representations are reducible with two constituents called $Wp_+(\rho)$ and $Wp_-(\rho)$. Both $Wp_\pm(\rho)$ are limits of discrete series representations. For cuspidal ρ they were found by Waldspurger. For special representations ρ they are also denoted T_+, T_- in [**W**]. In the notation of *loc. cit.* they are the two constituents of $1 \times \rho$, where ρ is an irreducible discrete series representation of $\mathrm{Gl}(2, F_v)$ and it is shown, that they occur as local components of weak endoscopic lifts. The multiplicity statement of hyp. A(6) implies, that these representations occur as local constituents Π_v of suitably chosen global cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_F)$ for some number field F. Hence we can argue as above in the case of cuspidal representations Π_v.

Via the Langlands classification the case of an arbitrary irreducible admissible representations finally is reduced to the tempered case (example 6). □

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