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NEWTON POLYGONS AND p-DIVISIBLE GROUPS: A CONJECTURE BY GROTHENDIECK

by

Frans Oort

Abstract. — In my talk in 2000 I discussed a conjecture in 1970 by Grothendieck concerning deformations of p-divisible groups; a proof of this conjecture give access to finding properties of Newton polygon strata in the moduli spaces of polarized abelian varieties in positive characteristic.

***Résumé* (Polygones de Newton et groupes p-divisibles: une conjecture de Grothendieck)**
En 1970 Grothendieck a formulé une conjecture concernant les déformations de groupes p-divisibles (groupes de Barsotti-Tate). Nous décrivons une démonstration de cette conjecture. Cela donne une information sur des strates définies par le polygone de Newton dans les espaces de modules des variétés abéliennes en caractéristique positive.

Introduction

0.1. We consider p-divisible groups (also called Barsotti-Tate groups) in characteristic p, abelian varieties, their deformations, and we draw some conclusions.

For a p-divisible group (in characteristic p) we can define its Newton polygon. This is invariant under isogeny. For an abelian variety the Newton polygon of its p-divisible group is "symmetric". We are interested in the strata defined by Newton polygons in local deformation spaces, or in the moduli space of polarized abelian varieties.

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Key words and phrases. — Abelian varieties, Newton polygons, p-divisible groups, Barsotti-Tate groups, moduli spaces, Grothendieck conjecture.

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0.2. Grothendieck showed that Newton polygons "go up" under specialization, see [**4**], page 149, see [**11**], Th. 2.3.1 on page 143; we obtain Newton polygon strata as closed subsets in the deformation space of a p-divisible group or in the moduli space of polarized abelian varieties in positive characteristic.

In 1970 Grothendieck conjectured the converse. In [**4**], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: *"...The wishful conjecture I have in mind now is the following: the necessary conditions* [...] *that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation* [...] *we want to know if every sequence of rational numbers satisfying* [...] *these numbers occur as the sequence of slopes of a fiber of G as some point of S."*

0.3. In this talk we study this conjecture by Grothendieck for p-divisible groups, for abelian varieties, for quasi-polarized p-divisible groups and for polarized abelian varieties. Then we draw conclusions for NP-strata. These results can be found in [**8, 21, 23**].

0.4. We give a proof of this conjecture by Grothendieck. This is done by combining various methods (below we explain the string of ideas leading to this proof); in various stages of the process we need quite different ideas and methods. Hence, in spirit, the proof of a straight statement is not uniform. We have not been able to unify these in one straightforward method. We wonder what Grothendieck would have substituted for our proof.

1. Notations

1.1. We fix some notations. All base fields will be of characteristic $p > 0$. The p-divisible group of an abelian variety X will be denoted by $X[p^\infty]$. We will use *covariant* Dieudonné modules.

We follow [**15**] by writing $G_{m,n}$ for the following p-divisible group (defined over $\mathbb{F}_p$, and considered over every field of characteristic p): this is a p-divisible group of dimension m, with Serre-dual of dimension n; here $m, n \in \mathbb{Z}_{\geqslant 0}$ are coprime integers; we have $G_{1,0} = \mathbb{G}_m[p^\infty]$, and we write $G_{0,1}$ for its Serre dual; for coprime $m, n \in \mathbb{Z}_{>0}$ the formal p-divisible group $G_{m,n}$ is given in the covariant Dieudonné module theory by

$$\mathbb{D}(G_{m,n}) = W[[F,V]]/W[[F,V]] \cdot (F^m - V^n)$$

(in this ring $W[[F,V]]$ we have the relations $FV = p = VF$ and for all $a \in W = W_\infty(K)$, where K is a perfect field, we have $Fa = a^\sigma F$ and $Va^\sigma = aV$; in case $K = \mathbb{F}_p$ this results in a commutative ring). We use $H_{m,n}$ as in [**9**], 5.3; this is a p-divisible group isogenous with $G_{m,n}$; it can be characterized by saying that moreover its endomorphism ring over $\overline{\mathbb{F}_p}$ is the maximal order in its endomorphism algebra.

We need some combinatorial notation concerning Newton polygons:

Throughout the paper we fix a prime number p. We apply notions as defined and used in [**22**], and in [**9**]. For a p-divisible group G, or an abelian variety X, over a field of positive characteristic we use its Newton polygon, abbreviated by NP, denoted by $\mathcal{N}(G)$, respectively $\mathcal{N}(X)$. For dimension d and height $h = d + c$ of G (respectively dimension $g = d = c$ of X) this is a lower convex polygon in $\mathbb{R} \times \mathbb{R}$ starting at $(0, 0)$ ending at (h, c) with integral break points, such that every slope is non-negative and at most equal to one. *We write $\beta \prec \gamma$ if every point of γ is on or below β* (the locus defined by γ contains the one defined by β). For further details we refer to [**22**]. For example, the Newton polygon $\mathcal{N}(G_{m,n})$ consists of $m + n$ slopes equal to $n/(m + n)$. We see that we use the notion of slope as the "V-slope" on p-divisible groups, which amounts to using the "F-slope" on covariant Dieudonné modules.

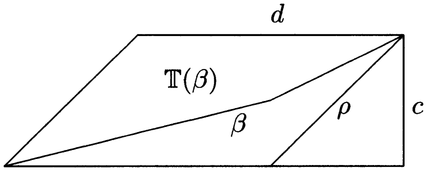
1.2. We use the following notation: we fix integers $h \geqslant d \geqslant 0$, and we write $c := h-d$. We consider Newton polygons ending at (h, c). For a point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we write $(x, y) \prec \gamma$ for the property "*the point (x, y) is on or above the Newton polygon γ*". For a Newton polygon β we write:

$$\mathbb{T}(\beta) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < c,\ y < x,\ (x, y) \prec \beta\},$$

and we define

$$\dim(\beta) := \#(\mathbb{T}(\beta)).$$

Note that for the "ordinary" Newton polygon $\rho := d \cdot (1, 0) + c \cdot (0, 1)$ the set of points $\mathbb{T} = \mathbb{T}(\rho)$ is a parallelogram; this explains our notation. Note that $\#(\mathbb{T}(\rho)) = d \cdot c$.



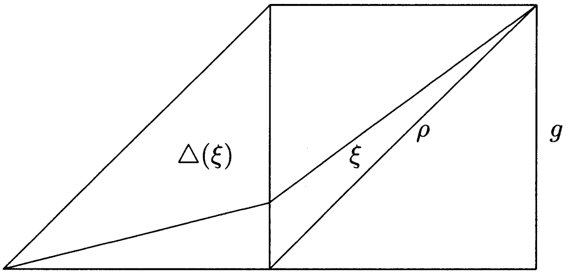
1.3. We fix an integer g. For every *symmetric* Newton polygon ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leqslant g,\ (x, y) \prec \xi\},$$

and we write

$$\mathrm{sdim}(\xi) := \#(\triangle(\xi)).$$

For the ordinary symmetric Newton polygon $\rho = g \cdot ((1, 0) + (0, 1))$ indeed $\triangle = \triangle(\rho)$ is a triangle; this explains our notation. But you can rightfully complain that the "triangle" $\triangle(\xi)$ in general is not a triangle. Note that $\#(\triangle(\rho)) = g(g + 1)/2$.



1.4. A theorem by Grothendieck and Katz, see [**12**], 2.3.2, says that for any family $\mathcal{G} \to S$ of p-divisible groups (in characteristic p) and for any Newton polygon γ there is a unique closed set $W \subset S$ containing all points s at which the fiber has a Newton polygon equal to or lying above γ:

$$s \in W \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathcal{N}(\mathcal{G}_s) \prec \gamma.$$

This set will be denoted by

$$\mathcal{W}_\gamma(\mathcal{G} \to S) \subset S.$$

In case of symmetric Newton polygons we write

$$\mathcal{W}_\gamma(\mathcal{A}_g \otimes \mathbb{F}_p) =: W_\gamma$$

for the Newton polygon stratum given in the moduli space of polarized abelian varieties in characteristic p. We will study this mainly inside $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, the moduli space of *principally* polarized abelian varieties in characteristic p.

1.5. We study formal abelian schemes, and formal p-divisible groups over formal schemes, and we study abelian schemes and p-divisible groups. Without further comments we use the following ideas.

Formal p-divisible groups. — As finite group schemes are "algebraizable", the same holds for certain limits; if $\mathcal{G} \to \mathrm{Spf}(A)$ is a formal p-divisible group, it comes from a p-divisible group over $\mathrm{Spec}(A)$, see [**6**], 2.4.4. We use the passage from formal p-divisible groups over $\mathrm{Spf}(A)$ to p-divisible groups over $\mathrm{Spec}(A)$ without further comments (here A is a complete local ring).

Serre-Tate theory. — Suppose given an abelian variety X_0 over a field K, and its p-divisible group $G_0 := X_0[p^\infty]$. A theorem by Serre and Tate gives an equivalence between formal deformations of (polarized) abelian schemes and the corresponding (quasi-polarized) p-divisible groups, see [**12**], Th. 1.2.1: any formal deformation of G_0 induces uniquely a formal deformation of X_0.

Formal abelian schemes. — In general a formal abelian scheme $\mathcal{A} \to \mathrm{Spf}(A)$ is not algebraizable. However polarized abelian schemes can algebraized: by the Chow-Grothendieck algebraization method for polarized formal schemes ("formal GAGA"), see [**5**], III^1.5.4, it follows that from a polarized formal abelian scheme $(\mathcal{X}, \mu) \to \mathrm{Spf}(A)$ we obtain an actual polarized abelian scheme over $\mathrm{Spec}(A)$.

1.6. Suppose $\mathcal{X} \to S$ is a scheme over an integral scheme S. Let $\eta \in S$ be its generic point, and let $0 \in S$ be a closed point. In this situation we will say that "$\mathcal{X}_\eta$ *specializes to* $\mathcal{X}_0$", and we say that "$\mathcal{X}_0$ *deforms to* $\mathcal{X}_\eta$" sometimes without specifying the base scheme S and the family $\mathcal{X} \to S$.

1.7. Displays. — Given a Dieudonné module M of a p-divisible group, and a W-base for the W-free module, the map $F : M \to M$ is given by a matrix, called a display. Mumford showed that deformations of certain p-divisible groups can be given by writing out a display over a more general base ring. What we need is contained in [**30**], [**29**]; also see [**17**], [**18**]. Below we construct deformations of local-local p-divisible groups. We shall write out the display, and use several times (without further mention) that this defines a deformation, see [**30**], Chapter 3, in particular his Corollary 3.16. Deformations of polarized formal p-divisible groups can be described with the help of displays, see [**18**], Section 1.

1.8. For an abelian variety X over a field K we write $f = f(X)$ for its p-rank, *i.e.* the integer such that $\mathrm{Hom}(\mu_p, X \otimes k) \cong (\mathbb{Z}/p)^f$, where $k = \overline{K}$.

For a group scheme N over a field $K \supset \mathbb{F}_p$ we write $a(N) = \dim_L \mathrm{Hom}(\alpha_p, N \otimes L)$, where $L \supset K$ is a perfect field containing K.

Note that there exist examples in which

$$\dim_K(\mathrm{Hom}(\alpha_p, G)) < \dim_L(\mathrm{Hom}(\alpha_p, G_L)).$$

However, if $a(G) = 1$, then $\dim_K(\mathrm{Hom}(\alpha_p, G)) = 1 = \dim_L(\mathrm{Hom}(\alpha_p, G_L))$.

Note that $\mathrm{Hom}(\alpha_p, G) \neq 0$ iff the local-local part of G is non-trivial, *i.e.* iff G is not ordinary. Hence if we write $a(G) \leqslant 1$ we intend to say: either G is ordinary, or $a(G) = 1$.

We use the notation k for an algebraically closed field (of characteristic p).

2. Results: deformations of p-divisible groups

***2.1. Theorem* (conjectured by Grothendieck, Montreal 1970).** — *Let K be a field of characteristic p, and let G_0 be a p-divisible group over K. We write $\mathcal{N}(\mathcal{G}_0) =: \beta$ for its Newton polygon. Suppose given a Newton polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation G_η of G_0 such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*

We write $\mathrm{Def}(G_0)$ for the universal deformation space (in equal characteristic p) of a p-divisible group G_0.

***2.2. Theorem* (properties of Newton polygon strata).** — *Suppose given a p-divisible group G_0 over a field K. Let γ be a Newton polygon with $\gamma \succ \mathcal{N}(G_0) =: \beta$. Consider the closed formal subset $\mathcal{W}_\gamma(\mathrm{Def}(G_0)) =: V_\gamma \subset \mathrm{Def}(G_0)$. The dimension of every component of V_γ equals $\dim(\gamma) = \#(\mathbb{T}(\gamma))$ and generically on every component of V_γ the Newton polygon is γ and the a-number generically is at most one.*

(In fact on V_γ the a-number generically is equal to one iff $\gamma \neq \rho := d\cdot(1,0)+c\cdot(0,1)$.)

3. Results: deformations of polarized p-divisible groups and of abelian varieties

***3.1. Theorem* (the principally polarized analog of the conjecture by Grothendieck)**
Let K be a field of characteristic p, and let (G_0, λ_0) be a principally *quasi-polarized p-divisible group over K. We write $\mathcal{N}(G_0) = \beta$ for its Newton polygon. Suppose given a symmetric Newton polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation (G_η, λ) of (G_0, λ_0) such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*

3.2. Corollary. — *Let K be a field of characteristic p, and let (X_0, λ_0) be a* principally *polarized abelian variety over K. We write $\mathcal{N}(X_0) = \beta$ for its Newton polygon. Suppose given a symmetric Newton polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation $(\mathcal{X}_\eta, \lambda)$ of (X_0, λ_0) such that $\mathcal{N}(\mathcal{X}_\eta) = \gamma$.*

Indeed, using the Serre-Tate theorem we deduce this corollary from the previous theorem.

3.3. Theorem. — *Suppose given a* principally *quasi-polarized p-divisible group (G_0, λ_0) over a field K. Let γ be a* symmetric *Newton polygon with $\gamma \succ \mathcal{N}(G_0) =: \beta$. Consider the closed formal subset $\mathcal{W}_\gamma(\mathrm{Def}(G_0, \lambda_0)) =: V_\gamma \subset \mathrm{Def}(G_0, \lambda_0)$. The dimension of every component of V_γ equals* $\mathrm{sdim}(\gamma) = \#(\triangle(\gamma))$ *and generically on every component of V_γ the Newton polygon is γ and the a-number generically is at most one.*

***3.4. Theorem* (see [20]).** — *For every p, and g and every symmetric Newton polygon β we have:*

(a) *For every irreducible component W of $W_\beta := \mathcal{W}_\beta(\mathcal{A}) \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ we have*

$$\mathcal{N}(-, W \subset \mathcal{A}) = \beta \quad \textit{and} \quad a(-, W \subset \mathcal{A}) \leqslant 1,$$

i.e. generically on W the Newton polygon of $\mathcal{X} \to W$ equals β, and generically the a-number is at most one.

(b) *The dimension of every irreducible component W of W_β equals* $\mathrm{sdim}(\beta) = \#(\triangle(\beta))$.

***3.5. Corollary* (a conjecture by Manin, see [15], p. 76).** — *Suppose given a prime number p and a* symmetric *Newton polygon ξ. Then there exists an abelian variety X defined over $\overline{\mathbb{F}_p}$ with $\mathcal{N}(X) = \xi$.*

This was proved in the Honda-Serre-Tate theory (via reduction modulo p of a well-chosen CM abelian variety in characteristic zero, see [**28**], page 98). Here we give a proof which uses only geometry in characteristic p.

3.6. Remark. — Theorem 3.4 applied to the locus $\mathcal{S}_{g,1}$ of principally polarized supersingular abelian varieties says that this locus is pure of dimension $\triangle(\sigma)$; this number turns out to be equal to $\triangle(\sigma) = [g^2/4]$ (were [ ] indicates the integral part); moreover on every component of $\mathcal{S}_{g,1}$ the a-number generically equals one. These results where conjectured by T. Oda and the present author, see [**19**], pp. 615/616. These results where proved by T. Katsura and the present author for g=3, see [**10**], and by K.-Z. Li and the present author for all g; for results and references see [**14**]. The method described here, and given in the three publication mentioned in the introduction, provides a new, different proof for these results.

3.7. Remark. — In 3.1 and in 3.2 the condition that the (quasi-)polarization is principal is essential. In fact, in [**10**], 6.10.b, we find an example of a component V of dimension 3 of the moduli space of polarized abelian 3-folds, such that every point of V corresponds with a supersingular abelian variety; we see that for such a point $v = [(X_0, \lambda_0)]$ with $\mathcal{N}(X_0) = 3 \cdot (1,1)$, *i.e.* X_0 is supersingular, there is no deformation as polarized abelian variety to (X, λ) with $\mathcal{N}(X) = (2,1) + (1,2)$. In [**10**], 6.10.c we find an example where generically the a-number is not one on an irreducible component of a supersingular Newton polygon stratum. More examples can be found in [**14**], Section 10 and Section 12. In fact it seems that much more is true, that "many" counter-examples to the analog of Grothendieck's conjecture can be given in the non-principally polarized case, see 7.8.

4. Methods: deformations to $a \leqslant 1$

***4.1. Theorem* (the "Purity theorem").** — *If in a family of p-divisible groups (say, over an irreducible scheme) the Newton polygon jumps, then it already jumps in codimension one.*

See [**9**], Th. 4.1. This very non-trivial result will be one of the main tools.

4.2. Catalogues. Let us fix a prime number p, and coprime $m, n \in \mathbb{Z}_{>0}$. We try to "classify" all p-divisible groups isogenous with $G_{m,n}$.

In general there is no good theory of moduli spaces for p-divisible groups (and there are various ways to remedy this). We use the (new) notion of a "catalogue". In our case this is a family $\mathcal{G} \to S$, *i.e.* a p-divisible group over some base scheme S, such that every $G \sim G_{m,n}$ defined over an algebraically closed field appears as at least one geometric fiber in $G \to S$. You can rightfully complain that this is a rather vague notion, that a catalogue is not unique (*e.g.* the pull back by a surjective morphism again is a catalogue), etc. However this notion has some advantages:

4.3. Theorem. — *Suppose given p, m, n as above. There exists a catalogue $\mathcal{G} \to T$ over $\mathbb{F}_p$ for the collection of p-divisible groups isogenous with $G_{m,n}$ such that T is geometrically irreducible.*

See [**9**], Theorem 5.11.

4.4. Theorem. — *Suppose G_0 is a p-divisible group; there exists a deformation to G_η such that*

$$\mathcal{N}(G_0) = \mathcal{N}(G_\eta) \quad \textit{and} \quad a(G_\eta) \leqslant 1.$$

4.5. We sketch a proof of 4.3, using 4.1, see [**9**]. We write $H = H_{m,n}$. We write $r := (m-1)(n-1)/2$. We see that for every $G \sim G_{m,n}$ there exists an isogeny $\varphi : H \to G$ of degree exactly $\deg(\varphi) = p^r$, see [**9**], 5.8. We construct $\mathcal{G} \to T$ as the representing object of isogenies $\varphi : H \times S \to G/S$ of this degree (it is easy to see that such a functor is representable).

Using this definition we see that the formal completion at $[(G_0, \varphi)] = s \in T$ embeds in $\mathrm{Def}(G_0)$, *i.e.* $T^{\wedge s} \hookrightarrow \mathrm{Def}(G_0)$. Furthermore we compute the longest chain of Newton polygons between $\mathcal{N}(G_{m,n})$ and the ordinary one: this equals $mn - r$ (an easy combinatorial fact). From these two properties, using 4.1, we deduce: *every component of T has dimension at least r.*

We make a stratification of T (using combinatorial data, such a thing like "semi-modules"). We show (using explicit equations) that every stratum is geometrically irreducible, and that there is one stratum, characterized by $a(G) = 1$, of dimension r, and *that all other strata have dimension less than r.* These considerations do not contain deep arguments, but the proofs are rather lengthy and complicated.

From these two aspects the proof follows: any component of T on which generically we would have $a > 1$ would have dimension strictly less than r, which contradicts "Purity". Hence the locus where $a = 1$ is dense in T, and we see that T is geometrically irreducible.

4.6. We sketch a proof of 4.4, see [**24**]. By 4.3 we conclude this deformation property 4.4 for iso-simple groups. Then we study groups filtered by iso-simple subfactors, and deformation theory of such objects. By the previous result we can achieve a deformation where all iso-simple subfactors are deformed within the isogeny class to $a \leqslant 1$. Then we write down an explicit deformation ("making extensions between iso-simple subfactors non-trivial") in order to achieve $a(G_\eta) \leqslant 1$, see [**24**], Section 2 for details.

4.7. Remark. — This method of catalogues for p-divisible groups works fine for simple groups. However the use of "catalogues" for non-isoclinic groups does not seem to give what we want; it is even not clear that nice catalogues exist in general. Note that we took isogenies of the form $\varphi : H \times S \to G/S$; however over a global base scheme monodromy groups need not be trivial, and this obstructs the existence of one catalogue which works in all cases (to be considered in further publications).

5. Methods: Cayley-Hamilton

This section is taken entirely from [**22**]. In general it is difficult to read off from a description of a p-divisible group (*e.g.* by its Dieudonné module) its Newton polygon. However in the particular case that its a-number is at most one this can be done. This we describe in this section. The marvel is a new idea which produces for a given element in a given Dieudonné module a polynomial (in constants and in F) which annihilates this element (but, in general, is does not annihilate other elements of the Dieudonné module). This idea for constructing this polynomial comes from the elementary theorem in linear algebra: every endomorphism of a vector spaces is annihilated by its characteristic polynomial. As we work in our case with an operator which does not commute with constants things are not that elementary. The method we propose works for $a(G_0) = 1$, but it breaks down in an essential way in other cases.

***5.1. Theorem* (of Cayley-Hamilton type)**. — *Let G_0 be a p-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$ with $a(G_0) \leqslant 1$. In $\mathcal{D} = \mathrm{Def}(G_0)$ there exists a coordinate system $\{t_j \mid j \in \mathbb{T}(\rho)\}$ and an isomorphism $\mathcal{D} \cong \mathrm{Spf}(k[[t_j \mid j \in \mathbb{T}(\rho)]])$ such that for any $\gamma \succ \mathcal{N}(G_0)$ we have*

$$\mathcal{W}_\gamma(\mathcal{D}) = \mathrm{Spf}(R_\gamma), \quad \textit{with } R_\gamma := k[[t_j \mid j \in \mathbb{T}(\gamma)]] = k[[t_j \mid j \in \mathbb{T}(\rho)]]/(t_j \mid j \notin \mathbb{T}(\gamma)).$$

5.2. Corollary. — *Let G_0 be a p-divisible group over a field K with $a(G_0) \leqslant 1$. In* $\mathrm{Def}(G_0)$ *every Newton polygon $\gamma \succ \mathcal{N}(G_0)$ is realized.*

5.3. These methods allow us to give a *proof for the Grothendieck conjecture*. In fact, starting with G_0 we use 4.4 in order to obtain a deformation to a p-divisible group with the same Newton polygon and with $a \leqslant 1$. For that group the method 5.1 of Cayley-Hamilton type can be applied, which shows that it can be deformed to a p-divisible group with a given lower Newton polygon. Combination of these two specializations shows that the Grothendieck conjecture 2.1 is proven. □

6. Polarized abelian varieties and quasi-polarized p-divisible groups

6.1. Methods described in the previous two sections for p-divisible groups also work (in almost the same way) for *principally* quasi-polarized p-divisible groups and for principally polarized abelian varieties. In fact we have the following tools:

6.2. Theorem. — *Let (G_0, λ_0) be a principally quasi-polarized p-divisible group over a field K, and let ξ be a symmetric Newton polygon, $\xi \succ \mathcal{N}(G_0)$. There is a deformation (G_η, λ_η) of (G_0, λ_0) such that $\xi = \mathcal{N}(G_\eta)$ and $a(G_\eta) \leqslant 1$.*

***6.3. Theorem* (of Cayley-Hamilton type).** — *Let (G_0, λ) be a principally quasi-polarized p-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$ with $a(G_0) \leqslant 1$. In $\mathcal{D} := \mathrm{Def}(G_0, \lambda)$ there exists a coordinate system $\{t_j \mid j \in \triangle(\rho)\}$ and an isomorphism*

$$\mathcal{D} \cong \mathrm{Spf}(k[[t_j \mid j \in \triangle(\rho)]])$$

such that for any symmetric $\xi \succ \mathcal{N}(X_0)$ we have

$$\mathcal{W}_\xi(\mathcal{D}) = \mathrm{Spf}(R_\xi), \quad \textit{with } R_\xi := k[[t_j \mid j \in \triangle(\xi)]] = k[[t_j \mid j \in \triangle(\rho)]]/(t_j \mid j \notin \triangle(\xi)).$$

6.4. By the Serre-Tate theorem these results imply the analogous statements for principally polarized abelian varieties. This provides proofs for 3.1, 3.3, 3.2, and 3.4.

6.5. Remark. — The conjecture by Manin that *every symmetric Newton polygon appears as the Newton polygon of an abelian variety*, see [**15**], page 76, see 3.5 follows from the polarized version of the Cayley-Hamilton method, see 6.3 (and we do not need the much deeper results of Section 4). In Section 5 of [**22**] this is described. Here are the essentials of that proof. We observe that the Manin conjecture holds for the supersingular Newton polygon $\sigma = g \cdot (1, 1)$ (here is the first algebraization fact). Then we observe that there exists a principally polarized supersingular abelian variety of given dimension g; this follows from [**14**], (4.9), but that difficult result is not necessary to prove this rather easy result, see [**22**], Section 4. Then, by 6.3 we show that its quasi-polarized formal group can be deformed to achieve a given symmetric Newton polygon. Then, by the Serre-Tate theorem, and by the Chow-Grothendieck algebraization (here is the second algebraization fact) we conclude the same for deformations of principally polarized abelian schemes starting from the supersingular σ, arriving at a a given symmetric ξ, thus proving the Manin conjecture.

6.6. Remark. — In many cases points of $W_\beta(a > 1)$ are singular points of W_β, and (if enough level structure is considered) the open set $W_\beta(a \leqslant 1) \subset W_\beta$ consists of non-singular points. Hence what we are doing in 4.4 is: move from a point in W_β to the regular interior. Then 5.1 tells us that the NP-strata around such a point are nested as coordinate hyperspaces, and we see that in the neighborhood of such a point every lower Newton polygon does appear (and we derive dimension statements). This explains our strategy: deformation theory in a singular point of the problem is difficult in general (and we could only progress via "Purity-catalogues", and not via deformation theory directly); then we arrive at regular points of the strata (in our case this is ensured by $a = 1$), and a fairly general argument (of Cayley-Hamilton type) finishes the proof. An analogous remark holds for the proof of the Grothendieck conjecture 2.1 and of 2.2 via 4.4 and 5.1.

7. Some questions and conjectures

7.1. For every Newton polygon β (and every g and every p) we obtain $W_\beta \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. For $\beta = \sigma$, the *supersingular* Newton polygon, this locus has "many" components (for $p >> 0$; in fact this number is a class number, asymptotically going to ∞ with $p \to \infty$).

Conjecture. — *Given p, g, and $\beta \neq \sigma$ we conjecture that the locus W_β is* geometrically irreducible.

7.2. We consider complete subvarieties of moduli spaces. It is known that for any field K, and any *complete* subvariety $W \subset \mathcal{A}_g \otimes K$, the dimension of W is at most $(g(g+1)/2) - g$, see [**3**], Coroll. 2.7 on page 70. We wonder is this maximum ever achieved? If yes, in which cases?

Conjecture. — *Let $g \geqslant 3$. Suppose W is a complete subvariety $W \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of dimension equal to $(g(g+1)/2) - g$* (the maximal possible dimension for complete subvarieties). *We expect that under these conditions W is equal to the locus V_0 of principally polarized abelian varieties with p-rank equal to zero.* (This locus is complete and has the right dimension.)

If this is true, we have a proof for:

7.3. Conjecture. — *Let $g \geqslant 3$. Let $W \subset \mathcal{A}_g \otimes \mathbb{C}$ be a* complete *subvariety. We expect that under these conditions:*

$$\dim(W) < (g(g+1)/2) - g.$$

Hecke orbits are dense in $\mathcal{A}_g \otimes \mathbb{C}$. Chai proved the same for Hecke orbits of *ordinary* polarized abelian varieties in positive characteristic, see [**1**]. In his case only ℓ-power isogenies need to be considered for one prime $\ell \neq p$.

7.4. Conjecture. — *Fix a polarized abelian variety $[(X,\lambda)] = x \in \mathcal{A}_g \otimes_{\mathbb{Z}} \mathbb{F}_p$. Consider the Hecke orbit of x. We conjecture that this Hecke orbit is everywhere dense in the Newton polygon stratum $W_{\mathcal{N}(X)}$.*

This will be studied in [**2**].

***7.5. Conjecture* (Foliations, see [25]).** — *We expect that the following facts to be true. For every Newton polygon β there should exist integers $i(\beta)$, $c(\beta), \in \mathbb{Z}_{\geqslant 0}$, and for every point $[(X,\lambda)] = x \in \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ with $\mathcal{N}(X) = \beta$ there should exist a closed subset $x \in \mathcal{I}(x) = \mathcal{I}_\beta(x) \subset W_\beta \subset \mathcal{A}$, contained in W_β^0, and a closed subset $x \in \mathcal{C}(x) = \mathcal{C}_\beta(x) \subset W_\beta^0 \subset \mathcal{A}$ in the open Newton polygon stratum W_β^0 such that:*

– $\dim(\mathcal{I}(x)) = i(\beta)$ *and* $\dim(\mathcal{C}(x)) = c(\beta)$.

– For every geometric point $[(Z, \zeta)] = z \in \mathcal{C}(x)(k)$ there is an isomorphism $(Z[p^\infty], \zeta) \cong (X[p^\infty], \lambda)$. All irreducible components of the locally closed set $\mathcal{C}(x)$ contain x, and it is the maximal closed set with this and the property just mentioned.

– For every geometric point $[(Y, \mu)] = y \in \mathcal{I}(x)$ there is a Hecke-correspondence using only iterates of α_p-isogenies relating $[(X, \lambda)]$ and $[(Y, \mu)]$. All irreducible components of the closed set $\mathcal{I}(x)$ contain x, and it is the maximal closed set with this and the property just mentioned.

– The dimensions are complementary: $i(\beta) + c(\beta) = \mathrm{sdim}(\beta)$, and locally at x their intersection is zero dimensional.

– For $\beta \precneqq \gamma$ we have $i(\beta) \geqslant i(\gamma)$ and $c(\beta) < c(\gamma)$.

– If moreover $a(X) \leqslant 1$, the (locally) closed sets $\mathcal{I}(x)$ and $\mathcal{C}(x)$ are regular at $x \in \mathcal{A}$, intersect transversally at x, and together their tangent spaces span the tangent space of $x \in W_\beta$.

– Examples:
for the supersingular locus we have $i(\sigma) = \mathrm{sdim}(\sigma) = [g^2/4]$ and $c(\sigma) = 0$;
for the ordinary locus we have $i(\rho) = 0$, and $c(\rho) = \mathrm{sdim}(\sigma) = (g(g+1))/2$;
for the case the p-rank equals one, i.e. $\beta = g \cdot (1,0) + (1,1) + g \cdot (0,1)$ we have $i(\beta) = 0$, and $c(\beta) = \mathrm{sdim}(\sigma) = ((g(g+1))/2) - 1$.
We have: p-rank $f(\beta) < g - 1$ iff $i(\beta) > 0$.
We have: $\beta \neq \sigma$ iff $c(\beta) > 0$.

– There is an easy combinatorial argument by which the numbers $i(\beta)$ and $c(\beta)$ can be read off from the Newton polygon diagram of β.

The sets $\mathcal{I}(x)$ will be called "isogeny leaves", and the $\mathcal{C}(x)$ will be called "central leaves".

7.6. Conjecture. — *Let ℓ be a prime number different from p, and $[(X, \lambda)] = x \in \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. The closure of the Hecke-ℓ-orbit of $[(X, \lambda)] = x$ inside $W^0_{\mathcal{N}(X)} \cap \mathcal{A}$ equals $\mathcal{C}(x)$.*

If this conjecture is true, then it follows that the conjecture 7.4 is true.

7.7. In general $G[p]$ does not determine a p-divisible group G. But in some cases it does. Let β be a symmetric Newton polygon. For a pair of relatively prime integers (m, n) we have defined in [**9**], Section 5 a p-divisible group $H_{m,n}$; it is characterized by: $H_{m,n} \sim G_{m,n}$, and for an algebraically closed field $k \supset \mathbb{F}_p$, the ring $\mathrm{End}(H_{m,n} \otimes k)$ is a maximal order in $\mathrm{End}^0(G_{m,n} \otimes k)$. We define H_β to be the direct sum of all $H_{m,n}$ ranging over all slopes of β. We expect:

Conjecture. — *Suppose G is a p-divisible group over an algebraically closed field k, such that $G[p] \cong H_\beta[p]$; then (?) we should conclude $G \cong H_\beta$.*

Note that in the special cases $\beta = \rho$ (the ordinary case), and $\beta = \sigma$ (supersingular) this conjecture is true; the conjectural statement above seems the natural generalization of this. Special cases have been proved.

***7.8. Conjecture* (Newton polygon strata, the non-principally polarized case)**

Let ξ be a symmetric Newton polygon and consider all possible polarized abelian varieties, where the polarization need not be principal. This gives a stratum $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$. Let $f = f(\xi)$ be the p-rank of ξ, i.e. this Newton polygon has exactly f slopes equal to zero. We expect: under these conditions, there is an irreducible component

$$W \subset \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \textit{with } \dim(W) = ((g(g+1)/2) - (g-f),$$

i.e. we expect that there is a component of every Newton polygon stratum which is a whole component of its p-rank stratum.

If this is the case, we see that there are "many" pairs of polarized abelian variety (X, λ) and a Newton polygon $\gamma \succ \mathcal{N}(X)$ such that there exist no deformation of (X, λ) to a polarized abelian variety with Newton polygon equal to γ, namely consider $\beta \prec \gamma$ with $\beta \neq \gamma$ and $f(\beta) = f(\gamma)$.

7.9. Postscript, November 2004. — After my talk in 2000, several of the conjectures above were proved. Here is a survey of relevant information which I know now.

Conjecture 7.1 has been proved by C-L. Chai and F. Oort. Details will appear in [**2**].

Conjecture 7.2 seems still unproven. However, Conjecture 7.3 has been proved by S. Keel and L. Sadun; see [**13**].

All statements in 7.5 have been established and published, see [**25**].

Conjecture 7.4 and conjecture 7.6 have been proved by C-L. Chai and F. Oort; details will appear in [**2**].

Conjecture 7.7 has been established; see [**27**]; also see [**26**].

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