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GALOIS REPRESENTATIONS, DIFFERENTIAL EQUATIONS, AND q-DIFFERENCE EQUATIONS : SKETCH OF A p-ADIC UNIFICATION

by

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Abstract. — This is a broad introduction to the following, more technical, paper [**AdV**]. We explain how [**AdV**] relates to two major themes of J.-P. Ramis' work, which eventually become unified in the p-adic world.

Résumé **(Représentations galoisiennes, équations différentielles et aux q-différences: esquisse d'une unification p-adique)**

Ce texte est une introduction développée à l'article suivant, plus technique [**AdV**]. Nous expliquons comment [**AdV**] est lié à deux thèmes majeurs de l'œuvre de J.-P. Ramis, et comment ceux-ci trouvent leur unification en passant au monde p-adique.

Introduction

Two remarkable analogies haved played an important role in Jean-Pierre Ramis' work:

– the analogy between linear complex differential equations and coverings in characteristic p (reported in D. Bertrand's contribution to this volume),
– the analogy between linear differential equations and q-difference equations (reported in J. Sauloy's contribution).

Our aim is to explain the analogs of these analogies in the p-adic world. We will see that once transposed into that context, these analogies become much more precise, and eventually lead to some equivalences of categories!

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1. A mysterious analogy: linear complex differential equations and coverings in characteristic p, tame and wild

1.1. A dictionary. — This analogy grew out of discussions between J.-P. Ramis and M. Raynaud during the "Nuit de la Musique 1993"[1]. Let us recall it in the form of a "dictionary":

Differential side	Characteristic-p side
$X = \overline{X} \setminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \setminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module / X	unramified Galois covering of X
singular point (in S)	branch point (in S)
regular singular point	tame branch point, *i.e.*, the ramification index at s is prime to p
irregular singular point	wild branch point
local differential Galois group at $s \in S$	inertia group at $s \in S$
(global) differential Galois group G (a linear alg. group / $\mathbb{C}$)	covering group G (a finite group)
torus in G	*p-Sylow subgroup of G*
$L(G)$: normal subgroup generated by all tori	$p(G)$: normal subgroup generated by all p-Sylow's
monodromy map $\mu : \pi_1(X) \to G/L(G)$	monodromy map $\mu : \pi_1^{(p')}(X) \to G/p(G)$
μ has Zariski-dense image (Ramis condition for the existence of a diff. module on X with diff. Galois group G, all singularities $s \in S$ being regular but one)	μ is surjective (Harbater condition for the existence of an unramified G-covering of X, all branch points $s \in S$ being tame but one).

Comment. — In the right-hand column, $\overline{\mathbf{F}}_p$ denotes a fixed algebraic closure of the field $\mathbf{F}_p$ with p-elements, and $\pi_1^{(p')}(X)$ denotes the profinite group which classifies unramified coverings of X of degree prime to p, *i.e.*, the prime-to-p quotient of Grothendieck's algebraic fundamental group $\pi_1(X)$ of X. According to Grothendieck,

[1] Older sources, in the ℓ-adic context, will be evoked in the next section.

$\pi_1^{(p')}(X)$ is a free prime-to-p profinite group on $2g+|S|-1$ generators (g denotes the genus of $\overline{X}$ and S is assumed to be non-empty).(2)

1.2. ℓ-adic linearized variant ($\ell \neq p$). — There is a somewhat older and more standard version of this dictionary (*cf. e.g.* the end of [**K**]) in which objects in the right-hand column are replaced by more linear ones (in fact $\mathbb{Z}_\ell$-linear(3) ones, for some fixed (but arbitrary) prime number $\ell \neq p$). It consists essentially in considering at once the whole tower of unramified coverings of X of degree a power of ℓ. In that way, finite groups are replaced, in the right-hand column, by ℓ-adic Lie groups, or even by algebraic groups over $\mathbb{Q}_\ell$ (by taking a suitable algebraic envelope).

Differential side	Characteristic-p side
$X=\overline{X}\smallsetminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X=\overline{X}\smallsetminus S$ affine curve / $k\subset\overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module M on X	lisse ℓ-adic sheaf $\mathcal{L}$ on X (ℓ-adic continuous representation of $\pi_1(X)$)
differential Galois group (an algebraic group / $\mathbb{C}$)	monodromy group (image of $\pi_1(X)$ or its Zariski closure, an algebraic group / $\mathbb{Q}_\ell$)
local differential Galois group	image of inertia group $\mathcal{I}$ (or its Zariski closure)
de Rham cohomology groups $H^i_{\mathrm{dR}}(X,M)$	étale cohomology groups $H^i_{\mathrm{\acute{e}t}}(X,\mathcal{L})$
$\chi(M)=\sum(-1)^i\dim H^i_{\mathrm{dR}}(X,M)$	$\chi(\mathcal{L})=\sum(-1)^i\dim H^i_{\mathrm{\acute{e}t}}(X,\mathcal{L})$
Deligne-Malgrange irregularity $\mathrm{irr}(M,s)$ *at* s	*Swan conductor* $\mathrm{sw}(M,s)$ *at* $s\in S$
Deligne's formula for $\chi(M)$ in terms of $\mathrm{rk}\,M$ and irregularities	Grothendieck's formula for $\chi(\mathcal{L})$ in terms of $\mathrm{rk}\,M$ and Swan conductors.

(2)Referee's remark. Earlier presentations of the Ramis-Raynaud dictionary can be found in M. van der Put's Bourbaki talk: Recent work on differential Galois theory (Exposé 849, Astérisque 252 (1998), 341-367), as well as in van der Put and Singer's book *Galois Theory of Linear Differential Equations*, Springer-Verlag (2003).

(3)Recall that the ring of ℓ-adic integers $\mathbb{Z}_\ell$ is the limit of the system $\cdots\to\mathbb{Z}/\ell^{n+1}\mathbb{Z}\to\mathbb{Z}/\ell^n\mathbb{Z}\to\cdots\mathbb{Z}/\ell\mathbb{Z}=\mathbf{F}_\ell$, so that any ℓ-adic integer can be expressed as a series $\sum_0^\infty a_n\ell^n$ where $a_n\in\{0,1,\ldots,\ell-1\}$. The field of fractions of $\mathbb{Z}_\ell$ is $\mathbb{Q}_\ell=\mathbb{Z}_\ell[\frac{1}{\ell}]$. In the sequel, we denote by $\overline{\mathbb{Q}}_\ell$ a fixed algebraic closure of $\mathbb{Q}_\ell$.

1.3. The ℓ-adic local monodromy theorem ($\ell \neq p$). — Let us recall the structure of the absolute Galois groups which play a role in the "characteristic-p side". We now assume that $k = \mathbf{F}_{p^n} \subset \overline{k} = \overline{\mathbf{F}}_p$ is the field with p^n elements. Then,

$$G_k := \mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}} = \prod_{\ell'} \mathbb{Z}_{\ell'} \quad \text{and}$$

$G_{k((x))} := \mathrm{Gal}\left(k((x))^{\mathrm{sep}}/k((x))\right)$ can be unscrewed via two exact sequences:

$$1 \longrightarrow \mathcal{I} \longrightarrow G_{k((x))} \longrightarrow G_k \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I} \longrightarrow \mathbb{Z}_\ell \times \prod_{\ell' \neq p,\ell} \mathbb{Z}_{\ell'} \longrightarrow 1$$

where $\mathcal{I} = G_{\overline{k}((x))}$ is the *inertia group*, and $\mathcal{P}$ is a pro-p-group called the *wild inertia group*.

This reflects the fact that in contrast to the char. 0 case, the algebraic closure of $\overline{k}((x))$ contains many more elements than just Puiseux series. For instance, roots z of the Artin-Schreier equation $z^{-p} - z^{-1} = x^{-1}$ cannot be expressed as Puiseux series.

Correspondingly, one has a tower of Galois extensions

$$k((x)) \subset \overline{k}((x)) \overset{\text{tame}}{\subset} \bigcup_{p \nmid n} \overline{k}((x^{1/n})) \overset{\text{wild}}{\subset} (k((x)))^{\mathrm{sep}},$$

with respective Galois groups $G_k, \mathcal{I}/\mathcal{P}$ and $\mathcal{P}$.

***Theorem 1.1* (Grothendieck [G]).** — *Every ℓ-adic representation of $G_{k((x))}$ is* quasi-unipotent, *i.e., a suitable open subgroup of $\mathcal{I}$ acts (through its quotient in $\mathbb{Z}_\ell$) by unipotent matrices.*

This can also be formulated, in the "Tannakian vein", as an equivalence of $\otimes$-categories

$$Rep_{\overline{\mathbb{Q}}_\ell}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{continuous } \overline{\mathbb{Q}}_\ell\text{-reps. of } \mathcal{I} \text{ which extend to reps. of } G_F\}$$

where $\mathcal{I}$ appears in the left-hand side as a constant group-scheme (and representations are understood in the scheme-theoretic sense), and in the right-hand side as a profinite topological group.

2. The p-adic analog of this analogy. An equivalence of categories.

2.1. Some motivation. Frobenius and overconvergence.— At least two aspects of the above dictionary 1.2 are unsatisfactory: the arbitraryness of the auxiliary prime number ℓ, and the very different natures of the cohomologies occurring in the left-hand (De Rham) and right-hand (étale) columns.

Both drawbacks would disappear, and the analogy would become much closer, if one could replace ℓ by p, and étale cohomology by some appropriate cohomology of De Rham type.

It turns out that this is indeed possible, provided one lifts the geometric situation from characteristic p to characteristic 0 (p-adic lifting), and replaces ℓ-adic sheaves by some analytic differential modules. The basic ideas here are due to Dwork[(4)].

i) The relevant "lifting" of X is constructed as follows: one fixes a p-adic field K (a finite extension of $\mathbb{Q}_p$) with residue field $k = \mathbf{F}_{p^n}$, and one fixes a smooth complete curve $\overline{\mathcal{X}}/K$ whose reduction modulo p is our given complete curve $\overline{X}/k$. Removing the open disks of radius 1 in $\overline{\mathcal{X}}$ which reduce mod. p to the points in S, one gets a p-adic analytic curve $\mathcal{X}$ which lifts X.

As was pointed out by Dwork, one should actually remove "infinitesimally more": the relevant space is the limit $\mathcal{X}^\dagger$ of $\overline{\mathcal{X}}$ deprived from disks of radius $1-\varepsilon$ around the singularities, when $\varepsilon \to 1$ ($\mathcal{X}^\dagger$ is a pro-ringed space[(5)]).

ii) According to Dwork again, the relevant p-adic differential modules should have two features:

1) *Frobenius structure*: after change of variable $x \mapsto x^{p^n}$, the new differential module is isomorphic to the old one,

2) *Overconvergence*: the differential module (and its Frobenius structure) should be also defined in some annulus inside each singular disk; in other words, should be defined over $\mathcal{X}^\dagger$.

Examples

a) $\overline{X} = \mathbb{P}^1_k$, $S = \{0\}$, $\mathcal{X} =$ outer unit disk $|x| \geqslant 1$. Let π be Dwork's constant, *i.e.*, a fixed root of the equation $\pi^{p-1} = -p$ in $\overline{\mathbb{Q}}_p$. Then the differential equation $y' = -\pi/x^2 y$ has the required properties (overconvergence of Frobenius means that $y(x^p)/y(x) = e^{(\pi/x^p)-(\pi/x)}$ has a p-adic radius of convergence > 1 in $1/x$).

b) Let us consider the differential module $x^{-\alpha}e^{1/x}\mathcal{O}$ (endowed with the derivation where xd/dx) where $\mathcal{O}$ is some ring of analytic functions away from the origin.

Let us first consider the complex case. The corresponding "period" is

$$\int_\gamma x^{-\alpha}e^{1/x}\,\frac{dx}{x} = -\frac{2\pi i}{\Gamma(1-\alpha)} \sim \Gamma(\alpha).^{(6)}$$

According to Ramis' precise Gevrey theory, an optimal choice for $\mathcal{O}$ is

$$\mathbb{C}[[1/x]]_{-1,1^-} := \big\{ \textstyle\sum a_n x^{-n} \mid \exists\,\kappa > 0,\ \exists\, r \in\,]0,1[\,;\ |a_n| \leqslant \kappa r^n/n! \big\}.^{(7)}$$

(4) We refer to [**R**] for a nice introduction to this circle of ideas.

(5) Working with $\mathcal{X}$ instead of $\mathcal{X}^\dagger$ would provide unwanted infinite-dimensional cohomology spaces in general.

(6) Here π is the usual one! The symbol $\sim$ means equality up to some factor in $\overline{\mathbb{Q}}^*$. The chosen loop comes from $-\infty$ and returns to $-\infty$ after turning once counterclockwise around the origin.

(7) $(-1, 1^-)$ is a characteristic index for which $\dim H^1 = 1$.

Let us now turn to the p-adic case. Overconvergence is satisfied after rescaling $1/x \mapsto \pi/x$. However strange this condition may look at first sight, it is nothing else than a Gevrey condition: indeed, in the p-adic case,

$$\big\{ \sum a_n x^{-n} \mid \exists \kappa > 0,\ \exists r \in]0,1[;\ |a_n| \leqslant \kappa r^n/|n!| \big\}$$

is precisely the ring of *analytic functions* on a disk of radius $> |\pi|$, which gives the ring of analytic functions on $\mathcal{X}^\dagger$ after rescaling $1/x \mapsto \pi/x$. On the other hand, one can evaluate Frobenius, and it turns out that its eigenvalues in some appropriate sense are, up to some algebraic factor, special values $\sim \Gamma_p(\alpha)$ of Morita's p-adic gamma function.

This is a general phenomenon, and the new version of the dictionary, with p-adic right-hand column, now looks as follows:

Differential side	Characteristic-p side
$X = \overline{X} \smallsetminus S$ affine curve / $\mathbb{C}$	$X = \overline{X} \smallsetminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ overconvergent lifting $\mathcal{X}^\dagger$ of X over a p-adic field
differential module M on X	differential module $\mathcal{M}^\dagger$ on $\mathcal{X}^\dagger$ (admitting a Frobenius structure)
$H^i_{\mathrm{dR}}(X, M)$	$H^i_{\mathrm{dR}}(X^\dagger, \mathcal{M}^\dagger)$
periods	*eigenvalues of Frobenius.*

Remark. — At the referee's suggestion, let us mention that there exists a completely different approach to the Ramis-Raynaud analogies, also involving p-adic differential equations, namely the theory of iterative differential Galois groups developped by H. Matzat and M. van der Put ([**MvP1, MvP2**]). This variant of differential Galois theory applies to function fields of characteristic p, and relies on the notion of an iterative derivation (Hasse, F.-K. Schmidt). The differential Galois groups attached to iterative differential modules are linear algebraic groups (not necessarily finite) over the field of (iterative) constants k, and one has a Galois correspondence. In special cases, Matzat and van der Put establish an iterative differential analog of the Abhyankar conjecture.

Iterative differential modules can be lifted to global p-adic differential equations of a very special kind, but the relationship with the above theory remains unclear.

2.2. The p-adic local monodromy theorem (Crew's conjecture). — Let again K be a finite extension of $\mathbb{Q}_p$, with residue field $k = \mathbf{F}_{p^n}$.

The lifting process associates to X over the finite field k the p-adic pro-space $\mathcal{X}^\dagger$ over K. In this process, localization around a point $s \in S$ becomes localization on an "infinitely thin annulus".

One is thus led to consider the so-called *Robba ring*, *i.e.*, the ring of K-analytic functions[8] on arbitrarily thin annuli $\mathcal{A}_{]1-\varepsilon,1[}$: $1-\varepsilon < |x| < 1$ (x denotes a local coordinate in the singular unit disk above s).

$$\mathcal{R} = \mathcal{R}_K = \bigcup_{\varepsilon} \{K\text{-analytic functions on } \mathcal{A}_{]1-\varepsilon,1[}\}.$$

The subring $\mathcal{E}^\dagger$ of *bounded* functions also plays an important role, because it turns out to be an *henselian field* with residue field $k((x))$; in other words, giving a finite separable extension of $k((x))$ amounts to giving a finite unramified extension $\mathcal{E}^{\dagger'}$ of $\mathcal{E}^\dagger$ (and this provides a finite etale extension $\mathcal{R}'$ of $\mathcal{R}$).

Theorem 2.1. — *Every differential module over $\mathcal{R}$ which admits a Frobenius structure is* quasi-unipotent, *i.e., has a basis of solutions in $\mathcal{R}'[\log x]$, where $\mathcal{R}'$ is the finite etale extension of $\mathcal{R}$ attached to a finite separable extension of $k((x))$.*

This is the p-adic (deeper) analog of Grothendieck's ℓ-adic local monodromy theorem 1.1, as the following "Tannakian formulation" puts in evidence:

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{differential modules}/\mathcal{R}_{\overline{\mathbb{Q}}_p} \text{ admitting a Frobenius structure}\}.$$

Example. — The 1-dimension representation of $\mathcal{I} = G_{\overline{k((x))}}$ attached to Artin-Scheier equation $z^{-p} - z^{-1} = x^{-1}$ corresponds to the differential module $y' = -\frac{\pi}{x^2} y$.

2.3. Hasse-Arf filtrations. — There are several approaches to Thm. 2.1. My own approach [**A2**] is based on the notion of *Hasse-Arf filtration* in a Tannakian category $\mathcal{T}$[9].

Data. — For every object $M \in \mathcal{T}$, a separated decreasing filtration $(F^{>\lambda}M)_{\lambda \geqslant 0}$ functorial and exact in M, satisfying

$$F^{>\lambda}(\mathbf{1}) = 0, \ F^{>\lambda}(M) = F^{>\lambda}(N) = 0 \quad \Longrightarrow \quad F^{>\lambda}(M \otimes N) = F^{>\lambda}(M^\vee) = 0.$$

One shows that every M then admits a canonical finite decomposition $M = \oplus \mathrm{gr}^{\lambda_i} M$, where $\mathrm{gr}^{\lambda_i} M$ is "of pure slope λ_i". This allows to attach to M its Newton polygon $NP(M)$ following the usual recipe. The "height" of $NP(M)$ is denoted by $h(M)$.

We say that the functorial filtration $(F^{>\lambda})$ is a *Hasse-Arf filtration* if $\forall M$, $h(M) \in \mathbb{N}$ (equivalently, if all Newton polygons have integral vertices).

Examples

1) The oldest example comes from arithmetic. $\mathcal{T} = Rep\, G_{k((x))}$[10]. The classical theory of ramification provides a non-decreasing sequence of normal subgroups $G^\mu \subset G_{k((x))}$ (the higher ramification groups), and one defines a filtration as follows:

$$F^{>\lambda}M = 0 \iff G^\mu \text{ acts trivially on } M \ \forall \mu > \lambda.$$

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[8] We refer to [**R'**] for a nice introduction to the theory of p-adic analytic functions.
[9] See also [**A4**] for a more detailed introduction to this topic.
[10] One could replace $k((x))$ by any local field.

In this context, $h(M)$ is called the Swan conductor of M and is denoted by $\mathrm{sw}(M)$. Its integrality is a classical theorem by Hasse and Arf.

2) $\mathcal{T}$ = {differential modules $M/\mathbb{C}[[x]]$}. Every object is endowed with the Turrittin-Levelt slope filtration. In this context, $h(M)$ is the irregularity $irr(M)$, and its integrality follows from the definition.

3) $\mathcal{T}$ = {differential modules $M/\mathcal{R}$ admitting a Frobenius structure}. Looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{]1-\varepsilon,1[}$, Christol and Mebkhout have defined the (analytic) *filtration by p-adic slopes* of M. In this context, $h(M)$ is called the p-adic irregularity of M and is denoted by $\mathrm{irr}_p(M)$. Christol and Mebkhout have shown that $\mathrm{irr}_p(M)$ is always an integer [**CM**].

It turns out that, despite their very different natures, examples 1) and 3) correspond to each other via the equivalence of categories 2.1:

***Theorem 2.2* (Matsuda, Tsuzuki; Crew [C]).** — *The canonical ⊗-equivalence:*

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\textit{differential modules } /\mathcal{R}_{\overline{\mathbb{Q}}_p} \textit{ admitting a Frobenius structure}\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p-adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\mathrm{sw} = \mathrm{irr}_p}$$

3. Another analogy: linear differential equations and q-difference equations; confluence

3.1. The q-world. — Let us provisionally abandon p in favor of q. The q-calculus has a long history (Euler, Gauss, Jacobi, Heine, ..., Ramanujan, ...[11]). It is based on the replacement of ordinary integers n by their q-analogs

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

The usual derivation d/dx is then replaced by the q-derivation

$$d_q : f(x) \longmapsto \frac{f(x) - f(qx)}{(1-q)x}$$

which sends x^n to $[n]_q x^{n-1}$ (and the q-exponential $e_q^x = \sum x^n/[n]_q^!$ to itself).

Differential equations are thus replaced by q-difference equations. The phenomenon of *confluence* arises when $q \to 1$: then $n_q \rightsquigarrow n$, $d_q \rightsquigarrow d/dx$, and q-difference equations tend to differential equations under appropriate convergence conditions. Conversely, for q close to 1, q-difference equations may be considered as deformations of differential equations.

(11) See some highlights in [**E**].

3.2. Non-commutative connections and q-deformations. — Let R be some "ring of functions" stable under the dilatation $\sigma_q : f(x) \mapsto f(qx)$.

Recall that a more intrinsic version of differential equations is provided by differential modules, or even better, by connections. Similarly, a more intrinsic version of q-differences is provided by *q-difference modules*: R-module M (projective of finite rank) + σ_q-linear automorphism.

This setting has one drawback: it does not allow to obtain the limit differential module in the case of confluence ($q \to 1$). However, one can also present differential modules as *connections*:

$$\nabla : M \longrightarrow \Omega^1_q \otimes M, \quad (\nabla(rm) = r\nabla(m) + dr \otimes m),$$

Ω^1_q = *non-commutative* bimodule of rank one: $f.\omega = \omega \cdot \sigma_q(f)$, $d : R \to \Omega^1_q$, $f \mapsto \omega \cdot d_q(f)$.

This gives rise to a unified theory of Galois differential groups in the differential and q-difference cases [**A1**], [**A3**], and a relevant setting for the algebraic study of confluence.

3.3. Analytic theory. — Here, for many reasons (*e.g.*, to avoid difficult problems of small divisors), one assumes $|q| \neq 1$.

The analytic theory of q-difference equations has been initiated by Adams, Birkhoff etc., and was revived by J.-P. Ramis in the early 90's. The analogy with differential equations is often straightforward at the formal/combinatorial level, but rather subtle at the analytic level, especially when wild phenomena or confluence are involved, *cf.* [**S**].

4. The p-adic analog of this analogy. Another equivalence of categories [AdV]

4.1. Frobenius structure. — We fix $n > 0$, and a prime p. Recall that a Frobenius structure on a differential module is an isomorphism between M and its "pull-back" by the change of variable $\phi : x \mapsto x^{p^n}$.(12)

In the q-difference case, one has the relation

$$\boxed{\sigma_q\, \phi = \phi\, \sigma_{q^{p^n}}},$$

so that the pull-back of a σ_q-module M is a priori a $\sigma_{q^{1/p^n}}$-module. In order to make a σ_q-module out of it, it suffices however to iterate p^n times the action of $\sigma_{q^{1/p^n}}$. We denote by $\phi^! M$ this new σ_q-module M. A Frobenius structure may then be defined to be an isomorphism between $\phi^! M$ and M.(13)

(12) Actually, one also has to twist the coefficients by some power of the so-called Frobenius automorphism of K, but we neglect this fact here.

(13) In [**AdV**], another notion of Frobenius structure is also considered.

4.2. p-adic confluence and canonical q-deformation. — We come back to the p-adic situation: K is p-adic field with residue field $k \subset \mathbf{F}_{p^n}$. We fix $q \in K$, $q \neq$ root of unity. We are interested in "confluence" (note that here $|q-1| < 1 \quad \to \quad |q| = 1$, in contrast to the usual postulate in the complex case). Actually, it simplifies matters to assume

$$\boxed{|1-q| < p^{-\frac{1}{p-1}}}.$$

Let M be a σ_q-module over the Robba ring $\mathcal{R} = \mathcal{R}_K$.

Theorem 4.1. — *There is a canonical "functor of confluence"*

$$\{q\text{-}difference\ modules\ /\mathcal{R}_{\overline{K}}\ admitting\ a\ Frobenius\ structure\}$$
$$\longrightarrow \{differential\ modules\ /\mathcal{R}_{\overline{K}}\ admitting\ a\ Frobenius\ structure\}$$

which is an equivalence of tannakian categories.

Its construction uses *quasi-unipotence (q-analog of Crew's conjecture), cf.* [**AdV**]. In fact for any q-difference module over $\mathcal{R}_K$ admitting a Frobenius structure, there is a canonical sequence of $q^{p^{in}}$-difference structures on the same underlying $\mathcal{R}$-module (with $i \to \infty$, so that $q^{p^{in}} \to 1$) which converges to a differential structure.

4.3. Another Hasse-Arf filtration?— This subsection is tentative. By looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{]1-\varepsilon,1[}$, it seems (not all details have been checked) that one can define, à la Christol and Mebkhout, a *filtration by p-adic slopes* on M, whence a notion of p-adic q-irregularity $q\text{-irr}_p(M)$, and that one has the following q-analog of 2.2:

Conjecture 4.2. — *The canonical $\otimes$-equivalence:*

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{q\text{-}difference\ modules\ /\mathcal{R}_{\overline{\mathbb{Q}}_p}\ admitting\ a\ Frobenius\ structure\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p-adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\mathrm{sw} = q\text{-irr}_p}.$$

Remark. — In this context, it is interesting to notice that the formal slope filtration for complex q-difference modules with $|q| \neq 1$ (*cf.* [**S**]) is *not* a Hasse-Arf filtration (in contrast both to the differential case and to the p-adic case), since negative slopes may occur, for instance. Once again, we see that the q-analogy is much tighter in the p-adic case.

References

[A1] Y. André – "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup.* **5** (2001), p. 1–55.

[A2] ______, "Filtrations de type Hasse-Arf et monodromie p-adique", *Invent. Math.* **148** (2002), no. 2, p. 285–317.

[A3] ______, "On Galois theory of q-deformations of differential equations", in *Deformations of differential equations and asymtotic analysis (Kyoto, 2002)* (Haraoka, ed.), RIMS report, vol. 1296, 2002, p. 35–46.

[A4] ______, "Singularities of p-adic differential equations; toward p-adic Stokes phenomena?", in *Differential equations and the Stokes phenomenon* (Braaksma, Immink, van der Put & Top, eds.), World scientific, 2003, p. 1–13.

[AdV] Y. André & L. Di Vizio – "q-difference equations and p-adic local monodromy", this volume.

[B] D. Bertrand – "Travaux de J.-P. Ramis sur les équations différentielles linéaires", this volume.

[CM] G. Christol & Z. Mebkhout – "Sur le théorème de l'indice des équations différentielles p-adiques III", *Ann. of Math.* **151** (2000), p. 385–457.

[C] R. Crew – "Canonical extensions, irregularities, and the Swan conductor", *Math. Ann.* **316** (2000), p. 19–37.

[E] T. Ernst – "The history of q-calculus and a new method", Ph.D. Thesis, University of Uppsala, 2001.

[G] A. Grothendieck – "Good reduction of abelian varieties", *Ann. of Math. (2)* **88** (1968), p. 492–517, appendix to J.P. Serre & J. Tate.

[K] N. Katz – "On the calculation of some differential Galois groups", *Invent. Math.* **87** (1987), p. 13–61.

[MvP2] B. Matzat & M. van der Put – "Constructive differential Galois theory", in *Galois Groups and Fundamental Groups* (L. Schneps, ed.), MSRI Publications, vol. 41, Cambridge University Press, 2003, p. 425–467.

[MvP1] ______, "Iterative differential equations and the Abhyankar conjecture", *J. reine angew. Math.* **257** (2003), p. 1–52.

[R] P. Robba – "Une introduction naïve aux cohomologies de Dwork", in *Introductions aux cohomologies p-adiques* (D. Barsky & P. Robba, eds.), Mém. Soc. math. France (N.S.), vol. 23, Société Mathématique de France, 1986, p. 61–105.

[R'] A. Robert – *A course in p-adic analysis*, Graduate Texts in Math., vol. 198, Springer-Verlag, New York, 2000.

[S] J. Sauloy – "Algebraic construction of the Stokes sheaf for irregular linear q-difference equations", this volume.

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