Astérisque

JACQUES SAULOY

Algebraic construction of the Stokes sheaf for irregular linear q-difference equations

Astérisque, tome 296 (2004), p. 227-251

<http://www.numdam.org/item?id=AST_2004__296__227_0>

Astérisque
296, 2004, p. 227–251

ALGEBRAIC CONSTRUCTION OF THE STOKES SHEAF FOR IRREGULAR LINEAR q-DIFFERENCE EQUATIONS

by

Jacques Sauloy

Je laisse aux nombreux avenirs (non à tous)
mon jardin aux sentiers qui bifurquent.
Jorge Luis Borges, *Fictions*

Abstract. — The local analytic classification of irregular linear q-difference equations has recently been obtained by J.-P. Ramis, J. Sauloy and C. Zhang. Their description involves a q-analog of the Stokes sheaf and theorems of Malgrange-Sibuya type and is based on a discrete summation process due to C. Zhang. We show here another road to some of these results by algebraic means and we describe the q-Gevrey devissage of the q-Stokes sheaf by holomorphic vector bundles over an elliptic curve.

***Résumé* (Construction algébrique du faisceau de Stokes pour les équations aux q-différences linéaires irrégulières)**

La classification analytique locale des équations aux q-différences linéaires irrégulières a été récemment réalisée par J.-P. Ramis, J. Sauloy et C. Zhang. Leur description fait intervenir un q-analogue du faisceau de Stokes et des théorèmes de type Malgrange-Sibuya et elle s'appuie sur la sommation discrète de C. Zhang. Nous montrons ici comment retrouver une partie de ces résultats par voie algébrique et nous décrivons le dévissage q-Gevrey du q-faisceau de Stokes par des fibrés vectoriels holomorphes sur une courbe elliptique.

1. Introduction and general conventions

1.1. Introduction. — This paper deals with Birkhoff's program of 1941 ([**3**], see also [**2**]) towards the local analytic classification of q-difference equations and some extensions stated by J.-P. Ramis in 1990 ([**13**]).

A full treatment of the Birkhoff program including the case of irregular q-difference equations is being given in [**16**]. The method used there closely follows the analytic procedure developed in the last decades by B. Malgrange, Y. Sibuya, J.-P. Ramis,... for the "classical" case, *i.e.*, the case of differential equations: adequate asymptotics,

2000 ***Mathematics Subject Classification.*** — Primary 39A13; Secondary 34M40, 32G34.
Key words and phrases. — q-difference equations, Stokes sheaf.

q-Stokes phenomenon, q-Stokes sheaves and theorems of Malgrange-Sibuya type; explicit cocycles are built using a *discrete summation process* due to C. Zhang ([**27**]) where the Jackson q-integral and theta functions are introduced in place of the Laplace integral and exponential kernels.

To get an idea of the classical theory for linear differential equations one should look at the survey [**25**] by V.S. Varadarajan, especially section 6, and to get some feeling of how the change of landscape from differential equations to q-difference equations operates, at the survey [**7**] by L. Di Vizio, J.-P. Ramis, J. Sauloy and C. Zhang.

The aim of this paper is to show how the harder analytic tools can, to some extent, be replaced by much simpler algebraic arguments. The problem under consideration being a transcendental one we necessarily keep using analytic arguments but in their most basic, "19th century style", features only. In particular, we avoid here using the discrete summation process.

Again our motivation is strongly pushed ahead by the classical model of which we recall three main steps: the *dévissage Gevrey* introduced by J.-P. Ramis ([**12**]) occured to be the fundamental tool for understanding the Stokes phenomenon. The underlying algebra was clarified by P. Deligne in [**4**], then put at work by D.G. Babbitt and V.S. Varadarajan in [**1**] (see also [**25**]) for moduli theoretic purposes. On the same basis, effective methods, a natural summation and galoisian properties were thoroughly explored by M. Loday-Richaud in [**9**].

Some specificities of our problem are due, on one hand, to the fact that the sheaves to be considered are quite similar to holomorphic vector bundles over an elliptic curve, whence the benefit of GAGA theorems, and, on another hand, to the existence for q-difference operators of an analytic factorisation without equivalent for differential operators. Such a factorisation originates in Birkhoff ([**3**]), where it was rather stated in terms of a triangular form of the system. It has been revived by C. Zhang ([**26**], [**10**]) in terms of factorisation and we will use it in its linear guise, as a filtration of q-difference modules ([**22**]).

In this paper, following the classical theory recalled above, we build a q-Gevrey filtration on the q-Stokes sheaf, thereby providing a q-analog of the Gevrey devissage in the classical case. This q-devissage jointly with a natural summation argument allows us to prove the q-analog of a Malgrange-Sibuya theorem (theorem 3 of [**25**]) in quite a direct and easy way; in particular, we avoid here the Newlander-Nirenberg structural theorem used in [**16**]. Our filtration is, in some way, easier to get than the classical one: indeed, due to the forementionned canonical filtration of q-difference modules, our systems admit a natural triangularisation which is independent of the choice of a Stokes direction and of the domination order of exponentials (here replaced by theta functions). Also, our filtration has a much nicer structure than the classical Gevrey filtration since the so-called elementary sheaves of the classical theory are here replaced by holomorphic vector bundles endowed with a very simple structure over an elliptic curve (they are tensor products of flat bundles by line bundles).

On the side of what this paper does not contain, there is neither a study of confluency when q goes to 1, nor any application to Galois theory. As for the former, we hope to extend the results in [**20**] to the irregular case, but this seems a difficult matter. Only partial results by C. Zhang are presently available, on significant examples. As for the latter, it is easier to obtain as a consequence of the present results that, under natural restrictions, "canonical Stokes operators are Galoisian" like in [**9**]. However, to give this statement its full meaning, we have to generalize the results of [**21**] and to associate vector bundles to arbitrary equations. This is a quite different mood that we will develop in a forthcoming paper ([**23**]; meanwhile, a survey is given in [**24**]). Here, we give some hints in remarks 3.11 and 4.5.

Also, let us point out that there has been little effort made towards systematisation and generalisation. The intent is to get as efficiently as possible to the striking specific features of q-difference theory. For instance, most of the results about morphisms between q-difference modules can be obtained by seing these morphisms as meromorphic solutions of other modules (internal Hom) and they can therefore be seen as resulting from more general statements. These facts, evenso quite often sorites, deserve to be written. In the same way, the many regularity properties of the homological equation $X(qz)A(z) - B(z)X(z) = Y(z)$ should retain some particular attention and be clarified in the language of functional analysis. They are implicitly or explicitly present in many places in the work of C. Zhang. Last, the q-Gevrey filtration should be translated in terms of factorisation of Stokes operators, like in [**9**].

Let us now describe the organization of the paper.

Notations and conventions are given in subsection 1.2.

Section 2 deals with the recent developments of the theory of q-difference equations and some improvements. In subsection 2.1, we recall the local classification of fuchsian systems by means of flat vector bundles as it can be found in [**21**] and its easy extension to the so-called "tamely irregular" q-difference modules. We then describe the filtration by the slopes ([**22**]). In subsections 2.2, we summarize results from [**16**] about the local analytic classification of irregular q-difference systems, based on the Stokes sheaf. The lemma 2.7 provides a needed improvement about Gevrey decay; proposition 2.8 and corollary 2.10 an improvement about polynomial normal forms.

In chapter 3, we first build our main tool, the algebraic summation process (theorem 3.7). Its application to the local classification is then developed in subsection 3.2. We state there and partially prove the second main result of this paper (theorem 3.18): a q-analog of the Malgrange-Sibuya theorem for the local analytic classification of linear differential equations.

Section 4 is devoted to studying the q-Gevrey filtration of the Stokes sheaf and proving the theorem 3.18. In subsection 4.1, we show how conditions of flatness (otherwise said, of q-Gevrey decay) of solutions near 0 translate algebraically and how to provide the devissage for the Stokes sheaf of a "tamely irregular" module. In subsection 4.2 we derive some cohomological consequences and we finish the proof of the theorem 3.18. Finally, in subsection 4.3, we sketch the Stokes sheaf of a general module.

The symbol $\square$ indicates the end of a proof or the absence of proof if considered straightforward. Theorems, propositions and lemmas considered as "prerequisites" and coming from the quoted references are not followed by the symbol $\square$.

Acknowledgements. — The present work is directly related with the paper [**16**], written in collaboration with Jean-Pierre Ramis and with Changgui Zhang. It has been a great pleasure to talk with them, confronting very different points of view and sharing a common excitement.

The epigraph at the beginning of this paper is intended to convey the happiness of wandering and daydreaming in Jean-Pierre Ramis' garden; and the overwhelming surprise of all its bifurcations. Like in Borges' story, pathes fork and then unite, the same landscapes are viewed from many points with renewed pleasure. This strong feeling of the unity of mathematics without any uniformity is typical of Jean-Pierre.

1.2. Notations and general conventions. — We fix once for all a complex number $q \in \mathbf{C}$ such that $|q| > 1$. We then define the automorphism σ_q on various rings, fields or spaces of functions by putting $\sigma_q f(z) = f(qz)$. This holds in particular for the field $\mathbf{C}(z)$ of complex rational functions, the ring $\mathbf{C}\{z\}$ of convergent power series and its field of fractions $\mathbf{C}(\{z\})$, the ring $\mathbf{C}[[z]]$ of formal power series and its field of fractions $\mathbf{C}((z))$, the ring $\mathcal{O}(\mathbf{C}^*, 0)$ of holomorphic germs and the field $\mathcal{M}(\mathbf{C}^*, 0)$ of meromorphic germs in the punctured neighborhood of 0, the ring $\mathcal{O}(\mathbf{C}^*)$ of holomorphic functions and the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions on $\mathbf{C}^*$; this also holds for all modules or spaces of vectors or matrices over these rings and fields.

For any such ring (resp. field) R, the σ_q-invariants elements make up the subring (resp. subfield) R^{σ_q} of constants. For instance, the field of constants of $\mathcal{M}(\mathbf{C}^*, 0)$ or that of $\mathcal{M}(\mathbf{C}^*)$ can be identified with a field of elliptic functions, the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the complex torus (or elliptic curve) $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. We shall use heavily the theta function of Jacobi defined by the following equality:

$$\theta_q(z) = \sum_{n \in \mathbf{Z}} q^{-n(n+1)/2} z^n.$$

This function is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-1; q]$, where we write $[a; q] = aq^{\mathbf{Z}}$, $(a \in \mathbf{C}^*)$. It satisfies the functional equation: $\sigma_q \theta_q = z\theta_q$. We shall also use its multiplicative translates $\theta_{q,c}(z) = \theta_q(z/c)$ (for $c \in \mathbf{C}^*$); the function $\theta_{q,c}$ is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-c; q]$ and satisfies the functional equation: $\sigma_q \theta_{q,c} = \frac{z}{c}\theta_{q,c}$.

As is customary for congruence classes, we shall write $\overline{a} = a \pmod{q^{\mathbf{Z}}}$ for the image of $a \in \mathbf{C}^*$ in the elliptic curve $\mathbf{E}_q$. This notation extends to a subset A of $\mathbf{C}^*$, so that $\overline{A}$ does *not* denote its topological closure. Then, for a divisor $D = \sum n_i[\alpha_i]$ over $\mathbf{E}_q$ (*i.e.*, the $n_i \in \mathbf{Z}$, the $\alpha_i \in \mathbf{E}_q$), we shall write $\mathrm{ev}_{\mathbf{E}_q}(D) = \sum n_i\alpha_i \in \mathbf{E}_q$ for its evaluation, computed with the group law on $\mathbf{E}_q$.

Let K denote any one of the forementioned fields of functions. Then, we write $\mathcal{D}_{q,K} = K\left\langle\sigma, \sigma^{-1}\right\rangle$ for the Öre algebra of non commutative Laurent polynomials characterized by the relation $\sigma \cdot f = \sigma_q(f) \cdot \sigma$. We now define the category of q-difference modules in three clearly equivalent ways:

$$\begin{aligned}
\mathrm{DiffMod}(K, \sigma_q) &= \{(E, \Phi) \mid E \text{ a } K\text{-vector space of finite rank,} \\
&\qquad\qquad \Phi : E \longrightarrow E \ \text{ a } \sigma_q\text{-linear map}\} \\
&= \{(K^n, \Phi_A) \mid A \in \mathrm{GL}_n(K), \Phi_A(X) = A^{-1}\sigma_q X\} \\
&= \text{ finite length left } \mathcal{D}_{q,K}\text{-modules.}
\end{aligned}$$

This is a $\mathbf{C}$-linear abelian rigid tensor category, hence a tannakian category. For basic facts and terminology about these, see [**21**], [**11**], [**6**], [**5**]. Last, we note that all objects in $\mathrm{DiffMod}(K, \sigma_q)$ have the form $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$.

2. Local analytic classification

2.1. Devissage of irregular equations ([21],[22])

Fuchsian and tamely irregular modules. For a q-difference module M over any of the fields $\mathbf{C}(z)$, $\mathbf{C}(\{z\})$, $\mathbf{C}((z))$, it is possible to define its Newton polygon at 0, or, equivalently, the slopes of M, which we write in descending order: $\mu_1 > \cdots > \mu_k \in \mathbf{Q}$, and their multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$. The module M is said to be pure of slope μ_1 if $k = 1$ and fuchsian if it is pure of slope 0. The latter condition is equivalent to M having the shape $M = (K^n, \Phi_A)$ with $A(0) \in \mathrm{GL}_n(\mathbf{C})$. There are also criteria of growth (or decay) of solutions near 0, see further below, in section 2.2, the subsection about flatness conditions.

Call $\mathcal{E}$ the category $DiffMod(\mathbf{C}(z), \sigma_q)$ of rational equations. Fuchsian modules at 0 and ∞ over $\mathbf{C}(z)$ make up a tannakian subcategory $\mathcal{E}_f$ of $\mathcal{E}$. In order to study them, one "localizes" these categories by extending the class of morphisms, precisely, by allowing morphisms defined over $\mathbf{C}(\{z\})$. This gives "thickened" categories $\mathcal{E}^{(0)}$ and $\mathcal{E}_f^{(0)}$. A classical lemma says that any fuchsian system is locally equivalent to one with constant coefficients. This suggests the introduction of the full subcategory $\mathcal{P}_f^{(0)}$ of $\mathcal{E}_f^{(0)}$ made up of "flat" objects, that is, the $(\mathbf{C}(z)^n, \Phi_A)$ with $A \in \mathrm{GL}_n(\mathbf{C})$. Thus, the inclusion of $\mathcal{P}_f^{(0)}$ into $\mathcal{E}_f^{(0)}$ is actually an isomorphism of tannakian categories.

To any $A \in \mathrm{GL}_n(\mathbf{C})$ one associates the holomorphic vector bundle F_A over $\mathbf{E}_q$ obtained by quotienting $\mathbf{C}^* \times \mathbf{C}^n$ by the equivalence relation $\sim_A$ generated by the relations $(z, X) \sim_A (qz, AX)$. This defines a functor from $\mathcal{P}_f^{(0)}$ to the category $\mathrm{Fib}_p(\mathbf{E}_q)$ of flat holomorphic vector bundles over $\mathbf{E}_q$. This is an equivalence of tannakian categories. Note that the classical lemma alluded above equally holds for any fuchsian q-difference module over $\mathbf{C}(\{z\})$ or over $\mathbf{C}((z))$, which implies that this local classification applies to $\mathrm{DiffMod}(\mathbf{C}(\{z\}), \sigma_q)$ and $\mathrm{DiffMod}(\mathbf{C}((z)), \sigma_q)$ as well. The galoisian aspects of this local correspondence and its global counterpart are detailed in [**21**].

A pure module of *integral* slope μ over $K = \mathbf{C}(\{z\})$ or $\mathbf{C}((z))$ has the shape $(K^n, \Phi_{z^{-\mu}A})$ with $A \in \mathrm{GL}_n(\mathbf{C})$. For such a module, the above construction of a vector bundle extends trivially, yielding the tensor product of a flat bundle by a line bundle. We shall call pure such a bundle.

Direct sums of pure modules play a special role in [**22**], [**16**] and in the present paper. We shall call them tamely irregular, in an intended analogy with tamely ramified extensions in algebraic number theory: for us, they are irregular objects without wild monodromy, as follows from [**16**]. The category of tamely irregular modules with integral slopes over $\mathbf{C}(\{z\})$ can, for the same reasons as above, be seen either as a subcategory of $\mathrm{DiffMod}(\mathbf{C}(\{z\}), \sigma_q)$ or of $\mathcal{E}^{(0)}$. We write it $\mathcal{E}_{\mathrm{mi},1}^{(0)}$ (1). It is generated (as a tannakian category) by the fuchsian modules and by the pure module $(\mathbf{C}(\{z\}), z^{-1}\sigma_q)$ of slope 1. We can thus associate to any such module a direct sum of pure modules, thereby defining a functor from $\mathcal{E}_{\mathrm{mi},1}^{(0)}$ to the category $\mathrm{Fib}(\mathbf{E}_q)$ of holomorphic vector bundles over $\mathbf{E}_q$. This functor is easily seen to be compatible with all linear operations (it is a functor of tannakian categories).

Filtration by the slopes. — The following theorem is proved in [**22**].

Theorem 2.1. — *Let the letter K stand for the field $\mathbf{C}(\{z\})$ (convergent case) or the field $\mathbf{C}((z))$ (formal case). In any case, any object M of* $\mathrm{DiffMod}(K, \sigma_q)$ *admits a unique filtration $(F^{\geqslant\mu}(M))_{\mu\in\mathbf{Q}}$ by subobjects such that each $F^{(\mu)}(M) = F^{\geqslant\mu}(M)/F^{>\mu}(M)$ is pure of slope μ. The $F^{(\mu)}$ are endofunctors of* $\mathrm{DiffMod}(K, \sigma_q)$ *and $\mathrm{gr} = \bigoplus F^{(\mu)}$ is a faithful exact $\mathbf{C}$-linear $\otimes$-compatible functor and a retraction of the inclusion of $\mathcal{E}_{\mathrm{mi}}^{(0)}$ into $\mathcal{E}^{(0)}$. In the formal case, gr is isomorphic to the identity functor.*

From now on, we only consider the full subcategory $\mathcal{E}_1^{(0)}$ of modules with integral slopes.

The notation $\mathcal{E}_1^{(0)}$ will be justified *a posteriori* by the fact that all its objects are locally equivalent to objects of $\mathcal{E}$ (existence of a normal polynomial form). This is an abelian tensor subcategory of $\mathcal{E}^{(0)}$ and the functor *gr* retracts $\mathcal{E}_1^{(0)}$ to $\mathcal{E}_{\mathrm{mi},1}^{(0)}$. We

(1) The subscript "mi" stands for "modérément irrégulier", the subscript 1 for restricting to slopes with denominator 1.

also introduce notational conventions which will be used all along this paper for a module M in $\mathcal{E}_1^{(0)}$ and its associated graded module $M_0 = \mathrm{gr}(M)$, an object of $\mathcal{E}_{\mathrm{mi},1}^{(0)}$.

The module M may be given the shape $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, with:

$$A = A_U \underset{\mathrm{def}}{=} \begin{pmatrix} z^{-\mu_1}A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & U_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{-\mu_k}A_k \end{pmatrix}, \tag{2.1.1}$$

where $\mu_1 > \cdots > \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ $(i = 1, \ldots, k)$ and

$$U = (U_{i,j})_{1 \leqslant i < j \leqslant k} \in \prod_{1 \leqslant i < j \leqslant k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})).$$

The associated graded module is then a direct sum $M_0 = P_1 \oplus \cdots \oplus P_k$, where, for $1 \leqslant i < j \leqslant k$, the module P_i is pure of rank r_i and slope μ_i and can be put into the form $P_i = (\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{-r_i}A_i})$. Therefore, one has $M_0 = (\mathbf{C}(\{z\})^n, \Phi_{A_0})$, where the matrix A_0 is block-diagonal:

$$A_0 = \begin{pmatrix} z^{-\mu_1}A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{-\mu_k}A_k \end{pmatrix}. \tag{2.1.2}$$

The set of analytic isoformal classes. — This section comes from [**16**]. The definitions here should be compared to those in [**25**], p. 29 or [**1**].

In $\mathrm{DiffMod}(\mathbf{C}((z)), \sigma_q)$, the canonical filtration of a module M is split; more precisely, the associated graded module $\mathrm{gr}(M)$ is the unique formal classifyer of M. The isoformal analytic classification is therefore the same as the isograded classification, whence the following definitions.

Definition 2.2. — Let $P_1, \ldots, P_k$ be pure modules with ranks $r_1, \ldots, r_k$ and with integral slopes $\mu_1 > \cdots > \mu_k$. The module $M_0 = P_1 \oplus \cdots \oplus P_k$ has rank $n = r_1 + \cdots + r_k$. We shall write $\mathcal{F}(M_0)$ for the set of equivalence classes of pairs (M, g) of a module M and an isomorphism $g : \mathrm{gr}(M) \to M_0$, where (M, g) is said to be equivalent to (M', g') if there exists a morphism $u : M \to M'$ such that $g = g' \circ \mathrm{gr}(u)$ (u is automatically an isomorphism).

We write $\mathfrak{G}$ for the algebraic subgroup of GL_n made up of matrices of the form

$$F = \begin{pmatrix} I_{r_1} & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & F_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & I_{r_k} \end{pmatrix}. \tag{2.2.1}$$

Its Lie algebra $\mathfrak{g}$ consists in matrices of the form

$$f = \begin{pmatrix} 0_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & f_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0_{r_k} \end{pmatrix}. \tag{2.2.2}$$

For F in $\mathfrak{G}$, we shall write $F[A] = (\sigma_q F)\, AF^{-1}$ for the result of the gauge transformation F on the matrix A.

We shall identify P_i with $(\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{-\mu_i} A_i})$, where $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. The datum of a pair (M, g) then amounts to that of a matrix A in the form 2.1.1. Two such matrices A, A' are equivalent iff there exists a matrix $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$ such that $F[A] = A'$.

Write $\mathfrak{G}^{A_0}(\mathbf{C}((z))) = \{F \in \mathfrak{G}(\mathbf{C}((z))) \mid F[A_0] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))\}$. The subgroup $\mathfrak{G}(\mathbf{C}(\{z\}))$ of $\mathfrak{G}(\mathbf{C}((z)))$ operates at left on the latter (by translation) and $\mathfrak{G}^{A_0}(\mathbf{C}((z)))$ is stable for that operation. The theory in the previous section entails:

$$\forall\, (U_{i,j})_{1 \leqslant i < j \leqslant k} \in \prod_{1 \leqslant i < j \leqslant k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})),\ \exists\,!\, \widehat{F} \in \mathfrak{G}(\mathbf{C}((z)))\ :\ \widehat{F}[A_0] = A_U.$$

This $\widehat{F}$ will be written $\widehat{F}(U)$. Its existence can also be proved by direct computation, solving by iteration the fixpoint equation of the z-adically contracting operator: $F \mapsto (A_U)^{-1} \left(\sigma_q \widehat{F}\right) A_0$. It follows that the unique formal gauge transformation of $\mathfrak{G}(\mathbf{C}((z)))$ taking A_U to A_V is $\widehat{F}(U, V) = \widehat{F}(V)\widehat{F}(U)^{-1}$. Besides, A_U is equivalent to A_V in the above sense if and only if $\widehat{F}(U, V) \in \mathfrak{G}(\mathbf{C}(\{z\}))$, or, equivalently, $\widehat{F}(V) \in \mathfrak{G}(\mathbf{C}(\{z\}))\widehat{F}(U)$. This translates into the following proposition.

Proposition 2.3. — *Sending A_U to $\widehat{F}(U)$ induces a one-to-one correspondence between $\mathcal{F}(M_0)$ and the left quotient $\mathfrak{G}(\mathbf{C}(\{z\}))\backslash\mathfrak{G}^{A_0}(\mathbf{C}((z)))$.*

One, thus, recognizes in the isoformal classification a classical problem of summation of divergent power series. In order to illustrate the possible strategies, we shall end this section by examining a specific example. We shall try, as far as possible, to mimic the methods and the terminology of the "classical" theory (Stokes operators for linear differential equations and summation in sectors along directions).

Example 2.4. — The module $M_u = (\mathbf{C}(\{z\})^2, \Phi_{A_u})$ corresponding to the matrix $A_u = \left(\begin{smallmatrix} 1 & u \\ 0 & z \end{smallmatrix}\right)$ is formally isomorphic to its associated graded module M_0. More precisely, there exists a formal gauge transformation F such that $F[A_0] = A_u$, that is, $F(qz)A_0(z) = A_u(z)F(z)$. If one moreover requires F to be compatible with the graduation, that is, to have the form $F = \left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$, then there is unicity of the formal series f, which must satisfy the functional equation

$$f(z) = -u(z) + zf(qz).$$

We call $\widehat{f}_u$ this unique formal solution (it can be computed by iterating the above fixpoint equation) and $\widehat{F}_u$ the corresponding formal gauge transformation. One checks that two such matrices A_u and A_v are analytically equivalent if and only if the formal power series $\widehat{f}_{u-v} = \widehat{f}_u - \widehat{f}_v$ is convergent. In this case (two slopes), the problem is additive.

For $u = 1$, the unique solution is

$$\widehat{f}_1 = -\sum_{n\geqslant 0} q^{n(n-1)/2} z^n,$$

the so-called Tschakaloff series (up to the sign). It is divergent and may be seen as a natural q-analog of the Euler series. Thus, A_1 is not equivalent to A_0.

In general, we apply the formal q-Borel-Ramis transform of level 1, defined by

$$\mathcal{B}_{q,1} \sum_n a_n z^n = \sum_n q^{-n(n-1)/2} a_n \xi^n.$$

It sends convergent series to series with an infinite radius of convergence. Our functional equation is transformed into

$$(1-\xi)\mathcal{B}_{q,1} f(\xi) = -\mathcal{B}_{q,1} u(\xi).$$

The existence of a convergent solution f has only one obstruction, the number $\nu = \mathcal{B}_{q,1}u(1)$. This number can therefore be considered as the unique analytic invariant of M_u within the formal class of M_0. It can also be considered as giving a normal form, since A_ν is the unique matrix in the analytic class of A_u such that $\nu \in \mathbf{C}$. It is a particular case of normal polynomial form (see further below).

The functional equation can also be solved by a variant of the method of "varying constants". We look for the solution in the form $g = \theta_{q,\lambda} f$. For convenience, we also write $v = \theta_{q,\lambda} u$, which is an element of $\mathcal{O}(\mathbf{C}^*, 0)$. We compare their Laurent series coefficientwise and we get

$$\forall\, n \in \mathbf{Z},\ (1-\lambda q^n) g_n = v_n.$$

If $\lambda \notin [1;q]$ (prohibited direction of summation), there is a unique solution $g \in \mathcal{O}(\mathbf{C}^*, 0)$ (it does converge where it should), thus a unique solution $f \in \mathcal{M}(\mathbf{C}^*, 0)$ such that $\theta_{q,\lambda} f$ has no poles in $\mathbf{C}^*$. We then get a unique solution $f_{\lambda,u}$ with (at most) simple poles over $[-\lambda;q]$: it is the summation of $\widehat{f}_u$ in the direction $\overline{\lambda} \in \mathbf{E}_q$ and its "sector" of validity is (the germ at 0) of $\mathbf{C}^* \smallsetminus [-\lambda;q]$, the preimage by the canonical projection $\mathbf{C}^* \to \mathbf{E}_q$ of the Zariski open set $\mathbf{E}_q \smallsetminus \{\overline{-\lambda}\}$.

There is another way of looking at this summation process, with a deeper analytical meaning. We can consider $\mathcal{B}_{q,1} f(\xi)$ as a meromorphic function ϕ over the ξ-plane and apply to it some q-analog of the Laplace transform. In our case, putting

$$\mathcal{L}^{\lambda}_{q,1} \phi(z) = \sum_{\xi \in [\lambda;q]} \frac{\phi(\xi)}{\theta_q(z/\xi)},$$

gives again $f_{\lambda,u}$. This *discrete summation* process is due to Changgui Zhang see ([**27**], also see [**7**]) and it is heavily used in [**16**]. In this work, we rather use the first more algebraic and more naive method.

2.2. Classification through the Stokes sheaf ([16],[15])

The Stokes sheaf and its Lie algebra. — First, we recall the relevant definitions about asymptotic expansions. The semigroup $\Sigma = q^{-\mathbf{N}}$ operates on $\mathbf{C}^*$ with quotient $\mathbf{E}_q$ (its *horizon*); in the classical setting, one would rather have an operation of the semigroup $\Sigma = e^{]-\infty,0]}$ with horizon the circle S^1 of directions. We consider as sectors the germs at 0 of invariant open subsets of $\mathbf{C}^*$. We introduce two sheaves of differential algebras over $\mathbf{C}^*$ by putting, for any sector U,

$$\mathcal{B}(U) = \{f \in \mathcal{O}(U) \mid f \text{ is bounded on all invariant relatively compact subset of } U\}$$
$$\mathcal{A}'(U) = \{f \in \mathcal{O}(U) \mid \exists\, \widehat{f} \in \mathbf{C}[[z]] \ : \ \forall n \in \mathbf{N},\ z^{-n}\big(f - S_{n-1}\widehat{f}\big) \in \mathcal{B}(U)\},$$

where, as usually, $S_{n-1}\widehat{f}$ stands for the truncation. For any sector U, we write $U_\infty = U/\Sigma$ for its horizon (an open subset of $\mathbf{E}_q$). We now define a sheaf of differential algebras over $\mathbf{E}_q$ by putting

$$\mathcal{A}(V) = \varinjlim \mathcal{A}'(U),$$

the direct limit being taken for the system of those open subsets U such that their horizon is $U_\infty = V$. There is a natural morphism from $\mathcal{A}$ to the constant sheaf with fibre $\mathbf{C}\{z\}$ over $\mathbf{E}_q$ and it is an epimorphism (q-analog of Borel-Ritt lemma). We call $\mathcal{A}_0$ its kernel, the sheaf of infinitely flat functions. For instance, it is easy to see that a solution of a Fuchsian equation divided by a product of theta functions is flat within its domain (more on this in the next subsection).

We, then, write $\Lambda_I = I_n + \mathrm{Mat}_n(\mathcal{A}_0)$ for the subsheaf of groups of $\mathrm{GL}_n(\mathcal{A})$ made up of matrices infinitely tangent to the identity and we put $\Lambda_I^{\mathfrak{G}} = \Lambda_I \cap \mathfrak{G}(\mathcal{A})$. This is a sheaf of matrices of the form 2.2.1 with all the $F_{i,j}$ flat. Last, for a module $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, we consider the subsheaf $\Lambda_I(M)$ of $\Lambda_I^{\mathfrak{G}}$ whose sections F satisfy the equality: $F[A] = A$ (automorphisms of M infinitely tangent to identity). This is the *Stokes sheaf* of the module M.

Note, for further use, that $\Lambda_I(M)$ is a sheaf of unipotent groups so that one can define algebraically the sheaf $\lambda_I(M)$ of their Lie algebras: we put $\lambda_I = \mathrm{Mat}_n(\mathcal{A}_0)$, $\lambda_I^{\mathfrak{g}} = \lambda_I \cap \mathfrak{g}(\mathcal{A})$ (see 2.2.2) and take as sections of $\lambda_I(M)$ those sections of $\lambda_I^{\mathfrak{g}}$ such that $(\sigma_q f)\, A = Af$. Obviously, f is a section of $\lambda_I(M)$ if and only if $I_n + f$ is a section of $\Lambda_I(M)$, or, equivalently, $\exp(f)$ is a section of $\Lambda_I(M)$. Indeed, the triangular form and the functional equations are easily checked, and the flatness properties stem from the well known fact that, for nilpotent matrices, f and $\exp(f) - I_n$ are polynomials in each other, without constant terms.

The q-analogs of Malgrange-Sibuya theorems. One can find in [**16**] the following q-analogs of classical theorems by Malgrange and Sibuya:

Theorem 2.5. *There are natural bijective mappings*

$$\mathfrak{G}(\mathbf{C}\{z\})\diagdown\mathfrak{G}^{A_0}(\mathbf{C}[[z]]) \longrightarrow \mathfrak{G}(\mathbf{C}(\{z\}))\diagdown\mathfrak{G}^{A_0}(\mathbf{C}((z))) \longrightarrow H^1(\mathbf{E}_q, \Lambda_I^{\mathfrak{G}}).$$

Actually, the following more general theorem is proven in *loc. cit.*, dealing with an arbitrary algebraic subgroup G of GL_n. Its proof relies on some heavy analysis (Newlander-Nirenberg theorem).

Theorem 2.6. *Let M_0 be as above. There are natural bijective mappings*

$$\mathcal{F}(M_0) \longrightarrow G(\mathbf{C}\{z\})\diagdown G^{A_0}(\mathbf{C}[[z]]) \longrightarrow G(\mathbf{C}(\{z\}))\diagdown G^{A_0}(\mathbf{C}((z))) \longrightarrow H^1(\mathbf{E}_q, \Lambda_I^G),$$

where $\Lambda_I^G = \Lambda_I \cap G(\mathcal{A})$.

The former theorem is deduced from the latter together with the existence of asymptotic solutions. One can explicitly build, by discrete resummation, privileged cocycles associated to a class in $\mathcal{F}(M_0)$ and to "Stokes directions". In the next chapter, I shall exhibit an algebraic variant of this construction. Morally, it is possible because the sheaf $\Lambda_I(M_0)$ is almost a vector bundle over the elliptic curve $\mathbf{E}_q$.

Flatness conditions. Details about the contents of this section can be found in [**17**] and [**16**]; see also the older references [**14**] and [**13**].

The above notion of flatness can be refined, introducing q-Gevrey levels. These may be characterized either in terms of growth (or decay) of functions near 0, or in terms of growth of coefficients of power series. We shall use here the following simple terminology and facts.

We start from a proper germ of $q^{-\mathbf{N}}$ invariant subset U of $(\mathbf{C}^*, 0)$. Then any solution of a fuchsian system that is holomorphic on U has polynomial growth at 0 (see for instance [**21**]); this is for instance true for a quotient of theta functions. We say that $f \in \mathcal{O}(U)$ has level of flatness $\geqslant t$ (where t is an integer) if, for one (hence any) theta function $\theta = \theta_{q,\lambda}$, the function $f|\theta|^t$ has polynomial growth near 0. We easily get the following implications.

Lemma 2.7

(i) *For $t > 0$, t-flatness implies flatness in the sense of asymptotics.*
(ii) *Solutions of pure systems of slope μ are μ-flat.*
(iii) *If a solution of a pure system of slope μ is t-flat with $t > \mu$, then it is 0.*

Normal polynomial forms. The computations will follow the same pattern as in [**16**], [**15**]. However, we shall need a slightly more general version afterwards (proposition 2.8).

We start with a computation with two slopes. Take integers $\mu > \mu'$, square invertible matrices $A \in \mathrm{GL}_r(\mathbf{C})$ and $A' \in \mathrm{GL}_{r'}(\mathbf{C})$. Just for this section, call $\mathcal{V}(r, r', \mu, \mu')$

the subspace of $\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$ spanned by matrices all of whose coefficients belong to $\sum_{\mu'\leqslant k<\mu}\mathbf{C}z^k$.

For $U\in\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$, write $B_U=\begin{pmatrix} z^{-\mu}A & U\\ 0 & z^{-\mu'}A'\end{pmatrix}$. Then, for any such U, there exists a unique pair (F,V) with $F\in\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$ and $V\in\mathcal{V}(r,r',\mu,\mu')$ such that the matrix $\begin{pmatrix} I_r & F\\ 0 & I_{r'}\end{pmatrix}$ defines an isomorphism from B_U to B_V. This amounts to solving:

$$(2.7.1)\qquad (\sigma_q F)(z^{-\mu'}A')-(z^{-\mu}A)F=V-U.$$

Successive reductions boil the problem down to example 2.4. We shall write $\mathrm{Red}(\mu,A,\mu',A',U)$ for the pair (F,V).

Now, we come back to our usual notations 2.1.1 and 2.1.2. We consider the matrix A_U associated to $U=(U_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$. Then, there is a unique pair $(\underline{F},V)$ with $\underline{F}=(F_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$ and $V=(V_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{V}(r_i,r_j,\mu_i,\mu_j)$ such that the associated gauge transformation $F\in\mathfrak{G}(\mathbf{C}(\{z\}))$ defines an isomorphism from A_U to A_V. The pair $(\underline{F},V)$ can be computed by solving iteratively a system of equations of the type 2.7.1 for $1\leqslant i<j\leqslant k$. This is done by inductively with the help of the formula:

$$(F_{i,j},V_{i,j})=\mathrm{Red}\left(\mu_i,A_i,\mu_j,A_j,U_{i,j}+\sum_{i<l<j}(\sigma_q F_{i,l})U_{l,j}-\sum_{i<l<j}V_{i,l}F_{l,j}\right).$$

What we get is, in essence, the canonical form of Birkhoff and Guenther. Standing alone, this statement confirms our earlier contention in section 2.1, to the effect that all objects of $\mathcal{E}_1^{(0)}$ are locally equivalent to objects of $\mathcal{E}$.

Now, we shall have use for an extension of these results allowing for coefficients in $\mathcal{O}(\mathbf{C}^*,0)$ (instead of $\mathbf{C}(\{z\})$).

Proposition 2.8. — *Let A_U be as above, but with $U=(U_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$. Then, there exists a unique pair $(\underline{F},V)$ with $\underline{F}=(F_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$ and $V=(V_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{V}(r_i,r_j,\mu_i,\mu_j)$ such that the associated gauge transformation $F\in\mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$ defines an isomorphism from A_U to A_V.*

Proof. — The same induction as before can be used, and the proof boils down to the following lemma.

Lemma 2.9. — *Let $\mu>\mu'$ in $\mathbf{Z}$, $A\in\mathrm{GL}_r(\mathbf{C})$, $A'\in\mathrm{GL}_{r'}(\mathbf{C})$ and $U\in\mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$. There exists a unique pair (F,V) with $F\in\mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$ and $V\in\mathcal{V}(r,r',\mu,\mu')$ satisfying 2.7.1.*

Proof. — The same reductions as in *loc. cit.* entail that we may as well assume from the beginning that $\mu=0$ and $\mu'=-1$. The equation as written has unknown F and right hand side $V-U$ in a space of rectangular matrices. Call s the rank of this space and call B the matrix of its automorphism $F\mapsto AFA'^{-1}$ relative to some basis. Multiplying both sides of 2.7.1 by A'^{-1} at right, we get an equivalent equation of the shape $z\,\sigma_q X - BX = Y - Y^{(0)}$, for which we want to show that, for arbitrary $B \in \mathrm{GL}_s(\mathbf{C})$ and $Y \in \mathcal{O}(\mathbf{C}^*,0)^s$, there is a unique solution $(X, Y^{(0)}) \in \mathcal{O}(\mathbf{C}^*,0)^s \times \mathbf{C}^s$. Note that, replacing $X = \sum_{n\in\mathbf{Z}} X_n z^n$, $Y = \sum_{n\in\mathbf{Z}} Y_n z^n$ and $Y^{(0)}$ respectively by $\sum_{n\in\mathbf{Z}} B^n X_n z^n$, $\sum_{n\in\mathbf{Z}} B^{n-1} Y_n z^n$ and $B^{-1}Y^{(0)}$, we do not change the conditions on X, Y, $Y^{(0)}$, and we are led to study a similar problem with $B = I_s$. The latter problem can be tackled componentwise: we are to show that, for any $u \in \mathcal{O}(\mathbf{C}^*,0)^s$, there is a unique pair $(f,\nu) \in \mathcal{O}(\mathbf{C}^*,0) \times \mathbf{C}$ such that $z\sigma_q f - f = u - \nu$ (compare to example 2.4).

We apply the q-Borel-Ramis transform of level 1. This clearly sends $\mathcal{O}(\mathbf{C}^*,0)$ to $\mathcal{O}(\mathbf{C}^*)$: indeed, for any $A > 0$, $A^n q^{-n(n-1)/2}$ tends to 0 when $n \to \pm\infty$. From the computations in example 2.4, we deduce that we have to take $\nu = \mathcal{B}_{q,1}u(1) = \sum_{n\in\mathbf{Z}} q^{-n(n-1)/2}u_n$; we must then prove the existence and unicity of f. Replacing u by $u - \nu$, we may assume that $\mathcal{B}_{q,1}u(1) = 0$. We write $f'_n = q^{-n(n-1)/2} f_n$ and $u'_n = q^{-n(n-1)/2}u_n$ the coefficients of the q-Borel-Ramis transforms $\mathcal{B}_{q,1}f$ and $\mathcal{B}_{q,1}u$. We know that $\sum_{-\infty}^{+\infty} u'_k = 0$ and we require that $\forall\, n \in \mathbf{Z}, f'_{n-1} - f'_n = u'_n$. The only possibility allowing $f'_n \to 0$ for $n \to \pm\infty$ is given by the two equivalent definitions

$$f'_n \underset{\text{def}}{=} \sum_{k=n+1}^{+\infty} u'_k$$
$$\underset{\text{def}}{=} -\sum_{k=-\infty}^{n} u'_k.$$

For $n \to +\infty$, we thus take (using the first definition of f'_n)

$$f_n = q^{n(n-1)/2} \sum_{k=n+1}^{+\infty} \frac{u_k}{q^{k(k-1)/2}}.$$

By assumption on u, there exists $A > 0$ and $C > 0$ such that, $\forall\, n \geqslant 0$, $|u_n| \leqslant CA^n$. Then,

$$|f_n| \leqslant \frac{CA^{n+1}}{|q|^n}\left(1 + \frac{A}{|q|^{n+1}} + \frac{A^2}{|q|^{(n+1)+(n+2)}} + \cdots\right),$$

whence $|f_n| = \mathrm{O}\left((A/|q|)^n\right)$ when $n \to +\infty$.

On the side of negative powers, putting, for convenience, $g_n = f_{-n}$ and $v_k = u_{-k}$ and, using the second definition for f'_n, we see that

$$g_n = q^{n(n+1)/2} \sum_{k=n}^{+\infty} \frac{v_k}{q^{k(k+1)/2}}.$$

By assumption on u, we have, for *any* $B > 0$, $|v_k| = \mathrm{O}(B^k)$ when $k \to +\infty$ and a similar computation as before then yields that, for *any* $B > 0$, $|g_n| = \mathrm{O}(B^n)$ when $n \to +\infty$, allowing one to conclude that $f \in \mathcal{O}(\mathbf{C}^*,0)$ as desired. □

We shall actually need only the following consequence of the proposition.

Corollary 2.10. — *Let $A = A_U$ in the canonical form 2.1.1, with $U = (U_{i,j}) \in \prod_{1 \leqslant i<j \leqslant k} \mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$. Then, there exists $F \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$ such that $A_V = F[A_U]$ has the same form, but with $V = (V_{i,j}) \in \prod_{1 \leqslant i<j \leqslant k} \mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$.* □

Obviously the same properties hold if one replaces $\mathcal{O}(\mathbf{C}^*,0)$ by $\mathcal{O}(\mathbf{C}^*)$.

3. Algebraic summation

3.1. The algorithm. — We keep the notations M_0, A_0, A_U of 2.1.1 and 2.1.2 and the corresponding conventions from section 2.1. Also, we shall, for $1 \leqslant i < j \leqslant k$, use the abreviation $\mu_{i,j} = \mu_i - \mu_j \in \mathbf{N}^*$.

Definition 3.1

(i) A summation divisor adapted to A_0 is a family $(D_{i,j})_{1 \leqslant i<j \leqslant k}$ of effective divisors over the elliptic curve $\mathbf{E}_q$, each $D_{i,j}$ having degree $\mu_{i,j}$, the family satisfying moreover the following compatibility condition:

$$\forall\, i,l,j \text{ such that } 1 \leqslant i < l < j \leqslant k,\ D_{i,j} = D_{i,l} + D_{l,j}.$$

Obviously, it amounts to the same thing to give only the $k-1$ divisors $D_{i,i+1}$, $i = 1, \ldots, k-1$.

(ii) We say that the adapted summation divisor $(D_{i,j})_{1 \leqslant i<j \leqslant k}$ is allowed if it satisfies the following conditions:

$$\forall\, i,j \text{ such that } 1 \leqslant i < j \leqslant k,\ \mathrm{ev}_{\mathbf{E}_q}(D_{i,j}) \notin \overline{(-1)^{\mu_{i,j}} \frac{\mathrm{Sp}(A_i)}{\mathrm{Sp}(A_j)}}.$$

Here, for $S, T \subset \mathbf{C}^*$, we put $S/T = \{s/t \mid s \in S, t \in T\}$; $\overline{X}$ and $\mathrm{ev}_{\mathbf{E}_q}$ were defined in the introduction.

Note that, for an adapted summation divisor, the condition of being allowed is a generic one.

Example 3.2. — A special case is that of an adapted summation divisor concentrated on a point $\alpha \in \mathbf{E}_q$, that is, each $D_{i,j} = \mu_{i,j}[\alpha]$. Then the condition that D is allowed is equivalent to:

$$\forall\, i,j \text{ such that } 1 \leqslant i < j \leqslant k,\ \mu_{i,j}\, \alpha \notin \overline{\mathrm{Sp}(A_i)} - \overline{\mathrm{Sp}(A_j)}.$$

It is generically (that is, over a non empty Zariski open subset) satisfied by $\alpha \in \mathbf{E}_q$.

Now, let $(D_{i,j})_{1 \leqslant i<j \leqslant k}$ be a summation divisor adapted to A_0. We choose points $a_l \in \mathbf{C}^*$ for $\mu_k < l \leqslant \mu_1$ such that, for $1 \leqslant i < j \leqslant k$,

$$D_{i,j} = \sum_{\mu_j < l \leqslant \mu_i} [\overline{a_l}].$$

These certainly exist. We then put:

$$t_i = \theta_q^{\mu_k} \prod_{\mu_k < l \leqslant \mu_i} \theta_{q,-a_l}.$$

Lemma 3.3

(i) *The functions* $t_1, \dots, t_k \in \mathcal{M}(\mathbf{C}^*)$ *are such that:*

(i1) *For* $i = 1, \dots, k$, $\sigma_q t_i = \alpha_i z^{\mu_i} t_i$, *where* $\alpha_i \in \mathbf{C}^*$.

(i2) *For* $1 \leqslant i < j \leqslant k$, $\operatorname{div}_{\mathbf{E}_q}(t_i) - \operatorname{div}_{\mathbf{E}_q}(t_j) = D_{i,j}$ *(the notation is explained in the course of the proof).*

(i3) *For* $1 \leqslant i < j \leqslant k$, *the function* $t_{i,j} = \sigma_q t_i / t_j$ *belongs to* $\mathcal{O}(\mathbf{C}^*)$.

(ii) *If the summation divisor* $(D_{i,j})_{1 \leqslant i < j \leqslant k}$ *is moreover allowed, for* $1 \leqslant i < j \leqslant k$, *the spectra of* $\alpha_i A_i$ *and* $\alpha_j A_j$ *have empty intersection on* $\mathbf{E}_q$:

$$\overline{\mathrm{Sp}(\alpha_i A_i)} \cap \overline{\mathrm{Sp}(\alpha_j A_j)} = \varnothing.$$

Proof. — It is an immediate consequence of the properties recalled in the introduction that these functions t_i indeed satisfy (i1). Moreover, the functional equation implies that the divisor $\operatorname{div}_{\mathbf{C}^*}(t_i)$ of zeroes and poles of t_i on $\mathbf{C}^*$ is invariant under the action of $q^{\mathbf{Z}}$, so that it makes sense to consider it as a divisor $\operatorname{div}_{\mathbf{E}_q}(t_i)$ on $\mathbf{E}_q$ (alternatively, one can consider t_i as a section of a line bundle over $\mathbf{E}_q$ and the notation is then classical). Again, because of the properties of theta functions, one clearly gets (i2). Assertion (i3) comes from the equalities

$$\begin{aligned} t_{i,j} &= \frac{\sigma_q t_i}{t_i} \times \frac{t_i}{t_j} \\ &= \alpha_i z^{\mu_i} \times \text{ a function with positive divisor.} \end{aligned}$$

The function $\theta_q^{\mu_i}\,\theta_q/\theta_{q,\alpha_i}$ satisfies the same functional equation as t_i, which means that their quotient is elliptic so that its divisor on $\mathbf{E}_q$ has trivial evaluation. Therefore,

$$\operatorname{div}_{\mathbf{E}_q}(t_i) = \overline{\frac{(-1)^{\mu_i}}{\alpha_i}}.$$

The conclusion (ii) then follows from the definition of an allowed divisor. □

We now introduce a temporary and slightly ambiguous notation. For an adapted summation divisor $D = (D_{i,j})_{1 \leqslant i < j \leqslant k}$, we write Θ_D for the following block-diagonal matrix:

$$\Theta_D = \begin{pmatrix} t_1 I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & t_k I_{r_k} \end{pmatrix}.$$

Of course, it does not only depend on D, but on a particular choice of the functions $t_1, \dots, t_k$ whose existence has just been proved. However, the summation process we are defining will produce a result that only depends on D. For a family of rectangular blocks $U'_{I,j}$, we shall use the following abreviation:

$$A'_{U'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix}.$$

Lemma 3.4

(i) *The effect of the gauge transformation Θ_D is to "regularize" the diagonal blocks of $A = A_U$: $\Theta_D[A] = A'_{U'}$, where, for $1 \leqslant i < j \leqslant k$, $U'_{i,j} = t_{i,j}U_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$ and, for $i = 1,\dots,k$, $A'_i = \alpha_i A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. If moreover the adapted summation divisor D is allowed, then, for $1 \leqslant i < j \leqslant k$, $\overline{\mathrm{Sp}(A'_i)} \cap \overline{\mathrm{Sp}(A'_j)} = \varnothing$.*

(ii) *Suppose we started with A_U in polynomial normal form. Then, we get $A'_{U'}$ such that $U'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*))$.*

Proof. The computations are immediate. □

We shall now take two matrices A_U and A_V in the formal class of A_0, flatten their slopes through the gauge transformation Θ_D, and then link the resulting matrices $A'_{U'}$ and $A'_{V'}$ by an isomorphism defined over $\mathbf{C}^*$. This relies on the following

Proposition 3.5

(i) *Let*

$$A'_{U'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix} \quad \text{and} \quad A'_{V'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & V'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix},$$

where, for $i = 1,\dots,k$, $A'_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ are such that, for $1 \leqslant i < j \leqslant k$, $\overline{\mathrm{Sp}(A'_i)} \cap \overline{\mathrm{Sp}(A'_j)} = \varnothing$ and, for $1 \leqslant i < j \leqslant k$, $U'_{i,j}, V'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$.

Then, there exists a unique $F' \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$ such that $F'[A'_{U'}] = A'_{V'}$.

(ii) *If, for $1 \leqslant i < j \leqslant k$, $U'_{i,j}, V'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*))$, then $F' \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*))$.*

Proof. — We have to solve inductively the system of equations

$$(\sigma_q F'_{i,j}) - A'_i F'_{i,j} = V'_{i,j} - U'_{i,j} + \sum_{i<l<j} V'_{i,l} F'_{l,j} - \sum_{i<l<j} (\sigma_q F'_{i,l}) U'_{l,j}.$$

The induction is the same as the one we met when building normal polynomial forms. The proposition then follows from the following lemma.

Lemma 3.6

(i) *Let* $B \in \mathrm{GL}_s(\mathbf{C})$ *and* $C \in \mathrm{GL}_t(\mathbf{C})$ *be invertible complex matrices such that* $\overline{\mathrm{Sp}(B)} \cap \overline{\mathrm{Sp}(C)} = \varnothing$. *Then, for* $Y' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*, 0))$, *the equation*

$$(\sigma_q X')\, B - CX' = Y'$$

has a unique solution $X' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*, 0))$.

(ii) *If* $Y' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*))$, *then* $X' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*))$.

Proof. — We write the Laurent series:

$$X' = \sum_{n \in \mathbf{Z}} X'_n z^n \quad \text{and} \quad Y' = \sum_{n \in \mathbf{Z}} Y'_n z^n.$$

By identification, we obtain $X'_n = \Phi_{q^n B, C}^{-1}(Y'_n)$, where $\Phi_{q^n B, C}$ is the automorphism $M \mapsto M(q^n B) - CM$ of $M_{t,s}(\mathbf{C})$; that it is indeed an automorphism comes from the assumption that $q^n B$ and C have non intersecting spectra. For $n \to +\infty$, $\Phi_{q^n B, C}^{-1} \sim q^{-n} \Phi_{B,0}^{-1}$ and, for $n \to -\infty$, $\Phi_{q^n B, C}^{-1} \to \Phi_{0,C}^{-1}$. Taking $]r, R[\times e^{i\mathbf{R}}$ to be the annulus of convergence of Y', we conclude that the annulus of convergence of X' is $]r, |q|R[\times e^{i\mathbf{R}}$. Annuli of definition actually grow, again an illustration of the good regularity properties of the homological equation. □

Putting all together, we now get our first fundamental theorem.

Theorem 3.7

(i) *Let* A_U, A_V *be defined as above, in the formal class of* A_0. *Then, there exists a unique* $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ *such that* $F[A_U] = A_V$ *and, for* $1 \leqslant i < j \leqslant k$, $\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geqslant -D_{i,j}$ *(the notation is explained in the course of the proof).*

(ii) *If* A_U, A_V *are in polynomial normal form,* $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$.

Proof. — We put $A'_{U'} = \Theta_D[A_U]$ and $A'_{V'} = \Theta_D[A_V]$, then $F[A_U] = A_V$ is equivalent to $F'[A'_{U'}] = A'_{V'}$, where $F' = \Theta_D F \Theta_D^{-1}$. The matrices F and F' together are upper-triangular with diagonal blocks $I_{r_1}, \ldots, I_{r_k}$ and their over-diagonal blocks are related by the relations: $F_{i,j} = \frac{t_j}{t_i} F'_{i,j}$. This implies the unicity of $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$) subject to the constraint that the coefficients of $F_{i,j}$ belong to $\frac{t_j}{t_i}\mathcal{O}(\mathbf{C}^*, 0)$ (resp. $\frac{t_j}{t_i}\mathcal{O}(\mathbf{C}^*)$). Since $\mathrm{div}_{\mathbf{E}_q}(t_i) - \mathrm{div}_{\mathbf{E}_q}(t_j) = D_{i,j}$, this proves (and explains) the given condition. □

As a matter of notation, we shall write $F_D(U, V)$ for the F obtained in the theorem: it does indeed depend only on D. We see it as the canonical resummation of $\widehat{F}(U, V)$ along the "direction" D. We shall write, in particular, $F_D(U) = F_D(0, U)$.

Let us call $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*))$) the subset of $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. of $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$) defined by the constraints $\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geqslant -D_{i,j}$ for $1 \leqslant i < j \leqslant k$.

Corollary 3.8. $F_D(U, V) = F_D(V) F_D(U)^{-1}$.

Proof. — Actually, $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*,0)) = \Theta_D\mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))\Theta_D^{-1}$ and $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*)) = \Theta_D\mathfrak{G}(\mathcal{O}(\mathbf{C}^*))\Theta_D^{-1}$, so that these subsets are subgroups. Then, the statement follows from the unicity property in the theorem. □

Corollary 3.9. — *The conclusion of the theorem still holds if one only assumes that* $U = (U_{i,j}) \in \prod_{1\leqslant i<j\leqslant k} \mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$.

Proof. — This immediately follows from corollary 2.10. □

Remark 3.10. — It is not difficult to prove that $\widehat{F}(U,V)$ is the aymptotic expansion of $F_D(U,V)$ in the sense of section 2.2. One first has to extend the definitions so as to allow for a pole at 0. The proof then proceeds in two steps.

(i) First, one proves that, in its domain of definition, $F_D(U,V)$ is a section of the sheaf $z^d\mathcal{B}$ for some $d \in \mathbf{Z}$. This is done using only the functional equation that it satisfies, and studying inductively its upper diagonal blocks $F_{i,j}$.

(ii) Then, one proves that the operator $F \mapsto A_V^{-1}(\sigma_q F)A_U$, sends $\mathfrak{G}(z^d\mathcal{B})$ to $\mathfrak{G}(z^{d+1}\mathcal{B})$. Starting from $F_D(U,V)$ and iterating yields the conclusion.

Actually, in [**16**], a stronger result is proved. It relies on a refined definition of asymptotics taking in account the position of poles: this is essential to get summation by discrete integral formulas.

Remark 3.11. — To give our theorem its functorial meaning, one should proceed as follows. One generalizes the construction of a vector bundle F_M from a q-difference module M. This defines a fibre functor ω over $\mathbf{E}_q$. Then, for each D, if one restricts to an appropriate subcategory of $\mathcal{E}_1^{(0)}$, $M \rightsquigarrow F_D(M)$ is an isomorphism from the fibre functor $\omega\circ\mathrm{gr}$ to ω. On the other hand, endowing F_M with the filtration coming from that of M, one defines an enriched functor and the underlying principle of all our uses of the homological equation is that this functor is fully faithful. This is exploited in [**23**].

3.2. Applications to classification

One direction of summation. — Let A_U, A_V be defined over $\mathbf{C}(\{z\})$. Suppose A_U and A_V are analytically equivalent. Then the power series $\widehat{F}(U,V)$ is convergent and satisfies the conclusion of theorem 3.7, so that, by unicity, $F_D(U,V) = \widehat{F}(U,V)$ for any allowed summation divisor. Conversely,

Proposition 3.12. — *Suppose* $F_D(U,V) \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$. *Then* A_U *and* A_V *are analytically equivalent (and all the above holds).*

Proof. — The gauge transform $F = F_D(U,V)$ is obtained by solving the system of equations

$$z^{-\mu_j}(\sigma_q F_{i,j})A_j - z^{-\mu_i}A_iF_{i,j} = V_{i,j} - U_{i,j} + \sum_{i<l<j} V_{i,l}F_{l,j} - \sum_{i<l<j}(\sigma_q F_{i,l})U_{l,j}.$$

By induction, we are reduced to the following lemma:

Lemma 3.13. — *Let $\mu > \mu'$ in $\mathbf{Z}$, $A \in \mathrm{GL}_r(\mathbf{C})$,$A' \in \mathrm{GL}_{r'}(\mathbf{C})$ and $Y \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$. Let $X \in \mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*, 0))$ be a solution of the equation:*

$$(\sigma_q X)\,(z^{-\mu'} A') - (z^{-\mu} A)X = Y.$$

Then, one actually has $X \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$.

Proof. — Going to the Laurent series and identifying coefficients, one finds

$$\forall n \in \mathbf{Z},\ q^{n+\mu'} X_{n+\mu'} A' - A X_{n+\mu} = Y_n.$$

Since $Y \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$, $Y_n = 0$ for $n \ll 0$. Therefore, for $n \ll 0$, writing $d = \mu - \mu' \in \mathbf{N}^*$, one has $X_n = q^{-n} A X_{n+d} A'^{-1}$. Since $|q| > 1$, either $X_n = 0$ for $n \ll 0$, or the coefficients of X are rapidly growing for indices near $-\infty$ prohibiting convergence and contradicting the assumption that $X \in \mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*, 0))$. □

In order to make this a statement about classification, we introduce one more notation. We write

$$\mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*, 0)) = \{F \in \mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*, 0)) \mid F[A_0] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))\}.$$

Clearly, the subset $\mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*, 0))$ of the group $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*, 0))$ is stable under the action by left translations of the subgroup $\mathfrak{G}(\mathcal{O}(\mathbf{C}^*, 0))$. Now, the above proposition immediately entails

Proposition 3.14. — *Mapping A_U to $F_D(U)$ yields a bijection*

$$\mathcal{F}(M_0) \longrightarrow \mathfrak{G}(\mathcal{O}(\mathbf{C}^*))\diagdown\mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*, 0)). \qquad \square$$

This is strikingly similar to the corresponding "formal modulo analytic" description in proposition 2.3.

Varying the direction of summation. — Let $D = (D_{i,j})_{1 \leqslant i < j \leqslant k}$ be an allowed summation divisor for M_0, A_0. We consider as its support and write $\mathrm{Supp}(D)$ the union $\bigcup_{1 \leqslant i < j \leqslant k} \mathrm{Supp}(D_{i,j})$ and define the following Zariski open subset of $\mathbf{E}_q$: $V_D = \mathbf{E}_q \setminus \mathrm{Supp}(D)$. We also write U_D for the preimage of V_D in $\mathbf{C}^*$. Thus, the elements of $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*, 0))$ are holomorphic germs over $(U_D, 0)$.

Now let D' be another allowed summation divisor. Then, for any A_U in the formal class of A_0, the gauge transformation $F_{D,D'}(U) \underset{\text{def}}{=} F_D(U)^{-1} F_{D'}(U)$ sends A_0 to itself. It is holomorphic on the open subset $(U_D \cap U_{D'}, 0)$. We call $U_{D,D'}$ the sector $U_D \cap U_{D'}$, which is the preimage of the open subset $V_{D,D'} = V_D \cap V_{D'}$ of $\mathbf{E}_q$. Note that, if D and D' have non intersecting supports (which is easy to realize), then U_D and $U_{D'}$ cover $\mathbf{C}^*$ and V_D and $V_{D'}$ cover $\mathbf{E}_q$.

Lemma 3.15. — *$F_{D,D'}(U)$ is a section of the sheaf $\Lambda_I(M_0)$ over $V_{D,D'}$.*

Proof. — One only has to prove that the upper-diagonal part of $F = F_{D,D'}(U)$ is flat. But its rectangular blocks satisfy: $(\sigma_q F_{i,j})\,(z^{-\mu_j} A_j) = (z^{-\mu_i} A_i) F_{i,j}$. This is a pure system of slope $\mu_{i,j} > 0$, hence $F_{i,j}$ is indeed flat by lemma 2.7. □

We now call $\mathfrak{U}$ (resp. $\mathfrak{V}$) the covering of $\mathbf{C}^*$ (resp. of $\mathbf{E}_q$) by the open subsets U_D (resp. V_D), where D runs among all the allowed summation divisors for A_0, M_0. The following is immediate.

Corollary 3.16. — *The family $(F_{D,D'}(U))_{D,D'}$ is a Cech cocycle of the sheaf $\Lambda_I(M_0)$ for the covering $\mathfrak{V}$ of $\mathbf{E}_q$.* □

Proposition 3.17. — *Mapping A_U to the cocycle $(F_{D,D'}(U))$ defines a one-to-one mapping:*

$$\mathcal{F}(M_0) \hookrightarrow Z^1(\mathfrak{V}, \Lambda_I(M_0)).$$

Proof. — If A_U is analytically equivalent to A_V, we have $A_V = F[A_U]$ for a unique $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$ and it is clear (by unicity) that $F_D(V) = F F_D(U)$ for all allowed divisors D, whence $F_{D,D'}(V) = F_{D,D'}(U)$ by immediate computation. This shows that the above mapping is well defined.

Conversely, just assume that $F_{D,D'}(V) = F_{D,D'}(U)$ for two allowed divisors with non intersecting supports. This equality gives $F_D(U,V) = F_{D'}(U,V)$. Hence, both sides are holomorphic over $U_D \cup U_{D'} = \mathbf{C}^*$ (near 0), and we already saw in proposition 3.12 that this implies the analytic equivalence of A_U and A_V. □

We now come to the second fundamental result of this paper.

Theorem 3.18. — *The above mapping yields a bijective correspondence:*

$$\mathcal{F}(M_0) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

Proof. — Let A_U and A_V have the same image in $H^1(\mathbf{E}_q, \Lambda_I(M_0))$, hence in $H^1(\mathfrak{V}, \Lambda_I(M_0))$ (in Cech cohomology, the H^1 of a covering embeds into the direct limit). There is, for each allowed divisor D, a matrix $G(D) \in \mathfrak{G}(\mathcal{O}(U_D)) \cap \mathrm{Aut}(M_0)$ in such a way that:

$$\forall\, D, D' \,:\, F_{D,D'}(U) = (G(D))^{-1}\, F_{D,D'}(V) G(D').$$

One draws that $F_D(V)G(D)\,(F_D(U))^{-1}$ does not depend on D, so that it is holomorphic on $\bigcup U_D = \mathbf{C}^*$; call Φ their common value. As a gauge transformation, it sends A_U to A_V. By proposition 3.12, A_U and A_V are analytically equivalent, which proves the injectivity.

We now prove that our mapping from $\mathcal{F}(M_0)$ to $H^1(\mathfrak{V}, \Lambda_I(M_0))$ is onto. For that, we take $(\Phi_{D,D'})_{D,D'} \in Z^1(\mathfrak{V}, \Lambda_I(M_0))$. By definition of the sheaf $\Lambda_I(M_0)$, each component $\Phi_{D,D'}$ is an element of $\mathfrak{G}(\mathcal{O}(U_{D,D'}))$, so that our cocycle can be considered as describing a vector bundle over $\mathbf{C}^*$, trivialized by the covering $\mathfrak{U}$ and with structural group in $\mathfrak{G}$. By [**19**], theorem 1.0 (see also [**18**]) it is trivial in the following sense: there is, for each D, a $\Phi_D \in \mathfrak{G}(\mathcal{O}(U_D))$ in such a way that, for all D, D', $\Phi_{D,D'} = \Phi_D^{-1}\Phi_{D'}$. Since the $\Phi_{D,D'}$ are automorphisms of A_0, the $\Phi_D[A_0]$ are all equal to a same matrix $A_{U'}$. Moreover, this is holomorphic over $\mathbf{C}^*$. By corollary 2.10, there is a $\Psi \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*, 0))$ such that $A_U = \Psi[A_{U'}]$ is meromorphic at 0. Then $\Psi\Phi_D$ is holomorphic on $(U_D, 0)$ and sends A_0 to A_U. Put $G_D = F_D(U)^{-1}\Psi\Phi_D$. This is a section of $\Lambda_I(M_0)$ over V_D. The equalities $F_D(U)G_D = \Psi\Phi_D$ entail $\Phi_D^{-1}\Phi_{D'} = G_D^{-1}F_{D,D'}(U)G_{D'}$, that is, the cocycle $(\Phi_{D,D'})_{D,D'}$ is equivalent to the cocycle $(F_{D,D'}(U))_{D,D'}$ which ends the proof of our statement.

There remains to check that the natural mapping from $H^1(\mathfrak{V}, \Lambda_I(M_0))$ to $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ is onto (we already said it was one-to-one). This is the content of proposition 4.4, to be proved after the discussion on the q-Gevrey filtration of the Stokes sheaf the in next chapter. □

4. The q-Gevrey filtration on the Stokes sheaf

The sources of inspiration for the contents of this chapter are [**12**], [**1**], [**9**], [**25**] and [**4**].

4.1. The filtration for the Stokes sheaf of a tamely irregular module

We stick to the conventions of section 2.1, in particular, the notations of 2.1.1 and 2.1.2.

Conditions of flatness. Let F be a section of the sheaf $\Lambda_I(M_0)$. Then, for $1 \leqslant i < j \leqslant k$, the block $F_{i,j}$ is solution of the equation

$$\sigma_q F_{i,j}\left(z^{-\mu_j}A_j\right) = \left(z^{-\mu_i}A_i\right)F_{i,j}.$$

From this and lemma 2.7, we deduce that $F_{i,j}$ is $(\mu_i - \mu_j)$-flat and that, if it is t-flat for some $t > \mu_i - \mu_j$, then it vanishes.

We now introduce a filtration of the Stokes sheaf and a filtration of the sheaf of its Lie algebras. For real non-negative t, we call $\lambda_I^t(M_0)$ the subsheaf of $\lambda_I(M_0)$ made of t-flat sections and $\Lambda_I^t(M_0)$ the subsheaf $I_n + \lambda_I^t(M_0)$ of $\Lambda_I(M_0)$. The latter is a sheaf of unipotent subgroups, while the former is the sheaf of its Lie algebras (see the discussion in section 2.2). Both filtrations are decreasing and exhaustive (the 0-term is the total sheaf, the t-term is the trivial sheaf for $t > \mu_1 - \mu_k$).

From the previous argument, we see that $\lambda_I^t(M_0)$ has a very simple concrete description in terms of matrices: its sections have non trivial blocks only over the "curved over-diagonal" consisting of those (i, j)-blocks such that $\mu_i - \mu_j = t$. There is a similar description for $\Lambda_I^t(M_0)$ (taking in account the block-diagonal of identities). We shall however have use for a more intrinsic definition of these filtrations. We first describe the filtration of $\lambda_I(M_0)$.

Proposition 4.1

(i) *The sheaf* $\lambda_I(M_0)$ *is the sheaf of sections of the vector bundle associated (see section 2.1) to the tamely irregular module* $\mathrm{End}^{>0}(M_0)$.

(ii) *The above filtration on the sheaf* $\lambda_I(M_0)$ *is the decreasing filtration associated to the graduation inherited from* $\mathrm{End}^{>0}(M_0)$.

Proof. — We have $M_0 = P_1 \oplus \cdots \oplus P_k$, whence:

$$\mathrm{End}(M_0) = \bigoplus_{1 \leqslant i,j \leqslant k} \mathrm{Hom}(P_j, P_i).$$

The internal Hom $\mathrm{Hom}(P_j, P_i)$ is a pure module of slope $\mu_i - \mu_j$. Therefore, $\mathrm{End}^{>0}(M_0)$ is the sum of those $\mathrm{Hom}(P_j, P_i)$ such that $\mu_i > \mu_j$, *i.e.* $i < j$.

On the other hand, the vector bundle associated to $\mathrm{Hom}(P_j, P_i)$ has as sections on an open subset V of $\mathbf{E}_q$ the morphisms from P_j to P_i that are holomorphic on the preimage U of V in $\mathbf{C}^*$. This implies that the sheaf of sections of the vector bundle associated to the module $\mathrm{End}^{>0}(M_0)$ is indeed $\lambda_I(M_0)$; that it is the direct sum of its subsheaves $\lambda_I^{(t)}(M_0)$, where $\lambda_I^{(t)}(M_0)$ is the sheaf of sections of the pure vector bundle associated to the pure module

$$\mathrm{End}^{(t)}(M_0) = \bigoplus_{\mu_i - \mu_j = t} \mathrm{Hom}(P_j, P_i);$$

and that $\lambda_I^t(M_0)$ is the direct sum of the $\lambda_I^{(t')}(M_0)$ for all $t' \geqslant t$. □

Actually the whole structure only depends on the filtrations and the properties of internal Homs, so that it can be extended to an arbitrary tannakian category.

Proposition 4.2. — *Let t be a nonnegative integer.*

(i) $\Lambda_I^t(M_0)$ *is a sheaf of normal subgroups of* $\Lambda_I(M_0)$.

(ii) *The map* $f \mapsto 1 + f$ *induces an isomorphism:*

$$\lambda_I^{(t)}(M_0) \simeq \frac{\Lambda_I^t(M_0)}{\Lambda_I^{t+1}(M_0)}.$$

Proof. — Actually, these are purely algebraic properties: for a nilpotent two sided ideal I of a non commutative algebra A, the subgroups $1 + I^t$ of the unit group are normal and their successive quotients are isomorphic to the quotient modules I^t/I^{t+1}. □

We now are in position to reconstruct the Stokes sheaf by successive exact sequences:

$$1 \longrightarrow \Lambda_I^{t+1}(M_0) \longrightarrow \Lambda_I^t(M_0) \longrightarrow \lambda_I^{(t)}(M_0) \longrightarrow 0. \tag{4.2.1}$$

Note also that, again from general algebraic considerations, we have a sequence of central extensions:

$$0 \longrightarrow \lambda_I^{(t)}(M_0) \longrightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^{t+1}(M_0)} \longrightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^t(M_0)} \longrightarrow 1. \tag{4.2.2}$$

4.2. Cohomological consequences

Lemma 4.3. — *Let V be a proper open subset of $\mathbf{E}_q$. Then $H^1(V, \Lambda_I(M_0))$ is trivial.*

Proof. — We apply theorem I.2 of [**8**] to the exact sequence 4.2.1. This gives an exact sequence of pointed sets:

$$H^1(V, \Lambda_I^{t+1}(M_0)) \longrightarrow H^1(V, \Lambda_I^t(M_0)) \longrightarrow H^1(V, \lambda_I^t(M_0)).$$

If the extreme terms are trivial, so must be the central one (this, by the very definition of an exact sequence of pointed sets). The rightmost term is the first cohomology group of a vector bundle (after proposition 4.1) over an open Riemann surface. Such a bundle being a trivial bundle, its H^1 is trivial. The leftmost term is trivial for $t > \mu_1 - \mu_k$. By descending induction, the inner term is trivial for all t, hence for $t = 0$. □

Proposition 4.4. — *The covering $\mathfrak{V}$ is good.*

Proof. — This means that the map from $H^1(\mathfrak{V}, \Lambda_I(M_0))$ to $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ is an isomorphism. After [**1**], cor. 1.2.4, p. 113, this follows from the lemma. □

Note that this ends the proof of theorem 3.18.

4.3. The Stokes sheaf of a general module. — We briefly sketch here how the previous results extend to the Stokes sheaf of a module M. We take M in the formal class of M_0 and identify it with $(\mathbf{C}(\{z\})^n, A_U)$, according to the conventions of section 2.1.

The mapping $\Phi \mapsto F_D(U)\Phi F_D(U)^{-1}$ defines an isomorphism from $\Lambda_I(M_0)$ to $\Lambda_I(M)$ over V_D. Therefore, the two sheaves are locally isomorphic. Actually, the latter is obtained from the former by the operation of twisting by the cocycle $(F_{D,D'}(U))_{D,D'}$, described in [**8**], prop. 4.2 (also see [**1**], II.1 or [**25**], pp. 30-31). According to the same references, their H^1 are isomorphic. Moreover, the same operations provide a local isomorphism of the sheaves of Lie algebras, so that $\lambda_I(M)$ is a vector bundle. Last, these isomorphisms preserve the filtrations by levels of flatness.

Remark 4.5. — One should also note that the mapping $X \mapsto F_D(U)X$ defines an isomorphism from the space of solutions of A_0 holomorphic over $(U_D, 0)$ to the same space for A_U. This means that their sheaves of solutions are locally isomorphic. Since the former is a vector bundle, so is the latter. This yields an explicit way of associating a vector bundle with an arbitrary module with integral slopes (for arbitrary slopes, see [**23**], [**24**]).

References

[1] D.G. Babbitt & V.S. Varadarajan – *Local Moduli for Meromorphic Differential Equations*, Astérisque, vol. 169-170, Société Mathématique de France, 1989.

[2] G.D. Birkhoff – The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Amer. Acad. Arts and Sci.* **49** (1913), p. 521–568.

[3] G.D. Birkhoff & P.E. Guenther – Note on a Canonical Form for the Linear q-Difference System, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), no. 4, p. 218–222.

[4] P. Deligne – Lettres à J.-P. Ramis, janvier & février 1986.

[5] ______, Catégories Tannakiennes, in *Grothendieck Festschrift, Vol. II* (P. Cartier et al., eds.), Progress in Math., Birkhäuser, 1990.

[6] P. Deligne & J. Milne – Tannakian Categories, in *Hodge Cycles, Motives and Shimura Varieties* (P. Deligne et al., eds.), Lect. Notes in Math., vol. 900, Springer Verlag, 1989.

[7] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – Equations aux q-différences, *Gaz. Math.* **96** (2003), p. 20–49.

[8] J. Frenkel – Cohomologie non abélienne et espaces fibrés, *Bull. Soc. math. France* **85** (1957), p. 135–220.

[9] M. Loday-Richaud – Stokes phenomenon, multisummability and differential Galois groups, *Ann. Inst. Fourier (Grenoble)* **44** (1994), no. 3, p. 849–906.

[10] F. Marotte & C. Zhang – Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 6, p. 1859–1890.

[11] M. van der Put & M.F. Singer – *Galois theory of difference equations*, Lect. Notes in Math., vol. 1666, Springer Verlag, 1997.

[12] J.-P. Ramis – Dévissage Gevrey, in *Journées singulières de Dijon*, Astérisque, vol. 59-60, Société Mathématique de France, 1978, p. 173–204.

[13] ______, Fonctions θ et équations aux q-différences, non publié, Strasbourg, 1990.

[14] ______, About the growth of entire functions solutions to linear algebraic q-difference equations, *Ann. Fac. Sci. Toulouse Math. (6)* **I** (1992), no. 1, p. 53–94.

[15] J.-P. Ramis, J. Sauloy & C. Zhang – La variété des classes analytiques d'équations aux q-différences dans une classe formelle, *C. R. Math.* **338** (2004), no. 4, p. 277–280.

[16] ______, Local analytic classification of irregular q-difference equations, Article in preparation.

[17] J.-P. Ramis & C. Zhang – Développement asymptotique q-Gevrey et fonction thêta de Jacobi, *C. R. Acad. Sci. Paris Sér. I Math.* **335** (2002), p. 899–902.

[18] H. Röhrl – Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen, *Math. Ann.* **133** (1957), p. 1–25.

[19] ______, Holomorphic fiber bundles over Riemann surfaces, *Bull. Amer. Math. Soc. (N.S.)* **68** (1962), p. 125–160.

[20] J. Sauloy – Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 4, p. 1021–1071.

[21] ______, Galois theory of fuchsian q-difference equations, *Ann. scient. Éc. Norm. Sup. 4e série* **36** (2003), no. 6, p. 925–968.

[22] ———, La filtration canonique par les pentes d'un module aux q-différences et le gradué associé, *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 1, p. 181–210.

[23] ———, Local Galois theory of irregular q-difference equations, article in preparation.

[24] ———, Isomonodromy for q-difference equations, article to be submitted.

[25] V.S. Varadarajan – Linear meromorphic differential equations: a modern point of view, *Bull. Amer. Math. Soc. (N.S.)* **33** (1996), no. 1, p. 1–42.

[26] C. Zhang – Développements asymptotiques q-Gevrey et séries Gq-sommables, *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 1, p. 227–261.

[27] ———, Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples, in *Differential equations and the Stokes phenomenon* (Braaksma, Immink, van der Put & Top, eds.), World scientific, 2003.

J. Sauloy, Laboratoire Émile Picard, CNRS UMR 5580, U.F.R. M.I.G., 118, route de Narbonne, 31062 Toulouse Cedex 4 (France) • *E-mail :* sauloy@picard.ups-tlse.fr
Url : http://picard.ups-tlse.fr/~sauloy/