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# KRICHEVER MODULES FOR DIFFERENCE AND DIFFERENTIAL EQUATIONS

*by*

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*Dedicated to Jean-Pierre Ramis on the occasion of his 60th birthday*

***Abstract.*** — In order to understand an analogue of Kronecker-Weber and some abelian Langlands theory for function fields, an explicit comparison between Krichever differential (or difference) modules of rank one and abelian differential (or difference) equations is carried out.

***Résumé* (Modules de Krichever pour les équations aux différences et différentielles)**
On compare explicitement les modules de Krichever différentiels ou aux différences de rang 1 aux extensions abéliennes différentielles ou aux différences. Ceci permet de donner, dans ces situations, un analogue du théorème de Kronecker-Weber.

## Introduction

The observation that the "elliptic module" of Drinfel'd has an analogue for difference and differential operators was made around 1977 by Krichever, Drinfel'd, Mumford et al. (see [**Dr, Kr, Mu, L1**]). Geometric Langlands theory (see [**L2, F-G-K-V, F-G-V**]) for differential equations is developed on the basis of this observation. In this paper we investigate some rather basic questions for the analogues of Drinfel'd modules for difference and differential equations. The main question is the relation between abelian differential (resp. difference) equations and Krichever differential (resp. difference) modules of rank one. By *abelian differential* (resp. *difference*) *equation* we mean a differential (resp. difference) module such that its differential (resp. difference) Galois group is an abelian (linear algebraic) group. For this purpose, explicit calculations of the universal Picard-Vessiot ring and the universal differential (resp. difference) Galois group for abelian differential (resp. difference) equations are carried out. In other words, we investigate Langlands theory for function fields where

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the Frobenius endomorphism is replaced by a differential operator or a difference operator. This is seen as a first step towards $\mathrm{GL}_1$-theory for Krichever modules. The differential field extension defined by the "torsion" of a rank one Krichever differential module has a vague resemblance to the torsion of rank one Drinfel'd modules. In contrast to this, the rank one Krichever difference modules provide abelian extensions in perfect analogy with rank one Drinfel'd modules. For the theory of ($q$-)difference equations we refer to [**P-S**] and the recent survey [**D-R-S-Z**].

## 1. Abelian differential equations

Let $K$ be a differential field. We suppose that the field of constants $C$ of $K$ is algebraically closed and has characteristic 0. The Tannakian category of all differential modules over $K$ is called $\mathrm{Diff}_K$. One associates to $\mathrm{Diff}_K$ a universal differential Picard-Vessiot ring $\mathrm{Univ}_K$. This is the direct limit of Picard-Vessiot rings for all linear differential equations over $K$. The universal differential Galois group $G_{\mathrm{univ},K}$ is the group of the differential automorphisms of $\mathrm{Univ}_K/K$. This is an affine group scheme. For a full Tannakian subcategory $\mathcal{C}$ of $\mathrm{Diff}_K$, there is also a universal Picard-Vessiot ring $\mathrm{Univ}_{\mathcal{C}}$ and a universal differential Galois group $G_{\mathcal{C}}$. From the Tannakian formulation of differential Galois theory it is clear that $\mathrm{Univ}_{\mathcal{C}}$ is a subring of $\mathrm{Univ}_K$ and that $G_{\mathcal{C}}$ is a quotient of $G_{\mathrm{univ},K}$. In this section, we present explicit calculations of $\mathrm{Univ}_{\mathcal{C}}$ and $G_{\mathcal{C}}$ for the case when $\mathcal{C}$ consists of the differential modules for which the differential Galois group is commutative. We write, in this situation, $G_{\mathrm{univ},K,\mathrm{ab}}$ for $G_{\mathcal{C}}$ and this group scheme is equal to $(G_{\mathrm{univ},K})_{\mathrm{ab}}$, the abelianized group scheme obtained from $G_{\mathrm{univ},K}$.

***Lemma 1.1.*** — *Let $M$ be a differential module over $K$ with a differential Galois group $G$ that is the product of two (algebraic) groups $H_1$ and $H_2$. Let $R$ be the Picard-Vessiot ring of $M$. Then $R = R_1 \otimes_K R_2$ where $R_i$, $i = 1, 2$ is the Picard-Vessiot ring for a differential module $M_i$ over $K$ with differential Galois group $H_i$.*

*Proof.* — Let $L$ be the field of fractions of $R$ and put $L_1 = L^{H_2}$, $L_2 = L^{H_1}$. By Galois correspondence $L_i$ is the Picard-Vessiot field of some differential module $M_i$ over $K$ with differential Galois group $H_i$. Let $R_i \subset L_i$ denote the $K$-subalgebra of $L_i$ consisting of the elements $f \in L_i$ such that the $K$-vector subspace of $L_i$, generated by $f$ and all its derivatives is finite dimensional. Then $R_i$ is the Picard-Vessiot ring of $M_i$ (compare [**P-S03**], Corollary 1.38). One considers the natural map

$$\psi : R_1 \otimes_K R_2 \longrightarrow L_1 \otimes_K L_2 \xrightarrow{\alpha} L.$$

This is a morphism of differential algebras over $K$. Moreover, $\psi$ is equivariant for the action of $H_1 \times H_2$. Let $L_3 \subset L$ denote the field of fractions of the image of $\alpha$. This is the Picard-Vessiot field for the differential module $L_1 \otimes_K M_2$ over $L_1$ since (compare [**P-S03**], Proposition 1.22):

(a) $L_3$ is generated over $L_1$ by the entries of a fundamental matrix for the differential module $L_1 \otimes M_2$ over $L_1$.

(b) the field of constants of $L_3$ is $C$.

Let $H_3$ denote the group of the differential automorphisms of $L_3/L_1$. The embedding $L_3 \subset L$ induces a morphism $H_2 \to H_3$. From the equalities $L_3^{H_3} = L_1$, $L^{H_2} = L_1$ and the Galois correspondence of differential Galois theory one deduces that $L_3 = L$ and $H_2 \to H_3$ is a bijection. The Picard-Vessiot ring $R_3$ of $L_1 \otimes_K M_2$ over $L_1$ is equal to the image $L_1 \otimes_K R_2/I$ of $L_1 \otimes_K R_2$ in $L_3$. In particular $I$ is a maximal differential ideal of $L_1 \otimes_K R_2$. The group of differential automorphisms of $R_2/K$ is $H_2$. If the ideal $I$ is not zero, then the group $H_3$ of the differential automorphisms of $L_1 \otimes_K R_2/I$ is a proper subgroup of $H_2$. This contradicts the equality $H_2 = H_3$. We conclude that $\psi$ is injective and that $L$ is the field of fractions of the image of $\psi$. Any element $f$ in the image of $\psi$ has the property that the $K$-vector space generated by $f$ and all its derivatives is finite dimensional. Thus, we have obtained an injective $\psi : R_1 \otimes_K R_2 \to R$ which induces a bijection for the corresponding fields of fractions. We will regard $\psi$ as an inclusion $R_1 \otimes_K R_2 \subset R$. Consider an element $f \in R$. Let $W$ be the $K$-vector space generated by $f$ and all its derivatives. The dimension of $W$ over $K$ is finite. Define the ideal $J \subset R_1 \otimes_K R_2$ to consist of the elements $g \in R_1 \otimes_K R_2$ satisfying $gW \subset R_1 \otimes_K R_2$. Then, $J$ is a non zero differential ideal and hence, equal to $R_1 \otimes_K R_2$. Thus, $R_1 \otimes_K R_2 = R$ □

***Corollary 1.2.*** *The universal Picard-Vessiot ring* $\mathrm{Univ}_{K,\mathrm{ab}}$ *for the category of the differential modules with abelian differential Galois group has the form* $K[\{s(a)_{a\in K}\}, \{y(b)\}_{b\in K}]$ *where the "symbols"* $s(a)$, $y(b)$ *satisfy:*

$$s(0) = 1, \quad s(a_1)s(a_2) = s(a_1 + a_2),$$
$$y(0) = 0, \quad y(b_1) + y(b_2) = y(b_1 + b_2),$$
$$s(a)' = as(a), \quad y(b)' = b.$$

*The relations between the symbols depend on the field $K$. They are generated by:*

$$s(f'/f) = c(f)f \quad \text{for } f \in K^* \text{ and suitable } c(f) \in C^*;$$
$$y(f') = f + d(f) \quad \text{for suitable } d(f) \in C.$$

*Proof.* — Any abelian linear algebraic group $G$ (over $C$) is a finite product of copies of the groups $\mathbf{G}_m$ (the multiplicative group), $\mathbf{G}_a$ (the additive group) and finite cyclic groups. According to the lemma we have only to consider differential modules $M$ over $K$ with differential Galois group in $\{\mathbf{G}_m, \mathbf{Z}/n\mathbf{Z}, \mathbf{G}_a\}$. For the first two classes of groups, the differential module is direct sum of 1-dimensional modules. Indeed, the solution space $V$ of $M$, is a direct sum of 1-dimensional spaces, invariant under the differential Galois group. Hence, for every equation $y' = ay$ (with $a \in K$) the universal Picard-Vessiot ring $\mathrm{Univ}_{K,\mathrm{ab}}$ should contain a non zero solution which we call $s(a)$. Let $M$ be an indecomposable differential module of dimension $m$ having differential Galois group $\mathbf{G}_a$. The corresponding solution space $V$ of dimension $m$ over $C$ has an indecomposable $\mathbf{G}_a$-action. This action has the form $t \mapsto \exp(tN)$, where $N : V \to V$ is a nilpotent map such that $N^{m-1} \neq 0$ and $N^m = 0$. Let $V_i = \ker(N^i)$ for $i = 1, 2, \ldots$. There are submodules $M_i \subset M$ corresponding to the $V_i$. One considers the 2-dimensional $M_2$ with its trivial 1-dimensional subspace. This produces an inhomogenous equation $y' = b$ for some $b \in K$. Let $L \supset K$ be the Picard-Vessiot field for $M$. Then there is a differential subfield $L_2 = K(y(b))$ of $L$, with $y(b)' = b$. This is the Picard-Vessiot field of $M_2$. Since $L_2/K$ and $L/K$ have the same differential Galois group, namely $\mathbf{G}_a$, one has (by the Galois correspondence for differential fields) that $L_2 = L$. Therefore, $\mathrm{Univ}_{K,\mathrm{ab}}$ must contain an element $y(b)$ with $y(b)' = b$ for every $b \in K$. Moreover, $\mathrm{Univ}_{K,\mathrm{ab}}$ is generated over $K$ by the elements $\{s(a)\}_{a \in K}$ and $\{y(b)\}_{b \in K}$. One can normalize the $y(b)$, such that $y(0) = 0$ and $y(b_1 + b_2) = y(b_1) + y(b_2)$ for all $b_1, b_2 \in K$. This is done by considering a basis $\{B_i\}_{i \in I}$ of $K$ as vector space over $\mathbf{Q}$. One chooses symbols $y(B_i)$ for every $i \in I$. Any $b$ is a finite sum $\sum_i \lambda_i B_i$ (with all $\lambda_i \in \mathbf{Q}$. Define now $y(b) := \sum_i \lambda_i y(B_i)$.

For every integer $n \geqslant 1$ and every $i \in I$ we choose an invertible symbol $s(\frac{1}{n!}B_i)$. This can be done such that $s(\frac{1}{n!}B_i)^n = s(\frac{1}{(n-1)!}B_i)$ for all $i \in I$ and $n \geqslant 2$. An arbitrary element $a \in K$ can be written as a finite sum $\sum_i \frac{t_i}{(n_i)!}B_i$ with $t_i \in \mathbf{Z}$ (almost all equal to zero) and $n_i \geqslant 1$. One defines $s(a) := \prod_i s(\frac{1}{(n_i)!}B_i)^{t_i}$. Clearly $s(0) = 1$ and $s(a_1 + a_2) = s(a_1)s(a_2)$ for all $s_1, s_2$.

Relations between symbols clearly depend on the field $K$. Between the symbols $\{s(a)\}$ and the symbols $\{y(b)\}$ there are no relations since the corresponding differential equations $y' = ay$ and $y' = b$ have unrelated differential Galois groups (namely a subgroup of $\mathbf{G}_m$ and $\mathbf{G}_a$). The symbols $\{y(b)\}$ form a group for the addition. Hence it suffices to introduce a relation $y(b) = f$ with $f \in K$ whenever the equation $y' = b$ has a solution in $K$. Consider the exact sequence $0 \to C \to K \to K' \to 0$, where $K' := \{f' \mid f \in K\}$ and the second map is $f \mapsto f'$. This sequence splits and we fix an additive (or even $C$-linear) map $\gamma : K' \to K$ satisfying $\gamma(b)' = b$ for every $b \in K'$. Then the relations that we want are $y(b) = \gamma(b)$ for every $b \in K'$.

Consider the exact sequence

$$1 \longrightarrow C^* \longrightarrow K^* \longrightarrow K^*/C^* \longrightarrow 1.$$

This exact sequence splits since the group $C^*$ is divisible. One identifies $K^*/C^*$ with the group $\{f'/f \mid f \in K^*\}$. Let $\delta : \{f'/f \mid f \in K^*\} \to K^*$ be a homomorphism of groups satisfying $\delta(a)'/\delta(a) = a$. Then the relations that we want are $s(a) = \delta(a) \in K^*$ for every $a \in \{f'/f \mid f \in K^*\}$. □

Define the group $\mathrm{Isom}_{K,1}$ and the $C$-vector space $A(K)$ by the exact sequences

$$1 \longrightarrow K^*/C^* \xrightarrow{\alpha} K \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow 0,$$

$$0 \longrightarrow K/C \xrightarrow{\beta} K \longrightarrow A(K) \longrightarrow 0,$$

where $\alpha$ is the map $f \mapsto f'/f$ and $\beta$ is given by $f \mapsto f'$. Then $\mathrm{Isom}_{K,1}$ is equal to the group of the isomorphy classes of the 1-dimensional differential modules over $K$. Indeed, consider the 1-dimensional module $M(a) := Ke$ with $\partial e = ae$ and $a \in K$. Then $M(a) \cong M(b)$ if and only if $a - b = f'/f$ for some $f \in K^*$. Moreover, $M(a) \otimes_K M(b) \cong M(a+b)$.

Let $C[\mathrm{Isom}_{K,1}]$ denote the group algebra of the group $\mathrm{Isom}_{K,1}$ over the field $C$. We recall that the elements of this algebra are the finite formal expressions $\sum_{a \in \mathrm{Isom}_{K,1}} \lambda_a[a]$ with all $\lambda_a \in C$. The addition is the obvious one. The multiplication is induced by the rule $[a] \cdot [b] = [a+b]$. Take a basis $\{b_i\}_{i \in I}$ of $A(K)$ over $C$. One forms the $C$-algebra $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]$ where the $\{t_i\}_{i \in I}$ is a family of indeterminates. This algebra is made into a Hopf algebra by the formulas $[a] \mapsto [a] \otimes [a]$ and $t_i \mapsto t_i \otimes 1 + 1 \otimes t_i$.

***Corollary 1.3.*** *The affine group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is equal to*

$$\mathrm{Spec}(C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]).$$

*Proof.* An element $g$ in $G_{\mathrm{univ},K,\mathrm{ab}}(C)$ is a differential automorphism of the differential ring $\mathrm{Univ}_{K,\mathrm{ab}}/K$. Then $g$ is given by elements $c(g,a) \in C^*$ (for $a \in K$) and elements $d(g,b)$ (for $b \in K$) satisfying $g(s(a)) = c(g,a) \cdot s(a)$ and $g(y(b)) = y(b) + d(g,b)$. Now $a \mapsto c(g,a)$ is a homomorphism from $K$ to $C^*$ satisfying $c(g,a) = 1$ if the equation $y' = ay$ has a solution in $K^*$. In other words $a \mapsto c(g,a)$ is any homomorphism $\mathrm{Isom}_{K,1} \to C^*$. The map $b \mapsto d(g,b)$ is a $C$-linear map and $d(g,b) = 0$ if there exists a $y \in K$ with $y' = b$. In other words, $b \mapsto d(g,b)$ is any $C$-linear map $A(K) \to C$. A $C$-linear map $\ell : A(K) \to C$ is determined by the collection $\{\ell(b_i)\}_{i \in I}$. Hence, $g$ corresponds to a $C$-algebra homomorphism $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}] \to C$. The multiplication of elements in $G_{\mathrm{univ},K,\mathrm{ab}}(C)$ is induced by the Hopf algebra structure of $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]$. This description remains valid if $C$ is replaced by any $C$-algebra (commutative and with identity). This proves the statement of the corollary. □

**1.1. $K$ is the field of formal Laurent series $C((x))$.** — One provides $K = C((x))$ with the usual differentiation $f \mapsto df/dx$. An easy calculation shows that $\mathrm{Isom}_{K,1}$ is equal to $x^{-1}C[x^{-1}]/\mathbf{Z}x^{-1}$. Indeed, the image of the map $f \mapsto f'/f$ consists of the Laurent series $m/x + C[[x]]$ with $m \in \mathbf{Z}$. As an additive group $\mathrm{Isom}_{K,1}$ is the direct sum of $\mathbf{Q}/\mathbf{Z}$ and an infinite dimensional vector space over $\mathbf{Q}$. Furthermore, we need only to introduce the symbols $s(a)$ for $a \in x^{-1}C[x^{-1}]$. The only relation is $s(1/x) = x$.

For the"additive" equations $y' = b$ we need only to consider $b = x^{-1}$. Indeed, the map $f \mapsto f'$ from $K$ to itself has a 1-dimensional cokernel, generated by the image of $x^{-1}$. We will write $\log x$ for the symbol $y(x^{-1})$. The conclusion is:

$$\mathrm{Univ}_{K,\mathrm{ab}} = K[\{s(a)\}_{a \in x^{-1}C[x^{-1}]}, \log x]$$

and the only relation between the symbols is given by $s(1/x) = x$. The affine group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is the Spec of the Hopf algebra $C[\mathrm{Isom}_{K,1}, t]$ over $C$. We note that the part $C[t]$ of the Hopf algebra comes from the one factor $\mathbf{G}_a$ corresponding to $\log x$. Finally, as it should be, $\mathrm{Spec}(\mathrm{Univ}_{K,\mathrm{ab}})$ is a torsor over $G_{\mathrm{univ},K,\mathrm{ab}}$. This torsor is not trivial, due to the torsion subgroup $\mathbf{Q}/\mathbf{Z}$ of $\mathrm{Isom}_{K,1}$.

**1.2. $K$ is the field $C(x)$.** — We follow the same procedure as in the last subsection. The group $\mathrm{Isom}_{K,1}$ is equal to $K/V$, where $V$ consists of the elements $f \in K = C(x)$ which have at most poles of order one and such that all the residues of $f dx$ are in $\mathbf{Z}$. The surjective map $K \to K/V$ has no section because the group $K/V$ has torsion elements. One way to make $\mathrm{Isom}_{K,1}$ more explicit is to consider partial fraction decompostion of the elements in $K$. An element $f \in C(x)$ is written as a finite sum $f_\infty + \sum_{c\in C} f_c$, with $f_\infty \in C[x]$ and $f_c \in (x-c)^{-1}C[(x-c)^{-1}]$ for each $c \in C$. This yields a direct sum decomposition $\oplus_{v\in\mathbf{P}^1(C)} \mathrm{Isom}_{K,1,v}$ of $\mathrm{Isom}_{K,1}$. Furthermore, $\mathrm{Isom}_{K,1,\infty} = C[x]$ and for every $c \in C$ one has $\mathrm{Isom}_{K,1,c} = (x-c)^{-1}C[(x-c)^{-1}]/\mathbf{Z}\frac{1}{x-c}$.

For the "additive" equations $y' = b$ with $b \in K$, we use also partial fractions. The only equations that one has to consider are $y' = 1/(x-c)$ with $c \in C$. We will write $\log(x-c)$ for the symbol $y(1/(x-c))$. The $K$-algebra $\mathrm{Univ}_{K,\mathrm{ab}}$ can now be written as

$$K[\{s(a)\}_{a\in K}, \{\log(x-c)\}_{c\in C}]$$

with, as before, the relations $s(0) = 1$, $s(a_1 + a_2) = s(a_1)s(a_2)$ and the new relations $s(1/(x-c)) = x - c$ for every $c \in C$. Furthermore $s(a)' = as(a)$ for all $a \in K$ and $\log(x-c)' = 1/(x-c)$ for all $c \in C$. The group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is the Spec of the Hopf algebra $C[\mathrm{Isom}_{K,1}, \{t_c\}_{c\in C}]$.

One can give this the following interpretation. The universal Picard-Vessiot ring for the global abelian situation has as symbols the union of all local symbols for the completions $C((x-c))$ (all $c \in C$) and $C((x^{-1}))$ of $C(x)$. We note that the completion $C((x^{-1}))$ of $C(x)$ at $\infty$ behaves somewhat differently because its differentiation $d/dx$ is not the one associated to the local parameter $x^{-1}$. A better way to formulate the differential equations that we are dealing with:

(a) the multiplicate equation in the form $dy/y = \omega$ with $\omega$ a differential form for $K$ (or for the curve $\mathbf{P}^1_C$).

(b) the additive equation in the form $dy = \omega$ with $\omega$ a differential form.

We will make this more explicit for the case of a function field over $C$.

**1.3. $K$ is a function field over $C$.** — $K$ is the function field of a smooth, irreducible, projective curve $X$ over $C$. Let $\Omega$ denote the $C$-vector space of all differential forms for $K$. In other words, $\Omega$ is the set of meromorphic 1-forms on $X$. We have to consider two types of equations:

(a) $dy/y = \omega$ with $\omega \in \Omega$ (multiplicative equations) and

(b) $dy = \omega$ with $\omega \in \Omega$ (additive equations).

For any (closed) point $x \in X$ one considers the field $K_x$, which is the completion of $K$ w.r.t. the discrete valuation attached to $x$. Let $\Omega_x$ denotes the universal finite differential module of $K_x$ over $C$. This means that after identifying $K_x$ with $C((t))$ one has $\Omega_x = C((t))dt$. For the equations of type (a) one has an exact sequence

$$0 \longrightarrow \mathrm{Jac}(X) \longrightarrow \Omega/\{dy/y \mid y \in K^*\} \xrightarrow{L} \oplus_{x\in X}\Omega_x/\{dy/y \mid y \in K_x^*\} \longrightarrow C/\mathbf{Z} \longrightarrow 0.$$

By $\mathrm{Jac}(X)$ we mean in fact the group of the $C$-valued points of the Jacobian variety of $X$. The term $\Omega/\{dy/y \mid y \in K^*\}$ is equal to $\mathrm{Isom}_{K,1}$. The term $\Omega_x/\{dy/y \mid y \in K_x^*\}$ is equal to $\mathrm{Isom}_{K_x,1}$. The last map is induced by the map $\oplus_{x\in X}\Omega_x \to C$, given by the formula $(\omega_x)_{x\in X} \mapsto \sum_{x\in X}\mathrm{Res}_x(\omega_x)$. The map $L$ is induced by the obvious map $\Omega \to \prod_{x\in X}\Omega_x$. Let $\omega \in \Omega$, $\omega \neq 0$ represent an element in the kernel of $L$. Then $\omega$ has at most poles of order 1 and all the residues of $\omega$ are in $\mathbf{Z}$. One associates to $\omega$ the divisor $D = \sum_{x\in X}\mathrm{Res}_x(\omega)x$. Then $\omega \in \{dy/y \mid y \in K^*\}$ if and only if $D$ is a principal divisor. We will omit the verification that the above sequence is exact. This sequence provides the relation between the global multiplicative symbols (*i.e.*, for $\mathrm{Univ}_{K,\mathrm{ab}}$) with the local multiplicative symbols (*i.e.*, for $\mathrm{Univ}_{K_x,\mathrm{ab}}$). One observes that the collection of the multiplicative local symbols have relations induced by "the sum of the residues is zero". For $X = \mathbf{P}^1_C$ an example of these relations is

$$s_c\Big(\frac{\lambda}{x-c}\Big)\cdot s_\infty\Big(\frac{\lambda x}{1-cx^{-1}}\Big) = 1.$$

Here $s_c$ and $s_\infty$ denote local symbols at the points $c$ and $\infty$. Further $\lambda \in C^*$. The above relation is obtained by transforming

$$\frac{d}{d(x-c)}\,y = \frac{\lambda}{x-c}\,y$$

into the equation

$$\frac{d}{dx^{-1}}\,y = \frac{-\lambda x}{1-cx^{-1}}\,y.$$

If the curve $X$ has genus $g > 0$, then apart from the local multiplicative symbols one needs also symbols for the group $\mathrm{Jac}(X)$ in order to obtain all global symbols.

For the equations of type (b) one has an exact sequence

$$0 \longrightarrow H^1_{\mathrm{DR}}(X,C) \longrightarrow \Omega/\{dy \mid y \in K\} \xrightarrow{L} \oplus_{x\in X}\Omega_x/\{dy \mid y \in K_x\} \longrightarrow C \longrightarrow 0.$$

The map $L$ is induced by the obvious map $\Omega \to \prod_{x\in X}\Omega_x$. Let $\omega \in \Omega$ represent an element in the kernel of $L$. Then the residue of $\omega$ at every point $x \in X$ is zero. The group $H^1_{\mathrm{DR}}(X,C)$ is the De Rham cohomology group with coefficients in $C$ is equal to $\{\omega \in \Omega \mid \text{all residues } 0\}/\{dy \mid y \in K\}$. This cohomology group has dimension $2g$ over $C$ where $g$ is the genus of $X$. The group $A(K)$ is equal to $\Omega/\{dy \mid y \in K\}$ and $A(K_x)$ is equal to $\Omega_x/\{dy \mid y \in K_x\}$. Therefore the exact sequence provides the relation between the global additive symbols (*i.e.*, for $\mathrm{Univ}_{K,\mathrm{ab}}$) and the local additive symbols (*i.e.*, for $\mathrm{Univ}_{K_x,\mathrm{ab}}$). For every $x \in X$ there is in fact one local symbol, namely $\log t_x$ where $t_x$ is a local parameter. Between these local symbols there is just one $C$-linear relation, induced by " the sum of the residues is zero". We note that for $X = \mathbf{P}^1_C$, this relation is obviously $\log x + \log x^{-1} = 0$. Furthermore, if the genus $g$ of $X$ is strictly positive then one needs $2g$ new additive symbols coming from a basis of $H^1_{\mathrm{DR}}(X, C)$ over $C$.

**1.4. Krichever differential modules of rank one.** — Let $X$ be a smooth, irreducible, projective curve over the algebraically closed field $C$ of characteristic 0. Choose a (closed) point $\infty \in X$ and let $A$ be the coordinate ring of $X \smallsetminus \{\infty\}$. Let $K$ denote a differential field (or more generally, differential ring) with field of constants $C$. A *differential Krichever module over* $K$ is an injective $C$-algebra homomorphism $\phi : A \to K[\partial]$ such that $\phi(A) \not\subset K$. We denote by $\mathrm{Univ}_K$ the universal Picard-Vessiot ring (or field) of $K$. The following result is sketched in [**L1**]. Here we provide a more complete proof.

***Proposition 1.4***. — *Let $A, K, \phi$ be as above.*

(1) *There is a positive integer $r$, called the* rank of $\phi$ *such that for every $a \in A$, $a \neq 0$ the degree of the operator $\phi(a)$ is equal to $r \cdot \deg(a)$.*

(2) *For every non zero ideal $I \subset A$ the $C$-vector space*

$$V(I) := \{v \in \mathrm{Univ}_K \mid \phi(f)v = 0 \textit{ for all } f \in I\}$$

*has a natural structure of $A$-module. For this structure $V(I)$ is isomorphic to $(A/I)^r$. Moreover, the torsion $A$-module*

$$W(\phi) := \{v \in \mathrm{Univ}_K \mid \exists\, a \in A,\ a \neq 0 \textit{ with } \phi(a)(v) = 0\}$$

*is isomorphic to $(\mathrm{Qt}(A)/A)^r$, where $\mathrm{Qt}(A)$ denotes the field of fractions of $A$.*

(3) *The differential Galois group of the set of operators $\{\phi(f) \mid f \in I\}$ is an algebraic subgroup of $\mathrm{GL}_r(A/I)$.*

*Proof*. — For any non zero ideal $I \subset A$, the set of operators $\{\phi(f) \mid f \in I\}$ generates a left ideal in $K[\partial]L_I$ where $L_I$ is a (monic) operator. Indeed, $K[\partial]$ is a (left and right) Euclidean ring. This observation clarifies the statements (2) and (3).

The map from $a \in A \smallsetminus \{0\}$ to $\mathbf{Z}$, given by $a \mapsto -\deg(\phi(a))$, extends to a discrete valuation on the field of fractions of $A$. This valuation has negative values on $A$ and is therefore equivalent to the discrete valuation attached to the point $\infty$. It follows that there is an rational number $r > 0$ such that $\deg(\phi(a)) = r \cdot \deg(a)$ holds for every $a \in A$, $a \neq 0$.

Consider an element $v \in V(I)$ and an element $a \in A$. The operator $\phi(a)$ acts on $\mathrm{Univ}_K$ in the obvious way. The element $\phi(a)v$ belongs again to $V(I)$ because for $f \in I$ one has $\phi(f)\phi(a)v = \phi(fa)v$ and $fa \in I$. In this way $V(I)$ is an $A$-module and also an $A/I$-module.

Let $\underline{P}$ be a non zero prime ideal of $A$. Then $V(\underline{P}^n)$ is a torsion module over the valuation ring $B = A_{\underline{P}}$. Choose $\pi \in A$ such that $\pi$ generates the maximal ideal $\underline{P}B$ of $B$. The map $\phi(\pi) : V(\underline{P}^{n+1}) \to V(\underline{P}^n)$ is surjective. It follows that

$W := \cup_{n\geqslant 1} V(\underline{P}^n)$ is a divisible torsion module over $B$. Let $\mathrm{Qt}(B)$ denote the field of fractions of $B$. Then $W$ is as $B$-module isomorphic to $(\mathrm{Qt}(B)/B)^{N(\underline{P})}$ for some integer $N(\underline{P}) \geqslant 1$. Hence $V(\underline{P}^n) \cong (A/\underline{P}^n)^{N(\underline{P})}$ for all $n \geqslant 1$. In particular, $N(\underline{P})$ is the dimension of $V(\underline{P})$ as vector space over $A/\underline{P}$. Using that $A$ is a Dedekind ring one finds that $V(IJ) = V(I)\oplus V(J)$, if $I$ and $J$ are relatively prime ideals. For $I = \underline{P}_1^{n_1}\cdots \underline{P}_s^{n_s}$ (with distinct prime ideals $\underline{P}_1,\dots,\underline{P}_s$) one has that the dimension of $V(I)$ over $C$ is equal to $\sum_{i=1}^s n_i\cdot N(\underline{P}_i)\cdot \dim_C A/\underline{P}_i$. This is also equal to $\sum_{i=1}^s N(\underline{P}_i)\cdot\dim_C A/\underline{P}_i^{n_i}$. For a principal ideal $I=(a)$ one has $\dim_C V((a)) = r\cdot\deg(a) = r\cdot\dim_C A/(a) = \sum_{i=1}^s r\cdot \dim_C A/\underline{P}_i^{n_i}$. It follows that all $N(\underline{P}_i)$ are equal to $r$. If one varies $a$ then one finds that all $N(\underline{P})$ are equal to $r$. In particular, $r$ is a positive integer and $V(I)\cong (A/I)^r$ for every non zero ideal $I$.

Finally, the elements of the differential Galois group $G$ of the set of differential operators $\phi(I)$, act on $V(I)$. This action commutes with the action of $\phi(a)$ for $a\in A$. Indeed, $g\in G$ commutes with $\partial$ and the multiplication by elements in $K$. Therefore $G$ is an algebraic subgroup of $\mathrm{GL}_r(A/I)$. □

***Remarks 1.5*** **(Isogenies and isomorphisms)**

(1) An *isogeny* $u:\phi\to\phi'$ between two Krichever differential modules $\phi,\phi' : A\to K[\partial]$ is a non zero element of $K[\partial]$ satisfying $u\cdot\phi(a) = \phi'(a)\cdot u$ for all $a\in A$. This formula implies that $\deg\phi(a) = \deg\phi'(a)$ for all $a\in A$. Therefore $\phi$ and $\phi'$ have the same rank. An isogeny $u:\phi\to\phi'$ is called an *isomorphism* if $u\in K^*$.

(2) We will prove the following result:

*Let $u:\phi\to\phi'$ be an isogeny. There exists a $v\in K[\partial]$, $v\neq 0$ and an $a\in A$ such that $v\cdot u = \phi(a)$.*

Consider $\ker(u) := \{f\in \mathrm{Univ}_K \mid u(f)=0\}$. For any $b\in A$ and any $f\in\ker(u)$ one has $\phi(b)(f)\in\ker(u)$. Indeed $u\cdot\phi(b) = \phi'(b)\cdot u$. Hence $\ker(u)$ is an $A$-module of finite dimension over $C$. Therefore there is an $a\in A$, $a\neq 0$ with $\phi(a)(\ker(u)) = 0$. In particular, $\ker(u)\subset\ker(\phi(a))$. This implies that $\phi(a) = v\cdot u$ for some $v\in K[\partial]$.

***Examples 1.6*** **(The rank one Krichever modules $\phi: C[t]\to C(x)[\partial]$)**

Put $\phi(t) = a_0 + a_1\partial$ with $a_0,a_1\in C(x)$ and $a_1\neq 0$. For every $f\in C[t]\smallsetminus C$, the differential operator $\phi(f)$ has an abelian differential Galois group. We want to determine the Picard-Vessiot ring $\mathrm{Univ}_{C(x),\phi}$ obtained from this category of differential operators. In $\mathrm{Univ}_{C(x)}$ we consider the following elements:

(a) an invertible $f$, solution of the operator $\phi(t)$.

(b) invertible solutions $y(c)$ of $y(c)' = \frac{c}{a_1}y(c)$ for all $c\in C$. We impose the rules (as we may) $y(0)=1$ and $y(c_1+c_2) = y(c_1)y(c_2)$.

(c) a solution $z$ of $z' = 1/a_1$.

*We claim that*

$$\mathrm{Univ}_{C(x),\phi} = C(x)[f, f^{-1}, \{y(c)\}_{c\in C}, z].$$

We observe that $fy(c)$ is a solution of the operator $\phi(t-c)$ and $fz$ is a solution of the operator $\phi(t^2)$. This proves one inclusion. The elements $fy(c)z^i$, $i=0,\dots,n-1$ are solutions of the operator $\phi((t-c)^n)$. We will show, by induction on $n$, that the elements $1,z,\dots,z^n$ are linearly independent over $C$ for every $n\geqslant 1$. For $n=1$ this is obvious. Suppose that $1,\dots,z^{n-1}$ are linearly independent over $C$ and consider an expression $c_0+c_1z+\cdots+c_nz^n=0$ with $c_0,\dots,c_n\in C$. Differentiation yields that $\frac{1}{a_1}(c_1+2c_2z+\cdots+nc_nz^{n-1})=0$ and so $c_1=\cdots=c_n=0$. Hence $c_0=0$, too. This shows the other inclusion and proves the claim.

The structure of the differential ring $C(x)[f,f^{-1},\{y(c)\}_{c\in C},z]$ depends of course on $a_0,a_1$. For "generic" $a_0,a_1$, the only relations is this ring are the ones imposed above, namely $y(0)=1$ and $y(c_1+c_2)=y(c_1)y(c_2)$. The differential Galois group is then a product of $\mathbf{G}_m$ (this factor comes from $f$), $\mathbf{G}_a$ (this factor comes from $z$) and $\mathrm{Spec}(C[C])$ (this factor comes from $\{y(c)\}_{c\in C}$), where $C[C]$ is the group algebra of the group $C$ over the field $C$.

In special cases, the factor $\mathbf{G}_m$ can become a finite group if the equation $y'=\frac{-n\cdot a_0}{a_1}y$ has a non zero solution in $C(x)$ for some integer $n\geqslant 1$. If $a_0$ happens to be an element of $C$, then the factor $\mathbf{G}_m$ disappears. Furthermore, the factor $\mathbf{G}_a$ disappears if $y'=1/a_1$ happens to have a solution in $C(x)$. Finally, it is possible that $y'=\frac{c}{a_1}y$ has a non zero solution in $C(x)$ for some $c\in C$, $c\neq 0$. In this case the group $\{c\in C\mid y'=\frac{c}{a_1}y$ has a solution in $C(x)\}$ is generated by some element $c_0\neq 0$. One finds a relation $y(c_0)=g$ for a suitable $g\in C(x)^*$.

The set $V\subset \mathrm{Univ}_{C(x)}$ of all solutions of all $\phi(f)$ has a $C[t]$-action and according to proposition 1.4, $V$ is isomorphic to $C(t)/C[t]$. The differential Galois group $G$ for the collection of all these differential equations is an algebraic subgroup of the automorphism group of $C(t)/C[t]$. The latter is equal to $\widehat{C[t]}^*$, where $\widehat{C[t]}$ is the projective limit of all $C[t]/(f)$. One observes that $G$, which is the group of the differential automorphisms of $\mathrm{Univ}_{C(x),\phi}$, is much smaller (even in the generic case) than $\widehat{C[t]}^*$. Indeed, only one additive equation is needed and there is (at most) one factor $\mathbf{G}_a$ in this group.

It is clear that the "torsion" of a single Krichever module $\phi$ of rank one cannot generate the full $\mathrm{Univ}_{C(x),\mathrm{ab}}$. One reason for this is that the differential operators $\phi(f)$ (with $f\in C[t]$) have their singularities in the union of the set of poles of $a_0/a_1$, the set of zeros of $a_1$ and possibly $\infty$. If one considers the differential operators $\phi(f)$ for all $f\in C[t]$ and all $\phi$, then one obtains the Picard-Vessiot ring $\mathrm{Univ}_{C(x),\mathrm{ab}}$. Even in the local situation, *i.e.*, $K=C((x))$, the set of differential operators $\phi(f)$ (all $f\in C[t]\smallsetminus C$) and a single Krichever module $\phi$, given by $\phi(t)=a_0+a_1\partial$, will not produce the local universal abelian Picard-Vessiot ring $\mathrm{Univ}_{C((x)),\mathrm{ab}}$. Indeed, the set of equations $y'=\frac{-a_0+c}{a_1}y$ (all $c\in C$) and $y'=1/a_1$, is too small for that.

We compare this with the Drinfel'd situation, where $\phi:\mathbf{F}_q[t]\to\mathbf{F}_q(t)[\tau]$ is a Drinfel'd module of rank one. Then $\phi(t)=t+a\tau$. Let $\phi'$ be another rank one

Drinfel'd module, given by $\phi'(t) = t + a'\tau$. Then $\phi$ and $\phi'$ become isomorphic after taking the cyclic extension $\mathbf{F}_q(t)(b)$ of $\mathbf{F}_q(t)$ of degree dividing $q-1$, given by the equation $ab^{q-1} = a'$. Hence the torsion points of $\phi$ and $\phi'$ produce essentially the same infinite abelian extension of $\mathbf{F}_q(t)$. For the case $\phi(t) = t - t\tau$, this extension is characterized as the maximal abelian extension of $\mathbf{F}_q(t)$ which is totally split at the place $\infty$. □

***Remarks 1.7*** **(Some interpretations of** $\mathrm{Univ}_{C(x),\phi}$**).** — The term $a_0$ in the operator $\phi(t) = a_0 + a_1\partial$ is of minor importance for $\mathrm{Univ}_{C(x),\phi}$. In the following examples we will take $a_0 = 0$.

(1) For the case $a_1 = x$, all $\phi(f)$ are regular singular at the two points $x = 0$ and $x = \infty$ (and regular outside $0, \infty$). The differential ring $C((x)) \otimes_{C(x)} \mathrm{Univ}_{C(x),\phi}$ is the universal extension of $C((x))$ for the collection of all regular differential equations over $C((x))$ (the same holds with $x$ replaced by $x^{-1}$).

(2) Suppose that $a_1 = (x - p_1)\cdots(x - p_s)$ with distinct $p_1, \ldots, p_s$. For each $i$ the differential ring $C((x - p_i)) \otimes_{C(x)} \mathrm{Univ}_{C(x),\phi}$ is again the universal extension of $C((x - p_i))$ for the collection of all regular singular equations at $p_i$. However, the various local solutions, like $(x - p_i)^{1/2}$, $\log(x - p_i)$ et cetera, are not independently present in $\mathrm{Univ}_{C(x),\phi}$.

(3) In the general case, the interpretation of the differential ring $\mathrm{Univ}_{C(x),\phi}$ remains unclear. Take for instance $a_1 = x^2$. Then $\mathrm{Univ}_{C(x),\phi} = C(x)[\{y(c)\}_{c\in C}]$ where the $y(c)$ satisfy $y(c)' = \frac{c}{x^2}\,y(c)$. The $y(c)$ are solutions of some irregular equations at $x = 0$.

## 2. Abelian difference equations

The field $C$ is algebraically closed and has characterstic 0. The two types of difference equations, namely ordinary difference equations and $q$-difference equations, that we will study correspond to the two automorphisms $\sigma$ and $\sigma_q$ of $C(x)$ given by $\sigma x = x + 1$ and $\sigma_q x = qx$. It is assumed that $q \in C^*$ is not a root of unity. Difference Galois theory, as developed in **[P-S]**, provides an adequate Picard-Vessiot theory and difference Galois groups. We note that ($q$-)difference equations is a *multiplicative* theory. Let us make this explicit. A ($q$-)difference module is a vector space $M$ over $K$ of finite dimension (and $K = C(x)$ or $C((x^{-1}))$ or $C((x))$), provided with an additive invertible operator $\Phi : M \to M$ satisfying $\Phi(fm) = \sigma(f)\Phi(m)$ (or $\Phi(fm) = \sigma_q(f)\Phi(m)$) for $m \in M$ and $f \in K$. The tensor product of two ($q$)-difference modules $M$ and $N$ is the vector space $M \otimes_K N$ provided with the operation $\Phi$ given by $\Phi(m \otimes n) = \Phi(m) \otimes \Phi(n)$.

An important difference with differential Galois theory is that Picard-Vessiot rings are reduced but may have zero divisors. The role of the differential Picard-Vessiot field is taken over by the total ring of quotients of the Picard-Vessiot ring. Lemma 1.1

and its proof remain valid for ($q$-)difference equations. Indeed, the one ingredient for this proof, which is not available in [**P-S**], is the following theorem.

***Theorem 2.1***. — *Let $K$ be a field of characteristic zero provided with an automorphism $\sigma$ of infinite order. The field of constants $C = \{a \in K \mid \sigma a = a\}$ is supposed to be algebraically closed. Let $M$ be a difference module over $K$ with Picard-Vessiot ring $R$ and difference Galois group $G$. Let $\mathrm{Qt}(R)$ denote the total ring of quotients of $R$. Then the following properties for an element $f \in \mathrm{Qt}(R)$ are equivalent:*

(1) *$f \in R$.*
(2) *The $C$-vector space generated by the orbit $\{gf \mid g \in G\}$ is finite dimensional.*
(3) *The $K$-vector space generated by $\{\sigma^n f \mid n \in \mathbf{Z}\}$ is finite dimensional.*

For the differential case, this theorem is proved in [**P-S03**], Corollary 1.38. This proof can be adapted for the difference case.

We will make the universal Picard-Vessiot ring $\mathrm{Univ}_{\mathrm{ab}}$ for the category of the ($q$-)difference modules with abelian difference Galois group explicit as well as the structure of the universal abelian ($q$-)difference Galois group $G_{\mathrm{univ,ab}}$.

**2.1. Abelian $q$-difference equations.** — As in section 1, we have to consider separately, two types of equations:

(a) (multiplicative equations) $\sigma_q y = ay$ with $a \in K^*$.
(b) (additive equations) $\sigma_q y = y + b$ with $b \in K$.

For $K$ we have the possibilities: $K = C(x)$, $K = C((x))$ and $K = C((x^{-1}))$. For the multiplicative equations we consider the exact sequence

$$1 \longrightarrow K^*/C^* \xrightarrow{\alpha} K^* \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow 1,$$

where $\alpha$ is given by $f \mapsto \sigma_q f/f$. As before, $\mathrm{Isom}_{K,1}$ is the group of the isomorphism classes of the 1-dimensional $q$-difference modules. For the field $K = C((x))$ one calculates that the image of $\alpha$ is equal to the subgroup $q^{\mathbf{Z}} \times (1 + xC[[x]])$ of $K^*$. Hence $\mathrm{Isom}_{K,1} \cong x^{\mathbf{Z}} \times C^*/q^{\mathbf{Z}}$. Hence the local multiplicative equations that contribute to the universal abelian Picard-Vessiot ring are $\sigma_q y = xy$ and $\sigma_q y = cy$ with $c \in C^*$.

The same description holds for the field $K = C((x^{-1}))$.

For the field $K = C(x)$ the description of $\mathrm{Isom}_{K,1}$ is more involved. One considers the exact sequence of groups

$$1 \longrightarrow C^* x^{\mathbf{Z}} \longrightarrow C(x)^* \longrightarrow \oplus_{\overline{c} \in C^*/q^{\mathbf{Z}}} \mathrm{Div}(\overline{c}) \longrightarrow 0.$$

Here $\overline{c}$ is seen as a subset $cq^{\mathbf{Z}}$ of $C^*$ and $\mathrm{Div}(\overline{c})$ is the group of the divisors on this subset. An element $f \in C(x)^*$ is mapped to $(\mathrm{div}_{\overline{c}}(f))_{\overline{c}}$, where $\mathrm{div}_{\overline{c}}(f)$ is the restriction of the divisor of $f$ to the set $\overline{c}$. We note that $\oplus_{\overline{c}} \mathrm{Div}(\overline{c})$ is equal to the group of the divisors on $C^*$. Every divisor $\sum_i n_i[\alpha_i]$ on $C^*$ is the divisor of $\prod_i \big((x-\alpha_i)/x\big)^{n_i} \in C(x)^*$. In particular, the above sequence splits an this leads to an exact sequence

$$1 \longrightarrow C^*/q^{\mathbf{Z}} \times x^{\mathbf{Z}} \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow \oplus_{\overline{c}} \mathbf{Z} \longrightarrow 0.$$

The main calculation takes place for the group $\mathrm{Div}(\overline{c})$. One has to calculate the cokernel of the operator $-1+\sigma_q$. One observes that $\mathrm{Div}(\overline{c})$ is in fact a free module over $\mathbf{Z}[\sigma_q, \sigma_q^{-1}]$, generated by the divisor $[c]$. Hence the cokernel of $-1+\sigma_q$ is isomorphic to $\mathbf{Z}[\sigma_q, \sigma_q^{-1}]/(-1+\sigma_q) \cong \mathbf{Z}$. Since the above exact sequence splits one has $\mathrm{Isom}_{K,1} \cong C^*/q^{\mathbf{Z}} \times x^{\mathbf{Z}} \oplus_{\overline{c} \in C^*/q^{\mathbf{Z}}} \mathbf{Z}$. More explicitly, the multiplicative equations that one has to consider for the construction of $\mathrm{Univ}_{K,\mathrm{ab}}$ are:

(i) $\sigma_q y = xy$,
(ii) $\sigma_q y = \dfrac{x-c}{x} y$ for $c$ in a set of representatives of $C^*/q^{\mathbf{Z}}$ and
(iii) $\sigma_q y = cy$ with $c \in C^*$, again in a set of representatives of $C^*/q^{\mathbf{Z}}$.

For later use we introduce names for the invertible solutions in $\mathrm{Univ}_{C(x),\,\mathrm{ab}}$ of these equations, namely $f_0$ for (i), $f_c$ for (ii) and $y(c)$ for (iii). They only relations between these elements are:

$\sigma f_c = f_{c/q}$ for $c \in C^*$,
$y(q) = x$ and $y(c_1 c_2) = y(c_1) \cdot y(c_2)$ for all $c_1, c_2 \in C^*$.

Define the $C$-vector space $A(K)$ by the exactness of the sequence

$$0 \longrightarrow K/C \xrightarrow{\ \beta\ } K \longrightarrow A(K) \longrightarrow 0,$$

where $\beta(f) = \sigma_q(f) - f$. Take $K = C(x)$. For the investigation of $A(K)$, which is the cokernel of the map form $K$ to itself, given by $f \mapsto \sigma_q(f) - f$, we use partial fractions. Write $C(x) = \oplus_{m \in \mathbf{Z}} Cx^m \oplus \oplus_{\overline{c} \in C^*/q^{\mathbf{Z}},\, n \geqslant 1} C(x)_{\overline{c},n}$. The space $C(x)_{\overline{c},n}$ consists of the partial fractions $\sum_{m \in \mathbf{Z}} c(m)/(q^m x - c)^n$, where $c \in C^*$ is chosen such that $c \mod q^{\mathbf{Z}} = \overline{c}$ and where all $c(m) \in C$. One observes that the spaces $Cx^m$ and $C(x)_{\overline{c},n}$ are invariant under the action of $\sigma_q$. Hence the cokernel of the operator $-1+\sigma_q$ is the direct sum of the cokernels for each of these spaces. For the spaces $Cx^m$ with $m \neq 0$, the cokernel is 0. On $C \cdot 1$ the action of $-1+\sigma_q$ is trivial. The action of $\sigma_q$ on the space $C(x)_{\overline{c},n}$ makes $C(x)_{\overline{c},n}$ into a module over the ring $C[\sigma_q, \sigma_q^{-1}]$. This module is in fact the free module generated by one element, namely $1/(x-c)^n$. Hence the cokernel of $-1+\sigma_q$ on this vector space is isomorphic to $C[\sigma_q, \sigma_q^{-1}]/(-1+\sigma_q) \cong C$. The cokernel of $-1+\sigma_q$ on $C(x)_{\overline{c},n}$ is represented by the element $1/(x-c)^n$. Therefore we have found a basis for $A(K)$ over $C$, namely the images of the elements 1 and $1/(x-c)^n$ with $n \geqslant 1$ and $c$ in a set of representatives of $C^*/q^{\mathbf{Z}}$. The corresponding additive equations are $\sigma_q y = y + 1$ and $\sigma_q y = y + 1/(x-c)^n$ (for $n \geqslant 1$ and $c$ in a set of representatives of $C^*/q^{\mathbf{Z}}$).

For the field $K = C((x))$ the cokernel of $-1+\sigma_q$ is 1-dimensional and the corresponding additive equation is $\sigma_q y = y + 1$. The same holds for the field $C((x^{-1}))$.

One can combine the above information in a straightforward way to obtain an explicit descriptions of $\mathrm{Univ}_{K,\mathrm{ab}}$ and $G_{\mathrm{univ},K,\mathrm{ab}}$ for the fields $K = C(x)$, $C((x))$, $C((x^{-1}))$.

**2.2. Abelian ordinary difference equations.** — $K = C(x)$ and $\sigma(x) = x + 1$. The methods used in the last subsection lead to explicit formulas for $\mathrm{Isom}_{K,1}$, $A(K)$, $\mathrm{Univ}_{K,\mathrm{ab}}$ and $G_{\mathrm{univ},K,\mathrm{ab}}$.

$$1 \longrightarrow C^* \longrightarrow C(x)^* \longrightarrow \oplus_{\overline{c}\in C/\mathbf{Z}} \mathrm{Div}(\overline{c}) \longrightarrow 0.$$

Here $\overline{c}$ is seen as the subset $c + \mathbf{Z}$ of $C$ and $\mathrm{Div}(\overline{c})$ is the set of the divisors on $\overline{c}$.

$$1 \longrightarrow C^* \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow \oplus_{\overline{c}\in C/\mathbf{Z}} \mathbf{Z} \longrightarrow 0.$$

The multiplicate equations that contribute to the formation of $\mathrm{Univ}_{K,\mathrm{ab}}$ are $\sigma y = (x - c)y$ with $c$ in a set of representatives of $C/\mathbf{Z}$ and moreover the equations $\sigma y = cy$ with $c \in C^*$. We prefer to split up the first collection of multiplicative equations as $\sigma y = \frac{x-c}{x}\, y$ for $c \in C^*$ and $\sigma y = xy$.

For the calculation of the cokernel of $-1 + \sigma$ on $C(x)$ one considers the decomposition

$$C(x) = C[x] \oplus \oplus_{\overline{c}\in C/\mathbf{Z},\ n\geqslant 1} C(x)_{\overline{c},n},$$

where $C(x)_{\overline{c},n}$ consists of the finite expressions $\sum_{m\in\mathbf{Z}} d(m)/(x + m - c)^n$ with all $d(m) \in C$ and with a fixed choice of $c \in \overline{c}$. It is easily seen that $C(x)_{\overline{c},n}$ is a free $C[\sigma, \sigma^{-1}]$-module generated by $1/(x - c)^n$. Hence the cokernel of $-1 + \sigma$ on $C(x)_{\overline{c},n}$ has dimension 1. The corresponding additive equation is $\sigma y = y + 1/(x - c)^n$. Furthermore, $-1 + \sigma$ is surjective on $C[x]$ and has kernel $C \cdot 1$. This describes the additive part of $\mathrm{Univ}_{C(x),\mathrm{ab}}$.

Finally, we consider the difference field $C((x^{-1}))$. The cokernel of $f \mapsto \sigma f/f$ on $C((x^{-1}))^*$ is easily seen to be

$$x^{\mathbf{Z}} \times C^* \times (1 + x^{-1}C[[x^{-1}]])/(1 + x^{-1})^{\mathbf{Z}}(1 + x^{-2}C[[x^{-1}]]).$$

The third group in this product can be identified with $C/\mathbf{Z}$. The corresponding multiplicative equations are: $\sigma y = xy$, $\sigma y = cy$ with $c \in C^*$ and $\sigma y = (1 + cx^{-1})y$ with $c$ in a class of representatives of $C/\mathbf{Z}$. The cokernel of $-1 + \sigma$ on $C((x^{-1}))$ has dimension 1. The corresponding additive equation is $\sigma y = y + x^{-1}$.

**2.3. Krichever difference modules.** — $C$ is an algebraically closed field of characteristic 0. One considers as before a smooth, irreducible, projective curve $X$ over $C$ with a specified point $\infty \in X$. Then $A$ denotes the $C$-algebra of the regular functions on $X \smallsetminus \{\infty\}$. One considers a $C$-algebra $R$ (commutative and with a unit element) provided with a specified $C$-algebra homomorphism $\gamma : A \to R$ and an endomorphism of infinite order $\sigma$. The skew polynomial ring $R[\sigma]$ is defined as usual. The elements are finite formal sums $\sum_{i\geqslant 0} r_i \cdot \sigma^i$ with all $r_i \in R$. The definition of the addition is obvious. The multiplication is induced by the formula $\sigma \cdot r = \sigma(r) \cdot \sigma$. A *Krichever difference module of rank $r$ over $R$* is a $C$-algebra homomorphism $\phi : A \to R[\sigma]$ satisfying:

(a) For $a \in A$, $a \neq 0$, the element $\phi(a)$ has degree $r \cdot \deg(a)$, and the leading coefficient of $\phi(a)$ is invertible,

(b) $\phi(a) = \gamma(a) + *\sigma + *\sigma^2 + \cdots$

With the above data, one considers $R[\sigma]$ as a left $R[\sigma] \otimes_C A$-module by the formula $(f \otimes a)g = f \cdot g \cdot \phi(a)$ for $f, g \in R[\sigma]$ and $a \in A$. For any non zero ideal $I \subset A$, the subset $R[\sigma]\phi(I)$ of $R[\sigma]$ is also a left $R[\sigma] \otimes_C A$-module. Hence $R[\sigma]/R[\sigma]\phi(I)$ is a again a left $R[\sigma] \otimes_C A$-module.

A *level $I$-structure for $\phi$* is an isomorphism of left $R[\sigma] \otimes_C A$-modules

$$\lambda : R \otimes_C (I^{-1}/A)^r \longrightarrow R[\sigma]/R[\sigma]\phi(I).$$

One regards $R$ as a left $R[\sigma]$-module by the formula $(\sum r_i \sigma^i)r = \sum r_i \sigma^i(r)$. This explains also the left $R[\sigma] \otimes_C A$-module structure on $R \otimes_C (I^{-1}/A)^r$.

The two basic cases are $A = C[x]$ and $(R, \sigma)$ is an extension of

(a) $(C(x), \sigma)$ with $\sigma(x) = x + 1$ or

(b) $(C(x), \sigma_q)$ with $\sigma_q(x) = qx$ ($q \neq 0$ and $q$ not a root of unity).

In the above cases, $\gamma : A = C[x] \to R$ is injective. We note that more general situations, e.g., $\ker(\gamma) \neq \{0\}$ or $A \neq C[x]$, seem of interest.

As for Drinfel'd modules with a suitable level structure, one may try to show the existence of a universal Krichever difference module of a fixed rank and provided with a suitable level structure. In other words, one may want to construct a fine moduli space.

We consider the basic example: $\phi$ of rank 1, $(R, \sigma)$ of type (a) or (b) and with level structure $\lambda$ for the ideal $(x)$. Put $\phi(x) = x + a\sigma$ with $a \in R^*$. The level structure $\lambda$ prescribes an element (say) $b \in R^*$ with $xb + a\sigma(b) = 0$. The universal object that one can make for this situation is a ring $U = C(x)[\{b_n\}_{n \geqslant 0}, \{b_n^{-1}\}]$ (*i.e.*, $U$ is a polynomial ring over $C(x)$ in the variables $\{b_n\}_{n \geqslant 0}$, localized at the set of all $b_n$). The operation $\sigma$ on $U$ is given by $\sigma(b_n) = b_{n+1}$ for every $n \geqslant 0$. The universal $\phi$ is given by $\phi(x) = x - x\frac{b_0}{b_1}\sigma$ and $b_0$ is the prescribed invertible solution of $\phi(x)y = 0$. We observe that $U$ is not finitely generated over the field $C(x)$ and so the "fine moduli space" $\mathrm{Spec}(U)$ is not of finite type. It is however of "finite difference type." The definition of this concept can be given as follows. A *free difference algebra of finite type over* $C(x)$ is an the polynomial ring $R = C(x)[\{b(i)_n\}_{i=1,\ldots,s,\ n \geqslant 0}]$ in the variables $\{b(i)_n\}$. Moreover, the endomorphism $\sigma$ on $R$ extends the given endomorphism of $C(x)$ (*i.e.*, either $\sigma(x) = x + 1$ or $\sigma(x) = qx$) and satisfies $\sigma(b(i)_n) = b(i)_{n+1}$ for all $n \geqslant 0$. A *difference algebra of finite type over* $C(x)$ is the quotient of a free difference algebra of finite type over $C(x)$ by an ideal that is invariant under $\sigma$.

***Theorem 2.2***. — *Let the ring $A$, the ideal $I \subset A$ (with $I \neq A, \{0\}$) and the rank $r$ be given. There exists a universal Krichever difference module of rank $r$ and level $I$ (of* *type (a) or (b)). The corresponding universal ring $U$ is of finite difference type over $C(x)$.*

*Proof.* — The proof is analogous to the one given in [**L1**] for the case of differential Krichever modules. We only sketch the proof for the difference case. One wants to obtain a universal $U, \sigma, \phi, \lambda$. Write $A = C[f_1, \ldots, f_s] = C[F_1, \ldots, F_s]/(G_1, \ldots, G_t)$. For the definition of the universal $\phi$ one has to introduce variables for the coefficients of all $\phi(f_i)$. The variables corresponding to the leading coefficients should be invertible. Moreover one has to introduce the $\sigma^i$-images of all these variables. For the formulation of the level structure $\lambda : U \otimes_C (I^{-1}/A)^r \to U[\sigma]/U[\sigma]\phi(I)$ one has to introduce again a set of variables and all their $\sigma^i$-images. This produces a free difference algebra of finite type over $C(x)$ (suitably localized). In this free object one has to divide out the ideal given by all relations $G_1, \ldots, G_t$ and the relations which make the map $\lambda$ into an isomorphism of left modules over $U[\sigma] \otimes_C A$. We note that $I \neq A$ garantees that the Krichever difference modules with level $I$ structure have no automorphisms. $\square$

**2.4. Krichever $q$-difference modules of rank one.** — We consider the rank one Krichever module $\phi : C[x] \to C(x)[\sigma]$ with $\phi(x) = x + a\sigma$ for the case $\sigma = \sigma_q$. The interesting question is the structure of the subring $\mathrm{Univ}_\phi$ of $\mathrm{Univ}_{C(x),\mathrm{ab}}$, generated by all the solutions of all $\phi(f)$ (with $f \in C[x] \smallsetminus C$). In the sequel we will use the notations of subsection 2.1.

First we consider the multiplicative equations involved in $\phi$. They are $\sigma_q(y) = \frac{-x}{a} y$ and $\sigma(y) = \frac{c-x}{a} y$ for all $c \in C^*$. Let $g \in \mathrm{Univ}_{C(x),\,\mathrm{ab}}$ be an invertible solution of the first equation. Then the solution of the other equations are $g \cdot f_c$ with $c \in C^*$. Hence $\{f_c\}_{c \in C^*} \subset \mathrm{Univ}_\phi$. Using the $f_c$'s one can modify the first equation into an equation of the form $\sigma_q y = dx^n y$ for certain $d \in C^*$ and $n \in \mathbf{Z}$. One concludes that $\mathrm{Univ}_\phi$ contains almost the complete multiplicative part of $\mathrm{Univ}_{C(x),\,\mathrm{ab}}$. The only possible exceptions are the elements $f_0$ and the $\{y(c)\}$. In case $a = -x$ this is precisely the part of the multiplicative symbols of $\mathrm{Univ}_{C(x),\mathrm{ab}}$ that is missing in $\mathrm{Univ}_\phi$.

Now we investigate the additive equations involved in $\phi$. They are incorporated in the difference operators $\phi((x-c)^n)$. An invertible solution $h$ of the operator $\phi(x-c)$ is $gf_c$ if $c \in C^*$ and $g$ otherwise. Define $z_0 = 1$ and for $n \geqslant 1$ the elements $z_n \in \mathrm{Univ}_{C(x),\,\mathrm{ab}}$ by the formula $\phi(x-c)hz_n = hz_{n-1}$. Then all $z_n \in \mathrm{Univ}_\phi$. One has the following recurrence relation

$$z_0 = 1 \quad \text{and} \quad (x-c)(1-\sigma_q)z_n = z_{n-1} \text{ for } n \geqslant 1.$$

For every $n \geqslant 1$, there are polynomial expression $A_n$ in $z_1, \ldots, z_n$ such that $\sigma_q A_n = A_n + 1/(x-c)^n$. A calcuation yields

$$A_1 = z_1, \quad A_2 = z_1^2 - 2z_2, \quad A_3 = z_1^3 - 3z_1z_2 + 3z_3.$$

For the general case we consider a "generating function" $F := \sum_{n \geqslant 0} z_n T^n$ which belongs to $\mathrm{Univ}_\phi[[T]]$. The action of $\sigma_q$ on $T$ is supposed to be trivial, *i.e.*, $\sigma_q T = T$.

One observes that $(x-c)(1-\sigma_q)F = T\cdot F$ and thus $\sigma_q F/F = (1-T/(x-c))$. Write $F=\prod_{n\geqslant 1}(1+a_nT^n)$ (with all $a_n\in \mathrm{Univ}_\phi$). Then

$$\prod_{n\geqslant 1}\Big(1-\frac{(1-\sigma_q)a_nT^n}{(1+a_nT^n)}\Big) = 1-\frac{T}{(x-c)}.$$

Taking $-\log$ on both sides of this formula one obtains

$$\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{1}{m}\Big(\frac{(1-\sigma_q)a_nT^n}{1+a_nT^n}\Big)^m = \sum_{m\geqslant 1}\frac{T^m}{m(x-c)^m}.$$

Comparing the coefficient of $T^s$ on both sides implies that:
$(1-\sigma_q)a_s+$ a polynomial expression in the $a_1,\ldots,a_{s-1}$ and their images under $\sigma_q$ is equal to $1/(x-c)^s$. We conclude from this that $\mathrm{Univ}_\phi$ contains all the additive symbols of $\mathrm{Univ}_{C(x),\mathrm{ab}}$ with the exception of the symbol corresponding to $\sigma_q y = y+1$.

For the formulation of the theorem below we introduce the following terminology. A ring $R$ with $C(x)\subset R\subset C((x^{-1}))$ will be called an *abelian difference subring* if:

(a) $\sigma_q(R)=R$,

(b) For every finite dimensional $C(x)$-vector space $V\subset R$ the $C(x)$-vector space $\widetilde{V}$ generated by $\{\sigma_q^n r \mid n\in\mathbf{Z},\ r\in V\}$ is finite dimensional, and

(c) the group of the $C(x)$-automorphisms of the field $C(x)(\widetilde{V})$ which commute with $\sigma_q$ is abelian.

It follows from this definition that $C(x)(\widetilde{V})$ is the field of fractions of the Picard-Vessiot ring of a $q$-difference equation over $C(x)$ with abelian difference Galois group. Furthermore there is a unique maximal abelian difference subring of $C((x^{-1}))$, namely the one generated by all multiplicative equations $\sigma_q y = ay$ and all additive equations $\sigma_q y = y+b$ over $C(x)$ that have a (non trivial) solution in $C((x^{-1}))$. The next proposition easily follows from the preceding calculations

***Theorem 2.3***. — *The rank one Krichever q-difference module*

$$\phi : C[x]\longrightarrow C(x)[\sigma_q] \text{ with } \phi(x) = x - x\sigma_q \text{ satisfies:}$$

$\mathrm{Univ}_\phi$ *has an* $\sigma_q$*-equivariant embedding in* $C((x^{-1}))$. *Its image is the maximal abelian difference subring of* $C((x^{-1}))$.

In the general case $\psi(x)=x+a\sigma_q$ with $a\in C(x)^*$, one observes that $\mathrm{Univ}_\psi$ contains an invertible element $b$ with $\sigma_q b = \frac{-x}{a}\,b$. Moreover $b^{-1}\psi(x)b=\phi(x)$. Hence $\mathrm{Univ}_\psi = \mathrm{Univ}_\phi[b,b^{-1}]$. Using the elements in $\mathrm{Univ}_\phi$ one can transform the equation $\sigma_q y = \frac{-x}{a}\,y$ in to an equation $\sigma_q y = cx^n y$ for suitable $c\in C^*$ and $n\in\mathbf{Z}$. The Picard-Vessiot ring of this equation over $C(x)$ is $C(x)[b,b^{-1}]$ and $\mathrm{Univ}_\psi = C(x)[b,b^{-1}]\otimes_{C(x)}\mathrm{Univ}_\phi$. We note that the difference Galois group $G$ of $\sigma_q y = cx^n y$ over $C(x)$ is a subgroup of $\mathbf{G}_m$. If $cx^n$ is not a root of unity, then $G=\mathbf{G}_m$ and $b$ is transcendental over $C(x)$. If $cx^n = d$ where $d$ is an $m^{\mathrm{th}}$root of unity, then $G$ is a cyclic group of order $m$. Moreover the ring $C(x)[b,b^{-1}] = C(x)[T,T^{-1}]/(T^m-1)$ has zero divisors if $m>1$.

We conclude that $\mathrm{Univ}_\psi$ is equal to $C(x)[b,b^{-1}]\otimes_{C(x)}U$ with $C(x)[b,b^{-1}]$ as above and $U$ the maximal abelian difference subring of $C((x^{-1}))$.

***Remarks 2.4* (Comparison with rank one Drinfel'd modules).** — The rank one Drinfel'd module $\phi : \mathbf{F}_q[t] \to \mathbf{F}_q(t)[\tau]$, given by $\phi(t) = t - t\tau$, has the property that the field extension of $\mathbf{F}_q(t)$ obtained by all the torsion elements for $\phi$ is the maximal abelian extension of $\mathbf{F}_q(t)$ which is totally split at $t=\infty$ (see [**G-P-R-V**]). Theorem 2.3 is the perfect analogue of this.

**2.5. Krichever difference modules of rank 1.** — The same questions as in the last subsection are studied but now for $\sigma$ given by $\sigma(x)=x+1$. Let $\phi(x)=x+a\sigma$. The multiplicative equations involved in $\mathrm{Univ}_\phi$ are $\sigma y = \frac{c-x}{a}y$. This can be separated into the equations $\sigma y = \frac{-x}{a}y$ and $\sigma y = \frac{x-c}{x}y$ for all $c\in C^*$. The last collection of equations produces all multiplicative equations for $C(x)$ with the exception of $\sigma y = xy$ and $\sigma y = cy$ with $c\in C^*$. For the additive equations, incorporated in the difference operators $\phi((x-c)^n)$, one finds, as in the $q$-difference case, all the additive equations $\sigma y = y + 1/(x-c)^n$ with $c\in C$ and $n\geqslant 1$. One observes that $\mathrm{Univ}_\phi$ involves equations that have no (non trivial) solution in $C((x^{-1}))$. The equations are $\sigma y = (1+cx^{-1})y$ for $c\in C^*$ and $\sigma y = y + x^{-1}$. We enlarge $C((x^{-1}))$ to $C((x^{-1}))[\{x^c\}_{c\in C},\log(x)]$. This algebra is well known from the theory of differential equations. It is the universal Picard-Vessiot extension for the collection of all regular singular differential equations over $C((x^{-1}))$. The formal rules are $x\frac{d}{dx}(x^c)=cx^c$, $x\frac{d}{dx}\log(x)=1$ and relations $x^{c_1}x^{c_2}=x^{c_1+c_2}$, $x^c=x^n\in C((x^{-1}))$ if $c$ is equal to the integer $n$. Their behaviour under $\sigma$ is given by $\sigma(x^c)=(1+x^{-1})^c x^c$ and $\sigma\log(x)=\log(x)+\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n}x^{-n}$. As for $q$-difference equations one defines abelian difference subrings of $C((x^{-1}))[\{x^c\}_{c\in C},\log(x)]$. The maximal abelian difference subring is denoted by $U$.

***Theorem 2.5.*** — *The rank one Krichever difference module*

$$\phi : C[x] \longrightarrow C(x)[\sigma] \text{ with } \phi(x) = x - x\sigma \text{ satisfies:}$$

$\mathrm{Univ}_\phi$ *has a $\sigma$-equivariant embedding in $C((x^{-1}))[\{x^c\}_{c\in C},\log(x)]$. Its image is the maximal abelian difference subring $U$.*

For a general rank one Krichever difference module $\psi$, given by $\psi(x)=x+a\sigma$, one has that $\mathrm{Univ}_\psi = C(x)[b,b^{-1}]\otimes_{C(x)}U$, where $C(x)[b,b^{-1}]$ is the Picard-Vessiot ring for the equation $\sigma y = Ay$ (suitable $A\in C(x)^*$) and $U$ is the maximal abelian difference subring defined above.

## References


[D-R-S-Z] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – "Équations aux $q$-différences", *Gaz. Math.* **96** (2003), et laboratoire Émile Picard, U.F.R. M.I.G. Toulouse, (2003).

[Dr] V.G. Drinfel'd – "Commutative subrings of certain non commutative rings", *Functional Anal. Appl.* **10** (1977), p. 9–12.

[F-G-K-V] E. Frenkel, D. Gaitsgory, D. Kazhdan & K. Vilonen – "Geometric realization of Whittaker functions and the Langlands conjecture", *J. Amer. Math. Soc.* **11** (1988), p. 451–484.

[F-G-V] E. Frenkel, D. Gaitsgory & K. Vilonen – "On the geometric Langlands conjecture", *J. Amer. Math. Soc.* **15** (2002), p. 367–417.

[G-P-R-V] U.E. Gekeler, M. van der Put, M. Reversat & J. van Geel (eds.) – *Drinfeld Modules, Modular Schemes and Applications*, World Scientific Publications Co, Inc., River Edge, 1997.

[Kr] I.M. Krichever – "Methods of algebraic geometry in the theory of non-linear equations", *Russian Math. Surveys* **32** (1977), no. 6, p. 185–213.

[L1] G. Laumon – Preprint Orsay, 1986.

[L2] ______, "Correspondance de Langlands géométrique pour les corps de fonctions", *Duke Math. J.* **54** (1987), no. 2, p. 309–351.

[Mu] D. Mumford – "An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations", in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977)*, 1977, p. 115–153.

[P-S] M. van der Put & M.F. Singer – *Galois Theory of Difference Equations*, Lect. Notes in Math., vol. 1666, Springer-Verlag, Berlin, 1997.

[P-S03] ______, *Galois Theory of Linear Differential Equations*, Ergebnisse der Mathematik, Springer Verlag, 2003.



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