Astérisque

CARLOS GUTIERREZ
ALBERTO SARMIENTO

Injectivity of C^1 Maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity and planar vector fields

Astérisque, tome 287 (2003), p. 89-102
<http://www.numdam.org/item?id=AST_2003__287__89_0>

Astérisque
287, 2003, p. 89–102

INJECTIVITY OF C^1 MAPS $\mathbb{R}^2 \to \mathbb{R}^2$ AT INFINITY AND PLANAR VECTOR FIELDS

by

Carlos Gutierrez & Alberto Sarmiento

Abstract. — Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map, where $\sigma > 0$ and $\overline{D}_\sigma = \{p \in \mathbb{R}^2 : \|p\| \leqslant \sigma\}$.
(i) If for some $\varepsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of $DX(p)$ belongs to $(-\varepsilon, \infty)$, there exists $s \geqslant \sigma$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;
(ii) If for some $\varepsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of $DX(p)$ belongs to $(-\varepsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geqslant 0\}$, there exists $p_0 \in \mathbb{R}^2$ such that the point ∞, of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$, is either an attractor or a repellor of $x' = X(x) + p_0$.

1. Introduction

The study of planar vector fields around singularities has somehow motivated the present work. A sample of this study is the work done by C. Chicone, F. Dumortier, J. Sotomayor, R. Roussarie, F. Takens. See for instance [**Chi, DRS, Rou, Tak**]. Here we study the behavior of a vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ around infinity. While a C^1 vector field around a singularity is quite regular, we work under conditions that do not imply, a priori, any regularity of the vector field around infinity. Given an open subset U of $\mathbb{R}^2$ and a C^1 map $Y : U \to \mathbb{R}^2$, we shall denote by $\mathrm{Spec}(Y) = \{\text{eigenvalues of } DY(p) : p \in U\}$. Our main result is the following

Theorem 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map, where $\sigma > 0$ and $\overline{D}_\sigma = \{p \in \mathbb{R}^2 : \|p\| \leqslant \sigma\}$. The following is satisfied:*

(i) *if for some $\varepsilon > 0$, $\mathrm{Spec}(X)$ is disjoint of $(-\varepsilon, \infty)$, then there exists $s \geqslant \sigma$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;*

2000 ***Mathematics Subject Classification.*** — 37E35, 37C10.
Key words and phrases. — Injectivity, Reeb component, vector fields.

C.G.: Supported in part by PRONEX/FINEP/MCT - grant number 76.97.1080.00.

(ii) *if for some* $\varepsilon > 0$, $\operatorname{Spec}(X)$ *is disjoint of* $(-\varepsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geqslant 0\}$, *then, there exists* $p_0 \in \mathbb{R}^2$ *such that the point* ∞, *of the Riemann sphere* $\mathbb{R}^2 \cup \{\infty\}$,*is either an attractor or a repellor of* $x' = X(x) + p_0$.

To give an idea of the proof of this result, let us introduce the following definition.

Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Since $f : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}$ is a C^1 submersion, $q \in \mathbb{R}^2 \to \nabla f^\#(q) = (-f_y(q), f_x(q))$, the Hamiltonian vector field of f, has no singularities. Let $g_0(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leqslant 2\}.$$

We will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *HRC* (*Half-Reeb Component*) of $\nabla f^\#$ (see figure 1) if there is a homeomorphism $h : B \to \mathcal{A}$ which is a topological equivalence between $\nabla f^\#|_{\mathcal{A}}$ and $\nabla {g_0}^\#|_B$, and such that

(1) $h(\{(x, y) \in B : x + y = 2\})$ (called *the compact edge of* $\mathcal{A}$) is a smooth segment transversal to $\nabla f^\#$ in the complement of $h(1, 1)$, and

(2) both $h(\{(x, y) \in B : x = 0\})$ and $h(\{(x, y) \in B : y = 0\})$ are full half–trajectories of $\nabla f^\#$.

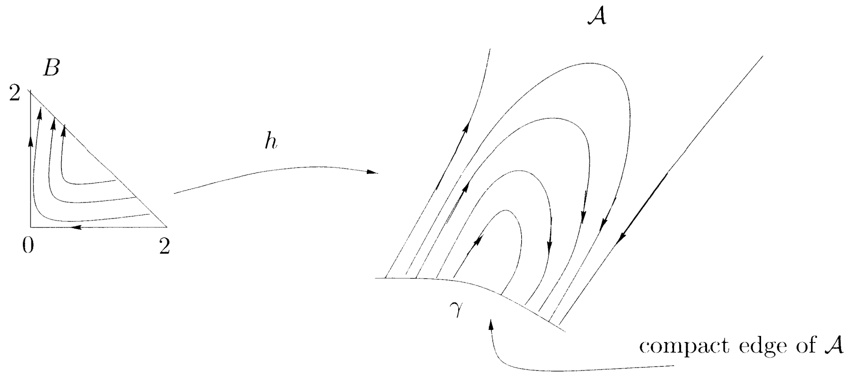

FIGURE 1. A half-Reeb component.

Observe that $\mathcal{A}$ may not be a closed subset of $\mathbb{R}^2$.

Proceed to give an idea of the proof of Theorem 1. First, we shall prove that:

Proposition 1. — *if* $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ *is a* C^1 *map as in Theorem 1, then any HRC of* $\nabla^\# f$ *is a bounded subset of* $\mathbb{R}^2$.

This is used to prove

Theorem 2. — *if* $Y = (\widetilde{f}, \widetilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^1 *map such that, for some* $\varepsilon > 0$, $\operatorname{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \varnothing$, *then* Y *is injective.*

Roughly speaking about Theorem 2, if the foliation induced by $\nabla \widetilde{f}^{\#}$ has no half-Reeb components then, $\nabla \widetilde{f}^{\#}$ is topologically equivalent to the foliation, on the (x, y)-plane, induced by the form dx (the foliation is made up by all the vertical straight lines). The injectivity of X will follow from the fact that $\nabla \widetilde{f}^{\#}$ and $\nabla \widetilde{g}^{\#}$ are linearly independent everywhere.

Sections 3 and 4 are devoted to prove

Corollary 2. — *if $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ is a C^1 map as in Theorem 1, then there exists a smooth compact disc E such that $\nabla f^{\#}$, restricted to $\mathbb{R}^2 \setminus E$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \setminus \overline{D}_1$, induced by dx.*

Observe that the foliation, on $\mathbb{R}^2 \setminus \overline{D}_1$, induced by dx has exactly two tangencies with $\partial \overline{D}_1$ (at $(1, 0)$ and $(0, 1)$) which are "quadratic" and "external". Let us say a little more about what is proved in Section 3 and 4: We show, in Section 3, that given any generic smooth compact disc $F \supset \overline{D}_\sigma$ the number of "external" tangencies of ∇f with ∂F is equal to 2 plus the number of "internal" tangencies of ∇f with ∂F. We show, in Section 4, that the disc F can be deformed to a smooth compact disc E so that the referred "external" and "internal" tangencies cancel in pairs yielding exactly 2 tangencies which are "external".

Using Theorem 2 we obtain

Proposition 2. — *Let X be as in Corollary 2. If X takes ∂E diffeomorphically to a circle then $X|_{\mathbb{R}^2 \setminus E}$ may be extended to a map which satisfies conditions of Theorem 2 and so it is injective.*

The proof of item (ii) of Theorem 1 is finished in Sections 5 and 6 by showing that, under conditions of Corollary 2, the disc E can be deformed so that, for the resulting new disc, still denoted by E, $\nabla f^{\#}|_{\mathbb{R}^2 \setminus E}$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \setminus \overline{D}_1$, induced by dx and moreover X takes ∂E diffeomorphically to a circle. Then the result follows from Proposition 2.

The item (ii) of Theorem 1 follows from the corresponding item (i) and some previous Gutierrez and Teixeira work [**G-T**].

Throughout this article, given an embedded circle $C \subset \mathbb{R}^2$, the compact disc (resp. open disc) bounded by C will be denoted by $\overline{D}(C)$ (resp. $D(C)$). Also, we will freely use the fact that the assumptions of the theorem are open in the Whitney C^1–topology. In this way, when possible and necessary, we will assume that X is smooth and that it satisfies some generic property which will be made precise at the proper place.

Acknowledgements. — We wish to thank referee's comments which have been appreciated and incorporated into this work.

2. A global injectivity result

We shall need the following lemma which is contained in the proof of [**Gut**, Lemma 2.5]. For $\theta \in \mathbb{R}$: let R_θ denote the linear rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Lemma 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Suppose that $\nabla^\# f$ has an HRC which is unbounded (as a subset of $\mathbb{R}^2$) but whose projection on the x-axis is a compact interval. Then, there exists $\varepsilon > 0$ such that, for all $\theta \in (-\varepsilon, 0) \cup (0, \varepsilon)$ $\nabla^\# f_\theta$ has a HRC whose projection on the x-axis is an interval of infinite length; here $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$.*

The proof of Proposition 1 and Theorem 2 can be found in [**CGL**] but, as we have already said and for sake of completeness, they are included here.

Proposition 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Then any HRC of $\nabla^\# f$ is a bounded subset of $\mathbb{R}^2$.*

Proof. — Let $\mathcal{A}$ be a half Reeb component for f. Let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. By composing with a rotation if necessary, in the way that is stated in Lemma 1, we may suppose that $\Pi(\mathcal{A})$ is an interval of infinite length, say $[b, \infty)$. We may also assume that X is smooth and —by Thom's Transversality Theorem for jets [**G-G**]— that

(a1) the set

$$T = \{(x, y) \in \mathbb{R}^2 : f_y(x, y) = 0\}$$

is made up of regular curves;

(a2) There is a discrete subset Δ of T such that if $p \in T \setminus \Delta$ (resp. $p \in \Delta$), $\nabla^\# f$ has quadratic contact (resp. cubic contact) with the vertical foliation of $\mathbb{R}^2$.

Then, if $a > b$ is large enough,

(b) for any $x \geqslant a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\nabla f^\#|_\mathcal{A}$ such that $\Pi(\alpha_x) \cap (x, \infty) = \varnothing$; in other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

It follows that

(c) if $x \geqslant a$ and $p \in \alpha_x \cap \Pi^{-1}(x)$ then $p \in T \cap \mathcal{A} \setminus \Delta$.

Let T_m be the set of $p \in \mathcal{A}$ such that, for some $x \geqslant a$, $p \in \alpha_x \cap \Pi^{-1}(x)$. Notice that, for every $x \geqslant a$, $\alpha_x \cap \Pi^{-1}(x)$ is a finite set; nevertheless, by (b), (c) and by using Thom's Transversality Theorem for jets, we may get the following stronger statement:

(d) There is a sequence $F = \{a_1, a_2, \ldots, a_i, \cdots\}$ in $[a, \infty)$, which may be either empty or finite or else countable, such that if $x \in F$ (resp. $x \in [a, \infty) \setminus F$), then $\Pi^{-1}(x) \cap T_m$ is a two-point-set (resp. a one-point-set).

If $x \in [a, \infty) \setminus F$, define $\eta(x) = (x, \eta_2(x)) = \Pi^{-1}(x) \cap T_m$. Observe that $\eta : [a, \infty) \setminus F \to T_m$ is a smooth embedding. As $f|_{\mathcal{A}}$ is bounded,

(e) $F \circ \eta$ extends continuously to a strictly increasing bounded map defined in $[a, \infty)$ such that, for all $x \in [a, \infty) \setminus F$, $f_x(\eta(x))$ has constant sign.

Therefore, there exists a real constant K such that

$$\begin{aligned} K = \int_{a_1}^{\infty} \frac{d}{dx} f(\eta(x))\, dx &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} \frac{d}{dx} f(\eta(x))\, dx \\ &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} f_x(\eta(x)) \end{aligned}$$

This and (e) imply that, for some sequence $x_n \to \infty$, $\lim_{n\to\infty} f_x(\eta(x_n)) = 0$. This is the required contradiction. □

Theorem 2. — *Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map. Suppose that, for some $\varepsilon > 0$, $\mathrm{Spec}(X) \cap (-\varepsilon, \varepsilon) = \varnothing$. Then X is injective.*

Proof. — By Proposition 1, the Hamiltonian vector fields induced by the coordinate functions of $X = (f, g)$ have no Reeb component. Therefore X is injective. □

3. Index of a vector field along a circle

We shall say that a collar neighborhood U of an embedded circle $C \subset R^2 \setminus \overline{D}_\sigma$ is *interior* (resp. *exterior*) if U is contained in $\overline{D}(C)$ (resp. $\mathbb{R}^2 \setminus D(C)$).

Proposition 2. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Let $C \subset R^2 \setminus \overline{D}_\sigma$ be a smooth circle surrounding the origin. Suppose that $X(C)$ is an embedded circle and that there exists an exterior collar neighborhood $U \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ of C such that $X(U)$ is also an exterior collar neighborhood of $X(C)$. Then X is an embedding.*

Proof. — By the assumptions, X can be extended to a C^1 map $Y : R^2 \to R^2$ which takes $\overline{D}(C)$ diffeomorphically onto the $\overline{D}(X(C))$. See [**Hir**]. Under these conditions we may apply either Theorem 2 or Gutierrez and Fessler Injectivity Theorem [**Gut, Fes**] to conclude that Y is an embedding and, a fortiori, that X is an embedding too. □

The theorem below on indexes of singularities of vector fields will be used to prove theorem 1. The proof can be found in [**Har**, Theorem 9.2]

Let C be a simple closed curve of R^2. A C^1 vector field $Y : R^2 \to R^2$ is said to be *internally (or externally) tangent to* C at $x_0 \in C$, if there exists an $\varepsilon > 0$ such that the solution arc $\phi(t)$ of the equation

$$x' = Y(x), \qquad x(0) = x_0,$$

is interior (or exterior) to C for $0 < |t| \leqslant \varepsilon$. We shall denote by $j_Y(C)$ the index of Y along C.

Theorem 3. — *Let Y be a C^1 vector field on a connected open set $E \subset R^2$. Let C be a positive oriented Jordan curve of class C^1 in E with the property that $Y(x) \neq 0$ on C and that Y is tangent to C at only a finite number of points $x_1, \ldots, x_n$ of C. Let n^i (resp. n^e be the number of these points x_j where the solution arc $\phi(t)$ of $x' = Y(x)$, $\phi(0) = x_j$ for small $|t|$ is internally (resp. externally) tangent to C at x_j (so that $n^i + n^e \leqslant n$). Then $2j_Y(C) = 2 + n^i - n^e$.*

Corollary 1. — *Let assume the notation and conditions of Theorem 1. In particular let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map. If $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is a smooth circle surrounding the origin, then $j_{\nabla f^\#}(C) = j_{\nabla f}(C) = 0$.*

Proof. — If $j_{\nabla f}(C) \neq 0$ there would exist a point $p \in C$ and a real $a > 0$ such that $\nabla f(p) = (a, 0)$. In particular a will be an eigenvalue of $DX(p)$. This contradiction with the assumptions of Theorem 1 proves the corollary. $\square$

4. Avoiding internal tangencies

We say that $\nabla f^\#$ is *in general position with* an embedded circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ if there exists a subset F of C, at most finite such that (i) $\nabla f^\#$ is transversal to $C \setminus F$, (ii) $\nabla f^\#$ has a quadratic tangency with C at each point of F, and (iii) a trajectory of $\nabla f^\#$ can meet tangentially C at most at one point.

Lemma 2. — *Suppose that $\nabla f^\#$ is in general position with a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ which surrounds the origin. Suppose also that a trajectory γ of $\nabla f^\#$ meets C transversally somewhere and with an external tangency at a point p. Then the trajectory γ contains a closed subinterval $[p, r]_f$ which meets C exactly at $\{p, r\}$ (doing it transversally at r) and the following is satisfied:*

(i) *If $[p, r]$ denotes the closed subinterval of C such that $\Gamma = [p, r] \cup [p, r]_f$ bounds a compact disc $\overline{D}(\Gamma)$ contained in $\mathbb{R}^2 \setminus D(C)$, then points of $\gamma \setminus [p, r]_f$ nearby p do not belong to $\overline{D}(\Gamma)$;*

(ii) *Let $(\widetilde{p}, \widetilde{r})$ and $[\widetilde{p}, \widetilde{r}]$ be subintervals of C satisfying $[p, r] \subset (\widetilde{p}, \widetilde{r}) \subset [\widetilde{p}, \widetilde{r}]$. If $\widetilde{p}$ and $\widetilde{r}$ are close enough to p and r, respectively, then we may deform C into a smooth circle C_1 in such a way that the deformation fixes $C \setminus (\widetilde{p}, \widetilde{r})$ and takes $[\widetilde{p}, \widetilde{r}] \subset C$ to a closed interval $[\widetilde{p}, \widetilde{r}]_1 \subset C_1$ which is close to $[p, r]_f$. Furthermore, $\nabla f^\#$ is in general position with C_1 and the number of tangencies of $\nabla f^\#$ with C_1 is smaller than that of $\nabla f^\#$ with C.*

Proof. — Certainly, γ contains a closed subinterval $[p, r]_f \subset \gamma$ which meets C exactly at $\{p, r\}$. As $\nabla f^\#$ is in general position with C, γ meet C transversally at r. Let

$[p,r]$ be the closed subinterval of C such that $\Gamma = [p,r] \cup [p,r]_f$ bounds a compact disc $\overline{D}(\Gamma)$ contained in $\mathbb{R}^2 \setminus D(C)$.

If $[p,r]_f$ does not satisfy (i) then the points of $\gamma \setminus [p,r]_f$ nearby p belong to $\overline{D}(\Gamma)$. Hence, as $\nabla f^{\#}$ has no singularities in $\mathbb{R}^2 \setminus D(C)$, points of $\gamma \setminus [p,r]_f$ nearby p belong to $\overline{D}(\Gamma)$. Therefore, there must be a closed subinterval $[q,p]_f \subset \gamma \cap \overline{D}(\Gamma)$ (see fig. 2.a) such that:

(a1) the union Γ_1 of the closed interval $[p,q] \subset [p,r]$ and $[p,q]_f$ bounds a compact disc $\overline{D}(\Gamma_1)$ contained in $(\mathbb{R}^2 \setminus D(C)) \cap \overline{D}(\Gamma)$;

(a2) $[q,p]_f$ meets C exactly at $\{p,q\}$, doing it transversally at q (with an external tangency at p); also, points of $\gamma \setminus [q,p]_f$ nearby p do not belong to $\overline{D}(\Gamma_1)$.

Summarizing either $[p,r]_f$ or $[q,p]_f$ satisfies (i). Therefore we may denote by $[p,r]_f$ the arc which satisfies (i).

We claim that (i) implies (ii). In fact, let us choose a small flow box B of $\nabla^{\#} f$ whose interior contains $[p,r]_f$. By the assumptions, we may suppose that $\widetilde{p}, \widetilde{r} \in B \cap C$. We may see that there exists a closed interval $[\widetilde{p},\widetilde{r}]_T \subset B$ transversal to $\nabla f^{\#}$ (drawn as a dotted line in fig. 2.b) and such that $C_1 = (C \setminus [\widetilde{p},\widetilde{r}]) \cup [\widetilde{p},\widetilde{r}]_T$ is a smooth circle, surrounding the origin, contained in $\mathbb{R}^2 \setminus D(C)$. Moreover, $\nabla f^{\#}$ meets C_1 with a smaller number of tangencies than it does it with C. The remaining conclusions of (ii) can easily be checked. Therefore (i) implies (ii). □

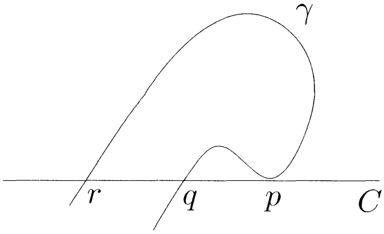

FIGURE 2.a

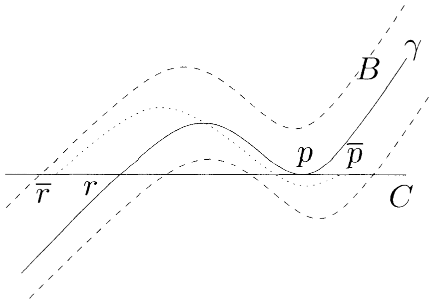

FIGURE 2.b

Remark 1. — Let suppose that $\nabla f^{\#}$ is in general position with a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ which surrounds the origin. Suppose also that $\nabla f^{\#}$ has an internal tangency with C at the point q; then, by observing the trajectories of $\nabla f^{\#}$ around q, we may see that there exist closed subintervals $[p,q]$ $[q,r]$ of C, with $[p,q] \cap [q,r] = \{q\}$, and a homeomorphism $T : [p,q] \to [q,r]$ such that,

(a1) $Tp = r, Tq = q$ and, for every $x \in (p,q]$, there is an arc of trajectory $[x,Tx]_f \subset \mathbb{R}^2 \setminus D(C)$ of $\nabla f^{\#}$, starting at x, ending at Tx and meeting C exactly and transversally at $\{x,Tx\}$, (a2) the family $[x, Tx]_f : x \in (p, q]$ depends continuously on x and tends to $\{q\}$ as $x \to q$.

Lemma 3. — *Let suppose that $\nabla f^\#$ is in general position with a smooth circle $C \subset \mathbb{R}^2 \smallsetminus \overline{D}_\sigma$ which surrounds the origin. Suppose also that $\nabla f^\#$ has an internal tangency with C at the point q. Given any pair of subintervals $[p, q]$, $[q, r]$ of C (generated by q) as in Remark 1, the family $\{[x, Tx]_f : x \in (p, q]\}$ tends continuously to the compact arc of trajectory $[p, Tp]_f \subset \mathbb{R}^2 \smallsetminus D(C)$ which either meets C exactly and transversally at $\{p, Tp\}$ or meets C with a quadratic external tangency; this second alternative happens if, and only if, $(p, q]$ is the maximal interval with properties (a1) and (a2) of Remark 1.*

Proof. — If $(p, q]$ is not the maximal interval with properties (a1) and (a2) of Remark 1, then $[p, Tp]_f \subset \mathbb{R}^2 \smallsetminus D(C)$ meets C exactly and transversally at $\{p, Tp\}$.

Otherwise, as $\mathbb{R}^2 \smallsetminus \overline{D}(C)$ is not bounded, the closure of $(p, q] \cup [p, r)$ cannot be the whole circle. Therefore, there are two possibilities. The first one is that the positive (resp. negative) half-trajectory γ_p^+ (resp. γ_r^-) of $\nabla f^\#$ starting at p (resp. at r) does not meet C and so it must accumulate at the point ∞ of the Riemann sphere $\mathbb{R}^2 \cup \infty$. Under these circumstances, the subinterval $[p, q] \cup [q, r]$ is the compact edge of non-bounded *HRC* of $\nabla f^\#$ made up of $\gamma_p^+ \cup \gamma_r^-$ together with the union of the arcs $[x, Tx]_f$, with $x \in (p, q]$. This contradiction with Proposition 1 shows that the second possibility must happen: this lemma is true. □

Lemma 4. — *There exists a smooth circle $C \subset \mathbb{R}^2 \smallsetminus \overline{D}_\sigma$, surrounding the origin, in general position with $\nabla f^\#$, and such that*

(i) *if a trajectory γ of $\nabla f^\#$ meets C, with an external tangency, say p, then $\gamma \cap C = \{p\}$;*

(ii) *As a consequence of (i), every tangency of the Hamiltonian vector field $\nabla f^\#$ with C is quadratic and external. In particular, there exists a correspondence between tangencies and HRCs (which —by Proposition 1— are contained in the disc of $\mathbb{R}^2$ bounded by C).*

Proof. — By a small C^2 perturbation of f, we may assume that $\nabla f^\#$ is in general position with C; in particular, every tangency of $\nabla f^\#$ with C is quadratic. If (i) of this lemma is not satisfied, we may use Lemma 2 to obtain a new circle C_1 such that $\nabla f^\#$ is in general position with C_1 and the number of tangencies of $\nabla f^\#$ with C_1 is smaller than that of $\nabla f^\#$ with C. Using this procedure, as many times as necessary, we will be able to obtain a circle as required to prove (i)

As (i) is true, (ii) follows from Lemma 3. □

Corollary 2. — *There exists a smooth circle $C \subset \mathbb{R}^2 \smallsetminus \overline{D}_\sigma$, surrounding the origin and there are two points $a, b \in C$, with $f(a) < f(b)$, such that $\nabla f^\#$ is tangent to C exactly at a and b; moreover, these tangencies are quadratic and external.*

Proof. — In fact, by Lemma 4, we may take a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$, surrounding the origin, such that every tangency of the Hamiltonian vector field $\nabla f^\#$ with C is quadratic and external. Therefore, by the index formula of Theorem 3 and by Corollary 1, there are two points $a, b \in C$, with $f(a) < f(b)$, satisfying this corollary. $\square$

5. Main Proposition

This section is devoted to the proof of the following

Proposition 3. — *There exists a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$, surrounding the origin, such that $X(C)$ is also an embedded circle and, for some exterior collar neighborhood $U \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ of C, $X(U)$ is also an exterior collar neighborhood of $X(C)$.*

The proof of this proposition will be completed at the end of this section after some preparatory lemmas.

We say that a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is *of ETT (i.e. external tangency type) for* $\nabla f^\#$ if the following is satisfied: C surrounds the origin, there are two points $a, b \in C$, with $f(a) < f(b)$, and there are points $a_1, a_2, \ldots, a_n \in C_-$ and $b_1, b_2, \ldots, b_n \in C_+$, where C_- and C_+ are the connected components of $C \setminus \{a, b\}$, such that:

(a1) $\nabla f^\#$ is tangent to C exactly at a and b; also, these tangencies are quadratic and external;

(a2) $f(a) = \inf\{f(x) : x \in C\} < \sup\{f(x) : x \in C\} = f(b)$;

(a3) f takes diffeomorphically each C_i, with $i \in \{-, +\}$, onto the open interval $(f(a), f(b))$ (i.e., $X(C_i)$ is the graph of a map $(f(a), f(b)) \to \mathbb{R}$);

(a4) X restricted to $C \setminus \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ is an embedding, and also, $X(C_-)$ and $X(C_+)$ meet transversally to each other

(a5) $(X(a_1), X(a_2), \ldots, X(a_n)) = (X(b_1), X(b_2), \ldots, X(b_n))$ and
$f(a) < f(a_1) = f(b_1) < f(a_2) = f(b_2) < \cdots < f(a_n) = f(b_n) < f(b)$.

(a6) there are sequences $x_n \to a$ and $y_n \to b$ of points x_n and y_n in $\mathbb{R}^2 \setminus \overline{D}(C)$ such that, for all n, $f(x_n) < f(a) < f(b) < f(y_n)$. This means that the local exterior of C around a (resp. around b) is taken to the unbounded connected component of $\mathbb{R}^2 \setminus X(C)$. In particular, $n \geqslant 0$ is an even number.

(a7) If $x \in \mathbb{R}^2 \setminus D(C)$ is close enough to $y \in C_+$ (resp. $y \in C_-$) and $f(x) = f(y)$, then $g(y) < g(x)$ (resp. $g(y) > g(x)$).

(a8) If $\overline{a}_1, \overline{a}_n \in C_-$ and $\overline{b}_1, \overline{b}_n \in C_+$ are close enough to a_1, a_n and b_1, b_n, respectively, and $[a_1, a_n] \subset (\overline{a}_1, \overline{a}_n)$, $[b_1, b_n] \subset (\overline{b}_1, \overline{b}_n)$ then, $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ is below $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$ (i.e. if $a' \in [\overline{a}_1, a_1) \cup (a_n, \overline{a}_n]$ and $b' \in [\overline{b}_1, b_1) \cup (b_n, \overline{b}_n]$ are such that $f(a') = f(b')$ then $g(a') < g(b')$).

Lemma 5. — *There is a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ of ETT for $\nabla f^\#$.*

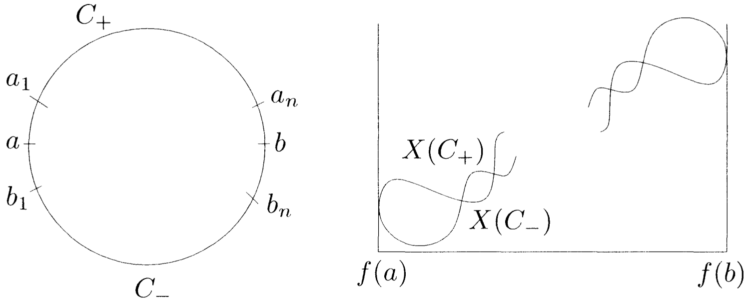

FIGURE 3

Proof. — In fact, by Corollary 2, we may take a smooth circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$, surrounding the origin, such that there are two points $a, b \in C$, with $f(a) < f(b)$, and so that (a1) above is satisfied. This implies that (a2) and (a3) of definition above are also satisfied. Furthermore, by a small perturbation of X, we may assume that (a4) and (a5) of definition above are satisfied too. Item (a6) follows directly from the preceding properties. As $X(C)$ is tangent to the vertical foliation at the points $X(a)$ and $X(b)$, and by using (a6), The connected components C_- and C_+ can be named to satisfy (a7). Item (a7) implies (a8). □

In the following of this section, C will be a smooth circle of ETT for $\nabla f^{\#}$ and we shall use all corresponding introduced notation.

Given $\alpha, \beta \in C_-$ (resp. $\alpha, \beta \in C_+$), $[\alpha, \beta], (\alpha, \beta), [\alpha, \beta)$ will denote subintervals of C_- (resp. of C_+) with endpoints α, β. Let L denote the straight line which passes through the points $X(a_1)$ e $X(a_n)$. Let $\mathcal{L}$ be the foliation of $\mathbb{R}^2$ made up by all the straight lines parallel to the line L. By a small perturbation of $X(C)$ with support in $X([a_1, a_n] \cup [b_1, b_n])$, we may assume that

(b) every point of tangency of $X([a_1, a_n])$ with $\mathcal{L}$ is quadratic, $X([a_1, a_n])$ and $X([b_1, b_n])$ are transversal to L.

Also, by taking $\overline{a}_1, \overline{a}_n \in C_-$ and $\overline{b}_1, \overline{b}_n \in C_+$ close to a_1, a_n and b_1, b_n, respectively, and $[a_1, a_n] \subset (\overline{a}_1, \overline{a}_n)$, $[b_1, b_n] \subset (\overline{b}_1, \overline{b}_n)$, we may suppose as well that

(c) $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ and $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$ are disjoint of L.

Let $\theta \in \mathbb{R}$ be such that $R_\theta(\mathcal{L})$ is made up of vertical lines, where R_θ is the rigid rotation of angle θ. Recall that $(f_\theta, g_\theta) = X_\theta = R_\theta \circ X \circ R_\theta^{-1}$. By means of a small C^2−perturbation of f, we may assume that

(d) $\nabla f_\theta{}^{\#}$ is in general position with $R_\theta(C)$.

Then we have that

Lemma 6. — *Remark 1 and Lemma 3 are also valid when referred to the vector field*

$$Y = (R_{-\theta})_* \nabla {f_\theta}^{\#} = R_{-\theta} \circ \nabla {f_\theta}^{\#} \circ R_\theta.$$

Also X takes any trajectory of Y into a subinterval of a leaf of $\mathcal{L}$.

Proof. — If γ is a trajectory of Y then $R_\theta(\gamma)$ is a trajectory of $\nabla {f_\theta}^{\#}$. Therefore, $X_\theta \circ R_\theta(\gamma)$ is a subinterval of a vertical line and so $R_{-\theta} \circ X_\theta \circ R_\theta(\gamma)$ is a subinterval of a leaf of $\mathcal{L}$. However,

$$R_{-\theta} \circ X_\theta \circ R_\theta = X.$$

This implies that X takes any trajectory of Y into a subinterval of a leaf of $\mathcal{L}$. On the other hand, as $(f_\theta, g_\theta) = X_\theta = R_\theta \circ X \circ R_\theta^{-1}$ satisfies the assumptions of Theorem 1, Remark 1 and Lemma 3 are valid for the vector field $\nabla {f_\theta}^{\#}$. By definition of Y, we obtain the remaining conclusion of this lemma. □

We claim that

Lemma 7. — *If $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ is below L and $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$ is above L, then there is a smooth circle $C_1 \subset \mathbb{R}^2 \smallsetminus D(C)$, surrounding the origin, obtained from C by a deformation which fixes $C \smallsetminus ((\overline{a}_1, \overline{a}_n) \cup (\overline{b}_1, \overline{b}_n))$ and takes $[\overline{a}_1, \overline{a}_n] \subset C$ and $[\overline{b}_1, \overline{b}_n] \subset C$) to the closed sub-intervals $[\overline{a}_1, \overline{a}_n]_{C_1} \subset C_1$ and $[\overline{b}_1, \overline{b}_n]_{C_1} \subset C_1$, respectively, which satisfy $X([\overline{a}_1, \overline{a}_n]_{C_1})$ is below L and $X([\overline{b}_1, \overline{b}_n]_{C_1})$ is above L. In particular, C_1 is as requested to prove Proposition 3.*

Proof. — Suppose that Y has an internal tangency with C at the point $q \in (a_1, a_n)$. By (d) and Lemma 6 we may proceed as in Remark 1 (applied to Y and considering the notation introduced there) to obtain sub-intervals $[p, q]$, $[q, r]$ of C (generated by q), determined by the condition that $(p, q]$ is the maximal subinterval of $[a_1, a_n]$ satisfying properties (a1) and (a2) of Remark 1. By Lemma 3, every element of the family $\{[x, Tx]_Y : x \in (p, q]\}$ is an arc of trajectory of Y. Notice that the maximality criterium for $(p.q]$ right above is different from that of Lemma 3.

To perform a sequence of adequate deformations, we meet three possible cases:

The first one is that $\{p, r\} \cap \{a_1, a_n\} \neq \varnothing$. Consider only the case in which $p = a_1$ and $r \neq a_n$. We may deform C into a new circle C_1 in such a way that: the deformation fixes $C \smallsetminus (\overline{a}_1, \overline{a}_n)$ and takes $[\overline{a}_1, \overline{a}_n] \subset C$ to a closed sub-interval $[\overline{a}_1, \overline{a}_n]_{C_1} \subset C_1$ such that

(e) the cardinality of $L \cap [\overline{a}_1, \overline{a}_n]_{C_1}$ is less than that of (the finite set) $L \cap [\overline{a}_1, \overline{a}_n]$; and, concerning tangencies with $\mathcal{L}$, that are above L, $[\overline{a}_1, \overline{a}_n]_{C_1}$ has less ones than $[\overline{a}_1, \overline{a}_n]$.

In this deformation the arc $[p, Tp] \subset C$ has been taken to an interval whose image by X is below L. This deformation takes place inside a small neighborhood of $\{[x, Tx]_Y : x \in [p, q]\}$) and so C_1 surrounds the origin. Also as both $[\overline{a}_1, \overline{a}_n] \subset C$ and $\mathcal{L}$ are transversal to the vertical foliation, $[\overline{a}_1, \overline{a}_n]_{C_1} \subset C_1$ can be obtained to be transversal to the vertical foliation. We do not care if C_1 has more self-intersections than C.

The second case happens when $\{p,r\} \subset (a_1,a_n)$ and the arc of trajectory $[p,r]_Y$ of Y meets C according to the conditions (i) of Lemma 3. The arguments of such lemma imply that we may deform of C into a new circle C_1 according to the following conditions. The deformation fixes $C \setminus (\overline{a}_1,\overline{a}_n)$ and takes $[\overline{a}_1,\overline{a}_n] \subset C$ to a closed sub-interval $[\overline{a}_1,\overline{a}_n]_{C_1} \subset C_1$ such that

(f) $L \cap [\overline{a}_1,\overline{a}_n]_{C_1}$ has the same number of elements than $L \cap [\overline{a}_1,\overline{a}_n]$; and, concerning tangencies with $\mathcal{L}$, that are above L, $X([\overline{a}_1,\overline{a}_n]_{C_1}])$ has one less than $X([\overline{a}_1,\overline{a}_n])$.

As above, this deformation takes place inside a small neighborhood of $\cup\{[x,Tx]_Y : x \in [p,q]\})$ and so C_1 surrounds the origin. Also as $X([\overline{a}_1,\overline{a}_n])$ and $\mathcal{L}$ are transversal to the vertical foliation, $[\overline{a}_1,\overline{a}_n]_{C_1} \subset C_1$ can be obtained so that $X([\overline{a}_1,\overline{a}_n]_{C_1})$ is transversal to the vertical foliation. Again, as in case above, we do not care if C_1 has more self- intersections than C.

The third case occurs when $\{p,r\} \subset (a_1,a_n)$ and the arc of trajectory $[p,r]_Y$ of Y does not meet C according to the condition (i) of Lemma 3. We shall show now that this case is not possible. In fact, otherwise, this supposition and (d) imply that the open subinterval of trajectory (p,r) meets tangentially C exactly once, say at s.

Let $[p,s]$ and $[s,r]$ be the subintervals of C such that $[p,s]_Y \cup [p,s]$ and $[s,r]_Y \cup [s,r]$ bound discs $\overline{D}([p.s]_Y \cup [p,s])$ and $\overline{D}([s,r]_Y \cup [s,r])$ contained in $\mathbb{R}^2 \setminus D(C)$. Then, either $[s,r]_Y$ is contained in $\overline{D}([p.s]_Y \cup [p,s])$ or $[p,s]_Y$ is contained in $\overline{D}([s,r]_Y \cup [s,r])$. If $[s,r]_Y$ is contained in $\overline{D}([p.s]_Y \cup [p,s])$, then the circle $(C \setminus [p,s]) \cup [p,s]_Y$ can be approximated by a circle C_1 such that $X(C_1)$ has exactly two tangencies with the vertical foliation: $\{X(a),X(s)\}$. It follows from (a7) that X meets C_1 with an internal tangency at s. As X meets C_1 with an external tangency at a, we conclude, by Theorem 3 that $j_X(C_1) = 1$. This contradiction with Corollary 1 shows that $[s,r]_Y$ is not contained in $\overline{D}([p.s]_Y \cup [p,s])$. Similarly, $[p,s]_Y$ is not contained in $\overline{D}([s,r]_Y \cup [s,r])$. This contradiction implies that the third case is not possible.

As cases 1 and 2 above are the only possible ones, and thanks to (e)-(f), we only need to perform finitely many times the process (just described above) of obtaining new circles, of ETT for $\nabla f^{\#}$, in order to finally obtain a smooth circle, say C_2, such that $X([\overline{a}_1,\overline{a}_n]_{C_2})$ is below L. Similarly, by a deformation that fixes $C_2 \setminus [\overline{b}_1,\overline{b}_n]$ we shall finally obtain one circle as requested in this lemma. $\square$

Proof of Proposition 3. — By (a8), $X([\overline{a}_1,a_1) \cup (a_n,\overline{a}_n])$ is below $X([\overline{b}_1,b_1) \cup (b_n,\overline{b}_n])$. It is easy to see that we may deform C, locally around $\{a_1,b_1,a_n,b_n\}$ so that the new circle of ETT for $\nabla f^{\#}$ satisfies the conditions of Lemma 7 and so it can be deformed into one as requested to prove this proposition. $\square$

As a direct consequence of this proposition and Proposition 2 we obtain:

Corollary 3. — *Under the assumptions of Theorem 1, there exists an embedded circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ such that X restricted to $\mathbb{R}^2 \setminus \overline{D}(C)$ can be extended to an orientation preserving embedding from $\mathbb{R}^2$ into $\mathbb{R}^2$.*

6. Proof of Theorem 1

Item (i) of Theorem 1 follows directly from Corollary 3.

We shall need the following result of Gutierrez and Teixeira [**G-T**]

Theorem 4. — *Let $\sigma > 0$ and $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field satisfying the following conditions:*

(i) *Y has a singularity, say S;*
(ii) *for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of $DY(p)$ belongs to $\{z \in \mathbb{C} : \Re(z) \geqslant 0\}$;*
(iii) *for all $p \in \mathbb{R}^2$, $\mathrm{Det}(DY(p)) > 0$.*

If $\mathcal{I}(Y) = \displaystyle\int_{\mathbb{R}^2} \mathrm{Trace}(DY)$ is less than 0 (resp. greater of equal than 0), then the point ∞ of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ is a repellor (resp. an attractor) of Y.

Proceed to prove item (ii) of Theorem 1. Under the terms of Corollary 3, X can be extended to an orientation preserving embedding $\widetilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$. Choose $p_0 \in \mathbb{R}^2$ such that $\widetilde{X} + p_0$ has a singularity. By applying Theorem 4, we shall obtain that the point ∞ of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ is either a repellor or an attractor of $\widetilde{X} + p_0$. This proves item (ii) of Theorem 1 because, around infinity, $\widetilde{X} + p_0$ and $X + p_0$ coincide.

We should comment that there are vector fields of $\mathbb{R}^2$, as in Theorem 4, having either attracting or repelling behavior at ∞ [**G-T**]. For sake of completeness we present in next section the example of Gutierrez and Teixeira of a vector field, as in Theorem 4, having attracting behavior at ∞.

7. An example

The purpose of this section is to exhibit a vector field X satisfying the conditions of Theorem 4 and such that the unstable manifold $W^u(0)$, of 0, is $\mathbb{R}^2$. In particular "∞" is an attractor of X. The required vector field is given by:

$$X(x, y) = g(r)(e^{-r}x - y, x + e^{-r}y)$$

where $r = \sqrt{x^2 + y^2}$ and $g(r) = \dfrac{1 - e^{-r}}{r\sqrt{1 + e^{-2r}}}$.

The following expressions can be obtained by a symbolic computer system like Mathematica:

(a) $\mathrm{Det}(DX) = \dfrac{e^r - 1}{re^{2r}}$

(b) $\mathrm{Trace}(DX) = \dfrac{e^r + (r-1)(1 + 2e^{2r} - e^{3r})}{re^{4r}(1 + e^{-2r})^{3/2}}$

(c) $\mathcal{I}(X) = \displaystyle\int_{\mathbb{R}^2} \mathrm{Trace}(DX) = 0.$

It is clear that $\mathrm{Det}(DX) > 0$ everywhere and that $\mathrm{Trace}(DX) < 0$ in the region $\{r > K\}$ for some large K.

It follows, from (a)–(c) and Theorem 4, that "∞" is an attractor of X. To obtain a stronger conclusion, we may observe that the inner product

$$\langle (x,y), X(x,y)\rangle = g(r)r^2e^{-r}$$

is greater than 0, for all $\sigma > 0$; therefore, the vector field X points outside all discs whose boundary has the form $\{r = \text{constant}\}$. This implies that $W^u(0) = \mathbb{R}^2$.

References

[CGL] M. Cobo, C. Gutierrez & J. Llibre. *On injectivity of C^1–maps of the real plane.* Preprint.

[Chi] C. Chicone. *Ordinary Differential Equations with Applications.* Springer Graduate Texts in Applied Mathematica, 34 (1999).

[DRS] F. Dumortier, R. Roussarie & J. Sotomayor. *Bifurcations of Planar Vector Fields: Nilpotent Singularities and Abelian Integrals.* Springer Lecture Notes in Mathematics, V. 1480 (1991).

[Fes] R. Fessler. *A proof of the two dimensional Markus–Yamabe Stability Conjecture and a generalization.* Annales Polinici Mathematici LXII.1, pp. 45–74 (1995).

[G-G] G. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities.* Springer-Verlag New York Inc. (1980).

[Gut] C. Gutierrez. *A solution to the bidimensional Global Asymptotic Stability Conjecture.* Ann. Inst. Henri Poincaré. V. 12 No 6, 1995, p 627-671.

[G-T] C. Gutierrez and M. Teixeira. *Asymptotic stability at infinity of planar vector fields.* Bol. Soc. Bras. Mat. Vol. 26, No. 1, pp 57-66 (1995).

[Har] P. Hartman. *Ordinary Differential Equations*, John Wiley and Sons, Inc. (1964).

[Hir] M.W. Hirsch. *Differential Topology* Graduate texts in mathematics, vol. 33; Springer-Verlag New York Inc. (1976)

[Rou] R. Roussarie. *Bifurcation of Planar Vector Fields and Hilbert's* 16^{th} *Problem.* Birkhauser Verlag, Basel (1998).

[Tak] F. Takens. *Singularities of Vector Fields.* Publ. Math. IHES, 43 (1974).

C. Gutierrez, ICMC-USP, São Carlos & IMPA, Rio de Janeiro, ICMC-USP: Av. Dr. Carlos Botelho, 1465; Caixa Postal 668, CEP- 13560-970, Sao Carlos-SP, Brazil
E-mail : gutp@icmc.sc.usp.br

A. Sarmiento, UFMG-ICEx-Dpto. de Matemática, Av. Antônio Carlos 6627, Pampulha, CEP 30161-970 - Belo Horizonte-MG-Brazil • *E-mail :* sarmient@mat.ufmg.br
Url : www.mat.ufmg.br/~sarmiento/