Astérisque

VICTOR J. DONNAY
CHARLES C. PUGH

Anosov geodesic flows for embedded surfaces

Astérisque, tome 287 (2003), p. 61-69
<http://www.numdam.org/item?id=AST_2003__287__61_0>

Astérisque
287, 2003, p. 61–69

ANOSOV GEODESIC FLOWS FOR EMBEDDED SURFACES

by

Victor J. Donnay & Charles C. Pugh

Abstract. — In this paper we embed a high genus surface in $\mathbb{R}^3$ so that its geodesic flow has no conjugate points and is Anosov, despite the fact that its curvature cannot be everywhere negative.

1. Introduction

At the International Conference on Dynamical Systems held in Rio de Janeiro in July, 2000, Michael Herman asked whether the geodesic flow for an embedded surface in $\mathbb{R}^3$ can be uniformly hyperbolic, i.e., Anosov. Using techniques from our paper [**5**] and a suggestion of John Franks and Clark Robinson, we answer Herman's question affirmatively. The embedded surface looks like a spherical shell with many holes drilled through it. See Figures 1 and 2.

The Lobachevsky-Hadamard Theorem states that if a Riemann manifold has negative sectional curvature then its geodesic flow is Anosov. The celebrated thesis of Anosov [**1**] shows that this implies ergodicity, in fact the Bernoulli property, a stronger form of ergodicity.

In [**2**], Burns and Donnay showed that every surface M embeds in $\mathbb{R}^3$ so that its geodesic flow is Bernoulli; however, this cannot be a consequence of M having negative curvature. For a compact surface $M \subset \mathbb{R}^3$ necessarily has regions of positive curvature, the standard explanation being that there is a smallest sphere S which contains M, and there are points at which S is tangent to M. At these points the

2000 ***Mathematics Subject Classification.*** — 37D40.
Key words and phrases. — Anosov geodesic flow, no conjugate points, isometrically embedded surface, high genus.

The authors thank the Instituto de Matemática Pura e Aplicada for hospitality and support during part of the time this paper was written.

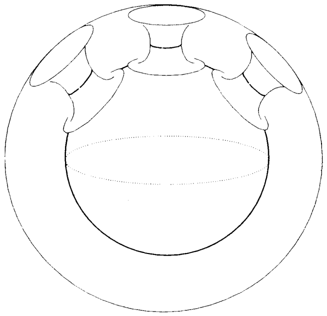

FIGURE 1. An embedded surface formed by connecting two concentric spheres with many tubes.

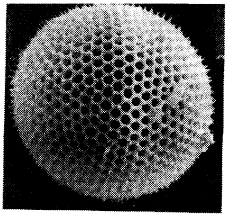

FIGURE 2. The radiolarian *Aulonia hexagona*, a marine micro-organism, as it appears through an electron microscope, by S.A. Kling.

curvature of M is positive. By continuity, the curvature of M is positive at nearby points too. The Bernoulli geodesic flows constructed by Burns and Donnay employ "focusing caps" to control the positive curvature. However, the caps are bounded by closed geodesics on which the curvature is zero, preventing uniform hyperbolicity. If the caps are perturbed to destroy these parabolic orbits the system can become non-ergodic [**3, 4**].

Instead of using caps, we use tubes of negative curvature together with the notion of a finite horizon geometry, which we introduced in [**5**], and are thereby able to show

Theorem A. — *There exist embedded surfaces in $\mathbb{R}^3$ for which the geodesic flows are Anosov.*

As an extension of Theorem A we discuss the immersed case, which has interest when the surface is not orientable.

Theorem B. — *There exist immersed non-orientable surfaces in $\mathbb{R}^3$ for which the geodesic flows are Anosov.*

The basic ingredient in our construction is illustrated in Figure 3; connect two flat tori (they are not embedded in $\mathbb{R}^3$) via a tube of negative curvature. The geodesic flow for this genus two surface is Bernoulli but not uniformly hyperbolic - since there

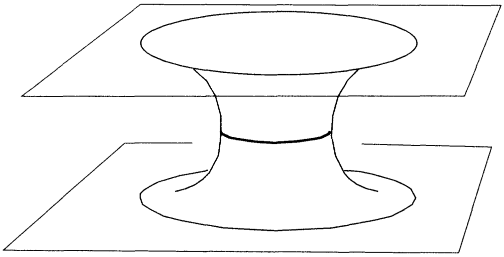

FIGURE 3. Two flat tori joined by a negatively curved tube.

are periodic geodesics lying completely in a flat region. If we now connect the two tori by enough tubes to produce a finite horizon pattern (see Section 2), i.e. every geodesic enters a tube in a bounded time, then the geodesic flow for this high genus surface is Anosov. To make an embedded Anosov example, we follow the suggestion of Franks and Robinson: reproduce the construction using very large and nearly flat concentric spheres instead of tori, again in a finite horizon pattern of tubes.

Remark. — Theorems A and B give the existence of high genus surfaces in $\mathbb{R}^3$ with Anosov geodesic flows, but we do not know a good lower bound on the genus. In [**6**], Wilhelm Klingenberg shows that no surface whose Riemann structure has conjugate points, which are produced by a surfeit of positive curvature, can have an Anosov geodesic flow. Hence our construction also provides examples of embedded surfaces without conjugate points. By Klingenberg's result, the sphere and torus never have Riemann structures whose geodesic flows are Anosov. So in particular, these surfaces cannot embed in $\mathbb{R}^3$ in such a way that their geodesic flows are Anosov. But what about the bitorus? Can it embed in $\mathbb{R}^3$ so that its geodesic flow is Anosov? Is it at least possible to embed the bitorus so that its metric has no conjugate points?

2. Finite Horizon

Let M be a surface equipped with a Riemann structure. A family $\mathcal{C}$ of curves $C_1, \ldots, C_k$ in M gives M *ϕ-finite horizon* if every unit length geodesic crosses at least one curve in $\mathcal{C}$ at an angle $\geqslant \phi$. In [**5**] we show in detail how to choose $\mathcal{C}$ that gives M finite horizon, when M is a surface embedded in $\mathbb{R}^3$ and its Riemann structure is the one it inherits from the embedding. Here is an outline of the construction.

We first construct a fine, smooth triangulation of M whose triangles have uniformly bounded eccentricity and nearly geodesic edges. (The eccentricity of a triangle is the reciprocal of its smallest vertex angle.) We then draw small geodesic discs at the vertices of the triangulation, and a string of N "pearl discs" along each edge of the triangulation outside the vertex discs. Finally, we draw $2N + 2$ "wing discs" parallel to the string of pearl discs. Altogether this gives $9(N + 1)$ discs per triangle. The pearl and wing discs have radius r, which is much less than the radius R of the vertex discs, and this makes the pearl and wing discs along one edge of a triangle disjoint from those along a different edge.

Technically, once we have a bound on the eccentricity of the triangles that appear in our triangulations, we choose R and N. We then keep R and N fixed, while we dilate the surface by a factor of 2^n, $n \to \infty$, making ever finer triangulations of the dilated surface that have nearly linear triangles of roughly unit size. The radii r of the pearl and wing disks vary depending on the length of the edge of the triangle but lie in a compact interval.

With respect to the flat Riemann structure, the disc pattern for a triangle is shown in Figure 4. Every unit segment starting inside the flat triangle must cross the boundary circles of these discs at some positive angle. By compactness, they cross at some uniformly positive angle ϕ, a fact that remains true under small perturbations. For example, if we shrink all the discs by a factor $\mu < 1$, where $1-\mu$ is small, they still give the finite horizon property for unit segments. Similarly, the finite horizon property still holds if the flat metric is replaced by a nearly flat metric.

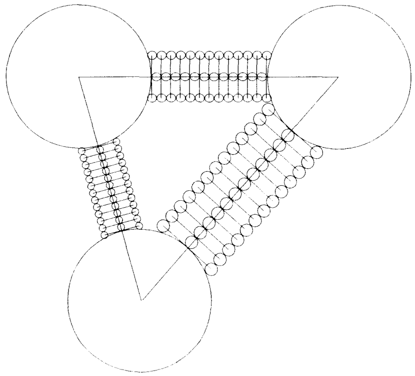

FIGURE 4. The pattern of discs for a linear triangle that gives the finite horizon property.

Denote by $2^n M$ the surface gotten by dilating M by a factor 2^n. The Riemann structure of $2^n M$ restricted to a nearly linear triangle T of roughly unit size is nearly flat. Thus, the geodesic discs of radius μr and μR laid down in the pattern of Figure 4 are disjoint and give the finite horizon property for unit geodesics on $2^n M$ when n is large.

We then flatten these disjoint geodesic discs by pushing each into the tangent plane at its center. Slightly smaller round discs lie in the flattened geodesic discs, and they still give the finite horizon property. The net effect is that the given surface M is replaced by a new one, $2^n M$, with diameter roughly 2^n, and having a great number of disjoint, flat *plateau discs* such that any unit geodesic crosses the boundary of at least one plateau disc at an angle $\geqslant \phi > 0$ for n large.

3. Dispersing Tubes

In [**5**] we glue "focusing caps" in place of the plateau discs above to make the geodesic flow non-uniformly hyperbolic. Here we glue tubes between pairs of plateau discs to make it Anosov.

Definition. — A *dispersing tube* T is a surface of revolution

$$T = \{(r, \theta, z) : r = h(z)\}$$

such that $h : [-1, 1] \to (0, 1]$ satisfies

(a) $h(z) = h(-z)$.

(b) $h(\pm 1) = 1$.

(c) If $|z| < 1$ then h is smooth and $h''(z) > 0$.

(d) The graph of h is infinitely tangent to the lines $z = \pm 1$. In particular $\lim_{z \to \pm 1} h'(z) = \pm\infty$.

Thus, T is a catenoid-like surface with its ends made planar. It has negative curvature. See Figure 3. The geodesics on a dispersing tube are simple to describe. There is the closed geodesic Γ around the "waist" of the tube, and there are geodesics asymptotic to it. Every other geodesic either enters and exits T without meeting Γ, or it crosses Γ once on its way from one end of T to the other. The entry and exit angles are equal because the tube is symmetric.

Note that independent linear scalings of z and r preserve the properties of T: it has negative curvature, it is infinitely tangent to the planes containing its boundary circles, and it contains a unique, closed waist geodesic Γ. Thus, we can make T long and thin, or short and broad. To avoid bending T, which may introduce positive curvature, we must be sure to keep its boundary circles in parallel planes.

4. The Perforated Sphere

Here is the proof of Theorem A. Take a sphere in $\mathbb{R}^3$ and make the finite horizon construction described in Section 2. (Any surface could be used instead of the sphere.) This gives a sequence of spheroids S_n of radius 2^n that contain many disjoint plateau discs of roughly unit radius. Each plateau disc lies in the plane normal to the radius vector from the origin. Then take a concentric copy of S_n, say S'_n, which is S_n shrunk by the factor $1 - 1/2^n$. The spheroids have radii that differ by 1. As $n \to \infty$, this makes the plateau discs nearly equal in radius and parallel in pairs. Replace each pair of parallel plateau discs by a dispersing tube. The boundary circles of the dispersing tubes have radius equal to the plateau disc in S'_n, which is slightly less than the corresponding radius in S_n. But as $n \to \infty$, the difference tends to 0, and consequently the slightly smaller discs on the outer spheroid continue to give the finite horizon property there. The dispersing tubes are roughly of unit size, so they all have roughly the same effect on geodesics passing through them.

The spheroids S_n, S'_n, with plateau disc pairs replaced by dispersing tubes is the perforated sphere $M = M_n$. There are three types of geodesics on M. The closed geodesic Γ around the waist of each tube, the geodesics that are asymptotic to these closed geodesics, and the geodesics that regularly enter and exit dispersing tubes at an angle $\geqslant \phi$.

Let φ be the geodesic flow for M. Its phase space is the unit tangent bundle SM. To show that φ is Anosov we consider the normal bundle N to the flow direction X. Then $T(SM) = N \oplus X$ is a $T\varphi$-invariant splitting. The normal bundle is given by $N = H \oplus V$, where H is the horizontal subspace and V is the vertical subspace. For $x \in SM$, let $P(x)$ be the standard, closed positive cone that consists of lines through the origin of $N(x)$ lying in the first and third quadrants with respect to $N = H \oplus V$. We claim that the positive cone field P is contracted uniformly into itself by the time one map $T\varphi_1$.

For $\xi \in N_x$, $\xi \neq 0$, let $u(t)$ denote the slope of the vector $T\varphi_t(\xi)$ with respect to the splitting $N = H \oplus V$ at $\varphi_t(x)$. Then u solves the Riccati equation

$$u' = -K(t) - u^2.$$

The vertical edge of the cone field is easily seen to be mapped inside the positive cone by a uniform amount under the time one map. Thus we need only examine the horizontal edge of the cone which corresponds to solutions of the Riccati equation with initial condition $u(0) = 0$. Henceforth, we restrict our attention to Riccati solutions with this initial condition.

Every unit geodesic has the following life. It experiences strictly negative curvature $K \leqslant \nu_0 < 0$ for at least a fixed time $t_0 > 0$ because it enters at least one tube at an angle $\geqslant \phi$, and it experiences curvature $K \leqslant \kappa_0$ for the rest of the time, where κ_0 is the maximum of the curvature on the surface. The positive curvature bound κ_0 becomes uniformly small when we take n large enough, while the negative bound ν_0 stays fixed, and the time bound t_0 stays fixed.

First, let us assume that the curvature on the surface is non-positive, so that $\kappa_0 = 0$. Then we can make the estimate that $u' = -K - u^2 \geqslant 0 - u^2$ which implies that $u(t = 1) > u_0 > 0$ for $u(t)$ any solution of the the Riccati solution along a geodesic on the surface M. The bound u_0 equals the value of the solution at $t = 1$ of the piecewise constant Riccati equation with $K(t) = \nu_0$ for $t \in [0, t_0]$ and $K(t) = 0$, for $t \in (t_0, 1]$.

By continuous dependence on parameters, if $\kappa_0 > 0$ is sufficiently small (i.e., if n is large), then $u(t = 1) > u_0/2 > 0$. Hence, the bottom edge of the cone is mapped into the cone by a uniform amount. We conclude that P is uniformly contracted into itself by the time one map $T\varphi_1$. This implies that

$$x \mapsto E^u(x) = \bigcap_{n=1}^{\infty} T\varphi_n(P(\varphi_{-n}x))$$

is a line field, and the restriction of $T\varphi_1$ to E^u is a uniform expansion. Symmetry implies that $T\varphi_{-1}$ contracts the negative cone field, and that it contains a line field E^s which is contracted by $T\varphi_1$. Thus, $T(SM) = E^u \oplus X \oplus E^s$ is an Anosov splitting for $T\varphi$, which completes the proof of Theorem A.

As a consequence of Theorem A we get the following stability result.

Corollary. — *There is a high genus surface M such that the set $\mathcal{E}$ of embeddings $M \to \mathbb{R}^3$ for which the geodesic flow is Anosov is non-empty and open. In particular there exist such embeddings of M that are analytic.*

Remark. — The earlier examples of ergodic geodesic flows for embedded surfaces [**2**], [**5**] are not stably ergodic as one can perturb the focusing cap to produce a "light-bulb" shaped cap which traps a positive measure set of trajectories and hence prevents ergodicity [**3**]. (Stable ergodicity means that the system and all small pertrurbations of it are ergodic.) Thus, our Anosov example above is the first geodesic flow for an embedded surface known to be stably ergodic.

Proof. — Theorem A asserts that $\mathcal{E} \neq \varnothing$. Uniform hyperbolicity is an open condition, so $\mathcal{E}$ is open with respect to the C^3 topology. The proof is completed by recalling that embeddings are open in the space of mappings, and analytic mappings are dense in the C^∞ topology. □

5. Non-orientable Surfaces

Here we show out how to construct a non-orientable immersed surface in $\mathbb{R}^3$ whose geodesic flow is Anosov, thereby proving Theorem B.

The simplest idea is to attach a Klein bottle or a Klein handle to the surface M constructed in Section 4. Doing so produces a certain amount of positive curvature, and it becomes unclear whether negative curvature continues to dominate.

A second idea is this. Take the previous pair of spheroids with the tubes joining them and select a pair of points p, q such that p is on the outer spheroid, q is on the inner spheroid, the points p, q are not near any of the tubes (they lie in the unused, middle portions of the triangles), and the segment $[p, q]$ passes through the origin. Then make plateaus at p and q and draw a long thin tube T of negative curvature from one plateau to the other, as shown in Figure 5. This causes the new surface to be non-orientable. (With just one perforation and the long tube, the surface is ambiently diffeomorphic to the standard immersion of the Klein bottle.) Every geodesic has the same type of behavior as before, except now it may spend a fair portion of its time in the flank of T where the curvature is barely negative. As $n \to \infty$, the negativity

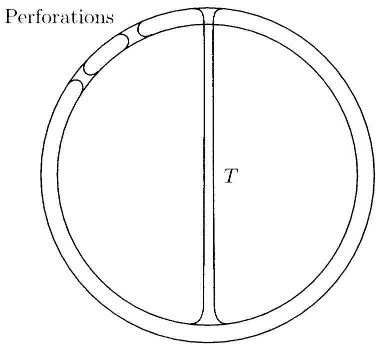

FIGURE 5. A long thin dispersing tube joining plateaus at p and q, shown in cross-section.

in T is on the order of $1/2^n$, which is the same as the worst positivity on the surface. Thus, it is not clear that negativity outweighs positivity enough to make the geodesic flow Anosov. More care may in fact validate this construction.

Here is what we do instead.

Choose a pair of adjacent triangles on the spheroid S_n constructed in Section 4. Form plateaus at their two centers based on a common plane, rather than different tangent planes at each center. Arbitrarily choose one of the two tangent planes to use. Do the same on the parallel spheroid S'_n. Hence all four plateaus are based on parallel planes. Make the tube connections between S_n and S'_n at all but these four new plateaus, and call the resulting surface M_1. Take a copy of M_1, say M_2, and rotate it to line up the four new unconnected plateaus from each surface, in parallel. Then draw tubes from one plateau disc to the other as shown in Figure 6.

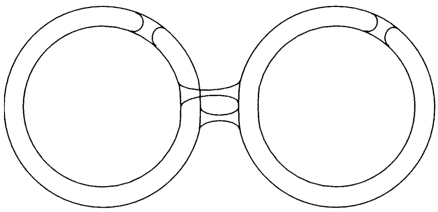

FIGURE 6. Cross-sectional view of connecting plateaus with tubes to make an immersed non-orientable surface whose geodesic flow is Anosov.

The resulting surface is non-orientable and the tubes all have roughly the same size. By the same sort of estimates as in the orientable case, negativity continues to dominate positivity and the geodesic flow remains Anosov.

Acknowledgement. — Figure 2 is reprinted from Haq et al, Editors, INTRODUCTION TO MARINE MICROPALEONTOLOGY, 1998, pp 203-244, S.A. King: "Radiolaria", with permission from Elsevier Science.

References

[1] D.V. Anosov (1967) "Geodesic Flows on Closed Riemann Manifolds with Negative Curvature", Proceedings of the Steklov Institute of Mathematics, Number 90.

[2] K. Burns and V.J. Donnay (1997) "Embedded surfaces with ergodic geodesic flow", International Journal of Bifurcation and Chaos, Volume 7, Number 7, pages 1509-1527.

[3] V.J. Donnay (1988) "Geodesic flow on the two-sphere, part I: positive measure entropy", Ergodic Theory and Dynamical Systems 8, pages 531-553.

[4] V.J. Donnay (1996) "Elliptic Islands in Generalized Sinai Billiards", Ergodic Theory and Dynamical Systems 16, pages 975-1010.

[5] V.J. Donnay and C.C. Pugh (2003) "Finite Horizon Riemann Structures and Ergodicity", to appear in Ergodic Theory and Dynamic Systems.

[6] W. Klingenberg (1974) "Riemannian Manifolds With Geodesic Flow of Anosov Type", The Annals of Mathematics, Second Series, Volume 99, Number 1, pages 1-13.

V.J. DONNAY, Mathematics Department, Bryn Mawr College, Bryn Mawr, Pennsylvania, 19010
E-mail : `vdonnay@brynmawr.edu`

C.C. PUGH, Mathematics Department, University of California, Berkeley California, 94720
E-mail : `pugh@math.berkeley.edu` • *Url :* `http://people.brynmawr.edu/vdonnay/`