Astérisque

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Astérisque, tome 287 (2003), p. 231-249

<http://www.numdam.org/item?id=AST_2003__287__231_0>

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ON THE DIVERGENCE OF GEODESIC RAYS IN MANIFOLDS WITHOUT CONJUGATE POINTS, DYNAMICS OF THE GEODESIC FLOW AND GLOBAL GEOMETRY

by

Rafael Oswaldo Ruggiero

Dedicated to J. Palis, on his 60th. birthday

Abstract. — Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the horospheres in $(\widetilde{M}, g)$ depend continuously on their normal directions. Then we show that geodesic rays diverge uniformly in the universal covering $(\widetilde{M}, g)$. We give some applications of this result to the study of the dynamics of the geodesic flow and the global geometry of manifolds without conjugate points.

Introduction

The problem of the divergence of geodesic rays in manifolds without conjugate points is one of the most natural, yet unsolved, questions of the theory. Recall that a C^∞ Riemannian, n-dimensional manifold (M, g) has no conjugate points if the exponential map is nonsingular at every point. The universal covering $\widetilde{M}$ of a manifold (M, g) is diffeomorphic to R^n, and the metric spheres in $(\widetilde{M}, g)$ — the universal covering endowed with the pullback of g — are diffeomorphic to the standard sphere in R^n. Given a point $p \in \widetilde{M}$, and two geodesics γ, β in $(\widetilde{M}, g)$ parametrized by arclength such that $p = \gamma(0) = \beta(0)$, we say that these geodesics diverge if $\lim_{t\to+\infty} d(\gamma(t), \beta(t)) = \infty$. Although two different geodesic rays starting from a point in $\widetilde{M}$ diverge in all well-known examples of manifolds without conjugate points (e.g., nonpositive curvature, no focal points, bounded asymptote), there is no general proof of this fact so far. The problem has been already considered by L. Green [**11**] in the late 50's, where Green deals with the divergence of radial Jacobi fields. Later, P. Eberlein [**6**] proves that radial Jacobi fields diverge along any geodesic in

2000 ***Mathematics Subject Classification.*** — 37D40, 53C23, 53C70.
Key words and phrases. — Conjugate points, horospheres, divergence of geodesic rays, local product structure, Gromov hyperbolic space, expansive flow.

Partially supported by CNPq, FAPERJ, and TWAS.

(M, g), but observes that the divergence might not be uniform, it could depend on the geodesic (in the same work [**6**], Eberlein points out a gap in Green's paper). The divergence of rays and Jacobi fields is related with many important geometric properties of manifolds without conjugate points, like the continuity of the horospherical foliations and Green bundles, and the existence of good compactifications of $\widetilde{M}$. This motivated somehow the introduction of some categories of manifolds without conjugate points in the literature (see for instance [**5**], [**6**], [**8**], for the so-called bounded asymptote condition, [**16**] for the Axiom of asymptoticity, [**14**] for the proof of the superlinear divergence of radial Jacobi fields in manifolds with bounded asymptote). The usual approach to the proofs of the continuity of horospheres, Green bundles, and divergence of rays, relies on strong assumptions on the asymptotic behaviour of geodesics and Jacobi fields (e.g., convexity in the case of nonpositive curvature; uniformly bounded asymptotic behaviour of Green Jacobi fields in the case of manifolds without focal points and manifolds with bounded asymptote). We shall present in this paper a more topological approach to the problem of the divergence of rays, based on simple variational properties of horospheres. Given $\theta = (p, v)$ in the unit tangent bundle $T_1\widetilde{M}$ of $\widetilde{M}$, we shall denote by $\gamma_\theta(t)$ the geodesic parametrized by arclength whose initial conditions are $\gamma_\theta(0) = p$, $\gamma'_\theta(0) = v$. We shall denote by $H_\theta(t)$ the horosphere of the geodesic γ_θ containing the point $\gamma_\theta(t)$. We say that the map $\theta \mapsto H_\theta(0)$ is continuous (in the compact open topology) if given a compact ball $B_r(q) \subset \widetilde{M}$ of radius r, and $\varepsilon > 0$, there exists $\delta = \delta(r, q, \varepsilon)$ such that if $\| \theta - \alpha \| \leqslant \delta$ then the Hausdorff distance d_H between the sets

$$d_H(H_\theta(0) \cap B_r(q), H_\alpha(0) \cap B_r(q)) \leqslant \varepsilon.$$

The introduction of this notion is motivated by the works of Pesin [**16**], Eschenburg [**8**], and Ballmann, Brin, and Burns [**1**]. Observe that, if M is compact, the number δ above does not depend on the point q, since every horosphere has an isometric image that meets a compact fundamental domain of $\widetilde{M}$ (horospheres are preserved by isometries of $(\widetilde{M}, g)$). In all known examples of manifolds without conjugate points the map $\theta \mapsto H_\theta(0)$ is continuous. Moreover, the assumption of the continuity of horospheres does not carry (a priori) any restrictions on either the convexity of the metric or the behaviour of Jacobi fields. The main result of the paper is the following:

Theorem 1. — *Let (M, g) be a compact, C^∞ Riemannian manifold without conjugate points. Assume that the map $\theta \mapsto H_\theta(0)$ is continuous in $T_1\widetilde{M}$. Then, every two different geodesics $\gamma(t)$, $\beta(t)$ with $\gamma(0) = \beta(0)$ in $\widetilde{M}$ diverge.*

The proof of Theorem 1 is done in Sections 1 and 2, where we also study some general problems concerning asymptoticity properties of geodesics which were introduced by Croke and Schroeder in [**4**]. Namely, consider the relation $\boldsymbol{R}$ between geodesics in $\widetilde{M}$ defined by: $\gamma \; \boldsymbol{R} \; \beta$ if and only if γ is a Busemann asymptote of β. We show in

Section 1 that, under our continuity hypothesis, this relation is an equivalence relation. In the remaining sections we give some applications of Theorem 1. The results in Section 3 are inspired in the following classical result of Eberlein: Let (M, g) be a compact, C^∞ Riemannian manifold without conjugate points. Assume that the Green subbundles $E^s(\theta)$, $E^u(\theta)$ are linearly independent at every point $\theta \in T_1M$. Then the geodesic flow of (M, g) is Anosov. Recall that the geodesic flow $\phi_t : T_1M \to T_1M$ is defined by $\phi_t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$. We obtain in Section 3 a sort of topological version of Eberlein's result. Recall that $(\widetilde{M}, g)$ is a *Gromov hyperbolic space* if there exists $\delta > 0$ such that every geodesic triangle formed by the union of three geodesic segments $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_0]$ satisfies the following property: the distance from any $p \in [x_i, x_{i+1}]$ to $[x_{i+1}, x_{i+2}] \cup [x_{i+2}, x_i]$ is bounded above by δ (the indices are taken mod. 3). The main Theorem of Section 3 is the following.

Theorem 2. — *Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the map $\theta \mapsto H_\theta(0)$ is continuous in the compact open topology in $\widetilde{M}$. Then, if $H_{(p,v)}(0) \cap H_{(p,-v)}(0) = \{p\}$ for every $(p, v) \in T_1\widetilde{M}$, the universal covering $(\widetilde{M}, g)$ is a Gromov hyperbolic space.*

Using some results in [**18**] we shall show that Theorem 2 is equivalent to the following result:

Theorem 3. — *Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the canonical liftings in T_1M of the submanifolds $H_{(p,v)}(0)$, $H_{(p,-v)}(0)$ give rise to continuous foliations H^s, H^u having a local product structure. Then $(\widetilde{M}, g)$ is a Gromov hyperbolic space.*

For the definition of the canonical liftings of the horospheres we refer to Section 3. A pair of ϕ_t-invariant foliations F_1, F_2 in T_1M has a *local product structure* if there exists an atlas $\{\Phi_i : U_i \subset T_1M \to R^{2n-1}\}$ of T_1M such that

(1) Every Φ_i is continuous.

(2) Each local chart is of the form $\Phi_i = (x^i, y^i, t)$, $t \in (-\varepsilon, \varepsilon)$, where the level sets $x^i = constant$, $y^i = constant$ are connected components of the foliations F_1, F_2 respectively.

In virtue of Theorems 2 and 3, we can say that the topological transversality (meaning local product structure) of the horospherical foliations in T_1M implies that $\widetilde{M}$ is a Gromov hyperbolic space. Notice that Theorem 1 is true for manifolds of nonpositive curvature, because the hypotheses in the Theorem imply that there are no flat planes in $\widetilde{M}$ ([**5**]). It also holds for manifolds without focal points, but if we allow focal points many key facts of the theory (convexity, bounded asymptotic behaviour of Jacobi fields and geodesics, etc.) might not hold.

In Section 4 we get some results concerning the boundary of a Gromov hyperbolic group that covers a compact manifold without conjugate points. Suppose that the

map $\theta \mapsto H_\theta(0)$ is continuous. Then we show that, if the fundamental group of M is Gromov hyperbolic, its ideal boundary is a sphere. This fact is well known for compact manifolds of nonpositive curvature whose fundamental group is Gromov hyperbolic. However, if we drop the assumption on the curvature it is not clear whether the ideal boundary of the fundamental group is a sphere.

Finally, in Section 5, we apply the results of Sections 1, 2 to manifolds satisfying the so-called Axiom of Asymptoticity, introduced by Pesin [**16**]. This notion is perhaps the first one in the literature of the research about continuity of horospheres which does not consider any assumptions on the C^2 features of the metric (convexity, Jacobi fields).

1. Horospheres and Busemann flows in $\widetilde{M}$

Throughout the paper, (M, g) will be a C^∞, compact Riemannian manifold without conjugate points. All the geodesics will be parametrized by arc length. We shall often call by $[p, q]$ the geodesic segment joining two points in $\widetilde{M}$. A very special property of manifolds with no conjugate points is the existence of the so-called *Busemann functions*: given $\theta = (p, v) \in T_1\widetilde{M}$ the *Busemann function* $b^\theta : \widetilde{M} \to R$ associated to θ is defined by

$$b^\theta(x) = \lim_{t \to +\infty} (d(x, \gamma_\theta(t)) - t)$$

The level sets of b^θ are the *horospheres* $H_\theta(t)$ where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$. Notice that $\gamma_\theta(t)$ intersects each level set of b^θ perpendicularly at only one point in $H_\theta(t)$, and that $b^\theta(H_\theta(t)) = -t$ for every $t \in R$. Next, we list some basic properties of horospheres and Busemann functions that will be needed in the forthcoming sections (see [**16**], [**4**] for instance, for details).

Lemma 1.1

(1) *b^θ is a C^1 function for every θ.*

(2) *The gradient ∇b^θ has norm equal to one at every point.*

(3) *Every horosphere is a C^{1+K}, embedded submanifold of dimension $n-1$ (C^{1+K} means K-Lipschitz normal vector field), where K is a constant depending on curvature bounds.*

(4) *The orbits of the integral flow of $-\nabla b^\theta$, $\psi_t^\theta : \widetilde{M} \to \widetilde{M}$, are geodesics which are everywhere perpendicular to the horospheres H_θ. In particular, the geodesic γ_θ is an orbit of this flow and we have that*

$$\psi_t^\theta(H_\theta(s)) = H_\theta(s + t)$$

for every $t, s \in R$.

A geodesic β is *asymptotic* to a geodesic γ in $\widetilde{M}$ if there exists a constant $C > 0$ such that $d(\beta(t), \gamma(t)) \leqslant C$ for every $t \geqslant 0$. We shall denote by *Busemann asymptotes* of γ_θ

the orbits of the flow ψ_t^θ. Busemann asymptotes of γ_θ might not be asymptotic to γ_θ, so the relation between geodesics γ $\boldsymbol{R}$ β if and only if "γ is a Busemann asymptote of β" might not be an equivalence relation. Observe that in all known examples of manifolds without conjugate points (nonpositive curvature, no focal points, metrics on surfaces without conjugate points [**1**]), the relation $\boldsymbol{R}$ is an equivalence relation. Lemma 1.1, item 4, implies that the horospheres $H_\theta(t)$ are equidistant, i.e., given any point $p \in H_\theta(s)$, then the distance $d(p, H_\theta(t))$ is equal to $|t-s|$. The canonical lifting in $T_1\widetilde{M}$ of $H_\theta(t)$ is the set $\widetilde{H}_\theta(t) = \{(p, -\nabla_p b^\theta), p \in H_\theta(t)\}$. Another way of defining the horosphere $H_\theta(0)$ is

$$H_\theta(0) = \lim_{r \to +\infty} S_r(\gamma_\theta(r)),$$

where $S_r(p)$ is the sphere of radius r centered at p, and the limit is uniform on compact subsets of $\widetilde{M}$. In other words, given $D > 0$, $q \in \widetilde{M}$, and $\varepsilon > 0$, there exists $T > 0$ such that the Haussdorf distance between the restrictions of $H_\theta(0)$ and $S_r(\gamma_\theta(r))$ to the ball $B_D(q)$ of radius D centered at q is less than ε for every $r \geqslant T$. We shall denote by d_H the Haussdorf distance between subsets in $\widetilde{M}$. Actually, given $\theta \in T_1\widetilde{M}$, the spheres $S_r(\gamma_\theta(r))$ converge to $H_\theta(0)$ in the C^1 topology uniformly of compact subsets (see [**16**]) as a consequence of the bounded geometry of $\widetilde{M}$. The notion of continuity of $\theta \mapsto H_\theta(0)$ given in the introduction is equivalent to the following: let θ_n converge to θ, then $H_{\theta_n}(0)$ converges to $H_\theta(0)$ uniformly on compact subsets of $\widetilde{M}$. Although it is clear that $H_\theta(t)$ depends continuously on $t \in R$, it is not known whether $H_\theta(0)$ depends continuously on θ. The continuity of $\theta \mapsto H_\theta$ is equivalent to the continuity in the C^1 topology of the map $\theta \mapsto b^\theta$ uniformly on compact subsets of $\widetilde{M}$. For the purposes of this section, it will be more convenient to state our results in terms of Busemann functions instead of using horospheres, in general the notation becomes simpler. Let $b^{\theta,t}(p) = d(p, \gamma_\theta(t)) - t$, so $b^\theta(p) = \lim_{t \to +\infty} b^{\theta,t}(p)$, for every $p \in \widetilde{M}$. We shall denote by $d_{T_1M}(\cdot,\cdot)$ the Sasaki metric in T_1M. The following result tells us that the continuity of $\theta \mapsto H_\theta(0)$ implies that horospheres can be uniformly approached by large spheres.

Lemma 1.2. — *Let (M, g) be a compact manifold without conjugate points, such that the map $\theta \mapsto b^\theta$ is continuous. Then, given $D > 0$, $\varepsilon > 0$, there exists $T > 0$ such that for every $\theta = (p_0, v_0) \in T_1\widetilde{M}$, and every ball B_D of radius D containing p_0, we have*

$$|b^\theta(p) - b^{\theta,t}(p)| \leqslant \varepsilon,$$

for every $p \in B_D$ and $t \geqslant T$.

Proof. — Let us first recall that the family of functions $b^{\theta,t}$ converges monotonically to b^θ, i.e., the difference $b^\theta(p) - b^{\theta,t}(p)$ decreases with respect to t for every p. This is due to the fact that the spheres $S_t(\gamma_\theta(t))$ converge monotonically to $H_\theta(0)$, i.e., the Hausdorff distance between the restrictions of $S_t(\gamma_\theta(t))$ and $H_\theta(0)$ to compact sets

decreases to 0 as t goes to $+\infty$. Since $b^\theta(p) = \lim_{t\to+\infty} b^{\theta,t}(p)$, there exists $T_{\theta,p,\varepsilon} > 0$ such that

$$|b^\theta(p) - b^{\theta,t}(p)| \leqslant \varepsilon$$

for every $t \geqslant T_{\theta,p,\varepsilon}$.

Claim 1. — *The number $T_{\theta,p,\varepsilon}$ can be made independent of p in B_D. Moreover, it depends on θ, D, ε.*

Indeed, this is a straighforward application of Dini's Lemma about uniform convergence of monotone sequences of functions: the family of functions

$$f_t : B_D \longrightarrow R, \quad f_t(p) = |b^\theta(p) - b^{\theta,t}(p)|$$

converges monotonically to zero at each point $p \in B_D$; so by Dini's Lemma the family converges uniformly to zero in the compact set B_D.

Next, we would like to relate $T_{\theta,D,\varepsilon}$ and $T_{\alpha,D,\varepsilon}$ for α close to θ. At this point we need the continuity of horospheres, we have not used this hypothesis so far. Recall that

$$|b^{\theta,t}(p) - b^{\alpha,t}(p)| = |d(p,\gamma_\theta(t)) - d(p,\gamma_\alpha(t))| \leqslant d(\gamma_\theta(t),\gamma_\alpha(t)).$$

By hypothesis, $\theta \mapsto b^\theta$ depends continuously on θ uniformly on compact subsets, so there exists $\delta > 0$ such that if $d_{T_1\widetilde{M}}(\theta,\alpha) < \delta$ then $|b^\theta(p) - b^\alpha(p)| \leqslant \varepsilon$ for every $p \in B_D$. On the other hand, by continuity of geodesics with respect to initial conditions, given $\varepsilon > 0$ there exists $\delta' = \delta'(\theta,\varepsilon) > 0$ such that if $d_{T_1\widetilde{M}}(\theta,\alpha) \leqslant \delta'$ then

$$|b^{\theta,t}(p) - b^{\alpha,t}(p)| \leqslant d(\gamma_\theta(t),\gamma_\alpha(t)) \leqslant \varepsilon,$$

for every $0 < t \leqslant T_{\theta.D,\varepsilon}$, and every $p \in B_D$. This implies that

$$\begin{aligned} |b^\alpha(p) - b^{\alpha,T}(p)| &\leqslant |b^\alpha(p) - b^\theta(p)| + |b^\theta(p) - b^{\theta,T}(p)| + |b^{\theta,T}(p) - b^{\alpha,T}(p)| \\ &\leqslant \varepsilon + \varepsilon + \varepsilon = 3\varepsilon, \end{aligned}$$

for every $d_{T_1\widetilde{M}}(\theta,\alpha) \leqslant \min\{\delta,\delta'\}$, for $T = T_{\theta.D,\varepsilon}$, and every $p \in B_D$. Therefore, by the monotonicity of the limit $b^\alpha(p) = \lim_{t\to+\infty} b^{\alpha,t}(p)$ we have that $|b^\alpha(p) - b^{\alpha,t}(p)| \leqslant 3\varepsilon$ for every $t \geqslant T = T_{\theta,D,\delta}$ and $d_{T_1\widetilde{M}}(\theta,\alpha) < \varepsilon_\theta = \min\{\varepsilon,\varepsilon'\}$. This means that

$$T_{\alpha,D,3\varepsilon} \leqslant T_{\theta.D,\varepsilon},$$

for every $d_{T_1\widetilde{M}}(\theta,\alpha) < \varepsilon_\theta$. Now, take a compact fundamental domain M_0 of the manifold M, let $K = \{\theta = (x,w), x \in M_0, w \in T_1\widetilde{M}\}$, and let us cover K by open neighborhoods $V_{\varepsilon_\theta}(\theta)$ where $\theta \in K$. Take a finite covering $\cup_{i=1}^m V_{\varepsilon_{\theta_i}}(\theta_i)$ of K by these neighborhoods. Let $T = \max_{i=1.2....m}\{T_{\theta_i.D.\delta}\}$, and assume without loss of generality that the ball B_D contains the fundamental domain M_0. Then we get that $|b^\alpha(p) - b^{\alpha,t}(p)| \leqslant 3\delta$ for every $p \in B_D$, $\alpha \in K$, and every $t \geqslant T$. Since balls, horospheres, and Busemann functions are preserved by isometries in $\widetilde{M}$, we deduce that $|b^\alpha(p) - b^{\alpha,t}(p)| \leqslant 3\varepsilon$ for every $\alpha = (q,w) \in T_1\widetilde{M}$, $t \geqslant T$, and p in every ball of radius D containing q, as we wished to show. □

Corollary 1.1. — *Let M be a compact Riemannian manifold without conjugate points, and assume that the map $\theta \mapsto H_\theta(0)$ is continuous. Then, if $q \in H_\theta(0)$, and $\alpha = (q, -\nabla_q b^\theta)$, we have that $b^\alpha = b^\theta$. In particular, $H_\alpha(0) = H_\theta(0)$ and the relation $\boldsymbol{R}$ is an equivalence relation between geodesics in $T_1\widetilde{M}$.*

Proof. — Let γ_α be the Busemann asymptote of θ through α. Let us denote by $[x, y]$ the geodesic segment joining the points x, y in $\widetilde{M}$. By definition, the geodesic γ_α is the limit of the geodesic segments $[q, \gamma_\theta(t)]$, where $t \to +\infty$. Let γ_{α_t} be the geodesic containing the segment $[q, \gamma_\theta(t)]$, with $\gamma_{\alpha_t}(0) = q$ and $\gamma_{\alpha_t}(r_t) = \gamma_\theta(t)$ for some positive r_t. Then, $\lim_{t\to+\infty} \alpha_t = \alpha$, and by the continuity hypothesis, $\lim_{t\to+\infty} b^{\alpha_t} = b^\alpha$, this limit being uniform on compact subsets of $\widetilde{M}$. Let $\theta = (p_0, v_0)$, and consider a compact ball B containing the foot points p_0 and q of the vectors θ and α respectively. By Lemma 1.2, the functions b^θ, b^α, b^{α_t}, $t > 0$, can be uniformly approached by radial functions $b^{\theta,T}$, $b^{\alpha,T}$, $b^{\alpha_t,T}$ in the compact ball B. Namely, given $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for every $T > T_\varepsilon$, every $p \in B$, we have that

$$|b^\theta(p) - b^{\theta,T}(p)| \leqslant \varepsilon, \quad |b^\alpha(p) - b^{\alpha,T}(p)| \leqslant \varepsilon, \quad |b^{\alpha_t}(p) - b^{\alpha_t,T}(p)| \leqslant \varepsilon,$$

for every $t > 0$. Since the functions b^{α_t} converge uniformly in B to the function b^α, there exists $S_\varepsilon > 0$ such that

$$|b^\alpha(p) - b^{\alpha,t}(p)| \leqslant \varepsilon,$$

for every $p \in B$ and $t \geqslant S_\varepsilon$. Recall that the number r_t is defined by $\gamma_{\alpha_t}(r_t) = \gamma_\theta(t)$.

Claim 1. — $\lim_{t\to+\infty} |r_t - t| = 0$.

This follows easily by definition: $r_t = d(\gamma_{\alpha_t}(0), \gamma_\theta(t)) = d(q, \gamma_\theta(t))$; which implies that

$$r_t - t = d(q, \gamma_\theta(t)) - t = b^{\theta,t}(q).$$

And, since $q \in H_\theta(0) = (b^\theta)^{-1}(0)$, we get

$$0 = b^\theta(q) = \lim_{t\to+\infty} b^{\theta,t}(q) = \lim_{t\to+\infty} |r_t - t|.$$

Claim 2. — $b^{\alpha_t,r_t}(p) = b^{\theta,t}(p) + t - r_t$ *for every* $p \in \widetilde{M}$.

Just check the definitions:

$$\begin{aligned} b^{\alpha_t,r_t}(p) &= d(p, \gamma_{\alpha_t}(r_t)) - r_t \\ &= d(p, \gamma_\theta(t)) - r_t \\ &= b^{\theta,t}(p) + t - r_t. \end{aligned}$$

Hence, if $t, r_t \geqslant \sup\{T_\varepsilon, S_\varepsilon\}$ we obtain,

$$\begin{aligned} |b^\alpha(p) - b^\theta(p)| &\leqslant |b^\alpha(p) - b^{\alpha_t}(p)| + |b^{\alpha_t}(p) - b^{\alpha_t,r_t}(p)| \\ &\quad + |b^{\alpha_t,r_t}(p) - b^{\theta,t}(p)| + |b^{\theta,t}(p) - b^\theta(p)| \\ &\leqslant 3\varepsilon + |r_t - t|, \end{aligned}$$

for every $p \in B$. Since ε and B are arbitrary, and by Claim 2, $\lim_{t\to+\infty} |r_t - t| = 0$, we deduce that $b^\alpha(p) = b^\theta(p)$ for every $p \in \widetilde{M}$. Since the level sets of Busemann functions are horospheres, then the horospheres of γ_θ are the same horospheres of γ_α. It is clear that the relation $\gamma \boldsymbol{R} \beta$ if and only if γ is a Busemann asymptote of β is an equivalence relation: their horospherical foliations will be the same, and therefore, their Busemann flows will coincide. □

We would like to point out that Croke and Schroeder in [**4**] posed the question of whether the relation $\boldsymbol{R}$ is an equivalence relation in the universal covering of a compact manifold without conjugate points. They show in fact that, if M is analytic, then $b^\alpha(p) - b^\theta(p)$ is constant if γ_θ, γ_α are axes of the same deck transformation in $\pi_1(M)$. Corollary 1.1 asserts that, under the assumption of continuity of horospheres, $b^\alpha(p) - b^\theta(p)$ is constant whenever γ_α is a Busemann asymptote of γ_θ.

2. The divergence of geodesic rays

The main result of the section is the following:

Theorem 2.1. — *Let (M, g) be a compact manifold without conjugate points such that the map $\theta \mapsto H_\theta(0)$ is continuous. Then the geodesic rays diverge uniformly in $\widetilde{M}$, namely, given $\varepsilon > 0$, $L > 0$, there exists $s_{\varepsilon,L} > 0$ such that any two geodesic rays $\gamma_{(p,v)}$, $\gamma_{(p,w)}$, where $p \in \widetilde{M}$ and v, w form an angle $\angle(v, w) \geqslant \varepsilon$, satisfy*

$$d(\gamma_{(p,v)}(t), \gamma_{(p,w)}(t)) \geqslant L$$

for every $t \geqslant s_{\varepsilon,L}$.

We begin by recalling the first variation formula.

Lemma 2.1. — *Let $\gamma(t)$ be a geodesic of (M, g) parametrized by arclength. Let $f : [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a differentiable variation of $\gamma[a, b]$, i.e., $f(t, 0) = \gamma(t)$. Then, the length $L(x)$ of the curve $f_x(t) = f(t, x)$ satisfies*

$$L'(0) = \left\langle \frac{\partial f}{\partial x}(t, 0), \gamma'(t) \right\rangle \Big|_a^b.$$

Now, let γ_θ, γ_α be two geodesics in $\widetilde{M}$ with $\gamma_\theta(0) = \gamma_\alpha(0) = p$. Assume that $\theta \neq \alpha$. Take $s < 0$, and let $f : [s, 0] \times (-a, a) \to \widetilde{M}$ be the variation of $\gamma_\theta[s, 0]$ defined by

– $f(t, 0) = \gamma_\theta(t)$ for every $t \in [s, 0]$.

– $f_x(t) = f(t, x)$ is the geodesic segment joining $\gamma_\theta(s)$ and $\gamma_\alpha(x)$ for each $|x| < a$ (observe that t might not be the arclength parameter of the geodesic $f_x(t)$).

If $|s|$ and a are small enough, the geodesic segments $f_x(t)$ are unique and minimizing. Hence, the variation f is differentiable because of the smooth dependence of small geodesic segments with respect to initial conditions. Notice that $\frac{\partial f}{\partial x}(s, x) = 0$ for every $x \in (-a, a)$. Also, we have that $\frac{\partial f}{\partial x}(0, x) = \gamma'_\alpha(x)$, and $\frac{\partial f}{\partial t}(t, 0) = \gamma'_\theta(t)$.

Corollary 2.1. — *Let* $\varepsilon = \angle(\gamma'_\theta(0), \gamma'_\alpha(0)) = \| \theta - \alpha \|_{T_1\widetilde{M}}$. *Then, there exist* $s_0 > 0$, $K_0 = K_0(\| K \|_\infty)$, *such that for every variation* $f : [s, 0] \times (-a, a) \to M$ *as above, with* $|s| \leqslant s_0$, $|a| \leqslant s$; *there exists* $\overline{\delta} = \overline{\delta}(\varepsilon, s, \| K \|_\infty)$, *such that for every* $x \in (-\overline{\delta}, \overline{\delta})$ *we have*

$$|L(x) - (L(0) + L'(0)x)| \leqslant K_0 x^2,$$

where $\| K \|_\infty$ *is the supremum of the sectional curvatures of* (M, g).

The proof of Corollary 2.1 is straightforward from the first and second variation formulas and the compactness of M. Corollary 2.1 can be viewed as a shortcut lemma, and we shall use it to give a sort of lower estimate of the distance between the horospheres $H_\theta(t)$, $H_\alpha(t)$ when γ_θ, γ_α are two different geodesic rays starting at the same point.

Lemma 2.2. — *Let* γ_θ, γ_α *be two geodesic rays with* $\gamma_\theta(0) = \gamma_\alpha(0) = p$. *Let* $\theta = (p, v)$, $\rho = \| \theta - \alpha \|$. *Then, for every* $\rho > 0$ *there exists* $\delta_1 = \delta_1(\rho, \| K \|_\infty) > 0$, *such that for every* t, t', *satisfying* $\gamma_\alpha(t') = \gamma_\alpha \cap H_{(p,-v)}(-t)$ *we have that*

$$|t' - t| \geqslant \delta_1(1 - \cos(\rho)).$$

Proof. — Let s_0, $\delta_0 = \overline{\delta}(\varepsilon, s_0, \| K \|_\infty)$, K_0 be the numbers defined in Corollary 2.1. We know that $t' \geqslant t$, because

$$\begin{aligned} t' = d(\gamma_\alpha(0), \gamma_\alpha(t')) &= d(\gamma_\theta(0), \gamma_\alpha(t')) \\ &\geqslant d(\gamma_{(p,-v)}(0), H_{(p,-v)}(-t)) \\ &= d(\gamma_\theta(0), \gamma_\theta(t)) = t, \end{aligned}$$

where we used that $\gamma_\theta(t) = \gamma_{(p,-v)}(-t)$ is the geodesic that realizes the distance between $p = \gamma_{(p,-v)}(0)$ and $H_{(p,-v)}(-t)$. Thus, $s_0 + t' \geqslant s_0 + t$. By Corollary 2.1,

$$\begin{aligned} d(\gamma_\theta(-s_0), \gamma_\alpha(\delta)) + d(\gamma_\alpha(\delta), \gamma_\alpha(t')) &= d(\gamma_\theta(-s_0), \gamma_\theta(0)) + \delta\cos(\rho) + O(\delta^2) + t' - \delta \\ &= s_0 + \delta\cos(\rho) + t' - \delta + O(\delta^2), \end{aligned}$$

for every $|\delta| \leqslant \delta_0$; which implies that

$$s_0 + t = d(\gamma_\theta(-s_0), \gamma_\theta(t)) < s_0 + \delta\cos(\rho) + t' - \delta + O(\delta^2).$$

for every $|\delta| < \delta_0$ (see figure 1).

So we get, by Corollary 2.1,

$$t' - t > \delta - \delta\cos(\rho) + O(\delta^2) = \delta(1 - \cos(\rho)) + O(\delta^2),$$

for every $|\sigma| < \delta_0$. On the other hand, there exists $0 < \delta_1 = \delta_1(\rho, K_0) \leqslant \delta_0$ such that $\delta_1(1 - \cos(\rho)) + O(\delta_1^2) > \frac{\delta_1}{2}(1 - \cos(\rho))$, therefore $t' - t > \frac{\delta_1}{2}(1 - \cos(\rho))$ as we wished to show. □

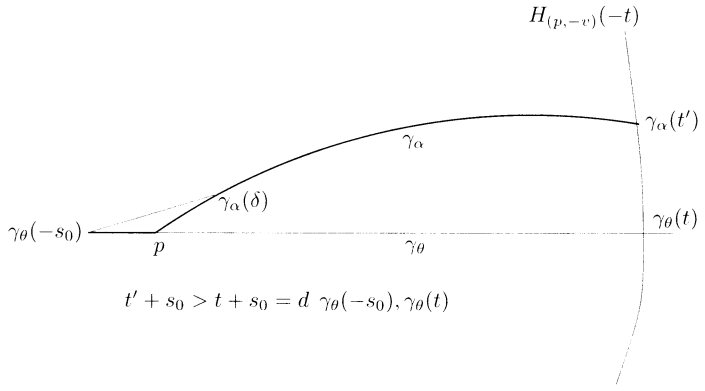

FIGURE 1

Proof of Theorem 2.1. — Let (M, g) be a compact manifold without conjugate points such that the map $\theta \mapsto H_\theta(0)$ is continuous. Let $C > 0$, and consider two geodesic rays γ_θ, γ_α with $\gamma_\theta(0) = \gamma_\alpha(0) = p$. Let us define

$$T_{\alpha,\theta,C} = \sup\{t > 0, d(\gamma_\theta(t), \gamma_\alpha(t)) \leqslant C\}.$$

We are going to show that there exists $T(\| \theta - \alpha \|, C) > 0$ such that $T_{\alpha,\theta,C} \leqslant T(\| \theta - \alpha \|, C)$ for every α, θ in $T_1\widetilde{M}$ with the same foot point p. In particular, $T(\| \theta - \alpha \|, C)$ depends on C and on the angle between θ and α. So let $t > 0$ be such that $d(\gamma_\theta(t), \gamma_\alpha(t)) \leqslant C$.

Claim 1. — *There exists a number t' such that*

$$\gamma_\alpha(t') = \gamma_\alpha \cap H_{(p,-v)}(-t + C).$$

In fact, recalling that $b^{(p,-v)}$ is the Busemann function of $\gamma_{(p,-v)}$, the hypotheses on $\gamma_\alpha(t)$ implies that

$$t - C \leqslant b^{(p,-v)}(\gamma_\alpha(t)) \leqslant t + C,$$

which is easy to check by the definition of the Busemann function (recall that $\gamma_\theta(t) = \gamma_{(p,-v)}(-t)$). On the other hand, observe that $H_{(p,-v)}(-t) = (b^{(p,-v)})^{-1}(t)$, and since $\gamma_\alpha(0) = \gamma_\theta(0) \in H_{(p,-v)}(0)$ we have that $b^{(p,-v)}(\gamma_\alpha(0)) = 0$, $b^{(p,-v)}(\gamma_\alpha(t)) \geqslant t - C$. Since the function $g(t) = b^{(p,-v)}(\gamma_\alpha(t))$ is continuous, it assumes all the values in the interval $[0, t - C]$. Hence, there exists t' such that $g(t') = t - C$. Or equivalently, $\gamma_\alpha(t') \in (b^{(p,-v)})^{-1}(t - C) = H_{(p,-v)}(-t + C)$. It is clear that $t' > t - C$, because $t - C$ is the distance between $p = \gamma_{(p,-v)}(0)$ and $H_{(p,-v)}(-t + C)$.

We can assume without loss of generality, by changing t by $t - C$, that γ_α intersects $H_{\gamma_\theta}(t)$. In this case we have $t' > t$, and we can assume that

$$t' = \inf\{s > t, \gamma_\alpha(s) \in H_{(p,-v)}(-t)\}.$$

Claim 2. — *Given $\varepsilon > 0$, there exists $T = T(\varepsilon, C) > 0$ such that $|t - t'| \leqslant \varepsilon$ for every $t \geqslant T$.*

This follows from Lemma 1.2. Indeed, since $\gamma_\theta(0) = \gamma_\alpha(0) = p$, the points $\gamma_\theta(t), \gamma_\alpha(t)$ belong to the sphere $S_t(p)$ for every t. Let $\phi_t : T_1\widetilde{M} \to T_1\widetilde{M}$ be the geodesic flow of $\widetilde{M}$. Let $\beta = \phi_{-t}(p, -v)$. The geodesic γ_β is a reparametrization of γ_θ satisfying $\gamma_\beta(0) = \gamma_\theta(t)$, and $\gamma_\beta(t) = \gamma_\theta(0) = p$. By the choice of t', we have that the ball $B_{3C}(\gamma_\beta(0))$ contains $\gamma_\alpha(t)$ and $\gamma_\alpha(t')$. Moreover, the points $\gamma_\theta(t) = \gamma_\beta(0)$ and $\gamma_\alpha(t)$ belong to the sphere $S_t(p) = S_t(\gamma_\beta(t))$. Thus, by Lemma 1.2, for $D = 3C$ and $\varepsilon > 0$, there exists $T > 0$ such that

$$d_H(H_\beta(0) \cap B_D(\gamma_\theta(t)), S_t(p) \cap B_D(\gamma_\theta(t))) \leqslant \varepsilon,$$

for every $t \geqslant T$. In particular, the point $\gamma_\alpha(t)$, that belongs to $S_t(p) \cap B_D(\gamma_\theta(t))$, is within a distance ε from $H_\beta(0)$ (see figure 2).

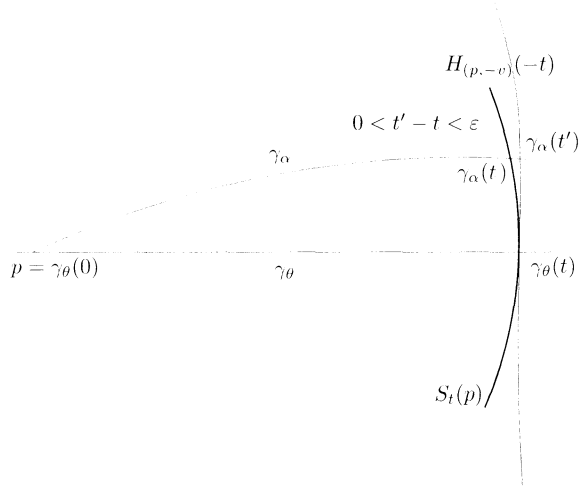

FIGURE 2

Hence, choosing ε small enough, the first time $s = t' > t$ where occurs a transversal intersection between $\gamma_\alpha(s)$ and $H_\beta(0) = H_{(p,-v)}(-t)$ satisfies

$$d(\gamma_\alpha(t'), \gamma_\alpha(t)) = |t' - t| \leqslant \varepsilon,$$

for every $t \geqslant T$. This finishes the proof of the Claim.

Applying Lemma 2.2 and Claim 2 to the geodesics γ_θ, γ_α, we have that for every $t \geqslant T(\varepsilon, C)$,

$$\varepsilon \geqslant |t' - t| \geqslant \delta_1(1 - \cos(\rho)),$$

where ρ is the angle between θ and α. Therefore, if $\rho > 0$, the number $\varepsilon_\rho = \delta_1(1 - \cos(\rho))$ is strictly greater than 0. Hence, the above inequality holds only if

$\varepsilon \geqslant \varepsilon_\rho$, and therefore, t has to be smaller than $T(\varepsilon_\rho/2, C)$ for instance. This implies that $T_{\theta,\alpha,C} \leqslant T(\varepsilon_\rho/2, C)$, as we wished to prove.

3. Topological transversality of horospheres and expansiveness are equivalent

The goal of this section is to combine the divergence of geodesic rays with some ideas connecting topological dynamics and global geometry. We shall obtain a sort of topological version of Eberlein's theorem about the characterization of Anosov geodesic flows by the transversality of the Green bundles. We begin by showing one of the main consequences of the divergence of geodesic rays.

Lemma 3.1. — *Let M be a compact manifold without conjugate points, such that the map $\theta \mapsto H_\theta(0)$ is continuous. If a geodesic β is asymptotic to a geodesic γ_θ in $\widetilde{M}$, then β is a Busemann asymptote of γ_θ.*

Proof. — Let us assume that $d(\gamma_\theta(t), \beta(t)) \leqslant C$ for every $t \geqslant 0$. Consider the geodesic segments $[\beta(0), \gamma_\theta(t)]$ (using the notation of Corollary 1.1), where $t \geqslant 0$. Let β_t be the geodesic defined by $\beta_t(0) = \beta(0)$, $\beta_t(T_t) = \gamma_\theta(t)$ for some positive T_t. Since $d(\beta_t(T_t), \beta(t)) \leqslant C$ for every $t \geqslant 0$, letting $t \to +\infty$ we have, by the uniform divergence of geodesic rays, that

$$\lim_{t \to +\infty} \beta_t = \beta,$$

uniformly on compact sets. This implies that β is a Busemann asymptote of γ_θ by definition. □

Lemma 3.2. — *Let M be a compact manifold without conjugate points, such that the map $\theta \mapsto H_\theta(0)$ is continuous. Then, if $\gamma_{(p,v)}$ and γ_α are bi-asymptotic, we have that, up to a reparametrization of γ_α, $\gamma_\alpha(0) \in H_{(p,v)}(0) \cap H_{(p,-v)}(0)$.*

Proof. — Let $\alpha = (q, w)$. Since γ_α is bi-asymptotic to $\gamma_{(p,v)}$, then, according to Lemma 3.1 and Corollary 1.1, γ_α is Busemann asymptotic to $\gamma_{(p,v)}$, and $\gamma_{(q,-w)}$ is Busemann asymptotic to $\gamma_{(p,-v)}$. Let us assume that $q = \gamma_\alpha(0) = \gamma_\alpha \cap H_{(p,v)}(0)$. In this case, we have $b^\alpha(\gamma_\alpha(0)) = 0$, $b^{(p,v)}(\gamma_\alpha(0)) = 0$, and by Corollary 1.1, $b^\alpha = b^{(p,v)}$. If we also had that $b^{(q,-w)} = b^{(p,-v)}$, then Lemma 3.2 would hold.

Claim. — $\gamma_\alpha(0) = q \in H_{(p,-v)}(0)$.

In fact, let t_0 be such that $\gamma_\alpha(t_0) = \gamma_\alpha \cap H_{(p,-v)}(0)$. Since $-t_0 = b^{(p,v)}(\gamma_\alpha(t_0))$, and $H_{(p,-v)}(0)$ is in the region $(b^{(p,v)})^{-1}[0, +\infty)$, we have that $t_0 \leqslant 0$. Let ϕ_t be the geodesic flow of $\widetilde{M}$. Notice that

$$\phi_{-t_0}(q, -w) = (\gamma_\alpha(t_0), -\gamma'_\alpha(t_0)),$$

so we have, by Corollary 1.1, that $b^{\phi_{-t_0}(q,-w)} = b^{(p,-v)}$ as long as $\gamma_\alpha(t_0) \in H_{(p,-v)}(0)$. Hence, $\psi_t^{(p,v)} = \psi_t^{(q,w)}$, $\psi_t^{(p,-v)} = \psi_t^{(q,-w)}$ for every $t \in R$ — where ψ^θ is the Busemann

flow of θ — and the foliations $\{H_{(q,-w)}(t), t \in R\}$, $\{H_{(p,-v)}(t), t \in R\}$, coincide. We get, by the choice of t_0, that

$$H_{(q,-w)}(-t_0) = H_{\phi_{-t_0}(q,-w)}(0) = H_{(p,-v)}(0).$$

But

$$H_{(q,-w)}(-t_0) = \psi_{-t_0}^{(q,-w)}(H_{(q,-w)}(0)),$$

(see figure 3) which implies that

$$\psi_{-t_0}^{(q,-w)}(H_{(q,-w)}(0)) = \psi_{-t_0}^{(p,-v)}(H_{(q,-w)}(0)) = H_{(p,-v)}(0).$$

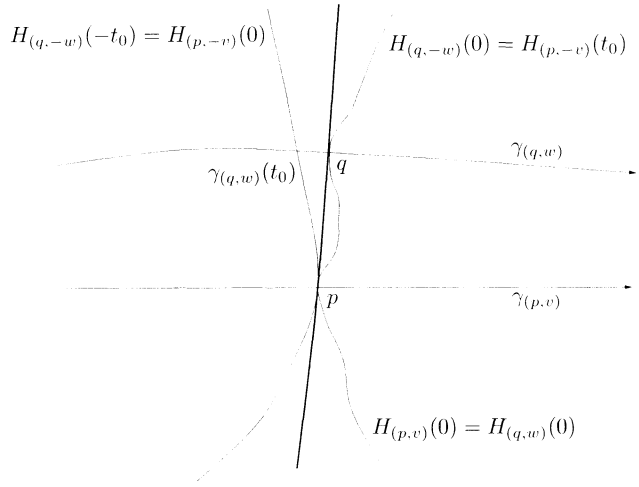

FIGURE 3

Therefore, we get

$$H_{(q,-w)}(0) = \psi_{t_0}^{(p,-v)}(H_{(p,-v)}(0)) = H_{(p,-v)}(t_0).$$

Since $H_{(q,-w)}(0)$ is in the region $(b^{(q,w)})^{-1}[0,+\infty) = (b^{(p,v)})^{-1}[0,+\infty)$, and $\gamma_{(p,v)}(-t_0) = \gamma_{(p,-v)}(t_0) \in H_{(p,-v)}(t_0)$, we have that

$$b^{(p,v)}(\gamma_{(p,v)}(-t_0)) = t_0 \geqslant 0.$$

Since t_0 was already nonpositive, we conclude that $t_0 = 0$, thus proving the Claim and the Lemma. □

Lemmas 3.1, 3.2 are concerned with one basic question of the theory of manifolds without conjugate points: Are the geodesics asymptotic to γ Busemann asymptotic to γ? In all known examples of manifolds without conjugate points the answer to this question is affirmative. However, there is no proof of this fact, as far as I know,

without strong assumptions on either the curvature, the convexity of the metric, or on the asymptotic behaviour of Jacobi fields.

Next, let us recall a notion that appears very often in topological dynamics. Given a C^∞ Riemannian manifold (N, g), a differentiable flow $f_t : N \to N$ without singularities is said to be *expansive* if there exists $\varepsilon > 0$ such that the following holds: let $p \in N$, and suppose that there exist $q \in N$, and a continuous, surjective reparametrization $\rho : R \to R$, with $\rho(0) = 0$, of the orbit of q such that $d(f_t(p), f_{\rho(t)}(q)) \leqslant \varepsilon$ for every $t \in R$; then q belongs to the orbit of p. The following results are proved in [**17**], [**18**].

Theorem 3.1. — *Let M be a compact manifold without conjugate points. If the geodesic flow is expansive, the universal covering of M endowed with the pullback of the metric of M is a Gromov hyperbolic space.*

Lemma 3.3. — *Let (M, g) be a compact Riemannian manifold without conjugate points. Then the geodesic flow is expansive if and only if for every pair of geodesics γ, β in $(\widetilde{M}, g)$ with $d(\gamma, \beta) \leqslant D$ we have that $\gamma = \beta$.*

So expansiveness of the geodesic flow is equivalent to the nonexistence of bi-asymptotic geodesics. Hence, to show Theorem 2 it is enough to show the following:

Lemma 3.4. — *Let (M, g) be a compact manifold without conjugate points such that $\theta \mapsto H_\theta(0)$ is continuous. If $H_{(p,v)}(0) \cap H_{(p,-v)}(0) = \{p\}$ for every $(p, v) \in T_1\widetilde{M}$, then there are no bi-asymptotic geodesics in $\widetilde{M}$.*

The proof of Lemma 3.4 is immediate from Lemma 3.2: bi-asymptotic geodesics are Busemann bi-asymptotic, and there is an injection between geodesics which are bi-asymptotic to $\gamma_{(p,v)}$ and the set $H_{(p,v)}(0) \cap H_{(p,-v)}(0)$.

For the proof of Theorem 3, we shall need some other definitions. The stable horosphere $H^s(p, v)$ of (p, v) in T_1M is defined by the following canonical lift:

$$H^s(p, v) = \Pi\{(q, w), q \in H_{(\widetilde{p},\widetilde{v})}(0), w = -\nabla_q b^{(\widetilde{p},\widetilde{v})}\},$$

where $\Pi : T_1\widetilde{M} \to T_1M$ is the canonical projection, and $\Pi(\widetilde{p}, \widetilde{v}) = (p, v)$. The unstable horosphere of (p, v) is defined by

$$H^u(p, v) = \Pi\{(q, w), q \in H_{(\widetilde{p},-\widetilde{v})}(0), w = \nabla_q b^{(\widetilde{p},-\widetilde{v})}\}.$$

When the geodesic flow is Anosov, these sets coincide with the dynamical stable and unstable sets of (p, v). The collection of the stable horospheres is denoted by H^s, and the collection of unstable horospheres is denoted by H^u.

Lemma 3.5. — *Let M be a compact manifold without conjugate points such that the collections H^s, H^u are continuous foliations of T_1M having a local product structure (as defined in the introduction). Then the geodesic flow is expansive.*

Proof. — The definition of local product structure includes the continuity of the map $\theta \mapsto H_\theta(0)$, so our hypothesis implies that geodesic rays diverge in $\widetilde{M}$. Observe also that the local product structure provides a number $r > 0$, such that for each (p, v) in T_1M there exists an open ball $V(p, v)$ of radius r where

$$V(p, v) \cap H^s(p, v) \cap H^u(p, -v) = \{(p, v)\}.$$

The number $r > 0$ is uniform in T_1M by the compactness of M and the continuity of the invariant foliations. Thus, the natural projection of the neighborhood $V(p, v)$ in M, lifted to $\widetilde{M}$, gives us an open neighborhood $W(\widetilde{p})$ of a lift $\widetilde{p}$ of p in $\widetilde{M}$ where

$$W(\widetilde{p}) \cap H_{(\widetilde{p},v)}(0) \cap H_{(\widetilde{p},-v)}(0) = \{\widetilde{p}\}.$$

Moreover, $W(\widetilde{p})$ contains an open ball $B_\delta(\widetilde{p})$ centered at $\widetilde{p}$, and δ does not depend on $\widetilde{p}$. Let us prove that this local transversality implies expansiveness. Arguing by contradiction, suppose that the geodesic flow is not expansive. Then, given any $\varepsilon > 0$ there would exist a pair of different geodesics $\gamma_{\theta_\varepsilon}$, $\gamma_{\beta_\varepsilon}$ in T_1M such that $d(\gamma_{\theta_\varepsilon}(t), \gamma_{\beta_\varepsilon}(\rho(t))) \leqslant \varepsilon$, for every $t \in R$, where $\rho : R \to R$ is a reparametrization of $\gamma_{\beta_\varepsilon}$ satisfying the conditions in the definition of expansiveness. It is clear that for ε small enough, we can lift the above geodesics to $\widetilde{M}$ and get a pair of geodesics $\widetilde{\gamma}_{\overline{\theta}_\varepsilon}$, $\widetilde{\gamma}_{\overline{\beta}_\varepsilon}$ such that $d_H(\widetilde{\gamma}_{\overline{\theta}_\varepsilon}, \widetilde{\gamma}_{\overline{\beta}_\varepsilon}) \leqslant \varepsilon$. Denote $(p_\varepsilon, v_\varepsilon) = \overline{\theta}_\varepsilon$. By Lemma 3.2 and Corollary 1.1, $\widetilde{\gamma}_{\overline{\theta}_\varepsilon}$ and $\widetilde{\gamma}_{\overline{\beta}_\varepsilon}$ would be Busemann bi-asymptotic to each other and hence, the set

$$H_{(p_\varepsilon, v_\varepsilon)}(0) \cap H_{(p_\varepsilon, -v_\varepsilon)}(0)$$

would contain a point within a very small distance from $(p_\varepsilon, v_\varepsilon) = \overline{\theta}_\varepsilon$. This clearly contradicts the existence of the neighborhood $B_\delta(p_\varepsilon)$. □

4. Visibility and the ideal boundary of $\pi_1(M)$

Recall that $\widetilde{M}$ is a visibility manifold if given $p \in \widetilde{M}$, $\varepsilon > 0$, there exists $L = L(p, \varepsilon) > 0$ such that if the distance from every point of a geodesic segment $[x, y]$ in $\widetilde{M}$ to p is greater than L, then the angle formed by the geodesic segments $[p, x]$ and $[p, y]$ at the point p is less than ε. When L does not depend on p, $\widetilde{M}$ is said to be a uniform visibility manifold. Visibility manifolds were introduced by Eberlein [**6**], and their geometric properties were extensively studied by Eberlein and O'Neill in the 70's (see [**7**] for instance). Visibility manifolds of nonpositive curvature enjoy many properties of negatively curved manifolds. In fact, if M is compact and has nonpositive curvature, then $\widetilde{M}$ is a visibility manifold if and only if $\widetilde{M}$ is a Gromov hyperbolic space. In [**17**] is stated that if M is compact and has no conjugate points, then $\widetilde{M}$ is a visibility manifold if and only if $\widetilde{M}$ is a Gromov hyperbolic space. It is first shown that the visibility property implies Gromov hyperbolicity, however, the proof of the converse statement has a gap based precisely in the (implicit) assumption

of the divergence of geodesic rays. In the light of the results in Section 2, what we have is the following:

Lemma 4.1. — *Let M be a compact manifold without conjugate points. Assume that the map $\theta \mapsto H_\theta(0)$ is continuous. If $\widetilde{M}$ is Gromov hyperbolic then $\widetilde{M}$ is a visibility manifold.*

Proof. — We just make a sketch of the proof pointing out the role of the divergence of geodesic rays in the argument. We want to show that the Gromov hyperbolicity of $\widetilde{M}$ implies visibility. Let $\delta > 0$ be such that every geodesic triangle in $\widetilde{M}$ is δ-thin. It is easy to see that there exists $D = D(\delta)$ such that in every geodesic triangle $[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_0]$ there exists three points $y_i \in [x_i, x_{i+1}]$ (indices taken mod. 3) with $d(y_i, y_{i+1}) \leqslant D$ (see figure 4).

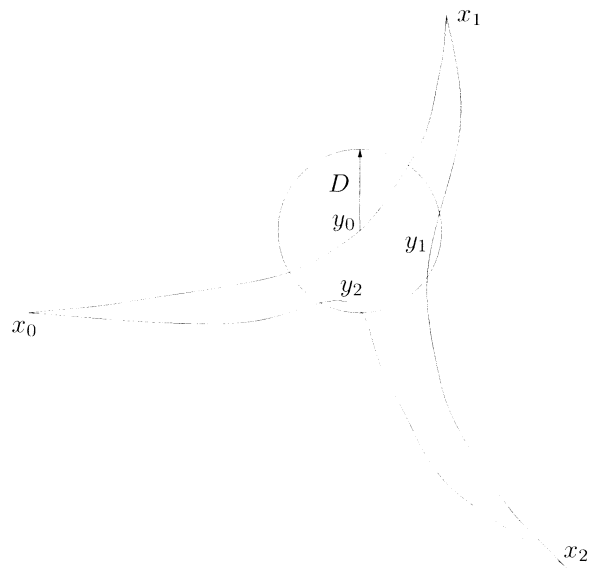

FIGURE 4

Let us suppose that the distance between x_0 and every point in $[x_1, x_2]$ is greater than $L > 0$. By the triangle inequality, we have that

$$\inf\{d(x_0, y_0), d(x_0, y_2)\} \geqslant L - D.$$

So we have two geodesic rays γ_0, γ_2 starting at ($x_0 = \gamma_0(0) = \gamma_2(0)$, namely, the geodesic rays containing the geodesic segments $[x_0, x_1]$, $[x_0, x_2]$ respectively, having points $y_0 \in \gamma_0$, $y_2 \in \gamma_2$, such that

(1) $d(\gamma_i(0), y_i) \geqslant L - D$, for $i = 0, 2$.
(2) $d(y_0, y_2) \leqslant D$.

The continuity of $\theta \mapsto H_\theta(0)$ implies the uniform divergence of rays in $\widetilde{M}$. So given $\varepsilon > 0$, there exists $T >$ such that if $L - D \geqslant T$, the angle formed by γ_0 and γ_2 at x_0 is less than ε. This clearly implies Lemma 4.1. □

Once we have that $\widetilde{M}$ is a visibility manifold, the theory of Eberlein and O'Neil [**7**] grants the existence of a well defined compactification of $\widetilde{M}$, similar to the compactification of the universal covering of negatively curved manifolds. The boundary of the compactification, called the ideal boundary, is homeomorphic to a sphere of dimension $n-1$, if n is the dimension of M. The action of $\pi_1(M)$ extends to the boundary and the complexity of the dynamics of the action is comparable with the complexity of the actions of Kleinian groups in the sphere. A similar theory is made for Gromov hyperbolic groups. For the definitions and proofs of statements we refer to [**12**], [**10**]. Indeed, the group $\pi_1(M)$ has a compactification as a metric space, via its Cayley graph endowed with the word metric on a finite set of generators. There is an ideal boundary for this compactification, and action of $\pi_1(M)$ induces an action in this ideal boundary. The point is that these two boundaries, the first one of geometric nature and the second one of algebraic nature, are homeomorphic. This is a straightforward consequence of the following fact: the Cayley graph endowed with the word metric and $\widetilde{M}$ are quasi-isometric spaces. Therefore, we have proven the following:

Lemma 4.2. — *Let M be a compact manifold without conjugate points such that the map $\theta \mapsto H_\theta(0)$ is continuous. If $\pi_1(M)$ is Gromov hyperbolic, then its ideal boundary is homeomorphic to a sphere.*

We would expect that Lemma 4.2 holds for every compact manifold without conjugate points, with no extra assumptions on the manifold. It is not known if the $n-1$ sphere is the ideal boundary of a Gromov hyperbolic group covering a compact, n-dimensional manifold, for $n \geqslant 3$ (a good survey of results and conjectures can be found in [**13**]).

5. Is the divergence of geodesic rays equivalent to the continuity of horospheres?

We would like to finish with some remarks about a class of manifolds without conjugate points introduced by Pesin in [**16**]. We say that a manifold M without conjugate points satisfies the so-called Axiom of Asymptoticity, if given any $\theta = (p, v)$ in $T_1\widetilde{M}$, a point $q \in \widetilde{M}$, sequences $\theta_n \to \theta$, $q_n \to q$, and $t_n \to +\infty$, then the sequence of geodesics $[q_n, \gamma_{\theta_n}(t_n)]$ converges to a geodesic β that is asymptotic to γ_θ. Pesin in [**16**] claimed that if M is a compact manifold without conjugate points satisfying the Axiom of Asymptoticity, then the map $\theta \mapsto H_\theta(0)$ is continuous in the sense defined in Section 1. However, the argument is based in Green's result [**11**] about the divergence

of geodesic rays in manifolds without conjugate points which has a gap in its proof, as we already mentioned in the introduction. Therefore, what is actually proved in [**16**] is the following:

Lemma 5.1. — *Let M be a compact manifold without conjugate points such that geodesic rays diverge in $\widetilde{M}$. If $\widetilde{M}$ satisfies the Axiom of Asymptoticity, then the map $\theta \mapsto H_\theta(0)$ is continuous.*

According to the results in Section 2 we have,

Corollary 5.1. — *Let M be a compact manifold without conjugate points such that $\widetilde{M}$ satisfies the Axiom of Asymptoticity. Then, geodesic rays diverge in $\widetilde{M}$ if and only if the map $\theta \mapsto H_\theta(0)$ is continuous.*

It is natural to expect that the divergence of geodesic rays implies the continuity of the horospheres. Straighforward generalizations of the proofs of Lemmas 3.1 and 3.2 apply to manifolds satisfying the Axiom of Asymptoticity.

Corollary 5.2. — *Let M be a compact manifold without conjugate points such that the map $\theta \mapsto H_\theta(0)$ is continuous. If $\widetilde{M}$ satisfies the Axiom of Asymptoticity then*

(1) *The relation between geodesics defined by $\gamma \, \boldsymbol{R} \, \beta$ if and only if β is a Busemann asymptote of γ, is an equivalence relation.*

(2) *If γ_α is asymptotic to γ_β, then $b^\theta - b^\alpha$ is constant.*

(3) *A geodesic β is bi-asymptotic to γ_θ if and only if, up to a reparametrization of β, $\beta(0) \in H_{(p,v)}(0) \cap H_{(p,-v)}(0)$.*

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