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ON SOME APPROXIMATIONS OF THE QUASI-GEOSTROPHIC EQUATION

by

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Abstract. — For two-dimensional quasi-geostrophic equation in Fourier space, we propose a new type approximation representing itself some quasi-linear equation. Natural finite dimensional approximations of this equation are investigated in the article.

1. Introduction

The main difficulty in the proof of existence and uniqueness of solutions of hydrodynamical equations is the lack of understanding of the role played by non-linear, or Eulerian, terms. In Fourier space these terms describe the expansion of initial excitations of Fourier modes but the way how this process goes is in general unclear.

In this paper we propose an approach which leads to some simplifications of the original equations with the belief that the processes of expansion remain the same. Our equations have natural finite-dimensional approximations which are systems of ODE and are easier to tackle.

We restrict ourselves to the two-dimensional quasi-geostrophic equation (QGE) for an unknown function $u(k,t), k = (k_1, k_2) \in R^2$ which in Fourier space has the form (see [**1**], [**2**])

$$\frac{\partial u(k,t)}{\partial t} = \int_{R^2} \frac{((k')^{\perp}, k-k')}{|k-k'|} u(k',t)u(k-k',t)dk' - \nu|k|^{2\alpha}u(k,t) \tag{1}$$

Here $|k| = (k_1^2 + k_2^2)^{1/2}$, $k^{\perp} = (-k_2, k_1)$, $\nu \geqslant 0$ is the viscosity and we are interested in even solutions $u(-k,t) = u(k,t)$. It is well-known that the mathematical difficulties

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related to (1) are in many respects similar to the well-known difficulties for the $3D$-Navier-Stokes system.

The main case is $\alpha = 1$. For $0 < \alpha < 1$ we obtain the so-called generalized QGE, which we also consider in this paper.

We are interested in solutions which are smooth in k and decay at infinity rather slowly. Our main assumption says that for such solutions the main contribution to the integral in (1) comes from $|k'| \ll |k|$ or $|k-k'| \ll |k|$. For $|k'| \ll |k|$ we can write

$$\frac{((k')^\perp, k-k')}{|k-k'|} = ((k')^\perp, k)\left(\frac{1}{|k|} - (\nabla\frac{1}{|k|}, k')\right) + \cdots$$

where dots mean terms of a smaller order of magnitude. Thus for $|k'| \ll |k|$ we keep the term

$$\begin{aligned}&\int_{R^2} ((k')^\perp, k)\left(\frac{1}{|k|} - (\nabla\frac{1}{|k|}, k')\right) u(k',t)(u(k,t) - (\nabla u(k,t), k'))dk' \\ &\quad= \int_{R^2} \frac{((k')^\perp, k)}{|k|} u(k',t)u(k,t)dk' - u(k,t)\int_{R^2} ((k')^\perp, k)(\nabla\frac{1}{|k|}, k')u(k',t)dk' \\ &-\int_{R^2} \frac{((k')^\perp, k)}{|k|}(\nabla u(k,t), k')u(k',t)dk' + \int_{R^2} ((k')^\perp, k)(\nabla\frac{1}{|k|}, k')(\nabla u(k,t), k'))u(k',t)dk'\end{aligned}$$

The first and the last integrals are zero because the integrands are odd functions of k'. For $|k-k'| \ll k$ put $k'' = k - k'$. Then

$$\begin{aligned}&\int_{R^2} \frac{((k-k'')^\perp, k)}{|k''|} u(k-k'', t)u(k'', t)dk'' \\ &\quad= \int_{R^2} \frac{((k^\perp), k'')}{|k''|}(u(k,t) - (\nabla u(k,t), k''))u(k'', t)dk'' + \cdots \\ &= \int_{R^2} \frac{((k^\perp), k'')}{|k''|} u(k,t)u(k'', t)dk'' - \int_{R^2} \frac{((k^\perp), k'')}{|k''|}(\nabla u(k,t), k'')u(k'', t)dk'' + \cdots\end{aligned}$$

Again dots mean terms of a smaller order of magnitude. The first integral is zero by the same reasons as above, i.e. the parity of the integrand. Thus our approximating equation takes the form

$$\begin{aligned}(2)\qquad \frac{\partial u(k,t)}{\partial t} &= -u(k,t)\int_{R^2} ((k')^\perp, k)(\nabla\frac{1}{|k|}, k')u(k',t)dk' \\ &\qquad - \int_{R^2} \frac{((k')^\perp, k)}{|k|} u(k',t)\,(k', \nabla u(k,t))\,dk' \\ &\qquad - \int_{R^2} \frac{((k)^\perp, k')}{|k'|} u(k',t)\,(k', \nabla u(k,t))\,dk' - \nu|k|^{2\alpha}u(k,t)\end{aligned}$$

The equation (2) does not satisfy the energy estimate but apparently remains dissipative because of viscosity. Let us rewrite (2) as follows:

$$(3)\quad \frac{\partial u(k,t)}{\partial t} = -u(k,t)\int_{R^2}((k')^\perp,k)(\nabla\frac{1}{|k|},k')u(k',t)dk' \\ -\int_{R^2}((k')^\perp,k)\left[\frac{1}{|k|}+\frac{1}{|k'|}\right]u(k',t)\,(k',\nabla u(k,t))\,dk' - \nu|k|^{2\alpha}u(k,t)$$

The equation (3) is a first order quasi-linear equation whose coefficients are global functions of u. Take the first term in (3):

$$I_1(t) = \int_{R^2}((k')^\perp,k)\left(\nabla\frac{1}{|k|},k'\right)u(k',t)dk'$$

We have

$$(k')^\perp = (-k_2',k_1');\ \nabla\frac{1}{|k|} = \left(-\frac{k_1}{(k_1^2+k_2^2)^{3/2}},-\frac{k_2}{(k_1^2+k_2^2)^{3/2}}\right).$$

Therefore

$$\begin{aligned} I_1(t) &= -\int(k_1'k_2-k_2'k_1)\frac{k_1k_1'+k_2k_2'}{|k|^3}\,u(k':t)\,dk' \\ &= -\frac{k_2^2}{|k|^3}\int k_1'k_2'u(k',t)dk' + \frac{k_1^2}{|k|^3}\int k_1'k_2'u(k',t)dk' \\ &\quad \frac{k_1k_2}{|k|^3}\int((k_1')^2-(k_2')^2)u(k',t)dk'. \end{aligned}$$

Denote

$$a_1(t) = \int k_1^2u(k,t)dk;\quad a_2(t) = \int k_2^2u(k,t)dk;\quad a_3(t) = \int k_1k_2u(k,t)dk;$$

Then

$$I_1(t) = \frac{k_1^2-k_2^2}{|k|^3}a_3 - \frac{k_1k_2}{|k|^3}(a_1-a_2)$$

Consider

$$I_2(t) = \int((k')^\perp,k)\frac{1}{|k|}u(k',t)(k',\nabla u(k,t))dk'$$

We have

$$\begin{aligned} I_2(t) &= \int dk'\frac{k_2k_1'-k_1k_2'}{|k|}u(k',t)(k_1'\frac{\partial u(k,t)}{\partial k_1}+k_2'\frac{\partial u(k,t)}{\partial k_2}) \\ &= \frac{\partial u(k,t)}{\partial k_1}\left[-\frac{k_1}{|k|}\int k_1'k_2'u(k',t)dk' + \frac{k_2}{|k|}\int(k_1')^2u(k',t)dk'\right] \\ &\quad + \frac{\partial u(k,t)}{\partial k_2}\left[-\frac{k_1}{|k|}\int(k_2')^2u(k',t)dk' + \frac{k_2}{|k|}\int k_1'k_2'u(k',t)dk'\right] \\ &= \frac{\partial u(k,t)}{\partial k_1}\left[-\frac{k_1}{|k|}a_3+\frac{k_2}{|k|}a_1\right] + \frac{\partial u(k,t)}{\partial k_2}\left[-\frac{k_1}{|k|}a_2+\frac{k_2}{|k|}a_3\right] \end{aligned}$$

The last term

$$\begin{aligned}I_3(t) &= \int \frac{((k')^{\perp}, k)}{|k'|} u(k', t) \left(k_1' \frac{\partial u(k,t)}{\partial k_1} + k_2' \frac{\partial u(k,t)}{\partial k_2} \right) dk' \\ &= \frac{\partial u(k,t)}{\partial k_1} \left(-k_1 \int \frac{k_1' k_2'}{|k'|} u(k',t) dk' + k_2 \int \frac{(k_1')^2}{|k'|} u(k',t) dk' \right) \\ &\quad + \frac{\partial u(k,t)}{\partial k_2} \left(-k_1 \int \frac{(k_2')^2}{|k'|} u(k',t) dk' + k_2 \int \frac{k_1' k_2'}{|k'|} u(k',t) dk' \right)\end{aligned}$$

Denote

$$b_1(t) = \int \frac{(k_1')^2}{|k'|} u(k',t) dk', \quad b_2(t) = \int \frac{(k_2')^2}{|k'|} u(k',t) dk', \quad b_3(t) = \int \frac{k_1' k_2'}{|k'|} u(k',t) dk',$$

$$\begin{aligned}h_0(k,t) &= \left[\frac{k_2^2 - k_1^2}{|k|^3} a_3 + \frac{k_1 k_2}{|k|^3} (a_1 - a_2) - \nu |k|^{2\alpha} \right], \\ h_1(k,t) &= -\left[\frac{k_1}{|k|} a_3 - \frac{k_2}{|k|} a_1 + k_1 b_3 - k_2 b_1 \right], \\ h_2(k,t) &= -\left[\frac{k_1}{|k|} a_2 - \frac{k_2}{|k|} a_3 + k_1 b_2 - k_2 b_3 \right].\end{aligned}$$

Then the equation (3) takes its final form:

$$\frac{\partial u(k,t)}{\partial t} + \frac{\partial u(k,t)}{\partial k_1} h_1(k,t) + \frac{\partial u(k,t)}{\partial k_2} h_2(k,t) = h_0(k,t) u$$

or

$$\frac{du(k,t)}{dt} = h_0(k,t) u(k,t) \tag{4}$$

where

$$\begin{aligned}\frac{dk_1}{dt} &= h_1(k,t) \\ \frac{dk_2}{dt} &= h_2(k,t).\end{aligned} \tag{5}$$

However, we should not forget that the coefficients a_i, b_i are also functions of unknowns k and u. (5) are the equations for characteristics of our quasi-linear equation. We can think about them as curves along which the non-linearity spreads. Denote by S^{t_1,t_2} the family of shifts along solutions of (5). Then

$$u(k, t_2) = u(k(t_1), t_1) \exp\left(\int_{t_1}^{t_2} h_0(k(\tau), \tau)) d\tau \right), \tag{6}$$

where $k(\tau) = S^{t_1,\tau} k(t_1)$.

Corollary 1. *The sign of u is preserved along the characteristics of* (5).

Proof. — Follows immediately from (6). □

This property is special for our approximation. Presumably it is not true in a general case.

Corollary 2. — *If* $u(-k,t)=u(k,t)$ *then* $h_1(-k,t)=-h_1(k,t)$, $h_2(-k,t)=-h_2(k,t)$.

This property has an important interpretation. Consider $h_1(k,t), h_2(k,t)$ as the components of our vector field (5). These components are odd functions of k. Therefore the trajectories of the symmetric (with respect to the origin) points are symmetric.

A similar approximation can be constructed for the Navier-Stokes system. It will be discussed in another paper.

2. Finite-dimensional Approximations

Assume that $u(k,0)$ is non-zero only for finitely many k, i.e. $u(k^{(i)},0)=u^{(i)}$ for $i=1,2,\ldots,I$. Then $u(k,t)$ is non-zero at I points $k^{(i)}(t)$. In this case

$$a_1=a_1(t)=\sum_{i=1}^{I}(k_1^{(i)})^2u(k^{(i)},t)$$
$$a_2=a_2(t)=\sum_{i=1}^{I}(k_2^{(i)})^2u(k^{(i)},t)$$
$$a_3=a_3(t)=\sum_{i=1}^{I}k_1^{(i)}k_2^{(i)}u(k^{(i)},t)$$

and

$$b_1(t)=\sum_{i=1}^{I}\frac{(k_1^{(i)})^2}{|k^{(i)}|}u(k^{(i)},t)$$
$$b_2(t)=\sum_{i=1}^{I}\frac{(k_2^{(i)})^2}{|k^{(i)}|}u(k^{(i)},t)$$
$$b_3(t)=\sum_{i=1}^{I}\frac{k_1^{(i)}k_2^{(i)}}{|k^{(i)}|}u(k^{(i)},t)$$

The system of equations of dynamics of the points $k^{(i)}$ takes the form

$$(7)\qquad \begin{aligned}\frac{dk_1^{(i)}}{dt}&=-\left[\frac{k_1^{(i)}}{|k^{(i)}|}a_3-\frac{k_2^{(i)}}{|k^{(i)}|}a_1+k_1^{(i)}b_3-k_2^{(i)}b_1\right]\\ \frac{dk_2^{(i)}}{dt}&=-\left[\frac{k_1^{(i)}}{|k^{(i)}|}a_2-\frac{k_2^{(i)}}{|k^{(i)}|}a_3+k_1^{(i)}b_2-k_2^{(i)}b_3\right]\end{aligned}$$

Let $I = 1$. Then

$$a_1 = (k_1^{(1)})^2 u(k^{(1)}, t), \quad a_2 = (k_2^{(1)})^2 u(k^{(1)}, t), \quad a_3 = k_1^{(1)} k_2^{(1)} u(k^{(1)}, t),$$
$$b_1 = \frac{(k_1^{(1)})^2}{|k^{(1)}|} u(k^{(1)}, t), \quad b_2 = \frac{(k_2^{(1)})^2}{|k^{(1)}|} u(k^{(1)}, t), \quad b_3 = \frac{k_1^{(1)} k_2^{(1)}}{|k^{(1)}|} u(k^{(1)}, t).$$

We immediately see that $h_1 = h_2 = 0$, i.e. the point stays fixed and $u(k^{(i)}, t) \to 0$ as $t \to \infty$.

If $I = 2$ and $k^{(2)} = -k^{(1)}, u(k^{(2)}) = u(k^{(1)})$ then in view of the symmetry (see Corollary 2) both points stay fixed. The first non-trivial case arises for an arbitrary configuration of two points. According to the Corollary 2 it is equivalent to the case of four points consisting of two symmetric pairs. Denote $u_1 = u(k^{(1)}), u_2 = u(k^{(2)})$.

We come to the following remarkable system of ODE:

$$(8) \quad \begin{aligned}
\frac{dk_1^{(1)}}{dt} &= -u_2 k_1^{(2)} \left[\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right] \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \\
\frac{dk_2^{(1)}}{dt} &= -u_2 k_2^{(2)} \left[\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right] \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \\
\frac{dk_1^{(2)}}{dt} &= u_1 k_1^{(1)} \left[\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right] \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \\
\frac{dk_2^{(2)}}{dt} &= u_1 k_2^{(1)} \left[\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right] \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \\
\frac{du_1}{dt} &= \left[-\frac{u_2}{|k^{(1)}|^3} \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \left(k_1^{(1)} k_2^{(2)} + k_2^{(1)} k_1^{(2)} \right) - \nu |k^{(1)}|^{2\alpha} \right] u_1 \\
\frac{du_2}{dt} &= \left[\frac{u_1}{|k^{(2)}|^3} \left(k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)} \right) \left(k_1^{(1)} k_2^{(2)} + k_2^{(1)} k_1^{(2)} \right) - \nu |k^{(2)}|^{2\alpha} \right] u_2
\end{aligned}$$

Lemma 1. — *$S = k_1^{(1)} k_2^{(2)} - k_2^{(1)} k_1^{(2)}$ is the first integral of* (8).

Proof. — Direct checking. □

It is not difficult to see that

$$(9) \quad \begin{aligned}
\frac{d|k^{(1)}|}{dt} &= -\frac{u_2 S(k^{(1)}, k^{(2)})}{|k^{(1)}|} \left(\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right) \\
\frac{d|k^{(2)}|}{dt} &= \frac{u_1 S(k^{(1)}, k^{(2)})}{|k^{(2)}|} \left(\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right) \\
\frac{d(k^{(1)}, k^{(2)})}{dt} &= S \left(-u_2 |k^{(2)}|^2 + u_1 |k^{(1)}|^2 \right) \left(\frac{1}{|k^{(1)}|} + \frac{1}{|k^{(2)}|} \right)
\end{aligned}$$

In view of (9) the last two equations of the system (8) can be rewritten in the form

$$
(10)\qquad
\begin{aligned}
\frac{d\ln|u_1|}{dt} &= -\frac{d}{dt}\left(\frac{1}{|k^{(1)}|}\right)\left(\frac{1}{|k^{(1)}|}+\frac{1}{|k^{(2)}|}\right)^{-1} - \nu|k^{(1)}|^{2\alpha}\\
\frac{d\ln|u_2|}{dt} &= -\frac{d}{dt}\left(\frac{1}{|k^{(2)}|}\right)\left(\frac{1}{|k^{(1)}|}+\frac{1}{|k^{(2)}|}\right)^{-1} - \nu|k^{(2)}|^{2\alpha}
\end{aligned}
$$

Lemma 2. — *The function* $F(u,k) = \ln\left(|u_1||u_2|\left(\frac{1}{|k^{(1)}|}+\frac{1}{|k^{(2)}|}\right)\right)$ *is a Lyapunov function of the system* (8).

Proof. — With the help of a simple transformation we obtain from(10) that along any trajectory of the system (8) the derivative of $F(t)$ is less than 0:

$$\frac{d}{dt}F(t) = -\nu\left(|k^{(1)}|^{2\alpha}+|k^{(2)}|^{2\alpha}\right). \qquad \square$$

Corollary. *For* $S \neq 0$ *and* $t > 0$

$$|u_1(t)u_2(t)|\left(\frac{1}{|k^{(1)}(t)|}+\frac{1}{|k^{(2)}(t)|}\right) \leqslant |u_1(0)u_2(0)|\left(\frac{1}{|k^{(1)}(0)|}+\frac{1}{|k^{(2)}(0)|}\right)\exp(-2\nu|S|^{\alpha}t).$$

Proof. — The statement follows from the inequality:

$$|k^{(1)}(t)|^{2\alpha}+|k^{(2)}(t)|^{2\alpha} \geqslant 2|k^{(1)}(t)|^{\alpha}|k^{(2)}(t)|^{\alpha} > 2|S|^{\alpha}. \qquad \square$$

The system (8) has an invariant four-dimensional manifold

$$\Gamma = \{k^{(1)}, k^{(2)}, u_1, u_2 \mid u_1 = 0, u_2 = 0\}$$

which is locally stable. Our next result shows that this manifold is also globally stable.

Theorem 1. *For any solution of* (8),

$$\lim_{t\to\infty} u_1(t) = \lim_{t\to\infty} u_2(t) = 0, \qquad \lim_{t\to\infty} k_j^{(i)} = \overline{k}_j^{(i)}$$

where $k_j^{(i)}$ *are limiting values of* $\overline{k}_j^{(i)}(t)$ *which certainly depend on the initial conditions.*

A priori there can be solutions which escape to infinity in finite time, i.e. $k_j^{(i)} \to \infty$ as $t \to t_0$, which means some blow up. The theorem shows that this does not happen in our case and each solution approaches some point of Γ.

Proof. — For $S = 0$ the statement of the theorem is obvious. Therefore, we can restrict ourselves to the case $S \neq 0$. It is sufficient to consider positive S, because the case of negative S can be reduced to positive S by changing the order of points. First, we prove the theorem for $\alpha = 1$ and in the Appendix we sketch the main steps for $\alpha < 1$.

Put

$$L(t) = |u_1(t)u_2(t)|\left(\frac{1}{|k^{(1)}(t)|} + \frac{1}{|k^{(2)}(t)|}\right) \quad \text{and} \quad \overline{L}(t) = sgn(u_1(0)u_2(0))L(t).$$

Using these notations we can write $u_1(t)$ in the following way:

$$u_1(t) = \left[-\int_{t_0}^{t} \frac{Su_1(\tau)u_2(\tau)}{|k^{(1)}(\tau)|^3}(k^{(1)}(\tau), k^{(2)}(\tau)) \exp\left(\nu\int_{t_0}^{\tau}|k^{(1)}(s)|^2 ds\right)dt + u_1(t_0)\right] \times \exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(s)|^2 ds\right)$$

$$= \left[-\int_{t_0}^{t}\overline{L}(\tau)\frac{S|k_2(\tau)|}{|k^{(1)}(\tau)|^2(|k^{(1)}(\tau)| + |k^{(2)}(\tau)|)}(k^{(1)}(\tau), k^{(2)}(\tau)) \exp\left(\nu\int_{t_0}^{\tau}|k^{(1)}(s)|^2 ds\right)d\tau + u_1(t_0)\right] \times \exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(s)|^2 ds\right).$$

Using the relations

$$\begin{aligned} &\overline{L}(\tau) = \overline{L}(\tau_0)\exp\left(-\nu\int_{t_0}^{\tau}(|k^{(1)}(s)|^2 + |k^{(2)}(s)|^2)ds\right), \\ &|k^{(1)}(\tau)|(|k^{(1)}(\tau)| + |k^{(2)}(\tau)|) > |k^{(1)}(\tau)||k^{(2)}(\tau)| \geqslant S, \\ &|(k^{(1)}(\tau), k^{(2)}(\tau))| \leqslant |k^{(1)}(\tau)||k^{(2)}(\tau)| \end{aligned} \tag{11}$$

we obtain from the previous expressions that

$$\begin{aligned} &\left|u_1(t) - u_1(t_0)\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(s)|^2 ds\right)\right| \\ &< \left(L(t_0)\int_{t_0}^{t}|k^{(2)}(\tau)|^2\exp\left(-\nu\int_{t_0}^{\tau}|k^{(2)}(s)|^2 ds\right)d\tau\right)\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(\tau)|^2 d\tau\right) \\ &\leqslant \nu^{-1}L(t_0)\left(1 - \exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(\tau)|^2 d\tau\right)\right)\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(\tau)|^2 d\tau\right). \end{aligned} \tag{12}$$

In the same way $u_2(t)$ satisfies the inequality

$$\begin{aligned} &\left|u_2(t) - u_2(t_0)\exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(s)|^2 ds\right)\right| \\ &< \nu^{-1}L(t_0)\left(1 - \exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(\tau)|^2 d\tau\right)\right)\exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(\tau)|^2 d\tau\right). \end{aligned}$$

It is not difficult to verify that for the functions $v_1(t) = u_1(t)/|k^{(1)}(t)|$ and $v_2(t) = u_2(t)/|k^{(2)}(t)|$ the following inequalities hold:

(13)

$$\left|v_1(t) - v_1(t_0)\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(s)|^2 ds\right)\right| < (2\nu\sqrt{S})^{-1}L(t_0)\left(1-\exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(\tau)|^2d\tau\right)\right)\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(\tau)|^2d\tau\right),$$

$$\left|v_2(t) - v_2(t_0)\exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(s)|^2 ds\right)\right| < (2\nu\sqrt{S})^{-1}L(t_0)\left(1-\exp\left(-\nu\int_{t_0}^{t}|k^{(1)}(\tau)|^2d\tau\right)\right),\exp\left(-\nu\int_{t_0}^{t}|k^{(2)}(\tau)|^2d\tau\right)$$

The inequalities (13) can be obtained from the system of differential equations for $v_1(t)$ and $v_2(t)$:

$$\begin{aligned}\frac{dv_1}{dt} &= v_1(t)\left(\frac{v_2(t)S(k^{(1)}(t),k^{(2)}(t))}{|k^{(1)}(t)|^2} - \nu|k^{(1)}(t)|^2\right)\\ \frac{dv_2}{dt} &= v_2(t)\left(\frac{-v_1(t)S(k^{(1)}(t),k^{(2)}(t))}{|k^{(2)}(t)|^2} - \nu|k^{(2)}(t)|^2\right).\end{aligned}\tag{14}$$

with the help of the above mentioned arguments for $u_1(t)$ and $u_2(t)$. The system (14) follows directly from (8).

At least one of the two integrals $\int_{t_0}^{\infty}|k^{(1)}(\tau)|^2d\tau$ and $\int_{t_0}^{\infty}|k^{(2)}(\tau)|^2d\tau$ diverges because their sum diverges. Assume for example that the first one diverges. Then from (12) it follows that $\lim_{t\to\infty} u_1(t) = 0$ because $\lim_{t_0\to\infty} L(t_0) = 0$. If $\int_{t_0}^{\infty}|k^{(2)}(\tau)|^2d\tau$ also diverges then $\lim_{t\to\infty} u_2(t) = 0$.

In the case of convergence of the last integral $\lim_{t\to\infty} u_2(t)$ exists and is nonzero. Indeed, for any given $\varepsilon > 0$ find such t_0 that $\nu^{-1}L(t_0) < \varepsilon/3$ and then choose $\bar{t}(t_0)$ so that for $t_2 > t_1 > \bar{t}$

$$\left|u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_2}|k^{(2)}(\tau)|^2d\tau\right) - u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_1}|k^{(2)}(\tau)|^2d\tau\right)\right| < \varepsilon/3$$

We have

$$\begin{aligned}|u_2(t_2) - u_2(t_1)| \leqslant &\left|u_2(t_2) - u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_2}|k^{(2)}(\tau)|^2d\tau\right)\right|\\ &+\left|u_2(t_1) - u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_1}|k^{(2)}(\tau)|^2d\tau\right)\right|\\ &+\left|u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_2}|k^{(2)}(\tau)|^2d\tau\right) - u_2(t_0)\exp\left(-\nu\int_{t_0}^{t_1}|k^{(2)}(\tau)|^2d\tau\right)\right|\\ &< +2\nu^{-1}L(t_0) + \varepsilon/3 < \varepsilon.\end{aligned}$$

This gives the existence of the desired limit.

The same statement is true for $\lim_{t\to\infty} v_2(t)$ (see (13) and (14)). Thus $|k^{(2)}(t)|$ tends to a positive number or to $+\infty$ when $t \to \infty$. This contradicts the convergence of $\int_{t_0}^{\infty} |k^{(2)}(\tau)|^2 d\tau$, i.e. $\lim_{t\to\infty} u_2(t) = 0$.

Now we shall study the behavior of the vectors $k^{(i)}(t)$, $i = 1, 2$, for $t \to \infty$. The first equality in (11) implies that there exist positive constants C_1, C_2 and suffiently large t_0 depending on the inital data for which

(15)

$$C_1 \exp\left(-\nu \int_{t_0}^{t} |k^{(i)}(\tau)|^2 d\tau \right) < |u_i(t)| < C_2 \exp\left(-\nu \int_{t_0}^{t} |k^{(i)}(\tau)|^2 d\tau \right), \quad i = 1, 2.$$

Substituting (15) in (11) we obtain after simple calculations that

$$A_1 < \frac{1}{|k^{(1)}(t)|} + \frac{1}{|k^{(2)}(t)|} < A_2 \tag{16}$$

for some positive A_i depending on the initial data.

From (15) and from the system (8) for components of the vectors $k^{(1)}(t), k^{(2)}(t)$ one can conclude that $k_j^{(i)}(t), i, j = 1, 2$ have finite limits when t tends $\to \infty$. Let us check this for $k_1^{(1)}(t)$:

$$k_1^{(1)}(t) - k_1^{(1)}(t_0) = -S \int_{t_0}^{t} u_2(\tau) k_1^{(2)}(\tau) \left(\frac{1}{|k^{(1)}(\tau)|} + \frac{1}{|k^{(2)}(\tau)|} \right) d\tau \tag{17}$$

The integral in the right-hand side converges because its absolute value is less than the integral

$$\int_{t_0}^{\infty} |u_2(\tau)|\, |k^{(2)}(\tau)| \left(\frac{1}{|k^{(1)}(\tau)|} + \frac{1}{|k^{(2)}(\tau)|} \right) d\tau$$

which is finite.

Theorem 1 is proven for $\alpha = 1$. □

3. Numerical experiments: results and discussion

In the previous section we considered the finite dimensional systems for $I = 1, 2$. For $I > 2$ we do not have rigorous results but did only some numerical experiments to understand the behavior of solutions in some cases. Several results are represented in Figures 1-11.

In Figures 1-8 solutions for $I = 4$ are shown, and in Figures 9-11 for $I = 64$. The initial data for $I = 4$ were taken as follows:

1	0.01	0.0	100.0
2	0.0	0.2	200.0
3	0.02	0.3	−50.0
4	0.04	0.5	−1.0

Here in the table the first column represents the number of a point, the second (third) column represents the first (second) coordinate of the point, the forth is $u_i(0)$.

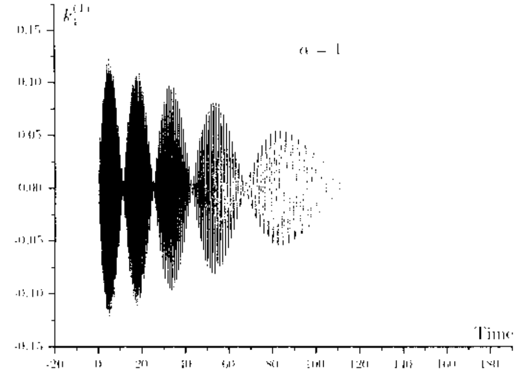

Figure 1

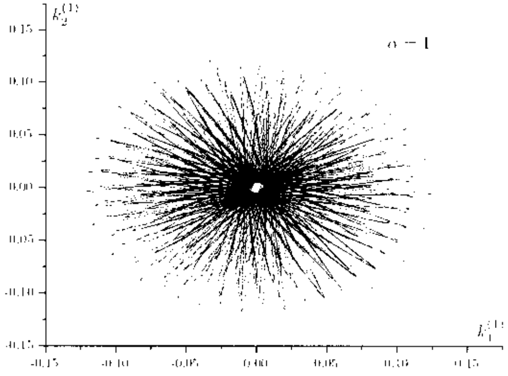

Figure 2

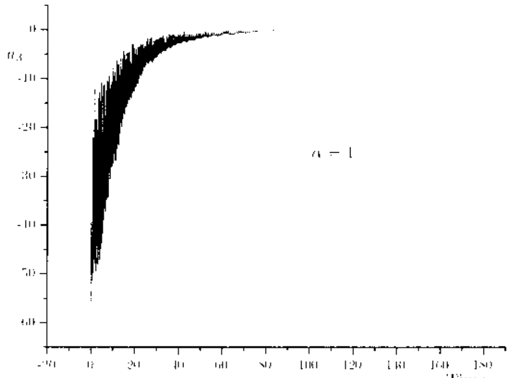

Figure 3

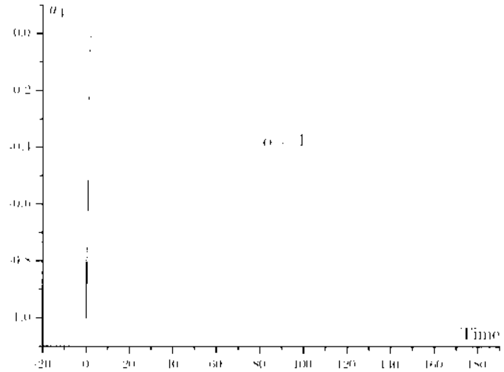

Figure 4

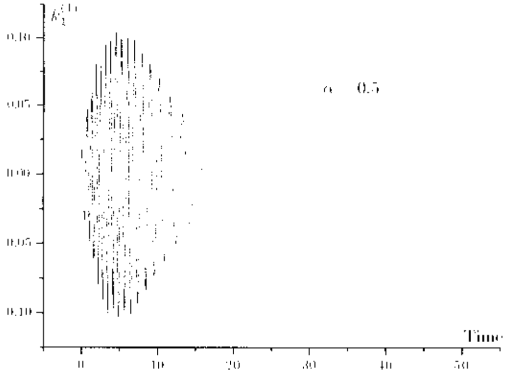

Figure 5

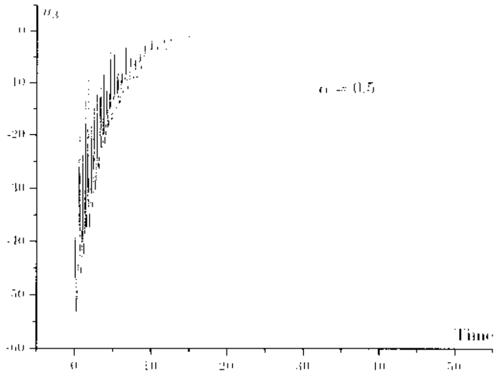

Figure 6

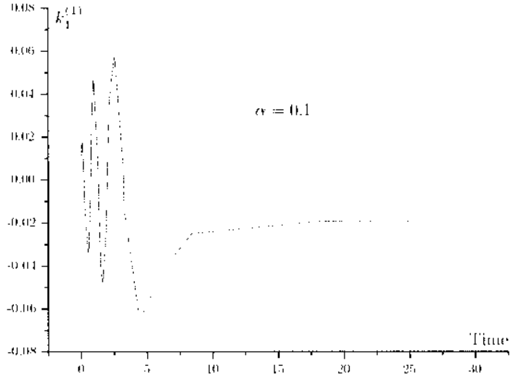

Figure 7

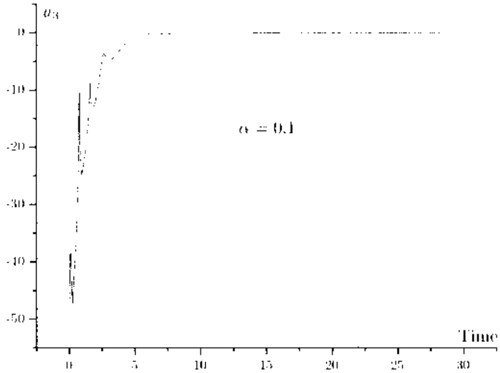

Figure 8

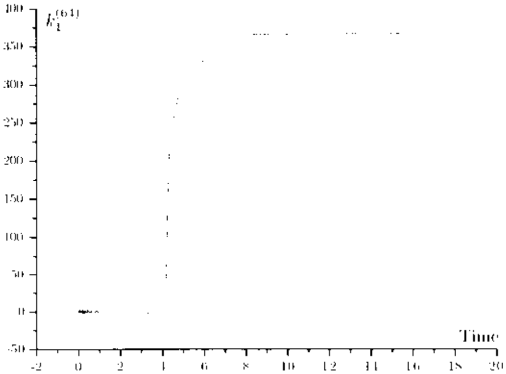

Figure 9

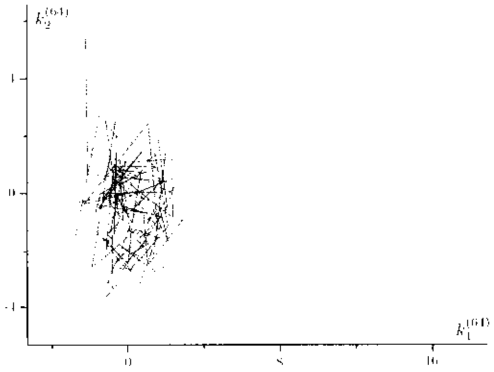

Figure 10

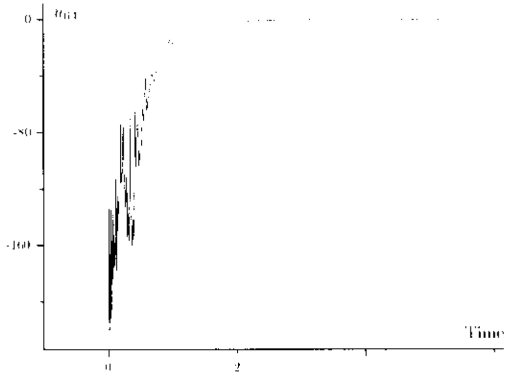

Figure 11

Figures 1, 2, 3, and 4 demonsrate the graphs of the function $k_1^{(1)}(t)$, phase portrait $k_2^{(1)}(k_1^{(1)})$, functions $u_3(t)$ and $u_4(t)$, respectively, for $\alpha = 1$. Figures 5 and 6 show $k_1^{(1)}(t)$ and $u_3(t)$, respectively, for $\alpha = 0.5$. Figures 7 and 8 demonstrate graphs of the same functions for $\alpha = 0.1$. At last on Figures 9, 10 and 11 $k_1^{(64)}(t)$, $k_2^{(64)}(k_1^{(64)})$ and $u_{64}(t)$ respectively are shown. For $I = 64$ initial data were choosen in the following way. Points were distributed uniformely in the square $[0.1, 2] \times [0.1, 2]$ and $u_i(0) = C_1 + C_2 \sin(\beta k_1^{(i)}) + C_3 \sin(\gamma k_2^{(i)})$, where $C_1 = -100$; $C_2 = 50$; $C_3 = 200$: $\beta = 5$; $\gamma = 6$. We carried out several hundreds of numerical experiments with different initial data. In all cases the behavior of solutions was similar. Namely, $u_i(t) \to 0$, $k_j^{(i)}(t)$ converge to limiting values as $t \to \infty$.

Appendix. Sketch of the proof of Theorem 1 for $\alpha < 1$

Put

$$w_i(t) = u_i(t) \exp\left(\nu \int_{t_0}^{t} |k^{(i)}(\tau)|^{2\alpha} d\tau\right) \quad (i = 1, 2)$$

and after simple transformations rewrite the last two equations in (8) in the following form:

$$(18) \qquad \begin{aligned} \frac{dw_1}{dt} &= -\frac{\overline{L(t)} S(k^{(1)}, k^{(2)}) |k^{(2)}|}{|k^{(1)}|^2 (|k^{(1)}| + |k^{(2)}|)} \exp\left(\nu \int_{t_0}^{t} |k^{(1)}(\tau)|^{2\alpha} d\tau\right) \\ \frac{dw_2}{dt} &= \frac{\overline{L(t)} S(k^{(1)}, k^{(2)}) |k^{(1)}|}{|k^{(2)}|^2 (|k^{(1)}| + |k^{(2)}|)} \exp\left(\nu \int_{t_0}^{t} |k^{(2)}(\tau)|^{2\alpha} d\tau\right) \end{aligned}$$

It is not difficult to see that

$$(19) \qquad \begin{aligned} \left|\frac{dw_1}{dt}\right|^{\alpha} &\leqslant \left(L(t_0) |k^{(2)}|^2 \exp\left(-\nu \int_{t_0}^{t} |k^{(2)}(\tau)|^{2\alpha} d\tau\right)\right)^{\alpha} \\ \left|\frac{dw_2}{dt}\right|^{\alpha} &\leqslant \left(L(t_0) |k^{(1)}|^2 \exp\left(-\nu \int_{t_0}^{t} |k^{(1)}(\tau)|^{2\alpha} d\tau\right)\right)^{\alpha} \end{aligned}$$

Integrating both parts of the inequalities (19) from t_0 to t we obtain:

$$\int_{t_0}^{t} \left|\frac{dw_1(\tau)}{d\tau}\right|^{\alpha} d\tau \leqslant (L(t_0))^{\alpha} (\alpha\nu)^{-1} \left(1 - \exp\left(-\alpha\nu \int_{t_0}^{t} |k^{(2)}(\tau)|^{2\alpha} d\tau\right)\right)$$

$$\int_{t_0}^{t} \left|\frac{dw_2(\tau)}{d\tau}\right|^{\alpha} d\tau \leqslant (L(t_0))^{\alpha} (\alpha\nu)^{-1} \left(1 - \exp\left(-\alpha\nu \int_{t_0}^{t} |k^{(1)}(\tau)|^{2\alpha} d\tau\right)\right)$$

Since $\lim_{t_0 \to \infty} L(t_0) = 0$ the norms in the $L^{\alpha}(t_0, \infty)$ of the functions $\dfrac{dw_i}{dt}$ tend to zero in the space $L^{\alpha}(t_0, \infty)$ when $t_0 \to \infty$.

In the same way, one can show that the norms in the space $L^{\alpha}(t_0, \infty)$ of the derivatives of functions $w_i(t)/|k^{(i)}(t)|$ also tend to zero when $t_0 \to \infty$.

As in the case $\alpha = 1$, previous arguments give that $\lim_{t \to \infty} u_i(t) = 0, i = 1, 2$. Hence all statments of Theorem 1 follow easily.

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