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ON BASIC PIECES OF AXIOM A DIFFEOMORPHISMS ISOTOPIC TO PSEUDOANOSOV MAPS

by

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Abstract. — We consider Axiom A diffeomorphisms g in the isotopy class of a pseudoanosov map f. It is shown that they have a unique "large" basic piece Λ, and necessary and sufficient conditions for g to be semiconjugated to f, that only involve conditions on Λ, are obtained. As a consequence, it is proved that if Λ is exteriorly situated, stable and unstable half-leaves of points of Λ boundedly deviate from geodesics.

1. Introduction

In this paper we consider Axiom A diffeomorphisms in the isotopy class of pseudoanosov maps. It is known (see [**H1, L1**]) that any pseudoanosov map f is persistent in its isotopy class i.e. for any homeomorphism g isotopic to f there exists a closed invariant set J_g such that g restricted to J_g is semiconjugated to f. When g verifies Axiom A, the first author (in [**L2**]) introduced the definition of "small" and "large" basic pieces of g. Roughly speaking, a basic piece B is small if an adequate lift to the universal cover of a B-stable (unstable) manifold is bounded. Otherwise, B is large. In [**L2**] it is proved that it is necessary and sufficient for a large basic piece Λ to be contained in J_g that the above mentioned lift lies at a bounded distance of an f-stable set of a point. In this work we prove that an Axiom A diffeomorphism g has a unique large basic piece Λ and, using that if g is not semiconjugated to f the Nielsen classes of g^n grow exponentially faster than the Nielsen classes of f^n (see [**H2**]), we prove that it is necessary and sufficient for g being semiconjugated to f that $\Lambda \subset J_g$. These results are proved in sections 3 and 4.

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In section 5 we lead with the case when the large basic set Λ is exteriorly situated i.e. there are no nul-homotopic loops that consist in the union of a stable and an unstable arc of a point of Λ. We show that, in this case, g is semiconjugated to f. As a consequence we obtain that half-leaves of stable and unstable manifolds of Λ boundedly deviates from geodesics that have the same asymptotic direction. Let us say that V. Grines (see [**G1**] and the survey [**G2**]) obtained this bounded deviation property for unstable (stable) manifolds of exteriorly situated nontrivial attractors (resp. repellers) of any diffeomorphism of a closed surface of genus larger than one and also for both, unstable and stable manifolds, provided the attractor (repeller) is a basic piece of a diffeomorphism that verifies Axiom A and that satisfies the strong transversality condition. In [**RW**] R. C. Robinson and R. Williams constructed an example of an Axiom A diffeomorphism on a surface of genus > 1 such that its nonwandering set consists exactly of an exteriorly situated attractor and an exteriorly situated repeller and that does not verify the strong transversality condition. V. Grines ([**G3**]) also showed that the stable manifold of the attractor (unstable for the repeller) of this example does not have the bounded deviation from geodesics property. Our results show that this kind of behaviour is impossible in the isotopy class of pseudoanosov maps (the fact that the Robinson-Williams example is not isotopic to a pseudoanosov map can be checked directly)

Finally, let us say that to understand the proofs, some familiarity with the theory of pseudoanosov maps is needed. We do not include this background material that the reader may find in [**FLP, CB, M, HT, T**].

2. Preliminaries

Let f be a pseudoanosov map of a compact connected oriented boundaryless surface M, let F be a lift of f to the universal cover $\mathcal{M}$ of M and $\pi : \mathcal{M} \to M$ the covering projection. Then, there exist (see [**T, FLP**]) equivariant pseudometrics D_S, D_U, and $\lambda > 1$ such that, for $\xi, \eta \in \mathcal{M}$,

$$D_S(F^{-1}(\xi), F^{-1}(\eta)) = \lambda D_S(\xi, \eta)$$
$$D_U(F(\xi), F(\eta)) = \lambda D_U(\xi, \eta)$$

and $D = D_S + D_U$ is an equivariant metric on $\mathcal{M}$.

For the remainder of this paper g will be a homeomorphism isotopic to f.

Definition 2.1. — *We say that $x, y \in M$ are equivalent iff for some (and then for any) lift G of g there exist $\xi \in \pi^{-1}(x)$, $\eta \in \pi^{-1}(y)$ and $K > 0$ such that $D(G^n(\xi), G^n(\eta)) \leqslant K$ for all $n \in \mathbb{Z}$.*

Obviously, this is an equivalence relation. In the following proposition we state some properties of this relation. For completeness we include the proofs that are essentially contained in [**H1, L1, CS**].

Proposition 2.2. — *The following statements hold:*

(1) *The constant K of the definition above only depends on g i.e. $\exists K_g$ such that if x is equivalent to y then $D(G^n(\xi), G^n(\eta)) < K_g$ for all $n \in \mathbb{Z}$. Moreover, K_g tends to 0 as g tends to f (see* [**L1, H1**]*).*

(2) *Let $[x]_g$ be the equivalence class of $x \in M$. Then, $[x]_g$ is a compact set and $g([x]_g) = [g(x)]_g$.*

(3) *The quotient space under the equivalence relation, $\overline{M}_g$, is a compact metrizable space and, by part 2, g induces a homeomorphism $\overline{g} : \overline{M}_g \to \overline{M}_g$. When $g = f$, $\overline{M}_f = M$ and $\overline{f} = f$, due to the infinite expansivity of any lift of f.*

Proof. — In order to prove the first part of the proposition take F, G to be adequate lifts of f, g and let $R > 0$ be such that $D(F, G) < R$ and $D(F^{-1}, G^{-1}) < R$.

Suppose that $D(\xi, \eta) > 2K$, then $D_U(\xi, \eta) > K$ or $D_S(\xi, \eta) > K$. With no loose of generality we can assume that the first inequality holds. If $K > 2R(\lambda - 1)^{-1}$ there exists $1 < \alpha < \lambda$ such that $K > 2R(\lambda - \alpha)^{-1} > 2R(\lambda - 1)^{-1}$ which implies $D_U(G(\xi), G(\eta)) > \lambda K - 2R > \alpha K$.

Thus, $D_U(G^n(\xi), G^n(\eta)) > \alpha^n K$. Since $\alpha^n K$ tends to infinite with n, this implies that if x and y are equivalent there exist ξ, η so that

$$D(G^n(\xi), G^n(\eta)) \leqslant 4R(\lambda - 1)^{-1} \quad \forall n \in \mathbb{Z}.$$

Choose $K_g = 4R(\lambda - 1)^{-1}$.

To prove the second part, take x_k equivalent to x for all $k \in \mathbb{N}$ and $x_k \to_{k \to +\infty} x^*$. Then, given $\xi \in \pi^{-1}(x)$, there are $\xi_k \in \pi^{-1}(x_k)$ such that $D(G^n(\xi_k), G^n(\xi)) < K_g$ $\forall n \in \mathbb{Z}$.

By taking, if necessary, a convergent subsequence, we may assume that $\xi_k \to_{k \to +\infty} \xi^* \in \pi^{-1}(x^*)$. Then, $D(G^n(\xi^*), G^n(\xi)) < K_g \ \forall n \in \mathbb{Z}$.

Now we prove the third part of the proposition. The proof is similar to the one included in [**CS**] where the same result for g C^0-close enough to an expansive homeomorphisms is shown.

It is not difficult to see that if $\{x_n\} \subset M$ is a sequence such that $x_n \to x$, $\limsup [x_n]_g \subset [x]_g$. Then, given an open set $U \subset M$, the set $\{y \in M; [y]_g \subset U\}$ is open. This easily implies that $\overline{M}_g$ is Haussdorf and metrizable. □

We want to study the connection between the dynamics of $\overline{g}$ and f; to this end we find conditions on g to be semiconjugated to f.

Define

Definition 2.3. — *The g-orbit of x is shadowed by the f-orbit of y iff there exist $\xi \in \pi^{-1}(x), \eta \in \pi^{-1}(y)$ such that $\{D(G^n(\xi), F^n(\eta)) : n \in \mathbb{Z}\}$ is bounded, for G, F equivariantly homotopic lifts of g, f.*

Observe that if the g-orbit of x is shadowed by the f-orbit of y, then y is unique and every g-orbit of a point of $[x]_g$ is shadowed by the f-orbit of y. Moreover, there is a uniform bound (independent of x and y) for $\{D(G^n(\xi), F^n(\eta)) : n \in \mathbb{Z}\}$ that tends to 0 as g approaches f (see [**H1, L1**]).

It is known that for any $y \in M$ there exists $x \in M$ such that the g-orbit of x is shadowed by the f-orbit of y(see [**H1, L1**]).

Definition 2.4. — *Call J_g the set of $x \in M$ such that the g-orbit of x is shadowed by the f-orbit of some y.*

We remark that J_g consists of *all* points that are f-shadowed.

It is not difficult to see that $J_g = J_{g^n}$, if we consider f^n instead of f. J_g is a compact g-invariant set that not necessarily equals M. Moreover, there exists a continuous surjection homotopic to the inclusion, $h : J_g \to M$, such that $f \circ h = h \circ g|_{J_g}$ (see [**L1, H1**]). This implies that, as the equivalence classes on J_g coincide with $h^{-1}(y)$, $y \in M$, the quotient of J_g under the equivalence relation we are interested in, is homeomorphic to M and $\overline{g}$ restricted to this set is conjugated to f.

Assume now that g is Axiom A; we look for conditions in order to have semiconjugacy to f ($J_g = M$).

Given $\xi \in \mathcal{M}$, let

$$W_S^F(\xi) = \{\eta \in \mathcal{M}; D(F^n(\xi), F^n(\eta)) \to 0 \; as \; n \to +\infty\}$$
$$W_U^F(\xi) = \{\eta \in \mathcal{M}; D(F^n(\xi), F^n(\eta)) \to 0 \; as \; n \to -\infty\}$$

denote the F-stable and unstable sets of ξ.

Analogously, denote by $W_S^G(\xi)$ and $W_U^G(\xi)$ the G-stable and unstable manifolds of ξ.

When g is an Axiom A diffeomorphism, for $\xi \in \mathcal{M}$, B a basic piece of g and $\pi(\xi) \in B$, we shall denote

$${}^B W_S^G(\xi) = \{\eta \in W_S^G(\xi); \pi(\eta) \in B\}$$
$${}^B W_U^G(\xi) = \{\eta \in W_U^G(\xi); \pi(\eta) \in B\}.$$

Definition 2.5. — *We shall say that B is "small" iff ${}^B W_S^G(\xi)$ is bounded for some (and then for every) $\xi \in \pi^{-1}(B)$. In case that ${}^B W_S^G(\xi)$ is unbounded we say that B is "large".*

Observe that B being small implies that the quotient of B under the equivalence relation is a periodic orbit for $\overline{g}$ (all the points in a topologically mixing component of B are equivalent).

In [**L2**] was proved that large basic pieces for g always exist in the case that g is C^0-near enough to f. The same proof works for g isotopic to f because it is sufficient to take $x \in J_g$ such that its g-orbit is shadowed by an f-orbit dense in the future. Then, the basic piece containing the ω-limit set of x ($\omega_g(x)$) is large.

We will say that a closed curve is essential if it is not nul-homotopic.

Lemma 2.6. — *Let Λ be a large basic piece of g. Then, there exists an essential closed curve consisting of an arc of $W^s(x)$ and an arc of $W^u(x)$, where $x \in \Lambda$.*

Proof. — Suppose that $\gamma = \gamma_s \cup \gamma_u$ is a nul-homotopic closed curve such that $\gamma_s \subset W^s(x)$, $\gamma_u \subset W^u(x)$, $x \in \Lambda$ (we may suppose that $x \in \gamma_s \cap \gamma_u$).

Then, there exists a closed curve $\widetilde{\gamma} \subset \mathcal{M}$ such that $\pi(\widetilde{\gamma}) = \gamma$. On the other hand, $\widetilde{\gamma} = \widetilde{\gamma}_s \cup \widetilde{\gamma}_u$ where $\widetilde{\gamma}_s$ and $\widetilde{\gamma}_u$ are arcs of $W_S^G(\xi)$ and $W_U^G(\xi)$ with $\xi \in \pi^{-1}(x)$. This implies that any point in $\gamma_s \cap \gamma_u$ is equivalent to x.

Thus, if there are no essential curves as in the statement, all points in the closure of $W^s(x)$ are equivalent, and since Λ is a finite union of iterates of this closure it would be small. □

In the next two sections we will use some basic notions about Nielsen classes of fixed points that we define below.

Definition 2.7. — *Let h be an homeomorphism of M and $p, q \in Fix(h)$; p is Nielsen equivalent to q iff for some (and then for any) lift H of h to $\mathcal{M}$ there exist a covering transformation τ and P, Q lifts of p, q such that $H(P) = \tau P$ and $H(Q) = \tau Q$. A Nielsen class of h is essential iff the Lefschetz index of the class is different from 0.*

The Nielsen class of a point $p \in Fix(h)$ is *not removable* if for any homeomorphism k isotopic to h there exist $p' \in Fix(k)$, equivariantly homotopic lifts H, K of h, k, P, P' lifts of p, p' and a covering transformation τ such that $H(P) = \tau P$ and $K(P') = \tau P'$. Essential Nielsen classes are not removable (see [**B**] chapter IV for more details).

Lemma 2.8. — *Two fixed points of g^n are equivalent iff they are Nielsen equivalent (see* [**H1**]*)*

Proof. — The if part follows easily from the definition of Nielsen equivalence.

Now, if p and q are equivalent we have that $G^n(P) = \sigma P$ and $G^n(Q) = \tau Q$ for σ, τ covering transformations and

$$D(P, (\sigma^{-1}G^n)^k Q) = D((\sigma^{-1}G^n)^k P, (\sigma^{-1}G^n)^k Q) < K_g$$

Then, there exists k_0 such that $(\sigma^{-1}G^n)^{k_0} Q = Q$ and, on the other hand, $\sigma^{-1}\tau Q = \sigma^{-1}G^n Q$.

This implies that $(\sigma^{-1}G^n)^{k_0}$ commutes with $\sigma^{-1}\tau$ and, as g is in the isotopy class of a pseudoanosov map, we obtain that $\sigma^{-1}\tau$ is the identity. □

We obtain the following corollary.

Corollary 2.9. — *Let g satisfy Axiom A then there exists $T > 0$ such that for all $n \in \mathbb{Z}$ the number of Nielsen classes of fixed points of g^n included in small basic pieces is less than T.*

Proof. — Since the set of equivalence classes of points in small basic pieces is finite, the result follows from Lemma 2.8. □

3. Uniqueness of large basic pieces

In this section we will use some results of the type of those obtained by Nielsen. The proofs may be found in [**CB, HT, M**].

In what follows we identify $\mathcal{M}$ with the Poincaré disk $\mathbb{D}$. We call S_∞ the boundary of $\mathbb{D}$ as an euclidean subset of $\mathbb{R}^2$ and $\overline{\mathbb{D}} = \mathbb{D} \cup S_\infty$.

Proposition 3.1 **([CB, HT, M])**. — *Let g be a homeomorphism isotopic to a pseudoanosov map and G any lift of g.*

i) *G has an even number of fixed points in S_∞, alternatively attracting and repelling in $\overline{\mathbb{D}}$.*

ii) *Each of these fixed points has the property that given any simple closed geodesic τ in M it has a nested base of neighbourhoods bounded by lifts of τ.*

This follows from the fact that, associated to the isotopy class of a pseudoanosov map f, there exist two transversal minimal geodesic laminations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that both of them transversely intersects infinitely many times any simple closed geodesic. As a consequence, it is not difficult to prove that any end-point of a lift to $\mathbb{D}$ of a geodesic of $\mathcal{F}^s$ or $\mathcal{F}^u$, has a nested sequence of neighbourhoods bounded by lifts of τ converging to this end-point. Finally let us say that the attracting (repeller) fixed points mentioned above are end-points of geodesics of $\mathcal{F}^u$ (resp. $\mathcal{F}^s$). See [**CB**] Chapter 5 and [**HT**] Proposition 4.3.

Theorem 3.2. — *Let g be an Axiom A diffeomorphism in the isotopy class of a pseudoanosov map then, g has a unique large basic piece.*

Proof. — Let $NE_n(g)$ be the number of essential Nielsen classes of g^n. By [**H1**] all the periodic points in essential classes of g^n are in J_g for all n and for each essential class of g^n, there exists a unique periodic point of f that shadows all the points of this Nielsen class.

As
$$\lim_{n\to\infty} \frac{\log NE_n(g)}{n} = \log\lambda$$
(the topological entropy of f, see [**FLP**]), by using Corollary 2.9 we obtain that there are periodic points of essential Nielsen classes in some large basic pieces.

In order to prove the theorem, we shall show that if Λ is a large basic piece and p is a periodic point of an essential Nielsen class of g^n contained in a large basic piece then $p \in \Lambda$. Obviously, this implies the theorem.

Let Λ be a large basic piece, then by Lemma 2.6 there exists an essential simple closed curve γ consisting of an arc of $W^s(q)$ and an arc of $W^u(q)$ where q is a periodic point of Λ.

Let G_n be a lift of g^{2n} such that $G_n(P) = P$ ($\pi(P) = p$) and fixes (setwise) the separatices of $W^U_G(P)$, where G is a lift of g. Now, if p is an essential fixed point of g^n

contained in a large basic set, by 3.1, $W^U_{G_n}(P) = W^U_G(P)$ accumulates in an attracting fixed point (sink) of the extension of G_n to S_∞ (we also call G_n its extension to $\overline{\mathbb{D}}$).

Since γ is freely homotopic to a simple closed geodesic, any lift of it has the same end-points at S_∞ of the corresponding lift of the geodesic. Then, again by 3.1, $W^U_G(P)$ cuts a lift of γ and this implies that $W^u(p)$ cuts $W^s(q)$. Analogously, $W^s(p)$ cuts $W^u(q)$. Thus $p \in \Lambda$. □

Corollary 3.3. — *$\Omega(\overline{g})$ consists of the quotient of Λ and a finite set of periodic orbits.*

Proof. — First of all observe that, by the density of periodic points, the quotient of $\Omega(g)$ is contained in $\Omega(\overline{g})$.

Now, if $[x]_g \subset \Omega(g)^c$, on account of the fact that each y in $[x]_g$ has a neighbourhood W such that $g^n(W)$ lies very close to the nonwandering set of g for sufficiently large n, it is easy to show that there exists an open set U, $[x]_g \subset U$ such that $g^n(U) \cap U = \varnothing$, $\forall n \in \mathbb{N}$. As in the proof of the third part of Proposition 2.2, by using that $V = \{y \in M; [y]_g \subset U\}$ is an open set, it is easy to see that $[x]_g$ is a wandering point of $\overline{g}$. □

4. Conditions for semiconjugacy

From now on let Λ be the large basic set of g given by Theorem 3.2.

Theorem 4.1. — *$\Lambda \subset J_g$ if and only if g is semiconjugated to f.*

Proof. — The "if" part is obvious because, since J_g consists of all points f-shadowed, $J_g = M$.

Let $N_n(h)$ be the number of distinct Nielsen classes of h^n, $h = f, g$. Then, if $\Lambda \subset J_g$, the number of Nielsen classes of g^n intersecting Λ is less or equal to $N_n(f)$ (see [**H1**]). Since, by Lemma 2.9, $N_n(g) \leqslant N_n(f) + T$, this implies that

$$\limsup \frac{\log N_n(g)}{n} \leqslant \log \lambda$$

(in fact the equality holds because f minimizes this number).

Then, by Corollary 3.6 of [**H2**], g is semiconjugated to f. □

Corollary 4.2. — *g is semiconjugated to f iff there exist $\xi, \eta \in \mathcal{M}$, $\pi(\xi) \in \Lambda$, and $V > 0$ such that for every $\zeta \in {}^{\Lambda}W^G_S(\xi)$, $D(\zeta, W^F_S(\eta)) < V$.*

Proof. — See Theorem 4.1 of [**L2**] and observe that the proof works for g in the isotopy class of f. □

5. Exteriorly situated basic pieces

Lemma 5.1. — *Let* $p \in Fix(g^n)$ *and* G_n *be a lift of* g^n *to* $\mathbb{D}$ *such that* $G_n(P) = P$, $\pi(P) = p$ *and* $\overline{W_U^{G_n}(P)}$ *contains two different fixed points of* $G_n|_{S_\infty}$. *Then,* $p \in J_g$.

Proof. — By [**H1**] it is enough to prove that the Nielsen class of p is essential. In order to prove this, first observe that, since the fixed points of G_n at S_∞ are sinks or sources, the fixed points accumulated by $W_U^{G_n}(P)$ are sinks. Then, $G_n|_{S_\infty}$ has at least four fixed points and, since the Lefschetz index of any of them is $1/2$ (when the fixed point is in S_∞ we have to take into account half of the index of the homeomorphisms on the double of $\overline{\mathbb{D}}$, which agrees with G_n on each copy of $\overline{\mathbb{D}}$) and the Euler's characteristic of $\mathbb{D}$ is 1, we obtain that the index of the Nielsen class of p is strictly negative. □

To state Theorem 5.3 we need another definition.

Definition 5.2. — *A nontrivial basic set (different from a periodic point) is called exteriorly situated if there is no nul-homotopic closed curve consisting of an arc of the stable and an arc of the unstable manifold of some point of it (see, for instance,* [**G1**]*).*

Observe that it is not difficult to prove that exteriorly situated basic pieces are large, since otherwise the whole stable (unstable) manifold of a periodic point would be included in a disk centered at the periodic point.

Theorem 5.3. — *If* Λ *is exteriorly situated then* g *is semiconjugated to* f.

Proof. — By Theorem 3.2 it is enough to show that $\Lambda \subset J_g$.

Take a periodic $p \in \Lambda$ such that it is not a boundary point of Λ. Now, take a simple closed curve $\gamma = \gamma_U \cup \gamma_S$ formed by oriented arcs of stable and unstable manifolds of Λ (not containing boundary points) intersecting at two points of Λ with the same oriented intersection number. The lift of γ is formed by a disjoint union of curves with different end-points at S_∞ which corresponds to the end-points of the connected lifts of the simple closed geodesic homotopic to γ. Let $P \in \pi^{-1}(p)$. Then, we claim that $W_U^G(P)$ cuts a connected lift of γ, say $\widetilde{\gamma}$, in at most one point. To see this we first observe that, as Λ is exteriorly situated, $W_U^G(P)$ cannot intersect twice the same lift of γ_S. Thus, if $W_U^G(P)$ cuts twice $\widetilde{\gamma}$, an arc of it and an arc of $\widetilde{\gamma}$ bounds a disk D. The boundary of D contains at least one lift $\widetilde{\gamma}_U$ of γ_U and l arcs contained in lifts of γ_S. By the form we choose γ we can continue $\widetilde{\gamma}_U$ inside D along the lift of the unstable manifold containing γ_U. As this curve is unbounded, it should cut the boundary of D at a lift of γ_S. In this way we obtain a new disk bounded by an arc of a lift of a unstable manifold of Λ, arcs of lifts of γ_U and, at most, $l-1$ arcs of lifts of γ_S. Repeating this argument, we will obtain a disk bounded by an arc of unstable manifold of Λ and one arc in a lift of γ_S which contradicts the fact of Λ being exteriorly situated (see [**G1**] for a similar argument). As $W^u(p)$ intersects infinitely many times γ, it is not difficult to see that $W_U^G(P)$ has exactly two end-points at S_∞.

Now, if G_n is a lift of g^n that fixes P and the separatrices of $W_U^G(P)$ we obtain that the end-points of $W_U^{G_n}(P) = W_U^G(P)$ are fixed and, by Lemma 5.1, $p \in J_g$. The density of non-boundary periodic points in Λ implies the statement. □

Definition 5.4. — *Suppose that a curve $\gamma : [0, +\infty) \to M$ has a lift $\widetilde{\gamma}$ with one end-point $\lim_{t\to+\infty} \widetilde{\gamma}(t) = \widetilde{\gamma}^+ \in S_\infty$. Let τ be a geodesic that also has $\widetilde{\gamma}^+$ as end-point. From $\widetilde{\gamma}(t)$ take the perpendicular τ_t^{per} to τ and call $\tau_t = \tau \cap \tau_t^{per}$. We say that γ has the property of bounded deviation from geodesic if $\{d(\widetilde{\gamma}(t), \tau_t); t \in [0, +\infty)\}$ is bounded. Here d is the metric of constant negative curvature (see* [**G1**]*) Observe that the property is independent of the geodesic with end-point $\widetilde{\gamma}^+$.*

Corollary 5.5. — *If x beelongs to the exteriorly situated basic piece Λ, the connected components of $W_g^s(x) \smallsetminus \{x\}$ and $W_g^u(x) \smallsetminus \{x\}$ not containing boundary points verifies the property of bounded deviation from geodesics.*

Proof. — Call an f-stable (unstable) half-leaf of x an arcwise connected component of $W_f^s(x) \smallsetminus \{x\}$ ($W_f^u(x) \smallsetminus \{x\}$) when it does not contain singularities. In case that $W_f^s(x)$ ($W_f^u(x)$) contains a singularity all the half-leaves of x are defined, in a natural way, by choosing one at each time the half-leaves of the singularity.

Half-leaves of f have the property of bounded deviation from geodesics (see, for instance, the construction of pseudoanosov maps of [**M**] or [**HT**] or the results about foliations of [**A**])

Then, as images of the g-stable and unstable manifolds by the semiconjugation are contained, respectively, in the stable and unstable sets of f and the semiconjugation is homotopic to the identity, the corollary is proved. □

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