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ON THE SCALING STRUCTURE FOR PERIOD DOUBLING

by

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Abstract. — We describe an order on the set of scaling ratios of the generic universal smooth period doubling Cantor set and prove that this set of ratios forms itself a Cantor set, a Conjecture formulated by Coullet and Tresser in 1977. This result establishes explicitly the geometrical complexity of the universal period doubling Cantor set. We also show a convergence result for the two period doubling renormalization operators, acting on the codimension one space of period doubling maps. In particular they form an iterated function system whose limit set contains a Cantor set.

1. Definitions and Statement of the Results

A *unimodal map with critical exponent* $\alpha > 1$ is an interval map that can be written in the form $f = \psi \circ q_t \circ \phi$, where ψ and ϕ are orientation preserving C^3 diffeomorphisms of $[0,1]$, and $q_t : [0,1] \to [0,1]$ with $t \in (0, \frac{1}{2}]$ is the *standard folding map (with critical exponent* $\alpha > 1$) defined by

$$q_t(x) = 1 - \frac{|x-t|^\alpha}{|1-t|^\alpha},$$

that "folds" the interval at its unique critical point t, $q_t(t) = 1$ and $q_t^2(t) = 0$.

The space of orientation preserving diffeomorphisms of the interval $[0,1]$ with fixed smoothness is denoted by $\mathrm{Diff}^k([0,1])$. The space of unimodal maps with fixed critical

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exponent $\alpha > 1$ and fixed smoothness can be represented by

$$\mathcal{U} = \mathrm{Diff}^k([0,1]) \times (0, \tfrac{1}{2}] \times \mathrm{Diff}^k([0,1]).$$

It carries what we call *C^k-distances d_k*, $k \geqslant 3$, which combines the two C^k distances on each of the two diffeomorphisms ψ and ϕ with the distance between the parameters t of the folding parts. Notice that in general, the critical point of f is $c_f = \phi^{-1}(t) \neq t$. Let p_f be the unique fixed point of $f \in \mathcal{U}$. A map on the interval is *renormalizable* if it exchanges some number N_1 of subintervals. The return map on one of these subintervals can again be renormalizable, exchanging this time N_2 intervals. If the process continues forever, one says the map is *infinitely renormalizable.* For precise definitions and an account of the theory, see for instance [**dMvS**]. Except otherwise specified when we say renormalizable, we mean renormalizable in the sense of period doubling, *i.e.*, the map exchanges two intervals. We will only consider infinitely renormalizable maps with $N_1 = N_2 = \cdots = 2$.

Fix a critical exponent $\alpha > 1$. We consider the set W of maps $f : [0,1] \to [0,1]$ with $f(c_f) = 1$ and $f(1) = 0$ which are infinitely renormalizable. The critical point defines two invariant intervals

$$U_f = [f^2(c_f), f^4(c_f)] \quad \text{and} \quad V_f = [f^3(c_f), f(c_f)].$$

To these two intervals correspond two *renormalization operators* $R_0 : W \to W$ and $R_1 : W \to W$ defined by:

$$R_0 f = [f^2|V_f], \quad \text{and} \quad R_1 f = [f^2|U_f],$$

where $[\cdot]$ means *affine rescaling to obtain a unimodal map on* $[0,1]$ *that sends its critical point to 1 and 1 to 0.*

Observe, both operators preserve W and R_1 is the critical point period doubling renormalization operator which has been most studied in the literature (see in particular [**La**], [**Ly**], [**Mc**], [**dMvS**], [**S2**], and references therein for the case when α is an even integer, and [**E1**], [**E2**] and [**Ma2**] for arbitrary $\alpha > 1$).

Let T_n be the set of all words of length n over the alphabet $\{0,1\}$. We denote by T the set of all infinite words of the form $w1^\infty$ over the alphabet $\{0,1\}$, and by $\overline{T}$ the set of all infinite words over the alphabet $\{0,1\}$, equipped with the usual metric. Notice that each T_n naturally embeds into T. For any word $\tau \in \overline{T}$, we will write $\tau_{\{n\}} \in T_n$ for the initial segment of length n of τ. We are going to consider the iterated function system generated by R_0 and R_1. To this end, we define:

$$R_{\tau_{\{n\}}} = R_{\tau(1)} \circ \cdots \circ R_{\tau(n)} : W \longrightarrow W,$$

and we will prove the following convergence result for this iterated function system.

Theorem 1.1. — *For any fixed point f_0 of R_0, there is a Hölder-continuous map $h : \overline{T} \to W$ such that for any $\tau \in \overline{T}$*

$$\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0 = h(\tau).$$

Moreover, the convergence of the sequence $\{R_{\tau_{\{n\}}} f_0\}$ *is exponential in the* C^2*-metric.*

A similar statement holds for any fixed point f_1 *of* R_1*.*

Remark 1.2. — For any $\alpha > 1$, the existence of a fixed point f_1 of R_1 is proven in [**E1, E2**] and [**Ma2**]. We will show (see Lemma 2.4) that the existence of a fixed point f_1 for R_1 is equivalent to the existence of a fixed point f_0 for R_0. The uniqueness of f_1 in the case when α is an even integer was proven in [**S2**]. In the sequel we will fix f_0 and f_1 to be fixed points of respectively R_0 and R_1.

Remark 1.3. — The set $h(\overline{T})$ of limits $\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0$ is denoted by $A \subset W$. Here the notation A represents the fact that we believe, but do not prove, that the set A is indeed the attractor of the iterated function system generated by R_0 and R_1, and in particular does not depend on the initial point, chosen here to be f_0.

The second Main result, Theorem 1.10, describes the structure of the set A in the case when $\alpha = 2$. It relies on convexity properties of f_0 and $R_1(f_0)$.

Convexity Conditions 1.4. — *We assume that:*

C1 $f_0|[(f_0)^3(c_{f_0}), 1)$ *is strictly convex,*

C2 $R_1(f_0)|[(R_1(f_0))^3(c_{R_1(f_0)}), 1]$ *is strictly convex.*

Remark 1.5. — In section 4 we will show that **C1** actually holds true in the case when successive R_1 renormalizations of a convex function converge to f_1: this is known to be the case when α is an even integer. Furthermore, as we will explain, one can check that both **C1** and **C2** hold true in the most important case of generic (quadratic) critical points, $\alpha = 2$.

Recall that a Cantor set is a perfect and totally disconnected compact metric space.

Proposition 1.6. — *If the Convexity Conditions C1 and C2 hold true, then the limit set* A *of orbits of* f_0 *under the interated function system defined by* R_0 *and* R_1 *is a Cantor set.*

For completeness and to fix notations and definitions, we include some basic discussion of the scaling function, whose origin is rather diffuse: first conjectures about a form of it appeared in [**CT**], the name and a form of it come from [**F**], while what was arguably the first theorem about it was in a never circulated work by Feigenbaum and Sullivan cited in [**S1**]. The literature on scaling functions is extensive and discusses scaling functions beyond the context of dynamics. In particular, in [**KSV**] a relation with the thermodynamic formalism appeared.

Let Λ be the invariant Cantor set of f_0. In the sequel we will remind the dynamical construction of covers of Λ by finitely many intervals. These covers, called cycles, form a refining nest of covers of this Cantor set. The scaling function contains the infinitesimal geometrical information on how these covers refine. It will be shown that the Cantor set Λ is, from a geometrical point of view, very different from the well known middle third Cantor set, in which each refinement is done everywhere in the same manner.

Although, the Cantor set Λ is the invariant set of a non expanding map, it is also the invariant Cantor set of an expanding interval map, the so-called *presentation function* [**R**], [**S1**], a great remark that Rand attributes to Misiurewicz. As we next recall, this directly follows from f_0 being a renormalization fixed point that is expanding to the right of p_{f_0}.

Let $U = U_{f_0}$ and $V = V_{f_0} = [1-v, 1]$. The affine (scaling) map $s : [0,1] \to [0,1]$ defined by $s : x \mapsto v \cdot (x-1) + 1$ is a homeomorphism from Λ to $\Lambda \cap V$. This is a direct consequence of the fact that s conjugates $f_0 = R_0(f_0) = s^{-1} \circ f_0^2 \circ s$ to f_0^2. Also the restriction,

$$f_0|V : \Lambda \cap V \longrightarrow \Lambda \cap U,$$

is a homeomorphism so that the map $g : [0,1] \to U$ defined by $g = (f_0|V) \circ s$ is a homeomorphism from Λ to $\Lambda \cap U$. Let $F : [0,1] \to [0,1]$ be the multivalued function defined by the two branches

$$F_0 = s : [0,1] \longrightarrow [0,1] \quad \text{and} \quad F_1 = g : [0,1] \longrightarrow [0,1].$$

The branch $F_0 = s$ is affine, contracting, and orientation preserving while the branch $F_1 = g$ is orientation reversing. Furthermore, the absolute value of the derivative of F_1 strictly increases as a consequence of the Convexity Condition C1, so that F_1 is also contracting (as p_{f_0} is an expanding fixed point). It follows that the invariant set of the iterated function system $F = \{F_0, F_1\}$ is Λ, the invariant Cantor set of f_0.

The cover $\{U, V\}$ of Λ is called *the cycle of the first generation.* The two intervals of this cycle are permuted by the map f_0. The Cantor set Λ is the intersection of a decreasing sequence of covers we call respectively the *cycles of generation* n: the cycle of generation n is the cover of Λ consisting of 2^n intervals which are permuted by f_0. The intervals that form the n^{th} cycle can be described as follows.

The construction of the cycles is made by using the iterated function system generated by F_0 and F_1. We will use a notation for the words describing sequences of compositions of these maps that will be different from the one we used in the definition of the iterated function system generated by R_0 and R_1. Namely, we write Σ_n for the set of words $w = w(1)w(2)\dots w(n)$ of length $|w| = n$ over the alphabet $\{0,1\}$, and Σ for the set of infinite sequences over the alphabet $\{0,1\}$ with the usual metric. Let

$$I_w = F_{w(n)} \circ \cdots \circ F_{w(1)}([0,1]).$$

The n^{th} cycle consists of the intervals I_w with w a word of length n.

Lemma 1.7. — *The way f_0 permutes these intervals is described by addition* mod 2^n *on the words indexing the intervals. In particular, if c is the critical point of f_0 then*

$c \in I_{1^n}$ and $f_0(c) \in I_{0^n}$. Moreover, $f(I_{1^n}) = I_{0^n}$ and

$$f_0 : I_w \longrightarrow I_{w+1},$$

is a diffeomorphism for each word not equal to 1^n, $n \geqslant 1$.

Proof. — Let w be a word of length $n-1$. Then

$$I_{w1} = F_1(I_w) = f_0 \circ s(I_w) = f_0(I_{w0}),$$

which proves that f_0 permutes the intervals as stated. □

The *orientation* of an interval I_w is defined to be the number

$$o(w) = (-1)^{\#(w)},$$

where $\#(w)$ is the number of 1's in w. The shift of a word $w = w(1)w(2)\dots w(n)$ is defined as

$$\sigma(w) = w(2)w(3)\dots w(n).$$

Observe, that

$$I_w \subset I_{\sigma(w)}.$$

In particular, the n^{th} cycle has two intervals in each interval of the $(n-1)^{th}$ cycle:

$$I_{0w}, I_{1w} \subset I_w.$$

The scaling function $q_n : w \mapsto (0,1)$ assigns to each word w of length n the ratio

$$q_n(w) = \frac{|I_w|}{|I_{\sigma(w)}|}.$$

The a priori bounds on the possible values of q_n, as presented in [**Ma1**] for example imply

$$|I_w| \leqslant \rho^{|w|}$$

for some fixed $\rho < 1$. From this and the smoothness of f_0 it follows that the sequence q_n converges to a Hölder function $q : \Sigma = \{0,1\}^{\mathbf{N}} \to (0,1)$. This function q is what we call the *scaling function*, in minor departure from some previous authors.

The next proposition describes properties of the scaling function. To formulate this proposition we need an order on Σ: with w standing for the maximal word such that $w_1 = ww^1$ and $w_2 = ww^2$, we say that w_1 is strictly smaller than w_2 (or $w_1 \prec w_2$)if and only if

$$(-1)^{\#(w)} \cdot w^1(1) < (-1)^{\#(w)} \cdot w^2(1).$$

Proposition 1.8. — *If the Convexity Conditions hold true then q is strictly monotone.*

Furthermore, under the same hypothesis, there exists constants $C > 0$ and $r < 1$ such that if $w_1 \prec w_2$ and $w_1(k) = w_2(k)$ whenever $k \leqslant n$ then

$$q(w_2) \geqslant q(w_1) + Cr^n$$

Remark 1.9. — If the Convexity Conditions C1 and C2 hold true, Proposition 1.8 confirms the 1977 Conjecture in [**CT**] that the limit set of the ratios $q_n(w)$ defining the period doubling Cantor set is itself a Cantor set.

In particular, we thus have the following

Theorem 1.10. — *In the case of quadratic critical point, $\alpha = 2$, we have the following.*

– *The Convexity Conditions holds true.*

– *The universal period doubling scaling function q is strictly monotone and the range forms a Cantor set.*

– *The limit set A of orbits of f_0 under the interated function system defined by R_0 and R_1 is a Cantor set.*

This Theorem establishes explicitly the geometrical complexity of the universal period doubling Cantor set: for related matters, see [**GT**] and [**T**].

Acknowledgements. — H. Epstein and O.E. Lanford discovered a relation between the fixed points of R_0 and R_1. Roughly speaking this relation states that if $f(x) = h(x^2)$ represents the fixed point of R_1 then $g(x) = (h(x))^2$ represents the fixed point of R_0. This result was not published. However, it was the main inspiration for Section 2. In particular, Lemma 2.4 contains this result.

2. Decompositions and Convergence

The notion of decomposition, introduced in [**Ma2**], is a tool to describe the combinatorial aspects of universality. In this section, after some background on decompositions, we prove the convergence properties stated in Theorem 1.1.

The set T_n is ordered by the embedding into the natural numbers defined by

$$\tau(1)\tau(2)\dots\tau(n) \longmapsto \sum_{i=1}^{n} \tau(i)\cdot 2^{n-i}.$$

Consider also the embedding $j_n : T_n \to T_{n+1}$ defined by

$$j_n : \tau \longmapsto \tau 1.$$

This embedding preserves the order. Observe that T inherites an order from the orders on the sets T_n, which extends to the order on $\overline{T}$ such that $\tau^1 \leqslant \tau^2$ iff $\tau^1_{\{n\}} \leqslant \tau^2_{\{n\}}$ for all $n \geqslant 1$. The elements of $\overline{T}$ are called *decomposition times.*

For the order $<$, the successor in T_n of $1^n \in T_n$ is $0^n \in T_n$ and the predecessor in T_n of $0^n \in T_n$ is 1^n. The successor of $\tau \in T$ in T_n is denoted by τ^{n+} and the predecessor is denoted by τ^{n-}.

The nonlinearity of an orientation preserving diffeomorphism $\phi \in \mathrm{Diff}^2([0,1])$ is

$$\eta_\phi = D \ln D\phi \in C^0([0,1]).$$

A *decomposed unimodal map* is a map

$$\widetilde{f} : T \longrightarrow \mathrm{Diff}^3([0,1]) \cup (0, \tfrac{1}{2}]$$

with the following properties

– $\widetilde{f}(1^\infty)$, the *folding* part of $\widetilde{f}$ represents an element q_t of the standard folding family, so we have $\widetilde{f}(1^\infty) = t \in (0, \frac{1}{2}]$,

– $\widetilde{f}(\tau) \in \mathrm{Diff}^3([0,1])$ for $\tau \neq 1^\infty$, (the *diffeomorphic parts* of $\widetilde{f}$).

– $\sum_{\tau \in T \smallsetminus \{1^\infty\}} |\eta_{\widetilde{f}(\tau)}|_0 < \infty$.

– $\sum_{\tau \in T \smallsetminus \{1^\infty\}} |D\eta_{\widetilde{f}(\tau)}|_0 < \infty$.

The set U of decomposed unimodal maps carries the metric d defined by

$$d(\widetilde{f}, \widetilde{g}) = \sum_{\tau \in T \smallsetminus \{1^\infty\}} |\eta_{\widetilde{f}(\tau)} - \eta_{\widetilde{g}(\tau)}|_1 \ + \ |\widetilde{f}(1^\infty) - \widetilde{g}(1^\infty)|.$$

The two summability conditions for decomposed unimodal maps allow to define what we call *compositions* associated to decomposed unimodal maps. Namely, if one considers a finite set T_n of decomposition times, the composition associated to $\widetilde{f}$ and T_n is defined as

$$O(\widetilde{f}, n) = \widetilde{f}(1^{n-1}0) \circ \cdots \circ \widetilde{f}(0^{n-1}1) \circ \widetilde{f}(0^n) \circ q_{\widetilde{f}(1^n)},$$

otherwise speaking, the folding part followed by the diffeomorphic parts in the order of the decomposition times (so that the end result of the composition is a unimodal map). In [**Ma2**] it is shown that this composition, when defined for decomposed unimodal maps over the sets T_n, extends to a composition operator still denoted O:

$$O : U \longrightarrow \mathcal{U},$$

where $\mathcal{U}$ is equipped with the C^2 metric, which is a Lipschitz map. This composition operator is based on a choice. Namely, the composition starts with the folding part $q_{\widetilde{f}(1^n)}$. We could as well start at any decomposition time $\tau \in T_N$, $N \geqslant 1$ and consider for each $n \geqslant N$ the compositions defined by

$$O(\tau, \widetilde{f}, n) = \widetilde{f}(\tau^{n-}) \circ \cdots \circ \widetilde{f}(0^{n-1}1) \circ \widetilde{f}(0^n) \circ q_{\widetilde{f}(1^n)} \circ \widetilde{f}(1^{n-1}0) \circ \cdots \circ \widetilde{f}(\tau^{n+}) \circ \widetilde{f}(\tau).$$

The same proof which was used in [**Ma2**] to construct $O(\widetilde{f})$ shows the pointwise convergence of the sequence $O(\tau, \widetilde{f}, n)$ as $n \to \infty$, thus defining a map denoted O again:

$$O : T \times U \longrightarrow \mathcal{U}.$$

Observe that $O(1^\infty, \widetilde{f})$ is the operator studied in [**Ma2**].

This construction can be generalized even more. Fix $\widetilde{f} \in U$ and choose $\tau_2 > \tau_1$ in T_N. For each $n \geqslant N$ define the diffeomorphism

$$O_{\tau_1}^{\tau_2}(\widetilde{f}, n) = \widetilde{f}(\tau_2^{n-}) \circ \cdots \circ \widetilde{f}(\tau^{n+}) \circ \widetilde{f}(\tau) \circ \cdots \circ \widetilde{f}(\tau_1^{n+}) \circ \widetilde{f}(\tau_1).$$

It follows from [**Ma2**] that these maps converge, and we set

$$O_{\tau_1}^{\tau_2}(\widetilde{f}) = \lim_{n \to \infty} O_{\tau_1}^{\tau_2}(\widetilde{f}, n).$$

Moreover, there is a constant $K_{\widetilde{f}}$ such that

$$|O_{\tau_1}^{\tau_2}(\widetilde{f}) - \mathrm{id}|_2 \leqslant K_{\widetilde{f}} \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geqslant \tau_1\}} |\eta_{\widetilde{f}(\tau)}|_0.$$

Lemma 2.1. — *The operator O extends continuously to an operator*

$$O : \overline{T} \times U \longrightarrow \mathcal{U}.$$

In particular, for each $\widetilde{f} \in U$ there exists a constant $K_{\widetilde{f}} > 0$ such that for any pair $\tau_2, \tau_1 \in \overline{T}$ with $\tau_2 \geqslant \tau_1$,

$$d_2(O(\tau_2, \widetilde{f}), O(\tau_1, \widetilde{f})) \leqslant K_{\widetilde{f}} \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geqslant \tau_1\}} |\eta_{\widetilde{f}(\tau)}|_0.$$

Moreover for each $\tau_3 > \tau_2 > \tau_1 \in \overline{T}$ and $\widetilde{f} \in U$

$$O_{\tau_1}^{\tau_3}(\widetilde{f}) = O_{\tau_2}^{\tau_3}(\widetilde{f}) \circ O_{\tau_1}^{\tau_2}(\widetilde{f}).$$

Proof. — Fix $\widetilde{f} \in U$ and choose $\tau_2 > \tau_1$ in T_N. Let $h = O_{\tau_1}^{\tau_2}(\widetilde{f})$. The construction of h implies directly

$$h \circ O(\tau_1, \widetilde{f}) = O(\tau_2, \widetilde{f}) \circ h.$$

This construction can be done for every pair of $\tau_1', \tau_2' \in [\tau_2, \tau_1] \cap T$. Hence, there is a constant which only depends on $\widetilde{f}$ such that

$$d_2(O(\tau_2', \widetilde{f}), O(\tau_1', \widetilde{f})) \leqslant \text{ Const } \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geqslant \tau_1\}} |\eta_{\widetilde{f}(\tau)}|_0.$$

From this we get the continuous extension of O to $\overline{T} \times U$, together with the estimate stated in the Lemma. The composition rule clearly holds for the operators $O_{\tau_1}^{\tau_2}(\widetilde{f}, n)$ and hence for the continuous extension of O. □

We will also write $O_\tau(\cdot)$ for $O(\tau, \cdot)$. Let $\mathcal{U}_0$ be the set of renormalizable unimodal maps and $U_0 = (O_{1^\infty})^{-1}(\mathcal{U}_0)$. A renormalization operator $R : U_0 \to U$ is constructed in [**Ma2**] such that

$$O_{1^\infty} \circ R = R_1 \circ O_{1^\infty}.$$

A decomposed unimodal map $\widetilde{f} \in U_0$ is said to be n *times renormalizable* iff $f = O(\widetilde{f}) \in \mathcal{U}$ is n times renormalizable: we then set $f = \phi \circ q_t$ with $t \in (0, \frac{1}{2}]$. This means there are pairwise disjoint intervals $I_\tau^{f,n}$, $\tau \in T_n$, forming the n^{th} cycle of f, such that

- $t \in I_{1^n}^{f,n}$,
- $f : I_\tau^{f,n} \to I_{\tau^{n+}}^{f,n}$ is a diffeomorpishm, whenever $\tau \neq 1^n$,
- $f : I_{1^n}^{f,n} \to I_{0^n}^{f,n}$ is onto.

Let $g : I \to J$ be an endormorphism which has either one or zero critical point. Then $[g] : [0,1] \to [0,1]$ is a either a unimodal map or an orientation preserving diffeomorphism obtained by affine scaling of the domain and image of g.

Lemma 2.2. — *Let $\widetilde{f} \in U$ be n times renormalizable and $O(\widetilde{f}) = f = \phi \circ q_t \in \mathcal{U}_0$ with $t \in (0, \frac{1}{2}]$. For $n \geqslant 1$ and $\tau \in T_n \subset T$*

- $O_\tau^{\tau^{n+}}(R^n \widetilde{f}) = [f | I_\tau^{f,n}]$,
- $O_\tau^{\tau^{n+}0^\infty}(R^n \widetilde{f}) = [q_t | I_\tau^{f,n}]$,
- $O_{\tau^{n+}0^\infty}^{\tau^{n+}}(R^n \widetilde{f}) = [\phi | q_t(I_\tau^{f,n})]$.

The reader is refered to [**Ma2**] for the precise definition of the renormalization operator $R : U_0 \to U$, from which the Lemma immediately follows. This lemma indeed captures all the properties of the renormalization operator R that we will need.

Proposition 2.3. — *For every $\tau \in T_n \subset T$*

$$O_\tau \circ R^n = R_\tau \circ O_{1^\infty}, \quad \textit{and} \quad O_{\tau 0^\infty} \circ R^n = R_\tau \circ O_{0^\infty}.$$

Proof. — Let $\widetilde{f} \in U$ be $n \geqslant 1$ times renormalizable and

$$O(\widetilde{f}) = O_{1^\infty}(\widetilde{f}) = f = \phi \circ q_t \in \mathcal{U}_0$$

with $t \in (0, \frac{1}{2}]$. As in the proof of Lemma 3.1 shows that for every $n \geqslant 1$ and $\tau \in T_n \subset T$

$$R_\tau(f) = [f^{2^n} | I_\tau^{f,n}].$$

Let $\tau_1 = \tau$, $\tau_k = \tau_{k-1}^{n+}$, for $k = 2, 3, \ldots, 2^n$. The composition rule for the operators $O_{\tau_1}^{\tau_2}$ and Lemma 2.2 imply

$$\begin{aligned} O_\tau \circ R^n(\widetilde{f}) &= O_{\tau_{2^n}}^{\tau_1}(R^n \widetilde{f}) \circ \cdots \circ O_{\tau_2}^{\tau_3}(R^n \widetilde{f}) \circ O_{\tau_1}^{\tau_2}(R^n \widetilde{f}) \\ &= [f | I_{\tau_{2^n}}^{f,n}] \circ \cdots \circ [f | I_{\tau_2}^{f,n}] \circ [f | I_{\tau_1}^{f,n}] \\ &= [f^{2^n} | I_{\tau_1}^{f,n}] \\ &= R_{\tau_1}(f) \\ &= R_\tau \circ O_{1^\infty}(\widetilde{f}). \end{aligned}$$

The second equation is proved similarly. □

Lemma 2.4. — *The operators R_0 and R_1 have fixed points. Furthermore, for any even integer α, both operators R_0 and R_1 have a unique fixed point.*

Proof. — It was shown in [**Ma2**] that the operator R has a fixed point. The previous proposition implies that a fixed point $\widetilde{f} \in U_0$ of R produces fixed points of R_0 and R_1. Namely,

$$R_1(O_{1^\infty}(\widetilde{f})) = O_{1^\infty}(\widetilde{f}) \quad \text{and} \quad R_0(O_{0^\infty}(\widetilde{f})) = O_{0^\infty}(\widetilde{f}).$$

Claim 2.5. — *For each fixed point $f \in \mathcal{U}$ of R_1 (or R_0) there exists a unique fixed point of R, say $\widetilde{f} \in U$ such that $O_{1^\infty}(\widetilde{f}) = f$ (or $O_{0^\infty}(\widetilde{f}) = f$).*

Proof. — Let $f = \phi \circ q_t \in \mathcal{U}$ be a fixed point of R_1 (the case of a fixed point for R_0 can be treated the similarly). Choose $\widetilde{f} \in U$ such that

$$O_{1^\infty}(\widetilde{f}) = f.$$

For example, consider $\widetilde{f} \in U$ defined by

- $\widetilde{f}(1^\infty) = q_t$,
- $\widetilde{f}(01^\infty) = \phi$,
- $\widetilde{f}(\tau) = \mathrm{id}$ for $\tau \neq 1^\infty, 01^\infty$.

The definition of $\widetilde{f}$ and the fact that $O_{1^\infty} \circ R = R_1 \circ O_{1^\infty}$, implies

$$O_{1^\infty}(R^n \widetilde{f}) = f, n \geqslant 1.$$

We will show

$$\lim_{n\to\infty} R^n \widetilde{f} = \widehat{f} \in U,$$

with

$$R\widehat{f} = \widehat{f} \text{ and } O_{1^\infty}(\widehat{f}) = f.$$

Let $n \geqslant 1$ and $\tau_3 > \tau_2 > \tau_1 \in T_{n+1}$ three consecutive decomposition times in T_{n+1} with $\tau_3, \tau_1 \in T_n$. Observe, that τ_3 and τ_1 are consecutive points in T_n. From Lemma 2.2 we get

$$\begin{aligned} O^{\tau_3}_{\tau_1}(R^{n+1}\widetilde{f}) &= O^{\tau_3}_{\tau_2}(R^{n+1}\widetilde{f}) \circ O^{\tau_2}_{\tau_1}(R^{n+1}\widetilde{f}) \\ &= [f|I^{f,n+1}_{\tau_2}] \circ [f|I^{f,n+1}_{\tau_1}] \\ &= [f^2|I^{f,n+1}_{\tau_1}] \\ &= [f|I^{f,n}_{\tau_1}], \end{aligned}$$

where we used that f is a fixed point of R_1. Again from Lemma 2.2 we get $[f|I^{f,n}_{\tau_1}] = O^{\tau_3}_{\tau_1}(R^n\widetilde{f})$. Hence,

$$O^{\tau_3}_{\tau_1}(R^{n+1}\widetilde{f}) = O^{\tau_3}_{\tau_1}(R^n\widetilde{f}).$$

This should be interpreted as $R^{n+1}\widetilde{f}$ being a refinement of $R^n\widetilde{f}$. In [**AMM**] it has been shown that there is a constant $K > 0$ and $\rho < 1$ such that

$$\sum_{\tau_1 \in T_n} |(O^{\tau_3}_{\tau_2}(R^{n+1}\widetilde{f}) - \mathrm{id})|_2 \leqslant K \cdot \rho^n.$$

This implies that $\lim_{n\to\infty} R^n\widetilde{f} = \widehat{f} \in U$. In particular, this implies that $\widehat{f}$ is a fixed point of R which projects by O_{1^∞} to f. This concludes the existence part of the Claim.

We can use Lemma 2.2 to identify $\widetilde{f}(\tau)$, $\tau \in T_N$. Namely,

$$\begin{aligned}\widetilde{f}(\tau) &= \lim_{n\to\infty} R^n\widetilde{f}(\tau)\\ &= \lim_{n\to\infty} O_\tau^{\tau^n+0^\infty}(R^n\widetilde{f})\\ &= \lim_{n\to\infty}[q_t|I_\tau^{f,n}]\\ &= [q_t|I_\tau^{f,N}],\end{aligned}$$

where we used that f is a fixed point of R_1 to obtain the last equality. This implies the uniqueness part of the Claim. □

It has been shown in [**S2**] that the operator R_1 has a unique fixed point when α is an even integer. Now the uniqueness part of Lemma 2.4 follows by using the Claim. □

Proof of Theorem 1.1. — Let f_0 be a fixed point of R_0 and $\widetilde{f}_0 \in U$ the unique fixed of R with $O_{0^\infty}(\widetilde{f}_0) = f_0$. Let $h : \overline{T} \to W$ be defined by

$$h(\tau) = O_\tau(\widetilde{f}_0).$$

For any $\tau_1, \tau_2 \in \overline{T}$ let $|\tau_2 - \tau_1|$ be the maximal length for which initial segments of the word τ_1 and τ_2 of that length agree. In [**AMM**] it has been shown that there is a constant $K > 0$ and $\rho < 1$ such that

$$\sum_{\tau_2>\tau>\tau_1} |\eta_{\widetilde{f}_0(\tau)}|_0 \leqslant K\cdot\rho^{|\tau_2-\tau_1|}.$$

Recall that $\tau_{\{n\}}$ is the word consisting of the first n symbols of a word $\tau \in \overline{T}$. From Lemma 2.1 we get

$$d_2(h(\tau_{\{n\}}0^\infty), h(\tau)) \leqslant K\cdot\rho^n.$$

Theorem 1.1 follows from Proposition 2.3. Namely,

$$\begin{aligned}R_{\tau_{\{n\}}} f_0 &= R_{\tau_{\{n\}}} \circ O_{0^\infty}\widetilde{f}_0\\ &= O_{\tau_{\{n\}}0^\infty} \circ R^n\widetilde{f}_0\\ &= O_{\tau_{\{n\}}0^\infty}\widetilde{f}_0\\ &= h(\tau_{\{n\}}0^\infty) \longrightarrow h(\tau),\end{aligned}$$

where the convergence is exponential.

3. The monotonicity of the scaling function

The monotonicity of the scaling function q, as formulated in Proposition 1.8 is based on the following combinatorial Lemmas. First we will concentrate on these Lemmas and prove Proposition 1.8. Secondly, Proposition 1.8 is used to prove Proposition 1.6.

Although decomposition times and the words used to define the intervals I_w are conceptually different, the following Lemma shows that they are strongly related.

Lemma 3.1. — *For every word w of length n*

$$R_w(f_0) = [f_0^{2^n}|I_w].$$

Proof. — The proof is by induction in n. For $n = 1$ the Lemma restates the definition of R_0 and R_1. Assume the Lemma holds for some $n \geqslant 1$. Choose a word w of length n and consider the two intervals I_{0w} and I_{1w}. These intervals are contained in I_w and each contains a boundary point of I_w. Using the induction hypothesis $R_w(f_0) = [f_0^{2^n}|I_w]$ and the fact that $f_0^{2^n}|I_w$ permutes I_{0w} and I_{1w} we get that $R_{0w}(f_0) = R_0(R_w(f_0))$ and $R_{1w}(f_0) = R_1(R_w(f_0))$ correspond to either of $f^{2^{n+1}}|I_{0w}$ or $f^{2^{n+1}}|I_{1w}$.

It is left to identify which of the two intervals corresponds to $U_{R_w(f_0)}$ (resp. to $V_{R_w(f_0)}$). The map f_0 permutes the intervals $I_{w'}$ with $|w'| = n+1$ according to addition mod.2^n on the words indexing the intervals, as described in Lemma 1.7. Observe that

$$1w = 0w + 2^n \cdot 1.$$

This means that $f_0^{2^n}|I_{0w}$ is monotone because $0w + k\cdot 1$, $k < 2^n$ never equals the word 1^{n+1} and $f_0|I_{1^{n+1}}$ is the only place where monotonicity of f_0 fails. Hence,

$$R_{0w}(f_0) = R_0([f_0^{2^n}|I_w]) = [(f_0^{2^n})^2|I_{0w}]$$

and

$$R_{1w}(f_0) = R_1([f_0^{2^n}|I_w]) = [(f_0^{2^n})^2|I_{1w}]. \qquad \square$$

In the sequel we will identify $R_w(f_0)$ with $f_0^{2^n}|I_w$.

Lemma 3.2. — *For every pair of words w and w^0, the map*

$$R_{w^0}(f_0) : I_{w0w^0} \longrightarrow I_{w1w^0},$$

is monotone and onto.

Proof. — Let $|w^0| = n$. The action of f_0 on the intervals of length $|w| + 1 + |w^0|$ is described by addition mod.2^n on the words indexing the intervals (see Lemma 1.7). In particular,

$$w1w^0 = w0w^0 + 2^n \cdot 1.$$

Hence

$$f_0^{2^n}(I_{w0w^0}) = I_{w1w^0}.$$

By construction we have

$$I_{w1w^0}, I_{w0w^0} \subset I_{w^0}.$$

Now the Lemma follows from $R_{w^0}(f_0) = [f_0^{2^n}|I_{w^0}]$, which we know from Lemma 3.1.

$\square$

Lemma 3.3. — *If the Convexity Condition holds then there exist constants $C > 0$ and $r \in (0,1)$ with the following property. Let w be a word of with $|w| = n$.*

If $o(w) = +1$ then

– $w0 < w1$ *and* $w00 < w01 < w11 < w10$
– $q_{n+1}(w0) < q_{n+1}(w1)$
– $q_{n+2}(w00) < q_{n+2}(w01) < q_{n+2}(w11) < q_{n+2}(w10)$
– $q_{n+2}(w11) > q_{n+2}(w01) + Cr^n$

If $o(w) = -1$ *then*

– $w1 < w0$ *and* $w10 < w11 < w01 < w00$
– $q_{n+1}(w1) < q_{n+1}(w0)$
– $q_{n+2}(w10) < q_{n+2}(w11) < q_{n+2}(w01) < q_{n+2}(w00)$
– $q_{n+2}(w01) > q_{n+2}(w11) + Cr^n$

Proof. — The construction of the intervals I_w imply immediately the following. If $o(w) = +1$ then the interval I_w contains the right boundary point of $I_{\sigma(w)}$. And if $o(w) = -1$ then I_w contains the left boundary point of $I_{\sigma(w)}$. Using this, the convexity of F_1 and the fact that F_0 is affine we get

Claim 3.4. — $o(w) \cdot q_{n+1}(w0) < o(w) \cdot q_{n+1}(w1)$, *for every word* w *with* $|w| = n$.

The case when $o(w) = -1$ of the Lemma can be proved similarly as the first case. We will only present the proof in the case $o(w) = +1$. The first statement is merely the definition of the order on the symbol space. The second follows directly from Claim 3.4. This Claim also implies

$$q_{n+2}(w00) < q_{n+2}(w01), \quad \text{and} \quad q_{n+2}(w11) < q_{n+2}(w10).$$

To study the middle inequality, observe that

$$I_{\sigma(w)01} \cup I_{\sigma(w)11} \subset I_1.$$

First observe that $o(w01) = -1$ (and $o(w11) = 1$). In particular the negatively oriented interval I_{w01} contains the left boundary point of the interval $I_{\sigma(w)01}$. Moreover,

$$I_{\sigma(w)01} \subset I_{01} \subset [0, f_0^4(c_{f_0})],$$

where $0 \in I_1$ is the left boundary point of I_1.

By Lemma 3.2 we have

$$R_1(f_0) : I_{\sigma(w)01} \longrightarrow I_{\sigma(w)11}.$$

The Convexity Condition states that the absolute value of the derivative of this map decreases strictly on the interval $[0, f_0^6(c_{f_0})]$. Now using

$$I_{w01} \subset I_{\sigma(w)01} \subset [0, f_0^6(c_{f_0})]$$

and that the interval $I_{w01} \subset I_{\sigma(w)01}$ contains the left boundary point of $I_{\sigma(w)01}$, we get

$$q_{n+2}(w01) < q_{n+2}(w11).$$

From the a priori bounds described for example in [**Ma1**], we know that there are constants $C > 0$ and $r \in (0,1)$ such that

$$|I_w| \geqslant C \cdot r^{|w|}$$

for all words w. This implies the final estimate of Lemma 3.3. □

Let w be a word with $|w| = k$. Then define the interval

$$J_w = [q_{k+1}(w0), q_{k+1}(w1)].$$

Proof of Proposition 1.8. — The proposition 1.8 is reformulated in

Claim 3.5. — *Let w be a word with $|w| = k$ and $|wh| = n$. Then*

$$q_n(wh) \in J_w.$$

In particular,

$$J_{wh} \subset J_w.$$

Moreover, if w^1 and w^2 are distinct words of length k then $J(w^1)$ and $J(w^2)$ are disjoint and the distance between them is larger than Cr^k.

Proof. — The proof of the first part of the Claim is by induction in n. For $n = 2$ the statement follows from the Lemma 3.3. Assume the Claim holds for all words wh with $|wh| \leqslant n$.

Consider a word $wh = w\widehat{h}h^1h^2$ with $|wh| = n + 1$ and $|h^1| = |h^2| = 1$. Then Lemma 3.3 implies that for every pair of symbol x, y

$$q_{n+1}(w\widehat{h}xy) \in [q_{n+1}(w\widehat{h}10), q_{n+1}(w\widehat{h}00)].$$

In particular,

$$\begin{aligned} q_{n+1}(wh) &\in [q_{n+1}(w\widehat{h}10), q_{n+1}(w\widehat{h}00)] \\ &= [q_n(w\widehat{h}1), q_n(w\widehat{h}0)] \\ &\subset J_w, \end{aligned}$$

The above equality follows from the fact that $q_{n+1}(w\widehat{h}10) = q_n(w\widehat{h}1)$ because the interval $I_{w\widehat{h}10}$ is obtained from $I_{w\widehat{h}1}$ by applying the affine branch F_0. The other boundary is treated similarly. The last inclusion follows from the induction hypothesis.

The proof of the second part of the Claim is by induction in $k = |w|$. For $k = 1$ the Claim considering the distance between J_0 and J_1 is a reformulation of the previous Lemma. Assume, the Claim is proved up to some $k \geqslant 1$. Let w^1 and w^2 be two words of length $k+1$, say $w^1 = \widetilde{w}^1 x$ and $w^2 = \widetilde{w}^2 y$ with $|\widetilde{w}^1| = |\widetilde{w}^2| = k$.

If $\widetilde{w}^1$ differs from $\widetilde{w}^2$ then the Claim follows because

$$J_{w^1} \subset J_{\widetilde{w}^1}, J_{w^2} \subset J_{\widetilde{w}^2}$$

and the induction hypothesis. So we may assume that

$$w^1 = w0, w^2 = w1.$$

Apply Lemma 3.3 again to conclude that J_{w^1} and J_{w^2} are disjoint with the appropriate distance between them. □

Proof of Proposition 1.6. — The proof of Proposition 1.6 relies on the relation between the two iterated function systems generated by respectively $\{R_0, R_1\}$ and $\{F_0, F_1\}$ as formulated in Lemma 3.1. Notice, the only difference between Σ and $\overline{T}$ is that they carry different orders. The order does not play any role in the proof of Proposition 1.6. We will use the symbol w for words which are in $\Sigma = \overline{T}$. In Section 2 we constructed the continuous map $h : \Sigma \to A$ (see Remark 1.3). Namely, for $w \in \Sigma$, let

$$h(w) = \lim_{n\to\infty} R_{w_{\{n\}}}(f_0).$$

In particular, this map is onto. It is left to show that h is injective.

Observe that every word w with $|w| = n$

$$R_{0w}(f_0) = R_0(R_w(f_0)).$$

In particular,

$$q_{n+1}(0w) = |V_{R_w(f_0)}|.$$

Recall that for $w \in \Sigma$ we denote the word consisting of the first n symbols of $w \in \Sigma$ by $w_{\{n\}}$. Let $w^1, w^2 \in \Sigma$ be such that $h(w^1) = h(w^2)$. Then

$$\begin{aligned}
|q(0w^1) - q(0w^2)| &= \lim_{n\to\infty} |q_{n+1}(0w^1_{\{n\}}) - q_{n+1}(0w^2_{\{n\}})| \\
&= \lim_{n\to\infty} ||V_{R_{w^1_{\{n\}}}(f_0)}| - |V_{R_{w^2_{\{n\}}}(f_0)}|| \\
&\leqslant \text{Const} \lim_{n\to\infty} \text{dist}(R_{w^1_{\{n\}}}(f_0), R_{w^2_{\{n\}}}(f_0)) \\
&= \text{Const} \cdot \text{dist}(h(w^1), h(w^2)) = 0.
\end{aligned}$$

The strict monotonicity of the scaling function, Proposition 1.8, implies $w^1 = w^2$. This proves that $h : \Sigma \to A$ is a homeomorphism.

4. The Convexity Condition

In this section the Convexity Condition will be studied.

Lemma 4.1. — *Let $f : (-1, 1) \to (-1, 1)$ be C^2. If*

- $f(0) = 0$,
- $Df(0) < -1$,
- $D^2 f(0) < 0$

then

$$D^2(f^2)(0) < 0.$$

Proof. — The chain rule applied to f^2 gives

$$D^2(f^2)(x) = D^2f(f(x)) \cdot (Df(x))^2 + Df(f(x)) \cdot D^2f(x).$$

Using the properties of f in $x = 0$ we get

$$D^2(f^2)(0) = D^2f(0) \cdot Df(0) \cdot [Df(0) + 1] < 0. \qquad \square$$

Lemma 4.2. — *Let $C \subset W$ consisting of unimodal maps $f \in W$, with negative Schwarzian derivative (see* [**dMvS**] *for the definition), and the following property: $f|[0,c]$ is convex, where c is the critical point of f, and $f|[c,1]$ is strictly convex (The derivative of f is decreasing over $[0,1]$ but strictly decreasing on $[c,1]$). Then*

$$R_0(C) \subset C.$$

Proof. — Let $f \in C$ with critical point $c \in [0,1]$ and let p_f be its fixed point. Let $V_f = P \cup Q$, where P, Q are the two intervals on which $R_0 f$ is monotone. Choose $Q \subset V_f$ such that $f(Q) \subset [0,c]$. The convexity property of f implies directly the strict convexity of $R_0(f)|Q$.

The Schwarzian derivative of f is negative. This implies that p_f is an expanding fixed point, otherwise it would attract the critical point (see [**dMvS**]). Hence, $Df(p_f) < -1$. The convexity condition of f allows us to apply the previous Lemma:

$$D^2R_0(f)(p_f) < 0,$$

i.e. the derivative of f^2 is decreasing in p_f. Now, the Minimum Principle for maps with negative Schwarzian derivative (again see [**dMvS**]), implies that Df^2 is decreasing monotonically to zero on the interval $[p_f, P]$, hence $R_0 f \in C$. $\square$

Lemma 4.3. — *The convexity condition* **C1** *holds true for any even critical exponent α, the map $f_0|[p_{f_0}, 1]$ is strictly convex.*

Proof. — Let $q_t \in W$ be a standard folding map. Clearly, $q_t \in C$. From [**S2**] we have

$$\lim_{n\to\infty} R_1^n q_t = f_1.$$

Let $\widehat{f}$ be the unique fixed point of R (with $O_{1\infty}(\widehat{f}) = f_1$). As in the proof of Claim 2.5 we get for every $\widetilde{f} \in U$ with $O_{1\infty}(\widetilde{f}) = q_t$ that

$$\lim_{n\to\infty} R^n \widetilde{f} = \widehat{f}.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} R_0^n q_t &= \lim_{n\to\infty} O_{0\infty} R^n \widetilde{f} \\ &= O_{0\infty}(\widehat{f}) \\ &= f_0,\end{aligned}$$

where f_0 is the fixed point of R_0. This implies that the derivative of f_0 is decreasing because of the previous Lemma . The renormalization fixed point f_0 is real analytic.

Hence, the set $E \subset [0,1]$ consisting of the flat points of f_0, points where $D^2 f_0$ vanishes, is finite.

The map f_0 is the fixed point of R_0. Hence, $s(E)$, the map s is the affine scaling of the interval $[0,1]$ to V_{f_0}, is the set of flat points of $R_0 f_0 (= f_0)$. Let $Q \subset V_{f_0}$ be the maximal interval such that $f_0(Q) \subset [0,c]$. Any non-flat point $x \in Q$ will be a non-flat point of $R_0 f_0$, this follows from the convexity, maybe not strict, of f_0. Hence

$$s(E) \cap Q \subset E.$$

Assume, $E \cap [c,1) \neq \varnothing$ and let $x \in E \cap [c,1)$ be the rightmost point. The fact that f_0 is a renormalization fixed point implies that $s(c)$ is the left boundary point of Q. In particular we get

$$x < s(x) \in s(E) \cap s([c,1]) \subset E \cap Q,$$

contradicting the fact that x was chosen to be the right most point in $E \cap [c,1)$. This proves that f_0 does not have flat points in $[p_{f_0}, 1) \subset [c,1)$. □

Lemma 4.4. — *The convexity condition* **C2** *holds true for* $\alpha = 2$.

Proof. — In the case $\alpha = 2$, an approximation of f_1 can be found in [**La**]. We will use the notation of [**La**]. The fixed point f_1 is represented as $g(z) = h(z^2)$ where $|z|^2 \leqslant 1.5$. Actually, the map h defined on the disk $D_0 = \{z \| z| \leqslant 1.5\}$ where it is analytic. The map h is approximated by a polynomial of degree 40.

$$h_0(z) = 1 + \sum_{n=1}^{40} g_n^{(0)} z^n$$

where

$$|g_n^{(0)}| \leqslant 10^{-(n-2)}$$

It is also shown in [**La**] that

$$|h(z) - h_0(z)| \leqslant 1.5 \cdot 10^{-23}, \quad z \in D_0.$$

From Lemma 2.4 we get that the map $f(z) = (h(z))^2$, $z \in D_0$ represents the fixed point f_0 of R_0, the maps are equal up to an affine scaling. The map $P(z) = (h_0(z))^2$, $z \in D_0$, approximates this fixed point. For both maps the dynamically relevant interval is $[0,1]$: $f([0,1]) = [0,1]$ and $P([0,1]) = [0,1]$.

To prove the convexity condition **C2** it suffices to show the strict convexity of f^2 restricted to the interval $[f^6(c_f), 1]$.

Claim 4.5. — *The derivative of of* h_0 *restricted to* D_0 *satisfies*

$$|Dh_0(z)| \leqslant 14.0$$

This estimate follows from the bounds on the coefficients of the polynomial h_0.

Claim 4.6. — *For $z \in D_0$*

$$|P(z) - f(z)| \leqslant 1.0 \cdot 10^{-20}$$

and the derivative of P restricted to the disk D_0 satisfies

$$|DP(z)| \leqslant 700.$$

The bound on the coefficients of the polynomial h_0 imply that $h_0(\{z||z| \leqslant 1.5\})$ is contained in a disk of radius 23 around 0. The bounds on the distance between h and h_0 and the fact that the derivative of the map $z \mapsto z^2$ is bounded by 50 on the disk of radius 23 around 0, finishes the proof of this Claim.

Let D be the $\frac{1}{1500}$−neighborhood of the interval $[0,1]$.

Claim 4.7. — *The map f^2 is defined on D (and is analytic). Moreover*

$$|f^2(z) - P^2(z)| \leqslant 10^{-17}, \quad z \in D.$$

The fact that f^2 is well defined on D follows from the fact that P maps D well inside the disk of radius D_0 and that P and f are close on D_0. The estimate on the distance between f^2 and P^2 on D follows from the bound on the derivative of P and the very small distance between f and P.

Claim 4.8. — *For every $z \in [0,1]$*

$$|D^2P^2(z) - D^2f^2(z)| \leqslant 1.0 \cdot 10^{-7}$$

and

$$|D^3P^2(z) - D^3f^2(z)| \leqslant 1.0 \cdot 10^{-4}$$

These bounds follow by applying the Cauchy integral formula for derivatives. Let $z_0 \in [0,1]$. Then

$$\begin{aligned}|D^2P^2(z) - D^2f^2(z)| &\leqslant \frac{1}{2\pi}\int_{\partial D}\frac{|P^2(z) - f^2(z)|}{|z - z_0|^3}dz\\ &\leqslant \frac{1}{2\pi}\cdot 10^{-20}\cdot\frac{1}{0.001^3}\cdot 2\pi(1.5) \leqslant 1.0\cdot 10^{-10}.\end{aligned}$$

The third derivative is treated similarly.

It is left to find a lower bound for $|D^2P^2(z)|$ larger than 10^{-9}. We will use traditional cross ratio technology [**dMvS**] to reduce this question to a calculation in finitely many points. Let $h : T \to h(T)$ be a diffeomorphism of the interval T to its image and suppose it has negative Schwarzian derivative. Let $M \subset T$ be a subinterval and let $L, R \subset T \setminus M$ be the two connected components of $T \setminus M$. Let

$$\tau = \min\left\{\frac{|h(L)|}{|h(M)|}, \frac{|h(R)|}{|h(M)|}\right\}.$$

Then

$$|Dh(x)| \leqslant \frac{1+\tau}{\tau}\cdot\frac{|h(M)|}{|M|}, \quad x \in M.$$

The nonlinearity of h is $\eta = D \ln Dh = D^2h/Dh$. We have the following estimate

$$|\eta(x)| \leqslant 2\frac{(1+\tau)|h(M)|}{\tau^2|M|^2}, \quad x \in M.$$

The third inequality we will use is

$$D^3h \geqslant \frac{3}{2}(\frac{D^2h}{Dh})^2 \cdot Dh$$

in the case when Dh is negative.

We will apply these three estimates to the map f^2 restricted to the interval $[c_f, 1]$ with $M = [f^6(c_f), 1]$. The period two point of f and the position of $f^4(c_f)$ can be precisely estimated with the help of P. Using estimates for these two points gives the following estimates

$$\tau \geqslant 0.2 \quad \text{and} \quad |M| \geqslant 0.1, \ |f^2(M)| \leqslant 0.6.$$

This implies

$$|D(D^2P^2)(x)| \geqslant -2.2 \cdot 10^8, \quad x \in M.$$

Claim 4.9. — *For every* $z \in [f^6(c_f), 1]$

$$|D^2P^2(z)| \geqslant 0.5.$$

This is shown by numerical analysis. The second derivative of P^2 is calculated in a sequence of points with increment 10^{-9} over the interval $[f^6(c_f), 1]$. In these point the second derivative of P^2 is smaller than -1. The derivative estimate of D^2P^2 leads to the lower bound as stated in the Claim. □

The quadratic case, $\alpha = 2$, as described in Theorem 1.10 follows from propositions 1.6 and 1.8, and lemmas 4.3 and 4.4.

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