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DUALITY FOR SMOOTH FAMILIES IN EQUIVARIANT STABLE HOMOTOPY THEORY

Po Hu

Abstract. — In this paper, we formulate and prove a duality theorem for the equivariant stable homotopy category, using the language of Verdier duality from sheaf theory. We work with the category of G-equivariant spectra (for a compact Lie group G) parametrized over a G-space X, and consider a smooth equivariant family $f : X \to Y$, which is a G-equivariant bundle whose fiber is a smooth compact manifold, and with actions of subgroups of G varying smoothly over Y. Then our main theorem is a natural equivalence between a certain direct image functor f_* and a "direct image with proper support functor" $f_!$, in the stable equivariant homotopy category over Y. In particular, the Wirthmüller and Adams isomorphisms in equivariant stable homotopy theory turn out to be special cases of this duality theorem.

Résumé **(Dualité pour les familles lisses en théorie de l'homotopie stable équivariante)**
Dans cet article, nous énonçons et démontrons un théorème de dualité pour la catégorie de l'homotopie stable équivariante, en utilisant le langage de la dualité de Verdier provenant de la théorie des faisceaux. Nous travaillons avec la catégorie des spectres G-équivariants (pour un groupe de Lie compact G) paramétrés par un G-espace X, et nous considérons une famille lisse équivariante $f : X \to Y$, c'est-à-dire un fibré G-équivariant de fibre une variété lisse compacte, et avec des actions de sous-groupes de G variant de manière lisse sur Y. Notre résultat principal est alors une équivalence naturelle entre un foncteur image directe f_* et un foncteur « image directe à support propre $f_!$ », dans la catégorie de l'homotopie stable équivariante sur Y. Les isomorphismes de Wirthmüller et Adams en théorie de l'homotopie stable équivariante apparaissent comme des cas particuliers de ce théorème de dualité.

CONTENTS

INTRODUCTION

The purpose of this paper is to formulate and prove a stable homotopy duality theorem for smooth equivariant families of manifolds, using a relationship of the stable homotopy language with sheaf theory. We work with G-equivariant spaces and spectra parametrized over G-equivariant spaces, where G is a compact Lie group. To relate this to the language of sheaves and Verdier duality from algebraic geometry (see *e.g.* [**2, 6**]), we introduce the notions of sheaves of spaces and of spectra. The Grothendieck site we use here is the most basic case, where the category is the comma category $G\mathrm{Top}\,/X$ of all G-equivariant topological spaces mapping to a given G-equivariant base space X. The coverings in this category are all colimits. This makes the results of this paper more directly related to classical stable homotopy theory [**8**] than its generalizations (*e.g.* [**1, 12**]), although our methods in principle also seem to apply to those more general contexts.

In our context, the main theorem is that for a map $f : X \to Y$ of base spaces satisfying certain conditions, there is a natural equivalence in the stable homotopy categories

$$f_* \simeq f_! \tag{0.1}$$

between a certain direct image functor f_* and a direct image with proper support functor $f_!$. This is an analogue of a classical result for proper maps of schemes, and abelian sheaves. A complementary statement for smooth maps relate the inverse image functor f^* to $f^!$, an inverse image with proper support functor in the derived category of abelian sheaves. We also have an analogue of this statement. As one would expect, our theorem implies Poincaré duality for equivariant manifolds (see [**8**]). It may perhaps be more surprising that it also includes other results of equivariant stable homotopy theory, namely the Wirthmüller and Adams isomorphisms [**8**].

We will work with maps f that are what we call equivariant smooth families of manifolds. Essentially, a G-equivariant map $f : X \to Y$ is an equivariant smooth family if it is an equivariant bundle whose fiber is a smooth compact manifold, and actions of subgroups of G on the fiber vary smoothly over the base space Y in a suitable sense (See Definition 4.2).

It turns out that in our case, instead of directly describing the direct image with proper support functor $f_!$, it is easier to define a left adjoint $f_\sharp$ to the inverse image functor f^*, and identify $f_!$ with $f_\sharp$ up to a shift by the *dualizing object* associated to the equivariant smooth family $f : X \to Y$. The dualizing object is a spectrum parametrized over X, which is invertible under the smash product in the homotopy category. A main part of the content of the theorem is to identify this dualizing object as the stable tangent bundle of X in the parametrized category over Y.

Another ingredient on which the meaning of our theorem depends heavily is the closed model structure on the categories of parametrized G-spaces and G-spectra. The duality theorem takes place in the homotopy category associated with the model structure on parametrized G-spectra. In Chapter 3, we give definitions of the model structures in detail. An important aspect of the model structure on parametrized spaces is that a G-space Z parametrized over X is fibrant if and only if the structure map $Z \to X$ is a fibration in the standard model structure on G-equivariant spaces (*i.e.* for fibrations, use Serre fibrations on H-fixed point sets for all closed subgroups H of G). A similar statement holds for parametrized G-spectra. Thus, one can think of the homotopy categories of parametrized G-spaces and spectra as dealing with objects that are in some sense bundle-like over the base space. (In particular, it does not capture objects such as skyscraper sheaves.)

We will show that the Wirthmüller and Adams isomorphisms are special instances of our duality theorem. Recall from [**8**] Theorem II.6.2 that for a (closed) subgroup H of G, the Wirthmüller isomorphism is that for an H-equivariant spectrum E

$$G \ltimes_H \Sigma^{-L} E \simeq F_H[G, E) \tag{0.2}$$

in the homotopy category of G-equivariant spectra. The two sides of the equivalence are the left and right adjoints to the forgetful functor from G-spectra to H-spectra, and the H-representation L is the tangent space of G/H at eH, with H-action by translation. If H is a normal subgroup of G, then the Adams isomorphism ([**8**] Theorem II.7.1) states that for an H-free G-spectrum E indexed on the H-fixed points $\mathcal{U}^H$ of a complete G-universe $\mathcal{U}$,

$$E/H \simeq (i_* E \wedge S^{-A})^H \tag{0.3}$$

in the homotopy category of G/H-equivariant spectra. Here, the two sides are the left and right adjoints to the functor from G/H-spectra to G-spectra that takes a G/H-spectrum to be an H-fixed G-spectrum. An H-free G-spectrum is a G-spectrum which has a cellular approximation, such that every cell is H-free, *i.e.* of the form $G/N_+ \wedge S^n$, where N is a subgroup of G such that $N \cap H = \{e\}$. The functor i_* from G-spectra indexed on $\mathcal{U}^H$ to G-spectra indexed on $\mathcal{U}$ is the universe change functor associated to the inclusion of universes $i : \mathcal{U}^H \to \mathcal{U}$ (see [**8**] Section II.1). Also, A is the adjoint representation of G, *i.e.* the tangent space of H at e, with G-action by conjugation.

The statement (0.2) of Wirthmüller isomorphism translates to the case of our duality theorem for the equivariant smooth family $f : G/H \to *$, via an equivalence of categories between H-equivariant spectra and G-equivariant spectra parametrized over G/H. The case of the Adams isomorphism is more complicated. The equivariant smooth family to which the duality theorem applies is the quotient map $f : E\mathcal{F} \to E\mathcal{F}/H$, where $E\mathcal{F}$ is the universal contractible H-free G-space, and $E\mathcal{F}/H$ its orbit space by H ([**8**] Section II.2). The closed model structures give an equivalence of homotopy categories between H-free G-spectra and G-spectra parametrized over $E\mathcal{F}$. Via this equivalence and composition with certain other functors, the duality theorem gives (0.3).

The organization of the paper is as follows. In Chapter 1, we give a formulation of Verdier duality from the theory of sheaves, to give motivations for bringing in the language of sheaves. The next two chapters give the foundations on G-equivariant spaces and spectra over a base space that we need for the main theorem. Namely, in Chapter 2, we recall the definitions of G-equivariant spaces and spectra over a base space X, and show that they are equivalent to the categories of sheaves on $G\mathrm{Top}/X$. We also give certain basic constructions such as the smash product, and define the base change functors, which are associated with a map $f : X \to Y$ of base spaces. Chapter 3 gives a self-contained definition of the closed model structures on the categories of G-spaces and spectra parametrized over X.

In Chapter 4, we state the main theorem of the paper, given in terms of equivalences between base change functors in the stable homotopy categories, up to a shift by a certain dualizing object, for a class of "good" maps $f : X \to Y$. This class of maps is the class of smooth families, which are G-equivariant bundles whose fibers are smooth manifolds. We also define the dualizing object, and prove some preliminary results towards proving the main theorem. The main part of the proof of the theorem is given in Chapter 5. For a smooth family $f : X \to Y$ where Y is compact, we define natural transformations between the base change functors on the level of spaces, which turn out to be homotopy inverses. Stabilizing gives the theorem in the case of a compact Y, and the general case is obtained via a colimit argument. In Chapter 6, we show that both the Wirthmüller and the Adams isomorphisms are examples of the main duality theorem. Finally, in Chapter 7, we give the proofs of some technical results on the closed model structure for G-spectra parametrized over X.

CHAPTER 1

MOTIVATION

We begin by recalling the classical statements of duality in the theory of sheaves (see for instance [**2, 6**]). Let X, Y be schemes, with a suitable topology, *e.g.* etale, Nisnevich, analytic, etc., and let $\mathcal{A}$ be a tensor category. Let $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ denote the categories of sheaves on X and Y into $\mathcal{A}$, respectively. For a map $f : X \to Y$ of schemes, there are various functors associated with f between the categories $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$, defined in the standard theory of sheaves. Specifically, there is the pullback or inverse image functor

$$f^* : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X).$$

Its right adjoint is the direct image functor

$$f_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y).$$

In addition to the pair of adjoints (f^*, f_*), we also have the direct image "with proper support"

$$f_! : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$$

whose right adjoint is

$$f^! : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X).$$

One way to phrase the statements of duality is as follows. Consider $D(\mathrm{Sh}(X))$ and $D(\mathrm{Sh}(Y))$, the derived categories of chain complexes of sheaves into $\mathcal{A}$ on X and Y. Let C_Y denote the constant sheaf on Y into the unit object of $\mathcal{A}$. Then in these derived categories:

(1) If f is a smooth map of schemes, then for any $Z \in \mathrm{Sh}(Y)$,

$$f^!(Z) \simeq f^*(Z) \otimes f^!(C_Y). \tag{1.1}$$

(2) If f is a proper map of schemes, then for any $T \in \mathrm{Sh}(X)$,

$$f_!(T) \simeq f_*(T). \tag{1.2}$$

The sheaf $f^!(C_Y)$ over X is called the dualizing object associated with f.

Our purpose is to replace the abelian category $\mathcal{A}$ by the equivariant stable homotopy category, and give general conditions for analogous statements to hold in topology.

CHAPTER 2

SPACES AND SPECTRA OVER A BASE SPACE

Let G be a compact Lie group, and let X be a compactly generated weak Hausdorff G-space. For simplicity, denote by $G\mathrm{Top}$ the category of compactly generated weak Hausdorff G-spaces and continuous G-maps (called $G\mathcal{U}$ in [**8**] Section I.1). Consider the comma category $G\mathrm{Top}\,/X$, an object of which is a compactly generated weak Hausdorff G-space Z, together with a given G-map $p : Z \to X$. The morphisms of $G\mathrm{Top}\,/X$ are continuous G-maps that commute with the maps to X. We can give $G\mathrm{Top}\,/X$ the structure of a Grothendieck site, by defining the coverings to be given by all colimits. Namely, if $\mathcal{I}$ is a diagram in $G\mathrm{Top}\,/X$, and Z is an object, such that $Z \cong \mathrm{colim}_{\mathcal{I}}$, then $\mathcal{I}$ is a covering diagram of Z. Let $\mathrm{Sh}(G\mathrm{Top}\,/X)$ be the category of sheaves of sets over $G\mathrm{Top}\,/X$ with this topology. It is however not a small site. So a sheaf of sets over this Grothendieck site is a contravariant functor $F : G\mathrm{Top}\,/X \to \mathrm{Sets}$, which takes all colimits to inverse limits. By Freyd's adjoint functor theorem, modulo set-theoretical difficulties, such a functor has a left adjoint $L : \mathrm{Sets} \to (G\mathrm{Top}\,/X)^{\mathrm{op}}$. In particular, F is represented by the object $L(*)$, in the sense that for any $Z \in G\mathrm{Top}(X)$,

$$\begin{aligned} F(Z) &\cong \mathrm{Hom}_{\mathrm{Sets}}(*, F(Z)) \\ &\cong \mathrm{Hom}_{(G\mathrm{Top}\,/X)^{\mathrm{op}}}(L(*), Z) \\ &= \mathrm{Hom}_{G\mathrm{Top}\,/X}(Z, L(*)). \end{aligned}$$

Conversely, for any $T \in G\mathrm{Top}\,/X$, the contravariant functor

$$Z \longmapsto \mathrm{Hom}_{G\mathrm{Top}\,/X}(Z, T)$$

takes all colimits to inverse limits, so representable presheaves on $G\mathrm{Top}\,/X$ are sheaves. Hence, our definition of $\mathrm{Sh}(G\mathrm{Top}\,/X)$ with respect to this topology is just $G\mathrm{Top}\,/X$ itself. Thus, in discussing sheaves of sets on the site $G\mathrm{Top}\,/X$, we are just considering the parametrized, or fiberwise homotopy theory of G-spaces over X.

The based version of the above also holds: recall that a based G-space over X is a G-space Z with maps $p : Z \to X$ and $i : X \to Z$, such that $p \cdot i = \mathrm{Id}_X$. The constant sheaf C_X of sets on $G\mathrm{Top}/X$, given by $C_X(Z) = *$ for every $Z \in G\mathrm{Top}/X$, is represented by $X \in G\mathrm{Top}/X$. So by arguments similar as above, the category $G\mathrm{Top}_\bullet/X$ of based G-spaces over X is naturally equivalent to the category of sheaves F of sets over $G\mathrm{Top}/X$, together with a morphism of sheaves $C_X \to F$. This is also equivalent to the category of sheaves of based sets over $G\mathrm{Top}/X$. Therefore, we can work with parametrized homotopy theory over X. In particular, a sheaf of spectra on $G\mathrm{Top}/X$ is a spectrum parametrized over X, where suspensions and loops are done in $G\mathrm{Top}_\bullet(X)$.

In a sense, this is the simplest example of a category of sheaves. However, we will find the language of sheaves and their standard functors, closely analogous to the case of derived abelian sheaves, helpful even in this basic case. It seems that a large part of this paper might apply to more advanced categories of sheaves. For instance, Voevodsky's category of algebraic spaces behaves in many ways similar to topological spaces, but algebraic spaces are defined as Nisnevich sheaves over schemes, which is the reason behind many of their properties [**12**].

We recall certain basic constructions in the category of based G-spaces over X. For an unbased G-space Z over X, we write Z_+ for $Z \amalg X$, which is a based G-space over X, where the basepoint maps into the disjoint copy of X by the identity. If Z, T are unbased G-spaces over X, and $j : Z \to T$ is a map over X, then their quotient $T/_X Z$ over X is a based G-space over X defined by the following pushout diagram:

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "p_Z"'] & T \arrow[d] \\ X \arrow[r] & T/_X Z \end{tikzcd}$$

where p_Z is the structure map of Z. Also, if Z, T are based G-spaces over X, with basepoints $i_Z : X \to Z$, $i_T : X \to T$, then the wedge sum of Z and T over X is a based G-space over X defined by the following pushout diagram:

$$\begin{tikzcd} X \arrow[r, "i_Z"] \arrow[d, "i_T"'] & Z \arrow[d] \\ T \arrow[r] & Z \vee_X T. \end{tikzcd}$$

The G-space Z maps to itself by the identity and to T by $Z \to X \xrightarrow{i_T} T$, so we have a map $Z \to Z \times_X T$ over X. Likewise, T maps to $Z \times_X T$ over X. This gives a map $Z \vee_X T \to Z \times_X T$. The smash product of Z and T over X is

$$Z \wedge_X T = (Z \times_X T)/_X(Z \vee_X T).$$

The 0-dimensional sphere over X is $S^0_X = X_+ = X \amalg X$. It is the unit object in the category of based G-spaces over X with respect to the smash product. Finally, by [**7**], if the structure map $Z \to X$ is open, then the functor $Z \wedge_X -$ has a right adjoint functor $\underline{\mathrm{Hom}}_X(Z, -)$. For a G-space T over X,

$$\underline{\mathrm{Hom}}_X(Z, T) = \amalg_{x \in X} \underline{\mathrm{Hom}}(Z_x, T_x)$$

as a set. Here, Z_x and T_x are the fibers of Z and T over x respectively, and $\underline{\mathrm{Hom}}(Z_x, T_x)$ is the set of nonequivariant maps from Z_x to T_x. The group G acts on the set of partial maps that make up $\underline{\mathrm{Hom}}_X(Z, T)$ by conjugation.

Recall that for a compact Lie group G, a G-universe is an infinite-dimensional G-representation $\mathcal{U}$ which contains the trivial representation, and if V is a finite-dimensional subrepresentation of $\mathcal{U}$, then $\mathcal{U}$ contains infinitely many copies of V. A G-universe is said to be complete if it contains every irreducible representation of G, and it is said to be trivial if it is a direct sum of infinitely many copies of the trivial representation. Let $\mathcal{U}$ be a G-universe. A parametrized G-prespectrum E over X is a collection $\{E_V\}$ of based G-spaces over X, together with structure maps over X

$$(X \times S^{W-V}) \wedge_X E_V \longrightarrow E_W$$

for all finite-dimensional representations $V \subset W$ in $\mathcal{U}$. Here, $W - V$ denotes the orthogonal complement of V in W, and $X \times S^{W-V}$ is a based G-space over X via the first projection map and the basepoint of S^{W-V}. Since the map

$$X \times S^V \longrightarrow X$$

is open for each V, the functor $\Sigma_X^V = (X \times S^V) \wedge_X -$ has a right adjoint

$$\Omega_X^V = \underline{\mathrm{Hom}}_X(X \times S^V, -).$$

A prespectrum E over X is a spectrum over X if for every pair of finite-dimensional representations $V \subset W$ in $\mathcal{U}$, the adjoint structure map

$$E_V \longrightarrow \Omega_X^{W-V} E_W \tag{2.1}$$

is a homeomorphism over X. Similarly as for prespectra and spectra over a point, there is a spectrification functor L from prespectra over X to spectra over X, which is the left adjoint to the forgetful functor (see [**8**], Section I.2). In particular, a prespectrum D over X is an *inclusion prespectrum* over X if for every pair of finite-dimensional representations $V \subset W$ in $\mathcal{U}$, the adjoint structure map 2.1 is an inclusion map. When D is an inclusion prespectrum over X, its spectrification LD is given by

$$(LD)_V = \mathrm{colim}_{W \subset \mathcal{U}} \Omega_X^{W-V} D_W$$

for each finite-dimensional representation V in $\mathcal{U}$, where the colimit is taken over the finite-dimensional representations W in $\mathcal{U}$ containing V.

In particular, for each $x \in X$, let $G_x \subseteq G$ be the isotropy subgroup of x. If E is a G-spectrum parametrized over X, then for each $x \in X$, we have $E_V \xrightarrow{\cong} \Omega_X^{W-V} E_W$ for all finite-dimensional $V \subseteq W$ in $\mathcal{U}$, so

$$(E_V)_x \xrightarrow{\cong} (\Omega_X^{W-V}(E_W))_x.$$

But

$$\Omega_X^{W-V} E_W = \underline{\mathrm{Hom}}_X(X \times S^{W-V}, E_W) = \amalg_{x\in X} \underline{\mathrm{Hom}}(S^{W-V}, (E_W)_x)$$

so $(\Omega_X^{W-V} E_W)_x = \Omega^{W-V}(E_W)_x$. Thus, the fibers $\{(E_V)_x\}$ form a G_x-spectrum in the classical sense.

Let $f : X \to Y$ be an open map. Then the base change functors can be described simply in the fiberwise context. For the inverse image functor f^*, we have that

$$f^*(Z) = X \times_Y Z$$

for a based G-space Z over Y, and the basepoint of $f^*(Z)$ is the pullback along f of the basepoint of Z. The right adjoint of f^* is

$$f_* : \mathrm{Top}_\bullet /X \longrightarrow \mathrm{Top}_\bullet /Y.$$

For $T \in \mathrm{Top}_\bullet /X$, $f_*(T)$ is the G-space of sections from X to T, fiberwise over the points of Y. Namely, consider X as a G-space over Y via f. Recall from [**7**] that by the openness of f, $X \times_Y -$, as a functor from G-spaces over Y to itself, has a right adjoint $\underline{\mathrm{Hom}}_Y(X, -)$, which is the space of nonequivariant partial sections from the fibers of X over Y. For Z a space over Y, $\underline{\mathrm{Hom}}_Y(X, Z) = \amalg_{y\in Y} \mathrm{Hom}(X_y, Z_y)$ as a set, but with an appropriate topology, where X_y and Z_y are the fibers over y in X and Z respectively. The G-action on this space is induced by the conjugation of G on the partial sections from X_y to Z_y. There is a map $\alpha : Y \to \underline{\mathrm{Hom}}_Y(X, X)$, which is adjoint to the identity on X. For a G-space T over X, with structure map $p : T \to X$, we can think of T as a G-space over Y by $f \cdot p$. Then $f_*(T)$ is defined by the following pullback square in the category of G-spaces over Y.

$$\begin{tikzcd}
f_*(T) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\
\underline{\mathrm{Hom}}_Y(X,T) \arrow[r, "\underline{\mathrm{Hom}}_Y(X,p)"'] & \underline{\mathrm{Hom}}_Y(X,X).
\end{tikzcd}$$

Thus, we have that $f_*(T) = \amalg_{y\in Y} \mathrm{Sec}(X_y, T_y)$ with an appropriate topology, where $\mathrm{Sec}(X_y, T_y) \subseteq \mathrm{Hom}(X_y, T_y)$ are the sections of $p|_{T_y} : T_y \to X_y$. If $i : X \to T$ is the basepoint of T, then there is a natural basepoint $Y \to f_*(T)$, which takes each $y \in Y$ to $i|_{X_y} : X_y \to T_y$. From now on, we always assume that $f : X \to Y$ is an open G-map.

Rather than $f_!$ and $f^!$, it is more natural in this case to define $f_\sharp$, the left adjoint to f^*. In the unbased case, $f_\sharp$ is just the forgetful functor, *i.e.* for an unbased G-space T over X with structure map $p : T \to X$, $f_\sharp(T)$ is T thought of as a G-space over Y via $f \cdot p$. In the based category, $f_\sharp$ is given by collapsing the basepoint. Namely, if $i : X \to T$ is the basepoint of T over X, then $f_\sharp(T)$ is given by the following pushout diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & T \arrow[d] \\ Y \arrow[r] & f_\sharp(T). \end{tikzcd}$$

We also have the stable versions of these functors. For a spectrum D over Y, $f^*(D)$ is obtained by applying f^* on the category of based G-spaces to each space of D. Likewise, for a G-spectrum E over X, $f_*(E)$ is obtained by applying f_* spacewise. For $f_\sharp(E)$, one first applies $f_\sharp$ to each space of E to get a prespectrum over Y, then apply the spectrification functor from the category of prespectra over Y to the category of spectra over Y (see [**8**], Section I.2).

We record the following lemma.

Lemma 2.2. — *Let $f : X \to Y$ be a map of G-spaces, and $i_K : K \to Y$ an inclusion (not necessarily open). Consider the pullback diagram*

$$\begin{tikzcd} f^{-1}(K) \arrow[r, "i"] \arrow[d, "f_K"'] & X \arrow[d, "f"] \\ K \arrow[r, "i_K"'] & Y. \end{tikzcd}$$

Then for a spectrum E over X, we have natural isomorphisms

$$i_K^* f_* E \cong f_{K*} i^* E$$

and

$$i_K^* f_\sharp E \cong f_{K\sharp} i^* E.$$

Proof. — Let Z be a based G-space over X. We have that $f_*(Z)$ is

$$\amalg_{y\in Y} \operatorname{Sec}(X_y, Z_y)$$

with an appropriate topology. Then

$$i_K^* f_*(Z) = K \times_Y (\amalg_{y\in Y} \operatorname{Sec}(X_y, Z_y)) = \amalg_{y\in K} \operatorname{Sec}(X_y, Z_y)$$

as a subspace of $f_*(Z)$ (with an appropriate topology), whereas

$$\begin{aligned} f_{K*} i^*(Z) &= f_{K*}(f^{-1}(K) \times_X Z) \\ &= \amalg_{y\in K} \operatorname{Sec}(f^{-1}(K)_y, (f^{-1}(K) \times_X Z)_y) \\ &= \amalg_{y\in K} \operatorname{Sec}(X_y, Z_y). \end{aligned}$$

Hence, the first statement holds on the level of based G-spaces. Thus, it holds for spectra over X as well, since the functors i_K^*, f_*, f_{K*} and i^* on spectra are all defined by just applying the corresponding functors spacewise.

For the second statement, again consider a based G-space Z over X. Then $f_\sharp(Z)$ is defined by the pushout square

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & f_\sharp(Z). \end{array}$$

Since i_K^* is a left adjoint, it commutes with pushouts, so

$$i_K^* f_\sharp(Z) = (Z \times_Y K)/_{Z \times_Y Y}(Z \times_Y X) = (i^*(Z))/_Z f^{-1}(Z)$$

which is just $f_{K\sharp} i^*(Z)$. Hence, the second statement holds on the level of G-spaces. For a spectrum E, $f_\sharp(E)$ is defined by first applying $f_\sharp$ on each space of E, then applying the spectrification functor L. Consider the diagram of categories

$$\begin{array}{ccc} \text{Spectra}/K & \xrightarrow{\text{Forget}} & \text{Prespectra}/K \\ {\scriptstyle (i_k)_*}\downarrow & & \downarrow{\scriptstyle (i_K)_*} \\ \text{Spectra}/X & \xrightarrow[\text{Forget}]{} & \text{Prespectra}/X. \end{array}$$

This diagram commutes since $(i_K)_*$ on spectra is just applied spacewise. Hence, the left adjoints i_K^* and L commute. Therefore, the functors $i_K^* f_\sharp$ and $f_{K\sharp} i^*$ on spectra are obtained by first applying $i_K^* f_\sharp$ and $f_{K\sharp} i^*$ spacewise, then applying L to both sides. Hence, the second statement also holds on the level of spectra. $\square$

To define smash products of spectra over X, we need to give some consideration to change of universe functors for spectra over X. For G-universes $\mathcal{U}$ and $\mathcal{V}$, let $\mathcal{I}(\mathcal{U}, \mathcal{V})$ be the space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$, not necessarily G-equivariant. Then G acts on $\mathcal{I}(\mathcal{U}, \mathcal{V})$ by conjugation. A G-linear isometry over X is an X-point of the space of linear isometries $\mathcal{I}(\mathcal{U}, \mathcal{V})$, *i.e.* a G-map

$$a : X \longrightarrow \mathcal{I}(\mathcal{U}, \mathcal{V}).$$

Equivalently, it is a G-map over X

$$a : X \longrightarrow X \times \mathcal{I}(\mathcal{U}, \mathcal{V})$$

where the target is a G-space over X via the first projection. By abuse of notation, we use a for both formulations. So for every point $x \in X$, $a(x)$ is a G_x-fixed point of $\mathcal{I}(\mathcal{U}, \mathcal{V})$, where G_x denotes the isotropy subgroup of x. This is the same as a G_x-equivariant linear isometry from $\mathcal{U}$ to $\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are G_x-universes by forgetting the G-actions on them to G_x. For such an a, we define the universe change functors with respect to a

$$a^* : G\text{-Spectra}/X \text{ on } \mathcal{V} \longrightarrow G\text{-Spectra}/X \text{ on } \mathcal{U}$$
$$a_* : G\text{-Spectra}/X \text{ on } \mathcal{U} \longrightarrow G\text{-Spectra}/X \text{ on } \mathcal{V}.$$

These functors will have the property that for G-spectra E over X indexed on $\mathcal{U}$, E' over X indexed on $\mathcal{V}$, and any $x \in X$, we have

$$(a_*E)_x = a(x)_*(E_x) \tag{2.3}$$

and

$$(a^*E')_x = a(x)^*(E'_x) \tag{2.4}$$

as G_x-equivariant spectra. On the right hand side, $a(x)_*$ and $a(x)^*$ are the universe change functors with respect to the G_x-linear isometry $a(x) : \mathcal{U} \to \mathcal{V}$.

To define the universe change functors, we use methods analogous to those of [**4**]. We first consider the case where X is compact. By adjunction, $a : X \to \mathcal{I}(\mathcal{U}, \mathcal{V})$ can also be written as a G-map over X

$$\overline{a} : X \times \mathcal{U} \longrightarrow X \times \mathcal{V}.$$

Let $U \subset \mathcal{U}$ be a finite-dimensional G-representation. Since X is compact, there is some finite-dimensional G-representation V in $\mathcal{V}$, such that $\overline{a}(X \times U) \subseteq X \times V$, *i.e.* a gives an embedding of bundles over X from $X \times U$ into $X \times V$. Let $\nu_{U,V}$ be the orthogonal complement of $\overline{a}(X \times U)$ in $X \times V$, and let $S(\nu_{U,V})$ be the sphere bundle of this bundle over X, which is a based G-space over X. For any finite-dimensional G-representation U in $\mathcal{U}$, suppose that $V \subseteq W$ are finite-dimensional G-representations contained in $\mathcal{V}$, and $\overline{a}(X \times U) \subseteq X \times V \subseteq X \times W$, then we have

$$\Sigma_X^{W-V} S(\nu_{U,V}) \xrightarrow{\cong} S(\nu_{U,W}).$$

Hence, $\{S(\nu_{U,V})\}$ form a G-prespectrum over X indexed on $\mathcal{V}$. Let $\mathcal{M}_U$ be the spectrification of this prespectrum. In particular, $\mathcal{M}_U \cong \Sigma_V^\infty S(\nu_{U,V})$ canonically for every finite-dimensional $V \subset \mathcal{V}$ such that $\overline{a}(X \times U) \subset (X \times V)$. If $U \subseteq U'$ are finite-dimensional G-representations in $\mathcal{U}$, then there is a canonical isomorphism of G-spectra over X indexed on V

$$\Sigma_X^{U'-U} \mathcal{M}_{U'} \xrightarrow{\cong} \mathcal{M}_U. \tag{2.5}$$

In (2.5), $\Sigma_X^{U'-U}$ denotes smashing with $S_X^{U'-U}$. Namely, for each finite-dimensional representation V in $\mathcal{V}$, such that $\overline{a}(X \times U) \subseteq \overline{a}(X \times U') \subseteq X \times V$, we map

$$\Sigma_X^{U'-U} S(\nu_{U',V}) = (X \times S^{U'-U}) \wedge_X S(\nu_{U',V}) \longrightarrow S(\nu_{U,V})$$

by applying the map $\overline{a}$ to $X \times S^{U'-U}$. Since $\overline{a}(X \times U) \oplus \overline{a}(X \times (U'-U)) = \overline{a}(X \times U')$, this is an isomorphism for every such V. Therefore, (2.5) is an isomorphism (see also [**4**], Appendix, Section 2). Hence, for a G-spectrum E over X indexed on $\mathcal{U}$, define

$$a_*E = \operatorname{colim}_{U \subset \mathcal{U}} E_U \wedge_X \mathcal{M}_U.$$

Here, the colimit ranges over all finite-dimensional G-representations U contained in $\mathcal{U}$: for $U \subseteq U'$ in $\mathcal{U}$, the map is

$$E_U \wedge_X \mathcal{M}_U \cong E_U \wedge_X \Sigma_X^{U'-U} \mathcal{M}_{U'} \cong \Sigma_X^{U'-U} E_U \wedge_X \mathcal{M}_{U'} \longrightarrow E_{U'} \wedge_X \mathcal{M}_{U'}.$$

For its right adjoint, we define for a G-spectrum E' over X indexed on $\mathcal{V}$

$$(a^*E')_U = \mathrm{Hom}_X(\mathcal{M}_U, E')$$

for any finite-dimensional G-representation U that is contained in $\mathcal{U}$, where $\mathrm{Hom}_X(\mathcal{M}_U, E')$ denotes the based G-space over X of (nonequivariant) morphisms of spectra over X indexed on $\mathcal{V}$. (This is defined similarly as $\underline{\mathrm{Hom}}_X(-,-)$ of G-spaces over X: as a set, it is the disjoint union of the maps on the fibers over all $x \in X$, with an appropriate topology and G-action by conjugation.) In particular, for every finite-dimensional representation V contained in V such that $\overline{a}(X \times U) \subseteq X \times V$, we have a canonical isomorphism

$$(a^*E')_U \cong \underline{\mathrm{Hom}}_X(S(\nu_{U,V}), E'_V)$$

by adjunction. For finite-dimensional representations $U \subseteq U'$ in $\mathcal{U}$, choose a finite-dimensional representation $V \subset \mathcal{V}$ such that

$$\overline{a}(X \times U) \subseteq \overline{a}(X \times U') \subseteq X \times V.$$

We define the structure isomorphism of a^*E' to be

$$\begin{aligned}(a^*E')_U &\cong \underline{\mathrm{Hom}}_X(S(\nu_{V,U}), E'_V)\\ &\cong \underline{\mathrm{Hom}}_X((X \times S^{U'-U}) \wedge_X S(\nu_{U',V}), E'_V)\\ &\cong \Omega_X^{U'-U}\underline{\mathrm{Hom}}_X(S(\nu_{U',V}), E'_V)\\ &\cong \Omega_X^{U'-U}(a^*E')_{U'}.\end{aligned} \tag{2.6}$$

It is easy to check that this is independent of the choice of V, via the structure maps of E', and that a_* is the left adjoint of a^*.

We need to check that the functors satisfy conditions (2.3) and (2.4). Let $x \in X$. For (2.4), let U and V be finite-dimensional representations in $\mathcal{U}$ and $\mathcal{V}$ respectively, with $\overline{a}(X \times U) \subseteq X \times V$. Then we have canonical isomorphisms

$$((a^*E')_U)_x \cong (\underline{\mathrm{Hom}}_X(S(\nu_{U,V}), E'_V))_x = \underline{\mathrm{Hom}}_*((S(\nu_{U,V}))_x, (E'_V)_x).$$

But the fiber over x of $S(\nu_{U,V})$ is $S^{V-a(x)(U)}$, so this is

$$\Omega^{V-a(x)(U)}(E'_x)_V \cong (E'_x)_{a(x)(U)}.$$

This gives that the fiber of a^*E' over x is $a(x)^*(E'_x)$. For (2.3), note that for each $x \in X$, finite-dimensional $U \subset \mathcal{U}$, and finite-dimensional $V \subset \mathcal{V}$ such that $\overline{a}(X \times U) \subseteq X \times V$, we have

$$(\mathcal{M}_U)_x \cong (\Sigma_V^\infty S(\nu_{U,V}))_x = \Sigma_V^\infty S^{V-a(x)(U)}$$

canonically. So for a G-spectrum E over X indexed on $\mathcal{U}$,

$$\begin{aligned}(a_*E)_x &= (\mathrm{colim}_{U\subset\mathcal{U}}\mathcal{M}_U \wedge_X E_U)_x\\ &\cong \mathrm{colim}_{U\subset\mathcal{U}}((\mathcal{M}_U)_x \wedge (E_x)_U)\\ &\cong \mathrm{colim}_{U\subset\mathcal{U},\ a(X\times U)\subseteq X\times V}(\Sigma_V^\infty S^{V-a(x)(U)} \wedge (E_x)_U).\end{aligned}$$

In the last line, the colimit ranges over all finite-dimensional representations $U \subset \mathcal{U}$ and $V \subset \mathcal{V}$ such that $\overline{a}(X \times U) \subseteq X \times V$. On the other hand, $a(x)_*E$ is obtained by applying the spectrification functor on the universe $\mathcal{V}$ to the inclusion G-prespectrum indexed on $\mathcal{V}$ whose V-th space is

$$\Sigma^{V-a(x)(a(x)^{-1}(V))} E_{a(x)^{-1}(V)}.$$

The spectrification functor takes colimits over finite-dimensional representations $V \subseteq \mathcal{V}$, so comparing the colimits, we see that the two are canonically isomorphic.

For general X, we will glue a^* and a_* over a covering of X by compact subspaces. Given $a : X \to \mathcal{I}(\mathcal{U}, \mathcal{V})$, for any compact G-subspace $K \subseteq X$, we get $a|_K : K \to \mathcal{I}(\mathcal{U}, \mathcal{V})$. We will show that the functors $(a|_K)_*$ and $(a|_K)^*$ are natural with respect to K. Suppose $K \subseteq K'$ are compact G-subspaces of X. For finite-dimensional G-representation U in $\mathcal{U}$, let $\mathcal{M}_U(K)$ and $\mathcal{M}_U(K')$ be the spectra respectively over K and K' indexed on $\mathcal{V}$ constructed above. Then for large enough finite-dimensional G-representation V in $\mathcal{V}$, we have that $\overline{a}(K' \times U) \subseteq K' \times V$, so $\overline{a}(K \times U) \subseteq K \times V$ as well. In particular, the orthogonal complement of $\overline{a}(K \times U)$ in $K \times V$ is the restriction to K of the orthogonal complement of $\overline{a}(K' \times U)$ in $K' \times V$ over K'. Hence, after taking spectrifications over K and over K', we get a canonical map of G-spaces

$$(\mathcal{M}_U(K))_V \longrightarrow (\mathcal{M}_U(K'))_V$$

over the inclusion $K \to K'$ for any finite-dimensional G-representation V in $\mathcal{V}$. We define

$$(\mathcal{M}_U(X))_V = \mathrm{colim}_{K \subseteq X \text{ compact}} (\mathcal{M}_U(K))_V$$

over all compact G-subspaces K of X and their inclusions. Then $\mathcal{M}_U(X)$ is a G-spectrum over X indexed on $\mathcal{V}$, and for all $U \subseteq U'$ in $\mathcal{U}$, there is a canonical isomorphism

$$\Sigma_X^{U'-U} \mathcal{M}_{U'}(X) \cong \mathcal{M}_U(X).$$

Therefore, we can define for a G-spectrum E over X indexed on $\mathcal{U}$

$$a_*E = \mathrm{colim}_{U \in \mathcal{U}} E_U \wedge \mathcal{M}_U(X).$$

For a G-spectrum E' over X indexed on $\mathcal{V}$, define

$$(a^*E')_U = \mathrm{Hom}(\mathcal{M}_U(X), E')$$

for every finite-dimensional G-representation U contained in $\mathcal{U}$. Equivalently, let $i_K : K \to X$ be the inclusion of K in X for each compact subspace K of X. Then

$$a_*E = \mathrm{colim}_{K \subseteq X} (a|_K)_*(i_K^* E)$$

and

$$a^*E' = \lim_{K \subseteq X} (a|_K)^*(i_K^* E').$$

It is again straightforward to check that (a_*, a^*) form a pair of adjoint functors. The fact that they satisfy conditions (2.3) and (2.4) follows from the compact case.

Example 2.7. — Suppose that the G-linear isometry $a : X \to \mathcal{I}(\mathcal{U}, \mathcal{V})$ is a constant map into a point $a \in \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then for a G-spectrum E' over X indexed on $\mathcal{V}$, the G-spectrum over X a^*E' indexed on $\mathcal{U}$ is just given by

$$(a^*E')_U = E'_{a(U)} \tag{2.8}$$

for each finite-dimensional representation $U \subset \mathcal{U}$. Similarly, if E is a G-spectrum over X indexed on $\mathcal{U}$, then a_*E is the spectrification of the G-prespectrum $a_*^{\mathrm{pre}}E$ over X indexed on $\mathcal{V}$ given by

$$(a_*^{\mathrm{pre}}E)_V = \Sigma_X^{V - a(a^{-1}(V))} E_{a^{-1}(V)} \tag{2.9}$$

for each finite-dimensional representation V contained in $\mathcal{V}$.

More generally, suppose A is a G-space over X such that the structure map $A \to X$ is open, and

$$\alpha : A \longrightarrow X \times \mathcal{I}(\mathcal{U}, \mathcal{V})$$

is a G-map over X. Equivalently, this is just any G-map $A \to \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then we can define the twisted half-smash product

$$A \ltimes_\alpha - : G\text{-spectra over } X \text{ on } \mathcal{U} \longrightarrow G\text{-spectra over } X \text{ on } \mathcal{V}$$

and its right adjoint, the twisted function spectrum functor

$$F_\alpha[A, -) : G\text{-spectra over } X \text{ on } \mathcal{V} \longrightarrow G\text{-spectra over } X \text{ on } \mathcal{U}.$$

For each point $x \in X$, consider the map $\alpha_x : A_x \to \mathcal{I}(\mathcal{U}, \mathcal{V})_x$ on the fibers over x. This is equivariant with respect to the isotropy subgroup G_x of x. For a G-spectrum E over X indexed on $\mathcal{U}^H$, the functor $A \ltimes_\alpha -$ will have the property that

$$(A \ltimes_\alpha E)_x = A_x \ltimes_{\alpha_x} E_x \tag{2.10}$$

where the right hand side is the twisted half-smash product of G_x-spectra defined in [**8**], Chapter VI, and [**4**]. Likewise, for a G-spectrum E' over X indexed on $\mathcal{U}$, the twisted function spectrum functor will have the property that

$$F_\alpha[A, E')_x = F_{\alpha_x}[A_x, E'_x). \tag{2.11}$$

If $A = X$, then $\alpha : A \to \mathcal{I}(\mathcal{U}, \mathcal{V})$ is a G-linear isometry over X, and we will have that $A \ltimes_\alpha - = \alpha_*$, $F_\alpha[A, -) = \alpha^*$.

The construction of $A \ltimes_\alpha -$ and $F_\alpha[A, -)$ are similar to that of [**5**]. Let $p_A : A \to X$ be the structure map of A over X. The map $\alpha : A \to X \times \mathcal{I}(\mathcal{U}, \mathcal{V})$ is equivalent to any G-map $A \to \mathcal{I}(\mathcal{U}, \mathcal{V})$, so it also corresponds to a G-map over A

$$\alpha : A \longrightarrow A \times \mathcal{I}(\mathcal{U}, \mathcal{V}).$$

Note that by an abuse of notation, we will denote this map also by α. Then our definition is as follows.

Definition 2.12. — For $\alpha : A \to X \times \mathcal{I}(\mathcal{U}, \mathcal{V})$, as above, define the functor $A \ltimes_\alpha -$ to be the composition

$$\begin{aligned} A \ltimes_\alpha - : \text{Spectra}/X \text{ on } \mathcal{U} &\xrightarrow{(p_A)^*} \text{Spectra}/A \text{ on } \mathcal{U} \\ &\xrightarrow{\alpha_*} \text{Spectra}/A \text{ on } \mathcal{V} \\ &\xrightarrow{(p_A)_\sharp} \text{Spectra}/X \text{ on } \mathcal{V}. \end{aligned}$$

Here, $(p_A)_\sharp$ and $(p_A)^*$ are the base change functors with respect to $p_A : A \to X$, and α_* is the universe change functor of spectra over A with respect of α thought of as a G-map over A. Similarly, define the functor $F_\alpha[A, -)$ to be the composition

$$\begin{aligned} F_\alpha[A, -) : \text{Spectra}/X \text{ on } \mathcal{V} &\xrightarrow{(p_A)^*} \text{Spectra}/A \text{ on } \mathcal{V} \\ &\xrightarrow{\alpha^*} \text{Spectra}/A \text{ on } \mathcal{U} \\ &\xrightarrow{(p_A)_*} \text{Spectra}/X \text{ on } \mathcal{U}. \end{aligned}$$

From the definitions, it is clear that when $A = X$, the twisted half-smash product and the twisted function spectrum functors are just the change of universe functors. The proofs of (2.10) and (2.11) are similar as for (2.3) and (2.4).

Also, the twisted half-smash product and twisted function spectrum are functorial with respect to A in the following sense. Suppose that A and B are spaces over X, with open structure maps $p_A : A \to X$ and $p_B : B \to X$ respectively. Also, let $g : A \to B$ be a (not necessarily open) G-map over X. Suppose that $\alpha_B : B \to \mathcal{I}(\mathcal{U}, \mathcal{V})$ is any G-map, and let $\alpha_A = \alpha_B \cdot g : A \to \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then we claim that there are natural transformations

$$\begin{gathered} g \ltimes - : A \ltimes_{\alpha_A} - \longrightarrow B \ltimes_{\alpha_B} - \\ F[g, -) : F_{\alpha_B}[B, -) \longrightarrow F_{\alpha_A}[A, -) \end{gathered}$$

that are compatible with respect to compositions of G-maps over X. For the first statement, note that $p_A = p_B \cdot g$. So we have for a G-spectrum E over X indexed on $\mathcal{U}$,

$$A \ltimes_{\alpha_A} = (p_A)_\sharp (\alpha_A)_* (p_A)^* E = (p_B)_\sharp g_\sharp (\alpha_A)_* g^* (p_B)^* E.$$

It is straightforward to check that the diagram of functors

$$\begin{tikzcd} \text{Spectra}/B \text{ on } \mathcal{U} \arrow[r, "(\alpha_B)_*"] \arrow[d, "g^*"'] & \text{Spectra}/B \text{ on } \mathcal{V} \arrow[d, "g^*"] \\ \text{Spectra}/A \text{ on } \mathcal{U} \arrow[r, "(\alpha_A)_*"'] & \text{Spectra}/A \text{ on } \mathcal{V} \end{tikzcd}$$

commutes up to natural isomorphism. Hence, we get a canonical map

$$g \ltimes E : A \ltimes_{\alpha_A} E \cong (p_B)_\sharp g_\sharp g^*(\alpha_B)_*(p_B)^* E \xrightarrow{c} (p_B)_\sharp(\alpha_B)_*(p_B)^* E = B \ltimes_{\alpha_B} E.$$

Here, the map c is the counit of the adjunction pair $(g_\sharp, g^*)$. The map $F[g, -)$ on the twisted function spectra follows by adjunction. More specifically, for any G-spectrum E over X indexed on $\mathcal{U}$, and G-spectrum E' over X indexed on $\mathcal{V}$, there is a canonical map of morphism sets of spectra

$$\begin{aligned}\mathrm{Hom}_{\text{Spectra on } \mathcal{U}}(E, F_{\alpha_B}[B, E')) &\cong \mathrm{Hom}_{\text{Spectra on } \mathcal{V}}(B \ltimes_{\alpha_B} E, E') \\ &\longrightarrow \mathrm{Hom}_{\text{Spectra on } \mathcal{V}}(A \ltimes_{\alpha_A} E, E') \\ &\cong \mathrm{Hom}_{\text{Spectra on } \mathcal{U}}(E, F_{\alpha_A}[A, E'))\end{aligned}$$

where the middle map is induced by $g \ltimes E$. Setting $E = F_{\alpha_B}[B, E')$ and starting with the identity map on $F_{\alpha_B}[B, E')$ gives $F[g, E')$ in

$$\mathrm{Hom}_{\text{Spectra on } \mathcal{U}}(F_{\alpha_B}[B, E'), F_{\alpha_A}[A, E')).$$

Now for G-spectra E and E' over X indexed on $\mathcal{U}$, we can define the external smash product $E \overline{\wedge} E'$ as a G-spectrum over X indexed on $\mathcal{U} \oplus \mathcal{U}$. Namely, for finite-dimensional representations V and V' in $\mathcal{U}$, we define

$$(E \overline{\wedge} E')_{V \oplus V'} = E_V \wedge E'_{V'}.$$

Choose a G-linear isometry $a : X \to \mathcal{I}(\mathcal{U} \oplus \mathcal{U}, \mathcal{U})$ over X. Then define the internal smash product of E and E' to be

$$E \wedge E' = a_*(E \overline{\wedge} E').$$

Let $\mathcal{L}(n) = \mathcal{I}(\mathcal{U}^{\oplus n}, \mathcal{U})$. In [**8**], Lemma II.1.5, it is shown that $\mathcal{L}$ is a contractible G-equivariant operad. Hence, the internal smash product of G-spectra over X is well-defined up to coherent homotopies.

Similarly, given a spectrum E indexed on $\mathcal{U}$ and a spectrum E'' indexed on $\mathcal{U} \oplus \mathcal{U}$, we can define the external function spectrum $\overline{F}(E, E'')$, which is a spectrum indexed on $\mathcal{U}$. Namely, for a finite-dimensional representation V contained in $\mathcal{U}$, we have that

$$\overline{F}(E, E'')_V = \mathrm{Hom}_{(\mathcal{U} \oplus \mathcal{U})\text{-spectra}}(\Sigma^\infty S^V \overline{\wedge} E, E'')$$

where $\mathrm{Hom}_{(\mathcal{U} \oplus \mathcal{U})\text{-spectra}}(-, -)$ denotes the G-space of maps in the category of spectra indexed on $\mathcal{U} \oplus \mathcal{U}$. Then for spectra E and E' indexed on $\mathcal{U}$, the (internal) function spectrum $F(E, E')$ is defined as

$$F(E, E') = \overline{F}(E, a^* E')$$

for a linear isometry $a : X \to \mathcal{I}(\mathcal{U} \oplus \mathcal{U}, \mathcal{U})$.

The proof of the following lemma is similar to the case of G-spectra over a point ([**8**], II.3.12). For a finite-dimensional G-representation V, we let Σ^∞_V denote the V-th shift desuspension of the suspension spectrum functor, and let $\Sigma^{-V}_{\text{shift}}$ denote the shift desuspension spectrum functor, similar to those defined in [**8**], Section I.4.

Lemma 2.13. — *If E is a G-spectrum over X indexed on $\mathcal{U}$, and Z is a based G-space over X, then for any finite-dimensional G-representation V contained in $\mathcal{U}$, there is a natural homotopy equivalence*

$$E \wedge_X \Sigma_V^\infty Z \simeq \Sigma_{\mathrm{shift}}^{-V}(E \wedge_X Z).$$

Here, the right hand side is the smash product of a spectrum with a space, which has a canonical definition, and the left hand side is the smash product of spectra indexed on $\mathcal{U}$, using any linear isometry $X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$ over X.

CHAPTER 3

CLOSED MODEL STRUCTURE ON SPECTRA OVER A BASE

The model category structure on spectra parametrized over a base G-space X is defined in [**11**], similarly as for unparametrized spectra (for basic definitions on model categories, see [**3**], see also [**4**]). In this chapter, we give a self-contained description of the model structure. First, we define the model structure on the category of G-spaces over X. We begin by recalling the model structure on the category of G-spaces. A map $f : X \to Y$ of G-spaces is a weak equivalence in the category of G-spaces if for every closed subgroup H of G, the map $f^H : X^H \to Y^H$ is a weak equivalence nonequivariantly. The map f is a fibration if f^H is a Serre fibration for every H. It is a cofibration if it is a retract of relative G-cell complexes, which are obtained by attaching cells of the form $G/H \times D^{n+1}$ along $G/H \times S^n$. In particular, the acyclic cofibrations are retracts of deformation retracts obtained by attaching cells of the form $(G/H \times D^{n+1}) \times I$ along $G/H \times D_{n+1}$.

Recall that a map of nonequivariant spaces is a (Serre) fibration if and only if it has the right lifting property with respect to the inclusions $S^n \to D^{n+1}$ for all n. So the G-equivariant map $f : X \to Y$ is a fibration if and only if for all subgroups H in G, the dotted arrow exists for all squares of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X^H \arrow[d, "f^H"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & Y^H \end{tikzcd}$$

in the category of nonequivariant spaces. The functor $(-)^H$ from the category of H-spaces to nonequivariant spaces has a left adjoint, which is regarding a nonequivariant space as a fixed H-space. So this is equivalent to having the dotted arrow in all squares of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

in the category of H-spaces, for all closed subgroups H of G, where S^n and D^{n+1} are regarded as fixed H-spaces. Also, $f : X \to Y$ is a map of H-spaces by forgetting the actions on X and Y from G to H. The forgetful functor from G-spaces to H-spaces has a left adjoint $G/H \times -$, so the above diagram is in turn equivalent to

$$\begin{tikzcd} G/H \times S^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ G/H \times D^{n+1} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in the category of G-spaces. Hence, the map f is a fibration of G-spaces if and only if it has the right lifting property with respect to all inclusions of the form $G/H \times S^n \to G/H \times D^{n+1}$, for all subgroups H of G.

For based G-spaces over X, we define the model category structure as follows.

Definition 3.1. — A map $g : T \to Z$ of based G-spaces over X is a cofibration/weak equivalence/fibration if it belongs to the corresponding class of maps in the category of unbased G-spaces.

It is easy to see that this gives the structure of a model category to $G\mathrm{Top}_\bullet /X$. We include the following result for motivation and future reference.

Lemma 3.2. — *For $f : X \to Y$, suppose Z_1, Z_2 are fibrant G-spaces over Y, and $g : Z_1 \to Z_2$ is a weak equivalence over Y. Then*

$$f^*g : f^*(Z_1) \longrightarrow f^*(Z_2)$$

is a weak equivalence over X.

We defer the proof of Lemma 3.2 to Chapter 7.

For $f : X \to Y$, by Lemma 3.2, f^* preserves fibrations and weak equivalences between fibrant objects. Thus, the functors $(f_\sharp, f^*)$ form a pair of Quillen adjoint functors, so they pass to a pair of adjoint functors on the homotopy categories of based G-spaces over X and Y. If Y is a cell complex, and $f : X \to Y$ is a fiber bundle whose fiber is also a cell complex, then f^* preserves attachment of cells, so it preserves cofibrations and acyclic cofibrations in addition. In that case, the adjoint functors (f^*, f_*) are also a pair of Quillen adjoint functors.

We define the following model structure on the category of parametrized spectra over X. We say that a map $i : E \to E'$ of parametrized G-spectra over X is a relative G-cell complex if Y is obtained by attaching cells of the form $(\Sigma_V^\infty)_X((G/H \times D^{n+1}) \amalg X)$ to X along $(\Sigma_V^\infty)_X((G/H \times S^n) \amalg X)$ in the category over X. Here, $(\Sigma_V^\infty)_X$ denotes the V-th shift desuspension of the suspension spectrum in the category of spectra over X. As a based G-space over X, $(G/H \times D^{n+1}) \amalg X$ may have structure map induced by any map $G/H \times D^{n+1} \to X$, and $(G/H \times S^n) \amalg X$ is a based G-space over X via the restriction.

Definition 3.3. — Let $f : E \to E'$ be a map of parametrized G-spectra over X.

(1) f is a fibration if for every finite-dimensional $V \subset \mathcal{U}$, $f_V : E_V \to E'_V$ is a Serre fibration of G-spaces;

(2) f is a weak equivalence if for every finite-dimensional $V \subset \mathcal{U}$, $f_V : E_V \to E'_V$ is a weak equivalence of G-spaces;

(3) f is a cofibration if f is a retract of a relative G-cell complex.

The following proposition is an analogue of Proposition 6.9 of [**10**].

Proposition 3.4. — *The classes of cofibrations, weak equivalences and fibrations, as in Definition 3.3, define a closed model structure on the category of parametrized spectra over X.*

Proof. — We will first define cofibrations and weak equivalences of spectra over X as in Definition 3.3, and define a class of "R-fibrations" by the right lifting property with respects to all acyclic cofibrations, and show that this is a model structure by arguments similar to those of [**4**] and [**10**]. By definition, we have the lifting axiom for a square with an acyclic cofibration and an R-fibration. By the small object argument (see [**3**]), for any map $f : E \to E'$, we can attach cells of the form

$$(\Sigma_V^\infty)_X(G/H \times D^n) \amalg X \longrightarrow (\Sigma_V^\infty)_X(((G/H \times D^n) \amalg X) \wedge_X ((X \times I) \amalg X))$$

and factor f to a composition of an acyclic cofibration and an R-fibration. For the other factorization, we again use the small object argument and attach cells of the form

$$(\Sigma_V^\infty)_X(G/H \times S^n) \amalg X \longrightarrow (\Sigma_V^\infty)_X(G/H \times D^{n+1}) \amalg X.$$

This factors f into a composition of a cofibration and a map that has the right lifting property with respect to all cofibrations.

Now let $p : E \to B$ be any map that has the right lifting property with respect to all cofibrations. Then p is certainly an R-fibration. Also, for any V, the diagram of G-spectra over X

$$\begin{tikzcd}
(\Sigma_V^\infty)_X(G/H \times S^n) \amalg X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
(\Sigma_V^\infty)_X(G/H \times D^{n+1}) \amalg X \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

is equivalent to a diagram in the category of G-spaces

$$\begin{tikzcd}
(G/H \times S^n) \amalg X \arrow[r] \arrow[d] & E_V \arrow[d, "p_V"] \\
(G/H \times D^{n+1}) \amalg X \arrow[r] \arrow[ur, dashed] & B_V.
\end{tikzcd}$$

The dotted arrow exists in the diagram of spectra, so it exists in the diagram of G-spaces as well. Hence p_V is an acyclic fibration of G-spaces, and thus a weak equivalence of G-spaces, for each V. This gives that p is an acyclic R-fibration. The lifting axiom for a square with a cofibration and an acyclic R-fibration follows formally.

Now we will show that the class of R-fibrations is exactly the same as the class of fibrations as given in Definition 3.3. Let $f : E \to E'$ be a map of spectra over X. If f has the right lifting property with respect to all acyclic cofibrations over X, then for any finite-dimensional V in the universe, we can consider the lifting diagram of f with any map from the acyclic cofibration

$$(\Sigma_V^\infty)_X(G/H \times D^n) \amalg X \longrightarrow (\Sigma_V^\infty)_X((G/H \times D^n) \amalg X \wedge_X ((X \times I) \amalg X)).$$

Then by applying the adjunction between $(\Sigma_V^\infty)_X$ and taking the V-th space, we see that f_V is a fibration of G-spaces for every finite dimensional V in the universe. Conversely, if f is fibration of G-spaces on each finite-dimensional V, then each testing diagram of spectra of the form

$$\begin{tikzcd}
(\Sigma_V^\infty)_X(G/H \times D^n) \amalg X \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
(\Sigma_V^\infty)_X(G/H \times D^n) \amalg X \wedge_X ((X \times I) \amalg X) \arrow[r] \arrow[ur, dashed] & E'
\end{tikzcd} \tag{3.5}$$

is equivalent by adjunction to a diagram of G-spaces of the form

$$\begin{tikzcd}
(G/H \times D^n) \amalg X \arrow[r] \arrow[d] & E_V \arrow[d, "f_V"] \\
((G/H \times D^n) \amalg X) \wedge_X ((X \times I) \amalg X) \arrow[r] \arrow[ur, dashed] & E'_V.
\end{tikzcd} \tag{3.6}$$

The dotted arrow exists in the diagram of G-spaces, so it exists in the diagram of spectra as well. It is easy to check that it is automatically a map of spectra over X. Thus, a map of spectra over X has the right lifting property with respect to all relative cell complexes over X which are also weak equivalences if and only if it is a fibration on each space. This gives Proposition 3.4. □

We also have a parametrized version of the relative Whitehead theorem. We state it as a parametrized version of the HELP (homotopy extension and lifting property) lemma (Theorem I.5.9 of [**8**]).

Lemma 3.7. — *Let $j : E \to E'$ be a relative G-cell complex over X, and let $e : N \to P$ be a weak equivalence of fibrant spectra over X. We can write $E \wedge I_+$ as the parametrized spectrum $E \wedge_X ((X \times I) \amalg X)$ over X. Then if we have maps $f : E' \to P$, $g : E \to N$ and $h : E \wedge I_+ \to P$ over X such that $f \cdot j = h \cdot i_0$ and $e \cdot g = h \cdot i_1$, then there are maps $\overline{g} : E' \to N$ and $\overline{h} : E' \wedge I_+ \to P$ over X such that the following* *diagram commutes.*

$$
\begin{tikzcd}
E \arrow[rr, "i_0"] \arrow[dd, "j"'] & & E \wedge I_+ \arrow[dl, "h"'] \arrow[dd, "j"] & & E \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "j"] \\
& P & & N \arrow[ll, "e"'] & \\
E' \arrow[rr, "i_0"'] \arrow[ur, "f"] & & E' \wedge I_+ \arrow[ul, dashed, "\overline{h}"'] & & E' . \arrow[ll, "i_1"] \arrow[ul, dashed, "\overline{g}"']
\end{tikzcd}
\tag{3.8}
$$

Again, we will prove Lemma 3.7 in Chapter 7. The parametrized version of the Whitehead theorem follows from Lemma 3.7, similarly as in the classical case.

Proposition 3.9. — *Let X be a cell complex. If E is a relative cell spectrum over X, and $e : N \to P$ is a weak equivalence of fibrant spectra over X, then in the homotopy category of spectra over X, the induced map of morphism sets* $\mathrm{Hom}(E, N) \to \mathrm{Hom}(E, P)$ *is a bijection. In particular, a weak equivalence between relative cell spectra over X which are also fibrant over X is a homotopy equivalence.*

We will show that the model structure given by Definition 3.3 is equal to the model structure of [**11**], where the class of stable fiberwise weak equivalences of parametrized spectra over X is defined as follows. For each $x \in X$, let $G_x \subseteq G$ be the isotropy subgroup of x. Let E is a parametrized spectrum over X. For $x \in X$, the fibers $\{(E_V)_x\}$ over x form a G_x-spectrum in the classical sense. If E, E' are fibrant (in the sense of Definition 3.3) parametrized G-spectra over X, a map $f : E \to E'$ is said to be a *stable fiberwise weak equivalence* if for every $x \in X$, the map on fibers $f_x : E_x \to E'_x$ is a weak equivalence of G_x-spectra. For maps $f : E \to E'$ between general parametrized G-spectra over X, f is a stable fiberwise weak equivalence if it is a stable fiberwise weak equivalence after we apply fibrant replacement to E and E' in the model structure of Definition 3.3. Together with the same cofibrations as those given in Definition 3.3 and fibrations determined by the right lift property, this gives a closed model structure on parametrized spectra.

Proposition 3.10. — *A map $f : E \to E'$ of parametrized G-spectra over X is a stable fiberwise weak equivalence in the sense of* [**11**] *if and only if it is a weak equivalence in the sense of Definition 3.3.*

We will need the following lemma.

Lemma 3.11. — *Let $f : E \to E'$ be a map of parametrized G-spectra over X, which are fibrant in the sense of Definition 3.3. Then f is a stable fiberwise weak equivalence if and only if for every finite-dimensional $V \subset \mathcal{U}$ and $x \in X$, the map $(f_V)_x : (E_V)_x \to (E'_V)_x$ is a weak equivalences of based G_x-spaces.*

Proof. — For each $x \in X$, the fiber spectra E_x and E'_x are just taken spacewise, so it suffices to show that for any subgroup H of G, a map of H-spectra $g : D \to D'$ is a weak equivalence if and only if for every finite-dimensional $V \subset \mathcal{U}$, $g_V : D_V \to D'_V$ is a weak equivalence of based H-spaces. This follows from Theorem 3.4 of [**9**]. Now we apply it to the case where $H = G_x$, $D = E_x$ and $D' = E'_x$, for all $x \in X$. □

Given Lemma 3.11, to show that stable fiberwise weak equivalences are the same as weak equivalences of Definition 3.3, it suffices to work on the level of fibrant based G-spaces.

Lemma 3.12. — *Let $f : T \to Z$ be a map of fibrant parametrized G-spaces over X. Then f is a weak equivalence in the category of G-spaces if and only if for every $x \in X$, the map on fibers over x*

$$f_x : T_x \longrightarrow Z_x$$

is a weak equivalence in the category of G_x-equivariant spaces, where G_x is the isotropy subgroup of x.

Proof. — The map $f : T \to Z$ is a weak equivalence of G-spaces if and only if for every subgroup $H \subseteq G$, $f^H : T^H \to Z^H$ is a nonequivariant weak equivalence. Likewise, for $x \in X$, the map on fibers $f_x : T_x \to Z_x$ is a weak equivalence of G_x-spaces if and only if for every $H \subseteq G_x$, $(f_x)^H : (T_x)^H \to (Z_x)^H$ is a nonequivariant weak equivalence. But note that $(T_x)^H = (T^H)_x$, $(Z_x)^H = (Z^H)_x$, and

$$(f_x)^H = (f^H)_x : (T^H)_x \longrightarrow (Z^H)_x.$$

Also, for a pair $(x \in X,\ H \subseteq G)$, the condition that $H \subseteq G_x$ is equivalent to the condition that $x \in X^H$. So it suffices to show that each of the nonequivariant maps $f^H : T^H \to Z^H$ over X^H is a weak equivalence if and only if for every $x \in X^H$, $(f^H)_x : (T^H)_x \to (Z^H)_x$ is a nonequivariant weak equivalence. Note that as nonequivariant spaces, T^H and Z^H are fibrant over B^H. The fact that f^H is a weak equivalence implies that $(f^H)_x$ is a weak equivalence for all x follows from Lemma 3.2. The converse statement follows from standard arguments using the long exact sequence in homotopy groups. □

Lemmas 3.11 and 3.12 show that between parametrized G-spectra over X whose spaces are all fibrant, stable fiberwise weak equivalences are the same as weak equivalences given in Definition 3.3. Let Γ be the fibrant replacement functor with respect to the model structure given in Definition 3.3, For a general map $f : E \to E'$, the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f"'] & \Gamma E \arrow[d, "\Gamma f"] \\ E' \arrow[r] & \Gamma E' \end{tikzcd}$$

commutes. By the above argument, f is a stable fiberwise weak equivalence if and only if Γ_f is a weak equivalence on each space. But the maps $E \to \Gamma E$ and $E' \to \Gamma E'$ are weak equivalences on each space. So f is a stable fiberwise weak equivalence if and only if it is a weak equivalence on each space.

Similarly as in the case of G-spaces over X, for a map $f : X \to Y$, the functors $(f_\sharp, f^*)$ on the categories of G-spectra over X and Y form a Quillen adjoint pair. This is because the functor f^* for parametrized G-spectra is defined spacewise, and so are fibrations and weak equivalences of parametrized G-spectra. If Y is a G-cell complex, and f is a fiber bundle whose fiber is also a G-cell complex, then the functors (f^*, f_*) on the level of parametrized G-spectra also form a Quillen adjoint pair, again since f_* for parametrized G-spectra is defined spacewise, as are fibrations and acyclic fibrations of parametrized G-spectra.

We record the following lemma, whose proof we will defer to Chapter 7.

Lemma 3.13. — *Let $X \to Y$ be a map of G-spaces, and let T and Z be fibrant and cofibrant based G-spaces over X, with structure maps $p_T : T \to X$ and $p_Z : Z \to X$. Suppose $f : T \to Z$ is a map of based G-spaces over X, such that f forgets to a homotopy equivalence over Y. Then f is a homotopy equivalence over X.*

The proof of the following lemma is similar to that of Proposition 7.1 in the Appendix of [**4**].

Lemma 3.14. — *Suppose that X is a G-cell complex, and $a : X \to \mathcal{I}(\mathcal{U}, \mathcal{V})$ is a G-map. Then the adjoint functors (a_*, a^*) on the categories of G-spectra over X indexed on $\mathcal{U}$ and on $\mathcal{V}$ form a Quillen adjoint pair. Hence, they pass to an adjoint pair of functors on the homotopy categories of G-spectra over X indexed on $\mathcal{U}$ and on $\mathcal{V}$.*

CHAPTER 4

THE EQUIVARIANT DUALITY THEOREM

The goal of this chapter is to formulate and prove a general duality theorem that combines (1.1) and (1.2) in the context of equivariant topology. For a map $f : X \to Y$ of G-spaces, it is more natural in topology to define $f_\sharp$, the left adjoint to f^*, than $f^!$ and $f_!$ directly. So for an appropriate condition of smoothness of f, if we can define the dualizing object C_f of f, (1.1) states that we can define

$$f^! = f^*(-) \wedge C_f.$$

If in addition, we have that C_f is invertible in the stable homotopy category over X, then $- \wedge C_f$ is an invertible functor on the stable homotopy category over X, so we can define $f_!$ in the stable homotopy categories to be

$$f_! = f_\sharp(- \wedge C_f^{-1}).$$

Then $f_!$ is the left adjoint functor to $f^!$ in the stable homotopy categories. Thus, the appropriate statement of duality, which puts together (1.1) and (1.2), is that for the right class of maps $f : X \to Y$, with conditions that are analogous to smoothness and properness for schemes, we can define an invertible spectrum C_f over X, such that for a spectrum E over X

$$f_\sharp(E \wedge C_f^{-1}) \simeq f_*(E) \tag{4.1}$$

in the stable homotopy category over Y.

We now consider more precisely the conditions in the equivariant context in order for (4.1) to hold. Let G be a fixed compact Lie group. Let $f : X \to Y$ be a map of G-spaces. Assume that Y is a cofibrant G-space, *i.e.* a G-cell complex. If Y is a single point, then "smoothness" and "properness" say that X is a smooth compact G-manifold. This suggests that (4.1) should hold for a class of "families of manifolds", which are some kind of fiber bundles whose fiber is a compact smooth manifold. We give the following definition.

Let M be a compact C^∞-manifold, not necessarily with a G-action. Let $\mathcal{S} = \mathrm{Diff}(M)$ be the group of diffeomorphisms of M, with the C^1-topology (see [**13**]). We will define the universal equivariant smooth family with fiber M. Define the family $\mathcal{F}_{\mathrm{sm}}$ of subgroups of G "with smooth action on M", a member of which is a closed subgroup $K \subseteq G$, together with a group homomorphism $\theta : K \to \mathcal{S}$ such that K acts smoothly on M via θ. For $(K,\theta) \in \mathcal{F}_{\mathrm{sm}}$, we can also think of K as a subgroup in the cartesian product $G \times \mathcal{S}$ via $k \mapsto (k, \theta(k)) \in G \times \mathcal{S}$ for each $k \in K$. So equivalently, $\mathcal{F}_{\mathrm{sm}}$ is the collection of subgroups $K \subset G \times \mathcal{S}$, such that $K \cap \mathcal{S} = \{e\}$, and the second projection map $K \to \mathcal{S}$ gives a smooth action of K on M. For such a subgroup $K \subset G \times \mathcal{S}$, the first projection map $K \to G$ is injective, and makes K into a subgroup of G, and the smooth action θ of $K \subseteq G$ on M is induced by the section projection to $\mathcal{S}$. We can make $\mathcal{F}_{\mathrm{sm}}$ into a topological category by defining the morphisms to be subconjugations, similarly as in the definition of the orbit category of a compact Lie group (see [**8**], V.9). The topology on the object set of $\mathcal{F}_{\mathrm{sm}}$ is the discrete topology. There is a functor

$$\mathrm{Orb} : \mathcal{F}_{\mathrm{sm}} \longrightarrow (G \times \mathcal{S})\text{-spaces}$$

which sends an orbit $(G \times \mathcal{S})/(K,\theta)$ to itself as a $(G \times \mathcal{S})$-space. Then by taking the simplicial $(G\times\mathcal{S})$-space $B(*, \mathcal{F}_{\mathrm{sm}}, \mathrm{Orb})$, where B denotes the 2-sided bar construction, with a $(G\times\mathcal{S})$-action induced by the action of $G\times\mathcal{S}$ on the last coordinate, we obtain a universal $(G \times \mathcal{S})$-space $E\mathcal{F}_{\mathrm{sm}}$, which depends on G and M, although we suppress them in the notation here.

Recall that the 2-sided bar construction of a category, originally introduced by J.P. May, is defined as follows. Let $\mathcal{C}$ be a small category, $\mathcal{D} : \mathcal{C} \to \mathrm{Sets}$, $\mathcal{E} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be functors, then the simplicial set $B(\mathcal{E}, \mathcal{C}, \mathcal{D})$ (the 2-sided bar construction) is defined to have the n-th stage

$$\mathcal{E} \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \cdots \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathcal{D}$$

(with n copies of $\mathrm{Mor}(\mathcal{C})$). The degeneracies are provided by inserting the map $\mathrm{Id} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, and faces are given by structure maps

$$\begin{aligned} \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) &\longrightarrow \mathrm{Mor}(\mathcal{C}) && \text{(composition)} \\ \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathcal{D} &\longrightarrow \mathcal{D} && \text{(functoriality)} \\ \mathcal{C} \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) &\longrightarrow \mathcal{C} && \text{(functoriality)} \end{aligned}$$

where $\mathcal{D}$ is identified with $\amalg_{x\in\mathrm{Obj}(\mathcal{C})}\mathcal{D}(x)$, and similarly for $\mathcal{E}$. In our setting, we need the generalization of this concept to (equivariant) topological categories, which is standard when $\mathrm{Obj}(\mathcal{C})$ is discrete and $\mathcal{D}$, $\mathcal{E}$ are continuous functors.

The $(G \times \mathcal{S})$-space $E\mathcal{F}_{\mathrm{sm}}$ is not necessarily a $(G \times \mathcal{S})$-cell complex. However, we can factor out the action of $\mathcal{S}$, and apply G-cell approximation. So as a G-space, the G-cell approximation of $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$ is constructed from cells of the form

$$((G \times \mathcal{S}/(K,\theta))/\mathcal{S} \times S^n \cong G/K \times S^n$$

where $(K, \theta) \in \mathcal{F}_{\mathrm{sm}}$. We are using the fact that the Diff(M)-conjugacy classes of the pairs (K, θ) form a discrete set [**14**]. In particular, the G-orbits of $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$ are of the form G/K, where $(K, \theta) \in \mathcal{F}_{\mathrm{sm}}$. We define the universal equivariant smooth family with fiber M to be the G-equivariant map

$$\gamma(G, M) : E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} M \longrightarrow E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}.$$

Over each orbit $G/K \cong ((G \times \mathcal{S})/(K, \theta))/\mathcal{S}$ (this is a canonical isomorphism) of $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$, the fiber of $\gamma(G, M)$ is

$$(G \times \mathcal{S}/(K, \theta)) \times_{\mathcal{S}} M \cong G \times_{(K,\theta)} M.$$

So as a G-space, $E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} M$ is constructed as a colimit of strata of the form

$$(G \times_{(K,\theta)} M) \times S^n \cong (G/K \times S^n) \times_{(K,\theta)} M.$$

Definition 4.2. — We say that a G-equivariant map $f : X \to Y$, where Y is a G-cell complex with countably many cells, is an equivariant smooth family of manifolds if it is a pullback of the universal equivariant smooth family $\gamma(G, M)$ via some G-map $Y \to E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$, for a smooth compact manifold M.

Lemma 4.3. — *A smooth family of manifolds, as in Definition 4.2, is a fibration of G-spaces as defined above in Chapter 3.*

Proof. — Since fibrations are closed under pullbacks, it suffices to show the universal family of manifolds

$$\gamma(G, M) : E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} M \longrightarrow E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$$

is a G-equivariant fibration. Note that whether a map $f : X \to Y$ of G-spaces is a fibration in our sense can be tested by diagrams of the form

$$\begin{tikzcd} G/H \times D^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ G/H \times D^n \times I \arrow[r] & Y \end{tikzcd}$$

for all closed subgroups H of G. By the adjunction between $G/H \times -$ and the forgetful functor, this is equivalent to a square in the category of H-spaces

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \times I \arrow[r] & Y \end{tikzcd}$$

where D^n and $D^n \times I$ are fixed H-spaces. Then by the adjunction between giving a fixed action to a nonequivariant space and $(-)^H$, this is in turn equivalent to the square in the category of nonequivariant spaces

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & X^H \arrow[d] \\ D^n \times I \arrow[r] & Y^H. \end{tikzcd}$$

It follows that f is a fibration if and only if for each closed subgroup H of G, the map of H-fixed points $f^H : X^H \to Y^H$ is a fibration of nonequivariant spaces.

Now by arguments similar as above, we can consider $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$ as the simplicial G-space $B(*, \mathcal{F}_{\mathrm{sm}}, \mathrm{Orb}')$, where Orb' is a functor from the topological category $\mathcal{F}_{\mathrm{sm}}$ to the category of G-spaces, which takes an orbit $(G \times \mathcal{S})/(K, \theta)$ to G/K. Although G/K does not take into account the map $\theta : K \to \mathcal{S}$, a copy of it occurs for every smooth action of K on M $\theta : K \to \mathcal{S}$. Again, the G-action on $B(*, \mathcal{F}_{\mathrm{sm}}, \mathrm{Orb}')$ is induced by the action on G on the last coordinate Orb'. Then to show that the map

$$B(*, \mathcal{F}_{\mathrm{sm}}, \mathrm{Orb})^H \longrightarrow B(*, \mathcal{F}_{\mathrm{sm}}, \mathrm{Orb}')^H$$

is a nonequivariant fibration, consider the map for a given (K, θ)

$$((G \times \mathcal{S})/(K, \theta))^H = \mathrm{Orb}(G \times \mathcal{S}/(K, \theta))^H \longrightarrow \mathrm{Orb}'(G \times \mathcal{S}/(K, \theta))^H = (G/K)^H.$$

Recall that $(G/K)^H$ is equal to the space of maps $G/H \to G/K$ in the orbit category, so it is empty if H is not subconjugate to K in G. If H is subconjugate to K in G, then

$$(G/K)^H = \{gK \mid g^{-1}Hg \subseteq K\}.$$

For any such g, we have $(hg)^{-1}H(hg) = g^{-1}Hg \subseteq K$, so $hgK = gK$ in $(G/K)^H$ for any $h \in H$. We have a canonical embedding $H \subseteq G \subset G \times \mathcal{S}$, by embedding H into the first variable. Then $((G \times \mathcal{S})/(K, \theta))^H$ is empty if H is not subconjugate to (K, θ) in $G \times \mathcal{S}$, if we think of (K, θ) as a subgroup of $G \times \mathcal{S}$ via θ. For any $(g, s) \in G \times \mathcal{S}$, $(g, s)^{-1}H(g, s) = g^{-1}Hg \times \{e\}$ in $G \times \mathcal{S}$, where $e \in \mathcal{S}$ is the unit element. So (g, s) takes H into (K, θ) in $G \times \mathcal{S}$ if and only if $g^{-1}Hg \subseteq K$ in G, and $\theta|_{g^{-1}Hg}$ is the trivial map into $\mathcal{S}$. Thus, if H is subconjugate to (K, θ) in $G \times \mathcal{S}$, then

$$(4.4)\quad ((G \times \mathcal{S})/(K, \theta))^H = \{(g, s)(K, \theta) \mid g^{-1}Hg \subseteq K,\ \theta|_{g^{-1}Hg} = e : g^{-1}Hg \to \mathcal{S}\}.$$

Note that if $gK = g'K$ in $(G/K)^H$, and $\theta|_{g'^{-1}Hg'} = e$, then $g = g'k$ for some $k \in K$, and for any $h \in H$,

$$\theta(g^{-1}hg) = \theta(k^{-1}{g'}^{-1}hg'k) = \theta(k)^{-1}\theta({g'}^{-1}hg')\theta(k) = \theta(k)^{-1}e\theta(k) = e.$$

So $\theta|_{g^{-1}Hg} = e$. The map $((G \times \mathcal{S})/(K, \theta))^H \to (G/K)^H$ takes the class of (g, s) to the class of g. The fiber over the class of g in $(G/K)^H$ is thus either empty or $\mathcal{S}$. Furthermore, the condition of (4.4) clearly remains unchanged when K is replaced by K' with some $g'' \in G$ such that ${g''}^{-1}Kg'' \subseteq K'$.

Thus, we only need to show that for gK and $g'K$ in $(G/K)^H$, if the fiber over gK in $((G \times \mathcal{S})/(K, \theta))^H$ is nonempty and the fiber over $g'K$ is empty, then gK and $g'K$

are in different components of $(G/K)^H$. Suppose they are in the same component of $(G/K)^H$, then the path between gK and $g'K$ in $(G/H)^K$ breaks up into paths in G and multiplications by elements of K. So without loss of generality, we can assume there exists a path $\{g_t\}$ in G, such that for every $t \in I$, $g_t^{-1}Hg \subseteq K$, and $g_0 = g$, $g_1 = g'$. Further, $\theta|_{g^{-1}Hg}$ is the trivial map into $\mathcal{S}$, whereas $\theta|_{g'^{-1}Hg'} : g'^{-1}Hg' \to \mathcal{S}$ is a nontrivial map. So we can define a continuous path of smooth actions

$$\rho_t : H \longrightarrow \mathcal{S}$$

by

$$\rho_t(h) = \theta(g_t^{-1}hg_t).$$

Then $\rho_0 : H \to \mathcal{S}$ gives the trivial action of H on M, whereas $\rho_1 : H \to \mathcal{S}$ gives a nontrivial smooth action of H on M. However, recall that the smooth actions of the compact Lie group H on M, modulo conjugations by elements of $\mathcal{S} = \mathrm{Diff}(M)$, form a discrete set [**14**]. Since the trivial action of H on M is in its own conjugation class, it cannot be continuously deformed to any nontrivial action. This is a contradiction. □

Given a smooth family of manifolds $f : X \to Y$, we define the dualizing object C_f as follows. Let $\Delta : X \to X \times_Y X$ be the diagonal map. We think of $X \times_Y X$ as a G-space over X by π_1, projection to the first coordinate. Then Δ is a map over X. Then we would like to put

$$\text{“}C_f = X \times_Y X/_X(X \times_Y X \smallsetminus \Delta)\text{”}.$$

To make C_f Hausdorff, we need to replace $X \times_Y X \smallsetminus \Delta$ by $X \times_Y X \smallsetminus U$, where U is a G-equivariant tubular neighborhood of Δ in $X \times_Y X$. We require $\Delta \subset U$ to be a G-deformation retract over X via the first projection π_1. This exists when Y is compact. Then C_f is independent of the choice of U up to homotopy. Alternatively, we can define a model of C_f to be

$$(X \times_Y X) \amalg_{X \times_Y X \smallsetminus \Delta} C(X \times_Y X \smallsetminus \Delta)$$

i.e. attaching a cone onto $X \times_Y X \smallsetminus \Delta$, in the category of G-spaces over X, which gives a homotopy equivalent construction which is also canonical. In fact, when $Y = *$ and X is a G-equivariant smooth manifold, C_f is naturally equivalent to the sphere bundle of the tangent bundle τ_X of X.

We have the following essential fact about the dualizing object when f is a smooth equivariant family of manifolds as above.

Lemma 4.5. — *Let $\mathcal{U}$ be a complete G-universe. Let $f : X \to Y$ an equivariant smooth family of manifolds, where Y is compact. Then there exists a spectrum C_f^{-1} over X indexed on $\mathcal{U}$, such that $C_f \wedge_X C_f^{-1}$ is homotopy equivalent to the sphere spectrum S_X over X. (Here, C_f is thought of as a based G-space over X, so $C_f \wedge_X C_f^{-1}$ is well-defined on the point-set level category of spectra.) For general smooth family of* *manifolds $f : X \to Y$, $\Sigma^\infty_X C_f$ is invertible in the homotopy category of spectra over X indexed on $\mathcal{U}$, with respect to the model structure of Definition 3.3.*

Remark 4.6. — By formal arguments, the inverse C_f^{-1} of C_f in the stable homotopy category over X must be $DC_f = F(C_f, S^0)$, the Spanier-Whitehead dual of C_f. On the point-set level, the function spectrum functor depends on a linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$. Hence, for each such linear isometry, we get a model for C_f^{-1} in the point-set category of spectra over X.

We will postpone the proof of Lemma 4.5. First, note that the functor $f_\sharp$ has the following property. For any finite-dimensional representation $V \subset \mathcal{U}$, let Σ^V_{shift} and $\Sigma^{-V}_{\text{shift}}$ denote the shift suspension and desuspension functors by V on the category of spectra parametrized over a G-space Y.

Lemma 4.7. — *For any G-spectrum Z over Y, based G-space T over X, and finite-dimensional G-representation $V \subset \mathcal{U}$, there is a natural isomorphism*

$$\Sigma^{-V}_{\text{shift}}(Z \wedge_Y f_\sharp T) \xrightarrow{\cong} \Sigma^{-V}_{\text{shift}} f_\sharp(f^*(Z) \wedge_X T). \tag{4.8}$$

Proof. — Recall that the functor f^* is defined spacewise, and the functor $f_\sharp$ is defined by first taking $f_\sharp$ spacewise, then taking the spectrification functor L. We first consider the level of based G-spaces. Let Z be a based G-space over Y, and T a based G-space over X. Then

$$\begin{aligned} f_\sharp(f^*Z \wedge_X T) &= f_\sharp((X \times_Y Z) \wedge_X T) \\ &= f_\sharp((X \times_Y Z) \times_X T/_X(X \times_Y Z) \cup T) \\ &= (X \times_Y Z) \times_X T/_Y(X \times_Y Z) \cup T). \end{aligned}$$

Here, $(X \times_Y Z) \times_X T$ is a G-space over Y via

$$(X \times_Y Z) \times_X T \longrightarrow X \xrightarrow{f} Y.$$

On the other hand,

$$\begin{aligned} Z \wedge_Y f_\sharp(T) &= Z \wedge_Y (T/_Y X) \\ &= (Z \times_Y T)/_Y((Z \times_Y X) \cup T). \end{aligned}$$

Here, in the numerator $Z \times_Y T$, T is thought of as a G-space over Y via the structure map $T \to X \xrightarrow{f} Y$. There is a natural isomorphism over Y

$$(X \times_Y Z) \times_X T \longrightarrow Z \times_Y T$$

which takes $((x, z), t)$ to (z, t), whose inverse takes $(z, t) \in Z \times_Y T$ to $((p_T(t), z), t) \in (X \times_Y Z) \times_X T$. It induces a natural isomorphism

$$Z \wedge_Y f_\sharp(T) \xrightarrow{\cong} f_\sharp(f^*Z \wedge_X T).$$

Now for any spectrum Z over Y and finite-dimensional $W \subset \mathcal{U}$, the natural isomorphism over Y

$$Z_W \wedge_Y f_\sharp T \xrightarrow{\cong} f_\sharp(f^* Z_W \wedge_X T)$$

commutes with the adjoint prespectrum structure maps, so the prespectrum over Y consisting of the G-spaces $\{Z_W \wedge_Y f_\sharp T\}$ is isomorphic to the prespectrum over Y consisting of the G-spaces $\{f_\sharp(f^* Z_W \wedge_X T)\}$. Applying the spectrification functor L from prespectra over Y to spectra over Y to both sides gives the isomorphism of spectra

$$Z \wedge_Y f_\sharp(T) \xrightarrow{\cong} L\{f_\sharp(f^* Z_W \wedge_X T)\}.$$

Now the spectrum $f_\sharp(f^* Z \wedge_X T)$ is given by

$$f_\sharp(f^* Z \wedge_X T) = L(f_\sharp(L\{f^* Z_W \wedge_X T\})).$$

The diagram of functors

$$\begin{tikzcd}
\text{Spectra}/Y \arrow[r, "\text{Forget}"] \arrow[d, "f^*"'] & \text{Prepectra}/Y \arrow[d, "f^*"] \\
\text{Spectra}/X \arrow[r, "\text{Forget}"'] & \text{Prespectra}/X
\end{tikzcd}$$

commutes, since f^* is just defined spacewise on prespectra and on spectra. Thus, the left adjoints $f_\sharp$ and L commute, so $f_\sharp(f^* Z \wedge_X T)$ is isomorphic to $L\{f_\sharp(f^* Z_W \wedge_X T)\}$. Now applying the shift desuspension functor $\Sigma^{-V}_{\text{shift}}$ to the isomorphism gives the statement of the lemma. □

In particular, for the case where $T = S^0_X = X \amalg X$, $\mathrm{f}_\sharp(T) = X \amalg Y$, so for a spectrum Z over Y, Lemma 4.7 gives that $f_\sharp f^*(Z) \simeq (X \amalg Y) \wedge_Y Z$ naturally in the category of spectra over Y.

The following is our main duality theorem in the equivariant topological context.

Theorem 4.9. — *Let $f : X \to Y$ be a smooth family of manifolds, and $\mathcal{U}$ be a complete G-universe. Then in the category of spectra over Y indexed on $\mathcal{U}$, for any fibrant and cofibrant spectrum E over X, we have a natural weak equivalence in the category of spectra over Y indexed on $\mathcal{U}$*

$$f_*(E) \simeq f_\sharp(E \wedge_X C_f^{-1}).$$

Here, C_f^{-1} is given in Lemma 4.5, using any choice of linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$.

Remark 4.10. — This is how, in the present language, one arrives at the concept of equivariant orientations and Poincaré duality as given in [**8**], Section III.6. Assume for simplicity that Y is a point, and X is a smooth G-manifold, with $f : X \to *$. We write C for C_f. Let e be a cofibrant and fibrant G-spectrum which is a ring spectrum, and V a G-representation. Then an e-orientation of X in dimension V is a map

$$\eta : f_\sharp(C) \longrightarrow \Sigma^V e$$

satisfying the following condition. (Note that since $f_\sharp(C)$ is the Thom space of the tangent bundle τ_X of X, η is a V-dimensional class in the e-cohomology of the Thom space of τ_X.) We have that

$$f_\sharp(C \wedge_X f^*e) \simeq f_\sharp C \wedge e$$

naturally by Lemma 4.7. Define the composition

$$f_\sharp(C \wedge_X f^*e) \simeq f_\sharp C \wedge e \xrightarrow{\eta \wedge \mathrm{Id}_e} \Sigma^V e \wedge e \longrightarrow \Sigma^V e \tag{4.11}$$

where the last map is the ring structure on e. Our condition for η to be an e-orientation is that the adjoint map to (4.11)

$$C \wedge_X f^*e \longrightarrow f^*(\Sigma^V e) \tag{4.12}$$

be a weak equivalence of spectra over X.

A V-dimensional e-orientation η of X determines a Poincaré duality isomorphism as follows. Since $f_\sharp$ preserves weak equivalences between spectra over X that are cofibrant and fibrant, by Theorem 4.9 and the discussion after Lemma 4.19 below, so does f_*. Hence, applying f_* to (4.12) gives a weak equivalence of G-spectra

$$f_*(f^*(\Sigma^V e)) \simeq f_*(f^*e \wedge_X C).$$

By Theorem 4.9, the right hand side is weakly equivalent to $f_\sharp(f^*e)$. Thus, we have

$$f_*(f^*(\Sigma^V e)) \simeq f_\sharp f^*(e).$$

By Lemma 4.7, the right hand side is just $X_+ \wedge e$, whereas the right hand side is $F(X_+, \Sigma^V e)$, so we get a weak equivalence

$$F(X_+, \Sigma^V e) \simeq X_+ \wedge e.$$

To prove Theorem 4.9, we begin with the following lemmas.

Lemma 4.13. — *For a G-equivariant space X, let $g : E \to E'$ be a map of spectra over X. For a G-subspace $K \subset X$, let $i_K : K \to X$ be the inclusion map. If for a cover $\{K_r\}$ of X by G-subspaces, $i^*_{K_r} g : i^*_{K_r} E \to i^*_{K_r} E'$ is a weak equivalence of spectra over K, then g is a weak equivalence of spectra over X.*

Proof. — By Proposition 3.10, a map of G-spectra over X is a weak equivalence if and only if for every $x \in X$, the map on fibers f_x is a weak equivalence of G_x-spectra. The condition that $i^*_{K_r} g$ is a weak equivalence for a cover $\{K_r\}$ of X clearly implies this fiberwise condition. □

Lemma 4.14. — *Let $\mathcal{U}$ be a G-universe. Let E be a cofibrant and fibrant G-spectrum over X, such that for a cover $\{K_r\}$ of X by G-equivariant subspaces, with $i_{K_r} : K_r \to X$ the inclusion for each K_r, $i^*_{K_r}(E)$ is invertible in the stable homotopy category of spectra over K indexed on $\mathcal{U}$. Then E is invertible in the stable homotopy category over X indexed on $\mathcal{U}$.*

Proof. — Let $D_X E$ denote the Spanier-Whitehead dual of E in the category of spectra over X. Recall that the internal smash product of two spectra E and E' indexed on $\mathcal{U}$ is given by first taking the external smash product $E \overline{\wedge} E'$, which is a spectrum indexed on $\mathcal{U} \oplus \mathcal{U}$, then applying a change of universe functor coming from a chosen linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$. For cofibrant E and E', this is independent of the choice of linear isometry, up to weak equivalences with all coherences. Likewise, the linear isometry determines a point-set model for $D_X E = F(E, S^0_X)$, the Spanier-Whitehead dual of E in the category of spectra over X. Then E is invertible if and only if the evaluation map

$$\varepsilon_X : E \wedge_X D_X E \longrightarrow S^0_X$$

is an isomorphism in the homotopy category of G-spectra over X. Since i^* is a left adjoint for any $i = i_{K_r}$, it commutes with smash products of G-spaces. Also, for any given linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$, i^* commutes with the change of universe functor $a_* :$ Spectra on $\mathcal{U} \oplus \mathcal{U} \to$ Spectra on $\mathcal{U}$. Hence, if we define the internal smash product of spectra and the Spanier-Whitehead dual using a, then

$$i^*(E \wedge_X D_X E) \cong i^* E \wedge_{K_r} i^*(D_X E).$$

Similarly, i^* commutes with the external function spectrum functor and with a^*, so

$$i^*(D_X E) = i^* F_X(E, S^0_X) \cong F_K(i^* E, i^* S^0_X) = D_K i^* E$$

since $i^* S^0_X = S^0_K$. By the naturality of the evaluation map, we get the map over K_r

$$i^*_{K_r} \varepsilon_X = \varepsilon_: i^*_{K_r} E \wedge_{K_r} D_X(i^*_{K_r} E) \longrightarrow S^0_{K_r}.$$

This is a weak equivalence for every K_r. Thus, ε_X is itself a weak equivalence over X by Lemma 4.13. □

We now use Lemma 4.14 to prove Lemma 4.5.

Proof of Lemma 4.5. — We first assume that Y is compact. We will embed X in $V \times Y$ in the category of G-spaces over Y, for some finite dimensional G-representation V. The map f is given by the pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} M \arrow[d, "\gamma(G,M)"] \\ Y \arrow[r] & E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}. \end{tikzcd}$$

The image of Y in $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$ is contained in a finite G-cell subcomplex of $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$. Denote the preimage of this finite G-cell subcomplex in $E\mathcal{F}_{\mathrm{sm}}$ by Z. So it suffices to consider the pullback square

$$\begin{array}{ccc} X & \longrightarrow & Z \times_{\mathcal{S}} M \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z/\mathcal{S}. \end{array}$$

If we have an embedding $Z \times_{\mathcal{S}} M \to V \times Z/\mathcal{S}$ over $Z/\mathcal{S}$, applying $Y \times_{Z/\mathcal{S}} -$ gives an embedding over Y

$$X = Y \times_{Z/\mathcal{S}} (Z \times_{\mathcal{S}} M) \longrightarrow Y \times_{Z/\mathcal{S}} (V \times Z/\mathcal{S}) = V \times Y.$$

Thus, it suffices to consider the universal case, for $f = \gamma_Z(G, M) : Z \times_{\mathcal{S}} M \to Z/\mathcal{S}$.

We first consider an orbit G/K in $Z/\mathcal{S}$, where $K \subset G \times \mathcal{S}$, $K \cap \mathcal{S} = \{e\}$, and K acts smoothly on the fiber M. Again, we can equivalently consider K as a subgroup of G, with a smooth action $\theta : K \to \mathcal{S}$ on M. Also, the fiber over G/K in $Z \times_{\mathcal{S}} M$ is $G \times_K M$. We can embed M K-equivariantly into a K-representation $W_{(K,\theta)}$, and embed $W_{(K,\theta)}$ into a G-representation $V_{(K,\theta)}$. Then we get G-equivariant embeddings

$$(4.15) \qquad G \times_K M \xrightarrow{\subset} G \times_K W_{(K,\theta)} \xrightarrow{\subset} G \times_K V_{(K,\theta)} \cong G/K \times V_{(K,\theta)}.$$

Now using the $(G \times \mathcal{S})$-cell structure of Z, each $(G \times \mathcal{S})$-orbit $(G \times \mathcal{S})/K$ of Z has an open $(G \times \mathcal{S})$-invariant neighborhood U such that $(G \times \mathcal{S})/K$ is a $(G \times \mathcal{S})$-equivariant retract of U (this is a general property of equivariant cell complexes). Using the compactness of $Z/\mathcal{S}$, we can cover Z by finitely many such neighborhoods $U_1, \dots, U_n$. Now using the retractions, each $U_i \times_{\mathcal{S}} M$ embeds G-equivariantly into $U_i/\mathcal{S} \times V_i$ for some finite-dimensional G-representation V_i. Using a G-equivariant partition of unity, we can then embed $Z \times_{\mathcal{S}} M$ into $Z/\mathcal{S} \times (\oplus_{i=1}^n V_i)$.

Let S^V be the one-point compactification of $V = \oplus_{i=1}^n V_i$. It can be shown that $Z/\mathcal{S} \times S^V$ has a structure of a G-cell complex, $Z \times_{\mathcal{S}} M$ is a G-cell subcomplex, and is Spanier-Whitehead dual to its complement. So let ν be the normal bundle of the embedding of $Z \times_{\mathcal{S}} M$ into $Z/\mathcal{S} \times V$. Then the sphere bundle $S(\nu)$ of ν, when smashed with C_f over $Z \times_{\mathcal{S}} M$, is homotopy equivalent to the based space $(Z \times_{\mathcal{S}} M) \times S^V$ over $Z \times_{\mathcal{S}} M$. Since $(Z \times_{\mathcal{S}} M) \times S^V$ is invertible in the stable homotopy category over $Z \times_{\mathcal{S}} M$, so is C_f. Specifically, define

$$C_f^{-1} = \Sigma_{\mathrm{shift}}^{-V} \Sigma^\infty S(\nu)$$

where $\Sigma_{\mathrm{shift}}^{-V}$ is the V-th shift desuspension functor in the category of spectra over $Z \times_{\mathcal{S}} M$. Then $C_f \wedge_{Z \times_{\mathcal{S}} M} C_f^{-1} \simeq \Sigma_{\mathrm{shift}}^{-V}(C_f \wedge_{Z \times_{\mathcal{S}} M} \Sigma^\infty S(\nu))$, by Lemma 2.13, is naturally homotopy equivalent to $\Sigma_{\mathrm{shift}}^{-V} \Sigma^\infty((Z \times_{\mathcal{S}} M) \times S^V) = S^0_{Z \times_{\mathcal{S}} M}$.

This gives that C_f is invertible for a smooth family of manifolds $f : X \to Y$, where Y is compact. For a general smooth family $f : X \to Y$, let $K \to Y$ be a compact G-subspace, so $f|_K : f^{-1}K \to Y$, and let $i_K : f^{-1}(K) \to X$ be the inclusion. We claim that $i_K^*(C_f)$ is naturally homotopy equivalent to $C_{f|_{f^{-1}K}}$, the dualizing object with respect to $f|_{f^{-1}(K)} : f^{-1}(K) \to K$, as a spectrum over $f^{-1}(K)$. Thus, Lemma 4.14 and the compact case give the the invertibility of C_f, since the equivariant smooth family $f : X \to Y$ is surjective, so $\{X \times_Y K\}$ over all compact subspaces $K \subseteq Y$ gives a covering of X. To see the claim, recall that up to natural homotopy equivalences, C_f is the suspension spectrum of the G-space over X

$$(X \times_Y X)/_X(X \times_Y X \smallsetminus \Delta(X))$$

which means that one attaches a cone of $X \times_Y X \smallsetminus \Delta$ (in the category of G-spaces over X) on $X \times_Y X$, where $\Delta : X \to X \times_Y X$ is the diagonal. The functor $i^* = f^{-1}(K) \times_X -$ commutes with colimits, so $i_K^* C_f$ is obtained by attaching a cone of $f^{-1}(K) \times_X (X \times_Y X \smallsetminus \Delta(X))$ (in the category of G-spaces over $f^{-1}(K)$) on $f^{-1}(K) \times_X (X \times_Y X)$. We have the obvious isomorphism

$$f^{-1}(K) \times_X (X \times_Y X) \cong f^{-1}(K) \times_Y X.$$

But if $(x, x') \in f^{-1}(K) \times_Y X$, then $f(x') = f(x) \in K \subset Y$, so $x' \in f^{-1}(K)$, and so

$$f^{-1}(K) \times_X (X \times_Y X) \cong f^{-1}(K) \times_K f^{-1}(K)$$

as G-spaces over $f^{-1}(K)$. Similarly, the G-subspace $f^{-1}(K) \times_X (X \times_Y X \smallsetminus \Delta(X))$ is isomorphic to $f^{-1}(K) \times_K f^{-1}(K) \smallsetminus \Delta(f^{-1}(K))$ over $f^{-1}(K)$. Thus, $i_K^*(C_f)$ is obtained by attaching a cone of $f^{-1}(K) \times_K f^{-1}(K) \smallsetminus \Delta(f^{-1}(K))$ in the category of G-spaces over $f^{-1}(K)$ on $f^{-1}(K) \times_K f^{-1}(K)$, which is the definition of $C_{f|_{f^{-1}(K)}}$. □

We will need the following notion of bundle-like objects over a base space. As we will see, in some situations, it is better behaved than the notion of cofibrant and fibrant objects.

Definition 4.16. — Let X be a G-space. A G-space Z over X, with structure map $p : Z \to X$, is said to be a homotopy cell bundle over X if for every subgroup H of G, every point $x \in X^H$ has a nonequivariant open neighborhood U in X^H, and an H-space F_U with the homotopy type of an H-cell complex, such that

$$p^{-1}(U) \cong U \times F_U$$

as H-spaces over U. Here, U is thought of as a fixed H-space, and $U \times F_U$ is an H-space over U via the first projection.

In the based category, we say that a based G-space Z over X is a based homotopy cell bundle if the same condition is satisfied, but F_U is now a based H-space with the homotopy type of a based H-cell complex.

By adjunction, a nonequivariant map $U \to X^H$ is equivalent to an H-equivariant map $U \to X$, where U is thought of as a fixed H-space. In turn, this is equivalent to a G-equivariant map

$$G/H \times U \cong G \times_H U \longrightarrow X.$$

Likewise, a point in X^H corresponds to a G-orbit G/H in X. We have G-equivariant maps $G \times_H p^{-1}(U) \to Z$ and $G \times_H p : G \times_H p^{-1}(U) \to G/H \times U$, which agree when we map Z and $G/H \times U$ to X, so we get a G-equivariant map

$$G \times_H p^{-1}(U) \longrightarrow (G/H \times U) \times_X Z.$$

This takes a point $(g, z) \in G \times_H p^{-1}(U)$, where $z \in p^{-1}(U) \subseteq Z$, to $((g, p(z)), gz) \in (G/H \times U) \times_X Z$, where $(g, p(z)) \in G/H \times U$, and $gz \in Z$. It is routine to check that this map is in fact a G-equivariant isomorphism. Therefore, the square

$$\begin{tikzcd} G \times_H p^{-1}(U) \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ G/H \times U \arrow[r] & X \end{tikzcd}$$

is a pullback in the category of G-equivariant spaces. Hence, the condition of Definition 4.16 is equivalent to the condition that for every G-orbit G/H in X, there is some G-equivariant open neighborhood of the form $G/H \times U$ of G/H in X, such that

$$(G/H \times U) \times_X Z \cong G \times_H (F_U \times U)$$

as G-spaces over $G/H \times U$, where U is thought of as a fixed H-space.

In particular, by equivariant cell subdivision, a homotopy cell bundle Z over a G-cell complex X is both fibrant and cofibrant in the category of based G-spaces over X. Similarly as in the case of based spaces, we can also define homotopy cell bundle spectra over X.

Definition 4.17. — For a G-space X, a G-spectrum E over X indexed on a G-universe $\mathcal{U}$ is a homotopy cell bundle spectrum if for every subgroup H of G and $x \in X^H$, x has a nonequivariant open neighborhood U in X^H, and an H-spectrum E_U of the homotopy type of a cell H-spectrum (indexed on $\mathcal{U}$ which is thought of as an H-universe), such that for the inclusion $i : U \to X^H$, and $\pi : U \to *$,

$$i^*(E) \cong \pi^*(E_U)$$

as H-spectra over U. Here, U is thought of as a fixed H-space.

Again, this is equivalent to the condition that for every G-orbit G/H in X, there is a G-equivariant open neighborhood of the form $G/H \times U$ of G/H in X, such that if we write $i : G/H \times U \to X$ for the inclusion, and $\pi : G/H \times U \to *$, then

$$i^*(E) \cong G \times_H (\pi^* E_U)$$

as G-spectra over $G/H \times U$. If X is a G-cell complex, then a homotopy cell bundle spectrum E over X is cofibrant and fibrant in the category of G-spectra over X.

We have the following special case of homotopy cell bundles.

Lemma 4.18. — *Let Y be a G-cell complex, and $f : X \to Y$ be an equivariant smooth family of manifolds. Then X is a homotopy cell bundle over Y.*

Proof. — Let the fiber of the smooth family be the manifold M. It is easy to see that the condition that a map be a homotopy cell bundle is closed under pullbacks. So it suffices to consider the universal case $f : E\mathcal{F}_{\mathrm{sm}} \times_{\mathrm{Diff}(M)} M \to E\mathcal{F}_{\mathrm{sm}}/\operatorname{Diff}(M)$. Locally, if H is a subgroup of G, and $\theta : H \to \operatorname{Diff}(M)$ is a smooth action of H on M, then we have a cell $G/H \times D^n$ in $E\mathcal{F}_{\mathrm{sm}}/\operatorname{Diff}(M)$, and the map f over this cell is

$$(G \times_{(H,\theta)} M) \times D^n \longrightarrow G/H \times D^n.$$

By [**14**], for each H, the space of smooth H-actions on M is discrete after we take the orbit space of the action of $\operatorname{Diff}(M)$ by conjugation. Hence, for each θ giving a smooth H-action on M, let $\operatorname{Diff}(M)_\theta$ be the isotropy subgroup of θ in $\operatorname{Diff}(M)$ with respect to the conjugation action, *i.e.* $\operatorname{Diff}(M)_\theta$ is the subgroup of H-equivariant diffeomorphisms on M when H acts on M by θ. Then

$$\operatorname{Diff}(M) \longrightarrow \operatorname{Diff}(M)/\operatorname{Diff}(M)_\theta$$

is a fibration. The target of this map is the component of θ in the space of smooth H-actions on M. Hence, suppose we have a given H and an H-fixed point x in $Y = E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$, such that the fiber over x is M with a smooth H-action via θ. Then by taking a section of the fibration from $\operatorname{Diff}(M)$ to the space of smooth H-actions on M, we can find an open neighborhood U of x in Y^H, such that for every $y \in U$, the fiber over U is M with a smooth H-action by $\alpha_y \theta \alpha_y^{-1}$, where $\alpha_y \in \operatorname{Diff}(M)$ varies continuously with y. This allows us to define an H-equivariant isomorphism between $X \times_Y U$ and $U \times M$ over U, where M is an H-space via θ, by conjugating by α_y^{-1} on the fiber over y for each $y \in U$. □

We will also need the following result.

Lemma 4.19. — *Let X be a G-cell complex, $p : Z \to X$ a based homotopy cell bundle over X. Then for any cofibrant spectrum E over X indexed on a complete G-universe $\mathcal{U}$, $E \wedge_X Z$ is cofibrant.*

Proof. — A cofibrant spectrum E is the retract of a G-cell spectrum over X. Since $- \wedge_X Z$ preserves retracts, we can assume that E is a G-cell spectrum over X, *i.e.* it is constructed by attaching cells of the form $\Sigma_V^\infty(G/H \times D^n) \amalg X$ onto X, where V can be any finite-dimensional G-representation in $\mathcal{U}$, and $(G/H \times S^n) \amalg X$ can be a based space over X via any map $G/H \times S^n \to X$. Since cofibrancy is preserved by pushouts and directed colimits, by gluing the cells of E, it suffices to show that $E \wedge_X Z$ is cofibrant for the the case of $E = \Sigma_V^\infty((G/H \times D^n) \amalg X)$. In this case, $E \wedge_X Z$ is naturally homotopy equivalent to $\Sigma_V^\infty(((G/H \times D^n) \amalg X) \wedge_X Z)$. Hence, it suffices to show that if T is a based space over X, such that the basepoint $X \to T$ is a relative cell complex over X, then $T \wedge_X Z$ is cofibrant as a based space over X.

In this case, X is a G-cell complex, and T is a relative cell complex over X, so T is a G-cell complex as well. Let $i : X \to Z$ be the basepoint of Z, then i is also homotopy equivalent a relative cell complex. We have the pullback maps over X

$$
\begin{tikzcd}
T \arrow[r, "T \times_X i"] \arrow[d] & T \times_X Z \arrow[r, "T \times_X p"] \arrow[d] & T \arrow[d] \\
X \arrow[r, "i"'] & Z \arrow[r, "p"'] & X
\end{tikzcd}
\tag{4.20}
$$

where $T \times_X p$ is the pullback bundle of p, with the same fibers as $p : Z \to X$. By subdividing X and T to make the cells of X and the cells of T over X small enough, we can assume that the bundle $T \times_X p$ is of the form

$$(G \times_H F_C) \times D^n \longrightarrow G/H \times D^n$$

when restricted to each cell $C = G/H \times D^n$ of the G-cell complex T, where F_C, the fiber over C, is a based H-space of cell homotopy type. Since the fiber F_C has cell homotopy type, by equivariant cell subdivision, we can give a G-cell structure to $(G \times_H F_C) \times D^n = (G/H \times D^n) \times_X Z$, such that

$$(G/H \times D^n) \xrightarrow{(G/H \times D^n) \times_X i} (G \times_H F_C) \times D^n$$

is homotopy equivalent to a relative G-cell complex. After gluing together the cells of T, we get a G-cell structure on $T \times_X Z$, such that $T \times_X i : T \to T \times_X Z$ is a relative G-cell complex. We can do the cell subdivision over the cells of X and the cells of T over X separately, so after gluing the cells of T, $Z \to T \times_X Z$ is again a relative cell complex. Hence, the top map of the pushout diagram

$$
\begin{tikzcd}
T \vee_X Z \arrow[r] \arrow[d] & T \times_X Z \arrow[d] \\
X \arrow[r] & T \wedge_X Z
\end{tikzcd}
$$

is also a relative G-cell complex over X. Hence, the bottom map is a relative G-cell complex over X as well. But this is the basepoint of $T \wedge_X Z$, *i.e.* $T \wedge_X Z$ is a cofibrant based space over X. □

The case in which we are interested is the following. Let $f : X \to Y$ be a G-equivariant smooth family of manifolds, where Y is a compact G-cell complex. Consider the model of the dualizing object C_f given by

$$(X \times_Y X) \amalg_{X \times_Y X \setminus \Delta} C(X \times_Y X \setminus \Delta),$$

i.e. attaching a cone onto the complement of the diagonal Δ in $X \times_Y X$. Since $f : X \to Y$ is a G-bundle whose fiber is a cell complex, X is a G-cell complex by equivariant cell subdivision. So C_f is homotopy equivalent to the sphere bundle $S(\tau|_X)$ of the tangent bundle of X. Choose any linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$. As shown in the proof of Lemma 4.5, by the compactness of Y, the spectrum $C_f^{-1} = DC_f$ is homotopy equivalent to $\Sigma^{-V}_{\text{shift}} S(\nu|_X)$ for some finite-dimensional G-representation Y, where $\nu|_X$ is the normal bundle of an embedding of X into $V \times Y$ over Y. As the sphere bundle of a G-equivariant vector bundle, the map $S(\nu|_X) \to X$ is a cell homotopy bundle over X. So for any cofibrant spectrum E over X indexed on $\mathcal{U}$, we have a natural homotopy equivalence

$$E \wedge_X C_f^{-1} \simeq \Sigma^{-V}_{\text{shift}}(E \wedge_X S(\nu|_X))$$

where the right hand side is the smash product of a spectrum with a space over X. By Lemma 4.19, $E \wedge_X S(\nu|_X)$ is cofibrant. Hence, $E \wedge_X C_f^{-1}$ is naturally homotopy equivalent to a cofibrant G-spectrum over X.

CHAPTER 5

PROOF OF THE MAIN THEOREM

We will now proceed with the proof of Theorem 4.9. We first prove the theorem in the case where Y is compact. The general case follows from applying Lemma 4.14 to the dualizing object C_f, and a colimit argument. We will write C for the dualizing object C_f. For the compact case, we will define inverse natural equivalences between f_* and $f_\sharp(- \wedge_X C^{-1})$.

We first define the maps on the level of based G-spaces. Let $f : X \to Y$ be an equivariant smooth family of manifolds, with Y compact. Fix an embedding of X into $Y \times S^V$ for some G-representation V as in the proof of Lemma 4.5. Let $S(\tau_X)$ denote the sphere bundle of the tangent bundle τ_X of X, and let $S(\nu_X)$ denote the sphere bundle of the normal bundle ν_X of X in $Y \times S^V$. By the proof of Lemma 4.5, for any choice of linear isometries $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$, C^{-1} is naturally homotopy equivalent to $\Sigma^{-V}_{\text{shift}} \Sigma^\infty_X S(\nu_X)$. In particular, $f_\sharp(S(\nu_X)) \cong \text{Th}(\nu_X)$ is the Thom space of ν_X in the category of based G-spaces over Y, so $f_\sharp(C^{-1})$ is naturally homotopy equivalent to $\Sigma^{-V}_{\text{shift}} \Sigma^\infty \text{Th}(\tau_X)$. The Pontryagin-Thom construction gives a map

$$S^V_Y \longrightarrow \text{Th}(\nu_X) \tag{5.1}$$

which collapses a complement of a normal tubular neighborhood of X in $Y \times V$ to Y. For a cofibrant and fibrant based G-space T over X, by smashing (5.1) with $f_*(T)$, we get the map of G-spaces over Y

$$\varphi : \Sigma^V_Y f_*(T) \longrightarrow f_*(T) \wedge f_\sharp(S(\nu)) \xrightarrow{\cong} f_\sharp(T \wedge_X f^* f_* S(\nu)) \longrightarrow f_\sharp(T \wedge S(\nu)). \tag{5.2}$$

Here, the isomorphism is by Lemma 4.7, and the last map is the counit of the adjunction pair (f^*, f_*).

We will also give the "inverse" ψ to φ. let T be a cofibrant and fibrant G-space over X, with structure map $p_T : T \to X$ and basepoint $i_T : X \to T$. We would like to define a natural map

$$\overline{\psi} : f^* f_\sharp(T) \longrightarrow T \wedge_X C$$

in the category of G-spaces over X. Then the adjoint to $\overline{\psi}$ would give

$$\psi : f_\sharp(T) \longrightarrow f_*(T \wedge_X C)$$

in the category of G-spaces over Y.

To define $\overline{\psi}$, we give some consideration to its source and target spaces. Recall that $f_\sharp(T)$ is defined as a based G-space over Y by the following diagram

$$\begin{tikzcd} X \arrow[r, "i_T"] \arrow[d, "f"'] & T \arrow[r, "p_T"] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & f_\sharp(T) \arrow[r] & Y \end{tikzcd}$$

where the left hand side square is a pushout, and the top and bottom compositions are identities. Since f^* preserves colimits, applying f^* to this diagram gives

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{Id} \times i_T"] \arrow[d, "\pi_1"'] & X \times_Y T \arrow[r, "\mathrm{Id} \times p_T"] \arrow[d] & X \times_Y X \arrow[d, "\pi_1"'] \\ X \arrow[r] & f^* f_\sharp(T) \arrow[r] & X. \end{tikzcd}$$

Again, the left hand side square is a pushout, and the top and bottom compositions are identities. So $f^*f_\sharp(T)$ is naturally isomorphic over X to $(X \times_Y T)/_X(X \times_Y X)$, the quotient in the category of G-spaces over X of $f^*(T) = X \times_Y T$ by $f^*(X) = X \times_Y X$, both of which have structure maps via the first projection π_1.

On the other hand, recall that in the category of based G-spaces over X, the dualizing object C can be defined up to homotopy equivalence to be

$$C = X \times_Y X/_X(X \times_Y X \smallsetminus U)$$

where Δ is the image of the diagonal $X \to X \times_Y X$, and U is a G-equivariant tubular neighborhood of Δ. The smash product $T \wedge_X C$ is defined to be

$$\begin{aligned} T \wedge_X C &= (T \times_X C)/_X(T \vee_X C) \\ &= (T \times_X (X \times_Y X))/_X(T \times_X (X \times_Y X \smallsetminus U) \cup i_T(X) \times_X (X \times_Y X)). \end{aligned}$$

There is a natural map from $X \times_Y T$ to $T \times_X (X \times_Y X)$, which takes (x, t) to $(t, p_T(t), x)$. At first glance, one might want to define $\overline{\psi}$ as the map on the quotients induced by this map. However, this is not a map over X. As a G-space over X, the structure map of $X \times_Y T = f^*(T)$ is just the first projection, whereas $T \times_X (X \times_Y X) \cong T \times_Y X$ as a G-space over X, where the structure map is via p_T. The two are in general not isomorphic.

To overcome this problem, we will "thicken" T as a G-space over X by making the following construction. The product $X \times_Y X$ is a G-space over X by the first projection π_1. Assume that the closure of the tubular neighborhood U of Δ in $X \times_Y X$ over X is contained in another such G-equivariant tubular neighborhood U'. We also assume that for each $x \in X$, the fiber $U'_x = U' \cap \pi_1^{-1}(x)$ over x in U' is an open contractible neighborhood of x in $\pi_1^{-1}(x)$. Such tubular neighborhoods U and U' exist if Y is compact. We define the following unbased G-space $\overline{T}$ over X.

$$\overline{T} = \{(x, t) \in X \times_Y T \mid p_T(t) \in U'_x\}.$$

The structure map $\overline{T} \to X$ is just the first projection, *i.e.* $(x,t) \mapsto x$. Since U' is G-equivariant, so is $\overline{T}$. There is a natural G-map over X

$$g_T : T \longrightarrow \overline{T}$$

which takes t to $(p_T(t), t)$.

To handle the basepoint, we also define the following based G-space $\widetilde{T}$ over X. There is an injective G-map $U' \to \overline{T}$ over X, which takes (x, x') to $(x, i_T(x'))$ Define the based space $\widetilde{T}$ over X by

$$\widetilde{T} = \overline{T}/_X U'.$$

If T is cofibrant in the category of based G-spaces over X, then the basepoint map $i_T : X \to T$ is a closed injection. Thus, so is the inclusion $\mathrm{Id}_X \times_Y i_T : X \times_Y X \to X \times_Y T$. So

$$U' = \overline{T} \cap (\mathrm{Id}_X \times_Y i_T)(X \times_Y X)$$

is closed in $\overline{T}$, which gives that $\widetilde{T}$ is a weak Hausdorff G-space. We define in the based category over X

$$g_T : T \longrightarrow \overline{T} \longrightarrow \widetilde{T}$$

to be the composition of the unbased g_T and the quotient map $\overline{T} \to \widetilde{T}$. Since the unbased $g_T : T \to \overline{T}$ and the quotient map $\overline{T} \to \widetilde{T}$ are G-maps over X, $g_T : T \to \widetilde{T}$ is a G-map over X. It is straightforward to check that it is a based map over X.

We make the following observation.

Lemma 5.3. — *If T is a fibrant based G-space over X, then $g_T : T \to \widetilde{T}$ is a weak equivalence over X. If T is also cofibrant or of G-cell homotopy type, then g_T is a homotopy equivalence over X. In particular, suppose $T = f^*(T')$ for some fibrant T' over Y. Define $r_T : \overline{T} \to T$ by $(x,(x',t')) \mapsto (x,t')$ for $(x',t') \in X \times_Y T' = T$. Then r_T is a homotopy inverse to g_T over X.*

Proof. — We first consider the unbased $g_T : T \to \overline{T}$. Consider the diagram over X

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "\Delta"'] & & U' \arrow[dl, "\subset"] \\ & X \times_Y X & \end{tikzcd}$$

where $X \times_Y X$ is a G-space over X by the first projection. Then $\Delta : X \to U'$ is a weak equivalence since it is the inclusion of a G-equivariant deformation retract. Also, consider $\mathrm{Id}_X \times_Y p_T : X \times_Y T \to X \times_Y X$. This is a fibration since p_T is a fibration. We have that

$$T \cong (X \times_Y T) \times_{X \times_Y X} X$$

via the G-equivariant isomorphism $t \mapsto ((p_T(t), t), p_T(t))$, and

$$\overline{T} \cong (X \times_Y T) \times_{X \times_Y X} U'$$

via the G-equivariant isomorphism $(x,t) \mapsto ((x,t),(x,p_T(t)))$ for $(x,t) \in \overline{T} \subseteq X \times_Y T$. The map $g_T : T \to \overline{T}$ is obtained by pulling back $\Delta : X \to U'$ along $\mathrm{Id}_X \times_Y p_T$. Since pulling back along fibrations preserve weak equivalences, g_T is a weak equivalence. Also, the quotient map $\overline{T} \to \widetilde{T}$ is a weak equivalence over X, since the tubular neighborhood U' is G-equivariantly homotopy equivalent to X. Thus, $g_T : T \to \widetilde{T}$ is a weak equivalence over X.

Now suppose the basepoint map $i_T : X \to T$ is a cofibration. We claim that after forgetting to the category of unbased G-spaces over Y, $g_T : T \to \widetilde{T}$ is a homotopy equivalence. To see this, we first consider the unbased $g_T : T \to \overline{T}$. Define $s_T : \overline{T} \to T$ over Y by $(x,t) \mapsto t$. Then $s_T \cdot g_T = \mathrm{Id}_T$. Also, let $\pi_2 : U' \to X$ be the second projection. It is not a map over X, but as a G-space over Y, we have

$$\overline{T} \cong U \times_X T$$

where $U' \to X$ by π_2. The isomorphism is $(x,t) \mapsto ((x,p_T(t)),t)$. Also, we have that $T \cong X \times_X T$, and $s_T = \pi_2 \times_X \mathrm{Id}_T : \overline{T} \cong U' \times_X T \to T$. The composition

$$U' \xrightarrow{\pi_2} X \xrightarrow{\Delta} U'$$

is homotopic to the identity on U via a homotopy $H : U' \times I \to U'$, such that for every $r \in I$,

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ U' \arrow[rr, "H_r"] \arrow[dr, "\pi_2"'] & & U' \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commutes. So $H \times_X \mathrm{Id}_T : \overline{T} \times I \to \overline{T}$ is a homotopy over Y between $g_T \cdot s_T$ and the identity on $\overline{T}$. Thus, $g_T : T \to \overline{T}$ is a homotopy equivalence over Y. Further, if we consider the map $X \to \overline{T}$ which takes $x \in X$ to $(x, i_T(x))$, then for each $r \in I$, the map $H_r \times_X \mathrm{Id}_T : \overline{T} \to \overline{T}$ is the identity on X.

For the quotient map $\overline{T} \to \widetilde{T}$, note that X is a G-deformation retract of U', via a homotopy $U' \times I \to U'$ that preserves the diagonal for all $r \in I$. Now since X is a homotopy cell bundle over Y, $X \times_Y -$ preserves cofibrations by Lemma 4.19. So $\mathrm{Id}_X \times_Y i_T : X \times_Y X \to X \times_Y T$ is a cofibration. Also, X is a G-cell complex. Since U' is the total space of the tangent bundle of X, U' is also a G-cell complex, as is $X \times_Y X$. We can divide the cells of U' and $X \times_Y X$ so that the inclusion $U' \to X \times_Y X$ is a relative G-cell complex. Thus, the map $\mathrm{Id}_X \times_Y i_T : U' \to \overline{T}$ is a cofibration, so the quotient map $\overline{T} \to \widetilde{T}$ is a homotopy equivalence, via homotopies that preserves X in $\overline{T}$ and $\widetilde{T}$ for all $r \in I$.

Thus, the composition $g_T : T \to \widetilde{T}$ is a homotopy equivalence in the unbased category over Y, via homotopies that is the identity on the basepoint copies of X in

T and $\widetilde{T}$ for all $r \in I$. Also, the basepoint of $\widetilde{T} = \overline{T}/_X U'$ is given by the pushout diagram

$$\begin{array}{ccc} U' & \longrightarrow & \overline{T} \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & \widetilde{T}. \end{array}$$

The top map is a cofibration, so $\widetilde{T}$ is cofibrant over X. Thus, by Lemma 3.13, we can lift the homotopy inverse of g_T over Y to a based homotopy inverse of g_T over X.

In particular, suppose $T = f^*(T') = X \times_Y T'$ for some fibrant G-space T' over Y. Then we have

$$\overline{T} = \{(x, (x', t')) \in X \times_Y (X \times_Y T') \mid x' \in U'_x\} = U' \times_Y T'$$

where U' is thought of as a G-space over Y by $U' \xrightarrow{\pi_1} X \to Y$. Then by the methods of Lemma 3.13, we can choose the lifting of the homotopy inverse s_T to a map over X to be

$$r_T : \overline{T} = U' \times_Y T' \longrightarrow T = X \times_Y T'$$

by $\pi_1 : U' \to X$, so $(x, (x', t')) \mapsto (x, t')$. Then $r_T \cdot g_T = \mathrm{Id}_T$. Also, If $H : U' \times I \to U'$ is a homotopy over X between $\pi_1 : U' \to X$ and the identity on U', then $H \times_Y \mathrm{Id}_{T'}$ gives a homotopy over X between $g_T \cdot r_T$ and the identity on $\overline{T}$. For the basepoint, note that $r_T : \overline{T} \to T$ passes to a based map $r_T : \widetilde{T} \to T$ over X. One can check that the homotopy between $g_T \cdot r_T$ and the identity on $\overline{T}$ passes to a based homotopy over X between $g_T \cdot r_T$ and the identity on $\widetilde{T}$. Thus, for $T = f^*(T')$, r_T is an explicit based homotopy inverse to g_T over X.

Finally, suppose T is of G-cell homotopy type over X. So there is some based space T_0 over X, such that the basepoint $X \to T_0$ of T_0 is a relative G-cell complex, and there is a G-equivariant homotopy equivalence $f : T \to T_0$ over X. The thickening construction taking T to $\overline{T} = (X \times_Y T) \times_{X \times_Y X} U'$ is functorial on the category of unbased G-spaces over X. For a based G-space T over X, $\widetilde{T} = \overline{T}/_X(\mathrm{Id}_X \times_Y i_T)(U')$, so it is also functorial on the category of based G-spaces over X. Thus, we have a commutative diagram of G-spaces over X

$$\begin{array}{ccc} T & \xrightarrow{f} & T_0 \\ {\scriptstyle g_T}\big\downarrow & & \big\downarrow{\scriptstyle g_{T_0}} \\ \widetilde{T} & \xrightarrow[\widetilde{f}]{} & \widetilde{T_0}. \end{array}$$

By the above, g_{T_0} is a homotopy equivalence. The functor $T \mapsto \widetilde{T}$ preserves homotopies, so $\widetilde{f}$ is a homotopy equivalence as well. Hence, g_T is also a homotopy equivalence. □

The following lemma gives the relation between the thickening of G-spaces over X with smash products.

Lemma 5.4. — *Let T and Z be fibrant based G-spaces over X.*

(1) *There is a natural weak equivalence over X*

$$\alpha : \widetilde{T \wedge_X Z} \xrightarrow{\simeq} \widetilde{T} \wedge_X \widetilde{Z}.$$

If T is also cofibrant over X, and Z is a homotopy cell bundle over X, then α is a homotopy equivalence over X.

(2) *If $T = f^*(T')$ for some fibrant based G-space T' over Y, then there is a natural isomorphism*

$$T \wedge_X \widetilde{Z} \xrightarrow{\cong} \widetilde{T \wedge_X Z}.$$

Proof. — The based G-space $\widetilde{T \wedge_X Z}$ is obtained as a quotient space of the unbased G-space $\overline{T \times_X Z} = \{(x,(t,z)) \in X \times (T \times_X Z) \mid p_T(t) = p_Z(z) \in U_x\}$. Similarly, $\widetilde{T} \wedge_X \widetilde{Z}$ is a quotient space of $\overline{T} \times_X \overline{Z} = \{((x,t),(x,z)) \mid p_T(t) \in U_x,\ p_Z(z) \in U_x\}$. So define the map

$$\alpha : \overline{T \times_X Z} \longrightarrow \overline{T} \times_X \overline{Z}$$

by $(x,(t,z)) \mapsto ((x,t),(x,z))$. It is routine to check that this map induces a based G-map on the quotient spaces

$$\alpha : \widetilde{T \wedge_X Z} \longrightarrow \widetilde{T} \wedge_X \widetilde{Z}.$$

This is a weak equivalence by the following commutative diagram, where the two vertical arrows are weak equivalences by Lemma 5.3, and the top arrow is the identity map.

$$\begin{tikzcd}
T \wedge_X Z \arrow[r, "="] \arrow[d, "g_{T\wedge_X Z}"'] & T \wedge_X Z \arrow[d, "g_T \wedge_X g_Z"] \\
\overline{T \times_X Z} \arrow[r, "\alpha"'] & \overline{T} \times_X \overline{Z}.
\end{tikzcd}$$

If T is also cofibrant over X and Z is a homotopy cell bundle over X, then by Lemma 4.19, $T \wedge_X Z$ is cofibrant over X. So by Lemma 5.3, the two vertical maps of the diagram are homotopy equivalences. Hence, so is α.

Now suppose that $T = f^*(T') = X \times_Y T'$ for some fibrant based G-space T' over Y. We define a map of unbased G-spaces

$$a : T \times_X \overline{Z} \longrightarrow \overline{T \times_X Z}.$$

A point of $T \times_X \overline{Z}$ is of the form $((x,t'),(x,z))$, where $(x,t') \in X \times_Y T' = T$, and $p_Z(z) \in U_x$. We define the map a to take this point to $(x,((p_Z(z),t'),z))$, where $(p_Z(z),t') \in T$, so $((p_Z(z),t'),z)$ is in $T \times_X Z$. One can check that this map induces a map of based G-spaces on the quotient spaces

$$a : T \wedge_X \widetilde{Z} \longrightarrow \widetilde{T \wedge_X Z}.$$

The inverse of a is induced by the following unbased map

$$b : \overline{T \times_X Z} \longrightarrow T \times_X \overline{Z}.$$

A point of $\overline{T \times_X Z}$ is of the form $(x, ((y, t'), z))$, where $(y, t') \in T = X \times_Y T'$, $y = p_Z(z) \in U_x$. We define b to take this point to $((x, t'), (x, z)) \in T \times_X \overline{Z}$. Again, it is straightforward to check that this gives a based map on the quotient spaces

$$b : \widetilde{T \wedge_X Z} \longrightarrow T \wedge_X \widetilde{Z}.$$

It is now easy to check that a and b are inverse maps of based G-spaces over X. $\square$

Now we can define the map ψ on the level of based G-spaces as follows. We define the model of the dualizing object C to be $C = X \times_Y X/_X(X \times_Y X \smallsetminus U)$. We first define ψ on the level of G-spaces. For a based G-space T over X, define a map over X

$$\begin{aligned} \overline{\psi_0} : X \times_Y T \longrightarrow (X \times_Y T/_X U') \wedge_X C & \\ &\cong (X \times_Y T/_X U') \wedge_X (X \times_Y X/(X \times_Y X \smallsetminus U)) \\ &\cong \frac{(X \times_Y T) \times_X (X \times_Y X)}{(U' \times_X (X \times_Y X) \cup (X \times_Y T) \times_X (X \times_Y X \smallsetminus U))} \end{aligned}$$

(Here, the last quotient is in the category over X). Namely, $\overline{\psi_0} : (x, t) \mapsto ((x, t), (x, p_T(t)))$ in the numerator of the target $(X \times_Y T) \times_X (X \times_Y X)$. This is a continuous map over X, and for $(x, t) \in X \times_Y T$, such that $p_T(t) \notin U'_x$, (x, t) lands in $(X \times_Y T) \times_X (X \times_Y X \smallsetminus U)$, so its image is in fact x in the basepoint $X \to (X \times_Y T/_X U') \wedge_X C$. Hence, the image of $\overline{\psi_0}$ is in fact contained in $(\overline{T}/_X U') \wedge_X C = \widetilde{T} \wedge_X C$. Also, suppose $t = i_T(x')$ for some $x' \in X$. If $x' \notin U'_x$, the image of $(x, i_T(x'))$ is x in the basepoint $X \to \widetilde{T} \wedge_X C$ by the above. If $x' \in U'_x$, then $\overline{\psi_0}$ takes $(x, i_T(x'))$ to $((x, i_T(x'), x, x) \in U' \times_X (X \times_Y X)$. Hence, $(x, i_T(x'))$ also maps to x in the basepoint. So $\overline{\psi_0}$ factors through a based map over X

$$\overline{\psi} : f^* f_\sharp(T) = (X \times_Y T)/_X(X \times_Y X) \longrightarrow \widetilde{T} \wedge_X C. \tag{5.5}$$

If T is cofibrant and fibrant, then the target $\widetilde{T} \wedge_X C$ is naturally homotopy equivalent to $T \wedge_X C$, since by Lemma 5.3, $g_T : T \to \widetilde{T}$ is a homotopy equivalence.

Our strategy for proving Theorem 4.9 is as follows. We will show that φ and ψ are "homotopy inverses" to each other on the level of G-spaces in a certain sense, in the case when Y is compact. Then we will define φ and ψ on the level of spectra, still for the case when Y is compact, and prove that the homotopy inverses on the level of spaces give that the spectra-level φ and ψ are inverse weak equivalences in this case. Finally, for general $f : X \to Y$ where Y is not necessarily compact, we cannot define the inverse map ψ since there may not be a suitable tubular neighborhood U' of the diagonal in $X \times_Y X$. But we can define φ on the level of spectra by a colimit argument over the compact skeleta of Y, and show that it is a weak equivalence of spectra over Y.

We have the natural homotopy equivalence

$$S^V_X = X \times S^V \xrightarrow{\simeq} S(\nu) \wedge_X S(\tau).$$

Also, recall that by Lemma 3.13, for T cofibrant and fibrant over X, the maps

$$g_T : T \longrightarrow \widetilde{T}$$

and

$$g_{T\wedge_X S(\nu)} : T \wedge_X S(\nu) \longrightarrow \widetilde{T \wedge_X S(\nu)}$$

are natural homotopy equivalences over X. The statement on the level of spaces is the following proposition.

Proposition 5.6. — *Let $f : X \to Y$ be an equivariant smooth family of manifolds, where Y is compact. Let T be a cofibrant and fibrant based G-space over X.*

(1) *The composition*

$$\Sigma^V_X f^* f_*(T) \cong f^* \Sigma^V_X f_*(T) \xrightarrow{f^*\varphi} f^* f_\sharp(T \wedge_X S(\nu)) \xrightarrow{\overline{\psi}} (\widetilde{T \wedge_X S(\nu)}) \wedge_X S(\tau)$$

is naturally homotopic to the composition

$$\Sigma^V_X f^* f_*(T) \xrightarrow{\Sigma^V_X c} \Sigma^V_X(T) \xrightarrow{\simeq} T \wedge_X S(\nu) \wedge_X S(\tau) \xrightarrow{g_{T\wedge_X S(\nu)}} (\widetilde{T \wedge_X S(\nu)}) \wedge_X S(\tau).$$

Here, $c : f^* f_*(T) \to T$ is the counit of the adjunction pair (f^*, f_*).

(2) *The composition*

$$\begin{aligned} \Sigma^V_X f_\sharp(T) \xrightarrow{\Sigma^V_X \psi} \Sigma^V_X f_*(\widetilde{T} \wedge_X S(\tau)) &\xrightarrow{\varphi} f_\sharp(\widetilde{T} \wedge_X S(\tau) \wedge_X S(\nu)) \\ &\xrightarrow{\simeq} f_\sharp(\Sigma^V_X \widetilde{T}) \xrightarrow{\cong} \Sigma^V_X f_\sharp(\widetilde{T}) \end{aligned}$$

is naturally homotopic to

$$\Sigma^V_X f_\sharp(g_T) : \Sigma^V_X f_\sharp(T) \longrightarrow \Sigma^V_X f_\sharp(\widetilde{T}).$$

Proof. — We first consider the case where $Y = *$ is a point, so X is a G-equivariant smooth manifold, and $f : X \to *$. To prove statement (1) of the proposition, we need to show that in the diagram

$$\begin{tikzcd} f^*\Sigma^V f_* T \arrow[r, "f^*\varphi"] \arrow[ddrr, dashed, "h"'] & f^* f_\sharp(T \wedge_X S(\nu)) \arrow[r, "\overline{\psi}"] & (\widetilde{T \wedge_X S(\nu)}) \wedge_X S(\tau) \\ & & T \wedge_X S(\nu) \wedge_X S(\tau) \arrow[u, "g_{T\wedge_X S(\nu)}"'] \\ & & \Sigma^V_X(T) \arrow[u, "\simeq"'] \end{tikzcd}$$

the dotted arrow h exists such that the diagram commutes up to homotopy, and h is homotopic to the counit of the adjunction pair (f^*, f_*). By Lemma 5.3, the vertical map $g_{T\wedge_X S(\nu)}$ is a homotopy equivalence. Now by Lemmas 5.3 and 5.4, we have the natural homotopy equivalences over X

$$\alpha : \widetilde{T \wedge_X S(\nu)} \xrightarrow{\simeq} \widetilde{T} \wedge_X \widetilde{S(\nu)}$$

and

$$g_T \wedge_X \mathrm{Id} : T \wedge_X \widetilde{S(\nu)} \longrightarrow \widetilde{T} \wedge_X \widetilde{S(\nu)}.$$

We also define a map

$$\gamma : S^V_X = X \times S^V \longrightarrow \widetilde{S(\nu)} \wedge_X S(\tau)$$

as follows. We have the embedding of X inside S^V. Let U be a G-equivariant normal tubular neighborhood of X in S^V, so we have a G-map $p : U \cong E(\nu) \to X$, and $S(\nu)$ is the one-point compactification of U in the category of G-spaces over X.

Also, we define the dualizing object C to be $S(\tau) \simeq (X \times X)/_X(X \times X \smallsetminus U')$, where U' is a tubular neighborhood of the diagonal in $X \times X$, chosen as follows. Since X is a compact smooth manifold, there is an equivariant Riemannian metric ρ on X. By the compactness of X, there is some $\varepsilon > 0$, such that

$$U' = \{(x, y) \in X \times X \mid \rho(x, y) < \varepsilon\}$$

is a tubular neighborhood of the diagonal in $X \times X$. In particular, consider $X \times X$ as a G-space over X via the first projection. Then U' has the property that for any $x \in X$, the fiber U'_x is a contractible neighborhood of x in X.

Let U' be a G-equivariant tubular neighborhood of the diagonal in $X \times X$, such that for every $x \in X$, the fiber U'_x over x via the first projection is a contractible neighborhood of x in X. We have that

$$S(\tau) \cong (X \times X)/_X(X \times X \smallsetminus U').$$

We define the thickening $\widetilde{S(\nu)}$ of $S(\nu)$ using the tubular neighborhood U'. For $x \in X$ and $v \in S^V$, suppose $v \in U$ and $p(v) \in U'_x$, then we define γ to take $(x, v) \in X \times S^V = S^V_X$ to $((x, v), (x, p(v)))$. In the target, (x, v) is a point in $\widetilde{S(\nu)}$, which is a quotient of $\overline{S(\nu)} \subseteq X \times S(\nu)$, and $(x, p(v))$ is a point of $S(\tau)$, which is a quotient of $E(\tau) \cong U' \subseteq X \times X$. If $p(v) \notin U'_x$, then we define $\gamma(x, v)$ to be x in the basepoint copy of X in $\widetilde{S(\nu)} \wedge S(\tau)$. It is straightforward to check that γ is a continuous G-equivariant map over X. By arguments similar to that of Lemma 5.4, $\mathrm{Id}_T \wedge_X \gamma$ is naturally homotopic to the composition of homotopy equivalences

$$\Sigma^V_X T \xrightarrow{\simeq} T \wedge_X S(\nu) \wedge_X S(\tau) \xrightarrow{\mathrm{Id} \wedge_X g_{S(\nu)}} T \wedge_X \widetilde{S(\nu)} \wedge_X S(\tau)$$

over X. The diagram of homotopy equivalences

$$\begin{tikzcd}
T \wedge_X S(\nu) \arrow[r, "\mathrm{Id} \wedge_X g_{S(\nu)}"] \arrow[d, "g_{T\wedge_X S(\nu)}"'] \arrow[dr, "g_T \wedge_X g_{S(\nu)}"] & T \wedge_X \widetilde{S(\nu)} \arrow[d, "g_T \wedge_X \mathrm{Id}"] \\
\widetilde{T \wedge_X S(\nu)}\alpha \arrow[r] & \widetilde{T} \wedge_X \widetilde{S(\nu)}
\end{tikzcd}$$

commutes. Hence, it suffices to show that the dotted arrow h exists making the following diagram commute up to homotopy.

$$\begin{tikzcd}
f^* f_\sharp(T \wedge_X S(\nu)) \arrow[r, "\overline{\psi}"] & (\widetilde{T \wedge_X S(\nu)}) \wedge_X S(\tau) \arrow[r, "\alpha", "\simeq"'] & \widetilde{T} \wedge_X \widetilde{S(\nu)} \wedge_X S(\tau) \\
 & & T \wedge_X S(\nu) \wedge_X S(\tau) \arrow[u, "\simeq"', "g_T \wedge_X g_{S(\nu)}"] \\
f^*\Sigma^V f_* T \arrow[uu, "f^*\varphi"] \arrow[r, dashed, "h"'] & \Sigma^V_X T \arrow[uur, "g_T \wedge_X \gamma", "\simeq"'] \arrow[ur, "\simeq"'] &
\end{tikzcd}.$$

The top row of the diagram is

$$\begin{aligned}
\Sigma^V_X f^* f_*(T) \cong f^*\Sigma_V f_*(T) &\xrightarrow{f^*\varphi} f^* f_\sharp(T \wedge_X S(\nu)) \\
&\xrightarrow{\overline{\psi}} (T \widetilde{\wedge_X S(\nu)}) \wedge_X S(\tau) \\
&\xrightarrow{\alpha} \widetilde{T} \wedge_X \widetilde{S(\nu)} \wedge_X S(\tau).
\end{aligned} \tag{5.7}$$

Since the target of f is a single point, $f_*(T) = \mathrm{Sec}(X,T)$ is the G-space of sections from X to T. Let $i : X \to T$ be the basepoint of T, and let $q : T \to X$ be the structure map of T. We have the normal tubular neighborhood U of X in V, so the sphere bundle $S(\nu)$ of ν is the one-point compactification of U in the category of G-spaces over X. Also, let X^ν be the Thom space of ν, so $X^\nu = f_\sharp(S(\nu))$. Then $f^*\varphi$ is

$$\begin{aligned}
(S^V \wedge \mathrm{Sec}(X,T)) \times X &\longrightarrow (X^\nu \wedge \mathrm{Sec}(X,T)) \times X \\
&= f_\sharp S(\nu) \wedge \mathrm{Sec}(X,T) \times X \\
&\cong f_\sharp(S(\nu) \wedge_X X \times \mathrm{Sec}(X,T)) \times X \\
&\longrightarrow f_\sharp(S(\nu) \wedge_X T) \times X \\
&= (S(\nu) \times_X T)/((S(\nu) \smallsetminus U) \times_X T \cup S(\nu) \times_X i(X)) \times X.
\end{aligned} \tag{5.8}$$

The first map is induced by the Pontryagin-Thom map $S^V \to X^\nu$. For the second map that is not an isomorphism, consider the projection $p : S(\nu) \to X$. By Lemma 4.7, for $v \in S(\nu)$ and $a \in \mathrm{Sec}(X,T)$, the natural isomorphism

$$f_\sharp S(\nu) \wedge \mathrm{Sec}(X,T) \cong f_\sharp(S(\nu) \wedge_X X \times \mathrm{Sec}(X,T))$$

takes (v,a) to $(v,p(v),a)$. Thus, the second map of the composition evaluates the section a at $p(v)$. The last copy of X carries along identically. Thus, for $v \in S^V$, $a \in \mathrm{Sec}(X,T)$ and $x \in X$, the composition (5.8) takes $((v,a),x) \in (S^V \wedge \mathrm{Sec}(X,T)) \times X$ to $(v, a(p(v)), x)$ in $f_\sharp(S(\nu) \wedge_X T) \times X$ if $v \in U$, and to x in the basepoint copy of X if $v \notin U$.

We have that C is the sphere bundle of the tangent bundle τ of X. Let $j : X \to S(\nu) \wedge_X T$ be the basepoint map over X. Recall the thickening $\widetilde{S(\nu) \wedge_X T}$ of $S(\nu) \wedge_X T$ used to define $\overline{\psi}$, with

$$\overline{S(\nu) \wedge_X T} = \{(x,(v,t)) \in X \times (S(\nu) \wedge_X T) \mid p(v) = q(t) \in U'_x\}$$

which is a G-space over X by the first projection, and

$$\widetilde{S(\nu) \wedge_X T} = \overline{S(\nu) \wedge_X T}/_X(X \times j(X)).$$

The second map $\overline{\psi}$ of (5.7) is

$$\begin{aligned} f_\sharp(S(\nu) \wedge_X T) \times X &\longrightarrow (\widetilde{S(\nu) \wedge_X T}) \wedge_X C \\ &= \frac{(\widetilde{S(\nu) \wedge_X T}) \times_X (X \times X)}{(\widetilde{S(\nu) \wedge_X T}) \times_X (X \times X \smallsetminus U')}. \end{aligned}$$

Here, the last quotient is taken in the category of G-spaces over X. For $v \in S^V$, $a \in \mathrm{Sec}(X,T)$, and $x \in X$, if $v \in U$, then $\overline{\psi}$ takes $((v, a(p(v))), x) \in f_\sharp(S(\nu) \wedge_X T) \times X$ to $((x, v, a(p(v))), (x, p(v))) \in (\widetilde{S(\nu) \wedge_X T}) \wedge_X (X \times X/_X(X \times X \smallsetminus U'))$. Here, $(v, a(p(v)))$ is a point of $S(\nu) \wedge_X T$, so $(x, v, a(p(v)))$ is an element of $\widetilde{S(\nu) \wedge_X T}$, and $(x, p(v))$ is in $C = (X \times X)/_X(X \times X \smallsetminus U')$. If $v \notin U$, then $\overline{\psi}$ takes $((v, a(p(v))), x)$ to x in the basepoint copy of X in $(\widetilde{S(\nu) \wedge_X T}) \wedge_X C$. Also, for any $t \in T$ and $x \in X$, if $t \notin U'_x$, then $\overline{\psi}$ maps $((v,t),x)$ to x in the basepoint of $\widetilde{S(\nu) \wedge_X T} \wedge_X C$. Since $a : X \to T$ is a section, for any $x \in X$, $q(a(x)) = x$. In particular, for $v \in S(\nu)$, and $x \in X$, if $q(a(p(v))) = p(v) \notin U'_x$, then (5.7) maps $((v,a),x)$ to x in the basepoint.

Also, by Lemma 5.4, the homotopy equivalence

$$\alpha : \widetilde{S(\nu) \wedge_X T} \longrightarrow \widetilde{S(\nu)} \wedge_X \widetilde{T}$$

takes $(x,(v,t))$ to $((x,v),(x,t))$, where $(x,v) \in \widetilde{S(\nu)}$, and $(x,t) \in \widetilde{T}$. Thus, (5.7) is

$$(5.9) \qquad (S^V \wedge \mathrm{Sec}(X,T)) \times X \longrightarrow (\widetilde{S(\nu) \wedge_X T}) \wedge_X S(\tau) \xrightarrow{\simeq} \widetilde{S(\nu)} \wedge_X \widetilde{T} \wedge_X S(\tau)$$

which takes $((v,a),x)$ to $((x,v),(x,a(p(v))),(x,p(v)))$ if $v \in U$ and $p(v) \in U'_x$ (*i.e.* if $v \in U \cap p^{-1}(U'_x)$ in S^V). Here, $(x,v) \in \widetilde{S(\nu)}$, $(x,a(p(v))) \in \widetilde{T}$, and $(x,p(v)) \in S(\tau) = X \times X/(X \times X \smallsetminus U')$. Otherwise, the map takes $((v,a),x)$ to x in the basepoint copy of X. By contracting the tubular neighborhood U' to X, the map (5.9) is homotopic to a map

$$(5.10) \qquad (S^V \wedge \mathrm{Sec}(X,T)) \times X \longrightarrow \widetilde{S(\nu)} \wedge_X \widetilde{T} \wedge_X S(\tau)$$

which takes $((v,a),x)$ to $((x,v),(x,a(x)),(x,x))$ if $v \in U \cap p^{-1}(U'_x)$, and to x in the basepoint copy of X otherwise. Namely, let $H : U' \times I \to U'$ be a linear homotopy over X, where H_0 is the identity on U', and H_1 is the first projection onto X. Then the homotopy between (5.7) and (5.10) is given by applying H to $p(v)$. The map (5.10) factors through to

$$\begin{aligned} (S^V \wedge \mathrm{Sec}(X,T)) \times X &\xrightarrow{h} (X \times S^V) \wedge_X T \\ &\xrightarrow{\simeq} T \wedge_X S(\nu) \wedge_X S(\tau) \\ &\xrightarrow{\alpha} \widetilde{T} \wedge_X \widetilde{S(\nu)} \wedge_X S(\tau) \end{aligned} \tag{5.11}$$

where h takes $((v,a),x)$ to $((x,v),a(x))$ if $v \in U \cap p^{-1}(U'_x)$, and to x in the basepoint otherwise. This is first evaluating the section a at x, then factoring out the complement of $U \cap p^{-1}(U'_x)$ in S^V. The tubular neighborhood U of X is isomorphic to the total space $E(\nu)$ of the normal bundle ν of X in V. So for each $x \in X$, $U \cap p^{-1}U'_x \cong E(\nu|_{U'_x})$ is the total space of ν restricted to the contractible neighborhood U'_x of x in X, so it is a contractible neighborhood of x in S^V. Hence, collapsing its complement is naturally homotopic to the identity on S^V. These homotopies vary continuously with respect to x. Hence, the map h is homotopic to just evaluating the section a at x, which is the counit of adjunction for (f^*, f_*). This gives the first statement of Proposition 5.6 for the case where $Y = *$.

The proof of statement (2) of Proposition 5.6 is similar, for the case where $Y = *$. We need to show that the dotted arrow h exists making the following diagram commute, such that h is naturally homotopic to $\Sigma^V f_\sharp(g_T)$.

$$\begin{tikzcd} \Sigma^V f_\sharp(T) \arrow[r, "\Sigma^V \psi"] \arrow[ddrr, dashed, "h"'] & \Sigma^V f_*(\widetilde{T} \wedge_X S(\tau)) \arrow[r, "\varphi"] & f_\sharp(\widetilde{T} \wedge_X S(\tau) \wedge_X S(\nu)) \\ & & f_\sharp(\Sigma^V_X \widetilde{T}) \arrow[u, "\simeq"'] \\ & & \Sigma^V f_\sharp(\widetilde{T}). \arrow[u, "\cong"'] \end{tikzcd}$$

There is an equivariant Riemannian metric ρ on X, and as in the proof of statement (1) of Proposition 5.6, we can define the tubular neighborhood U' of the diagonal in $X \times X$ so that for every $x \in X$, the fiber U'_x is an open ball in X centered at x. We have $C = S(\tau) \cong X \times X/(X \times X \setminus U')$ as before. Also, we have the G-equivariant tubular neighborhood U of X in S^V, and the projection map $p : U \to X$. Also, let $q : T \to X$ be the structure map of T. For $x \in X$ and $t \in T$,

$$\overline{\psi} : f^* f_\sharp T \longrightarrow \widetilde{T} \wedge_X S(\tau)$$

takes $(x,t) \in f^*f_\sharp(T) = (X \times T)/_X(X \times X)$ to $((x,t),(x,q(t)))$ if $q(t) \in U'_x$, and to x in the basepoint copy of X otherwise. Here, (x,t) is thought of as a point in $\widetilde{T}$, which is a quotient of $\overline{T} \subseteq X \times T$, and $(x,q(t))$ is a point in $S(\tau)$, which is a quotient of $E(\tau) \cong U' \subseteq X \times X$. Thus, for $v \in S^V$ and $t \in T$, Σ^V of its adjoint is

$$\Sigma^V \psi : \Sigma^V f_\sharp T \longrightarrow \Sigma^V \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)).$$

For $v \in S^V$ and $t \in T$, this takes (v,t) to $(v, \psi(t))$, where $\psi(t) : X \to \widetilde{T} \wedge_X S(\tau)$ is a section which takes $x \in X$ to $((x,t),x,q(t))$ if $q(t) \in U'_x$, and to x in the basepoint copy of X otherwise.

Now the composition $\Sigma^V(\varphi \cdot \psi)$ is

$$\begin{aligned}
\Sigma^V f_\sharp T &\xrightarrow{\Sigma^V \psi} S^V \wedge \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)) \\
&\longrightarrow X^\nu \wedge \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)) \\
&= f_\sharp(S(\nu)) \wedge \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)) \\
&\cong f_\sharp(S(\nu) \wedge_X (X \times \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)))) \\
&\longrightarrow \frac{S(\nu) \times_X (\widetilde{T} \wedge_X S(\tau))}{(S(\nu) \smallsetminus U) \times_X (\widetilde{T} \wedge_X S(\tau))} \\
&= f_\sharp(S(\nu) \wedge_X (\widetilde{T} \wedge_X S(\tau))).
\end{aligned} \tag{5.12}$$

The second map is induced by the Pontryagin-Thom map $S^V \to X^\nu$. By Lemma 4.7, for $v \in S(\nu)$ and $a \in \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau))$, the isomorphism

$$f_\sharp S(\nu) \wedge \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)) \xrightarrow{\cong} f_\sharp(S(\nu) \wedge_X (X \times \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau))))$$

takes (v,a) to $(v,(p(v),a))$. The last map is induced by evaluating the sections

$$X \times \operatorname{Sec}(X, \widetilde{T} \wedge_X S(\tau)) \longrightarrow \widetilde{T} \wedge_X S(\tau).$$

Thus, for $v \in S^V$ and $t \in T$, if $v \in U$ and $q(t) \in U'_{p(v)}$, the composition (5.12) takes (v,t) to $(v,(p(v),t),(p(v),q(t))) \in f_\sharp(S(\nu) \wedge_X \widetilde{T} \wedge_X S(\tau))$, where v is thought of as an element in $S(\nu)$, $(p(v),t)$ is in $\widetilde{T}$, and $(p(v),q(t))$ is in $S(\tau)$. Otherwise, (5.12) takes (v,t) to the basepoint.

By the definition of U', for each $v \in U$ and $t \in T$, the condition that $q(t) \in U'_{p(v)}$ is equivalent to the condition that $\rho(q(t),p(v)) < \varepsilon$ for a fixed $\varepsilon > 0$. This is symmetric with respect to $p(v)$ and $q(t)$, so we have that $p(v) \in U'_{q(t)}$ if and only if $q(t) \in U'_{p(v)}$. By contracting U' to X, we get that the composition (5.12) is homotopic to a map

$$\Sigma^V f_\sharp T \longrightarrow f_\sharp(S(\nu) \wedge_X \widetilde{T} \wedge_X S(\tau)) \tag{5.13}$$

which takes (v,t) to $(v,(p(v),t),(p(v),p(v)))$ if $v \in U$ and $p(v) \in B(q(t),\varepsilon)$, and to the basepoint otherwise. The map 5.13 factors through to

$$\Sigma^V f_\sharp(T) \xrightarrow{h} \Sigma^V f_\sharp(\widetilde{T}) \cong f_\sharp((X \times S^V) \wedge_X \widetilde{T}) \xrightarrow{\simeq} f_\sharp(S(\nu) \wedge_X \widetilde{T} \wedge_X S(\tau)) \tag{5.14}$$

where h takes (v,t) to $(v,(p(v),t))$ if $v \in U$ and $p(V) \in U'_{q(t)}$, and to the basepoint otherwise. By collapsing U' to the diagonal, we get that h is homotopic to a map h' that takes (v,t) to $(v,(q(t),t))$ if $v \in U$ and $p(v) \in U'_{q(t)}$, and to the basepoint otherwise. So over each $t \in T$, h' collapses the complement in S^V of the open neighborhood $U \cap p^{-1}(U'_{q(t)})$ of $q(t)$, then applies q_T. Again, U, U' are defined in a way such that $U \cap p^{-1}(U'_{q(t)})$ is a contractible neighborhood, so collapsing the complement of $U \cap p^{-1}(U'_{q(t)})$ is naturally homotopic to the identity on S^V. Also, U and $U'_{q(t)}$ vary continuously with respect to v and t, so h' is homotopic to $\Sigma^V f_\sharp(g_T)$.

This gives the proof of Proposition 5.6 in the case where $Y = *$. Now suppose $f : X \to Y$ is an equivariant smooth family of manifolds, where Y is compact. Then we can divide the cells of Y such that for each cell $G/H \times D^n$ of Y, the fiber of f over $G/H \times D^n$ is $(G \times_H M) \times D^n$, where M, the fiber of f, is a smooth compact manifold with some smooth H-action (depending on the cell of Y). For any point $y \in G/H \times D^n$, the proof for the case $Y = *$ gives H-equivariant homotopies between the maps in the statement of the proposition, for $f_y : M \to \{y\}$, the restriction of f to the fiber over the point y. These are in fact independent of the choice of $y \in G/H \times D^n$. By applying $(G \times_H -) \times D^n$ to the homotopies, we get that the proposition holds for the map

$$(G \times_H M) \times D^n \longrightarrow G/H \times D^n.$$

These homotopies are natural over the cells of Y, so by gluing the homotopies over the cells of Y, we get that Proposition 5.6 holds for any equivariant smooth family of manifolds where Y is compact. □

We now define the natural maps φ and ψ on the level of spectra, again for the case where Y is compact, and show that Proposition 5.6 implies that the spectra-level versions of φ and ψ are inverse weak equivalences. Let E be a cofibrant and fibrant spectrum over X. Recall the Pontryagin-Thom map (5.1). In the category of spectra over Y, taking the shift desuspension by V gives

$$t : S^0_Y = \Sigma^{-V}_{\text{shift}} S^V_Y \longrightarrow \Sigma^{-V}_{\text{shift}} \operatorname{Th}(\nu_X) \cong f_\sharp(C^{-1}). \tag{5.15}$$

Here, the isomorphism $\Sigma^{-V}_Y \operatorname{Th}(\nu_X) \cong f_\sharp(C^{-1})$ comes from Lemma 4.7. We define the natural transformation $\varphi : f_* \to f_\sharp(- \wedge C^{-1})$ as follows. Let E be a spectrum over X. In the homotopy category of spectra over Y, smashing the map t with $f_*(E)$ gives

$$\begin{aligned} \varphi_E : f_*(E) &\longrightarrow f_*(E) \wedge_Y f_\sharp(C^{-1}) \\ &\xrightarrow{\cong} f_\sharp(f^* f_*(E) \wedge_X C^{-1}) \\ &\longrightarrow f_\sharp(E \wedge_X C^{-1}). \end{aligned} \tag{5.16}$$

Here, the first map is $\mathrm{Id}_{f_*(E)} \wedge_Y t$, the equivalence is by Lemma 4.7 (since C^{-1} is homotopy equivalent to the shift desuspension of the suspension spectrum of a space), and the last map is the counit of the adjunction pair (f^*, f_*).

Note that since the smash product of spectra is only defined up to weak equivalences, the map (5.16) is only defined uniquely in the homotopy category of spectra over Y. We would like to have a model of it that is defined on the point-set level. One way to do this is to take a choice of linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$, so the point set level model of smash product is defined via a. We also give a canonical model for $f_\sharp(E \wedge_X C^{-1})$. To this end, we take the map (5.1) of G-spaces, and smash it with $f_*(E)$ to get the map in the point-set category of spectra

$$(5.17)\quad \Sigma^V_Y f_*(E) \to f_*(E) \wedge_Y \mathrm{Th}(\nu_X) \xrightarrow{\simeq} f_\sharp(f^* f_*(E) \wedge_X S(\nu_X)) \to f_\sharp(E \wedge_X S(\nu_X)).$$

Since E is cofibrant, we can take the model

$$E \wedge_X C^{-1} = \Sigma^{-V}_{\mathrm{shift}}(E \wedge_X S(\nu_X)).$$

Then the target of (5.17) is naturally homotopy equivalent to

$$f_\sharp(\Sigma^V_{\mathrm{shift}} E \wedge_X C^{-1}) \simeq \Sigma^V_{\mathrm{shift}} f_\sharp(E \wedge_X C^{-1}).$$

However, $f_*(E)$ may not be cofibrant, so we do not have homotopical control on the source of (5.17). To remedy this, we use the cofibrant replacement functor Γ on the category of spectra over Y, with respect to the model structure given in Definition 3.3. We have natural maps

$$(5.18)\qquad S^V_Y \wedge_Y \Gamma f_*(E) \xrightarrow{a} S^V_Y \wedge_Y f_*(E) \xrightarrow{5.17} \Sigma^V_{\mathrm{shift}} f_\sharp(E \wedge_X C^{-1}).$$

Here, a is an acyclic fibration, and the source of (5.18) is naturally homotopy equivalent to $\Gamma(S^V_Y f_*(E))$. Recall that for a cofibrant spectrum D, there is a natural homotopy equivalence between $\Sigma^V_{\mathrm{shift}}(D)$ and $S^V \wedge D$. So we can replace $S^V_Y \wedge_Y -$ in (5.18) by $\Sigma^V_{\mathrm{shift}}$ to get

$$\Sigma^V_{\mathrm{shift}} \Gamma f_*(E) \longrightarrow \Sigma^V_{\mathrm{shift}} f_\sharp(E \wedge_X C^{-1}).$$

Taking $\Sigma^{-V}_{\mathrm{shift}}$ then gives

$$(5.19)\qquad \varphi'_E : \Gamma f_*(E) \longrightarrow f_\sharp(E \wedge_X C^{-1}).$$

The following diagram commutes after we pass to the homotopy category of spectra over Y for any choice of linear isometry $a : X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$.

$$(5.20)\qquad \begin{tikzcd} \Gamma f_*(E) \arrow[r, "\varphi'_E"] \arrow[d, "a"'] & f_\sharp(E \wedge_X C^{-1}) \\ f_*(E) \arrow[ur, "\varphi_E"'] & \end{tikzcd}$$

So (5.19) is a point-set model of the map φ_E.

For the inverse map ψ, note that by part 2 of Lemma 5.4, the thickening construction $T \mapsto \widetilde{T}$ on G-spaces commutes with the suspension functor $\Sigma_X^V = (X \times S^V) \wedge_X -$ for all finite-dimensional $V \subset \mathcal{U}$. This allows us to define the thickening on the category of spectra. Let E be a spectrum over X. Then for all finite-dimensional $V \subseteq W$ in $\mathcal{U}$, let $\rho_V^W : \Sigma_X^{W-V} E_V \to E_W$ be the structure map. Then we have

$$r_V^W : \Sigma_X^{W-V} \widetilde{E_V} \xrightarrow{\cong} \widetilde{\Sigma_X^{W-V} E_V} \xrightarrow{\widetilde{\rho_V^W}} \widetilde{E_W}.$$

To check that the composition of these maps are compatible for finite-dimensional representations $V \subseteq W_1 \subseteq W_2$ in $\mathcal{U}$, we need the following diagram of isomorphisms to commute:

$$\begin{tikzcd}
\Sigma_X^{W_2-V} \widetilde{E_V} \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & \widetilde{\Sigma_X^{W_2-V} E_V} \arrow[dd, "\cong"] \\
\Sigma_X^{W_2-W_1} \Sigma_X^{W_1-V} \widetilde{E_V} \arrow[d, "\cong"'] & & \\
\Sigma_X^{W_2-W_1} (\widetilde{\Sigma_X^{W_1-V} E_V}) \arrow[rr, "\cong"'] & & \widetilde{\Sigma_X^{W_2-W_1} \Sigma_X^{W_1-V} E_V}.
\end{tikzcd}$$

It is routine to check that this diagram commutes by the definition of the isomorphisms in the proof of Lemma 5.4. Thus, $\{\widetilde{E_V}\}$ form a prespectrum over X. Applying the spectrification functor gives the thickening $\widetilde{E}$ of E in the category of spectra.

Lemma 5.21. — *If E is a cofibrant and fibrant G-spectrum over X, then we have a natural weak equivalence of G-spectra over X*

$$g_E : E \longrightarrow \widetilde{E}$$

induced by the maps g_{E_V} on the level of spaces.

Proof. — For E a fibrant spectrum, each space of E is fibrant over X. So we have a weak equivalence of based G-spaces $g_{E_V} : E_V \to \widetilde{E_V}$ over X for every finite-dimensional $V \subset \mathcal{U}$. Note that for all finite-dimensional $V \subseteq W$ in $\mathcal{U}$, the diagram

$$\begin{tikzcd}
& \Sigma_X^{W-V} E_V \arrow[dl, "\Sigma_X^{W-V} g_{E_V}"'] \arrow[dr, "g_{\Sigma_X^{W-V} E_V}"] & \\
\Sigma_X^{W-V} \widetilde{E_V} \arrow[rr, "\cong"'] & & \widetilde{\Sigma_X^{W-V} E_V}
\end{tikzcd}$$

commutes, so the maps give a spacewise weak equivalence from E to the prespectrum $\{\widetilde{E_V}\}$.

Now suppose E is also cofibrant over X. Since the map g_E is natural with respect to retracts, we can assume without loss of generality that E is a G-cell spectrum over X. Then by arguments similar to those of I.8.14 of [8], each space E_V of E has the homotopy type of a relative G-cell complex over X. So by Lemma 5.3, each $g_{E_V} : E_V \to \widetilde{E_V}$ is a homotopy equivalence over X, and we get a spacewise homotopy equivalence of prespectra $g : E \to \{\widetilde{E_V}\}$ over X. Also, the prespectrum $\{\widetilde{E_V}\}$ is an inclusion prespectrum, and taking the spectrification functor from inclusion prespectra to spectra takes a spacewise homotopy equivalence to a weak equivalence of spectra. $\square$

With the thickening of a cofibrant and fibrant spectrum E over X, we can now define the inverse map ψ to φ on the level of G-spectra. To define $\overline{\psi}$ for spectra, let E be a cofibrant and fibrant spectrum over X indexed on the universe $\mathcal{U}$. By the definition of the model structure on the category of parametrized spectra, E_V is a fibrant G-space over X for every finite-dimensional $V \subset \mathcal{U}$. We define $\widetilde{E}$ by taking $\widetilde{E_V}$ spacewise, and we define $\overline{\psi}$ on the spectra level first, then taking its adjoint. More specifically, we define the map $\overline{\psi}$ on a spectrum E by applying $\overline{\psi}$ for spaces, as in (5.5), to E_V for each finite-dimensional representation $V \subset \mathcal{U}$. To make this work, we need to check that the following diagram commutes for all finite-dimensional $V \subset W$ in $\mathcal{U}$.

$$\begin{tikzcd}
\Sigma_X^{W-V} f^* f_\sharp(E_V) \arrow[r, "\Sigma_X^{W-V}\overline{\psi}"] \arrow[d, "\cong"'] & \Sigma_X^{W-V} \widetilde{E_V} \wedge_X C \arrow[d, "\cong"] \\
f^* f_\sharp(\Sigma_X^{W-V} E_V) \arrow[d, "f^* f_\sharp \rho_V^W"'] & (\widetilde{\Sigma_X^{W-V} E_V}) \wedge_X C \arrow[d, "\widetilde{\rho_V^W} \wedge_X C"] \\
f^* f_\sharp E_W \arrow[r, "\overline{\psi}"'] & \widetilde{E_W} \wedge_X C.
\end{tikzcd}$$

Going back to the definition of $\overline{\psi}$ for spaces and using the fact that the structure map ρ_V^W is a map over X, one checks that this diagram commutes. By taking $\overline{\psi}$ from (5.5) spacewise, we get a map of prespectra

$$\overline{\psi} : \{f^* f_\sharp(E_V)\} \longrightarrow \{\widetilde{E_V} \wedge_X C\}.$$

Applying the spectrification functor to both the source and the target gives the map of spectra

$$\overline{\psi} : f^* f_\sharp(E) \longrightarrow \widetilde{E} \wedge_X C.$$

Now taking the adjoint gives the map of spectra

$$\psi : f_\sharp(E) \longrightarrow f_*(\widetilde{E} \wedge_X C).$$

Also, if we take

$$\psi : f_\sharp(E_V) \longrightarrow f_*(\widetilde{E_V} \wedge_X C)$$

on each space E_V of E, and then spectrify, we get a map

$$\psi_{\text{spacewise}} : f_\sharp(E) \longrightarrow L\{f_*(\widetilde{E_V} \wedge_X C)\}.$$

Since the functor f_* does not commute with the spectrification functor L, the target is not the same as $f_*(\widetilde{E} \wedge_X C)$.

We will now show that Proposition 5.6 implies that the spectra-level maps φ and ψ are inverse weak equivalences. Since the map φ is natural on spectra over X, it is preserved by retracts. So it suffices to consider the case where E is a fibrant G-cell spectrum over X. Note that the space-level homotopies constructed in Proposition 5.6 only depend on a deformation retraction of the tubular neighborhood U' onto X, all the maps and homotopies are natural, they commute with suspensions and loops. Thus, for a cofibrant and fibrant spectrum E over X, by applying Proposition 5.6 to each space of E, and checking that the homotopies commute with the structure maps, we get that the statements of Proposition 5.6 still hold if we replace the based G-space T over X by the spectrum E. However, again since f_*E may not be a cofibrant spectrum, we do not have homotopical control over $\Sigma^V_Y f_*(E)$.

To solve this problem, we consider the cofibrant replacement functor Γ in the category of spectra over Y. For any spectrum D over Y, there is a cofibrant spectrum ΓD over Y, such that there is an acyclic fibration $a : \Gamma D \to D$. We can make Γ into a functor, and a into a natural transformation $\Gamma \to \mathrm{Id}$. For E a cofibrant and fibrant spectrum over X, the V-th suspension of the map (5.19) is

$$\varphi' : \Sigma^V_Y \Gamma f_*(E) \xrightarrow{\Sigma^V_Y a} \Sigma^V_Y f_*(E) \longrightarrow f_\sharp(E \wedge_X S(\nu)) \tag{5.22}$$

where the second map is obtained by taking the space-level φ on each space E_W of E, then spectrifying both source and target. We also define

$$\psi' : f_\sharp(E) \longrightarrow \Gamma f_*(\widetilde{E} \wedge_X S(\tau)). \tag{5.23}$$

Namely, ψ' is the lifting in the square

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Gamma f_*(\widetilde{E} \wedge_X S(\tau)) \arrow[d, "a"] \\
f_\sharp(E) \arrow[r, "\psi"'] \arrow[ur, dashed, "\psi'"] & f_*(\widetilde{E} \wedge_X S(\tau))
\end{tikzcd}$$

which exists since $f_\sharp(E)$ is cofibrant, and $\Gamma f_*(\widetilde{E} \wedge_X S(\tau)) \to f_*(\widetilde{E} \wedge_X S(\tau))$ is an acyclic fibration. Similarly, from $\psi_{\text{spacewise}} : f_\sharp(E) \to L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\}$, we get

$$\psi'_{\text{spacewise}} : f_\sharp(E) \longrightarrow \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\}.$$

The diagram

$$\begin{tikzcd}
\Sigma^V_Y f_\sharp(E) \arrow[r, "\Sigma^V_Y \psi'"] \arrow[dr, "\Sigma^V_Y \psi"'] & \Sigma^V_Y \Gamma f_*(\widetilde{E} \wedge_X S(\tau)) \arrow[r, "\Sigma^V_Y \varphi'"] \arrow[d, "\Sigma^V_Y a"] & \Sigma^V_Y f_\sharp(\widetilde{E}) \\
& \Sigma^V_Y f_*(\widetilde{E} \wedge_X S(\tau)) \arrow[ur, "\Sigma^V_Y \varphi"'] &
\end{tikzcd}$$

commutes, so the second statement of Proposition 5.6 gives that

$$(5.24) \qquad \Sigma^V_Y f_\sharp(E) \xrightarrow{\Sigma^V_Y \psi'} \Sigma^V_Y \Gamma f_*(\widetilde{E} \wedge_X S(\tau)) \xrightarrow{\Sigma^V_Y \varphi'} \Sigma^V_Y f_\sharp(\widetilde{E})$$

is homotopic to $\Sigma^V_Y f_\sharp(g_E)$. Likewise, we have the commutative diagram

$$\begin{tikzcd}
\Sigma^V_Y f_\sharp(E) \arrow[r, "\Sigma^V_Y \psi'_{\text{spacewise}}"] \arrow[dr, "\Sigma^V_Y \psi_{\text{spacewise}}"'] & \Sigma^V_Y \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[r, "\Sigma^V_Y \varphi'_{\text{spacewise}}"] \arrow[d, "\Sigma^V_Y a"] & \Sigma^V_Y f_\sharp(E') \\
& \Sigma^V_Y L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[ur, "\Sigma^V_Y \varphi_{\text{spacewise}}"'] &
\end{tikzcd}.$$

Hence, we also get from Proposition 5.6 that

$$(5.25) \quad \Sigma^V_Y f_\sharp(E) \xrightarrow{\Sigma^V_Y \psi'_{\text{spacewise}}} \Sigma^V_Y \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \xrightarrow{\Sigma^V_Y \varphi'_{\text{spacewise}}} \Sigma^V_Y f_\sharp(\widetilde{E})$$

is homotopic to $\Sigma^V_Y f_\sharp(g_E)$. Now $g_E : E \to \widetilde{E}$ is the spectrification of a spacewise homotopy equivalence of prespectra. The functors Σ^V_Y and $f_\sharp$ commute with the spectrification functor and preserve spacewise homotopy equivalences of prespectra. Hence, $\Sigma^V f_\sharp(g_E)$ is the spectrification of a spacewise homotopy equivalence of prespectra, so it is a weak equivalence of spectra. This gives that (5.24) and (5.25) are weak equivalences of spectra over Y.

Next, we would like to desuspend (5.25) by V. Recall that if D is a cofibrant spectrum, then there is a natural homotopy equivalence between $\Sigma^V D$ and the shift suspension $\Sigma^V_{\text{shift}} D$ of D. A similar statement holds for cofibrant spectra over Y. In (5.24), the spectra $\Sigma^V_Y f_\sharp(E)$ and $\Sigma^V_Y \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\}$ are cofibrant, since S^V_Y is a homotopy cell bundle over Y. However, we do not know about $\Sigma^V f_\sharp(\widetilde{E})$. To get around this problem, recall the cylinder construction K [**8**] Section I.6 and [**4**], which replaces a prespectrum by one that is Σ-cofibrant. There is an analogous construction in the category of prespectra over Y. (For more details on the cylinder construction over a base space, see Chapter 6 below.) For any prespectrum D over Y, we have a functorial map $r_D : KD \to D$, where KD is a Σ-cofibrant spectrum over Y. Define the cylinder construction functor Z on spectra over Y to be LD, where L is the spectrification functor from G-prespectra over X to G-spectra over X. Applying L to the natural transformation $r : K \to \text{Id}$ on prespectra gives a natural transformation $Z \to \text{Id}$ on the category of spectra over Y, which we will denote also by r. By arguments analogous to those of Proposition X.5.4 of [4], smashing with S^V_Y commutes with K on the level of G-prespectra over Y. So it commutes with Z as well. Hence, we have the commutative diagram of G-spectra over Y

$$\begin{tikzcd} \Sigma^V_Y Z f_\sharp(E) \arrow[r] \arrow[d, "\Sigma^V_Y r"'] & \Sigma^V_Y Z\Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[r] \arrow[d, "\Sigma^V_Y r"] & \Sigma^V_Y Z f_\sharp(\widetilde{E}) \arrow[d, "\Sigma^V_Y r"] \\ \Sigma^V_Y f_\sharp(E) \arrow[r] & \Sigma^V_Y \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[r] & \Sigma^V_Y f_\sharp(\widetilde{E}) \end{tikzcd} \tag{5.26}$$

where the bottom row is (5.25), and the top row is Z applied to it. By arguments similar to that of Proposition I.8.14 of [8], the left vertical map of this diagram is a homotopy equivalence, since $f_\sharp(E)$ is a G-cell spectrum over Y. Now the bottom right corner of the diagram is the spectrification of the prespectrum $\{\Sigma^V_Y f_\sharp(\widetilde{E_V})\}$ over Y. By arguments similar to that of Construction I.6.8 of [8], the map

$$r : K\{\Sigma^V_Y f_\sharp(\widetilde{E_V})\} \longrightarrow \{\Sigma^V_Y f_\sharp(\widetilde{E_V})\}$$

is a spacewise homotopy equivalence. Passing to spectra gives that the right vertical map of diagram (5.26) is a weak equivalence of spectra over Y. Hence, the top row of (5.26) is a weak equivalence of G-spectra over Y.

Now by choosing G-cell decompositions of the tubular neighborhood U' of the diagonal in $X \times_Y X$ so that the inclusion $U' \subseteq X \times_Y X$ is a relative G-cell complex, one sees that the thickening of a relative G-cell complex over X is also a relative G-cell complex over X, so each $\widetilde{E_V}$ is also of G-cell homotopy type. Therefore, each space $f_\sharp\widetilde{E_V}$ is of G-cell homotopy type over Y. This gives that the upper right corner of diagram (5.26) is of the homotopy type of a G-cell spectrum over Y. Hence, in the top row of (5.26) we can replace the Σ^V_Y by Σ^V_{shift} up to homotopy equivalences. But Σ^V_{shift} is an invertible functor, which gives that

$$Z f_\sharp(E) \xrightarrow{Z\psi'_{\text{spacewise}}} Z\Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \xrightarrow{Z\varphi'_{\text{spacewise}}} Z f_\sharp(\widetilde{E}) \tag{5.27}$$

is a weak equivalence of G-spectra over Y. But we also have the commutative diagram

$$\begin{tikzcd} Z f_\sharp(E) \arrow[r, "Z\psi'_{\text{spacewise}}"] \arrow[d, "r"'] & Z\Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[r, "Z\varphi'_{\text{spacewise}}"] \arrow[d, "r"] & Z f_\sharp(\widetilde{E}) \arrow[d, "r"] \\ f_\sharp(E) \arrow[r, "\psi'_{\text{spacewise}}"'] & \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \arrow[r, "\varphi'_{\text{spacewise}}"'] & f_\sharp(\widetilde{E}). \end{tikzcd}$$

The top row and the vertical maps of this diagram are weak equivalences, so we get that the composition

$$f_\sharp(E) \xrightarrow{\psi'_{\text{spacewise}}} \Gamma L\{f_*(\widetilde{E_V} \wedge_X S(\tau))\} \xrightarrow{\varphi'_{\text{spacewise}}} f_\sharp(\widetilde{E}) \tag{5.28}$$

is a weak equivalence of G-spectra over Y. Now we substitute $E \wedge_X C^{-1}$ for E, where $C^{-1} = \Sigma^{-V}_{\text{shift}} S(\nu)$. Since by Lemma 4.5, $E \wedge_X C^{-1} \wedge_X S(\tau)$ is homotopy equivalent to E as spectra over X, we get that

$$(5.29) \qquad f_\sharp(E \wedge_X C^{-1}) \xrightarrow{\psi_{\text{spacewise}}} \Gamma L\{f_*(\widetilde{E_V})\} \xrightarrow{\varphi'_{\text{spacewise}}} f_\sharp(\widetilde{E \wedge_X C^{-1}})$$

is a weak equivalence of spectra. But now we have a spacewise homotopy equivalence of prespectra $f_*(g_E) : \{f_*(\widetilde{E_V})\} \to f_*(E)$, where the source is an inclusion prespectrum, and the target is a spectrum. Hence, applying the spectrification functor and then Γ gives a weak equivalence of spectra

$$\Gamma L\{f_*(\widetilde{E_V})\} \longrightarrow \Gamma f_*(E).$$

By similar arguments, we have a weak equivalence of spectra

$$f_\sharp(\widetilde{E \wedge_X C^{-1}}) \longrightarrow f_\sharp(E \wedge_X C^{-1}).$$

By the definition of φ' and $\varphi'_{\text{spacewise}}$, the diagram

$$(5.30) \qquad \begin{tikzcd} f_\sharp(E \wedge_X C^{-1}) \arrow[r, "\psi'_{\text{spacewise}}"] \arrow[d, "="'] & \Gamma L\{f_*(\widetilde{E_V})\} \arrow[r, "\varphi'_{\text{spacewise}}"] \arrow[d, "\simeq"] & f_\sharp(\widetilde{E \wedge_X C^{-1}}) \arrow[d, "\simeq"] \\ f_\sharp(E) \arrow[r] & \Gamma f_*(E) \arrow[r, "\varphi'"'] & f_\sharp(E \wedge_X C^{-1}) \end{tikzcd}$$

commutes. Hence, the bottom row of the diagram is a weak equivalence of spectra, and we get that $\varphi' : \Gamma f_*(E) \to f_\sharp(E \wedge_X C^{-1})$ has a left inverse in the homotopy category of G-spectra over Y.

For the other composition, consider the map

$$\bar{c} : \Sigma^V_Y f_*(E) \longrightarrow f_*(\Sigma^V_X E)$$

adjoint to the counit of adjunction

$$f^* \Sigma^V f_*(E) \cong \Sigma^V_X f^* f_* E \xrightarrow{c} \Sigma^V_X E.$$

We have the square

$$(5.31) \qquad \begin{tikzcd} * \arrow[rr] \arrow[d] & & \Gamma f_*(\Sigma^V_X E) \arrow[d, "a"] \\ \Sigma^V_Y \Gamma f_*(E) \arrow[r, "\Sigma^V_Y a"'] \arrow[urr, dashed, "u"] & \Sigma^V_Y f_*(E) \arrow[r, "c"'] & f_*(\Sigma^V_X E). \end{tikzcd}$$

Since $\Sigma^V_Y \Gamma f_*(E)$ is cofibrant and $a : \Gamma f_*(\Sigma^V_X E) \to f_*(\Sigma^V_X E)$ is an acyclic fibration, the dotted arrow u exists. The adjoint to the first statement of Proposition 5.6 gives that the diagram in the category of G-prespectra over Y

$$(5.32)\qquad \begin{tikzcd} \Sigma_Y^V f_*(E) \arrow[r, "\varphi"] \arrow[d, "\bar{c}"'] & f_\sharp(E \wedge_X S(\nu)) \arrow[d, "\psi_{\text{spacewise}}"] \\ f_*(\Sigma_X^V E) \arrow[r, "f_*(g_{E\wedge_X S(\nu)})"'] & f_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)) \end{tikzcd}$$

commutes up to homotopy. The bottom map is a spacewise homotopy equivalence of prespectra. Again, let K be the cylinder construction in the category of prespectra over X, and $Z = LK$ be the cylinder construction in the category of spectra over X. Then we have a spacewise homotopy equivalence of prespectra

$$a : Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)) \longrightarrow f_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)).$$

Applying the prespectra-level cylinder construction to $\psi_{\text{spacewise}}$ gives a map of G-prespectra over Y

$$K(\psi_{\text{spacewise}}) : Kf_\sharp(E \wedge_X S(\nu)) \longrightarrow Kf_\sharp(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)).$$

It follows from a diagram chase, using diagrams (5.31) and (5.32), as well as the definitions of φ' and ψ', that the big square of the following diagram in the category of G-prespectra over Y commutes up to homotopy.

$$\begin{tikzcd} K\Sigma_Y^V \Gamma f_*(E) \arrow[r, "\Sigma_Y^V K\varphi'"] \arrow[d, "u"'] & Kf_\sharp(E \wedge_X S(\nu)) \arrow[r, "K\psi_{\text{spacewise}}"] & Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)) \arrow[d, "a"] \\ K\Gamma f_*(\Sigma_X^V E) \arrow[r, "Kg_{E\wedge_X S(\nu)}"'] & Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)) \arrow[r, "a"'] \arrow[ur, "="'] & f_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)) \end{tikzcd}$$

Since the map a is a spacewise homotopy equivalence, we can lift the homotopy to

$$Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)).$$

Hence, the composition of prespectra

(5.33)

$$K\Sigma_Y^V \Gamma f_*(E) \xrightarrow{K\Sigma_Y^V \varphi'} f_\sharp(E \wedge_X S(\nu)) \xrightarrow{K\psi_{\text{spacewise}}} Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau))$$

is naturally spacewise homotopic to the composition of prespectra

(5.34)

$$K\Sigma_Y^V \Gamma f_*(E) \xrightarrow{K(u)} K\Gamma f_*(\Sigma_X^V E) \xrightarrow{Kf_*(g_{E\wedge_X S(\nu)})} Kf_*(\{E_V \widetilde{\wedge_X S}(\nu)\} \wedge_X S(\tau)).$$

Since K preserves spacewise homotopy equivalences of prespectra, the second map of (5.34) is a spacewise homotopy equivalence. The functor Γ also commutes with shift suspensions. So we can take u to be the composition

$$\begin{aligned}\Sigma^V_Y\Gamma f_*(E) &\xrightarrow{\simeq} \Sigma^V_{\text{shift}}\Gamma f_*(E)\\ &\xrightarrow{\cong} \Gamma\Sigma^V_{\text{shift}} f_*(E)\\ &\xrightarrow{\cong} \Gamma f_*(\Sigma^V_{\text{shift}} E)\\ &\xrightarrow{\simeq} \Gamma f_*(\Sigma^V_X E).\end{aligned}$$

All the maps of this composition are isomorphisms or homotopy equivalences, so u is a homotopy equivalence. Thus, the first map Ku of (5.34) is also a spacewise homotopy equivalence. Hence, (5.33) is a spacewise homotopy equivalence of prespectra. Now all the spectra concerned in (5.33) are inclusion prespectra, so applying the spectrification functor to it gives a weak equivalence of spectra

$$Z\Sigma^V_Y\Gamma f_*(E) \xrightarrow{Z\Sigma^V_Y\varphi'} Zf_\sharp(E\wedge_X S(\nu)) \xrightarrow{Z\psi_{\text{spacewise}}} Zf_*(\{E_V \widetilde{\wedge_X S(\nu)}\}\wedge_X S(\tau)).$$

But since $\Sigma^V_Y\Gamma f_*(E)$ and $f_\sharp(E \wedge_X S(\nu))$ are cofibrant, each is naturally homotopy equivalent to its cylinder construction. Hence, we get that the composition of spectra

$$\Sigma^V_Y\Gamma f_*(E) \xrightarrow{\Sigma^V_Y\varphi'} f_\sharp(E\wedge_X S(\nu)) \xrightarrow{Z\psi_{\text{spacewise}}} Zf_*(\{E_V \widetilde{\wedge_X S(\nu)}\}\wedge_X S(\tau))$$

is a weak equivalence.

This gives that

$$\Sigma^V_Y\varphi' : \Sigma^V\Gamma f_*(E) \longrightarrow f_\sharp(E\wedge_X S(\nu)) = \Sigma^V_{\text{shift}} f_\sharp(E\wedge_X C^{-1})$$

has a right inverse in the homotopy category of G-spectra over Y. But since both its source and target are cofibrant spectra, we can replace Σ^V_Y in the source by Σ^V_{shift}. Now taking shift desuspension gives that

$$\varphi' : \Gamma f_*(E) \longrightarrow f_\sharp(E\wedge_X C^{-1})$$

has a right inverse in the homotopy category of G-spectra over Y. But also, by (5.30), it also has a left inverse in the homotopy category of G-spectra over Y. Therefore, we get that the map (5.19)

$$\varphi' : \Gamma f_*(E) \longrightarrow \Sigma^{-V}_{\text{shift}} f_\sharp(E\wedge_X S(\nu)) = f_\sharp(E\wedge_X C^{-1})$$

is a natural weak equivalence of spectra. By the commutative diagram (5.20) this gives that $\varphi : f_*(E) \to f_\sharp(E\wedge_X C^{-1})$ is a natural weak equivalence of spectra, for all equivariant smooth families of manifolds $f : X \to Y$, Y compact, and all cofibrant and fibrant spectra E over X.

For general equivariant smooth family of manifolds $f : X \to Y$, where Y is any G-cell complex with countably many cells, we use a colimit argument on the finite subcomplexes of Y. We observe the following fact. Suppose $i : K' \subset K$ are compact G-cell complexes, and $f : X_K \to K$ is a smooth family of manifolds over K. Let $X_{K'} = X \times_K K' = f^{-1}(K')$. Also, write $f' = f|_{X_{K'}} : X_{K'} \to K'$, making $X_{K'}$ a smooth family of manifolds over K'. Let $i : X_{K'} \to X_K$ be the inclusion map. For a cofibrant and fibrant spectrum E_K over X_K, we defined

$$\varphi_{X_K} : f_*(E_K) \xrightarrow{\simeq} f_\sharp(E_K \wedge_X C_f^{-1}).$$

By Lemma 2.2, the map $i^*(\varphi_{X_K})$ is a map

$$\begin{aligned} f'_* i^*(E_K) \cong i^* f_*(E_K) &\longrightarrow i^* f_\sharp(E_K \wedge_{X_K} C_f^{-1}) \\ &\cong f'_\sharp i^*(E_K \wedge_{X_K} C_f^{-1}) \\ &\simeq f'_\sharp(i^* E_K \wedge_{X_{K'}} C_{f'}^{-1}) \end{aligned}$$

where the last map is a homotopy equivalence natural in K', with respect to inclusions. By the naturality of the construction of φ, one can check that this is just $\varphi_{X_{K'}}$, with respect to the map $f' : X_{K'} \to K'$. The diagram

$$\begin{tikzcd} f_* i^*(E_K) \arrow[r, "\varphi_{X_K}"] \arrow[d] & f_\sharp i^*(E_K \wedge_{X_K} C_f^{-1}) \arrow[d] \\ f_*(E_{K'}) \arrow[r, "\varphi_{X_{K'}}"'] & f_\sharp(E_{K'} \wedge_{X_{K'}} C_f^{-1}) \end{tikzcd}$$

commutes. So given a general G-cell complex Y, with $f : X \to Y$ a smooth family of manifolds, for each compact G-subcomplex $K \subset Y$, let $f_K : X \times_Y K \to K$ be the pullback of f with respect to the inclusion $K \subset Y$. We have a stable map

$$t_K : S_K^0 \longrightarrow f_\sharp(C_{f_K^{-1}}^{-1}) = f_\sharp(i^* C_f^{-1}).$$

Now let $f : X \to Y$ be any equivariant smooth family of manifolds, and let K be a finite subcomplex of Y, with inclusion $i_K : K \to Y$. By equivariant cell subdivision, $X_K = X \times_Y K$ is also a finite subcomplex of X. We will denote the inclusion $X_K \to X$ also by i_K. We have that for each $i_K : K \to Y$,

$$(i_K)_\sharp(S_K^0) = S_K^0 \amalg_K X$$

as spectra over Y. Hence, we in fact have

$$S_Y^0 \cong \mathrm{colim}_K (i_K)_\sharp S_K^0$$

as spectra over Y, over all finite subcomplexes $K \subset X$. Similarly, defining $C_f^{-1} = DC_f$ by any choice of linear isometry $X \to \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U})$, we get that

$$f_\sharp(C_f^{-1}) \cong f_{K\sharp}(\mathrm{colim}_K (i_K)_\sharp C_{f_K}^{-1}) \cong \mathrm{colim}_K (i_K)_\sharp (f_{K\sharp}(C_{f_K}^{-1})).$$

So passing to the colimits, we get a stable map in the category of spectra over Y

$$t_Y : S_Y^0 \longrightarrow f_\sharp(C_f^{-1}).$$

This allows us to define the natural map $\varphi_X : f_*(E) \to f_\sharp(E \wedge_X C_f^{-1})$ for E a cofibrant and fibrant spectrum over X, similarly as in the case when Y is compact.

Now suppose E a cofibrant and fibrant spectrum over X. For any finite subcomplex K of Y, we now denote both the inclusion maps $K \to Y$ and $X_K \to X$ by i_K, and the restriction $X_K \to K$ of f by f_K. By Lemma 2.2, we have natural isomorphisms

$$i_K^* f_*(E) \cong (f_K)_* i_K^*(E)$$

and

$$i_K^* f_\sharp(E \wedge_X C_f^{-1}) \cong (f_K)_\sharp i_K^*(E \wedge_X C_f^{-1}).$$

Since i_K^* commutes with external smash products of spectra, and $i_K^*(C_f^{-1})$ is naturally homotopy equivalent to $C_{f_K}^{-1}$, we get that for any choice of linear isometry

$$X \longrightarrow \mathcal{I}(\mathcal{U}^{\oplus 2}, \mathcal{U}),$$

$(f_K)_\sharp i_K^*(E \wedge_X C_f^{-1})$ is naturally homotopy equivalent to $(f_K)_\sharp(i_K^*(E) \wedge_{X_K} C_{f_K}^{-1})$. By the definition of φ_X for the noncompact case, it is straightforward to check that the diagram

$$\begin{tikzcd} i_K^* f_*(E) \arrow[r, "i_K^*\varphi_X"] \arrow[d, "\cong"'] & i_K^* f_\sharp(E \wedge_X C_f^{-1}) \arrow[d, "\simeq"] \\ f_*(i_K^* E) \arrow[r, "\varphi_{X_K}"'] & f_\sharp(i_K^* E \wedge_{X_K} C_{f_K}^{-1}) \end{tikzcd}$$

commutes for every finite subcomplex K of Y. However, although $i_K^*(E)$ is a fibrant spectrum over X_K, it is not necessarily cofibrant, so we do not have that the bottom map of the diagram is a weak equivalence. To solve this problem, we make the following construction.

Definition 5.35. — Let K' be a G-cell complex, and K a subcomplex, with inclusion $i : K \to K'$. For a G-spectrum E' over K' and a G-spectrum E over K indexed on a G-universe $\mathcal{U}$, a map $g : E \to i^* E'$ gives a map of unbased G-spaces

$$g_V : E_V \longrightarrow i^* E'_V \xrightarrow{\subseteq} E'_V$$

for each finite-dimensional representation $V \subset \mathcal{U}$. We say that a map $g : E \to i^* E'$ of G-spectra over K is a map over $i : K \xrightarrow{\subseteq} K'$, and for every finite-dimensional representation V contained in $\mathcal{U}$, the diagram of unbased G-spaces

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d] & K' \arrow[d] \\ E_V \arrow[r, "g_V"'] \arrow[d] & E'_V \arrow[d] \\ K \arrow[r, "i"'] & K' \end{tikzcd}$$

commutes. We write a map of spectra over the inclusion $K \subseteq K'$ just as $g : E \to E'$, even though strictly speaking, E and E' are in different categories. A map of spectra over the inclusion $K \subseteq K'$ is an inclusion over $K \subseteq K'$ if it is a spacewise inclusion.

For our G-cell complex Y, let $\{K_j\}$ be an increasing sequence of finite subcomplexes, such that $Y = \bigcup_j K_j$. For the equivariant smooth family of manifolds $f : X \to Y$, write X_j for $X_{K_j} = X \times_Y K_j$, and $f_j : X_j \to K_j$ for the restriction of f. Then $\{X_j\}$ is also an increasing sequence of finite subcomplexes of X, and $X = \bigcup_j X_j$. If E is a spectrum over X, then for every j, we have an inclusion of spectra over the inclusion $X_j \subseteq X_{j+1}$

$$i_j^*(E) \xrightarrow{\subseteq} i_{j+1}^*(E).$$

In particular, for each finite-dimensional representation $V \subset \mathcal{U}$, $E_V = \bigcup_j i_j^*(E_V)$. In this situation, we say that

$$E = \bigcup_j i_j^*(E)$$

is the (spacewise) union of the spectra $i_j^*(E)$ over the sequence of inclusions $X_j \subseteq X_{j+1}$. Conversely, suppose we have a sequence of inclusions of spectra $D_j \to D_{j+1}$ over the inclusions $X_j \subseteq X_{j+1}$, where D_j is a spectrum over X_j. Then their spacewise union forms a spectrum D over X, and we write

$$D = \bigcup_j D_j.$$

We will use the following lemma to show that the compact case of Theorem 4.9 leads to the general case.

Lemma 5.36. — *Suppose E is a cofibrant and fibrant G-spectrum over X. Then there is some cofibrant and fibrant spectrum E' over X, such that E is weakly equivalent to E', and*

$$E' = \bigcup_j E_j'$$

where each E_j' is a cofibrant and fibrant spectrum over X_j, with an inclusion of spectra $E_j' \to E_{j+1}'$ over the inclusion $X_j \to X_{j+1}$ for every j.

Given the lemma, we will show that

$$\varphi : f_*(E') \longrightarrow f_\sharp(E' \wedge_X C_f^{-1}) \tag{5.37}$$

is a weak equivalence of spectra over Y. To this end, we will show that both f_* and $f_\sharp(- \wedge_X C_f^{-1})$ commutes with unions of spectra. Namely, we claim that

$$f_*(E') = f_*\Big(\bigcup_j E_j'\Big) \cong \bigcup_j ((f_j)_*(E'(j))) \tag{5.38}$$

and

$$f_\sharp(E' \wedge_X C_f^{-1}) = f_\sharp\Big(\Big(\bigcup_j E_j'\Big) \wedge_X C_f^{-1}\Big) \simeq \bigcup_j ((f_j)_\sharp(E_j' \wedge_{X_j} C_{f_j}^{-1})) \tag{5.39}$$

is a natural homotopy equivalence. For (5.38), note that since f_* is taken spacewise, it suffices to show that if T is a based space over X, such that $T = \bigcup_j T_j$ for an increasing sequence of based spaces T_j over X_j, such that the diagram

$$\begin{tikzcd}
X_j \arrow[r, "\subseteq"] \arrow[d] & X_{j+1} \arrow[d] \\
T_j \arrow[r, "\subseteq"'] \arrow[d] & T_{j+1} \arrow[d] \\
X_k \arrow[r, "\subseteq"'] & X_{j+1}
\end{tikzcd}$$

commutes for every j, then $f_*(T) \cong \bigcup_j (f_j)_*(T_j)$. We have

$$f_*(T) = \mathrm{Sec}_Y(X, Z) = \coprod_{y\in Y} \mathrm{Sec}(X_y, T_y)$$

as sets, where X_y and T_y are the fibers of X and T over a point y of Y, and $\mathrm{Sec}(X_y, T_y)$ denotes the space of (nonequivariant) sections to the structure map $T_y \to X_y$, with G-action by conjugation. Again, here $\coprod_{y\in Y} \mathrm{Sec}(X_y, T_y)$ is topologized as in [**7**]. Note that since $X \to Y$ is an equivariant smooth family of manifolds, $X_y \cong M$ as nonequivariant spaces for every $y \in Y$. So $f_*(Y)$ is (G-equivariantly) the same as

$$\coprod_{y\in Y} \mathrm{Sec}(M, T_y) \cong \coprod_{y\in Y} \mathrm{Sec}(M, \bigcup_j (T_j)_y).$$

But M is compact, and for every j, $(T_j)_y \to (T_{j+1})_y$ is an inclusion. Hence, we also have a G-equivariant isomorphism

$$\mathrm{Sec}\left(M, \bigcup_j ((T_r)_y)\right) \cong \bigcup_j \mathrm{Sec}(M, (T_j)_y)$$

where for each r, $\mathrm{Sec}(M, (T_j)_y) \to \mathrm{Sec}(M, (T_{j+1})_y)$ is an inclusion. Hence, we get

$$f_*(T) \cong \coprod_{y\in Y} \Big(\bigcup_j \mathrm{Sec}(M, (T_j)_y)\Big).$$

On the other hand,

$$\bigcup_j (f_j)_*(T_j) = \bigcup_j (\coprod_{y\in K_j} \mathrm{Sec}(M, (T_j)_y))$$

where the right hand side is given an appropriate topology. It is easy to see that these two are G-equivariantly isomorphic, which gives (5.38). To prove (5.39), note that up to natural homotopy equivalences, we can define $C_{f_j}^{-1}$ to be $i_j^*(C_f^{-1})$. Then

$$C_f^{-1} = \bigcup_j C_{f_j}^{-1}$$

is a union of spectra over the sequence of inclusions $X_j \subseteq X_{j+1}$. By arguments similar as above, unions of spaces over the sequence of inclusions $X_j \to X_{j+1}$ commutes with taking loops. As a directed colimit, it also commutes with colimits. Hence, unions of spectra commutes with the spectrification functor. So again, it suffices to show that for a based G-space T over X, such that $T = \bigcup_j T_j$ for an increasing sequence T_j of based G-spaces over X_j,

$$f_\sharp(T \wedge_X C_f^{-1}) \cong \bigcup_j f_\sharp(T_j \wedge_{X_j} C_{f_j}^{-1}).$$

Now if Z is a based G-space over X, it is easy to see that

$$T \wedge_X Z \cong \bigcup_j (T_j \wedge_{X_j} i_j^*(Z))$$

G-equivariantly. This is because for each j, there is an inclusion

$$T_j \wedge_{X_j} i_j^* Z \longrightarrow T \wedge_X Z$$

over the inclusion $X_j \subseteq X$. This induces a map $\bigcup_j (T_j \wedge_{X_j} i_j^* Z) \to T \wedge_X Z$, which is an G-equivariant isomorphism. But $C_{f_j}^{-1} = i_j^*(C_f^{-1})$ is obtained by applying i_j^* to C_f^{-1} spacewise, so we get that

$$T \wedge_X C_f^{-1} \cong \bigcup_j (T_j \wedge_{X_j} C_{f_j}^{-1}). \tag{5.40}$$

Similarly, since $X = \bigcup_j X_j$, $Y = \bigcup_j K_j$, and the union commutes with pushouts, we get that for a space $T = \bigcup_{T_j}$ as above, $f_\sharp(T) \cong \bigcup_j ((f_j)_\sharp T_j)$. Passing to spectra, we get that for any spectrum $D = \bigcup_j D_j$ over X which is a union of an increasing sequence of spectra D_j over X_j over the inclusions $X_j \subseteq X_{j+1}$, $f_\sharp(D) \cong \bigcup_j ((f_j)_\sharp D_j)$. Applying this to (5.40) gives (5.39).

This gives the commutative diagram in the category of G-spectra over Y

$$\begin{tikzcd} f_*(E') \arrow[r, "\varphi_X"] \arrow[d, "\cong"'] & f_\sharp(E' \wedge_X C_f^{-1}) \arrow[d, "\simeq"] \\ \bigcup_j (f_j)_*(E'_j) \arrow[r, "\bigcup_j \varphi_{X_j}"'] & \bigcup_j (f_j)_\sharp(E'_j \wedge_{X_j} C_{f_j}^{-1}). \end{tikzcd} \tag{5.41}$$

Each $\varphi_{X_j} : (f_j)_*(E'_j) \to (f_j)_\sharp(E'_j \wedge_{X_j} C_{f_j}^{-1})$ is now a weak equivalence of G-spectra over K_j, so on each space, φ_{X_j} is a weak equivalence of unbased G-spaces. The union of weak equivalences of unbased G-spaces is a weak equivalence, so the bottom map $\bigcup_j \varphi_{X_j}$ of (5.41) is a weak equivalence of unbased G-spaces, *i.e.* a weak equivalence of G-spectra over Y. Therefore, the top map of (5.41) is a weak equivalence of G-spectra over Y, which gives Theorem 4.9 for E'.

Hence, we have that

$$\varphi : f_*(E') \longrightarrow f_\sharp(E' \wedge_X C_f^{-1})$$

is a weak equivalence of spectra over Y. But we have the following diagram in the homotopy category of spectra over Y.

$$\begin{tikzcd} f_*(E') \arrow[r, "\varphi", "\simeq"'] \arrow[d] & f_\sharp(E' \wedge_X C_f^{-1}) \arrow[d] \\ f_*(E) \arrow[r, "\varphi"'] & f_\sharp(E \wedge_X C_f^{-1}). \end{tikzcd}$$

By the functoriality of the constructions to obtain E' and the definition of φ, it is routine to check that this diagram commutes. But E and E' are now both cofibrant and fibrant over X. Since f_* preserves weak equivalences of fibrant spectra, the left vertical map of this diagram is an isomorphism in the homotopy category of spectra over Y. Also, since C_f^{-1} is a homotopy cell bundle spectrum over X, both $E\wedge_X C_f^{-1}$ and $E'\wedge_X C_f^{-1}$ are cofibrant over X by Lemma 4.19. Since $f_\sharp$ preserves weak equivalences between cofibrant spectra, the right vertical map is also an isomorphism in the homotopy category of spectra over Y. Hence, the bottom map of the diagram is an isomorphism in the homotopy category of spectra over Y, *i.e.* a weak equivalence of spectra over Y.

To finish the proof of Theorem 4.9, it remains to prove Lemma 5.36.

Proof of Lemma 5.36. — For each j and any finite-dimensional representation V in the universe $\mathcal{U}$, the diagram of unbased G-spaces

$$\begin{tikzcd} X_j \arrow[r, "\subseteq"] \arrow[d] & X_{j+1} \arrow[d] \\ i_j^*(E_V) \arrow[r, "\subseteq"'] \arrow[d] & i_{j+1}^*(E_V) \arrow[d] \\ X_j \arrow[r] & X_{j+1} \end{tikzcd}$$

commutes. Hence, we have an inclusion of spectra $i_j^*E \to i_{j+1}^*E$ over the inclusion $X_j \subseteq X_{j+1}$, and $E = \bigcup_j i_j^*E$ is the union over the sequence of these inclusions. Let Γ_j be the cofibrant replacement functor of spectra over X_j. Then the functors Γ_j are also natural with respect to inclusions of spectra over the inclusions $X_j \subseteq X_{j+1}$. To see this, note that the functor Γ_j is obtained by attaching to X_j all cells of the form $\Sigma_V^\infty((G/H \times D^n) \amalg X_j)$ such that there is a commutative diagram of G-spectra over X_j

$$\begin{tikzcd} \Sigma_V^\infty((G/H \times S^{n-1}) \amalg X_j) \arrow[r] \arrow[d] & X_j \arrow[d] \\ \Sigma_V^\infty((G/H \times D^n) \amalg X_j) \arrow[r] & i_j^*E. \end{tikzcd}$$

By adjunction, this is equivalent to the diagram in the category of unbased G-spaces

$$\begin{tikzcd} G/H \times S^{n-1} \arrow[r] \arrow[d] & X_j \arrow[d] \\ G/H \times D^n \arrow[r] & i_j^*(E_V). \end{tikzcd}$$

By composing with the inclusion $i_j^*(E_V) \to i_{j+1}^*(E_V)$ over $X_j \subseteq X_{j+1}$, we get a diagram of unbased G-spaces

$$\begin{tikzcd} G/H \times S^{n-1} \arrow[r] \arrow[d] & X_{j+1} \arrow[d] \\ G/H \times D^n \arrow[r] & i_{j+1}^*(E_V) \end{tikzcd}$$

which gives a cell $\Sigma_V^\infty((G/H \times D^n) \amalg K_{j+1})$ in the category of G-spectra over X_{j+1}. (Note that here, Σ_V^∞ now denotes shift desuspension of the suspension spectrum over X_{j+1} instead of over X_j.) Therefore, each stage of the small objects constructions constructing Γ_j and Γ_{j+1} are natural with respect to the inclusion of spectra $i_j^*E \subseteq i_{j+1}^*E$. This gives that Γ_j is natural with respect to inclusions of spectra. For every j, we have a map of spectra $\Gamma_j i_j^*E \to \Gamma_{j+1} i_{j+1}^*E$ over the inclusion $X_j \to X_{j+1}$, and each $\Gamma_j i_j^*E$ is a cofibrant spectrum over X_j. It is also in fact fibrant over X_j, since i_j^* takes a fibrant spectrum over X to a fibrant spectrum over X_j, and $\Gamma_j i_j^*E \to i_j^*E$ is an acyclic fibration.

Now for each j, let $E_j(0)$ be the telescope of the sequence of maps of spectra

$$\begin{tikzcd} X_0 \arrow[r, "\subseteq"] \arrow[d] & X_1 \arrow[r, "\subseteq"] \arrow[d] & \cdots \arrow[r, "\subseteq"] & X_j \arrow[d] \\ \Gamma_0 i_0^*E \arrow[r] \arrow[d] & \Gamma_1 i_1^*E \arrow[r] \arrow[d] & \cdots \arrow[r] & \Gamma_j i_j^*E \arrow[d] \\ X_0 \arrow[r, "\subseteq"'] & X_1 \arrow[r, "\subseteq"'] & \cdots \arrow[r, "\subseteq"'] & X_j \end{tikzcd}$$

over the inclusions $X_r \subseteq X_{r+1}$, with $r < j$. This is obtained by taking the telescope construction spacewise (in the category of unbased G-spaces), then taking the spectrification functor over X_j. Then $E_j(0)$ is cofibrant over X_j, $E_j(0)$ is weakly equivalent to i_j^*E, and we have an inclusion of spectra $E_j(0) \to E_{j+1}(0)$ over the inclusion $X_j \subseteq X_{j+1}$. Let ΓE be the telescope of the infinite sequence of maps of spectra

$$\Gamma_0 i_0^*E \longrightarrow \Gamma_1 i_1^*E \longrightarrow \Gamma_2 i_2^*E \longrightarrow \cdots$$

over the infinite sequence of inclusions

$$X_0 \xrightarrow{\subseteq} X_1 \xrightarrow{\subseteq} X_2 \xrightarrow{\subseteq} \cdots .$$

Then ΓE is a cofibrant spectrum over X, and $\Gamma E \to E$ is a weak equivalence of spectra over X. Also, $\Gamma E \cong \bigcup_j E_j(0)$.

However, ΓE and the $E_j(0)$'s are now no longer fibrant, so we need to apply fibrant replacement to them again. We will use the fibrant replacement functors on the $E_j(0)$'s to obtain the E'_j's inductively. Since $E_0(0) = \Gamma_0 i_0^* E$, it is fibrant over X_0, so we define $E'_0 = E_0(0)$. Now suppose we have constructed spectra $E'_0, \ldots, E'_j$ over $X_0, \ldots, X_j$, such that each E'_r is cofibrant and fibrant over X_r, $E_r(0) \to E'_r$ is an acyclic cofibration. Also, suppose we have cofibrant spectra $E_r(j)$ over X_r for every $r > j$, with acyclic cofibrations $E_r(0) \to E_r(j)$, such that there is the following diagram of inclusions of spectra over the inclusions $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_j$.

$$\begin{tikzcd}
X_0 \arrow[r, "\subseteq"] \arrow[d] & X_1 \arrow[r, "\subseteq"] \arrow[d] & \cdots \arrow[r, "\subseteq"] & X_j \arrow[r, "\subseteq"] \arrow[d] & X_{j+1} \arrow[r, "\subseteq"] \arrow[d] & X_{j+2} \cdots \arrow[d] \\
E_0(0) \arrow[r, "\subseteq"] \arrow[d, "\simeq"'] & E_1(0) \arrow[r, "\subseteq"] \arrow[d, "\simeq"] & \cdots \arrow[r, "\subseteq"] & E_j(0) \arrow[r, "\subseteq"] \arrow[d, "\simeq"'] & E_{j+1}(0) \arrow[r, "\subseteq"] \arrow[d, "\simeq"] & E_{j+2}(0) \cdots \arrow[d, "\simeq"] \\
E'_0 \arrow[r, "\subseteq"'] \arrow[d] & E'_1 \arrow[r, "\subseteq"'] \arrow[d] & \cdots \arrow[r, "\subseteq"] & E'_j \arrow[r, "\subseteq"'] \arrow[d] & E_{j+1}(j) \arrow[r, "\subseteq"'] \arrow[d] & E_{j+2}(j) \cdots \arrow[d] \\
X_0 \arrow[r, "\subseteq"'] & X_1 \arrow[r, "\subseteq"'] & \cdots \arrow[r, "\subseteq"'] & X_j \arrow[r, "\subseteq"'] & X_{j+1} \arrow[r, "\subseteq"'] & X_{j+2} \cdots .
\end{tikzcd} \tag{5.42}$$

Here, all the horizontal maps are spacewise inclusions. The first row of vertical maps are cofibrations, the second row of vertical maps are acyclic cofibrations, and the first j maps of the bottom row of vertical maps are fibrations. We construct E'_{j+1} to be the fibrant replacement of $E_{j+1}(j)$, obtained by attaching to $E_{j+1}(j)$ all cells of the form $\Sigma_V^\infty((G/H \times D^n \times I) \amalg X_{j+1})$ such that there is a commutative diagram of the form

$$\begin{tikzcd}
\Sigma_V^\infty((G/H \times D^n) \amalg X_{j+1}) \arrow[r] \arrow[d] & E_{j+1}(j) \arrow[d] \\
\Sigma_V^\infty((G/H \times D^n \times I) \amalg X_{j+1}) \arrow[r] & X_{j+1}
\end{tikzcd}$$

in the category of G-spectra over X_{j+1}. So E'_{j+1} is a fibrant spectrum over X_{j+1}, and there is an acyclic cofibration of spectra $E_{j+1}(j) \to E'_{j+1}$. Composing with the acyclic cofibration $E_{j+1}(0) \to E_{j+1}(j)$ gives an acyclic cofibration $E_{j+1}(0) \to E'_{j+1}$. Also, an acyclic cofibration of spectra is a spacewise inclusion. So we also have the composition

$$E'_j \xrightarrow{\subseteq} E_{j+1}(j) \xrightarrow{\subseteq} E'_{j+1}$$

which is an inclusion of spectra over the inclusion $X_j \subseteq X_{j+1}$. Therefore, we can replace $E_{j+1}(j)$ in diagram (5.42) by E'_{j+1}. The first $j+1$ maps in the bottom row of the diagram are now fibrations. We still need to construct $E_r(j+1)$ for all $r \geqslant j+2$. For such an r, we have the acyclic cofibration $E_{j+1}(j) \to E'_{j+1}$, and the inclusion of spectra $E_{j+1}(j) \to E_r(j)$ over the inclusion $X_{j+1} \subseteq X_r$. Define the spectrum $E_r(j+1)$ over X_{j+1} to be the spectrification of the prespectrum $E_r^{\mathrm{pre}}(j+1)$ over X_{j+1}, whose V-th space is the pushout in the category of unbased G-spaces given by

$$\begin{tikzcd} E_{j+1}(j)_V \arrow[r] \arrow[d] & E_r(j)_V \arrow[d] \\ (E'_{j+1})_V \arrow[r] & E_r^{\mathrm{pre}}(j+1)_V. \end{tikzcd}$$

Then it is straightforward to check that we have an acyclic cofibration $E_r(j) \to E_r(j+1)$ of spectra over X_r. Also, there are inclusions of spectra $E'_{j+1} \to E_r(j+1)$ over the inclusions $X_{j+1} \subseteq X_r$, as well as inclusions of spectra $E_r(j+1) \to E_{r+1}(j+1)$ over the inclusions $X_r \subseteq X_{r+1}$, which are compatible with each other. This allows us to replace $E_r(j)$ by $E_r(j+1)$ in the third row of diagram (5.42) for $r \geqslant j+2$, which gives the inductive step.

Finally, we define $E' = \bigcup_j E'_j$. Then we have maps of spectra over X

$$\Gamma E \longrightarrow E' \xrightarrow{p} X. \tag{5.43}$$

The first map is the union over j of the acyclic cofibrations $E_j(0) \to E'_j$, so it is an acyclic cofibration. Thus, E' is cofibrant and weakly equivalent to E. Also, the second map of (5.43) is the union over j of the fibrations $E'_j \to X_j$. We claim that p is a fibration. To see this, suppose that we have a testing diagram of the form

$$\begin{tikzcd} \Sigma_V^\infty((G/H \times D^n) \amalg X) \arrow[r] \arrow[d] & E' \arrow[d, "p"] \\ \Sigma_V^\infty((G/H \times D^n \times I) \amalg X) \arrow[r] & X \end{tikzcd} \tag{5.44}$$

in the category of G-spectra over X. By adjunction, this is equivalent to a diagram in the category of unbased G-spaces

$$\begin{tikzcd} G/H \times D^n \arrow[r] \arrow[d] & E'_V = \bigcup_j (E'_j)_V \arrow[d] \\ G/H \times D^n \times I \arrow[r] & X. \end{tikzcd}$$

Since the union E'_V is the colimit of the $(E'_J)_V$ over a sequence of inclusions, and X is the colimit of the X_j over a sequence of inclusions, this factors to

$$
\begin{tikzcd}
G/H \times D^n \arrow[r] \arrow[d] & (E'_j)_V \arrow[d] \\
G/H \times D^n \times I \arrow[r] \arrow[ur, dashed] & X_j
\end{tikzcd}
\tag{5.45}
$$

for some j. The right vertical map of (5.45) is a fibration, so a dotted arrow exists in (5.45) making the diagram commute. Therefore, a lifting exists in (5.44) as well. □

This concludes the proof of Theorem 4.9.

CHAPTER 6

THE WIRTHMÜLLER AND ADAMS ISOMORPHISMS

In this chapter, we will show that the Wirthmüller and Adams isomorphisms in equivariant homotopy theory are instances of the duality theorem 4.9. We begin with the Wirthmüller isomorphism.

Let G be a compact Lie group, and H a closed subgroup of G. Let L denote the tangent space of G/H at eH. The group G acts on G/H by translation, inducing an action on the tangent bundle of G/H. The subgroup H fixes the fiber at eH, which is L. Hence, L is an H-representation via the translation action. The Wirthmüller isomorphism [8] Theorem II.6.2 states that for an H-spectrum E of H-cell homotopy type,

$$G \ltimes_H (E \wedge S^{-L}) \simeq F_H[G, E] \tag{6.1}$$

in the category of G-spectra. To see this as an example of Theorem 4.9, consider the G-orbit G/H. There is a natural equivalence between the categories of based H-spaces and based G-spaces over G/H. For a based H-space Z, $G \times_H Z$ is a G-space, with a natural map $G \times_H Z \to G/H$ induced by the collapse map $Z \to *$. Likewise, the basepoint $G/H \to G \times_H Z$ is induced by the basepoint of Z. Conversely, if $G/H \xrightarrow{i} T \xrightarrow{p} G/H$ is a based G-space over G/H, then the fiber $p^{-1}(eH)$ is an H-space, with the basepoint $i(eH)$. It is easy to check that these two functors are inverse to each other. Stabilizing, we get an equivalence of categories between H-spectra indexed on a G-universe $\mathcal{U}$, thought of as an H-universe, and G-spectra over G/H over $\mathcal{U}$. Also, this equivalence of categories takes H-spectra of H-cell homotopy type to G-spectra of G-cell homotopy type over G/H.

We claim that the map $f : G/H \to *$ is a family of manifolds in the sense defined above. In fact, for any compact manifold M with a smooth G-action, consider the map of G-spaces

$$f : M \longrightarrow *.$$

By the G-action on M, G is contained in the family $\mathcal{F}_{\mathrm{sm}}$, so $E\mathcal{F}_{\mathrm{sm}}$ has a cell of the form $(G \times \mathcal{S})/G$. In the G-space $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$, therefore, there is a cell of the form $G/G = *$, giving a canonical map $i_M : * \to E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$. The following square is a pullback

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} M \arrow[d] \\ * \arrow[r, "i_M"'] & E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}. \end{tikzcd}$$

Hence, f is an equivariant smooth family of manifolds, so Theorem 4.9 holds for f. In particular, for $M = G/H$, by Theorem 4.9, we have

$$f_\sharp(E \wedge C_f^{-1}) \simeq f_*(E)$$

as G-spectra, for a G-spectrum E over G/H. It is straightforward to check that the composition functor

$$G\text{-spectra} \xrightarrow{f^*} G\text{-spectra}/(G/H) \xrightarrow{\simeq} H\text{-spectra}$$

is just the forgetful functor, so the right adjoint $F_H[G,-)$ coincides with f_*, and the left adjoint $G \ltimes_H -$ coincides with $f_\sharp$, via the equivalence of categories between H-spectra and G-spectra over G/H. Recall also that the dualizing object C_f is the sphere bundle of the tangent bundle of G/H, which is in this case

$$G \times_H L \longrightarrow G/H.$$

Thus, by definition, C_f^{-1} corresponds to the H-spectrum S^{-L} by the equivalence of categories between H-spectra and G-spectra over G/H, and the duality theorem gives (6.1) exactly. One can say a map from a G-manifold M to a single point is the simplest kind of equivariant smooth family of manifolds, and a single orbit G/H is the simplest manifold in the equivariant world. In this sense, the Wirthmüller isomorphism is the simplest case of the general equivariant duality theorem.

A more interesting example is that of the Adams isomorphism. Let H be a normal (closed) subgroup of G, and let $J = G/H$. Let A be the adjoint representation, *i.e.* the tangent space of H at e, with a G-action by conjugation. Also, let $\mathcal{U}$ be a complete G-universe. Let $i : \mathcal{U}^H \to \mathcal{U}$ be the inclusion, and i_*, i^* denote the change of universe functors between G-spectra indexed on $\mathcal{U}^H$ and on $\mathcal{U}$. (Unfortunately, there is some opportunity for confusion from the similarity between the classical notation for the universe change functors and the base change functors. Note that for a map f of base spaces, f_* is the right adjoint to f^*, but for a linear isometry i of universes, the universe change functor i_* is the left adjoint to i^*.)

Recall from [**8**], Section II.2 that a G-equivariant spectrum E is said to be H-free, if E has a G-cell approximation E', such that the cells of E' are of the form $\Sigma_V^\infty G/N_+ \wedge D^n$, where $N \cap H = \{e\}$. Hence, every G-cell approximation of E is of this form, and any G-spectrum that is weakly equivalent to any H-free G-cell spectrum is H-free. The H-free G-spectra form a full subcategory of the category of G-spectra. There is no model category structure on this full subcategory of H-free G-spectra, since it is not closed under point-set level colimits and limits. Nevertheless, we can consider the full subcategory of the homotopy category of G-spectra whose objects are H-free G-spectra. We call this the homotopy category of H-free G-spectra, even though it does not come from a model category structure on the point-set level subcategory of H-free G-spectra.

Recall from Theorem II.2.8 of [8] that the functor i_* from the category of H-free G-spectra indexed on $\mathcal{U}^H$ to the category of H-free G-spectra indexed on $\mathcal{U}$ induces an equivalence of homotopy categories. Then the Adams isomorphism is the following statement.

***Theorem 6.2* (Adams Isomorphism [8], Theorem II.7.1).** — *If E is an H-free G-spectrum of G-cell homotopy type indexed on $\mathcal{U}^H$, then*

$$E/H \simeq (i_*E \wedge S^{-A})^H.$$

in the category of J-spectra indexed on $\mathcal{U}^H$.

Note that A is not contained in the H-fixed universe $\mathcal{U}^H$, which is one of the reasons that necessitate the use of change of universe functors in the statement.

To see the Adams isomorphism in the context of duality, we need to understand H-free G-spectra. Recall the construction of the universal H-free G-space. Let $\mathcal{F}$ be the family of subgroups of G, consisting of all subgroups $N \subset G$ such that $N \cap H = \{e\}$. Then there is an universal contractible H-free G-space $E\mathcal{F}$ (see [8], Section II.2). Consider the map of G-spaces

$$j : E\mathcal{F} \longrightarrow *.$$

We have a pair of Quillen adjoint functors $(j_\sharp, j^*)$ between the categories of G-spectra and G-spectra over $E\mathcal{F}$. In particular, $j_\sharp$ lands in H-free G-spectra, so we in fact have a pair of adjoint functors between H-free G-spectra and G-spectra over $E\mathcal{F}$.

The following lemma holds for spectra indexed on $\mathcal{U}$ and on $\mathcal{U}^H$.

Lemma 6.3. — *If E is an H-free G-spectrum, the counit of the adjoint pair $(j_\sharp, j^*)$ is a homotopy equivalence $j_\sharp j^*E \simeq E$. If E is a cofibrant G-spectrum over $E\mathcal{F}$, then the unit of this adjunction pair $E \to j^*j_\sharp E$ is a weak equivalence.*

Proof. — For an H-free G-spectrum E, the counit of adjunction is

$$c : j_\sharp j^*E \longrightarrow E.$$

It is easy to see that $j_\sharp j^*E \simeq E\mathcal{F}_+ \wedge E$, and the map c is obtained by collapsing $E\mathcal{F}$. By the freeness of E, this is a natural homotopy equivalence.

Conversely, let E be a cofibrant G-spectrum over $E\mathcal{F}$. The functors j^* and $j_\sharp$ preserve colimits, so it suffices to consider the case where E is the suspension spectrum of a single orbit $G/N_+ = G/N \amalg E\mathcal{F}$, where $N \in \mathcal{F}$. Let $p : G/N \to E\mathcal{F}$ be the structure map, and $\Gamma_p : G/N \to G/N \times E\mathcal{F}$ be the graph of p. Then

$$j^* j_\sharp(G/N \amalg E\mathcal{F}) = (G/N \times E\mathcal{F}) \amalg E\mathcal{F}$$

and the unit of adjunction is

$$\Gamma_p \amalg E\mathcal{F} : G/N \amalg E\mathcal{F} \longrightarrow (G/N \times E\mathcal{F}) \amalg E\mathcal{F}$$

This is a G-map over $E\mathcal{F}$. By classical equivariant homotopy theory, Γ_p is a homotopy equivalence, hence a weak equivalence in the category of G-spaces. Thus, $\Gamma_p \amalg E\mathcal{F}$ is a weak equivalence in the category of based G-spaces over $E\mathcal{F}$. □

Thus, j^* and $j_\sharp$ pass to inverse equivalences between the homotopy categories of H-free G-spectra and G-spectra over $E\mathcal{F}$. This allows us to think of H-free G-spectra in the context suited to the duality theorem.

Consider the map of G-spaces

$$f : E\mathcal{F} \longrightarrow E\mathcal{F}/H.$$

We will show that f is an equivariant smooth family of manifolds. Given this, we get functors $f_\sharp$, f_* from G-spectra over $E\mathcal{F}$ to G-spectra over $E\mathcal{F}/H$. Also, let $h : E\mathcal{F}/H \to *$, so $j = h \cdot f : E\mathcal{F} \to *$. Also, let $i : \mathcal{U}^H \to \mathcal{U}$ be the obvious inclusion of universes. We have the following diagram of functors on the point-set level categories.

$$\begin{tikzcd}
H\text{-free } G\text{-spectra on } \mathcal{U}^H \arrow[d, "i_*"'] & \\
H\text{-free } G\text{-spectra on } \mathcal{U} \arrow[r, "j^*"'] & G\text{-spectra}/E\mathcal{F} \text{ on } \mathcal{U} \arrow[d, shift right, "f_\sharp(-\wedge_{E\mathcal{F}} C_f^{-1})"'] \arrow[d, shift left, "f_*"] \\
& G\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U} \arrow[d, "(-)^H"] \\
& J\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U}^H \arrow[d, "h_\sharp"] \\
& J\text{-spectra on } \mathcal{U}^H.
\end{tikzcd} \tag{6.4}$$

We claim that the compositions from H-free G-spectra on $\mathcal{U}^H$ to J-spectra on $\mathcal{U}^H$, using the two functors $f_\sharp(-\wedge_{E\mathcal{F}} C_f^{-1})$ and f_*, agree up to weak equivalences with the functors that occur in the classical statement of the Adams isomorphism.

Proposition 6.5. — *Let E be an H-free G-spectrum of cell homotopy type indexed on $\mathcal{U}^H$. Then*

(1) *The composition $h_\sharp(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))^H$ is naturally weakly equivalent to $(i_*E \wedge S^{-A})^H$ in the category of J-spectra indexed on $\mathcal{U}^H$.*

(2) *The composition $h_\sharp(f_*j^*i_*E)^H$ is naturally weakly equivalent to E/H in the category of J-spectra indexed on $\mathcal{U}^H$.*

First, we have the following lemmas.

Lemma 6.6. — *Let $i : \mathcal{U}^H \to \mathcal{U}$ be the inclusion map. Then for a homotopy cell bundle spectrum E over $E\mathcal{F}$, the unit of the adjunction $u : E \to i^*i_*E$ is a spacewise homotopy equivalence.*

Proof. — Similarly as in the case for H-free G-spectra over a point. Let $\mathcal{I}(\mathcal{U}^H, \mathcal{U})$ be the space of linear isometries from $\mathcal{U}^H$ to $\mathcal{U}$, with a G-action by conjugation. In particular, if A is a G-space over $E\mathcal{F}$, and we have a G-map $\alpha : A \to \mathcal{I}(\mathcal{U}^H, \mathcal{U})$, then we have from Chapter 2 the twisted half-smash product

$$A \ltimes_\alpha - : G\text{-spectra over } E\mathcal{F} \text{ on } \mathcal{U}^H \longrightarrow G\text{-spectra over } E\mathcal{F} \text{ on } \mathcal{U}.$$

In particular, define

$$\alpha_0 : E\mathcal{F} \longrightarrow \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

which takes x to (x, i) for every $x \in E\mathcal{F}$. Then for a G-spectrum E over $E\mathcal{F}$ indexed on $\mathcal{U}^H$, $E\mathcal{F} \ltimes_{\alpha_0} E = (\alpha_0)_*E \cong i_*E$. Also, for a G-spectrum E' over $E\mathcal{F}$ indexed on $\mathcal{U}$, $F_{\alpha_0}[E\mathcal{F}, E') = (\alpha_0)^*E' \cong i^*E'$.

We claim there exists a G-map

$$\alpha_1 : E\mathcal{F} \longrightarrow \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

such that for every $x \in E\mathcal{F}$, $\alpha_1(x) : \mathcal{U}^H \to \mathcal{U}$ is a isomorphism. Also, α_0 and α_1 are path connected to each other as $E\mathcal{F}$-points in the G-space $\mathcal{I}(\mathcal{U}^H, \mathcal{U})$. Namely, there is a G-map

$$\alpha : E\mathcal{F} \times I \longrightarrow \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

such that $\alpha \cdot i_0 = \alpha_0$, and $\alpha \cdot i_1 = \alpha_1$. This is done by the acyclic models argument over the cells of $E\mathcal{F}$. Recall the cells of $E\mathcal{F}$ are of the form $G/N \times D^n$, where N is a subgroup of G, and $H \cap N = \{e\}$. Also, for such an N, the universes $\mathcal{U}^H$ and $\mathcal{U}$ are N-equivariantly isomorphic. There is a path $I \to \mathcal{I}(\mathcal{U}^H, \mathcal{U})$ connecting this N-equivariant isomorphism and the inclusion $i : \mathcal{U}^H \to \mathcal{U}$, which is an N-equivariant linear isometry for every $t \in I$. Since the action of N on $\mathcal{I}(\mathcal{U}^H, \mathcal{U})$ is by conjugation, a linear isometry from $\mathcal{U}^H$ to $\mathcal{U}$ is N-equivariant if and only if it is in $\mathcal{I}(\mathcal{U}^H, \mathcal{U})^N$. Thus, we have a path $I \to \mathcal{I}(\mathcal{U}^H, \mathcal{U})^N$, *i.e.* an N-equivariant path $I \to \mathcal{I}(\mathcal{U}^H, \mathcal{U})$, where I is thought of as having the trivial N-action. Applying the functor $G/N \times -$ then gives a G-map

$$G/N \times I \longrightarrow \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

which at time 0 is i over every point of G/N, and at time 1 is an isomorphism over each point of G/N. Let $E\mathcal{F}_{(n)}$ denote the n-th skeleton of $E\mathcal{F}$. Let $j_0,\ j_1 : * \to I$ be the inclusions of the point at 0 and 1. Suppose that we have compatible maps $\alpha_{(n)} : E\mathcal{F}_{(n)} \times I \to E\mathcal{F}_{(n)} \times \mathcal{I}(\mathcal{U}^H, \mathcal{U})$, such that $\alpha_{(n)} \cdot j_0 = \alpha_0|_{E\mathcal{F}_{(n)}}$, and $\alpha_{(n)} \cdot j_1$ is an isomorphism over each point of $E\mathcal{F}_{(n)}$. Suppose $G/N \times D^{n+1}$ is a cell of $E\mathcal{F}$ of dimension $n+1$, with an attaching map $G/N \times S^n \to E\mathcal{F}_{(n)}$. Then there is a map

$$\alpha_{(n)}|_{G/N\times S^n} : I \longrightarrow (G/N \times S^n) \times \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

such that $\alpha_{(n)}|_{G/N\times S^n} \cdot j_0 = \alpha_0$, and $\alpha_{(n)}|_{G/N\times S^n} \cdot j_1$ is an isomorphism over every point of $G/N \times S^n$. By acyclic models, one can extend this map to

$$(G/N \times D^{n+1}) \times I \longrightarrow \mathcal{I}(\mathcal{U}^H, \mathcal{U})$$

with the same properties at times 0 and 1. This gives the homotopy between α_0 and a map which is an isomorphism in each fiber, over $E\mathcal{F}_{(n)}$ with the cell $G/N \times D^{n+1}$ attached. Thus, induction over the skeleta of $E\mathcal{F}$ gives α and α_1.

The map $j_0 : E\mathcal{F} \to E\mathcal{F} \times I$ is a homotopy equivalence over $E\mathcal{F}$. Then by arguments similar to Theorem 7.4 in Appendix A of [**4**], for a homotopy cell bundle spectrum E over $E\mathcal{F}$ on $\mathcal{U}^H$, there is an induced homotopy equivalence

$$i_*E = E\mathcal{F} \ltimes_{\alpha_0} E \longrightarrow (E\mathcal{F} \times I) \ltimes_\alpha E.$$

The functor i^* preserves homotopy equivalences of spectra. Also, there is a spacewise homotopy equivalence

$$F_{\alpha_0}[E\mathcal{F}, (E\mathcal{F} \times I) \ltimes_\alpha E) \longrightarrow F_\alpha[E\mathcal{F} \times I, (E\mathcal{F} \times I) \ltimes E).$$

Hence, we have a spacewise homotopy equivalence

$$\begin{aligned}\beta_0 : i^*i_*E \longrightarrow i^*((E\mathcal{F} \times I) \ltimes_\alpha E) &= F_{\alpha_0}[E\mathcal{F}, (E\mathcal{F} \times I) \ltimes_\alpha E) \\ &\longrightarrow F_\alpha[E\mathcal{F} \times I, (E\mathcal{F} \times I) \ltimes_\alpha E).\end{aligned}$$

Similarly, there is a spacewise homotopy equivalence

$$\beta_1 : F_{\alpha_1}[E\mathcal{F}, E\mathcal{F} \ltimes_{\alpha_1} E) \longrightarrow F_\alpha[E\mathcal{F} \times I, (E\mathcal{F} \times I) \ltimes_{\alpha_1} E).$$

We have the diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, "u"] & i^*i_*E \arrow[d, "\beta_0"] \\ F_{\alpha_1}[E\mathcal{F}, E\mathcal{F} \ltimes_{\alpha_1} E) \arrow[r, "\beta_1"'] & F_\alpha[E\mathcal{F} \times I, (E\mathcal{F} \times I) \ltimes_\alpha E) \end{tikzcd}$$

where the u's denotes the units of adjunction. However, since α_1 is an isomorphism over each point of $E\mathcal{F}$, the unit of adjunction

$$u : E \longrightarrow F_{\alpha_1}[E\mathcal{F}, E\mathcal{F} \ltimes \alpha_1 E)$$

is an isomorphism. The maps β_0 and β_1 are spacewise homotopy equivalences. Hence, $u : E \to i^*i_*E$ is a spacewise homotopy equivalence. □

Lemma 6.7. — *Let H be a normal subgroup of G, and $J = G/H$. The fixed point functors $(-)^H$ from G-spectra indexed on $\mathcal{U}$ and from G-spectra indexed on $\mathcal{U}^H$ to J-spectra indexed on $\mathcal{U}^H$ preserve weak equivalences.*

Proof. — Let $e : E \to E'$ be a map of G-spectra indexed on $\mathcal{U}$. Then e is a weak equivalence of spectra if and only if for every finite-dimensional $V \subset \mathcal{U}$, $e_V : E_V \to E'_V$ is a weak equivalence of G-spaces. Thus, the change of universes functor i^* preserves weak equivalences of spectra, and it suffices to show that the fixed point functor from G-spaces to J-spaces preserves weak equivalences. Let $e : T \to Z$ be now a weak equivalence of G-spaces. Then for every subgroup N of G, $e^N : T^N \to Z^N$ is a nonequivariant weak equivalence. Let N' be a subgroup of J, then $N' = N/H$ for a subgroup N of G containing H. The action of J on T^H and Z^H is induced by the action of G on T and Z, so $(T^H)^{N'} = T^N$ and $(Z^H)^{N'} = Z^N$. Therefore, e induces a nonequivariant weak equivalence $(T^H)^{N'} \to (Z^H)^{N'}$ for every subgroup N' of J, so $e^H : T^H \to Z^H$ is a weak equivalence of J-spaces. □

Lemma 6.8. — *The diagram of functors*

$$\begin{array}{ccc} H\text{-free } G\text{-spectra on } \mathcal{U}^H & \xrightarrow{j^*} & G\text{-spectra over } E\mathcal{F} \text{ on } \mathcal{U}^H \\ {\scriptstyle i_*}\big\downarrow & & \big\downarrow{\scriptstyle i_*} \\ H\text{-free } G\text{-spectra on } \mathcal{U} & \xrightarrow[j^*]{} & G\text{-spectra over } E\mathcal{F} \text{ on } \mathcal{U} \end{array}$$

commutes up to natural isomorphism.

Proof. — Let E be a H-free G-spectrum indexed on $\mathcal{U}^H$. Then i_*E is obtained by spectrifying the prespectrum $i_*^{\mathrm{pre}}E$ on $\mathcal{U}$, whose V-th space is $\Sigma^{V-(V\cap\mathcal{U}^H)}E_{V\cap\mathcal{U}^H}$ for each finite-dimensional V in $\mathcal{U}$. The right adjoint of j^* is j_*, which commutes with the forgetful functor from spectra on $\mathcal{U}$ to prespectra on $\mathcal{U}$. So j^* commutes with the spectrification functor L. Hence, it suffices to show that on the level of prespectra over $E\mathcal{F}$ indexed on $\mathcal{U}$,

$$j^* i_*^{\mathrm{pre}} E \cong i_*^{\mathrm{pre}} j^* E.$$

The V-th space of the left hand side is

$$E\mathcal{F} \times \Sigma^{V-(V\cap\mathcal{U}^H)} E_{V\cap\mathcal{U}^H}$$

and the V-th space of the right hand side is

$$\Sigma_{E\mathcal{F}}^{V-(V\cap\mathcal{U}^H)}(E\mathcal{F} \times E_{V-(V\cap\mathcal{U}^H)}).$$

These two are naturally isomorphic as G-spaces over $E\mathcal{F}$. □

We also have the following observation, whose proof we defer.

Lemma 6.9. — *For maps $f : E\mathcal{F} \to E\mathcal{F}/H$ and $j : E\mathcal{F} \to *$, we have that the dualizing object C_f of f is isomorphic to $j^*(S^A)$ as G-spaces, where A is the adjoint representation.*

We now prove Proposition 6.5, which identifies the compositions of (6.4) with the two sides of the Adams isomorphism.

Proof of Proposition 6.5. — For the first statement, consider the composition of functors from H-free G-spectra indexed on $\mathcal{U}$ to J-spectra indexed on $\mathcal{U}^H$ using $f_\sharp(- \wedge_{E\mathcal{F}} C_f^{-1})$. Let Z be a based G-space over $E\mathcal{F}/H$. Since $E\mathcal{F}/H$ is fixed as an H-space, taking the fixed point functor preserves the pushout square

$$\begin{tikzcd} E\mathcal{F}/H \arrow[r] \arrow[d] & Z \arrow[d] \\ * \arrow[r] & Z/(E\mathcal{F}/H). \end{tikzcd}$$

So $(Z^H)/(E\mathcal{F}/H) \cong (Z/(E\mathcal{F}/H))^H$. On the level of spectra, consider the diagram of functors

$$\begin{tikzcd} G\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U} \arrow[r, "h_\sharp"] \arrow[d, "i^*"'] & G\text{-spectra on } \mathcal{U} \arrow[d, "i^*"] \\ G\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U}^H \arrow[r, "h_\sharp"] \arrow[d, "(-)^H"'] & G\text{-spectra on } \mathcal{U}^H \arrow[d, "(-)^H"] \\ J\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U}^H \arrow[r, "h_\sharp"'] & J\text{-spectra on } \mathcal{U}^H. \end{tikzcd} \tag{6.10}$$

The functor $h_\sharp$ on spectra is obtained by first applying $h_\sharp$ spacewise, which gives inclusion prespectra, then applying the spectrification functor. The functors $(-)^H$ and Ω^V commute if V is a finite-dimensional H-fixed G-representation, so $(-)^H$ on spectra indexed on $\mathcal{U}^H$ commutes with the spectrification functor from inclusion prespectra indexed on $\mathcal{U}^H$ to spectra indexed on $\mathcal{U}^H$. Hence, the bottom square of the diagram commutes in the point set category up to canonical isomorphism.

We make the following claim.

Proposition 6.11. — *The top square of diagram (6.10) commutes up to natural weak equivalences if applied to $f_\sharp(j^*i_*E\wedge_{E\mathcal{F}}C_f^{-1})$, where E is a cofibrant H-free G-spectrum indexed on $\mathcal{U}^H$. Namely, there is a natural weak equivalence of G-spectra indexed on $\mathcal{U}^H$*

$$h_\sharp i^*(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})) \simeq i^*h_\sharp(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})). \tag{6.12}$$

We will defer the proof of Proposition 6.11. By Lemma 6.7, $(-)^H$ preserves weak equivalences. So given (6.12), the entire diagram (6.10) commutes up to natural weak equivalence, in the sense that

$$\begin{aligned} h_\sharp(i^*(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C - f^{-1})))^H &\cong (h_\sharp i^*(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))^H \\ &\simeq (i^*h_\sharp(f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))^H \end{aligned}$$

naturally for a H-free G-spectrum E of G-cell homotopy type indexed on $\mathcal{U}^H$.

Thus, the composition of functors

$$\begin{aligned} & H\text{-free } G\text{-spectra on } \mathcal{U} \xrightarrow{j^*} G\text{-spectra}/E\mathcal{F} \text{ on } \mathcal{U} \\ & \xrightarrow{f_\sharp(- \wedge_{E\mathcal{F}} C_f^{-1})} G\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U} \\ (6.13) \qquad & \xrightarrow{(-)^H} J\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U}^H \\ & \xrightarrow{h_\sharp} J\text{-spectra on } \mathcal{U}^H \end{aligned}$$

is weakly equivalent to $(h_\sharp f_\sharp(j^*(-) \wedge_{E\mathcal{F}} C_f^{-1}))^H$.

Since the functor j^* commutes with smash products, for a cofibrant H-free G-spectrum E on $\mathcal{U}^H$, $j^*(i_*E) \wedge_{E\mathcal{F}} C_f^{-1}$ is weakly equivalent to $j^*(i_*E \wedge S^{-A})$ in the category of G-spectra over $E\mathcal{F}$ indexed on $\mathcal{U}$. Since E is cofibrant, both $j^*(i_*E) \wedge_{E\mathcal{F}} C_f^{-1}$ and $j^*(i_*E \wedge S^{-A})$ are cofibrant, the former by the discussion after Lemma 4.19. The functor $h_\sharp f_\sharp$ preserves weak equivalences between cofibrant objects, and $(-)^H$ preserves weak equivalences by Lemma 6.7. So by Lemma 6.9, the composition of functors (6.13), applied to i_*E, is weakly equivalent to $(h_\sharp f_\sharp j^*(i_*E \wedge S^{-A}))^H = (j_\sharp j^*(i_*E \wedge S^{-A}))^H$. But we also have that $j_\sharp j^*(i_*E \wedge S^{-A})$ is weakly equivalent to $i_*E \wedge S^{-A}$, and again, $(-)^H$ preserves this weak equivalence. Thus, for a cofibrant H-free G-spectrum E indexed on $\mathcal{U}^H$, the composition (6.13), applied to i_*E, is naturally weakly equivalent to $(i_*E \wedge S^{-A})^H$.

For the second part of the Proposition 6.5, we need to consider the composition

$$\begin{aligned} & H\text{-free } G\text{-spectra on } \mathcal{U} \xrightarrow{j^*} G\text{-spectra}/E\mathcal{F} \text{ on } \mathcal{U} \\ & \xrightarrow{f_*} G\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U} \\ (6.14) \qquad & \xrightarrow{(-)^H} J\text{-spectra}/(E\mathcal{F}/H) \text{ on } \mathcal{U}^H \\ & \xrightarrow{h_\sharp} J\text{-spectra on } \mathcal{U}^H. \end{aligned}$$

We have the following lemma.

Lemma 6.15. — *For a G-spectrum E over $E\mathcal{F}$ indexed on $\mathcal{U}^H$,*

$$(f_*(E))^H \cong (f_\sharp(E))/H$$

naturally.

Proof. — It suffices to prove the lemma on the level of G-spaces. Since the functor $(-)/H$ is a left adjoint, for a G-space T over $E\mathcal{F}$, it takes the pushout diagram

$$\begin{tikzcd} E\mathcal{F} \arrow[r] \arrow[d] & T \arrow[d] \\ E\mathcal{F}/H \arrow[r] & f_\sharp(T) \end{tikzcd}$$

to the pushout diagram

$$\begin{tikzcd} E\mathcal{F}/H \arrow[r] \arrow[d, "="'] & T/H \arrow[d] \\ E\mathcal{F}/H \arrow[r] & f_\sharp(T)/H. \end{tikzcd}$$

So $(f_\sharp(T)/H = T/H$. Let $p : T \to E\mathcal{F}$ be the structure map of T. Recall that set-theoretically, $f_*(T) = \amalg_y \operatorname{Sec}(E\mathcal{F}_y, T_y)$ over the points $y \in E\mathcal{F}/H$, and $(f_*(T))^H$ consists of the H-equivariant sections. Since $E\mathcal{F}/H$ is fixed by H, each H-orbit of T is contained in T_y for a single $y \in E\mathcal{F}/H$. Thus, it suffices to consider a single point of $E\mathcal{F}/H$, and compare $\operatorname{Sec}(E\mathcal{F}_y, T_y)$ and T_y/H. Choose $x \in E\mathcal{F}_y$, then $E\mathcal{F}_y = Hx$ is homeomorphic to H as an H-space, so the image of x in a section determines the entire section. If $k : E\mathcal{F}_y \to T_y$ is a section of p, then the image of k is an H-orbit in T_y. But for every H-orbit O in T_y, there is an unique $z \in p^{-1}(x) \cap O$, which determines a section $E\mathcal{F}_y = Hx \to O \subseteq T_y$ that takes x to z. Therefore, we have that $(f_*T)^H \cong T/H \cong (f_\sharp(T))/H$.

Thus, the statement of the lemma holds for the prespectra-level functors. Applying the spectrification functor L to both sides gives the lemma for spectra. □

For an H-free G-spectrum E indexed on $\mathcal{U}$ of G-cell homotopy type, we need to apply the composition (6.14), to $i_*(E)$. This is $h_\sharp(f_*j^*(i_*E))^H$. In taking the H-fixed points of a spectrum indexed on $\mathcal{U}$, we first forget to the universe $\mathcal{U}^H$, *i.e.* apply i^*, then take H-fixed points spacewise. Hence, the composition (6.14) for $i_*(E)$ is really $h_\sharp(i^*f_*j^*i_*E)^H$, where $(-)^H$ is taken spacewise, since the spectrum $i^*f_*j^*i_*E$ is now indexed on $\mathcal{U}^H$. Now it is easy to check that i^* and f_* commute, since f_* on spectra is obtained by applying the space-level f_* on each space of the spectrum. Also, j^* commutes with i_* by Lemma 6.8. Hence, this is $h_\sharp(f_*i^*i_*(j^*E))^H$. Now j^*E is spacewise homotopy equivalent to $i^*i_*(j^*E)$ by Lemma 6.6, and the fact that j^*E is trivially a homotopy cell bundle spectrum over $E\mathcal{F}$, since E is of G-cell homotopy type. The functor $(f_*(-))^H$ on spectra is obtained by applying $(f_*(-))^H = (f_\sharp(-))/H$ spacewise. Since $(f_\sharp(-))/H$ preserves homotopies, the spectra-level functor $(f_*(-))^H$ preserves spacewise homotopy equivalences. Also, $h_\sharp$ takes a spacewise homotopy equivalence to a weak equivalence of spectra. This is because applying $h_\sharp$ spacewise takes a spacewise homotopy equivalence of spectra to a spacewise homotopy equivalence of inclusion prespectra, and the spectrification functor from inclusion prespectra to prespectra takes a spacewise homotopy equivalence to a weak equivalence. Hence, (6.14) is naturally weakly equivalent in the category of J-spectra on $\mathcal{U}^H$ to $h_\sharp(f_* j^* E)^H$, which is $h_\sharp(f_\sharp j^*(E)/H)$ by Lemma 6.15. But the functors $h_\sharp$ and $(-)/H$ commute since their right adjoints commute, so (6.14) of $i_* E$ is weakly equivalent to $(h_\sharp f_\sharp j^*(E))/H = (j_\sharp j^*(E))/H$. We have a weak equivalence $E \to j_\sharp j^* E$. Since $E\mathcal{F}$ is a cell complex, the functor j^* preserves cofibrant objects, so both E and $j_\sharp j^* E$ are cofibrant. Also, $(-)/H$ preserves weak equivalences of cofibrant spectra. Thus, for an H-free G-spectrum E indexed on $\mathcal{U}^H$, (6.14) is naturally weakly equivalent to E/H. □

Now we prove Proposition 6.11. We will begin with the following lemma.

Lemma 6.16. — *If E is a homotopy cell bundle spectrum over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$, then for any finite-dimensional G-representation V,*

$$h_\sharp(\Omega^V_{E\mathcal{F}/H} E) \simeq \Omega^V h_\sharp E$$

naturally as G-spectra indexed on $\mathcal{U}^H$.

Proof. — Consider a cell $C = G/NH \times D^n$ of $E\mathcal{F}/H$, where N is a subgroup of G such that $N \cap H = \{e\}$. Let E_C be the fiber of E over C, so E_C is a G-spectrum over C indexed on $\mathcal{U}^H$. Since E is a homotopy cell bundle spectrum over $E\mathcal{F}/H$, by subdividing the cells of $E\mathcal{F}/H$, we can assume that over each cell $C = G/NH \times D^n$ of $E\mathcal{F}/H$, the fiber E_C is $G \times_{NH} (E'_C \times D^n)$ as a G-spectrum indexed in $\mathcal{U}^H$ over C. Here, E'_C is an NH-spectrum indexed on $\mathcal{U}^H$. Also, we can assume that E'_C is of the homotopy type of an NH-cell spectrum.

Now for a based NH-space X, consider the based G-space $G \times_{NH} (X \times D^n)$ over $G/NH \times D^n$, whose structure map $p : G \times_{NH} (X \times D^n) \to G/NH \times D^n$ is induced from collapsing X to a point, and whose basepoint map $i : G/NH \times D^n \to G \times_{NH} (X \times D^n)$ is induced from the basepoint of X. Then

$$\begin{aligned} G_+ \wedge_{NH} (X \wedge D^n_+) &\cong G_+ \wedge_{NH} ((X \times D^n)/(* \times D^n)) \\ &\cong (G \times_{NH} (X \times D^n))/(G/NH \times D^n) \end{aligned}$$

which is naturally isomorphic to $(h|_C)_\sharp(G \times_{NH} (X \times D^n))$. Passing to spectra, we get that similarly

$$(h|_C)_\sharp(E_C) \cong G \ltimes_{NH} (E'_C \wedge D^n_+)$$

naturally as G-spectra indexed on $\mathcal{U}$. Again, $E'_C \wedge D^n_+$ is also of the homotopy type of an NH-cell spectrum. Let L be the tangent space of G/NH at eNH, with a NH-action by translation. Note that since H is normal in G, the G-action on G/NH

by translation, when restricted to H, is trivial. So L is in fact an H-fixed NH-representation. Thus, S^{-L} exists as a G-spectrum indexed on $\mathcal{U}^H$, and the Wirthmüller isomorphism holds for G-spectra indexed in $\mathcal{U}^H$. In particular,

$$G \ltimes_{NH} (E'_C \wedge D^n_+) \simeq F_{NH}[G, E'_C \wedge D^n_+).$$

However, the functor $F_{NH}[G,-)$ commutes with Ω^V for any finite-dimensional G-representation V. Thus, $G \ltimes_{NH} -$ commutes with Ω^V up to weak equivalence. Let Ω^V_C denote the V-th loop functor for G-spectra over C indexed on $\mathcal{U}^H$. Then

$$\begin{aligned}
(h|_C)_\sharp(\Omega^V_C E_C) &\cong (h|_C)_\sharp \Omega^V_C (G \times_{NH} (E'_C \times D^n)) \\
&\cong (h|_C)_\sharp (G \times_{NH} (\Omega^V E'_C \times D^n)) \\
&\cong G \ltimes_{NH} (\Omega^V E'_C \wedge D^n_+) \\
&\simeq \Omega^V (G \ltimes_{NH} (E'_C \wedge D^n_+)) \\
&\cong \Omega^V ((h|_C)_\sharp E_C).
\end{aligned}$$

Now $h_\sharp(E)$ is obtained by gluing together $(h|_C)_\sharp(E_C)$'s using cofiber sequences and directed colimits in the category of G-spectra indexed on $\mathcal{U}^H$. These constructions are instances of homotopy colimits, which also commute with Ω^V on the derived category in the sense above. Hence, $h_\sharp$ commutes with Ω^V up to natural weak equivalences of G-spectra indexed on $\mathcal{U}^H$, *i.e.* for a homotopy cell bundle spectrum E over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$ with fibers E'_C of cell homotopy type,

$$\begin{aligned}
h_\sharp \Omega^V_{E\mathcal{F}/H} E &\simeq \mathrm{hocolim}_C (h|_C)_\sharp \Omega^V_C E_C \\
&\simeq \mathrm{hocolim}_C \Omega^V (h|_C)_\sharp E_C \\
&\simeq \Omega^V \mathrm{hocolim}_C (h_C)_\sharp E_C \\
&\simeq \Omega^V h_\sharp E.
\end{aligned}$$

□

Proof of Proposition 6.11. — We define the following category, called the category of $(\mathcal{U},\mathcal{U}^H)$-presystems.

Definition 6.17. — An $(\mathcal{U},\mathcal{U}^H)$-presystem is a collection $\{E(V)\}$ of G-spectra indexed on $\mathcal{U}^H$, where the V's range over all finite-dimensional representations V of $\mathcal{U}$, such that $V \cap \mathcal{U}^H = \{0\}$. We require structure maps

(6.18) $$E(V) \longrightarrow \Omega^{W-V} E(W)$$

whenever $V \subseteq W$, $W \cap \mathcal{U}^H = \{0\}$. The structure maps satisfy the obvious composition relations. Morphisms of $(\mathcal{U},\mathcal{U}^H)$-presystems are collections of maps $\{E(V) \to E'(V)\}$, where each $E(V) \to E'(V)$ is a map of G-spectra indexed on $\mathcal{U}^H$, and the maps commute with the structure maps. Also, define the category of $(\mathcal{U},\mathcal{U}^H)$-systems to be the full subcategory of $(\mathcal{U},\mathcal{U}^H)$-presystems, with objects all $\{E(V)\}$'s whose structure maps are all isomorphisms.

In fact, this category of $(\mathcal{U},\mathcal{U}^H)$-systems is naturally equivalent to the category of G-spectra indexed on $\mathcal{U}$. Namely, suppose E is a G-spectrum indexed on $\mathcal{U}$. Let V be a finite-dimensional G-representation contained in $\mathcal{U}$, then we define $E(V)$ by

$$E(V)_Z = E_{V\oplus Z}$$

for any finite-dimensional G-representation Z contained in $\mathcal{U}^H$. The structure maps of $E(V)$ are those of E. It is easy to check that we get maps (6.17), which are isomorphisms of G-spectra indexed on $\mathcal{U}^H$, so we get a $(\mathcal{U},\mathcal{U}^H)$-system $\{E(V)\}$ from E. Conversely, given a $(\mathcal{U},\mathcal{U}^H)$-system $\{E(V)\}$, define a G-spectrum E indexed on $\mathcal{U}$ by

$$E_W = E(W - (W \cap \mathcal{U}^H))_{W\cap\mathcal{U}^H}.$$

It is straightforward to check that this gives inverse equivalences of categories between $(\mathcal{U},\mathcal{U}^H)$-systems and G-spectra indexed on $\mathcal{U}$. On the level of prespectra, although the categories of G-prespectra indexed on $\mathcal{U}$ and $(\mathcal{U},\mathcal{U}^H)$-presystems are not equivalent, a $(\mathcal{U},\mathcal{U}^H)$-presystem gives a G-prespectrum indexed on $\mathcal{U}$.

Likewise, for any G-space X, we can define the categories of $(\mathcal{U},\mathcal{U}^H)$-presystems and systems over X, whose objects are collections $\{E(V)\}$ for all finite-dimensional $V \subset \mathcal{U}$, where each $E(V)$ is now a G-spectrum over X indexed on $\mathcal{U}^H$. Then a similar equivalence of categories holds for G-spectra over X indexed on $\mathcal{U}$ and $(\mathcal{U},\mathcal{U}^H)$-systems over X. In the following, we will use the categories of G-spectra over X indexed on $\mathcal{U}$ and $(\mathcal{U},\mathcal{U}^H)$-systems over X interchangeably.

In our case, the input spectrum is $f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$, where E is an H-free G-spectrum of cell homotopy type indexed on $\mathcal{U}^H$. By Lemma 6.9, for any choice of universe change $a : E\mathcal{F} \to \mathcal{I}(\mathcal{U}^2,\mathcal{U})$, we get a homotopy equivalence

$$C_f^{-1} \simeq j^*\Sigma^{-A}_{\text{shift}} S^0 \cong \Sigma^{-A}_{\text{shift}} S^0_{E\mathcal{F}}.$$

So we have

$$\begin{aligned}
f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}) &\simeq f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} \Sigma^{-A}_{\text{shift}} S^0_{E\mathcal{F}}) \\
&\simeq f_\sharp(\Sigma^{-A}_{\text{shift}}(j^*i_*E)) \\
&\cong f_\sharp(j^*(\Sigma^{-A}_{\text{shift}} i_*E)) \\
&\simeq f_\sharp j^*(i_*E \wedge S^{-A}) \\
&\cong f_\sharp f^* h^*(i_*E \wedge S^{-A}) \\
&\cong (E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} (h^*(i_*E \wedge S^{-A})).
\end{aligned}$$

All the maps of this composition are isomorphisms or homotopy equivalences. The map $f : E\mathcal{F} \to E\mathcal{F}/H$ is a G-equivariant bundle with fiber H, and $h^*(i_*E \wedge S^{-A})$ is trivially a homotopy cell bundle spectrum over $E\mathcal{F}/H$, with fiber $i_*E \wedge S^{-A}$, which is a G-spectrum of cell homotopy type. Therefore, $f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$ is a homotopy cell bundle spectrum over $E\mathcal{F}/H$.

We will denote $f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$ by E' for short. Write E' as a $(\mathcal{U},\mathcal{U}^H)$-system $\{E'(V)\}$ over $E\mathcal{F}/H$. Suppose we can replace E by a $(\mathcal{U},\mathcal{U}^H)$-presystem $\{\overline{E'}_{\mathrm{pre}}(V)\}$ over $E\mathcal{F}/H$, such that

(1) There is a natural spacewise homotopy equivalence $\overline{E'}_{\mathrm{pre}} \xrightarrow{\simeq} E$ of G-prespectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}$;

(2) each $\overline{E'}_{\mathrm{pre}}(V)$ is a homotopy cell bundle spectrum over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$;

(3) for all $V \subseteq W$, $W \cap \mathcal{U}^H = \{0\}$, the adjoint structure map of the $(\mathcal{U},\mathcal{U}^H)$-presystem over $E\mathcal{F}/H$

$$\Sigma^{W-V}_{E\mathcal{F}/H}\overline{E'}_{\mathrm{pre}}(V) \longrightarrow \overline{E'}_{\mathrm{pre}}(W)$$

is a cofibration of G-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$.

Then for all $V \subset W$, such that $W \cap \mathcal{U}^H = \{0\}$, the diagram of G-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$

$$\begin{tikzcd}
\overline{E'}_{\mathrm{pre}}(V) \arrow[r, "\simeq"] \arrow[d] & E(V) \arrow[d, "\cong"] \\
\Omega^{W-V}_{E\mathcal{F}/H}\overline{E'}_{\mathrm{pre}}(W) \arrow[r, "\simeq"'] & \Omega^{W-V}_{E\mathcal{F}/H}E(W)
\end{tikzcd}$$

has spacewise homotopy equivalences for the horizontal maps, so its left vertical map is also a spacewise homotopy equivalence. Therefore, by Lemma 6.16,

$$h_\sharp\overline{E'}_{\mathrm{pre}}(V) \simeq h_\sharp\Omega^{W-V}_{E\mathcal{F}/H}\overline{E'}_{\mathrm{pre}}(W) \simeq \Omega^{W-V}h_\sharp\overline{E'}_{\mathrm{pre}}(W). \tag{6.19}$$

Since $h_\sharp(\overline{E'}_{\mathrm{pre}}(V))$ and $h_\sharp(\overline{E'}_{\mathrm{pre}}(W))$ have the homotopy types of G-cell spectra, and by Remark I.6.4 of [**8**], Ω^{W-V} preserves cell homotopy types, (6.19) is in fact a homotopy equivalence.

Now there is a stabilization functor $L(\mathcal{U},\mathcal{U}^H)$ from $(\mathcal{U},\mathcal{U}^H)$-presystems to $(\mathcal{U},\mathcal{U}^H)$-systems. It is the left adjoint to the forgetful functor from $(\mathcal{U},\mathcal{U}^H)$-systems to $(\mathcal{U},\mathcal{U}^H)$-presystems, and is similar to the spectrification functor. If $\{D(W)\}$ is a $(\mathcal{U},\mathcal{U}^H)$-presystem, whose structure maps are spacewise inclusions, then

$$(L(\mathcal{U},\mathcal{U}^H)\{D(W)\})(V) = \mathrm{colim}_{V\subseteq W}\Omega^{W-V}D(W).$$

Here, the colimit is taken over all finite-dimensional representations $W \subset \mathcal{U}$ containing V, and $W \cap \mathcal{U}^H = \{0\}$. If we think of a $(\mathcal{U},\mathcal{U}^H)$-presystem as a G-prespectrum indexed on $\mathcal{U}$, and a $(\mathcal{U},\mathcal{U}^H)$-system as a G-spectrum indexed on $\mathcal{U}$, then $L(\mathcal{U},\mathcal{U}^H)$ coincides with the spectrification functor from G-prespectra indexed on $\mathcal{U}$ to G-spectra indexed on $\mathcal{U}$. In particular, let $\overline{E'} = L(\mathcal{U},\mathcal{U}^H)\{\overline{E'}(V)\}$. Then by Condition 2 for $\{\overline{E'}(V)\}$ and arguments similar to that of I.8.10 of [**8**], there is a natural homotopy equivalence of G-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$

$$\overline{E'}_{\mathrm{pre}}(V) \xrightarrow{\simeq} \overline{E'}(V)$$

for every finite-dimensional G-representation V with $V \cap \mathcal{U}^H = \{0\}$. Hence, we also have a natural homotopy equivalence of G-spectra indexed on $\mathcal{U}^H$

$$h_\sharp(\overline{E'}(V)) \xrightarrow{\simeq} h_\sharp(\overline{E'}_{\mathrm{pre}}(V)). \tag{6.20}$$

Note that both $\{h_\sharp(\overline{E'}(V))\}_V$ and $\{h_\sharp(\overline{E'}_{\mathrm{pre}}(V))\}_V$ form $(\mathcal{U},\mathcal{U}^H)$-presystems.

Thinking of $\overline{E'} = \{\overline{E'}(V)\}_V$ as a G-spectrum over $E\mathcal{F}/H$ indexed on $\mathcal{U}$, we have

$$h_\sharp(\overline{E'}) = L(\mathcal{U},\mathcal{U}^H)\{h_\sharp\overline{E'}(V)\}.$$

Since spectrification takes spacewise homotopy equivalences to weak equivalences, by (6.20), this is naturally weakly equivalent to $L(\mathcal{U},\mathcal{U}^H)\{h_\sharp\overline{E'}_{\mathrm{pre}}(V)\}$. Since $h_\sharp$ of spectra preserves cofibrations, the adjoint structure maps

$$\Sigma^{W-V}h_\sharp(\overline{E'}_{\mathrm{pre}}(V)) \longrightarrow h_\sharp(\overline{E'}_{\mathrm{pre}}(W))$$

are cofibrations of G-spectra indexed on $\mathcal{U}^H$. By this and (6.19), using arguments similar to Lemma I.8.10 of [**8**], we get that

$$h_\sharp(\overline{E'}_{\mathrm{pre}}(V)) \longrightarrow (L(\mathcal{U},\mathcal{U}^H)\{h_\sharp\overline{E'}_{\mathrm{pre}}(W)\}_W)(V)$$

is a natural homotopy equivalence for every finite-dimensional V such that $V \cap \mathcal{U}^H = \{0\}$.

For every V with $V \cap \mathcal{U}^H = \{0\}$, $\overline{E'}_{\mathrm{pre}}(V) \to E'(V)$ is a spacewise homotopy equivalence. Thus, $\overline{E'}(V)$ is also spacewise homotopy equivalent to $E'(V)$. Also, $h_\sharp$ preserves homotopy equivalences of spaces, and the spectrification functor takes spacewise homotopy equivalences to weak equivalences. Therefore, we get

$$\begin{aligned} i^*h_\sharp(E) \simeq i^*(h_\sharp(\overline{E'}) &\simeq i^*L(\mathcal{U},\mathcal{U}^H)\{\overline{E'}_{\mathrm{pre}}(V)\} \\ &= L(\mathcal{U},\mathcal{U}^H)\{\overline{E'}_{\mathrm{pre}}(V)\}(0) \\ &\simeq h_\sharp(\overline{E'}_{\mathrm{pre}}(0)) \\ &\simeq h_\sharp(E(0)) \\ &= h_\sharp i^*(E). \end{aligned}$$

This is the statement of Proposition 6.11.

It remains to construct the replacement $\{\overline{E'}_{\mathrm{pre}}(V)\}$ of $E' = f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$. For this, we use the cylinder construction [**8**], Section I.6. Suppose we have a G-spectrum E of G-cell homotopy type indexed on $\mathcal{U}$, recall from [**8**] that each space of E is of cell homotopy type. The cylinder construction KE is the prespectrum on $\mathcal{U}$ given by

$$KE_V = \mathrm{hocolim}_{W\subset V}\Sigma^{V-W}E_W.$$

The $\mathcal{U}$-prespectrum KE is Σ-cofibrant, and there is a natural spacewise homotopy equivalence of G-prespectra indexed on $\mathcal{U}$

$$KE \xrightarrow{\simeq} E.$$

Let $L_{\mathcal{U}^H}$ be the spectrification from G-prespectra indexed on $\mathcal{U}^H$ to G-spectra indexed on $\mathcal{U}^H$. For each finite-dimensional V contained in $\mathcal{U}$, such that $V \cap \mathcal{U}^H = \{0\}$,

$\{KE_{V\oplus Z}\}_{Z\subset \mathcal{U}^H}$ is a Σ-cofibrant G-prespectrum indexed on $\mathcal{U}^H$, which is spacewise homotopy equivalent to $E(V)$. By Proposition I.8.13 of [**8**], we get natural spacewise homotopy equivalences of G-spectra indexed on $\mathcal{U}^H$

$$L_{\mathcal{U}^H}\{KE_{V\oplus Z}\} \xrightarrow{\simeq} E(V).$$

Define $\overline{E}_{\mathrm{pre}}(V) = L_{\mathcal{U}^H}\{KE_{V\oplus Z}\}$. Then each $\overline{E}_{\mathrm{pre}}(V)$ is of cell homotopy type. Further, we have structure maps

$$\Sigma^{W-V}\overline{E}_{\mathrm{pre}}(V) \longrightarrow \overline{E}_{\mathrm{pre}}(W)$$

whenever $V \subseteq W$, $W \cap \mathcal{U}^H = \{0\}$. These are cofibrations of G-spectra indexed on $\mathcal{U}^H$, since they are spectrifications of spacewise cofibrations of prespectra.

Now we have

$$E' = f_\sharp(j^* i_* E \wedge_{E\mathcal{F}/H} C_f^{-1}) \simeq (E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} (h^*(i_* E \wedge S^{-A}))$$

over $E\mathcal{F}/H$, where E is our H-free G-spectrum of G-cell homotopy type indexed on $\mathcal{U}^H$. Thus, $i_*E \wedge S^{-A}$ is a G-spectrum of cell homotopy type indexed on $\mathcal{U}$. Applying the above discussion to $i_*E \wedge S^{-A}$, we get a G-prespectrum $K(i_*E \wedge S^{-A})$, which is spacewise homotopy equivalent to $i_*E \wedge S^{-A}$. Also, each space of $K(i_*E \wedge S^{-A})$ has the homotopy type of a G-cell complex, and each adjoint prespectrum structure map is a cofibration of G-spaces. For a fixed finite-dimensional G-representation V such that $V \cap \mathcal{U}^H = \{0\}$, consider the G-prespectrum over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$, whose Z-th space is

$$(E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} h^* K(i_*E \wedge S^{-A})_{V\oplus Z}$$

for each finite-dimensional G-representation Z contained in $\mathcal{U}^H$. Since h^* and $(E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} -$ both preserve homotopy equivalences, this prespectrum is spacewise homotopy equivalent to $E'(V)$. Let $L_{\mathcal{U}^H}$ denote the spectrification functor from G-prespectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$ to G-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$. We define $\overline{E'}_{\mathrm{pre}}(V)$ to be $L_{\mathcal{U}^H}$ of this G-prespectrum over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$. Then $\{\overline{E'}_{\mathrm{pre}}(V)\}$ form a $(\mathcal{U},\mathcal{U}^H)$-presystem over $E\mathcal{F}/H$.

The G-space $E\mathcal{F}/H$ is a G-cell complex, so h^* preserves cofibrations. Also, as we will see in detail later, the map $E\mathcal{F} \to E\mathcal{F}/H$ is a smooth family with fiber H, so it is also a homotopy cell bundle. Hence, by Lemma 4.19, the functor

$$(E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} -$$

also preserves cofibrations of G-spaces over $E\mathcal{F}/H$. So for all $V \subseteq W$ in $\mathcal{U}$, with $W \cap \mathcal{U}^H = \{0\}$, the structure map

$$\begin{aligned}\Sigma^{W-V}_{E\mathcal{F}/H}&(E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} h^* K(i_*E \wedge S^{-A})_{V\oplus Z} \\ &\longrightarrow (E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} h^* K(i_*E \wedge S^{-A})_{W\oplus Z}\end{aligned}$$

is a cofibration for any $Z \subset \mathcal{U}^H$. Applying the functor $L_{\mathcal{U}^H}$, we get that the structure maps

$$\Sigma^{W-V}_{E\mathcal{F}/H}\overline{E'}_{\mathrm{pre}}(V) \longrightarrow \overline{E'}_{\mathrm{pre}}(W)$$

are cofibrations of G-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$. Also, since the functor $L_{\mathcal{U}^H}$ commutes with smashing with a space and also with h^*, each $\overline{E'}_{\mathrm{pre}}(V)$ is in fact

$$(6.21) \qquad (E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} h^* L_{\mathcal{U}^H} K((i_*E \wedge S^{-A})(V)).$$

But $L_{\mathcal{U}^H} K((i_*E \wedge S^{-A})(V))$ is a G-spectrum of cell homotopy type indexed on $\mathcal{U}^H$, so (6.21) is a homotopy cell bundle spectrum over $E\mathcal{F}/H$. Finally,

$$K((i_*E \wedge S^{-A})(V)) \longrightarrow L_{\mathcal{U}^H} K((i_*E \wedge S^{-A})(V))$$

is a spacewise homotopy equivalence, so $\overline{E'}(V)$ is spacewise homotopy equivalent to the G-prespectrum $(E\mathcal{F} \amalg E\mathcal{F}/H) \wedge_{E\mathcal{F}/H} K((i_*E \wedge S^{-A})(V))$ over $E\mathcal{F}/H$, which is in turn spacewise homotopy equivalent to $E'(V)$. This gives that the replacement $\{\overline{E'}_{\mathrm{pre}}(V)\}_V$ of E' satisfies the necessary conditions. This concludes the proof of Proposition 6.11. □

To show that Theorem 4.9 holds for $f : E\mathcal{F} \to E\mathcal{F}/H$, we need to show that f is an equivariant smooth family of manifolds. We recall briefly the way to think of the Adams isomorphism from [**8**] Section II.7. Let $\Gamma = H \ltimes G$, where G acts on H by conjugation. So there is the short exact sequence of groups

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

In particular, $H \cong \Gamma/G$ as a Γ-space. Let $\theta : \Gamma \to G$ be the map which takes (h, g) to $hg \in G$. Then $\theta^* E\mathcal{F}$ is $E\mathcal{F}$ as a Γ-space, whose Γ-action comes from the surjection θ. The map $f : E\mathcal{F} \to E\mathcal{F}/H$ is a fiber bundle with fiber $H = \Gamma/G$, when we think of it as

$$E\mathcal{F} \cong \theta^* E\mathcal{F} \times_H \Gamma/G \longrightarrow \theta^* E\mathcal{F} \times_H * \cong E\mathcal{F}/H.$$

The fiber of f is the manifold $H = \Gamma/G$, and $\mathcal{S} = \mathrm{Diff}(H)$ in the language of equivariant smooth families. In particular, there is an embedding $i : H \to \mathcal{S}$ since H acts on itself smoothly by translation. There is also a map $j : G \to \mathcal{S}$ since G acts smoothly on H by conjugation. So we can define an embedding of groups

$$\iota : \Gamma \longrightarrow G \times \mathcal{S}$$

where $g \in G \subset \Gamma$ maps to $(g, j(g))$, and $h \in H \subset \Gamma$ maps to $(e, i(h))$. Let N be a subgroup of G such that $N \cap H = \{e\}$. Define the subgroup $H \odot N \subset \Gamma$ by

$$H \odot N = \{(h, h^{-1}n) \in \Gamma \mid h \in H,\ n \in N\}.$$

To see that $H \odot N$ is a subgroup of Γ, write $\Gamma = \{(h, g) \mid h \in H,\ g \in G\}$, with the multiplication

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 h_2 g_1^{-1}), g_1 g_2).$$

Then for $(h_1, h_1^{-1}n_1)$ and $(h_2, h_2^{-1}n_2)$ in $H \odot N$,

$$\begin{aligned}(h_1, h_1^{-1}n_1)(h_2, h_2^{-1}n_2) &= (h_1(h_1^{-1}n_1h_2n_1^{-1}h_1), (h_1^{-1}n_1)(h_2^{-1}n_2)) \\ &= (n_1h_2n_1^{-1}h_1, h_1^{-1}n_1h_2^{-1}n_2) \\ &= (n_1h_2n_1^{-1}h_1, (h_1^{-1}n_1h_2^{-1}n_1^{-1})(n_1n_2)) \\ &= (n_1h_2n_1^{-1}h_1, (n_1h_2n_1^{-1}h_1)^{-1}(n_1n_2)).\end{aligned}$$

Since H is normal in G, $n_1h_2n_1^{-1}h_1$ is contained in H. Then $\iota(H \odot N)$ is a subgroup of $G \times \mathcal{S}$ that acts smoothly on Γ/G, and $\iota(H \odot N) \cap \mathcal{S} = \{e\}$. Thus, $\iota(H \odot N)$ is in $\mathcal{F}_{\mathrm{sm}}$. So $\iota(H \odot -)$ gives a functor from the category $\mathcal{F}$ to $\mathcal{F}_{\mathrm{sm}}$. Note that as a Γ-space,

$$\theta^*(G/N) = \Gamma/(H \odot N).$$

So the cells of $E\mathcal{F}/H = (\theta^* E\mathcal{F})/H$ are of the form

$$(\Gamma/(H \odot N))/H \cong G/(H \cdot N).$$

Here $H \cdot N$ is the smallest subgroup of G that contains both H and N. Since H is normal, this is just the set-theoretical cartesian product of H and N in G. On the other hand, the corresponding cell of $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$ is of the form

$$((G \times \mathcal{S}/\iota(H \odot N))/\mathcal{S} \cong G/(H \cdot N).$$

The map ι induces a natural isomorphism between the two. Taking colimit of the cells of $E\mathcal{F}/H$, we have a map

$$\bar{\iota} : E\mathcal{F}/H \longrightarrow E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}.$$

Over each cell $\Gamma/(H \odot N)/H$ in $E\mathcal{F}/H$, the fiber in $E\mathcal{F} = \theta^* E\mathcal{F} \times_H \Gamma/G$ is

$$\Gamma/(H \odot N) \times_H \Gamma/G.$$

On the other hand, over the corresponding cell $((G \times \mathcal{S})/\iota(H \odot N))/\mathcal{S}$ in $E\mathcal{F}_{\mathrm{sm}}/\mathcal{S}$, the fiber in $E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} \Gamma/G$ is

$$(G \times \mathcal{S})/\iota(H \odot N) \times_{\mathcal{S}} \Gamma/G \cong G \times_{H \cdot N} \Gamma/G.$$

Again, ι induces a natural isomorphism between these two. So we also have a map

$$\tilde{\iota} : E\mathcal{F} = \theta^* E\mathcal{F} \times_H \Gamma/G \longrightarrow E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} \Gamma/G$$

and the diagram

$$\begin{tikzcd} E\mathcal{F} \arrow[r, "\tilde{\iota}"] \arrow[d, "f"'] & E\mathcal{F}_{\mathrm{sm}} \times_{\mathcal{S}} \Gamma/G \arrow[d] \\ E\mathcal{F}/H \arrow[r, "\bar{\iota}"'] & E\mathcal{F}_{\mathrm{sm}}/\mathcal{S} \end{tikzcd}$$

commutes and is a pullback square, since the fibers on the left and right hands are the same. Hence, $f : E\mathcal{F} \to E\mathcal{F}/H$ is a family of manifolds in our sense.

It remains to prove Lemma 6.9.

Proof of Lemma 6.9. — The dualizing object C_f is the sphere bundle of the tangent bundle of $E\mathcal{F}$ in the category over $E\mathcal{F}/H$. By [8], Section II.7 the tangent bundle of Γ/G is just $\Gamma/G \times A$, where A is the adjoint representation. So the tangent bundle of $E\mathcal{F}$ is

$$\theta^* E\mathcal{F} \times_H (\Gamma/G \times A) \longrightarrow \theta^* E\mathcal{F} \times_H \Gamma/G \cong E\mathcal{F}.$$

The total space of the bundle is $(\theta^* E\mathcal{F} \times (\Gamma/G \times A))/H$, where $\Gamma/G \times A$ is a Γ-space via the action of Γ on the first coordinate. So as a G-space over $E\mathcal{F}$, $\theta^* E\mathcal{F} \times_H (\Gamma/G \times A)$ is isomorphic to $(\theta^* E\mathcal{F} \times \Gamma/G)/H \times A \cong E\mathcal{F} \times A$, *i.e.* the tangent bundle of $E\mathcal{F}$ is trivial. Thus, its sphere bundle is $E\mathcal{F} \times S^A = j^*(S^A)$. □

Thus, we have that the conditions for Theorem 4.9 are satisfied by

$$f : E\mathcal{F} \longrightarrow E\mathcal{F}/H.$$

The two compositions of (6.4) coincide up to weak equivalences with the two sides of the Adams isomorphism. So for an H-free G-spectrum E of G-cell homotopy type indexed on $\mathcal{U}^H$, Theorem 4.9 gives a weak equivalence between $f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1})$ and $f_*(j^* i_* E)$. It remains to show that this gives a weak equivalence between $h_\sharp(f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1}))^H$ and $h_\sharp(f_*(j^* i_* E))^H$. By Lemma 6.7, the functor $(-)^H$ preserves weak equivalences, but the functor $h_\sharp$ does not preserve weak equivalences in general. To get around this, we will show the following.

Proposition 6.22. — *For an H-free G-spectrum E of G-cell homotopy type indexed on $\mathcal{U}^H$, the spectra $(f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1}))^H$ and $(f_*(j^* i_* E))^H$ are each spacewise homotopy equivalent to a cell spectrum in the category of J-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$.*

Proof. — We first consider the case of $(f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1}))^H$. Recall Theorem I.1.1 of [8], which states that if T is a compact G-space, and Z is a G-space with the homotopy type of a G-cell complex, then $F(T, Z)$ also has the homotopy type of a G-cell complex. In particular, for every finite-dimensional V in $\mathcal{U}$, $\Omega^V Z$ has the homotopy type of a G-cell complex. We have a version of this statement for fibrant G-spaces over X. Suppose X is a G-cell complex, and Z is a G-space over X which is a homotopy cell bundle over X. Then

$$\Omega^V_X Z = \underline{\mathrm{Hom}}_X(S^V \times X, Z) = \amalg_{x \in X} \mathrm{Hom}(S^V, Z_x)$$

as sets. We can give a cell structure to X such that over each cell, the fibers Z_x are constant. Let $G/N \times D^n$ be such a cell. Then over $G/N \times D^n$, we have that

$$(\Omega^V_X)_{G/N \times D^n} = \amalg_{x \in G/N \times D^n} \underline{\mathrm{Hom}}(S^V, Z_x) = (G/N \times D^n) \times \underline{\mathrm{Hom}}(S^V, Z_x)$$

for any $x \in G/N \times D^n$. The fiber Z_x has the homotopy type of a G-cell complex, thus, so does $\underline{\mathrm{Hom}}(S^V, Z_x)$ and $(G/N \times D^n) \times \underline{\mathrm{Hom}}(S^V, Z_x)$. By gluing over the cells of X, we then get that $\Omega^V_X Z$ is of G-cell homotopy type if Z is fibrant and of G-cell homotopy type over X.

Now the spectrum $f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$ is a homotopy cell bundle spectrum over $E\mathcal{F}/H$. Thus, by applying the arguments of Proposition I.8.14 of [**8**] to the fibers of the homotopy cell bundle spectrum, each space of $f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$ has the homotopy type of a homotopy cell bundle over $E\mathcal{F}/H$. Hence, so does each space of $i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$. Now recall the cylinder construction KD for a spectrum D (see [**8**] Section I.6 and [**4**], Section X.5). By an analogous argument, one also has the cylinder construction $K(-)$ in the category of spectra over $E\mathcal{F}/H$. We apply it to $i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$. By arguments similar to Proposition X.5.3 of [**4**], $K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))$ has the homotopy type of a relative G-cell spectrum over $E\mathcal{F}/H$. Also, for any spectrum D over $E\mathcal{F}/H$, there is a weak equivalence of spectra $r : KD \to D$. Thus, for each finite-dimensional V in the universe, r_V is a weak equivalence of G-spaces. In our case, for each finite-dimensional V in $\mathcal{U}^H$, $(K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))_V$ and $(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))_V$ both have homotopy types of relative cell complexes over $E\mathcal{F}/H$, so the weak equivalence

$$r_V : (K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))_V \longrightarrow (i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))_V$$

is a homotopy equivalence, *i.e.* the map of spectra over $E\mathcal{F}/H$

$$r : K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})) \longrightarrow i^*f_\sharp({}^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})$$

is a spacewise homotopy equivalence. The fixed point functor $(-)^H$ on spectra indexed on the H-fixed universe $\mathcal{U}^H$ preserves cell structure, so

$$r^H : (K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))^H \longrightarrow (i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C - f^{-1}))^H$$

is a spacewise homotopy equivalence, and $(K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))^H$ has the homotopy type of a G-cell spectrum over $E\mathcal{F}/H$.

For the other spectrum $(i^*f_*(j^*i_*E))^H$, we have that j^* commutes with i_* by Lemma 6.8, so it is in fact isomorphic to $(i^*f_*i_*j^*E)^H$. But the functors i^* and f_* also commute, so this is isomorphic to $(f_*i^*i_*j^*E)^H$. Since j^*E is a homotopy cell bundle spectrum over $E\mathcal{F}$, by Lemma 6.6, the unit of adjunction $j^*E \to i^*i_*j^*E$ is a spacewise homotopy equivalence. The functor $(f_*(-))^H$ is taken spacewise, and on a G-space, $(f_*(-))^H \cong (f_\sharp(-))/H$, so it preserves homotopies of G-spaces. Hence, $(f_*(-))^H$ on spectra preserves spacewise homotopy equivalences. So there is a spacewise homotopy equivalence

$$(f_*(j^*E))^H \simeq (i^*f_*(j^*i_*E))^H.$$

But $(f_*(j^*E))^H \cong (f_\sharp(j^*E))/H$, which is a cell spectrum. □

Therefore, we have a weak equivalence of J-spectra over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$

$$(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))^H \simeq (i^*f_*(j^*i_*E))^H$$

and spacewise homotopy equivalences

$$(K(i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1})))^H \simeq (i^*f_\sharp(j^*i_*E \wedge_{E\mathcal{F}} C_f^{-1}))^H$$

and

$$(i^* f_*(j^* i_* E))^H \simeq (f_\sharp j^* E)/H.$$

This gives a weak equivalence of J-spectra of J-cell homotopy type over $E\mathcal{F}/H$ indexed on $\mathcal{U}^H$

$$(K(i^* f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1})))^H \simeq (f_\sharp j^* E)/H.$$

Since $h_\sharp$ preserves weak equivalences between spectra of cell homotopy type (*i.e.* homotopy equivalences) and also takes a spacewise homotopy equivalence of spectra to a weak equivalence of spectra, Proposition 6.22 gives the weak equivalence

$$\begin{aligned} h_\sharp(i^* f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1}))^H &\simeq h_\sharp(K(i^* f_\sharp(j^* i_* E \wedge_{E\mathcal{F}} C_f^{-1})))^H \\ &\simeq h_\sharp((f_\sharp j^* E)/H) \\ &\simeq h_\sharp(i^* f_*(j^* i_* E))^H. \end{aligned}$$

Hence, the main duality Theorem 4.9 implies the Adams isomorphism.

CHAPTER 7

PROOF OF RESULTS ON THE MODEL STRUCTURE OVER A BASE

In this chapter, we prove some results that are stated in Chapter 3. The first such result is Lemma 3.2, which gives that weak equivalences between fibrant G-spaces over a base are preserved by pullbacks.

Proof of Lemma 3.2. — Let $p_1 : Z_1 \to Y$ and $p_2 : Z_2 \to Y$ be the structure maps of Z_1 and Z_2 respectively, so p_1 and p_2 are fibrations. Also, we have the weak equivalence $g : Z_1 \to Z_2$ over Y. We can factor the map $f : X \to Y$ in the category of G-spaces to

$$X \xrightarrow{f_1} X' \xrightarrow{f_2} Y$$

such that f_1 is an acyclic cofibration and f_2 is a fibration. Then $f^*(g) = f_1^*(f_2^*(g))$. So we have the following diagram of G-spaces.

$$\begin{tikzcd}
& X' \times_Y Z_2 \arrow[rr] \arrow[dd] & & Z_2 \arrow[dd, "p_2"] \\
X' \times_Y Z_1 \arrow[ur, "f_2^* g"] \arrow[rr] \arrow[dr] & & Z_1 \arrow[ur, "g", "\simeq"'] \arrow[dr, "p_1"'] & \\
& X' \arrow[rr, "f_2"'] & & Y.
\end{tikzcd}$$

Recall that the pullback of a fibration of G-spaces is a fibration. Thus, the maps $X' \times_Y Z_1 \to Z_1$ and $X' \times_Y Z_2 \to Z_2$ are fibrations. The top square of the diagram is a pullback. Recall also that the model category structure of G-spaces is proper, so pullbacks along fibrations preserve weak equivalences. Thus, $f_2^*(g)$ is a weak equivalence.

We also have the diagram

$$\begin{tikzcd}
& X \times_Y Z_2 \arrow[rr] \arrow[dd, "f^*(p_2)"] & & X' \times_Y Z_2 \arrow[dd, "f_2^*(p_2)"] \\
X \times_Y Z_1 \arrow[ur, "f^*g"] \arrow[rr] \arrow[dr, "f^*(p_1)"'] & & X' \times_Y Z_1 \arrow[ur, "f_2^*g"', "\simeq"] \arrow[dr, "f_2^*(p_1)"] & \\
& X \arrow[rr, "\simeq", "f_1"'] & & X'.
\end{tikzcd}$$

The maps $f_2^*(p_1)$ and $f_2^*(p_2)$ are fibrations, and the map f_1 is an acyclic cofibration. So by the properness of the model structure on G-spaces, the maps $X \times_Y Z_1 \to X' \times_Y Z_1$ and $X \times_Y Z_2 \to X' \times_Y Z_2$ are weak equivalences. Thus, in the top square of the diagram, three of the maps are weak equivalences, so the fourth map $f^*(g)$ is also an weak equivalence. $\square$

The next result from Section 3 we need to prove is Lemma 3.7, the parametrized version of the homotopy extension and lifting property.

Proof of Lemma 3.7. — Let p_E, $p_{E'}$, p_N and p_P denote the structure maps of the spectra to X, respectively. Similarly as in [**8**] Theorem I.5.9, it suffices to consider the case when $E = (\Sigma_V^\infty)_X(G/H \times S^{n-1}) \amalg X$ and $E' = (\Sigma_V^\infty)_X(G/H \times D^n) \amalg X$, with structure maps coming from any map $G/H \times D^n \to X$. We think of $I = [0,1]$. By Theorem I.5.9 of [**8**], we can obtain maps $\widetilde{g} : E' \to N$ and $\widetilde{h} : E' \wedge I_+ \to P$ such that diagram (3.8) commutes, but $\widetilde{g}$ and $\widetilde{h}$ may not be maps over X.

We write $E' \wedge [1,2]_+$ for $E' \wedge_X ((X \times [1,2]) \amalg X)$. Define the map $\widetilde{h}^{\mathrm{op}} : E' \wedge [1,2]_+ \to P$ as follows. For any $t \in [0,1]$, write $\widetilde{h}_t = \widetilde{h} \cdot i_t : E' \to E' \wedge I_+ \to P$. Then for any $t \in [1,2]$, set

$$(\widetilde{h}^{\mathrm{op}})_t = \widetilde{h}_{2-t} : E' \longrightarrow P.$$

We have that $p_P \cdot \widetilde{h}|_{E \wedge I_+} = p_P \cdot h = p_{E \wedge I_+}$ is constant with respect to the coordinate in I, *i.e.* for every $t \in I$,

$$p_P \cdot \widetilde{h}|_{E \wedge I_+} \cdot i_t = p_P \cdot \widetilde{h}|_{E \wedge I_+} \cdot i_0 : E \longrightarrow E \wedge I_+ \longrightarrow P \longrightarrow X.$$

Thus, $p_P \cdot \widetilde{h}^{\mathrm{op}}|_{E \wedge [1,2]_+} : E \wedge [1,2]_+ \to X$ is also constant with respect to the the coordinate in $[1,2]$. Also, the composition

$$E' \xrightarrow{i_1} E' \wedge I_+ \xrightarrow{\widetilde{h}} P$$

factors to $e \cdot \widetilde{g} : E' \to N \to P$. We define $(E \wedge [1,2]_+) \cup E'$ by the following pushout diagram in the category of spectra over X.

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "i_1"'] & E' \arrow[d] \\ E \wedge [1,2]_+ \arrow[r] & (E \wedge [1,2]_+) \cup E'. \end{tikzcd}$$

We have a map

$$j \cup i_1 : (E \wedge [1,2]_+) \cup E' \longrightarrow E' \wedge [1,2]_+$$

induced by $j : E \wedge [1,2]_+ \to E' \wedge [1,2]_+$ and $i_1 : E' \to E' \wedge [1,2]_+$. Then $j \cup i_1$ is an acyclic cofibration of spectra over X. Define a map

$$\alpha : (E \wedge [1,2]_+) \cup E' \longrightarrow N.$$

On E', α maps to N by $\widetilde{h}_1 = \widetilde{g}$, and for any $t \in [1,2]$, $\alpha_t = \alpha \cdot i_t : E \to E \wedge [1,2]_+ \to N$ is equal to $\widetilde{g}|E = g$. We have the commutative diagram

$$\begin{tikzcd} E' \arrow[r, "\widetilde{g}"] \arrow[d, "i_1"] & N \arrow[d, "e"] \\ E' \wedge [1,2]_+ \arrow[r, "\widetilde{h}^{\mathrm{op}}"'] & P. \end{tikzcd}$$

Also, consider the diagram

$$\begin{tikzcd} E \wedge [1,2]_+ \arrow[rr, "\alpha"] \arrow[d, "j"'] & & N \arrow[d, "p_N"] \\ E' \wedge [1,2]_+ \arrow[r, "\widetilde{h}^{\mathrm{op}}"'] & P \arrow[r, "p_P"'] & X. \end{tikzcd}$$

For every $t \in [1,2]$, $p_P \cdot (\widetilde{h}^{\mathrm{op}})_t \cdot j = p_P \cdot (\widetilde{h}^{\mathrm{op}})_1 \cdot j = p_P \cdot e \cdot g$, whereas $p_N \cdot \alpha_t = p_N \cdot g$. Since e is a map over X, the diagram commutes. Thus, we have the diagram

$$\begin{tikzcd} (E \wedge [1,2]_+) \cup E' \arrow[rr, "\alpha"] \arrow[d, "j"'] & & N \arrow[d, "p_N"] \\ E' \wedge [1,2]_+ \arrow[r, "\widetilde{h}^{\mathrm{op}}"'] \arrow[urr, dashed, "\beta"] & P \arrow[r, "p_P"'] & X. \end{tikzcd}$$

The square commutes, left vertical map j is an acyclic cofibration, and p_N is a fibration, so the dotted arrow β exists. For the map $\beta_2 : E' \to N$, $p_N \cdot \beta_2$ is equal to $p_P \cdot (\widetilde{h}^{\mathrm{op}})_2$. But $(\widetilde{h}^{\mathrm{op}})_2 = \widetilde{h}_0 = f : E' \to P$ is a map over X, so β_2 is a map over X.

We define a map

$$\gamma : E' \wedge [0,2]_+ \longrightarrow X$$

by $\gamma_t = p_P \cdot \widetilde{h}_t$ for $t \in [0,1]$, and $\gamma_t = p_P \cdot \widetilde{h}_t^{\mathrm{op}} = p_N \cdot \beta_t$ for $t \in [1,2]$. We have that $\widetilde{h}_1^{\mathrm{op}} = \widetilde{h}_1$, so γ is a continuous map. For each $t \in [0,2]$, $\gamma_t = \gamma_{2-t}$. Define also

$$c : E' \wedge [0,2]_+ \longrightarrow X$$

where $c \cdot i_0 : E' \to E' \wedge [0,2]_+ \to X$ is equal to $p_{E'}$, and c is constant with respect to the coordinate in $[0,2]$. We have that γ is homotopic to c. Namely, the homotopy is

$$H : E' \wedge [0,2]_+ \wedge [0,1]_+ \longrightarrow X.$$

For $t \in [0,2]$ and $s \in [0,1]$, $H_{t,s} : E' \to X$ is given by

$$\begin{aligned} H_{t,s} &= \gamma_{t(1-s)} \text{ for } 0 \leqslant t \leqslant 1 \\ &= \gamma_{2-(2-t)(1-s)} \text{ for } 1 \leqslant t \leqslant 2. \end{aligned}$$

If $t = 1$, for every $s \in [0,1]$, $\gamma_{2-(2-1)(1-s)} = \gamma_{1+s} = \gamma_{1-s} : E' \to X$, so H is a continuous map. We have $H_{t,0} = \gamma_t$ and $H_{t,1} = \gamma_0 = \gamma_2 = p_{E'}$ for any $t \in [0,2]$.

We write $\delta(E' \wedge [0,2]_+) = (E' \wedge \{0\}_+) \cup (E' \wedge \{2\}_+) \cup (E \wedge [0,2]_+)$ to be the following pushout:

$$\begin{tikzcd} E \vee_X E \arrow[r, "j \vee_X j"] \arrow[d, "i_0 \vee_X i_2"'] & E' \vee_X E' \arrow[d] \\ E \wedge [0,2]_+ \arrow[r] & \delta(E' \wedge [0,2]_+). \end{tikzcd}$$

We have $\iota : \delta(E' \wedge [0,2]_+) \to E' \wedge [0,2]_+$. For every $s \in [0,1]$, $H_{0,s} = \gamma_0 = p_{E'}$, $H_{2,s} = \gamma_2 = p_{E'}$. For $0 \leqslant t \leqslant 1$, $H_{t,s}|_E = \gamma_{t(1-s)}|E = p_P \cdot h = p_E$. Similarly, $H_{t,s}|_E = p_E$ for $1 \leqslant t \leqslant 2$. So the homotopy H is constant on $\delta(E' \wedge [0,2]_+)$. Let $(E' \wedge [0,2]_+) \cup (\delta(E' \wedge [0,2]_+) \wedge [0,1]_+)$ be defined by the pushout diagram

$$\begin{tikzcd} \delta(E' \wedge [0,2]_+) \arrow[r, "i_0"] \arrow[d, "\iota"'] & \delta(E' \wedge [0,2]_+) \wedge [0,1]_+ \arrow[d] \\ E' \wedge [0,2]_+ \arrow[r] & (E' \wedge [0,2]_+) \cup (\delta(E' \wedge [0,2]_+) \wedge [0,1]_+) \end{tikzcd}$$

in the category of spectra over X. There is a map

$$i_0 \cup (\iota \wedge [0,1]_+) : (E' \wedge [0,2]_+) \cup (\delta(E' \wedge [0,2]_+) \wedge [0,1]_+) \longrightarrow E' \wedge [0,2]_+ \wedge [0,1]_+.$$

Also, define the maps

$$\varepsilon : E' \wedge [0,2]_+ \longrightarrow P$$

and

$$\varepsilon' : \delta(E' \wedge [0,2]_+) \wedge [0,1]_+ \longrightarrow P.$$

For $t \in [0,1]$, set $\varepsilon_t = \widetilde{h}_t : E' \to P$. For $t \in [1,2]$, set $\varepsilon_t = e \cdot \beta_t : E' \to N \to P$. $\varepsilon' \cdot i_0 : \delta(E' \wedge [0,2]_+)$ is equal to $\varepsilon|_{\delta(E' \wedge [0,2]_+)}$, and it is constant with respect to the coordinate in $[0,1]$.

Thus, we have the following diagram.

$$\begin{tikzcd}
(E' \wedge [0,2]_+) \cup (\delta(E' \wedge [0,2]_+) \wedge [0,1]_+) \arrow[r, "\varepsilon \cup \varepsilon'"] \arrow[d, "i_0 \cup (\iota \wedge [0,1]_+)"'] & P \arrow[d, "p_P"] \\
E' \wedge [0,2]_+ \wedge [0,1]_+ \arrow[r, "H"'] \arrow[ur, dashed, "\overline{H}"] & X.
\end{tikzcd}$$

Since

$$\iota : \delta(E' \wedge [0,2]_+) \longrightarrow E' \wedge [0,2]_+$$

is a cofibration, the left vertical arrow $i_0 \cup (i \wedge [0,1]_+)$ is a deformation retract, thus an acyclic cofibration. Also, p_P is a fibration, so the dotted arrow $\overline{H}$ exists, making the diagram commute. Consider

$$\overline{H}_1 = \overline{H} \cdot i_1 : E' \wedge [0,2]_+ \longrightarrow E' \wedge [0,2]_+ \wedge [0,1]_+ \longrightarrow P.$$

Then $\overline{H}_1|_{E \wedge [1,2]_+}$ is constant with respect to the coordinate in $[1,2]$. So $\overline{H}_1$ factors to

$$\overline{h} : (E' \wedge [0,2]_+)/_X(E \wedge [1,2]_+) \longrightarrow P.$$

The source is homeomorphic over X to $E' \wedge [0,1]_+$, so we have $\overline{h} : E' \wedge [0,1]_+ \to P$. Also,

$$\overline{h}|_{E' \wedge \{1\}_+} = \overline{H}_1 \cdot i_2 = \overline{H}_0 \cdot i_2 = e \cdot \beta_2$$

so it lifts to $\beta_2 : E' \to N$. So define $\overline{g} = \beta_2$. It is then straightforward to check that $\overline{g}$ and $\overline{h}$ are maps over X, and that they make the diagram (3.8) commute. □

Finally, we prove Lemma 3.13.

Proof of Lemma 3.13. — Let $i_Z : X \to Z$ and $i_T : X \to T$ be the basepoints of Z and T. Since i_Z and i_T are cofibrations, we can find a homotopy inverse $g : Z \to T$ to f over Y, so that $(\mathrm{Id}, g) : (X,Z) \to (X,T)$ is a homotopy inverse to (Id, f) in the category of pairs over Y, and that the homotopy is the identity on X for every $t \in I$. But g is not necessarily a map over X. Consider the diagram in the category over Y

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[dr, "p_Z"'] & T \arrow[r, "f"] \arrow[d, "p_T"] & Z \arrow[dl, "p_Z"] \\
& X &
\end{tikzcd}$$

The left triangle does not commute, but the right triangle does. Also, the composition of the top row is homotopic to the identity, so the entire large triangle commutes up to homotopy in the category over Y, with a homotopy that is the identity on the basepoint X for every $t \in I$. Hence, the left triangle also commutes up to homotopy in the category over Y, with a homotopy that is the identity on X for every $t \in I$. This means that there is a homotopy in the category of G-spaces over Y

$$H : Z \wedge_X (X \times I \amalg X) \longrightarrow X,$$

such that $H_0 = p_Z : Z \to X$, and $H_1 = p_T \cdot g : Z \to X$. We have the commutative diagram in the category of G-spaces over Y

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& Z \arrow[r, "g"] \arrow[d, "i_1"'] & T \arrow[d, "p_T"] \\
& Z \wedge_X (X \times I \amalg X) \arrow[r, "H"'] \arrow[ur, dashed, "\overline{H}"] & X.
\end{tikzcd}$$

Since Z is cofibrant over X, the map i_1 is an acyclic cofibration. Also, p_T is a fibration, so there exists a lifting $\overline{H} : Z \wedge I_+ \to T$ making the diagram commute. Since $p_T \cdot \overline{H}_0 = H_0 = p_Z : Z \to X$, $\overline{H}_0 : Z \to T$ is a map over X. Also, the diagram

$$\begin{tikzcd}
X \arrow[r, "i_Z"] \arrow[d, "i_Z"'] & Z \arrow[d, "i_1"] \\
Z \arrow[r, "i_0"'] & Z \wedge_X (X \times I \amalg X)
\end{tikzcd}$$

commutes, so $\overline{H}_0$ is based over X. Now $\overline{H}_0$ is homotopic to $g = \overline{H}_1$, so

$$f \cdot \overline{H} : Z \wedge_X I_+ \longrightarrow T \longrightarrow Z$$

is a homotopy between $f \cdot \overline{H}_0$ and $f \cdot g$, which is in turn homotopic to Id_Z. Hence, $f \cdot \overline{H}_0$ is homotopic to the identity on Z. Similarly,

$$\overline{H} \cdot (f \wedge \mathrm{Id}) : T \wedge I_+ \longrightarrow Z \wedge I_+ \longrightarrow T$$

is a homotopy between $\overline{H}_0 \cdot f$ and $g \cdot f$, which is in turn $\overline{H}_0 \cdot f$ and $g \cdot f$, which is in turn homotopic to Id_T, so $\overline{H}_0 \cdot f$ is homotopic to Id_T. These homotopies are not over X, but by arguments similar to those of Lemma 3.7, we can correct them to based homotopies over X. Hence, $\overline{H}_0$ is a based homotopy inverse to f over X. $\square$

BIBLIOGRAPHY

[1] J. BLOCK & A. LAZAREV – Homotopy theory and generalized duality for spectral sheaves, *Internat. Math. Res. Notices* (1996), no. 20, p. 983–996.

[2] A. BOREL & N. SPALTENSTEIN – Sheaf theoretic intersection cohomology, in *Intersection cohomology (Bern, 1983)*, Progress in Math., vol. 50, Birkhäuser, 1984, p. 47–182.

[3] W.G. DYWER & J. SPALINSKI – Homotopy theories and model categories, in *Handbook of algebraic topology* (I.M. James, ed.), Elsevier, 1995, p. 1–56.

[4] A.D. ELMENDORF, I. KRIZ, M.A. MANDELL & J.P. MAY – *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, 1997, with an appendix by M. Cole.

[5] P. HU – *S-modules in the category of schemes*, Mem. Amer. Math. Soc., vol. 767, American Mathematical Society, 2003.

[6] B. IVERSEN – *Cohomology of sheaves*, Universitext, Springer-Verlag, 1986.

[7] L.G. LEWIS – Open maps, colimits, and a convenient category of fibered spaces, *Topology and its applications* **19** (1985), p. 75–89.

[8] J.P. M. L.G. LEWIS & M. STEINBERGER – *Equivariant stable homotopy theory*, Lect. Notes in Math., vol. 1213, Springer-Verlag, 1986.

[9] M.A. MANDELL & J.P. MAY – *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc., vol. 159, American Mathematical Society, 2002.

[10] M.A. MANDELL, J.P. MAY, S. SCHWEDE & B. SHIPLEY – Model categories of diagram spectra, *Proc. London Math. Soc. (3)* **82** (2001), no. 2, p. 441–512.

[11] J.P. MAY – Notes on ex-spaces and towards ex-spectra, Preprint, 2000.

[12] F. MOREL & V. VOEVODSKY – $\mathbb{A}^1$-homotopy theory of schemes, *Inst. Hautes Études Sci. Publ. Math.* **90** (2001), p. 45–143.

[13] R.S. Palais – Local triviality of the restriction map for embeddings, *Comment. Math. Helv.* **34** (1960), p. 305–312.

[14] ———, Equivalences of nearby differentiable actions of a compact group, *Bull. Amer. Math. Soc.* **67** (1961), p. 362–364.