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LOGARITHMIC SOBOLEV INEQUALITY AND SEMI-LINEAR DIRICHLET PROBLEMS FOR INFINITELY DEGENERATE ELLIPTIC OPERATORS

by

Yoshinori Morimoto & Chao-Jiang Xu

Abstract. — Let $X = (X_1, \dots, X_m)$ be an infinitely degenerate system of vector fields, we prove firstly the logarithmic Sobolev inequality for this system on the associated Sobolev function spaces. Then we study the Dirichlet problem for the semilinear problem of the sum of square of vector fields X.

***Résumé* (Inégalité de Sobolev logarithmique et problèmes de Dirichlet semi-linéaires pour des opérateurs elliptiques infiniment dégénérés)**

Soit $X = (X_1, \dots, X_m)$ un système de champs de vecteurs infiniment dégénérés. On montre d'abord l'inégalité de Sobolev logarithmique pour ce système de champs de vecteurs sur les espaces de fonctions associés, puis on étudie le problème de Dirichlet semi-linéaire pour des opérateurs somme de carrés de champs de vecteurs X.

1. Introduction

In this work, we consider a system of vector fields $X = (X_1, \dots, X_m)$ defined on an open domain $\widetilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following logarithmic regularity estimate,

$$\|(\log \Lambda)^s u\|_{L^2}^2 \leqslant C \left\{ \sum_{j=1}^{m} \|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}, \quad \forall\, u \in C_0^\infty(\widetilde{\Omega}); \tag{1.1}$$

where $\Lambda = (e + |D|^2)^{1/2} = \langle D \rangle$. We shall give some sufficient conditions for this estimates in the Appendix, see also [**5, 10, 12, 14, 15, 21**]. The typical example is the system in $\mathbb{R}^2$ such as $X_1 = \partial_{x_1}, X_2 = e^{-|x_1|^{-1/s}} \partial_{x_2}$ with $s > 0$. Remark that if $s > 1$, the estimate (1.1) implies the hypoellipticity of the infinitely degenerate elliptic operators of second order $\triangle_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j.

If Γ is a smooth surface of $\widetilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X, if for any point $x_0 \in \Gamma$, there exists at least one vector field of

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$X_1,\dots,X_m$ which is transversal to Γ at x_0. Let now $\Gamma=\cup_{j\in J}\Gamma_j$ be the union of a family of smooth surface in $\widetilde{\Omega}$. We say that Γ is non characteristic for X, if for any point $x_0\in\Gamma$, there exists at least one vector field of $X_1,\dots,X_m$ which traverses Γ_j at x_0 for all $j\in J_0=\{k\in J; x_0\in\Gamma_k\}$. For this second case, the typical example is $X_1=\partial_{x_1}, X_2=\exp(-(x_1^2\sin^2(\pi/x_1))^{-1/2s})\partial_{x_2}$, we have $\Gamma_j=\{x_1=1/j\}$, $j\in\mathbb{Z}\setminus\{0\}, \Gamma_0=\{x_1=0\}$, and X_1 is transverse to all $\Gamma_j, j\in\mathbb{Z}$.

Associated with the system of vector fields $X=(X_1,\dots,X_m)$, we define the following function spaces:

$$H^1_X(\widetilde{\Omega})=\Big\{u\in L^2(\widetilde{\Omega}); X_ju\in L^2(\widetilde{\Omega}), j=1,\dots,m\Big\}.$$

Take now $\Omega\subset\subset\widetilde{\Omega}$, we suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X. We define $H^1_{X,0}(\Omega)=\big\{u\in H^1_X(\Omega); u|_{\partial\Omega}=0\big\}$, we shall prove in the second section (see Lemma 2.1) that this is a Hilbert space.

Our first result is the following logarithmic Sobolev inequality.

Theorem 1.1. — *Suppose that the system of vector fields $X=(X_1,\dots,X_m)$ verifies the estimate (1.1) for some $s>1/2$. Then there exists $C_0>0$ such that*

$$\int_\Omega |v|^2\log^{2s-1}\Big(\frac{|v|}{\|v\|_{L^2}}\Big)\leqslant C_0\Big\{\sum_{j=1}^m\|X_jv\|^2_{L^2}+\|v\|^2_{L^2}\Big\},\tag{1.2}$$

for all $v\in H^1_{X,0}(\Omega)$.

Comparing this inequality with that of finite degenerate case of Hörmander's system, for example, for the system $X_1=\partial_{x_1}, X_2=x_1^k\partial_{x_2}$ on $\mathbb{R}^2$, we have (see [**4, 7, 24**])

$$\|v\|_{L^p}\leqslant C\left(\|\partial_1v\|^2_{L^2}+\|x_1^k\partial_2v\|^2_{L^2}+\|v\|^2_{L^2}\right)^{1/2}$$

for all $v\in C_0^\infty(\Omega)$, with $p=2+4/k$. Consequently, if k go to infinity, we can only expect to gain the logarithmic estimates as (1.2). That means that we are not in the elliptic case of [**17**].

Similarly to the elliptic and subelliptic case (see [**3, 24**]), by using the Sobolev's inequality, we study the following semi-linear Dirichlet problems

$$\begin{aligned}&\triangle_X u=au\log|u|+bu,\\&u|_{\partial\Omega}=0,\end{aligned}\tag{1.3}$$

where $a,b\in\mathbb{R}$. We have the following theorem.

Theorem 1.2. — *We suppose that the system of vector fields $X=(X_1,\dots,X_m)$ satisfies the following hypotheses:*

H-1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X ;

H-2) the system of vector fields X satisfies the finite type of Hörmander's condition on $\widetilde{\Omega}$ except an union of smooth surfaces Γ which are non characteristic for X.

H-3) the system of vector fields X verifies the estimate (1.1) for $s>3/2$.

Suppose $a \neq 0$ in (1.3). Then the semi-linear Dirichlet problem (1.3) posses at least one non trivial weak solution $u \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$. Moreover, if $a > 0$, we have $u \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\overline{\Omega} \setminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \setminus \Gamma$.

As in the elliptic case, we do not know the uniqueness of solutions (see [**3**]). The regularity of this weak solution near to the infinitely degenerate point of Γ is a more complicated problem, which will be studied in our future works.

The structure of the paper is as follows: The second section consists of the proof of Theorem 1.1. The third section is devoted to the proof for the existence of weak solution of Theorem 1.2, we introduce a variational problem and prove that the associated Euler-Lagrange equation is (1.3). In the fourth section we study the boundedness of weak solution of variational problems, which is a difficult step as in the classical case for the critical semilinear elliptic equations (see [**20**]). In the appendix we give some sufficient conditions for the logarithmic regularity estimates.

2. Logarithmic Sobolev inequality

We study now the function spaces $H^1_{X,0}(\Omega)$, see the similar results in [**22**].

Lemma 2.1. — *Suppose that $\partial\Omega$ is C^∞ and non characteristic for the system X, then $H^1_{X,0}(\Omega)$ is well-defined, and a Hilbert space. Moreover the extension of an element of $H^1_{X,0}(\Omega)$ by 0 belongs to $H^1_X(\widetilde{\Omega})$.*

Proof. — For the well-definedness, we need to prove the existence of trace for $v \in H^1_X(\Omega)$. We know that the trace problem is a local problem, so after the localization and straightened, we transfer the problem to the case: $v \in L^2(\mathbb{R}^d_+), \partial_{x_d} v \in L^2(\mathbb{R}^d_+)$ with support of v is a subset of $\{|(x', x_d)| < c, x_d \geqslant 0\}$, of course we can take the smooth function approximate to v, then we have

$$v(x', x_d) - v(x', c) = \int_c^{x_d} \partial_{x_d} v(x', t)dt,$$

which prove that

$$\|v(\cdot, x_d)\|^2_{L^2} \leqslant c\|\partial_{x_d} v\|^2_{L^2}, \tag{2.1}$$

for all $0 \leqslant x_d \leqslant c$. This shows that the trace $v(x', 0) \in L^2(\mathbb{R}^{d-1})$.

We shall prove now $H^1_{X,0}(\Omega)$ is a closed subspace of $H^1_X(\Omega)$. Let $\{v_j\}$ be a Cauchy sequence of $H^1_{X,0}(\Omega)$. Since it is also a Cauchy sequence of $H^1_X(\Omega)$, there exists a limit $v_0 \in H^1_X(\Omega)$, and so it suffices to show that $v|_{\partial\Omega} = 0$. Applying (2.1) to $v_j - v_0$, we have

$$\|v_j(\cdot, 0) - v_0(\cdot, 0)\|^2_{L^2} \leqslant c\|\partial_{x_d}(v_j - v_0)\|^2_{L^2},$$

which implies $\|v_0(\cdot, 0)\|_{L^2} = 0$. We have proved that $H^1_{X,0}(\Omega)$ is a Hilbert space. The extension problem is the same as classic case. This is also a local problem, if we extend v by 0 to $x_d < 0$ and denote that function by $\overline{v}$, then $v, \partial_{x_d} v \in L^2(\mathbb{R}^d_+), v|_{x_d=0} = 0$

implies that $\overline{v}, \partial_{x_d}\overline{v} \in L^2(\mathbb{R}^d)$, and the tangential derivation has nothing to change. So we have proved the Lemma.

Since $L\log L$ is not a normed space, we need the following Lemma, see also [**19**] for some detail of function space $L\log L$.

Lemma 2.2. — *Let $\sigma_2 > 0$, $B > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in L^2 satisfying*

$$\int |v_j|^2 |\log|v_j||^{\sigma_2} \leqslant B.$$

Then $\{|v_j|^2|\log|v_j||^{\sigma_1}\}$ is uniformly integrable for any $0 \leqslant \sigma_1 < \sigma_1$. Therefore there exists a convergent sub-sequence $\{v_{j_k}\}$ such that

$$\lim_{k\to\infty} \int |v_{j_k}|^2 |\log|v_{j_k}||^{\sigma_1} = \int |v_0|^2|\log|v_0||^{\sigma_1},$$

and

$$\int |v_0|^2|\log|v_0||^{\sigma_2} \leqslant B.$$

Proof. — We prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \subset \Omega$, $\mu(E) < \delta$, then

$$\int_E |v_j|^2|\log|v_j||^{\sigma_1} < \varepsilon, \quad \forall j.$$

But for any $\varepsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log^{\sigma_2-\sigma_1} t} < \varepsilon, \qquad \forall t \geqslant t_0.$$

Take now $\delta = \varepsilon(t_0^2 \log^{\sigma_1} t_0)^{-1}$, $\mu(E) < \delta$, and

$$A_j = E \cap \{|v_j| \leqslant t_0\},\ B_j = E \cap \{|v_j| > t_0\}.$$

then

$$\int_{A_j} |v_j|^2|\log|v_j||^{\sigma_1} \leqslant t_0^2 \log^{\sigma_1} t_0 \mu(A_j) < \varepsilon,$$

and

$$\int_{B_j} |v_j|^2|\log|v_j||^{\sigma_1} \leqslant \varepsilon \int_{B_j} |v_j|^2|\log|v_j||^{\sigma_2} \leqslant \varepsilon M$$

where $M = \sup_j \int_\Omega |v_j|^2|\log|v_j||^{\sigma_2}$. The proof of the Lemma is complete.

Proof of Theorem 1.1. — We are following the idea of [**4**]. Take $v \in H^1_{X,0}(\Omega)$, we use the same notation for the extension by 0, As in the classical case, there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that $\rho_\varepsilon * v \in C_0^\infty$, $\lim_{\varepsilon\to 0} \rho_\varepsilon * v = v$ in L^2 and $\|X(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}$, $\|(\log\Lambda)^s(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|(\log\Lambda)^s v\|_{L^2} + \|v\|_{L^2}\}$ with C independent on ε. By using (1.1) and Lemma 2.2, we need only to prove the following estimate:

$$\int_\Omega |v|^2 \log^{2s-1}\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0 \|(\log\Lambda)^s v\|^2_{L^2}, \tag{2.2}$$

for all for $v \in C_0^\infty(\Omega)$.

By the homogenization, we prove (2.2) for $v \in C_0^\infty(\Omega)$ and $\|v\|_{L^2} = 1$. Since $2s-1>0$, we have

$$\int_\Omega |v|^2 |\log |v||^{2s-1} \leqslant C|\Omega| + \int_{|v|\geqslant e} |v|^2 \log^{2s-1}\langle |v|\rangle$$
$$\leqslant C_0 + \int_\Omega |v|^2 \log^{2s-1}\langle |v|\rangle.$$

Since Ω is bounded, $v \in L^\infty(\Omega)$ and $2s-1>0$, we have by the definition of Lebesgue integration

$$\int_\Omega |v|^2 \log^{2s-1}\langle |v|\rangle = -\int_0^\infty \lambda^2 \log^{2s-1}\langle\lambda\rangle d\mu\{|v|>\lambda\}$$
$$= \int_0^\infty \Big(2\lambda \log^{2s-1}\langle\lambda\rangle + (2s-1)\frac{\lambda^3}{\langle\lambda\rangle^2}\log^{2s-2}\langle\lambda\rangle\Big)\,\mu(|v|>\lambda)d\lambda,$$

where $\mu(\cdot)$ is the Lebesgue measure. Since $\lambda^3/\langle\lambda\rangle^2 \leqslant \lambda$, $\log\langle\lambda\rangle \geqslant 1$, we have that

$$\int_\Omega |v|^2|\log|v||^{2s-1} \leqslant C_0 + C_s \int_0^\infty \lambda \log^{2s-1}\langle\lambda\rangle \mu(|v|>\lambda)d\lambda. \tag{2.3}$$

So we need to estimate the second term of right hand side of (2.3). For $A>0$ we set $v = v_{1,A} + v_{2,A}$ with $\widehat{v}_{1,A} = \widehat{v}(\xi)\mathbf{1}_{\{|\xi|\leqslant e^A\}}$. Then

$$\mu\{|v|>\lambda\} \leqslant \mu\{|v_{1,A}|>\lambda/2\} + \mu\{|v_{2,A}|>\lambda/2\}.$$

For the first term we have

$$\|v_{1,A}\|_{L^\infty} \leqslant \|\widehat{v}_{1,A}\|_{L^1} \leqslant \|v\|_{L^2}\|\mathbf{1}_{\{|\xi|\leqslant e^A\}}\|_{L^2} \leqslant C_d e^{\frac{d}{2}A}.$$

Choose now $A_\lambda = \frac{2}{d}\log(\lambda/4C_d)$, we have $\mu\{|v_{1,A_\lambda}|>\lambda/2\} = 0$, hence

$$\int_0^\infty \lambda \log^{2s-1}\langle\lambda\rangle \mu(|v|>\lambda)d\lambda \leqslant C_0 + C_s\int_e^\infty \lambda\log^{2s-1}\lambda\mu(|v|>\lambda)d\lambda$$
$$\leqslant C_0 + C_s\int_e^\infty \lambda\log^{2s-1}\lambda\mu(|v_{2,A_\lambda}|>\lambda/2)d\lambda$$
$$\leqslant C_0 + 2C_s\int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda}\|v_{2,A_\lambda}\|_{L^2}^2 d\lambda$$
$$\leqslant C_0 + 2C_s\int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda}\int_{\{\xi\in\mathbb{R}^d;|\xi|\geqslant e^{A_\lambda}\}} |\widehat{v}(\xi)|^2 d\xi d\lambda.$$

Now $|\xi| \geqslant e^{A_\lambda}$ implies that $\lambda \leqslant 4C_d \langle |\xi| \rangle^{d/2}$. By using Fubini theorem we have

$$\begin{aligned}\int_0^\infty \lambda \log^{2s-1}\langle\lambda\rangle \mu(|v| > \lambda) d\lambda &\leqslant C_0 + 2C_s \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 \int_e^{4C_d\langle|\xi|\rangle^{d/2}} \frac{\log^{2s-1}\lambda}{\lambda} d\lambda d\xi \\ &\leqslant C_0 + 2C_s \int_{\mathbb{R}^d} \log^{2s}(4C_d\langle|\xi|\rangle^{d/2})|\widehat{v}(\xi)|^2 d\xi \\ &\leqslant C_s \int_{\mathbb{R}^d} \log^{2s}\langle|\xi|\rangle |\widehat{v}(\xi)|^2 d\xi = C_s \|(\log \Lambda)^s v\|^2_{L^2(\Omega)}.\end{aligned}$$

Here we have used the fact

$$\int_{\mathbb{R}^d} \log^{2s}\langle|\xi|\rangle |\widehat{v}(\xi)|^2 d\xi \geqslant \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 d\xi = 1.$$

Thus we have proved (2.2) by using (2.3).

In the proof of existence of weak solution for the variational problem of section 3, we need also the first Poincaré's inequality. We study the following Dirichlet eigenvalue problems:

$$\begin{aligned} \triangle_X u &= \lambda u, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{2.4}$$

We have

Lemma 2.3. — *Under the hypotheses H-1), H-2) and H-3), the first eigenvalue λ_1 of problems (2.4) is strictly positive. This is equivalent to*

$$\|\varphi\|^2_{L^2} \leqslant \frac{1}{\lambda_1} \sum_{j=1}^m \|X_j\varphi\|^2_{L^2}, \quad \forall\, \varphi \in H^1_{X,0}(\Omega). \tag{2.5}$$

By using this lemma, in $H^1_{X,0}(\Omega)$, we can use $\|X\varphi\|_{L^2} = \left(\sum_{j=1}^m \|X_j\varphi\|^2_{L^2}\right)^{1/2}$ as norm.

Proof. — We set

$$\lambda_1 = \inf_{\|\varphi\|_{L^2}=1,\, \varphi\in H^1_{X,0}(\Omega)} \left\{ \|X\varphi\|^2_{L^2} \right\}.$$

Suppose that $\lambda_1 = 0$, then there exists $\{\varphi_j\} \subset H^1_{X,0}(\Omega)$ such that $\|X\varphi_j\|_{L^2} \to 0$ and $\|\varphi_j\|_{L^2} = 1$. By using (1.1), $H^1_{X,0}(\Omega)$ is compactly embedding into $L^2(\Omega)$. The variational calculus deduce that there exists $\varphi_0 \in H^1_{X,0}(\Omega), \|\varphi_0\|_{L^2} = 1, \varphi_0 \geqslant 0$ verifies

$$\triangle_X \varphi_0 = 0.$$

Since $\triangle_X$ is hypoelliptic on $\widetilde{\Omega}$ and $\partial\Omega$ is non characteristic for X, we have $\varphi_0 \in C^\infty(\overline{\Omega}), \varphi_0|_{\partial\Omega} = 0$ (see [**6, 9, 11, 16**]). Under the hypothesis H-2), Bony's maximum principle (see [**2**]) implies that φ_0 has not the maximum point in $\Omega \setminus \Gamma$, and the maximum of φ_0 propagates along the integral curves of $X_1, \dots, X_m$ in the interior of Ω. Since Γ is non characteristic for the system $X_1, \dots, X_m$, for any point of Γ, there exists at least one vector field of $X_1, \cdots, X_m$ which is transversal to Γ. Hence if the maximum of φ_0 attains at a point of Γ in the interior of Ω, then the maximum of φ_0 propagates along the integral curve of that vector field which traverses Γ, that means the maximum of φ_0 attains at a point of $\Omega \setminus \Gamma$, so it is impossible. Now it is only possible that the maximum of φ_0 attains at $\partial\Omega$, but $\varphi_0|_{\partial\Omega} = 0$, which implies that $\varphi_0 \equiv 0$ on Ω. This is impossible because $\|\varphi_0\|_{L^2} = 1$, so that we prove finally $\lambda_1 > 0$.

3. Variational problems

For $a \in \mathbb{R}$, we study now the following variational problems

$$I_a = \inf_{\|v\|_{L^2}=1,\, v\in H^1_{X,0}(\Omega)} I_a(v), \tag{3.1}$$

with

$$I_a(v) = \|Xv\|^2_{L^2(\Omega)} - a\int_\Omega |v|^2 \log|v|.$$

We have firstly the existence of minimizer of $I_a(v)$.

Proposition 3.1. — *Under the hypotheses H-1), H-2) and H-3), I_a is an attained minimum in $H^1_{X,0}(\Omega)$.*

Proof. — We prove firstly $I_a(v)$ is bounded below on $\{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. Hypothesis H-3) and Theorem 1.1 give that

$$\int_\Omega |v|^2 \log^2\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0\left(\|Xv\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right), \tag{3.2}$$

for all $v \in H^1_{X,0}(\Omega)$. Now if $a = 0$, we have $I_0(v) \geqslant \lambda_1$ for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. If $a \neq 0$, we have

$$a\int_\Omega |v|^2\,|\log|v|| \leqslant \frac{1}{2C_0}\int_\Omega |v|^2\,|\log|v||^2 + \frac{C_0|a|^2}{2} \leqslant \frac{1}{2}\|Xv\|^2_{L^2(\Omega)} + \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right),$$

for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. We have that

$$\begin{aligned} I_a(v) &= \|Xv\|^2_{L^2} - |a|\int_\Omega |v|^2|\log|v|| \\ &\geqslant \|Xv\|^2_{L^2} - \frac{1}{2}\|Xv\|^2_{L^2} - \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right) \\ &\geqslant \frac{1}{2}\lambda_1 - \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right), \end{aligned}$$

for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$.

Let now $\{v_j\} \subset \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$ be a minimizer sequence of I_a, then

$$\left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right) + I_a(v_j) \geqslant \frac{1}{2}\|Xv_j\|^2_{L^2}.$$

It follows that $\{v_j\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$. Then there exists a subsequence (denote still by $\{v_j\}$) such that $v_j \rightharpoonup v_0$ in $H^1_{X,0}(\Omega)$ and $v_j \to v_0$ in $L^2(\Omega)$ which give that

$$\liminf_{j\to\infty} \|Xv_j\|^2_{L^2(\Omega)} \geqslant \|Xv_0\|^2_{L^2(\Omega)}, \quad \lim_{j\to\infty} \|v_j\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)} = 1.$$

By using (3.2), $\{\int_\Omega |v_j|^2 \log^2 v_j\}$ is bounded, the Lemma 2.2 implies that there exists a subsequence of $\{v_j\}$ such that

$$\lim_{j\to\infty} \int_\Omega |v_j|^2 \log v_j = \int_\Omega |v_0|^2 \log v_0.$$

But we have also a direct proof of this convergence

$$\begin{aligned}
&\Big|\int_\Omega |v_j|^2 \log v_j - \int_\Omega |v_0|^2 \log v_0\Big| \\
&\quad = \Big|\int_\Omega (v_j - v_0) \int_0^1 v_t(2\log v_t + 1)dtdx\Big| \\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \Big(\int_\Omega |v_t|^2(\log^2 |v_t| + 1)dx\Big)^{1/2} dt \\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \Big(\|v_t\|_{L^2} + \int_0^1 \Big(\int_\Omega |v_t|^2 \left|\log^2 |v_t|\right| dx\Big)^{1/2}\Big)dt \\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \Big(\|v_t\|_{L^2} + \big(\|Xv_t\|^2_{L^2} + \|v_t\|^2_{L^2} + \|v_t\|^2_{L^2} \log^2 \|v_t\|^2_{L^2}\big)^{1/2}\Big)dt,
\end{aligned}$$

where $v_t = v_j + t(v_j - v_0)$, and we have used (3.2) for the function $v_t \in H^1_{X,0}(\Omega)$. Since $\{v_j\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$, and $\|v_j - v_0\|_{L^2} \to 0$, the right hand side of above estimate go to 0 if $j \to \infty$. We have proved finally Proposition 3.1.

We study now the Euler-Lagrange equation of variational problems (3.1).

Proposition 3.2. — *The minimizer u of variational problem (3.1) is a non trivial weak solution of the following semilinear Dirichlet problem*

$$\begin{aligned} &\triangle_X u = au \log |u| + I_a u, \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{3.3}$$

Proof. — The minimizer u obtained in Proposition 3.1 is in $\{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$ and $u \geqslant 0$. u is a weak solution of (3.3) is equivalent to

$$\int_\Omega \sum_{j=1}^m X_j u X_j \varphi - a\int_\Omega u\varphi \log |u| - I_a \int_\Omega u\varphi = 0, \tag{3.4}$$

for all $\varphi \in H^1_{X,0}(\Omega)$. For fixed $\varphi \in H^1_{X,0}(\Omega)$ and $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ small enough, we put

$$u_\varepsilon = u + \varepsilon\varphi, \quad \widetilde{u}_\varepsilon = u_\varepsilon/\|u_\varepsilon\|_{L^2},$$

then $\widetilde{u}_\varepsilon \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$, so that

$$H(\varepsilon) = I_a(\widetilde{u}_\varepsilon) \geqslant I_a(u) = I_a,$$

and

$$H(\varepsilon) = \frac{1}{\|u_\varepsilon\|_{L^2}^2} I_a(u_\varepsilon) + a \log \|u_\varepsilon\|_{L^2}.$$

By direct calculus,

$$H'(\varepsilon) = -\frac{2}{\|u_\varepsilon\|_{L^2}^4} I_a(u_\varepsilon) \int_\Omega u_\varepsilon \varphi + \frac{a}{\|u_\varepsilon\|_{L^2}^2} \int_\Omega u_\varepsilon \varphi$$
$$+ \frac{1}{\|u_\varepsilon\|_{L^2}^2} \left(2 \int_\Omega X u_\varepsilon X \varphi - 2a \int_\Omega u_\varepsilon \varphi \log |u_\varepsilon| - a \int_\Omega u_\varepsilon \varphi \right).$$

We have to prove the continuity of $H'(\varepsilon)$ at $\varepsilon = 0$, since $u_\varepsilon, Xu_\varepsilon \in L^2(\Omega)$, we need only to prove

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \varphi \log |u_\varepsilon| = \int_\Omega u \varphi \log |u|.$$

this can be deduced by Lebesgue dominant theorem if we use the fact $|t \log t| \leqslant t^2 + e^{-1}, \forall\, t \geqslant 0$ and φ can be approximated by bounded functions. So that we have, for any $\varepsilon \in \mathbb{R}$, with $|\varepsilon|$ small enough

$$I_a(\widetilde{u}_\varepsilon) = H(\varepsilon) = H(0) + H'(0)\varepsilon + \delta(\varepsilon)\varepsilon \geqslant I_a(u) = H(0),$$

where $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$. We get finally $H'(0) = 0$, this is true for all $\varphi \in H^1_{X,0}(\Omega)$, we have proved Proposition 3.2.

Theorem 3.1. — *Let $a, b \in \mathbb{R}, a \neq 0$, under the hypotheses H-1), H-2) and H-3), the Dirichlet problems (1.3) has at least one non trivial weak solution $u \in H^1_{X,0}(\Omega)$, $u \geqslant 0$, $\|u\|_{L^2} > 0$.*

In fact, if $\widetilde{u}$ is a weak solution of problem (3.3), for $c > 0$ we set $u = c\widetilde{u}$, then $\|u\|_{L^2} = c > 0$, $u \geqslant 0$, $u \in H^1_{X,0}(\Omega)$ and in the weak sense

$$\triangle_X u = au \log |u| + (I_a - \log c) u.$$

Choose $c = e^{I_a - b} > 0$, we get (1.3).

Following this direction, we can study the high order nonlinear eigenvalue problems. Suppose that we have the logarithmic Sobolev inequality

$$\int_\Omega |v|^2 \log^{k+1} \left(\frac{|v|}{\|v\|_{L^2}} \right) \leqslant C_0 \left(\|Xv\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right).$$

For $a_1, \ldots, a_k \in \mathbb{R}$, we study the variational problems

$$I^k_{a_1,\ldots,a_k} = \inf_{\|v\|_{L^2}=1,\, v \in H^1_{X,0}(\Omega)} I^k_{a_1,\ldots,a_k}(v),$$

with

$$I^k_{a_1,\ldots,a_k}(v) = \|Xv\|^2_{L^2(\Omega)} - \sum_{j=1}^k a_j \int_\Omega |v|^2 \log^j |v|.$$

As in the proof of Proposition 3.1, we need to prove that there exists a subsequence of $\{v_j\}$ of minimizer sequence such that

$$\lim_{j\to\infty}\int_\Omega |v_j|^2\log^k v_j = \int_\Omega |v_0|^2\log^k v_0,$$

which was already shown in the Lemma 2.2.

By similar calculus as in Proposition 3.2, we can prove that for any $a_1,\dots,a_k\in\mathbb{R}$, there exists $I^k_{a_1\cdots,a_k}$ such that the following semilinear Dirichlet problems

$$\begin{aligned}&\triangle_X u = \textstyle\sum_{j=1}^k a_j u\log^j|u| + I^k_{a_1\cdots,a_k}u,\\ &u|_{\partial\Omega}=0,\end{aligned}$$

has at least one non trivial solution in $H^1_{X,0}(\Omega)$, with $u\geqslant 0$ and $\|u\|_{L^2}=1$. Moreover, we have similar regularity results as Theorem 1.2.

4. Boundedness and regularity of weak solutions

By using the interpolation inequality, the condition H-3) and the Logarithmic Sobolev inequality (1.2) give that, for any $N\geqslant 1$, there exists C_N such that

$$\int_\Omega v^2\log^2\left(\frac{|v|}{\|v\|_{L^2}}\right)\leqslant \frac{1}{N}\|Xv\|^2_{L^2}+C_N\|v\|^2_{L^2}, \tag{4.1}$$

for all $v\in H^1_{X,0}(\Omega)$.

Theorem 4.1. — *Let $u\in H^1_{X,0}(\Omega), u\geqslant 0, \|u\|_{L^2}\neq 0$ be a weak solutions of equation*

$$\triangle_X u = au\log u + bu. \tag{4.2}$$

Then $u\in L^\infty(\Omega)$.

It suffices to show that there exists $\overline{A}>0$ such that the estimate

$$\|u\|_{L^p}\leqslant \overline{A} \tag{4.3}$$

holds for any $p\geqslant 2$. In fact, if $\Omega_\varepsilon=\{x\in\Omega; |u(x)|\geqslant \overline{A}+\varepsilon\}$ for $\varepsilon>0$ then it follows from (4.3) that $|\Omega_\varepsilon|\leqslant\left(\frac{\overline{A}}{\overline{A}+\varepsilon}\right)^p\to 0\ \ (p\to\infty)$ and hence we have $||u||_{L^\infty}\leqslant\overline{A}$.

We prove this by the following three propositions. To get the estimate as (4.3), we shall use u^{2p-1} or $u^{2p-1}\log^{2m}(u^p)$ as test function for the equation (4.2) for $p\geqslant 1$, $m\in\mathbb{N}$, but we don't know if $u^{2p-1}\log^{2m}(u^p)\in H^1_{X,0}(\Omega)$, so we replace the function u by $u_{(k)}$ with $u_{(k)}(x)=u(x)$ if $x\in\{x\in\Omega; |u(x)|<k\}$ and $u_{(k)}(x)=k$ if $x\in\{x\in\Omega; |u(x)|\geqslant k\}$ for $k>1$, $p>1$. Then it is easy to check (see [**22**] and Theorem 7.8 of [**8**]) that $u^{2p-1}_{(k)}\log^{2m}(u^p_{(k)})\in H^1_{X,0}(\Omega)$ for all $p>1$, $m\in\mathbb{N}$. If $p=1$, we use $u\,(\log^m u)^2_{(k)}\in H^1_{X,0}(\Omega)$ as test function. To simplify the notation, we shall drop the subscript and use $u^{2p-1}\log^{2m}(u^p)$ as test function.

Proposition 4.1. — *Let $u \in H^1_{X,0}(\Omega)$, $u \geqslant 0$, $\|u\|_{L^2} \neq 0$ be a weak solution of equation (4.2). Suppose that for some $p_0 \geqslant 1$, there exists A_0 such that*

$$\|u\|_{L^{2p_0}} \leqslant A_0.$$

Then

$$\int_\Omega |X(\widetilde{u})^{p_0}|^2 + \int_\Omega (\widetilde{u})^{2p_0} \log^2((\widetilde{u})^{p_0}) \leqslant 2C_2 + |a|^2 + 2p_0(|b| + \log A_0), \tag{4.4}$$

where the constant C_2 is given in (4.1) and $\widetilde{u} = u/\|u\|_{L^{2p_0}}$.

Proof. — We have $\widetilde{u} \in H^1_{X,0}(\Omega)$, $\|\widetilde{u}\|_{L^{2p_0}} = 1$, and $\widetilde{u}$ is a weak solution of equation

$$\triangle_X \widetilde{u} = a\widetilde{u}\log\widetilde{u} + (b - \log\|u\|_{L^{2p_0}})\widetilde{u}. \tag{4.5}$$

Take u^{2p_0-1} as test function, we have

$$\frac{2p_0-1}{p_0^2}\int_\Omega |X\widetilde{u}^{p_0}|^2 = \frac{a}{p_0}\int_\Omega \widetilde{u}^{2p_0}\log\widetilde{u}^{p_0} + (b - \log\|u\|_{L^{2p_0}})\int_\Omega \widetilde{u}^{2p_0}$$

which shows that

$$\int_\Omega |X\widetilde{u}^{p_0}|^2 \leqslant \frac{1}{2}\int \widetilde{u}^{2p_0}\log^2\widetilde{u}^{p_0} + (\frac{1}{2}|a|^2 + p_0|b| + p_0\log A_0). \tag{4.6}$$

On the other hand, the logarithmic Sobolev inequality (4.1) gives

$$\int_\Omega (u^{p_0})^2 \log^2\left(\frac{|u^{p_0}|}{\|u^{p_0}\|_{L^2}}\right) \leqslant \frac{1}{2}\|X(u^{p_0})\|^2_{L^2} + C_2\|u^{p_0}\|^2_{L^2}.$$

Note that $\|u^{p_0}\|_{L^2} = \|u\|^{p_0}_{L^{2p_0}}$ and $\widetilde{u} = u/\|u\|_{L^{2p_0}}$, we have

$$\int_\Omega \widetilde{u}^{2p_0}\log^2(\widetilde{u}^{p_0}) \leqslant \frac{1}{2}\|X(\widetilde{u}^{p_0})\|^2_{L^2} + C_2. \tag{4.7}$$

Adding (4.6) and (4.7), we have the desired estimate (4.4).

Proposition 4.2. — *We have for any $m \in \mathbb{N}$*

$$\int_\Omega |X(\widetilde{u}^{p_0})|^2 \log^{2m-2}(\widetilde{u}^{p_0}) + \int_\Omega \widetilde{u}^{2p_0}\log^{2m}(\widetilde{u}^{p_0}) \leqslant M_1^{2m} P(m,p_0)(m!)^2, \tag{4.8}$$

where $P(m,p_0) = p_0^m$ if $m \leqslant \sqrt{p_0}$, $P(m,p_0) = p_0^{\sqrt{p_0}}$ if $m > \sqrt{p_0}$, and

$$M_1 \geqslant (2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 8\log A_0)^{1/2}.$$

Proof. — For $m = 1$, this is (4.4). We prove now (4.8) by induction, suppose that (4.8) is true for some $m \in \mathbb{N}$, then we prove it for $m+1$. From now on we drop the tilde of u and subscript of p to simplify the notation. Take $u^{2p-1}\log^{2m}(u^p)$ as test function in (4.5), we have

$$\frac{2p-1}{p^2}\int_\Omega |Xu^p|^2\log^{2m}(u^p) + \frac{2m}{p}\int_\Omega |Xu^p|^2\log^{2m-1}(u^p)$$
$$= \frac{a}{p}\int_\Omega u^{2p}\log^{2m+1}(u^p) + (b - \log\|u\|_{L^{2p}})\int_\Omega u^{2p}\log^{2m}(u^p),$$

which gives

$$\int_\Omega |Xu^p|^2 \log^{2m}(u^p) \leqslant \frac{1}{2}\int_\Omega |Xu^p|^2 \log^{2m}(u^p) + 2m^2 \int_\Omega |Xu^p|^2 \log^{2m-2}(u^p) + \frac{1}{4}\int_\Omega u^{2p}\log^{2m+2}(u^p) + (|a|^2 + p|b| + p\log A_0)\int_\Omega u^{2p}\log^{2m}(u^p)$$

so that

$$\int_\Omega |Xu^p|^2 \log^{2m}(u^p) \leqslant \frac{1}{2}\int_\Omega u^{2p}\log^{2m+2}(u^p) + (4m^2 + 2(|a|^2 + p|b| + p\log A_0))M_1^{2m}P(m,p)(m!)^2. \tag{4.9}$$

We study now the term $\int_\Omega u^{2p}\log^{2m+2}(u^p)$, we cut $\Omega = \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-$ with $\Omega_1 = \{x \in \Omega; u(x) \leqslant 1\}$ and

$$\Omega_2^+ = \{x\in\Omega; u(x) > 1, |\log^m(u(x)^p)| \leqslant \|u^p\log^m(u^p)\|_{L^2}\},$$
$$\Omega_2^- = \{x\in\Omega; u(x) > 1, |\log^m(u(x)^p)| > \|u^p\log^m(u^p)\|_{L^2}\}.$$

Then

$$\int_{\Omega_1} u^{2p}\log^{2m+2}(u^p) \leqslant |\Omega|((m+1)!)^2.$$

For the second term, (4.4) give

$$\int_{\Omega_2^+} u^{2p}\log^{2m+2}(u^p) \leqslant \|u^p\log^m(u^p)\|_{L^2}^2 \int_\Omega u^{2p}\log^2(u^p) \leqslant (2C_2 + |a|^2 + 2p|b| + 2p\log A_0)M_1^{2m}P(m,p)(m!)^2,$$

and for the third term, we use the logarithmic Sobolev inequality (4.1) for $N=4$,

$$\begin{aligned}\int_{\Omega_2^-} u^{2p}\log^{2m+2}(u^p) &\leqslant \int_{\Omega_2^-} (u^p\log^m u^p)^2 \log^2\left(\frac{u^p\log^m(u^p)}{\|u^p\log^m(u^p)\|_{L^2}}\right)\\
&\leqslant \frac{1}{4}\|X(u^p\log^m u^p)\|_{L^2}^2 + C_4\|u^p\log^m u^p\|_{L^2}^2\\
&\leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + m^2\int_\Omega |X(u^p)|^2\log^{2m-2}(u^p) + C_4\int_\Omega u^{2p}\log^{2m}(u^p)\\
&\leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + (C_4+m^2)M_1^{2m}P(m,p)(m!)^2.\end{aligned}$$

Adding those three terms, we get

$$\int_\Omega u^{2p}\log^{2m+2}(u^p) \leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + |\Omega|((m+1)!)^2 + (2C_2 + C_4 + m^2 + |a|^2 + 2p|b| + 2p\log A_0)M_1^{2m}P(m,p)(m!)^2. \tag{4.10}$$

Adding (4.9) and (4.10), we get

$$(4.11)\quad \int_\Omega u^{2p}\log^{2m+2}(u^p)+\int_\Omega |X(u^p)|^2\log^{2m}(u^p)\leqslant (2|\Omega|+4C_2+2C_4 \\ +10+6|a|^2+8|b|+8\log A_0)M_1^{2m}P(m+1,p)((m+1)!)^2.$$

We have proved Proposition 4.2.

Proposition 4.3. — *Let $u\in H^1_{X,0}(\Omega), u\geqslant 0, \|u\|_{L^2}\neq 0$ be a weak solution of equation (4.2). Suppose that for some $p_0\geqslant 1$ and $A_0\geqslant e^{12}$ we have*

$$\|u\|_{L^{2p_0}}\leqslant A_0.$$

Then for

$$M_1\geqslant (2|\Omega|+4C_2+2C_4+10+6|a|^2+8|b|+8\log A_0)^{1/2},$$

and $\delta=1/2M_1$, we have

$$(4.12)\quad \int_\Omega u^{2p_0(1+\delta)}\leqslant A_0^{2p_0(1+\delta)\left(1+\left(\frac{1}{p_0(1+\delta)}\right)^{1/3}\right)}.$$

Proof. — For any $\delta>0$, the estimate (4.8) gives that

$$\Big(\int_\Omega|\widetilde{u}^{p_0(1+\delta)}|^2dx\Big)^{1/2}=\Big(\int_\Omega|\widetilde{u}^{p_0}\widetilde{u}^{\delta p_0}|^2dx\Big)^{1/2}=\Big(\int_\Omega\Big|\widetilde{u}^{p_0}e^{\delta\log(\widetilde{u}^{p_0})}\Big|^2dx\Big)^{1/2}$$
$$=\Big(\int_\Omega\Big|\widetilde{u}^{p_0}\sum_{m=0}^{\infty}\frac{(\delta\log(\widetilde{u}^{p_0}))^m}{m!}\Big|^2dx\Big)^{1/2}\leqslant\sum_{m=0}^{\infty}\Big(\int_\Omega\widetilde{u}^{2p_0}\frac{(\delta\log(\widetilde{u}^{p_0}))^{2m}}{(m!)^2}dx\Big)^{1/2}$$
$$\leqslant\sum_{m=0}^{\infty}\frac{\delta^m}{m!}\Big(\int_\Omega\widetilde{u}^{2p_0}\log^{2m}(\widetilde{u}^{p_0})dx\Big)^{1/2}\leqslant\sum_{m=0}^{\infty}\delta^mM_1^mP(m,p_0)\leqslant p_0{}^{\sqrt{p_0}}\sum_{m=0}^{\infty}(\delta M_1)^m.$$

For $\delta=1/2M_1$, we have finally

$$\int_\Omega u^{2p_0(1+\delta)}dx\leqslant 4p_0^{2\sqrt{p_0}}A_0^{2p_0(1+\delta)}.$$

Since for any $p_0>1$,

$$4p_0^{2\sqrt{p_0}}=4e^{2\sqrt{p_0}\log p_0}\leqslant\left(e^{12}\right)^{2p_0^{2/3}}.$$

We have proved (4.12) if $A_0\geqslant e^{12}$, and Proposition 4.3.

The same calculus give also

$$(4.13)\quad \int_\Omega|X(u^{p_0(1+\delta)})|^2dx\leqslant(1+\delta)^2(4M_1)^2A_0^{2p_0(1+\delta)\left(1+\left(\frac{1}{p_0(1+\delta)}\right)^{1/3}\right)}.$$

We put now for $k\in\mathbb{N}$,

$$p_k=p_0(1+\delta)^k, A_k=A_0^{1+p_0^{-1/3}\sum_{j=1}^k\left(\frac{1}{(1+\delta)}\right)^{j/3}},$$

then Proposition 4.3 implies that

$$\int_\Omega u^{2p_0(1+\delta)^{k+1}} a = \int_\Omega u^{2p_k(1+\delta)} \leqslant A_k^{2p_k(1+\delta)\left(1+\left(\frac{1}{p_k(1+\delta)}\right)^{1/3}\right)}$$
$$\leqslant A_0^{2p_0(1+\delta)^{k+1}\left(1+p_0^{-1/3}\sum_{j=1}^{k+1}\left(\frac{1}{(1+\delta)}\right)^{j/3}\right)},$$

with $\delta = 1/2M_1$ and

$$M_1 \geqslant \left(2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 8\log A_k\right)^{1/2}. \tag{4.14}$$

We have now for $\delta = 1/2M_1 \leqslant 1/4$,

$$\frac{\log A_k}{\log A_0} = 1 + p_0^{-1/3}\sum_{j=1}^{k}\left(\frac{1}{(1+\delta)}\right)^{j/3} \leqslant 1 + p_0^{-1/3}\sum_{j=1}^{\infty}\left(\frac{1}{(1+\delta)}\right)^{j/3}$$
$$= 1 + p_0^{-1/3}\frac{\left(\frac{1}{1+\delta}\right)^{1/3}}{1-\left(\frac{1}{1+\delta}\right)^{1/3}} \leqslant 1 + 4p_0^{-1/3}M_1 \leqslant 5M_1.$$

So we can choose M_1 independent on k

$$M_1 = \left(2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 40\log A_0\right). \tag{4.15}$$

We have proved for any $k \in \mathbb{N}$,

$$\int_\Omega u^{2p_0(1+\delta)^k} \leqslant \left(A_0^{5M_1}\right)^{2p_0(1+\delta)^k}.$$

For $p_0 = 1$, we have $A_0 = e^{12}$. So we have proved (4.3) with $\overline{A} = e^{60M_1}$ if $\|u\|_{L^2} = 1$.

Now the proof of the Theorem 4.1 is complete.

Theorem 4.2. — *Let $u \in H^1_{X,0}(\Omega), u \geqslant 0, \|u\|_{L^2} \neq 0$ be a weak solution of equation (4.2), suppose that $a > 0$, Γ and $\partial\Omega$ is non characteristic. Then $u \in C^\infty(\Omega \smallsetminus \Gamma) \cap C^0(\overline{\Omega} \smallsetminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \smallsetminus \Gamma$.*

Proof. — Suppose $x_0 \in \Omega \smallsetminus \Gamma$, then there exists a neighborhood $V_0 \subset \Omega \smallsetminus \Gamma$ of x_0, for $\varphi \in C_0^\infty(V_0)$ we shall prove that $v = \varphi u \in C^\infty(V_0)$. It follows from equation (4.2) that,

$$\triangle_X v = a\varphi u \log u + b\varphi u + \sum_{j=1}^{m}\varphi_j X_j u + \varphi_0 u = f_0 + \sum_{j=1}^{m} X_j f_j,$$

with $\varphi_j \in C^\infty(V_0), f_j \in L^\infty(V_0), j = 0,\dots,m$. Since the system of vector fields X satisfies the finitely type Hörmander's condition on V_0, the regularity result of [**23**] (see also [**22, 24**]) implies that $u \in C^\varepsilon(V_0)$ for some $\varepsilon > 0$. If $u(x) \geqslant \alpha > 0$ for $x \in V_0$, we have $u\log u \in C^\varepsilon(V_0)$ since $t\log t \in C^\infty(t \geqslant \alpha)$. Then we prove by recurrence that $u \in C^\infty(V_0)$. For $x_0 \in \partial\Omega \smallsetminus \Gamma$, we have also $u \in C^\varepsilon(V_0 \cap \overline{\Omega})$, but we know only $u\log u \in C^0(V_0 \cap \overline{\Omega})$, so we can't get the C^∞ regularity of u near to the boundary $\partial\Omega$. Now we finish the interior regularity of Theorem 4.2 by the following lemma.

Lemma 4.1. — *Suppose that* $u \in C^0(\Omega_1), u \geqslant 0$ *is a non trivial weak solution of equation (4.2) on an open set* $\Omega_1 \subset \Omega$, *let* $a > 0$, *then* $u(x) > 0$ *for all* $x \in \Omega_1$.

Proof. — Suppose that $u(x_0) = 0$ for some $x_0 \in \Omega_1$, then we have $f = au \log u + bu$ continuous on Ω_1, and $f(x_0) = 0$, then for any $\varepsilon > 0$, there exists a small neighborhood $U_0 \subset \Omega_1$ of x_0 such that $0 \leqslant u(x) \leqslant \varepsilon$ on U_0. Since $a > 0$, we have for ε small enough, $f(x) \leqslant 0$ on U_0, so that $\triangle_X u \leqslant 0$ on U_0, but x_0 is a minimum point of u, as in the proof of Lemma 2.3, the maximum principle of Bony ([**2**]) implies that $u \equiv 0$ on U_0, so that u is a trivial solution by continuous of u in Ω_1.

5. Appendix: Logarithmic regularity estimate

In this section we shall give sufficient conditions in order that the sum of squares of real vector fields

$$\triangle_X = \sum_{j=1}^{m} X_j^* X_j,$$

satisfies the logarithmic regularity estimate (1.1). We start by the following simple model operator in $\mathbb{R}^2$

$$L_0 = D_{x_1}^2 + D_{x_2}(g(x_1)D_{x_2}),$$

where $C^\infty \ni g(t) > 0$ if $t \neq 0$ and $g(0) = 0$. In what follows we do not require that $g(x)$ is written as $g = \varphi^2$ for some $\varphi \in C^\infty$, and we consider a little more general logarithmic regularity estimate than (1.1). The following proposition is essentially due to the device of Wakabayashi (see Example 5.1 of [**21**]).

Proposition 5.1. — *Let* $f(t)$ *and* $g(t)$ *be non-negative continuous functions and satisfy* $f(t), g(t) > 0$ *if* $t \neq 0$. *Assume that there exists an* $\varepsilon \geqslant 0$ *such that*

$$\limsup_{t \to 0} \left| \frac{\int_0^t f(\tau)d\tau}{\sqrt{f(t)}} \right|^{1/s} |\log g(t)| \leqslant \varepsilon. \tag{5.1}$$

Then for any compact set K *in* $\mathbb{R}^2$ *there exist constants* $C_0 > 0$ *independent of* ε *and* $C_\varepsilon > 0$ *such that*

$$||\sqrt{f(x_1)}(\log \Lambda)^s u||^2 \leqslant C_0 \varepsilon^{2s}(L_0 u, u) + C_\varepsilon ||u||^2 \tag{5.2}$$

for all $u \in C_0^\infty(K)$.

Remark. — The typical example satisfying (5.1) is $g(t) = \exp(-2|t|^{-1/s})$, stated in Introduction with $f \equiv 1$. It is known that (5.1) is also necessary for (5.2) with neglecting constant factor of ε if $f(t)$ and $g(t)$ are monotone in each half axis $\mathbb{R}_\pm$.

The necessity is shown by way of another sufficient condition for (5.1), given by Koike [**10**], as follows:

$$\limsup_{t\to 0} \mu(f;t)^{1/s}|\log g(t)| \leqslant \varepsilon,$$

where $\mu(f;t) = \sup_{0\leqslant \pm\tau\leqslant \pm t} \sqrt{f(\tau)}|t-\tau|$ if $\pm t > 0$. This condition is equivalent with (5.1) except for constant factor of ε under the monotonous condition. We refer [**14**] and references therein concerning details for the estimate (5.2).

Proof. — If $F(t) = \int_0^t f(\tau)d\tau$ then it follows from (5.1) that there exists a $t_0 > 0$ such that

$$g(t) < 1 \text{ and } |F(t)|(-\log g(t))^s \leqslant 2\varepsilon^s\sqrt{f(t)} \qquad \text{if } |t| < t_0. \tag{5.3}$$

Since $g(t) > 0$ for $t \neq 0$, one can find a $\lambda_0 > 0$ such that

$$\text{if } \lambda \geqslant \lambda_0 \text{ then } \Omega_\lambda := \{t; g(t)\lambda \leqslant 1\} \subset \{t; |t| < t_0\}. \tag{5.4}$$

Note that for $v(t) \in C_0^\infty(\mathbb{R}^1)$ we have

$$\begin{aligned}\|\sqrt{f(t)}(\log\lambda)^s v\|^2 &= ([D_t, F(t)](\log\lambda)^{2s}v, v)\\ &\leqslant 2|(D_t v, F(t)(\log\lambda)^{2s}v)|\\ &\leqslant 8\varepsilon^{2s}\|D_t v\|^2 + \frac{1}{8\varepsilon^{2s}}\|F(t)(\log\lambda)^{2s}v\|^2\end{aligned}$$

by the Schwartz inequality. Choosing another sufficiently large $\lambda_0 > 0$ if necessary, we may assume

$$\frac{1}{8\varepsilon^{4s}}F(t)^2(\log\lambda)^{4s} \leqslant \lambda \leqslant g(t)\lambda^2 \text{ in } \Omega_\lambda^c \cap \operatorname{supp} v \text{ if } \lambda \geqslant \lambda_0.$$

If $\lambda \geqslant \lambda_0$ then it follows from (5.3) and (5.4) that

$$F(t)^2(\log\lambda)^{4s} \leqslant F(t)^2(-\log g(t))^{2s}(\log\lambda)^{2s} \leqslant 4\varepsilon^{2s}f(t)(\log\lambda)^{2s} \quad \text{in } \Omega_\lambda.$$

Above two estimates give

$$\frac{1}{8\varepsilon^{2s}}\|F(t)(\log\lambda)^{2s}v\|^2 \leqslant \frac{1}{2}\int_{\Omega_\lambda} f(t)(\log\lambda)^{2s}|v|^2dt + \varepsilon^{2s}\int_{\Omega_\lambda^c} g(t)\lambda^2|v|^2dt.$$

Therefore we have

$$\|\sqrt{f(t)}(\log\lambda)^s v\|^2 \leqslant 16\varepsilon^{2s}(\|D_t v\|^2 + (g(t)\lambda^2 v, v))$$

if $\lambda \geqslant \lambda_0$. The estimate (5.2) is obvious if we consider the partial Fourier transform $v(x_1, \lambda)$ of $u(x_1, x_2)$ with respect to x_2 variable.

In the rest of this section we shall give a sufficient condition for general operator $\triangle_X$, by using Sawyer's lemma (see below), as in [**15**]. For the sake of simplicity, we confirm ourself to the logarithmic regularity estimate (1.1). Let X_J denote the repeated commutator

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}]\cdots]]]$$

for $J=(j_1,\dots,j_k), j_i\in\{1,\dots,m\}$, (and set $|J|=k$). For $k\geqslant 1$ put

$$G(x;k)=\min_{\xi\in\mathbb{S}^{d-1}}\sum_{|J|\leqslant k}|X_J(x,\xi)|^2,\qquad g(t;j,k,x_0)=G((\exp tX_j)(x_0);k),$$

where $(\exp tX_j)(x_0)$ is the integral curve of X_j starting from $x_0\in\Gamma$. Here we recall that $\Gamma=\{x\in\widetilde{\Omega}\,;\,\exists\,\xi\in\mathbb{S}^{d-1}$ satisfying $X_J(x,\xi)=0,\forall\,J\}$. Let $g_I^{j,k}(x_0)$ denote the mean value $\frac{1}{|I|}\int_I g(t;j,k,x_0)dt$ on the interval I. Then we have the following:

Proposition 5.2. — *If $s>0$ and if there exists an $\varepsilon>0$ such that*

$$\inf_{\substack{\delta>0,k\in N\\ \mu>0,1\leqslant j\leqslant m}}\Big(\sup\Big\{|I|^{1/s}|\log g_I^{j,k}(x_0)|\,;\,I\subset(-\mu,\mu)\ \text{and}\ g_I^{j,k}(x_0)<\delta\Big\}\Big)<\varepsilon \tag{5.5}$$

for any $x_0\in\Gamma$, then there exist constants $C_0>0$ independent of ε and $C_\varepsilon>0$ such that

$$\|(\log\Lambda)^s u\|_{L^2}^2\leqslant C_0\varepsilon^{2s}\,(\Delta_X u,u)+C_\varepsilon\|u\|_{L^2}^2, \tag{5.6}$$

for any $u\in C_0^\infty(\widetilde{\Omega})$.

Remark. — The condition (5.5) admits the case where all integral curves of X_j intersect Γ in any small neighborhood of x_0, such as the following:

$$X_1=\partial_{x_1}\ X_2=\exp\big(-\big(x_1^2\sin^2(\pi/x_1)\big)^{-1/2s}\big)\partial_{x_2}.$$

In this example, Γ is composed of hypersurfaces $\Gamma_j=\{x_1=1/j\}$ $(j\in\mathbb{Z}\smallsetminus\{0\})$ and $\Gamma_0=\{x_1=0\}$. Since $|x_1\sin\pi/x_1|$ is approximated to $\pi j|x_1-1/j|$ near Γ_j by Taylor's formula, (5.5) is satisfied for $x_0\in\Gamma_j$. Let $x_0\in\Gamma_0$. If the interval I contains the point $1/j$ and its length is larger than a half of $1/j$, then $g_I^{1,k}(x_0)$ is comparable to that with X_2 replaced by $\exp(-|x_1|^{-1/s})\partial_{x_2}$. If the length of I is not larger than a half of $1/j$, we can use the preceding result in the case of $x_0\in\Gamma_j$.

Proof of Proposition 5.2. — It follows from (5.5) that there exist some $j\in\{1,\dots,m\}$, $\delta>0$, $k\in\mathbb{N}$ and $\mu>0$ such that

$$\Big|\log g_I^{j,k}(x_0)\Big|^{2s}\leqslant(2\varepsilon)^{2s}|I|^{-2}\quad\text{if } I\subset(-\mu,\mu)\ \text{and}\ g_I^{j,k}(x_0)<\delta.$$

Take the new local coordinates $x=(x_1,x')$ in a neighborhood of x_0 such that $x_0=(0,0)$ and the line $x'=$ constant vector in $\mathbb{R}^{d-1}$ is the integral curve of X_j starting from $(0,x')$. Since $G(x;k)$ is continuous, we have

$$\Big|\log g_I^{j,k}(0,x')\Big|^{2s}\leqslant(4\varepsilon)^{2s}|I|^{-2}\quad\text{if } I\subset(-\mu,\mu)\,|x'|<\mu\ \text{and}\ g_I^{j,k}(0,x')<\delta$$

by taking other small $\mu,\delta>0$ if necessary. For a moment we consider x' as parameters. Let λ be a large parameter satisfying $\lambda\geqslant 1/\delta$. If $g_I^{j,k}(0,x')\lambda<1$ then we have $-\log g_I^{j,k}(0,x')\geqslant\log\lambda$ and hence

$$(\log\lambda)^{2s}\leqslant(4\varepsilon)^{2s}(|I|^{-2}+g_I^{j,k}(0,x')\lambda^2)\ \text{for}\ \forall\,I\subset(-\mu,\mu). \tag{5.7}$$

When $g_I^{j,k}(0,x')\lambda \geqslant 1$, this is also true for $\lambda \geqslant \lambda_0$ if λ_0 is chosen sufficiently large, depending on ε. By means of the following lemma of Sawyer, we see that (5.7) implies

$$\int (\log \lambda)^{2s} |v(t)|^2 dt \leqslant C_0 \varepsilon^{2s} \int (|D_t v(t)|^2 + G(t,x';k)\lambda^2 |v(t)|^2) dt, \tag{5.8}$$

for all $v(t) \in C_0^1((-\mu,\mu))$, where $C_0 > 0$ is a constant independent of ε.

***Sawyer's lemma* (see Remark 5 in [18]).** — *Let I_0 be an open interval in $\mathbb{R}_x^1$ and let $V(t), W(t) \geqslant 0$ belong to $L^1_{\mathrm{loc}}(I_0)$. Then we have the estimate*

$$\int_{I_0} V(t)|v(t)|^2 dt \leqslant C \int_{I_0} \left(W(t)|v(t)|^2 + |v'(t)|^2\right) dt \tag{5.9}$$

for all $v \in C_0^1(I_0)$ with a constant $C > 0$ if and only if

$$V_I \leqslant A\{3W_{3I} + 2|I|^{-2}\} \text{ for any interval } I \text{ with } 3I \subset I_0. \tag{5.10}$$

holds with a constant $A > 0$. Moreover, if C and A are the best constants (5.9) and (5.10) then $A < C < 100A$. Here $3I$ denotes the interval with the same center as I but with length three times.

In fact, if we set $V(t) = (\log \lambda)^{2s}$ and $W(t) = g(t;j,k,(0,x'))\lambda^2 = G(t,x';k)\lambda^2$, it is obvious that (5.8) follows from (5.7) if we replace $3I$ by I. It is well-known that

$$\sum_{|J|\leqslant k} \|\Lambda^{\sigma-1} X_J u\|^2 \leqslant C\{(\Delta_X u, u) + \|u\|^2\} \tag{5.11}$$

for some $0 < \sigma = \sigma(k) \leqslant 1/2$. If we set

$$q(x_1, x', \xi') = \Big(\sum_{|J|\leqslant k} |X_J(x,\xi)|^2 |\xi|^{-2+2\sigma}\Big)\Big|_{\xi_1=0},$$

in our local coordinates near x_0, then we have $q(x_1,x',\xi') - G(x;k) \geqslant 0$ on $\xi' \in \mathbb{S}^{d-2}$ and

$$\|D_t u\|^2 + (q^w(t,x',D')u,u) \leqslant C\{(\Delta_X u,u) + \|u\|^2\},$$

where q^w denotes the pseudo-differential operator of Weyl symbol in $\mathbb{R}^{d-1}_{x'}$. If $\widetilde{q}(t,x',\xi') = q(t,x',\xi')|\xi'|^{-2\sigma}$, then in view of the Littlewood-Paley decomposition in $\mathbb{R}^{d-1}_{\xi'}$ we may replace the second term by $(\widetilde{q}^w(t,x',D')\lambda^2 u, u)$, provided that the support of the partial Fourier transform of $u(t,x')$ with respect to x' is contained in $\{\lambda^{1/\sigma} \leqslant |\xi'| \leqslant 2\lambda^{1/\sigma}\}$. Though G is not smooth enough in general, the Wick approximation of $\widetilde{q}^w$ gives

$$(\widetilde{q}^w(t,x,D')\lambda^2 u, u) \geqslant (G(t,x';k)\lambda^2 u, u) - C\|u\|^2,$$

(see Proposition 2.1 of [**13**] and Proposition 1.1 of [**1**]). Hence (5.8) leads us to (5.6) for u with $\operatorname{supp} u$ contained in a small neighborhood of x_0. Finally, the usual covering argument shows (5.6) for the general u.

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