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SERGE ALINHAC

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AN EXAMPLE OF BLOWUP AT INFINITY FOR A QUASILINEAR WAVE EQUATION

by

Serge Alinhac

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Abstract. — We consider an example of a Quasilinear Wave Equation which lies between the genuinely nonlinear examples (for which finite time blowup is known) and the null condition examples (for which global existence and free asymptotic behavior is known). We show global existence, though geometrical optics techniques show that the solution does not behave like a free solution at infinity. The method of proof involves commuting with fields depending on u, and uses ideas close to that of the paradifferential calculus.

***Résumé* (Explosion à l'infini pour un exemple d'équation d'ondes quasi-linéaire)**
Nous considérons un exemple d'équation d'ondes quasi-linéaire qui se situe entre les exemples vraiment non-linéaires (pour lesquels l'explosion en temps fini est connue) et les exemples vérifiant la condition nulle (pour lesquels la solution existe globalement et est asymptotiquement libre). Nous montrons l'existence globale, bien que des arguments d'optique géométrique non-linéaire indiquent un comportement non libre de la solution à l'infini. La méthode de la preuve fait intervenir la commutation avec des champs dépendant de u, et utilise des idées proches de celles du calcul paradifférentiel.

In this text, Theorems, Propositions etc. are numbered according to the section where they appear, without any mention of the Chapter. When quoted in a different chapter, they appear with the additional mention of the Chapter. For instance, in Chapter III, section 2, there is Lemma 2. In Chapter IV, section 4, the same Lemma is quoted as Lemma III.2.

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Introduction

We prove in this paper the global existence (for ε small enough) of smooth solutions to the equation in $\mathbf{R}^3_x \times \mathbf{R}_t$

$$\partial_t^2 u - c^2(u)\Delta_x u = 0, c(u) = 1 + u,$$

with smooth and compactly supported initial data of size ε.

This result has been proved before only in the *radially symmetric case* by Lindblad [**13**], who also pointed out to some evidence that the nonradial solutions should have a very large lifespan. It turns out that the solutions do not behave at $t = +\infty$ like solutions of the free wave equation (that is, $u \sim \varepsilon/(1+t)$); most derivatives of u have, apart from the factor $\varepsilon/(1+t)$, an exponential growth $\exp C\tau$ at infinity, where $\tau = \varepsilon \log(1+t)$ is the slow time. This explains the title of this paper.

The method of proof is that of Klainerman [**11**], combining energy inequalities and commutations with appropriate "**Z**" fields. Because of the blowup at infinity, the fields we use have to be adapted to the geometry of the problem (as in Christodoulou-Klainerman [**7**]), and their coefficients smoothed out. This is very close to the paradifferential calculus of Bony [**6**], or, equivalently, to a Nash-Moser process.

I. Main result and ideas of the proof

We consider in $\mathbf{R}^3_x \times \mathbf{R}_t$ the equation

$$F(u) \equiv \partial_t^2 u - c^2(u)\Delta_x u = 0, \qquad (1.1)_a$$

where we will take for simplicity $c = c(u) = 1 + u$, since higher powers of u produce only easily handled terms. The coordinates will be

$$x = (x_1, x_2, x_3), \quad t = x_0,$$

and

$$\partial u = (\partial_1 u, \partial_2 u, \partial_3 u, \partial_t u).$$

The initial data are

$$u(x,0) = \varepsilon u_1^0(x) + \varepsilon^2 u_2^0(x) + \cdots, \quad (\partial_t u)(x,0) = \varepsilon u_1^1(x) + \varepsilon^2 u_2^1(x) + \cdots, \qquad (1.1)_b$$

for real C^∞ functions u_i^j, supported in the ball $|x| \leqslant M$.

We will use the usual polar coordinates $r = |x|$, $x = r\omega$, and define the rotation fields

$$R_1 = x_2\partial_3 - x_3\partial_2, \quad R_2 = x_3\partial_1 - x_1\partial_3, \quad R_3 = x_1\partial_2 - x_2\partial_1.$$

By Z_0 we will denote one of the standard Klainerman's fields

$$\partial_i, R_j, \quad S = t\partial_t + r\partial_r, \quad h_i = x_i\partial_t + t\partial_i. \qquad (1.2)$$

For the Laplace operator, we have then

$$\Delta_x = \partial_r^2 + (2/r)\partial_r + (1/r^2)\Delta_\omega,$$

where the Laplace operator on the sphere Δ_ω is $\Delta_\omega = R_1^2 + R_2^2 + R_3^2$.

We define two linear operators

$$P \equiv c^{-1}\partial_t^2 - c\Delta, \quad P_1 \equiv c^{-1}\partial_t^2 - c(\partial_r^2 + r^{-2}\Delta_\omega), \tag{1.3}$$

such that, setting $u = \varepsilon/rU$, we have $Pu = 0$, $P_1 U = 0$. We also set

$$L \equiv c^{-1/2}\partial_t + c^{1/2}\partial_r, \quad L_1 \equiv c^{-1/2}\partial_t - c^{1/2}\partial_r,$$

for which we have

$$[L, L_1] = (L_1 u/2c)L_1 - (Lu/2c)L, \quad P_1 = LL_1 - cr^{-2}\Delta_\omega + (Lu/2c)L. \tag{1.4}$$

Remark that, since $c = c(u)$, iterated use of the fields $L, L_1, \partial_j, R_j, S$ will generate a considerable number of terms depending again on u. To master this phenomenon, we will have to construct an appropriate "Calculus". Finally, we set

$$\sigma_1 = M + 1 - r + t, \tag{1.5}$$

which is positive and roughly equivalent to the distance to the boundary of the light cone.

Our main result is the following Theorem.

Theorem. — *Let $s_0 \in \mathbf{N}$. For ε small enough, the Cauchy problem* (1.1) *has a global smooth solution u. Moreover, we have the estimates*

$$|Z_0^\alpha \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\alpha| \leqslant s_0,$$

$$|\partial u| \leqslant C\varepsilon(1+t)^{-1}, \quad |Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0 - 2.$$

In the case of radially symmetric data, the solution u is a smooth function of (r^2, t). For this case, Lindblad [**13**] has proved global existence. We explain now the main ideas of the proof. In the whole paper, all constants will be denoted by C, unless otherwise specified.

I.1. A first insight using nonlinear geometrical optics

a. If w denotes the solution of the linearized problem on zero

$$(\partial_t^2 - \Delta)w = 0, \quad w(x,0) = u_1^0(x), \quad (w_t)(x,0) = u_1^1(x),$$

we know (see [**10**]) that, for some smooth F_0,

$$w \sim 1/rF_0(\omega, r-t), \quad r \to +\infty.$$

Taking εw as a rough approximation of u, we observe as in [**10**], [**1**] that the quadratic nonlinearity $u\Delta u$ produces a *slow time* effect, for the slow time $\tau \equiv \varepsilon\log(1+t)$. This means that, for large time, we expect formally u to be better approximated by

$$\varepsilon/rV(r-t, \omega, \tau),$$

for a smooth V satisfying $V(r-t,\omega,0) = F_0(\omega, r-t)$. Substituting the above expression of u in (1.1), we obtain

$$(1.6) \qquad V_{\sigma\tau} + VV_{\sigma\sigma} = 0, \quad V(\sigma,\omega,0) = F_0(\omega,\sigma), \quad \sigma \equiv r-t.$$

As pointed out already in [**13**], this is in sharp contrast with what happens, for instance, for the equation $\partial_t^2 u - (1+u_t)\Delta u = 0$. In this case, a similar approach yields for V the equation $2V_{\sigma\tau} - V_\sigma V_{\sigma\sigma} = 0$, which is essentially Burgers'equation and blows up in finite time. Here, one easily sees that (1.6) has *global* solutions: this gives a hint that the lifespan of u could be very large (though not necessarily $+\infty$, see for instance the case of the null condition in two space dimensions [**1**]); the consequences of this fact are precisely stated in Theorem II.1.

b. Looking more closely, we see that the solution V of (1.6) satisfies

$$|V_\sigma| \leqslant C, \quad |\partial^\alpha_{\sigma,\omega,\tau} V| \leqslant Ce^{C\tau}.$$

Since we are willing to use Klainerman's method [**11**], we have to apply products Z_0^α to $(1.1)_a$, and use an energy inequality for P to control $|\partial Z_0^\alpha u|_{L^2}$. On the one hand, the boundedness of V_σ yields

$$|\partial u| \leqslant C\varepsilon/(1+t).$$

In the standard energy inequality for P (see [**10**] Prop. 6.3.2), this will cause an *amplification factor* of the initial energy of the form

$$\exp C\varepsilon \int_0^t ds/(1+s) = (1+t)^{C\varepsilon}.$$

Thus the best one can expect, using the energy method and Klainerman's inequality, is

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2},$$

which is the result we obtain. On the other hand, if we believe that u and its derivatives actually behave like ε/rV, we see that derivatives like $R_i u$ or $\partial_t^2 u$, etc. do behave like $\varepsilon/r(1+t)^{C\varepsilon}$, which matches with what we just obtained from the energy method. This is why we say that we have *blowup at infinity*: the solution u exists globally, but does not behave like a solution of the linear equation. This phenomenon has been observed already, for instance in the study by Delort [**8**] of the Klein-Gordon equation.

I.2. Commuting Klainerman's fields

a. If we apply for instance a rotation field R_i to $(1.1)_a$, we obtain

$$PR_i u - 2(R_i u)(\Delta u) = 0.$$

Writing the energy inequality for P, it is not possible to reasonably absorb the term $(R_i u)(\Delta u)$ using Gronwall's lemma since

$$\exp \int_0^t |R_i u|_{L^\infty} \sim \exp[C^{-1}(1+t)^{C\varepsilon}]$$

is far too big. On the other hand, applying Z_0^α to $(1.1)_a$ produces a term $(Z_0^\alpha u)(\Delta u)$ in the equation for $PZ_0^\alpha u$, which is a zero order term: to handle this term will require some type of Poincaré lemma, controlling $Z_0^\alpha u$ by $\partial Z_0^\alpha u$. Note that even in a finite strip $|r-t| \leqslant C$ close to the boundary of the light cone, such a term cannot be reasonably controlled since again Δu behaves exponentially in τ at infinity.

b. Hence we have to modify the standard fields Z_0 to get better commutation properties. Following the geometric approach of Christodoulou-Klainerman [**7**], we define an optic function (in fact, only an *approximate* optic function) $\psi = \psi(r, \omega, t)$ by

$$L\psi = 0, \quad \psi(0, \omega, t) = -M - 1 - t.$$

This is a substitute for the standard optic function $r - t + C$, whose level surfaces are the light cones $r = t + C$. To write down the modified fields Z_m, we first adapt Z_0 to the geometry of the operator by defining $H_0 = ct\partial_r + r/c\partial_t$. For some $a(R_i)$, $a(S)$, $a(H_0)$ to be defined, we set now

$$R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1.$$

Let us pause to explain how this compares with the approach of [**7**]. In [**7**], the authors introduce an exact optic function, whose level surfaces give a foliation of outgoing cones. The rotation fields and L are defined to be tangent to these cones. This way of taking into account the exact geometry of the symbol has the advantage of producing in the computations relatively easily understandable geometric objects. On the other hand, it leads to rather tedious computations: may be, one is demanding too much. Here, since Lu and $(R_i/r)u$ are expected to behave much better than other derivatives of u, we consider that the effect of taking more complicated perturbations (of the standard fields) involving L or R_i/r would be negligible. The choice of the perturbation coefficients a is dictated only by *commutation* properties with L. Ideally, taking

$$(1.7)_a \qquad La(R_i) + a(R_i)(L_1 u/(2c)) = -R_i u/(2c),$$

$$(1.7)_b \qquad La(S) + a(S)(L_1 u/(2c)) = -Su/(2c), a(H_0) = -a(S),$$

would give

$$[R_i^m, L] = *L, \quad [S^m, L] = *L, \quad [H_0^m, L] = *L.$$

To avoid singularities at $r = 0$, we introduce in fact a cutoff $\overline{\chi} = \overline{\chi}(r/(1+t))$ in (1.7) (see III.1 and the commutation relations of Lemma III.3.1).

I.3. Induction on time. — The proof is by "induction on time" (see [**10**] for instance). We first make the induction hypothesis

$$\textbf{(IH)} \qquad |Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+\eta}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0, \quad \eta = 10^{-2}, \quad s_0 \geqslant 10.$$

This is a pointwise estimate, which is supposed to be valid up to some time T. The strategy of the proof is the following:

Step 1. — From (IH), we deduce (still for $t \leqslant T$) the better behavior in L^∞ norm of a small number of derivatives of u (see Proposition III.7)

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu-1}, \quad |\alpha| \leqslant s_0 - 4.$$

Here, $\mu = 1/2 + 10^{-1}$.

Step 2. — Using the energy method of Klainerman, we bound in L^2 norm (still for $t \leqslant T$) a large number of derivatives of u (see VII.3)

$$|Z_0^\alpha \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\alpha| \leqslant 2(s_0 - 4).$$

Step 3. — Using Klainerman's inequality, we obtain

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0 \leqslant 2s_0 - 10.$$

If $C\varepsilon \leqslant \eta/2$ and t is large, this is much better than (IH) and Theorem II.1 allows us to prove that for small enough ε, (IH) never stops being true and u exists globally.

To prove the L^∞ estimates of Step 1, we write the equation in the form

$$LL_1U = c/r^2\Delta_\omega U - (Lu/2c)LU,$$

and apply products Z_m^α to the left. In particular, we get $|L_1U| \leqslant C$, which eventually gives $|\partial u| \leqslant C\varepsilon/(1+t)$.

To prove L^2 estimates without loosing derivatives, we have to commute Z_m with P, which causes new problems we analyze now.

I.4. Smoothing

a. In the expression of $[Z_m, P]u$ necessarily appears the term $(Pa)L_1u$, containing $(r^{-2}\Delta_\omega a)L_1u$ and $(LL_1a)L_1u$. Since, from (1.7), we expect to control $R_i^k a$ in terms of $R_i^k Z_m u$ only, we see that we are missing *two* derivatives if we want to keep the full r^{-2} decay, or missing *one* if we rather write

$$r^{-2}\Delta_\omega = r^{-1}\sum(R_i/r)R_i.$$

In both cases, we have to put a *smoothing operator* S_θ in front of a. Here, θ is a big parameter, and $S_\theta v$ is roughly the smooth truncation of $\widehat{v}(\xi)$ for $|\xi| \leqslant \theta$. This is very close to the paradifferential approach introduced by Bony [**6**], where symbols say $a(x)\xi$ correspond to operators T_aD_x and not to aD_x. A typical application of these ideas is given in Alinhac [**4**], where instead of using true vectors fields $\sum a_i\partial_i$ tangent to some (non smooth) surface, we use $\sum T_{a_i}\partial_i$. In other words, we have to commute to the equation vector fields (here, the Z_m) tangent to characteristic surfaces of the operator (here, essentially the modified cones $\psi = const$), but these vector fields have to be smoothed first. Alternatively, one can say that we use a Nash-Moser procedure (see for instance [**5**]). As shown by Hörmander [**9**], the two approaches are essentially equivalent.

Since La is already known, we hope to neglect the term LL_1a and concentrate on $\Delta_\omega a$. If we take S_θ to be smoothing in the ω variables only, we have (with another S_θ on the right)

$$R_i S_\theta v \sim \theta S_\theta v.$$

Choosing $\theta = \theta(t)$, we hope for the decay factor $1/r$ to compensate for the growth $\theta(t)$ in the term such as $1/rR_iS_\theta v$. Unfortunately, since L and L_1 have variable coefficients, commutators arise in $LL_1S_\theta a$ which display second order derivatives of a with respect to ∂_r and ∂_t also. We are thus forced to introduce S_θ as a smoothing operator *both* in the variables ω and r, say

$$S_\theta = S^{(r)}_{\theta_1} S^{(\omega)}_{\theta_2},$$

where the two parameters $\theta_1(t)$ and $\theta_2(t)$ have to be determined.

b. According to the analysis of **a.**, we use now the smoothed modified fields

$$\widetilde{R}^m_i = R_i + \widetilde{a}(R_i)L_1, \quad \widetilde{S}^m = S + \widetilde{a}(S)L_1, \quad \widetilde{H}^m_i = H_i + \widetilde{a}(H_i)L_1,$$

where $H_i = ct\partial_i + x_i/c\partial_t$ and

$$\widetilde{a}(R_i) = S_\theta a(R_i), \quad \widetilde{a}(S) = S_\theta a(S), \quad \widetilde{a}(H_i) = -\omega_i\widetilde{a}(S) - (\omega \wedge \widetilde{a}(R))_i.$$

These fields are denoted by $\widetilde{Z}_m$. Of course, we have to develop a calculus for these fields and their commutators with S_θ, etc., which is very similar to the calculus of paradifferential operators. Needless to say, this part of the paper, corresponding to sections IV.3, IV.4, IV.5, is quite tedious, and should be skipped by the reader.

c. On the one hand, we have the formula (cf. Lemma IV.5.1)

$$[\partial_t, S_\theta] = \theta_1'/\theta_1 s_\theta + \theta_2'/\theta_2 s_\theta.$$

On the other hand, we need in our estimates to have $\theta_i'/\theta_i = O(\varepsilon(1+t)^{-1})$. Hence we are forced to take

$$\theta_i = \theta_i^0(1+t)^{\varepsilon\beta_i}.$$

It turns out that the two speeds β_i will have to be chosen different: β_1 and $\beta_2 - \beta_1$ have to be big enough. This reflects the dissymmetry between the first order derivatives of u: $\varepsilon^{-1}(1+t)u_r$ is bounded while $\varepsilon^{-1}(1+t)R_iu$ may grow like $(1+t)^{C\varepsilon}$.

I.5. Structure of $[\widetilde{Z}_m, P]u$. — This is the heart of the matter. Since the Z_0 fields have been modified so as to improve the commutation with L (see **2.b**), we expect good formulas for $[\widetilde{Z}_m, LL_1]$ also. In contrast, computing the term $[\widetilde{Z}_m, \Delta_\omega]$ and taking the smoothing operator S_θ into account is rather tedious. The result is described in Proposition VI.1. It turns out that the most delicate terms to control are the ones containing a, especially

$$r^{-2}L_1\widetilde{a}\Delta_\omega u, \quad L\widetilde{a}L_1^2u, \quad L_1L\widetilde{a}\partial u, \quad (1+t)^{-1}L\widetilde{a}\partial u. \tag{1.8}$$

These terms are handled in part **C** of the proof of Proposition VII.1. Formulas for the higher order commutators $[\widetilde{Z}_m^\alpha, P]$ are also established, and require the full calculus for the fields $\widetilde{Z}_m$.

I.6. Energy inequalities. — Writing $P\widetilde{Z}_m u = -[\widetilde{Z}_m, P]u$, $v = \widetilde{Z}_m u$ and using an energy inequality for P, we have to check that the various terms of $[\widetilde{Z}_m, P]u$ can be absorbed from right to left in the inequality. To handle the first term in (1.8), we need an inequality displaying a better control of the special derivatives $(R_i/r)v$. Such inequalities have been already discussed and used in [**2**], [**3**]: the idea is to establish an energy inequality with a "ghost weight" $e^{b(r-t)}$, where b is bounded. Here, we use ψ instead of $r-t$, and take a weight

$$\exp(\tau+1)b(\psi), \quad b(s) = B(-s)^{-\nu}, \quad \nu > 0,\ B > 0,$$

where ν and B^{-1} have to be chosen small enough (see Proposition V.3.1). This weight does not disappear, but is bounded below and above by $C(1+t)^{C\varepsilon}$, which is allowed in our context.

I.7. Poincaré Lemma. — As explained in **2.a**, we need a Poincaré Lemma to control the zero order term $(\Delta u)v$ in the linearized operator acting on v. In the context of the weighted L^2 norms explained in § 6, we obtained roughly the formula (see Proposition V.2)

$$\int_{r\geqslant t/2} e^p (\Delta u)^2 v^2 dx \leqslant C\varepsilon^2 (1+t)^{-2} \int_{r\geqslant t/2} e^p v_r^2 dx, \quad p = (\tau+1)b(\psi).$$

The *miracle* here is that we only know

$$|\Delta u| \leqslant C\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu-2}$$

and still get the estimate we would obtain if we had $C_1 = 0$. This is due to the special structure of $L_1^2 U$ displayed in Lemmas II.3.3 and II.3.5.1, which say roughly

$$L_1^2 U \sim \psi_r h(\psi), \quad |h(s)| \leqslant C(1+|s|)^{-3/2+4\eta}.$$

To prove the inequality, we make the change of variable $s = \psi(r,\omega,t)$ in the integrals, and proceed as usual in the s variables.

I.8. Calculus for systems of modified Z_0 fields. — In the course of this paper, we use in fact several systems of modified fields, each of which giving birth to a special calculus. For instance, besides the two main systems of the Z_m of Chapter III.1 and the $\widetilde{Z}_m$ of Chapter IV.1 mentioned above, we have

i) The enlarged calculus for Z_m and the system $\overline{Z}_0$ in the proof of Proposition III.7,

ii) The new system Z_m and the system $\overline{Z}_0$ in the proof of Proposition IV.1,

iii) The system $\overline{Z}_m$ in the proof of Proposition VII.2.

We deliberately made the following choice: rather than building before the proofs of these results a tight wall of Lemmas that no reader can cross, we chose to rather write *"Scheherazade type"* of proofs, where the needed Lemmas are displayed and proved exactly when one needs them. This allows the reader to view Proposition III.7, Proposition IV.1, Proposition VII.2 as black boxes which need not be opened in a first approach, and avoids confusion between the different systems of fields.

The plan of the paper is as follows: in part II, we prove the large time existence theorem (needed to start the induction) and discuss the first consequences of the induction hypothesis, in particular the boundedness of $\varepsilon^{-1}(1+t)\partial u$ and the special structure of $L_1^2 U$. Chapter III is devoted to obtain the improved L^∞ estimates on u. In part IV, the smooth modified fields $\widetilde{Z}_m$ are defined and many lemmas display the calculus for these fields. The weighted energy norms, the energy inequality and the Poincaré Lemma are proved in Chapter V. The structure of the commutators $[\widetilde{Z}_m, P]$ and $[\widetilde{Z}_m^\alpha, P]$ are discussed in VI. Finally, using V and VI, simultaneous weighted L^2 estimates of $\widetilde{Z}_m^{k+1}\partial u$ and $\widetilde{Z}_m^k \partial a$ are obtained in VII, allowing us to finish the proof of the main result in VII.3.

II. Large time existence, induction hypothesis and first consequences

II.1. Large time existence. — We consider the Cauchy problem I.1.1. Our first result displays a very large lifespan of the solution.

Theorem 1. — *Let $\overline{\tau} > 0$ and $s_0 \in \mathbf{N}$. Then, if ε is small enough, the solution u to the Cauchy problem (1.1) exists and is C^∞ for $\tau \equiv \varepsilon \log(1+t) \leqslant \overline{\tau}$. Moreover, we have for some C the estimates*

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0. \tag{1.1}$$

Proof. — We only sketch the proof, since it is very close to the proof of Theorem 6.5.3 in [**10**], using "induction on time". There are two main differences:

i) The approximate solution u_a can be constructed without time limitation.

ii) The structure of the equation on the difference $\dot{u} = u - u_a$ is slightly different.

Let us review this more closely.

i) Construction of an approximate solution

a. Let w satisfy

$$w_{tt} - \Delta w = 0, \quad w(x,0) = u_1^0(x), \quad w_t(x,0) = u_1^1(x).$$

Then w can be written

$$w = 1/rF(\omega, 1/r, r-t),$$

where F is defined in [**10**], (6.2.5). Note that here F is supported, like w, in $-M \leqslant r-t \leqslant M$. When $t \to +\infty$,

$$w \sim 1/rF_0(\omega, r-t), \quad F_0(\omega,\sigma) = F(\omega, 0, \sigma).$$

We consider now, for $\omega \in S^2$, $\tau \geqslant 0$, $\sigma \equiv r-t \leqslant M$, the Cauchy problem

$$\partial^2_{\sigma\tau} V + V\partial^2_{\sigma\sigma} V = 0, \quad V(\sigma,\omega,0) = F_0(\omega,\sigma).$$

We claim that this problem has a smooth solution for $0 \leqslant \tau \leqslant \overline{\tau}$, supported for $\sigma \leqslant M$. In fact, set

$$\sigma = \phi(s,\omega,\tau), \quad W(s,\omega,\tau) = V(\phi,\omega,\tau).$$

We have

$$W_s = \phi_s V_\sigma, W_{s\tau} = \phi_{s\tau} V_\sigma + \phi_s(V_{\sigma\tau} + \phi_\tau V_{\sigma\sigma}),$$
$$\partial_\tau(W_s/\phi_s) = (V_{\sigma\tau} + \phi_\tau V_{\sigma\sigma}).$$

We choose now ϕ defined by

$$\phi_s = \exp(\tau\partial_\sigma F_0), \quad \phi(M,\omega,\tau) = M,$$

and set $W = \phi_\tau$. Note that $\phi(s,\omega,0) = s$, and W_s is zero for $|s| \geqslant M$. Since $W_s/\phi_s = \partial_\sigma F_0(\omega,\sigma)$, we have

$$0 = \partial_\tau(W_s/\phi_s) = (V_{\sigma\tau} + VV_{\sigma\sigma})(\phi,\omega,\tau).$$

Moreover, for $\tau = 0$, $W_s = \partial_s F_0$, $W(M,\omega,0) = F_0(\omega,M) = 0$, hence

$$W(s,\omega,0) = F_0(\omega,s), \quad V(\sigma,\omega,0) = F_0(\omega,\sigma).$$

Finally, for $\sigma \leqslant \phi(-M,\omega,\tau)$, V is a smooth function of (ω,τ). In particular, $|V| \leqslant C$.

b. We introduce now two smooth real cutoff functions

$$\chi_1 = \chi_1(\varepsilon t), \quad \chi_2 = \chi_2(r/(1+t)),$$

where $\chi_1(s)$ is zero for $s \geqslant 2$ and one for $s \leqslant 1$, while $\chi_2(s)$ is zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$. We define the approximate solution by

$$u_a = \varepsilon\chi_1 w + \varepsilon/r(1-\chi_1)\chi_2 V(r-t,\omega,\tau).$$

As in [**10**], we have for all α the estimates $|Z_0^\alpha u_a| \leqslant C\varepsilon/(1+t)$. We set also $J_a = \partial_t^2 u_a - (1+u_a)^2\Delta u_a$. To prove the analogue to Lemma 6.5.5 of [**10**], we have to note that

$$\partial_t^2(\chi_2 V) = \chi_2\partial_t^2 V + 2(\partial_t\chi_2)(-V_\sigma + \varepsilon/(1+t)V_\tau) + (\partial_t^2\chi_2)V,$$
$$\partial_r^2(\chi_2 V) = \chi_2\partial_r^2 V + 2(\partial_r\chi_2)(V_\sigma) + (\partial_r^2\chi_2)V.$$

In these expressions, note that

$$\partial\chi_2 = O(1/(1+t)), \quad \partial^2\chi_2 = O(1/(1+t)^2), \quad \chi_2' V_\sigma \equiv 0.$$

For $t \geqslant 2/\varepsilon$, we obtain

$$J_a = -2\varepsilon^2/r^2(V_{\sigma\tau} + VV_{\sigma\sigma}) + O(\varepsilon/(1+t)^3).$$

Thanks to the equation on V, we finally obtain in this region, for all α,

$$|Z_0^\alpha J_a| \leqslant C\varepsilon(1+t)^{-3}.$$

In the first period $\varepsilon t \leqslant 1$ or in the transition region $1 \leqslant \varepsilon t \leqslant 2$, the discussion is the same as in [**10**], and we obtain

$$|Z_0^\alpha J_a| \leqslant C\varepsilon^2 |\log\varepsilon|(1+t)^{-2}.$$

The main difference here with [**10**] is that V is no longer zero for $\sigma \leqslant -M$. Hence the support of J_a is only contained in the region $(1+t)/2 \leqslant r \leqslant M+t$, and

$$\begin{aligned} &|Z_0^\alpha J_a|_{L^2} \leqslant C\varepsilon^2|\log\varepsilon|(1+t)^{-1/2}, \quad t \leqslant 2/\varepsilon,\\ &|Z_0^\alpha J_a|_{L^2} \leqslant C\varepsilon(1+t)^{-3/2}, \quad t \geqslant 2/\varepsilon. \end{aligned}$$

We obtain finally

$$\int_{\tau\leqslant\overline{\tau}} |Z_0^\alpha J_a|_{L^2} dt \leqslant C\varepsilon^{3/2}|\log\varepsilon|.$$

ii) The induction argument. — We write the equation on $u = u_a + \dot{u}$ in the form

$$(cP)\dot{u} = \partial_t^2\dot{u} - (1+u_a+\dot{u})^2\Delta\dot{u} = -J_a + (\Delta u_a)(2(1+u_a)+\dot{u})\dot{u}.$$

We make the induction hypothesis

$$|Z_0^\alpha \partial\dot{u}| \leqslant \varepsilon\sigma_1^{-1/2}/(1+t), \quad |\alpha| \leqslant s_0.$$

This means that this pointwise estimate is supposed to hold for $t \leqslant T$, for some T. We will eventually prove that T satisfies $\varepsilon\log(1+T) \geqslant \overline{\tau}$. First, since $|\partial u_a + \partial\dot{u}| \leqslant C\varepsilon(1+t)^{-1}$, we can use the standard energy inequality for the operator cP to evaluate $|\partial\dot{u}|_{L^2}$. We wish to apply Z_0^α to the left to the equation on $\dot{u}$, with $|\alpha| \leqslant 2s_0$. Since we have, for constants $C_{\alpha\beta}$,

$$[Z_0^\alpha, (\partial_t^2 - \Delta)] = \sum_{|\beta|<|\alpha|} C_{\alpha\beta} Z_0^\beta (\partial_t^2 - \Delta),$$

we write the equation in the form

$$(\partial_t^2 - \Delta)\dot{u} = ((1+u)^2 - 1)\Delta\dot{u} - J_a + (\Delta u_a)(2(1+u_a)+\dot{u})\dot{u} \equiv G.$$

Applying Z_0^α, we obtain $(\partial_t^2 - \Delta)Z_0^\alpha\dot{u} = Z_0^\alpha G - \sum C_{\alpha\beta}Z_0^\beta G$. In $Z_0^\alpha G$, we distinguish the term $((1+u)^2-1)\Delta Z_0^\alpha\dot{u}$ which we take back to the left-hand side to get $(cP)(Z_0^\alpha\dot{u})$.

a. We ignore the factor $(2(1+u_a)+\dot{u})$ accompanying $(\Delta u_a)\dot{u}$ in G. For terms

$$(Z_0^\gamma \Delta u_a)(Z_0^\delta \dot{u}), \quad |\gamma| + |\delta| \leqslant |\alpha|,$$

we use the inequality $|\sigma_1^{-1}v|_{L^2} \leqslant C|\partial v|_{L^2}$. Since $\sigma_1|Z_0^\gamma\Delta u_a| \leqslant C\varepsilon/(1+t)$, such terms are absorbed using Gronwall's inequality.

b. We ignore the factor $2(1+(u_a+\dot{u})/2)$ accompanying $(u_a+\dot{u})\Delta\dot{u}$ in G. We have to deal with terms

1) $$(u_a+\dot{u})[Z_0^\alpha, \Delta]\dot{u},$$

2) $$(Z_0^\gamma(u_a+\dot u))(Z_0^\delta\Delta\dot u),\quad |\delta|<|\alpha|,\quad |\gamma|+|\delta|\leqslant|\alpha|.$$

We use (the stars denoting irrelevant coefficients)

$$[Z_0^\alpha,\Delta]=\sum_{|\beta|\leqslant|\alpha|-1}*\partial^2 Z_0^\beta,\partial=\sigma_1^{-1}\sum *Z_0.$$

Hence

$$[Z_0^\alpha,\Delta]\dot u=\sigma_1^{-1}\sum_{|\gamma|\leqslant|\alpha]}*\partial Z_0^\gamma\dot u.$$

On the other hand,

$$\sigma_1^{-1}|u_a+\dot u|\leqslant C\varepsilon/(1+t),$$

thus the term 1) will be controlled using Gronwall's inequality.

For 2), we remark first that the part $(Z_0^\gamma u_a)(Z_0^\delta\Delta\dot u)$ is easily handled. For the other part, we distinguish which factor we are going to evaluate in L^2 norm. If $|\gamma|\leqslant s_0$, we write as before

$$|Z_0^\gamma\dot u Z_0^\delta\Delta\dot u|_{L^2}\leqslant\sum_{|\beta|\leqslant|\alpha|}C|\sigma_1^{-1}Z_0^\gamma\dot u|_{L^\infty}|\partial Z_0^\beta\dot u|_{L^2}$$

and use Gronwall's inequality. If $|\gamma|\geqslant s_0+1$, we write

$$|(\sigma_1^{-1}Z_0^\gamma\dot u)(\sigma_1 Z_0^\delta\Delta\dot u)|_{L^2}\leqslant\sum_{|\beta|\leqslant s_0}C|\partial Z_0^\beta\dot u|_{L^\infty}|\partial Z_0^\gamma\dot u|_{L^2}$$

and use once again Gronwall's inequality.

Finally, we obtain

$$|Z_0^\alpha\partial\dot u|_{L^2}\leqslant C\varepsilon^{3/2}|\log\varepsilon|,\quad |\alpha|\leqslant 2s_0.$$

Using Klainerman's inequality, we obtain for $|\alpha|\leqslant 2s_0-2$

$$|Z_0^\alpha\partial\dot u|\leqslant C\varepsilon^{3/2}|\log\varepsilon|\sigma_1^{-1/2}(1+t)^{-1}.$$

If $2s_0-2\geqslant s_0$, that is $s_0\geqslant 2$ and ε is small enough, we obtain the statement by the usual induction argument. □

II.2. The optic function. — We assume in what follows that u is defined and C^∞ for $t\leqslant T'$; u is also defined in any finite strip $-C\leqslant t\leqslant 0$ for small enough ε. We extend the integral curve $r=t+M$ of L for negative time until it reaches the t-axis. All objects and estimates related to u will implicitly be considered as defined in the corresponding region. We define the optic function $\psi=\psi(r,\omega,t)$ by

$$L\psi=0,\quad \psi(0,\omega,t)=-M-1-t.$$

Then $\psi\leqslant C<0$ in the region of interest. As in [**12**], the function ψ is a substitute for the usual phase $r-t-M-1$. The cones $\psi=$ const will be considered as deformations of the standard cones $\sigma_1=$ const, and later on, the geometry of the fields Z_0 will be adapted to these new cones.

Lemma 2. — *For $\tau\leqslant\overline{\tau}$, we have for C big enough*

i) $|\psi| \leqslant C(\sigma_1 + \tau^2)$,
ii) *for* $\sigma_1 \geqslant C\tau^2$, $C|\psi| \geqslant \sigma_1$.

Proof. — From a given point $M_0 = (r_0, \omega_0, t_0)$ we draw backward, for some big enough C, the integral curve Γ of L, along with the curves Γ_1 and Γ_2 respectively defined by

$$\sigma_1' = -C\varepsilon(1+t)^{-1}\sigma_1^{1/2}, \quad \sigma_1' = C\varepsilon(1+t)^{-1}\sigma_1^{1/2}.$$

According to the bound of u deduced from (1.1), the first curve is above, the second below Γ. The three curves meet $r=0$ at t_1, θ, t_2, with $t_1 \geqslant -M-1-\psi \geqslant t_2$. By integration,

$$2\sigma_1^{1/2}(t) + C\varepsilon\log(1+t_0) = 2\sigma_1^{1/2}(t_1) + C\varepsilon\log(1+t_1),$$

hence

$$|\psi| \leqslant \sigma_1(t_1) \leqslant C(\sigma_1 + (\varepsilon\log(1+t))^2).$$

From the second differential equation, we get, if $\sigma_1 \geqslant (C\varepsilon\log(1+t))^2$, $4|\psi| \geqslant \sigma_1$. □

II.3. Induction hypothesis and its consequences. — We already know that u exists as a C^∞ function for $t \leqslant T'$, $\varepsilon\log(1+T') \geqslant \overline{\tau}$. For some $s_0 \in \mathbf{N}$ and some small $\eta > 0$ to be fixed later independently of ε (we will take in fact $s_0 \geqslant 10$ and, say, $\eta = 10^{-2}$), we assume now

$$\textbf{(IH)} \qquad |Z_0^\alpha \partial u| \leqslant \overline{C}\varepsilon(1+t)^{-1+\eta}\sigma_1^{-1/2}, \quad t \leqslant T \leqslant T', \quad |\alpha| \leqslant s_0.$$

From now on, all estimates will take place for $t \leqslant T$, and will use the induction hypothesis (IH). We will eventually prove that $T = T'$, thus getting global existence.

II.3.1. Estimates on the optic function

Lemma 3.1. — *For C big enough, we have*
i) $|\psi| \leqslant C(\sigma_1 + \varepsilon^2(1+t)^{2\eta})$.
ii) *For* $\sigma_1 \geqslant C\varepsilon^2(1+t)^{2\eta}$, $C|\psi| \geqslant \sigma_1$.
iii) *Everywhere for* $\tau \geqslant \overline{\tau}$, *we have*

$$\sigma_1 \leqslant C\varepsilon^2(1+t)^{2\eta}|\psi|, \quad |\psi| \leqslant C\varepsilon^2(1+t)^{2\eta}\sigma_1.$$

The proof is exactly the same as the proof of Lemma 2.

II.3.2. Structure of L_1U. — Since, from (IH), ∂u is much smaller than $\varepsilon(1+t)^{-1}$ as soon as $\sigma_1 \geqslant \gamma(1+t)$ (for any $\gamma > 0$), most of the estimates we need will take place in the "exterior" region R_e defined by

$$r \geqslant M + t/2, \quad \overline{\tau} \leqslant \tau.$$

The part of the boundary of R_e which is the union of $r = M + t/2$ and $\tau = \overline{\tau}$ will be denoted by γ. First of all, to establish later an energy inequality, we need to prove that $|\partial U|$ is bounded.

Lemma 3.2. — *In R_e, we have*

$$L_1U = V_T(\psi, \omega) + \rho_1,$$

with

$$|V_T(s,\omega)| \leqslant C(1+|s|)^{-1/2+2\eta}, \quad |\rho_1| \leqslant C\varepsilon|\psi|^{1/2}(1+t)^{-1+2\eta}.$$

Proof. — First, we obtain from (IH) the estimates

$$|Z_0^\alpha \partial U| \leqslant C(1+t)^\eta \sigma_1^{-1/2}.$$

Set now

$$f = cr^{-2}\Delta_\omega U - (Lu/2c)LU,$$

for which $LL_1U = f$. Noting that $(r+t)(\partial_t + \partial_r) = \sum \omega_i h_i + S$, we get

$$|Lu| \leqslant C|(\partial_t + \partial_r)u| + C|u||\partial u| \leqslant \varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2},$$

$$|LU| \leqslant C|(\partial_t + \partial_r)U| + C|u||\partial U| \leqslant (1+t)^{-1+2\eta}\sigma_1^{1/2},$$

hence

$$|f| \leqslant (1+t)^{-2+\eta}\sigma_1^{1/2} \leqslant C\varepsilon(1+t)^{-2+2\eta}|\psi|^{1/2}.$$

If we draw from a point M in R_e the integral curve Γ of L, meeting γ at M', we denote by Γ_- and Γ_+ respectively the backward and forward parts of Γ in R_e. We set then

$$V_T(\psi,\omega) = (L_1U)(M') + \int_\Gamma f, \rho_1 = -\int_{\Gamma_+} f.$$

Here, the integrals are taken along Γ. From the estimates on f, we get (uniformly in T)

$$|\rho_1| \leqslant C\varepsilon|\psi|^{1/2}\int_t^{+\infty}(1+s)^{-2+2\eta}ds \leqslant C\varepsilon|\psi|^{1/2}(1+t)^{-1+2\eta}.$$

To estimate V_T, we compare both sides on γ, using Lemma 2. □

In the rest of the paper, to simplify notations, we drop the dependence of V on (T,ω).

II.3.3. The quantities a_1, b_1. — Let us define and fix in the sequence a cutoff function $\overline{\chi}$ by

$$\overline{\chi} = \overline{\chi}(r/(C+t)),$$

where $0 \leqslant \overline{\chi} \leqslant 1$ is smooth, zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$ and $C = 2(M+1)$. We define now a_1, b_1 by

$$Lb_1 = -\overline{\chi}L_1u/2c, \quad b_1(0,t) = 0, \quad a_1 = \exp b_1.$$

The following Lemma indicates the precise structure of b_1.

Lemma 3.3. — *We have*

i) $b_1 = -(\tau/2)V(\psi) + \rho_2, |\rho_2| \leqslant C,$

ii) $a_1\psi_r = 1 + \rho_3,\ |\rho_3| \leqslant C\varepsilon.$

Proof

a. By definition, using Lemma 3.2,

$$g = L(b_1 + (\tau/2)V(\psi)) = -(\varepsilon/2rc)\rho_1 - (\varepsilon\overline{\chi}/2c^{1/2}r^2)U + (\varepsilon(1-\overline{\chi})/2rc)L_1U + \frac{\varepsilon}{1+t}V(c^{-1/2}/2 - 1/2c) + \varepsilon(V/2c)((1+t)^{-1} - r^{-1}),$$

hence

$$\begin{aligned}|g| &\leqslant C\varepsilon(1+t)^{-2+\eta}\sigma_1^{1/2} + C\varepsilon^2(1+t)^{-2+2\eta}|\psi|^{1/2} \\ &\quad + C\varepsilon^2(1+t)^{-2+\eta}\sigma_1^{1/2}|\psi|^{-1/2+2\eta} + C\varepsilon\sigma_1(1+t)^{-2}|\psi|^{-1/2+2\eta} \\ &\leqslant C\varepsilon^2(1+t)^{-2+2\eta}|\psi|^{1/2+2\eta}.\end{aligned}$$

Thus,

$$b_1 + \tau/2V(\psi) = \rho_2^1(\psi) + \rho_2^2,$$

with

$$|\rho_2^2| \leqslant C\varepsilon^2(1+t)^{-1+2\eta}|\psi|^{1/2+2\eta}.$$

Since $b_1 + \tau/2V$ is bounded on γ, ρ_2^1 is bounded, which proves i).

b. We have $L\psi_r + c^{-1/2}u_r\psi_r = 0$. Hence

$$L\log(a_1\psi_r) = -(L_1u/2c + c^{-1/2}u_r) + (1-\overline{\chi})L_1u/2c.$$

Now $L_1u + 2c^{1/2}u_r = Lu$,

$$|L_1u + 2c^{1/2}u_r| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2}, \quad |L\log(a_1\psi_r)| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2}.$$

Since $a_1\psi_r(0,t) = 1/c = 1 + O(\varepsilon)$, we obtain ii). □

II.3.4. Improved estimates on the optic function

Lemma 3.4. — *For C big enough, we have the estimates*

i) $|\psi| \leqslant C\sigma_1 + C(1+t)^{C\varepsilon}$,

ii) *If $\sigma_1 \geqslant C(1+t)^{C\varepsilon}$, then $C|\psi| \geqslant \sigma_1$. In all cases, we have*

$$\sigma_1 \leqslant C|\psi|(1+t)^{C\varepsilon}, \quad |\psi| \leqslant C\sigma_1(1+t)^{C\varepsilon}.$$

Proof

a. From Lemma 3.2, we obtain $|L_1U| \leqslant C|\psi|^{-1/2+2\eta}$, since $|\psi| \leqslant C(1+t)$. Hence, using (IH),

$$|\partial_rU| \leqslant C(1+t)^{-1+\eta}\sigma_1^{1/2} + C|\psi|^{-1/2+2\eta} \leqslant |\psi|^{-1/2+2\eta}.$$

Using the estimates $a_1 \leqslant C(1+t)^{C\varepsilon}$ and $a_1\psi_r \geqslant 1/2$ from Lemma 3.3, we have

$$|\partial_rU| \leqslant C(1+t)^{C\varepsilon}|\psi|^{-1/2+2\eta}\psi_r,$$

and by integration

$$|U| \leqslant C_0(1+t)^{C_0\varepsilon}|\psi|^{1/2+2\eta}.$$

b. Just as in Lemma 2, let us consider the integral curve Γ of L through a point M_0, and denote by Γ_1 and Γ_2 respectively the curves

$$\sigma_1' = - \pm C\varepsilon/(1+t)(1+t)^{C\varepsilon} A_0^{1/2+2\eta},$$

where $A_0 = |\psi(M_0)|$ and C is big enough. Let us call respectively σ_1^1, $\overline{\sigma}_1$, σ_1^2 the values of σ_1 at the points where Γ_1, Γ, Γ_2 intersect γ. Since σ_1 is decreasing along γ, and Γ is above Γ_2 and below Γ_1, we have

$$\sigma_1^2 \leqslant \overline{\sigma}_1 \leqslant \sigma_1^1.$$

Integrating the equation for Γ_1, we thus get

$$A_0 \leqslant C\overline{\sigma}_1 \leqslant C\sigma_1^1 \leqslant C\sigma_1(M_0) + CA_0^{1/2+2\eta}(1+t_0)^{C\varepsilon},$$

which gives i).

c. Using Γ_2, we get

$$\sigma_1(M_0) \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon} A_0^{1/2+2\eta} \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon}(\sigma_1(M_0))^{1/2+2\eta},$$

hence $\sigma_1(M_0) \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon}$. If

$$\sigma_1(M_0) \geqslant 2C(1+t_0)^{C\varepsilon},$$

we obtain $\sigma_1(M_0) \leqslant 2\sigma_1^2$. Since $|\psi| \geqslant 1/2\sigma_1$ on γ, we have finally

$$A_0 \geqslant (1/2)\overline{\sigma}_1 \geqslant (1/2)\sigma_1^2 \geqslant (1/4)\sigma_1(M_0),$$

which is ii). □

We conclude from this Lemma that $|\psi|$ is not quite equivalent to σ_1: there exists a *blind zone*

$$\sigma_1 \leqslant C(1+t)^{C\varepsilon}$$

in which we cannot ensure that $|\psi|$ is big even is σ_1 is. This is due to a possible drift of the integral curves of L toward the cone $r = t + M$. Inside this blind zone, we can only prove $|L_1 u| \leqslant C\varepsilon(1+t)^{-1}$, while $|L_1 u| \leqslant C\varepsilon(1+t)^{-1}\sigma_1^{-1/2+0}$ outside.

II.3.5. Structure of $L_1^2 U$. — To prove later the Poincaré Lemma, we need to elucidate the special structure of $L_1^2 U$.

Lemma 3.5.1. — *In R_e, we have $a_1 L_1^2 U = h_T(\psi,\omega) + \rho_4$, with*

$$|h_T(s,\omega)| \leqslant C(1+|s|)^{-3/2+4\eta}, \quad |\rho_4| \leqslant C(1+t)^{-3/2+4\eta}.$$

Proof. — We have first

$$[L, a_1 L_1] = -(a_1/2c)LuL + (1-\chi)a_1(L_1 u/2c)L_1,$$

hence

$$g = L(a_1 L_1^2 U) = -(a_1/2c)LuLL_1 U + a_1 L_1(LL_1 U) + (1-\chi)a_1(L_1 u/2c)L_1^2 U.$$

But

$$L_1(LL_1U) = -L_1u/r^2\Delta_\omega U - 2c^{3/2}/r^3\Delta_\omega U - c/r^2(L_1\Delta_\omega U) + L_1Lu/2cLU - LuL_1u/2c^2LU + (Lu)^2/4c^2LU - LuL_1u/4c^2L_1U + Lu/2cLL_1U,$$

thus $|g| \leqslant C(1+t)^{-5/2+4\eta}$. By integrating L, we get the structure of $a_1L_1^2U$ with the estimate on ρ_4; comparing then both sides on γ yields the estimate on h. □

Finally, we have to evaluate the smallness of ψ_{rr}.

Lemma 3.5.2. — *We have, for $r \geqslant M + t/2$, the estimate*

$$|\psi_{rr}|/\psi_r^2 \leqslant C\tau(1+|\psi|)^{-3/2+4\eta} + C\varepsilon(1+|\psi|)^{-3/2+4\eta}.$$

Proof. — First $\psi_{tt} = c^2\psi_{rr} + (cu_r - u_t)\psi_r, (\partial_t + c\partial_r)\psi_t + u_t\psi_r = 0,$

$$(\partial_t + c\partial_r)\psi_{tt} = -u_{tt}\psi_r - 2u_t\psi_{rt}.$$

Hence

$$(\partial_t + c\partial_r)(\psi_{tt}/\psi_t^2) = u_{tt}/(c\psi_t) - 2u_t^2/(c^2\psi_t).$$

For $r \leqslant M + t/2$, the right hand side is less than $C\varepsilon(1+t)^{-5/2+2\eta}$; since $\psi_{tt}/\psi_t^2(0,t) = 0$, we obtain by integration

$$|\psi_{tt}/\psi_t^2| \leqslant C\varepsilon|\psi|^{-3/2+2\eta}.$$

Now, for $r \geqslant M + t/2$,

$$u_{tt} = c\varepsilon/(4r)L_1^2U + O(\varepsilon|\psi|^{-1/2})(1+t)^{-2+4\eta},$$

hence

$$|u_{tt}/(c\psi_t) - 2u_t^2/(c^2\psi_t^2)| \leqslant C\varepsilon(1+t)^{-1}|h(\psi)| + O(\varepsilon|\psi|^{-1/2}(1+t)^{-2+4\eta}),$$

which gives by integration from $r = M + t/2$ the desired estimate. □

III. Improved L^∞ estimates on u

In this chapter, we will prove that the L^∞ estimates (IH) on u imply in fact the much better estimates of Proposition 7.

III.1. Modified vector fields. — In order to control u and its derivatives in the spirit of Klainerman [**11**], we will need modified vectors fields Z_m ("m" for modified), which are perturbations of the standard vectors fields Z_0 defined in (II.1.2). First, we set

$$H_0 = c(u)t\partial_r + \frac{r}{c(u)}\partial_t, \quad H_i = c(u)t\partial_i + \frac{x_i}{c(u)}\partial_t, \quad 1 \leqslant i \leqslant 3,$$

thus defining hyperbolic rotations adapted to the operator P. Note that

$$H_0 = \sum \omega_i H_i, \quad H_i = \omega_i H_0 + ct(\partial_i - \omega_i\partial_r).$$

For each of the fields R_i, S, H_0 we define now $a(R_i), a(S), a(H_0)$ by

$$(1.1)_a \qquad La(R_i) + \overline{\chi}a(R_i)(L_1u/(2c)) = -\overline{\chi}R_iu/(2c),$$

$$(1.1)_b \qquad La(S) + \overline{\chi}a(S)(L_1u/(2c)) = -\overline{\chi}Su/(2c), \quad a(H_0) = -a(S),$$

$$(1.1)_c \qquad a(R_i)(0,t) = 0, \quad a(R_i)(x,0) = 0, \quad a(S)(0,t) = 0, \quad a(S)(x,0) = 0.$$

Remember that $\overline{\chi}$ is a standard cutoff defined in II.3.3. Thus the coefficients a are smooth functions (as long as u exists), vanishing for $r \geqslant t + M$ or $r \leqslant t/2 + M + 1$. The set of the coefficients

$$a(R_i), a(S), a_1$$

will be denoted by (Coeff$'$). We then define the *modified fields* R_i^m, S^m, H_0^m and K by

$$(1.2) \quad R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1, \quad K = a_1L_1.$$

We will write these equalities simply as $Z_m = Z + aL_1$, where Z will be one of the adapted vector fields R_i, S, H_0 or 0, and a will stand for the corresponding coefficient as in (1.2). Remark that

$$(r + ct)L = \sqrt{c}(H_0 + S) = \sqrt{c}(H_0^m + S^m).$$

We finally define the family Φ' as the collection of the fields $Z_m = R_i^m, S^m, H_0^m, K$. As usual, Z_m^k will simply denote a product of k fields taken among Φ'. It is always understood here that some of the fields in Φ' are singular at $r = 0$, and they will be considered only for $r \geqslant \gamma_0(1 + t)$ ($\gamma_0 > 0$).

In what follows, we will simply write f to denote a real C^∞ function of the (finitely many) variables

$$\varepsilon, u, \omega, \sigma_1(1 + t)^{-1}, (1 + t)^{-\nu_i}, \sigma_1^{-\nu_i}, \nu_i > 0.$$

Remark that $\overline{\chi} = f$. Finally, we denote by N_k one of the quantities

$$\varepsilon^{-1}(1 + t)\sigma_1^{-1}Z_m^k u, \quad \varepsilon^{-1}(1 + t)Z_m^k Lu, \quad \varepsilon^{-1}(1 + t)Z_m^k L_1u,$$

$$\sigma_1^{-1}Z_m^{k-1}a, \quad Z_m^{k-1}La, \quad Z_m^{k-1}L_1a, \quad a \in (\text{Coeff}').$$

We add the convention that 1 is also a N_0. We need now develop a calculus for these modified fields. To simplify the notation, we dispense in general with writing sums of terms of the same kind. For instance, we will write N_k for a sum of various N_k, Z_m for a sum of Z_m, etc.

III.2. Some calculus Lemma

Lemma 2. — *We have the following identities:*

i) $Z_m^k f = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
ii) $Z_m^k N_p = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k + p$, $k_i \geqslant p$ *for some* i,
iii) $Z_m^k t = t \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
iv) $Z_m^k \sigma_1 = \sigma_1 \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots k_j \leqslant k$.

Proof. — In view of the structure of the formulas, it is enough to prove them for $k = 1$ and any p.

We have

$$R_i \omega = f, \quad R_i \sigma_1 = 0, \quad R_i t = 0,$$
$$S\omega = 0,\ S\sigma_1 = -M - 1 + \sigma_1,\ St = t, S(\sigma_1/1+t) = f,\ S(1+t)^{-\nu} = f,\ S\sigma_1^{-\nu} = f,$$
$$H_0\omega = 0, \quad H_0\sigma_1 = f\sigma_1 N_0, \quad H_0 t = ft, \quad H_0(\sigma_1/1+t) = fN_0.$$

On the other hand,

$$L_1\omega = 0, \quad L_1\sigma_1 = f, \quad L_1 t = f, \quad L_1(\sigma_1/1+t) = f/1+t,$$
$$L_1(1+t)^{-\nu} = f/1+t, \quad L_1(\sigma_1^{-\nu}) = f/\sigma_1.$$

Hence

$$Z_m u = fN_0 + fN_0N_1, Z_m(\sigma_1/1+t) = fN_0 + fN_1,$$
$$Z_m(1+t)^{-\nu} = fN_0 + fN_1, Z_m\sigma_1^{-\nu} = fN_0 + fN_1,$$

and $Z_m f = f + fN_0 + fN_1$. Thus, i), iii) and iv) are proved. Now

$$L_1((1+t)/\sigma_1) = \sigma_1^{-1} f((1+t)/\sigma_1), \quad Z((1+t)/\sigma_1) = ((1+t)/\sigma_1) fN_0,$$

hence

$$Z_m((1+t)/\sigma_1) = ((1+t)/\sigma_1)(fN_0 + fN_1).$$

Thus, with $A = L$ or $A = L_1$,

$$Z_m[\varepsilon^{-1}((1+t)/\sigma_1) Z_m^p u] = (fN_0 + fN_1)N_p + N_{p+1},$$
$$Z_m[\varepsilon^{-1}(1+t) Z_m^p A u] = (fN_0 + fN_1)N_p + N_{p+1},$$
$$Z_m[\sigma_1^{-1} Z_m^p a] = (fN_0 + fN_1)N_{p+1} + N_{p+2},$$

which proves ii). □

III.3. Commutation Lemmas. — For fields X_i, Y, we will note

$$(adX)Y = [X, Y], \quad (adX^k)Y = [X_1, [X_2, \ldots Y] \ldots].$$

The following Lemmas justify the introduction of the modified fields Z_m: they just commute better with L than the standard fields Z_0.

Lemma 3.1. — *We have*

i) $$[Z_m, L] = fdL_1 + fN_0N_1L,$$

ii) $$[Z_m, L_1] = fN_0N_1L + fN_0N_1L_1,$$

iii) $$(adZ_m^k)L = \sum fN_{k_1}\cdots N_{k_j}L + \sum f(Z_m^q d)N_{l_1}\cdots N_{l_i}L_1,\quad k_1+\cdots+k_j \leqslant k,\quad q+l_1+\cdots+l_i \leqslant k-1,$$

iv) $$(adZ_m^k)L_1 = \sum fN_{k_1}\cdots N_{k_j}L + \sum fN_{l_1}\cdots N_{l_i}L_1,\quad k_1+\cdots+k_j \leqslant k,\quad l_1+\cdots+l_i \leqslant k.$$

Here, d denotes one of the quantities $d = (1-\overline{\chi})Z_m u = \varepsilon f N_1$.

Proof. — Since $d = \varepsilon f N_1$, thanks to Lemma 2, it is enough to prove the formulas for $k = 1$. We have

$$\begin{aligned}
[R_i, L] &= -R_i u/(2c)L_1, \qquad [R_i, L_1] = -R_i u/(2c)L,\\
[S, L] &= -L - Su/(2c)L_1, \quad [S, L_1] = -L_1 - Su/(2c)L,\\
[H_0, L] &= \Big(-1 + \frac{r-ct}{2c\sqrt{c}}Lu\Big)L + \Big(\frac{r+ct}{2c\sqrt{c}}Lu - H_0u/(2c)\Big)L_1,\\
[H_0, L_1] &= \Big(\frac{r-ct}{2c\sqrt{c}}L_1u - H_0u/(2c)\Big)L + \Big(1 + \frac{r+ct}{2c\sqrt{c}}L_1u\Big)L_1.
\end{aligned}$$

Remark here that

$$(r-ct)L_1 = \sqrt{c}(H_0 - S), \quad (r+ct)L = \sqrt{c}(H_0 + S),$$

hence the above formulas simplify to

$$\begin{aligned}
[H_0, L] &= \Big(-1 + \frac{r-ct}{2c\sqrt{c}}Lu\Big)L + Su/(2c)L_1,\\
[H_0, L_1] &= -Su/(2c)L + \Big(1 + \frac{r+ct}{2c\sqrt{c}}L_1u\Big)L_1.
\end{aligned}$$

Since

$$[aL_1, L] = -(La)L_1 - aL_1u/(2c)L_1 + aLu/(2c)L, \quad [aL_1, L_1] = -(L_1a)L_1,$$

we obtain, thanks to the choices of the a for each Z,

$$\begin{aligned}
[R_i^m, L] &= -(1-\overline{\chi})R_i^m u/(2c)L_1 + aLu/(2c)L,\\
[S^m, L] &= -(1-\overline{\chi})S^m u/(2c)L_1 + (aLu/(2c) - 1)L,\\
[H_0^m, L] &= (1-\overline{\chi})S^m u/(2c)L_1 + \Big(\frac{r-ct}{2c\sqrt{c}}Lu + aLu/(2c) - 1\Big)L,\\
[K, L] &= aLu/(2c)L - (1-\overline{\chi})Ku/(2c)L_1.
\end{aligned}$$

Similarly,

$$[R_i^m, L_1] = -1/(2c)(R_i^m u - aL_1u)L - (L_1a)L_1,$$
$$[S^m, L_1] = -1/(2c)(S^m u - aL_1u)L - (1 + L_1a)L_1,$$
$$[H_0^m, L_1] = -1/(2c)(S^m u - aL_1u)L + \Big(1 + \frac{r+ct}{2c\sqrt{c}}L_1u - L_1a\Big)L_1.$$

If we remark that

$$aLu = \sigma_1^{-1}a\varepsilon\sigma_1/(1+t)\varepsilon^{-1}(1+t)Lu = fN_0N_1,$$
$$(r-ct)Lu = r - ct/(r+ct)\sqrt{c}(H_0^m u + S^m u) = fN_1,$$

we can write

$$[Z_m, L] = f(1-\overline{\chi})(Z_m u)L_1 + fN_0N_1L = fdL_1 + fN_0N_1L,$$
$$[Z_m, L_1] = fN_0N_1L + fN_0N_1L_1.$$

□

Lemma 3.2. — *We have*

i) $$[Z_m^k, L] = \sum fN_{k_1}\cdots N_{k_j}Z_m^pL + \sum fZ_m^q dN_{l_1}\cdots N_{l_i}Z_m^rL_1,$$
$$p \leqslant k-1,\quad p + k_1 + \cdots + k_j \leqslant k,\quad r \leqslant k-1,\quad q + r + l_1 + \cdots + l_i \leqslant k-1.$$

ii) $$[Z_m^k, L_1] = \sum fN_{k_1}\cdots N_{k_j}Z_m^pL + \sum fN_{l_1}\cdots N_{l_i}Z_m^rL_1,$$
$$p \leqslant k-1,\quad p + k_1 + \cdots + k_j \leqslant k,\quad r \leqslant k-1,\quad r + l_1 + \cdots + l_i \leqslant k.$$

iii) $$[Z_m^k, L_1] = \sum_{\substack{p\leqslant k-1\\ p+\sum k_i\leqslant k}} fN_{k_1}\cdots N_{k_j}Z_m^pL + \sum_{(\sum l_j\leqslant k-1)} fN_{l_1}\cdots N_{l_i}Z_m^{l_{i+1}}L_1aZ_m^{l_{i+2}}L_1$$
$$+ \sum_{(p+\sum k_i\leqslant k-1)} fN_{k_1}\cdots N_{k_j}Z_m^pL_1 + \sum_{(\sum l_j+q+r\leqslant k-1)} fN_{l_1}\cdots N_{l_i}Z_m^q dZ_m^rL_1.$$

Proof. — For $k=1$, the formulas i) and ii) follow from Lemma 3.1. For iii) we write

$$[Z_m, L_1] = fN_0N_1L + (fN_0 - L_1a)L_1.$$

Since

$$[Z_m^{k+1}, A] = Z_m^k[Z_m, A] + [Z_m^k, A]Z_m,$$

we obtain easily the Lemma by induction, using Lemma 2. □

For technical reasons, we will need the following variant of Lemma 3.2.

Lemma 3.3. — *If $Lw = g$, we have*

$$LZ_m^kw = \sum fN_{l_1}\cdots N_{l_i}Z_m^{l_{i+1}}g + \sum fZ_m^{q_1}d\cdots Z_m^{q_i}dN_{k_1}\cdots N_{k_j}L_1Z_m^{k_{j+1}}w$$
$$+ \sum(1+t)^{-1}fZ_m^{q_1}d\cdots Z_m^{q_i}dN_{k_1}\cdots N_{k_j}Z_m^{k_{j+1}}w$$
$$= \textstyle\sum_1 + \sum_2 + \sum_3$$

In $\sum_1$, $\sum l_j \leqslant k$. *In* $\sum_2$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k, k_{j+1} \leqslant k-1$. *In* $\sum_3$,

$$i \geqslant 1, \quad i + \sum q_j + \sum k_i \leqslant k+1, \quad 1 \leqslant k_{j+1} \leqslant k-1.$$

Proof. — For $k=1$,

$$LZ_m w = Z_m g - [Z_m, L]w = Z_m g + fN_0N_1 g + fN_0 dL_1 w.$$

Hence the formula is correct, with $\sum_3 = 0$. Now

$$Z_m LZ_m^k w = fN_0N_1 LZ_m^k w + fN_0 dL_1 Z_m^k w + LZ_m^{k+1} w,$$
$$LZ_m^{k+1} w = Z_m \textstyle\sum_1 + Z_m \sum_2 + Z_m \sum_3 + fN_0N_1(\sum_1 + \sum_2 + \sum_3) + fN_0 dL_1 Z_m^k w.$$

The last term belongs to $\sum_2$ for $k+1$. The terms involving $\sum_1$ again belong to $\sum_1$ for $k+1$. The terms involving $\sum_3$ again belong to $\sum_3$ for $k+1$, and $fN_0N_1\sum_2$ belongs to $\sum_2$ for $k+1$. In $Z_m\sum_2$, the only nontrivial term is the one containing

$$Z_m L_1 Z_m^{k_{j+1}} w = fN_0N_1 LZ_m^{k_{j+1}} w + fN_0N_1 L_1 Z_m^{k_{j+1}} w + L_1 Z_m^{k_{j+1}+1} w.$$

The last two terms give terms belonging to $\sum_2$ for $k+1$. For the first, we write

$$LZ_m^{k_{j+1}} w = f(1+t)^{-1} \textstyle\sum Z_m^{k_{j+1}+1} w,$$

and the corresponding terms belong to $\sum_3$ for $k+1$. □

III.4. A computation of Z_m^k

Lemma 4. — *We have, Z_0 denoting the standard fields defined in I.1.2*

$$Z_m^k = \textstyle\sum fN_0^{l_0} N_1^{l_1} N_{k_1} \cdots N_{k_j} Z_0^p, \quad 1 \leqslant p \leqslant k, k_i \geqslant 2, \quad p + \sum (k_i - 1) \leqslant k.$$

Proof. — We use the formula $\partial_i = \sigma_1^{-1} \sum f Z_0$. We get by inspection $Z_m = \sum fN_0N_1Z_0$, which implies the Lemma for $k=1$, and the Lemma follows in general by induction, using Lemma 2. □

III.5. Estimates of the N_k

Proposition 5. — *We have, for $k \leqslant s_0 - 3$*

$$|N_k|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

Proof

a. We have $La = -\overline{\chi}/(2c) Z_m u \equiv F_0 = \varepsilon f N_1$. Hence

$$L(\sigma_1^{-1} a) = -\overline{\chi}/(2c)\sigma_1^{-1} Z_m u + fu\sigma_1^{-2} a = \varepsilon(1+t)^{-1} fN_0N_1 \equiv F_1,$$
$$LL_1 a = [L, L_1]a + L_1 La = L_1 u/(2c) L_1 a - Lu/(2c) F_0 + L_1 F_0 \equiv F_2.$$

Also $LL_1 U = c/r^2 \Delta_\omega U - Lu/(2c) LU \equiv G$.

b. From Lemma 2, we get

$$Z_m^l F_1 = \varepsilon(1+t)^{-1} \textstyle\sum' fN_{k_1} \cdots N_{k_j}, \sum k_i \leqslant l+1,$$

where here and later $\sum'$ means that not all N_{k_i} are one. We now evaluate F_2:

$$F_2 = f\varepsilon(1+t)^{-1} N_0N_1 + L_1 F_0,$$

$$\begin{aligned} L_1F_0 &= fL_1uZ_mu + f(1+t)^{-1}Z_mu + fN_0N_1Lu + fN_0N_1L_1u + fZ_mL_1u \\ &= f\varepsilon(1+t)^{-1}N_0^2N_1. \end{aligned}$$

We thus obtain

$$Z_m^kF_2 = \varepsilon(1+t)^{-1}\sum{}' fN_{k_1}\cdots N_{k_j}, \quad k_1+\cdots+k_j \leqslant k+1.$$

c. We have in fact

$$Z_mr = fr + fa, \varepsilon Z_mU = fru + fau + rZ_mu,$$

hence

$$\begin{aligned} \varepsilon Z_m^{k+1}U = rZ_m^{k+1}u + r\sum_{\substack{1\leqslant p\leqslant k \\ k_1+\cdots k_j+p\leqslant k+1}} fN_{k_1}\cdots N_{k_j}Z_m^pu + r\sum_{p+k_1+\cdots+k_j\leqslant k} fN_{k_1}\cdots N_{k_j}Z_m^pu \\ + \sum fN_{l_i}\cdots N_{l_i}Z_m^qaZ_m^pu, \\ p+q+l_i+\cdots+l_i \leqslant k. \end{aligned}$$

Thus

$$\begin{aligned} \varepsilon^{-1}\sigma_1^{-1}(1+t)Z_m^{k+1}u = f\sigma_1^{-1}Z_m^{k+1}U + fN_0\sigma_1^{-1}Z_m^ka + \sum{}' fN_{k_1}\cdots N_{k_j}, \\ k_1+\cdots+k_j \leqslant k+1, \quad k_i\leqslant k. \end{aligned}$$

Similarly, we obtain, with $A = L$ or $A = L_1$

$$\varepsilon AU = \pm c^{1/2}u + rAu, \quad \varepsilon Z_mAU = fN_0Z_mu + frAu + faAu + rZ_mAu.$$

The last three terms are handled as before. For the first term, we write

$$\begin{aligned} Z_m^k(fN_0Z_mu) = fN_0Z_m^{k+1}u + \sum fN_{k_1}\cdots N_{k_j}Z_m^pu, \\ p+k_1+\cdots+k_j \leqslant k+1, \quad 1\leqslant p\leqslant k. \end{aligned}$$

Thus

$$\begin{aligned} \varepsilon^{-1}(1+t)Z_m^{k+1}Au = fZ_m^{k+1}AU + fN_0\varepsilon^{-1}Z_m^{k+1}u + fN_0\sigma_1^{-1}Z_m^ka \\ + \sum{}' fN_{k_1}\cdots N_{k_j}, \\ k_1+\cdots+k_j \leqslant k+1, \quad k_i\leqslant k. \end{aligned} \tag{5.1}$$

d. Using Lemma 3.3 for $w = \sigma_1^{-1}a$ and $g = F_1$ or $w = L_1a$ et $g = F_2$, we obtain

$$LZ_m^k(\sigma_1^{-1}a) = F_1^k, \quad LZ_m^kL_1a = F_2^k, \quad F_i^0 = F_i..$$

To estimate the right hand sides, we need the following Lemma.

Lemma 5.1. — *In any region $r \leqslant \gamma(1+t), \gamma < 1$, we have*

$$(1+t)|L_1w| \leqslant C\sum|Z_mw|.$$

Proof. — We have the identity

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0^m - S^m + 2aL_1.$$

On the other hand, a rough estimate of a shows that

$$\sigma_1^{-1}|a| \leqslant C\varepsilon\sigma_1^{-1/2}(1+t)^{C\varepsilon+\eta}.$$

Hence, in the region we consider, $\sigma_1^{-1}|a|$ is as small as we want, and the Lemma follows. □

We have, with the notations of Lemma 3.3, applied for the index k with $w = a\sigma_1$, $g = F_1$,

$$F_1^k = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .$$

From the structure of F_1, we get

$$\sum\nolimits_1 = \varepsilon(1+t)^{-1}\sum fN_{l_1}\cdots N_{l_i}, \quad \sum l_j \leqslant k+1.$$

Using Lemma 3.3 and the structure of $d = \varepsilon fN_1$, we have

$$|\sum\nolimits_2| \leqslant C|d|(1+t)^{-1}|N_{k+1}| + C\varepsilon(1+t)^{-1}\sum|N_{k_1}|\cdots|N_{k_{j+1}}|, \quad k_i \leqslant k.$$

Note that $|d| \leqslant C\varepsilon(1+t)^{-\eta}$. We have a similar estimate for $\sum_3$. The computations are completely similar for F_2^k.

e. We have now to control the values of $Z_m^k(\sigma_1^{-1}a)$, $Z_m^kL_1a$, $Z_m^{k+1}L_1U$ on the boundary $r = M + t/2$.

Lemma 5.2. — *On the boundary $r = t/2 + M$, for $k \leqslant s_0 - 1$,*

i) $Z_m^kL_1a = 0$, $|Z_m^k(\sigma_1^{-1}a)| \leqslant C$,

ii) $|Z_m^{k+1}L_1U| \leqslant C$.

Proof. — Close to this boundary, a is either identically one or zero: the value of $Z_m^kL_1a$ is zero. For the U term, we remark that we can replace $Z_m \neq K$ by the corresponding Z, K by L_1. For such fields Z (including L_1), we have

$$Z = Z_0 + ftu/\sigma_1 Z_0 = fN_0Z_0.$$

Denoting only here by N_k the terms

$$\varepsilon^{-1}\sigma_1^{-1}(1+t)Z^ku, \quad \varepsilon^{-1}(1+t)Z^kLu, \quad \varepsilon^{-1}(1+t)Z^kL_1u,$$

we get as before

$$Z^k = \sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}, \quad \sum l_j \leqslant k, \quad l_{i+1} \geqslant 1.$$

Hence, with $A = 1, L, L_1$,

$$Z^kAu = \sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}Au.$$

By induction, starting from $|N_0| \leqslant C$ by the induction hypothesis, we get $|N_k| \leqslant C$ for $k \leqslant s_0$. Finally

$$Z^{k+1}L_1U = \sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}L_1U,$$

hence the conclusion. The proof is similar for $Z_m^k \sigma_1^{-1}$. □

f. We now set

$$\phi_l = \sum |N_l|_{L^\infty},$$

and assume by induction $\phi_l \leqslant C(1+t)^{C\varepsilon}, l \leqslant k$ (we have already shown and used that $|N_0| \leqslant C$). Because of the structure of Lemma 4, we need first control ϕ_1 without using Lemma 4. Let

$$G = c/r^2 \Delta_\omega U - Lu/(2c)LU,$$

$$G_1 = LZ_m L_1 U = Z_m G + f dL_1^2 U + f N_0 N_1 G = f N_0 N_1 G + f N_0 N_1 Z_0 G + f dL_1^2 U.$$

It is clear that

$$|Z_0^l (cr^{-2}\Delta_\omega U)| \leqslant C(1+t)^{-2+\eta}\sigma_1^{1/2}.$$

On the other hand,

$$L = c^{-1/2}(\partial_t + \partial_r) + (c^{1/2} - c^{-1/2})\partial_r = f(1+t)^{-1}\sum Z_0 + fu\partial,$$

and as usual

$$fu\partial = fu\sigma_1^{-1}\sum Z_0 = \varepsilon(1+t)^{-1} f N_0 \sum Z_0.$$

Finally $L = (1+t)^{-1}\sum f N_0 Z_0$. Hence

$$Lu/(2c)LU = (1+t)^{-2} f N_0^2 Z_0 u Z_0 U,$$

$$|Z_0^l (Lu/(2c)LU)| \leqslant C(1+t)^{-2+2\eta}.$$

Adding, we get

$$|Z_0^l G| \leqslant C(1+t)^{-3/2+\eta}.$$

We also have

$$|dL_1^2 U| \leqslant C(1+t)^{-2+2\eta}(1+\phi_1),$$

hence

$$|G_1| \leqslant C(1+t)^{-3/2+\eta}\phi_1 + C(1+t)^{-1-\eta}.$$

From this estimate, we get by integrating

$$|Z_m L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$

Integrating the equations on $\sigma_1^{-1}a$ and $L_1 a$, we get, using the estimates on F_i^0 established in **d.**,

$$\sigma_1^{-1}|a| + |L_1 a| \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\int_0^t \phi_1 ds/(1+s) + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$

Now,

$$\begin{aligned}|(1+t)(\varepsilon\sigma_1)^{-1} Z_m u| &\leqslant C(1+t)\varepsilon^{-1}|\partial_r Z_m u|_{L^\infty} \\ &\leqslant C(1+t)\varepsilon^{-1}|LZ_m u|_{L^\infty} + C(1+t)\varepsilon^{-1}|L_1 Z_m u|_{L^\infty}.\end{aligned}$$

At this point we need the refinement iii) in Lemma 3.2:

$$(1+t)\varepsilon^{-1}|[Z_m, L_1]u| \leqslant C(1+t)\varepsilon^{-1}(|Lu|\phi_1 + |L_1 u|(1+|L_1 a|)).$$

Since $(1+t)\varepsilon^{-1}|Lu| \leqslant C(1+t)^{-\eta}$, we have

$$|(1+t)(\sigma_1\varepsilon)^{-1}Z_m u| \leqslant C(1+t)\varepsilon^{-1}|LZ_m u| + C(1+t)\varepsilon^{-1}|Z_m L_1 u| + C(1+t)^{-\eta}\phi_1 + C(1+t)^{C\varepsilon} + C|L_1 a|.$$

We use Lemma 3.1 to evaluate the first term:

$$(1+t)\varepsilon^{-1}|LZ_m u| \leqslant (1+t)\varepsilon^{-1}|Z_m Lu| + C|d|(1+t)\varepsilon^{-1}|L_1 u| + C|N_1|(1+t)\varepsilon^{-1}|Lu|.$$

But

$$|Z_m Lu| \leqslant C\phi_1|Z_0 Lu| + C(1+t)^{C\varepsilon}|Z_0 Lu|.$$

Since

$$(1+t)\varepsilon^{-1}|Z_0^q Lu| \leqslant C(1+t)^{-\eta},$$

we get

$$(1+t)\varepsilon^{-1}|Z_m Lu| \leqslant C + C(1+t)^{-\eta}\phi_1,$$

and finally

$$(1+t)\varepsilon^{-1}|LZ_m u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_1 + C\varepsilon\phi_1.$$

From (5.1) we get now

$$(1+t)\varepsilon^{-1}|Z_m L_1 u| \leqslant C|Z_m L_1 U| + \varepsilon^{-1}|Z_m u| + C|\sigma_1^{-1}a| + C(1+t)^{C\varepsilon}.$$

Since

$$\varepsilon^{-1}|Z_m u| \leqslant C + C(1+t)^{-\eta}\phi_1,$$

and, from the very definition of a, $|La| \leqslant C\varepsilon\phi_1$, we get finally

$$\phi_1 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_1 + C\varepsilon\phi_1 + C\int_0^t \phi_1\varepsilon ds/(1+s) + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$

The conclusion follows by Gronwall's Lemma, since $|\phi_1| \leqslant C$ for finite t.

g. To control ϕ_k, $k \geqslant 2$, we essentially have to repeat the argument of **f.**, using Lemma 4 when necessary. Setting $LZ_m^{k+1}L_1 U = G_{k+1}$, we estimate first G_{k+1} using Lemma 3.3, which requires controlling $Z_m^l G, l \leqslant k+1$. Thanks to Lemma 4,

$$Z_m^l G = \sum f N_0^{l_0} N_1^{l_1} N_{k_1} \cdots N_{k_j} Z_0^p G,$$

and we already know $|Z_0^l G| \leqslant C(1+t)^{-3/2+\eta}$. Hence

$$|Z_m^l G| \leqslant C(1+t)^{-1-\eta}, \quad l \leqslant k,$$
$$|Z_m^{k+1} G| \leqslant C(1+t)^{-1-\eta}\phi_{k+1} + C(1+t)^{-1-\eta}.$$

We obtain from Lemma 3.3, applied for the index $k+1$ with $w = L_1 U, g = G$, and the induction hypothesis on ϕ_l

$$|G_{k+1}| \leqslant C(1+t)^{-1-\eta} + C(1+t)^{-1-\eta}\phi_{k+1} + C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1-\eta}|Z_m^{k+1}L_1 U|.$$

From this estimate, we get by integration down to the boundary $r = t/2 + M$

$$|Z_m^{k+1} L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t |Z_m^{k+1} L_1 U| ds/(1+s)^{1+\eta} + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta},$$

hence

$$|Z_m^{k+1} L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.$$

Integrating the equations on $Z_m^k(\sigma_1^{-1} a)$ and $Z_m^k L_1 a$ we get, using the estimates on F_i^k established in **d.**,

$$|Z_m^k(\sigma_1^{-1} a)| + |Z_m^k L_1 a| \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\int_0^t \phi_{k+1} ds/(1+s) + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.$$

Now,

$$\begin{aligned} |(1+t)(\varepsilon\sigma_1)^{-1} Z_m^{k+1} u| &\leqslant C(1+t)\varepsilon^{-1} |\partial_r Z_m^{k+1} u|_{L^\infty} \\ &\leqslant C|(1+t)\varepsilon^{-1} L Z_m^{k+1} u|_{L^\infty} + C|(1+t)\varepsilon^{-1} L_1 Z_m^{k+1} u|_{L^\infty}. \end{aligned}$$

At this point we need the refinement iii) in Lemma 3.2:

$$\begin{aligned} |(1+t)\varepsilon^{-1}[Z_m^{k+1}, L_1] u| &\leqslant |(1+t)\varepsilon^{-1} L u| \phi_{k+1} + C(1+t)^{C\varepsilon} \\ &+ C|Z_m^k L_1 a||(1+t)\varepsilon^{-1} L_1 u| + C(1+t)^{C\varepsilon} + C\sum_{q\leqslant k} (1+t)^{C\varepsilon} |Z_m^q d|. \end{aligned}$$

We have $|(1+t)\varepsilon^{-1} L u| \leqslant C(1+t)^{-\eta}$. Using Lemma 2 and Lemma 4, we get

$$Z_m^q d = \sum_{\substack{q_1+q_2=q \\ \sum l_i \leqslant q_1}} f N_{l_1} \cdots N_{l_j} Z_m^{q_2+1} u,$$

$$Z_m^l u = \sum f N_0^{l_0} N_1^{l_1} N_{k_1} \cdots N_{k_j} Z_0^p u.$$

Since $|Z_0^p u| \leqslant C\varepsilon(1+t)^{-2\eta}$ we obtain

$$(1+t)^{C\varepsilon} |Z_m^q d| \leqslant C + C(1+t)^{-\eta} \phi_{k+1}, \quad q \leqslant k.$$

Finally

$$\begin{aligned} |(1+t)(\sigma_1\varepsilon)^{-1} Z_m^{k+1} u| &\leqslant C|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| + C|(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \\ &+ C(1+t)^{-\eta}\phi_{k+1} + C(1+t)^{C\varepsilon} + C|Z_m^k L_1 a|. \end{aligned}$$

We use Lemma 3.3 to evaluate the first term:

$$|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| \leqslant \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

We obtain

$$\sum\nolimits_2 + \sum\nolimits_3 \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\phi_{k+1},$$

$$\sum\nolimits_1 \leqslant C|(1+t)\varepsilon^{-1} L u||N_{k+1}| + C(1+t)^{C\varepsilon} \sum_{p\leqslant k+1} (1+t)\varepsilon^{-1} |Z_m^p L u|.$$

Using again Lemma 4, we obtain for $p \leqslant k+1$,

$$|Z_m^p L u| \leqslant C|N_{k+1}||Z_0^q L u| + C(1+t)^{C\varepsilon} |Z_0^q L u|.$$

Since $(1+t)\varepsilon^{-1}|Z_0^q Lu| \leqslant C(1+t)^{-\eta}$ by the induction hypothesis, we get

$$\textstyle\sum_1 \leqslant C + C(1+t)^{-\eta}\phi_{k+1},$$

and finally

$$|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} + C\varepsilon\phi_{k+1}.$$

From (5.1) we get now

$$|(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \leqslant C|Z_m^{k+1} L_1 U| + C\varepsilon^{-1}|Z_m^{k+1} u| + C|\sigma_1^{-1} Z_m^k a| + C(1+t)^{C\varepsilon}.$$

From Lemma 4 we have

$$|\varepsilon^{-1} Z_m^{k+1} u| \leqslant C + C(1+t)^{-\eta}\phi_{k+1}, |Z_m^k La| \leqslant C + C(1+t)^{-\eta}\phi_{k+1}.$$

From Lemma 2 we have

$$|\sigma_1^{-1} Z_m^k a| \leqslant C|Z_m^k(\sigma_1^{-1} a)| + C(1+t)^{C\varepsilon},$$

thus finally

$$\begin{aligned}
\phi_{k+1} &\leqslant |\sigma_1^{-1} Z_m^k a| + |Z_m^k La| + |Z_m^k L_1 a| + |(1+t)(\varepsilon\sigma_1)^{-1} Z_m^{k+1} u| \\
&\qquad + |(1+t)\varepsilon^{-1} Z_m^{k+1} Lu| + |(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \\
&\leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} + C\varepsilon\phi_{k+1} \\
&\qquad + C|Z_m^k(\sigma_1^{-1} a)| + C|Z_m^k L_1 a| + C|Z_m^{k+1} L_1 U| \\
&\leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} C\varepsilon\phi_{k+1} \\
&\qquad + C\int_0^t \phi_{k+1}\varepsilon ds/(1+s) + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.
\end{aligned}$$

The conclusion follows by Gronwall's Lemma. □

III.6. Improved estimates of the N_k. — We will need later to know that the N_k have a better behavior inside the light cone.

Proposition 6. — *Let $\mu > 1/2$. For $\eta > 0$ small enough, we have for $k \leqslant s_0 - 3$, with the exception of $N_0 = 1$, the estimates*

$$|N_k| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

Proof. — We follow here the proof of Proposition 5 and use the notations there.

a. We have

$$L(\sigma_1^{1-\mu} L_1 U) = \sigma_1^{1-\mu} G + f N_0 \varepsilon (1+t)^{-1}(\sigma_1^{1-\mu} L_1 U), \quad |\sigma_1^{1-\mu} G| \leqslant C(1+t)^{-1-\eta}.$$

Since $|\sigma_1^{1-\mu} L_1 U| \leqslant C$ on $r = t/2 + M$, we get by integrating the equation

$$|L_1 U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

On the other hand, we know $|LU| \leqslant C\sigma_1^{1/2}(1+t)^{-1+\eta} \leqslant C\sigma_1^{-1/2+\eta}$. Hence $|\partial_r U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}$, which implies

$$|U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu}, \quad (1+t)(\sigma_1\varepsilon)^{-1}|u| \leqslant C(+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

Finally, with $A = L$ or $A = L_1$,

$Au = \varepsilon/rAU - (Ar/r)u,\ (1+t)\varepsilon^{-1}|Au| \leqslant C|AU| + C(1+t)^{-1}|U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$

This proves the estimate for N_0.

b. We assume now

$$|N_l| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}, \quad l \leqslant k,$$

and set $\psi_k = \sum |\sigma_1^{1-\mu} N_k|_{L^\infty}$. We follow the proof of Proposition 5, g, just looking more closely to the powers of σ_1. Set $V_{k+1} = \sigma_1^{1-\mu} Z_m^{k+1} L_1 U$. We have

$$LV_{k+1} = \sigma_1^{1-\mu} G_{k+1} + fN_0\varepsilon(1+t)^{-1}V_{k+1}.$$

We see that

$$|\sigma_1^{1-\mu} Z_0^l G| \leqslant C(1+t)^{-1-2\eta},$$

hence

$$\sigma_1^{1-\mu}|Z_m^l G| \leqslant C(1+t)^{-1-\eta}, \quad l \leqslant k,$$
$$\sigma_1^{1-\mu}|Z_m^{k+1} G| \leqslant C(1+t)^{-1-\eta}(1+|N_{k+1}|).$$

Using Lemma 3.3 with $w = L_1 U, g = G$, we get

$$G_{k+1} = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

We have from the above estimates

$$\begin{aligned}\sigma_1^{1-\mu}|\textstyle\sum_1| &\leqslant C(1+t)^{-1-\eta}(1+|N_{k+1}|) + C(1+t)^{-1-\eta}|N_{k+1}| + C(1+t)^{-1-\eta}\\ &\leqslant C(1+t)^{-1-\eta/2}.\end{aligned}$$

Since we get easily

$$\sigma_1^{1-\mu}|Z_m^l L_1 U| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k,$$

we have, using $|d| \leqslant C\varepsilon(1+t)^{-\eta}$ and the estimate on $|Z_m^l d|$ already established,

$$\begin{aligned}\sigma_1^{1-\mu}|\textstyle\sum_2| &\leqslant C|d|(1+t)^{-1}|V_{k+1}| + C|Z_m^k d|(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta}\\ &\leqslant C(1+t)^{-1-\eta}(1+|V_{k+1}|)\end{aligned}$$

and a similar estimate for $\sum_3$. Finally

$$\sigma_1^{1-\mu}|G_{k+1}| \leqslant C(1+t)^{-1-\eta/2}(1+|V_{k+1}|).$$

We already know that $|V_{k+1}| \leqslant C$ on the boundary $r = t/2 + M$, hence by integration we obtain

$$|V_{k+1}| \leqslant C(1+t)^{C\varepsilon}.$$

c. We have, still with the notations of Proposition 5,

$$L(\sigma_1^{-\mu}a) = \sigma_1^{1-\mu}F_1 + fN_0\varepsilon(1+t)^{-1}(\sigma_1^{-\mu}a),$$
$$L(\sigma_1^{1-\mu}L_1 a) = \sigma_1^{1-\mu}F_2 + fN_0\varepsilon(1+t)^{-1}(\sigma_1^{1-\mu}L_1 a).$$

Set now

$$LZ_m^k(\sigma_1^{-\mu}a) = \overline{F}_1^k, \quad LZ_m^k(\sigma_1^{1-\mu}L_1 a) = \overline{F}_2^k.$$

To estimate $\overline{F}_1^k$, we use Lemma 3.3 with

$$w = \sigma_1^{-\mu} a, \quad g = \sigma_1^{1-\mu} F_1 + f N_0 \varepsilon (1+t)^{-1} (\sigma_1^{-\mu} a).$$

We find

$$\overline{F}_1^k = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

We have

$$Z_m^l g = \sigma_1^{1-\mu} \sum_{\substack{l_1+l_2=l \\ \sum k_i \leqslant l_1}} f N_{k_1} \cdots N_{k_j} Z_m^{l_2} F_1 + \varepsilon(1+t)^{-1} \sum_{\substack{l_1+l_2=l \\ \sum k_i \leqslant l_1}} f N_{k_1} \cdots N_{k_j} Z_m^{l_2} (\sigma_1^{-\mu} a).$$

Since

$$\sigma_1^{1-\mu} |Z_m^k F_1| \leqslant C\varepsilon(1+t)^{-1} |\sigma_1^{1-\mu} N_{k+1}| + C\varepsilon(1+t)^{-1+C\varepsilon},$$
$$\sigma_1^{1-\mu} |Z_m^{l_2} F_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l_2 \leqslant k-1,$$

the first sum is less than

$$C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1} |\sigma_1^{1-\mu} N_{k+1}|.$$

The second sum is less than

$$C\varepsilon(1+t)^{-1} |Z_m^k(\sigma_1^{-\mu} a)| + C\varepsilon(1+t)^{-1+C\varepsilon},$$

and finally

$$\textstyle |\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}(|\sigma_1^{1-\mu} N_{k+1}| + |Z_m^k(\sigma_1^{-\mu} a)|).$$

Just as before, we also get

$$\textstyle |\sum_2| + |\sum_3| \leqslant C(1+t)^{-1-\eta}(1 + |Z_m^k(\sigma_1^{-\mu} a)|),$$

hence

$$|\overline{F}_1^k| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta} + C\varepsilon(1+t)^{-1}\psi_{k+1}$$
$$+ (C\varepsilon(1+t)^{-1} + C(1+t)^{-1-\eta}) |Z_m^k(\sigma_1^{-\mu} a)|$$

and a similar estimate for $\overline{F}_2^k$. Integration along L, we get

$$|Z_m^k(\sigma_1^{-\mu} a)| + |Z_m^k(\sigma_1^{1-\mu} L_1 a)| \leqslant C(1+t)^{C\varepsilon} + C \int_0^t \psi_{k+1} \varepsilon ds/(1+s).$$

d. We have

$$|(1+t)\varepsilon^{-1} \sigma_1^{-\mu} Z_m^{k+1} u| \leqslant C(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} L Z_m^{k+1} u|_{L^\infty} + C(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} L_1 Z_m^{k+1} u|_{L^\infty}.$$

Just as before, using point iii) in Lemma 3.2, we obtain

$$(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} [Z_m^{k+1}, L_1] u| \leqslant (1+t)\varepsilon^{-1} |Lu| \psi_{k+1} + C(1+t)^{C\varepsilon}$$
$$+ C(1+t)\varepsilon^{-1} |L_1 u| |\sigma_1^{1-\mu} Z_m^k L_1 a| + C,$$

hence

$$(1+t)\varepsilon^{-1}|\sigma_1^{-\mu}Z_m^{k+1}u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu}LZ_m^{k+1}u| + C(1+t)^{-\eta}\psi_{k+1} + C|\sigma_1^{1-\mu}Z_m^kL_1a| + C(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu}Z_m^{k+1}L_1u|.$$

Exactly as before, we get

$$(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu}LZ_m^{k+1}u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\psi_{k+1}.$$

From (5.1) we get now

$$(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu}Z_m^{k+1}L_1u| \leqslant C|\sigma_1^{1-\mu}Z_m^{k+1}L_1U| + C\varepsilon^{-1}|\sigma_1^{1-\mu}Z_m^{k+1}u| + C|\sigma_1^{-\mu}Z_m^ka| + C(1+t)^{C\varepsilon}.$$

Using Lemma 4, we obtain

$$|\sigma_1^{1-\mu}Z_m^{k+1}u| \leqslant C, \quad |\sigma_1^{1-\mu}Z_m^kLa| \leqslant C.$$

We also have from Lemma 2

$$|\sigma_1^{-\mu}Z_m^ka| \leqslant |Z_m^k(\sigma_1^{-\mu}a)| + C(1+t)^{C\varepsilon},$$
$$|\sigma_1^{1-\mu}Z_m^kL_1a| \leqslant |Z_m^k(\sigma_1^{1-\mu}L_1a)| + C(1+t)^{C\varepsilon}.$$

Finally,

$$\psi_{k+1} \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\psi_{k+1} + C\int_0^t \psi_{k+1}\varepsilon ds/(1+s),$$

which yields the result by Gronwall Lemma. □

III.7. Back to the standard fields. — In this section, we will transform the estimates on u given in terms of the fields Z_m into estimates given in terms of the standard fields Z_0. Remember that we have fixed $\mu > 1/2$ (μ as close as we want to $1/2$).

Proposition 7. — *We have, for $k \leqslant s_0 - 4$, the estimates*

$$|Z_0^ku| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu},$$
$$|Z_0^k\partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu-1}.$$

Proof

1. First, we need control b_1.

Lemma 7.1. — *We have, for $\alpha \leqslant s_0 - 3$,*

$$|Z_m^\alpha b_1| \leqslant C(1+t)^{C\varepsilon}.$$

Proof. — We use Lemma 3.3 with $w = b_1$, $g = -\overline{\chi}/2cL_1u = f\varepsilon/(1+t)N_0$. We obtain

$$LZ_m^kb_1 = \textstyle\sum_1 + \sum_2 + \sum_3.$$

Since

$$\textstyle\sum_1 = \varepsilon/(1+t)\sum fN_{l_1}\cdots N_{l_i}N_{k_i}\cdots N_{k_j}, \quad \sum l_j + \sum k_l \leqslant k,$$

we obtain $|\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}$.

Exactly as before, using Lemma 5.1, we have

$$|\textstyle\sum_2| \leqslant C(1+t)^{-1-\eta}|Z_m^k b_1| + C\varepsilon(1+t)^{-1+C\varepsilon} \displaystyle\sum_{l\leqslant k-1} |Z_m^l b_1|.$$

For $\sum_3$, we get simply $|\sum_3| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sum_{l\leqslant k-1}|Z_m^l b_1|$.

We already know that $|b_1| \leqslant C(1+t)^{C\varepsilon}$. By induction, assuming already

$$\sum_{l\leqslant k-1} |Z_m^l b_1| \leqslant C(1+t)^{C\varepsilon}$$

we obtain

$$|LZ_m^k b_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta}|Z_m^k b_1|.$$

Integrating yields the desired estimate. □

2. We have $Z_m = Z + aL_1$, but we have only a good control of a/σ_1, not of a. This forces us to display the fact that L_1 is a better field than the Z_m. To motivate some technical definitions which will be given in 3., we present the following attempt to express $\sigma_1 L_1$ in terms of the Z_m. We first write

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0 - S = H_0^m - S^m + 2a(S)L_1.$$

We introduce now a cutoff in the blind zone. For this, set $q = q_0\sigma_1^{-1}\exp C_0\tau$, and define $\chi_1 = \chi_1(q)$, where $\chi_1(s)$ is zero for $s \leqslant 1$ and one for $s \geqslant 2$. We write then

$$\begin{aligned}
\frac{r-t}{\sqrt{c}}L_1 &= H_0^m - S^m + 2\chi_1 a(S)L_1 + \frac{tu}{\sqrt{c}}L_1 + 2a(S)(1-\chi_1)L_1,\\
\sigma_1 DL_1 &= (C_1 - 2a(S)\sqrt{c}\chi_1)L_1 - \sqrt{c}(H_0^m - S^m),\\
D &= (1 - \sigma_1^{-1}tu - 2\sqrt{c}(1-\chi_1)\sigma_1^{-1}a(S)).
\end{aligned}$$

Since

$$|a/\sigma_1| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1},$$

we have

$$|(1-\chi_1)a(S)/\sigma_1| \leqslant Cq_0^{\mu-1}(1+t)^{C\varepsilon - C_0\varepsilon(1-\mu)}.$$

If we choose q_0 and C_0 large enough, we obtain

$$|(1-\chi_1)\sigma_1 a(S)| \leqslant 1/4.$$

Hence, for ε small enough, D^{-1} will be a smooth function of

$$u, \quad tu/\sigma_1, \quad (1-\chi_1)\sigma_1^{-1}a(S).$$

We fix now this choice of q_0, C_0.

3. We have now to develop a calculus analogous to that of Chapter III, and enlarged so as to contain the cutoff in q we have just introduced. We denote by N_0 as before one of the quantities

$$1, \quad \varepsilon^{-1}(1+t)\sigma_1^{-1}u, \quad \varepsilon^{-1}(1+t)Lu, \quad \varepsilon^{-1}(1+t)L_1 u.$$

When we want to emphasize the fact that N_0 is not 1 but actually involves u, we write N_0'. We denote now by N_k, for $k \geqslant 1$, one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}Z_m^k u,\quad \varepsilon^{-1}(1+t)Z_m^k Lu,\quad \varepsilon^{-1}(1+t)Z_m^k L_1 u,$$
$$\sigma_1^{-1}Z_m^{k-1}a,\quad g_0(q)Z_m^{k-1}a,\quad Z_m^{k-1}La,\quad Z_m^{k-1}L_1 a,\quad a\in(\mathrm{Coeff}).$$

Here, g_0 is any smooth function, vanishing for $q \leqslant 1/2$, whose derivative belongs to C_0^∞. This is of course a slight abuse of notation, since the g_0 actually used in the whole computation are generated by χ_1 and finitely many derivatives of χ_1. Hence, for these enlarged N_l, we still have

$$|N_l| \leqslant C(1+t)^{C\varepsilon},\quad l \leqslant s_0-3.$$

In fact,

$$|g_0(q)Z_m^{l-1}a| \leqslant C|q^{-1}g_0(q)|(1+t)^{C_0\varepsilon}|\sigma_1^{-1}a|$$

and $q \geqslant 1/2$ on the support of g_0.

In view of **2.**, we enlarge a little the definition of f. We will denote by f a smooth function of

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ N_0.$$

Here, g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$. Finally, we need to introduce nonlinear analogues to N_1, denoted by ν_1. We define ν_1 as any smooth function of

$$\varepsilon, u, \omega, \sigma_1(1+t)^{-1}, (1+t)^{-\mu}, \sigma_1^{-\mu}, g(q), N_0, (1-\chi_1(q))\sigma_1^{-1}a.$$

In some sense, we see that ν_1 is a generalization of f to order one derivatives. Of course, the quantity D^{-1} from **2** is a ν_1.

4. Some calculus Lemmas

We have to prove that the analogue to Lemma III.2 for the enlarged quantities is correct.

Lemma 2'. — *We have the following identities:*

i) $Z_m^k f = \sum f N_{k_1}\cdots N_{k_j}$, $k_1+\cdots+k_j \leqslant k$,
ii) $Z_m^k N_p = \sum f N_{k_1}\cdots N_{k_j}$, $k_1+\cdots+k_j \leqslant k+p$, *and, for some* i, $k_i \geqslant p$,
iii) $Z_m^k t = t\sum f N_{k_1}\cdots N_{k_j}$, $k_1+\cdots+k_j \leqslant k$,
iv) $Z_m^k \sigma_1 = \sigma_1 \sum f N_{k_1}\cdots N_{k_j}$, $k_1+\cdots+k_j \leqslant k$.
v) $Z_m^k \nu_1 = \sum \nu_1 N_1^l N_{l_1}\cdots N_{l_j}$, $l_i \geqslant 2$, $\sum(l_i-1) \leqslant k$.

Proof

a. We try first i) for $k=1$. With $q = q_0\sigma_1^{-1}\exp C_0\tau$,

$$R_i(q)=0,\quad S(q)=fq,\quad H_0(q)=fq\widetilde{N}_0,\quad L_1(q)=fq\sigma_1^{-1},\quad L(q)=fq\sigma_1^{-1},$$

hence

$$Z_m(q) = Z(q)+aL_1(q) = fqN_0 + fqN_1,$$
$$Z_m(g(q)) = qg'(q)(f+fN_1) = f+fN_1.$$

Finally, $Z_m N_0 = f + fN_1$.

b. From **a.**, we have $Z_m g_0(q) = g_0(q) f N_1$, hence

$$Z_m \widetilde{N}_p = fN_1 N_p + N_{p+1}$$

and ii) is proved for $k = 1$ and any p.

c. iii) and iv) are clear for $k = 1$. Thus, by induction, i)-iv) are proved.

d. To prove v) for $k = 1$, we just have to check the factor $(1 - \chi_1)\sigma_1^{-1} a$:

$$Z_m[(1 - \chi_1)\sigma_1^{-1} a] = fN_1^2 + fN_2.$$

Now, by induction,

$$Z_m^{k+1}\nu_1 = \sum \nu_1 (N_1^{l+2} + N_1^l N_2) N_{l_1} \cdots N_{l_j} + \sum l \nu_1 N_1^{l-1}(fN_2 + fN_1^2) N_{l_1} \cdots N_{l_j} + \sum_{(\sum k_i \leqslant l_i + 1)} \nu_1 N_1^l N_{l_1} \cdots (\sum f N_{k_1} \cdots N_{k_r}) \cdots N_{l_j}.$$

For a term $N_2 N_{l_1} \cdots N_{l_j}$, the sum of indexes is less than or equal to $k + j + 2 = k + 1 + j + 1$, as desired. For a term in the last sum, we note that $Z_m N_p$ contains at least one factor N_q, $q \geqslant p$ if $p \geqslant 2$. Let r' be the number of k_i greater than or equal to two: $1 \leqslant r' \leqslant r$. The sum of indexes corresponding to these terms is less than the sum of all indexes, which is less than or equal to $(\sum l_i) + 1 \leqslant k + j + 1 \leqslant k + 1 + j - 1 + r'$ as desired. □

We define, for $k \geqslant 1$,

$$M_k = \nu_1 N_1^l N_{l_1} \cdots N_{l_j}, \quad l \geqslant 0, \quad l_i \geqslant 2, \quad \sum (l_i - 1) \leqslant k - 1.$$

This definition is justified by Lemma 2', v). Remark that

$$M_1 = \nu_1 N_1^l, \quad M_1 M_k = M_k, \quad M_k M_l = M_{k+l-1},$$

and

$$\sum_{(\sum k_i \leqslant k)} \nu_1 N_{k_1} \cdots N_{k_j} = M_k, \quad Z_m M_k = \sum M_{k+1}, \quad Z_m^p M_k = \sum M_{k+p}.$$

5. We are now ready to prove Proposition 7. Denote by $\overline{Z}_0$ the fields

$$R_i, \quad S, \quad h_0 = t\partial_r + r\partial_t, \quad \partial_t.$$

Lemma 7.5. — *We have*

$$\overline{Z}_0^k = \sum M_q a_1^{-l} (Z_m^{r_1} b_1) \cdots (Z_m^{r_i} b_1) Z_m^p,$$

with

$$p \geqslant 1, \quad 0 \leqslant l \leqslant k, \quad r_i \geqslant 1, \quad q - 1 + \sum r_j + p \leqslant k.$$

Proof

a. Consider first $k = 1$. We write, according to **2.**,

$$\sigma_1 L_1 = \nu_1((f + fN_1)L_1 + fZ_m) = M_1 Z_m + M_1 a_1^{-1} Z_m.$$

Then

$$R_i = R_i^m - \sigma_1^{-1} a(R_i)(M_1 Z_m + M_1 a_1^{-1} Z_m),$$

and similarly for S and H_0. Then

$$H_0 = ct\partial_r + r/c\partial_t = h_0 + ftuL + ftuL_1,$$
$$h_0 = H_0 + f(H_0^m + S^m) + M_1 Z_m + M_1 a_1^{-1} Z_m.$$

Finally,

$$\partial_t = 2\sqrt{c}(L + L_1) = fZ_m + fa_1^{-1} Z_m.$$

b. Now

$$\overline{Z}_0^{k+1} = (M_1 + M_1 a_1^{-1}) Z_m(\overline{Z}_0^k),$$

and the formula follows at once by induction, since

$$M_1 M_q = M_q, \quad Z_m M_l = M_{l+1}, \quad Z_m a_1^{-l} = -l a_1^{-l} Z_m b_1. \qquad \square$$

From this Lemma, we get, for $l \leqslant s_0 - 3$,

$$|\varepsilon^{-1}(1+t)\sigma_1^{-1}\overline{Z}_0^l u| + |\varepsilon^{-1}(1+t)\overline{Z}_0^l L u| + |\varepsilon^{-1}(1+t)\overline{Z}_0^l L_1 u| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

For $l \leqslant s_0 - 4$, we can in fact enlarge this estimate to have also

$$|\varepsilon^{-1}(1+t)\overline{Z}_0^l \partial u| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

To prove this, we write

$$\partial_t = 2\sqrt{c}(L + L_1), \quad \partial_i = (\omega_i/2\sqrt{c})(L - L_1) - 1/r(\omega \wedge R)_i, \quad R_i = \overline{Z}_0.$$

From the weak control

$$|\overline{Z}_0^l u| \leqslant C(1+t)^{C\varepsilon}$$

already proved, we get

$$|\overline{Z}_0^l f(\omega, u)| + |r\overline{Z}_0^l(\omega/r)| \leqslant C(1+t)^{C\varepsilon}.$$

6. Finally, we want to replace, in the above formula, the fields $\overline{Z}_0$ by Z_0. But all fields Z_0 can be expressed in terms of $\overline{Z}_0$. In fact, R_i, S and ∂_t are already $\overline{Z}_0$, and

$$h_i = t\partial_i + x_i\partial t = \omega_i h_0 - t/r(\omega \wedge R)i,$$

$$\partial_i = \omega_i(-\partial_t + (r+t)^{-1}(h_0 + S)) - 1/r(\omega \wedge R)_i.$$

Thus

$$Z_0 = \sum f(\omega, (1+t)^{-1}, r(1+t)^{-1})\overline{Z}_0.$$

This implies that we have the desired estimates of Proposition 7. $\square$

IV. A calculus of modified Klainerman's vector fields

IV.1. Definitions and L^∞ estimates of the perturbation coefficients. — In the previous chapter III, we have already used modified fields

$$Z_m = Z + aL_1$$

where the a have been defined by III.1.1. Our final result in Chapter III was the estimates, for $k \leqslant s_0 - 4$,

$$|Z_0^k u| \leqslant C_1\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu},$$
$$|Z_0^k \partial u| \leqslant C_1\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu-1}.$$

For aesthetic as well as technical reasons, we will start again from scratch and define new, and better supported coefficients a, by the formula

$$\begin{aligned} La(R_i) + \chi(q)a(R_i)(L_1u/2c) &= -\chi(q)R_iu/2c, \\ La(S) + \chi(q)a(S)(L_1u/2c) &= -\chi(q)Su/2c, \\ a(H_0) = -a(S),\ a(R_i)(0,t) &= 0,\ a(R_i)(x,0) = 0, \\ a(S)(0,t) &= 0,\ a(S)(x,0) = 0. \end{aligned} \tag{1.1}$$

Here $q = q_0\sigma_1^{-1}\exp C_0\tau$, where q_0 is taken to be

$$q_0 = 1/2\exp(-C_0\varepsilon\log 2)$$

in such a way that the boundary of the support of $\chi(q)$ intersects $r = t + M$ at $t = 1$. The big constant C_0 is still to be determined. The function $\chi(s)$ is a real C^∞ function being zero for $s \leqslant 1/2$ and one for $s \geqslant 1$. The aesthetic reason is to perturb as little as possible the standard (adapted) fields Z. It turns out that it is enough to take perturbation coefficients a supported in a logarithmic zone $\sigma_1 \leqslant C(1+t)^{C\varepsilon}$. The technical reason will appear in the proof of Proposition VII.1, where powers of σ_1 on support of a have to be bounded by factors $(1+t)^{\gamma_i\varepsilon}$ for appropriate γ_i.

Proposition 1. — *The coefficients $a(R_i)$ and $a(S)$ defined by (1.1) are zero for t small, for $r \geqslant M + t$ or $q \leqslant 1/2$. Moreover, we can choose C_0 such that, for $k \leqslant s_0 - 5$, we have*

$$|\sigma_1^{-1}Z_0^k a| + |Z_0^k\partial a| \leqslant C(1+t)^{C\varepsilon}.$$

Proof

a. To prove the claim about the supports, we have to check that the domain left to the curve

$$\sigma_1 - 2q_0\exp C_0\tau = 0$$

is an influence domain of the t-axis (where a is zero) for L. But, on this curve,

$$\begin{aligned} L(\sigma_1 - 2q_0\exp C_0\tau) &= L\sigma_1 - (q_0/\sqrt{c})(\exp C_0\tau)C_0\varepsilon/(1+t) \\ &= -\sigma_1\varepsilon/(\sqrt{c}(1+t))((1+t)u/(\varepsilon\sigma_1) + C_0). \end{aligned}$$

If C_0 is big enough, this is negative, proving the claim.

b. To estimate a and its derivatives, we will use the same method as in Chapter III, except that we already know estimates on u. Exactly as in III.1, we define

$$R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1.$$

We forget about K now, and take the family Φ' of the fields Z_m as the collection of the fields

$$R_i^m, S^m, \quad H_0^m, L_1.$$

We will write f to denote a real C^∞ function of the variables

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ N_0,$$

where

$$N_0 = 1,\ \varepsilon^{-1}(1+t)\sigma_1^{-1}u,\ \varepsilon^{-1}(1+t)Lu,\ \varepsilon^{-1}(1+t)L_1u,$$

and g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$. We denote by N_k, $k \geqslant 1$ one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}Z_m^k u,\ \varepsilon^{-1}(1+t)Z_m^k Lu,\ \varepsilon^{-1}(1+t)Z_m^k L_1 u,$$
$$\sigma_1^{-1}Z_m^{k-1}a,\ Z_m^{k-1}La,\ Z_m^{k-1}L_1a.$$

This machinery is the same as in III, except that we have enlarged f with $g(q)$ and N_0. As we can see from the proof of Lemma 2' in section III.7, all the calculus and commutation Lemmas of III (that is, Lemma 2, Lemma 3.1, 3.2, 3.3 and Lemma 4) remain valid with these new definitions. We will refer to these calculus lemmas just as Lemma 2, Lemma 3.1, etc. The only difference in the commutation relations is that

$$[L_1, L] = -L_1u/(2c)L_1 + Lu/(2c)L,$$

which means that, in Lemma 3.1, 3.2 or 3.3, we have either $d = (1-\chi(q))Z_m u$ or $d = L_1 u$.

c. We will need the following correspondence between the fields Z_m and the standard fields Z_0.

Lemma 1.1. — *We have*

$$Z_m^k = \sum f N_{k_1}\cdots N_{k_j} Z_0^p + \sum f N_{l_1}\cdots N_{l_i} Z_m^{r_1}(a/\sigma_1)\cdots Z_m^{r_q}(a/\sigma_1)Z_0^p.$$

In the first sum, $p \geqslant 1$, $\sum k_i + p \leqslant k$. In the second sum, $p \geqslant 1$, $q \leqslant k$ and $\sum l_j + \sum r_i + p \leqslant k$.

Proof. — We have $\partial = f\sigma_1 Z_0$. For $k=1$, we write

$$L_1 = f\partial_i + f\partial_t = f\sigma_1 Z_0,$$
$$R_i^m = R_i + aL_1 = Z_0 + fa/\sigma_1 Z_0, \quad S^m = S + fa/\sigma_1 Z_0.$$

Then

$$H_0 = t\partial_r + r\partial_t + f(1+t)u\partial = \sum \omega_i(t\partial_i + x_i\partial_t) + ftu/\sigma_1 Z_0 = fZ_0,$$
$$H_0^m = fZ_0 + fa/\sigma_1 Z_0,$$

which proves the claim. For $k \geqslant 2$, we write $Z_m^{k+1} = Z_m Z_m^k$, and the Lemma follows by induction, since a term

$$Z_m Z_0^p = f Z_0^{p+1} + f a/\sigma_1 Z_0^{p+1}$$

adds one term in each sum. □

d. The following Lemma will be crucial in the whole construction.

Lemma 1.2. — *Assume $1/2 < \mu < 2/3$. Then we have*

$$|a_t| + |a_r| + |a/\sigma_1^\mu| \leqslant C_3(1+t)^{C_3\varepsilon},$$

where C_3 depends only on C_1 and u and not on C_0. Moreover, if C_0 is big enough, we have, on the support of $1-\chi$, the estimates

$$|a/\sigma_1| \leqslant C, \quad |Z_m u| \leqslant C|Z_0 u|, \quad |Z_m \partial u| \leqslant C|Z_0 \partial u|,$$

$$|Z_0 u/\sigma_1|(1 + |\partial a| + \varepsilon^{-1}(1+t)(|Z_0 \partial u| + |Z_0 u/\sigma_1|) \leqslant C\varepsilon(1+t)^{-1-\varepsilon}.$$

Proof

a. In fact, with $b = a/\sigma_1^\mu$,

$$Lb = -\mu b(L\sigma_1/\sigma_1) - \chi b L_1 u/(2c) - \chi/(2c)\sigma_1^{-\mu} Z_0 u, \quad L\sigma_1 = -u/\sqrt{c},$$

hence

$$|Lb| \leqslant C_2\varepsilon(1+t)^{-1} b + C_2\varepsilon(1+t)^{-1+C_1\varepsilon}.$$

By integration, we get

$$|b| \leqslant (C_2/C_1)(1+t)^{(C_1+C_2)\varepsilon}.$$

We have now

$$L_1 La = -L_1(\chi/(2c))(aL_1 u + Z_0 u) - \chi/(2c)(L_1 a L_1 u + f\partial Z_0 u + f(a/\sigma_1) Z_0 L_1 u).$$

But, since $L_1 q = f q \sigma_1^{-1}$,

$$L_1(\chi/(2c)) = 1/(2c)\chi'(q) f q \sigma_1^{-1} - \chi/(2c^2) L_1 u = f/\sigma_1,$$

$$L_1 La = f(a/\sigma_1) L_1 u + f Z_0 u/\sigma_1 + f\varepsilon(1+t)^{-1} L_1 a + f\partial Z_0 u + f(a/\sigma_1) Z_0 L_1 u.$$

$$LL_1 a = f\varepsilon(1+t)^{-1} L_1 a + f(Z_0 L_1 u)(a/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) \\ + f Z_0 u/\sigma_1 + f\partial Z_0 u + f(Z_0 u/\sigma_1)(a/\sigma_1) = g_1.$$

We deduce that

$$|LL_1 a| \leqslant C\varepsilon/(1+t)|L_1 a| + C\varepsilon/(1+t)(1+t)^{C\varepsilon},$$

where again C does not depend on C_0. Since La is bounded independently of C_0, we get by integration the first part of the Lemma.

b. From **a.**, we get $|a/\sigma_1| \leqslant C$ as soon as $C_0(1-\mu) \geqslant C_3$. Then, for any v,

$$|Z_m v| \leqslant C|Z_0 v| + C|a/\sigma_1||Z_0 v| \leqslant C|Z_0 v|.$$

Since

$$|Z_0 u(a/\sigma_1)^2| \leqslant C\varepsilon(1+t)^{-1+C_1\varepsilon+2C_3\varepsilon}\sigma_1^{3\mu-2},$$

we obtain on the support of $1-\chi$

$$|Z_0 u/\sigma_1| + |Z_0 u(a/\sigma_1)^2| \leqslant C\varepsilon(1+t)^{-1+C_4\varepsilon}(1+t)^{-(2-3\mu)C_0\varepsilon},$$

where C_4 does not depend on C_0. This completes the proof. □

e. From now on we assume that C_0 has been fixed big enough for the estimates of Lemma 1.2 to hold. We now assume by induction

$$|N_l| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k,$$

which is true for $k=0$. In particular, in view of Lemma 2,

$$|Z_m^l(a/\sigma_1)| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k-1.$$

Using Lemma 1.1 for the index k, we obtain

$$|Z_m^k \partial u| + |\sigma_1^{-1} Z_m^k Z_0 u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

We write now

$$L(a/\sigma_1) = -\chi/(2c)(a/\sigma_1)L_1 u - \chi/(2c)(Z_0 u/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) = g.$$

Applying Lemma 3.3 for the index k with $w = a/\sigma_1$, we get

$$LZ_m^k(a/\sigma_1) = \textstyle\sum_1 + \sum_2 + \sum_3.$$

Since

$$g = f\varepsilon(1+t)^{-1}(a/\sigma_1) + f(Z_0 u/\sigma_1),$$

we have

$$|Z_m^l g| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l \leqslant k-1,$$
$$|Z_m^k g| \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1)|.$$

Hence

$$\textstyle|\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1)|.$$

In $\sum_3$, all terms are controlled by induction, and $|\sum_3| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}$. In $\sum_2$, if $k_{j+1} \leqslant k-2$, we just write $L_1 Z_m^{k_{j+1}} w = Z_m^{k_{j+1}+1} w$ and the term in controlled by induction. If $k_{j+1} = k-1$, the corresponding term is just $fdL_1 Z_m^{k-1} w$. If $d = L_1 u$, we remember $L_1 = Z_m$ and keep the term as it is. If $d = (1-\chi)Z_m u$, we need to use that L_1 is a better field than the Z_m. We write as in **2**, Proposition III.7,

$$r - t/\sqrt{c}L_1 = H_0^m - S^m + 2aL_1 + tu/\sqrt{c}L_1,$$
$$\sigma_1 L_1 = fZ_m + faL_1 + fL_1.$$

Iterating this, we obtain

$$\sigma_1 L_1 = fZ_m + f(a/\sigma_1)Z_m + fa^2/\sigma_1 L_1.$$

Using the corresponding inequality to estimate the term at hand, we get

$$|fdL_1Z_m^{k-1}w| \leqslant C|d|/\sigma_1[(C+C|a/\sigma_1|)|Z_m^k w| + Ca^2/\sigma_1|L_1Z_m^{k-1}w|].$$

From Lemma 1.2, we obtain finally in all cases,

$$|\sum\nolimits_2| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1|.$$

Integrating the equation on $Z_m^k(a/\sigma_1)$, we obtain

$$|Z_m^k(a/\sigma_1)| \leqslant C(1+t)^{C\varepsilon}, \quad \sigma_1^{-1}|Z_m^k a| \leqslant C(1+t)^{C\varepsilon}.$$

f. Since

$$|Z_0pLu| + |Z_0^pL_1u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad p \leqslant k+1,$$

we obtain, using now Lemma 1.1 with the index $k+1$, applied to u, Lu or L_1u,

$$|\sigma_1^{-1}Z_m^{k+1}u| + |Z_m^{k+1}Lu| + |Z_m^{k+1}L_1u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

Similarly, since $La = fZ_0u + f(a/\sigma_1)Z_0u$, we obtain directly $|Z_m^kLa| \leqslant C$.

g. Remember that

$$LL_1a = f\varepsilon(1+t)^{-1}L_1a + f(Z_0L_1u)(a/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) + fZ_0u/\sigma_1 + f\partial Z_0u + f(Z_0u/\sigma_1)(a/\sigma_1) = g_1.$$

Applying Lemma 3.3 for the index k and $w = L_1a$, we obtain

$$LZ_m^kL_1a = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .$$

As before, we get first

$$|Z_m^l g_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l \leqslant k-1,$$
$$|Z_m^k g_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^kL_1a|,$$

which gives

$$|\sum\nolimits_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^kL_1a|.$$

The analysis of $\sum_2$ and $\sum_3$ are strictly identical to the ones we have done for controlling $Z_m^k(a/\sigma_1)$. Finally

$$|LZ_m^kL_1a| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^kl_1a|,$$

which gives by integration the desired estimate, and proves that $|N_{k+1}| \leqslant C(1+t)^{C\varepsilon}$.

h. It remains now to translate this result using the standard fields Z_0. As in **5**, Proposition III.7, we denote by $\overline{Z}_0$ the fields

$$R_i,\ S,\ h_0 = r\partial_t + t\partial_r,\ \partial_t.$$

Lemma 1.3. — *We have*

$$\overline{Z}_0^k = \sum fN_{k_1}\cdots N_{k_i}(Z_m^{r_1}a)\cdots(Z_m^{r_j}a)Z_m^p,$$

with

$$p \geqslant 1, \quad j \leqslant k, \quad \sum k_j + \sum r_i + p \leqslant k.$$

Proof. — The argument is not the same as in Lemma 7.5, since we have defined no ν_1 here. We have

$$R_i = R_i^m - aZ_m, \quad S = S^m - aZ_m, \quad H_0 = H_0^m + aZ_m,$$

$$H_0 = h_0 + fZ_m + f(1+t)uL_1.$$

Remembering that

$$\sigma_1 L_1 = fZ_m + fL_1 + faL_1,$$

we get

$$f(1+t)uL_1 = f((1+t)u/\sigma_1)(fZ_m + fL_1 + faL_1) = fZ_m + faZ_m.$$

This proves the Lemma for $k = 1$, and the general case follows by induction. □

Now, since support a is contained in $q \geqslant 1/2$,

$$|Z_m^r a| \leqslant |\sigma_1^{-1} Z_m^r a| \sigma_1 \leqslant C(1+t)^{C\varepsilon},$$

hence this Lemma, applied to a, La or $L_1 a$, yields

$$|\sigma_1^{-1} \overline{Z}_0^l a| + |\overline{Z}_0^l La| + |\overline{Z}_0^l L_1 a| \leqslant C(1+t)^{C\varepsilon}.$$

The transition from $\overline{Z}_0$ to Z_0 is now identical with 7.5 c, and this completes the proof. □

IV.2. Smoothing operators

IV.2.1. Smoothing operators on the sphere. — We will need, in the spirit of the paradifferential calculus of J.M. Bony [**6**], smoothing operators S_θ^2 acting on functions on the unit sphere S^2. To define these S_θ^2, we will fix

$$\phi_2 \in C_0^\infty(\mathbf{R}^2), \quad 0 \leqslant \phi_2 \leqslant 1, \quad \int \phi_2 = 1,$$

and a partition of unity on S^2

$$\chi_+ + \chi_- = 1,$$

where $\chi_\pm$ is one for $\pm x_3 \geqslant 0$ and vanishes near the pole $(0, 0, -\pm 1)$. For w defined on the sphere, we set

$$S_\theta^2 w = \sum_{(+,-)} (\phi_{2,\theta} * [(\chi_\pm w)(p_{-\pm}^{-1})])(p_{-\pm}),$$

where $p_\pm$ are the stereographic projections from the poles $(0, 0, \pm 1)$, and

$$\phi_{2,\theta}(y) = \theta^2 \phi_2(\theta y).$$

The operators S^2_θ enjoy the usual properties

$$(2.1.1)_a \qquad ||S^2_\theta w|| \leqslant C||w||,$$

$$(2.1.1)_b \qquad ||S^2_\theta w - w|| \leqslant \theta^{-k} \sum_{l\leqslant k} ||R^l w||,$$

$$(2.1.1)_c \qquad ||R^k S^2_\theta w|| \leqslant C\theta^k ||w||.$$

Here, $||.||$ stands for the L^2 or L^∞ norm on the sphere, and

$$R^k = R_{i_1} \cdots R_{i_k}.$$

When computing with the S_θ, we think of them as if they were only the convolution with $\phi_{2,\theta}$, omitting for simplicity the cutoff functions etc. Note that if we abandon the property $\int \phi_2 = 1$, properties $(2.1.1)_a$ and $(2.1.1)_c$ remain.

IV.2.2. Smoothing operators. — We choose now

$$\phi_1 \in C_0^\infty(\mathbf{R}), \quad 0 \leqslant \phi_1 \leqslant 1, \quad \int \phi_1 = 1, \quad \operatorname{supp} \phi_1 \subset \{r \leqslant 0\},$$

and set

$$S^1_\theta w(r,\omega,t) = \int \theta\phi_1(\theta(r-r'))w(r',\omega,t)dr'.$$

This is the standard smoothing operator in the r-variable. We will use it only in a fixed domain on the form

$$r \geqslant \gamma_1(1+t), \quad \gamma_1 > 0,$$

acting on functions supported in $r \leqslant M + t$. With to different (big) parameters θ_1 and θ_2 to be chosen later, and $\theta = (\theta_1, \theta_2)$, we define finally

$$S_\theta w(r,\omega,t) = S^1_{\theta_1} S^2_{\theta_2} w.$$

It is clear that, for some C (independent of t) we have the inequality

$$|S_\theta w(.,t)|_{L^2_x} \leqslant C|w(.,t)|_{L^2_x}.$$

This inequality holds also if the integrals of the ϕ_i are not normalized to be one, in which case, to avoid confusion, we denote the corresponding operators by s_θ.

Computing commutators of S_θ with various fields, we will also need operators

$$s^{k,l}_\theta[p;q]w \equiv s_\theta[p;q]w, \quad p = (p_1,\dots,p_k), \ q = (q_1,\dots,q_l)$$

defined by

$$\begin{aligned} s_\theta[p;q]w = \Big\{ \int & \theta_1^{1+k}\phi_1(\theta_1(r-r'))\theta_2^{2+l}\phi_2(\theta_2(y-y')) \\ & [p_1(r,p_+^{-1}(y),t) - p_1(r',p_+^{-1}(y),t)] \cdots [p_k(r,p_+^{-1}(y),t) - p_k(r',p_+^{-1}(y),t)] \\ & [q_1(r',p_+^{-1}(y),t) - q_1(r',p_+^{-1}(y'),t)] \cdots [q_l(r',p_+^{-1}(y),t) - q_l(r',p_+^{-1}(y'),t)] \\ & \chi(y')w(r',y',t)dr'dy' \Big\}(p_+), \end{aligned}$$

or similar integral involving p_-. Here, ϕ_1 and ϕ_2 need not have integral one, and χ is an arbitrary function in $C_0^\infty(\mathbf{R}^2)$. Note that $s_\theta[p;q]$ is automatically normalized to take into account the effects of the factors

$$p_i(r) - p_i(r'), \quad q_j(y) - q_j(y').$$

The continuity of these operators is given in the following Lemma.

Lemma 2.2. — *We have (uniformly in t)*

i) $|s_\theta[p;q]w|_{L^\infty} \leqslant C|w|_{L^\infty}\Pi|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
ii) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^2}\Pi|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
iii) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^\infty}|\partial_r p_1|_{L^2}\Pi_{i\geqslant 2}|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
iv) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^\infty}|Rq_1|_{L^2}\Pi|\partial_r p_i|_{L^\infty}\Pi_{i\geqslant 2}|Rq_i|_{L^\infty}$.

Proof. — The first two points are obvious. To prove iii) or iv), it is enough to consider, for instance, an integral

$$\int \theta^3\phi_2(\theta(y-y'))|b_1(y) - b_1(y')|dy'.$$

Since $|b_1(y) - b_1(y')| \leqslant |y-y'| \int_0^1 |\partial b_1|(y' + s(y-y'))ds$,

$$\Big|\int \theta^3\phi_2(\theta(y-y'))(b_1(y) - b_1(y'))dy'\Big|_{L^2} \leqslant C\int_0^1 ds \int \theta^2\psi(\theta z)|\partial b_1|^2(y'+sz)dy'dz \leqslant C|\partial b_1|_{L^2}^2,$$

which gives the result. □

IV.3. Modified Klainerman's fields

a. We define now fields $\widetilde{Z}_m$, analogous to the fields Z_m used in chapter III, but with two important differences:

i) $\widetilde{Z}_m$ has to have smooth coefficients everywhere and not only outside $r=0$.

ii) The perturbation coefficients a from $Z_m = Z + aL_1$ have to be smoothed by S_θ, so as to bear extra derivatives (as occurs typically in a Nash-Moser scheme, see [**5**] for instance).

From now on, we fix, for some

$$\beta_1 > 0,\ \beta_2 > 0,\ \beta_2 \geqslant \beta_1,\ \theta_2^0 \geqslant \theta_1^0 \geqslant 1,\ \varepsilon\beta_2 \leqslant 1$$

to be chosen later,

$$\theta_i = \theta_i(t) = \theta_i^0(1+t)^{\beta_i\varepsilon}.$$

The coefficients $a(R_i), a(S)$ have already been defined in IV. 1. We define $a = a(H_i)$ by

$$a(H_i) = -\omega_i a(S) - (\omega\wedge a(R))_i.$$

We set now (recalling $H_i = ct\partial_i + x_i/c\partial_t$)

$$(3.1)_a \qquad \widetilde{R}_i^m = R_i + \widetilde{a}(R_i)L_1,$$

$$(3.1)_b \qquad \widetilde{S}^m = S + \widetilde{a}(S)L_1,$$

$$(3.1)_c \qquad \widetilde{H}_i^m = H_i + \widetilde{a}(H_i)L_1,$$

$$(3.1)_d \qquad \widetilde{K} = L_1 + L = (2/\sqrt{c})\partial_t.$$

Here

$$\widetilde{a}(R_i) = S_\theta a(R_i), \widetilde{a}(S) = S_\theta a(S),$$

and, for technical reasons,

$$(3.2) \qquad \widetilde{a}(H_i) = -\omega_i \widetilde{a}(S) - (\omega \wedge \widetilde{a}(R))_i.$$

We do not use H_0 since it does not satisfy i). Remark also that

$$(3.3) \qquad \sum \omega_i a(H_i) = -a(S), \quad \sum \omega_i \widetilde{a}(H_i) = -\widetilde{a}(S).$$

Thanks to these choices, we get

$$\frac{r+ct}{\sqrt{c}} L = \sum \omega_i H_i + S = \sum \omega_i \widetilde{H}_i^m + \widetilde{S}^m.$$

The set of the coefficients

$$a(R_i),\ a(S),\ a(H_i)$$

will be denoted by (Coeff), while the set of

$$\widetilde{a}(R_i),\ \widetilde{a}(S),\ \widetilde{a}(H_i)$$

will be denoted by $\widetilde{(\mathrm{Coeff})}$. We will denote by $\widetilde{\Phi}$ the collection of the fields

$$\widetilde{R}_i^m,\ \widetilde{S}^m,\ \widetilde{H}_i^m,\ \widetilde{K},$$

and call $\widetilde{Z}_m$ any of them. Except for $\widetilde{K}$, we will write simply

$$\widetilde{Z}_m = Z + \widetilde{a}L_1,$$

where Z means one of R_i, S, H_i.

b. We denote by $\widetilde{N}_0$ one of the quantities

$$1,\ \varepsilon^{-1}(1+t)\sigma_1^{-1}u,\ \varepsilon^{-1}(1+t)\partial u.$$

Remark that $|\widetilde{N}_0| \leqslant C$. When we want to emphasize the fact that $\widetilde{N}_0$ is not 1 but actually involves u, we write $\widetilde{N}_0'$. We denote by $\widetilde{N}_k$, for $k \geqslant 1$, one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}\widetilde{Z}_m^k u, \quad \varepsilon^{-1}(1+t)\widetilde{Z}_m^k \partial u,$$

$$\sigma_1^{-1}\widetilde{Z}_m^{k-1}\widetilde{a}, \quad \widetilde{Z}_m^{k-1}\widetilde{a}, \quad \widetilde{Z}_m^{k-1}\partial\widetilde{a}, \quad \widetilde{a} \in \widetilde{(\mathrm{Coeff})}.$$

As before, we enlarge a little the definition of f. We will denote by f a smooth function of

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ \widetilde{N}_0,\ \nu_i > 0.$$

Here, g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$.

c. We can now express L_1 in terms of the $\widetilde{Z}_m$.

Lemma 3.1. — *We have the relations*

i) $L_1 = f\widetilde{Z}_m$, $L = f/(1+t)\widetilde{Z}_m$,

ii) $Z = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$,

iii) $\sigma_1 L_1 = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$,

iv) $\sigma_1\partial_t = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$, $\sigma_1\partial_i = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$.

Proof

a. From the definition of $\widetilde{K}$, $L_1 = \widetilde{K} - L$. But $L = f\widetilde{Z}_m$, hence i). Writing $Z = \widetilde{Z}_m - \widetilde{a}L_1$ and using i), we get ii).

b. Once again

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0 - S = \sum \omega_i \widetilde{H}_i^m - \widetilde{S}^m + 2\widetilde{a}(S)L_1.$$

As before, we deduce from this $\sigma_1 L_1 = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$, which is iii).

Finally,

$$\partial_i = f/(1+t)R + fL + fL_1 = \sigma_1^{-1}f(\widetilde{Z}_m - \widetilde{a}L_1) + f/(1+t)\widetilde{Z}_m + f\sigma_1^{-1}(\sigma_1 L_1),$$

which gives iv). □

IV.4. Some calculus Lemmas for the modified fields. — We have to prove the analogue to Lemma III.2.

Lemma 4.1. — *We have the following identities:*

i) $\widetilde{Z}_m^k f = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,

ii) $\widetilde{Z}_m^k\widetilde{N}_p = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1 + \cdots + k_j \leqslant k+p$, *and, for some* i, $k_i \geqslant p$,

iii) $\widetilde{Z}_m^k t = t\sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,

iv) $\widetilde{Z}_m^k \sigma_1 = \sigma_1\sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1 + \cdots + k_j \leqslant k$.

Proof

a. We try first i) for $k=1$. For the variables

$$\varepsilon,\ u,\ \omega,\ \sigma_1/(1+t),\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i}$$

in f, we only have to check the action of H_i and L. But, analogously to H_0,

$$H_i\omega = f, \quad H_i t = ft, \quad H_i\sigma_1 = \sigma_1 f\widetilde{N}_0, \quad H_i(\sigma_1/(1+t)) = f\widetilde{N}_0$$

and the action of L is at least as good as that of L_1.

Now, with $q = q_0\sigma_1^{-1}\exp C_0\tau$,

$$R_i(q) = 0,\ S(q) = fq,\ H_i(q) = fq\widetilde{N}_0,\ L_1(q) = fq\sigma_1^{-1},\ L(q) = fq\sigma_1^{-1},$$

hence

$$\widetilde{Z}_m(q) = Z(q) + \widetilde{a}L_1(q) = fq\widetilde{N}_0 + fq\widetilde{N}_1,$$

$$\widetilde{Z}_m(g(q)) = qg'(q)f\widetilde{N}_1 = f\widetilde{N}_1.$$

Finally, $\widetilde{Z}_m\widetilde{N}_0 = f + f\widetilde{N}_1$.

b. We have

$$\widetilde{Z}_m\widetilde{N}_p = f\widetilde{N}_1\widetilde{N}_p + \widetilde{N}_{p+1}$$

and ii) is proved for $k = 1$ and any p. **c.** iii) and iv) are clear for $k = 1$. Thus, by induction, i)-iv) are proved. □

We define, for $k \geqslant 1$,

$$M_k = f\widetilde{N}_1^l\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_j}, \quad l \geqslant 0, \quad l_i \geqslant 2, \quad \sum(l_i - 1) \leqslant k - 1.$$

Remark that

$$M_1 = f\widetilde{N}_1^l, \quad M_1M_k = M_k, \quad M_kM_l = M_{k+l-1},$$

and

$$\sum_{(\sum k_i \leqslant k)} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j} = M_k.$$

As in **4** of Proposition III.7, we get easily

$$\widetilde{Z}_mM_k = \sum M_{k+1}, \widetilde{Z}_m^pM_k = \sum M_{k+p}.$$

We will state here for further reference the following commutation Lemmas.

Lemma 4.2. — *We have the formula*

i) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p\partial,$$

ii) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\partial\widetilde{Z}_m^p.$$

In both sums, we have $p \leqslant k - 1$, $\sum k_j + p \leqslant k$.

iii) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p\partial + \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}(\widetilde{Z}_m^{r_1}A)\cdots(\widetilde{Z}_m^{r_q}A)\widetilde{Z}_m^p\partial.$$

Here, $A = \partial\widetilde{a}$ *or* $A = \sigma_1^{-1}\widetilde{a}$. *In the first sum, we have* $\sum k_j + p \leqslant k - 1$. *In the second sum, we have* $q \geqslant 1$, $\sum k_j + \sum(r_i + 1) + p \leqslant k$.

Proof

a. Consider first $k = 1$. While $[R_i, \partial]$, $[S, \partial]$ are just ∂ multiplied by constants, we have

$$[H_i, \partial_j] = -\partial_j(ct)\partial_i - \partial_j(x_i/c)\partial_t = f\partial, [\widetilde{K}, \partial] = f\partial.$$

Now

$$[L_1, \partial] = f\partial u\partial + f(1 + t)^{-1}\partial = f(1 + t)^{-1}\partial.$$

Hence

$$[\widetilde{a}L_1, \partial] = f(1 + t)^{-1}\widetilde{a}\partial + f\partial\widetilde{a}\partial = fA\partial.$$

This proves the Lemma for $k = 1$.

b. We write now

$$[\widetilde{Z}_m^{k+1}, \partial] = \widetilde{Z}_m[\widetilde{Z}_m^k, \partial] + [\widetilde{Z}_m, \partial]\widetilde{Z}_m^k,$$

and consider formula i). We see that the first term yields obviously terms of the desired form, while the second is

$$(f + f\widetilde{N}_1)([\partial, \widetilde{Z}_m^k] + \widetilde{Z}_m^k \partial).$$

This proves i). To prove ii), we write instead

$$\widetilde{Z}_m \partial \widetilde{Z}_m^p = [\widetilde{Z}_m, \partial] \widetilde{Z}_m^p + \partial \widetilde{Z}_m^{p+1}.$$

To prove iii), we see that $\widetilde{Z}_m([\widetilde{Z}_m^k, \partial])$ yields automatically good terms. For $[\widetilde{Z}_m, \partial]\widetilde{Z}_m^k$, we write this term as

$$(f + fA)([\partial, \widetilde{Z}_m^k] + \widetilde{Z}_m^k \partial),$$

which yields only terms of the desired form. □

Lemma 4.3. — *We have*

i) $$[\widetilde{Z}_m, R_j/r] = M_1/(1+t)\widetilde{Z}_m + M_1\sigma_1/(1+t)\partial + f/(1+t)(R_j\widetilde{a})\partial,$$

ii) $$[\widetilde{Z}_m^k, R_j/r] = (1+t)^{-1} M_l \widetilde{Z}_m^{p_1+1} + \sigma_1 (1+t)^{-1} M_l \partial \widetilde{Z}_m^{p_1} + \theta_2 (1+t)^{-1} M_l (\widetilde{Z}_m^{p_1} s_\theta a) \partial \widetilde{Z}_m^{p_2}.$$

In all terms of formula ii), we have $l - 1 + \sum p_i \leqslant k - 1$.

Proof

a. We have

$$[R_i, R_j] = -\varepsilon_{ijk} R_k, [S, R_j] = 0, [h_i, R_j] = -\varepsilon_{ijk} h_k.$$

Now

$$H_i = h_i + tu\partial_i - x_i u/c\partial_t, \quad [H_i, R_j] = -\varepsilon_{ijk} h_k + ftu\partial + ftRu\partial.$$

But

$$h_k = H_k - tu\partial_k + x_k u/c\partial_t = fR + \omega_k H_0 + ftu\partial, H_0 = f\sigma_1\partial + f\widetilde{Z}_m,$$

hence

$$[H_i, R_j/r] = fR/r + M_1\sigma_1/(1+t)\partial + f/(1+t)\widetilde{Z}_m$$

and the same is true for the other Z as well. Finally,

$$[\widetilde{a}L_1, R_j/r] = -R_j\widetilde{a}/rL_1 + \widetilde{a}[L_1, R_j/r],$$
$$[L_1, R_j/r] = fR/r^2 + f\varepsilon\sigma_1/(1+t)^2\widetilde{N}_1\partial,$$

which gives i), which is also ii) for $k = 1$, since $R_j\widetilde{a} = f\theta_2 s_\theta a$.

b. Since $[\widetilde{Z}_m, \partial] = M_1\partial$, ii) follows by induction from the properties of the M_k. □

IV.5. Some commutation Lemmas for the modified fields

Lemma 5.1. — *We have the formula*

$$\text{i)}\qquad [\partial_t, S_\theta]w = \frac{\theta_1'}{\theta_1}s_\theta w + \frac{\theta_2'}{\theta_2}s_\theta w,$$

$$\text{ii)}\qquad [\partial_t, S_\theta]w = \frac{\theta_1'}{\theta_1^2}s_\theta \partial_r w + \frac{\theta_2'}{\theta_2^2}(s_\theta w + s_\theta f R w),$$

$$\text{iii)}\qquad [b, S_\theta]w = \theta_1^{-1}s_\theta[b;]w + \theta_2^{-1}s_\theta[;b]w,$$

$$\text{iv)}\quad \theta_1[\partial_i, S_\theta]w = fs_\theta f\partial w + fs_\theta[;h(\omega)]f\partial w + fs_\theta[;h(\omega)]fw(1+t)^{-1} + fs_\theta[;h(\omega)w(1+t)^{-1}]1,$$

$$\text{v)}\qquad [\widetilde{Z}_m, S_\theta]w = f\theta_1^{-1}s_\theta[f\widetilde{N}_1^l;]M_1\widetilde{Z}_m w + f\theta_2^{-1}s_\theta[;f\widetilde{N}_1^l]M_1\widetilde{Z}_m w + f\widetilde{N}_1\theta_1^{-1}s_\theta f\partial w + f\widetilde{N}_1\theta_2^{-1}s_\theta w + f\widetilde{N}_1\theta_2^{-1}s_\theta M_1\widetilde{Z}_m w + f\theta_2^{-1}s_\theta[;fw]1.$$

Proof

a. We have

$$\begin{aligned}[\partial_t, S_\theta^1]w &= \frac{\theta'}{\theta}\int \theta(r\phi_1)_r(\theta(r-r'))w(r')dr' \\ &= -\frac{\theta'}{\theta}\int \partial_{r'}[(r\phi_1)(\theta(r-r'))]w(r')dr' \\ &= \frac{\theta'}{\theta^2}\int \theta(r\phi_1)(\theta(r-r'))\partial_r w(r')dr'.\end{aligned}$$

Similarly,

$$\begin{aligned}[\partial_t, S_\theta^2]w &= \theta'/\theta\{\int \theta^2[2\phi_2 + y\partial\phi_2](\theta(y-y'))(\chi_+ w)(p_-^{-1}(y'))dy'\}(p_-) + \cdots \\ &= \theta'/\theta\{\int \theta\sum \partial_j[(y_j\phi_2)(\theta(y-y'))](\chi_+ w)(p_-^{-1}(y'))dy'\}(p_-) + \cdots \\ &= \theta'/\theta^2\{\sum\int \theta^2(y_j\phi_2)(\theta(y-y'))\partial_j[(\chi_+ w)(p_-^{-1}(y'))]dy'\}(p_-) + \cdots .\end{aligned}$$

This gives the formula i) and ii). **b.** Let $p_-'R_i = \sum \alpha_i^j \partial_{y_j}$. We have

$$\partial_j \int \phi_{2,\theta}h(y')dy' = \int \phi_{2,\theta}(\partial_j h)(y')dy',$$

$$\alpha_i^j(y)\partial_j \int \phi_{2,\theta}h(y')dy' = \int \phi_{2,\theta}(\alpha_i^j(y) - \alpha_i^j(y'))(\partial_j h)(y')dy' + \int \phi_{2,\theta}(\alpha_i^j\partial_j h)(y')dy',$$

hence, with $h(y) = (\chi_+ w)(p_-^{-1}(y))$,

$$[R_i, S_\theta^2]w = \theta^{-1}\Big\{\int \theta\phi_{2,\theta}(\alpha_i^j(y) - \alpha_i^j(y'))\partial_j h(y')dy'\Big\}(p_-)$$
$$- \theta^{-1}\Big\{\int \theta\phi_{2,\theta}[(wR_i(\chi_+))(p_-^{-1}(y)) - (wR_i(\chi_+))(p_-^{-1}(y'))]dy'\Big\}(p_-) + \cdots$$
$$+ wR_i(\chi_+ + \chi_-),$$

the dots meaning a similar term with p_+ and the last term being zero since we have a partition of unity. Since R commutes with S_θ^1, we obtain

$$\theta_2[R_i, S_\theta]w = s_\theta[;h]w + s_\theta[;h]fRw + s_\theta[;hw]1,$$

where $h = h(\omega)$ stands for various smooth functions of ω.

c. Now remark that, for any function b,

$$b(r,y)\int \theta_1\phi_1(\theta_1(r-r'))\theta_2^2\phi_2(\theta_2(y-y'))w(r',y')dr'dy'$$
$$= \int \cdots [(b(r,y) - b(r',y)) + (b(r',y) - b(r',y')) + b(r',y')]w(r',y')dr'dy',$$

which gives iii).

d. Since

$$\partial_i = \omega_i\partial_r - 1/r(\omega \wedge R)_i,$$

we have

$$[\partial_i, s_\theta]w = [\omega_i, s_\theta]w_r + f/(1+t)[R, s_\theta]w + f/(1+t)[\omega, s_\theta]Rw + [1/r, s_\theta]fRw.$$

Since

$$1/r - 1/r' = -(r-r')/rr', \quad [1/r, s_\theta]w = (\theta_1 r)^{-1}s_\theta(w/r),$$

we obtain iv). **e.** With the above formula, we also obtain, using Lemma 2,

$$[\widetilde{a}L_1, S_\theta]w = \widetilde{a}/\sqrt{c}[\partial_t, S_\theta]w + [\widetilde{a}/\sqrt{c}, S_\theta]\partial_t w - [\widetilde{a}\sqrt{c}, S_\theta]\partial_r w.$$

Now, since

$$(b(r,y) - b(r',y))/\sigma_1(r') = b(r,y)/\sigma_1(r) - b(r',y)/\sigma_1(r') + (r'-r)b(r,y)/\sigma_1(r)\sigma_1(r'),$$

$$s_\theta[p;q](w/\sigma_1) = s_\theta[p_1,\ldots,p_{i-1},p_i/\sigma_1,p_{i+1},\ldots,p_k;q]w$$
$$+ p_i/\sigma_1 s_\theta[p_1,\ldots,\widehat{p_i},\ldots,p_k;q](w/\sigma_1),$$
$$s_\theta[p;q](w/\sigma_1) = s_\theta[p;q_1,\ldots,q_i/\sigma_1,\ldots,q_l]w.$$

Here, $\widehat{p_i}$ means as usual that p_i is omitted. We may write sometimes

$$s_\theta[\ldots,p_i/\sigma_1,\ldots;q]$$

instead of the correct

$$s_\theta[p_1,\ldots,p_i/\sigma_1,\ldots,p_k;q],$$

the ... meaning that the non-written terms are unchanged. We thus obtain for instance

$$[b, S_\theta]\partial w = \theta_1^{-1} s_\theta[b;]\partial w + \theta_2^{-1} s_\theta[;b]\partial w$$
$$= \theta_1^{-1} s_\theta[b/\sigma_1;]\sigma_1\partial w + b/\sigma_1 s_\theta \partial w + \theta_2^{-1} s_\theta[;b/\sigma_1]\sigma_1\partial w.$$

To summarize, using ii), we get

$$[\widetilde{a}L_1, S_\theta]w = f/\theta_1 s_\theta[f\widetilde{N}_1;]M_1\widetilde{Z}_m w + f/\theta_2 s_\theta[;f\widetilde{N}_1]M_1\widetilde{Z}_m w$$
$$+ f\widetilde{N}_1/\theta_1 s_\theta f\partial w + f/\theta_2 s_\theta M_1\widetilde{Z}_m w.$$

f. We also have

$$[S, S_\theta]w = t[\partial_t, S_\theta]w + [r, S_\theta]\partial_r w.$$

Since $[r, S_\theta]w = \theta_1^{-1} s_\theta w$,

$$[S, S_\theta]w = f/\theta_1 s_\theta f\partial w + f/\theta_2(s_\theta w + s_\theta M_1\widetilde{Z}_m w).$$

Similarly

$$[H_0, S_\theta]w = [c, S_\theta]t\partial_r w + [c^{-1}, S_\theta]r\partial_t w + r/c[\partial_t, S_\theta]w + 1/c[r, S_\theta]\partial_t w$$
$$= f/\theta_1 s_\theta[tu;]\sigma_1^{-1} f\sigma_1\partial w + f/\theta_2 s_\theta[;f]M_1\widetilde{Z}_m w + f/\theta_1 s_\theta f\partial w$$
$$+ f/\theta_2(s_\theta w + s_\theta fRw).$$

Since, using the formula of **d.**,

$$s_\theta[tu;]v/\sigma_1 = s_\theta[f;]v + fs_\theta(v/\sigma_1),$$

we obtain again

$$[H_0, S_\theta]w = f/\theta_1 s_\theta[f;]M_1\widetilde{Z}_m w + f/\theta_1 s_\theta f\partial w + f/\theta_2 s_\theta[;f]M_1\widetilde{Z}_m w$$
$$+ f/\theta_2(s_\theta w + s_\theta fRw).$$

Since $H_i = \omega_i H_0 - ct/r(\omega \wedge R)_i$, we get the same formula for $[H_i, S_\theta]$, with the additional term $f/\theta_2 s_\theta[;fw]1$. This completes the proof. □

Lemma 5.2. — *We have the formula*

i) $$[\widetilde{Z}_m, s_\theta[p;q]]w = M_1 s_\theta[p;q]w + M_1/\theta_1 s_\theta[p;q]f\partial w + f/\theta_1 s_\theta[p;q]M_1\widetilde{Z}_m w$$
$$+ \sum M_1 s_\theta[p_1, \dots, M_1\widetilde{Z}_m p_j, \dots, p_k; q]w + \sum M_1 s_\theta[p; q_1, \dots, M_1\widetilde{Z}_m q_j, \dots, q_l]w$$
$$+ f/\theta_1 s_\theta[p, M_1; q]M_1\widetilde{Z}_m w + f/\theta_2 s_\theta[p; q, M_1]M_1\widetilde{Z}_m w$$
$$+ \sum M_1 s_\theta[p_1, \dots, \widehat{p}_j, \dots p_k; q]f(\partial p_j)w$$
$$+ \sum M_1/\theta_1 s_\theta[p; q_1, \dots, f\partial q_j, \dots, q_l]w + \sum M_1/\theta_2 s_\theta[p_1, \dots, \widehat{p}_j, \dots, p_k; q, f\partial p_j]w.$$

ii) $$\widetilde{Z}_m s_\theta w = M_1 s_\theta M_1 \widetilde{Z}_m^r w, r \leqslant 1,$$

iii) $$\widetilde{Z}_m s_\theta[p;]w = \sum_{r_1+r_2\leqslant 1} M_1 s_\theta[M_1\widetilde{Z}_m^{r_1}p;]M_1\widetilde{Z}_m^{r_2}w + M_1 s_\theta(f\partial p)w + M_1 s_\theta[;f\partial p]w,$$

iv) $$\widetilde{Z}_m s_\theta[;q]w = \sum_{r_1+r_2\leqslant 1} s_\theta[;M_1\widetilde{Z}_m^{r_1}q]M_1\widetilde{Z}_m^{r_2}w + M_1 s_\theta(f\partial q)w + M_1 s_\theta[;f\partial q]w.$$

v) $$\widetilde{Z}_m^k s_\theta[;q]w = s_\theta[;M_{l_0}\widetilde{Z}_m^{r_1+p_1}q]M_{l_1}\widetilde{Z}_m^{r_2+p_2}w + M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{r_1+p_1}q\widetilde{Z}_m^{r_2+p_2}w.$$

In formula v), we have for both terms

$$r_1 + r_2 \leqslant 1, \quad \sum(l_i - 1) + p_1 + p_2 \leqslant k - 1.$$

Proof. — We prove only the delicate formula i), and v), the other formula ii), iii) and iv) being proved more easily along the same lines. We need only to get terms involving $\widetilde{Z}_m p, \widetilde{Z}_m q$ or terms in $\widetilde{Z}_m w$ with a small factor in front. **a.** We have easily the formula

$$\begin{aligned}[]
[\partial_t, s_\theta[p;q]]w &= f/(1+t)s_\theta[p;q]w + \sum s_\theta[p_1,\dots,\partial_t p_i,\dots,p_k;q]w \\
&\qquad + \sum s_\theta[p;q_1,\dots,\partial_t q_i,\dots,q_l]w, \\
[\partial_r, s_\theta[p;q]]w &= \sum s_\theta[p_1,\dots,\partial_r p_i,\dots,p_k;q]w + \sum s_\theta[p;q_1,\dots,\partial_r q_i,\dots,q_l]w.
\end{aligned}$$

Also

$$[b, s_\theta[p;q]]w = \theta_1^{-1}s_\theta[b,p;q]w + \theta_2^{-1}s_\theta[p;b,q]w.$$

b. Since

$$\begin{aligned}
&\partial_{y_j}\int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(q_l(y)-q_l(y'))h(y')dy' = \\
&\sum\int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(\partial_j q_i(y)-\partial_j q_i(y'))\cdots(q_l(y)-q_l(y'))h(y')dy' \\
&\qquad + \int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(q_l(y)-q_l(y'))\partial_j h(y')dy',
\end{aligned}$$

we need only push coefficients α_i^j through the last integral. We thus obtain as in **b**, Lemma 5.1,

$$\begin{aligned}[]
[R_i, s_\theta[p;q]]w &= f s_\theta[p;q]w + \sum f s_\theta[p_1,\dots,fRp_i,\dots p_k;q]w \\
&\qquad + \sum f s_\theta[p;q_1,\dots,fRq_i,\dots q_l]w + \theta_2^{-1}s_\theta[p;h(\omega),q]fRw.
\end{aligned}$$

This is of the desired form.

c. Similarly, since

$$r(p(r,y)-p(r',y)) = rp(r,y) - r'p(r',y) + (r'-r)(p(r',y)-p(r',y')+p(r',y')),$$

we obtain

$$\begin{aligned}
rs_\theta[p;q]w &= s_\theta[p_1,\dots,rp_i,\dots,p_k;q]w \\
&\qquad + \theta_2^{-1}s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q,p_i]w + s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q]p_i w,
\end{aligned}$$

and trivially

$$rs_\theta[p;q]w = \theta_1^{-1}s_\theta[p;q]w + s_\theta[p;q_1,\dots,rq_i,\dots,q_l]w.$$

Hence

$$[S, s_\theta[p;q]]w = fs_\theta[p;q]w + \theta_1^{-1}s_\theta[p;q]w_r + \sum\Big\{s_\theta[p_1,\dots,Sp_i,\dots p_k;q]w + s_\theta[p;q_1,\dots,Sq_j,\dots,q_l]w + s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q](\partial_r p_i)w + \theta_2^{-1}s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q,\partial_r p_i]w\Big\}.$$

This is of the desired form.

d. We have now

$$H_0 s_\theta[p;q]w = cs_\theta[p;q]t\partial_r w + r/cs_\theta[p;q]\partial_t w + fs_\theta[p;q]w + \sum\Big\{cs_\theta[\dots t\partial_r p_j\dots;q]w + r/cs_\theta[\dots\partial_t p_j\dots;q]w + cs_\theta[p;\dots t\partial_r q_j\dots]w + r/cs_\theta[p;\dots\partial_t q_j\dots]w\Big\}.$$

Since it is technically awkward to commute $1/c$ with s_θ, we proceed slightly differently. We write for instance

$$(r/c)s_\theta[\dots\partial_t p_j\dots;q]w = rcs_\theta[\dots\partial_t p_j\dots;q]w + f(1+t)us_\theta[\dots\partial_t p_j\dots;q]w.$$

Using the formula

$$\sigma_1 s_\theta[p;q]v = s_\theta[\dots,\sigma_1 p_j,\dots;q]v + s_\theta[\dots,\widehat{p}_j,\dots;q]p_j v + 1/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,p_j]v,$$
$$\sigma_1 s_\theta[p;q]v = s_\theta[p;\dots,\sigma_1 q_j,\dots]v + 1/\theta_1 s_\theta[p;q]v,$$

we obtain for the same typical term, remembering that $(1+t)u/\sigma_1$ is an f,

$$f(1+t)us_\theta[\dots,\partial_t p_j,\dots;q]w = fs_\theta[\dots,\sigma_1\partial_t p_j,\dots;q]w + fs_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_t p_j]w.$$

Similarly, we have

$$f(1+t)us_\theta[p;\dots,\partial_t q_j,\dots]w = fs_\theta[p;\dots,\sigma_1\partial_t q_j,\dots]w + f/\theta_1 s_\theta[p;\dots,\partial_t q_j,\dots]w.$$

Using the formula of **c.** to move r to one of the factors p or q, we get

$$H_0 s_\theta[p;q]w = s_\theta[p;q]H_0 w + \sum fs_\theta[\dots,M_1\widetilde{Z}_m p_j,\dots;q]w + \sum fs_\theta[p;\dots,M_1\widetilde{Z}_m q_j,\dots]w + fs_\theta[p;q]w + f/\theta_1 s_\theta[p;q]\partial_t w + \sum\Big\{fs_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f/\theta_1 s_\theta[p;\dots,\partial_t q_j,\dots]w + f/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_t p_j]w\Big\} + f/\theta_1 s_\theta[u,p;q]f(1+t)\partial w + f/\theta_2 s_\theta[p;q,u]f(1+t)\partial w.$$

To see $(1+t)u/\sigma_1$ instead of u in the last two terms, we use the formula

$$s_\theta[\dots,\sigma_1 p_j,\dots;q]v = p_j s_\theta[\dots,\widehat{p}_j,\dots;q]v + s_\theta[p;q]\sigma_1 v,$$
$$s_\theta[p;\dots,\sigma_1 q_j,\dots]v = s_\theta[p;q]\sigma_1 v.$$

We thus obtain the desired form for $[H_0, s_\theta[p;q]]$.

e. Finally, we write

$$\begin{aligned}\widetilde{a}L_1 s_\theta[p;q]w = f\widetilde{N}_1 s_\theta[p;q]w + \widetilde{a}/\sqrt{c}s_\theta[p;q]\partial_t w - \widetilde{a}\sqrt{c}s_\theta[p;q]\partial_r w \\ + \sum\Big\{\widetilde{a}/\sqrt{c}s_\theta[\dots,\partial_t p_j,\dots;q]w - \widetilde{a}\sqrt{c}s_\theta[\dots,\partial_r p_j,\dots;q]w \\ + \widetilde{a}/\sqrt{c}s_\theta[p;\dots,\partial_t q_j,\dots]w - \widetilde{a}\sqrt{c}s_\theta[p;\dots,\partial_r q_j,\dots]w\Big\}.\end{aligned}$$

To see the term $\widetilde{a}L_1 w$, we will consider for instance

$$[\widetilde{a}/\sqrt{c}, s_\theta[p;q]]\partial_t w = 1/\theta_1 s_\theta[\widetilde{a}/\sqrt{c}, p;q]\partial_t w + 1/\theta_2 s_\theta[p;q,\widetilde{a}/\sqrt{c}]\partial_t w.$$

In order to see $\widetilde{a}/\sigma_1 = \widetilde{N}_1$ instead of $\widetilde{a}$, we have to move around σ_1 using the formula of **d.** We get

$$\widetilde{N}_1/\theta_1 s_\theta[p;q]\partial_t w + 1/\theta_1 s_\theta[M_1,p;q]M_1\widetilde{Z}_m w + 1/\theta_2 s_\theta[p;q,M_1]M_1\widetilde{Z}_m w.$$

The computation is analogous with the term containing $\partial_r w$. To handle a term like

$$\widetilde{a}/\sqrt{c}s_\theta[\dots,\partial_t p_j,\dots;q]w,$$

we again have to move around σ_1: this term is equal to

$$\begin{aligned}f\widetilde{N}_1\sigma_1 s_\theta[\dots,\partial_t p_j,\dots;q]w = f\widetilde{N}_1 s_\theta[\dots,\sigma_1\partial_t p_j,\dots;q]w \\ + f\widetilde{N}_1 s_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f\widetilde{N}_1/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_r p_j]w,\end{aligned}$$

and a similar expression for the terms involving $\partial_r p_j, \partial_t q_j, \partial_r q_j$.

To complete the proof, we note that $H_i = \omega_i H_0 + fR$, hence

$$[H_i, s_\theta[p;q]] = \omega_i[H_0, s_\theta[p;q]] + [\omega_i, s_\theta[p;q]]H_0 + f[R, s_\theta[p;q]] + [f, s_\theta[p;q]]R,$$

which yields only terms of the desired form. Finally,

$$\widetilde{K}s_\theta[p;q]w = 2/\sqrt{c}\{s_\theta[p;q]\partial_t w + s_\theta[\dots,\partial_t p_j,\dots;q]w + s_\theta[p;\dots,\partial_t q_j,\dots]w\}.$$

We could write $\partial_t w = f\widetilde{K}w$ and this would be enough for what we have in mind, but since we want a commutator, we proceed differently. We write

$$2/\sqrt{c}s_\theta[p;q]\partial_t w = s_\theta[p;q]\widetilde{K}w + (2/\sqrt{c}-2)s_\theta[p;q]\partial_t w + s_\theta[p;q](2-2/\sqrt{c})\partial_t w,$$

and again move around σ_1 in the first term to see u/σ_1 and $\sigma_1\partial w$. We obtain terms of the desired form with a gain of $1/(1+t)$ instead of θ_i^{-1}, and this is enough to complete the proof of i).

f. To prove v), we note that it is true for $k=1$, since $f\partial = M_1\widetilde{Z}_m$. Applying $\widetilde{Z}_m$ to v) and using ii) and iv), we get the formula by induction. □

Lemma 5.3. — *For all k, we have the formula*

$$\begin{aligned}[\widetilde{Z}_m^k, S_\theta]w = \theta_1^{-1}\Big\{\sum M_{l_0}s_\theta[M_{l_1};]M_{l_2}\widetilde{Z}_m^{p+r}w + M_{l_0}s_\theta[;M_{l_1}]M_{l_2}\widetilde{Z}_m^{p+r}w \\ + M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{p+r}w + M_{l_0}s_\theta[;M_{l_1}\widetilde{Z}_m^{p}w]M_{l_2}\Big\}.\end{aligned}$$

In all terms, we have

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

Proof

a. We will prove by induction that $\widetilde{Z}_m^{k-1}[\widetilde{Z}_m, S_\theta]w$ is equal to the right hand side with the same conditions

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

For $k = 1$, by inspection of the formula for $[\widetilde{Z}_m, S_\theta]$ in Lemma 5.1, we see that this is true, since $\theta_1 \leqslant \theta_2$.

Assuming that this is true for k, we write $\widetilde{Z}_m^k = \widetilde{Z}_m \widetilde{Z}_m^{k-1}$ and examine the various terms. We have, using formula iii) of Lemma 5.2,

$$\widetilde{Z}_m s_\theta[M_{l_1};]M_{l_2}\widetilde{Z}_m^{p+r}w = M_1 s_\theta[M_1\widetilde{Z}_m^{p_1}(M_{l_1});]M_1\widetilde{Z}_m^{p_2}(M_{l_2}\widetilde{Z}_m^{p+r}w) + M_1 s_\theta(M_1\widetilde{Z}_m(M_{l_1})M_{l_2}\widetilde{Z}_m^{p+r}w) + M_1 s_\theta[; M_1\widetilde{Z}_m(M_{l_1})]M_{l_2}\widetilde{Z}_m^{p+r}w.$$

Since $\widetilde{Z}_m^q M_p = M_{p+q}$, we get

$$= M_1 s_\theta[M_{l_1+p_1};](M_{l_2+p_2}\widetilde{Z}_m^{p+r}w + M_{l_2}\widetilde{Z}_m^{p+p_2+r}w) + M_1 s_\theta(M_{l_1+l_2}\widetilde{Z}_m^{p+r}w) + M_1 s_\theta[; M_{l_1+1}]M_{l_2}\widetilde{Z}_m^{p+r}w.$$

Taking into account that

$$\widetilde{Z}_m(M_l\theta_1^{-1}) = \theta_1^{-1}M_{l+1},$$

we see that the action of $\widetilde{Z}_m$ on the first term of the right-hand side yields terms of the desired form. The issue is completely similar with the second term, and easy for the third. For the last term, we write

$$\widetilde{Z}_m s_\theta[; M_{l_1}\widetilde{Z}_m^p w]M_{l_2} = M_1 s_\theta[; M_{l_1+p_1}\widetilde{Z}_m^p w + M_{l_1}\widetilde{Z}_m^{p+p_1}w]M_{l_2} + M_1 s_\theta(M_{l_1+l_2}\widetilde{Z}_m^p w + M_{l_1+l_2-1}\widetilde{Z}_m^{p+1}w) + M_1 s_\theta[; M_{l_1+1}\widetilde{Z}_m^p w + M_{l_1}\widetilde{Z}_m^{p+1}w]M_{l_2},$$

and see that all terms are of the desired form.

b. Since

$$[\widetilde{Z}_m^k, S_\theta]w = \sum_{1\leqslant l\leqslant k} \widetilde{Z}_m^{k-l}[\widetilde{Z}_m, S_\theta]\widetilde{Z}_m^{l-1}w,$$

we see that this term is a sum of terms of the desired form with

$$\sum(l_i - 1) + p + l_1 \leqslant k - l + l - 1 = k - 1.$$

This completes the proof. □

Later on, we will need the following pseudo-commutator formula.

Lemma 5.4. — *We have the formula*

$$\widetilde{Z}_m^k s_\theta w = f s_\theta f \widetilde{Z}_m^k w + \theta_1\sigma_1 M_{l_0} s_\theta M_{l_1}\widetilde{Z}_m^p w + \theta_2^{-1}M_{l_0}s_\theta[; M_{l_1}]M_{l_2}\widetilde{Z}_m^{p+r}w + \theta_2^{-1}M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{p+r}w + \theta_2^{-1}M_{l_0}s_\theta[; M_{l_1}\widetilde{Z}_m^p w]M_{l_2}.$$

In these sums,

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

Proof

a. We consider first $k = 1$, and set $\partial_t w + c\partial_r w = g$. As in the proof of Lemma 5.1, we have

$$[S, s_\theta]w = t[\partial_t, s_\theta]w + \theta_1^{-1} s_\theta w_r = f s_\theta w + s_\theta w,$$

since, by integration by parts, $s_\theta w_r = \theta_1 s_\theta w$. Next,

$$[H_0, s_\theta]w = [c, s_\theta]tw_r + [c^{-1}, s_\theta]rw_t + r/(ct)s_\theta w + c^{-1}\theta_1^{-1} s_\theta w_t.$$

First,

$$u s_\theta w_r = u\theta_1 s_\theta w, s_\theta u w_r = s_\theta(uw)_r - s_\theta u_r w = \theta_1 s_\theta uw - s_\theta u_r w,$$
$$[c, s_\theta]w_r = [u, s_\theta]w_r = \theta_1[u, s_\theta]w + s_\theta u_r w.$$

Second,

$$[1/c, s_\theta]h = -1/c[u, s_\theta](h/c),$$
$$[u, s_\theta]rw_t/c = [u, s_\theta](rg/c - (rw)_r + w) = f(1+t)[u, s_\theta]fg + \theta_1 f(1+t)[u, s_\theta]fw.$$

Finally,

$$\theta_1^{-1} s_\theta w_t = \theta_1^{-1} s_\theta(g - (cw)_r + u_r w) = \theta_1^{-1} s_\theta g + s_\theta f w + \theta_1^{-1} s_\theta u_r w.$$

Collecting the terms, we obtain

$$[H_0, s_\theta]w = f s_\theta f w + f s_\theta f g + f\theta_1(1+t)[u, s_\theta]fw + f(1+t)[u, s_\theta]fg.$$

On the other hand,

$$\begin{aligned}\widetilde{a}L_1 s_\theta w &= f\widetilde{a}[\partial_t, s_\theta]w + f\widetilde{a}s_\theta(g - (cw)_r + u_r w) + f\widetilde{a}\theta_1 s_\theta w\\ &= \theta_1 M_1 s_\theta f w + f\widetilde{N}_1 s_\theta g,\\ s_\theta(\widetilde{a}L_1 w) &= s_\theta(f\widetilde{N}_1 g) + f\theta_1 s_\theta(f\widetilde{N}_1 w).\end{aligned}$$

Finally,

$$H_i s_\theta w = f[H_0, s_\theta]w + f s_\theta H_0 w + f[R, s_\theta]w + f s_\theta R w.$$

Collecting the terms, we obtain

$$\begin{aligned}\widetilde{Z}_m s_\theta w = f s_\theta f \widetilde{Z}_m w &+ \theta_1 M_1 s_\theta M_1 w + \theta_1 M_1(1+t)[u, s_\theta]M_1 w\\ &+ f\widetilde{N}_1 s_\theta g + s_\theta(f\widetilde{N}_1^r g) + f(1+t)[u, s_\theta]fg\\ &\qquad + f\theta_2^{-1}(s_\theta[;f]M_1\widetilde{Z}_m^r w + s_\theta[;fw]1).\end{aligned}$$

Opening the commutator term on w, we find

$$\theta_1 M_1(1+t)[u, s_\theta]M_1 w = \theta_1\sigma_1 M_1 s_\theta M_1 w.$$

Remember now that

$$g = \sqrt{c}Lw = f(1+t)^{-1}\widetilde{Z}_m w.$$

Hence

$$f(1+t)[u,s_\theta]fg = fs_\theta f\widetilde{Z}_m w,$$

and we replace also g by this value in the two other terms containing g. We thus obtain

$$\widetilde{Z}_m s_\theta w = fs_\theta f\widetilde{Z}_m w + (1+t)^{-1} f\widetilde{N}_1^{r_1} s_\theta f\widetilde{N}_1^{r_2}\widetilde{Z}_m w \\ + \theta_1\sigma_1 M_1 s_\theta M_1 w + f\theta_2^{-1}(s_\theta[;f]M_1\widetilde{Z}_m^r w + s_\theta[;fw]1).$$

Thus the result is true for $k=1$, the second term in the right-hand side being of the form

$$\theta_2^{-1}M_1 s_\theta M_1\widetilde{Z}_m w$$

since $(1+t)^{-1} = f\theta_2^{-1}$.

b. Let us assume now the formula for $\widetilde{Z}_m^l s_\theta w$, $l\leqslant k$. We obtain

$$\widetilde{Z}_m^{k+1}s_\theta w = \widetilde{Z}_m(\widetilde{Z}_m^k s_\theta w) = M_1 s_\theta f\widetilde{Z}_m^k w + f(fs_\theta f\widetilde{Z}_m(f\widetilde{Z}_m^k w) \\ + \theta_1\sigma_1 M_1 s_\theta M_1(f\widetilde{Z}_m^k w)) + f\theta_2^{-1}(M_1 s_\theta[;M_1]M_1\widetilde{Z}_m^r(f\widetilde{Z}_m^k w) \\ + M_1 s_\theta M_1 f\widetilde{Z}_m^k w + M_1 s_\theta[;M_1\widetilde{Z}_m^k w]M_1) + \theta_1\sigma_1(M_1 M_{l_0} + M_{l_0+1})s_\theta M_{l_1}\widetilde{Z}_m^p w \\ + \theta_1\sigma_1 M_{l_0}(M_1 s_\theta M_1\widetilde{Z}_m^r(M_{l_1}\widetilde{Z}_m^p w)) + \widetilde{Z}_m(\cdots+\cdots+\cdots).$$

The last three terms are identical to the corresponding terms in the proof of Lemma 5.3, we need not redo the computation. All other terms are easily seen to be of the desired form. □

IV.6. L^∞ estimates of the quantities $\widetilde{N}_k$

Proposition 6. — *Fix $\mu > 1/2$. For η small enough, θ_1^0 and β_1 big enough, we have the estimates (except of course for $\widetilde{N}_0 = 1$)*

i) $$|\widetilde{N}_k| \leqslant C(1+t)^{C_1\varepsilon}\sigma_1^{\mu-1}, \quad k\leqslant s_0-4.$$

ii) $$|\widetilde{Z}_m^{k-1}a| + |\widetilde{Z}_m^{k-1}\partial a| \leqslant C(1+t)^{C_1\varepsilon}, \quad k\leqslant s_0-4.$$

Here, C_1 does not depend on the θ_i.

Proof. — From Proposition III.7 and Proposition IV.1, we know that, for $k\leqslant s_0-4$,

$$\varepsilon^{-1}(1+t)|\sigma_1^{-1}Z_0^k u|, \quad \varepsilon^{-1}(1+t)|Z_0^k\partial u|, \quad \sigma_1^{-1}|Z_0^{k-1}a|, |Z_0^{k-1}\partial a|$$

are bounded by $C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}$, for $a = a(R_i), a(S)$. These estimates extend easily also to $a = a(H_i) = -\omega_i a(S) - (\omega\wedge a(R))_i$. Remark that, since a are supported in

$$\sigma_1 \leqslant C(1+t)^{C_0\varepsilon},$$

we can ignore the powers of σ_1 in estimates involving a.

a. First we estimate $\widetilde{N}_1$. From the properties of S_θ and Lemma 5.1, we get

$$|\widetilde{a}| \leqslant C(1+t)^{C\varepsilon}, \quad |\partial\widetilde{a}| \leqslant C|a|_{L^\infty} + C|\partial a|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

Now

$$\widetilde{Z}_m = fN_0Z_0 + f\sigma_1^{-1}\widetilde{a}Z_0$$

gives the control of the terms of $\widetilde{N}_1$ involving u. Remark also

$$|\widetilde{Z}_m a| \leqslant C(1+t)^{C\varepsilon}|Z_0a|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

b. We need to establish an analogue to Lemma 1.1, including some refinement using the fact that u and a do not play the same role in the process of estimating the $\widetilde{N}_k$.

Lemma 6.1. — *We have the formula*

i) $\widetilde{Z}_m^k = \sum f\widetilde{N}_1^l\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}Z_0^p,$

ii) $\displaystyle \widetilde{Z}_m^k = \sum_{\substack{p\geqslant 1\\ \sum k_i+p\leqslant k}} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}Z_0^p + \sum_{q\geqslant 1} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^{r_1}(\widetilde{a}/\sigma_1)\cdots\widetilde{Z}_m^{r_q}(\widetilde{a}/\sigma_1)Z_0^p.$

In the sum of i), we have $p\geqslant 1$, $k_i\geqslant 2$, $\sum(k_i-1)+p\leqslant k$. In the second sum of ii), we have $p\geqslant 1$, $q\leqslant k$, $\sum k_j+\sum r_i+p\leqslant k$.

c. We assume $1\leqslant k\leqslant s_0-5$ and

$$|\widetilde{N}_l| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}, \quad l\leqslant k,$$
$$|\widetilde{Z}_m^l a| + |\widetilde{Z}_m^l\partial a| \leqslant C(1+t)^{C\varepsilon}, \quad l\leqslant k-1.$$

Using first Lemma 6.1 i) for the index k, applied to a or ∂a, we get

$$|\widetilde{Z}_m^k a| + |\widetilde{Z}_m^k\partial a| \leqslant C(1+t)^{C\varepsilon}.$$

Using Lemma 5.3 for the index k and $w=a$, we see that the only terms which are not already controlled (using the induction hypothesis), are the terms

$$M_1s_\theta[M_k;]M_1\widetilde{Z}_m^r a,\ M_1s_\theta[;M_k]M_1\widetilde{Z}_m^r a,\ M_1s_\theta[;M_ka]M_1.$$

It is important to check the way θ_i enters the constants (that is, f) in Lemma 5.3: β_i and θ_i^0 enter the computation only through formula i) or ii) of Lemma 4.1. In these formula, θ_i^0 do not appear, and β_i appear only through $\beta_i\varepsilon$; replacing θ_2^{-1} by $f\theta_1^{-1}$, or θ_1^{-1} by f, gives f containing $\theta_1^0/\theta_2^0\leqslant 1$ or $(\theta_1^0)^{-1}\leqslant 1$ as constants, and negative powers of $(1+t)$ expressed with $\beta_i\varepsilon$. Hence, thanks to the constraints $\varepsilon\beta_i\leqslant 1$, all f entering the computation are bounded independently of the choices of the quantities θ_i^0, β_i. We thus obtain

$$|\widetilde{Z}_m^k\widetilde{a}| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}\theta_1^{-1}|\widetilde{N}_{k+1}|,$$

where here and later numbered constants C_2 and C_3 do not depend on θ_i. We obtain similarly

$$\theta_1|[\widetilde{Z}_m^k, s_\theta]\partial a| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}|\widetilde{N}_{k+1}|.$$

We have

$$\widetilde{Z}_m^k\partial\widetilde{a} = \widetilde{Z}_m^k[\partial, S_\theta]a + [\widetilde{Z}_m^k, S_\theta]\partial a + S_\theta\widetilde{Z}_m^k\partial a.$$

To evaluate the first term in the right-hand side, we use Lemma 5.1 ii) or iv) and Lemma 5.2. With the same reasoning as before, we obtain

$$\theta_1|\widetilde{Z}_m^k[\partial, S_\theta]a| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}|\widetilde{N}_{k+1}|.$$

Finally,

$$|\widetilde{Z}_m^k\partial\widetilde{a}| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}\theta_1^{-1}|\widetilde{N}_{k+1}|.$$

Using now Lemma 6.1 ii) for the index $k+1$, applied to u or ∂u, we get

$$\sigma_1^{-\mu}(1+t)\varepsilon^{-1}|\widetilde{Z}_m^{k+1}u| \leqslant C(1+t)^{C\varepsilon} + C_4(1+t)^{C_5\varepsilon}|\widetilde{Z}_m^k\widetilde{a}|,$$

and a similar formula for ∂u. Finally,

$$|\sigma_1^{1-\mu}\widetilde{N}_{k+1}| \leqslant C(1+t)^{C\varepsilon} + C_6(1+t)^{C_7\varepsilon}\theta_1^{-1}|\sigma_1^{1-\mu}\widetilde{N}_{k+1}|,$$

where as before the constants C_6 and C_7 are independent of θ_i. We choose then

$$\beta_1 \geqslant C_7, \quad \theta_1^0 \geqslant 2C_6$$

to obtain the desired estimate. □

V. Weighted L^2 norms, Poincaré Lemma and Energy Inequalities

V.1. Weighted norms. — For small $\nu > 0$ and big $B > 0$ to be chosen later, we set

$$b(s) = B(-s)^{-\nu}, \quad s \leqslant C < 0, \quad p = (\tau+1)b(\psi).$$

Remark that $p_r > 0$, since $b' > 0$ and $\psi_r > 0$. For fixed t, we define the L^2 weighted norm by

$$|w|_0^2 = \int (\exp p)|w|^2 dx.$$

We first have to clarify the control of $\sigma_1^{-1}w$ by ∂w in this norm.

Lemma 1.1. — *We have, for any smooth w supported in $|x| \leqslant M+t$,*

$$|\sigma_1^{-1}w|_0 \leqslant C|w_r|_0.$$

Proof. — For fixed ω, t, omitting these variables for simplicity, we write

$$w(r) = -\int_r^{M+t} w_r(s)ds,$$

$$w(r)^2 \leqslant C(\sigma_1(r))^{1-\mu}\int_r^{M+t}(\sigma_1(s))^\mu w_r^2(s)ds, \quad 0<\mu<1.$$

Hence, since p is increasing,

$$\int_0^{M+t} e^{p(r)}(\sigma_1(r))^{-2}w(r)^2r^2dr \leqslant C\int_0^{M+t} e^{p(s)}(\sigma_1(s))^\mu w_r^2(s)ds\int_0^s r^2(\sigma_1(r))^{-1-\mu}dr.$$

We split the right-hand side integral in

$$\int_0^{(M+t)/2} + \int_{(M+t)/2}^{M+t}.$$

In the first integral,

$$\sigma_1(r) \geqslant 1 + (M+t)/2,$$

hence it is less than

$$C\int_0^{(M+t)/2} e^{p(s)} s^2 w_r^2(s)(\sigma_1(s)(1+(M+t)/2)^{-1})^\mu (s(1+(M+t)/2)^{-1})ds \leqslant C\int_0^{(M+t)/2} e^{p(s)} w_r^2(s) s^2 ds.$$

In the second integral, we write

$$\int_0^s r^2(\sigma_1(r))^{-1-\mu} dr \leqslant C(M+t)^2(\sigma_1(s))^{-\mu},$$

and obtain that it is less than

$$\int_{(M+t)/2}^{M+t} e^{p(s)} w_r^2(s)(M+t)^2 ds \leqslant C\int_{(M+t)/2}^{M+t} e^{p(s)} w_r^2(s) s^2 ds.$$

Collecting the two bounds and integrating in ω finishes the proof. □

We now have to make sure that the smoothing operators behave properly.

Lemma 1.2. — *If β_1 is big enough, we have the formula*

i) $$|s_\theta[p;q]b|_0 \leqslant C|b|_0 \Pi|\partial_r p_i|_{L^\infty} \Pi |Rq_j|_{L^\infty},$$

ii) $$|s_\theta[p;q]b|_0 \leqslant C|\partial_r p_1|_0 |b|_{L^\infty} \Pi_{i\geqslant 2}|\partial_r p_i|_{L^\infty} \Pi |Rq_j|_{L^\infty},$$

iii) $$|s[p;q]b|_0 \leqslant C|Rq_1|_0 |b|_{L^\infty} \Pi|\partial_r p_i|_{L^\infty} \Pi_{j\leqslant 2}|Rq_j|_{L^\infty}.$$

Proof. — We prove only iii), which is the more difficult. With

$$q_1(r',y) - q_1(r',y') = \Big(\int_0^1 (\partial_y q_1)(sy + (1-s)y')ds\Big)(y-y'),$$

we can rewrite $s_\theta[p;q]b$ (assuming k factors p_i and l factors q_j) as sums of

$$\int_0^1 ds \int \theta_1^{1+k}\theta_2^{2+l-1}\phi_1(\theta_1(r-r'))\phi_2(\theta_2(y-y'))(p_1(r,y)-p_1(r',y))\cdots(p_k(r,y)-p_k(r',y))$$
$$(\partial_{y_j} q_1)(r', sy+(1-s)y')(q_2(r',y)-q_2(r',y'))\cdots(q_l(r',y)-q_l(r',y'))(\chi b)(r',y')dr'dy'.$$

To introduce e^p into this integral, we write

$$e^{p(r,y)} = e^{p(r,y)} - e^{p(r',y)} + e^{p(r',y)} - e^{p(r',sy+(1-s)y'))} + e^{p(r',sy+(1-s)y')}.$$

For the integral corresponding to the last term, the previously proved L^2 estimate works, yielding the quantity $|Rq_1|_0$. The first two terms are bounded by

$$|e^p|_{L^\infty}(|r-r'||p_r|_{L^\infty} + |y-y'||p_y|_{L^\infty}),$$

and

$$|e^p| \leqslant C(1+t)^{C\varepsilon}, \quad |p_r| \leqslant C(1+t)^{C\varepsilon}.$$

From the equation $\psi_t + c\psi_r = 0$, we get

$$(\partial_t + c\partial_r)(\sigma_1^{-\mu} R_i \psi) = -\sigma_1^{-\mu} R_i u \psi_r + \mu u/\sigma_1 (\sigma_1^{-\mu} R_i \psi),$$

and we already know that

$$|\sigma_1^{-\mu} R_i u \psi_r| \leqslant C\varepsilon (1+t)^{-1+C\varepsilon}.$$

Hence, while $\partial\psi$ is bounded for $r \leqslant t/2$, we get for $r \geqslant t/2$ and thus everywhere

$$|\sigma_1^{-\mu} R_i \psi| \leqslant C(1+t)^{C\varepsilon}.$$

This shows

$$|Rp| \leqslant C(\tau+1)|\psi|^{-1-\nu}|R\psi| \leqslant C(1+t)^{C\varepsilon}.$$

Thus the integrals corresponding to the first two terms we have just bounded are bounded by

$$C\theta_1^{-1}(1+t)^{C\varepsilon}|\partial_y q_1|_{L^2}|b|_{L^\infty}\Pi|\partial_r p_i|_{L^\infty}\Pi_{j\geqslant 2}|\partial_y q_j|_{L^\infty}.$$

Putting the weight e^p inside $|\partial_y q_1|_{L^2}$ costs only an extra factor $C(1+t)^{C\varepsilon}$, hence if β_1 is big enough, the claim is proved. □

V.2. The Poincaré Lemma. — The Poincaré Lemma is what we need to control the zero order term $(\Delta u)v$ in the linearized operator on u acting on v.

Proposition 2. — *Fix ν, $0 < \nu \leqslant 1/4$, and let $b(s) = B(-s)^{-\nu}$, $B > 0$. Then we can choose B such that, for any smooth v supported for $r \leqslant M + t$, we have, with $p = (\tau+1)b(\psi)$,*

$$\int_{r\geqslant t/2} (\exp p)(L_1^2 u)^2 v^2 dx \leqslant C\varepsilon^2 (1+t)^{-2} \int_{r\geqslant t/2} (\exp p) v_r^2 dx + C\varepsilon^2 \int_{r\geqslant t/2} (\exp p)(1+t)^{-7/2}\sigma_1^{-1} v^2 dx.$$

The point of this Lemma is this: the factor $L_1^2 u$ is well localized near the boundary of the light cone, but behaves only like $C\varepsilon(1+t)^{-1+C_1\varepsilon}$ there. In this Lemma, we get the inequality we would easily get if C_1 were zero.

Proof. — Using Lemma II.3.5.1, we get first, with $I = \int_{r\geqslant t/2} (\exp p)(L_1^2 U)^2 v^2 dx$

$$I \leqslant C \int_{r\geqslant t/2} (\exp p)(1+t)^{-3/2}\sigma_1^{-1} v^2 dx + 2\int_{r\geqslant t/2} (\exp p) a_1^{-2} h(\psi)^2 v^2 dx.$$

We perform a change of variables in the last integral, setting

$$s = \psi(r,\omega,t), \quad r = \phi(s,\omega,t), \quad \psi(\phi,\omega,t) = s.$$

The domain $t/2 \leqslant r \leqslant t + M$ is sent on the domain

$$\psi(t/2,\omega,t) \leqslant s \leqslant \psi(t+M,\omega,t) \equiv C(\omega),$$

since ψ is constant along any ray $r = t + M$. Hence, with $w(s,\omega,t) = v(\phi,\omega,t)$,

$$\int_{r\geqslant t/2} (\exp p) a_1^{-2} h(\psi)^2 v^2 dr = \int_{\psi(t/2)\leqslant s\leqslant C(\omega)} e^{\tau b(s)} a_1^{-2}(\phi) h^2(s) w^2 \phi_s ds.$$

We also have, from Lemma II.3.3

$$\phi_s / a_1^2(\phi) \leqslant 2/\phi_s,$$

hence

$$a_1^{-2}(\phi) h^2(s) \phi_s \leqslant C h^2(s)/\phi_s.$$

Now, with $\widetilde{b}(s) = e^{(\tau+1)b(s)}(\phi_s)^{-1}$,

$$\widetilde{b}(s)'/\widetilde{b}(s) = (\tau+1)b'(s) - \phi_{ss}/\phi_s.$$

But

$$\phi_s \psi_{rr}(\phi) = -\phi_{ss}/(\phi_s)^2,$$

and Lemma II.3.5.2 implies

$$|\phi_{ss}/\phi_s| \leqslant [|\psi_{rr}|/(\psi_r)^2](\phi) \leqslant C\tau(1+|s|)^{-3/2+4\eta} + C\varepsilon(1+|s|)^{-3/2+4\eta}.$$

Since $0 < \nu \leqslant 1/2 - 4\eta$, we can choose B big enough to ensure $\widetilde{b}' \geqslant 0$.

Proceeding as usual we write now

$$w(s) = \int_{C(\omega)}^{s} w_s(s')ds',$$

$$|w(s)|^2 \leqslant \Big(\int_s^{C(\omega)} w_s^2(s')\widetilde{b}(s')ds'\Big)\Big(\int_s^{C(\omega)} \frac{ds'}{\widetilde{b}(s')}\Big),$$

and, since $\widetilde{b}$ is increasing, the last integral is less than $(C(\omega)-s)/\widetilde{b}(s)$. Hence

$$\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s)h(s)^2 w^2(s)ds \leqslant \Big(\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s')w_s^2(s')ds'\Big)\Big(\int_{\psi(t/2)}^{C(\omega)} h(s)^2(C(\omega)-s)ds\Big),$$

and the last integral is bounded by

$$\int_{-\infty}^{C(\omega)} (1+|s|)^{-2+8\eta} ds \leqslant C.$$

Noting that $w_s = \phi_s v_r(\phi)$ and $\phi_s \widetilde{b}(s) = e^{\tau b(s)}$, we obtain by changing back the variables

$$\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s) w_s^2(s) ds \leqslant \int_{t/2}^{t+M} (\exp \tau b(\psi)) v_r^2 dr.$$

Finally, we obtain easily

$$|L_1^2 u - \varepsilon/r L_1^2 U| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{-1/2},$$

which completes the proof. □

V.3. The energy inequalities. — We present here one of the many possible variations on the ideas of [**3**].

Proposition 3.1. — *Let* $P = c^{-1}\partial_t^2 - c\Delta_x$ *and* $p = (\tau+1)b(\psi)$ *as in Proposition 2,* $T_i v = \partial_i v + (\omega_i/c)\partial_t v$. *Assuming that* u *satisfies the induction hypothesis* (IH) *on* $[0,T]$, *we have, for* $t \leqslant T$,

$$|(\partial v)(.,t)|_0^2 + C\int_0^t\int_{r\geqslant t'/2}(\exp p)(\tau+1)b'(\psi)\sum(T_i v)^2 dx dt'$$
$$\leqslant C|(\partial v)(.,0)|_0^2 + C\int_0^t\int_{\mathbf{R}^3}(\exp p)|Pv||v_t| dx dt'$$
$$+ C\varepsilon\int_0^t dt'/(1+t')|(\partial v)(.,t')|_0^2.$$

Proof. — We have

$$(\exp p)Pvv_t = \partial_t(1/2(\exp p)(v_t^2/c + c|v_x|^2)) - \sum\partial_i((\exp p)cv_i v_t) + (\exp p)Q,$$

with

$$Q = 1/(2c)(u_t/c - p_t)v_t^2 + \sum(u_i + cp_i)v_i v_t - 1/2(u_t + cp_t)|v_x|^2.$$

Writing explicitly the derivatives of p we get

$$Q = (\tau+1)/(2c)b'(\psi)[-c^2\psi_t\sum(v_i - (\psi_i/\psi_t)v_t)^2 - v_t^2/\psi_t(\psi_t^2 - c^2|\psi_x|^2)]$$
$$-\varepsilon(1+t)^{-1}b(\psi)/2(c^{-1}v_t^2 + c|v_x|^2) + u_t/2c^2 v_t^2 + \sum u_i v_i v_t - u_t/2|v_x|^2.$$

Integrating this identity in the strip $[0,t]\times\mathbf{R}^3$, we obtain as usual the control of the energy

$$E(t) = 1/2\int_{\mathbf{R}^3}(\exp p)(v_t^2/c + c|v_x|^2),$$

and the terms of the last line in Q are bounded by

$$C\varepsilon\int_0^t E(t')dt'/(1+t').$$

Now, ψ is not an exact phase function for P. For $r \leqslant t/2$, $\partial\psi$ is bounded, hence the terms of the first line of Q are bounded by

$$C(\tau+1)|b'(\psi)||\partial v|^2 \leqslant C(1+\tau)(1+t)^{-1-\nu}|\partial v|^2,$$

which are negligible terms. For $r \geqslant t/2$, we write

$$\psi_t^2 - c^2|\psi_x|^2 = -c^2/r^2\sum(R_i\psi)^2.$$

From the equation $\psi_t + c\psi_r = 0$, we get

$$(\partial_t + c\partial_r)(\sigma_1^{-\mu}R_i\psi) = -\sigma_1^{-\mu}R_i u\psi_r + \mu u/\sigma_1(\sigma_1^{-\mu}R_i\psi),$$

and we already know that

$$|\sigma_1^{-\mu}R_i u\psi_r| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

Hence

$$|\sigma_1^{-\mu} R_i\psi| \leqslant C(1+t)^{C\varepsilon}.$$

The error term

$$(\tau+1)b' v_t^2/\psi_t (R_i\psi)^2/r^2$$

is then bounded by

$$Cv_t^2(1+\varepsilon\log(1+t))(1+|\psi|)^{-1-\nu}\sigma_1^{2\mu}(1+t)^{-2+C\varepsilon} \leqslant C(1+t)^{-1-\eta}v_t^2,$$

which is negligible. Finally,

$$v_i - (\psi_i/\psi_t)v_t = v_i + (\omega_i/c)v_t - (v_t/\psi_t)(\psi_i + (\omega_i/c)\psi_t),$$

$$\psi_i + (\omega_i/c)\psi_t = \psi_i - \omega_i\psi_r.$$

Replacing $v_i - \psi_i/\psi_t v_t$ by $T_i v$ in Q gives an error term bounded by

$$(\tau+1)b'(\psi)|\psi_t|v_t^2(R_i\psi)^2/r^2,$$

which we have already seen to be negligible. □

In contrast with what could seem obvious, the energy inequality for L is non trivial.

Proposition 3.2. — *Let $p = (\tau+1)b(\psi)$ as in Proposition 2, and $\gamma > 0$. Then, for smooth functions v supported in $\gamma(1+t) \leqslant r \leqslant M+t$, we have the inequalities*

i) $$(1+t)^{-1}|e^{p/2}v(.,t)|_{L^2} \leqslant C\int_0^t (1+t')^{-1}|e^{p/2}(Lv)(.,t')|_{L^2}dt' + C\varepsilon\int_0^t (1+t')^{-2}|v(.,t')|_0 dt',\ v(x,0)=0,$$

ii) $$(1+t)^{-1}|(\partial v)(.,t)|_0 \leqslant C\int_0^t (1+t')^{-1}|(\partial Lv)(.,t')|_0 dt' + C\varepsilon\int_0^t (1+t')^{-2}|(\partial v)(.,t')|_0 dt',\ v(x,0)=v_t(x,0)=0.$$

Proof

a. We write

$$e^p\sqrt{c}Lvv = 1/2\partial_t(e^p v^2) + 1/2\partial_r(ce^p v^2) - (1/2)e^p v^2(p_t + cp_r) - (1/2)e^p u_r v^2,$$

and remark that

$$p_t + cp_r = (\tau+1)b'(\psi)(\psi_t + c\psi_r) + \varepsilon(1+t)^{-1}b(\psi).$$

Hence, integrating in r and t on $[0,+\infty[\times[0,t]$, we get

$$\int e^p\sqrt{c}Lvv\,drdt' = 1/2\int e^p v^2(r,\omega,t)dr - (1/2)\int e^p v^2(1+t')^{-1}[(1+t')u_r + \varepsilon b(\psi)]drdt',$$

which gives the bound

$$(1/2)\int e^p v^2(r,\omega,t)dr \leqslant C\varepsilon\int e^p v^2(1+t')^{-1}drdt' + C\int e^p|Lv||v|drdt'.$$

Integrating now also in ω and using again the support condition on v, we obtain

$$g(t)^2 \equiv ((1+t)^{-1}|e^{p/2}v(.,t)|_{L^2})^2 \leqslant C\varepsilon \int_0^t (1+t')^{-1}g^2(t')dt' + \int_0^t (1+t')^{-1}|e^{p/2}(Lv)(.,t')|_{L^2}g(t')dt',$$

which gives i).

b. With $Lv = h$, we have

$$h_t = Lv_t - (u_t/2c)L_1v, h_r = Lv_r - (u_r/2c)L_1v, R_ih = LR_iv - (R_iu/2c)L_1v.$$

Using the inequality of **a.** for v_t, v_r yields the desired terms. For R_iv, we obtain

$$\rho_i(t) \equiv (1+t)^{-1}|e^{p/2}R_iv(.,t)|_{L^2} \leqslant C\varepsilon \int_0^t (1+t')^{-1}\rho_i(t')dt' + C\int_0^t (1+t')^{-1}[|e^{p/2}R_ih|_{L^2} + |R_iu|_{L^\infty}|e^{p/2}(\partial v)|_{L^2}]dt'.$$

Dividing both sides by $(1+t)$, using the support condition and the fact that $t' \leqslant t$ in the integrals, we get

$$(1+t)^{-1}|e^{p/2}(R_i/r)v(.,t)|_{L^2} \leqslant C\int_0^t (1+t')^{-1}|e^{p/2}(R_i/r)h|_{L^2}dt' + C\varepsilon\int_0^t (1+t')^{-2}|e^{p/2}(\partial v)|_{L^2}dt'.$$

Since $\partial_i = \omega_i\partial_r - (\omega \wedge (R/r))_i$, this gives ii). □

VI. Commutations with the operator P

VI.1. Computation of $[\widetilde{Z}_m, P]$ and consequences. — Recall that

$$P = c^{-1}\partial_t^2 - c\Delta.$$

To establish formula describing $[\widetilde{Z}_m, P]$, we compute separately the two terms $[Z, P]$, which involves only u, and $[\widetilde{a}L_1, P]$.

Lemma 1.1. — *We have the formula $(1.2)_a$, $(1.2)_b$, $(1.2)_c$, $(1.2)_d$. Away from $r = 0$, we also have the formula*

$$(1.1)_a \quad [\widetilde{K}, P] = \widetilde{K}u/cP - \frac{1}{8c}(Lu + 3L_1u)L_1^2 - \frac{1}{8c}(3Lu + L_1u)L^2 - c^{-3/2}u_tL_1L - \frac{1}{\sqrt{c}r^2}R_juR_j\partial_t - (1/2c)[3/2c^3u_t^2 - 1/4c^2u_tu_r + 1/4cu_r^2 + 3(u_t^2/c^2 - |u_x|^2)]L_1 - (1/2c)[L_1u/2c(\sqrt{c}/2u_r - u_t/\sqrt{c}) + 3(u_t^2/c^2 - |u_x|^2)]L,$$

$$(1.1)_b \quad [R_i,P]=(R_iu/c)P-(R_iu/2c)L_1^2-(R_iu/2c)L^2-(R_iu/c)LL_1 \\ -R_iuLu/4c^2L_1-R_iu/2c^2(Lu+L_1u/2)L,$$

$$(1.1)_c \quad [S,P]=(Su/c-2)P-(Su/2c)L_1^2-(Su/2c)L^2-(Su/c)LL_1 \\ -SuLu/4c^2L_1-Su/2c^2(Lu+L_1u/2)L,$$

$$(1.1)_d \quad [H_i,P]=(H_iu/c)P+(-H_iu/2c+\frac{\omega_i(r+ct)}{2c\sqrt{c}}Lu)L_1^2 \\ +(-H_iu/2c+\frac{\omega_i(r-ct)}{2c\sqrt{c}}L_1u)L^2+(-H_iu/c+x_iu_t/c^2-t\omega_iu_r)LL_1 \\ -(tL_1uL+tLuL_1)(\partial_i-\omega_i\partial_r)+2ct/r^2R_juR_j\partial_i-2x_i/cr^2R_juR_j\partial_t-2u_t/c\partial_i \\ -(H_iuLu/4c^2-\frac{\omega_i(r-ct)}{4c^2\sqrt{c}}LuL_1u+x_i/c\sqrt{c}(u_t^2/c^2-|u_x|^2)+\partial_iu/\sqrt{c})L_1 \\ -(H_iuLu/2c^2+L_1u/4c^2(H_iu-\omega_iSu)+\partial_iu/\sqrt{c}+\omega_i/c\sqrt{c}(ctu_r^2-r|u_x|^2))L.$$

Proof

a. We have

$$(1.2)_a \qquad [\widetilde{K},P]=\widetilde{K}u/cP-\widetilde{K}u/c^2\partial_t^2-c^{-1/2}\sum u_j\partial_{jt}^2,$$

$$(1.2)_b \quad [R_i,P]=[R_i,c^{-1}\partial_t^2]-[R_i,c\Delta]=-R_iu/c^2\partial_t^2-R_iu\Delta=R_iu/cP-2R_iu/c^2\partial_t^2.$$

Similarly, since $[S,\partial_t^2]=-2\partial_t^2$, $[S,\Delta]=-2\Delta$,

$$(1.2)_c \qquad [S,P]=[S,c^{-1}\partial_t^2]-[S,c\Delta]=-Su/c^2\partial_t^2-Su\Delta-2/c\partial_t^2+2c\Delta \\ =(Su/c-2)P-2Su/c^2\partial_t^2.$$

b. We have

$$[\partial_t^2,H_i]=2((c+tu_t)\partial_{it}^2-x_iu_t/c^2\partial_t^2)+(2u_t+tu_{tt})\partial_i+x_i/c^2(2u_t^2/c-u_{tt})\partial_t,$$

$$[\Delta,H_i]=2(tu_j\partial_{ij}^2-x_i/c^2u_j\partial_{jt}^2+1/c\partial_{it}^2) \\ +t\Delta u\partial_i-2/c^2u_i\partial_t+x_i/c^2(2|u_x|^2/c-\Delta u)\partial_t.$$

Hence

$$[H_i,P]=H_iu/cP-2H_iu/c^2\partial_t^2-2u_t/c(t\partial_{it}^2-x_i/c^2\partial_t^2)+2u_j(ct\partial_{ij}^2-x_i/c\partial_{jt}^2) \\ -2u_t/c\partial_i-2/c^2(cu_i+x_i(u_t^2/c^2-|u_x|^2)\partial_t.$$

We can write

$$\begin{aligned}
ct\partial^2_{it} &= \partial_t H_i - x_i/c\partial_t^2 - (c+tu_t)\partial_i + x_iu_t/c^2\partial_t,\\
ct\partial^2_{ij} &= \partial_j H_i - x_i/c\partial^2_{jt} - tu_j\partial_i + (x_iu_j/c^2 - \delta_{ij}/c)\partial_t,\\
ct\partial^2_{ij} &= \partial_j H_i - (x_i/c^2t)\partial_t H_j + (x_ix_j/c^3t)\partial_t^2\\
&\qquad + (x_i/c^2t)(c+tu_t)\partial_j - tu_j\partial_i - c^{-1}(\delta_{ij} + (x_ix_j/c^3t)u_t - x_iu_j/c)\partial_t.
\end{aligned}$$

Using these identities to express all second order derivatives as ∂_t^2 modulo ∂H_k, we get

$$\begin{aligned}
(1.2)_d\quad [H_i,P] &= H_iu/cP - 2H_iu/c^2\partial_t^2 - 2u_t/c^2\partial_t H_i - 4(x_iu_j/c^2t)\partial_t H_j\\
&+ 2u_j\partial_j H_i + 4(x_i/c^3t)Su\partial_t^2 + 2t(u_t^2/c^2 - |u_x|^2)\partial_i + 4(x_iu_j/c^2t)(c+tu_t)\partial_j\\
&- 4x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 4/c(u_i + x_ix_ju_ju_t/c^3t)\partial_t.
\end{aligned}$$

If we are away from $r=0$, we can handle differently, using the identity

$$\sum v_j\partial_j = v_r\partial_r + 1/r^2\sum R_jvR_j.$$

We write then

$$\begin{aligned}
[H_i,P] &= H_iu/cP - 2H_iu/c^2\partial_t^2 + 2x_iu_t/c^3\partial_t^2\\
&- 2tu_t/c\partial_t(\partial_i - \omega_i\partial_r + \omega_i\partial_r) + 2ct(u_r\partial_r\partial_i + 1/r^2R_juR_j\partial_i)\\
&- 2x_i/c(u_r\partial^2_{rt} + 1/r^2R_juR_j\partial_t) - 2u_t/c\partial_i - 2x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 2u_i/c\partial_t\\
&= H_iu/cP - 2H_iu/c^2\partial_t^2 - 2(tu_t/c\partial_t - ctu_r\partial_r)(\partial_i - \omega_i\partial_r)\\
&+2/r^2(ctR_juR_j\partial_i - x_i/cR_juR_j\partial_t) - 2u_t/c\partial_i - 2x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 2u_i/c\partial_t + 2\sum,
\end{aligned}$$

where $\sum$ means here the sum of the following four terms

$$\sum = x_iu_t/c^3\partial_t^2 - t\omega_iu_t/c\partial^2_{rt} - x_iu_r/c\partial^2_{rt} + ctu_r\omega_i\partial_r^2.$$

Using the identities

$$\begin{aligned}
2/c\partial_t^2 &= 1/2L^2 + 1/2L_1^2 + LL_1 + Lu/4cL_1 + (L_1u + 2Lu)/4cL,\\
2c\partial_r^2 &= 1/2L^2 + 1/2L_1^2 - LL_1 + Lu/4cL_1 + (L_1u - 2Lu)/4cL,\\
4\partial^2_{rt} &= L^2 - L_1^2 + Lu/2cL_1 - L_1u/2cL,
\end{aligned}$$

we obtain from **a.** the desired forms for $[\widetilde{K},P]$, $[R_i,P]$ and $[S,P]$. In the present computation of $[H_i,P]$, we get

$$\begin{aligned}
2\sum &= \frac{\omega_i(r-ct)}{2c\sqrt{c}}L_1uL^2 + \frac{\omega_i(r+ct)}{2c\sqrt{c}}LuL_1^2 + (x_iu_t/c^2 - t\omega_iu_r)LL_1\\
&\qquad + (x_iu_t^2/c^3\sqrt{c} - t\omega_iu_r^2/\sqrt{c} + \omega_iL_1uSu/4c^2)L + \frac{\omega_i(r-ct)}{4c^2\sqrt{c}}LuL_1uL.
\end{aligned}$$

After some algebraic manipulations, we get the result for $[H_i,P]$. □

Lemma 1.2. — *We have*

$$
\begin{aligned}
[\widetilde{a}L_1, P] = (-L_1\widetilde{a} + \widetilde{a}L_1u/c)P - (L\widetilde{a} + \widetilde{a}L_1u/2c)L_1^2 + \widetilde{a}/c(Lu/2 - L_1u)L_1L \\
+ r^{-2}R_j\widetilde{a}R_jL_1 - \widetilde{a}r^{-2}R_juR_jL - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega \\
+ [-L_1L\widetilde{a} - L_1u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a} \\
- \widetilde{a}(-c/r^2 + (-L_1u)^2/2c^2 + LuL_1u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1 \\
+ [L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1uu_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L.
\end{aligned}
$$

Proof. — We have

$$
\begin{aligned}
[L_1, \partial_t^2] &= u_t/c\partial_tL + u_{tt}/2cL - 1/4c^{-5/2}u_t^2\partial_t - 3/4c^{-3/2}u_t^2\partial_r, \\
[L_1, \Delta] &= u_j/c\partial_jL + 2\sqrt{c}/r^3\Delta_\omega - 1/4c^{-5/2}|u_x|^2\partial_t \\
&\qquad - 3/4c^{-3/2}|u_x|^2\partial_r + \Delta u/2cL + 2\sqrt{c}/r^2\partial_r,
\end{aligned}
$$

hence, writing here $b = \widetilde{a}$,

$$
\begin{aligned}
[bL_1, P] = bL_1u/cP - 2b/c^2L_1u\partial_t^2 - (Pb)L_1 - 2b_t/c\partial_tL_1 + 2cb_r\partial_rL_1 \\
+ 2c/r^2R_jbR_jL_1 + bu_t/c^2\partial_tL - bu_r\partial_rL - b/r^2R_juR_jL - 2bc\sqrt{c}/r^3\Delta_\omega \\
- 2bc\sqrt{c}/r^2\partial_r - b/4c(u_t^2/c^2 - |u_x|^2)(2L - L_1).
\end{aligned}
$$

The strategy is the following: after some algebraic arrangements, we express LL_1 using P only in the term $(L_1b)LL_1$, and take a careful look at the first order terms. We have first

$$
\begin{aligned}
-2b_t/c\partial_tL_1 + 2cb_r\partial_rL_1 &= -(Lb)L_1^2 - (L_1b)LL_1, \\
u_t/c^2\partial_tL - u_r\partial_rL &= (Lu/2c)L_1L + (L_1u/2c)L^2.
\end{aligned}
$$

Next

$$
-2b/c^2L_1u\partial_t^2 = -b/cL_1u(LL_1 + 1/2L^2 + 1/2L_1^2 + Lu/4cL_1 + u_t/c\sqrt{c}L).
$$

Now we replace, in the term $(L_1b)LL_1$,

$$
LL_1 = P + c/r^2\Delta_\omega + 2c/r\partial_r - Lu/2cL,
$$

which gives

$$
\begin{aligned}
[bL_1, P] = (bL_1u/c - L_1b)P - (Lb + bL_1u/2c)L_1^2 + b(Lu/2c - L_1u/c)L_1L \\
+ 2c/r^2R_jbR_jL_1 - b/r^2R_juR_jL - cr^{-2}(2b\sqrt{c}/r + L_1b)\Delta_\omega + Q_1,
\end{aligned}
$$

where the first order terms Q_1 are

$$
\begin{aligned}
Q_1 &= 2c/r(b_rL_1 - (L_1b)\partial_r) + q_1L_1 + q_2L, \\
q_1 &= -b/2c^2(L_1u)^2 - b/4c^2LuL_1u + bc/r^2 - L_1u/2cL_1b \\
&\qquad - L_1Lb + c/r^2\Delta_\omega b - 1/4c(u_t^2/c^2 - |u_x|^2), \\
q_2 &= -b/c^2\sqrt{c}L_1uu_t - bc/r^2 + LuL_1b/2c + 1/2c(u_t^2/c^2 - |u_x|^2).
\end{aligned}
$$

It is important to remark that

$$b_rL_1 - L_1b\partial_r = b_r(L - 2\sqrt{c}\partial_r) - (Lb - 2\sqrt{c}b_r)\partial_r = b_rL - Lb\partial_r$$
$$= b_rL - (Lb)/2\sqrt{c}(L - L_1) = 1/2\sqrt{c}(-L_1bL + LbL_1).$$

Collecting the terms gives the result. □

Putting together the two above Lemmas yields the desired expression.

Lemma 1.3. — *We have the formula*

i) $$[\widetilde{R}_i^m, P] = (R_iu/c - L_1\widetilde{a} + \widetilde{a}L_1u/c)P - \widetilde{A}(R_i)L_1^2 - R_iu/2cL^2$$
$$+(-R_iu/c+\frac{\widetilde{a}}{c}(Lu/2-L_1u))L_1L+r^{-2}R_j\widetilde{a}R_jL_1-\widetilde{a}r^{-2}R_juR_jL-cr^{-2}(L_1\widetilde{a}+2\widetilde{a}\sqrt{c}/r)\Delta_\omega$$
$$+[-R_iu/2c^2(L_1u + Lu/2) - L_1L\widetilde{a} - L_1u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1u)^2/2c^2 + LuL_1u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+[-R_iu/4c^2L_1u+L_1\widetilde{a}(Lu/2c-\sqrt{c}/r)-\widetilde{a}(c/r^2+L_1uu_t/c^2\sqrt{c}+1/2c(u_t^2/c^2-|u_x|^2))]L,$$

ii) $$[\widetilde{S}^m, P] = (Su/c - 2 - L_1\widetilde{a} + \widetilde{a}L_1u/c)P - \widetilde{A}(S)L_1^2 - (Su/2c)L^2$$
$$+(-Su/c+\frac{\widetilde{a}}{c}(Lu/2-L_1u)))L_1L+r^{-2}R_j\widetilde{a}R_jL_1-\widetilde{a}r^{-2}R_juR_jL-cr^{-2}(L_1\widetilde{a}+2\widetilde{a}\sqrt{c}/r)\Delta_\omega$$
$$+[-Su/2c^2(L_1u + Lu/2) - L_1L\widetilde{a} - L_1u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1u)^2/2c^2 + LuL_1u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+[-Su/4c^2L_1u+L_1\widetilde{a}(Lu/2c-\sqrt{c}/r)-\widetilde{a}(c/r^2+L_1uu_t/c^2\sqrt{c}+1/2c(u_t^2/c^2-|u_x|^2))]L,$$

iii) $$[\widetilde{H}_i^m, P] = (H_iu/c - L_1\widetilde{a} + \widetilde{a}L_1u/c)P - \widetilde{A}(H_i)L_1^2 - \frac{r-ct}{2cr}(\omega \wedge Ru)_iL_1^2$$
$$+(-H_iu/c + x_iu_t/c^2 - t\omega_iu_r + \frac{\widetilde{a}}{c}(Lu/2 - L_1u))L_1L + (-H_iu/2c + \frac{\omega_i(r-ct)}{2c\sqrt{c}}L_1u)L^2$$
$$- (tL_1uL + tLuL_1)(\partial_i - \omega_i\partial_r) + 2ct/r^2R_juR_j\partial_i - 2x_i/cr^2R_juR_j\partial_t$$
$$+ r^{-2}R_j\widetilde{a}R_jL_1 - \widetilde{a}r^{-2}R_juR_jL - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega - 2u_t/c\partial_i$$
$$+ [-H_iu/2c^2(L_1u + Lu/2) + \frac{\omega_i}{4c^2\sqrt{c}}L_1u(-2c\sqrt{c}tu_r + (r-ct)Lu) - \partial_iu/\sqrt{c}$$
$$+ x_i/2c^2(u_tL_1u - 2\sqrt{c}(u_t^2/c^2 - |u_x|^2)) - L_1L\widetilde{a} - L_1u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1u)^2/2c^2 + LuL_1u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+[+L_1u/4c^2(\omega_iSu - H_iu) - \partial_iu/\sqrt{c} + \omega_i/c\sqrt{c}(r|u_x|^2 - ctu_r^2) + Lu/2c(t\omega_iu_r - x_iu_t/c^2)$$
$$+ L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1uu_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L.$$

Here,

$$\begin{aligned}\widetilde{A}(R_i) &= L\widetilde{a}(R_i) + L_1u/2c\widetilde{a}(R_i) + R_iu/2c = L\widetilde{a}(R_i) + \widetilde{R}_i^m u/(2c),\\ \widetilde{A}(S) &= L\widetilde{a}(S) + L_1u/2c\widetilde{a}(S) + Su/2c = L\widetilde{a}(S) + \widetilde{S}^m u/(2c),\\ \widetilde{A}(H_i) &= -\omega_i\widetilde{A}(S) - (\omega\wedge\widetilde{A}(R))_i.\end{aligned}$$

Proof. — The formula are obtained by just adding the formula of Lemma 1.1 and 1.2, and using $[L, L_1] = L_1u/2cL_1 - Lu/2cL$ to replace LL_1 by L_1L. The expressions of $\widetilde{A}(R_i)$ and $\widetilde{A}(S)$ are clear. We get

$$\widetilde{A}(H_i) = H_iu/2c - \frac{\omega_i(r+ct)}{2c\sqrt{c}}Lu + L\widetilde{a}(H_i) + L_1u/2c\widetilde{a}(H_i) - \frac{r-ct}{2cr}(\omega\wedge Ru)_i.$$

Since

$$H_i = \omega_iH_0 - ct/r(\omega\wedge R)_i, \omega_iH_0 = \frac{\omega_i(r+ct)}{\sqrt{c}}L - \omega_iS,$$

we obtain

$$\widetilde{A}(H_i) = L\widetilde{a}(H_i) + L_1u/2c\widetilde{a}(H_i) - \omega_iSu/2c - 1/2c(\omega\wedge Ru_i).$$

Using the definition of $\widetilde{a}(H_i)$, we get the result. □

We will dispatch the terms in Lemma 1.3 into three categories:

i) A term which can be written in the form

$$M_1\alpha\partial\widetilde{Z}_m, \quad M_1\alpha\sigma_1^{-1}\widetilde{Z}_m, M_1\alpha\partial$$

will be called "standard"(st.); otherwise, it will be called "special" (sp.).

ii) A standard term for which, for some $\gamma > 0$,

$$|\alpha| \leqslant C(1+t)^{-1-\gamma}$$

will be called integrable (int.). Otherwise, it will be called "just".

Rewriting appropriately the terms in Lemma 1.3, we obtain the following Proposition.

Proposition 1. — *We have*

$$\begin{aligned}[\widetilde{Z}_m, P] &= \delta P + \textstyle\sum_1 + \sum_2 + \sum_3,\\ \delta &= f\widetilde{Z}_mu + f\partial\widetilde{a} + f\partial u\widetilde{a},\end{aligned}$$

i) $\sum_1$ *is a sum of standard integrable terms with*

$$\alpha = \sigma_1/(1+t)^2(\varepsilon^{-1}(1+t)\partial u), \quad \alpha = \sigma_1/(1+t)^2\widetilde{N}_1.$$

ii) $\sum_2$ *is the sum of the just standard terms*

$$\textstyle\sum_2 = f\partial u\partial\widetilde{Z}_m + f\partial\widetilde{Z}_mu\partial + f\partial u\partial,$$

iii) $\sum_3$ *is the sum of the special terms*

$$\sum_3 = -\widetilde{A}L_1^2 + r^{-2}R_j\widetilde{a}R_jL_1 + fr^{-2}L_1\widetilde{a}\Delta_\omega + cr^{-2}\Delta_\omega\widetilde{a}L_1$$
$$+ f(1+t)^{-1}\partial u R_j\widetilde{a}\partial + fL_1L\widetilde{a}\partial + f\partial u L_1\widetilde{a}\partial + f(1+t)^{-1}L\widetilde{a}\partial.$$

Proof. — We proceed by inspecting the terms in Lemma 1.3, after an appropriate rewriting. We discuss only the terms in $[\widetilde{H}_i, P]$, which are the most difficult, examining the terms in the order they appear in the Lemma. The terms of the other $[\widetilde{Z}_m, P]$ have the same forms. The special terms will be discussed in the next Proposition.

1. The term $-\widetilde{A}L_1^2$ is special.
2. We have

$$r - ct = f + \sigma_1 f = f\sigma_1,$$

hence

$$\frac{r-ct}{2cr}(\omega \wedge Ru)_i = f\sigma_1/(1+t)R_ju,$$

and using Lemma IV.3.1,

$$\sigma_1/(1+t)R_juL_1^2 = M_1/(1+t)\widetilde{Z}_m u([\widetilde{Z}_m, L_1] + f\partial\widetilde{Z}_m).$$

Since $[\widetilde{Z}_m, L_1] = M_1\partial$, the term is

$$M_1\varepsilon\sigma_1/(1+t)^2\widetilde{N}_1(\partial + \partial\widetilde{Z}_m).$$

It is st. int. with $\alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1$.

3. Recalling that

$$L = \frac{\sqrt{c}}{r+ct}(\sum \omega_i\widetilde{H}_i + \widetilde{S}),$$

we note first

$$L_1(\frac{\sqrt{c}}{r+ct}) = \frac{r-ct}{2\sqrt{c}(r+ct)^2}L_1u, \quad L(\frac{\sqrt{c}}{r+ct}) = \frac{1}{2\sqrt{c}(r+ct)^2}((r-ct)Lu - 4c\sqrt{c}).$$

Hence

$$L_1L = \frac{f\sigma_1}{(1+t)^2}\partial u\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m, \tag{1.3}$$

$$L^2 = f/(1+t)^2\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m. \tag{1.4}$$

We write

$$M_1\widetilde{Z}_m uL_1L = M_1\varepsilon\sigma_1/(1+t)\widetilde{N}_1(\sigma_1/(1+t)^2(\partial u)\widetilde{Z}_m + 1/(1+t)\partial\widetilde{Z}_m,$$

hence both terms are st. int., with

$$\alpha = \varepsilon^2\sigma_1^3/(1+t)^4\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

We write

$$ft\partial uL_1L = f(\partial u)^2\sigma_1/(1+t)\widetilde{Z}_m + f\partial u\partial\widetilde{Z}_m,$$

the second term is (just) while the first is st. int. with

$$\alpha = \varepsilon^2\sigma_1^2/(1+t)^3((1+t)\partial u/\varepsilon).$$

We write

$$f\widetilde{a}\partial uL_1L = (\sigma_1^{-1}\widetilde{a})(f\sigma_1^2(\partial u)^2/(1+t)^2\widetilde{Z}_m + f\sigma_1/(1+t)\partial u\partial\widetilde{Z}_m),$$

showing that both terms are st. int. with

$$\alpha = \varepsilon^2\sigma_1^3(1+t)^4\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

We write

$$fH_iuL^2 = M_1\widetilde{Z}_mu(f/(1+t)^2\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m),$$

hence both terms are st. int., with

$$\alpha = \varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

4. To handle the term

$$tL_1uL(\partial_i - \omega_i\partial_r) = fL_1u\widetilde{Z}_m(\omega_iR_j/r) = M_1(\partial u)/(1+t)R_j + f\partial u[\widetilde{Z}_m, R_j/r] + f\partial u\partial\widetilde{Z}_m,$$

we need Lemma IV.4.3. The term

$$M_1((1+t)\partial u\varepsilon)\varepsilon/(1+t)^2\widetilde{Z}_m$$

is st. int. with

$$\alpha = \varepsilon\sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

According to Lemma IV.4.3, he middle-term is equal to

$$M_1\partial u/(1+t)\widetilde{Z}_m + M_1\sigma_1/(1+t)\partial u\partial + f\partial u/(1+t)R_j\widetilde{a}\partial.$$

The last term is sp., the first two are st. int. with

$$\alpha = \varepsilon\sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

5. We write the term $tLuL_1(\partial_i - \omega_i\partial_r)$ as

$$f\widetilde{Z}_muL_1(R/r) = f/(1+t)^2\widetilde{Z}_muR + f/(1+t)\widetilde{Z}_mu([L_1,R] + RL_1).$$

Since

$$[L_1,R] = fRuL, RL_1 = f\widetilde{N}_1^l\widetilde{Z}_mL_1, [\widetilde{Z}_m, L_1] = f\widetilde{N}_1\partial,$$

the term is

$$M_1/(1+t)^2\widetilde{Z}_mu\widetilde{Z}_m + M_1/(1+t)\widetilde{Z}_mu(\widetilde{Z}_muL + \partial + \partial\widetilde{Z}_m).$$

All three terms are st. int. with

$$\alpha = \varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

6. We write

$$f/(1+t)R_juR_j\partial_i = M_1/(1+t)\widetilde{Z}_mu([\widetilde{Z}_m,\partial_i] + \partial\widetilde{Z}_m).$$

In view of Lemma IV.4.2, both terms are st. int. with $\alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1$. 7. The term $r^{-2}R_j\widetilde{a}R_jL_1$ is sp.

We write

$$\widetilde{a}r^{-2}R_juR_jL = M_1\widetilde{a}/(1+t)^2\widetilde{Z}_mu([\widetilde{Z}_m, L] + L\widetilde{Z}_m),$$

showing that both terms are st. int. with $\alpha=\varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1$. The next term is sp., then we write

$$f\widetilde{a}r^{-3}\widetilde{R}_j^2=f\widetilde{a}/r^2R_j(R_j/r)=M_1\sigma_1/(1+t)^2\widetilde{N}_1([\widetilde{Z}_m,\partial]+\partial+\partial\widetilde{Z}_m),$$

which gives three st. int. terms with $\alpha=\sigma_1/(1+t)^2\widetilde{N}_1$.

8. We reach now the first order terms. While

$$f\partial u\partial,\quad f(1+t)(\partial u)^2\partial=f\varepsilon\frac{(1+t)\partial u}{\varepsilon}\partial u\partial=f\partial u\partial,$$

are just, we write

$$f(\partial u)Zu\partial=M_1\widetilde{Z}_mu\partial u\partial,$$

which is a st. int. term with $\alpha=\varepsilon^2\sigma_1/(1+t)^2\widetilde{N}_1$.

9. The next four terms containing $\widetilde{a}$ are special.

10. We write then

$$f/(1+t)^2\widetilde{a}\partial=f\sigma_1/(1+t)^2\widetilde{N}_1\partial,\quad f\widetilde{a}(\partial u)^2\partial=f\varepsilon^2\sigma_1/(1+t)^2((1+t)\partial u\varepsilon)\partial,$$

hence the two terms are st. int. with

$$\alpha=\sigma_1/(1+t)^2\widetilde{N}_1,\quad \alpha=\sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

Finally, the terms in L have exactly the same structure, with the exception of

$$f(1+t)^{-1}L_1\widetilde{a}L=fL_1\widetilde{a}(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m)$$

which is st. int. with $\alpha=\sigma_1/(1+t)^2\widetilde{N}_1$. □

The following Lemma displays the structure of the most delicate terms in $\sum_3$.

Lemma 1.4. — *We have*

$$\begin{aligned}
\sqrt{c}L\widetilde{a}&=-S_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_mu+(a-\widetilde{a})L_1u))+\varepsilon(1+t)^{-1}s_\theta a+[u,S_\theta]a_r,\\
\partial_r(\sqrt{c}L\widetilde{a})&=-\partial_rS_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_mu+(a-\widetilde{a})L_1u))\\
&\quad+\varepsilon(1+t)^{-1}s_\theta a_r+u_rS_\theta a_r+\theta_1[u,s_\theta]a_r,\\
\partial_t(\sqrt{c}L\widetilde{a})&=-\partial_tS_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_mu+(a-\widetilde{a})L_1u))\\
&\quad+\varepsilon/(1+t)^2s_\theta a+\varepsilon/(1+t)s_\theta a_t+u_tS_\theta a_r\\
&\quad+u/(1+t)s_\theta a_r+(1+t)^{-1}s_\theta ua_r-S_\theta u_ta_r+S_\theta u_ra_t+\theta_1[u,s_\theta]a_t,\\
r^{-2}R_i\widetilde{a}R_iL_1&=M_1(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m+M_1(1+t)^{-2}\theta_2(s_\theta a)\partial,\\
fr^{-2}L_1\widetilde{a}\Delta_\omega&=M_1(1+t)^{-1}(\partial\widetilde{a})(R/r)\widetilde{Z}_m\\
&\quad+M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m)+M_1(1+t)^{-2}(\partial\widetilde{a})(\sigma_1\partial+\theta_2(s_\theta a))\partial),\\
cr^{-2}\Delta_\omega\widetilde{a}\partial&=f(1+t)^{-2}\theta_2^2(s_\theta a)\partial,\ f(1+t)^{-1}\partial uR\widetilde{a}\partial=f(1+t)^{-2}\theta_2(s_\theta a)\partial.
\end{aligned}$$

Moreover, we have

$$|(1+|\partial_t\widetilde{a}|+|\partial_r\widetilde{a}|+\widetilde{a}^2/\sigma_1+\varepsilon^{-1}(1+t)(|Z_0\partial u|+|Z_0u/\sigma_1|))\sigma_1^{-1}\widetilde{A}|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1-\varepsilon}.$$

Proof

a. We have

$$\sqrt{c}L\widetilde{a}=(\partial_t+c\partial_r)\widetilde{a}=\varepsilon/(1+t)s_\theta a+[c,S_\theta]a_r-S_\theta(\sqrt{c}La),$$

which gives the first formula. The second formula follows, since

$$\partial_r s_\theta a=s_\theta a_r,\quad \partial_r S_\theta b=\theta_1 s_\theta b.$$

The third is also clear, using the formula of Lemma IV.5.1, since $S_\theta b_r=\theta_1 s_\theta b$.

b. From the definition of s_θ, we have

$$R_i s_\theta b=h(\omega)\theta_2 s_\theta b,\quad \Delta_\omega S_\theta b=f\theta_2^2 s_\theta b.$$

Hence

$$\begin{aligned} r^{-2}R_j\widetilde{a}R_jL_1&=h(\omega)r^{-2}\theta_2 s_\theta aR_jL_1=M_1(1+t)^{-2}\theta_2 s_\theta a(M_1\partial+L_1\widetilde{Z}_m),\\ cr^{-2}\Delta_\omega\widetilde{a}L_1&=f(1+t)^{-2}\theta_2^2 s_\theta a\partial,\\ f(1+t)^{-1}\partial uR_j\widetilde{a}\partial&=f(1+t)^{-2}\theta_2 s_\theta a\partial.\end{aligned}$$

c. We have

$$fr^{-2}L_1\widetilde{a}R_j^2=fr^{-1}L_1\widetilde{a}R_j(R_j/r)=M_1(1+t)^{-1}\partial\widetilde{a}([\widetilde{Z}_m,R_j/r]+R_j/r\widetilde{Z}_m).$$

Using Lemma IV.4.3, we get the result.

d. Using Lemma IV.5.1 to evaluate $[\partial_t,S_\theta]$ and Lemma IV.3.1 to express R, we can write

$$\begin{aligned}\sqrt{c}\widetilde{A}&=\varepsilon(1+t)^{-1}\theta_1^{-1}s_\theta a_r+\varepsilon(1+t)^{-1}\theta_2^{-1}s_\theta M_1\widetilde{Z}_m^r a+[u,S_\theta]a_r\\ &\quad-S_\theta(\chi/(2\sqrt{c})L_1u(a-\widetilde{a}))+\frac{1-\chi}{2\sqrt{c}}\widetilde{Z}_mu+\chi/(2\sqrt{c})\widetilde{Z}_mu-S_\theta(\chi/(2\sqrt{c})\widetilde{Z}_mu).\end{aligned}$$

Using the already established formula $s_\theta(\sigma_1 b)=\sigma_1 s_\theta b+\theta_1^{-1}s_\theta b$, we can bound the first three terms of $\sigma_1^{-1}\widetilde{A}$ by

$$C\varepsilon(1+t)^{-1}\theta_1^{-1}|\partial a|+C\varepsilon(1+t)^{-1}\theta_2^{-1}|M_1|(|\sigma_1^{-1}\widetilde{Z}_m^r a|+|a_r|).$$

Next, since

$$\begin{aligned}|b-S_\theta b|&\leqslant C\theta_1^{-1}|b_r|+C\theta_2^{-1}(|b|+|Rb|),\\ |a-\widetilde{a}|&\leqslant C\theta_1^{-1}|\partial a|+C\theta_2^{-1}(|a|+|M_1||\widetilde{Z}_m a|),\end{aligned}$$

$$|\chi/(2\sqrt{c})\widetilde{Z}_mu/\sigma_1-S_\theta(\chi/(2\sqrt{c})\widetilde{Z}_mu/\sigma_1)|\leqslant C\varepsilon(1+t)^{-1}(\theta_1^{-1}|M_1|+\theta_2^{-1}|M_2|).$$

Note that the error term produced when introducing σ_1^{-1} in S_θ is bounded by

$$C\varepsilon(1+t)^{-1}\theta_1^{-1}|M_1|.$$

Finally, observing that

$$|\widetilde{Z}_m u| \leqslant C|Z_0 u|, |\partial_t \widetilde{a}| + |\partial_r \widetilde{a}| \leqslant C|a|/(1+t) + C|a_t| + C|a_r|,$$

we see that we can use Lemma IV.1.2 to control the terms containing $(1-\chi)\sigma_1^{-1}\widetilde{Z}_m u$. Taking β_1 big enough with respect to $|M_2|$ yields the desired estimate. □

VI.2. Higher order commutators. — Taking a standard cutoff $\overline{\chi} = \overline{\chi}(r/(1+t))$ (that is $\overline{\chi}(s)$ is zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$), we write

$$[\widetilde{Z}_m, P] = \overline{\chi}[\widetilde{Z}_m, P] + (1-\overline{\chi})[\widetilde{Z}_m, P],$$

and use the formula of Lemma 1.3 for the first term, the formula (1.2) of Lemma 1.1 for the second.

We need now a Lemma describing the structure of $[\widetilde{Z}_m^k, P]$.

Lemma 2.1. — *Writing in short*

$$[\widetilde{Z}_m, P] = \delta P + Q,$$

we have

i) $$[\widetilde{Z}_m^k, P] = \sum \widetilde{Z}_m^{l_1}\delta \cdots \widetilde{Z}_m^{l_i}\delta \widetilde{Z}_m^p P + \sum \widetilde{Z}_m^{k_1}\delta \cdots \widetilde{Z}_m^{k_j}\delta \widetilde{Z}_m^q Q \widetilde{Z}_m^p.$$

By an abuse of notation, we do not put indexes for the δ and Q, though there is one for each $\widetilde{Z}_m$. In the first sum,

$$i \geqslant 1, \quad p + \sum(l_j + 1) \leqslant k.$$

In the second sum,

$$q + p + \sum(k_i + 1) \leqslant k - 1.$$

ii) $$\widetilde{Z}_m^q[\widetilde{Z}_m, P]\widetilde{Z}_m^p = \sum \widetilde{Z}_m^{l_1}\delta \cdots \widetilde{Z}_m^{l_i}\delta \widetilde{Z}_m^{p_1} P + \sum \widetilde{Z}_m^{k_1}\delta \cdots \widetilde{Z}_m^{k_j}\delta \widetilde{Z}_m^{p_1} Q \widetilde{Z}_m^{p_2}.$$

In the first sum,

$$i \geqslant 1, \quad p_1 + \sum(l_j + 1) \leqslant p + q + 1.$$

In the second sum,

$$p_1 + p_2 + \sum(k_i + 1) \leqslant p + q.$$

Proof. — For $k = 1$, i) is clear. We write now

$$[\widetilde{Z}_m^{k+1}, P] = \widetilde{Z}_m[\widetilde{Z}_m^k, P] + [\widetilde{Z}_m, P]\widetilde{Z}_m^k.$$

We see that $\widetilde{Z}_m$ acting on both sums yields only correct terms. On the other hand,

$$[\widetilde{Z}_m, P]\widetilde{Z}_m^k = \delta[P, \widetilde{Z}_m^k] + \delta \widetilde{Z}_m^k P + Q\widetilde{Z}_m^k,$$

and all three terms are of the desired form. This proves i). The proof of ii) is completely similar. □

VII. L^2 estimates of u and a

Using the structure of $P\widetilde{Z}_m^{k+1}u$ displayed in VI, and the energy inequality for P, we want to estimate now $|\partial\widetilde{Z}_m^{k+1}u|_0$. Similarly, we will estimate $|\partial\widetilde{Z}_m^k a|_0$. To this aim, we introduce some notations. We set, with $a = a(R_i)$ or $a = a(S)$,

$$A_k = (1+t)^{-1}(|\sigma_1^{-1}\widetilde{Z}_m^{k-1}a|_0 + |\widetilde{Z}_m^{k-1}\partial a|_0),\ k \geqslant 1, \quad \phi_k = \sum |\widetilde{N}_k|_0,\ k \geqslant 0,$$

$$\phi'_k = \varepsilon^{-1}(1+t)(|\widetilde{Z}_m^k\partial u|_0 + |\sigma_1^{-1}\widetilde{Z}_m^k u|_0) + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0 + |\sigma_1^{-1}\widetilde{Z}_m^{k-1}\widetilde{a}|_0,\ k \geqslant 1.$$

The point of these notations is that the "bad" $\widetilde{N}_k$ is $\widetilde{Z}_m^k\widetilde{a}$, that we were forced to introduce to have Lemma IV.3.1. The quantity ϕ'_k is just ϕ_k deprived of this bad term. Note that, since $\widetilde{a}$ is supported for $\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}$, we have

$$\phi_k \leqslant C(1+t)^{C_0\varepsilon}\phi'_k,$$

but this "small" amplification factor is very important in all this paper. According to Lemma V.1.1, we have

$$\phi'_k \sim \varepsilon^{-1}(1+t)|\widetilde{Z}_m^k\partial u|_0 + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0.$$

Since the energy inequality will control $\partial\widetilde{Z}_m^{k+1}u$, we introduce also

$$\phi''_k = \varepsilon^{-1}(1+t)|\partial\widetilde{Z}_m^k u|_0 + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0.$$

Thanks to Lemma IV.4.2, we see that, assuming

$$\phi_l \leqslant C(1+t)^{1+C\varepsilon}, \quad l \leqslant k,$$

we obtain

$$|[\widetilde{Z}_m^{k+1}, \partial]u|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}|\widetilde{Z}_m^k\partial\widetilde{a}|_0.$$

It follows that

$$\phi'_{k+1} \leqslant C(1+t)^{1+C\varepsilon} + C\phi''_{k+1}, \phi''_{k+1} \leqslant C(1+t)^{1+C\varepsilon} + C\phi'_{k+1}.$$

VII.1. L^2 estimates of u

Proposition 1. — *We can choose β_1 and $\beta_2 - \beta_1$ big enough to ensure the following implication: Assume that, for $0 \leqslant l \leqslant k \leqslant 2(s_0 - 4) - 1$, we have*

$$|\widetilde{N}_l|_0 \leqslant C(1+t)^{1+C\varepsilon}, \quad A_l \leqslant C(1+t)^{C\varepsilon}.$$

Then, for some $\gamma > 0$,

$$\begin{aligned} |P\widetilde{Z}_m^{k+1}u|_0 \leqslant{}& C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1} \\ &+ C\varepsilon^2(1+t)^{-2}\phi'_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1} + C\varepsilon^2(1+t)^{-1}A_{k+1} \\ &+ C(1+t)^{-1+C\varepsilon}\Big(\int_{\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}} e^p(T_i\widetilde{Z}_m^{k+1}u)^2 dx\Big)^{1/2}. \end{aligned}$$

Remark that the statement of the Proposition does not change if we replace ϕ'_{k+1} by ϕ''_{k+1}.

Proof. — Before proceeding, let us explain how we classify the various terms of $P\widetilde{Z}_m^{k+1}u$. We call SC (for subcritical) the terms which can be estimated by already known quantities, that is, using the induction hypothesis. We wish these SC terms to be bounded in weighted norm either by $\varepsilon(1+t)^{-1-\gamma}$, or by $\varepsilon^2(1+t)^{-1+C\varepsilon}$ (γ will denote here various strictly positive numbers). In both cases,

$$\int_0^t |\text{SC term}|_0 dt' \leqslant C\varepsilon(1+t)^{C\varepsilon}.$$

The other terms are called C (critical) terms, and are more delicate to handle, since we want them to be bounded by quantities we control directly through energy inequalities, in such a way that application of Gronwall's Lemma will be possible without damage. More precisely, the quantities we expect to control are

$$|\partial\widetilde{Z}_m^{k+1}u|_0, \quad |\partial\widetilde{Z}_m^k a|_0,$$

using the inequalities for P and for L respectively. The C terms for which we have an easy control will be bounded by $\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}$ or $\varepsilon(1+t)^{-1-\gamma}A_{k+1}$. The limiting case will be C terms bounded by $\varepsilon^2(1+t)^{-2}\phi'_{k+1}$ or $\varepsilon^2(1+t)^{-1}A_{k+1}$. Finally, one term involves the special derivatives T_i, and is expected to be handled using the control of these special derivatives given in Proposition V.3.1.

A. 1. According to Lemma 2.1, ignoring $\overline{\chi}$ here, we have

$$P\widetilde{Z}_m^{k+1}u = \sum \widetilde{Z}_m^{k_1}\delta\cdots\widetilde{Z}_m^{k_j}\delta\widetilde{Z}_m^q(\textstyle\sum_1+\sum_2+\sum_3)(\widetilde{Z}_m^r u),$$

with

$$q+r+\textstyle\sum(k_i+1) \leqslant k.$$

We are going to write down operators like $\widetilde{Z}_m^q Q$, and estimate the corresponding terms. With the notations of Proposition 1.1,

$$\widetilde{Z}_m^q Q = \widetilde{Z}_m^q \textstyle\sum_1 + \widetilde{Z}_m^q \sum_2 + \widetilde{Z}_m^q \sum_3 .$$

A.2. We have

$$\widetilde{Z}_m^q \textstyle\sum_1 = \displaystyle\sum_{q_1+q_2+q_3=q} \widetilde{Z}_m^{q_1} M_1 \widetilde{Z}_m^{q_2}\alpha[\widetilde{Z}_m^{q_3}\partial\widetilde{Z}_m + \widetilde{Z}_m^{q_3}(\sigma_1^{-1}\widetilde{Z}_m) + \widetilde{Z}_m^{q_3}\partial].$$

Now $\widetilde{Z}_m^{q_1} M_1 = M_{1+q_1}$,

$$\widetilde{Z}_m^{q_2}(\sigma_1/(1+t)^2(\varepsilon^{-1}(1+t)\partial u)) = \sigma_1/(1+t)^2 \textstyle\sum' f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}, \quad \sum l_j \leqslant q_2,$$

$$\widetilde{Z}_m^{q_2}(\sigma_1/(1+t)^2\widetilde{N}_1) = \sigma_1/(1+t)^2 \textstyle\sum' f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}, \quad \sum k_i \leqslant 1+q_2.$$

Using Lemma IV.4.2, we also have

$$\widetilde{Z}_m^{q_3}\partial = \partial\widetilde{Z}_m^{q_3} + \textstyle\sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\partial\widetilde{Z}_m^p, \quad \sum l_i + p \leqslant q_3, \ p \leqslant q_3-1,$$

$$\widetilde{Z}_m^{q_3}(\sigma_1^{-1}\widetilde{Z}_m) = \sigma_1^{-1}\textstyle\sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\widetilde{Z}_m^{p'}, \quad \sum l_j + p' \leqslant q_3+1.$$

We will often use the following standard remark: we have

$$M_r = f\widetilde{N}_1^l\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_j}, \quad l_i \geqslant 2, \quad \textstyle\sum(l_i-1) \leqslant r-1.$$

Either all l_i are $\leqslant s_0 - 4$, or one of them at least is $\geqslant s_0 - 3$; in the latter case, noting $\sum'(l_i - 1)$ the sum corresponding to the other indexes, we have

$$\sum{}'(l_i - 1) + s_0 - 4 \leqslant r - 1.$$

If $r \leqslant 2(s_0 - 4)$, this implies that for all other indexes, $l_i \leqslant s_0 - 4$.

Hence

$$|M_{k+1}|_0 \leqslant C(1+t)^{C\varepsilon}|\widetilde{N}_{k+1}|_0 + C(1+t)^{1+C\varepsilon},$$
$$|M_r|_0 \leqslant C(1+t)^{1+C\varepsilon}, \quad r \leqslant k.$$

If $q_1 = q = k$, the corresponding term in $\widetilde{Z}_m^q \sum_1 \widetilde{Z}_m^r u$ is bounded in weighted L^2 norm by

$$|M_{k+1}|_0 |\alpha\varepsilon(1+t)^{-1}\sigma_1^{\mu-1}|_{L^\infty} \leqslant C\varepsilon(1+t)^{-2-\gamma}(1+t+|\widetilde{N}_{k+1}|_0).$$

If $q_2 = q = k$, the corresponding term has the same bound, and also if $q_3 = q = k$ or $r = k$. In all other cases, the term is bounded by $C\varepsilon(1+t)^{-1-\gamma}$. Since $\delta = f\widetilde{N}_1$, $\widetilde{Z}_m^{k_i}\delta = M_{1+k_i}, 1 + k_i \leqslant k$, the term involving $\sum_1$ in $P\widetilde{Z}_m^{k+1}u$ is bounded in weighted L^2 norm by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}|\widetilde{N}_{k+1}|_0.$$

A.3. We turn now to the terms involving $\sum_2$. We have

$$\widetilde{Z}_m^q \sum{}_2 \widetilde{Z}_m^r u = \sum_{q_1+q_2=q} \widetilde{Z}_m^{q_1}(f\partial u)\widetilde{Z}_m^{q_2}\partial\widetilde{Z}_m^r u + \widetilde{Z}_m^q(f\partial\widetilde{Z}_m u\partial\widetilde{Z}_m^r u).$$

For the first term, we have as before,

$$\widetilde{Z}_m^{q_1}(f\partial u) = \varepsilon(1+t)^{-1}M_{q_1}, \quad q_1 \geqslant 1,$$
$$\widetilde{Z}_m^{q_2}\partial\widetilde{Z}_m^r u = \partial\widetilde{Z}_m^{q_2+r}u + \sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\partial\widetilde{Z}_m^{p+r}u, \quad \sum l_j + p \leqslant q_2,\ p \leqslant q_2 - 1.$$

If $q_2 + r = k$, necessarily $q_1 = 0$ and no δ terms are present, hence the corresponding term is bounded by

$$C\varepsilon(1+t)^{-1}|\widetilde{Z}_m^{k+1}u|_0.$$

If $q_2 + r \leqslant k - 1$, the weighted L^2 norm is bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$. For the second term, either $q = k$ and all derivatives fall on the middle term to give $f\partial\widetilde{Z}_m^{k+1}u\partial u$, or the powers of $\widetilde{Z}_m$ acting on u are all at most k. In the first case, the L^2 norm is bounded by

$$C\varepsilon/(1+t)|e^{p/2}\partial\widetilde{Z}_m^{k+1}u|_{L^2}.$$

In the second case, it is bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$. To summarize, the term involving $\sum_2$ in $P\widetilde{Z}_m^{k+1}u$ is bounded in weighted L^2 norm by

$$C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|e^{p/2}\partial\widetilde{Z}_m^{k+1}u|_{L^2}.$$

B. We turn now to the special terms $\widetilde{Z}_m^q \sum_3 \widetilde{Z}_m^p u$. We claim that, if $p + q \leqslant k - 1$, all these terms are SC. In particular, any term in $P\widetilde{Z}_m^{k+1}u$ containing at least one δ factor will be SC, since then

$$p + q \leqslant k - 1, \quad k_i \leqslant k - 1.$$

It is also important to remark that, for SC terms supported for $\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}$, powers of σ_1 are not crucial, since extra factors $(1+t)^{C\varepsilon}$ are admitted in the estimate of the Proposition. In what follows, the index r is always $r=0$ or $r=1$.

B.1. We have

$$\widetilde{Z}_m^q(r^{-2}R_j\widetilde{a}R_jL_1\widetilde{Z}_m^pu) = M_{l_1}(1+t)^{-2}\theta_2(\widetilde{Z}_m^{p_1}s_\theta a)\partial\widetilde{Z}_m^{p_2+p+r}u,$$

with $l_1 - 1 + p_1 + p_2 \leqslant q$. If $p+q \leqslant k-1$, the term is SC, and also for $p+q=k$, except for the C terms

$$M_1(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m^{k+1}u,\ M_{k+1}(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m^ru,\ M_1(1+t)^{-2}\theta_2(\widetilde{Z}_m^ks_\theta a)\partial\widetilde{Z}_m^ru.$$

Considering the last term, using Lemma IV.5.3, we see that all terms in $[\widetilde{Z}_m^k, s_\theta]a$ are SC, except

$$\theta_1^{-1}(M_1s_\theta[M_1;]M_1\widetilde{Z}_m^ka + M_1s_\theta[M_k;]M_1\widetilde{Z}_m^ra + M_1s_\theta[;M_1]M_1\widetilde{Z}_m^ka$$
$$+ M_1s_\theta[;M_k]M_1\widetilde{Z}_m^ra + M_1s_\theta M_1\widetilde{Z}_m^ka + M_1s_\theta[;M_ka + M_1\widetilde{Z}_m^{k-1}a]M_1).$$

These terms are bounded in weighted L^2 norm by $K_1 \times |\widetilde{Z}_m^ka|_0, K_2 \times |M_{k+1}|_0$. Here,

$$K_1 = \theta_1^{-1}|M_2|_{L^\infty}, \quad K_2 = \theta_1^{-1}(|\widetilde{Z}_m^ra|_{L^\infty} + |M_1|_{L^\infty}).$$

Let us explain here once for all the meaning of such expressions. The notation M_1, M_2 etc. is a commodity not to write explicitly the exact powers of $\widetilde{N}_1$ involved. The point is that these powers, in the finite computation we are doing here (once s_0 has been chosen), never exceed some number depending on s_0. The important fact is that, according to Proposition IV.6, $\widetilde{N}_l, l \leqslant s_0 - 4$ is bounded in L^∞ norm by $C(1+t)^{C_1\varepsilon}$, where C_1 does not depend on θ. Hence, here and in what follows, we can choose β_1 big enough to have

$$|K_i| \leqslant C(1+t)^{-C'\varepsilon},$$

with C' as big as we want.

Returning to our term, we see that its norm does not exceed

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1}.$$

The same analysis applies to the terms coming from

$$cr^{-2}\Delta_\omega\widetilde{a}\partial, \quad f(1+t)^{-1}\partial uR\widetilde{a}\partial,$$

with the same result.

B.2. We have

$$\widetilde{Z}_m^q(fr^{-2}L_1\widetilde{a}\Delta_\omega\widetilde{Z}_m^pu) = \widetilde{Z}_m^q(M_1(1+t)^{-1}(\partial\widetilde{a})(R/r)\widetilde{Z}_m^{p+1}u)$$
$$+ \widetilde{Z}_m^q(M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m^{p+1}u)) + \widetilde{Z}_m^q(M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)\partial\widetilde{Z}_m^pu)$$
$$+ \widetilde{Z}_m^q(M_1(1+t)^{-2}\theta_2(\partial\widetilde{a})(s_\theta a)\partial\widetilde{Z}_m^pu) = (1)+(2)+(3)+(4).$$

The term (4) is handled just as in **B.1.** The term (3) is analogous to (2), with one less derivative. We have

$$(2) = M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\sigma_1^{-1}\widetilde{Z}_m^{p_2+1}u,$$

with $l_1 - 1 + p_1 + p_2 \leqslant p + q$. The only C terms here are

$$M_1(\sigma_1/(1+t)^2)(\partial\widetilde{a})\sigma_1^{-1}\widetilde{Z}_m^{k+1}u, \quad M_1(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{k}\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m u),$$

$$M_{k+1}(\sigma_1/(1+t)^2)(\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m u).$$

They are bounded by $C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}$. The SC terms are bounded by $C\varepsilon(1+t)^{-1-\gamma}$. Using Lemma IV.4.3, we can write

$$\begin{aligned}(1) = &\sum_{l_1-1+p_1+p_2\leqslant q} M_{l_1}(1+t)^{-1}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(R/r)\widetilde{Z}_m^{p_2+p+1}u\\ &+ \sum_{l_1-1+p_1+p_2\leqslant q} M_{l_1}(1+t)^{-2}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\widetilde{Z}_m^{p_2+p+1}u\\ &+ \sum_{l_1-1+p_1+p_2\leqslant q-1} M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\partial\widetilde{Z}_m^{p_2+p+1}u\\ &+ \sum_{l_1-1+p_1+p_2+p_3\leqslant q-1} M_{l_1}(\theta_2/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(\widetilde{Z}_m^{p_2}s_\theta a)\partial\widetilde{Z}_m^{p_3+p+1}u.\end{aligned}$$

The last three terms come from the commutator of R/r with some power of $\widetilde{Z}_m$, and the last two are SC and bounded by $C\varepsilon(1+t)^{-1-\gamma}$. The second term is easily handled and bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}.$$

If $p_2 + p + 1 \leqslant k$, the first term can be rewritten as

$$M_{l_1}(1+t)^{-2}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\widetilde{Z}_m^{p_2+p+2}u.$$

The C terms are then

$$M_{k+1}(1+t)^{-2}(\partial\widetilde{a})\widetilde{Z}_m^2 u, \quad M_1(1+t)^{-2}(\widetilde{Z}_m^k\partial\widetilde{a})\widetilde{Z}_m^2 u,$$

$$M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m^{k+1}u), \quad l_1 + p_1 \leqslant 2.$$

All others are SC terms. The sum of all terms is bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}.$$

If $p_2 + p + 1 = k + 1$, we keep the first term as

$$M_1(1+t)^{-1}(\partial\widetilde{a})R/r\widetilde{Z}_m^{k+1}u.$$

If we think of R/r as $f\partial$, we cannot control this term. We have to keep in mind that $R/r = fT_i$ and keep the term as such for later treatment.

B.3. We have

$$\widetilde{Z}^q_m(f\partial u L_1\widetilde{a}\partial\widetilde{Z}^p_m u) = \widetilde{Z}^q_m(f\varepsilon(1+t)^{-1}\partial\widetilde{a}\partial\widetilde{Z}^p_m u)$$
$$= \sum_{p_1+p_2\leqslant p+q} f\varepsilon(1+t)^{-1}(\widetilde{Z}^{p_1}_m\partial\widetilde{a})\partial\widetilde{Z}^{p_2}_m u$$
$$+ \sum_{l_1-1+p_1+p_2\leqslant p+q-1} M_{l_1}\varepsilon(1+t)^{-1}(\widetilde{Z}^{p_1}_m\partial\widetilde{a})\partial\widetilde{Z}^{p_2}_m u.$$

All terms in the second sum are SC. The only C term in the first sum is

$$f\varepsilon(1+t)^{-1}\widetilde{Z}^k_m\partial\widetilde{a}\partial u,$$

which is bounded by $C\varepsilon^2(1+t)^{-2}\phi'_{k+1}$. All SC terms from both sums are bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$.

C. To understand the behavior of the last three special terms

$$-\widetilde{A}L_1^2,\quad fL_1L\widetilde{a}\partial,\quad f(1+t)^{-1}L\widetilde{a}\partial,$$

we cannot consider $L\widetilde{a}$ as an $\widetilde{N}_1$. We need make explicit its closeness to La, and, in particular, show that a factor ε is present in its estimates.

C.1 We prove the following estimates:

$$|\widetilde{Z}^l_m(\sqrt{c}L\widetilde{a})|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1+C\varepsilon},\quad l\leqslant s_0-5,$$
$$|\widetilde{Z}^l_m(\sqrt{c}L\widetilde{a})|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon},\quad l\leqslant k-1,$$
$$|\widetilde{Z}^k_m(\sqrt{c}L\widetilde{a})|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1+C\varepsilon}\phi_{k+1} + C\varepsilon(1+t)^{C\varepsilon}A_{k+1}.$$

From Lemma 1.4, we have with a self-defined E

$$\sqrt{c}L\widetilde{a} = -S_\theta E + \varepsilon(1+t)^{-1}s_\theta a + [u, S_\theta]a_r.$$

a. We have

$$\widetilde{Z}^l_m E = (\chi/(2\sqrt{c})(\widetilde{Z}^{l+1}_m u + (\widetilde{Z}^l_m a - \widetilde{Z}^l_m\widetilde{a})L_1 u + \sum_{l'\geqslant 1}(\widetilde{Z}^{l-l'}_m a - \widetilde{Z}^{l-l'}_m\widetilde{a})\widetilde{Z}^{l'}_m L_1 u))$$
$$+ \sum_{l'\leqslant l-1} M_{l-l'}(\widetilde{Z}^{l'+1}_m u + \sum(\widetilde{Z}^{l''}_m a - \widetilde{Z}^{l''}_m\widetilde{a})\widetilde{Z}^{l'-l''}_m L_1 u).$$

If $l\leqslant k-1$, all terms in

$$[\widetilde{Z}^l_m, S_\theta]E + S_\theta\widetilde{Z}^l_m E$$

are SC, and are easily seen to be bounded as indicated. If $l=k$, the only C terms in $\widetilde{Z}^k_m E$ are

$$f(\widetilde{Z}^{k+1}_m u + (\widetilde{Z}^k_m a - \widetilde{Z}^k_m\widetilde{a})L_1 u).$$

These terms, and also all other SC terms, are bounded as desired. Since we do not care about factors $(1+t)^{C\varepsilon}$ here, we see that the same bounds are true also for all terms in $[\widetilde{Z}^k_m, S_\theta]E$.

b. We have

$$\widetilde{Z}^l_m(\varepsilon(1+t)^{-1}s_\theta a)=\varepsilon(1+t)^{-1}([\widetilde{Z}^l_m,s_\theta]a+s_\theta(\widetilde{Z}^l_m a)+\sum_{l_1+l_2\leqslant l}M_{l_1}\widetilde{Z}^{l_2}_m s_\theta a).$$

The terms in the last sum are all SC, and bounded as desired. For $l\leqslant k-1$, all other terms are also SC and appropriately bounded. For $l=k$,

$$|\widetilde{Z}^k_m a|_0\leqslant C(1+t)^{1+C_0\varepsilon}A_{k+1},$$
$$[\widetilde{Z}^k_m,s_\theta]a|_0\leqslant C(1+t)^{1+C\varepsilon}A_{k+1}+C(1+t)^{C\varepsilon}\phi_{k+1}.$$

c. We write

$$\widetilde{Z}^l_m([u,S_\theta]a_r)=\sum_{l_1+l_2=l}\widetilde{Z}^{l_1}_m u\widetilde{Z}^{l_2}_m S_\theta a_r-[\widetilde{Z}^l_m,S_\theta]ua_r-S_\theta\widetilde{Z}^l_m(ua_r).$$

We do not use here the bracket structure, estimating each term separately.

C.2. We prove the following estimates, where $A=\partial_r$ or $A=\partial_t$:

$$|\widetilde{Z}^l_m AL\widetilde{a}|_{L^\infty}\leqslant C\varepsilon(1+t)^{-1+C\varepsilon},\quad l\leqslant s_0-5,$$
$$|\widetilde{Z}^l_m AL\widetilde{a}|_0\leqslant C\varepsilon(1+t)^{C\varepsilon},\quad l\leqslant k-1,$$
$$|\widetilde{Z}^k_m AL\widetilde{a}|_0\leqslant C\varepsilon(1+t)^{C\varepsilon}+C\varepsilon(1+t)^{-1}\phi'_{k+1}+C\varepsilon A_{k+1}.$$

We handle only $A=\partial_t$, the other case being similar and easier.

a. We have first

$$\partial_t S_\theta E=S_\theta E_t+\varepsilon s_\theta(1+t)^{-1}E,$$
$$E_t=\partial_t(\chi/(2\sqrt{c}))(\widetilde{Z}_m u+(a-\widetilde{a})L_1u)$$
$$+(\chi/(2\sqrt{c}))(\partial_t\widetilde{Z}_m u+(a_t-\partial_t\widetilde{a})L_1u+(a-\widetilde{a})\partial_t L_1u).$$

We note first that

$$\partial(\chi/(2\sqrt{c}))=f\sigma_1^{-1}.$$

We observe now, using Lemma 1.4, that all terms in $\partial_t(\sqrt{c}L\widetilde{a})$ are either

i) linear in $\partial_t^r\widetilde{Z}_m u$ $(r\leqslant 1)$,

ii) bilinear in $\partial^r u$ $(r\leqslant 1)$ and $\partial^{r'}a$ $(r'\leqslant 1)$ or $\partial^{r'}\widetilde{a}$ $(r'\leqslant 1)$, with the exception of $(a-\widetilde{a})\partial_t L_1 u$,

iii) linear in $\partial^r a$ $(r\leqslant 1)$ with a coefficient at least as good as $\varepsilon(1+t)^{-1}$.

Since we do not care about factors $(1+t)^{C\varepsilon}$ in the estimation of SC terms, we obtain that all SC terms in $\widetilde{Z}^l_m AL\widetilde{a}$ have the desired bound. We concentrate therefore on C terms, which can occur only for $l=k$. If we ignore at first the bracket terms in $\widetilde{Z}^k_m\partial_t S_\theta E$, we can consider only $S_\theta\widetilde{Z}^k_m E_t$, since the other term involving $(1+t)^{-1}E$ is similar and simpler. The C terms in $\widetilde{Z}^k_m E_t$ are

$$f\sigma_1^{-1}\widetilde{Z}^{k+1}_m u,\quad f\sigma_1^{-1}(\widetilde{Z}^k_m a-\widetilde{Z}^k_m\widetilde{a})L_1u,$$
$$\widetilde{Z}^k_m\partial_t\widetilde{Z}_m u,\quad(\widetilde{Z}^k_m a_t-\widetilde{Z}^k_m\partial_t\widetilde{a})L_1u,\quad(\widetilde{Z}^k_m a-\widetilde{Z}^k_m\widetilde{a})\partial_t L_1u,(a-\widetilde{a})\widetilde{Z}^k_m\partial_t L_1u.$$

Except for the last two terms, they are respectively bounded in weighted L^2 norm by

$$C\varepsilon(1+t)^{-1}\phi'_{k+1}, \quad C\varepsilon A_{k+1} + C\varepsilon(1+t)^{-1}\phi'_{k+1},$$
$$C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}\phi'_{k+1}, \quad C\varepsilon A_{k+1} + C\varepsilon(1+t)^{-1}\phi'_{k+1}.$$

We now write

$$|\partial_t L_1 u| \leqslant C|LL_1u| + C|L_1^2 u| \leqslant C\varepsilon\sigma_1^{\mu-1}(1+t)^{-2+C\varepsilon} + C|L_1^2 u|.$$

Thus, using the Poincaré Lemma, and Lemma IV.4.2, we obtain

$$|(\widetilde{Z}_m^k a - \widetilde{Z}_m^k \widetilde{a})\partial_t L_1 u|_0 \leqslant C\varepsilon(1+t)^{-1}\phi'_{k+1} + C\varepsilon A_{k+1} + C\varepsilon(1+t)^{C\varepsilon}.$$

For the last term, we write

$$|\widetilde{Z}_m^k \partial_t L_1 u|_0 \leqslant C|\widetilde{Z}_m^{k+1}\partial u|_0 + C\varepsilon(1+t)^{C\varepsilon}, \quad |a-\widetilde{a}| \leqslant C\theta_1^{-1}|M_2|.$$

Choosing β_1 big enough in the sense we have already explained will give $|a-\widetilde{a}| \leqslant C$, which finishes the estimate of $|S_\theta \widetilde{Z}_m^k E_t|_0$. Now, as explained before, if β_1 has been chosen big enough, the bracket terms

$$[\widetilde{Z}_m^k, \quad S_\theta]E_t, [\widetilde{Z}_m^k, \quad s_\theta](1+t)^{-1}E$$

generate terms having the same bound, except the terms involving M_{k+1}. This terms will be bounded by

$$C\theta_1^{-1}\varepsilon(1+t)^{-1+C\varepsilon}\phi_{k+1} \leqslant C\theta_1^{-1}\varepsilon(1+t)^{-1+C\varepsilon}\phi'_{k+1},$$

which have the desired bound if β_1 is big enough.

b. The term $\varepsilon/(1+t)^2 s_\theta a$ is much better than $\varepsilon/(1+t)s_\theta a_t$, and similarly the terms $u/(1+t)s_\theta a_r$, $(1+t)^{-1}s_\theta u a_r$ are much better than $u_t S_\theta a_r, S_\theta u_t a_r$. Considering only

$$\widetilde{Z}_m^l(\varepsilon/(1+t)s_\theta a_t + u_t S_\theta a_r - S_\theta u_t a_r + S_\theta u_r a_t),$$

we see that all terms are SC, except when $l = k$. In this latter case, the C terms may come only from

$$\varepsilon/(1+t)\widetilde{Z}_m^k s_\theta a_t + u_t \widetilde{Z}_m^k S_\theta a_r - \widetilde{Z}_m^k S_\theta u_t a_r + \widetilde{Z}_m^k S_\theta u_r a_t.$$

Ignoring first the bracket terms, we obtain the desired bound $C\varepsilon A_{k+1}$ for the C terms, and the bound $C\varepsilon(1+t)^{C\varepsilon}$ for the SC terms. For the bracket terms, we proceed as before, getting the same bound plus $C\varepsilon(1+t)^{-1}\phi'_{k+1}$.

c. The term $\theta_1[u, s_\theta]a_t$, being already amplified by θ_1, is the most delicate to handle. We write

$$\widetilde{Z}_m^l([u, s_\theta]a_t) = \sum_{l_1+l_2=l, l_1\geqslant 1} \widetilde{Z}_m^{l_1}u\widetilde{Z}_m^{l_2}s_\theta a_t + u\widetilde{Z}_m^l s_\theta a_t - \widetilde{Z}_m^l s_\theta u a_t.$$

If $l \leqslant k-1$, all terms are SC and bounded as desired. If $l = k$, we use Lemma IV.5.4 to express $\widetilde{Z}_m^k s_\theta$. This Lemma displays terms of three types:

i) A term $f s_\theta f \widetilde{Z}_m^k$, critical with no amplification,

ii) Subcritical terms,

iii) Possibly critical terms accompanied by a factor θ_2^{-1}.

We obtain

$$\theta_1 \widetilde{Z}_m^k[u, s_\theta]a_t = \theta_1 f[u, s_\theta] f \widetilde{Z}_m^k a_t + \theta_1 \times \text{SC terms} + \theta_1\theta_2^{-1}(\dots).$$

The first term is bounded in weighted L^2 norm by

$$C(|u_r|_{L^\infty} + \theta_1/\theta_2 |Ru|_{L^\infty})|\widetilde{Z}_m^k a_t|_0.$$

Having chosen β_1, we can choose β_2 big enough with respect to β_1 to obtain (on the support of a)

$$\theta_1/\theta_2|Ru|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1}.$$

The SC terms are bilinear in u and a and bounded by $C\varepsilon(1+t)^{C\varepsilon}$. The terms containing $\theta_1\theta_2^{-1}$ are either SC, or involve $\widetilde{Z}_m^k a_t$ or M_{k+1}. We handle them as usual, choosing $\beta_2 - \beta_1$ big enough if necessary, and get the bound

$$C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon A_{k+1} + C\varepsilon(1+t)^{-1}\phi'_{k+1}.$$

C.3. Consider now the term $\widetilde{Z}_m^q(f(1+t)^{-1}L\widetilde{a}\partial\widetilde{Z}_m^p u)$. All terms are SC, except if $p=0$, $q=k$ the only term

$$f(1+t)^{-1}\widetilde{Z}_m^k L\widetilde{a}\partial u.$$

According to the estimates of **C.1**, the weighted L^2 norm of these terms is bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1}.$$

C.4. We use the estimates of **C.2** to handle the term $\widetilde{Z}_m^q(fAL\widetilde{a}\partial\widetilde{Z}_m^p u)$. We obtain right away the bound

$$C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon^2(1+t)^{-2}\phi'_{k+1} + C\varepsilon^2(1+t)^{-1}A_{k+1}.$$

C.5. We consider finally

$$\widetilde{Z}_m^q(\widetilde{A}L_1^2\widetilde{Z}_m^p u) = \sum_{q_1+q_2=q}(\widetilde{Z}_m^{q_1}\widetilde{A})\widetilde{Z}_m^{q_2}L_1^2\widetilde{Z}_m^p u.$$

a. For $1 \leqslant q_1 \leqslant k-1$, all terms are SC. Remembering that $L_1 = f\widetilde{Z}_m$, we write

$$(\widetilde{Z}_m^{q_1}L\widetilde{a})\widetilde{Z}_m^{q_2}f\widetilde{Z}_m f\partial\widetilde{Z}_m^p u.$$

Using the estimates of **C.1**, we see that these SC terms are bounded as desired. For the other part of $\widetilde{A}$, we write, since $L_1^2 = f\partial^2 + f/(1+t)\partial$,

$$\widetilde{Z}_m^{q_1+1}u\widetilde{Z}_m^{q_2}L_1^2\widetilde{Z}_m^p u = M_l(1+t)^{-1}(\widetilde{Z}_m^{q_1+1}u)\widetilde{Z}_m^s\partial u + M_l(\widetilde{Z}_m^{q_1+1}u)\partial\widetilde{Z}_m^s\partial u,$$

where in both sums $l-1+s \leqslant p+q_2$. Using $\partial = M_1\sigma_1^{-1}\widetilde{Z}_m$ in the last term, we see that all these SC terms are bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$.

b. If $q_1 = k$, we have the term $\widetilde{Z}_m^k\widetilde{A}L_1^2 u$, which gives (apart from trivial SC terms) $\widetilde{Z}_m^k L\widetilde{a}L_1^2 u$, $\widetilde{Z}_m^{k+1}uL_1^2 u$. Using Poincaré Lemma, we see that the last term is bounded by

$$C\varepsilon/(1+t)|\partial\widetilde{Z}_m^{k+1}u|_0,$$

which is the desired bound. Similarly, the first term is bounded by

$$C\varepsilon/(1+t)|\partial_r \widetilde{Z}_m^k L\widetilde{a}|_0.$$

Here arises a slight technical difficulty: the commutation of ∂_r with $\widetilde{Z}_m^k$ yields non radial derivatives, and our **C.2** estimates are only for $A=\partial_t$ or $A=\partial_r$. We have easily, in the spirit of Lemma 4.2,

$$[\widetilde{Z}_m^k, \partial_r] = \sum_{l\leqslant k-1} M_{k-l}\widetilde{Z}_m^l \partial.$$

Hence

$$\partial_r \widetilde{Z}_m^k L\widetilde{a} = \widetilde{Z}_m^k \partial_r L\widetilde{a} + \sum M_{k-l}\widetilde{Z}_m^l \partial L\widetilde{a}.$$

If, in the last term, $\partial=\partial_t$, we use **C.2**. If not, we write

$$\partial_i = \omega_i\partial_r + f/(1+t)R = f\partial_r + M_1/(1+t)\widetilde{Z}_m,$$

and $\widetilde{Z}_m^l \partial L\widetilde{a}$ yields either SC terms involving $\widetilde{Z}_m^{l'}\partial_r L\widetilde{a}$ that we have already handled (in **C.4**), or terms

$$\sum_{l'\leqslant l\leqslant k-1} (1+t)^{-1} M_{l-l'+1}\widetilde{Z}_m^{l'+1} L\widetilde{a}$$

that we have already handled in **C.3**.

c. Finally, if $q_1=0$, apart from already discussed terms, we are left with

$$\widetilde{A}L_1^2\widetilde{Z}_m^k u.$$

We proceed now exactly as in the proof of Proposition IV.1, e), writing

$$\sigma_1 L_1 = f\widetilde{Z}_m + f(\widetilde{a}/\sigma_1)\widetilde{Z}_m + f\widetilde{a}^2/\sigma_1 L_1,$$

so our term is

$$(\widetilde{A}/\sigma_1)(f\widetilde{Z}_m L_1\widetilde{Z}_m^k u + f(\widetilde{a}/\sigma_1)\widetilde{Z}_m L_1\widetilde{Z}_m^k u + f(\widetilde{a}^2/\sigma_1)\widetilde{Z}_m L_1\widetilde{Z}_m^k u).$$

Using the estimate of Lemma 1.4, we obtain the bound

$$C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon^2(1+t)^{-2-\varepsilon}\phi'_{k+1}.$$

D. Taking $\overline{\chi}$ into account now, using Lemma 2.1, we obtain

$$[\widetilde{Z}_m^{k+1}, P] = \sum \widetilde{Z}_m^q[\widetilde{Z}_m, P]\widetilde{Z}_m^p$$

as the sum of the main term

$$\overline{\chi}\sum \widetilde{Z}_m^q[\widetilde{Z}_m, P]\widetilde{Z}_m^p,$$

and terms supported on the support of $1-\overline{\chi}$. The main term has been analyzed in **A, B, C** using the expression of $[\widetilde{Z}_m, P]$ given in Proposition 1.1. The other terms are easily analyzed using formula (1.2) of Lemma 1.1, and yield terms bounded by

$$C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}. \qquad \square$$

VII.2. L^2 estimates of a. — We estimate now the perturbation coefficients.

Proposition 2. — *We have the estimate*

$$(1+t)^{-1}|\widetilde{Z}_m^k\partial a|_0 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-1+C\varepsilon}|\partial\widetilde{Z}_m^{k+1}u|_0$$
$$+C\int_0^t \varepsilon A_{k+1}ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma}ds + C\int_0^t \varepsilon\phi'_{k+1}(s)ds/(1+s)^2.$$

Proof. — The new difficulty here is that the fields H_i do not commute very well with L: we have to use H_0 instead.

1. We construct a calculus just as in IV.3. We define f as before, and keep the fields

$$\widetilde{R}_i^m = R_i + \widetilde{a}(R_i)L_1, \quad \widetilde{S}^m = S + \widetilde{a}(S)L_1, \quad \widetilde{K} = L + L_1.$$

We replace $\widetilde{H}_i^m = H_i + \widetilde{a}(H_i)L_1$ by $\overline{H}_m = H_0 - \widetilde{a}(S)L_1$. We denote by $\overline{Z}_m$ any one of the fields

$$\widetilde{R}_i^m,\ \widetilde{S}^m,\ \widetilde{K},\ H_0 - \widetilde{a}(S)L_1,$$

and by $\overline{N}_k$ any of the quantities

$$(1+t)\varepsilon^{-1}\sigma_1^{-1}\overline{Z}_m^k u,\ (1+t)\varepsilon^{-1}\overline{Z}_m^k\partial u,\ \sigma_1^{-1}\overline{Z}_m^{k-1}\widetilde{a},\ \overline{Z}_m^{k-1}\widetilde{a},\ \overline{Z}_m^{k-1}\partial\widetilde{a},$$

where $\widetilde{a} = \widetilde{a}(R_i)$ or $\widetilde{a} = \widetilde{a}(S)$. We have for these fields the usual calculus Lemmas: Lemma 4.1 is straightforward. Note also

$$r + ct/\sqrt{c}L = H_0 + S = \overline{H}_m + S^m = \overline{Z}_m, \quad L_1 = \widetilde{K} - L = f\overline{Z}_m.$$

The analogue to Lemma 3.1 is also true:

$$R = f\overline{Z}_m + f\overline{N}_1\overline{Z}_m, \quad \sigma_1 L_1 = f\overline{Z}_m + f\overline{N}_1\overline{Z}_m.$$

We define as before quantities

$$\overline{M}_k = \sum f\overline{N}_1^l\overline{N}_{k_1}\cdots\overline{N}_{k_i}, \quad k_j\geqslant 2,\ \sum(k_j-1)\leqslant k-1.$$

The following Lemma gives the relations between the fields $\widetilde{Z}_m$ and $\overline{Z}_m$.

Lemma 1. — *We have the formula*

i) $$\overline{Z}_m^k = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p, \quad p\geqslant 1,\ \sum k_i + p\leqslant k,$$

ii) $$\widetilde{Z}_m^k = \sum f\overline{N}_1^l\overline{N}_{k_1}\cdots\overline{N}_{k_i}\overline{Z}_m^{p+1}.$$

Here, $k_j\geqslant 2$, $\sum(k_j-1)+p\leqslant k-1$.

iii) $$\overline{N}_k = \sum_{(\sum k_j\leqslant k)} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}, \quad \overline{M}_k = M_k,$$

iv) $$\widetilde{Z}_m^k = (f + f\sigma_1/(1+t)\widetilde{a})^k\overline{Z}_m^k + \sum_{0\leqslant p\leqslant k-2}\overline{M}_{k-p}\overline{Z}_m^{p+1}.$$

Note that $|\sigma_1/(1+t)\widetilde{a}|\leqslant C$.

Proof

a. We have

$$\overline{H}_m = H_0 - \widetilde{a}(S)L_1 = \sum \omega_i \widetilde{H}_i^m - (\sum \omega_i \widetilde{a}(H_i) + \widetilde{a}(S))L_1 = \sum \omega_i \widetilde{Z}_m,$$

which proves i) for $k = 1$. Conversely,

$$\begin{aligned}\widetilde{H}_i^m &= H_i + \widetilde{a}(H_i)L_1 = \omega_i H_0 - ct/r(\omega \wedge R)_i + \widetilde{a}(H_i)L_1 \\ &= \omega_i(H_0 - \widetilde{a}(S)L_1) - ct/r(\omega \wedge \widetilde{R})_i - (r-ct)/r(\omega \wedge \widetilde{a}(R))_i L_1 \\ &= f\overline{Z}_m + f\sigma_1/(1+t)\widetilde{a}\overline{Z}_m,\end{aligned}$$

which proves ii), iii) and iv) for $k = 1$.

b. Formula i) is immediate by induction. Formula ii) can be written

$$\widetilde{Z}_m^k = \sum_{p \leqslant k-1} \overline{M}_{k-p} \overline{Z}_m^{p+1}.$$

Hence the calculus on $\overline{M}_l$ proves ii) for all k, and the same reasoning applies to prove iv). Finally, iii) follows from i) and ii) by the very definitions of the quantities, since

$$M_1 = \overline{M}_1, \quad \overline{N}_k = M_k, \quad \widetilde{N}_k = \overline{M}_k. \qquad \square$$

2. We have the following commutation Lemma.

Lemma 2. — *We have*

i) $$[\overline{Z}_m, L_1] = (f + f\overline{N}_1)L + (f + f\overline{N}_1)L_1,$$

ii) $$[\overline{Z}_m, L] = f\overline{d}L_1 + (f + f\overline{N}_1)L.$$

Here, $\overline{d}$ means one of the three quantities

$$\overline{d} = L\widetilde{a}(R_i) + \widetilde{R}_i^m u/2c, \quad \overline{d} = L\widetilde{a}(S) + \widetilde{S}^m u/2c, \quad \overline{d} = L_1 u.$$

Thus the critical quantity $\overline{d}$ is just $\widetilde{A}$ (or $L_1 u$). We have, with $Lw = g$, the formula

iii) $$\begin{aligned}[L, \overline{Z}_m^k]w &= \sum f\overline{N}_{l_1} \cdots \overline{N}_{l_i} \overline{Z}_m^{l_{i+1}} g + \sum f\overline{Z}_m^{q_1}\overline{d} \ldots \overline{Z}_m^{q_i}\overline{dN}_{k_1} \cdots \overline{N}_{k_j} L_1 \overline{Z}_m^{k_{j+1}} w \\ &+ \sum (1+t)^{-1} f\overline{Z}_m^{q_1}\overline{d} \ldots \overline{Z}_m^{q_i}\overline{dN}_{k_1} \cdots \overline{N}_{k_j} \overline{Z}_m^{k_{j+1}} w = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .\end{aligned}$$

In $\sum_1$, $\sum l_j \leqslant k$, $l_{i+1} \leqslant k-1$. In $\sum_2$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k$, $k_{j+1} \leqslant k-1$. In $\sum_3$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k+1$, $1 \leqslant k_{j+1} \leqslant k-1$.

Proof. — Since i) is clear, we need only prove ii), the proof of iii) following then exactly as in Lemma III.3.3. We have

$$\begin{aligned}[\widetilde{R}_i^m, L] &= (f + f\overline{N}_1)L - (L\widetilde{a}(R_i) + \widetilde{R}_i^m u/2c)L_1, \\ [\widetilde{S}^m, L] &= (f + f\overline{N}_1)L - (L\widetilde{a}(S) + \widetilde{S}^m u/2c)L_1, \\ [\overline{H}_m, L] &= f + f\overline{N}_1)L + (L\widetilde{a}(S) + \widetilde{S}^m u/2c)L_1, \\ [\widetilde{K}, L] &= Lu/2cL - L_1 u/2cL_1.\end{aligned} \qquad \square$$

3. We write now

$$La = -\chi/(2c)(Zu + aL_1u) = -\chi/(2c)(\overline{Z}_m u + (a-\widetilde{a})L_1u),$$
$$LL_1a = g_1 + L_1u/(2c)L_1a = G_1, \quad LR/ra = g_2 - \sqrt{c}Ra/r^2 = G_2,$$

with

$$g_i = f/\sigma_1\overline{Z}_m u + f\partial\overline{Z}_m u + f\partial u(a/\sigma_1) + f\partial u\partial a + f\partial u(\widetilde{a}/\sigma_1) + f\partial u\partial\widetilde{a} + f(a-\widetilde{a})\partial L_1u.$$

Using the structure of the g_i, we see that all terms in $\overline{Z}^l_m g_i$ are SC (in the sense of Proposition 1) for $l \leqslant k-1$ and

$$|\overline{Z}^l_m g_i|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\overline{Z}^l_m g_i|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

For G_1, we have the same estimates as for g_1. If $l = k$, we can replace the fields $\overline{Z}_m$ by $\widetilde{Z}_m$ in the critical terms of $\overline{Z}^k_m g_i$, this substitution generating only SC terms with the already seen estimate. Hence

$$|\overline{Z}^k_m g_i|_0 + |\overline{Z}^k_m G_1|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}\phi'_{k+1} + C\varepsilon A_{k+1}.$$

The delicate part is the estimation of $\overline{Z}^l_m(\sqrt{c}Ra/r^2)$ in $\overline{Z}^l_m G_2$. We write

$$\overline{Z}^l_m(\sqrt{c}Ra/r^2) = \sqrt{c}/r\overline{Z}^l_m(Ra/r) + \sum_{1\leqslant l_1\leqslant l}(1+t)^{-1}M_{l_1}\overline{Z}^{l-l_1}_m(Ra/r).$$

Now $Ra = \overline{M}_1\overline{Z}_m a$,

$$\overline{Z}^{l-l_1}_m(M_1(1+t)^{-1}\overline{Z}_m a) = \sum_{0\leqslant l_2\leqslant l-l_1}(1+t)^{-1}M_{1+l_2}\overline{Z}^{l-l_1-l_2+1}_m a,$$
$$\overline{Z}^l_m(\sqrt{c}Ra/r^2) = \sqrt{c}/r\overline{Z}^l_m(Ra/r) + \sum_{(l_1\geqslant 1, l_1+l_2\leqslant l)}(1+t)^{-2}M_{l_1+l_2}\overline{Z}^{l-l_1-l_2+1}_m a.$$

If $l = k$, we keep the first term as it is, the second sum being bounded by

$$C(1+t)^{-\gamma} + C(1+t)^{-\gamma}A_{k+1}.$$

If $l = k-1$, we compute the first term as before, and obtain for the whole of $\overline{Z}^{k-1}_m(\sqrt{c}Ra/r^2)$ the above bound. If $l \leqslant k-2$, the bound is the same as before, without the critical part containing A_{k+1}.

4. With $w = L_1a$ or $w = Ra/r$ and $Lw = G$, we write the result of Lemma 2 in the form

$$L\overline{Z}^k_m w = \overline{Z}^k_m G + \sum_{l\leqslant k-1}\overline{M}_{k-l}\overline{Z}^l_m G + f\overline{d}L_1\overline{Z}^{k-1}_m w$$
$$+ \sigma_1^{-1}\sum_{1\leqslant q\leqslant k-1}\overline{M}_1\overline{Z}^q_m d\overline{Z}^{k-q}_m w + \sigma_1^{-1}\sum_{\substack{p\geqslant 1, q\geqslant 1\\ p+q\leqslant k-1}}\overline{M}_{k-p-q}\overline{Z}^q_m d\overline{Z}^p_m w.$$

All terms of the second line are SC terms, and we see using the **C.1** estimates of Proposition 1 that they are bounded by $C\varepsilon(1+t)^{C\varepsilon}$. We also have, using the estimates of **3.**,

$$\sum_{l\leqslant k-1} |\overline{M}_{k-l}\overline{Z}^l_m G|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C(1+t)^{-\gamma} + C(1+t)^{-\gamma}A_{k+1}.$$

We handle the critical term $\overline{d}L_1\overline{Z}^{k-1}_m w$ exactly as we have done with the term $\widetilde{A}L_1^2\widetilde{Z}^k_m u$ in **C.5** of the proof of Proposition 1. Using the energy inequality for L, we finally get

$$(1+t)^{-1}(|\overline{Z}^k_m L_1 a|_0 + |\overline{Z}^k_m(Ra/r)|_0) \leqslant C(1+t)^{C\varepsilon}$$
$$+C\int_0^t \varepsilon\phi_{k+1}ds/(1+s)^2 + C\varepsilon\int_0^t A_{k+1}ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma}ds.$$

From the very definition of a, we obtain

$$|\overline{Z}^k_m La|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{C\varepsilon}A_{k+1} + C(1+t)^{C\varepsilon}|\partial\widetilde{Z}^{k+1}_m u|_0.$$

Adding this to the preceding estimate, we get

$$(1+t)^{-1}|\overline{Z}^k_m\partial a|_0 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-1+C\varepsilon}|\partial\widetilde{Z}^{k+1}_m u|_0$$
$$+C\int_0^t \varepsilon\phi'_{k+1}ds/(1+s)^2 + C\varepsilon\int_0^t A_{k+1}ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma}ds.$$

Now, using Lemma 1, we can replace the fields $\overline{Z}_m$ by $\widetilde{Z}_m$ in the above estimate, obtaining the desired result. □

VII.3. End of the proof of the main result

a. We first use the energy inequality and Proposition 1. In doing so, we have to take care of the special quantity

$$E = \int_0^t (1+t')^{-1+C\varepsilon}|\partial\widetilde{Z}^{k+1}_m u|_0|T_i\widetilde{Z}^{k+1}_m u|_0 dt'$$

arising from

$$\iint e^p|P\widetilde{Z}^{k+1}_m u||\partial_t\widetilde{Z}^{k+1}_m u|dxdt'.$$

It is understood here, in accordance with Proposition 1, that the integral of $T_i\widetilde{Z}^{k+1}_m u$ is taken only on

$$\sigma_1 \leqslant C(1+t')^{C_0\varepsilon}.$$

Using Cauchy-Schwarz inequality, we obtain, with $\alpha>0, \beta>0$ to be chosen,

$$E \leqslant \alpha\int_0^t (1+t')^{-\beta\varepsilon}|T_i\widetilde{Z}^{k+1}_m u|_0^2 dt' + 1/(4\alpha)\int_0^t (1+t')^{-2+2C\varepsilon+\beta\varepsilon}|\partial\widetilde{Z}^{k+1}_m u|_0^2 dt'.$$

Since the energy inequality gives us a control of

$$\int_0^t\int_{r\geqslant t'/2} e^p(\tau+1)b'(\psi)\sum(T_i\widetilde{Z}^{k+1}_m u)^2 dxdt',$$

and $b'(\psi) = B\nu|\psi|^{-\nu-1}$, we have, by Lemma II.3.4, a control of

$$\int_0^t \int_{\sigma_1 \leqslant C(1+t')^{C_0\varepsilon}} e^p (1+t')^{-C_2\varepsilon} \sum (T_i \widetilde{Z}_m^{k+1} u)^2 dx dt'.$$

Taking $\beta = C_2$ and α small enough, we see that the first term of E is absorbed in the left-hand side of the inequality, while the second is smaller than terms already there.

b. We have now

$$\begin{aligned} |\partial \widetilde{Z}_m^{k+1} u|_0 &\leqslant C\varepsilon + C\varepsilon \int_0^t dt'/(1+t')|\partial \widetilde{Z}_m^{k+1} u|_0 + C\int_0^t |P\widetilde{Z}_m^{k+1} u|_0 dt' \\ &\leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon \int_0^t \phi'_{k+1} dt'/(1+t')^{2+\gamma} + C\varepsilon^2 \int_0^t \phi'_{k+1} dt'/(1+t')^2 \\ &\qquad + C\varepsilon \int_0^t A_{k+1} dt'/(1+t')^{1+\gamma} + C\varepsilon^2 \int_0^t A_{k+1} dt'/(1+t'). \end{aligned}$$

We set here for convenience

$$E_{k+1} = \varepsilon^{-1} |\partial \widetilde{Z}_m^{k+1} u|_0 + A_{k+1}.$$

We use now the formula

$$|\widetilde{Z}_m^k \partial \widetilde{a}|_0 \leqslant C(1+t)^{1+C\varepsilon} + C|\widetilde{Z}_m^k \partial a|_0 + C\varepsilon^{-1}(1+t)|\partial \widetilde{Z}_m^{k+1} u|_0.$$

To prove it, we go back to the formula

$$\widetilde{Z}_m^k \partial \widetilde{a} = \widetilde{Z}_m^k [\partial, S_\theta] a + [\widetilde{Z}_m^k, S_\theta] \partial a + S_\theta \widetilde{Z}_m^k \partial a.$$

As before, the first two terms in the right-hand side involve

i) Terms already bounded by the induction hypothesis,

ii) Terms bounded by $|\widetilde{Z}_m^k \partial a|_0$ with a coefficient of the form $\theta_1^{-1} C_2 (1+t)^{C_3\varepsilon}$, where C_2 and C_3 do not depend on θ_1.

iii) Terms involving $\widetilde{N}_{k+1}$, with a coefficient of the same form as in ii).

The part of $\widetilde{N}_{k+1}$ involving $\widetilde{a}$ will be absorbed in the left-hand side by choosing β_1 and θ_1^0 big enough. Keeping the part involving derivatives of u, we obtain the formula. Using it, we obtain

$$(1+t)^{-1} \phi'_{k+1} \leqslant C(1+t)^{C\varepsilon} + CE_{k+1}.$$

With these notations, the control of $\partial \widetilde{Z}_m^{k+1} u$ given by the energy inequality for P and the control of A_{k+1} given by the energy inequality for L, added together, give

$$E_{k+1} \leqslant C(1+t)^{C\varepsilon} + C\int_0^t E_{k+1} dt'/(1+t')^{1+\gamma} + C\varepsilon \int_0^t E_{k+1} dt'/(1+t'),$$

which yields by Gronwall Lemma $E_{k+1} \leqslant C(1+t)^{C\varepsilon}$. This proves the induction hypothesis for $l = k+1$

$$|\widetilde{N}_{k+1}|_0 \leqslant C(1+t)^{1+C\varepsilon}, \quad A_{k+1} \leqslant C(1+t)^{C\varepsilon}.$$

c. It remains now to obtain, for the standard fields $Z_0 = R_i, S, h_i, \partial$,

$$|Z_0^k \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad k \leqslant 2(s_0 - 4).$$

First, we obtain

$$Z_0 = f\widetilde{N}_1^r \widetilde{Z}_m, \quad r \leqslant 1.$$

Next, exactly as in Lemma 2, we get

$$Z_0^k = \sum f\widetilde{N}_1^l \widetilde{N}_{k_1} \cdots \widetilde{N}_{k_i} \widetilde{Z}_m^{p+1},$$

with

$$k_j \geqslant 2, \quad \sum(k_j - 1) + p \leqslant k - 1.$$

Applying this identity to ∂u, we obtain finally

$$|Z_0^k \partial u|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon}.$$

Since we have the inequality

$$|w|_0 \leqslant C(1+t)^{C\varepsilon} |w|_{L^2},$$

this gives the result.

d. From Klainerman's inequality, we obtain now

$$|Z_0^k \partial u| \leqslant C\varepsilon\sigma_1^{-1/2}(1+t)^{-1+C\varepsilon}, \quad k \leqslant 2(s_0 - 4) - 2.$$

Assuming that

$$2(s_0 - 4) - 2 \geqslant s_0,$$

for instance, $s_0 = 10$, we obtain the same control as the induction hypothesis, with η replaced by $C\varepsilon$.

Fix now $\overline{\tau} > 0$: we know from Theorem II.1 that, for ε small enough, there exists a smooth solution for $\tau \leqslant \overline{\tau} = \varepsilon \log(1 + \overline{t})$ with

$$|Z_0^k \partial u| \leqslant C^{(1)}\varepsilon\sigma_1^{-1/2}(1+t)^{-1}, \quad k \leqslant s_0.$$

In particular, u exists as a smooth function for $t < T'$ (with $T' > \overline{t}$), and satisfies for $t < T \leqslant T'$ (with $T > \overline{t}$) the inequality (say $\eta = 10^{-2}$)

$$|Z_0^k \partial u| \leqslant C^{(1)}\varepsilon\sigma_1^{-1/2}(1+t)^{-1+\eta}.$$

If $T < T'$, we obtain from this hypothesis, as we have seen, for $t \leqslant T$,

$$|Z_0^k \partial u| \leqslant C^{(2)}\varepsilon\sigma_1^{-1/2}(1+t)^{-1+C\varepsilon}.$$

If ε is small enough to verify $C\varepsilon \leqslant \eta/2$, we deduce from this

$$|Z_0^k \partial u| \leqslant C^{(2)}(1+\overline{t})^{-\eta/2}\varepsilon\sigma_1^{-1/2}(1+t)^{-1+\eta}, \quad \overline{t} \leqslant t \leqslant T.$$

If ε is such that

$$C^{(2)}(1+\overline{t})^{-\eta/2} \leqslant C^{(1)}/2,$$

we see that the supremum of such T cannot be strictly less than T', hence $T' = +\infty$ and our estimates are true for all t, which finishes the proof. □

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S. ALINHAC, Département de Mathématiques, Université Paris-Sud, 91405 Orsay Cedex, France
E-mail : `serge.alinhac@math.u-psud.fr`