Astérisque

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Astérisque, tome 281 (2002)
<http://www.numdam.org/item?id=AST_2002__281__R1_0>

ASTÉRISQUE 281

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Société Mathématique de France 2002
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 ***Mathematics Subject Classification.*** — 60J80, 60J30, 60J25, 60G57, 60F17, 60G52.

Key words and phrases. — Lévy process, branching process, continuous-state branching process, Galton-Watson tree, random tree, height process, exploration process, local time, reduced tree, functional limit theorem, time-reversal, Poissonnian mark, stable tree, Lévy snake, superprocess, exit measure, exit distribution, reduced spatial tree.

RANDOM TREES, LÉVY PROCESSES AND SPATIAL BRANCHING PROCESSES

Thomas Duquesne, Jean-François Le Gall

Abstract. — We investigate the genealogical structure of general critical or subcritical continuous-state branching processes. Analogously to the coding of a discrete tree by its contour function, this genealogical structure is coded by a real-valued stochastic process called the height process, which is itself constructed as a local time functional of a Lévy process with no negative jumps. We present a detailed study of the height process and of an associated measure-valued process called the exploration process, which plays a key role in most applications. Under suitable assumptions, we prove that whenever a sequence of rescaled Galton-Watson processes converges in distribution, their genealogies also converge to the continuous branching structure coded by the appropriate height process. We apply this invariance principle to various asymptotics for Galton-Watson trees. We then use the duality properties of the exploration process to compute explicitly the distribution of the reduced tree associated with Poissonnian marks in the height process, and the finite-dimensional marginals of the so-called stable continuous tree. This last calculation generalizes to the stable case a result of Aldous for the Brownian continuum random tree. Finally, we combine the genealogical structure with an independent spatial motion to develop a new approach to superprocesses with a general branching mechanism. In this setting, we derive certain explicit distributions, such as the law of the spatial reduced tree in a domain, consisting of the collection of all historical paths that hit the boundary.

***Résumé* (Arbres aléatoires, processus de Lévy et processus de branchement spatiaux)**

Nous étudions la structure généalogique de processus de branchement critiques ou sous-critiques à espace d'états continu. De manière analogue au codage d'un arbre discret par son contour, cette structure généalogique est codée par un processus aléatoire appelé le processus des hauteurs, qui est lui-même construit comme une fonctionnelle de type temps local d'un processus de Lévy sans saut négatif. Nous présentons une étude détaillée du processus des hauteurs et d'un processus à valeurs mesures associé appelé le processus d'exploration. Sous des hypothèses convenables, nous montrons que si une suite de processus de Galton-Watson convenablement changés d'échelle converge en loi, leurs généalogies convergent aussi vers la structure de branchement codée par le processus des hauteurs. Nous appliquons ce principe d'invariance à divers théorèmes limites pour les arbres de Galton-Watson. A l'aide des propriétés de dualité du processus d'exploration, nous calculons la loi de l'arbre réduit associé à des marques poissonniennes dans le processus des hauteurs, et les lois marginales de dimension finie de l'arbre continu stable. Ce dernier calcul généralise au cas stable un résultat d'Aldous pour l'arbre brownien continu. Finalement, en combinant la structure généalogique avec un déplacement spatial, nous développons une nouvelle approche des superprocessus avec un mécanisme de branchement général. Dans ce cadre, nous obtenons certaines distributions explicites, dont celle de l'arbre spatial réduit dans un domaine, qui décrit toutes les trajectoires historiques ayant atteint la frontière.

CONTENTS

INTRODUCTION

The main goal of this work is to investigate the genealogical structure of continuous-state branching processes in connection with limit theorems for discrete Galton-Watson trees. Applications are also given to the construction and various properties of spatial branching processes including a general class of superprocesses.

Our starting point is the recent work of Le Gall and Le Jan [**32**] who proposed a coding of the genealogy of general continuous-state branching processes via a real-valued random process called the height process. Recall that continuous-state branching processes are the continuous analogues of discrete Galton-Watson branching processes, and that the law of any such process is characterized by a real function ψ called the branching mechanism. Roughly speaking, the height process is a continuous analogue of the contour process of a discrete branching tree, which is easy to visualize (see Section 0.1, and note that the previous informal interpretation of the height process is made mathematically precise by the results of Chapter 2). In the important special case of the Feller branching diffusion ($\psi(u) = u^2$), the height process is reflected linear Brownian motion: This unexpected connection between branching processes and Brownian motion, or random walk in a discrete setting has been known for long and exploited by a number of authors (see e.g. [**3**], [**11**], [**18**], [**39**], [**42**]). The key contribution of [**32**] was to observe that for a general subcritical continuous-state branching process, there is an explicit formula expressing the height process as a functional of a spectrally positive Lévy process whose Laplace exponent ψ is precisely the branching mechanism. This suggests that many problems concerning the genealogy of continuous-state branching processes can be restated and solved in terms of spectrally positive Lévy processes, for which a lot of information is available (see e.g. Bertoin's recent monograph [**5**]). It is the principal aim of the present work to develop such applications.

In the first two sections below, we briefly describe the objects of interest in a discrete setting. In the next sections, we outline the main contributions of the present work.

0.1. Discrete trees

Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. If $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we set $|u| = n$, so that $|u|$ represents the "generation" of u. If $u = (u_1, \ldots u_m)$ and $v = (v_1, \ldots, v_n)$ belong to $\mathcal{U}$, we write $uv = (u_1, \ldots u_m, v_1, \ldots, v_n)$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$.

A (finite) rooted ordered tree $\mathcal{T}$ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \mathcal{T}$.

(ii) If $v \in \mathcal{T}$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$, then $u \in \mathcal{T}$.

(iii) For every $u \in \mathcal{T}$, there exists a number $k_u(\mathcal{T}) \geqslant 0$ such that $uj \in \mathcal{T}$ if and only if $1 \leqslant j \leqslant k_u(\mathcal{T})$.

We denote by $\mathbf{T}$ the set of all rooted ordered trees. In what follows, we see each vertex of the tree $\mathcal{T}$ as an individual of a population whose $\mathcal{T}$ is the family tree. The cardinality $\#(\mathcal{T})$ of $\mathcal{T}$ is the total progeny.

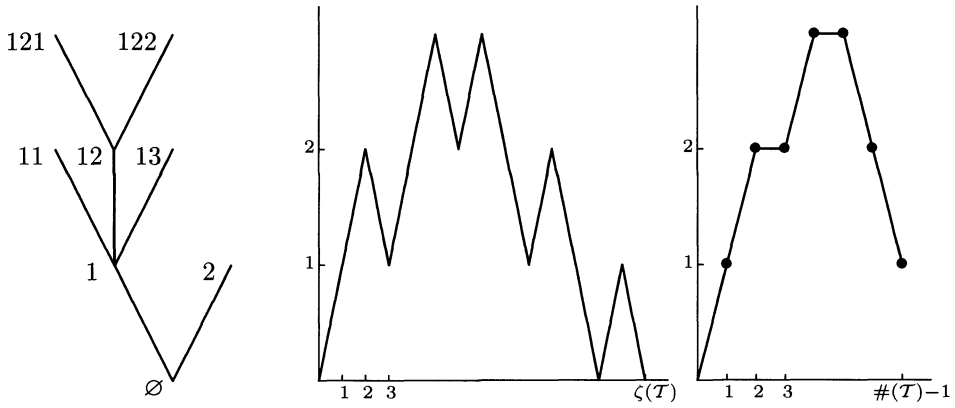
If $\mathcal{T}$ is a tree and $u \in \mathcal{T}$, we define the shift of $\mathcal{T}$ at u by $\theta_u\mathcal{T} = \{v \in U : uv \in \mathcal{T}\}$. Note that $\theta_u\tau \in \mathbf{T}$.

We now introduce the (discrete) *height function* associated with a tree $\mathcal{T}$. Let us denote by $u(0) = \varnothing, u(1), u(2), \ldots, u(\#(\mathcal{T}) - 1)$ the elements of $\mathcal{T}$ listed in lexicographical order. The height function $H(\mathcal{T}) = (H_n(\mathcal{T}); 0 \leqslant n < \#(\mathcal{T}))$ is defined by

$$H_n(\mathcal{T}) = |u(n)|, \quad 0 \leqslant n < \#(\mathcal{T}).$$

The height function is thus the sequence of the generations of the individuals of $\mathcal{T}$, when these individuals are visited in the lexicographical order (see Fig. 1 for an example). It is easy to check that $H(\mathcal{T})$ characterizes the tree $\mathcal{T}$.

The *contour function* gives another way of characterizing the tree, which is easier to visualize on a picture (see Fig. 1). Suppose that the tree is embedded in the half-plane in such a way that edges have length one. Informally, we imagine the motion of a particle that starts at time $t = 0$ from the root of the tree and then explores the tree from the left to the right, moving continuously along the edges at unit speed, until it comes back to its starting point. Since it is clear that each edge will be crossed twice in this evolution, the total time needed to explore the tree is $\zeta(\mathcal{T}) := 2(\#(\mathcal{T}) - 1)$. The value C_t of the contour function at time t is the distance (on the tree) between the position of the particle at time t and the root. By convention $C_t = 0$ if $t \geqslant \zeta(\mathcal{T})$. Fig. 1 explains the definition of the contour function better than a formal definition.



tree $\mathcal{T}$ contour function height function

FIGURE 1

0.2. Galton-Watson trees

Let μ be a critical or subcritical offspring distribution. This means that μ is a probability measure on $\mathbb{Z}_+$ such that

$$\sum_{k=0}^{\infty} k\mu(k) \leqslant 1.$$

We exclude the trivial case where $\mu(1) = 1$.

There is a unique probability distribution $\mathbf{Q}_\mu$ on $\mathbf{T}$ such that

(i) $\mathbf{Q}_\mu(k_\varnothing = j) = \mu(j), \quad j \in \mathbb{Z}_+$.

(ii) For every $j \geqslant 1$ with $\mu(j) > 0$, the shifted trees $\theta_1\mathcal{T}, \ldots, \theta_j\mathcal{T}$ are independent under the conditional probability $\mathbf{Q}_\mu(\cdot \mid k_\varnothing = j)$ and their conditional distribution is $\mathbf{Q}_\mu$.

A random tree with distribution $\mathbf{Q}_\mu$ is called a Galton-Watson tree with offspring distribution μ, or in short a μ-Galton-Watson tree.

Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of independent μ-Galton-Watson trees. We can associate with this sequence a *height process* obtained by concatenating the height functions of each of the trees $\mathcal{T}_1, \mathcal{T}_2, \ldots$. More precisely, for every $k \geqslant 1$, we set

$$H_n = H_{n-(\#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_{k-1}))}(\mathcal{T}_k) \text{ if } \#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_{k-1}) \leqslant n < \#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_k).$$

The process $(H_n, n \geqslant 0)$ codes the sequence of trees.

Similarly, we define a *contour process* $(C_t, t \geqslant 0)$ coding the sequence of trees by concatenating the contour functions

$$(C_t(\mathcal{T}_1), t \in [0, \zeta(\mathcal{T}_1) + 2]), \ (C_t(\mathcal{T}_2), t \in [0, \zeta(\mathcal{T}_2) + 2]), \text{ etc.}$$

Note that $C_t(\mathcal{T}_n) = 0$ for $t \in [\zeta(\mathcal{T}_n), \zeta(\mathcal{T}_n) + 2]$, and that we are concatenating the functions $(C_t(\mathcal{T}_n), t \in [0, \zeta(\mathcal{T}_n) + 2])$ rather than the functions $(C_t(\mathcal{T}_n), t \in [0, \zeta(\mathcal{T}_n)])$. This is a technical trick that will be useful in Chapter 2 below. We may also observe that the process obtained by concatenating the functions $(C_t(\mathcal{T}_n), t \in [0, \zeta(\mathcal{T}_n)])$ would not determine the sequence of trees.

There is a simple relation between the height process and the contour process: See Section 2.4 in Chapter 2 for more details.

Although the height process is not a Markov process, except in very particular cases, it turns out to be a simple functional of a Markov chain, which is even a random walk. The next lemma is taken from [**32**], but was obtained independently by other authors: See [**7**] and [**4**].

Lemma. — *Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of independent μ-Galton-Watson trees, and let $(H_n, n \geqslant 0)$ be the associated height process. There exists a random walk V on $\mathbb{Z}$ with initial value $V_0 = 0$ and jump distribution $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, 2, \ldots$, such that for every $n \geqslant 0$,*

$$H_n = \mathrm{Card}\{k \in \{0, 1, \ldots, n-1\} : V_k = \inf_{k \leqslant j \leqslant n} V_j\}. \tag{1}$$

A detailed proof of this lemma would be cumbersome, and we only explain the idea. By definition, H_n is the generation of the individual visited at time n, for a particle that visits the different vertices of the sequence of trees one tree after another and in lexicographical order for each tree. Write R_n for the quantity equal to the number of younger brothers (younger means greater in the lexicographical order) of the individual visited at time n plus the number of younger brothers of his father, plus the number of younger brothers of his grandfather etc. Then the random walk that appears in the lemma may be defined by

$$V_n = R_n - (j-1) \qquad \text{if } \#(\mathcal{T}_1) + \cdots + \#(\mathcal{T}_{j-1}) \leqslant n < \#(\mathcal{T}_1) + \cdots + \#(\mathcal{T}_j).$$

To verify that V is a random walk with jump distribution ν, note that because of the lexicographical order of visits, we have at time n no information on the fact that the individual visited at that time has children or not. If he has say $k \geqslant 1$ children, which occurs with probability $\mu(k)$, then the individual visited at time $n+1$ will be the first of these children, and our definitions give $R_{n+1} = R_n + (k-1)$ and $V_{n+1} = V_n + (k-1)$. On the other hand if he has no child, which occurs with probability $\mu(0)$, then the individual visited at time $n+1$ is the first of the brothers counted in the definition of R_n (or the ancestor of the next tree if $R_n = 0$) and we easily see that $V_{n+1} = V_n - 1$. We thus get exactly the transition mechanism of the random walk with jump distribution ν.

Let us finally explain formula (1). From our definition of R_n and V_n, it is easy to see that the condition $n < \inf\{j > k : V_j < V_k\}$ holds iff the individual visited at time n is a descendant of the individual visited at time k (more precisely, $\inf\{j > k : V_j < V_k\}$

is the time of the first visit after k of an individual that is not a descendant of individual k). Put in a different way, the condition $V_k = \inf_{k \leqslant j \leqslant n} V_j$ holds iff the individual visited at time k is an ascendant of the individual visited at time n. It is now clear that the right-hand side of (1) just counts the number of ascendants of the individual visited at time n, that is the generation of this individual.

0.3. The continuous height process

To define the height process in a continuous setting, we use an analogue of the discrete formula (1). The role of the random walk V in this formula is played by a Lévy process $X = (X_t, t \geqslant 0)$ without negative jumps. We assume that X does not drift to $+\infty$ (this corresponds to the subcriticality of μ in the discrete setting), and that the paths of X are of infinite variation a.s.: The latter assumption implies in particular that the process X started at the origin will immediately hit both $(0, \infty)$ and $(-\infty, 0)$. The law of X can be characterized by its Laplace functional ψ, which is the nonnegative function on $\mathbb{R}_+$ defined by

$$E[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)).$$

By the Lévy-Khintchine formula and our special assumptions on X, the function ψ has to be of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int \pi(dr)\,(e^{-\lambda r} - 1 + \lambda r),$$

where $\alpha, \beta \geqslant 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int \pi(dr)(r \wedge r^2) < \infty$. We write

$$S_t = \sup_{s \leqslant t} X_s\ , \qquad I_t = \inf_{s \leqslant t} X_s\,.$$

By analogy with the discrete case, we would like to define H_t as the "measure" of the set

$$\{s \leqslant t : X_s = \inf_{s \leqslant r \leqslant t} X_r\}. \tag{2}$$

However, under our assumptions on X, the Lebesgue measure of this set is always zero, and so we need to use some sort of local time. The key idea is to introduce for every fixed $t > 0$ the time-reversed process

$$\widehat{X}_s^{(t)} = X_t - X_{(t-s)-}\ , \quad 0 \leqslant s \leqslant t\,,$$

and its associated supremum

$$\widehat{S}_s^{(t)} = \sup_{r \leqslant s} \widehat{X}_r^{(t)}\,.$$

We observe that via time-reversal $s \to t - s$, the set (2) corresponds to $\{s \leqslant t : \widehat{S}_s^{(t)} = \widehat{X}_s^{(t)}\}$. This leads to the rigorous definition of H: H_t is defined as the local time at level 0, at time t of the process $\widehat{S}^{(t)} - \widehat{X}^{(t)}$. This definition makes sense because $\widehat{S}^{(t)} - \widehat{X}^{(t)}$ has the same law over $[0, t]$ as the so-called reflected process $S - X$ for

which 0 is a regular point under our assumptions. Note that the normalization of local time has to be specified in some way: See Section 1.1. The process $(H_t, t \geqslant 0)$ is called the *ψ-height process*, or simply the height process.

Why is the ψ-height process H an interesting object of study? In the same way as the discrete height process codes the genealogy of a sequence of independent Galton-Watson trees, we claim that the continuous height process represents the genealogical structure of continuous-state branching processes, which are the continuous analogues of Galton-Watson processes. This informal claim is at the heart of the developments of the present work. Perhaps the best justification for it can be found in the limit theorems of Chapter 2 that relate the discrete and continuous height processes (see Section 0.4 below). Another justification is the Ray-Knight theorem for the height process that will be discussed below.

The goal of Chapter 1 is to present a self-contained construction and to derive several new properties of the ψ-height process. Although there is some overlap with [**32**], our approach is different and involves new approximations. It is important to realize that H_t is defined as the local time at time t of a process which itself depends on t. For this reason, it is not clear whether the paths of H have any regularity properties. Also H is not Markov, except in the very special case where X has no jumps. To circumvent these difficulties, we rely on the important tool of the exploration process: For every $t \geqslant 0$, we define a random measure ρ_t on $\mathbb{R}_+$ by setting

$$\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_t^s \, f(H_s) \tag{3}$$

where

$$I_t^s = \inf_{s \leqslant r \leqslant t} X_r$$

and the notation $d_s I_t^s$ refers to integration with respect to the nondecreasing function $s \to I_t^s$. The *exploration process* $(\rho_t, t \geqslant 0)$ is a Markov process with values in the space $M_f(\mathbb{R}_+)$ of finite measures on $\mathbb{R}_+$. It was introduced and studied in [**32**], where its definition was motivated by a model of a LIFO queue (see [**35**] for some applications to queueing theory).

The exploration process has several interesting properties. In particular it is càdlàg (right-continuous with left limits) and it has an explicit invariant measure in terms of the subordinator with Laplace exponent $\psi(\lambda)/\lambda$ (see Proposition 1.2.5). Despite its apparently complicated definition, the exploration process is the crucial tool that makes it possible to answer most questions concerning the height process. A first illustration of this is the choice of a "good" lower-semicontinuous modification of H_t, which is obtained by considering for every $t \geqslant 0$ the supremum of the support of the measure ρ_t (beforehand, to make sense of the definition of ρ_t, one needs to use a first version of H that can be defined by suitable approximations of local times).

An important feature of both the height process and the exploration process is the fact that both H_t and ρ_t depend only on the values of X, or of $X-I$, on the excursion interval of $X-I$ away from 0 that straddles t. For this reason, it is possible to define and to study both the height process and the exploration process under the excursion measure of $X-I$ away from 0. This excursion measure, which is denoted by N, plays a major role throughout this work, and many results are more conveniently stated under N. Informally, the height process under N codes exactly one continuous tree, in the same way as each excursion away from 0 of the discrete height process corresponds to one Galton-Watson tree in the sequence (*cf.* Section 0.2).

As a typical application of the exploration process, we introduce and study the local times of the height process, which had not been considered in earlier work. These local times play an important role in the sequel, in particular in the applications to spatial branching processes. The local time of H at level $a \geqslant 0$ and at time t is denoted by L_t^a and these local times can be defined through the approximation

$$\lim_{\varepsilon \to 0} E\Big[\sup_{s \leqslant t} \Big| \varepsilon^{-1} \int_0^s 1_{\{a<H_r<a+\varepsilon\}} dr - L_s^a \Big| \Big] = 0$$

(Proposition 1.3.3). The proof of this approximation depends in a crucial way on properties of the exploration process derived in Section 1.3: Since H is in general not Markovian nor a semimartingale, one cannot use the standard methods of construction of local time.

The Ray-Knight theorem for the height process states that if

$$T_r = \inf\{t \geqslant 0 : X_t = -r\}, \quad \text{for } r > 0,$$

the process $(L_{T_r}^a, a \geqslant 0)$ is a continuous-state branching process with branching mechanism ψ (in short a ψ-CSBP) started at r. Recall that the ψ-CSBP is the Markov process $(Y_a, a \geqslant 0)$ with values in $\mathbb{R}_+$ whose transition kernels are characterized by their Laplace transform: For $\lambda > 0$ and $b > a$,

$$E[\exp -\lambda Y_b \mid Y_a] = \exp(-Y_a\, u_{b-a}(\lambda)),$$

where $u_t(\lambda)$, $t \geqslant 0$ is the unique nonnegative solution of the differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)) \ , \quad u_0(\lambda) = \lambda.$$

By analogy with the discrete setting, we can think of $L_{T_r}^a$ as "counting" the number of individuals at generation a in a Poisson collection of continuous trees (those trees coded by the excursions of $X-I$ away from 0 before time T_r). The Ray-Knight theorem corresponds to the intuitive fact that the population at generation a is a branching process.

The previous Ray-Knight theorem had already been derived in [**32**] although in a less precise form (local times of the height process had not been constructed). An important consequence of the Ray-Knight theorem, also derived in [**32**], is a criterion

for the path continuity of H: H has continuous sample paths iff

$$\int_1^\infty \frac{d\lambda}{\psi(\lambda)} < \infty. \tag{4}$$

This condition is in fact necessary and sufficient for the a.s. extinction of the ψ-CSBP. If it does not hold, the paths of H have a very wild behavior: The values of H over any nontrivial interval $[s,t]$ contain a half-line $[a,\infty)$. On the other hand, (4) holds if $\beta > 0$, and in the stable case $\psi(\lambda) = c\lambda^\gamma$, $1 < \gamma \leqslant 2$ (the values $\gamma \in (0,1]$ are excluded by our assumptions).

In view of applications in Chapter 4, we derive precise information about the Hölder continuity of H. We show that if

$$\gamma = \sup\{r \geqslant 0 : \lim_{\lambda\to\infty} \lambda^{-r}\psi(\lambda) = +\infty\},$$

then the height process H is a.s. Hölder continuous with exponent r for any $r \in (0, 1-\gamma^{-1})$, and a.s. not Hölder continuous with exponent r if $r > 1-\gamma^{-1}$.

0.4. From discrete to continuous trees

Chapter 2 discusses limit theorems for rescaled Galton-Watson trees. These results demonstrate that the ψ-height process is the correct continuous analogue of the discrete height process coding Galton-Watson trees.

It is well known [**27**] that continuous-state branching processes are the only possible scaling limits of discrete-time Galton-Watson branching processes. One may then ask for finer limit theorems involving the genealogy. Precisely, starting from a sequence of rescaled Galton-Watson processes that converge in distribution towards a continuous-state branching process, can one say that the corresponding discrete Galton-Watson trees also converge, in some sense, towards a continuous genealogical structure? The results of Chapter 2 show that the answer is yes.

To be specific, consider a sequence (μ_p) of (sub)critical offspring distributions. For every $p \geqslant 1$, let Y^p be a (discrete-time) Galton-Watson process with offspring distribution μ_p started at $Y_0^p = p$. Suppose that the processes Y^p converge after rescaling towards a ψ-CSBP, where ψ satisfies the conditions introduced in Section 0.3. Precisely, we assume that there is a sequence $\gamma_p \uparrow \infty$ such that

$$(p^{-1}Y^p_{[\gamma_p t]}, t \geqslant 0) \xrightarrow[p\to\infty]{\text{(d)}} (Y_t, t \geqslant 0), \tag{5}$$

where Y is a ψ-CSBP, and the symbol (d) indicates convergence in distribution in the Skorokhod space. Let H^p be the discrete height process associated with μ_p in the sense of Section 0.2. Then Theorem 2.2.1 shows that

$$(\gamma_p^{-1}H^p_{[p\gamma_p t]}, t \geqslant 0) \xrightarrow[p\to\infty]{\text{(fd)}} (H_t, t \geqslant 0), \tag{6}$$

where H is the ψ-height process and (fd) indicates convergence of finite-dimensional marginals. A key ingredient of the proof is the observation due to Grimvall [**21**] that

the convergence (5) implies the convergence in distribution (after suitable rescaling) of the random walks V^p with jump distribution $\nu_p(k) = \mu_p(k+1)$, $k = -1, 0, 1, \ldots$, towards the Lévy process with Laplace exponent ψ. The idea is then to pass to the limit in the formula for H^p in terms of V^p, recalling that the ψ-height process is given by an analogous formula in terms of the Lévy process X. In the special case $\beta = 0$ and under more restrictive assumptions, the convergence (6) had already appeared in [**32**].

In view of applications, the limiting result (6) is not satisfactory because the convergence of finite-dimensional marginals is too weak. In order to reinforce (6) to a functional convergence, it is necessary to assume some regularity of the paths of H. We assume that condition (4) ensuring the path continuity of H holds (recall that if this condition does not hold, the paths of H have a very wild behavior). Then, we can prove (Theorem 2.3.1) that the convergence (6) holds in the sense of weak convergence on the Skorokhod space, provided that the following condition is satisfied: For every $\delta > 0$,

$$\liminf_{p\to\infty} P[Y^p_{[\delta\gamma_p]} = 0] > 0. \tag{7}$$

Roughly speaking this means that the rescaled Galton-Watson process $(p^{-1}Y^p_{[\gamma_p t]})_{t\geqslant 0}$ may die out at a time of order 1, as its weak limit Y does (recall that we are assuming (4)). The technical condition (7) is both necessary and sufficient for the reinforcement of (6) to a functional convergence. Simple examples show that this condition cannot be omitted in general.

However, in the important special case where $\mu_p = \mu$ for every p, we are able to show (Theorem 2.3.2) that the technical condition (7) is always satisfied. In that case, ψ must be of the form $\psi(u) = c\,u^\gamma$ with $1 < \gamma \leqslant 2$, so that obviously (4) also holds. Thus when $\mu_p = \mu$ for every p, no extra condition is needed to get a functional convergence.

In Section 2.4, we show that the functional convergence derived for rescaled discrete height processes can be stated as well in terms of the contour processes (*cf.* Section 0.1). Let $C^p = (C^p_t, t \geqslant 0)$ be the contour process for a sequence of independent μ_p-Galton-Watson trees. Under the assumptions that warrant the functional convergence in (6), Theorem 2.4.1 shows that we have also

$$(\gamma_p^{-1} C^p_{p\gamma_p t}, t \geqslant 0) \xrightarrow[p\to\infty]{\text{(d)}} (H_{t/2}, t \geqslant 0).$$

Thus scaling limits are the same for the discrete height process and for the contour process.

In the remaining part of Chapter 2, we give applications of (6) assuming that the functional convergence holds. In particular, rather than considering a sequence of μ_p-Galton-Watson trees, we discuss the height process associated with a single tree

conditioned to be large. Precisely, let $\widetilde{H}^p$ be the height process for one μ_p-Galton-Watson tree conditioned to non-extinction at generation $[\gamma_p T]$, for some fixed $T > 0$. Then, Proposition 2.5.2 gives

$$(\gamma_p^{-1} \widetilde{H}^p_{[p\gamma_p t]}, t \geqslant 0) \xrightarrow[p\to\infty]{\text{(d)}} (\widetilde{H}_t, t \geqslant 0),$$

where the limiting process is an excursion of the ψ-height process conditioned to hit level T. This is of course reminiscent of a result of Aldous [**3**] who proved that in the case of a critical offspring distribution μ with finite variance, the contour process of a μ-Galton-Watson tree conditioned to have exactly p vertices converges after a suitable rescaling towards a normalized Brownian excursion (see also [**19**] and [**36**] for related results including the convergence of the height process in Aldous' setting). Note that in Aldous' result, the conditioning becomes degenerate in the limit, since the "probability" that a Brownian excursion has length exactly one is zero. This makes it more difficult to derive this result from our approach, although it seems very related to our limit theorems. See however Duquesne [**10**] for an extension of Aldous' theorem to the stable case using the tools of the present work (a related result in the stable case was obtained by Kersting [**26**]).

The end of Chapter 2 is devoted to reduced trees. We consider again a single μ_p-Galton-Watson tree conditioned to non-extinction at generation $[\gamma_p T]$. For every $k < [\gamma_p T]$, we denote by $Z_k^{(p),[\gamma_p T]}$ the number of vertices at generation k that have descendants at generation $[\gamma_p T]$. Under the assumptions and as a consequence of Proposition 2.5.2, we can prove that

$$(Z^{(p),[\gamma_p T]}_{[\gamma_p t]}, 0 \leqslant t < T) \xrightarrow[p\to\infty]{\text{(fd)}} (Z_t^T, 0 \leqslant t < T)$$

where the limit Z^T has a simple definition in terms of $\widetilde{H}$: Z_t^T is the number of excursions of $\widetilde{H}$ above level t that hit level T. Thanks to the properties of the height process and the exploration process that have been derived in Chapter 1, it is possible to calculate the distribution of the time-inhomogeneous branching process $(Z_t^T, t \geqslant 0)$. This distribution is derived in Theorem 2.7.1. Of course in the stable case, corresponding to $\mu_p = \mu$ for every p, the distribution of Z^T had been computed previously. See in particular Zubkov [**50**] and Fleischmann and Siegmund-Schultze [**17**].

0.5. Duality properties of the exploration process

In the applications developed in Chapters 3 and 4, a key role is played by the duality properties of the exploration process ρ. We first observe that formula (3) defining the exploration process can be rewritten in the following equivalent way

$$\rho_t(dr) = \beta 1_{[0,H_t]}(r)\, dr + \sum_{s \leqslant t, X_{s-} < I_t^s} (I_t^s - X_{s-}) \delta_{H_s}(dr)$$

where δ_{H_s} is the Dirac measure at H_s, and we recall that $I_t^s = \inf_{s \leqslant r \leqslant t} X_r$. We then define another measure η_t by setting

$$\eta_t(dr) = \beta 1_{[0,H_t]}(r)\, dr + \sum_{s \leqslant t, X_{s-} < I_t^s} (X_s - I_t^s)\delta_{H_s}(dr).$$

To motivate this definition, we may come back to the discrete setting of Galton-Watson trees. In that setting, the value at time n of the discrete height process H_n is the generation of the n-th visited vertex by a "particle" that visits vertices in lexicographical order one tree after another, and the analogue of ρ_t gives for every $k \leqslant H_n$ the number of younger (i.e. coming later in the lexicographical order) brothers of the ancestor at generation k of the n-th visited vertex. Then the analogue of η_t gives for every $k \leqslant H_n$ the number of older brothers of the ancestor at generation k of the n-th visited vertex.

It does not seem easy to study directly the Markovian properties or the regularity of paths of the process $(\eta_t, t \geqslant 0)$. The right point of view is to consider the pair (ρ_t, η_t), which is easily seen to be a Markov process in $M_f(\mathbb{R}_+)^2$. The process (ρ_t, η_t) has an invariant measure $\mathbb{M}$ determined in Proposition 3.1.3. The key result (Theorem 3.1.4) then states that the Markov processes (ρ, η) and (η, ρ) are in duality under $\mathbb{M}$. A consequence of this is the fact that $(\eta_t, t \geqslant 0)$ also has a càdlàg modification. More importantly, we obtain a crucial time-reversal property: Under the excursion measure N of $X - I$, the processes $(\rho_s, \eta_s; 0 \leqslant s \leqslant \sigma)$ and $(\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}; 0 \leqslant s \leqslant \sigma)$ have the same distribution (here σ stands for the duration of the excursion under N). This time-reversal property plays a major role in many subsequent calculations. It implies in particular that the law of H under N is invariant under time-reversal. This property is natural in the discrete setting, if we think of the contour process of a Galton-Watson tree, but not obvious in the continuous case.

0.6. Marginals of trees coded by the height process

Let us explain more precisely how an excursion of the ψ-height process codes a continuous branching structure. We consider first a deterministic continuous function $e : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $e(t) > 0$ iff $0 < t < \sigma$, for some $\sigma = \sigma(e) > 0$. For any $s, s' \geqslant 0$, set

$$m_e(s, s') = \inf_{s \wedge s' \leqslant t \leqslant s \vee s'} e(t).$$

Then e codes a continuous genealogical structure via the following simple prescriptions:

(i) To each $s \in [0, \sigma]$ corresponds a vertex at generation $e(s)$.

(ii) Vertex s is an ancestor of vertex s' if $e(s) = m_e(s, s')$. In general, $m_e(s, s')$ is the generation of the last common ancestor to s and s'.

(iii) We put $d(s, s') = e(s) + e(s') - 2m_e(s, s')$ and identify s and s' ($s \sim s'$) if $d(s, s') = 0$.

Formally, the tree coded by e can be defined as the quotient set $[0,\sigma]/\sim$, equipped with the distance d and the genealogical relation specified in (ii).

With these definitions, the line of ancestors of a vertex s is isometric to the segment $[0,e(s)]$. If we pick two vertices s and s', their lines of ancestors share a common part isometric to $[0,m_e(s,s')]$, and then become distinct. In general, if we consider p instants $t_1,\dots,t_p$ with $0 \leqslant t_1 \leqslant \cdots \leqslant t_p \leqslant \sigma$, we can associate with these p instants a genealogical tree $\theta(e,t_1,\dots,t_p)$, which consists of a discrete rooted ordered tree with p leaves, denoted by $\mathcal{T}(e,t_1,\dots,t_p)$ and marks $h_v(e,t_1,\dots,t_p) \geqslant 0$ for $v \in \mathcal{T}(e,t_1,\dots,t_p)$, that correspond to the lifetimes of vertices in $\mathcal{T}(e,t_1,\dots,t_p)$. See subsection 3.2.1 for a precise definition.

In the second part of Chapter 3, we use the duality results proved in the first part to calculate the distribution of the tree $\theta(H,\tau_1,\dots,\tau_p)$ under certain excursion laws of H and random choices of the instants $\tau_1,\dots,\tau_p$. We assume that the continuity condition (4) holds. We first consider Poissonnian marks with intensity λ, and the height process H under the excursion measure N of $X-I$. Let $\tau_1,\dots,\tau_M$ be the marks that fall into the duration interval $[0,\sigma]$ of the excursion. Theorem 3.2.1 shows that under the probability measure $N(\cdot \mid M \geqslant 1)$, the tree $\theta(H,\tau_1,\dots,\tau_M)$ is distributed as the family tree of a continuous-time Galton-Watson process starting with one individual at time 0 and where

- lifetimes have an exponential distribution with parameter $\psi'(\psi^{-1}(\lambda))$;
- the offspring distribution is the law of the variable ξ with generating function

$$E[r^\xi] = r + \frac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}.$$

In the quadratic case, we get a critical binary branching $E[r^\xi] = \frac{1}{2}(1+r^2)$. The result in that case had been obtained by Hobson [**22**].

We finally specialize to the stable case $\psi(\lambda) = \lambda^\gamma$, $\gamma \in (1,2]$. By scaling arguments, we can then make sense of the law $N_{(1)} = N(\cdot \mid \sigma = 1)$ of the normalized excursion of H. Using the case of Poissonnian marks, we compute explicitly the law of the tree $\theta(H,t_1,\dots,t_p)$ under $N_{(1)}$, when $(t_1,\dots,t_p)$ are chosen independently and uniformly over $[0,1]^p$. In the quadratic case $\psi(u) = u^2$, H is under $N_{(1)}$ a normalized Brownian excursion, and the corresponding tree is called the continuum random tree (see Aldous [**1**],[**2**],[**3**]). By analogy, in our more general case $\psi(u) = u^\gamma$, we may call the tree coded by H under $N_{(1)}$ the stable continuum random tree. Our calculations give what Aldous calls the finite-dimensional marginals of the tree. In the case $\gamma = 2$, these marginals were computed by Aldous (see also Le Gall [**31**] for a different approach closer to the present work). In that case, the discrete skeleton $\mathcal{T}(H,t_1,\dots,t_p)$ is uniformly distributed over all binary rooted ordered trees with k leaves. When $\gamma < 2$, things become different as we can get nonbinary trees (the reason why we get only binary trees in the Brownian case is the fact that local minima of Brownian motion are distinct). Theorem 3.3.3 shows in particular that if $\mathcal{T}$ is a tree with p leaves

such that $k_u(\mathcal{T}) \neq 1$ for every $u \in \mathcal{T}$ (this condition must be satisfied by our trees $\mathcal{T}(e, t_1, \dots, t_p)$) then the probability that $\mathcal{T}(H, t_1, \dots, t_p) = \mathcal{T}$ is

$$\frac{p!}{(\gamma-1)(2\gamma-1)\cdots((p-1)\gamma-1)} \prod_{v \in \mathcal{N}_\mathcal{T}} \frac{|(\gamma-1)(\gamma-2)\cdots(\gamma-k_v+1)|}{k_v!}$$

where $\mathcal{N}_\mathcal{T} = \{v \in \mathcal{T} : k_v > 0\}$ is the set of nodes of $\mathcal{T}$. It would be interesting to know whether this distribution on discrete trees has occurred in other settings.

0.7. The Lévy snake

Chapters 1 – 3 explore the continuous genealogical structure coded by the ψ-height process H. In Chapter 4, we examine the probabilistic objects obtained by combining this branching structure with a spatial motion given by a càdlàg Markov process ξ with state space E. Informally, "individuals" do not only reproduce themselves, but they also move in space independently according to the law of ξ. The (ξ, ψ)-superprocess is then a Markov process taking values in the space of finite measures on E, whose value at time t is a random measure putting mass on the set of positions of "individuals" alive at time t. Note that the previous description is very informal since in the continuous branching setting there are no individual particles but rather a continuum of infinitesimal particles. Recent accounts of the theory of superprocesses can be found in Dynkin [**13**], Etheridge [**15**] and Perkins [**40**].

Our coding of the genealogy by the height process leads to introducing a Markov process whose values will give the historical paths followed by the "individuals" in the population. This a generalization of the Brownian snake introduced in [**28**] and studied in particular in [**31**]. To give a precise definition, fix a starting point $x \in E$, consider the ψ-height process $(H_s, s \geqslant 0)$ and recall the notation $m_H(s, s') = \inf_{[s,s']} H_r$ for $s \leqslant s'$. We assume that the continuity condition (4) holds. Then conditionally on $(H_s, s \geqslant 0)$ we consider a time-inhomogeneous Markov process $(W_s, s \geqslant 0)$ whose distribution is described as follows:

– For every $s \geqslant 0$, $W_s = (W_s(t), 0 \leqslant t < H_s)$ is a path of ξ started at x and with finite lifetime H_s.

– If we consider two instants s and s', the corresponding paths W_s and $W_{s'}$ are the same up to time $m_H(s, s')$ and then behave independently.

The latter property is consistent with the fact that in our coding of the genealogy, vertices attached to s and s' have the same ancestors up to generation $m_H(s, s')$. See Section 4.1 for a more precise definition.

The pair (ρ_s, W_s) is then a Markov process with values in the product space $M_f(\mathbb{R}_+) \times \mathcal{W}$, where $\mathcal{W}$ stands for the set of all finite càdlàg paths in E. This process is called the *Lévy snake* (with initial point x). It was introduced and studied in [**33**], where a form of its connection with superprocesses was established. Chapter 4 gives much more detailed information about its properties. In particular, we prove

the strong Markov property of the Lévy snake (Theorem 4.1.2), which plays a crucial role in several applications.

We also use the local times of the height process to give a nicer form of the connection with superprocesses. Write $\widehat{W}_s$ for the left limit of W_s at its lifetime H_s (which exists a.s. for each fixed s), and recall the notation $T_r = \inf\{t \geqslant 0 : X_t = -r\}$. For every $t \geqslant 0$, we can define a random measure Z_t on E by setting

$$\langle Z_t, f \rangle = \int_0^{T_r} d_s L_s^t \, f(\widehat{W}_s) \, .$$

Then $(Z_t, t \geqslant 0)$ is a (ξ, ψ)-superprocess with initial value $r\delta_x$. This statement is in fact a special case of Theorem 4.2.1 which constructs a (ξ, ψ)-superprocess with an arbitrary initial value. For this more general statement, it is necessary to use excursion measures of the Lévy snake: Under the *excursion measure* $\mathbb{N}_x$, the process $(\rho_s, s \geqslant 0)$ is distributed according to its excursion measure N, and $(W_s, s \geqslant 0)$ is constructed by the procedure explained above, taking x for initial point.

As a second application, we use local time techniques to construct exit measures from an open set and to establish the integral equation satisfied by the Laplace functional of exit measures (Theorem 4.3.3). Recall that exit measures of superprocesses play a fundamental role in the connections with partial differential equations studied recently by Dynkin and Kuznetsov (a detailed account of these connections can be found in the book [**13**]).

We then study the continuity of the path-valued process W_s with respect to the uniform topology on paths. This question is closely related to the compact support property for superprocesses. In the case when ξ is Brownian motion in $\mathbb{R}^d$, Theorem 4.5.2 shows that the condition

$$\int_1^\infty \Big(\int_0^t \psi(u) \, du \Big)^{-1/2} dt < \infty$$

is necessary and sufficient for W_t to be continuous with respect to the uniform topology on paths. The proof relies on connections of the exit measure with partial differential equations and earlier work of Sheu [**45**], who was interested in the compact support property for superprocesses. In the case of a general spatial motion, assuming only that ξ has Hölder continuous paths, we use the continuity properties of H derived in Chapter 1 to give simple sufficient conditions ensuring that the same conclusion holds.

Although we do not develop such applications in the present work, we expect that the Lévy snake will be a powerful tool to study connections with partial differential equations, in the spirit of [**30**], as well as path properties of superprocesses (see [**34**] for a typical application of the Brownian snake to super-Brownian motion).

In the last two sections of Chapter 4, we compute certain explicit distributions related to the Lévy snake and the (ξ, ψ)-superprocess, under the excursion measures $\mathbb{N}_x$. We assume that the path-valued process W_s is continuous with respect to the

uniform topology on paths, and then the value $W_s(H_s)$ can be defined as a left limit at the lifetime, simultaneously for all $s \geqslant 0$. If D is an open set in E such that $x \in D$, we consider the first exit time

$$T_D = \inf\{s \geqslant 0 : \tau(W_s) < \infty\}$$

where $\tau(W_s) = \inf\{t \in [0, H_s] : W_s(t) \notin D\}$. Write $u(y) = \mathbb{N}_y(T_D < \infty) < \infty$ for every $y \in D$. Then the distribution of W_{T_D} under $\mathbb{N}_x(\cdot \cap \{T_D < \infty\})$ is characterized by the function u and the distribution Π_x of ξ started at x via the formula: For every $a \geqslant 0$,

$$\mathbb{N}_x\Big(1_{\{T_D<\infty\}} 1_{\{a<H_{T_D}\}} F(W_{T_D}(t), 0 \leqslant t \leqslant a)\Big)$$
$$= \Pi_x\Big[1_{\{a<\tau\}} u(\xi_a) F(\xi_r, 0 \leqslant r \leqslant a) \exp\Big(-\int_0^a \widetilde{\psi}(u(\xi_r))dr\Big)\Big],$$

where τ stands for the first exit time of ξ from D, and $\widetilde{\psi}(r) = \psi(r)/r$. Theorem 4.6.2 gives more generally the law of the pair (W_{T_D}, ρ_{T_D}) under $\mathbb{N}_x(\cdot \cap \{T_D < \infty\})$. In the special case when ξ is Brownian motion in $\mathbb{R}^d$, the function u can be identified as the maximal nonnegative solution of $\frac{1}{2}\Delta u = \psi(u)$ in D, and the law of W_{T_D} is that of a Brownian motion with drift $\nabla u/u$ up to its exit time from D. This considerably extends a result of [**29**] proved in the quadratic branching case by a very different method.

The last section of Chapter 4 investigates reduced spatial trees, again under the assumption that the path-valued process W_s is continuous with respect to the uniform topology on paths. We consider a spatial open set D with $x \in D$, and the Lévy snake under its excursion measure $\mathbb{N}_x$ (in the superprocess setting this means that we are looking at all historical paths corresponding to one given ancestor at time 0). We condition on the event that $\{T_D < \infty\}$, that is one at least of the paths W_s exits D, and we want to describe the spatial structure of all the paths that exit D, up to their respective exit times. This is an analogue (and in fact a generalization) of the reduced tree problem studied in Chapter 2. In the spatial situation, all paths W_s that exit D will be the same up to a certain time m_D at which there is a branching point with finitely many branches, each corresponding to an excursion of the height process H above level m_D during which the Lévy snake exits D. In each such excursion the paths W_s that exit D will be the same up to a level strictly greater than m_D, at which there is another branching point, and so on.

To get a full description of the reduced spatial tree, one only needs to compute the joint distribution of the path $W_0^D = W_{T_D}(. \wedge m_D)$, that is the common part to all paths that do exit D, and the number N_D of branches at the first branching point. Indeed, conditionally on the pair (W_0^D, N_D), the "subtrees of paths" that originate from the first branching point will be independent and distributed according to the full reduced tree with initial point $\widehat{W}_0^D = W_0^D(m_D)$ (see Theorem 4.7.2 for more precise statements). Theorem 4.7.2 gives explicit formulas for the joint distribution

of (W_0^D, N_D), again in terms of the function $u(y) = \mathbb{N}_y(T_D < \infty) < \infty$. Precisely, the law of the "first branch" W_0^D is given by

$$\mathbb{N}_x(1_{\{T_D<\infty\}} F(W_0^D)) = \int_0^\infty db\, \Pi_x\Big[1_{\{b<\tau\}} u(\xi_b)\, \theta(u(\xi_b))\, \exp\Big(-\int_0^b \psi'(u(\xi_r))dr\Big) F(\xi_r, 0 \leqslant r \leqslant b)\Big],$$

where $\theta(r) = \psi'(r) - \widetilde{\psi}(r)$. Furthermore the conditional distribution of N_D given W_0^D depends only on the branching point $\widehat{W}_0^D$ and is given by

$$\mathbb{N}_x[r^{N_D} \mid T_D < \infty, W_0^D] = r\, \frac{\psi'(U) - \gamma_\psi(U, (1-r)U)}{\psi'(U) - \gamma_\psi(U, 0)}\,, \qquad 0 \leqslant r \leqslant 1,$$

where $U = u(\widehat{W}_0^D)$ and $\gamma_\psi(a,b) = \frac{\psi(a)-\psi(b)}{a-b}$. In the stable case $\psi(u) = u^\gamma$, the variable N_D is independent of W_0^D and its generating function is $(\gamma-1)^{-1}((1-r)^\gamma - 1 + \gamma r)$.

In the special case $\psi(u) = u^2$, the previous result was used by Mselati [**38**] to investigate absolute continuity properties of the law of the exit measure of super-Brownian motion, in view of applications to the semilinear partial differential equation $\Delta u = u^2$. Again in the quadratic branching case, our description of the reduced spatial tree is reminiscent of the recent work of Salisbury and Verzani [**43**],[**44**] who study exit measures of super-Brownian motion conditioned to hit a number of specified points on the boundary. This conditioning leads to a spatial tree described by a branching particle backbone process with immigration of mass along the paths of the particles.

Acknowledgment. — We would like to thank Yves Le Jan for allowing us to use several ideas that originated in his work in collaboration with one of us. We also thank the referee for several useful remarks.

CHAPTER 1

THE HEIGHT PROCESS

1.1. Preliminaries on Lévy processes

1.1.1. Basic assumptions. — In this section, we introduce the class of Lévy processes that will be relevant to our study and we record some of their basic properties. For almost all facts about Lévy processes that we need, we refer to the recent book of Bertoin [**5**] (especially Chapter VII).

We consider a Lévy process X on the real line. It will be convenient to assume that X is the canonical process on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg (right-continuous with left limits) real-valued paths. Unless otherwise noted, the underlying probability measure P is the law of the process started at 0. We denote by $(\mathcal{G}_t, t \in [0, \infty])$ the canonical filtration, completed as usual by the class of all P-negligible sets of $\mathcal{G}_\infty$.

We assume that the following three properties hold a.s.:

(H1) X has no negative jumps.

(H2) X does not drift to $+\infty$.

(H3) The paths of X are of infinite variation.

Thanks to (H1), the "Laplace transform" $E[\exp -\lambda X_t]$ is well defined for every $\lambda \geqslant 0$ and $t \geqslant 0$, and can be written as

$$E[\exp -\lambda X_t] = \exp(t\psi(\lambda)),$$

with a function ψ of the form

$$\psi(\lambda) = \alpha_0 \lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-\lambda r} - 1 + 1_{\{r<1\}}\lambda r),$$

where $\alpha_0 \in \mathbb{R}$, $\beta \geqslant 0$ and the Lévy measure π is a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge r^2)\,\pi(dr) < \infty$.

Assumption (H2) then holds iff X has first moments and $E[X_1] \leqslant 0$. The first moment assumption is equivalent to saying that π satisfies the stronger integrability condition

$$\int_{(0,\infty)} (r \wedge r^2)\,\pi(dr) < \infty.$$

Then ψ can be written in the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-\lambda r} - 1 + \lambda r), \tag{8}$$

Note that ψ is then convex and that we have $E[X_t] = -t\,\psi'(0) = -t\alpha$. The condition $E[X_1] \leqslant 0$ thus holds iff $\alpha \geqslant 0$. The process X is recurrent or drifts to $-\infty$ according as $\alpha = 0$ or $\alpha > 0$.

Finally, according to [**5**] (Corollary VII.5), assumption (H3) holds iff at least one of the following two conditions is satisfied: $\beta > 0$, or

$$\int_{(0,1)} r\,\pi(dr) = \infty.$$

Summarizing, we assume that X is a Lévy process with no negative jumps, whose Laplace exponent ψ has the form (8), where $\alpha \geqslant 0$, $\beta \geqslant 0$ and π is a σ-finite measure on $(0,\infty)$ such that $\int(r \wedge r^2)\pi(dr) < \infty$, and we exclude the case where both $\beta = 0$ and $\int_{(0,1)} r\,\pi(dr) < \infty$.

Remark. — Only assumption (H1) is crucial to the connections with branching processes that are presented in this work. Assumption (H2) means that we restrict our attention to the critical or subcritical case. We impose assumption (H3) in order to concentrate on the most interesting cases: A simpler parallel theory can be developed in the finite variation case, see Section 3 of [**32**].

We will use the notation $T_y = \inf\{t \geqslant 0 : X_t = -y\}$ for $y \in \mathbb{R}$. By convention $\inf \varnothing = +\infty$.

Under our assumptions, the point 0 is regular for $(0,\infty)$ and for $(-\infty,0)$, meaning that $\inf\{t > 0 : X_t > 0\} = 0$ and $\inf\{t > 0 : X_t < 0\} = 0$ a.s. (see [**5**], Theorem VII.1 and Corollary VII.5). We sometimes use this property in connection with the so-called duality property: For every $t > 0$, define a process $\widehat{X}^{(t)} = (\widehat{X}^{(t)}_s, 0 \leqslant s \leqslant t)$ by setting

$$\widehat{X}^{(t)}_s = X_t - X_{(t-s)-}\ , \quad \text{if } 0 \leqslant s < t,$$

and $\widehat{X}^{(t)}_t = X_t$. Then $(\widehat{X}^{(t)}_s, 0 \leqslant s \leqslant t)$ has the same law as $(X_s, 0 \leqslant s \leqslant t)$.

Let S and I be respectively the supremum and the infimum process of X, defined by

$$S_t = \sup_{s \leqslant t} X_s\ , \quad I_t = \inf_{s \leqslant t} X_s.$$

If we combine the duality property with the regularity of 0 for both $(0,\infty)$ and $(-\infty,0)$, we easily get that the set

$$\{s > 0 : X_{s-} = I_s \text{ or } X_{s-} = S_{s-}\}$$

almost surely does not intersect $\{s \geqslant 0 : \Delta X_s \neq 0\}$. This property will be used implicitly in what follows.

1.1.2. Local times at the maximum and the minimum. — Both processes $X - S$ and $X - I$ are strong Markov processes, and the results recalled at the end of the previous subsection imply that the point 0 is regular for itself with respect to each of these two Markov processes. We can thus define the corresponding Markovian local times and excursion measures, which both play a fundamental role in this work.

Consider first $X - S$. We denote by $L = (L_t, t \geqslant 0)$ a local time at 0 for $X - S$. Observe that L is only defined up to a positive multiplicative constant, that will be specified later. Let N^* be the associated excursion measure, which is a σ-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. It will be important for our purposes to keep track of the final jump under N^*. This can be achieved by the following construction. Let (a_j, b_j), $j \in J$, be the excursion intervals of $X - S$ away from 0. In the transient case ($\alpha > 0$), there is exactly one value $j \in J$ such that $b_j = +\infty$. For every $j \in J$ let $\omega^j \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ be defined by

$$\omega^j(s) = X_{(a_j+s)\wedge b_j} - X_{a_j}\,, \qquad s \geqslant 0.$$

Then the point measure

$$\sum_{j\in J} \delta_{(L_{a_j}, \omega^j)}$$

is distributed as $1_{\{l\leqslant\eta\}}\mathcal{N}(dld\omega)$, where $\mathcal{N}$ denotes a Poisson point measure with intensity $dl\, N^*(d\omega)$, and $\eta = \inf\{l : \mathcal{N}([0,l] \times \{\sigma = +\infty\}) \geqslant 1\}$, if

$$\sigma(\omega) = \inf\{t > 0 : \omega(r) = \omega(t) \text{ for every } r \geqslant t\}$$

stands for the duration of the excursion ω. This statement characterizes the excursion measure N^*, up to the multiplicative constant already mentioned. Note that $X_0 = 0$ and $X_t = X_\sigma \geqslant 0$ for $t \geqslant \sigma$, N^* a.e.

Consider then $X - I$. It is easy to verify that the continuous increasing process $-I$ is a local time at 0 for the Markov process $X - I$. We will denote by N the associated excursion measure, which can be characterized in a way similar to N^* (with the difference that we have always $-I_\infty = +\infty$ a.s., in contrast to the property $L_\infty < \infty$ a.s. in the transient case). We already noticed that excursions of $X - I$ cannot start with a jump. Hence, $X_0 = 0$, N a.e. It is also clear from our assumptions on X that $\sigma < \infty$, $X_t > 0$ for every $t \in (0, \sigma)$ and $X_{\sigma-} = 0$, N a.e.

We will now specify the normalization of N^*, or equivalently of L. Let m denote Lebesgue measure on $\mathbb{R}$.

Lemma 1.1.1. — *We can fix the normalization of L, or equivalently of N^*, so that, for every Borel subset B of $(-\infty, 0)$,*

$$N^*\left(\int_0^\sigma ds\, 1_B(X_s)\right) = m(B). \tag{9}$$

Proof. — For every $x \in \mathbb{R}$, write P_x for the law of the Lévy process started at x. Also set $\tau = \inf\{s \geqslant 0 : X_s \geqslant 0\}$ and recall that $(X_t, t > 0)$ is Markovian under N^* with the transition kernels of the underlying Lévy process stopped when hitting $[0, \infty)$. Thanks to this observation, it is enough to prove that, for every $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that for every Borel subset B of $(-\infty, -\varepsilon)$,

$$E_{-\varepsilon}\Big[\int_0^\tau ds\, 1_B(X_s)\Big] = c(\varepsilon) m(B).$$

Consider first the transient case. By applying the strong Markov property at hitting times of negative values, it is easy to verify that the measure on $(-\infty, -\varepsilon)$ defined by

$$B \longrightarrow E_{-\varepsilon}\Big[\int_0^\infty ds\, 1_B(X_s)\Big]$$

must be a multiple of Lebesgue measure. However, writing $T_0^\varepsilon = 0, T_1^\varepsilon, \ldots, T_n^\varepsilon$, etc. for the successive visits of $-\varepsilon$ via $[0, \infty)$, we have

$$E_{-\varepsilon}\Big[\int_0^\infty ds\, 1_B(X_s)\Big] = \sum_{i=0}^\infty E_{-\varepsilon}\Big[1_{\{T_i^\varepsilon < \infty\}} \int_{T_i^\varepsilon}^{T_{i+1}^\varepsilon} ds\, 1_B(X_s)\Big] = \frac{E_{-\varepsilon}\Big[\int_0^\tau ds\, 1_B(X_s)\Big]}{P_{-\varepsilon}[\tau = \infty]}.$$

The desired result follows.

In the recurrent case, the ergodic theorem gives

$$\frac{1}{n}\int_0^{T_n^\varepsilon} ds\, 1_B(X_s) \xrightarrow[n\to\infty]{\text{a.s.}} E_{-\varepsilon}\Big[\int_0^\tau ds\, 1_B(X_s)\Big],$$

whereas the Chacon-Ornstein ergodic theorem implies

$$\frac{\int_0^{T_n^\varepsilon} ds\, 1_B(X_s)}{\int_0^{T_n^\varepsilon} ds\, 1_{(-2\varepsilon,-\varepsilon)}(X_s)} \xrightarrow[n\to\infty]{\text{a.s.}} \frac{m(B)}{\varepsilon}.$$

The conclusion easily follows. □

In what follows we always assume that the normalization of L or of N^* is fixed as in Lemma 1.1.1.

Let $L^{-1}(t) = \inf\{s, L_s > t\}$. By convention, $X_{L^{-1}(t)} = +\infty$ if $t \geqslant L_\infty$. The process $(X_{L^{-1}(t)}, t \geqslant 0)$ is a subordinator (the so-called ladder height process) killed at an independent exponential time in the transient case.

Lemma 1.1.2. — *For every* $\lambda > 0$,

$$E[\exp -\lambda X_{L^{-1}(t)}] = \exp(-t\widetilde{\psi}(\lambda)),$$

where

$$\widetilde{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda} = \alpha + \beta\lambda + \int_0^\infty (1 - e^{-\lambda r})\pi([r, \infty))\, dr.$$

Proof. — By a classical result of fluctuation theory (see e.g. [**6**] Corollary p.724), we have

$$E[\exp -\lambda X_{L^{-1}(t)}] = \exp(-ct\widetilde{\psi}(\lambda)),$$

where c is a positive constant. We have to verify that $c = 1$ under our normalization.

Suppose first that $\pi \neq 0$. Then notice that the Lévy measure $c\pi([r,\infty))dr$ of $X_{L^{-1}(t)}$ is the "law" of X_σ under $N^*(\cdot \cap \{X_\sigma > 0\})$. However, for any nonnegative measurable function f on $[0,\infty)^2$, we get by a predictable projection

$$\begin{aligned} N^*\Big(f(\Delta X_\sigma, X_\sigma)\, 1_{\{X_\sigma>0\}}\Big) &= N^*\Big(\sum_{0<s\leqslant\sigma} f(\Delta X_s, X_s)\, 1_{\{X_s>0\}}\Big) \\ &= N^*\Big(\int_0^\sigma ds \int \pi(dx)\, f(x, X_{s-}+x)\, 1_{\{X_{s-}+x>0\}}\Big) \\ &= \int_{-\infty}^0 dy \int \pi(dx)\, f(x, y+x)\, 1_{\{y+x>0\}}, \end{aligned}$$

using Lemma 1.1.1 in the last equality. It follows that

$$N^*\Big(f(\Delta X_\sigma, X_\sigma)\, 1_{\{X_\sigma>0\}}\Big) = \int \pi(dx) \int_0^x dz\, f(x,z), \tag{10}$$

and we get $c = 1$ by comparing with the Lévy measure of $X_{L^{-1}(t)}$.

In the case $\pi = 0$, X is a scaled linear Brownian motion with drift, and the same conclusion follows from direct computations. □

Note that we have in particular $P[L^{-1}(t) < \infty] = e^{-\alpha t}$, which shows that L_∞ has an exponential distribution with parameter α in the transient case.

When $\beta > 0$, we can get a simple expression for L_t. From well-known results on subordinators, we have a.s. for every $u \geqslant 0$,

$$m(\{X_{L^{-1}(t)}; t \leqslant u, L^{-1}(t) < \infty\}) = \beta(u \wedge L_\infty).$$

Since the sets $\{X_{L^{-1}(t)}; t \leqslant u, L^{-1}(t) < \infty\}$ and $\{S_r; r \leqslant L^{-1}(u)\}$ coincide except possibly for a countable set, we have also

$$m(\{S_r; r \leqslant t\}) = \beta\, L_t \tag{11}$$

for every $t \geqslant 0$ a.s.

The next lemma provides a useful approximation of the local time L_t.

Lemma 1.1.3. — *For every $x > 0$,*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^{L^{-1}(x)} 1_{\{S_s - X_s < \varepsilon\}} ds = x \wedge L_\infty$$

in the L^2-norm. Consequently, for every $t \geqslant 0$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^{t} 1_{\{S_s - X_s < \varepsilon\}} ds = L_t$$

in probability.

Proof. — It is enough to prove the first assertion. Let $\mathcal{N}$ be as previously a Poisson point measure on $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ with intensity $dl\, N^*(d\omega)$, and

$$\eta = \inf\{l : \mathcal{N}([0,l] \times \{\sigma = +\infty\}) \geqslant 1\}.$$

For every $x > 0$ set

$$J_\varepsilon(x) = \frac{1}{\varepsilon} \int \mathcal{N}(dl\, d\omega)\, 1_{\{l \leqslant x\}} \int_0^{\sigma(\omega)} 1_{(-\varepsilon,0]}(\omega(s))\, ds.$$

Then,

$$E[J_\varepsilon(x)] = \frac{x}{\varepsilon} N^*\Big(\int_0^\sigma 1_{(-\varepsilon,0]}(X_s) ds \Big) = x$$

by (9). Furthermore,

$$E[J_\varepsilon(x)^2] = (E[J_\varepsilon(x)])^2 + x\varepsilon^{-2} N^*\Big(\Big(\int_0^\sigma 1_{(-\varepsilon,0]}(X_s) ds \Big)^2 \Big),$$

and

$$\begin{aligned} N^*\Big(\Big(\int_0^\sigma 1_{(-\varepsilon,0]}(X_s) ds \Big)^2 \Big) &= 2\, N^*\Big(\int_{0 \leqslant s \leqslant t \leqslant \sigma} 1_{(-\varepsilon,0]}(X_s) 1_{(-\varepsilon,0]}(X_t) ds dt \Big) \\ &= 2\, N^*\Big(\int_0^\sigma ds\, 1_{(-\varepsilon,0]}(X_s) E_{X_s}\Big[\int_0^\tau dt\, 1_{(-\varepsilon,0]}(X_t) \Big] \Big) \\ &\leqslant 2\varepsilon \sup_{0 \geqslant y > -\varepsilon} E_y\Big[\int_0^\tau dt\, 1_{(-\varepsilon,0]}(X_t) \Big], \end{aligned}$$

using the same notation $\tau = \inf\{t \geqslant 0 : X_t \geqslant 0\}$ as previously. We then claim that

$$\sup_{0 \geqslant y > -\varepsilon} E_y\Big[\int_0^\tau dt\, 1_{(-\varepsilon,0]}(X_t) \Big] = o(\varepsilon) \tag{12}$$

as $\varepsilon \to 0$. Indeed, by applying the strong Markov property at T_y, we have for $y > 0$,

$$N^*(T_y < \infty)\, E_{-y}\Big[\int_0^\tau dt\, 1_{(-\varepsilon,0]}(X_t) \Big] \leqslant N^*\Big(\int_0^\sigma dt\, 1_{(-\varepsilon,0]}(X_t) \Big) = \varepsilon,$$

and the claim follows since $N^*[T_y < \infty] \uparrow +\infty$ as $y \downarrow 0$. From (12) and the preceding calculations, we get

$$\lim_{\varepsilon \to 0} E[(J_\varepsilon(x) - x)^2] = 0.$$

By Doob's inequality (or a monotonicity argument), we have also

$$\lim_{\varepsilon \to 0} E[\sup_{0 \leqslant z \leqslant x} (J_\varepsilon(z) - z)^2] = 0.$$

The lemma now follows, since the pair

$$\Big(\frac{1}{\varepsilon} \int_0^{L^{-1}(x)} 1_{\{S_s - X_s < \varepsilon\}} ds, L_\infty \Big)$$

has the same distribution as $(J_\varepsilon(x \wedge \eta), \eta)$. □

As a consequence of Lemma 1.1.3, we may choose a sequence $(\varepsilon_k, k = 1, 2, \dots)$ of positive real numbers decreasing to 0, such that

$$L_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{S_s - X_s < \varepsilon_k\}} ds \ , \qquad P \text{ a.s.} \tag{13}$$

Using monotonicity arguments and a diagonal subsequence, we may and will assume that the previous convergence holds simultaneously for every $t \geqslant 0$ outside a single set of zero probability. In particular, if we set for $\omega \in \mathbb{D}([0,t], \mathbb{R})$,

$$\Phi_t(\omega) = \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{\sup_{[0,s]} \omega(r) - \omega(s) < \varepsilon_k\}} ds,$$

we have $L_t = \Phi_t(X_s, 0 \leqslant s \leqslant t)$, for every $t \geqslant 0$, P a.s.

Recall the notation $\widehat{X}^{(t)}$ for the process X time-reversed at time t.

Proposition 1.1.4. — *For any nonnegative measurable functional F on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$,*

$$N\Big(\int_0^\sigma dt\, F(\widehat{X}^{(t)}_{s\wedge t}, s \geqslant 0) \Big) = E\Big[\int_0^{L_\infty} dx\, F(X_{s\wedge L^{-1}(x)}, s \geqslant 0) \Big].$$

Proof. — We may assume that F is bounded and continuous. Fix $t > 0$ and if $\omega \in \mathbb{D}([0,t], \mathbb{R})$, set $T_{\max}(\omega) = \inf\{s \in [0,t] : \sup_{[0,s]} \omega(r) = \sup_{[0,t]} \omega(r)\}$ and let $\theta\omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ be defined by $\theta\omega(t) = \omega(t \wedge T_{\max}(\omega))$. Let $z > 0$. Excursion theory for $X - I$ shows that, for every $\varepsilon > 0$,

$$N\Big(\int_0^\sigma dt\, 1_{\{\Phi_t(\widehat{X}^{(t)}) \leqslant z\}} F(\widehat{X}^{(t)}_{s\wedge t}, s \geqslant 0) \Big) = \frac{1}{\varepsilon} E\Big[\int_0^{T_\varepsilon} dt\, 1_{\{\Phi_t(\widehat{X}^{(t)}) \leqslant z\}} F \circ \theta(\widehat{X}^{(t)}) \Big].$$

In deriving this equality, we also apply to the time-reversed process $\widehat{X}^{(t)}$ the fact that the local time L_s does not increase after the (first) time of the maximum over $[0,t]$. Then,

$$\begin{aligned}
&\frac{1}{\varepsilon} E\Big[\int_0^{T_\varepsilon} dt\, 1_{\{\Phi_t(\widehat{X}^{(t)}) \leqslant z\}} F \circ \theta(\widehat{X}^{(t)}) \Big] \\
&\qquad = \frac{1}{\varepsilon} E\Big[\int_0^{\infty} dt\, 1_{\{I_t > -\varepsilon\}}\, 1_{\{\Phi_t(\widehat{X}^{(t)}) \leqslant z\}} F \circ \theta(\widehat{X}^{(t)}) \Big] \\
&\qquad = \frac{1}{\varepsilon} \int_0^{\infty} dt\, E[1_{\{S_t - X_t < \varepsilon\}}\, 1_{\{L_t \leqslant z\}} F \circ \theta(X_s, s \leqslant t)] \\
&\qquad = E\Big[\frac{1}{\varepsilon} \int_0^{L^{-1}(z)} dt\, 1_{\{S_t - X_t < \varepsilon\}} F \circ \theta(X_s, s \leqslant t) \Big].
\end{aligned}$$

We then take $\varepsilon = \varepsilon_k$ and pass to the limit $k \to \infty$, using the L^2 bounds provided by Lemma 1.1.3. Note that the measures

$$\frac{1}{\varepsilon_k} 1_{[0, L^{-1}(z)]}(t)\, 1_{\{S_t - X_t < \varepsilon_k\}} dt$$

converge weakly to the finite measure $1_{[0,L^{-1}(z)]}(t)dL_t$. Furthermore, $\theta(X_s, s \leqslant t) = (X_{s\wedge t}, s \geqslant 0)$, dL_t a.e., a.s., and it is easy to verify that the mapping $t \to F\circ\theta(X_s, s \leqslant t)$ is continuous on a set of full dL_t-measure. We conclude that

$$N\Big(\int_0^\sigma dt\, 1_{\{\Phi_t(\widehat{X}^{(t)})\leqslant z\}}\, F(\widehat{X}^{(t)}_{s\wedge t}, s \geqslant 0)\Big) = E\Big[\int_0^{L^{-1}(z)} F(X_{s\wedge t}, s \geqslant 0)\, dL_t\Big]$$
$$= E\Big[\int_0^{z\wedge L_\infty} F(X_{s\wedge L^{-1}(x)}, s \geqslant 0)\, dx\Big],$$

and the desired result follows by letting $z \to \infty$. □

1.2. The height process and the exploration process

We write $\widehat{S}^{(t)}_s = \sup_{[0,s]} \widehat{X}^{(t)}_r$ $(0 \leqslant s \leqslant t)$ for the supremum process of $\widehat{X}^{(t)}$.

Definition 1.2.1. — The height process is the real-valued process $(H_t, t \geqslant 0)$ defined as follows. First $H_0 = 0$ and for every $t > 0$, H_t is the local time at level 0 at time t of the process $\widehat{X}^{(t)} - \widehat{S}^{(t)}$.

The normalization of local time is of course that prescribed by Lemma 1.1.1.

Note that the existence of a modification of the process $(H_t, t \geqslant 0)$ with good continuity properties is not clear from the previous definition. When $\beta > 0$ however, we can use (11) to see that

$$H_t = \frac{1}{\beta}\, m(\{I^s_t; s \leqslant t\}), \tag{14}$$

where for $0 \leqslant s \leqslant t$,

$$I^s_t = \inf_{s\leqslant r\leqslant t} X_r.$$

Clearly the right-hand side of (14) gives a continuous modification of $(H_t, t \geqslant 0)$. When $\beta = 0$, this argument does not apply and we will see later that there may exist no continuous (or even càdlàg) modification of $(H_t, t \geqslant 0)$.

At the present stage, we will use the measurable modification of $(H_t, t \geqslant 0)$ with values in $[0, \infty]$ obtained by taking

$$H^o_t = \Phi_t(\widehat{X}^{(t)}_s, 0 \leqslant s \leqslant t) = \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{X_s < I^s_t + \varepsilon_k\}}\, ds, \tag{15}$$

where the ε_k's are as in (13). The liminf in (15) is a limit (and is finite) a.s. for every fixed $t \geqslant 0$. The following lemma shows that more is true.

Lemma 1.2.1. — *Almost surely for every $t \geqslant 0$, we have*

$$H^o_s = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^s 1_{\{X_r < I^r_s + \varepsilon_k\}}\, dr < \infty,$$

for every $s < t$ such that $X_{s-} \leqslant I^s_t$, and for $s = t$ if $\Delta X_t > 0$.

Proof. — Let s and t be as in the statement. Then there must exist a rational $u \in (s, \infty)$ such that $X_{s-} \leqslant I^s_u$. We can then apply to the time-reversed process $\widehat{X}^{(u)}$ the approximation result (13) at times u and $u - s$ respectively. The desired result follows. □

We denote by $M_f(\mathbb{R}_+)$ the space of all finite measures on $\mathbb{R}_+$, which is equipped with the topology of weak convergence.

Definition 1.2.2. — The exploration process is the process $(\rho_t, t \geqslant 0)$ with values in $M_f(\mathbb{R}_+)$ defined as follows. For every nonnegative measurable function f,

$$\langle \rho_t, f \rangle = \int_{[0,t]} d_s I^s_t \, f(H^o_s) \tag{16}$$

where the notation $d_s I^s_t$ refers to integration with respect to the nondecreasing function $s \to I^s_t$.

Since we did not exclude the value $+\infty$ for H^o_t (as defined by (15)), it may not be obvious that the measure ρ_t is supported on $[0, \infty)$. However, this readily follows from the previous lemma since the measure $d_s I^s_t$ is supported on the set $\{s < t : X_{s-} \leqslant I^s_t\}$ (to which we need to add the point t if $\Delta X_t > 0$).

Notice that if u and v belong to the set $\{s \leqslant t : X_{s-} \leqslant I^s_t\}$, and if $u \leqslant v$, then for every $r \in [0, u)$ the condition $X_r < I^r_u + \varepsilon_k$ implies $X_r < I^r_v + \varepsilon_k$, and by construction it follows that $H^o_u \leqslant H^o_v$. Using the previous remark on the support of the measure $d_s I^s_t$, we see that the measure ρ_t is supported on $[0, H^o_t]$, for every $t \geqslant 0$, a.s.

The total mass of ρ_t is

$$\langle \rho_t, 1 \rangle = I^t_t - I^0_t = X_t - I_t.$$

In particular $\rho_t = 0$ iff $X_t = I_t$.

It will be useful to rewrite the definition of ρ_t in terms of the time-reversed process $\widehat{X}^{(t)}$. Denote by $\widehat{L}^{(t)} = (\widehat{L}^{(t)}_s, 0 \leqslant s \leqslant t)$ the local time at 0 of $\widehat{X}^{(t)} - \widehat{S}^{(t)}$ (in particular $H^o_t = \widehat{L}^{(t)}_t$). Note that for $t \geqslant 0$ fixed, we have $H^o_s = \widehat{L}^{(t)}_t - \widehat{L}^{(t)}_s$ for every $s \in [0, t]$ such that $X_{s-} \leqslant I^s_t$, a.s. (compare (13) and (15)). Hence, for every $t \geqslant 0$,

$$\langle \rho_t, f \rangle = \int_{[0,t]} d\widehat{S}^{(t)}_s \, f(\widehat{L}^{(t)}_t - \widehat{L}^{(t)}_s), \qquad \text{a.s..} \tag{17}$$

If μ is a nonzero measure in $M_f(\mathbb{R}_+)$, we write $\operatorname{supp} \mu$ for the topological support of μ and set $H(\mu) = \sup(\operatorname{supp} \mu)$. By convention $H(0) = 0$. By a preceding remark, $H(\rho_t) \leqslant H^o_t$ for every $t \geqslant 0$, a.s.

Lemma 1.2.2. — *For every $t \geqslant 0$, $P[H(\rho_t) = H^o_t] = 1$. Furthermore, almost surely for every $t > 0$, we have*

(i) $\rho_t(\{0\}) = 0$;

(ii) $\operatorname{supp} \rho_t = [0, H(\rho_t)]$ *if* $\rho_t \neq 0$;

(iii) $H(\rho_s) = H^o_s$ *for every* $s \in [0, t)$ *such that* $X_{s-} \leqslant I^s_t$ *and for* $s = t$ *if* $\Delta X_t > 0$.

Proof. — It is well known, and easy to prove from the strong Markov property, that the two random measures dS_s and dL_s have the same support a.s. Then (17) implies that $\operatorname{supp} \rho_t = [0, H_t^o]$ a.s. for every fixed $t > 0$. In particular, $P[H_t^o = H(\rho_t)] = 1$. Similarly (17) implies that $P[\rho_t(\{0\}) > 0] = 0$ for every fixed $t > 0$. However, if we have $\rho_t(\{0\}) > 0$ for some $t \geqslant 0$, our definitions and the right-continuity of paths show that the same property must hold for some rational $r > t$. Property (i) follows.

Let us now prove (ii), which is a little more delicate. We already noticed that (ii) holds for every fixed t, a.s., hence for every rational outside a set of zero probability. Let $t > 0$ with $X_t > I_t$, and set

$$\gamma_t = \sup\{s < t : I_t^s < X_t\}.$$

We consider two different cases.

(a) Suppose first that $X_{\gamma_t -} < X_t$, which holds in particular if $\Delta X_t > 0$. Then note that

$$\langle \rho_t, f \rangle = \int_{[0,\gamma_t)} d_s I_t^s \, f(H_s^o) + (X_t - X_{\gamma_t -}) f(H_t^o).$$

Thus we can find a rational $r > t$ sufficienty close to t, so that ρ_r and ρ_t have the same restriction to $[0, H_t^o)$. The fact that property (ii) holds for r implies that it holds for t, and we see also that $H_t^o = H(\rho_t)$ in that case.

(b) Suppose that $X_{\gamma_t -} = X_t$. Then we set for every $\varepsilon > 0$,

$$\langle \rho_t^\varepsilon, f \rangle = \int_{[0,t]} d_s I_t^s \, 1_{\{I_t^s < X_t - \varepsilon\}} \, f(H_s^o).$$

From the remarks following the definition of ρ_t, it is clear that there exists some $a \geqslant 0$ such that ρ_t^ε is bounded below by the restriction of ρ_t to $[0, a)$, and bounded above by the restriction of ρ_t to $[0, a]$. Also note that $\rho_t = \lim \uparrow \rho_t^\varepsilon$ as $\varepsilon \downarrow 0$. Now, for every $\varepsilon > 0$, we can pick a rational $r > t$ so that $I_r^t > X_t - \varepsilon$, and we have by construction

$$\rho_t^\varepsilon = \rho_r^{\varepsilon + X_r - X_t}.$$

From the rational case, the support of $\rho_r^{\varepsilon + X_r - X_t}$ must be an interval $[0, a]$, and thus the same is true for ρ_t^ε. By letting $\varepsilon \downarrow 0$, we get (ii) for t.

We already obtained (iii) for $s = t$ when $\Delta X_s > 0$ (see (a) above). If $s \in (0, t)$ is such that $X_{s-} \leqslant I_t^s$, we will have also $X_{s-} \leqslant I_r^s$ for any rational $r \in (s, t)$. Then $H_s^o \leqslant H_r^o = H(\rho_r)$, and on the other hand, it is clear that the measures ρ_s and ρ_r have the same restriction to $[0, H_s^o)$. Thus (ii) implies that the support of ρ_s contains $[0, H_s^o)$, and so $H(\rho_s) \geqslant H_s^o$. This gives the desired result since the reverse inequality always holds. □

Proposition 1.2.3. — *The process $(\rho_t, t \geqslant 0)$ is a càdlàg strong Markov process in $M_f(\mathbb{R}_+)$. Furthermore, $(\rho_t, t \geqslant 0)$ is càdlàg with respect to the variation distance on finite measures.*

Proof. — We first explain how to define the process ρ started at an arbitrary initial value $\mu \in M_f(\mathbb{R}_+)$. To this end, we introduce some notation. Let $\mu \in M_f(\mathbb{R}_+)$ and $a \geqslant 0$. If $a \leqslant \langle \mu, 1\rangle$, we let $k_a\mu$ be the unique finite measure on $\mathbb{R}_+$ such that, for every $r \geqslant 0$,

$$k_a\mu([0,r]) = \mu([0,r]) \wedge (\langle \mu, 1\rangle - a).$$

In particular, $\langle k_a\mu, 1\rangle = \langle \mu, 1\rangle - a$. If $a \geqslant \langle \mu, 1\rangle$, we take $k_a\mu = 0$.

If $\mu \in M_f(\mathbb{R}_+)$ has compact support and $\nu \in M_f(\mathbb{R}_+)$, we define the concatenation $[\mu, \nu] \in M_f(\mathbb{R}_+)$ by the formula

$$\int [\mu,\nu](dr)\, f(r) = \int \mu(dr)\, f(r) + \int \nu(dr)\, f(H(\mu) + r).$$

With this notation, the law of the process ρ started at $\mu \in M_f(\mathbb{R}_+)$ is the distribution of the process ρ^μ defined by

$$\rho^\mu_t = [k_{-I_t}\mu, \rho_t]\ , \qquad t > 0. \tag{18}$$

Note that this definition makes sense because $k_{-I_t}\mu$ has compact support, for every $t > 0$ a.s.

We then verify that the process ρ has the stated properties. For simplicity, we consider only the case when the initial value is 0, that is when ρ is defined as in Definition 1.2.2. The right-continuity of paths is straightforward from the definition since the measures $1_{[0,t']}(s)d_sI^s_{t'}$ converge to $1_{[0,t]}(s)d_sI^s_t$ in the variation norm as $t' \downarrow t$. Similarly, we get the existence of left-limits from the fact that the measures $1_{[0,t']}(s)d_sI^s_{t'}$ converge to $1_{[0,t)}(s)d_sI^s_t$ in the variation norm as $t' \uparrow t$, $t' < t$. We see in particular that ρ and X have the same discontinuity times and that

$$\rho_t = \rho_{t-} + \Delta X_t\, \delta_{H^o_t}. \tag{19}$$

We now turn to the strong Markov property. Let T be a stopping time of the canonical filtration. We will express ρ_{T+t} in terms of ρ_T and the shifted process $X^{(T)}_t = X_{T+t} - X_T$. We claim that a.s. for every $t > 0$

$$\rho_{T+t} = [k_{-I^{(T)}_t}\rho_T, \rho^{(T)}_t] \tag{20}$$

where $\rho^{(T)}_t$ and $I^{(T)}_t$ obviously denote the analogues of ρ_t and I_t when X is replaced by $X^{(T)}$. When we have proved (20), the strong Markov property of the process ρ follows by standard arguments, using also our definition of the process started at a general initial value.

For the proof of (20), write

$$\langle \rho_{T+t}, f\rangle = \int_{[0,T]} d_sI^s_{T+t} f(H^o_s) + \int_{(T,T+t]} d_sI^s_{T+t} f(H^o_s).$$

We consider separately each term in the right-hand side. Introduce

$$u = \sup\{r \in (0,T] : X_{r-} < I^T_{T+t}\},$$

with the usual convention $\sup \varnothing = 0$. We have $I^s_{T+t} = I^s_T$ for $s \in [0,u)$ and $I^s_{T+t} = I^T_{T+t}$ for $s \in [u,T]$. Since $X_T - I^T_{T+t} = -I^{(T)}_t$, it then follows from our definitions that

$$(21)\quad \int_{[0,T]} d_s I^s_{T+t}\, f(H^o_s) = \int_{[0,u)} d_s I^s_T\, f(H^o_s) + 1_{\{u>0\}}(I^T_{T+t} - X_{u-}) f(H^o_u) = \langle k_{-I^{(T)}_t}\rho_T, f\rangle.$$

Also notice that the measures ρ_u and $k_{-I^{(T)}_t}\rho_T$ coincide, except possibly at the point H^o_u. In any case, $H(\rho_u) = H(k_{-I^{(T)}_t}\rho_T)$, and we have also $H^o_u = H(\rho_u)$ by Lemma 1.2.2 (iii).

Now observe that for $d_s I^s_{T+t}$ almost every $s \in (T, T+t]$, we have $H^o_s = H^o_u + H^{o,(T)}_{s-T}$, with an obvious notation. To see this, pick a rational $r > T+t$ such that $I^{T+t}_r > X_{s-}$ and argue on the time-reversed process $\widehat{X}^{(r)}$ as in the proof of Lemma 1.2.1. Hence,

$$(22)\quad \int_{(T,T+t]} d_s I^s_{T+t}\, f(H^o_s) = \int_{(T,T+t]} d_s I^s_{T+t}\, f(H^o_u + H^{o,(T)}_{s-T}) = \int \rho^{(T)}_t(dx)\, f(H^o_u + x).$$

Formula (20) follows by combining (21) and (22). □

We now come back to the problem of finding a modification of the height process with good continuity properties. By the first assertion of Lemma 1.2.2, $(H(\rho_t), t \geqslant 0)$ is a modification of $(H^o_t, t \geqslant 0)$. *From now on, we will systematically use this modification* and write $H_t = H(\rho_t)$. From Lemma 1.2.2 (iii), we see that formula (16) defining $(\rho_t, t \geqslant 0)$ remains true if H^o_s is replaced by H_s. The same applies to formula (19) giving the jumps of ρ. Furthermore, the continuity properties of the process ρ_t (and especially the form of its jumps) imply that the mapping $t \longrightarrow H(\rho_t) = H_t$ is lower semicontinuous a.s.

Let us make an important remark at this point. Write

$$g_t = \sup\{s \leqslant t : X_s = I_s\}$$

for the beginning of the excursion of $X - I$ that straddles t. Then a simple time-reversal argument shows that a.s. for every t such that $X_t - I_t > 0$, we have

$$\lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^{g_t} 1_{\{X_s < I^s_t + \varepsilon_k\}}\, ds = 0$$

and thus we can replace (15) by

$$H^o_t = \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_{g_t}^t 1_{\{X_s < I^s_t + \varepsilon_k\}}\, ds.$$

Recalling (16), we see that, a.s. for every $t \geqslant 0$ such that $X_t - I_t > 0$, we can write ρ_t and H_t as measurable functions of the excursion of $X - I$ that straddles t

(and of course $\rho_t = 0$ and $H_t = 0$ if $X_t = I_t$). We can thus define both the height process and the exploration process under the excursion measure N. Furthermore, if (α_j, β_j), $j \in J$, denote the excursion intervals of $X - I$, and if ω_j, $j \in J$, denote the corresponding excursions, we have $H_t = H_{t-\alpha_j}(\omega_j)$ and $\rho_t = \rho_{t-\alpha_j}(\omega_j)$ for every $t \in (\alpha_j, \beta_j)$ and $j \in J$, a.s.

Since 0 is a regular point for $X - I$, we also see that the measure 0 is a regular point for the exploration process ρ. It is immediate from the previous remark that the excursion measure of ρ away from 0 is the "law" of ρ under N. Similarly, the process $-I$, which is the local time at 0 for $X - I$, is also the local time at 0 for ρ.

We now state and prove a useful technical lemma about the process H.

***Lemma 1.2.4* (Intermediate values property).** — *Almost surely for every $t < t'$, the process H takes all values between H_t and $H_{t'}$ on the time interval $[t, t']$.*

Proof. — First consider the case when $H_t > H_{t'}$. By using the lower semi-continuity of H, we may assume that t is rational. From (20), we have $\rho_{t+s} = [k_{-I_s^{(t)}}\rho_t, \rho_s^{(t)}]$ for every $s > 0$, a.s. Hence, if

$$\gamma_r = \inf\{s > 0 : I_s^{(t)} = -r\},$$

we have $\rho_{t+\gamma_r} = k_r\rho_t$, and so $H_{t+\gamma_r} = H(k_r\rho_t)$ for every $r \geqslant 0$, a.s. However, Lemma 1.2.2 (ii) implies that the mapping $r \to H(k_r\rho_t)$ is continuous. Now note that for $r = 0$, $H_{t+\gamma_r} = H_t$, whereas for $r = X_t - I_{t'}^t = -I_{t'-t}^{(t)}$ we have $t + \gamma_r \leqslant t'$ and our definitions easily imply $\rho_{t+\gamma_r} \leqslant \rho_{t'}$ and $H_{t+\gamma_r} \leqslant H_{t'}$.

Consider then the case when $H_t < H_{t'}$. By lower semi-continuity again, we may assume that t' is rational. In terms of the process time-reversed at time t', we have $H_{t'} = \widehat{L}_{t'}^{(t')}$. Set

$$\sigma_r = \inf\{s \geqslant 0 : \widehat{S}_s^{(t')} \geqslant r\},$$

which is well defined for $0 \leqslant r \leqslant X_{t'} - I_{t'}$. Since the subordinator $S_{L^{-1}(t)}$ is strictly increasing, we see that the mapping $r \to \widehat{L}_{\sigma_r}^{(t')}$ is continuous for $r \in [0, X_{t'} - I_{t'}]$, a.s. Now note that

$$H_{t'-\sigma_r} = \widehat{L}_{t'}^{(t')} - \widehat{L}_{\sigma_r}^{(t')}$$

for every $r \in [0, X_{t'} - I_{t'}]$, a.s. For $r = X_{t'} - I_{t'}^t = \widehat{S}_{t'-t}^{(t')}$, we have $t' - \sigma_r \geqslant t$ and $H_{t'-\sigma_r} \leqslant H_t$ by construction. The desired result follows. □

The next proposition is a corollary of Proposition 1.1.4. We denote by U a subordinator defined on an auxiliary probability space (Ω^0, P^0), with Laplace exponent

$$E^0[\exp -\lambda U_t] = \exp\Big(-t\Big(\beta\lambda + \int_0^\infty (1 - e^{-\lambda r})\,\pi([r,\infty))\,dr\Big)\Big) = \exp(-t(\widetilde{\psi}(\lambda) - \alpha)).$$

For every $a \geqslant 0$, we let J_a be the random element of $M_f(\mathbb{R}_+)$ defined by $J_a(dr) = 1_{[0,a]}(r)\,dU_r$.

Proposition 1.2.5. — *For every nonnegative measurable function Φ on $M_f(\mathbb{R}_+)$,*

$$N\Big(\int_0^\sigma dt\,\Phi(\rho_t)\Big) = \int_0^\infty da\, e^{-\alpha a}\, E^0[\Phi(J_a)].$$

Let $b \geqslant 0$. Then $\rho_s(\{b\}) = 0$ for every $s \geqslant 0$, N a.e. or P a.s.

Proof. — We have $\rho_t = \Sigma(\widehat{X}^{(t)}_{s\wedge t}, s \geqslant 0)$, with a functional Σ that is made explicit in (17). We then apply Proposition 1.1.4 to obtain

$$N\Big(\int_0^\sigma dt\,\Phi(\rho_t)\Big) = E\Big[\int_0^{L_\infty} da\,\Phi\circ\Sigma(X_{s\wedge L^{-1}(a)}, s \geqslant 0)\Big].$$

However, for $a < L_\infty$,

$$\langle \Sigma(X_{s\wedge L^{-1}(a)}, s \geqslant 0), f\rangle = \int_0^{L^{-1}(a)} dS_s\, f(a - L_s).$$

The first assertion is now a consequence of Lemma 1.1.2, which shows that $P[a < L_\infty] = e^{-\alpha a}$ and that, conditionally on $\{L^{-1}(a) < \infty\}$, $S_{L^{-1}(r)} = X_{L^{-1}(r)}$, $0 \leqslant r \leqslant a$ has the same distribution as U (we also use the fact that $(U_a - U_{a-r}$, $0 \leqslant r \leqslant a)$ has the same distribution as $(U_r, 0 \leqslant r \leqslant a)$).

Consider now the second assertion. Note that the case $b = 0$ is a consequence of Lemma 1.2.2 (i). So we may assume that $b > 0$ and it is enough to prove the result under the excursion measure N. However, since b is a.s. not a jump time of U, the right side of the formula of the proposition vanishes for $\Phi(\mu) = \mu(\{b\})$. The desired result follows, using also the fact that ρ is càdlàg in the variation norm. □

We denote by $\mathbf{M}$ the measure on $M_f(\mathbb{R}_+)$ defined by:

$$\langle \mathbf{M}, \Phi\rangle = \int_0^\infty da\, e^{-\alpha a}\, E^0[\Phi(J_a)].$$

Proposition 1.2.5 implies that the measure $\mathbf{M}$ is invariant for ρ.

The last proposition of this section describes the potential kernel of the exploration process killed when it hits 0. We fix $\mu \in M_f(\mathbb{R}_+)$ and let ρ^μ be as in (18) the exploration process started at μ. We use the notation introduced in the proof of Proposition 1.2.3.

Proposition 1.2.6. — *Let $\tau_0 = \inf\{s \geqslant 0,\ \rho^\mu_s = 0\}$. Then,*

$$E\Big[\int_0^{\tau_0} ds\,\Phi(\rho^\mu_s)\Big] = \int_0^{\langle\mu,1\rangle} dr \int \mathbf{M}(d\theta)\,\Phi([k_r\mu, \theta]).$$

Proof. — First note that $\tau_0 = T_{\langle\mu,1\rangle}$ by an immediate application of the definition of ρ^μ. Then, denote by (α_j, β_j), $j \in J$ the excursion intervals of $X - I$ away from 0 before time $T_{\langle\mu,1\rangle}$, and by ω_j, $j \in J$ the corresponding excursions. As we observed

before Proposition 1.2.5, we have $\rho_t = \rho_{t-\alpha_j}(\omega_j)$ for every $t \in (\alpha_j, \beta_j)$, $j \in J$, a.s. Since $\{s \geqslant 0 : X_s = I_s\}$ has zero Lebesgue measure a.s., it follows that

$$E\Big[\int_0^{\tau_0} ds\, \Phi(\rho_s^\mu)\Big] = E\Big[\sum_{j\in J} \int_0^{\beta_j - \alpha_j} dr\, \Phi([k_{-I_{\alpha_j}}\mu, \rho_r(\omega_j)])\Big].$$

By excursion theory, the point measure

$$\sum_{j\in J} \delta_{I_{\alpha_j}, \omega_j}(dude)$$

is a Poisson point measure with intensity $1_{[-\langle\mu,1\rangle,0]}(u)du\, N(d\omega)$. Hence,

$$E\Big[\int_0^{\tau_0} ds\, \Phi(\rho_s^\mu)\Big] = \int_0^{\langle\mu,1\rangle} du\, N\Big(\int_0^\sigma dr\, \Phi([k_u\mu, \rho_r])\Big),$$

and the desired result follows from Proposition 1.2.5. □

1.3. Local times of the height process

1.3.1. The construction of local times. — Our goal is to construct a local time process for H at each level $a \geqslant 0$. Since H is in general not Markovian (and not a semimartingale) we cannot apply a general theory, but still we will use certain ideas which are familiar in the Brownian motion setting. In the case $a = 0$, we can already observe that $H_t = 0$ iff $\rho_t = 0$ or equivalently $X_t - I_t = 0$. Therefore the process $-I$ is the natural candidate for the local time of H at level 0.

Let us fix $a \geqslant 0$. Since $t \to \rho_t$ is càdlàg in the variation norm, it follows that the mapping $t \to \rho_t((a,\infty))$ is càdlàg. Furthermore, it follows from (19) that the discontinuity times of this mapping are exactly those times t such that $\Delta X_t > 0$ and $H_t > a$, and the corresponding jump is ΔX_t.

Set

$$\tau_t^a = \inf\{s \geqslant 0 : \int_0^s 1_{\{H_r > a\}}\, dr > t\} = \inf\{s \geqslant 0 : \int_0^s 1_{\{\rho_r((a,\infty))>0\}}\, dr > t\}.$$

From Proposition 1.2.5, we get that $\int_0^\infty 1_{\{H_r>a\}}\, dr = \infty$ a.s., so that the random times τ_t^a are a.s. finite.

When $a > 0$, we also set

$$\widetilde{\tau}_t^a = \inf\{s \geqslant 0 : \int_0^s 1_{\{H_r \leqslant a\}}\, dr > t\}$$

and we let $\mathcal{H}_a$ be the σ-field generated by the càdlàg process $(X_{\widetilde{\tau}_t^a}, \rho_{\widetilde{\tau}_t^a}; t \geqslant 0)$ and the class of P-negligible sets of $\mathcal{G}_\infty$. We also define $\mathcal{H}_0$ as the σ-field generated by the class of P-negligible sets of $\mathcal{G}_\infty$.

Proposition 1.3.1. — *For every $t \geqslant 0$, let ρ_t^a be the random measure on $\mathbb{R}_+$ defined by*

$$\langle \rho_t^a, f\rangle = \int_{(a,\infty)} \rho_{\tau_t^a}(dr)\, f(r-a).$$

The process $(\rho_t^a, t \geqslant 0)$ has the same distribution as $(\rho_t, t \geqslant 0)$ and is independent of $\mathcal{H}_a$.

Proof

First step. — We first verify that the process $(\langle \rho_t^a, 1\rangle, t \geqslant 0)$ has the same distribution as $(\langle \rho_t, 1\rangle, t \geqslant 0)$.

Let $\varepsilon > 0$. We introduce two sequences of stopping times S_ε^k, T_ε^k, $k \geqslant 1$, defined inductively as follows:

$$\begin{aligned} S_\varepsilon^1 &= \inf\{s \geqslant 0 : \rho_s((a,\infty)) \geqslant \varepsilon\}, \\ T_\varepsilon^k &= \inf\{s \geqslant S_\varepsilon^k : \rho_s((a,\infty)) = 0\}, \\ S_\varepsilon^{k+1} &= \inf\{s \geqslant T_\varepsilon^k : \rho_s((a,\infty)) \geqslant \varepsilon\}. \end{aligned}$$

It is easy to see that these stopping times are a.s. finite, and $S_\varepsilon^k \uparrow \infty$, $T_\varepsilon^k \uparrow \infty$ as $k \uparrow \infty$.

From (20) applied with $T = S_\varepsilon^k$, we obtain that, for every $k \geqslant 1$,

$$T_\varepsilon^k = \inf\{s > S_\varepsilon^k : X_s = X_{S_\varepsilon^k} - \rho_{S_\varepsilon^k}((a,\infty))\}. \tag{23}$$

Formula (20) also implies that, for every $s \in [0, T_\varepsilon^k - S_\varepsilon^k]$,

$$\begin{aligned} \rho_{S_\varepsilon^k + s}((a,\infty)) &= (\rho_{S_\varepsilon^k}((a,\infty)) + I_s^{(S_\varepsilon^k)}) + (X_s^{(S_\varepsilon^k)} - I_s^{(S_\varepsilon^k)}) \\ &= X_{S_\varepsilon^k + s} - (X_{S_\varepsilon^k} - \rho_{S_\varepsilon^k}((a,\infty))). \end{aligned} \tag{24}$$

We set

$$Y_s^{k,\varepsilon} = \rho_{(S_\varepsilon^k + s) \wedge T_\varepsilon^k}((a,\infty)).$$

As a straightforward consequence of (23) and (24), conditionally on $\mathcal{G}_{S_\varepsilon^k}$, the process $Y^{k,\varepsilon}$ is distributed as the underlying Lévy process started at $\rho_{S_\varepsilon^k}((a,\infty))$ and stopped at its first hitting time of 0.

We then claim that, for every $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} \sup_{\{k \geqslant 1, S_\varepsilon^k \leqslant t\}} \rho_{S_\varepsilon^k}((a,\infty)) = 0, \qquad \text{a.s.} \tag{25}$$

Indeed, by previous observations about the continuity properties of the mapping $s \longrightarrow \rho_s((a,\infty))$, we have

$$\sup_{\{k \geqslant 1, S_\varepsilon^k \leqslant t\}} \rho_{S_\varepsilon^k}((a,\infty)) \leqslant \varepsilon + \sup\{\Delta X_s; s \leqslant t, H_s > a, \rho_s((a,\infty)) \leqslant \varepsilon\}.$$

However, the sets $\{s \leqslant t; \Delta X_s > 0, H_s > a, \rho_s((a,\infty)) \leqslant \varepsilon\}$ decrease to $\varnothing$ as $\varepsilon \downarrow 0$, and so

$$\lim_{\varepsilon \to 0} \Big(\sup\{\Delta X_s; s \leqslant t, H_s > a, \rho_s((a,\infty)) \leqslant \varepsilon\} \Big) = 0,$$

a.s., which yields the desired claim.

Set

$$Z_s^{k,\varepsilon} = Y_s^{k,\varepsilon} - \inf_{0 \leqslant r \leqslant s} Y_r^{k,\varepsilon} \leqslant Y_s^{k,\varepsilon}.$$

Then, conditionally on $\mathcal{G}_{S_\varepsilon^k}$, $Z^{k,\varepsilon}$ is distributed as an independent copy of the reflected process $X - I$, stopped when its local time at 0 hits $\rho_{S_\varepsilon^k}((a,\infty))$.

Denote by $U^\varepsilon = (U^\varepsilon_s, s \geqslant 0)$ the process obtained by pasting together the paths $(Z^{k,\varepsilon}_s, 0 \leqslant s \leqslant T^k_\varepsilon - S^k_\varepsilon)$. By the previous remarks, U^ε is distributed as the reflected Lévy process $X - I$.

We then set

$$\tau^{a,\varepsilon}_s = \inf\{t \geqslant 0 : \int_0^t \sum_{k=1}^{\infty} 1_{[S^k_\varepsilon, T^k_\varepsilon]}(r)\, dr > s\}.$$

Observe that the time-changed process $(\rho_{\tau^{a,\varepsilon}_s}((a,\infty)), s \geqslant 0)$ is obtained by patching together the paths $(Y^{k,\varepsilon}_s, 0 \leqslant s \leqslant T^k_\varepsilon - S^k_\varepsilon)$. Moreover, we have for every $k \geqslant 1$,

$$\sup_{0 \leqslant s \leqslant T^k_\varepsilon - S^k_\varepsilon} (Y^{k,\varepsilon}_s - Z^{k,\varepsilon}_s) = \rho_{S^k_\varepsilon}((a,\infty)) = Y^{k,\varepsilon}_0.$$

From (25), we conclude that for every $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} \Big(\sup_{s \leqslant t} |U^\varepsilon_s - \rho_{\tau^{a,\varepsilon}_s}((a,\infty))| \Big) = 0. \tag{26}$$

Notice that $\tau^{a,\varepsilon}_s \downarrow \tau^a_s$ as $\varepsilon \downarrow 0$ and recall that for every $\varepsilon > 0$, U^ε is distributed as the reflected Lévy process $X - I$. We then get from (26) that the process $\langle \rho^a_s, 1\rangle = \rho_{\tau^a_s}((a,\infty))$ is distributed as the reflected proces $X - I$, which completes the first step.

Second step. — We will now verify that ρ^a can be obtained as a functional of the total mass process $\langle \rho^a, 1\rangle$ in the same way as ρ is obtained from $\langle \rho, 1\rangle$. It will be enough to argue on one excursion of $\langle \rho^a, 1\rangle$ away from 0. Thus, let (u, v) be the interval corresponding to one such excursion. We can associate with (u, v) a unique connected component (p, q) of the open set $\{s, H_s > a\}$, such that $\tau^a_{u+r} = p + r$ for every $r \in [0, v - u)$, and $q = \tau^a_{v-}$. By the intermediate values property, we must have $H_p = H_q = a$.

We also claim that $X_r > X_p$ for every $r \in (p, q)$. If this were not the case, we could find $r \in (p, q)$ such that $X_r = \inf\{X_s, p \leqslant s \leqslant r\}$, which forces $H_r \leqslant H_p = a$ and gives a contradiction.

The previous observations and the definition of the process ρ imply that, for every $r \in (p, q)$, the restriction of ρ_r to $[0, a]$ is exactly $\rho_p = \rho_q$. Define

$$\omega(r) = X_{(p+r)\wedge q} - X_p = \langle \rho_{(p+r)\wedge q}, 1\rangle - \langle \rho_p, 1\rangle = \langle \rho^a_{(u+r)\wedge v}, 1\rangle,$$

so that ω is the excursion of $\langle \rho^a, 1\rangle$ corresponding to (u, v). The construction of the process ρ implies that, for $0 < r < q - p = v - u$,

$$\rho_{p+r} = [\rho_p, \rho_r(\omega)],$$

and so, for the same values of r,

$$\rho^a_{u+r} = \rho_r(\omega).$$

This completes the second step of the proof.

Third step. — It remains to prove that ρ^a is independent of the σ-field $\mathcal{H}_a$. For $\varepsilon > 0$, denote by $\mathcal{H}_a^\varepsilon$ the σ-field generated by the processes

$$(X_{(T_\varepsilon^k + s) \wedge S_\varepsilon^{k+1}}, s \geqslant 0)$$

for $k = 0, 1, \ldots$ (by convention $T_\varepsilon^0 = 0$), and the negligible sets of $\mathcal{G}_\infty$. From our construction (in particular the fact that $X_s > X_{T_\varepsilon^k}$ for $s \in [S_\varepsilon^k, T_\varepsilon^k)$), it is easy to verify that the processes $(\rho_{(T_\varepsilon^k + s) \wedge S_\varepsilon^{k+1}}, s \geqslant 0)$ are measurable with respect to $\mathcal{H}_a^\varepsilon$, and since $H_t > a$ for $t \in (S_\varepsilon^k, T_\varepsilon^k)$, it follows that $\mathcal{H}_a \subset \mathcal{H}_a^\varepsilon$.

By the arguments of the first step, the processes $Z^{k,\varepsilon}$ are independent conditionally on $\mathcal{H}_a^\varepsilon$, and the conditional law of $Z^{k,\varepsilon}$ is the law of an independent copy of the reflected process $X - I$, stopped when its local time at 0 hits $\rho_{S_\varepsilon^k}((a, \infty))$. It follows that the process U^ε of the first step is independent of $\mathcal{H}_a^\varepsilon$, hence also of $\mathcal{H}_a$. By passing to the limit $\varepsilon \to 0$, we obtain that the total mass process $\langle \rho^a, 1 \rangle$ is independent of $\mathcal{H}_a$. As we know that ρ^a can be reconstructed as a measurable functional of its total mass process, this completes the proof. □

We let $l^a = (l^a(s), s \geqslant 0)$ be the local time at 0 of $\langle \rho^a, 1 \rangle$, or equivalently of ρ^a.

Definition 1.3.1. — The local time at level a and at time s of the height process H is defined by

$$L_s^a = l^a\Big(\int_0^s 1_{\{H_r > a\}}\, dr \Big).$$

This definition will be justified below: See in particular Proposition 1.3.3.

1.3.2. Some properties of local times. — The next lemma can be seen as dual to Lemma 1.1.3.

Lemma 1.3.2. — *For every $t \geqslant 0$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{\{H_s \leqslant \varepsilon\}}\, ds = -I_t,$$

in the L^1-norm.

Proof. — We use arguments similar to the proof of Lemma 1.1.3. Recall that $T_x = \inf\{t \geqslant 0 : X_t = -x\}$. We first establish that for every $x > 0$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{T_x} 1_{\{H_s \leqslant \varepsilon\}}\, ds = x, \tag{27}$$

in probability. Note that Proposition 1.2.5 gives for any nonnegative measurable function g

$$N\Big(\int_0^\sigma ds\, g(H_s) \Big) = \int_0^\infty da\, e^{-\alpha a}\, g(a).$$

Let ω^j, $j \in J$ denote the excursions of $X - I$ away from 0 and let (α_j, β_j) be the corresponding time intervals. We already noticed that $H_s = H_{s-\alpha_j}(\omega^j)$ for $s \in (\alpha_j, \beta_j)$. Hence, using also the previous displayed formula, we have

$$E\Big[\frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s \leqslant \varepsilon\}}\, ds\Big] = \frac{x}{\varepsilon} N\Big(\int_0^\sigma 1_{\{H_s \leqslant \varepsilon\}}\, ds\Big) = \frac{x}{\varepsilon}\Big(\frac{1 - e^{-\alpha\varepsilon}}{\alpha}\Big) \leqslant x, \tag{28}$$

and in particular,

$$\lim_{\varepsilon \to 0} E\Big[\frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s \leqslant \varepsilon\}}\, ds\Big] = x. \tag{29}$$

We then want to get a second moment estimate. To this end, it is necessary to introduce a suitable truncation. Fix $K > 0$. A slight modification of the proof of (29) gives

$$\lim_{\varepsilon \to 0} E\Big[\frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s - I_s \leqslant K\}}\, ds\Big] = x. \tag{30}$$

If $H^{(s)}$ denotes the height process for the shifted process $X_t^{(s)} = X_{s+t} - X_s$, the bound $H^{(s)}_{t-s} \leqslant H_t$ (for $0 \leqslant s \leqslant t$) is obvious from our construction. We can use this simple observation to bound

$$\begin{aligned}
&N\Big(\Big(\int_0^\sigma 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s \leqslant K\}}\, ds\Big)^2\Big) \\
&\quad \leqslant 2N\Big(\int_0^\sigma ds\, 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s \leqslant K\}} \int_s^\sigma dt\, 1_{\{H_t \leqslant \varepsilon\}}\Big) \\
&\quad \leqslant 2N\Big(\int_0^\sigma ds\, 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s \leqslant K\}} \int_s^\sigma dt\, 1_{\{H^{(s)}_{t-s} \leqslant \varepsilon\}}\Big) \\
&\quad = 2N\Big(\int_0^\sigma ds\, 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s \leqslant K\}}\, E_{X_s}\Big[\int_0^{T_0} dt\, 1_{\{H_t \leqslant \varepsilon\}}\Big]\Big) \\
&\quad \leqslant 2\varepsilon N\Big(\int_0^\sigma ds\, 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s \leqslant K\}}\, X_s\Big) \qquad \text{(by (28))} \\
&\quad = 2\varepsilon \int_0^\varepsilon dy\, E[X_{L^{-1}(y)}\, 1_{\{L^{-1}(y) < \infty, X_{L^{-1}(y)} \leqslant K\}}] \qquad \text{(Proposition 1.1.4)} \\
&\quad \leqslant 2\varepsilon^2 E[X_{L^{-1}(\varepsilon)} \wedge K] \\
&\quad = o(\varepsilon^2),
\end{aligned}$$

by dominated convergence. As in the proof of Lemma 1.1.3, we can conclude from (29) and the previous estimate that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s \leqslant \varepsilon\}}\, 1_{\{X_s - I_s \leqslant K\}}\, ds = x$$

in the L^2-norm. Since this holds for every $K > 0$, (27) follows.

From (27) and a monotonicity argument we deduce that the convergence of Lemma 1.3.2 holds in probability. To get L^1-convergence, we need a few other estimates. First

observe that

$$(31)\quad E\Big[\int_0^t 1_{\{H_s\leqslant\varepsilon\}}ds\Big] = \int_0^t dsP[H_s\leqslant\varepsilon] = \int_0^t dsP[L_s\leqslant\varepsilon] = E[L^{-1}(\varepsilon)\wedge t]\leqslant C\,\varepsilon,$$

with a constant C depending only on t (in the last bound we applied to L^{-1} an estimate valid for any subordinator). Then,

$$\begin{aligned}
E\Big[\Big(\int_0^t 1_{\{H_s\leqslant\varepsilon\}}ds\Big)^2\Big] &= 2\,E\Big[\iint_{\{0<r<s<t\}} drds 1_{\{H_r\leqslant\varepsilon\}}1_{\{H_s\leqslant\varepsilon\}}\Big]\\
&\leqslant 2\,E\Big[\iint_{\{0<r<s<t\}} drds 1_{\{H_r\leqslant\varepsilon\}}1_{\{H^{(r)}_{s-r}\leqslant\varepsilon\}}\Big]\\
&= 2\,E\Big[\int_0^t dr\,1_{\{H_r\leqslant\varepsilon\}}E\Big[\int_0^{t-r} ds\,1_{\{H_s\leqslant\varepsilon\}}\Big]\Big]\\
&\leqslant 2\Big(E\Big[\int_0^t dr\,1_{\{H_r\leqslant\varepsilon\}}\Big]\Big)^2.
\end{aligned}$$

As a consequence of the last estimate and (31), the variables $\varepsilon^{-1}\int_0^t 1_{\{H_s\leqslant\varepsilon\}}ds$, $\varepsilon>0$ are bounded in L^2. This completes the proof of Lemma 1.3.2. □

We can now give a useful approximation result for local times of the height process.

Proposition 1.3.3. — *For every $t\geqslant 0$,*

$$\lim_{\varepsilon\to 0}\sup_{a\geqslant 0} E\Big[\sup_{s\leqslant t}\Big|\varepsilon^{-1}\int_0^s 1_{\{a<H_r\leqslant a+\varepsilon\}}\,dr - L^a_s\Big|\Big] = 0.$$

Similarly, for every $t\geqslant 0$,

$$\lim_{\varepsilon\to 0}\sup_{a\geqslant \varepsilon} E\Big[\sup_{s\leqslant t}\Big|\varepsilon^{-1}\int_0^s 1_{\{a-\varepsilon<H_r\leqslant a\}}\,dr - L^a_s\Big|\Big] = 0.$$

There exists a jointly measurable modification of the collection $(L^a_s, a\geqslant 0, s\geqslant 0)$, which is continuous and nondecreasing in the variable s, and such that, a.s. for any nonnegative measurable function g on $\mathbb{R}_+$ and any $s\geqslant 0$,

$$(32)\qquad \int_0^s g(H_r)\,dr = \int_{\mathbb{R}_+} g(a)\,L^a_s\,da.$$

Proof. — First consider the case $a=0$. Then, $\rho^0=\rho$ and $L^0_t = l^0(t) = -I_t$. Lemma 1.3.2 and a simple monotonicity argument, using the continuity of L^0_s, give

$$(33)\qquad \lim_{\varepsilon\to 0} E\Big[\sup_{s\leqslant t}\Big|\varepsilon^{-1}\int_0^s 1_{\{0<H_r\leqslant\varepsilon\}}\,dr - L^0_s\Big|\Big] = 0.$$

For $a>0$, set $A^a_t = \int_0^t 1_{\{H_s>a\}}\,ds$. Note that

$$\{a<H_s\leqslant a+\varepsilon\} = \{\rho_s((a,\infty))>0\}\cap\{\rho_s((a+\varepsilon,\infty))=0\},$$

and so

$$\begin{aligned}\int_0^s 1_{\{a<H_r\leqslant a+\varepsilon\}}\,dr &= \int_0^t 1_{\{\rho_s((a,\infty))>0\}\cap\{\rho_s((a+\varepsilon,\infty))=0\}}\,ds\\ &= \int_0^{A_t^a} 1_{\{\rho_r^a((\varepsilon,\infty))=0\}}dr\\ &= \int_0^{A_t^a} 1_{\{0<H_r^a\leqslant\varepsilon\}}dr,\end{aligned}$$

where $H_t^a = H(\rho_t^a)$. The first convergence of the proposition then follows from (33), the trivial bound $A_t^a \leqslant t$ and the fact that ρ^a has the same distribution as ρ.

The second convergence is easily derived from the first one by elementary arguments. Let us only sketch the method. For any fixed $\delta > 0$, we can choose $\varepsilon_0 > 0$ sufficiently small so that for every $a \geqslant 0$, $\varepsilon \in (0, \varepsilon_0]$,

$$E\Big[\Big|\varepsilon^{-1}\int_0^t 1_{\{a<H_r\leqslant a+\varepsilon\}}\,dr - L_s^a\Big|\Big] \leqslant \delta. \tag{34}$$

Then, if $0 < \varepsilon < \varepsilon_0 \wedge a$,

$$E\Big[\Big|\varepsilon^{-1}\int_0^t 1_{\{a-\varepsilon<H_r\leqslant a\}}\,dr - \varepsilon_0^{-1}\int_0^t 1_{\{a-\varepsilon<H_r\leqslant a-\varepsilon+\varepsilon_0\}}\,dr\Big|\Big] < 2\delta.$$

However, if ε is very small in comparison with ε_0, one also gets the bound

$$E\Big[\Big|\varepsilon_0^{-1}\int_0^t 1_{\{a-\varepsilon<H_r\leqslant a-\varepsilon+\varepsilon_0\}}\,dr - (\varepsilon_0-\varepsilon)^{-1}\int_0^t 1_{\{a<H_r\leqslant a+\varepsilon_0-\varepsilon\}}\,dr\Big|\Big] \leqslant \delta.$$

We get the desired result by combining the last two bounds and (34).

The existence of a jointly measurable modification of the process $(L_s^a, a \geqslant 0, s \geqslant 0)$ that satisfies the density of occupation time formula (32) follows from the first assertion of the proposition by standard arguments. □

From now on, we will only deal with the jointly measurable modification of the local times $(L_s^a, a \geqslant 0, s \geqslant 0)$ given by Proposition 1.3.3. We observe that it is easy to extend the definition of these local times under the excursion measure N. First notice that, as an obvious consequence of the first assertion of Proposition 1.3.3, we have also for every $a \geqslant 0$, $t \geqslant 0$

$$\lim_{\varepsilon\to 0} E\Big[\sup_{0\leqslant r\leqslant s\leqslant t}\Big|\varepsilon^{-1}\int_r^s 1_{\{a<H_u\leqslant a+\varepsilon\}}\,du - (L_s^a - L_r^a)\Big|\Big] = 0. \tag{35}$$

Then, let V be a measurable subset of $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ such that $N[V] < \infty$. For instance we may take $V = \{\sup_{s\geqslant 0} X_s > \delta\}$ for $\delta > 0$. By considering the first excursion of $X - I$ that belongs to V, and then using (35), we immediately obtain the existence under N of a continuous increasing process, still denoted by $(L_s^a, s \geqslant 0)$, such that

$$\lim_{\varepsilon\to 0} N\Big(1_V \sup_{s\leqslant t}\Big|\varepsilon^{-1}\int_0^s 1_{\{a<H_r\leqslant a+\varepsilon\}}\,dr - L_s^a\Big|\Big) = 0. \tag{36}$$

The next corollary will now be a consequence of Proposition 1.1.4. We use the notation introduced before Proposition 1.2.5.

Corollary 1.3.4. — *For any nonnegative measurable function F on $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$, and every $a>0$,*

$$N\Big(\int_0^\sigma dL_s^a\, F(\widehat{X}^{(s)}_{r\wedge s}, r\geqslant 0)\Big) = E[1_{\{L^{-1}(a)<\infty\}}\, F(X_{r\wedge L^{-1}(a)}, r\geqslant 0)].$$

In particular, for any nonnegative measurable function F on $M_f(\mathbb{R}_+)$,

$$N\Big(\int_0^\sigma dL_s^a\, F(\rho_s)\Big) = e^{-\alpha a}\, E^0[F(J_a)].$$

Proof. — We may assume that F is bounded and continuous. Then let h be a nonnegative continuous function on $\mathbb{R}_+$, which vanishes outside $[\delta, A]$, for some $0<\delta<A<\infty$. For the first identity, it is enough to prove that

$$N\Big(\int_0^\sigma dL_s^a\, h(s)\, F(\widehat{X}^{(s)}_{r\wedge s}, r\geqslant 0)\Big) = E[1_{\{L^{-1}(a)<\infty\}}\, h(L^{-1}(a))\, F(X_{r\wedge L^{-1}(a)}, r\geqslant 0)].$$

Notice that the mapping $s\longrightarrow(\widehat{X}^{(s)}_t, t\geqslant 0)$ is continuous except possibly on a countable set that is not charged by the measure dL_s^a. From (36), applied with $V=\{\omega,\sigma(\omega)>\delta\}$, and then Proposition 1.1.4, we get

$$\begin{aligned}
&N\Big(\int_0^\sigma dL_s^a\, h(s)\, F(\widehat{X}^{(s)}_{r\wedge s}, r\geqslant 0)\Big)\\
&\qquad = \lim_{\varepsilon\to 0} N\Big(\frac{1}{\varepsilon}\int_0^\sigma ds\, 1_{\{a<H_s\leqslant a+\varepsilon\}}\, h(s)\, F(\widehat{X}^{(s)}_{r\wedge s}, r\geqslant 0)\Big)\\
&\qquad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\Big[\int_{a\wedge L_\infty}^{(a+\varepsilon)\wedge L_\infty} dx\, h(L^{-1}(x))\, F(X_{t\wedge L^{-1}(x)}, t\geqslant 0)\Big]\\
&\qquad = E[1_{\{L^{-1}(a)<\infty\}}\, h(L^{-1}(a))\, F(X_{r\wedge L^{-1}(a)}, r\geqslant 0)],
\end{aligned}$$

which completes the proof of the first assertion. The second assertion follows from the first one in the same way as Proposition 1.2.5 was derived from Proposition 1.1.4. □

We conclude this section with some remarks that will be useful in the applications developed below. Let $x>0$ and let (α_j,β_j), resp. ω_j, $j\in J$, denote the excursion intervals, resp. the excursions of $X-I$ before time T_x. For every $a>0$, we have P a.s.

$$L^a_{T_x} = \sum_{j\in J} L^a_{\sigma(\omega_j)}(\omega_j). \tag{37}$$

A first inequality is easily derived by writing

$$L^a_{T_x}\geqslant \int_0^{T_x} d_sL^a_s\, 1_{\{X_s>I_s\}} = \sum_{j\in J}(L^a_{\beta_j}-L^a_{\alpha_j}) = \sum_{j\in J} L^a_{\sigma(\omega_j)}(\omega_j)$$

where the last equality follows from the approximations of local time. The converse inequality seems to require a different argument in our general setting. Observe that, by excursion theory and then Proposition 1.2.5,

$$\begin{aligned}
E[L^a_{T_x}] &\leqslant \liminf_{k\to\infty} E\Big[\frac{1}{\varepsilon_k}\int_0^{T_x} ds\, 1_{\{a<H_s<a+\varepsilon_k\}}\Big] \\
&= \liminf_{k\to\infty} E\Big[\sum_{j\in J}\frac{1}{\varepsilon_k}\int_0^{\sigma(\omega_j)} ds\, 1_{\{a<H_s(\omega_j)<a+\varepsilon_k\}}\Big] \\
&= \liminf_{k\to\infty} x\, N\Big(\frac{1}{\varepsilon_k}\int_0^{\sigma} ds\, 1_{\{a<H_s<a+\varepsilon_k\}}\Big) \\
&= \liminf_{k\to\infty} \frac{x}{\varepsilon_k}\int_a^{a+\varepsilon_k} db\, e^{-\alpha b} \\
&= x\, e^{-\alpha a}
\end{aligned}$$

whereas Corollary 1.3.4 (with $F = 1$) gives $E[\sum_{j\in J} L^a_{\sigma(\omega_j)}(\omega_j)] = x\, N(L^a_\sigma) = x\, e^{-\alpha a}$. This readily yields (37).

Let us finally observe that we can extend the definition of the local times L^a_s to the process ρ started at a general initial value $\mu \in M_f(\mathbb{R}_+)$. In view of forthcoming applications consider the case when μ is supported on $[0, a)$, for $a > 0$. Then, the previous method can be used to construct a continuous increasing process $(L^a_s(\rho^\mu), s \geqslant 0)$ such that

$$L^a_s(\rho^\mu) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^s dr\, 1_{\{a<H(\rho^\mu_r)<a+\varepsilon\}}$$

in probability (or even in the L^1-norm). Indeed the arguments of the proof of Proposition 1.3.1 remain valid when ρ is replaced by ρ^μ, and the construction and approximation of $L^a_s(\rho^\mu)$ follow. Recall the notation $\tau_0 = \inf\{s \geqslant 0 : \rho^\mu_s = 0\}$ and observe that $\tau_0 = T_x$ if $x = \langle \mu, 1\rangle$. Let (α_j, β_j), ω_j, $j \in J$ be as above and set $r_j = H(k_{-I_{\alpha_j}}\mu)$. Then we have

$$L^a_{\tau_0}(\rho^\mu) = \sum_{j\in J} L^{a-r_j}_{\beta_j-\alpha_j}(\omega_j). \tag{38}$$

The proof is much similar to that of (37): The fact that the left side of (38) is greater than the right side is easy from our approximations of local time. The equality is then obtained from a first-moment argument, using Proposition 1.2.6 and Fatou's lemma to handle the left side.

1.4. Three applications

1.4.1. The Ray-Knight theorem. — Recall that the ψ-continuous-state branching process (in short the ψ-CSBP) is the Markov process $(Y_a, a \geqslant 0)$ with values in $\mathbb{R}_+$ whose transition kernels are characterized by their Laplace transform: For $\lambda > 0$

and $b > a$,

$$E[\exp -\lambda Y_b \mid Y_a] = \exp(-Y_a\, u_{b-a}(\lambda)),$$

where $u_t(\lambda)$, $t \geqslant 0$ is the unique nonnegative solution of the integral equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\, ds = \lambda. \tag{39}$$

Theorem 1.4.1. — *Let $x > 0$. The process $(L^a_{T_x}, a \geqslant 0)$ is a ψ-CSBP started at x.*

Proof. — First observe that $L^a_{T_x}$ is $\mathcal{H}_a$-measurable. This is trivial for $a = 0$ since $L^0_{T_x} = x$. For $a > 0$, note that, if

$$T^a_x = \inf\{s \geqslant 0 : X_{\widehat{\tau}^a_s} = -x\},$$

we have

$$\int_0^{T_x} ds\, 1_{\{a-\varepsilon<H_s\leqslant a\}} = \int_0^{T^a_x} ds\, 1_{\{a-\varepsilon<H_{\widehat{\tau}^a_s}\leqslant a\}}.$$

and the right-hand side is measurable with respect to the σ-field $\mathcal{H}_a$. The measurability of $L^a_{T_x}$ with respect to $\mathcal{H}_a$ then follows from the second convergence of Proposition 1.3.3.

We then verify that the function

$$u_a(\lambda) = N[1 - e^{-\lambda L^a_\sigma}] \quad (a > 0), \qquad u_0(\lambda) = \lambda$$

solves equation (39). From the strong Markov property of ρ under the excursion measure N, we get for $a > 0$

$$u_a(\lambda) = \lambda N\Big(\int_0^\sigma dL^a_s\, e^{-\lambda(L^a_\sigma - L^a_s)}\Big) = \lambda N\Big(\int_0^\sigma dL^a_s\, F(\rho_s)\Big),$$

where, for $\mu \in M_f(\mathbb{R}_+)$, $F(\mu) = E[\exp(-\lambda L^a_{\tau_0}(\rho^\mu))]$. By Corollary 1.3.4, we can concentrate on the case when μ is supported on $[0, a)$, and then (38) gives

$$\begin{aligned} F(\mu) &= \exp\Big(-\int_0^{\langle\mu,1\rangle} du\, N(1 - \exp(-\lambda L^{a-H(k_{-u}\mu)}_\sigma))\Big) \\ &= \exp\Big(-\int \mu(dr)\, N(1 - \exp(-\lambda L^{a-r}_\sigma))\Big). \end{aligned}$$

Hence, using again Corollary 1.3.4,

$$\begin{aligned} u_a(\lambda) &= \lambda\, N\Big(\int_0^\sigma dL^a_s \exp(-\int \rho_s(dr) u_{a-r}(\lambda))\Big) \\ &= \lambda\, e^{-\alpha a} E^0\Big[\exp\Big(-\int J_a(dr)\, u_{a-r}(\lambda)\Big)\Big] \\ &= \lambda\, \exp\Big(-\int_0^a dr\, \widetilde{\psi}(u_{a-r}(\lambda))\Big). \end{aligned}$$

It is a simple matter to verify that (39) follows from this last equality.

By (37) and excursion theory, we have

$$E[\exp(-\lambda L^a_{T_x})] = \exp(-x\, N(1-\exp(-\lambda L^a_\sigma))) = \exp(-x\, u_a(\lambda)). \tag{40}$$

To complete the proof, it is enough to show that for $0 < a < b$,

$$E[\exp(-\lambda L^b_{T_x}) \mid \mathcal{H}_a] = \exp(-u_{b-a}(\lambda)\, L^a_{T_x}). \tag{41}$$

Recall the notation ρ^a from Proposition 1.3.1, and denote by $\widetilde{L}^c_s$ the local times of $H^a_s = H(\rho^a_s)$. From our approximations of local times, it is straightforward to verify that

$$L^b_{T_x} = \widetilde{L}^{b-a}_{A^a_{T_x}},$$

where $A^a_s = \int_0^s dr\, 1_{\{H_r > a\}}$ as previously. Write $U = L^a_{T_x}$ to simplify notation. If $T^a_r = \inf\{t \geqslant 0 : l^a(t) > r\}$, we have $A^a_{T_x} = T^a_U$ (note that $l^a(A^a_{T_x}) = U$ by construction, and that the strong Markov property of X at time T_x implies $l^a(t) > l^a(A^a_{T_x})$ for every $t > A^a_{T_x}$). Hence,

$$E[\exp(-\lambda L^b_{T_x}) \mid \mathcal{H}_a] = E[\exp(-\lambda \widetilde{L}^{b-a}_{T^a_U}) \mid \mathcal{H}_a] = E[\exp(-\lambda \widetilde{L}^{b-a}_{T^a_u})]_{u=U},$$

where in the second equality, we use the fact that the process $(\widetilde{L}^{b-a}_{T^a_u}, u \geqslant 0)$ is a functional of ρ^a, and is thus independent of $\mathcal{H}_a$ (Proposition 1.3.1), whereas $U = L^a_{T_x}$ is $\mathcal{H}_a$-measurable. Since ρ^a has the same distribution as ρ, $\widetilde{L}^{b-a}_{T^a_u}$ and $L^{b-a}_{T_u}$ also have the same law, and the desired result (41) follows from (40). □

Corollary 1.4.2. — *For every $a \geqslant 0$, set*

$$v(a) = N\Big(\sup_{0 \leqslant s \leqslant \sigma} H_s > a\Big).$$

Then,

(i) *If $\int_1^\infty \frac{du}{\psi(u)} = \infty$, we have $v(a) = \infty$ for every $a > 0$.*

(ii) *If $\int_1^\infty \frac{du}{\psi(u)} < \infty$, the function $(v(a), a > 0)$ is determined by*

$$\int_{v(a)}^\infty \frac{du}{\psi(u)} = a.$$

Proof. — By the lower semi-continuity of H, the condition $\sup_{0 \leqslant s \leqslant \sigma} H_s > a$ holds iff $A^a_\sigma > 0$, and our construction shows that this is the case iff $L^a_\sigma > 0$. Thus,

$$v(a) = N(L^a_\sigma > 0) = \lim_{\lambda \to \infty} N(1 - e^{-\lambda L^a_\sigma}) = \lim_{\lambda \to \infty} u_a(\lambda),$$

with the notation of the proof of Theorem 1.4.1. From (39), we have

$$\int_{u_a(\lambda)}^\lambda \frac{du}{\psi(u)} = a,$$

and the desired result follows. □

1.4.2. The continuity of the height process. — We now use Corollary 1.4.2 to give a necessary and sufficient condition for the path continuity of the height process H.

Theorem 1.4.3. — *The process H has continuous sample paths P a.s. iff $\int_1^\infty \frac{du}{\psi(u)} < \infty$.*

Proof. — By excursion theory, we have

$$P\Big[\sup_{0\leqslant s\leqslant T_x} H_s > a\Big] = 1 - \exp(-xv(a)).$$

By Corollary 1.4.2 (i), we see that H cannot have continuous paths if $\int_1^\infty \frac{du}{\psi(u)} = \infty$.

Assume that $\int_1^\infty \frac{du}{\psi(u)} < \infty$. The previous formula and the property $v(a) < \infty$ imply that

$$\lim_{t\downarrow 0} H_t = 0 \qquad P \text{ a.s.} \tag{42}$$

Since $v(a) \downarrow 0$ as $a \uparrow \infty$, we also get that the process H is locally bounded, a.s. The path continuity of H will then follow from Lemma 1.2.4 if we can show that for every fixed interval $[a, a+h]$, $h > 0$, the number of upcrossings of H along $[a, a+h]$ is a.s. finite on every finite time interval. Set $\gamma_0 = 0$ and define by induction for every $n \geqslant 1$,

$$\delta_n = \inf\{t \geqslant \gamma_{n-1} : H_t \geqslant a + h\},$$
$$\gamma_n = \inf\{t \geqslant \delta_n : H_t \leqslant a\}.$$

Both δ_n and γ_n are $(\mathcal{G}_t)$-stopping times. Note that $H_{\gamma_n} \leqslant a$ by the lower semi-continuity of H. On the other hand, as a straightforward consequence of (20), we have a.s. for every $t \geqslant 0$,

$$H_{\gamma_n+t} \leqslant H_{\gamma_n} + H_t^{(\gamma_n)}.$$

where $H^{(\gamma_n)}$ stands for the height process associated with $X_t^{(\gamma_n)} = X_{\gamma_n+t} - X_{\gamma_n}$. Therefore $\delta_{n+1} - \gamma_n \geqslant \kappa_n$, if $\kappa_n = \inf\{t \geqslant 0 : H_t^{(\gamma_n)} \geqslant h\}$. The strong Markov property implies that the variables κ_n are i.i.d. . Furthermore, $\kappa_n > 0$ a.s. by (42). It follows that $\delta_n \uparrow \infty$ as $n \uparrow \infty$, which completes the proof. □

It is easy to see that the condition $\int_1^\infty \frac{du}{\psi(u)} < \infty$ is also necessary and sufficient for H to have continuous sample paths N a.e. On the other hand, we may consider the process ρ started at an arbitrary initial value $\mu \in M_f(\mathbb{R}^d)$, as defined by formula (18), and ask about the sample path continuity of $H(\rho_s)$. Clearly, the answer will be no if the support of μ is not connected. For this reason, we introduce the set M_f^0 which consists of all measures $\mu \in M_f(\mathbb{R}_+)$ such that $H(\mu) < \infty$ and $\operatorname{supp}\mu = [0, H(\mu)]$. By convention the zero measure also belongs to M_f^0.

From (18) and Lemma 1.2.2, it is easy to verify that the process ρ started at an initial value $\mu \in M_f^0$ will remain forever in M_f^0, and furthermore $H(\rho_s)$ will have continuous sample paths a.s. Therefore, we may restrict the state space of ρ to M_f^0. This restriction will be needed in Chapter 4.

1.4.3. Hölder continuity of the height process. — In view of applications in Chapter 4, we now discuss the Hölder continuity properties of the height process. We assume that the condition $\int_1^\infty du/\psi(u) < \infty$ holds so that H has continuous sample paths by Theorem 1.4.3. We set

$$\gamma = \sup\{r \geqslant 0 : \lim_{\lambda\to\infty} \lambda^{-r}\psi(\lambda) = +\infty\}.$$

The convexity of ψ implies that $\gamma \geqslant 1$.

Theorem 1.4.4. — *The height process H is P-a.s. locally Hölder continuous with exponent α for any $\alpha \in (0, 1 - 1/\gamma)$, and is P-a.s. not locally Hölder continuous with exponent α if $\alpha > 1 - 1/\gamma$.*

Proof. — We rely on the following key lemma. Recall the notation $\widehat{L}^{(t)}$ for the local time at 0 of $\widehat{X}^{(t)} - \widehat{S}^{(t)}$ (*cf.* Section 1.2).

Lemma 1.4.5. — *Let $t \geqslant 0$ and $s > 0$. Then P a.s.,*

$$H_{t+s} - \inf_{r\in[t,t+s]} H_r = H(\rho_s^{(t)}),$$

$$H_t - \inf_{r\in[t,t+s]} H_r = \widehat{L}^{(t)}_{t\wedge R},$$

where $R = \inf\{r \geqslant 0 : \widehat{X}^{(t)}_r > -I^{(t)}_s\}$ ($\inf\varnothing = \infty$).

Proof. — From (20), we get, a.s. for every $r > 0$,

$$H_{t+r} = H(k_{-I^{(t)}_r}\rho_t) + H(\rho_r^{(t)}). \tag{43}$$

From this it follows that

$$\inf_{r\in[t,t+s]} H_r = H(k_{-I^{(t)}_s}\rho_t)$$

and the minimum is indeed attained at the (a.s. unique) time $v \in [t, t+s]$ such that $X_v = I^t_{t+s}$. The first assertion of the lemma now follows by combining the last equality with (43) written with $r = s$.

Let us turn to the second assertion. If $I_t \geqslant I^t_{t+s}$, then on one hand $X_v = I_v$ and $\inf_{r\in[t,t+s]} H_r = H_v = 0$, on the other hand, $R = \infty$, and the second assertion reduces to $H_t = \widehat{L}^{(t)}_t$ which is the definition of H_t. Therefore we can assume that $I_t < I^t_{t+s}$. Let

$$u = \sup\{r \in [0,t] : X_{r-} < I^t_{t+s}\}.$$

As in the proof of Proposition 1.2.3, we have

$$H_u = H(k_{-I^{(t)}_s}\rho_t) = \inf_{r\in[t,t+s]} H_r.$$

On the other hand, the construction of the height process shows that the equality $H_r = \widehat{L}^{(t)}_t - \widehat{L}^{(t)}_{t-r}$ holds simultaneously for all $r \in [0,t]$ such that $X_{r-} \leqslant I^r_t$ (*cf.* Lemma 1.2.1). In particular for $r = u$ we get

$$H_t - \inf_{r\in[t,t+s]} H_r = H_t - H_u = \widehat{L}^{(t)}_t - (\widehat{L}^{(t)}_t - \widehat{L}^{(t)}_{t-u}) = \widehat{L}^{(t)}_{t-u}.$$

To complete the proof, simply note that we have $t - u = R$ on the event $\{I_t < I^t_{t+s}\}$. □

To simplify notation we set $\varphi(\lambda) = \lambda/\psi^{-1}(\lambda)$. The right-continuous inverse L^{-1} of L is a subordinator with Laplace exponent φ: See Theorem VII.4 (ii) in [**5**], and note that the constant c in this statement is equal to 1 under our normalization of local time (compare with Lemma 1.1.2).

Lemma 1.4.6. — *For every $t \geqslant 0$, $s > 0$ and $q > 0$,*

$$E[|H_{t+s} - \inf_{r\in[t,t+s]} H_r|^q] \leqslant C_q\, \varphi(1/s)^{-q}\,,$$

and

$$E[|H_t - \inf_{r\in[t,t+s]} H_r|^q] \leqslant C_q\, \varphi(1/s)^{-q}\,,$$

where $C_q = e\Gamma(q+1)$ is a finite constant depending only on q.

Proof. — Recall that $H(\rho_s) = H_s \overset{(\mathrm{d})}{=} L_s$. From Lemma 1.4.5 we have

$$E[|H_{t+s} - \inf_{r\in[t,t+s]} H_r|^q] = E[L_s^q] = q\int_0^{+\infty} x^{q-1} P[L_s > x]dx\,.$$

However,

$$P[L_s > x] = P[s > L^{-1}(x)] \leqslant e\, E[\exp(-L^{-1}(x)/s)] = e\exp(-x\varphi(1/s)).$$

Thus

$$E[|H_{t+s} - \inf_{[t,t+s]} H|^q] \leqslant eq\int_0^{+\infty} x^{q-1}\exp(-x\varphi(1/s))\,dx = C_q\varphi(1/s)^{-q}\,.$$

This completes the proof of the first assertion.

In order to prove the second one, first note that $I^{(t)}_s$ is independent of $\mathcal{G}_t$ and therefore also of the time-reversed process $\widehat{X}^{(t)}$. Writing $\tau_a = \inf\{r \geqslant 0 : S_r > a\}$, we get from the second assertion of Lemma 1.4.5

$$E[|H_t - \inf_{r\in[t,t+s]} H_r|^q] \leqslant \int_{[0,+\infty)} P[-I_s \in da]\, E[L^q_{\tau_a}].$$

Note that

$$E[L^q_{\tau_a}] = q\int_0^{+\infty} x^{q-1} P[L_{\tau_a} > x]\,dx\,,$$

and that $P[L_{\tau_a} > x] = P[S_{L^{-1}(x)} < a]$. It follows that

$$E[|H_t - \inf_{r\in[t,t+s]} H_r|^q] \leqslant q\int_{[0,+\infty)} P[-I_s \in da]\int_0^{+\infty} dx\, x^{q-1} P[S_{L^{-1}(x)} < a]\,.$$

An integration by parts leads to

$$E[|H_t - \inf_{r\in[t,t+s]} H_r|^q] \leqslant q\int_0^{+\infty} dx\, x^{q-1}\int_{[0,+\infty)} P[S_{L^{-1}(x)} \in db]P[-I_s > b]\,.$$

However

$$P[-I_s > b] = P[T_b < s] \leqslant e\, E[\exp(-T_b/s)] = e\, \exp(-b\psi^{-1}(1/s))$$

since we know ([**5**] Theorem VII.1) that $(T_b, b \geqslant 0)$ is a subordinator with exponent ψ^{-1}. Recalling Lemma 1.1.2, we get

$$\begin{aligned} E[|H_t - \inf_{r\in[t,t+s]} H_r|^q] &\leqslant eq \int_0^{+\infty} dx\, x^{q-1} E[\exp(-\psi^{-1}(1/s) S_{L^{-1}(x)})] \\ &= eq \int_0^{+\infty} dx\, x^{q-1}\, \exp(-x/(s\psi^{-1}(1/s))) \\ &= C_q \varphi(1/s)^{-q}. \end{aligned}$$

This completes the proof of Lemma 1.4.6. □

Proof of Theorem 1.4.4. — From Lemma 1.4.6 and an elementary inequality, we get for every $t \geqslant 0$, $s > 0$ and $q > 0$

$$E[|H_{t+s} - H_t|^q] \leqslant 2^{q+1} C_q\, \varphi(1/s)^q.$$

Let $\alpha \in (0, 1 - 1/\gamma)$. Then $(1-\alpha)^{-1} < \gamma$ and thus $\lambda^{-(1-\alpha)^{-1}}\psi(\lambda)$ tends to ∞ as $\lambda \to \infty$. It easily follows that $\lambda^{\alpha-1}\psi^{-1}(\lambda)$ tends to 0 and so $\lambda^{-\alpha}\varphi(\lambda)$ tends to ∞ as $\lambda \to \infty$. The previous bound then yields the existence of a constant C depending on q and α such that for every $t \geqslant 0$ and $s \in (0, 1]$,

$$E[|H_{t+s} - H_t|^q] \leqslant C\, s^{q\alpha}.$$

The classical Kolmogorov lemma gives the first assertion of the theorem.

To prove the second assertion, observe that for every $\alpha > 0$ and $A > 0$,

$$P[H_s < As^\alpha] = P[L_s < As^\alpha] = P[s < L^{-1}(As^\alpha)]\ .$$

Then use the elementary inequality

$$P[s < L^{-1}(As^\alpha)] \leqslant \frac{e}{e-1} E[1 - \exp(-L^{-1}(As^\alpha)/s)],$$

which leads to

$$P[H_s < As^\alpha] \leqslant \frac{e}{e-1} (1 - \exp(-As^\alpha \varphi(1/s))).$$

If $\alpha > 1 - 1/\gamma$, we can find a sequence (s_n) decreasing to zero such $s_n^\alpha \varphi(1/s_n)$ tends to 0. Thus, for any $A > 0$

$$\lim_{n\to\infty} P[H_{s_n} < As_n^\alpha] = 0\ ,$$

and it easily follows that $\limsup_{s\to 0} s^{-\alpha} H_s = \infty$, P a.s., which completes the proof. □

CHAPTER 2

CONVERGENCE OF GALTON-WATSON TREES

2.1. Preliminaries

Our goal in this chapter is to study the convergence in distribution of Galton-Watson trees, under the assumption that the associated Galton-Watson processes, suitably rescaled, converge in distribution to a continuous-state branching process. To give a precise meaning to the convergence of trees, we will code Galton-Watson trees by a discrete height process, and we will establish the convergence of these (rescaled) discrete processes to the continuous height process of the previous chapter. We will also prove that similar convergences hold when the discrete height processes are replaced by the contour processes of the trees.

Let us introduce the basic objects considered in this chapter. For every $p \geqslant 1$, let μ_p be a subcritical or critical offspring distribution. That is, μ_p is a probability distribution on $\mathbb{Z}_+ = \{0, 1, \ldots\}$ such that

$$\sum_{k=0}^{\infty} k\, \mu_p(k) \leqslant 1.$$

We systematically exclude the trivial cases where $\mu_p(1) = 1$ or $\mu_p(0) = 1$. We also define another probability measure ν_p on $\{-1, 0, 1, \ldots\}$ by setting $\nu_p(k) = \mu_p(k+1)$ for every $k \geqslant -1$.

We denote by $V^p = (V_k^p, k = 0, 1, 2, \ldots)$ a discrete-time random walk on $\mathbb{Z}$ with jump distribution ν_p and started at 0. We also denote by $Y^p = (Y_k^p, k = 0, 1, 2, \ldots)$ a Galton-Watson branching process with offspring distribution μ_p started at $Y_0^p = p$.

Finally, we consider a Lévy process $X = (X_t, t \geqslant 0)$ started at the origin and satisfying assumptions (H1) and (H2) of Chapter 1. As in Chapter 1, we write ψ for the Laplace exponent of X. We denote by $Y = (Y_t, t \geqslant 0)$ a ψ-continuous-state branching process started at $Y_0 = 1$.

The following variant of a result due to Grimvall [**21**] plays an important role in our approach. Unless otherwise specified the convergence in distribution of processes

is in the functional sense, that is in the sense of the weak convergence of the laws of the processes on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. We will use the notation $\xrightarrow{(\mathrm{fd})}$ to indicate weak convergence of finite-dimensional marginals.

For $a \in \mathbb{R}$, $[a]$ denotes the integer part of a.

Theorem 2.1.1. — *Let $(\gamma_p,\, p = 1, 2, \dots)$ be a nondecreasing sequence of positive integers converging to ∞. The convergence in distribution*

$$\left(p^{-1} Y^p_{[\gamma_p t]}\,,\, t \geqslant 0\right) \xrightarrow[p\to\infty]{(\mathrm{d})} (Y_t, t \geqslant 0) \tag{44}$$

holds if and only if

$$\left(p^{-1} V^p_{[p\gamma_p t]}\,,\, t \geqslant 0\right) \xrightarrow[p\to\infty]{(\mathrm{d})} (X_t, t \geqslant 0). \tag{45}$$

Proof. — By standard results on the convergence of triangular arrays (see e.g. Theorem 2.7 in Skorokhod [**46**]), the functional convergence (45) holds iff

$$p^{-1} V^p_{p\gamma_p} \xrightarrow[p\to\infty]{(\mathrm{d})} X_1. \tag{46}$$

Fix any sequence $p_1 < p_2 < \cdots < p_k < \cdots$ such that $\gamma_{p_1} < \gamma_{p_2} < \cdots$. If $j = \gamma_{p_k}$ for some $k \geqslant 1$, set $c_j = p_k$, $V^{(j)} = V^{p_k}$ and let θ_j be the probability measure on $\mathbb{R}$ defined by $\theta_j(\frac{n}{c_j}) = \nu_{p_k}(n)$ for every integer $n \geqslant -1$. Then (46) is equivalent to saying that

$$\frac{1}{c_j} V^{(j)}_{jc_j} \xrightarrow[j\to\infty]{(\mathrm{d})} X_1$$

for any choice of the sequence $p_1 < p_2 < \cdots$. Equivalently the convolutions $(\theta_j)^{*jc_j}$ converge weakly to the law of X_1. By Theorems 3.4 and 3.1 of Grimvall [**21**], this property holds iff the convergence (44) holds along the sequence (p_k). (Note that condition (b) in Theorem 3.4 of [**21**] is automatically satisfied here since we restrict our attention to the (sub)critical case.) This completes the proof. □

2.2. The convergence of finite-dimensional marginals

From now on, we suppose that assumption (H3) holds in addition to (H1) and (H2). Thus we can consider the height process $H = (H_t, t \geqslant 0)$ of Chapter 1.

For every $p \geqslant 1$, let $H^p = (H^p_k, k \geqslant 0)$ be the discrete height process associated with a sequence of independent Galton-Watson trees with offspring distribution μ_p (*cf.* Section 0.2). As was observed in Section 0.2, we may and will assume that the processes H^p and V^p are related by the formula

$$H^p_k = \mathrm{Card}\{j \in \{0, 1, \dots, k-1\} : V^p_j = \inf_{j \leqslant l \leqslant k} V^p_l\}. \tag{47}$$

The following theorem sharpens a result of [**32**].

Theorem 2.2.1. — *Under either of the convergences* (44) *or* (45), *we have also*

$$\left(\frac{1}{\gamma_p} H^p_{[p\gamma_p t]}\,,\, t \geqslant 0\right) \xrightarrow[p\to\infty]{\text{(fd)}} (H_t, t \geqslant 0). \tag{48}$$

Proof. — Let f_0 be a truncation function, that is a bounded continuous function from $\mathbb{R}$ into $\mathbb{R}$ such that $f_0(x) = x$ for every x belonging to a neighborhood of 0. By standard results on the convergence of rescaled random walks (see e.g. Theorem II.3.2 in [**23**]), the convergence (45) holds iff the following three conditions are satisfied:

$$\text{(C1)} \qquad \lim_{p\to\infty} p\gamma_p \sum_{k=-1}^{\infty} f_0(\frac{k}{p})\, \nu_p(k) = -\alpha + \int_0^\infty (f_0(r) - r)\, \pi(dr)$$

$$\text{(C2)} \qquad \lim_{p\to\infty} p\gamma_p \sum_{k=-1}^{\infty} f_0(\frac{k}{p})^2\, \nu_p(k) = 2\beta + \int_0^\infty f_0(r)^2\, \pi(dr)$$

$$\text{(C3)} \qquad \lim_{p\to\infty} p\gamma_p \sum_{k=-1}^{\infty} h(\frac{k}{p})\, \nu_p(k) = \int_0^\infty h(r)\, \pi(dr),$$

for any bounded continuous function h on $\mathbb{R}$ that vanishes on a neighborhood of 0.

By (47) and time-reversal, H^p_k has the same distribution as

$$\Lambda^{(p)}_k = \text{Card}\{j \in \{1, \ldots, k\} : V^p_j = \sup_{0\leqslant l\leqslant j} V^p_l\}.$$

Without loss of generality, the Skorokhod representation theorem allows us to assume that the convergence

$$\left(p^{-1} V^p_{[p\gamma_p t]}\,,\, t \geqslant 0\right) \xrightarrow[p\to\infty]{} (X_t, t \geqslant 0) \tag{49}$$

holds a.s. in the sense of Skorokhod's topology. Suppose we can prove that for every $t > 0$,

$$\lim_{p\to\infty} \gamma_p^{-1}\, \Lambda^{(p)}_{[p\gamma_p t]} = L_t \tag{50}$$

in probability. (Here $L = (L_t, t \geqslant 0)$ is the local time process of $X - S$ at level 0 as in Chapter 1.) Then a simple time-reversal argument implies that $\gamma_p^{-1} H^p_{[p\gamma_p t]}$ also converges in probability to $\widehat{L}^{(t)}_t = H_t$, with the notation of Chapter 1. Therefore the proof of Theorem 2.2.1 reduces to showing that (50) holds.

We first consider the case where $\int_{(0,1)} r\pi(dr) = \infty$. We introduce the stopping times $(\tau^p_k)_{k\geqslant 0}$ defined recursively as follows:

$$\tau^p_0 = 0,$$
$$\tau^p_{m+1} = \inf\{n > \tau^p_m : V^p_n \geqslant V^p_{\tau^p_m}\}.$$

Conditionally on the event $\{\tau^p_m < \infty\}$, the random variable $1_{\{\tau^p_{m+1}<\infty\}}(V^p_{\tau^p_{m+1}} - V^p_{\tau^p_m})$ is independent of the past of V^p up to time τ^p_m and has the same law as $1_{\{\tau^p_1<\infty\}} V^p_{\tau^p_1}$.

Also recall the classical equality (*cf.* (5.4) in [**32**]):

$$P[\tau_1^p < \infty,\, V^p_{\tau_1^p} = j] = \nu_p([j,\infty))\ , \qquad j \geqslant 0. \tag{51}$$

For every $u > \delta > 0$, set:

$$\kappa(\delta,u) = \int_0^\infty \pi(dr) \int_0^r dx\, 1_{(\delta,u]}(x) = \int_0^\infty \pi(dr)\, \Big((r-\delta)^+ \wedge (u-\delta)\Big)\,,$$

$$\kappa_p(\delta,u) = \frac{\displaystyle\sum_{p\delta<j\leqslant pu} \nu_p([j,\infty))}{\displaystyle\sum_{j\geqslant 0} \nu_p([j,\infty))} = P[p\delta < V^p_{\tau_1^p} \leqslant pu \mid \tau_1^p < \infty]\,,$$

$$L_t^{\delta,u} = \mathrm{Card}\{s \leqslant t : \Delta S_s \in (\delta,u]\}\,,$$

$$l_k^{p,\delta,u} = \mathrm{Card}\{j < k : \overline{V}^p_j + p\delta < V^p_{j+1} \leqslant \overline{V}^p_j + pu\}\,,$$

where $\overline{V}^p_j = \sup\{V^p_i, 0 \leqslant i \leqslant j\}$. Note that $\kappa(\delta,u) \uparrow \infty$ as $\delta \downarrow 0$, by our assumption $\int_{(0,1)} r\pi(dr) = \infty$. From the a.s. convergence of the processes $p^{-1}V_{[p\gamma_p t]}$, we have

$$\lim_{p\to\infty} l^{p,\delta,u}_{[p\gamma_p t]} = L_t^{\delta,u}\ , \quad \text{a.s.} \tag{52}$$

(Note that $P[\Delta S_s = a \text{ for some } s > 0] = 0$ for every fixed $a > 0$, by (10).) By applying excursion theory to the process $X - S$ and using formula (10), one easily gets for every $u > 0$

$$\lim_{\delta\to 0} \kappa(\delta,u)^{-1} L_t^{\delta,u} = L_t\ , \quad \text{a.s.} \tag{53}$$

We claim that we have also

$$\lim_{p\to\infty} \gamma_p\, \kappa_p(\delta,u) = \int_0^\infty \Big((r-\delta)^+ \wedge (u-\delta)\Big)\, \pi(dr) = \kappa(\delta,u). \tag{54}$$

To get this convergence, first apply (C3) to the function $h(x) = (x-\delta)^+ \wedge (u-\delta)$. It follows that

$$\lim_{p\to\infty} p\,\gamma_p \sum_{k=-1}^{\infty} \nu_p(k) \Big((\frac{k}{p} - \delta)^+ \wedge (u-\delta)\Big) = \kappa(\delta,u).$$

On the other hand, it is elementary to verify that

$$\left| p\,\gamma_p \sum_{k=-1}^{\infty} \nu_p(k) \Big((\frac{k}{p} - \delta)^+ \wedge (u-\delta)\Big) - \gamma_p \sum_{p\delta<j\leqslant pu} \nu_p([j,\infty)) \right| \leqslant \gamma_p \sum_{k\geqslant \delta p} \nu_p(k)$$

and the right-hand side tends to 0 by (C3). Thus we get

$$\lim_{p\to\infty} \gamma_p \sum_{p\delta<j\leqslant pu} \nu_p([j,\infty)) = \kappa(\delta,u).$$

Furthermore, as a simple consequence of (C1) and the (sub)criticality of μ_p, we have also

$$\sum_{j=0}^{\infty} \nu_p([j,\infty)) = 1 + \sum_{k=-1}^{\infty} k\nu_p(k) \underset{p\to\infty}{\longrightarrow} 1. \tag{55}$$

(This can also be obtained from (51) and the weak convergence (45).) Our claim (54) now follows.

Finally, we can also obtain a relation between $l_k^{p,\delta,u}$ and $\Lambda^{(p)}_{[p\gamma_p t]}$. Simply observe that conditional on $\{\tau_k^p < \infty\}$, $l^{p,\delta,u}_{\tau_k^p}$ is the sum of k independent Bernoulli variables with parameter $\kappa_p(\delta,u)$. Fix an integer $A > 0$ and set $A_p = \gamma_p A + 1$. From Doob's inequality, we easily get (see [**32**], p.249 for a similar estimate)

$$E\left[\sup_{0\leqslant j\leqslant \tau^p_{A_p}} \left|\frac{1}{\gamma_p}\left(\Lambda_j^{(p)} - \kappa_p(\delta,u)^{-1} l_j^{p,\delta,u}\right)\right|^2\right] \leqslant \frac{8(A+1)}{\gamma_p}\,\kappa_p(\delta,u)^{-1}.$$

Hence, using (54), we have

$$\limsup_{p\to\infty} E\Big[\sup_{j\leqslant \tau^p_{A_p}} \left|\frac{1}{\gamma_p}\left(\Lambda_j^{(p)} - \kappa_p(\delta,u)^{-1} l_j^{p,\delta,u}\right)\right|^2\Big] \leqslant \frac{8(A+1)}{\kappa(\delta,u)}. \tag{56}$$

To complete the proof, let $\varepsilon > 0$ and first choose A large enough so that $P[L_t \geqslant A - 3\varepsilon] < \varepsilon$. If $u > 0$ is fixed, we can use (53) and (56) to pick $\delta > 0$ small enough and then $p_0 = p_0(\delta)$ so that

$$P\Big[\Big|\kappa(\delta,u)^{-1} L_t^{\delta,u} - L_t\Big| > \varepsilon\Big] < \varepsilon \tag{57}$$

and

$$P\left[\sup_{j\leqslant \tau^p_{A_p}} \left|\frac{1}{\gamma_p}\left(\Lambda_j^{(p)} - \kappa_p(\delta,u)^{-1} l_j^{p,\delta,u}\right)\right| > \varepsilon\right] < \varepsilon, \qquad \text{if } p \geqslant p_0. \tag{58}$$

From (52) and (54), we can also find $p_1(\delta)$ so that for every $p \geqslant p_1$,

$$P\left[\left|\frac{1}{\gamma_p \kappa_p(\delta,u)}\, l^{p,\delta,u}_{[p\gamma_p t]} - \kappa(\delta,u)^{-1} L_t^{\delta,u}\right| > \varepsilon\right] < \varepsilon. \tag{59}$$

By combining the previous estimates (57), (58) and (59), we get for $p \geqslant p_0 \vee p_1$

$$P\Big[\Big|\frac{1}{\gamma_p}\,\Lambda^{(p)}_{[p\gamma_p t]} - L_t\Big| > 3\varepsilon\Big] \leqslant 3\varepsilon + P[[p\gamma_p t] > \tau^p_{A_p}]. \tag{60}$$

Furthermore, by using (58) and then (57) and (59), we have for p sufficiently large,

$$\begin{aligned} P[\tau^p_{A_p} < [p\gamma_p t]] &\leqslant \varepsilon + P\Big[\frac{1}{\gamma_p \kappa_p(\delta,u)}\, l^{p,\delta,u}_{[p\gamma_p t]} \geqslant A - \varepsilon\Big] \\ &\leqslant 3\varepsilon + P[L_t \geqslant A - 3\varepsilon] \\ &\leqslant 4\varepsilon \end{aligned}$$

from our choice of A. Combining this estimate with (60) completes the proof of (50) in the case $\int_0^1 r\pi(dr) = \infty$.

It remains to treat the case where $\int_0^1 r\pi(dr) < \infty$. In that case, (H3) implies that $\beta > 0$, and we know from (11) that

$$L_t = \frac{1}{\beta} m(\{S_s; s \leqslant t\}) .$$

Furthermore, (10) and the assumption $\int_0^1 r\pi(dr) < \infty$ imply that for any $t > 0$,

$$\mathrm{Card}\{s \in [0,t];\ \Delta S_s > 0\} < \infty , \quad \text{a.s.}$$

For every $\delta > 0$ and $t \geqslant 0$, we set

$$\widetilde{S}_t^\delta = S_t - \sum_{s\in[0,t]} \mathbf{1}_{(\delta,\infty)}(\Delta S_s)\Delta S_s .$$

By the previous remarks, we have a.s. for δ small enough,

$$\widetilde{S}_t^\delta = S_t - \sum_{s\in[0,t]} \Delta S_s = m(\{S_s; s \leqslant t\}) = \beta L_t . \tag{61}$$

Let us use the same notation τ_m^p, $\overline{V}_j^p$ as in the case $\int_0^1 r\pi(dr) = \infty$, and also set for any $m \geqslant 1$,

$$d_m^p = \mathbf{1}_{\{\tau_m^p < \infty\}}(V^p_{\tau_m^p} - V^p_{\tau_{m-1}^p})$$

and

$$\widetilde{S}_m^{\delta,p} = \sum_{n\geqslant 1} d_n^p \mathbf{1}_{\{d_n^p \leqslant p\delta,\ \tau_n^p \leqslant m\}} .$$

The convergence (49) implies that

$$\left(\frac{1}{p}\overline{V}^p_{[p\gamma_p s]}, s \geqslant 0\right) \underset{p\to\infty}{\longrightarrow} (S_s, s \geqslant 0) , \quad \text{a.s.,} \tag{62}$$

and, for every $t \geqslant 0$,

$$\frac{1}{p}\sum_{n\geqslant 1} d_n^p \mathbf{1}_{\{d_n^p > p\delta,\ \tau_n^p \leqslant [p\gamma_p t]\}} \underset{p\to\infty}{\longrightarrow} \sum_{s\in[0,t]} \mathbf{1}_{(\delta,\infty)}(\Delta S_s)\Delta S_s , \quad \text{a.s.}$$

Thus we have

$$\lim_{p\to\infty} \frac{1}{p}\widetilde{S}^{\delta,p}_{[p\gamma_p t]} = \widetilde{S}_t^\delta \quad \text{a.s.} \tag{63}$$

The desired convergence (50) is then a consequence of (61), (63) and the following result: For every $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{p\to\infty} P[|\frac{1}{p}\widetilde{S}^{\delta,p}_{[p\gamma_p t]} - \frac{\beta}{\gamma_p}\Lambda^{(p)}_{[p\gamma_p t]}| > \varepsilon] = 0 . \tag{64}$$

To prove (64), set

$$\alpha_1(p,\delta) = E\left[d_1^p \mathbf{1}_{\{d_1^p \leqslant \delta p\}} \mid \tau_1^p < \infty\right] ;$$

$$\alpha_2(p,\delta) = E\left[(d_1^p)^2 \mathbf{1}_{\{d_1^p \leqslant \delta p\}} \mid \tau_1^p < \infty\right] .$$

Observe that

$$E\left[(d_1^p \mathbf{1}_{\{d_1^p \leqslant \delta p\}} - \alpha_1(p,\delta))^2 \mid \tau_1^p < \infty\right] \leqslant \alpha_2(p,\delta) .$$

Let $A > 0$ be an integer and let $A_p = \gamma_p A + 1$ as above. By Doob's inequality,

$$E\Big[\sup_{\substack{1 \leqslant m \leqslant A_p \\ \tau_m^p < \infty}} |\widetilde{S}^{\delta,p}_{\tau_m^p} - m\alpha_1(p,\delta)|^2\Big] \leqslant 4A_p\alpha_2(p,\delta) .$$

Since

$$\sup_{\substack{1 \leqslant m \leqslant A_p \\ \tau_m^p < \infty}} |\widetilde{S}^{\delta,p}_{\tau_m^p} - m\alpha_1(p,\delta)| = \sup_{1 \leqslant j \leqslant \tau^p_{A_p}} |\widetilde{S}^{\delta,p}_j - \alpha_1(p,\delta)\Lambda_j^{(p)}| .$$

we have

$$E\Big[\sup_{0 \leqslant j \leqslant \tau^p_{A_p}} |\frac{1}{p}\widetilde{S}^{\delta,p}_j - \frac{\alpha_1(p,\delta)}{p}\Lambda_j^{(p)}|^2\Big] \leqslant \frac{4A_p}{p^2}\alpha_2(p,\delta) . \tag{65}$$

We now claim that

$$\lim_{p\to\infty} \frac{\gamma_p}{p}\alpha_1(p,\delta) = \beta + \frac{1}{2}\int_{(0,\infty)} (r \wedge \delta)^2 \pi(dr) \underset{\delta\to 0}{\longrightarrow} \beta , \tag{66}$$

and

$$\lim_{\delta\to 0} \limsup_{p\to\infty} \frac{\gamma_p}{p^2}\,\alpha_2(p,\delta) = 0 . \tag{67}$$

To verify (66), note that, by (51),

$$\frac{\gamma_p}{p}\alpha_1(p,\delta) = \frac{\gamma_p}{p}\frac{\sum_{j=0}^{[\delta p]} j\,\nu_p([j,\infty))}{\sum_{j=0}^{\infty}\nu_p([j,\infty))} = \frac{p\gamma_p}{2}\frac{\sum_{k=0}^{\infty}\nu_p(k)\Big(\frac{k}{p}\wedge\frac{[\delta p]}{p}\Big)\Big(\frac{k}{p}\wedge\frac{[\delta p]}{p}+\frac{1}{p}\Big)}{\sum_{j=0}^{\infty}\nu_p([j,\infty))} \tag{68}$$

We now apply (C1) and (C2) with the truncation function $f_0(x) = (x \wedge \delta) \vee (-\delta)$. Multiplying by p^{-1} the convergence in (C1) and adding the one in (C2), we get

$$\lim_{p\to\infty} p\gamma_p \sum_{k=0}^{\infty} \nu_p(k)\Big(\frac{k}{p}\wedge\delta\Big)\Big(\frac{k}{p}\wedge\delta+\frac{1}{p}\Big) = 2\beta + \int_{(0,\infty)} (r\wedge\delta)^2\pi(dr).$$

Comparing with (68) and using (55), we immediately get (66). The proof of (67) is analogous.

By (65) and an elementary inequality, we have

$$E\Big[\sup_{0 \leqslant j \leqslant \tau^p_{A_p}} |\frac{1}{p}\widetilde{S}^{\delta,p}_j - \frac{\beta}{\gamma_p}\Lambda_j^{(p)}|^2\Big] \leqslant \frac{8A_p}{p^2}\alpha_2(p,\delta) + 2(\frac{A_p}{\gamma_p})^2\,(\beta - \frac{\gamma_p}{p}\alpha_1(p,\delta))^2 .$$

Thus, (66) and (67) imply that for any $A > 0$

$$\lim_{\delta\to 0}\limsup_{p\to\infty} E\Big[\sup_{0 \leqslant j \leqslant \tau^p_{A_p}} |\frac{1}{p}\widetilde{S}^{\delta,p}_j - \frac{\beta}{\gamma_p}\Lambda_j^{(p)}|^2\Big] = 0 . \tag{69}$$

It follows that

$$\lim_{\delta\to 0}\limsup_{p\to\infty} P\Big[|\frac{1}{p}\widetilde{S}^{\delta,p}_{[p\gamma_p t]} - \frac{\beta}{\gamma_p}\Lambda^{(p)}_{[p\gamma_p t]}| > \varepsilon\Big] \leqslant \limsup_{p\to\infty} P[\tau^p_{A_p} < [p\gamma_p t]] .$$

However,

$$P\big[\tau^p_{A_p} < [p\gamma_p t]\big] \leqslant P\big[\tfrac{1}{p}\overline{V}^p_{[p\gamma_p t]} \geqslant \tfrac{1}{p}\widetilde{S}^{\delta,p}_{\tau^p_{A_p}}\ ,\ \tau^p_{A_p} < \infty\big]$$

and by (69) the right side is bounded above for p large by $P[\frac{1}{p}\overline{V}^p_{[p\gamma_p t]} \geqslant \frac{\beta}{\gamma_p}A_p - 1] + \varepsilon_\delta$, where $\varepsilon_\delta \to 0$ as $\delta \to 0$. In view of (62), this is enough to conclude that

$$\lim_{A\to\infty} \limsup_{p\to\infty} P\big[\tau^p_{A_p} < [p\gamma_p t]\big] = 0\ ,$$

and the desired result (64) follows. This completes the proof of (50) and of Theorem 2.2.1. □

2.3. The functional convergence

Our goal is now to discuss conditions that ensure that the convergence of Theorem 2.2.1 holds in a functional sense. We assume that the function ψ satisfies the condition

$$\int_1^\infty \frac{du}{\psi(u)} < \infty. \tag{70}$$

By Theorem 1.4.3, this implies that the height process $(H_t, t \geqslant 0)$ has continuous sample paths. On the other hand, if this condition does not hold, the paths of the height process do not belong to any of the usual functional spaces.

For every $p \geqslant 1$, we denote by $g^{(p)}$ the generating function of μ_p, and by $g^{(p)}_n = g^{(p)} \circ \cdots \circ g^{(p)}$ the n-th iterate of $g^{(p)}$.

Theorem 2.3.1. — *Suppose that the convergences* (44) *and* (45) *hold and that the continuity condition* (70) *is satisfied. Suppose in addition that for every* $\delta > 0$,

$$\liminf_{p\to\infty} g^{(p)}_{[\delta\gamma_p]}(0)^p > 0. \tag{71}$$

Then,

$$\Big(\gamma_p^{-1} H^p_{[p\gamma_p t]}\ ,\ t \geqslant 0\Big) \xrightarrow[p\to\infty]{\text{(d)}} (H_t, t \geqslant 0) \tag{72}$$

in the sense of weak convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

Let us make some important remarks. Condition (71) can be restated in probabilistic terms as follows: For every $\delta > 0$,

$$\liminf_{p\to\infty} P[Y^p_{[\delta\gamma_p]} = 0] > 0.$$

(As will follow from our results, this implies that the extinction time of Y^p, scaled by γ_p^{-1}, converges in distribution to the extinction time of Y, which is finite a.s. under (70).) It is easy to see that the condition (71) is necessary for the conclusion (72) to hold. Indeed, suppose that (71) fails, so that there exists $\delta > 0$ such that $P[Y^p_{[\delta\gamma_p]} = 0]$ converges to 0 as $p \to \infty$, at least along a suitable subsequence. Clearly, this convergence also holds (along the same subsequence) if Y^p starts at $[ap]$ instead

of p, for any fixed $a > 0$. From the definition of the discrete height process, we get that

$$P\Big[\sup_{k\leqslant T^p_{[ap]}} H^p_k \geqslant [\delta\gamma_p]\Big] \underset{p\to\infty}{\longrightarrow} 1\ ,$$

where $T^p_j = \inf\{k \geqslant 0 : V^p_k = -j\}$. From (45), we know that $(p\gamma_p)^{-1}T^p_{[ap]}$ converges in distribution to T_a. Since $T_a \downarrow 0$ as $a \downarrow 0$, a.s., we easily conclude that, for every $\varepsilon > 0$,

$$P\Big[\sup_{t\leqslant\varepsilon} \gamma_p^{-1} H^p_{[p\gamma_p t]} \geqslant \frac{[\delta\gamma_p]}{\gamma_p}\Big] \underset{p\to\infty}{\longrightarrow} 1\ ,$$

and thus (72) cannot hold.

On the other hand, one might think that the condition (71) is automatically satisfied under (44) and (70). Let us explain why this is not the case. Suppose for simplicity that ψ is of the type

$$\psi(\lambda) = \alpha\lambda + \int_{(0,\infty)} \pi(dr)\,(e^{-\lambda r} - 1 + \lambda r),$$

and for every $\varepsilon > 0$ set

$$\psi_\varepsilon(\lambda) = \alpha\lambda + \int_{(\varepsilon,\infty)} \pi(dr)\,(e^{-\lambda r} - 1 + \lambda r).$$

Note that $\psi_\varepsilon(\lambda) \leqslant C_\varepsilon\lambda$ and so $\int_1^\infty \psi_\varepsilon(\lambda)^{-1}d\lambda = \infty$. Thus, if Y^ε is a ψ_ε-CSBP started at 1, we have $Y^\varepsilon_t > 0$ for every $t > 0$ a.s. (Grey [**20**], Theorem 1). It is easy to verify that

$$(Y^\varepsilon_t, t \geqslant 0) \underset{\varepsilon\to 0}{\longrightarrow} (Y_t, t \geqslant 0)$$

at least in the sense of the weak convergence of finite-dimensional marginals. Let us fix a sequence (ε_k) decreasing to 0. Recall from [**27**] that every continuous-state branching process can be obtained as a weak limit of rescaled Galton-Watson branching processes. Thus for every k, we can find a subcritical or critical offspring distribution ν_k, and two positive integers $p_k \geqslant k$ and $\gamma_k \geqslant k$, in such a way that if $Z^k = (Z^k_j, j \geqslant 0)$ is a Galton-Watson process with offspring distribution ν_k started at $Z^k_0 = p_k$, the law of the rescaled process

$$Z^{(k)}_t = (p_k)^{-1} Z^k_{[\gamma_k t]}$$

is arbitrarily close to that of Y^{ε_k}. In particular, we may assume that $P[Z^{(k)}_k > 0] > 1 - 2^{-k}$, and that the rescaled processes $Z^{(k)}$ converge to Y in the sense of weak convergence of finite-dimensional marginals. By Theorem 3.4 of [**21**], this convergence also holds in the functional sense in the Skorokhod space. However, the extinction time of $Z^{(k)}$ converges in probability to $+\infty$, and so the condition (71) cannot hold.

There is however a very important special case where (71) holds.

Theorem 2.3.2. — *Suppose that $\mu_p = \mu$ for every p and that the convergence* (44) *holds. Then the condition* (71) *is automatically satisfied and the conclusion of Theorem 2.3.1 holds.*

As we will see in the proof, under the assumption of Theorem 2.3.2, the process X must be stable with index $\alpha \in (1,2]$. Clearly condition (70) holds in that case.

Proof of Theorem 2.3.1. — To simplify notation, we set $H_t^{(p)} = \gamma_p^{-1} H^p_{[p\gamma_p t]}$ and $V_t^{(p)} = p^{-1} V^p_{[p\gamma_p t]}$. In view of Theorem 2.2.1, the proof of Theorem 2.3.1 reduces to checking that the laws of the processes $(H_t^{(p)}, t \geqslant 0)$ are tight in the set of probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. By standard results (see e.g. Corollary 3.7.4 in [**14**]), it is enough to verify the following two properties:

(i) For every $t \geqslant 0$ and $\eta > 0$, there exists a constant $K \geqslant 0$ such that

$$\liminf_{p\to\infty} P[H_t^{(p)} \leqslant K] \geqslant 1 - \eta.$$

(ii) For every $T > 0$ and $\delta > 0$,

$$\lim_{n\to\infty} \limsup_{p\to\infty} P\Big[\sup_{1\leqslant i\leqslant 2^n} \sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} |H_t^{(p)} - H^{(p)}_{(i-1)2^{-n}T}| > \delta \Big] = 0.$$

Property (i) is immediate from the convergence of finite-dimensional marginals. Thus the real problem is to prove (ii). We fix $\delta > 0$ and $T > 0$ and first observe that

$$P\Big[\sup_{1\leqslant i\leqslant 2^n} \sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} |H_t^{(p)} - H^{(p)}_{(i-1)2^{-n}T}| > \delta \Big] \leqslant A_1(n,p) + A_2(n,p) + A_3(n,p)$$

where

$$A_1(n,p) = P\Big[\sup_{1\leqslant i\leqslant 2^n} |H^{(p)}_{i2^{-n}T} - H^{(p)}_{(i-1)2^{-n}T}| > \frac{\delta}{5} \Big]$$

$$A_2(n,p) = P\Big[\sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} H_t^{(p)} > H^{(p)}_{(i-1)2^{-n}T} + \frac{4\delta}{5} \text{ for some } 1 \leqslant i \leqslant 2^n \Big]$$

$$A_3(n,p) = P\Big[\inf_{t\in[(i-1)2^{-n}T, i2^{-n}T]} H_t^{(p)} < H^{(p)}_{i2^{-n}T} - \frac{4\delta}{5} \text{ for some } 1 \leqslant i \leqslant 2^n \Big]$$

The term A_1 is easy to bound. By the convergence of finite-dimensional marginals, we have

$$\limsup_{p\to\infty} A_1(n,p) \leqslant P\Big[\sup_{1\leqslant i\leqslant 2^n} |H_{i2^{-n}T} - H_{(i-1)2^{-n}T}| \geqslant \frac{\delta}{5} \Big]$$

and the path continuity of the process H ensures that the right-hand side tends to 0 as $n \to \infty$.

To bound the terms A_2 and A_3, we introduce the stopping times $\tau_k^{(p)}$, $k \geqslant 0$ defined by induction as follows:

$$\tau_0^{(p)} = 0$$
$$\tau_{k+1}^{(p)} = \inf\{t \geqslant \tau_k^{(p)} : H_t^{(p)} > \inf_{\tau_k^{(p)} \leqslant r \leqslant t} H_r^{(p)} + \frac{\delta}{5}\}.$$

Let $i \in \{1, \ldots, 2^n\}$ be such that

$$\sup_{t \in [(i-1)2^{-n}T, i2^{-n}T]} H_t^{(p)} > H_{(i-1)2^{-n}T}^{(p)} + \frac{4\delta}{5}. \tag{73}$$

Then it is clear that the interval $[(i-1)2^{-n}T, i2^{-n}T]$ must contain at least one of the random times $\tau_k^{(p)}$, $k \geqslant 0$. Let $\tau_j^{(p)}$ be the first such time. By construction we have

$$\sup_{t \in [(i-1)2^{-n}T, \tau_j^{(p)})} H_t^{(p)} \leqslant H_{(i-1)2^{-n}T}^{(p)} + \frac{\delta}{5},$$

and since the positive jumps of $H^{(p)}$ are of size γ_p^{-1}, we get also

$$H_{\tau_j^{(p)}}^{(p)} \leqslant H_{(i-1)2^{-n}T}^{(p)} + \frac{\delta}{5} + \gamma_p^{-1} < H_{(i-1)2^{-n}T}^{(p)} + \frac{2\delta}{5}$$

provided that $\gamma_p > 5/\delta$. From (73), we have then

$$\sup_{t \in [\tau_j^{(p)}, i2^{-n}T]} H_t^{(p)} > H_{\tau_j^{(p)}}^{(p)} + \frac{\delta}{5},$$

which implies that $\tau_{j+1}^{(p)} \leqslant i2^{-n}T$. Summarizing, we get for p large enough so that $\gamma_p > 5/\delta$

$$A_2(n,p) \leqslant P\Big[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < 2^{-n}T \text{ for some } k \geqslant 0\Big]. \tag{74}$$

A similar argument gives exactly the same bound for the quantity $A_3(n,p)$.

The following lemma is directly inspired from [**14**] p.134-135.

Lemma 2.3.3. — *For every $x > 0$ and $p \geqslant 1$, set*

$$G_p(x) = P\Big[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x \text{ for some } k \geqslant 0\Big]$$

and

$$F_p(x) = \sup_{k \geqslant 0} P\Big[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x\Big].$$

Then, for every integer $L \geqslant 1$,

$$G_p(x) \leqslant L\, F_p(x) + L\, e^T \int_0^\infty dy\, e^{-Ly}\, F_p(y).$$

Proof. — For every integer $L \geqslant 1$, we have

$$\begin{aligned} G_p(x) &\leqslant \sum_{k=0}^{L-1} P[\tau_k^{(p)} < T \text{ and } \tau_{k+1}^{(p)} - \tau_k^{(p)} < x] + P[\tau_L^{(p)} < T] \\ &\leqslant L F_p(x) + e^T E\Big[1_{\{\tau_L^{(p)} < T\}} \exp\Big(-\sum_{k=0}^{L-1} (\tau_{k+1}^{(p)} - \tau_k^{(p)})\Big)\Big] \\ &\leqslant L F_p(x) + e^T \prod_{k=0}^{L-1} E\Big[1_{\{\tau_L^{(p)} < T\}} \exp(-L(\tau_{k+1}^{(p)} - \tau_k^{(p)}))\Big]^{1/L}. \end{aligned}$$

Then observe that for every $k \in \{0, 1, \ldots, L-1\}$,

$$\begin{aligned} E\Big[1_{\{\tau_L^{(p)} < T\}} \exp(-L(\tau_{k+1}^{(p)} - \tau_k^{(p)}))\Big] &\leqslant E\Big[1_{\{\tau_k^{(p)} < T\}} \int_{\tau_{k+1}^{(p)} - \tau_k^{(p)}}^{\infty} dy\, L e^{-Ly}\Big] \\ &\leqslant \int_0^{\infty} dy\, L e^{-Ly} F_p(y). \end{aligned}$$

The desired result follows. □

Thanks to Lemma 2.3.3, the limiting behavior of the right-hand side of (74) will be reduced to that of the function $F_p(x)$. To handle $F_p(x)$, we use the next lemma.

Lemma 2.3.4. — *The random variables $\tau_{k+1}^{(p)} - \tau_k^{(p)}$ are independent and identically distributed. Under the assumptions of Theorem 2.3.1, we have*

$$\lim_{x \downarrow 0} \Big(\limsup_{p \to \infty} P[\tau_1^{(p)} \leqslant x] \Big) = 0.$$

We need a simple lemma.

Lemma 2.3.5. — *Let V be a random walk on $\mathbb{Z}$. For every $n \geqslant 0$, set*

$$H_n^o = \mathrm{Card}\{k \in \{0, 1, \ldots, n-1\} : V_k = \inf_{k \leqslant j \leqslant n} V_j\}. \tag{75}$$

Let τ be a stopping time of the filtration $(\mathcal{F}_n^o)$ generated by V. Then the process

$$\Big(H_{\tau+n}^o - \inf_{\tau \leqslant k \leqslant \tau+n} H_k^o, n \geqslant 0 \Big)$$

is independent of $\mathcal{F}_\tau^o$ and has the same distribution as $(H_n^o, n \geqslant 0)$.

Proof. — By considering the first time after τ where the random walk V attains its minimum over $[\tau, \tau + n]$, one easily gets

$$\inf_{\tau \leqslant k \leqslant \tau+n} H_k^o = \mathrm{Card}\{k \in \{0, 1, \ldots, \tau - 1\} : V_k = \inf_{k \leqslant j \leqslant \tau+n} V_j\}.$$

Hence,

$$\begin{aligned} H_{\tau+n}^o - \inf_{\tau \leqslant k \leqslant \tau+n} H_k^o &= \mathrm{Card}\{k \in \{\tau, \ldots, \tau + n - 1\} : V_k = \inf_{k \leqslant j \leqslant \tau+n} V_j\} \\ &= \mathrm{Card}\{k \in \{0, \ldots, n-1\} : V_k^\tau = \inf_{k \leqslant j \leqslant n} V_j^\tau\}, \end{aligned}$$

where V^τ denotes the shifted random walk $V_n^\tau = V_{\tau+n} - V_\tau$. Since V^τ is independent of $\mathcal{F}_\tau$ and has the same distribution as V, the desired result follows from the previous formula and (75). □

Proof of Lemma 2.3.4. — Fix $k \geqslant 1$ and set for every $t \geqslant 0$,

$$\widetilde{H}_t^{(p)} = H^{(p)}_{\tau_k^{(p)}+t} - \inf_{\tau_k^{(p)} \leqslant r \leqslant \tau_k^{(p)}+t} H_r^{(p)}.$$

As a consequence of Lemma 2.3.5, the process $(\widetilde{H}_t^{(p)}, t \geqslant 0)$ is independent of the past of $V^{(p)}$ up to the stopping time $\tau_k^{(p)}$ and has the same distribution as $(H_t^{(p)}, t \geqslant 0)$. Since by definition

$$\tau_{k+1}^{(p)} - \tau_k^{(p)} = \inf\{t \geqslant 0 : \widetilde{H}_t^{(p)} > \frac{\delta}{5}\}$$

the first assertion of the lemma follows.

Let us turn to the second assertion. To simplify notation, we write $\delta' = \delta/5$. For every $\eta > 0$, set

$$T_\eta^{(p)} = \inf\{t \geqslant 0 : V_t^{(p)} = -\frac{[p\eta]}{p}\}.$$

Then,

$$P[\tau_1^{(p)} \leqslant x] = P\Big[\sup_{s \leqslant x} H_s^{(p)} > \delta'\Big] \leqslant P\Big[\sup_{s \leqslant T_\eta^{(p)}} H_s^{(p)} > \delta'\Big] + P[T_\eta^{(p)} < x].$$

On one hand,

$$\limsup_{p\to\infty} P[T_\eta^{(p)} < x] \leqslant P[T_\eta \leqslant x],$$

and for any choice of $\eta > 0$, the right-hand side goes to zero as $x \downarrow 0$. On the other hand, the construction of the discrete height process shows that the quantity

$$\sup_{s \leqslant T_\eta^{(p)}} H_s^{(p)}$$

is distributed as $\gamma_p^{-1}(M_p - 1)$, where M_p is the extinction time of a Galton-Watson process with offspring distribution μ_p, started at $[p\eta]$. Hence,

$$P\Big[\sup_{s \leqslant T_\eta^{(p)}} H_s^{(p)} > \delta'\Big] = 1 - g^{(p)}_{[\delta'\gamma_p]+1}(0)^{[p\eta]},$$

and our assumption (71) implies that

$$\lim_{\eta\to 0}\Big(\limsup_{p\to\infty} P\Big[\sup_{s \leqslant T_\eta^{(p)}} H_s^{(p)} > \delta'\Big]\Big) = 0.$$

The second assertion of the lemma now follows. □

We can now complete the proof of Theorem 2.3.1. Set:

$$F(x) = \limsup_{p\to\infty} F_p(x)\ , \quad G(x) = \limsup_{p\to\infty} G_p(x).$$

Lemma 2.3.4 immediately shows that $F(x) \downarrow 0$ as $x \downarrow 0$. On the other hand, we get from Lemma 2.3.3 that for every integer $L \geqslant 1$,

$$G(x) \leqslant L\, F(x) + L\, e^T \int_0^\infty dy\, e^{-Ly}\, F(y).$$

It follows that we have also $G(x) \downarrow 0$ as $x \downarrow 0$. By (74), this gives

$$\lim_{n\to\infty} \Big(\limsup_{p\to\infty} A_2(n,p) \Big) = 0,$$

and the same property holds for $A_3(n,p)$. This completes the proof of (ii) and of Theorem 2.3.1. □

Proof of Theorem 2.3.2. — We now assume that $\nu_p = \nu$ for every p and so $g_n^{(p)} = g_n$. We first observe that the process X must be stable. This is not immediate, since the convergence (45) a priori implies only that ν belongs to the domain of partial attraction of the law of X_1, which is not enough to conclude that ν is stable. However, the conditions (C1) – (C3), which are equivalent to (45), immediately show that the sequence γ_p/γ_{p+1} converges to 1 as $p \to \infty$. Then Theorem 2.3 in [**37**] implies that ν belongs to the domain of attraction of the law of X_1, and by classical results the law of X_1 must be stable with index $\alpha \in (0,2]$. We can exclude $\alpha \in (0,1]$ thanks to our assumptions (H2) and (H3) (the latter is only needed to exclude the trivial case $\psi(\lambda) = c\lambda$). Thus $\alpha \in (1,2]$ and $\psi(\lambda) = c\,\lambda^\alpha$ for some $c > 0$. As a consequence of (39), we have $E[e^{-\lambda Y_\delta}] = \exp -(\lambda^{-\overline{\alpha}} + c\overline{\alpha}\delta)^{-1/\overline{\alpha}}$, where $\overline{\alpha} = \alpha - 1$. In particular, $P[Y_\delta = 0] = \exp -(c\overline{\alpha}\delta)^{-1/\overline{\alpha}} > 0$.

Let $g = g_1$ be the generating function of μ. We have $g'(1) = \sum k\,\mu(k) = 1$, because otherwise this would contradict (45). From Theorem 2 in [**16**], p.577, the function

$$\sum_{k\geqslant x} \mu(k)$$

must be regularly varying as $x \to \infty$, with exponent $-\alpha$. Then note that

$$g(e^{-\lambda}) - 1 + \lambda = \sum_{k=0}^\infty \mu(k)\,(e^{-\lambda k} - 1 + \lambda k) = \lambda \int_0^\infty dx(1 - e^{-\lambda x}) \sum_{k\geqslant x} \mu(k).$$

An elementary argument shows that $g(e^{-\lambda}) - 1 + \lambda$ is also regularly varying as $\lambda \to 0$ with exponent α. Put differently,

$$g(r) = r + (1-r)^\alpha L(1-r)\ , \quad 0 \leqslant r < 1\ ,$$

where the function $L(x)$ is slowly varying as $x \to 0$. This is exactly what we need to apply a result of Slack [**47**].

Let $Z_1^{(p)}$ be a random variable distributed as $(1 - g_{[\delta\gamma_p]}(0))$ times the value at time $[\delta\gamma_p]$ of a Galton-Watson process with offspring distribution μ started with one

individual at time 0 and conditioned to be non-extinct at time $[\delta\gamma_p]$. Theorem 1 of [**47**] implies that

$$Z_1^{(p)} \xrightarrow[p\to\infty]{(\mathrm{d})} U$$

where $U > 0$ a.s. In particular, we can choose positive constants c_0 and c_1 so that $P[Z_1^{(p)} > c_0] > c_1$ for all p sufficiently large. On the other hand, we have

$$\frac{1}{p} Y^p_{[\delta\gamma_p]} \overset{(\mathrm{d})}{=} \frac{1}{p(1 - g_{[\delta\gamma_p]}(0))} \left(Z_1^{(p)} + \cdots + Z_{M_p}^{(p)} \right)$$

where $Z_1^{(p)}, Z_2^{(p)}, \ldots$ are i.i.d., and M_p is independent of the sequence $(Z_j^{(p)})$ and has a binomial $B(p, 1 - g_{[\delta\gamma_p]}(0))$ distribution.

It is now easy to obtain the condition (71). Fix $\delta > 0$. Clearly it suffices to verify that the sequence $p(1-g_{[\delta\gamma_p]}(0))$ is bounded. If not the case, we can choose a sequence (p_k) such that $p_k(1 - g_{[\delta\gamma_{p_k}]}(0))$ converges to ∞. From the previous representation for the law of $\frac{1}{p} Y^p_{[\delta\gamma_p]}$, it then follows that

$$P\Big[\frac{1}{p_k} Y^{p_k}_{[\delta\gamma_{p_k}]} > c_0 c_1 \Big] \underset{k\to\infty}{\longrightarrow} 1.$$

From (44), we get that $P[Y_\delta \geqslant c_0 c_1] = 1$, which gives a contradiction since $P[Y_\delta = 0] > 0$. This completes the proof of (71).

Finally, since (70) holds, we can apply Theorem 2.3.1. □

2.4. Convergence of contour processes

In this section, we show that the limit theorems obtained in the previous section for rescaled discrete height processes can be formulated as well in terms of the contour processes of the Galton-Watson trees. The proof relies on simple connections between the height process and the contour process of a sequence of Galton-Watson trees.

To begin with, we consider a (subcritical or critical) offspring distribution μ, and a sequence of independent μ-Galton-Watson trees. Let $(H_n, n \geqslant 0)$ and $(C_t, t \geqslant 0)$ be respectively the height process and the contour process associated with this sequence of trees (see Section 0.2). We also set

$$K_n = 2n - H_n.$$

Note that the sequence K_n is strictly increasing and $K_n \geqslant n$.

Recall that the value at time n of the height process corresponds to the generation of the individual visited at time n, assuming that individuals are visited in lexicographical order one tree after another. It is easily checked by induction on n that $[K_n, K_{n+1}]$ is exactly the time interval during which the contour process goes from the individual n to the individual $n+1$. From this observation, we get

$$\sup_{t\in[K_n,K_{n+1}]} |C_t - H_n| \leqslant |H_{n+1} - H_n| + 1.$$

A more precise argument for this bound follows from the explicit formula for C_t in terms of the height process: For $t \in [K_n, K_{n+1}]$,

$$C_t = (H_n - (t - K_n))^+ \qquad \text{if } t \in [K_n, K_{n+1} - 1],$$
$$C_t = (H_{n+1} - (K_{n+1} - t))^+ \qquad \text{if } t \in [K_{n+1} - 1, K_{n+1}].$$

These formulas are easily checked by induction on n.

Define a random function $f : \mathbb{R}_+ \longrightarrow \mathbb{Z}_+$ by setting $f(t) = n$ iff $t \in [K_n, K_{n+1})$. From the previous bound, we get for every integer $m \geqslant 1$,

$$\sup_{t\in[0,m]} |C_t - H_{f(t)}| \leqslant \sup_{t\in[0,K_m]} |C_t - H_{f(t)}| \leqslant 1 + \sup_{n\leqslant m} |H_{n+1} - H_n|. \tag{76}$$

Similarly, it follows from the definition of K_n that

$$\sup_{t\in[0,m]} |f(t) - \frac{t}{2}| \leqslant \sup_{t\in[0,K_m]} |f(t) - \frac{t}{2}| \leqslant \frac{1}{2} \sup_{n\leqslant m} H_n + 1. \tag{77}$$

We now come back to the setting of the previous sections, considering for every $p \geqslant 1$ a sequence of independent Galton-Watson trees with offspring distribution μ_p. For every $p \geqslant 1$, we denote by $(C_t^p, t \geqslant 0)$ the corresponding contour process.

Theorem 2.4.1. — *Suppose that the convergences* (45) *and* (72) *hold. Then,*

$$\left(\gamma_p^{-1} C^p_{p\gamma_p t}\,,\ t \geqslant 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (H_{t/2}, t \geqslant 0). \tag{78}$$

In particular, (78) *holds under the assumptions of Theorem 2.3.1 or those of Theorem 2.3.2.*

Proof. — For every $p \geqslant 1$, write f_p for the analogue of the function f introduced above. Also set $\varphi_p(t) = (p\gamma_p)^{-1} f_p(p\gamma_p t)$. By (76), we have for every $m \geqslant 1$,

$$\sup_{t\leqslant m} \left| \frac{1}{\gamma_p} C^p_{p\gamma_p t} - \frac{1}{\gamma_p} H^p_{p\gamma_p \varphi_p(t)} \right| \leqslant \frac{1}{\gamma_p} + \frac{1}{\gamma_p} \sup_{n\leqslant mp\gamma_p} |H^p_{n+1} - H^p_n| \underset{p\to\infty}{\longrightarrow} 0 \tag{79}$$

in probability, by (72).

On the other hand, we get from (77)

$$\sup_{t\leqslant m} |\varphi_p(t) - \frac{t}{2}| \leqslant \frac{1}{2p\gamma_p} \sup_{k\leqslant mp\gamma_p} H^p_k + \frac{1}{p\gamma_p} \underset{p\to\infty}{\longrightarrow} 0 \tag{80}$$

in probability, by (72).

The statement of the theorem now follows from (72), (79) and (80). □

2.5. A joint convergence and an application to conditioned trees

The convergences in distribution (72) and (78) hold jointly with (44) and (45). This fact is useful in applications and we state it here as a corollary.

As previously, we consider for every p a sequence of independent μ_p-Galton-Watson trees and we denote by $(H_n^p, n \geqslant 0)$ the associated height process and by $(C_t^p, t \geqslant 0)$ the associated contour process. The random walk V^p with jump distribution $\nu_p(k) =$

$\mu_p(k+1)$ is related to H^p via formula (47). Finally, for every integer $k \geqslant 0$, we denote by Y_k^p the number of individuals at generation k in the first p trees of the sequence, so that, in agreement with the previous notation, $(Y_n^p, n \geqslant 0)$ is a Galton-Watson process with offspring distribution μ_p started at $Y_0^p = p$.

Recall that $(L_t^a, a \geqslant 0, t \geqslant 0)$ denote the local times of the (continuous-time) height process associated with the Lévy process X. From Theorem 1.4.1, we know that $(L_{T_1}^a, a \geqslant 0)$ is a ψ-CSBP and thus has a càdlàg modification.

Corollary 2.5.1. — *Suppose that the assumptions of Theorem 2.3.1 are satisfied. Then,*

$$\left(p^{-1}V^p_{[p\gamma_p t]}, \gamma_p^{-1}H^p_{[p\gamma_p t]}, \gamma_p^{-1}C^p_{2p\gamma_p t}\,;\, t \geqslant 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (X_t, H_t, H_t; t \geqslant 0)$$

in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$. *We have also*

$$\left(p^{-1}Y^p_{[\gamma_p a]}\,,\, a \geqslant 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (L^a_{T_1}, a \geqslant 0)$$

in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. *Furthermore, these two convergences hold jointly, in the sense that, for any bounded continuous function* F *on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$,

$$\lim_{p\to\infty} E\Big[F\Big((p^{-1}V^p_{[p\gamma_p t]}, \gamma_p^{-1}H^p_{[p\gamma_p t]}, \gamma_p^{-1}C^p_{2p\gamma_p t})_{t\geqslant 0}, (p^{-1}Y^p_{[\gamma_p a]})_{a\geqslant 0}\Big)\Big] = E[F((X_t, H_t, H_t)_{t\geqslant 0}, (L^a_{T_1})_{a\geqslant 0})].$$

Proof. — To simplify notation, write $V_t^{(p)} = p^{-1}V^p_{[p\gamma_p t]}$, $H_t^{(p)} = \gamma_p^{-1}H^p_{[p\gamma_p t]}$, $C_t^{(p)} = \gamma_p^{-1}C^p_{2p\gamma_p t}$ and $Y_a^{(p)} = p^{-1}Y^p_{[\gamma_p a]}$. By (45), (72) and (78), we know that each of the three sequences of the laws of the processes $V^{(p)}$, $H^{(p)}$, $C^{(p)}$ is tight, and furthermore $H^{(p)}$ and $C^{(p)}$ converge in distribution towards a continuous process. By a standard result (see e.g. Corollary II.3.33 in [**24**]), we get that the laws of the triples $(V^{(p)}, H^{(p)}, C^{(p)})$ are tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$. Let (X, H^*, H^{**}) be a weak limit point of this sequence of triples (with a slight abuse of notation, we may assume that the first component of the limiting triple is the underlying Lévy process X). By the Skorokhod representation theorem, we may assume that along a subsequence,

$$(V^{(p)}, H^{(p)}, C^{(p)}) \longrightarrow (X, H^*, H^{**})$$

a.s. in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$. However, the convergence (49) and a time-reversal argument imply that

$$\lim_{p\to\infty} H_t^{(p)} = \widehat{L}_t^{(t)} = H_t$$

in probability. This is enough to conclude that $H_t^* = H_t$. Similarly, the proof of Theorem 2.4.1 shows that

$$\lim_{p\to\infty} (C_t^{(p)} - H_t^{(p)}) = 0$$

in probability. This yields $H_t^{**} = H_t^* = H_t$ and we see that the limiting triple is equal to (X, H, H) and does not depend on the choice of the subsequence. The first convergence of the corollary now follows.

By (44), we know that

$$(Y_a^{(p)}, a \geqslant 0) \xrightarrow[p\to\infty]{(\mathrm{d})} (Y_a, a \geqslant 0)$$

where Y is a ψ-CSBP started at 1. Since we also know that $(L^a_{T_1}, a \geqslant 0)$ is a ψ-CSBP started at 1, the second convergence in distribution is immediate, and the point is to verify that this convergence holds jointly with the first one. To this end, note that the laws of the pairs $((V^{(p)}, H^{(p)}, C^{(p)}), Y^{(p)})$ are tight in the space of probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$. By extracting a subsequence and using the Skorokhod representation theorem, we may assume that

$$\Big((V^{(p)}, H^{(p)}, C^{(p)}), Y^{(p)}\Big) \underset{p\to\infty}{\longrightarrow} \Big((X, H, H), Z\Big),$$

a.s. in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$. The proof will be finished if we can verify that $Z_a = L^a_{T_1}$, the local time of H at level a and time T_1. To this end, let g be a Lipschitz continuous function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with compact support. The preceding convergence implies

$$\lim_{p\to\infty} \int_0^\infty g(a) Y_a^{(p)}\, da = \int_0^\infty g(a) Z_a\, da\ , \qquad \text{a.s.} \tag{81}$$

On the other hand, let T_p^p be the hitting time of $-p$ by V^p. The convergence of $V^{(p)}$ towards X easily implies

$$\lim_{p\to\infty} \frac{1}{p\gamma_p} T_p^p = \inf\{t \geqslant 0 : X_t = -1\} = T_1\ , \quad \text{a.s.} \tag{82}$$

Then, from the definition of the height process of a sequence of trees, we have

$$\begin{aligned}
\int_0^\infty g(a) Y_a^{(p)}\, da &= \int_0^\infty g(a)\, \frac{1}{p} Y^p_{[\gamma_p a]}\, da \\
&= \frac{1}{p} \sum_{k=0}^{\infty} \int_{\gamma_p^{-1}k}^{\gamma_p^{-1}(k+1)} g(a) \Big(\sum_{j=0}^{T_p^p - 1} 1_{\{H_j^p = k\}} \Big) da \\
&= \frac{1}{p} \sum_{j=0}^{T_p^p - 1} \int_{\gamma_p^{-1} H_j^p}^{\gamma_p^{-1}(H_j^p + 1)} g(a)\, da \\
&= \frac{1}{p\gamma_p} \sum_{j=0}^{T_p^p - 1} g(\gamma_p^{-1} H_j^p) + O(\frac{1}{p\gamma_p^2} T_p^p) \\
&= \int_0^{(p\gamma_p)^{-1} T_p^p} g(\gamma_p^{-1} H^p_{[p\gamma_p s]})\, ds + O(\frac{1}{p\gamma_p^2} T_p^p)
\end{aligned}$$

and in view of (82) this converges to

$$\int_0^{T_1} g(H_s)\, ds = \int_0^\infty g(a)\, L^a_{T_1}\, da.$$

Comparing with (81), we conclude that

$$\int_0^\infty g(a) Z_a \, da = \int_0^\infty g(a) \, L^a_{T_1} \, da.$$

This implies that $Z_a = L^a_{T_1}$ and completes the proof. □

As an application, we now discuss conditioned trees. Fix $T > 0$ and on some probability space, consider a μ_p-Galton-Watson tree conditioned on non-extinction at generation $[\gamma_p T]$, which is denoted by $\widetilde{\mathcal{T}}^p$. Let $\widetilde{H}^p = (\widetilde{H}^p_n, n \geqslant 0)$ be the associated height process, with the convention that $\widetilde{H}^p_n = 0$ for $n \geqslant \mathrm{Card}(\widetilde{\mathcal{T}}^p)$.

Proposition 2.5.2. — *Under the assumptions of Theorem 2.3.1, we have*

$$\left(\gamma_p^{-1} \widetilde{H}^p_{[p\gamma_p t]}, t \geqslant 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (\widetilde{H}_t, t \geqslant 0),$$

where the limiting process $\widetilde{H}$ is distributed as H under $N(\cdot \mid \sup H_s \geqslant T)$.

Remark. — We could have stated a similar result for the contour process instead of the discrete height process.

Proof. — Write $\widetilde{H}^{(p)}_s = \gamma_p^{-1} \widetilde{H}^p_{[p\gamma_p s]}$ to simplify notation. Also let $H^{(p)}_s = \gamma_p^{-1} H^p_{[p\gamma_p s]}$ be as above the rescaled height process for a sequence of independent μ_p-Galton-Watson trees. Set

$$\begin{aligned}
R^{(p)}_T &= \inf\{s \geqslant 0 : H^{(p)}_s = \frac{[\gamma_p T]}{\gamma_p}\}, \\
G^{(p)}_T &= \sup\{s \leqslant R^{(p)}_T : H^{(p)}_s = 0\}, \\
D^{(p)}_T &= \inf\{s \geqslant R^{(p)}_T : H^{(p)}_s = 0\}.
\end{aligned}$$

Then without loss of generality we may assume that

$$\widetilde{H}^{(p)}_s = H^{(p)}_{(G^{(p)}_T + s) \wedge D^{(p)}_T}, \qquad s \geqslant 0.$$

This is simply saying that the first tree with height at least $[\gamma_p T]$ in a sequence of independent μ_p-Galton-Watson trees is a μ_p-Galton-Watson tree conditioned on non-extinction at generation $[\gamma_p T]$.

Set

$$\begin{aligned}
R_T &= \inf\{s \geqslant 0 : H_s = T\}, \\
G_T &= \sup\{s \leqslant R_T : H_s = 0\}, \\
D_T &= \inf\{s \geqslant R_T : H_s = 0\},
\end{aligned}$$

and note that we may take $\widetilde{H}_s = H_{(G_T + s) \wedge D_T}$, by excursion theory for $X - I$.

We now claim that the convergence in distribution of $\widetilde{H}^{(p)}$ towards $\widetilde{H}$ follows from the previous corollary, and more precisely from the joint convergence

$$(V^{(p)}, H^{(p)}) \xrightarrow[p\to\infty]{\text{(d)}} (X, H).$$

It is again convenient to use the Skorokhod representation theorem and to assume that the latter convergence holds a.s. We can then prove that $\widetilde{H}^{(p)}$ converges a.s. towards $\widetilde{H}$.

To this end we need a technical lemma about the height process. We state it in greater generality than needed here in view of other applications.

Lemma 2.5.3. — *Let $b > 0$. Then P a.s. or N a.e. b is not a local maximum nor a local minimum of the height process.*

Proof. — Let

$$D = \{b > 0 : P[\sup_{p \leqslant s \leqslant q} H_s = b] > 0 \text{ for some rationals } q > p \geqslant 0\}.$$

Clearly D is at most countable. However, from Proposition 1.3.1 and the relation between the height process and the exploration process, it immediately follows that if $b \in D$ then $b - a \in D$ for every $a \in [0, b)$. This is only possible if $D = \varnothing$. The case of local minima is treated in the same way. □

It follows from the lemma that we have also $R_T = \inf\{s \geqslant 0 : H_s > T\}$. Then the a.s. convergence of $H^{(p)}$ towards H easily implies that $R_T^{(p)}$ converges to R_T a.s., and that

$$\limsup_{p \to \infty} G_T^{(p)} \leqslant G_T \ , \quad \liminf_{p \to \infty} D_T^{(p)} \geqslant D_T \ .$$

To get reverse inequalities, we may argue as follows. Recall that the support of the random measure dI_s is exactly the set $\{s : H_s = 0\}$, so that for every fixed $s \geqslant 0$, we have $I_s > I_{R_T}$ a.s. on the set $\{s < G_T\}$. If $I_s^{(p)} = \inf\{V_r^{(p)}, r \leqslant s\}$, the a.s. convergence of $V^{(p)}$ to X implies that $I_s^{(p)}$ converges to I_s uniformly on compact sets, a.s. It readily follows that a.s. on the set $\{s < G_T\}$ we have $I_s^{(p)} > I_{R_T^{(p)}}^{(p)}$ for all p sufficiently large. Hence a.s. for p large, we have $s < G_T^{(p)}$ on the set $\{s < G_T\}$. We conclude that $G_T^{(p)} \to G_T$ a.s., and a similar argument gives $D_T^{(p)} \to D_T$. From the preceding formulas for $\widetilde{H}^{(p)}$ and $\widetilde{H}$, it follows that $\widetilde{H}^{(p)} \to \widetilde{H}$ a.s. This completes the proof of the proposition. □

Remark. — The methodology of proof of Proposition 2.5.2 could be applied to other conditioned limit theorems. For instance, we could consider the rescaled height (or contour) process of the μ_p-Galton-Watson tree conditioned to have at least $p\gamma_p$ vertices and derive a convergence towards the excursion of the height process H conditioned to have length greater than 1. We will leave such extensions to the reader. We point out here that it is much harder to handle degenerate conditionings. To give an important example, consider the case where $\mu_p = \mu$ for every p. It is natural to ask for a limit theorem for the (rescaled) height or contour process of a μ-Galton-Watson tree conditioned to have a large fixed number of vertices. The previous results strongly suggest that the limiting process should be a normalized (i.e. conditioned to

have length equal to 1) excursion of the height process H. This is indeed true under suitable assumptions: When μ is critical with finite variance, this was proved by Aldous [**3**] in the case of the contour process and the limit is a normalized Brownian excursion as expected. Aldous' result has been extended by Duquesne [**10**] to the case when μ is in the domain of attraction of a stable law of index $\gamma \in (1,2]$. In this more general setting, the limit is the normalized excursion of the stable height process, which is discussed in Section 3.5 below.

2.6. The convergence of reduced trees

Consider a μ-Galton-Watson tree, which describes the genealogy of a Galton-Watson process with offspring distribution μ starting with one individual at time 0. For every integer $n \geqslant 1$, denote by $P^{(n)}$ the conditional probability knowing that the process is not extinct at time n, or equivalently the height of the tree is at least n. Under $P^{(n)}$, we can consider the reduced tree that consists only of those individuals in the generations up to time n that have descendants at generation n. The results of the previous sections can be used to investigate the limiting behavior of these reduced trees when n tends to ∞, even in the more general setting where the offspring distribution depends on n.

Here, we will concentrate on the population of the reduced tree at every generation. For every $k \in \{0,1,\dots,n\}$, we denote by Z_k^n the number of individuals in the tree at generation k which have descendants at generation n. Obviously, $k \to Z_k^n$ is nondecreasing, $Z_0^n = 1$ and Z_n^n is equal to the number of individuals in the original tree at generation n. If g denotes the generating function of μ and g_n, $n \geqslant 0$ are the iterates of g, it is easy to verify that $(Z_k^n, 0 \leqslant k \leqslant n)$ is a time-inhomogeneous Markov chain whose transition kernels are characterized by:

$$E^{(n)}[r^{Z_{k+1}^n} \mid Z_k^n] = \Big(\frac{g(r(1-g_{n-k-1}(0)) + g_{n-k-1}(0)) - g_{n-k}(0)}{1 - g_{n-k}(0)}\Big)^{Z_k^n}, \quad 0 \leqslant k < n\,.$$

The process $(Z_k^n, 0 \leqslant k \leqslant n)$ (under the probability measure $P^{(n)}$) will be called the reduced process of the μ-Galton-Watson tree at generation n. It is easy to see that for every $k \in \{0,1,\dots,n-1\}$, Z_k^n can be written as a simple functional of the height process of the tree: Z_k^n counts the number of excursions of the height process above level k that hit level n.

Consider as in the previous sections a sequence $(\mu_p, p = 1,2,\dots)$ of (sub)critical offspring distributions, and for every integer $n \geqslant 1$ let $Z^{(p),n} = (Z_k^{(p),n}, 0 \leqslant k \leqslant n)$ be the reduced process of the μ_p-Galton-Watson tree at generation n. For every $T > 0$, we denote by $N_{(T)}$ the conditional probability $N(\cdot \mid \sup\{H_s, s \geqslant 0\} \geqslant T)$ (this makes sense provided that the condition (70) holds, *cf.* Corollary 1.4.2).

Theorem 2.6.1. — *Suppose that the assumptions of Theorem 2.3.1 hold and let $T > 0$. Then,*

$$\left(Z^{(p),[\gamma_p T]}_{[\gamma_p t]}, 0 \leqslant t < T\right) \xrightarrow[p\to\infty]{(fd)} (Z^T_t, 0 \leqslant t < T) \ ,$$

where the limiting process $(Z^T_t, 0 \leqslant t < T)$ is defined under $N_{(T)}$ as follows: For every $t \in [0,T)$, Z^T_t is the number of excursions of H above level t that hit level T.

A more explicit description of the limiting process and of the associated tree will be given in the next section.

Proof. — We use the notation of the proof of Proposition 2.5.2. In particular, the height process of the μ_p-Galton-Watson tree conditioned on non-extinction at generation $[\gamma_p T]$ is $(\widetilde{H}^p_k, k \geqslant 0)$ and the associated rescaled process is $\widetilde{H}^{(p)}_s = \gamma_p^{-1} \widetilde{H}^p_{[p\gamma_p s]}$. We may and will assume that $\widetilde{H}^{(p)}_s$ is given by the formula

$$\widetilde{H}^{(p)}_s = H^{(p)}_{(G^{(p)}_T + s) \wedge D^{(p)}_T}$$

and that $(\widetilde{H}^{(p)}_s, s \geqslant 0)$ converges a.s. in the sense of the Skorokhod topology, towards the process $\widetilde{H}_s = H_{(G_T+s)\wedge D_T}$ whose law is the distribution of H under $N_{(T)}$.

Now we observe that the reduced process $Z^{(p),[\gamma_p T]}_{[\gamma_p t]}$ can be expressed in terms of $\widetilde{H}^{(p)}$. More precisely, it is clear by construction that for every $k \in \{0, 1, \ldots, [\gamma_p T] - 1\}$, $Z^{(p),[\gamma_p T]}_k$ is the number of excursions of $\widetilde{H}^p$ above level k that hit level $[\gamma_p T]$. Equivalently, for every t such that $[\gamma_p t] < [\gamma_p T]$,

$$\widetilde{Z}^{(p)}_t := Z^{(p),[\gamma_p T]}_{[\gamma_p t]}$$

is the number of excursions of $\widetilde{H}^{(p)}$ above level $[\gamma_p t]/\gamma_p$ that hit level $[\gamma_p T]/\gamma_p$.

Let $t > 0$. Using the fact that t, resp. T, is a.s. not a local minimum, resp. maximum, of H (Lemma 2.5.3), it is easy to deduce from the convergence $\widetilde{H}^{(p)} \to \widetilde{H}$ that the number of excursions of $\widetilde{H}^{(p)}$ above level $[\gamma_p t]/\gamma_p$ that hit level $[\gamma_p T]/\gamma_p$ converges a.s. to the number of excursions of $\widetilde{H}$ above level t that hit level T. In other words, $\widetilde{Z}^{(p)}_t$ converges a.s. to Z^T_t. This completes the proof. □

2.7. The law of the limiting reduced tree

In this section, we will describe the law of the process $(Z^T_t, 0 \leqslant t < T)$ of the previous section, and more precisely the law of the underlying branching tree. We suppose that the Lévy process X satisfies (70) in addition to (H1) – (H3). The random variable Z^T_t (considered under the probability measure $N_{(T)}$) counts the number of excursions of H above level t that hit level T.

Before stating our result, we recall the notation of Section 1.4. For every $\lambda > 0$ and $t > 0$,

$$u_t(\lambda) = N(1 - \exp(-\lambda L^t_\sigma))$$

solves the integral equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\, ds = \lambda$$

and

$$v(t) = u_t(\infty) = N(L_\sigma^t > 0) = N\Big(\sup_{s \geqslant 0} H_s > t\Big)$$

is determined by

$$\int_{v(t)}^{\infty} \frac{dx}{\psi(x)} = t.$$

Note the composition property $u_t \circ u_s = u_{t+s}$, and in particular $u_t(v(r)) = v(t+r)$.

Theorem 2.7.1. — *Under $N_{(T)}$, the process $(Z_t^T, 0 \leqslant t < T)$ is a time-inhomogeneous Markov process whose law is characterized by the following identities: For every $\lambda > 0$,*

$$N_{(T)}[\exp -\lambda Z_t^T] = 1 - \frac{u_t((1-e^{-\lambda})v(T-t))}{v(T)}. \tag{83}$$

and if $0 \leqslant t < t' < T$,

$$N_{(T)}[\exp -\lambda Z_{t'}^T \mid Z_t^T] = (N_{(T-t)}[\exp -\lambda Z_{t'-t}^{T-t}])^{Z_t^T} \tag{84}$$

Alternatively, we can describe the law of the process $(Z_t^T, 0 \leqslant t < T)$ under $N_{(T)}$ by the following properties.

– $Z_r^T = 1$ if and only if $r \in [0, \gamma_T)$, where the law of γ_T is given by

$$N_{(T)}[\gamma_T > t] = \frac{\widetilde{\psi}(v(T))}{\widetilde{\psi}(v(T-t))}\ , \qquad 0 \leqslant t < T, \tag{85}$$

where $\widetilde{\psi}(x) = x^{-1}\psi(x)$.

– The conditional distribution of $Z_{\gamma_T}^T$ knowing γ_T is characterized by

$$N_{(T)}[r^{Z_{\gamma_T}^T} \mid \gamma_T = t] = r\, \frac{\psi'(U) - \gamma_\psi(U, (1-r)U)}{\psi'(U) - \gamma_\psi(U,0)}, \qquad 0 \leqslant r \leqslant 1 \tag{86}$$

where $U = v(T-t)$ and for every $a, b \geqslant 0$,

$$\gamma_\psi(a,b) = \begin{cases} (\psi(a) - \psi(b))/(a-b) & \text{if} \quad a \neq b, \\ \psi'(a) & \text{if} \quad a = b\,. \end{cases}$$

– Conditionally on $\gamma_T = t$ and $Z_{\gamma_T}^T = k$, the process $(Z_{t+r}^T, 0 \leqslant r < T-t)$ is distributed as the sum of k independent copies of the process $(Z_r^{T-t}, 0 \leqslant r < T-t)$ under $N_{(T-t)}$.

Proof. — One can give several approaches to Theorem 2.7.1. In particular, the time-inhomogeneous Markov property could be deduced from the analogous result for discrete reduced trees by using Theorem 2.6.1. We will prefer to give a direct approach relying on the properties of the height process.

Before stating a key lemma, we introduce some notation. We fix $t \in (0,T)$. Note that the definition of Z_t^T also makes sense under the conditional probability $N_{(t)}$. We

denote by $(e_i^t, i = 1, \dots, Z_t^T)$ the successive excursions of H above level t that hit level $T - t$, shifted in space and time so that each starts from 0 at time 0. Recall the notation L_s^a for the local times of the height process. We also write $L_{(i)}^t$ for the local time of H at level t at the beginning of excursion e_i^t.

Lemma 2.7.2. — *Under $N_{(t)}$, conditionally on the local time L_σ^t, the point measure*

$$\sum_{i=1}^{Z_t^T} \delta_{(L_{(i)}^t, e_i^t)}$$

is Poisson with intensity $1_{[0,L_\sigma^t]}(\ell) d\ell\, N(de \cap \{\sup H_s > T - t\})$. In particular, under $N_{(t)}$ or under $N_{(T)}$, conditionally on Z_t^T, the excursions $(e_i^t, i = 1, \dots, Z_t^T)$ are independent with distribution $N_{(T-t)}$.

Proof. — We rely on Proposition 1.3.1 and use the notation of Chapter 1. Under the probability measure P, denote by f_i^t, $i = 1, 2, \dots$ the successive excursions of H above level t that hit T, and let ℓ_i^t be the local time of H at level t at the beginning (or the end) of excursion f_i^t. Then the f_i^t's are also the successive excursions of the process $H_s^t = H(\rho_s^t)$ that hit level $T - t$, and the numbers ℓ_i^t are the corresponding local times (of H^t) at level 0. By Proposition 1.3.1 and excursion theory, the point measure

$$\sum_{i=1}^{\infty} \delta_{(\ell_i^t, f_i^t)}$$

is Poisson with intensity $d\ell\, N(df \cap \{\sup H_s > T - t\})$ and is independent of the σ-field $\mathcal{H}_t$.

On the other hand, let λ_1 be the local time of H at level t at the end of the first excursion of H away from 0 that hits level t. From the approximation of local time provided by Proposition 1.3.3, it is easy to see that λ_1 is $\mathcal{H}_t$-measurable. By excursion theory for $X - I$, the law under $N_{(t)}$ of the pair

$$\Big(L_\sigma^t, \sum_{i=1}^{Z_t^T} \delta_{(L_{(i)}^t, e_i^t)}\Big)$$

is the same as the law under P of

$$\Big(\lambda_1, \sum_{\{i:\ell_i^t \leqslant \lambda_1\}} \delta_{(\ell_i^t, f_i^t)}\Big).$$

The first assertion of the lemma now follows from the preceding considerations.

The second assertion stated under $N_{(t)}$ is an immediate consequence of the first one. The statement under $N_{(T)}$ follows since $N_{(T)} = N_{(t)}(\cdot \mid Z_t^T \geqslant 1)$. □

We return to the proof of Theorem 2.7.1. Note that (84) is an immediate consequence of the second assertion of the lemma. Let us prove (83). By the first assertion

of the lemma, Z_t^T is Poisson with intensity $v(T-t)L_\sigma^t$, conditionally on L_σ^t, under $N_{(t)}$. Hence,

$$\begin{aligned} N_{(t)}[e^{-\lambda Z_t^T}] &= N_{(t)}\Big[e^{-L_\sigma^t v(T-t)(1-e^{-\lambda})}\Big] \\ &= 1 - \frac{1}{v(t)} N\Big(1 - e^{-L_\sigma^t v(T-t)(1-e^{-\lambda})}\Big) \\ &= 1 - \frac{1}{v(t)} u_t((1-e^{-\lambda})v(T-t)). \end{aligned}$$

Then observe that

$$N_{(t)}[1-e^{-\lambda Z_t^T}] = \frac{1}{v(t)} N(1-e^{-\lambda Z_t^T}) = \frac{v(T)}{v(t)} N_{(T)}[1-e^{-\lambda Z_t^T}].$$

Formula (83) follows immediately.

It is clear that there exists a random variable γ_T such that $Z_t^T = 1$ iff $0 \leqslant t < \gamma_T$, $N_{(T)}$ a.s. (γ_T is the minimum of the height process between the first and the last hitting time of T). Let us prove (85). By (83), we have,

$$N_{(T)}[\gamma_T > t] = \lim_{\lambda\to\infty} e^\lambda N_{(T)}[e^{-\lambda Z_t^T}] = \lim_{\lambda\to\infty} e^\lambda \Big(1 - \frac{u_t((1-e^{-\lambda})v(T-t))}{v(T)}\Big).$$

Recalling that $u_t(v(T-t)) = v(T)$, we have as $\varepsilon \to 0$,

$$u_t((1-\varepsilon)v(T-t)) = v(T) - \varepsilon v(T-t)\frac{\partial u_t}{\partial \lambda}(v(T-t)) + o(\varepsilon),$$

and it follows that

$$N_{(T)}[\gamma_T > t] = \frac{v(T-t)}{v(T)} \frac{\partial u_t}{\partial \lambda}(v(T-t)).$$

Formula (85) follows from that identity and the fact that, for $\lambda > 0$,

$$\frac{\partial u_t}{\partial \lambda}(\lambda) = \frac{\psi(u_t(\lambda))}{\psi(\lambda)}. \tag{87}$$

To verify (87), differentiate the integral equation for $u_t(\lambda)$:

$$\frac{\partial u_t}{\partial \lambda}(\lambda) = 1 - \int_0^t \frac{\partial u_s}{\partial \lambda}(\lambda)\, \psi'(u_s(\lambda))\, ds$$

which implies

$$\frac{\partial u_t}{\partial \lambda}(\lambda) = \exp\Big(-\int_0^t \psi'(u_s(\lambda))ds\Big).$$

Then note that $\frac{\partial}{\partial t}\log\psi(u_t(\lambda)) = -\psi'(u_t(\lambda))$ and thus

$$\int_0^t \psi'(u_s(\lambda))ds = \log\psi(u_t(\lambda)) - \log\psi(\lambda).$$

This completes the proof of (87) and (85).

We now prove the last assertion of the theorem. Recall the notation introduced before Lemma 2.7.2. Clearly it suffices to prove that the following property holds:

(P) Under $N_{(T)}$, conditionally on $\gamma_T = t$ and $Z^T_{\gamma_T} = n$, the excursions $e_1^{\gamma_T}, \dots, e_n^{\gamma_T}$ are i.i.d. according to the distribution $N_{(T-t)}$.

We can deduce property (P) from Lemma 2.7.2 via an approximation procedure. Let us sketch the argument. For any $p \geqslant 2$ and any bounded continuous functional F on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}_+)^p$,

$$(88)\quad N_{(T)}[1_{\{Z^T_{\gamma_T}=p\}} F(\gamma_T, e_1^{\gamma_T}, \dots, e_p^{\gamma_T})] = \lim_{n\to\infty} \sum_{j=1}^{n-1} N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;(j-1)T/n<\gamma_T\leqslant jT/n\}} F(jT/n, e_1^{jT/n}, \dots, e_p^{jT/n})\Big].$$

Note that the event $\{\gamma_T \leqslant jT/n\}$ contains $\{Z^T_{jT/n} = p\}$. As a consequence of the second part of Lemma 2.7.2 (applied with $t = jT/n$) we have

$$N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;\gamma_T\leqslant jT/n\}} F(jT/n, e_1^{jT/n}, \dots, e_p^{jT/n})\Big] = N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;\gamma_T\leqslant jT/n\}} \times \int N_{(T-jT/n)}(df_1)\dots N_{(T-jT/n)}(df_p)\, F(jT/n, f_1, \dots, f_p)\Big].$$

We want to get a similar identity where the event $\{\gamma_T \leqslant jT/n\}$ is replaced by $\{\gamma_T \leqslant (j-1)T/n\} = \{Z^T_{(j-1)T/n} \geqslant 2\}$. A slightly more complicated argument (relying on two applications of Lemma 2.7.2, the first one with $t = (j-1)T/n$ and then with $t = T/n$) shows similarly that

$$N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;\gamma_T\leqslant (j-1)T/n\}} F(jT/n, e_1^{jT/n}, \dots, e_p^{jT/n})\Big] = N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;\gamma_T\leqslant (j-1)T/n\}} \times \int N_{(T-jT/n)}(df_1)\dots N_{(T-jT/n)}(df_p)\, F(jT/n, f_1, \dots, f_p)\Big].$$

By making the difference between the last two displays, we see that the sum in the right side of (88) exactly equals

$$\sum_{j=1}^{n-1} N_{(T)}\Big[1_{\{Z^T_{jT/n}=p;(j-1)T/n<\gamma_T\leqslant jT/n\}} \times \int N_{(T-jT/n)}(df_1)\dots N_{(T-jT/n)}(df_p)\, F(jT/n, f_1, \dots, f_p)\Big].$$

Using an easy continuity property of the mapping $r \to N_{(r)}$, we get from this and (88) that

$$N_{(T)}[1_{\{Z^T_{\gamma_T}=p\}} F(\gamma_T, e_1^{\gamma_T}, \dots, e_p^{\gamma_T})] = N_{(T)}\Big[1_{\{Z^T_{\gamma_T}=p\}} \int N_{(T-\gamma_T)}(df_1)\dots N_{(T-\gamma_T)}(df_p) F(\gamma_T, f_1, \dots, f_p)\Big],$$

which completes the proof of property (P) and of the last assertion of the theorem.

We finally verify (86). First observe from (85) that the density of the law of γ_T under $N_{(T)}$ is given by

$$h_T(t) = \widetilde{\psi}(v(T))\, h(T-t)$$

where

$$h(t) = \frac{v(t)\psi'(v(t))}{\psi(v(t))} - 1.$$

On the other hand, fix $\delta \in (0,T)$, and note that $\{\gamma_T > \delta\} = \{Z_\delta^T = 1\}$. By the last assertion of Lemma 2.7.2 we have for any nonnegative function f,

$$N_{(T)}[f(\gamma_T, Z^T_{\gamma_T})\, 1_{\{\gamma_T > \delta\}} \mid \gamma_T > \delta] = N_{(T-\delta)}[f(\gamma_{T-\delta} + \delta, Z^{T-\delta}_{\gamma_{T-\delta}})].$$

Hence, if $(\theta_t^T(k), k = 2, 3, \dots)$, $0 < t < T$ denotes a regular version of the conditional law of $Z^T_{\gamma_T}$ knowing that $\gamma_T = t$, we have

$$\begin{aligned}\int_\delta^T dt\, h(T-t) \sum_{k=2}^\infty \theta_t^T(k) f(t,k) &= \int_0^{T-\delta} dt\, h(T-\delta-t) \sum_{k=2}^\infty \theta_t^{T-\delta}(k) f(t+\delta, k) \\ &= \int_\delta^T dt\, h(T-t) \sum_{k=2}^\infty \theta_{t-\delta}^{T-\delta}(k) f(t,k).\end{aligned}$$

This shows that we must have $\theta_t^T = \theta_{t-\delta}^{T-\delta}$ for a.a. $t \in (\delta, T)$. By simple arguments, we can choose the regular versions $\theta_t^T(k)$ in such a way that $\theta_t^T(k) = \theta_{T-t}(k)$ for every $k \geqslant 2$, $T > 0$ and $t \in (0,T)$.

We can then compute $N_{(T)}[e^{-\lambda L_\sigma^T}]$ in two different ways. First,

$$N_{(T)}[e^{-\lambda L_\sigma^T}] = 1 - \frac{N(1 - e^{-\lambda L_\sigma^T})}{v(T)} = 1 - \frac{u_T(\lambda)}{v(T)}.$$

Then, using property (P) once again,

$$\begin{aligned}N_{(T)}[e^{-\lambda L_\sigma^T}] &= N_{(T)}\Big[\Big(N_{(T-t)}[e^{-\lambda L_\sigma^{T-t}}]_{t=\gamma_T}\Big)^{Z^T_{\gamma_T}}\Big] \\ &= \int_0^T dt\, h_T(t) \sum_{k=2}^\infty \theta_{T-t}(k) \Big(1 - \frac{u_{T-t}(\lambda)}{v(T-t)}\Big)^k.\end{aligned}$$

By comparing with the previous display and using the formula for $h_T(t)$, we get

$$\int_0^T dt \Big(\frac{\psi'(v(t))v(t)}{\psi(v(t))} - 1\Big) \sum_{k=2}^\infty \theta_t(k) \Big(1 - \frac{u_t(\lambda)}{v(t)}\Big)^k = \frac{v(T) - u_T(\lambda)}{\psi(v(T))}.$$

We can now differentiate with respect to T (for a proper justification we should argue that the mapping $t \to \theta_t$ is continuous, but we omit details). It follows that

$$\Big(\frac{\psi'(v(T))v(T)}{\psi(v(T))} - 1\Big) \sum_{k=2}^{\infty} \theta_T(k) \Big(1 - \frac{u_T(\lambda)}{v(T)}\Big)^k = -1 + \frac{\psi'(v(T))v(T)}{\psi(v(T))} + \frac{\psi(u_T(\lambda)) - u_T(\lambda)\psi'(v(T))}{\psi(v(T))}.$$

Hence,

$$\sum_{k=2}^{\infty} \theta_T(k) \Big(1 - \frac{u_T(\lambda)}{v(T)}\Big)^k = 1 - \frac{\psi(u_T(\lambda)) - u_T(\lambda)\psi'(v(T))}{\psi(v(T)) - v(T)\psi'(v(T))}.$$

If we substitute $r = 1 - \frac{u_T(\lambda)}{v(T)}$ in this last identity we get

$$\sum_{k=2}^{\infty} \theta_T(k)\, r^k = 1 - \frac{\psi((1-r)v(T)) - (1-r)v(T)\psi'(v(T))}{\psi(v(T)) - v(T)\psi(v(T)}.$$

Formula (86) follows after straightforward transformations of the last expression.

The proof of Theorem 2.7.1 is now complete. Observe that the (time-inhomogeneous) Markov property of the process $(Z_t^T, 0 \leqslant t < T)$ is a consequence of the description provided in the second part of the theorem, and in particular of the special form of the law of γ_T and the fact that the law of $Z_{\gamma_T}^T$ under $N_{(T)}[\cdot \mid \gamma_T > \delta]$ coincides with the law of $Z_{\gamma_T - \delta}^{T-\delta}$ under $N_{(T-\delta)}$. □

Let us discuss special cases of the theorem. When $\psi(u) = cu^\alpha$, with $c > 0$ and $1 < \alpha \leqslant 2$, we have $v(t) = (c(\alpha-1)t)^{-1/(\alpha-1)}$, and formula (85) shows that the law of γ_T is uniform over $[0, T]$. This is the only case where this property holds: If we assume that γ_T is uniform over $[0, T]$, (85) implies that $\widetilde{\psi}(v(t)) = C/t$ for some $C > 0$. By differentiating $\log v(t)$, we then get that $v(t) = C' t^{-C}$ and it follows that ψ is of the desired form.

Also in the stable case $\psi(u) = cu^\alpha$, formula (86) implies that $Z_{\gamma_T}^T$ is independent of γ_T, and that its distribution is characterized by

$$N_{(T)}[r^{Z_{\gamma_T}^T}] = \frac{(1-r)^\alpha - 1 + \alpha r}{\alpha - 1}.$$

Of course when $\alpha = 2$, we recover the well known fact that $Z_{\gamma_T}^T = 2$. When $\alpha \in (1, 2)$, we get

$$N_{(T)}[Z_{\gamma_T}^T = k] = \frac{\alpha(2-\alpha)(3-\alpha)\cdots(k-1-\alpha)}{k!}, \quad k \geqslant 2.$$

To conclude let us mention that limiting reduced trees have been studied extensively in the literature. In the finite variance case, the uniform distribution for γ_T appears in Zubkov [**50**], and the full structure of the reduced tree is derived by Fleischmann and Siegmund-Schultze [**17**]. Analogous results in the stable case (and in the more general setting of multitype branching processes) can be found in Vatutin [**48**] and Yakymiv [**49**].

CHAPTER 3

MARGINALS OF CONTINUOUS TREES

3.1. Duality properties of the exploration process

In this section, we study certain duality properties of the process ρ. In view of forthcoming applications, the main result is the time-reversal property stated in Corollary 3.1.6 below. However the intermediate results needed to derive this property are of independent interest.

We work in the general setting of Chapter 1. In particular, the Lévy process X satisfies assumptions (H1) – (H3), and starts at 0 under the probability measure P. Since the subordinator $S_{L^{-1}(t)}$ has drift β (Lemma 1.1.2), it readily follows from formula (17) that the continuous part of ρ_t is $\beta 1_{[0,H_t]}(r)dr$. We can thus rewrite Definition 1.2.2 in an equivalent way as follows:

$$\rho_t(dr) = \beta 1_{[0,H_t]}(r)\, dr + \sum_{\substack{0<s\leqslant t \\ X_{s-}<I_t^s}} (I_t^s - X_{s-})\, \delta_{H_s}(dr). \tag{89}$$

We then introduce another measure-valued process $(\eta_t, t \geqslant 0)$ by setting

$$\eta_t(dr) = \beta 1_{[0,H_t]}(r)\, dr + \sum_{\substack{0<s\leqslant t \\ X_{s-}<I_t^s}} (X_s - I_t^s)\, \delta_{H_s}(dr). \tag{90}$$

In the same way as ρ_t, the measure η_t is supported on $[0, H_t]$. We will see below that η_t is a.s. a *finite* measure, a fact that is not obvious from the previous formula. In the queueing system interpretation of [**32**], the measure ρ_t accounts for the remaining service times for all customers present in the queue at time t. In this interpretation, η_t describes the services already accomplished for these customers.

We will see that in some sense, the process $(\eta_t, t \geqslant 0)$ is the dual of $(\rho_t, t \geqslant 0)$. It turns out that the study of $(\eta_t, t \geqslant 0)$ is significantly more difficult than that of $(\rho_t, t \geqslant 0)$. We start with a basic lemma.

Lemma 3.1.1. — *For each fixed value of $t > 0$, we have $\langle \eta_t, 1\rangle < \infty$, P a.s. or N a.e. The process $(\eta_t, t \geqslant 0)$, which takes values in $M_f(\mathbb{R}_+)$, is right-continuous in probability under P. Similarly, $(\eta_t, t > 0)$ is right-continuous in measure under N.*

Proof. — Let us first prove that $\langle \eta_t, 1\rangle < \infty$, P a.s. It is enough to verify that

$$\sum_{0<s\leqslant t} \Delta X_s \, 1_{\{X_{s-}<I^s_t\}} < \infty$$

P a.s. By time-reversal, this is equivalent to

$$\sum_{0<s\leqslant t} \Delta X_s \, 1_{\{X_s>S_{s-}\}} < \infty \tag{91}$$

P a.s. However, for every $a > 0$,

$$\begin{aligned} E\Big[\sum_{0<s\leqslant L^{-1}(a)} (\Delta X_s \wedge 1)\, 1_{\{X_s>S_{s-}\}}\Big] &= aN^*((\Delta X_\sigma)\wedge 1)1_{\{X_\sigma>0\}}) \\ &= a\int \pi(dx)\int_0^x dz(z\wedge 1) \\ &\leqslant a\int \pi(dx)\,(x\wedge x^2) \\ &< \infty \end{aligned}$$

using (10) in the second equality. This gives our claim (91) and the first assertion of the lemma under P. The property $\langle \eta_t, 1\rangle < \infty$, N a.e., then follows from arguments of excursion theory, using in particular the Markov property of X under N.

The preceding considerations also imply that

$$\lim_{t\downarrow 0} \sum_{0<s\leqslant t} \Delta X_s \, 1_{\{X_s>S_{s-}\}} = 0$$

in P-probability. Via time-reversal, it follows that the process η_t is right-continuous at $t = 0$ in probability under P. Then let $t_0 > 0$. We first observe that $\eta_{t_0}(\{H_{t_0}\}) = 0$ P a.s. This follows from the fact that there is a.s. no value of $s \in (0, t_0]$ with $X_s > S_{s-}$ and $L_s = 0$. Then, for $t > t_0$, write $u = u(t)$ for the (first) time of the minimum of X over $[t_0, t]$. Formula (90) implies that η_t is bounded below by the restriction of η_{t_0} to $[0, H_u)$, and bounded above by $\eta_{t_0} + \widetilde{\eta}^{(t)}_{t-t_0}$, where $\langle \widetilde{\eta}^{(t)}_{t-t_0}, 1\rangle$ has the same distribution as $\langle \eta_{t-t_0}, 1\rangle$ (more precisely, $\widetilde{\eta}^{(t)}_{t-t_0}$ is distributed as η_{t-t_0}, up to a translation by H_u). The right-continuity in P-probability of the mapping $t \to \eta_t$ at $t = t_0$ follows from this observation, the property $\eta_{t_0}(\{H_{t_0}\}) = 0$, the a.s. lower semi-continuity of H_t, and the case $t_0 = 0$.

The right-continuity in measure under N follows from the same arguments. □

Rather than investigating the Markovian properties of $(\eta_t, t \geqslant 0)$ we will consider the pair (ρ_t, η_t). We first introduce some notation. Let $(\mu, \nu) \in M_f(\mathbb{R}_+)^2$, and let

$a > 0$. Recall the notation of Proposition 1.2.3. In a way analogous to Chapter 1, we define $k_a(\mu, \nu) \in M_f(\mathbb{R}_+)^2$ by setting

$$k_a(\mu, \nu) = (\overline{\mu}, \overline{\nu})$$

where $\overline{\mu} = k_a\mu$ and the measure $\overline{\nu}$ is the unique element of $M_f(\mathbb{R}_+)$ such that

$$(\mu + \nu)_{|[0,H(k_a\mu)]} = k_a\mu + \overline{\nu}.$$

Note that the difference $\mu_{|[0,H(k_a\mu)]} - k_a\mu$ is a nonnegative multiple of the Dirac measure at $H(k_a\mu)$, so that $\overline{\nu}$ and $\nu_{|[0,H(k_a\mu)]}$ may only differ at the point $H(k_a\mu)$.

Then, if $\theta_1 = (\mu_1, \nu_1) \in M_f(\mathbb{R}_+)^2$ and $\theta_2 = (\mu_2, \nu_2) \in M_f(\mathbb{R}_+)^2$, and if $H(\mu_1) < \infty$, we define the concatenation $[\theta_1, \theta_2]$ by

$$[\theta_1, \theta_2] = ([\mu_1, \mu_2], \nu)$$

where $\langle \nu, f \rangle = \int \nu_1(ds) 1_{[0,H(\mu_1)]}(s) f(s) + \int \nu_2(ds) f(H(\mu_1) + s)$.

Proposition 3.1.2

(i) *Let $s \geqslant 0$ and $t > 0$. Then, for every nonnegative measurable function f on $M_f(\mathbb{R}_+)^2$,*

$$E[f(\rho_{s+t}, \eta_{s+t}) \mid \mathcal{G}_s] = \Pi_t^0 f(\rho_s, \eta_s)$$

where $\Pi_t^0((\mu, \nu), d\mu' d\nu')$ is the distribution of the pair

$$[k_{-I_t}(\mu, \nu), (\rho_t, \eta_t)]$$

under P. The collection $(\Pi_t^0, t > 0)$ is a Markovian semigroup on $M_f(\mathbb{R}_+)^2$.

(ii) *Let $s > 0$ and $t > 0$. Then, for every nonnegative measurable function f on $M_f(\mathbb{R}_+)^2$,*

$$N(f(\rho_{s+t}, \eta_{s+t})\, 1_{\{s+t<\sigma\}} \mid \mathcal{G}_s) = 1_{\{s<\sigma\}}\, \Pi_t f(\rho_s, \eta_s)$$

where $\Pi_t((\mu, \nu), d\mu' d\nu')$ is the distribution of the pair

$$[k_{-I_t}(\mu, \nu), (\rho_t, \eta_t)]$$

under $P(\cdot \cap \{T_{\langle \mu,1 \rangle} > t\})$. The collection $(\Pi_t, t > 0)$ is a submarkovian semigroup on $M_f(\mathbb{R}_+)^2$.

Proof

(i) Recall the notation of the proof of Proposition 1.2.3, and in particular formula (20). According to this formula, we have

$$\rho_{s+t} = [k_{-I_t^{(s)}} \rho_s, \rho_t^{(s)}] \tag{92}$$

where the pair $(I_t^{(s)}, \rho_t^{(s)})$ is defined in terms of the shifted process $X^{(s)}$, which is independent of $\mathcal{G}_s$. We then want to get an analogous expression for η_t. Precisely, we claim that

$$(\rho_{s+t}, \eta_{s+t}) = [k_{-I_t^{(s)}}(\rho_s, \eta_s), (\rho_t^{(s)}, \eta_t^{(s)})] \tag{93}$$

with an obvious notation. Note that (92) is the equality of the first components in (93).

To deal with the second components, recall the definition of η_{s+t}

$$\eta_{s+t}(du) = \beta 1_{[0,H_{s+t}]}(u)\, du + \sum_{\substack{0<r\leqslant s+t \\ X_{r-}<I^r_{s+t}}} (X_r - I^r_{s+t})\, \delta_{H_r}(du).$$

First consider the absolutely continuous part. By (92), we have

$$H_{s+t} = H(k_{-I^{(s)}_t}\rho_s) + H(\rho^{(s)}_t) = H(k_{-I^{(s)}_t}\rho_s) + H^{(s)}_t$$

and thus

$$\int du\, 1_{[0,H_{s+t}]}(u)\, f(u)$$
$$= \int du\, 1_{[0,H(k_{-I^{(s)}_t}\rho_s)]}(u)\, f(u) + \int du\, 1_{[0,H^{(s)}_t]}(u)\, f(H(k_{-I^{(s)}_t}\rho_s) + u).$$

This shows that the absolutely continuous part of η_{s+t} is the same as that of the second component of the right side of (93).

Then the singular part of η_{s+t} is equal to

$$\sum_{\substack{0<r\leqslant s \\ X_{r-}<I^r_{s+t}}} (X_r - I^r_{s+t})\, \delta_{H_r} + \sum_{\substack{s<r\leqslant s+t \\ X_{r-}<I^r_{s+t}}} (X_r - I^r_{s+t})\, \delta_{H_r}. \tag{94}$$

Note that, if $r \in (s, s+t]$ is such that $X_{r-} < I^r_{s+t}$, we have $H_r = H(k_{-I^{(s)}_t}\rho_s) + H^{(s)}_{r-s}$ (see the proof of Proposition 1.2.3). Thanks to this remark, we see that the second term of the sum in (94) is the image of the singular part of $\eta^{(s)}_t$ under the mapping $u \to H(k_{-I^{(s)}_t}\rho_s) + u$.

To handle the first term of (94), we consider two cases. Suppose first that $I_s < I^s_{s+t}$. Then set

$$v = \sup\{r \in (0,s] : X_{r-} < I^s_{s+t}\}.$$

In the first term of (94), we need only consider values $r \in (0, v]$. Note that $H_v = H(k_{-I^{(s)}_t}\rho_s)$ and that the measures ρ_v and $k_{-I^{(s)}_t}\rho_s$ are equal except possibly at the point H_v (see again the proof of Proposition 1.2.3). Then,

$$\sum_{\substack{0<r<v \\ X_{r-}<I^r_{s+t}}} (X_r - I^r_{s+t})\, \delta_{H_r} = \sum_{\substack{0<r<v \\ X_{r-}<I^r_s}} (X_r - I^r_s)\, \delta_{H_r}$$

coincides with the restriction of the singular part of η_s to $[0, H_v) = [0, H(k_{-I^{(s)}_t}\rho_s))$. On the other hand, $\eta_{s+t}(\{H_v\})$ is equal to

$$X_v - I^v_{s+t} = \eta_s(\{H_v\}) + \rho_s([0, H(k_{-I^{(s)}_t}\rho_s)]) - \langle k_{-I^{(s)}_t}\rho_s, 1\rangle$$

since by construction

$$\begin{aligned}
&\eta_s(\{H_v\}) = X_v - I_s^v\,,\\
&\rho_s([0,H(k_{-I_t^{(s)}}\rho_s)]) = \rho_s([0,H_v]) = I_s^v - I_s\,,\\
&\langle k_{-I_t^{(s)}}\rho_s, 1\rangle = X_s - I_s + I_t^{(s)} = I_{s+t}^s - I_s = I_{s+t}^v - I_s\,.
\end{aligned}$$

By comparing with the definition of $k_a(\mu,\nu)$, we see that the proof of (93) is complete in the case $I_s < I_{s+t}^s$.

The case $I_s \geqslant I_{s+t}^s$ is easier. In that case $k_{-I_t^{(s)}}\rho_s = 0$, and even $k_{-I_t^{(s)}}(\rho_s,\eta_s) = (0,0)$ (note that η_s gives no mass to 0, a.s.). Furthermore, the first sum in (94) vanishes, and it immediately follows that (93) holds.

The first assertion in (i) is a consequence of (93) and the fact that $X^{(s)}$ is independent of $\mathcal{G}_s$.

As for the second assertion, it is enough to verify that, for every $s,t>0$ we have

$$[k_{-I_t^{(s)}}[k_{-I_s}(\mu,\nu),(\rho_s,\eta_s)],(\rho_t^{(s)},\eta_t^{(s)})] = [k_{-I_{s+t}}(\mu,\nu),(\rho_{s+t},\eta_{s+t})]. \tag{95}$$

Note that the case $\mu=\nu=0$ is just (93). To prove (95), we consider the same two cases as previously.

If $I_{s+t}^s > I_s$, or equivalently $-I_t^{(s)} < \langle\rho_s,1\rangle$, then $I_s = I_{s+t}$ and so $k_{-I_s}(\mu,\nu) = k_{-I_{s+t}}(\mu,\nu)$. Furthermore, it is easy to verify that a.s.

$$k_{-I_t^{(s)}}[k_{-I_s}(\mu,\nu),(\rho_s,\eta_s)] = [k_{-I_s}(\mu,\nu),k_{-I_t^{(s)}}(\rho_s,\eta_s)].$$

Hence

$$\begin{aligned}
[k_{-I_t^{(s)}}[k_{-I_s}(\mu,\nu),(\rho_s,\eta_s)],(\rho_t^{(s)},\eta_t^{(s)})] &= [[k_{-I_s}(\mu,\nu),k_{-I_t^{(s)}}(\rho_s,\eta_s)],(\rho_t^{(s)},\eta_t^{(s)})]\\
&= [k_{-I_s}(\mu,\nu),[k_{-I_t^{(s)}}(\rho_s,\eta_s),(\rho_t^{(s)},\eta_t^{(s)})]],
\end{aligned}$$

and (95) follows from (93).

Finally, if $I_{s+t}^s \leqslant I_s$, or equivalently $-I_t^{(s)} > \langle\rho_s,1\rangle$, it easily follows from our definitions (and from the fact that $\eta_s(\{0\})=0$ a.s.) that

$$k_{-I_t^{(s)}}[k_{-I_s}(\mu,\nu),(\rho_s,\eta_s)] = k_{-I_{s+t}}(\mu,\nu)\,, \qquad \text{a.s.}$$

Furthermore, the property $I_{s+t}^s \leqslant I_s$ also implies that $(\rho_t^{(s)},\eta_t^{(s)}) = (\rho_{s+t},\eta_{s+t})$, and this completes the proof of (95).

(ii) First note that, for $s,t>0$, the identity (93) also holds N a.e. on $\{s+t<\sigma\}$ with the same proof (the argument is even simpler as we do not need to consider the case $I_{s+t}^s \leqslant I_s$). Also observe that N a.e. on $\{s<\sigma\}$, the condition $s+t<\sigma$ holds iff $-I_t^{(s)} < X_s = \langle\rho_s,1\rangle$, or equivalently $t < T_{\langle\rho_s,1\rangle}^{(s)} = \inf\{r\geqslant 0 : X_r^{(s)} = -\langle\rho_s,1\rangle\}$. The first assertion in (ii) follows from these observations and the Markov property under N.

The second assertion in (ii) follows from (95) and the fact that

$$\begin{aligned}\{T_{\langle\mu,1\rangle} > s+t\} &= \{I_{s+t} > -\langle\mu,1\rangle\}\\ &= \{I_s > -\langle\mu,1\rangle\} \cap \{I_t^{(s)} > -\langle\mu,1\rangle - X_s\}\\ &= \{I_s > -\langle\mu,1\rangle\} \cap \{T^{(s)}_{\langle k_{-I_s}\mu+\rho_s,1\rangle} > t\}\,.\end{aligned}$$

□

The previous proposition shows that the process (ρ_s, η_s) is Markovian under P. We now proceed to investigate its invariant measure.

Let $\mathcal{N}(dsd\ell dx)$ be a Poisson point measure on $(\mathbb{R}_+)^3$ with intensity

$$ds\,\pi(d\ell)\,1_{[0,\ell]}(x)dx.$$

For every $a > 0$, we denote by $\mathbb{M}_a$ the law on $M_f(\mathbb{R}_+)^2$ of the pair (μ_a, ν_a) defined by

$$\begin{aligned}\langle\mu_a, f\rangle &= \int \mathcal{N}(dsd\ell dx)\,1_{[0,a]}(s)\,xf(s) + \beta\int_0^a ds\,f(s)\\ \langle\nu_a, f\rangle &= \int \mathcal{N}(dsd\ell dx)\,1_{[0,a]}(s)\,(\ell - x)f(s) + \beta\int_0^a ds\,f(s).\end{aligned}$$

Note that $\mathbb{M}_a$ is invariant under the symmetry $(\mu,\nu) \to (\nu,\mu)$. We also set

$$\mathbb{M} = \int_0^\infty da\,e^{-\alpha a}\,\mathbb{M}_a.$$

The marginals of $\mathbb{M}$ coincide with the measure $\mathbf{M}$ of Chapter 1.

Proposition 3.1.3. — *Let Φ be a nonnegative measurable function on $M_f(\mathbb{R}_+)^2$. Then,*

$$N\Big(\int_0^\sigma dt\,\Phi(\rho_t,\eta_t)\Big) = \int \mathbb{M}(d\mu d\nu)\,\Phi(\mu,\nu).$$

Proof. — This is an extension of Proposition 1.2.5 and the proof is much analogous. Consider (under P) the countable collection of instants s_i, $i \in I$ such that $X_{s_i} > S_{s_i-}$. It follows from (10) that

$$\Big(L_\infty, \sum_{i\in I}\delta_{(L_{s_i},\Delta X_{s_i},X_{s_i}-S_{s_i-})}(dsd\ell dx)\Big) \overset{(\mathrm{d})}{=} \Big(\zeta, 1_{[0,\zeta]}(s)\mathcal{N}(dsd\ell dx)\Big) \tag{96}$$

where ζ is an exponential variable with parameter α independent of $\mathcal{N}$ ($\zeta = \infty$ if $\alpha = 0$). Recall from Chapter 1 the definition of the time-reversed process $\widehat{X}^{(t)}$. As in

(17), we can rewrite the definition of ρ_t and η_t in terms of the reversed process $\widehat{X}^{(t)}$:

$$\langle \rho_t, f\rangle = \beta \int_0^{\widehat{L}_t^{(t)}} dr\, f(r) + \sum_{\substack{0<s\leqslant t \\ \widehat{X}_s^{(t)} > \widehat{S}_{s-}^{(t)}}} (\widehat{X}_s^{(t)} - \widehat{S}_{s-}^{(t)})\, f(\widehat{L}_t^{(t)} - \widehat{L}_s^{(t)}),$$

$$\langle \eta_t, f\rangle = \beta \int_0^{\widehat{L}_t^{(t)}} dr\, f(r) + \sum_{\substack{0<s\leqslant t \\ \widehat{X}_s^{(t)} > \widehat{S}_{s-}^{(t)}}} (\widehat{S}_{s-}^{(t)} - \widehat{X}_{s-}^{(t)})\, f(\widehat{L}_t^{(t)} - \widehat{L}_s^{(t)}).$$

Hence we can write $(\rho_t, \eta_t) = \Gamma(\widehat{X}_{s\wedge t}^{(t)}, s \geqslant 0)$ with a measurable functional Γ that is made explicit in the previous formulas. Proposition 1.1.4 now gives

$$N\Big(\int_0^\sigma dt\, \Phi(\rho_t, \eta_t)\Big) = E\Big[\int_0^{L_\infty} da\, \Phi \circ \Gamma(X_{s\wedge L^{-1}(a)}, s \geqslant 0)\Big].$$

However, $\Gamma(X_{s\wedge L^{-1}(a)}, s \geqslant 0) = (\overline{\mu}_a, \overline{\nu}_a)$, with

$$\langle \overline{\mu}_a, f\rangle = \beta \int_0^a dr\, f(r) + \sum_{i\in I} 1_{[0,a]}(L_{s_i})\, (X_{s_i} - S_{s_i-})\, f(a - L_{s_i})$$

$$\langle \overline{\nu}_a, f\rangle = \beta \int_0^a dr\, f(r) + \sum_{i\in I} 1_{[0,a]}(L_{s_i})\, (\Delta X_{s_i} - (X_{s_i} - S_{s_i-}))\, f(a - L_{s_i}).$$

Now use (96) to complete the proof. □

For every $t > 0$, we denote by $\widehat{\Pi}_t$ the image of the kernel Π_t under the symmetry $(\mu, \nu) \to (\nu, \mu)$, that is

$$\widehat{\Pi}_t \Phi(\mu, \nu) = \int \Pi_t((\nu, \mu), d\nu' d\mu')\, \Phi(\mu', \nu').$$

Theorem 3.1.4. — *The kernels Π_t and $\widehat{\Pi}_t$ are in duality under $\mathbb{M}$.*

This means that for any nonnegative measurable functions Φ and Ψ on $M_f(\mathbb{R}_+)^2$,

$$\mathbb{M}(\Phi \Pi_t \Psi) = \mathbb{M}(\Psi \widehat{\Pi}_t \Phi).$$

Proof. — We first consider the potential kernels

$$U = \int_0^\infty dt\, \Pi_t \ , \ \widehat{U} = \int_0^\infty dt\, \widehat{\Pi}_t$$

and we prove that

$$\mathbb{M}(\Phi U \Psi) = \mathbb{M}(\Psi \widehat{U} \Phi). \tag{97}$$

This is equivalent to saying that the measure

$$\mathbb{M}(d\mu d\nu)\, U((\mu, \nu), d\mu' d\nu')$$

is invariant under the transformation $(\mu, \nu, \mu', \nu') \to (\nu', \mu', \nu, \mu)$.

To this end, we first derive an explicit expression for the kernel U. By the definition of the kernels Π_t, we have

$$U\Phi(\mu,\nu) = E\Big[\int_0^{T_{\langle\mu,1\rangle}} dt\, \Phi([k_{-I_t}(\mu,\nu),(\rho_t,\eta_t)])\Big].$$

This is computed in a way similar to the proof of Proposition 1.2.6, using Proposition 3.1.3 in place of Proposition 1.2.5. It follows that

$$U\Phi(\mu,\nu) = \int_0^{\langle\mu,1\rangle} dr \int \mathbb{M}(d\mu' d\nu')\, \Phi([k_r(\mu,\nu),(\mu',\nu')]). \tag{98}$$

We then need to get more information about the joint distribution of $((\mu,\nu),k_r(\mu,\nu))$ under $\mathbb{M}(d\mu d\nu)1_{[0,\langle\mu,1\rangle]}(r)dr$. Recall the notation $\mathcal{N}$, μ_a, ν_a introduced before the statement of Proposition 3.1.3. Write

$$\mathcal{N} = \sum_{i\in I} \delta_{(s_i,\ell_i,x_i)}$$

for definiteness, in such a way that

$$(\mu_a,\nu_a) = (\beta m_a + \sum_{s_i\leqslant a} x_i\,\delta_{s_i}, \beta m_a + \sum_{s_i\leqslant a}(\ell_i - x_i)\,\delta_{s_i}),$$

where m_a denotes Lebesgue measure on $[0,a]$. Since $\mathbb{M}_a$ is the law of (μ_a,ν_a), we get

$$\begin{aligned}
&\int \mathbb{M}_a(d\mu d\nu) \int_0^{\langle\mu,1\rangle} dr\, F((\mu,\nu),k_r(\mu,\nu)) \\
&\quad = E\Big[\int_0^{\langle\mu_a,1\rangle} dr\, F((\mu_a,\nu_a),k_r(\mu_a,\nu_a))\Big] \\
&\quad = E\Big[\beta\int_0^a ds\, F((\mu_a,\nu_a),(\mu_{a|[0,s]},\nu_{a|[0,s]}))\Big] \\
&\quad + E\Big[\sum_{s_i\leqslant a}\int_0^{x_i} dy\, F((\mu_a,\nu_a),(\mu_{a|[0,s_i)} + y\delta_{s_i},\nu_{a|[0,s_i)} + (\ell_i - y)\delta_{s_i}))\Big]
\end{aligned} \tag{99}$$

using the definition of k_r.

At this point, we recall the following well-known lemma about Poisson measures.

Lemma 3.1.5. — *Let E be a measurable space and let Δ be a σ-finite measure on E. Let $\mathcal{M}$ be a Poisson point measure on $[0,a]\times E$ with intensity $ds\,\Delta(de)$. Then, for any nonnegative measurable function Φ,*

$$E\Big[\int \mathcal{M}(dsde)\,\Phi((s,e),\mathcal{M})\Big] = E\Big[\int_0^a ds\int_E \Delta(de)\,\Phi((s,e),\mathcal{M}+\delta_{(s,e)})\Big].$$

Thanks to this lemma, the second term in the right side of (99) can be written as

$$\begin{aligned}
E\Big[\int_0^a ds \int \pi(d\ell)\int_0^\ell dx\int_0^x dy \\
F((\mu_a + x\delta_s,\nu_a + (\ell - x)\delta_s),(\mu_{a|[0,s)} + y\delta_s,\nu_{a|[0,s)} + (\ell - y)\delta_s))\Big].
\end{aligned}$$

We now integrate (99) with respect to $e^{-\alpha a}da$. After some easy transformations, we get

$$\begin{aligned}
&\int \mathbb{M}(d\mu d\nu) \int_0^{\langle \mu,1\rangle} dr\, F((\mu,\nu), k_r(\mu,\nu)) \\
&\quad = \beta \int \mathbb{M}(d\mu_1 d\nu_1)\mathbb{M}(d\mu_2 d\nu_2)\, F([(\mu_1,\nu_1),(\mu_2,\nu_2)],(\mu_1,\nu_1)) \\
&\quad + \int \mathbb{M}(d\mu_1 d\nu_1)\mathbb{M}(d\mu_2 d\nu_2) \int \pi(d\ell) \int_0^\ell dx \int_0^x dy \\
&\qquad\qquad F([(\mu_1,\nu_1),(x\delta_0+\mu_2,(\ell-x)\delta_0+\nu_2)],[(\mu_1,\nu_1),(y\delta_0,(\ell-y)\delta_0)]).
\end{aligned}$$

Recalling formula (98) for the potential kernel U, we see that the measure

$$\mathbb{M}(d\mu d\nu)\, U((\mu,\nu), d\mu' d\nu')$$

is the sum of two terms. The first one is the distribution under

$$\beta \mathbb{M}(d\mu_1 d\nu_1)\mathbb{M}(d\mu_2 d\nu_2)\mathbb{M}(d\mu_3 d\nu_3)$$

of the pair

$$(\mu,\nu) = [(\mu_1,\nu_1),(\mu_2,\nu_2)]\ ,\ (\mu',\nu') = [(\mu_1,\nu_1),(\mu_3,\nu_3)].$$

The second one is the distribution under

$$\mathbb{M}(d\mu_1 d\nu_1)\mathbb{M}(d\mu_2, d\nu_2)\mathbb{M}(d\mu_3 d\nu_3)\pi(d\ell) 1_{\{0<y<x<\ell\}} dx\, dy$$

of the pair

$$(\mu,\nu) = [(\mu_1,\nu_1),(x\delta_0+\mu_2,(\ell-x)\delta_0+\nu_2)]\ ,\ (\mu',\nu') = [(\mu_1,\nu_1),(y\delta_0+\mu_3,(\ell-y)\delta_0+\nu_3)].$$

In this form, it is clear that $\mathbb{M}(d\mu d\nu)\, U((\mu,\nu), d\mu' d\nu')$ has the desired invariance property. This completes the proof of (97).

Consider now the resolvent kernels

$$U_p((\mu,\nu), d\mu' d\nu') = \int_0^\infty dt\, e^{-pt} \Pi_t((\mu,\nu), d\mu', d\nu').$$

By a standard argument (see e.g. [**9**], p.54), (97) also implies that, for every $p > 0$, $\mathbb{M}(\Phi U_p \Psi) = \mathbb{M}(\Psi \widehat{U}_p \Phi)$, or equivalently

$$\int_0^\infty dt\, e^{-pt}\, \mathbb{M}(\Phi \Pi_t \Psi) = \int_0^\infty dt\, e^{-pt}\, \mathbb{M}(\Psi \Pi_t \Phi). \tag{100}$$

Recall that our goal is to prove the identity $\mathbb{M}(\Phi\Pi_t\Psi) = \mathbb{M}(\Psi\Pi_t\Phi)$ for every $t > 0$. We may assume that the functions Φ and Ψ are continuous and both dominated by $e^{-a\langle \mu,1\rangle}$ for some $a > 0$. The latter condition guarantees that $\mathbb{M}(\Phi) < \infty$ and $\mathbb{M}(\Psi) < \infty$. From the definition of Π_t and the right-continuity in probability of the mapping $t \to (\rho_t, \eta_t)$ (Lemma 3.1.1), it is easy to verify that $t \to \Pi_t\Psi(\mu,\nu)$ is right-continuous over $(0,\infty)$. The same holds for the mapping $t \to \mathbb{M}(\Phi\Pi_t\Psi)$, and the statement of the theorem follows from (100). □

For notational reasons, we make the convention that $\rho_s = \eta_s = 0$ if $s < 0$.

Corollary 3.1.6. — *The process $(\eta_s, s \geqslant 0)$ has a càdlàg modification under N or under P. Furthermore, the processes $(\rho_s, \eta_s; s \geqslant 0)$ and $(\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}; s \geqslant 0)$ have the same distribution under N.*

A consequence of the corollary is the fact that the processes $(H_t, t \geqslant 0)$ and $(H_{(\sigma-t)\vee 0}, t \geqslant 0)$ have the same distribution (say in the sense of finite-dimensional marginals when H is not continuous) under N. In view of the results of Chapter 2, this is not surprising, as the same time-reversal property obviously holds for the discrete contour process. The more precise statement of the corollary will be useful in the next sections.

Proof. — The second part of the corollary is essentially a consequence of the duality property stated in the previous theorem. Since we have still little information about regularity properties of the process η, we will proceed with some care. We first introduce the Kuznetsov measure $\mathbb{K}$, which is the σ-finite measure on $\mathbb{R} \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ defined by

$$\mathbb{K}(dr d\omega) = dr\, N(d\omega).$$

We then define $\gamma(r,\omega) = r$, $\delta(r,\omega) = r + \sigma(\omega)$ and, for every $t \in \mathbb{R}$,

$$\overline{\rho}_t(r,\omega) = \rho_{t-r}(\omega)\ ,\ \overline{\eta}_t(r,\omega) = \eta_{t-r}(\omega)$$

with the convention explained before the statement of the corollary. Note that $(\overline{\rho}_t, \overline{\eta}_t) \neq (0,0)$ iff $\gamma < t < \delta$.

It readily follows from Proposition 3.1.3 that, for every $t \in \mathbb{R}$, the distribution of $(\overline{\rho}_t, \overline{\eta}_t)$ under $\mathbb{K}(\cdot \cap \{(\overline{\rho}_t, \overline{\eta}_t) \neq (0,0)\})$ is $\mathbb{M}$. Let $t_1, \ldots, t_p \in \mathbb{R}$ with $t_1 < t_2 < \cdots < t_p$. Using Proposition 3.1.2 and induction on p, we easily get that the restriction to $(M_f(\mathbb{R}_+)^2 \backslash \{(0,0)\})^p$ of the distribution of the p-tuple $((\overline{\rho}_{t_1}, \overline{\eta}_{t_1}), \ldots, (\overline{\rho}_{t_p}, \overline{\eta}_{t_p}))$ is

$$\mathbb{M}(d\mu_1 d\nu_1)\, \Pi_{t_2-t_1}((\mu_1,\nu_1), d\mu_2 d\nu_2) \ldots \Pi_{t_p-t_{p-1}}((\mu_{p-1},\nu_{p-1}), d\mu_p d\nu_p).$$

By Theorem 3.1.4, this measure is equal to

$$\mathbb{M}(d\mu_p d\nu_p)\, \widehat{\Pi}_{t_p-t_{p-1}}((\mu_p,\nu_p), d\mu_{p-1} d\nu_{p-1}) \ldots \widehat{\Pi}_{t_2-t_1}((\mu_2,\nu_2), d\mu_1 d\nu_1).$$

Hence the two p-tuples $((\overline{\rho}_{t_1}, \overline{\eta}_{t_1}), \ldots, (\overline{\rho}_{t_p}, \overline{\eta}_{t_p}))$ and $((\overline{\eta}_{-t_1}, \overline{\rho}_{-t_1}), \ldots, (\overline{\eta}_{-t_p}, \overline{\rho}_{-t_p}))$ have the same distribution, in restriction to $(M_f(\mathbb{R}_+)^2 \backslash \{(0,0)\})^p$, under $\mathbb{K}$. Since $(\overline{\rho}_t, \overline{\eta}_t) \neq (0,0)$ iff $\gamma < t < \delta$, a simple argument shows that we can remove the restriction and conclude that these two p-tuples have the same distribution under $\mathbb{K}$. (This distribution is σ-finite except for an infinite mass at the point $(0,0)^p$.)

In particular, $(\overline{\rho}_{t_1}, \ldots, \overline{\rho}_{t_p})$ and $(\overline{\eta}_{-t_1}, \ldots, \overline{\eta}_{-t_p})$ have the same distribution under $\mathbb{K}$. Let F be a bounded continuous function on $M_f(\mathbb{R}_+)^p$, such that $F(0, \ldots, 0) = 0$.

Suppose that $0 < t_1 < t_2 < \cdots < t_p$ and let $u < v$. Then we have

$$\mathbb{K}(1_{[u,v]}(\gamma)F(\overline{\rho}_{\gamma+t_1},\ldots,\overline{\rho}_{\gamma+t_p})) = \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z},\, k\varepsilon\in[u,v]}\mathbb{K}\Big(1_{\{\overline{\rho}_{k\varepsilon}=0,\overline{\rho}_{(k+1)\varepsilon}\neq 0\}}F(\overline{\rho}_{k\varepsilon+t_1},\ldots,\overline{\rho}_{k\varepsilon+t_p})\Big)$$

and the similar formula

$$\mathbb{K}(1_{[u,v]}(-\delta)F(\overline{\eta}_{\delta-t_1},\ldots,\overline{\eta}_{\delta-t_p})) = \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z},\, k\varepsilon\in[u,v]}\mathbb{K}\Big(1_{\{\overline{\eta}_{-k\varepsilon}=0,\overline{\eta}_{-(k+1)\varepsilon}\neq 0\}}F(\overline{\eta}_{-k\varepsilon-t_1},\ldots,\overline{\eta}_{-k\varepsilon-t_p})\Big),$$

using the right-continuity in N-measure of η_t. Hence the vectors $(\gamma,\overline{\rho}_{\gamma+t_1},\ldots,\overline{\rho}_{\gamma+t_p})$ and $(-\delta,\overline{\eta}_{\delta-t_1},\ldots,\overline{\eta}_{\delta-t_p})$ have the same distribution under $\mathbb{K}$. It follows that the processes $(\rho_t, t\geqslant 0)$ and $(\eta_{\sigma-t}, t\geqslant 0)$ have the same finite-dimensional marginals under N. Since we already know that $(\rho_t, t\geqslant 0)$ is càdlàg, we obtain that $(\eta_t, t\geqslant 0)$ has a càdlàg modification under N. The time-reversal property of the corollary follows immediately from the previous identification of finite-dimensional marginals. This property implies in particular that $\eta_{0+} = \eta_{\sigma-} = 0$ N a.e.

It remains to verify that $(\eta_t, t\geqslant 0)$ has a càdlàg modification under P. On each excursion interval of $X - I$ away from 0, we can apply the result derived above under the excursion measure N. It remains to deal with instants t such that $X_t = I_t$, for which $\eta_t = 0$. To this end, we note that, for every $\varepsilon > 0$,

$$N\Big(\sup_{s\in[0,\sigma]}\langle\eta_s,1\rangle > \varepsilon\Big) = N\Big(\sup_{s\in[0,\sigma]}\langle\rho_s,1\rangle > \varepsilon\Big) < \infty.$$

Hence, for any fixed $x > 0$, we will have $\langle\eta_s,1\rangle\leqslant\varepsilon$ for all $s\in[0,T_x]$ except possibly for s belonging to finitely many excursion intervals of $X - I$. Together with the continuity of η at times 0 and σ under N, this implies that P a.s. for every t such that $X_t = I_t$, the right and left limits of η_s both exist at time t and vanish. □

3.2. The tree associated with Poissonnian marks

3.2.1. Trees embedded in an excursion. — We first give the definition of the tree associated with a continuous function $e : [a,b]\to\mathbb{R}_+$ and p instants $t_1,\ldots,t_p$ with $a\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_p\leqslant b$.

Recall from Section 0.1 the definition of a (finite) rooted ordered tree, and the notation $\mathbf{T}$ for the collection of these trees. If v is an individual (a vertex) in the tree $\mathcal{T}\in\mathbf{T}$, the notation $k_v(\mathcal{T})$ stands for the number of children of v. Individuals v without children, i.e. such that $k_v(\mathcal{T}) = 0$, are called leaves. For every $p\geqslant 1$, we denote by $\mathbf{T}_p$ the set of all (rooted ordered) trees with p leaves.

If $\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^k$ are k trees, the concatenation of $\mathcal{T}^1, \dots, \mathcal{T}^k$, which is denoted by $[\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^k]$, is defined in the obvious way: For $n \geqslant 1$, $(i_1, \dots, i_n)$ belongs to $[\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^k]$ if and only if $1 \leqslant i_1 \leqslant k$ and $(i_2, \dots, i_n)$ belongs to $\mathcal{T}^{i_1}$.

A *marked tree* is a pair $\theta = (\mathcal{T}, \{h_v, v \in \mathcal{T}\})$, where $h_v \geqslant 0$ for every $v \in \mathcal{T}$. The number h_v is interpreted as the lifetime of individual v, and $\mathcal{T}$ is called the skeleton of θ. We denote by $\mathbb{T}_p$ the set of all marked trees with p leaves.

Let $\theta^1 = (\mathcal{T}^1, \{h^1_v, v \in \mathcal{T}\}) \in \mathbb{T}_{p_1}, \dots, \theta^k = (\mathcal{T}^k, \{h^k_v, v \in \mathcal{T}^k\}) \in \mathbb{T}_{p_k}$, and $h \geqslant 0$. The concatenation $[\theta^1, \theta^2, \dots, \theta^k]_h$ is the element of $\mathbb{T}_{p_1+\cdots+p_k}$ whose skeleton is $[\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^k]$ and such that the lifetimes of vertices in $\mathcal{T}^i$, $1 \leqslant i \leqslant k$ become the lifetimes of the corresponding vertices in $[\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^k]$, and finally the lifetime of $\varnothing$ in $[\theta^1, \theta^2, \dots, \theta^k]_h$ is h.

Let $a, b \in \mathbb{R}_+$ with $a \leqslant b$ and let $e : [a, b] \longrightarrow \mathbb{R}_+$ be a continuous function. For every $a \leqslant u \leqslant v \leqslant b$, we set

$$m(u, v) = \inf_{u \leqslant t \leqslant v} e(t).$$

Let $t_1, \dots, t_p \in \mathbb{R}_+$ be such that $a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_p \leqslant b$. We will now construct a marked tree

$$\theta(e, t_1, \dots, t_p) = (\mathcal{T}(e, t_1, \dots, t_p), \{h_v(e, t_1, \dots, t_p), v \in \mathcal{T}\}) \in \mathbb{T}_p$$

associated with the function e and the times $t_1, \dots, t_p$. We proceed by induction on p. If $p = 1$, $\mathcal{T}(e, t_1) = \{\varnothing\}$ and $h_\varnothing(e, t_1) = e(t_1)$.

Let $p \geqslant 2$ and suppose that the tree has been constructed up to order $p - 1$. Then there exists an integer $k \in \{1, \dots, p-1\}$ and k integers $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant p-1$ such that $m(t_i, t_{i+1}) = m(t_1, t_p)$ iff $i \in \{i_1, \dots, i_k\}$. For every $\ell \in \{0, 1, \dots, k\}$, define e^ℓ by the formulas

$$\begin{aligned} e^0(t) &= e(t) - m(t_1, t_p), && t \in [t_1, t_{i_1}], \\ e^\ell(t) &= e(t) - m(t_1, t_p), && t \in [t_{i_\ell + 1}, t_{i_{\ell+1}}], \qquad 1 \leqslant \ell \leqslant k-1. \\ e^k(t) &= e(t) - m(t_1, t_p), && t \in [t_{i_k+1}, t_p]. \end{aligned}$$

We then set:

$$\theta(e, t_1, \dots, t_p) = [\theta(e^0, t_1, \dots, t_{i_1}), \theta(e^1, t_{i_1+1}, \dots, t_{i_2}), \dots, \theta(e^k, t_{i_k+1}, \dots, t_p)]_{m(t_1, t_p)}.$$

This completes the construction of the tree by induction. Note that $k + 1$ is the number of children of $\varnothing$ in the tree $\theta(e, t_1, \dots, t_p)$, and $m(t_1, t_p)$ is the lifetime of $\varnothing$.

3.2.2. Poissonnian marks. — We consider a standard Poisson process with parameter λ defined under the probability measure Q_λ. We denote by $\tau_1 \leqslant \tau_2 \leqslant \cdots$ the jump times of this Poisson process. Throughout this section, we argue under the measure $Q_\lambda \otimes N$, which means that we consider the excursion measure of $X - I$ together with independent Poissonnian marks with intensity λ on $\mathbb{R}_+$. To simplify notation however, we will systematically write N instead of $Q_\lambda \otimes N$.

Set $M = \sup\{i \geqslant 1 : \tau_i \leqslant \sigma\}$, which represents the number of marks that fall in the excursion interval (by convention, $\sup \varnothing = 0$). Then,

$$N(M \geqslant 1) = N(1 - e^{-\lambda\sigma}) = \psi^{-1}(\lambda),$$

where the second equality follows from the fact that the Laplace exponent of the subordinator T_x is $\psi^{-1}(\lambda)$ (see [**5**], Theorem VII.1).

From now on, we assume that the condition $\int_1^\infty \frac{du}{\psi(u)} < \infty$ holds, so that H has continuous sample paths (Theorem 1.4.3). We can then use subsection 3.2.1 to define the embedded tree $\theta(H, \tau_1, \ldots, \tau_M)$ under $N(\cdot \mid M \geqslant 1)$. Our main goal is to determine the law of this tree.

Theorem 3.2.1. — *Under the probability measure $N(\cdot|M \geqslant 1)$, the tree $\theta(H, \tau_1, \ldots, \tau_M)$ is distributed as the family tree of a continuous-time Galton-Watson process starting with one individual at time 0 and such that:*

- *Lifetimes of individuals have exponential distributions with parameter $\psi'(\psi^{-1}(\lambda))$;*
- *The offspring distribution is the law of the variable ξ with generating function*

$$E[r^\xi] = r + \frac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}.$$

Remark. — As the proof will show, the theorem remains valid without the assumption that H has continuous paths. We will leave this extension to the reader. Apart from some technical details, it simply requires the straightforward extension of the construction of subsection 3.2.1 to the case when the function e is only lower semi-continuous.

The proof of Theorem 3.2.1 requires a few intermediate results. To simplify notation, we will write $\tau = \tau_1$. We start with an important application of Corollary 3.1.6.

Lemma 3.2.2. — *For any nonnegative measurable function f on $M_f(\mathbb{R}_+)$,*

$$N(f(\rho_\tau)1_{\{M\geqslant 1\}}) = \lambda \int \mathbb{M}(d\mu d\nu)\, f(\mu)\, e^{-\psi^{-1}(\lambda)\,\langle \nu, 1\rangle}.$$

Proof. — We have

$$N(f(\rho_\tau)1_{\{M\geqslant 1\}}) = \lambda N\Big(\int_0^\sigma dt\, e^{-\lambda t}\, f(\rho_t)\Big) = \lambda N\Big(\int_0^\sigma dt\, e^{-\lambda(\sigma - t)}\, f(\eta_t)\Big),$$

using the time-reversal property of Corollary 3.1.6. At this point, we use the Markov property of X under N:

$$N\Big(\int_0^\sigma dt\, e^{-\lambda(\sigma-t)}\, f(\eta_t)\Big) = N\Big(\int_0^\sigma dt\, f(\eta_t)\, E_{X_t}[e^{-\lambda T_0}]\Big).$$

We have already noticed that for $x \geqslant 0$,

$$E_x[e^{-\lambda T_0}] = E_0[e^{-\lambda T_x}] = e^{-x\psi^{-1}(\lambda)}.$$

Since $X_t = \langle \rho_t, 1\rangle$ under N, it follows that

$$N(f(\rho_\tau)1_{\{M\geqslant 1\}}) = \lambda N\Big(\int_0^\sigma dt\, f(\eta_t)\, e^{-\langle \rho_t,1\rangle \psi^{-1}(\lambda)}\Big) = \lambda \int \mathbb{M}(d\mu d\nu)\, f(\nu)\, e^{-\langle \mu,1\rangle \psi^{-1}(\lambda)},$$

using Proposition 3.1.3. Since $\mathbb{M}$ is invariant under the mapping $(\mu,\nu) \to (\nu,\mu)$, this completes the proof. □

We now set

$$K = \begin{cases} \inf\{H_s : \tau_1 \leqslant s \leqslant \tau_M\} & \text{if } M \geqslant 2 \\ \infty & \text{if } M \leqslant 1 \end{cases}$$

Then K represents the lifetime of the ancestor in the tree $\theta(H, \tau_1, \ldots, \tau_M)$ (assuming that the event $\{M \geqslant 2\}$ holds). To give a formula for the number of children ξ of the ancestor, set

$$\tau_{(K)} = \inf\{t \geqslant \tau : H_t \leqslant K\}\ ,\ \tau'_{(K)} = \inf\{t \geqslant \tau : H_t < K\}.$$

Then, again on the event $\{M \geqslant 2\}$, $\xi - 1$ is the number of excursions of H above level K, on the time interval $[\tau_{(K)}, \tau'_{(K)}]$, which contain at least one of the Poissonnian marks. This identification follows readily from the construction of subsection 3.2.1.

The next proposition gives the joint distribution of the pair (K, ξ) under $N(\cdot \cap \{M \geqslant 2\})$.

Proposition 3.2.3. — *Let $r \in [0,1]$ and let h be a nonnegative measurable function on $[0,\infty]$, with $h(\infty) = 0$. Then,*

$$N(r^{\xi} h(K) \mid M \geqslant 1) = \Big(r\psi'(\psi^{-1}(\lambda)) + \frac{\psi((1-r)\psi^{-1}(\lambda)) - \lambda}{\psi^{-1}(\lambda)}\Big) \int_0^\infty db\, h(b)\, e^{-b\psi'(\psi^{-1}(\lambda))}.$$

The basic idea of the proof is to apply the Markov property to the process ρ at time τ. To this end, we need some notation. We write $\mathbf{P}^*_\mu$ for the probability measure under which ρ starts at an arbitrary measure $\mu \in \mathbb{M}_f(\mathbb{R}^d)$ and is stopped when it hits 0. As usual, $H_t = H(\rho_t)$. Under $\mathbf{P}^*_\mu$, the process $X_t = \langle \rho_t, 1\rangle$ is the underlying Lévy process started at $\langle \mu, 1\rangle$, stopped at $T_0 = \inf\{t \geqslant 0 : X_t = 0\}$. We keep the notation I_t for the minimum process of X. We let (a_j, b_j), $j \in J$ be the collection of excursion intervals of $X - I$ away from 0 and before time T_0. For every $j \in J$ we define the corresponding excursion by

$$\omega_j(t) = X_{(a_j+t)\wedge b_j} - I_{a_j}\ ,\quad t \geqslant 0.$$

From excursion theory, we know that the point measure

$$\sum_{j\in J} \delta_{(I_{a_j}, \omega_j)}$$

is Poisson under $\mathbf{P}^*_\mu$, with intensity $1_{[0,\langle \mu,1\rangle]}(u) du\, N(d\omega)$ (*cf.* the proof of Proposition 1.2.6). On the other hand, by properties of the exploration process derived in Chapter 1, we know that $\mathbf{P}^*_\mu$ a.s. for every $s \in [0, T_0]$ such that $X_s - I_s = 0$ (and in

particular for $s = a_j$, $j \in J$) we have $\rho_s = k_{\langle\mu,1\rangle - I_s}\mu$ and thus $H_s = H(k_{\langle\mu,1\rangle - I_s}\mu)$. Observe also that the image of the measure $1_{[0,\langle\mu,1\rangle]}(u)du$ under the mapping $u \to H(k_{\langle\mu,1\rangle - u}\mu)$ is exactly $\mu(dh)$. By combining these observations, we get:

(P) The point measure $\sum_{j\in J} \delta_{(H_{a_j},\omega_j)}$ is Poisson under $\mathbf{P}^*_\mu$, with intensity $\mu(dh)\,N(d\omega)$.

Finally, assume that we are also given a collection $\mathcal{P}_\lambda$ of Poisson marks with intensity λ, independently of ρ under $\mathbf{P}^*_\mu$, and set

$$L = \inf\{H_{a_j} : j \in J, (a_j, b_j) \cap \mathcal{P}_\lambda \neq \varnothing\}, \qquad (\inf \varnothing = \infty),$$
$$\zeta = \mathrm{Card}\{j \in J : H_{a_j} = L \text{ and } (a_j, b_j) \cap \mathcal{P}_\lambda \neq \varnothing\}.$$

Then the Markov property of the exploration process at time τ shows that, for any nonnegative measurable function h on $[0,\infty]$ such that $h(\infty) = 0$,

$$N(1_{\{M\geqslant 1\}} r^{\xi-1} h(K)) = N(1_{\{M\geqslant 1\}} \mathbf{E}^*_{\rho_\tau}[r^\zeta h(L)]). \tag{101}$$

To verify this equality, simply observe that those excursions of H above level K on the time interval $[\tau_{(K)}, \tau'_{(K)}]$ that contain one Poissonnian mark, exactly correspond to those excursions of the shifted process $\langle\rho_{\tau+t}, 1\rangle$ above its minimum that start from the height K and contain one mark.

The next lemma is the key step towards the proof of Proposition 3.2.3.

Lemma 3.2.4. — *Let $a \geqslant 0$ and let $\mu \in M_f(\mathbb{R}_+)$ be such that $\operatorname{supp}\mu = [0,a]$ and $\mu(dt) = \beta 1_{[0,a]}(t)dt + \mu_s(dt)$, where μ_s is a countable sum of multiples of Dirac point masses at elements of $[0,a]$. Then, if $r \in [0,1]$ and h is a nonnegative measurable function on $[0,\infty]$ such that $h(\infty) = 0$,*

$$\mathbf{E}^*_\mu[r^\zeta h(L)] = \beta r \psi^{-1}(\lambda) \int_0^a db\, e^{-\psi^{-1}(\lambda)\mu([0,b])}\, h(b) + \sum_{\mu(\{s\})>0} \left(e^{-(1-r)\mu(\{s\})\psi^{-1}(\lambda)} - e^{-\mu(\{s\})\psi^{-1}(\lambda)}\right) e^{-\mu([0,s))\psi^{-1}(\lambda)}\, h(s). \tag{102}$$

Proof. — First note that it is easy to derive the law of L under $\mathbf{P}^*_\mu$. Let $b \in [0,a]$. We have by property (P)

$$\begin{aligned}\mathbf{P}^*_\mu[L > b] &= \mathbf{P}^*_\mu[(a_j, b_j) \cap \mathcal{P}_\lambda = \varnothing \text{ for every } j \in J \text{ s.t. } H_{a_j} \leqslant b] \\ &= \mathbf{E}^*_\mu[\exp(-\mu([0,b])N(1 - e^{-\lambda\sigma}))] \\ &= \exp(-\mu([0,b])\psi^{-1}(\lambda)).\end{aligned}$$

In particular, atoms of the distribution of L in $[0,\infty)$ exactly correspond to atoms of μ, and the continuous part of the distribution of L is the measure

$$\beta\psi^{-1}(\lambda)\exp(-\mu([0,b])\psi^{-1}(\lambda))1_{[0,a]}(b)db.$$

We then need to distinguish two cases:

(1) Let $s \in [0,a]$ be an atom of μ. By the preceding formula,

$$\mathbf{P}^*_\mu[L = s] = (1 - e^{-\mu(\{s\})\psi^{-1}(\lambda))})e^{-\mu([0,s))\psi^{-1}(\lambda)}.$$

Note that the excursions ω_j that start at height s are the atoms of a Poisson measure with intensity $\mu(\{s\})N$. Using also the independence properties of Poisson measures, we get that, conditionally on $\{L = s\}$, ξ is distributed as a Poisson random variable with intensity $\mu(\{s\})\psi^{-1}(\lambda)$, conditioned to be greater than or equal to 1:

$$\mathbf{E}^*_\mu[r^\xi \mid L = s] = \frac{e^{-(1-r)\mu(\{s\})\psi^{-1}(\lambda))} - e^{-\mu(\{s\})\psi^{-1}(\lambda))}}{1 - e^{-\mu(\{s\})\psi^{-1}(\lambda))}}.$$

(2) If L is not an atom of μ, then automatically $\xi = 1$. This is so because the values H_{a_j} corresponding to indices j such that $\mu(\{H_{a_j}\}) = 0$ must be distinct, by (P) and standard properties of Poisson measures.

The lemma follows by combining these two cases with the distribution of L. □

Proof of Proposition 3.2.3. — By combining (101), Lemma 3.2.2 and (102), we obtain that

$$N(1_{\{M\geqslant 1\}} r^{\xi-1} h(K)) = A_1 + A_2$$

where

$$A_1 = \beta r \lambda \psi^{-1}(\lambda) \int_0^\infty da\, e^{-\alpha a} \int \mathbb{M}_a(d\mu d\nu) e^{-\langle \nu, 1\rangle \psi^{-1}(\lambda)} \int_0^a db\, e^{-\psi^{-1}(\lambda)\mu([0,b])}\, h(b),$$

and

$$A_2 = \lambda \int_0^\infty da\, e^{-\alpha a} \int \mathbb{M}_a(d\mu d\nu) e^{-\langle \nu, 1\rangle \psi^{-1}(\lambda)} \times \sum_{\mu(\{s\})>0} \Big(e^{-(1-r)\mu(\{s\})\psi^{-1}(\lambda)} - e^{-\mu(\{s\})\psi^{-1}(\lambda)}\Big) e^{-\mu([0,s))\psi^{-1}(\lambda)}\, h(s).$$

To compute A_1, we observe that for $u > 0$ and $0 \leqslant b \leqslant a$,

$$\begin{aligned}
\mathbb{M}_a(e^{-u(\mu([0,b])+\nu([0,a])}) &= \mathbb{M}_a(e^{-u(\mu+\nu)([0,b])}) \mathbb{M}_a(e^{-u\nu((b,a])}) \\
&= e^{-\beta u(a+b)} \exp\Big(-b \int \pi(d\ell)\ell(1 - e^{-u\ell})\Big) \\
&\quad \times \exp\Big(-(a-b) \int \pi(d\ell) \int_0^\ell dx (1 - e^{-ux})\Big) \\
&= e^{\alpha a} \exp(-b\psi'(u) - (a-b)\frac{\psi(u)}{u}),
\end{aligned}$$

using the easy formulas

$$\int \pi(d\ell)\ell(1 - e^{-u\ell}) = \psi'(u) - \alpha - 2\beta u\,,$$

$$\int \pi(d\ell) \int_0^\ell dx (1 - e^{-ux}) = \frac{1}{u}(\psi(u) - \alpha u - \beta u^2)\,.$$

It follows that

$$A_1 = \beta r \lambda \psi^{-1}(\lambda) \int_0^\infty da \int_0^a db\, h(b) e^{-b\psi'(\psi^{-1}(\lambda)) - (a-b)(\lambda/\psi^{-1}(\lambda))}$$
$$= \beta r \psi^{-1}(\lambda)^2 \int_0^\infty db\, h(b) e^{-b\psi'(\psi^{-1}(\lambda))}.$$

To evaluate A_2, first observe that, with the notation preceding Proposition 3.1.3, we have

$$A_2 = \lambda \int_0^\infty da\, e^{-\alpha a}\, E\Big[\int_{\{s \leqslant a\}} \mathcal{N}(ds d\ell dx)\, h(s)\, (e^{-(1-r)x\psi^{-1}(\lambda)} - e^{-x\psi^{-1}(\lambda)})$$
$$\times \exp\Big(-\psi^{-1}(\lambda) \Big(\int_{\{s' \leqslant a\}} \mathcal{N}(ds' d\ell' dx')(\ell' - x') + \int_{\{s' < s\}} \mathcal{N}(ds' d\ell' dx') x' \Big) \Big) \Big].$$

From Lemma 3.1.5, it follows that

$$A_2 = \lambda \int_0^\infty da\, e^{-\alpha a} \int_0^a db\, h(b)\, \mathbb{M}_a(e^{-(\mu([0,b]) + \nu([0,a]))\psi^{-1}(\lambda)})$$
$$\times \int \pi(d\ell) \int_0^\ell dx (e^{-(1-r)x\psi^{-1}(\lambda)} - e^{-x\psi^{-1}(\lambda)}) e^{-(\ell - x)\psi^{-1}(\lambda)}$$
$$= \psi^{-1}(\lambda) \int_0^\infty db\, h(b)\, e^{-b\psi'(\psi^{-1}(\lambda))}$$
$$\times \int \pi(d\ell) \int_0^\ell dx (e^{-(1-r)x\psi^{-1}(\lambda)} - e^{-x\psi^{-1}(\lambda)}) e^{-(\ell - x)\psi^{-1}(\lambda)}$$

where the last equality is obtained from the same calculations as those made in evaluating A_1. Furthermore, straightforward calculations give

$$\int \pi(d\ell) \int_0^\ell dx (e^{-(1-r)x\psi^{-1}(\lambda)} - e^{-x\psi^{-1}(\lambda)}) e^{-(\ell - x)\psi^{-1}(\lambda)}$$
$$= \psi'(\psi^{-1}(\lambda)) - \beta r \psi^{-1}(\lambda) + \frac{1}{r\psi^{-1}(\lambda)} (\psi((1-r)\psi^{-1}(\lambda)) - \lambda).$$

By substituting this in the previous display and combining with the formula for A_1, we arrive at the result of the proposition. □

Proof of Theorem 3.2.1. — It is convenient to introduce the random variable Λ defined by

$$\Lambda = \begin{cases} K & \text{if } M \geqslant 2 \\ H_\tau & \text{if } M = 1 \end{cases}$$

On the event $\{M = 1\}$ we also set $\xi = 0$. We can easily compute the law of the pair (Λ, ξ). Indeed, by applying the Markov property at τ as previously, we easily get

$$\begin{aligned} N(h(\Lambda)1_{\{M=1\}}) &= N(h(H_\tau)1_{\{M=1\}}) \\ &= N(h(H_\tau)\, e^{-<\rho_\tau, 1>\psi^{-1}(\lambda)}) \\ &= \lambda \int_0^\infty da\, e^{-\alpha a}\, h(a) \int \mathbb{M}(d\mu d\nu)\, e^{-(\langle \mu, 1\rangle + <\nu, 1>)\psi^{-1}(\lambda)}) \\ &= \lambda \int_0^\infty da\, h(a)\, e^{-\psi'(\psi^{-1}(\lambda))a} \end{aligned}$$

By combining with Proposition 3.2.3, we get

$$\begin{aligned} &N(r^\xi h(\Lambda)1_{\{M\geqslant 1\}}) \\ &\qquad = \Big(r\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda)) + \psi((1-r)\psi^{-1}(\lambda))\Big) \int_0^\infty db\, h(b)\, e^{-b\psi'(\psi^{-1}(\lambda))}. \end{aligned}$$

This formula entails that we have the following properties under $N(\cdot \mid M \geqslant 1)$: The variables Λ and ξ are independent, Λ is exponentially distributed with parameter $\psi'(\psi^{-1}(\lambda))$, and the generating function of ξ is as stated in Theorem 3.2.1. To complete the proof, it remains to verify the "recursivity property" of the tree $\theta(H, \tau_1, \ldots, \tau_M)$, that is to verify that under $N(\cdot \mid M \geqslant 2)$, the shifted trees corresponding to each individual in the first generation are independent and distributed as the whole tree under $N(\cdot \mid M \geqslant 1)$. This is a consequence of the following claim.

Claim. — *Let (α_j, β_j), $j = 1, \ldots, \xi$ be the excursion intervals of H above level Λ that contain at least one mark, ranked in chronological order, and for every $j = 1, \ldots, \xi$ let $h_j(s) = H_{(\alpha_j + s)\wedge\beta_j} - \Lambda$ be the corresponding excursion. Then, conditionally on the pair (Λ, ξ), the excursions $h_1, \ldots, h_\xi$ are independent and distributed according to the law of $(H_s, s \geqslant 0)$ under $N(\cdot \mid M \geqslant 1)$.*

To verify this property, we first argue under $\mathbf{P}^*_\mu$ as previously. Precisely, we consider the excursions ω_j for all $j \in J$ such that $H_{a_j} = L$ and $(a_j, b_j) \cap \mathcal{P}_\lambda \neq \varnothing$. We denote by $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_\zeta$ these excursions, ranked in chronological order. Then property (P) and familiar properties of Poisson measures give the following fact. For every $k \geqslant 1$, under the measure $\mathbf{P}^*_\mu(\cdot \mid \zeta = k)$, the excursions $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_k$ are independent, distributed according to $N(\cdot \mid M \geqslant 1)$, and these excursions are also independent of the measure

$$\sum_{H_{a_j} > L} \delta_{(I_{a_j}, \omega_j)}\,.$$

Let $\sigma_L := \inf\{s \geqslant 0 : H_s = L\}$. Excursion theory for $X - I$ allows us to reconstruct the process $(X_{s\wedge\sigma_L}, s \geqslant 0)$ as a measurable function of the point measure in the last display. Hence we can also assert that, under $\mathbf{P}^*_\mu(\cdot \mid \zeta = k)$, $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_k$ are independent of $(X_{s\wedge\sigma_L}, s \geqslant 0)$. In particular, they are independent of $L = H(k_{\langle\mu,1\rangle - I_{\sigma_L}}\mu)$.

We now apply these properties to the shifted process $X_{\tau+\cdot}$ under $N(\cdot \mid M \geqslant 1)$. We slightly abuse notation and keep denoting by $\widetilde{\omega}_1, \dots, \widetilde{\omega}_\zeta$ the excursions of $X_{\tau+\cdot} - I_{\tau+\cdot}$ that contain a mark (so that $\zeta = \xi - 1$ on the event $M \geqslant 2$). By construction, for every $j \in \{2, \dots, \xi\}$, the function h_j is the height process of $\widetilde{\omega}_{j-1}$. Hence it follows from the previous properties under $\mathbf{P}^*_\mu$ that under $N(\cdot \mid \xi = k)$ (for a fixed $k \geqslant 2$), the processes $h_2, \dots, h_k$ are independent, have the distribution given in the claim, and are also independent of the pair (h_1, Λ). Hence, for any test functions $F_1, \dots, F_k, G$, we get

$$\begin{aligned} &N(F_1(h_1) \dots F_k(h_k) G(\Lambda) \mid \xi = k) \\ &\qquad = N(F_2(H) \mid M \geqslant 1) \cdots N(F_k(H) \mid M \geqslant 1)\, N(F_1(h_1) G(\Lambda) \mid \xi = k). \end{aligned}$$

Now from Corollary 3.1.6, we know that the time-reversed process $(H_{(\sigma-s)_+}, s \geqslant 0)$ has the same distribution under N as the process $(H_s, s \geqslant 0)$. Furthermore, this time-reversal operation will leave Λ and ξ invariant and transform the excursion h_1 into the time-reversal of h_ξ, denoted by $\widehat{h}_\xi$ (provided we do simultaneously the similar transformation on the underlying Poissonnian marks). It follows that

$$\begin{aligned} N(F_1(h_1) G(\Lambda) \mid \xi = k) &= N(F_1(\widehat{h}_k) G(\Lambda) \mid \xi = k) \\ &= N(F_1(\widehat{h}_k) \mid \xi = k) N(G(\Lambda) \mid \xi = k) \\ &= N(F_1(H) \mid M \geqslant 1) N(G(\Lambda) \mid M \geqslant 1). \end{aligned}$$

By substituting this equality in the previous displayed formula, we obtain the claim. This completes the proof of Theorem 3.2.1. □

3.3. Marginals of stable trees

We first reformulate Theorem 3.2.1 in a way more suitable for our applications. Recall that $\mathbf{T}_p$ is the set of all (rooted ordered) trees with p leaves. If $\mathcal{T} \in \mathbf{T}_p$ we denote by $\mathcal{L}_\mathcal{T}$ the set of all leaves of $\mathcal{T}$, and set $\mathcal{N}_\mathcal{T} = \mathcal{T} \backslash \mathcal{L}_\mathcal{T}$. Recall the notation $k_v = k_v(\mathcal{T})$ for the number of children of an element v of $\mathcal{T}$. We write $\mathbf{T}^*_p$ for the subset of $\mathbf{T}_p$ composed of all trees $\mathcal{T}$ such that $k_v(\mathcal{T}) \geqslant 2$ for every $v \in \mathcal{N}_\mathcal{T}$. By construction, the skeleton of the marked trees $\theta(e, t_1, \dots, t_p)$ always belongs to $\mathbf{T}^*_p$.

Theorem 3.3.1. — *Let $p \geqslant 1$. Then, for any nonnegative measurable function Φ on $\mathbb{T}_p$, and every $\lambda > 0$,*

$$\begin{aligned} &N\Big(e^{-\lambda\sigma} \int_{\{t_1 < \dots < t_p < \sigma\}} dt_1 \dots dt_p\, \Phi(\theta(H, t_1, \dots, t_p))\Big) \\ &\qquad = \sum_{\mathcal{T} \in \mathbf{T}^*_p} \Big(\prod_{v \in \mathcal{N}_\mathcal{T}} \frac{|\psi^{(k_v)}(\psi^{-1}(\lambda))|}{k_v!} \Big) \\ &\qquad\qquad \times \int \prod_{v \in \mathcal{T}} dh_v\, \exp\Big(- \psi'(\psi^{-1}(\lambda)) \sum_{v \in \mathcal{T}} h_v \Big) \Phi(\mathcal{T}, (h_v)_{v \in \mathcal{T}}). \end{aligned}$$

Proof. — By elementary properties of the standard Poisson process, the left side is equal to

$$\lambda^{-p} N(\Phi(\theta(H,\tau_1,\dots,\tau_M))1_{\{M=p\}})$$

with the notation of the previous section. This quantity can be evaluated thanks to Theorem 3.2.1: From the generating function of the offspring distribution, we get

$$\begin{aligned}
P[\xi=0] &= \frac{\lambda}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}\\
P[\xi=1] &= 0\\
P[\xi=k] &= \frac{1}{k!}\,\frac{\psi^{-1}(\lambda)^{k-1}|\psi^{(k)}(\psi^{-1}(\lambda))|}{\psi'(\psi^{-1}(\lambda))}, \qquad \text{for every } k\geqslant 2.
\end{aligned}$$

Hence the probability under $N(\cdot \mid M\geqslant 1)$ that the skeleton of the tree $\theta(H,\tau_1,\dots,\tau_M)$ is equal to a given tree $\mathcal{T}\in\mathbf{T}_p^*$ is

$$\Big(\prod_{v\in\mathcal{N}_\mathcal{T}} \frac{1}{k_v!}\,\frac{\psi^{-1}(\lambda)^{k_v-1}|\psi^{(k_v)}(\psi^{-1}(\lambda))|}{\psi'(\psi^{-1}(\lambda))}\Big)\Big(\frac{\lambda}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}\Big)^p$$
$$= \frac{1}{\psi^{-1}(\lambda)}\,\frac{1}{\psi'(\psi^{-1}(\lambda))^{|\mathcal{T}|}}\,\lambda^p \prod_{v\in\mathcal{N}_\mathcal{T}} \Big(\frac{1}{k_v!}\,|\psi^{(k_v)}(\psi^{-1}(\lambda))|\Big).$$

Recalling that $N(M\geqslant 1)=\psi^{-1}(\lambda)$, and using the fact that the lifetimes h_v, $v\in\mathcal{T}$ are independently distributed according to the exponential distribution with parameter $\psi'(\psi^{-1}(\lambda))$, we easily arrive at the formula of the theorem. □

By letting $\lambda\to 0$ in the preceding theorem, we get the following corollary, which is closely related to Proposition 3.2 of [**33**].

Corollary 3.3.2. — *Suppose that $\int\pi(dr)\,r^p<\infty$. Then, for any nonnegative measurable function Φ on $\mathcal{T}_p$,*

$$N\Big(\int_{\{t_1<\cdots<t_p<\sigma\}} dt_1\dots t_p\,\Phi(\theta(H,t_1,\dots,t_p))\Big)$$
$$= \sum_{\mathcal{T}\in\mathbf{T}_p^*}\Big(\prod_{v\in\mathcal{N}_\mathcal{T}}\beta_{k_v}\Big)\int\prod_{v\in\mathcal{T}} dh_v\,\exp\Big(-\alpha\sum_{v\in\mathcal{T}}h_v\Big)\Phi(\mathcal{T},(h_v)_{v\in\mathcal{T}}),$$

where, for every $k=2,\dots,p$,

$$\beta_k=\frac{|\psi^{(k)}(0)|}{k!}=\beta 1_{\{k=2\}}+\frac{1}{k!}\int r^k\,\pi(dr).$$

Remark. — The formula of the corollary still holds without the assumption $\int\pi(dr)\,r^p<\infty$ but it has to be interpreted properly since some of the numbers β_k may be infinite.

From now on, we concentrate on the stable case $\psi(u) = u^\gamma$ for $1 < \gamma < 2$. Then the Lévy process X satisfies the scaling property

$$(X_{\lambda t}, t \geqslant 0) \overset{(d)}{=} (\lambda^{1/\gamma} X_t, t \geqslant 0)$$

under P. Thanks to this property, it is possible to choose a regular version of the conditional probabilities $N_{(u)} := N(\cdot \mid \sigma = u)$ in such a way that for every $u > 0$ and $\lambda > 0$, the law of $(\lambda^{-1/\gamma} X_{\lambda t}, t \geqslant 0)$ under $N_{(\lambda u)}$ is $N_{(u)}$. Standard arguments then show that the height process $(H_s, s \geqslant 0)$ is well defined as a continuous process under the probability measures $N_{(u)}$. Furthermore, it follows from the approximations of H_t (see Lemma 1.1.3) that the law of $(H_{\lambda s}, s \geqslant 0)$ under $N_{(\lambda u)}$ is equal to the law of $(\lambda^{1-\frac{1}{\gamma}} H_s, s \geqslant 0)$ under $N_{(u)}$.

The probability measure $N_{(1)}$ is called the law of the normalized excursion. The tree coded by the process $(H_t, 0 \leqslant t \leqslant 1)$ under $N_{(1)}$ (in the sense of Section 0.6) is called the stable continuum random tree. The next theorem identifies its finite-dimensional marginals, in the sense of Aldous [**3**].

Theorem 3.3.3. — *Suppose that $\psi(u) = u^\gamma$ for some $\gamma \in (1, 2)$. Then the law of the tree $\theta(H, t_1, \ldots, t_p)$ under the probability measure*

$$p!\, 1_{\{0<t_1<t_2<\cdots<t_p<1\}} dt_1 \ldots dt_p\, N_{(1)}(d\omega)$$

is characterized by the following properties:
(i) *The probability of a given skeleton $\mathcal{T} \in \mathbf{T}_p^*$ is*

$$\frac{p!}{\prod_{v \in \mathcal{N}_\mathcal{T}} k_v!} \, \frac{\prod_{v \in \mathcal{N}_\mathcal{T}} |(\gamma-1)(\gamma-2)\ldots(\gamma-k_v+1)|}{(\gamma-1)(2\gamma-1)\ldots((p-1)\gamma-1)}.$$

(ii) *If $p \geqslant 2$, then conditionally on the skeleton $\mathcal{T}$, the lifetimes $(h_v)_{v \in \mathcal{T}}$ have a density with respect to Lebesgue measure on $\mathbb{R}_+^\mathcal{T}$ given by*

$$\frac{\Gamma(p - 1/\gamma)}{\Gamma(\delta_\mathcal{T})} \gamma^{|\mathcal{T}|} \int_0^1 du\, u^{\delta_\mathcal{T} - 1}\, q\Big(\gamma \sum_{v \in \mathcal{T}} h_v, 1 - u\Big)$$

where $\delta_\mathcal{T} = p - (1 - 1/\gamma)|\mathcal{T}| - 1/\gamma > 0$, and $q(s, u)$ is the continuous density at time s of the stable subordinator with exponent $1 - 1/\gamma$, which is characterized by

$$\int_0^\infty du\, e^{-\lambda u}\, q(s, u) = \exp(-s\, \lambda^{1-1/\gamma}).$$

If $p = 1$, then $\mathcal{T} = \{\varnothing\}$ and the law of $h_\varnothing$ has density

$$\gamma\, \Gamma(1 - 1/\gamma)\, q(\gamma h, 1)$$

with respect to Lebesgue measure on $\mathbb{R}_+$.

Proof. — For every $u > 0$, let $\Theta_{(u)}$ be the law of the tree $\theta(H, s_1, \dots, s_p)$ under the probability measure

$$p! u^{-p} 1_{\{s_1 < s_2 < \cdots < s_p < u\}} \, ds_1 \dots ds_p \, N_{(u)}(d\omega).$$

By the scaling properties of the height process (see the remarks before Theorem 3.3.3), we have, for every $u > 0$ and every $\mathcal{T} \in \mathbf{T}_p^*$,

$$\Theta_{(u)}(\{\mathcal{T}\} \times \mathbb{R}_+^{\mathcal{T}}) = \Theta_{(1)}(\{\mathcal{T}\} \times \mathbb{R}_+^{\mathcal{T}}).$$

Hence, by conditioning with respect to σ in Theorem 3.3.1, we get

$$\frac{1}{p!} N(\sigma^p e^{-\lambda\sigma}) \, \Theta_{(1)}(\{\mathcal{T}\} \times \mathbb{R}_+^{\mathcal{T}}) = \psi'(\psi^{-1}(\lambda))^{-|\mathcal{T}|} \prod_{v \in \mathcal{N}_{\mathcal{T}}} \frac{|\psi^{(k_v)}(\psi^{-1}(\lambda))|}{k_v!}. \tag{103}$$

From this, we can compute $\Theta_{(1)}(\{\mathcal{T}\} \times \mathbb{R}_+^{\mathcal{T}})$ by observing that, for every $k \geqslant 1$,

$$\psi^{(k)}(\psi^{-1}(\lambda)) = \gamma(\gamma - 1) \cdots (\gamma - k + 1) \, \lambda^{1 - k/\gamma},$$

and

$$N(\sigma^p e^{-\lambda\sigma}) = \Big| \frac{d^p}{d\lambda^p} N(1 - e^{-\lambda\sigma}) \Big| = \Big| \frac{d^p}{d\lambda^p} \psi^{-1}(\lambda) \Big| = \Big| \frac{1}{\gamma}(\frac{1}{\gamma} - 1) \cdots (\frac{1}{\gamma} - p + 1) \Big| \lambda^{\frac{1}{\gamma} - p}.$$

If we substitute these expressions in (103), the terms in λ cancel and we get part (i) of the theorem.

To prove (ii), fix $\mathcal{T} \in \mathbf{T}_p^*$, and let D be a bounded Borel subset of $\mathbb{R}_+^{\mathcal{T}}$. Write $p_\sigma(du)$ for the law of σ under N. Then by applying Theorem 3.3.1 with $\Phi = 1_{\{\mathcal{T}\} \times D}$ and $\Phi = 1_{\{\mathcal{T}\} \times \mathbb{R}_+^{\mathcal{T}}}$, we get

$$\begin{aligned} &\int p_\sigma(du) \, e^{-\lambda u} \, u^p \, \Theta_{(u)}(\{h_v\}_{v \in \mathcal{T}} \in D \mid \mathcal{T}) \\ &\quad = \Big(\int p_\sigma(du) \, e^{-\lambda u} \, u^p \Big) \int_D \prod_{v \in \mathcal{T}} dh_v \, \psi'(\psi^{-1}(\lambda))^{|\mathcal{T}|} \exp\Big(- \psi'(\psi^{-1}(\lambda)) \sum_{v \in \mathcal{T}} h_v \Big). \end{aligned} \tag{104}$$

By scaling (or inverting $N(1 - e^{-\lambda\sigma}) = \lambda^{1/\gamma}$), we have $p_\sigma(du) = c \, u^{-1 - \frac{1}{\gamma}} \, du$. It follows that

$$\begin{aligned} &\int_0^\infty du \, e^{-\lambda u} \, u^{p - 1 - 1/\gamma} \, \Theta_{(u)}(\{h_v\}_{v \in \mathcal{T}} \in D \mid \mathcal{T}) \\ &\qquad = \frac{\gamma^{|\mathcal{T}|} \Gamma(p - 1/\gamma)}{\lambda^{\delta_{\mathcal{T}}}} \int_D \prod_{v \in \mathcal{T}} dh_v \, \exp\Big(- \gamma \lambda^{1 - 1/\gamma} \sum_{v \in \mathcal{T}} h_v \Big), \end{aligned}$$

where $\delta_{\mathcal{T}} = p - (1-1/\gamma)|\mathcal{T}| - 1/\gamma$ as in the theorem. Suppose first that $p \geqslant 2$. To invert the Laplace transform, observe that the right side can be written as

$$\frac{\gamma^{|\mathcal{T}|}\Gamma(p-1/\gamma)}{\Gamma(\delta_{\mathcal{T}})} \int_0^\infty du\, e^{-\lambda u}\, u^{\delta_{\mathcal{T}}-1} \times \int_0^\infty du'\, e^{-\lambda u'} \Big(\int_D \prod_{v\in\mathcal{T}} dh_v\, q(\gamma \sum_{v\in\mathcal{T}} h_v, u') \Big)$$
$$= \frac{\gamma^{|\mathcal{T}|}\Gamma(p-1/\gamma)}{\Gamma(\delta_{\mathcal{T}})} \int_0^\infty du\, e^{-\lambda u} \Big(\int_D \prod_{v\in\mathcal{T}} dh_v \int_0^u dr\, r^{\delta_{\mathcal{T}}-1}\, q(\gamma \sum_{v\in\mathcal{T}} h_v, u-r) \Big).$$

The first formula of (ii) now follows. In the case $p = 1$, we get

$$\int_0^\infty du\, e^{-\lambda u}\, u^{-1/\gamma}\, \Theta_{(u)}(h_\varnothing \in D) = \gamma\Gamma(1-1/\gamma) \int_D dh\, \exp(-\gamma\lambda^{1-1/\gamma} h),$$

and the stated result follows by inverting the Laplace transform. □

Remarks

(a) The previous proof also readily gives the analogue of Theorem 3.3.3 in the case $\psi(u) = u^2$, which corresponds to the finite-dimensional marginals of Aldous' continuum random tree (see Aldous [**3**], or Chapter 3 of [**31**]). In that case, the discrete skeleton of $\theta(H, t_1, \ldots, t_p)$ is with probability one a binary tree, meaning that $k_v = 2$ for every $v \in \mathcal{N}_{\mathcal{T}}$. The law of $\mathcal{T}(H, t_1, \ldots, t_p)$ is the uniform probability measure on the set of all binary trees in $\mathbf{T}_p^*$, so that the probability of each possible skeleton is

$$\frac{p!}{2^{p-1}\,(1\times 3\times\cdots\times(2p-3))}.$$

This formula can be deduced informally by letting γ tend to 2 in Theorem 3.3.3 (i).

To obtain the analogue of (ii), note that there is an explicit formula for $q(s,u)$ when $\psi(u) = u^2$:

$$q(s,u) = \frac{s}{2\sqrt{\pi}u^{3/2}}\, e^{-s^2/(4u)}.$$

Observe that when the skeleton is binary, we have always $|\mathcal{T}| = 2p-1$. It follows that the powers of λ cancel in the right side of (104), and after straightforward calculations, we obtain that the density of $(h_v)_{v\in\mathcal{T}}$ on $\mathbb{R}_+^{2p-1}$ is

$$2^{2p-1}\Gamma(p-1/2)\, q\Big(2\sum h_v, 1\Big) = 2^p\,(1\times 3\times\cdots\times(2p-3)) \Big(\sum h_v\Big) \exp\Big(-\Big(\sum h_v\Big)^2\Big).$$

Compare with Aldous [**3**] or Chapter 3 of [**31**], but note that constants are different because $\psi(u) = u^2$ corresponds to a Brownian motion with variance $2t$ (also the CRT is coded by twice the normalized Brownian excursion in [**3**]).

(b) We could get rid of the factor

$$\frac{p!}{\prod_{v\in\mathcal{N}_{\mathcal{T}}} k_v!}$$

in Theorem 3.3.3 by considering rooted (unordered) trees with p labelled leaves rather than rooted ordered trees : See the discussion at the end of Chapter 3 of [**31**].

CHAPTER 4

THE LÉVY SNAKE

4.1. The construction of the Lévy snake

Our goal is now to combine the branching structure studied in the previous chapters with a spatial displacement prescribed by a Markov process ξ. Throughout this chapter, we assume that H has continuous paths (the condition $\int^\infty du/\psi(u) < \infty$ holds) although many of the results can presumably be extended to the general case.

4.1.1. Construction of the snake with a fixed lifetime process. — We consider a Markov process ξ with càdlàg paths and values in a Polish space E, whose topology is defined by a metric δ. For simplicity, we will assume that ξ is defined on the canonical space $\mathbb{D}(\mathbb{R}_+, E)$ of càdlàg functions from $\mathbb{R}_+$ into E. For every $x \in E$, we denote by Π_x the distribution of ξ started at x. It is implicitly assumed in our definition of a Markov process that the mapping $x \to \Pi_x$ is measurable. We also assume that ξ is continuous in probability under Π_x (equivalently, ξ has no fixed discontinuities, $\Pi_x[\xi_s \neq \xi_{s-}] = 0$ for every $s > 0$). On the other hand, we do not assume that ξ is strong Markov.

For $x \in E$, we denote by $\mathcal{W}_x$ the space of all E-valued killed paths started at x. An element of $\mathcal{W}_x$ is a càdlàg mapping $\mathrm{w} : [0, \zeta) \longrightarrow E$ such that $\mathrm{w}(0) = x$. Here $\zeta \in (0, \infty)$ is called the lifetime of the path. When there is a risk of confusion we write $\zeta = \zeta_{\mathrm{w}}$. Note that we do not require the existence of the left limit $\mathrm{w}(\zeta-)$. By convention, the point x is also considered as a killed path with lifetime 0. We set $\mathcal{W} = \cup_{x \in E} \mathcal{W}_x$ and equip $\mathcal{W}$ with the distance

$$d(\mathrm{w}, \mathrm{w}') = \delta(\mathrm{w}(0), \mathrm{w}'(0)) + |\zeta - \zeta'| + \int_0^{\zeta \wedge \zeta'} dt\, (d_t(\mathrm{w}_{\leqslant t}, \mathrm{w}'_{\leqslant t}) \wedge 1),$$

where d_t is the Skorokhod metric on the space $\mathbb{D}([0,t], E)$, and $\mathrm{w}_{\leqslant t}$ denotes the restriction of w to the interval $[0,t]$. It is then elementary to check that the space $(\mathcal{W}, d)$ is a Polish space. The space (E, δ) is embedded isometrically in $\mathcal{W}$ thanks to the previous convention.

Let $x \in E$ and $\mathrm{w} \in \mathcal{W}_x$. If $a \in [0, \zeta_{\mathrm{w}})$ and $b \in [a, \infty)$, we can define a probability measure $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ on $\mathcal{W}_x$ by requiring that:

(i) $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s., $\mathrm{w}'(t) = \mathrm{w}(t)$, $\forall\, t \in [0, a)$;

(ii) $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s., $\zeta_{\mathrm{w}'} = b$;

(iii) the law of $(\mathrm{w}'(a+t), 0 \leqslant t < b-a)$ under $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ is the law of $(\xi_t, 0 \leqslant t < b-a)$ under $\Pi_{\mathrm{w}(a-)}$.

In (iii), $\mathrm{w}(0-) = x$ by convention. In particular, $R_{0,b}(\mathrm{w}, d\mathrm{w}')$ is the law of $(\xi_t, 0 \leqslant t < b)$ under Π_x, and $R_{0,0}(\mathrm{w}, d\mathrm{w}') = \delta_x(d\mathrm{w}')$.

When $\mathrm{w}(\zeta_{\mathrm{w}}-)$ exists, we may and will extend the previous definition to the case $a = \zeta_{\mathrm{w}}$.

We denote by $(W_s, s \geqslant 0)$ the canonical process on the product space $(\mathcal{W})^{\mathbb{R}_+}$. We will abuse notation and also write $(W_s, s \geqslant 0)$ for the canonical process on the set $C(\mathbb{R}_+, \mathcal{W})$ of all continuous mappings from $\mathbb{R}_+$ into $\mathcal{W}$. Let us fix $x \in E$ and $\mathrm{w}_0 \in \mathcal{W}_x$, and let $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ be such that $h(0) = \zeta_{\mathrm{w}_0}$. For $0 \leqslant s \leqslant s'$, we set

$$m_h(s, s') = \inf_{s \leqslant r \leqslant s'} h(r).$$

We assume that either $\mathrm{w}_0(\zeta_{\mathrm{w}_0}-)$ exists or $m_h(0, r) < h(0)$ for every $r > 0$. Then, the Kolmogorov extension theorem can be used to construct the (unique) probability measure $Q^h_{\mathrm{w}_0}$ on $(\mathcal{W}_x)^{\mathbb{R}_+}$ such that, for $0 = s_0 < s_1 < \cdots < s_n$,

$$\begin{aligned} &Q^h_{\mathrm{w}_0}[W_{s_0} \in A_0, \ldots, W_{s_n} \in A_n] \\ &= 1_{A_0}(\mathrm{w}_0) \int_{A_1 \times \cdots \times A_n} R_{m_h(s_0,s_1),h(s_1)}(\mathrm{w}_0, d\mathrm{w}_1) \ldots R_{m_h(s_{n-1},s_n),h(s_n)}(\mathrm{w}_{n-1}, d\mathrm{w}_n). \end{aligned}$$

Notice that our assumption on the pair (w_0, h) is needed already for $n = 1$ to make sense of the measure $R_{m_h(s_0,s_1),h(s_1)}(\mathrm{w}_0, d\mathrm{w}_1)$.

From the previous definition, it is clear that, for every $s < s'$, $Q^h_{\mathrm{w}_0}$ a.s.,

$$W_{s'}(t) = W_s(t), \qquad \forall\, t < m_h(s, s'),$$

and furthermore $\zeta_{W_s} = h(s)$, $\zeta_{W_{s'}} = h(s')$. Hence,

$$d(W_s, W_{s'}) \leqslant |h(s) - h(s')| + |(h(s) \wedge h(s')) - m_h(s, s')| = (h(s) \vee h(s')) - m_h(s, s').$$

From this bound, it follows that the mapping $s \longrightarrow W_s$ is $Q^h_{\mathrm{w}_0}$ a.s. uniformly continuous on the bounded subsets of $[0, \infty) \cap \mathbb{Q}$. Hence this mapping has $Q^h_{\mathrm{w}_0}$ a.s. a continuous extension to the positive real line. We abuse notation and still denote by $Q^h_{\mathrm{w}_0}$ the induced probability measure on $C(\mathbb{R}_+, \mathcal{W}_x)$. By an obvious continuity argument, we have $\zeta_{W_s} = h(s)$, for every $s \geqslant 0$, $Q^h_{\mathrm{w}_0}$ a.s., and

$$W_{s'}(t) = W_s(t), \qquad \forall\, t < m_h(s, s'), \quad \forall\, s < s', \quad Q^h_{\mathrm{w}_0} \text{ a.s.}$$

We will refer to this last property as the *snake property*. The process $(W_s, s \geqslant 0)$ is under $Q^h_{\mathrm{w}_0}$ a time-inhomogeneous continuous Markov process.

4.1.2. The definition of the Lévy snake. — Following the remarks of the end of Chapter 1, we now consider the exploration process ρ as a Markov process with values in the set

$$M_f^0 = \{\mu \in M_f(\mathbb{R}_+) : H(\mu) < \infty \text{ and } \operatorname{supp}\mu = [0, H(\mu)]\} \cup \{0\}.$$

We denote by $\mathbf{P}_\mu$ the law of $(\rho_s, s \geqslant 0)$ started at μ. We will write indifferently $H(\rho_s)$ or H_s.

We then define Θ as the set of all pairs $(\mu, \mathrm{w}) \in M_f^0 \times \mathcal{W}$ such that $\zeta_{\mathrm{w}} = H(\mu)$, and at least one of the following two properties hold:

(i) $\mu(\{H(\mu)\}) = 0$;
(ii) $\mathrm{w}(\zeta_{\mathrm{w}}-)$ exists.

We equip Θ with the product distance on $M_f^0 \times \mathcal{W}$. For every $y \in E$, we also set $\Theta_y = \{(\mu, \mathrm{w}) \in \Theta : \mathrm{w}(0) = y\}$.

From now on until the end of this section, we fix a point $x \in E$.

Notice that when $H(\mu) > 0$ and $\mu(\{H(\mu)\}) = 0$, we have $\inf_{[0,s]} H(\rho_r) < H(\mu)$ for every $s > 0$, $\mathbf{P}_\mu$ a.s. To see this, note that $I_s < 0$ for every $s > 0$, P-a.s., by the regularity of 0 for $(-\infty, 0)$, for the underlying Lévy process. Then, P-a.s. for every $s > 0$ we can pick $r \in (0, s)$ such that $X_r = I_r < 0$. Formula (18) then shows that $\rho_r^\mu = k_{-I_r}\mu$ and our assumption $\mu(\{H(\mu)\}) = 0$ implies that $H(\rho_r^\mu) < H(\mu)$, which gives the claim.

Using the last observation and the previous subsection, we can for every $(\mu, \mathrm{w}) \in \Theta_x$ define a probability measure $\mathbb{P}_{\mu,\mathrm{w}}$ on $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+) \times \mathcal{W})$ by the formula

$$\mathbb{P}_{\mu,\mathrm{w}}(d\rho\, dW) = \mathbf{P}_\mu(d\rho)\, Q_{\mathrm{w}}^{H(\rho)}(dW),$$

where in the right side $H(\rho)$ obviously stands for the function $(H(\rho_s), s \geqslant 0)$, which is continuous $\mathbf{P}_\mu$ a.s.

We will write $\mathbb{P}_x$ instead of $\mathbb{P}_{0,x}$ when $\mu = 0$.

Proposition 4.1.1. — *The process (ρ_s, W_s) is under $\mathbb{P}_{\mu,\mathrm{w}}$ a càdlàg Markov process in Θ_x.*

Proof. — We first verify that $\mathbb{P}_{\mu,\mathrm{w}}$ a.s. the process (ρ_s, W_s) does not visit Θ_x^c. We must check that $W_s(H_s-)$ exists whenever $\rho_s(\{H_s\}) > 0$. Suppose thus that $\rho_s(\{H_s\}) > 0$. Then, we have also $\rho_{s'}(\{H_s\}) > 0$, and so $H_{s'} \geqslant H_s$, for all $s' > s$ sufficiently close to s. In particular, we can find a rational $s_1 > s$ such that $H_{s_1} \geqslant H_s$ and $\inf_{[s,s_1]} H_r = H_s$, which by the snake property implies that $W_{s_1}(t) = W_s(t)$ for every $t \in [0, H_s)$. However, from the construction of the measures Q_{w}^h, it is clear that a.s. for every rational $r > 0$, the killed path W_r must have a left limit at every $t \in (0, H_r]$. We conclude that $W_s(H_s-) = W_{s_1}(H_s-)$ exists.

The càdlàg property of paths is obvious by construction. To obtain the Markov property, we consider nonnegative functions $f_1, \ldots, f_n$ on M_f^0 and $g_1, \ldots, g_n$ on $\mathcal{W}_x$.

Then, if $0 < s_1 < \cdots < s_n$,

$$\begin{aligned}
&\mathbb{E}_{\mu,\mathrm{w}}[f_1(\rho_{s_1})g_1(W_{s_1})\dots f_n(\rho_{s_n})g_n(W_{s_n})] \\
&\quad = \mathbf{E}_\mu\Big[f_1(\rho_{s_1})\dots f_n(\rho_{s_n})Q_{\mathrm{w}}^{H(\rho)}[g_1(W_{s_1})\dots g_n(W_{s_n})]\Big] \\
&\quad = \mathbf{E}_\mu\Big[f_1(\rho_{s_1})\dots f_n(\rho_{s_n})\int R_{m_{H(\rho)}(0,s_1),H(\rho_{s_1})}(\mathrm{w},d\mathrm{w}_1) \\
&\qquad \cdots R_{m_{H(\rho)}(s_{n-1},s_n),H(\rho_{s_n})}(\mathrm{w}_{n-1},d\mathrm{w}_n)\, g_1(\mathrm{w}_1)\dots g_n(\mathrm{w}_n)\Big] \\
&\quad = \mathbb{E}_{\mu,\mathrm{w}}\Big[f_1(\rho_{s_1})g_1(W_{s_1})\dots f_{n-1}(\rho_{s_{n-1}})g_{n-1}(W_{s_{n-1}}) \\
&\qquad \mathbf{E}_{\rho_{s_{n-1}}}\Big[f_n(\rho_{s_n-s_{n-1}})\int R_{m_{H(\rho)}(0,s_n-s_{n-1}),H(\rho_{s_n-s_{n-1}})}(W_{s_{n-1}},d\mathrm{w})g_n(\mathrm{w})\Big]\Big],
\end{aligned}$$

where in the last equality we used the Markov property for $(\rho_s, s \geqslant 0)$ at time s_{n-1}. We get the desired result with a transition kernel given by

$$\begin{aligned}
\mathbf{Q}_r G(\mu,\mathrm{w}) &= \int \mathbf{P}_\mu(d\rho)\int R_{m_{H(\rho)}(0,r),H(\rho_r)}(\mathrm{w},d\mathrm{w}')\, G(\rho_r,\mathrm{w}') \\
&= \int \mathbf{P}_\mu(d\rho)\int Q_{\mathrm{w}}^{H(\rho)}(dW)\, G(\rho_r,W_r). \qquad \square
\end{aligned}$$

In what follows we will often use the convenient notation $\overline{W}_s = (\rho_s, W_s)$. By our construction, the conditional distribution under $\mathbb{P}_{\mu,\mathrm{w}}$ of $(W_s, s \geqslant 0)$ knowing $(\rho_s, s \geqslant 0)$ is $Q_{\mathrm{w}}^{H(\rho)}$. In particular, if we write $\zeta_s = \zeta_{W_s}$ for the lifetime of W_s, we have

$$\zeta_s = H(\rho_s) = H_s \text{ for every } s \geqslant 0,\ \mathbb{P}_{\mu,\mathrm{w}} \text{ a.s.}$$

4.1.3. The strong Markov property. — We denote by $(\mathcal{F}_s)_{s\geqslant 0}$ the canonical filtration on $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+)\times\mathcal{W})$.

Theorem 4.1.2. — *The process $(\overline{W}_s, s \geqslant 0; \mathbb{P}_{\mu,\mathrm{w}}, (\mu,\mathrm{w}) \in \Theta_x)$ is strong Markov with respect to the filtration $(\mathcal{F}_{s+})$.*

Proof. — Let $(\mu,\mathrm{w}) \in \Theta_x$. It is enough to prove that, if T is a bounded stopping time of the filtration $(\mathcal{F}_{s+})$, then, for any bounded $\mathcal{F}_{T+}$-measurable functional F, for any bounded Lipschitz continuous function f on Θ_x, and for every $t > 0$,

$$\mathbb{E}_{\mu,\mathrm{w}}[F\, f(\overline{W}_{T+t})] = \mathbb{E}_{\mu,\mathrm{w}}[F\, \mathbb{E}_{\overline{W}_T}[f(\overline{W}_t)]].$$

First observe that

$$\begin{aligned}
\mathbb{E}_{\mu,\mathrm{w}}[F\, f(\overline{W}_{T+t})] &= \lim_{n\to\infty}\sum_{k=1}^{\infty}\mathbb{E}_{\mu,\mathrm{w}}[F\, 1_{\{\frac{k-1}{n}\leqslant T<\frac{k}{n}\}}\, f(\overline{W}_{\frac{k}{n}+t})] \\
&= \lim_{n\to\infty}\sum_{k=1}^{\infty}\mathbb{E}_{\mu,\mathrm{w}}[F\, 1_{\{\frac{k-1}{n}\leqslant T<\frac{k}{n}\}}\, \mathbf{Q}_t f(\overline{W}_{\frac{k}{n}})].
\end{aligned}$$

In the first equality, we used the right continuity of paths, and in the second one the ordinary Markov property. We see that the desired result follows from the next lemma. □

Lemma 4.1.3. — *Let $t > 0$, let T be a bounded stopping time of the filtration $(\mathcal{F}_{s+})$ and let f be a bounded Lipschitz continuous function on Θ_x. Then the mapping $s \longrightarrow \mathbf{Q}_t f(\overline{W}_s)$ is $\mathbb{P}_{\mu,\mathrm{w}}$ a.s. right-continuous at $s = T$.*

Proof of Lemma 4.1.3. — We use the notation $Y_s = \langle \rho_s, 1 \rangle$. Recall that Y is distributed under $\mathbb{P}_{\mu,\mathrm{w}}$ as the reflected Lévy process $X - I$ started at $\langle \mu, 1 \rangle$. Let $\varepsilon > 0$. By the right-continuity of the paths of Y, if $s > T$ is sufficiently close to T, we have

$$\varepsilon_1(s) = Y_T - \inf_{u \in [T,s]} Y_u < \varepsilon, \qquad \varepsilon_2(s) = Y_s - \inf_{u \in [T,s]} Y_u < \varepsilon.$$

On the other hand, we know from (20) that $\rho_{T+r} = [k_{\varepsilon_1}\rho_T, \rho_r^{(T)}]$, and it follows that $k_{\varepsilon_1}\rho_T = k_{\varepsilon_2}\rho_s$. Furthermore, $\inf_{[T,s]} H(\rho_u) = H(k_{\varepsilon_1}\rho_T) = H(k_{\varepsilon_2}\rho_s)$, and by the snake property,

$$W_s(u) = W_T(u), \qquad \forall\, u \in [0, H(k_{\varepsilon_1}\rho_T)).$$

Let us fix $\overline{\mathrm{w}} = (\mu, \mathrm{w}) \in \Theta_x$, and set

$$\mathcal{V}_\varepsilon(\overline{\mathrm{w}}) = \big\{ \overline{\mathrm{w}}' = (\mu', \mathrm{w}') \in \Theta_x;\ \exists \varepsilon_1, \varepsilon_2 \in [0, \varepsilon),\ k_{\varepsilon_1}\mu = k_{\varepsilon_2}\mu', \\ \text{and } \mathrm{w}'(u) = \mathrm{w}(u),\ \forall\, u \in [0, H(k_{\varepsilon_1}\mu))\big\}.$$

In view of the preceding observations, the proof of Lemma 4.1.3 reduces to checking that

$$\lim_{\varepsilon \to 0} \Big(\sup_{\overline{\mathrm{w}}' \in \mathcal{V}_\varepsilon(\overline{\mathrm{w}})} \big| \mathbf{Q}_t f(\overline{\mathrm{w}}') - \mathbf{Q}_t f(\overline{\mathrm{w}}) \big| \Big) = 0. \tag{105}$$

We will use a coupling argument to obtain (105). More precisely, if $\overline{\mathrm{w}}' \in \mathcal{V}_\varepsilon(\overline{\mathrm{w}})$, we will introduce two (random) variables $\overline{\mathrm{w}}_{(1)}$ and $\overline{\mathrm{w}}_{(2)}$ such that $\overline{\mathrm{w}}_{(1)}$, resp. $\overline{\mathrm{w}}_{(2)}$, is distributed according to $\mathbf{Q}_t(\overline{\mathrm{w}}, \cdot)$, resp. $\mathbf{Q}_t(\overline{\mathrm{w}}', \cdot)$, and $\overline{\mathrm{w}}_{(1)}$ and $\overline{\mathrm{w}}_{(2)}$ are close to each other. Let us fix $\overline{\mathrm{w}}' \in \mathcal{V}_\varepsilon(\overline{\mathrm{w}})$ and let $\varepsilon_1, \varepsilon_2 \in [0, \varepsilon)$ be associated with $\overline{\mathrm{w}}'$ as in the definition of $\mathcal{V}_\varepsilon(\overline{\mathrm{w}})$. For definiteness we assume that $\varepsilon_1 \leqslant \varepsilon_2$ (the other case is treated in a symmetric way). Let $X^{(1)}$ be a copy of the Lévy process X started at 0 and let $I^{(1)}$ and $\rho^{(1)}$ be the analogues of I and ρ for $X^{(1)}$. We can then define $\overline{\mathrm{w}}_{(1)} = (\mu_{(1)}, \mathrm{w}_{(1)})$ by

$$\mu_{(1)} = [k_{-I_t^{(1)}}\mu, \rho_t^{(1)}]$$

$$\mathrm{w}_{(1)}(r) = \begin{cases} \mathrm{w}(r) & \text{if } r < H(k_{-I_t^{(1)}}\mu), \\ \xi^{(1)}(r - H(k_{-I_t^{(1)}}\mu)) & \text{if } H(k_{-I_t^{(1)}}\mu) \leqslant r < H(\mu_{(1)}), \end{cases}$$

where, conditionally on $X^{(1)}$, $\xi^{(1)} = (\xi^{(1)}(t), t \geqslant 0)$ is a copy of the spatial motion ξ started at $\mathrm{w}(H(k_{-I_t^{(1)}}\mu)-)$. Clearly, $\overline{\mathrm{w}}_{(1)}$ is distributed according to $\mathbf{Q}_t(\overline{\mathrm{w}}, \cdot)$.

The definition of $\overline{\mathrm{w}}_{(2)}$ is analogous but we use another copy of the underlying Lévy process. Precisely, we let Z be a copy of X independent of the pair $(X^{(1)}, \xi^{(1)})$, and if $T_*(Z) := \inf\{r \geqslant 0 : Z_r = \varepsilon_1 - \varepsilon_2\}$, we set

$$X_s^{(2)} = \begin{cases} Z_s & \text{if } 0 \leqslant s \leqslant T_*(Z), \\ \varepsilon_1 - \varepsilon_2 + X^{(1)}_{s-T_*(Z)} & \text{if } s > T_*(Z). \end{cases}$$

We then take, with an obvious notation,

$$\mu_{(2)} = [k_{-I_t^{(2)}}\mu', \rho_t^{(2)}].$$

The definition of $\mathrm{w}_{(2)}$ is somewhat more intricate. Let $\tau^{(1)}$ be the (a.s. unique) time of the minimum of $X^{(1)}$ over $[0,t]$. Consider the event

$$A(\varepsilon, \varepsilon_1, \varepsilon_2) = \{T_*(Z) + \tau^{(1)} < t, I_t^{(1)} < -\varepsilon\}.$$

Notice that $T_*(Z)$ is small in probability when ε is small, and $I_t^{(1)} < 0$ a.s. It follows that $P[A(\varepsilon, \varepsilon_1, \varepsilon_2)] \geqslant 1 - \alpha(\varepsilon)$, where the function $\alpha(\varepsilon)$ satisfies $\alpha(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$. However, on the event $A(\varepsilon, \varepsilon_1, \varepsilon_2)$, we have $I_t^{(2)} = \varepsilon_1 - \varepsilon_2 + I_t^{(1)}$, and so

$$k_{-I_t^{(2)}}\mu' = k_{-I_t^{(1)}-\varepsilon_1} k_{\varepsilon_2}\mu' = k_{-I_t^{(1)}-\varepsilon_1} k_{\varepsilon_1}\mu = k_{-I_t^{(1)}}\mu.$$

Also recall that from the definition of $\mathcal{V}_\varepsilon(\overline{\mathrm{w}})$, we have $\mathrm{w}'(r) = \mathrm{w}(r)$ for every $r < H(k_{\varepsilon_1}\mu)$, hence for every $r < H(k_{-I_t^{(1)}}\mu)$ when $A(\varepsilon, \varepsilon_1, \varepsilon_2)$ holds.

We construct $\mathrm{w}_{(2)}$ by imposing that, on the set $A(\varepsilon, \varepsilon_1, \varepsilon_2)$,

$$\mathrm{w}_{(2)}(r) = \begin{cases} \mathrm{w}'(r) = \mathrm{w}(r) & \text{if } r < H(k_{-I_t^{(2)}}\mu') = H(k_{-I_t^{(1)}}\mu), \\ \xi^{(1)}(r - H(k_{-I_t^{(1)}}\mu)) & \text{if } H(k_{-I_t^{(1)}}\mu) \leqslant r < H(\mu_{(2)}), \end{cases}$$

whereas on $A(\varepsilon, \varepsilon_1, \varepsilon_2)^c$, we take

$$\mathrm{w}_{(2)}(r) = \begin{cases} \mathrm{w}'(r) & \text{if } r < H(k_{-I_t^{(2)}}\mu'), \\ \xi^{(2)}(r - H(k_{-I_t^{(2)}}\mu')) & \text{if } H(k_{-I_t^{(2)}}\mu') \leqslant r < H(\mu_{(2)}), \end{cases}$$

where, conditionally on $X^{(2)}$, $\xi^{(2)}$ is independent of $\xi^{(1)}$ and distributed according to the law of ξ started at $\mathrm{w}'(H(k_{-I_t^{(2)}}\mu')-)$. Note that, in the first case, we use the *same* process $\xi^{(1)}$ as in the definition of $\mathrm{w}_{(1)}$. It is again easy to verify that $\overline{\mathrm{w}}_{(2)}$ is distributed according to $\mathbf{Q}_t(\overline{\mathrm{w}}', \cdot)$.

To complete the proof, note that the distance in variation $d_{\mathrm{var}}(\mu_{(1)}, \mu_{(2)})$ is equal to $d_{\mathrm{var}}(\rho_t^{(1)}, \rho_t^{(2)})$ on the set $A(\varepsilon, \varepsilon_1, \varepsilon_2)$. Furthermore, from the construction of $X^{(2)}$, on the set $A(\varepsilon, \varepsilon_1, \varepsilon_2)$, we have also

$$\rho_t^{(2)} = \rho^{(1)}_{t-T_*(Z)}.$$

and thus $d_{\mathrm{var}}(\mu_{(1)}, \mu_{(2)}) = d_{\mathrm{var}}(\rho_t^{(1)}, \rho^{(1)}_{t-T_*(Z)})$ is small in probability when ε is small, because t is a.s. not a discontinuity time of $\rho^{(1)}$. In addition, again on the set

$A(\varepsilon, \varepsilon_1, \varepsilon_2)$, the paths $\mathrm{w}_{(1)}$ and $\mathrm{w}_{(2)}$ coincide on the interval $[0, H(\mu_{(1)}) \wedge H(\mu_{(2)}))$, and so

$$d(\mathrm{w}_{(1)}, \mathrm{w}_{(2)}) \leqslant |H(\mu_{(2)}) - H(\mu_{(1)})| = |H(\rho^{(1)}_{t-T_*(Z)}) - H(\rho^{(1)}_t)|$$

is small in probability when ε goes to 0. The limiting result (105) now follows from these observations and the fact that $P[A(\varepsilon, \varepsilon_1, \varepsilon_2)^c]$ tends to 0 as ε goes to 0. □

4.1.4. Excursion measures. — We know that $\mu = 0$ is a regular recurrent point for the Markov process ρ_s, and the associated local time is the process L^0_s of Section 1.3. It immediately follows that $(0, x)$ is also a regular recurrent point for the Lévy snake (ρ_s, W_s), with associated local time L^0_s. We will denote by $\mathbb{N}_x$ the corresponding excursion measure. It is straightforward to verify that

(i) the law of $(\rho_s, s \geqslant 0)$ under $\mathbb{N}_x$ is the excursion measure $N(d\rho)$;

(ii) the conditional distribution of $(W_s, s \geqslant 0)$ under $\mathbb{N}_x$ knowing $(\rho_s, s \geqslant 0)$ is $Q_x^{H(\rho)}$.

From these properties and Proposition 1.2.5, we easily get for any nonnegative measurable function F on $M_f(\mathbb{R}_+) \times \mathcal{W}$,

$$\mathbb{N}_x\Big(\int_0^\sigma ds\, F(\rho_s, W_s) \Big) = \int_0^\infty da\, e^{-\alpha a}\, E^0 \otimes \Pi_x[F(J_a, (\xi_r, 0 \leqslant r \leqslant a))] \tag{106}$$

Here, as in Chapter 1, $J_a(dr)$ stands for the measure $1_{[0,a]}(r) dU_r$, where U is under the probability measure P^0 a subordinator with Laplace exponent $\widetilde{\psi}(\lambda) - \alpha$, where $\widetilde{\psi}(\lambda) = \psi(\lambda)/\lambda$. Note that the right side of (106) gives an invariant measure for the Lévy snake (ρ_s, W_s).

The strong Markov property of the Lévy snake can be extended to the excursion measures in the following form. Let T be a stopping time of the filtration $(\mathcal{F}_{s+})$ such that $T > 0$, $\mathbb{N}_x$ a.e., let F be a nonnegative $\mathcal{F}_{T+}$-measurable functional on $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+) \times \mathcal{W})$, and let G be any nonnegative measurable functional on $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+) \times \mathcal{W})$. Then,

$$\mathbb{N}_x[F\, G(\overline{W}_{T+s}, s \geqslant 0)] = \mathbb{N}_x[F\, \mathbb{E}^*_{\overline{W}_T}[G]],$$

where $\mathbb{P}^*_{\mu,\mathrm{w}}$ denotes the law under $\mathbb{P}_{\mu,\mathrm{w}}$ of the process $(\overline{W}_s, s \geqslant 0)$ stopped at $\inf\{s \geqslant 0, \rho_s = 0\}$. This statement follows from Theorem 4.1.2 by standard arguments.

4.2. The connection with superprocesses

4.2.1. Statement of the result. — In this section, we state and prove the basic theorem relating the Lévy snake with the superprocess with spatial motion ξ and branching mechanism ψ. This connection was already obtained in a less precise form in [**33**].

We start with a few simple observations. Let $\kappa(ds)$ be a random measure on $\mathbb{R}_+$, measurable with respect to the σ-field generated by $(\rho_s, s \geqslant 0)$. Then, from the form of the conditional distribution of $(W_s, s \geqslant 0)$ knowing $(\rho_s, s \geqslant 0)$, it is easy to see that, for any nonnegative measurable functional F on $\mathcal{W}_x$,

$$\mathbb{E}_x\Big[\int \kappa(ds)\, F(W_s)\Big] = \mathbb{E}_x\Big[\int \kappa(ds)\, \Pi_x[F(\xi_r, 0 \leqslant r < H_s)]\Big],$$

and a similar formula holds under $\mathbb{N}_x$. This identity implies in particular that the left limit $W_s(H_s-)$ exists $\kappa(ds)$ a.e., $\mathbb{P}_x$ a.s. (or $\mathbb{N}_x$ a.e.). We will apply this simple observation to the random measure $d_s L^a_s$ associated with the local time of H at level a (*cf.* Chapter 1). To simplify notation, we will write $\widehat{W}_s = W_s(H_s-)$ when the limit exists, and when the limit does not exist, we take $\widehat{W}_s = \Delta$, where Δ is a cemetery point added to E.

In order to state the main theorem of this section, we denote by $\mathcal{Z}_a = \mathcal{Z}_a(\rho, W)$ the random measure on E defined by

$$\langle \mathcal{Z}_a, f\rangle = \int_0^\sigma d_s L^a_s\, f(\widehat{W}_s).$$

This definition makes sense under the excursion measures $\mathbb{N}_x$.

Theorem 4.2.1. — *Let $\mu \in M_f(E)$ and let*

$$\sum_{i\in I} \delta_{(x_i,\rho^i,W^i)}$$

be a Poisson point measure with intensity $\mu(dx)\mathbb{N}_x(d\rho dW)$. Set $Z_0 = \mu$ and for every $a > 0$

$$Z_a = \sum_{i\in I} \mathcal{Z}_a(\rho^i, W^i).$$

The process $(Z_a, a \geqslant 0)$ is a superprocess with spatial motion ξ and branching mechanism ψ, started at μ.

This means that $(Z_a, a \geqslant 0)$ is a Markov process with values in $M_f(E)$, whose semigroup is characterized by the following Laplace functional. For every $0 \leqslant a \leqslant b$ and every function $f \in \mathcal{B}_{b+}(E)$,

$$E[\exp -\langle Z_b, f\rangle \mid Z_a] = \exp -\langle Z_a, u_{b-a}\rangle$$

where the function $(u_t(y), t \geqslant 0, y \in E)$ is the unique nonnegative solution of the integral equation

$$u_t(y) + \Pi_y\Big(\int_0^t \psi(u_{t-r}(\xi_r))\, dr\Big) = \Pi_y(f(\xi_t)). \tag{107}$$

The proof of Theorem 4.2.1 is easily reduced to that of the following proposition.

Proposition 4.2.2. — *Let $0 < a < b$ and let $f \in \mathcal{B}_{b+}(E)$. Then,*

$$\mathbb{N}_x(\exp -\langle \mathcal{Z}_b, f\rangle \mid (\mathcal{Z}_r, 0 \leqslant r \leqslant a)) = \exp -\langle \mathcal{Z}_a, u_{b-a}\rangle \tag{108}$$

where for every $t > 0$ and $y \in E$,

$$u_t(y) = \mathbb{N}_y(1 - \exp -\langle \mathcal{Z}_t, f\rangle).$$

Furthermore, if we set $u_0(y) = f(y)$, the function $(u_t(y), t \geqslant 0, y \in E)$ is the unique nonnegative solution of the integral equation (107).

Remark. — Although $\mathbb{N}_x$ is an infinite measure, the conditioning in (108) makes sense because we can restrict our attention to the set $\{\mathcal{Z}_a \neq 0\} = \{L^a_\sigma > 0\}$ which has finite $\mathbb{N}_x$-measure (*cf.* Corollary 1.4.2). A similar remark applies in several places below, e.g. in the statement of Proposition 4.2.3.

Given Proposition 4.2.2, it is a straightforward exercise to verify that the process $(Z_a, a \geqslant 0)$ of Theorem 4.2.1 has the finite-dimensional marginals of the superprocess with spatial motion ξ and branching mechanism ψ, started at μ. In fact the statement of Propostion 4.2.2 means that the laws of $(\mathcal{Z}_a, a > 0)$ under $\mathbb{N}_y$, $y \in E$ are the canonical measures of the superprocess with spatial motion ξ and branching mechanism ψ, and given this fact, Theorem 4.2.1 is just the canonical representation of superprocesses.

The remaining part of this section is devoted to the proof of Proposition 4.2.2. We will proceed in two steps. In the first one, we introduce a σ-field $\mathcal{E}_a$ that contains $\sigma(\mathcal{Z}_u, 0 \leqslant u \leqslant a)$, and we compute $\mathbb{N}_x(\exp -\langle \mathcal{Z}_b, f\rangle \mid \mathcal{E}_a)$ in the form given by (108). In the second step, we establish the integral equation (107).

4.2.2. First step. — Recall the notation of Section 1.3

$$\widetilde{\tau}^a_s = \inf\{r : \int_0^r du\, 1_{\{H_u \leqslant a\}} > s\}.$$

Note that $\widetilde{\tau}^a_s < \infty$ for every $s \geqslant 0$, $\mathbb{N}_x$ a.e. For $a > 0$, we let $\mathcal{E}_a$ be the σ-field generated by the right-continuous process $(\rho_{\widetilde{\tau}^a_s}, W_{\widetilde{\tau}^a_s}; s \geqslant 0)$ and augmented with the class of all sets that are $\mathbb{N}_x$-negligible for every $x \in E$. From the second approximation of Proposition 1.3.3, it is easy to verify that L^a_σ is measurable with respect to the σ-field generated by $(\rho_{\widetilde{\tau}^a_s}, s \geqslant 0)$, and in particular with respect to $\mathcal{E}_a$ (*cf.* the beginning of the proof of Theorem 1.4.1).

We then claim that $\mathcal{Z}_a$ is $\mathcal{E}_a$-measurable. It is enough to check that, if g is bounded and continuous on $\mathcal{W}_x$,

$$\int_0^\sigma dL^a_s\, g(W_s)$$

is $\mathcal{E}_a$-measurable. However, by Proposition 1.3.3, this integral is the limit in $\mathbb{N}_x$-measure as $\varepsilon \to 0$ of

$$\frac{1}{\varepsilon}\int_0^\sigma ds\, 1_{\{a-\varepsilon < H_s \leqslant a\}}\, g(W_s).$$

For $\varepsilon < a$, this quantity coincides with

$$\frac{1}{\varepsilon}\int_0^\infty ds\, 1\{a-\varepsilon < H_{\widetilde{\tau}_s^a} \leqslant a\}\, g(W_{\widetilde{\tau}_s^a}),$$

and the claim follows from the definition of $\mathcal{E}_a$.

We then decompose the measure $\mathcal{Z}_b$ according to the contributions of the different excursions of the process H above level a. Precisely, we let (α_i, β_i), $i \in I$ be the excursion intervals of H above a over the time interval $[0, \sigma]$. We will use the following simple facts that hold N a.e.: For every $i \in I$ and every $t > 0$, we have

$$\int_0^{\beta_i+t} 1_{\{H_s \leqslant a\}} ds > \int_0^{\beta_i} 1_{\{H_s \leqslant a\}} ds$$

and

$$L^a_{\beta_i+t} > L^a_{\beta_i}.$$

The first assertion is an easy consequence of the strong Markov property of ρ, recalling that $\rho_s(\{a\}) = 0$ for every $s \geqslant 0$, N a.e. To get the second one, we can use Proposition 1.3.1 and the definition of the local time L^a to see that it is enough to prove that

$$\int_0^{\beta_i+t} 1_{\{H_s > a\}} ds > \int_0^{\beta_i} 1_{\{H_s > a\}} ds$$

for every $t > 0$ and $i \in I$. Via a time-reversal argument (Corollary 3.1.6), it suffices to verify that, if $\sigma^q_a = \inf\{s > q : H_s > a\}$, we have

$$\int_0^{\sigma^q_a+t} 1_{\{H_s \leqslant a\}} ds > \int_0^{\sigma^q_a} 1_{\{H_s \leqslant a\}} ds$$

for every $t > 0$ and every rational $q > 0$, N a.e. on the set $\{q < \sigma^q_a < \infty\}$. The latter fact is again a consequence of the strong Markov property of the process ρ.

As was observed in the proof of Proposition 1.3.1, for every $i \in I$, for every $s \in (\alpha_i, \beta_i)$, the restriction of ρ_s to $[0, a]$ coincides with $\rho_{\alpha_i} = \rho_{\beta_i}$. Furthermore, the snake property implies that, for every $i \in I$, the paths W_s, $\alpha_i < s < \beta_i$ take the same value x_i at time a, and this value must be the same as the left limit $\widehat{W}_{\alpha_i} = \widehat{W}_{\beta_i}$ (recall our assumption that ξ has no fixed discontinuities). We can then define the pair $(\rho^i, W^i) \in \mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+)) \times \mathcal{W})$ by setting

$$\begin{array}{ll} \langle \rho^i_s, g\rangle = \int_{(a,\infty)} \rho_{\alpha_i+s}(dr)\, g(r-a) & \text{if } 0 < s < \beta_i - \alpha_i \\ \rho^i_s = 0 & \text{if } s = 0 \text{ or } s \geqslant \beta_i - \alpha_i, \end{array}$$

and

$$\begin{array}{ll} W^i_s(r) = W_{\alpha_i+s}(a+r),\ \zeta_{W^i_s} = H_{\alpha_i+s} - a & \text{if } 0 < s < \beta_i - \alpha_i \\ W^i_s = x_i & \text{if } s = 0 \text{ or } s \geqslant \beta_i - \alpha_i. \end{array}$$

Proposition 4.2.3. — *Under $\mathbb{N}_x$, conditionally on $\mathcal{E}_a$, the point measure*

$$\sum_{i\in I} \delta_{(\rho^i, W^i)}$$

is a Poisson point measure with intensity

$$\int \mathcal{Z}_a(dy)\,\mathbb{N}_y(\cdot).$$

Proof. — Let the process ρ^a_t be defined as in Proposition 1.3.1. Note that under N the definition of ρ^a_t only makes sense for $t < \int_0^\sigma ds\,1_{\{H_s>a\}}$. For convenience, we take $\rho^a_t = 0$ if $t \geqslant \int_0^\sigma ds\,1_{\{H_s>a\}}$. We also set

$$\widetilde{\rho}_t = \rho_{\widetilde{\tau}^a_t}\ , \quad \widetilde{W}_t = W_{\widetilde{\tau}^a_t}.$$

With these definitions, the processes ρ^i, $i \in I$ are exactly the excursions of the process ρ^a away from 0. For every $i \in I$, introduce the local time at the beginning (or the end) of excursion ρ^i:

$$\ell^i = L^a_{\alpha_i}.$$

By Proposition 1.3.1 and standard excursion theory, we know that conditionally on the process $\widetilde{\rho}_t$, the point measure

$$\sum_{i\in I} \delta_{(\ell^i,\rho^i)}$$

is Poisson with intensity $1_{[0,L^a_\sigma]}(\ell)d\ell\,N(d\rho)$ (recall that L^a_σ is measurable with respect to the σ-field generated by $\widetilde{\rho}$). Note that Proposition 1.3.1 is formulated under $\mathbb{P}_x$: However, by considering the first excursion of ρ away from 0 that hits the set $\{\sup H_s > a\}$, we can easily derive the previous assertion from Proposition 1.3.1.

Define $\widetilde{L}^a_s = L^a_{\widetilde{\tau}^a_s}$ (note that this is a continuous process), and let $\gamma^a(r)$ be the right-continuous inverse of $\widetilde{L}^a$:

$$\gamma^a(r) = \inf\{s \geqslant 0 : \widetilde{L}^a_s > r\}.$$

Then, if f is any nonnegative measurable function on E, we have $\mathbb{N}_x$ a.e.

$$\langle \mathcal{Z}_a, f\rangle = \int_0^\infty dL^a_s\, f(\widehat{W}_s) = \int_0^\infty d\widetilde{L}^a_s\, f(\widehat{\widetilde{W}}_s) = \int_0^{L^a_\sigma} d\ell\, f(\widehat{\widetilde{W}}_{\gamma^a(\ell)}). \tag{109}$$

Notice that both processes $\widetilde{L}^a$ and γ^a are measurable with respect to the σ-field generated by $\widetilde{\rho}$ (for $\widetilde{L}^a$, this follows again from Proposition 1.3.3).

Consider now the processes $\widetilde{W}$ and W^i, $i \in I$. The following two properties are straightforward consequences of our construction:

(i) The law of $\widetilde{W}$ under $Q_x^{H(\rho)}$ is $Q_x^{H(\widetilde{\rho})}$.

(ii) Under $Q_x^{H(\rho)}$, conditionally on $\widetilde{W}$, the "excursions" W^i, $i \in I$ are independent and the conditional distribution of W^i is $Q_{x_i}^{H(\rho^i)}$, where $x_i = \widehat{W}_{\beta_i} = \widehat{\widetilde{W}}_{\gamma^a(\ell^i)}$.

To verify the second expression for x_i, note that if $\widetilde{A}^a_s = \int_0^s dr\,1_{\{H_r\leqslant a\}}$, we have $W_{\beta_i} = \widetilde{W}_{\widetilde{A}^a_{\beta_i}}$ (because $\widetilde{A}^a_{\beta_i+t} > \widetilde{A}^a_{\beta_i}$ for every $t > 0$) and $\widetilde{A}^a_{\beta_i} = \gamma^a(\ell^i)$ (because $L^a_{\beta_i+t} > L^a_{\beta_i} = \ell_i$ for every $t > 0$).

As a consequence of (i), the conditional distribution (under $\mathbb{N}_x$) of $\widetilde{W}$ knowing ρ depends only on $\widetilde{\rho}$. Hence, $\widetilde{W}$ and the point measure $\sum_{i\in I}\delta_{(\ell^i,\rho^i)}$ are conditionally independent given $\widetilde{\rho}$ under $\mathbb{N}_x$.

Let F be a nonnegative measurable function on $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+)\times\mathcal{W})$. We use the previous observations in the following calculation:

$$\begin{aligned}
&\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\,\exp(-\sum_{i\in I}F(\rho^i,W^i))\Big)\\
&\qquad=\int N(d\rho)\,Q_x^{H(\rho)}\Big(G(\widetilde{\rho},\widetilde{W})\,\exp(-\sum_{i\in I}F(\rho^i,W^i))\Big)\\
&\qquad=\int N(d\rho)\,Q_x^{H(\rho)}\Big(G(\widetilde{\rho},\widetilde{W})\prod_{i\in I}Q^{H(\rho^i)}_{\widehat{\widetilde{W}}_{\gamma^a(\ell^i)}}(e^{-F(\rho^i,\cdot)})\Big)\\
&\qquad=\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\prod_{i\in I}Q^{H(\rho^i)}_{\widehat{\widetilde{W}}_{\gamma^a(\ell^i)}}(e^{-F(\rho^i,\cdot)})\Big)\\
&\qquad=\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\,\exp\Big(-\int_0^{L^a_\sigma}d\ell\int N(d\rho)\,Q^{H(\rho)}_{\widehat{\widetilde{W}}_{\gamma^a(\ell)}}(1-e^{-F(\rho,\cdot)})\Big)\Big).
\end{aligned}$$

The second equality follows from (ii) above. In the last one, we used the conditional independence of $\widetilde{W}$ and of the point measure $\sum_{i\in I}\delta_{(\ell^i,\rho^i)}$, given $\widetilde{\rho}$, and the fact that the conditional distribution of this point measure is Poisson with intensity $1_{[0,L^a_\sigma]}(\ell)d\ell\,N(d\rho)$. Using (109), we finally get

$$\begin{aligned}
&\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\,\exp(-\sum_{i\in I}F(\rho^i,W^i))\Big)\\
&\qquad=\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\,\exp\Big(-\int_0^{L^a_\sigma}d\ell\,\mathbb{N}_{\widehat{\widetilde{W}}_{\gamma^a(\ell)}}(1-e^{-F(\rho,W)})\Big)\Big)\\
&\qquad=\mathbb{N}_x\Big(G(\widetilde{\rho},\widetilde{W})\,\exp(-\int\mathcal{Z}_a(dy)\,\mathbb{N}_y(1-e^{-F}))\Big).
\end{aligned}$$

This completes the proof. □

Let f be a nonnegative measurable function on E, and let $0\leqslant a<b$. With the preceding notation, it is easy to verify that $\mathbb{N}_x$ a.s.

$$\begin{aligned}
\langle\mathcal{Z}_b,f\rangle&=\int_0^\sigma dL^b_s\,f(\widehat{W}_s)\\
&=\sum_{i\in I}\int_{\alpha_i}^{\beta_i}dL^b_s\,f(\widehat{W}_s)\\
&=\sum_{i\in I}\int_0^\infty dL^{b-a}_s(\rho^i)\,f(\widehat{W}^i_s)\\
&=\sum_{i\in I}\langle\mathcal{Z}_{b-a}(\rho^i,W^i),f\rangle.
\end{aligned}$$

As a consequence of Proposition 4.2.3, we have then

$$\mathbb{N}_x[\exp -\langle \mathcal{Z}_b, f\rangle \mid \mathcal{E}_a] = \exp -\langle \mathcal{Z}_a, u_{b-a}\rangle,$$

where

$$u_r(y) = \mathbb{N}_y[1 - \exp -\langle \mathcal{Z}_r, f\rangle].$$

4.2.3. Second step. — It remains to prove that the function $(u_r(y), r \geqslant 0, y \in E)$ introduced at the end of the first step solves the integral equation (107). By definition, we have for $a > 0$,

$$\begin{aligned} u_a(y) &= \mathbb{N}_y\Big(1 - \exp -\int_0^\infty dL_s^a\, f(\widehat{W}_s)\Big) \\ &= \mathbb{N}_y\Big(\int_0^\infty dL_s^a\, f(\widehat{W}_s)\, \exp\Big(-\int_s^\infty dL_r^a\, f(\widehat{W}_r)\Big)\Big) \\ &= \mathbb{N}_y\Big(\int_0^\infty dL_s^a\, f(\widehat{W}_s)\, \mathbb{E}^*_{\rho_s, W_s}\Big[\exp -\int_0^\infty dL_r^a\, f(\widehat{W}_r)\Big]\Big) \qquad (110) \end{aligned}$$

where we recall that $\mathbb{P}^*_{\mu,\mathrm{w}}$ stands for the law of the Lévy snake started at (μ, w) and stopped when ρ_s first hits 0. In the last equality, we replaced $\exp -\int_s^\infty dL_r^a\, f(\widehat{W}_r)$ by its optional projection, using the strong Markov property of the Lévy snake to identify this projection.

We now need to compute for a fixed $(\mu, \mathrm{w}) \in \Theta_x$,

$$\mathbb{E}^*_{\mu,\mathrm{w}}\Big[\exp -\int_0^\infty dL_r^a\, f(\widehat{W}_r)\Big].$$

We will derive this calculation from a more general fact, that is also useful for forthcoming applications. First recall that $Y_t = \langle \rho_t, 1\rangle$ is distributed under $\mathbb{P}^*_{\mu,\mathrm{w}}$ as the underlying Lévy process started at $\langle \mu, 1\rangle$ and stopped when it first hits 0. We write $K_t = \inf_{r \leqslant t} Y_r$, and we denote by (α_i, β_i), $i \in I$ the excursion intervals of $Y_t - K_t$ away from 0. For every $i \in I$, we set $h_i = H_{\alpha_i} = H_{\beta_i}$. From the snake property, it is easy to verify that $W_s(h_i) = \mathrm{w}(h_i-)$ for every $s \in (\alpha_i, \beta_i)$, $i \in I$, $\mathbb{P}^*_{\mu,\mathrm{w}}$ a.s. We then define the pair (ρ^i, W^i) by the formulas

$$\begin{aligned} &\langle \rho^i_s, g\rangle = \textstyle\int_{(h_i,\infty)} \rho_{\alpha_i+s}(dr)\, g(r - h_i) && \text{if } 0 \leqslant s \leqslant \beta_i - \alpha_i \\ &\rho^i_s = 0 && \text{if } s > \beta_i - \alpha_i, \end{aligned}$$

and

$$\begin{aligned} &W^i_s(t) = W_{\alpha_i+s}(h_i + t), \quad \zeta^i_s = H_{\alpha_i+s} - h_i && \text{if } 0 < s < \beta_i - \alpha_i \\ &W^i_s = \mathrm{w}(h_i-) && \text{if } s = 0 \text{ or } s \geqslant \beta_i - \alpha_i. \end{aligned}$$

Lemma 4.2.4. — *Let $(\mu, \mathrm{w}) \in \Theta_x$. The point measure*

$$\sum_{i \in I} \delta_{(h_i, \rho^i, W^i)}$$

is under $\mathbb{P}^*_{\mu,\mathrm{w}}$ a Poisson point measure with intensity

$$\mu(dh)\, \mathbb{N}_{\mathrm{w}(h-)}(d\rho\, dW).$$

Proof. — Consider first the point measure

$$\sum_{i\in I}\delta_{(h_i,\rho^i)}.$$

If $I_s = K_s - \langle\mu,1\rangle$, we have $h_i = H(\rho_{\alpha_i}) = H(k_{-I_{\alpha_i}}\mu)$. Excursion theory for $Y_s - K_s$ ensures that

$$\sum_{i\in I}\delta_{(-I_{\alpha_i},\rho^i)}$$

is under $\mathbb{P}^*_{\mu,\mathrm{w}}$ a Poisson point measure with intensity $1_{[0,\langle\mu,1\rangle]}(u)\,du\,N(d\rho)$. Since the image measure of $1_{[0,\langle\mu,1\rangle]}(u)\,du$ under the mapping $u\longrightarrow H(k_u\mu)$ is precisely the measure μ, it follows that

$$\sum_{i\in I}\delta_{(h_i,\rho^i)}$$

is a Poisson point measure with intensity $\mu(dh)\,N(d\rho)$. To complete the proof, it remains to obtain the conditional distribution of $(W^i, i\in I)$ knowing $(\rho_s, s\geqslant 0)$. However, the form of the conditional law Q^H_{w} easily implies that under Q^H_{w}, the processes W^i, $i\in I$ are independent, and furthermore the conditional distribution of W^i is $Q^{H^i}_{\mathrm{w}(h_i-)}$, where $H^i_s = H(\rho^i_s)$. It follows that

$$\sum_{i\in I}\delta_{(h_i,\rho^i,W^i)}$$

is a Poisson measure with intensity

$$\mu(dh)\,N(d\rho)\,Q^{H(\rho)}_{\mathrm{w}(h-)}(dW) = \mu(dh)\,\mathbb{N}_{\mathrm{w}(h-)}(d\rho dW).$$

This completes the proof. □

We apply Lemma 4.2.4 to a pair (μ,w) such that $H(\mu)\leqslant a$ and $\mu(\{H(\mu)\})=0$. Then, it is easy to verify that $\mathbb{P}^*_{\mu,\mathrm{w}}$ a.s.

$$\int_0^\infty dL^a_r\,f(\widehat{W}_r) = \sum_{i\in I}\int_{\alpha_i}^{\beta_i} dL^a_r\,f(\widehat{W}_r) = \sum_{i\in I}\int_0^\infty dL^{a-h_i}_r(\rho^i)\,f(\widehat{W}^i_r),$$

and thus, by Lemma 4.2.4,

$$\text{(111)}\quad \mathbb{E}^*_{\mu,\mathrm{w}}\Big[\exp-\int_0^\infty dL^a_r\,f(\widehat{W}_r)\Big] = \exp\Big(-\int\mu(dh)\,\mathbb{N}_{\mathrm{w}(h-)}[1-\exp-\langle\mathcal{Z}_{a-h},f\rangle]\Big).$$

We now come back to formula (110). As a consequence of Proposition 1.2.5, we know that $\rho_s(\{a\})=0$ for every $s\geqslant 0$, $\mathbb{N}_y$ a.e. We can thus use (111) to get

$$\begin{aligned} u_a(y) &= \mathbb{N}_y\Big(\int_0^\infty dL^a_s\,f(\widehat{W}_s)\,\exp\Big(-\int\rho_s(dh)\,\mathbb{N}_{W_s(h-)}[1-\exp-\langle\mathcal{Z}_{a-h},f\rangle]\Big)\Big)\\ &= \mathbb{N}_y\Big(\int_0^\infty dL^a_s\,f(\widehat{W}_s)\,\exp\Big(-\int\rho_s(dh)\,u_{a-h}(W_s(h-))\Big)\Big). \qquad \text{(112)}\end{aligned}$$

Let J_a, P^0 be as in (106).

Lemma 4.2.5. — *For any nonnegative measurable function F on Θ_y,*

$$\mathbb{N}_y\Big(\int_0^\infty dL_s^a\, F(\rho_s, W_s)\Big) = e^{-\alpha a}\, E^0 \otimes \Pi_y[F(J_a, (\xi_r, 0 \leqslant r < a))].$$

Proof. — If $F(\rho_s, W_s)$ depends only on ρ_s, the result follows from Corollary 1.3.4. In the general case, we may take F such that $F(\rho_s, W_s) = F_1(\rho_s)F_2(W_s)$, and we use the simple observation of the beginning of this section. □

From (112) and Lemma 4.2.5, we get

$$\begin{aligned} u_a(y) &= e^{-\alpha a}\, E^0 \otimes \Pi_y\Big[f(\xi_a)\exp\Big(-\int J_a(dh)\, u_{a-h}(\xi_{h-})\Big)\Big] \\ &= \Pi_y\Big[f(\xi_a)\exp\Big(-\int_0^a \widetilde{\psi}(u_{a-r}(\xi_r))\, dr\Big)\Big]. \end{aligned}$$

The proof of (107) is now completed by routine calculations. We have

$$\begin{aligned} u_a(y) &= \Pi_y[f(\xi_a)] - \Pi_y\Big[f(\xi_a)\int_0^a db\,\widetilde{\psi}(u_{a-b}(\xi_b))\,\exp\Big(-\int_b^a \widetilde{\psi}(u_{a-r}(\xi_r))\, dr\Big)\Big] \\ &= \Pi_y[f(\xi_a)] - \Pi_y\Big[\int_0^a db\,\widetilde{\psi}(u_{a-b}(\xi_b)) \\ &\qquad\qquad \Pi_{\xi_b}\Big[f(\xi_{a-b})\exp\Big(-\int_0^{a-b}\widetilde{\psi}(u_{a-b-r}(\xi_r))\, dr\Big)\Big]\Big] \\ &= \Pi_y[f(\xi_a)] - \Pi_y\Big[\int_0^a db\,\widetilde{\psi}(u_{a-b}(\xi_b))\, u_{a-b}(\xi_b)\Big], \end{aligned}$$

which gives (107) and completes the proof of Proposition 4.2.2. □

Remark. — By combining the previous arguments, especially Lemma 4.2.4, with the duality properties of ρ (see the derivation of (137) below), we could have obtained the following result. Let f and g be nonnegative measurable functions defined respectively on E and on $M_f(E)$. Let V be a subordinator with Laplace exponent $\psi'(\lambda)-\alpha$ defined under the probability measure P^0. Then, for every $a > 0$,

$$\begin{aligned} (113)\quad &\mathbb{N}_x\Big(\int \mathcal{Z}_a(dy)\, f(y)\, g(\mathcal{Z}_a)\Big) \\ &\qquad = e^{-\alpha a}\, E^0 \otimes \Pi_x\Big[f(\xi_a)\, E^a_{V,\xi}\Big[g\Big(\int \mathcal{N}(dsd\rho dW)\, \mathcal{Z}_{a-s}(\rho, W)\Big)\Big]\Big], \end{aligned}$$

where $\mathcal{N}(dsd\rho dW)$ is under $P^a_{V,\xi}$ a Poisson point measure on $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+) \times \mathcal{W})$ with intensity

$$1_{[0,a]}(s)\, dV_s\, \mathbb{N}_{\xi_s}(d\rho dW).$$

Formula (113) identifies the Palm distributions associated with the random measure $\mathcal{Z}_a$ under $\mathbb{N}_x$. It should be compared with the results of Chapter 4 of [**8**], in particular Proposition 4.1.5. In the stable case considered in [**8**], V is a stable subordinator, which arises analytically in the derivation of the Palm measure formula. Here V can be interpreted probabilistically in terms of the measure-valued processes ρ and η. As

(113) will not be needed in the applications below, we will leave details of the proof to the reader.

4.3. Exit measures

Throughout this section, we consider an open set $D \subset E$, and we denote by τ the first exit time of ξ from D:

$$\tau = \inf\{t \geqslant 0 : \xi_t \notin D\},$$

where $\inf \varnothing = \infty$ as usual. By abuse of notation, we will also denote by $\tau(\mathrm{w})$ the exit time from D of a killed path $\mathrm{w} \in \mathcal{W}$,

$$\tau(\mathrm{w}) = \inf\{t \in [0, \zeta_{\mathrm{w}}) : \mathrm{w}(t) \notin D\}\,.$$

Let $x \in D$. The next result is much analogous to Proposition 1.3.1.

Proposition 4.3.1. — *Assume that* $\Pi_x(\tau < \infty) > 0$. *Then,*

$$\int_0^\infty ds\, 1_{\{\tau(W_s) < H_s\}} = \infty\,, \qquad \mathbb{P}_x \text{ a.s.}$$

Furthermore, let

$$\sigma_s^D = \inf\{t \geqslant 0 : \int_0^t dr\, 1_{\{\tau(W_r) < H_r\}} > s\},$$

and let $\rho_s^D \in M_f(\mathbb{R}_+)$ *be defined by*

$$\langle \rho_s^D, f\rangle = \int \rho_{\sigma_s^D}(dr)\, f(r - \tau(W_{\sigma_s^D}))\, 1_{\{r > \tau(W_{\sigma_s^D})\}}.$$

Then the process $(\rho_s^D, s \geqslant 0)$ *has the same distribution under* $\mathbb{P}_x$ *as* $(\rho_s, s \geqslant 0)$.

Remark. — We could have considered the more general situation of a space-time open set D (as a matter of fact, this is not really more general as we could replace ξ_t by (t, ξ_t)). Taking $D = [0, a) \times E$, we would recover part of the statement of Proposition 1.3.1. This proposition contains an independence statement that could also be extended to the present setting.

Proof. — To simplify notation, we set

$$A_s^D = \int_0^s dr\, 1_{\{\tau(W_r) < H_r\}}.$$

By using (106), excursion theory and our assumption $\Pi_x(\tau < \infty) > 0$, it is a simple exercise to verify that $A_\infty^D = \infty$, $\mathbb{P}_x$ a.s., and thus the definition of σ_s^D makes sense for every $s \geqslant 0$, a.s. The arguments then are much similar to the proof of

Proposition 1.3.1. For every $\varepsilon > 0$, we introduce the stopping times S_ε^k, T_ε^k, $k \geqslant 1$, defined inductively by:

$$S_\varepsilon^1 = \inf\{s \geqslant 0 : \tau(W_s) < \infty \text{ and } \rho_s((\tau(W_s), \infty)) \geqslant \varepsilon\},$$
$$T_\varepsilon^k = \inf\{s \geqslant S_\varepsilon^k : \tau(W_s) = \infty\},$$
$$S_\varepsilon^{k+1} = \inf\{s \geqslant T_\varepsilon^k : \tau(W_s) < \infty \text{ and } \rho_s((\tau(W_s), \infty)) \geqslant \varepsilon\}.$$

It is easy to see that these stopping times are a.s. finite, and $S_\varepsilon^k \uparrow \infty$, $T_\varepsilon^k \uparrow \infty$ as $k \uparrow \infty$.

From the key formula (20), we see that for

$$S_\varepsilon^k \leqslant s < \inf\{r \geqslant S_\varepsilon^k : \langle \rho_r, 1\rangle \leqslant \rho_{S_\varepsilon^k}([0, \tau(W_{S_\varepsilon^k})])\}$$

we have $H_s > \tau(W_{S_\varepsilon^k})$, and the paths W_s and $W_{S_\varepsilon^k}$ coincide over $[0, \tau(W_{S_\varepsilon^k})]$ (by the snake property), so that in particular $\tau(W_s) = \tau(W_{S_\varepsilon^k}) < \infty$. On the other hand, for

$$s = \inf\{r \geqslant S_\varepsilon^k : \langle \rho_r, 1\rangle \leqslant \rho_{S_\varepsilon^k}([0, \tau(W_{S_\varepsilon^k})])\}$$

the path W_s is the restriction of $W_{S_\varepsilon^k}$ to $[0, \tau(W_{S_\varepsilon^k}))$ and thus $\tau(W_s) = \infty$. From these observations, we see that

$$T_\varepsilon^k = \inf\{r \geqslant S_\varepsilon^k : \langle \rho_r, 1\rangle \leqslant \rho_{S_\varepsilon^k}([0, \tau(W_{S_\varepsilon^k})])\}$$

and that conditionally on the past up to time S_ε^k, the process

$$Y_s^{k,\varepsilon} = \rho_{(S_\varepsilon^k + s) \wedge T_\varepsilon^k}((\tau(W_{S_\varepsilon^k}), \infty))$$

is distributed as the underlying Lévy process started at $\rho_{S_\varepsilon^k}((\tau(W_{S_\varepsilon^k}), \infty))$ and stopped at its first hitting time of 0.

The same argument as in the proof of (25) shows that, for every $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} \sup_{\{k \geqslant 1, S_\varepsilon^k \leqslant t\}} \rho_{S_\varepsilon^k}((\tau(W_{S_\varepsilon^k}), \infty)) = 0, \qquad \text{a.s.} \tag{114}$$

The remaining part of the proof is very similar to the end of the proof of Proposition 1.3.1. Using (114) and the observations preceding (114), we get by a passage to the limit $\varepsilon \to 0$ that the total mass process $\langle \rho_s^D, 1\rangle = \rho_{\sigma_s^D}((\tau(W_{\sigma_s^D}), \infty))$ has the same distribution as the process $\langle \rho_s, 1\rangle$. Then the statement of the proposition follows by an argument similar to the second step of the proof of Proposition 1.3.1. □

Let $\ell^D = (\ell^D(s), s \geqslant 0)$ be the local time at 0 of the process $\langle \rho^D, 1\rangle$. Recall the notation A_s^D from the previous proof. We define the *exit local time* from D by the formula

$$L_s^D = \ell^D(A_s^D) = \ell^D\Big(\int_0^s dr\, 1_{\{\tau(W_r) < H_r\}}\Big).$$

Recall from (106) the notation J_a, P^0.

Proposition 4.3.2. — *For any nonnegative measurable function Φ on $M_f(\mathbb{R}_+) \times \mathcal{W}$,*

$$\mathbb{N}_x\Big(\int_0^\sigma dL_s^D\, \Phi(\rho_s, W_s)\Big) = E^0 \otimes \Pi_x\Big[1_{\{\tau < \infty\}} e^{-\alpha\tau}\, \Phi(J_\tau, (\xi_r, 0 \leqslant r < \tau))\Big].$$

Proof. — By applying Lemma 1.3.2 to the reflected Lévy process $\langle \rho^D, 1\rangle$, we get for every $s \geqslant 0$,

$$\ell^D(s) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^s dr\, 1_{\{0<H(\rho^D_r)\leqslant\varepsilon\}}$$

in $L^1(\mathbb{P}_x)$. From a simple monotonicity argument, we have then for every $t \geqslant 0$

$$\lim_{\varepsilon\to 0} \mathbb{E}_x \left[\sup_{s\leqslant t} \left|\ell^D(s) - \frac{1}{\varepsilon}\int_0^s dr\, 1_{\{0<H(\rho^D_r)\leqslant\varepsilon\}}\right|\right] = 0.$$

Using the formulas $L^D_s = \ell^D(A^D_s)$ and $H(\rho_{\sigma^D_r}) = \tau(W_{\sigma^D_r}) + H(\rho^D_r)$ (the latter holding on the set $\{H(\rho_{\sigma^D_r}) > \tau(W_{\sigma^D_r})\}$, by the definition of ρ^D), we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}_x \left[\sup_{s\leqslant t} \left|L^D_s - \frac{1}{\varepsilon}\int_0^s dr\, 1_{\{\tau(W_r)<H_r\leqslant\tau(W_r)+\varepsilon\}}\right|\right] = 0.$$

Arguing as in the derivation of (36), we get, for any measurable subset V of $\mathbb{D}(\mathbb{R}_+, M_f(\mathbb{R}_+)\times\mathcal{W})$ such that $\mathbb{N}_x(V) < \infty$,

$$\lim_{\varepsilon\to 0} \mathbb{N}_x \left(1_V \sup_{s\leqslant t} \left|L^D_s - \frac{1}{\varepsilon}\int_0^s dr\, 1_{\{\tau(W_r)<H_r\leqslant\tau(W_r)+\varepsilon\}}\right|\right) = 0. \tag{115}$$

We then observe that for any bounded measurable function F on $\mathbb{R}_+\times M_f(\mathbb{R}_+)\times\mathcal{W}$, we have

$$\mathbb{N}_x\Big(\int_0^\sigma ds\, F(s,\rho_s,W_s)\Big) = E\otimes\Pi_x\Big[\int_0^{L_\infty} da\, F(L^{-1}(a),\Sigma_a,(\xi_r, 0\leqslant r<a))\Big] \tag{116}$$

where the random measure Σ_a is defined under P by

$$\langle\Sigma_a, g\rangle = \int_0^{L^{-1}(a)} dS_s\, g(a-L_s).$$

Indeed, we observe that the special case where $F(s,\mu,\mathrm{w})$ does not depend on w,

$$N\Big(\int_0^\sigma ds\, F(s,\rho_s)\Big) = E\Big[\int_0^{L_\infty} da\, F(L^{-1}(a),\Sigma_a)\Big]$$

is a consequence of Proposition 1.1.4 (see the proof of Proposition 1.2.5), and it then suffices to use the conditional distribution of W knowing $(\rho_s, s\geqslant 0)$.

After these preliminaries, we turn to the proof of the proposition. We let F be a bounded continuous function on $\mathbb{R}_+\times M_f(\mathbb{R}_+)\times\mathcal{W}$, and assume in addition that there exist $\delta > 0$ and $A > 0$ such that $F(s,\mu,\mathrm{w}) = 0$ if $s\leqslant\delta$ or $s\geqslant A$. As a consequence

of (115) and (116), we have then

$$\begin{aligned}
&\mathbb{N}_x\Big(\int_0^\sigma dL_s^D\, F(s,\rho_s,W_s)\Big)\\
&\qquad=\lim_{\varepsilon\to 0}\mathbb{N}_x\Big(\frac{1}{\varepsilon}\int_0^\sigma dr\, F(r,\rho_r,W_r)\,1_{\{\tau(W_r)<H_r\leqslant\tau(W_r)+\varepsilon\}}\Big)\\
&\qquad=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\otimes\Pi_x\Big[\int_0^{L_\infty} da\, F(L^{-1}(a),\Sigma_a,(\xi_r,0\leqslant r<a))\,1_{\{\tau<a\leqslant\tau+\varepsilon\}}\Big]\\
&\qquad=E\otimes\Pi_x[1_{\{\tau<L_\infty\}}\,F(L^{-1}(\tau),\Sigma_\tau,(\xi_r,0\leqslant r<\tau))].
\end{aligned}$$

From this identity, we easily get

$$\mathbb{N}_x\Big(\int_0^\sigma dL_s^D\,\Phi(\rho_s,W_s)\Big)=E\otimes\Pi_x[1_{\{\tau<L_\infty\}}\,\Phi(\Sigma_\tau,(\xi_r,0\leqslant r<\tau))].$$

Recall that $P[L_\infty>a]=e^{-\alpha a}$ and, that conditionally on $\{L_\infty>a\}$, Σ_a has the same distribution as J_a. The last formula is thus equivalent to the statement of the proposition. □

We now introduce an additional assumption. Namely we assume that for every $x\in D$, the process ξ is Π_x a.s. continuous at $t=\tau$, on the event $\{\tau<\infty\}$. Obviously this assumption holds if ξ has continuous sample paths, but there are other cases of interest.

Under this assumption, Proposition 4.3.2 ensures that $\mathbb{N}_x$ a.e. the left limit $\widehat{W}_s$ exists dL_s^D a.e. over $[0,\sigma]$ and belongs to ∂D. We define under $\mathbb{N}_x$ the *exit measure* $\mathcal{Z}^D$ from D by the formula

$$\langle\mathcal{Z}^D,g\rangle=\int_0^\sigma dL_s^D\,g(\widehat{W}_s).$$

The previous considerations show that $\mathcal{Z}^D$ is a (finite) measure supported on ∂D. As a consequence of Proposition 4.3.2, we have for every nonnegative measurable function g on ∂D,

$$\mathbb{N}_x(\langle\mathcal{Z}^D,g\rangle)=\Pi_x(1_{\{\tau<\infty\}}e^{-\alpha\tau}g(\xi_\tau)).$$

Theorem 4.3.3. — *Let g be a bounded nonnegative measurable function on ∂D. For every $x\in D$ set*

$$u(x)=\mathbb{N}_x(1-\exp-\langle\mathcal{Z}^D,g\rangle).$$

Then u solves the integral equation

$$u(x)+\Pi_x\Big(\int_0^\tau dt\,\psi(u(\xi_t))\Big)=\Pi_x(1_{\{\tau<\infty\}}g(\xi_\tau)).$$

Proof. — Several arguments are analogous to the second step of the proof of Proposition 4.2.2 in Section 4, and so we will skip some details. By the definition of $\mathcal{Z}^D$,

we have

$$\begin{aligned} u(x) &= \mathbb{N}_x\Big(1 - \exp - \int_0^\sigma dL_s^D\, g(\widehat{W}_s)\Big) \\ &= \mathbb{N}_x\Big(\int_0^\sigma dL_s^D\, g(\widehat{W}_s)\, \exp\Big(-\int_s^\sigma dL_r^D\, g(\widehat{W}_r)\Big)\Big) \\ &= \mathbb{N}_x\Big(\int_0^\sigma dL_s^D\, g(\widehat{W}_s)\, \mathbb{E}^*_{\rho_s, W_s}\Big[\exp - \int_0^\infty dL_r^D\, g(\widehat{W}_r)\Big]\Big). \end{aligned}$$

Note that the definition of the random measure dL_r^D makes sense under $\mathbb{P}^*_{\mu,\mathrm{w}}$, provided that $\tau(\mathrm{w}) = \infty$, thanks to Lemma 4.2.4 and the approximations used in the proof of Proposition 4.3.2. Using Lemma 4.2.4 as in subsection 4.2.3, we get if $(\mu, \mathrm{w}) \in \Theta_x$ is such that $\tau(\mathrm{w}) = \infty$,

$$\begin{aligned} \mathbb{E}^*_{\mu,\mathrm{w}}\Big[\exp - \int_0^\infty dL_r^D\, g(\widehat{W}_r)\Big] &= \exp\Big(-\int \mu(dh)\, \mathbb{N}_{\mathrm{w}(h-)}\Big(1 - \exp - \int_0^\sigma dL_r^D\, g(\widehat{W}_r)\Big)\Big) \\ &= \exp\Big(-\int \mu(dh)\, \mathbb{N}_{\mathrm{w}(h-)}\Big(1 - e^{-\langle \mathcal{Z}^D, g\rangle}\Big)\Big) \\ &= \exp(-\int \mu(dh)\, u(\mathrm{w}(h-))). \end{aligned}$$

Hence, using also Proposition 4.3.2,

$$\begin{aligned} u(x) &= \mathbb{N}_x\Big(\int_0^\sigma dL_s^D\, g(\widehat{W}_s)\, \exp(-\int \rho_s(dh)\, u(W_s(h-)))\Big) \\ &= E^0 \otimes \Pi_x\Big[1_{\{\tau<\infty\}} e^{-\alpha\tau} g(\xi_\tau)\, \exp(-\int J_\tau(dh)\, u(\xi_{h-}))\Big] \\ &= \Pi_x\Big[1_{\{\tau<\infty\}}\, g(\xi_\tau)\, \exp\Big(-\int_0^\tau dh\, \widetilde{\psi}(u(\xi_h))\Big)\Big]. \end{aligned}$$

The integral equation of the theorem now follows by the same routine arguments used in the end of the proof of Proposition 4.2.2. □

4.4. Continuity properties of the Lévy snake

From now on until the end of this chapter we assume that the underlying spatial motion ξ has continuous sample paths. The construction of Section 4.1 applies with the following minor simplification. Rather than considering càdlàg paths, we can define $\mathcal{W}_x$ as the set of all E-valued killed continuous paths started at x. An element of $\mathcal{W}$ is thus a continuous mapping $\mathrm{w} : [0, \zeta) \longrightarrow E$, and the distance between w and w′ is defined by

$$d(\mathrm{w}, \mathrm{w}') = \delta(\mathrm{w}(0), \mathrm{w}'(0)) + |\zeta - \zeta'| + \int_0^{\zeta\wedge\zeta'} dt\, (\sup_{r\leqslant t} \delta(\mathrm{w}(r), \mathrm{w}'(r)) \wedge 1). \tag{117}$$

Without risk of confusion, we will keep the same notation as in Section 4.1. The construction developed there goes through without change with these new definitions.

Our goal is to provide conditions on ψ and ξ that will ensure that the process W_s is continuous with respect to a distance finer than d, which we now introduce. We need to consider stopped paths rather than killed paths. A stopped (continuous) path is a continuous mapping $\mathrm{w} : [0,\zeta] \longrightarrow E$, where $\zeta \geqslant 0$. When $\zeta = 0$, we identify w with $\mathrm{w}(0) \in E$. We denote by $\mathcal{W}^*$ the set of all stopped paths in E. The set $\mathcal{W}^*$ is equipped with the distance

$$d^*(\mathrm{w},\mathrm{w}') = |\zeta - \zeta'| + \sup_{t\geqslant 0} \delta(\mathrm{w}(t\wedge\zeta), \mathrm{w}(t\wedge\zeta')).$$

Note that $(\mathcal{W}^*, d^*)$ is a Polish space.

If $\mathrm{w} \in \mathcal{W}$ is a killed path such that $\zeta > 0$ and the left limit $\widehat{\mathrm{w}} = \mathrm{w}(\zeta-)$ exists, we write w^* for the corresponding stopped path $\mathrm{w}^*(t) = \mathrm{w}(t)$ if $t < \zeta$, and $\mathrm{w}^*(\zeta) = \widehat{\mathrm{w}}$. When $\zeta = 0$ we make the convention that $x^* = x$. Note that W_s^* is well defined $\mathbb{P}_x$ a.s., for every fixed $s \geqslant 0$.

As in Chapter 1, we set

$$\gamma = \sup\{a \geqslant 0 : \lim_{\lambda\to\infty} \lambda^{-a}\psi(\lambda) = \infty\} \geqslant 1.$$

Proposition 4.4.1. — *Suppose that there exist three constants $p > 0$, $q > 0$ and $C < \infty$ such that for every $t > 0$ and $x \in E$,*

$$\Pi_x\Big[\sup_{r\leqslant t} \delta(x,\xi_r)^p\Big] \leqslant C\, t^q. \tag{118}$$

Suppose in addition that

$$q(1-\frac{1}{\gamma}) > 1.$$

Then the left limit $\widehat{W}_s = W_s(H_s-)$ exists for every $s \geqslant 0$, $\mathbb{P}_x$ a.s. or $\mathbb{N}_x$ a.e. Furthermore the process $(W_s^*, s \geqslant 0)$ has continuous sample paths with respect to the distance d^*, $\mathbb{P}_x$ a.s. or $\mathbb{N}_x$ a.e.

Remark. — Only the small values of t are relevant in our assumption (118) since we can always replace the distance δ by $\delta \wedge 1$. Uniformity in x could also be relaxed, but we do not strive for the best conditions.

Proof. — It is enough to argue under $\mathbb{P}_x$. Let us fix $t \geqslant 0$ and $s \in (0,1)$. Then,

$$\mathbb{E}_x[d^*(W_t^*, W_{t+s}^*)^p] \leqslant 2^p\Big(\mathbb{E}_x[|H_{t+s}-H_t|^p] + \mathbb{E}_x\Big[\sup_{r\geqslant 0} \delta(W_t^*(r\wedge H_t), W_{t+s}^*(r\wedge H_{t+s}))^p\Big]\Big).$$

To simplify notation, set $m = m_H(t,t+s) = \inf_{[t,t+s]} H_r$. From the conditional distribution of the process W knowing H, we easily get

$$\begin{aligned}
&\mathbb{E}_x\Big[\sup_{u\geqslant 0} \delta(W_t^*(u\wedge H_s), W_{t+s}^*(u\wedge H_{t+s}))^p \,\Big|\, H_r, r\geqslant 0\Big] \\
&\quad\leqslant 2^p\Big(\Pi_x\Big[\Pi_{\xi_m}\Big[\sup_{u\leqslant H_t - m} \delta(\xi_0,\xi_u)^p\Big]\Big] + \Pi_x\Big[\Pi_{\xi_m}\Big[\sup_{u\leqslant H_{t+s}-m} \delta(\xi_0,\xi_u)^p\Big]\Big]\Big) \\
&\quad\leqslant C\, 2^p\left(|H_t - m|^q + |H_{t+s} - m|^q\right),
\end{aligned}$$

using our assumption (118) in the last bound. By combining the previous estimates with Lemma 1.4.6, we arrive at

$$\mathbb{E}_x[d^*(W_t^*, W_{t+s}^*)^p] \leqslant 2^{2p+1}\left(C_p\varphi(1/s)^{-p} + C\,C_q\varphi(1/s)^{-q}\right),$$

where $\varphi(\lambda) = \lambda/\psi^{-1}(\lambda)$. Now choose $\alpha \in (0, 1-\frac{1}{\gamma})$ such that $q\alpha > 1$. Notice that we may also assume $p\alpha > 1$ since by replacing the distance δ by $\delta \wedge 1$, we can take p as large as we wish. The condition $\alpha < 1 - \frac{1}{\gamma}$ and the definition of γ imply that $\varphi(\lambda) \geqslant c\lambda^\alpha$ for every $\lambda \geqslant 1$, for some constant $c > 0$. Hence, there exists a constant C' independent of t and s such that

$$\mathbb{E}_x[d^*(W_t^*, W_{t+s}^*)^p] \leqslant C'(s^{p\alpha} + s^{q\alpha}).$$

The Kolmogorov lemma then gives the existence of a continuous modification of the process $(W_s^*, s \geqslant 0)$ with respect to the distance d^*. The various assertions of the proposition follow easily, recalling that we already know that the process $(W_s, s \geqslant 0)$ has continuous paths for the distance d. □

4.5. The Brownian motion case

In this section, we concentrate on the case when the underlying spatial motion ξ is Brownian motion in $\mathbb{R}^d$. We will give a necessary and sufficient condition for the process W^* to have a modification that is continuous with respect to the distance d^*.

To this end, we introduce the following condition on ψ:

$$\int_1^\infty \Big(\int_0^t \psi(u)\,du\Big)^{-1/2} dt < \infty. \tag{119}$$

Note that this condition is stronger than the condition $\int_1^\infty du/\psi(u) < \infty$ for the path continuity of H. In fact, since ψ is convex, there exists a positive constant c such $\psi(t) \geqslant ct$ for every $t \geqslant 1$. Then, for $t \geqslant 1$,

$$\int_0^t \psi(u)du \leqslant t\psi(t) \leqslant c^{-1}\psi(t)^2$$

and thus

$$\int_1^\infty \frac{du}{\psi(u)} \leqslant c^{-1/2}\int_1^\infty \Big(\int_0^t \psi(u)\,du\Big)^{-1/2} dt.$$

Also note that (119) holds if $\gamma > 1$. On the other hand, it is easy to produce examples where (119) does not hold although H has continuous sample paths.

Condition (119) was introduced in connection with solutions of $\Delta u = \psi(u)$ in domains of $\mathbb{R}^d$. We briefly review the results that will be relevant to our needs (see [**25**],[**41**] and also Lemma 2.3 in [**45**]). We denote by B_r the open ball of radius r centered at the origin in $\mathbb{R}^d$.

A. If (119) holds, then, for every $r > 0$, there exists a positive solution of the problem

$$\begin{cases} \frac{1}{2}\Delta u = \psi(u) & \text{in } B_r \\ u_{|\partial B_r} = \infty\,. \end{cases} \tag{120}$$

Here the condition $u_{|\partial B_r} = \infty$ means that $u(x)$ tends to $+\infty$ as $x \to y, x \in B_r$, for every $y \in \partial B_r$.

B. If (119) does not hold, then for every $c > 0$, there exists a positive solution of the problem

$$\begin{cases} \frac{1}{2}\Delta u = \psi(u) & \text{in } \mathbb{R}^d \\ u(0) = c\,. \end{cases} \tag{121}$$

Connections between the Lévy snake and the partial differential equation $\frac{1}{2}\Delta u = \psi(u)$ follow from Theorem 4.3.3. Note that this is just a reformulation of the well-known connections involving superprocesses. We use the notation of Section 4.3. A domain D in $\mathbb{R}^d$ is regular if every point y of ∂D is regular for D^c, meaning that: $\inf\{t > 0 : \xi_t \notin D\} = 0$, Π_y a.s.

Proposition 4.5.1. — *Assume that ξ is Brownian motion in $\mathbb{R}^d$. Let D be a bounded regular domain in $\mathbb{R}^d$, and let g be a nonnegative continuous function on ∂D. Then the function*

$$u(x) = \mathbb{N}_x(1 - \exp -\langle \mathcal{Z}^D, g\rangle)$$

is twice continuously differentiable in D and is the unique nonnegative solution of the problem

$$\begin{cases} \frac{1}{2}\Delta u = \psi(u) & \text{in } D \\ u_{|\partial D} = g\,. \end{cases} \tag{122}$$

Proof. — This follows from Theorem 4.3.3 by standard arguments. In the context of superprocesses, the result is due to Dynkin [**12**]. See e.g. Chapter 5 in [**31**] for a proof in the case $\psi(u) = u^2$, which is readily extended. □

We can now state our main result.

Theorem 4.5.2. — *Assume that ξ is Brownian motion in $\mathbb{R}^d$. The following conditions are equivalent.*

(i) $\mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0) < \infty$ *for some* $r > 0$.

(ii) $\mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0) < \infty$ *for every* $r > 0$.

(iii) *The left limit $\widehat{W}_s = W_s(\zeta_s-)$ exists for every $s \geqslant 0$, $\mathbb{P}_0$ a.s., and the mapping $s \to \widehat{W}_s$ is continuous, $\mathbb{P}_0$ a.s.*

(iv) *The left limit $\widehat{W}_s = W_s(\zeta_s-)$ exists for every $s \geqslant 0$, $\mathbb{P}_0$ a.s., and the mapping $s \to W^*_s$ is continuous for the metric d^*, $\mathbb{P}_0$ a.s.*

(v) *Condition* (119) *holds.*

Remark. — The conditions of Theorem 4.5.2 are also equivalent to the a.s. compactness of the range of the superprocess with spatial motion ξ and branching mechanism ψ, started at a nonzero initial measure μ with compact support. This fact, that follows from Theorem 5.1 in [**45**], can be deduced from the representation of Theorem 4.2.1.

Proof. — The equivalence between (i), (ii) and (v) is easy given facts A. and B. recalled above. We essentially reproduce arguments of [**45**]. By fact B., if (v) does not hold, then we can for every $c > 0$ find a nonnegative function v_c such that $v_c(0) = c$ and $\frac{1}{2}\Delta v_c = \psi(v_c)$ in $\mathbb{R}^d$. Let $r > 0$ and $\lambda > 0$. By Proposition 4.5.1, the nonnegative function

$$u_{\lambda,r}(x) = \mathbb{N}_x(1 - \exp -\lambda\langle \mathcal{Z}^{B_r}, 1\rangle), \qquad x \in B_r$$

solves $\frac{1}{2}\Delta u_{\lambda,r} = \psi(u_{\lambda,r})$ in B_r with boundary condition λ on ∂B_r. By choosing λ sufficiently large so that $\sup\{v_c(y), y \in \partial B_r\} < \lambda$, and using the comparison principle for nonnegative solutions of $\frac{1}{2}\Delta u = \psi(u)$ (see Lemma V.7 in [**31**]), we see that $v_c \leqslant u_{\lambda,r}$ in B_r. In particular,

$$c = v_c(0) \leqslant u_{\lambda,r}(0) \leqslant \mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0).$$

Since c was arbitrary, we get $\mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0) = \infty$ and we have proved that (i) $\Rightarrow$ (v). Trivially (ii) $\Rightarrow$ (i).

Suppose now that (v) holds. Let $r > 0$. By fact A., we can find a function $u_{(r)}$ such that $\frac{1}{2}\Delta u_{(r)} = \psi(u_{(r)})$ in B_r with boundary condition $+\infty$ on ∂B_r. The maximum principle then implies that, for every $\lambda > 0$, $u_{\lambda,r} \leqslant u_{(r)}$. Hence

$$\mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0) = \lim_{\lambda\uparrow\infty} \uparrow u_{\lambda,r}(0) \leqslant u_{(r)}(0) < \infty$$

and (ii) holds. We have thus proved the equivalence of (i), (ii) and (v).

Let us prove that (iii) $\Rightarrow$ (ii). We assume that (iii) holds. Let $r > 0$. Then on the event $\{\mathcal{Z}^{B_r} \neq 0\}$, there exists $s \in (0, \sigma)$ such that $\widehat{W}_s \in \partial B_r$. It follows that

$$\mathbb{N}_0\Big(\sup_{s\in(0,\sigma)} |\widehat{W}_s| \geqslant r\Big) \geqslant \mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0).$$

Let $T_1 = \inf\{s \geqslant 0 : L_s^0 > 1\}$ (in agreement with the notation of Chapter 1). The path continuity of $\widehat{W}_s$ ensures that $\mathbb{P}_0$ a.s. there are only finitely many excursions intervals (α_i, β_i) of H_s away from 0, before time T_1, such that

$$\sup_{s\in(\alpha_i,\beta_i)} |\widehat{W}_s| \geqslant r.$$

On the other hand, excursion theory implies that the number of such intervals is Poisson with parameter

$$\mathbb{N}_0\Big(\sup_{s\in(0,\sigma)} |\widehat{W}_s| \geqslant r\Big).$$

We conclude that the latter quantity is finite, and so $\mathbb{N}_0(\mathcal{Z}^{B_r} \neq 0) < \infty$.

Note that (iv) $\Rightarrow$ (iii). Thus, to complete the proof of Theorem 4.5.2, it remains to verify that (ii) $\Rightarrow$ (iv). From now on until the end of the proof, we assume that (ii) holds.

We use the following simple lemma.

Lemma 4.5.3. — *Let D be a domain in $\mathbb{R}^d$ containing 0, and let*

$$S = \inf\{s \geqslant 0 : W_s(t) \notin D \text{ for some } t \in [0, H_s)\}.$$

Then $\mathbb{N}_0(\mathcal{Z}^D \neq 0) \geqslant \mathbb{N}_0(S < \infty)$.

Proof. — By excursion theory, we have

$$\mathbb{P}_0[S \geqslant T_1] = \exp(-\mathbb{N}_0(S < \infty)).$$

Then, let τ be as previously the exit time from D. If there exists $s < T_1$ such that $\tau(W_s) < H_s$, then the same property holds for every $s' > s$ such that $s' - s$ is sufficiently small, by the continuity of H and the snake property. Hence,

$$\{S < T_1\} \subset \{\int_0^{T_1} ds\, 1_{\{\tau(W_s)<H_s\}} > 0\}, \qquad \mathbb{P}_0 \text{ a.e.}$$

It follows that

$$\mathbb{P}_0[S \geqslant T_1] \geqslant \mathbb{P}_0\Big[\int_0^{T_1} ds\, 1_{\{\tau(W_s)<H_s\}} = 0\Big] = \mathbb{P}_0[L^D_{T_1} = 0],$$

where the second equality is a consequence of the formula

$$L^D_{T_1} = \ell^D(\int_0^{T_1} ds\, 1_{\{\tau(W_s)<H_s\}}),$$

together with the fact that $\ell^D(s) > 0$ for every $s > 0$, a.s.

Using again excursion theory and the construction of the exit measure under $\mathbb{N}_0$, we get

$$\mathbb{P}_0[S \geqslant T_1] \geqslant \mathbb{P}_0[L^D_{T_1} = 0] = \exp(-\mathbb{N}_0(\mathcal{Z}^D \neq 0)).$$

By comparing with the first formula of the proof, we get the desired inequality. □

Let $\varepsilon > 0$. We specialize the previous lemma to the case $D = B_\varepsilon$ and write $S = S^\varepsilon_1$. Then, for every $r > 0$,

$$\mathbb{P}_0[S^\varepsilon_1 \geqslant T_r] = \exp(-r\mathbb{N}_0(S^\varepsilon_1 < \infty)) \geqslant \exp(-r\mathbb{N}_0(\mathcal{Z}^{B_\varepsilon} \neq 0)).$$

From (ii), it follows that $S^\varepsilon_1 > 0$, $\mathbb{P}_0$ a.e. Also note that S^ε_1 is a stopping time of the filtration $(\mathcal{F}_{s+})$ and that $\{W_{S^\varepsilon_1}(t) : 0 \leqslant t < H_{S^\varepsilon_1}\} \subset \overline{B}_\varepsilon$ (if this inclusion were not true, the snake property would contradict the definition of S^ε_1).

Recall the notation $m_H(s, s') = \inf_{[s,s']} H_r$ for $s \leqslant s'$. We define inductively a sequence $(S^\varepsilon_n)_{n\geqslant 1}$ of stopping times (for the filtration $(\mathcal{F}_{s+})$) by setting

$$S^\varepsilon_{n+1} = \inf\{s > S^\varepsilon_n : |W_s(t) - W_s(t \wedge m_H(S^\varepsilon_n, s))| > \varepsilon \text{ for some } t \in [0, H_s)\}.$$

At this point we need another lemma.

Lemma 4.5.4. — *Let T be a stopping time of the filtration $(\mathcal{F}_{s+})$, such that $T < \infty$, $\mathbb{P}_0$ a.s. For every $s \geqslant 0$, define a killed path $\widetilde{W}_s$ with lifetime $\widetilde{H}_s$ by setting*

$$\widetilde{W}_s(t) = W_s(m_H(T,T+s)+t) - W_s(m_H(T,T+s)),\ 0 \leqslant t < \widetilde{H}_s := H_{T+s} - m_H(T,T+s)$$

with the convention that $\widetilde{W}_s = 0$ if $\widetilde{H}_s = 0$. Then the process $(\widetilde{W}_s, s \geqslant 0)$ is independent of $\mathcal{F}_{T+}$ and has the same distribution as $(W_s, s \geqslant 0)$ under $\mathbb{P}_0$.

This lemma follows from the strong Markov property of the Lévy snake, together with Lemma 4.2.4. The translation invariance of the spatial motion is of course crucial here.

As a consequence of the preceding lemma, we obtain that the random variables $S^\varepsilon_1, S^\varepsilon_2 - S^\varepsilon_1, \dots, S^\varepsilon_{n+1} - S^\varepsilon_n, \dots$ are independent and identically distributed. Recall that these variables are positive a.s. Also observe that

$$\{W_{S^\varepsilon_{n+1}}(t) - W_{S^\varepsilon_{n+1}}(t \wedge m_H(S^\varepsilon_n, S^\varepsilon_{n+1})) : t \in [0, H_{S^\varepsilon_{n+1}})\} \subset \overline{B}_\varepsilon,$$

by the same argument as used previously for $\{W_{S^\varepsilon_1}(t) : t \in [0, H_{S^\varepsilon_1})\}$.

Let $a > 0$. We claim that $\mathbb{N}_0$ a.s. we can choose $\delta_1 > 0$ small enough so that, for every $s, s' \in [0, T_a]$ such that $s \leqslant s' \leqslant s + \delta_1$,

$$|W_{s'}(t) - W_{s'}(m_H(s,s') \wedge t)| \leqslant 3\varepsilon, \qquad \text{for every } t \in [0, H_{s'}). \tag{123}$$

Let us verify that the claim holds if we take

$$\delta_1 = \inf\{S^\varepsilon_{n+1} - S^\varepsilon_n\,;\, n \geqslant 1,\ S^\varepsilon_n \leqslant T_a\} > 0.$$

Consider $s, s' \in [0, T_a]$ with $s \leqslant s' \leqslant s + \delta_1$. Then two cases may occur.

Either s, s' belong to the same interval $[S^\varepsilon_n, S^\varepsilon_{n+1}]$. Then, from the definition of S^ε_{n+1} we know that

$$|W_{s'}(t) - W_{s'}(t \wedge m_H(S^\varepsilon_n, s'))| \leqslant \varepsilon \text{ for every } t \in [0, H_{s'}). \tag{124}$$

Since $m_H(s,s') \geqslant m_H(S^\varepsilon_n, s')$ we can replace t by $t \wedge m(s,s')$ to get

$$|W_{s'}(t \wedge m_H(s,s')) - W_{s'}(t \wedge m_H(S^\varepsilon_n, s'))| \leqslant \varepsilon \text{ for every } t \in [0, H_{s'}),$$

and our claim (123) follows by combining this bound with the previous one.

Then we need to consider the case where $s \in [S^\varepsilon_{n-1}, S^\varepsilon_n]$ and $s' \in (S^\varepsilon_n, S^\varepsilon_{n+1}]$ for some $n \geqslant 1$ (by convention $S^\varepsilon_0 = 0$). If $m_H(s,s') = m_H(S^\varepsilon_n, s')$, then the same argument as in the first case goes through. Therefore we can assume that $m_H(s,s') < m_H(S^\varepsilon_n, s')$, which implies $m_H(S^\varepsilon_{n-1}, S^\varepsilon_n) < m_H(S^\varepsilon_n, s')$. Note that the bound (124) still holds. We also know that

$$|W_{S^\varepsilon_n}(t) - W_{S^\varepsilon_n}(t \wedge m_H(S^\varepsilon_{n-1}, S^\varepsilon_n))| \leqslant \varepsilon \text{ for every } t \in [0, H_{S^\varepsilon_n}). \tag{125}$$

We replace t by $t \wedge m_H(S^\varepsilon_n, s')$ in this bound, and note that $W_{S^\varepsilon_n}(t \wedge m_H(S^\varepsilon_n, s')) = W_{s'}(t \wedge m_H(S^\varepsilon_n, s'))$ for every $t \in [0, H_{S^\varepsilon_n} \wedge H_{s'})$, by the snake property. It follows that

$$|W_{s'}(t \wedge m_H(S^\varepsilon_n, s')) - W_{S^\varepsilon_n}(t \wedge m_H(S^\varepsilon_{n-1}, S^\varepsilon_n))| \leqslant \varepsilon \text{ for every } t \in [0, H_{s'}). \tag{126}$$

Similarly, we can replace t by $t \wedge m_H(s,s')$ in (125), using again the snake property to write $W_{S_n^\varepsilon}(t \wedge m_H(s,s')) = W_{s'}(t \wedge m_H(s,s'))$ (note that $m_H(S_{n-1}^\varepsilon, S_n^\varepsilon) \leqslant m_H(s,s') < m_H(S_n^\varepsilon, s')$). It follows that

$$|W_{s'}(t \wedge m_H(s,s')) - W_{S_n^\varepsilon}(t \wedge m_H(S_{n-1}^\varepsilon, S_n^\varepsilon))| \leqslant \varepsilon \text{ for every } t \in [0, H_{s'}). \tag{127}$$

Our claim (123) is now a consequence of (124), (126) and (127).

We can already derive from (123) the fact that the left limit $\widehat{W}_s$ exists for every $s \in [0, T_a]$, $\mathbb{P}_0$ a.s. We know that this left limit exists for every rational $s \in [0, T_a]$, $\mathbb{P}_0$ a.s. Let $s \in (0, T_a]$, and let s_n be a sequence of rationals increasing to s. Then the sequence $m_H(s_n, s)$ also increases to H_s. If $m_H(s_n, s) = H_s$ for some n, then the snake property shows that $W_s(t) = W_{s_n}(t)$ for every $t \in [0, H_s)$ and the existence of $\widehat{W}_s$ is an immediate consequence. Otherwise, (123) shows that for n large enough,

$$\sup_{t \in [0, H_s)} |W_s(t) - W_s(t \wedge m_H(s_n, s))| \leqslant 3\varepsilon$$

and by applying this to a sequence of values of ε tending to 0 we also get the existence of $\widehat{W}_s$.

We finally use a time-reversal argument. From Corollary 3.1.6, we know that the processes $(H_{t \wedge T_a}, t \geqslant 0)$ and $(H_{(T_a - t)^+}, t \geqslant 0)$ have the same distribution. By considering the conditional law of W knowing H, we immediately obtain that the processes $(W_{t \wedge T_a}, t \geqslant 0)$ and $(W_{(T_a - t)^+}, t \geqslant 0)$ also have the same distribution. Thanks to this observation and the preceding claim, we get that $\mathbb{P}_0$ a.s. there exists $\delta_2 > 0$ such that for every $s, s' \in [0, T_a]$ with $s \leqslant s' \leqslant s + \delta_2$,

$$|W_s(t) - W_s(m_H(s,s') \wedge t)| \leqslant 3\varepsilon, \qquad \text{for every } t \in [0, H_s). \tag{128}$$

To complete the proof, note that the snake property implies that

$$W_s^*(m_H(s,s')) = W_{s'}^*(m_H(s,s')),$$

using a continuity argument in the case $m_H(s,s') = H_s \wedge H_{s'}$. Thus, if $s \leqslant s' \leqslant T_a$ and $s' - s \leqslant \delta_1 \wedge \delta_2$,

$$\begin{aligned}
&\sup_{t \geqslant 0} |W_s^*(t \wedge H_s) - W_{s'}^*(t \wedge H_{s'})| \\
&\quad \leqslant \sup_{t \in [0, H_s]} |W_s^*(t) - W_s^*(t \wedge m_H(s,s'))| + \sup_{t \in [0, H_{s'}]} |W_{s'}^*(t) - W_{s'}^*(t \wedge m_H(s,s'))| \\
&\quad = \sup_{t \in [0, H_s)} |W_s(t) - W_s(t \wedge m_H(s,s'))| + \sup_{t \in [0, H_{s'})} |W_{s'}(t) - W_{s'}(t \wedge m_H(s,s'))| \\
&\quad \leqslant 6\varepsilon\,.
\end{aligned}$$

This gives the continuity of the mapping $s \to W_s^*$ with respect to the distance d^*, and completes the proof of (iv). □

4.6. The law of the Lévy snake at a first exit time

Our goal in this section is to give explicit formulas for the law of the Lévy snake at its first exit time from a domain. We keep assuming that H has continuous sample paths and in addition we suppose that the process W^* has continuous sample paths with respect to the metric d^*. Note that the previous two sections give sufficient conditions for this property to hold.

Let D be an open set in E and $x \in D$. We slightly abuse notation by writing $\tau(\mathrm{w}) = \inf\{t \in [0, \zeta_{\mathrm{w}}] : \mathrm{w}(t) \notin D\}$ for any stopped path $\mathrm{w} \in \mathcal{W}^*$. We also set

$$T_D = \inf\{s > 0 : \tau(W_s^*) < \infty\}.$$

The continuity of W^* with respect to the metric d^* immediately implies that $T_D > 0$, $\mathbb{N}_x$ a.e. or $\mathbb{P}_x$ a.e. Furthermore, on the event $\{T_D < \infty\}$ the path $W^*_{T_D}$ hits the boundary of D exactly at its lifetime. The main result of this section determines the law of the pair (ρ_{T_D}, W_{T_D}) under $\mathbb{N}_x(\cdot \cap \{T_D < \infty\})$.

Before stating this result, we need some notation and a preliminary lemma. For every $y \in D$, we set

$$u(y) = \mathbb{N}_y(T_D < \infty) < \infty.$$

Recall that, for every $a, b \geqslant 0$, we have defined

$$\gamma_\psi(a,b) = \begin{cases} (\psi(a) - \psi(b))/(a-b) & \text{if} \quad a \neq b, \\ \psi'(a) & \text{if} \quad a = b\,. \end{cases}$$

Note that $\gamma_\psi(a,0) = \widetilde{\psi}(a)$ (by convention $\widetilde{\psi}(0) = \psi'(0) = \alpha$). The following formulas will be useful: For every $a, b \geqslant 0$,

$$\int \pi(dr) \int_0^r d\ell\, (1 - e^{-a\ell - b(r-\ell)}) = \gamma_\psi(a,b) - \alpha - \beta(a+b) \tag{129}$$

$$\int \pi(dr) \int_0^r d\ell\, e^{-a\ell}(1 - e^{-b(r-\ell)}) = \gamma_\psi(a,b) - \widetilde{\psi}(a) - \beta b\,. \tag{130}$$

The first formula is easily obtained by observing that, if $a \neq b$,

$$\int_0^r d\ell (1 - e^{-\ell a - (r-\ell)b}) = \frac{1}{a-b}(r(a-b) + (e^{-ra} - e^{-rb})).$$

The second one is a consequence of the first one and the identity

$$\widetilde{\psi}(a) = \alpha + \beta a + \int \pi(dr) \int_0^r d\ell (1 - e^{-a\ell}).$$

Recall from Section 3.1 the definition of the probability measures $\mathbb{M}_a$ on $M_f(\mathbb{R}_+)^2$.

Lemma 4.6.1. — (i) *Let $a > 0$ and let F be a nonnegative measurable function on $M_f(\mathbb{R}_+) \times M_f(\mathbb{R}_+) \times \mathcal{W}$. Then,*

$$\mathbb{N}_x\Big(\int_0^\sigma dL_s^a\, F(\rho_s, \eta_s, W_s) \Big) = e^{-\alpha a} \int \mathbb{M}_a(d\mu\, d\nu)\, \Pi_x[F(\mu, \nu, (\xi_r, 0 \leqslant r < a))].$$

(ii) *Let f, g be two nonnegative measurable functions on $\mathbb{R}_+$. Then,*

$$N\Big(\int_0^\sigma dL_s^a \exp(-\langle\rho_s, f\rangle - \langle\eta_s, g\rangle)\Big) = \exp\Big(-\int_0^a \gamma_\psi(f(t), g(t))\, dt\Big).$$

Proof. — (i) As in the proof of Lemma 4.2.5, we may restrict our attention to a function $F(\rho_s, \eta_s, W_s) = F(\rho_s, \eta_s)$. Then the desired result follows from Corollary 1.3.4 in the same way as Proposition 3.1.3 was deduced from Proposition 1.1.4.
(ii) By part (i) we have

$$N\Big(\int_0^\sigma dL_s^a \exp(-\langle\rho_s, f\rangle - \langle\eta_s, g\rangle)\Big) = e^{-\alpha a}\int \mathbb{M}_a(d\mu\, d\nu) \exp(-\langle\mu, f\rangle - \langle\nu, g\rangle).$$

From the definition of $\mathbb{M}_a$ this is equal to

$$\exp\Big(-\alpha a - \beta\int_0^a (f(t)+g(t))dt - \int_0^a dt \int \pi(dr) \int_0^r d\ell(1 - e^{-\ell f(t) - (r-\ell)g(t)})\Big).$$

The stated result now follows from (129). □

Theorem 4.6.2. — *Assume that $u(x) > 0$. Let $a > 0$, let F be a nonnegative measurable function on $\mathcal{W}_x^*$ and let g be a nonnegative measurable function on $\mathbb{R}_+$ with support contained in $[0, a]$. Then*

$$\begin{aligned}(131)\quad &\mathbb{N}_x\Big(1_{\{T_D<\infty\}} 1_{\{a<H_{T_D}\}} F(W_{T_D}(t), 0 \leqslant t \leqslant a) \exp(-\langle\rho_{T_D}, g\rangle)\Big)\\ &\qquad = \Pi_x\Big[1_{\{a<\tau\}} u(\xi_a) F(\xi_r, 0 \leqslant r \leqslant a) \exp\Big(-\int_0^a \gamma_\psi(u(\xi_r), g(r))dr\Big)\Big].\end{aligned}$$

Alternatively, the law of W_{T_D} under $\mathbb{N}_x(\cdot \cap \{T_D < \infty\})$ is characterized by:

$$\begin{aligned}(132)\quad &\mathbb{N}_x\Big(1_{\{T_D<\infty\}} 1_{\{a<H_{T_D}\}} F(W_{T_D}(t), 0 \leqslant t \leqslant a)\Big)\\ &\qquad = \Pi_x\Big[1_{\{a<\tau\}} u(\xi_a) F(\xi_r, 0 \leqslant r \leqslant a) \exp\Big(-\int_0^a \widetilde{\psi}(u(\xi_r))dr\Big)\Big],\end{aligned}$$

and the conditional law of ρ_{T_D} knowing W_{T_D} is the law of

$$\beta 1_{[0,H_{T_D}]}(r)\, dr + \sum_{i\in I}(v_i - \ell_i)\, \delta_{r_i}$$

where $\sum \delta_{(r_i, v_i, \ell_i)}$ is a Poisson point measure on $\mathbb{R}_+^3$ with intensity

$$(133)\qquad 1_{[0,H_{T_D}]}(r) 1_{[0,v]}(\ell) e^{-\ell u(W_{T_D}(r))}\, dr\, \pi(dv) d\ell.$$

Proof. — We will rely on results obtained in Section 4.2 above. As in subsection 4.2.2, we denote by (ρ^i, W^i), $i \in I$ the "excursions" of the Lévy snake above height a. We let (α_i, β_i) be the time interval corresponding to the excursion (ρ^i, W^i) and $\ell^i = L_{\alpha_i}^a$. We also use the obvious notation

$$T_D(W^i) = \inf\{s \geqslant 0 : \tau(W_s^{i*}) < \infty\}.$$

For every $s \geqslant 0$, set

$$G_s = 1_{\{s<T_D\}} F(W_s^*) \exp(-\langle\rho_s, g\rangle).$$

Then it is easy to verify that

$$\sum_{i\in I} G_{\alpha_i}\, 1_{\{T_D(W^i)<\infty\}} = 1_{\{T_D<\infty\}} 1_{\{a<H_{T_D}\}} F(W_{T_D}(t), 0\leqslant t\leqslant a)\exp(-\langle \rho_{T_D}, g\rangle). \tag{134}$$

In fact, the sum in the left side contains at most one nonzero term, and exactly one iff $T_D<\infty$ and $a<H_{T_D}$. On this event, T_D belongs to one excursion interval above height a, say (α_j,β_j), and then the restriction of ρ_{T_D} to $[0,a]$ coincides with ρ_{α_j} (see the second step of the proof of Proposition 1.3.1), whereas the snake property ensures that the paths W_{T_D} and W_{α_i} are the same over $[0,a)$. Our claim (134) follows.

Recall the notation $\widetilde{W},\widetilde{\rho},\gamma^a(\ell)$ introduced in the proof of Proposition 4.2.3. The proof of this proposition shows that conditionally on the σ-field $\mathcal{E}_a$, the point measure

$$\sum_{i\in I}\delta_{(\ell^i,\rho^i,W^i)}$$

is Poisson with intensity

$$1_{[0,L^a_\sigma]}(\ell)\, d\ell\, \mathbb{N}_{\widehat{\widetilde{W}}_{\gamma^a(\ell)}}(d\rho dW).$$

Note that the statement of Proposition 4.2.3 is slightly weaker than this, but the preceding assertion follows readily from the proof.

We now claim that we can find a deterministic function Δ and an $\mathcal{E}_a$-measurable random variable Z such that, for every $j\in I$, we have

$$G_{\alpha_j} = \Delta(Z,\ell^j,(\ell^i,W^i)_{i\in I}). \tag{135}$$

Precisely, this relation holds if we take for every $\ell\geqslant 0$,

$$\Delta(Z,\ell,(\ell^i,W^i)_{i\in I}) = \Big(\prod_{i:\ell^i<\ell} 1_{\{T_D(W^i)=\infty\}}\Big)\, 1_{\{\widetilde{W}_r(t)\in D,\forall\, r\in[0,\gamma^a(\ell)],t\in[0,a]\}} \times F(\widetilde{W}^*_{\gamma^a(\ell)})\exp(-\langle\widetilde{\rho}_{\gamma^a(\ell)},g\rangle).$$

Note that the right side of the last formula depends on ℓ, on $(\ell^i,W^i)_{i\in I}$ and on the triple $(\widetilde{W},\widetilde{\rho},\gamma^a)$ which is $\mathcal{E}_a$-measurable, and thus can be written in the form of the left side. Then, to justify (135), note that

$$1_{\{\alpha_j<T_D\}} = \Big(\prod_{i:\ell^i<\ell^j} 1_{\{T_D(W^i)=\infty\}}\Big) 1_{\{\widetilde{W}_r(t)\in D,\forall\, r\in[0,\gamma^a(\ell^j)],t\in[0,a]\}},$$

since $\gamma^a(\ell^j)=\int_0^{\alpha_j} dr\, 1_{\{H_r\leqslant a\}}$ as observed in the proof of Proposition 4.2.3. The latter proof also yields the identities

$$W_{\alpha_j}=W_{\beta_j}=\widetilde{W}_{\gamma^a(\ell^j)}\ ,\qquad \rho_{\alpha_j}=\rho_{\beta_j}=\widetilde{\rho}_{\gamma^a(\ell^j)}$$

from which (135) follows.

Then, by an application of Lemma 3.1.5 to the point measure $\sum_{i\in I}\delta_{(\ell^i,\rho^i,W^i)}$, which is Poisson conditional on $\mathcal{E}_a$, we have

$$\begin{aligned}
&\mathbb{N}_x\Big(\sum_{j\in I} G_{\alpha_j}\,1_{\{T_D(W^j)<\infty\}}\,\Big|\,\mathcal{E}_a\Big)\\
&\qquad=\mathbb{N}_x\Big(\sum_{j\in I}\Delta(Z,\ell^j,(\ell^i,W^i)_{i\in I})\,1_{\{T_D(W^j)<\infty\}}\,\Big|\,\mathcal{E}_a\Big)\\
&\qquad=\mathbb{N}_x\Big(\int_0^{L^a_\sigma} d\ell\int \mathbb{N}_{\widehat{\widetilde{W}}_{\gamma^a(\ell)}}(d\rho' dW')\,1_{\{T_D(W')<\infty\}}\Delta(Z,\ell,(\ell^i,W^i)_{i\in I})\,\Big|\,\mathcal{E}_a\Big)\\
&\qquad=\mathbb{N}_x\Big(\int_0^{L^a_\sigma} d\ell\,\Delta(Z,\ell,(\ell^i,W^i)_{i\in I})\,\mathbb{N}_{\widehat{\widetilde{W}}_{\gamma^a(\ell)}}(T_D<\infty)\,\Big|\,\mathcal{E}_a\Big).
\end{aligned}$$

Now use the definition of $\Delta(Z,\ell,(\ell^i,W^i)_{i\in I})$ to get

$$\begin{aligned}
(136)\quad &\mathbb{N}_x\Big(\sum_{j\in I} G_{\alpha_j}\,1_{\{T_D(W^j)<\infty\}}\Big)\\
&\qquad=\mathbb{N}_x\Big(\int_0^{L^a_\sigma} d\ell\,u(\widehat{\widetilde{W}}_{\gamma^a(\ell)})\Big(\prod_{i:\ell^i<\ell}1_{\{T_D(W^i)=\infty\}}\Big)\,1_{\{T_D(\widetilde{W})>\gamma^a(\ell)\}}\\
&\qquad\qquad\qquad\times F(\widetilde{W}^*_{\gamma^a(\ell)})\exp(-\langle\widetilde{\rho}_{\gamma^a(\ell)},g\rangle)\Big)\\
&\qquad=\mathbb{N}_x\Big(\int_0^\sigma dL^a_s\,u(\widehat{W}_s)\,F(W^*_s)\,\exp(-\langle\rho_s,g\rangle)\,1_{\{s<T_D\}}\Big).
\end{aligned}$$

The last equality is justified by the change of variables $\ell=L^a_s$ and the fact that dL^a_s a.e.,

$$\widetilde{W}_{\gamma^a(L^a_s)}=\widetilde{W}_{\widetilde{A}^a_s}=W_s\ ,\qquad \widetilde{\rho}_{\gamma^a(L^a_s)}=\widetilde{\rho}_{\widetilde{A}^a_s}=\rho_s,$$

(where $\widetilde{A}^a_s=\int_0^s dr\,1_{\{H_r\leqslant a\}}$ as previously) and similarly, dL^a_s a.e.,

$$\begin{aligned}
\Big(\prod_{i:\ell^i<L^a_s}1_{\{T_D(W^i)=\infty\}}\Big)\,1_{\{T_D(\widetilde{W})>\gamma^a(L^a_s)\}}&=1_{\{W^*_r(t)\in D,\,\forall\,r\leqslant s,\ t\in[a,H_r]\}}\,1_{\{T_D(\widetilde{W})>\widetilde{A}^a_s\}}\\
&=1_{\{s<T_D\}}.
\end{aligned}$$

To evaluate the right side of (136), we use a duality argument. It follows from Corollary 3.1.6 and the construction of the Lévy snake that the triples

$$(\rho_s,L^a_s,W_s;0\leqslant s\leqslant\sigma)$$

and

$$(\eta_{(\sigma-s)-},L^a_\sigma-L^a_{\sigma-s},W_{\sigma-s};0\leqslant s\leqslant\sigma)$$

have the same distribution under $\mathbb{N}_x$. From this we get

$$\begin{aligned}
(137)\quad &\mathbb{N}_x\Big(\int_0^\sigma dL^a_s\,u(\widehat{W}_s)\,F(W^*_s)\,\exp(-\langle\rho_s,g\rangle)\,1_{\{s<T_D\}}\Big)\\
&\qquad=\mathbb{N}_x\Big(\int_0^\sigma dL^a_s\,u(\widehat{W}_s)\,F(W^*_s)\,\exp(-\langle\eta_s,g\rangle)\,1_{\{\tau(W^*_r)=\infty,\,\forall\,r\geqslant s\}}\Big).
\end{aligned}$$

Now we can use the strong Markov property of the Lévy snake (as in the second step of the proof of Proposition 4.2.2), and then Lemma 4.2.4, to get

$$\begin{aligned}(138)\quad &\mathbb{N}_x\Big(\int_0^\sigma dL_s^a\, u(\widehat{W}_s)\, F(W_s^*)\, \exp(-\langle\eta_s,g\rangle)\, 1_{\{\tau(W_r^*)=\infty,\,\forall\, r\geqslant s\}}\Big)\\ &\quad=\mathbb{N}_x\Big(\int_0^\sigma dL_s^a\, u(\widehat{W}_s)\, F(W_s^*)\, \exp(-\langle\eta_s,g\rangle)\, 1_{\{\tau(W_s^*)=\infty\}}\, \mathbb{P}^*_{\rho_s,W_s}[T_D=\infty]\Big)\\ &\quad=\mathbb{N}_x\Big(\int_0^\sigma dL_s^a\, u(\widehat{W}_s)\, F(W_s^*)\\ &\qquad\qquad\times\exp(-\langle\eta_s,g\rangle)\, 1_{\{\tau(W_s^*)=\infty\}}\, \exp\Big(-\int\rho_s(dt)\, u(W_s(t))\Big)\Big).\end{aligned}$$

Finally, we use Lemma 4.6.1 to write

$$\begin{aligned}&\mathbb{N}_x\Big(\int_0^\sigma dL_s^a\, u(\widehat{W}_s)\, F(W_s^*)\, 1_{\{\tau(W_s^*)=\infty\}}\, \exp(-\langle\eta_s,g\rangle)\, \exp\Big(-\int\rho_s(dt)\, u(W_s(t))\Big)\Big)\\ &=e^{-\alpha a}\int\mathbb{M}_a(d\mu d\nu)\,\Pi_x\Big[1_{\{a<\tau\}}u(\xi_a)F(\xi_r,0\leqslant r\leqslant a)\, e^{-\langle\nu,g\rangle}\exp(-\int\mu(dr)\, u(\xi_r))\Big]\\ &=e^{-\alpha a}\Pi_x\Big[1_{\{a<\tau\}}u(\xi_a)F(\xi_r,0\leqslant r\leqslant a)\int\mathbb{M}_a(d\mu d\nu)\, e^{-\langle\nu,g\rangle}\exp(-\int\mu(dr)\, u(\xi_r))\Big]\\ &=\Pi_x\Big[1_{\{a<\tau\}}u(\xi_a)F(\xi_r,0\leqslant r\leqslant a)\exp\Big(-\int_0^a\gamma_\psi(u(\xi_r),g(r))dr\Big)\Big].\end{aligned}$$

Formula (131) follows by combining this equality with (134), (136), (137) and (138).

Formula (132) is the special case $g=0$ in (131). To prove the last assertion, let $\zeta(W_{T_D},d\mu)$ be the law of the random measure

$$\beta 1_{[0,H_{T_D}]}(r)\, dr+\sum_{i\in I}(v_i-\ell_i)\,\delta_{r_i}$$

where $\sum\delta_{(r_i,v_i,\ell_i)}$ is a Poisson point measure on $\mathbb{R}_+^3$ with intensity given by formula (133). Then, for every $a>0$, we can use (132) to compute

$$\begin{aligned}&\mathbb{N}_x\Big(F(W_{T_D}(r),0\leqslant r\leqslant a)\, 1_{\{T_D<\infty\}}1_{\{a<H_{T_D}\}}\int\zeta(W_{T_D},d\mu)\, e^{-\langle\mu,g\rangle}\Big)\\ &\quad=\Pi_x\Big[1_{\{a<\tau\}}\, F(\xi_r,0\leqslant r\leqslant a)u(\xi_a)\, \exp\Big(-\int_0^a\widetilde{\psi}(u(\xi_r))\, dr\Big)\\ &\qquad\times\exp\Big(-\beta\int_0^a dr\, g(r)-\int_0^a dr\int\pi(dv)\int_0^v d\ell e^{-\ell u(\xi_r)}(1-e^{-(v-\ell)g(r)})\Big)\Big]\\ &\quad=\Pi_x\Big[1_{\{a<\tau\}}\, F(\xi_r,0\leqslant r\leqslant a)u(\xi_a)\, \exp\Big(-\int_0^a\gamma_\psi(u(\xi_r),g(r))\, dr\Big)\Big],\end{aligned}$$

using (130) in the last equality.

Set $\mathbb{N}_x^D=\mathbb{N}_x(\cdot\mid T_D<\infty)$ to simplify notation. By comparing with (131), we see that for any nonnegative measurable function g with support in $[0,a]$, we have

$$\mathbb{N}_x^D[e^{-\langle\rho_{T_D},g\rangle}\mid W_{T_D}]=\int\zeta(W_{T_D},d\mu)\, e^{-\langle\mu,g\rangle},$$

a.s. on the set $\{H_{T_D} > a\}$. This is enough to conclude that $\zeta(W_{T_D}, d\mu)$ is the conditional distribution of ρ_{T_D} knowing W_{T_D}, provided that we already know that $\rho_{T_D}(\{H_{T_D}\}) = 0$ a.s. The latter fact however is a simple consequence of (106). This completes the proof of the theorem. □

The case of Brownian motion. — Suppose that the spatial motion ξ is d-dimensional Brownian motion and that D is a domain in $\mathbb{R}^d$. Then, it is easy to see that the function $u(x) = \mathbb{N}_x(T_D < \infty)$, $x \in D$ is of class C^2 and solves $\frac{1}{2}\Delta u = \psi(u)$. In the context of superprocesses, this was observed by Dynkin [**12**]. We may argue as follows. First note that the set of nonnegative solutions of $\frac{1}{2}\Delta u = \psi(u)$ in a domain is closed under pointwise convergence (for a probabilistic proof, reproduce the arguments of the proof of Proposition 9 (iii) in [**31**]). Then let (D_n) be a sequence of bounded regular subdomains of D, such that $\overline{D}_n \subset D_{n+1}$ and $D = \lim \uparrow D_n$. For every $n \geqslant 0$, set

$$v_n(x) = \mathbb{N}_x(\mathcal{Z}^{D_n} \neq 0)\ , \quad u_n(x) = \mathbb{N}_x(T_{D_n} < \infty)\ , \quad x \in D_n.$$

From the properties of the exit measure, it is immediate to see that $v_n \leqslant u_n$. On the other hand, by writing

$$v_n(x) = \lim_{\lambda \uparrow \infty} \uparrow\ \mathbb{N}_x(1 - \exp -\lambda\langle \mathcal{Z}^{D_n}, 1\rangle),$$

we deduce from Proposition 4.5.1 and the stability of the set of nonnegative solutions under pointwise convergence that v_n is of class C^2 and solves $\frac{1}{2}\Delta v_n = \psi(v_n)$ in D. Since the function $x \longrightarrow \mathbb{N}_x(1 - \exp -\lambda\langle \mathcal{Z}^{D_n}, 1\rangle)$ has boundary value λ on ∂D_n (Proposition 4.5.1), we also see that v_n has boundary value $+\infty$ on ∂D_n.

Then, it follows from Lemma 4.5.3 and our assumption $\overline{D}_n \subset D_{n+1}$ that $v_n(x) \geqslant u_{n+1}(x)$ for $x \in D_n$. Since it is easy to see that $u_n(x)$ decreases to $u(x)$ as $n \to \infty$, for every $x \in D$, we conclude from the inequalities $u_{n+1}(x) \leqslant v_n(x) \leqslant u_n(x)$ that $v_n(x)$ also converges to $u(x)$ pointwise as $n \to \infty$. Hence u is a nonnegative solution of $\frac{1}{2}\Delta u = \psi(u)$ in D. The preceding argument gives more. Let v be any nonnegative solution of $\frac{1}{2}\Delta v = \psi(v)$ in D. Since $v_{n|\partial D_n} = +\infty$, the comparison principle (Lemma V.7 in [**31**]) implies that $v \leqslant v_n$ in D_n. By passing to the limit $n \to \infty$, we conclude that $v \leqslant u$. Hence u is the maximal nonnegative solution of $\frac{1}{2}\Delta u = \psi(u)$ in D.

Suppose that $u(x) > 0$ for some $x \in D$. It is easy to see that this implies $u(y) > 0$ for every $y \in D$ (use a suitable Harnack principle or a probabilistic argument relying on the fact that $u(\xi_t)\exp(-\int_0^t \widetilde{\psi}(u(\xi_r))dr)$ is a martingale). By applying Itô's formula to $\log u(\xi_t)$, we see that Π_x a.s. on $\{t < \tau\}$,

$$\begin{aligned}\log u(\xi_t) &= \log u(x) + \int_0^t \frac{\nabla u}{u}(\xi_r) \cdot d\xi_r + \frac{1}{2}\int_0^t \Delta(\log u)(\xi_r)dr \\ &= \log u(x) + \int_0^t \frac{\nabla u}{u}(\xi_r) \cdot d\xi_r + \int_0^t \Big(\widetilde{\psi}(u(\xi_r)) - \frac{1}{2}\Big|\frac{\nabla u}{u}\Big|^2(\xi_r)\Big)dr.\end{aligned}$$

We can then rewrite (132) in the form

$$\mathbb{N}_x^D\Big(1_{\{T_D<\infty\}}1_{\{t<H_{T_D}\}}F(W_{T_D}(r),0\leqslant r\leqslant t)\Big)$$
$$=\Pi_x\Big[1_{\{t<\tau\}}\exp\Big(\int_0^t\frac{\nabla u}{u}(\xi_r)\cdot d\xi_r-\frac{1}{2}\int_0^t\Big|\frac{\nabla u}{u}\Big|^2(\xi_r)\,dr\Big)F(\xi_r,0\leqslant r\leqslant t)\Big].$$

An application of Girsanov's theorem then shows that W_{T_D} is distributed as the solution of the stochastic differential equation

$$dx_t=dB_t+\frac{\nabla u}{u}(x_t)dt$$
$$x_0=x$$

(where B is a standard d-dimensional Brownian motion) which can be defined up to its first hitting time of ∂D. See [**29**] for a discussion and another interpretation of this distribution on paths in the case $\psi(u)=u^2$.

4.7. The reduced tree in an open set

We keep the notation and assumptions of the previous section. In particular, we assume that W^* has continuous sample paths with respect to the distance d^*, D is an open set in E, $x\in D$, $T_D=\inf\{s>0:\tau(W_s^*)<\infty\}$ and $u(x)=\mathbb{N}_x(T_D<\infty)<\infty$. To avoid trivialities, we assume that $u(x)>0$, and we recall the notation $\mathbb{N}_x^D=\mathbb{N}_x(\cdot\mid T_D<\infty)$. We will assume in addition that

$$\sup_{y\in K}u(y)<\infty \tag{139}$$

for every compact subset K of D. This assumption holds in particular when ξ is Brownian motion in $\mathbb{R}^d$, under the condition (119) (use translation invariance and the fact that $u(0)<\infty$ when D is an open ball centered at the origin).

We also set:

$$L_D=\sup\{s\geqslant 0:\tau(W_s^*)<\infty\},$$

and

$$m_D=\inf_{T_D\leqslant s\leqslant L_D}H_s.$$

As a consequence of the first lemma below, we will see that $m_D<H_{T_D}$, $\mathbb{N}_x^D$ a.s.

Our goal is to describe the genealogical structure of the paths W_s that exit D, up to their first exit time from D, under the probability measure $\mathbb{N}_x^D$. To be more precise, all paths W_s such that $\tau(W_s^*)<\infty$ must coincide up to level m_D. At level m_D there is a branching point with finitely many branches, each corresponding to an excursion of H above level m_D during which W hits D^c. In each such excursion, the paths W_s that hit D^c will be the same up to a level (strictly greater than m_D) at which there is another branching point, and so on.

We will describe this genealogical structure in a recursive way. We will first derive the law of the common part to the paths W_s that do exit D. This common part

is represented by a stopped path W_0^D in $\mathcal{W}_x$ with lifetime $\zeta_{W_0^D} = m_D$. Then we will obtain the distribution of the "number of branches" at level m_D, that is the number of excursions of W above height m_D that hit D^c. Finally, we will see that conditionally on W_0^D, these excursions are independent and distributed according to $\mathbb{N}_{\widehat{W}_0^D}(\cdot \mid T_D < \infty)$. This completes our recursive description since we can apply to each of these excursions the results obtained under $\mathbb{N}_x^D$.

Before coming to the main result of this section, we state an important lemma.

Lemma 4.7.1. — *The point T_D is not isolated in $\{s \geqslant 0 : \tau(W_s^*) < \infty\}$, $\mathbb{N}_x$ a.e. on $\{T_D < \infty\}$.*

Proof. — We start with some preliminary observations. Let $(\mu, \mathrm{w}) \in \Theta_x$ be such that $\mu(\{H(\mu)\}) = 0$ and $\mathrm{w}(t) \in D$ for every $t \in [0, H(\mu))$. As an application of Lemma 4.2.4, we have

$$\begin{aligned}\mathbb{P}^*_{\mu,\mathrm{w}}[T_D < \infty] &= 1 - \exp - \int_{[0,H(\mu))} \mathbb{N}_{\mathrm{w}(t)}(T_D < \infty)\, \mu(dt) \\ &= 1 - \exp - \int_{[0,H(\mu))} u(\mathrm{w}(t))\, \mu(dt).\end{aligned}$$

By the previous formula, the equality $\mathbb{P}^*_{\mu,\mathrm{w}}[T_D = 0] = 1$ can only hold if

$$\int_{[0,H(\mu))} u(\mathrm{w}(t))\, \mu(dt) = \infty. \tag{140}$$

Conversely, condition (140) also implies that $\mathbb{P}^*_{\mu,\mathrm{w}}[T_D = 0] = 1$. To see this, first note that our assumption (139) guarantees that for every $\varepsilon > 0$,

$$\int_{[0,H(\mu)-\varepsilon]} u(\mathrm{w}(t))\, \mu(dt) < \infty,$$

and thus we have also under (140)

$$\int_{(H(\mu)-\varepsilon,H(\mu))} u(\mathrm{w}(t))\, \mu(dt) = \infty.$$

Then write μ_ε for the restriction of μ to $[0, H(\mu) - \varepsilon]$, and set

$$S_\varepsilon = \inf\{s \geqslant 0 : \langle \rho_s, 1\rangle = \langle \mu_\varepsilon, 1\rangle\}.$$

Lemma 4.2.4 again implies that

$$\mathbb{P}^*_{\mu,\mathrm{w}}[T_D \leqslant S_\varepsilon] = 1 - \exp - \int_{(H(\mu)-\varepsilon,H(\mu))} u(\mathrm{w}(t))\, \mu(dt) = 1.$$

Since $S_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$, $\mathbb{P}^*_{\mu,\mathrm{w}}$ a.s., we get that $\mathbb{P}^*_{\mu,\mathrm{w}}[T_D = 0] = 1$, which was the desired result.

Let us prove the statement of the lemma. Thanks to the strong Markov property, it is enough to prove that $\mathbb{P}^*_{\rho_{T_D}, W_{T_D}}[T_D = 0] = 1$, $\mathbb{N}_x$ a.e. on $\{T_D < \infty\}$. Note that

we have $\rho_{T_D}(\{H_{T_D}\}) = 0$ and $W_{T_D}(t) \in D$ for every $t < H_{T_D}$, $\mathbb{N}_x$ a.e. on $\{T_D < \infty\}$. By the preceding observations, it is enough to prove that

$$\int_{[0,H_{T_D})} u(W_{T_D}(t))\, \rho_{T_D}(dt) = \infty\ , \qquad \text{a.e. on } \{T_D < \infty\}. \tag{141}$$

To this end, set for every $s > 0$,

$$M_s = \mathbb{N}_x(T_D < \infty \mid \mathcal{F}_s).$$

The Markov property at time s shows that we have for every $s > 0$, $\mathbb{N}_x$ a.e.,

$$\begin{aligned} M_s &= 1_{\{T_D \leqslant s\}} + 1_{\{s<T_D\}} \mathbb{P}^*_{\rho_s, W_s}[T_D < \infty] \\ &= 1_{\{T_D \leqslant s\}} + 1_{\{s<T_D\}} \Big(1 - \exp - \int u(W_s(t))\, \rho_s(dt)\Big) \end{aligned}$$

Since the process (ρ_s) is right-continuous for the variation distance on measures, it is easy to verify that the process $1_{\{s<T_D\}}(1 - \exp - \int u(W_s(t))\, \rho_s(dt))$ is right-continuous. Because $(M_s, s > 0)$ is a martingale with respect to the filtration $(\mathcal{F}_s)$, a standard result implies that this process also has left limits at every $s > 0$, $\mathbb{N}_x$ a.e. In particular the left limit at T_D

$$\lim_{s \uparrow T_D, s<T_D} M_s = \lim_{s \uparrow T_D, s<T_D} \Big(1 - \exp - \int u(W_s(t))\, \rho_s(dt)\Big)$$

exists $\mathbb{N}_x$ a.e. on $\{T_D < \infty\}$. It is not hard to verify that this limit is equal to 1: If $D_n = \{y \in D : \mathrm{dist}(y, D^c) > n^{-1}\}$ and $T_n = T_{D_n}$, we have $T_n < T_D$ and $T_n \uparrow T_D$ on $\{T_D < \infty\}$, and $M_{T_n} = \mathbb{N}_x(T_D < \infty \mid \mathcal{F}_{T_n})$ converges to 1 as $n \to \infty$ on the set $\{T_D < \infty\}$ because T_D is measurable with respect to the σ-field $\bigvee \mathcal{F}_{T_n}$.

Summarizing, we have proved that

$$\lim_{s \uparrow T_D, s<T_D} \int u(W_s(t))\, \rho_s(dt) = +\infty \tag{142}$$

$\mathbb{N}_x$ a.e. on $\{T_D < \infty\}$. Then, for every rational $a > 0$, consider on the event $\{T_D < \infty\} \cap \{H_{T_D} > a\}$, the number $\alpha_{(a)}$ defined as the left end of the excursion interval of H above a that straddles T_D. As a consequence of the considerations in subsection 4.2.2, the following two facts hold on $\{T_D < \infty\} \cap \{H_{T_D} > a\}$:

$$\begin{aligned} &\rho_{\alpha_{(a)}} \text{ is the restriction of } \rho_{T_D} \text{ to } [0, a) \\ &W_{\alpha_{(a)}}(t) = W_{T_D}(t)\ , \qquad \text{for every } t \in [0, a). \end{aligned}$$

Thus, we have also on the same event

$$\int u(W_{\alpha_{(a)}}(t))\, \rho_{\alpha_{(a)}}(dt) = \int_{[0,a)} u(W_{T_D}(t))\, \rho_{T_D}(dt).$$

Now on the event $\{T_D < \infty\}$ we can pick a sequence (a_n) of rationals strictly increasing to H_{T_D}. We observe that $\alpha_{(a_n)}$ also converges to T_D (if S is the increasing limit

of $\alpha_{(a_n)}$, the snake property implies that $\widehat{W}_S = \widehat{W}_{T_D} \in D^c$ and so we have $S \geqslant T_D$, whereas the other inequality is trivial). Therefore, using (142),

$$\infty = \lim_{n\to\infty} \int u(W_{\alpha_{(a_n)}}(t))\, \rho_{\alpha_{(a_n)}}(dt) = \lim_{n\to\infty} \int_{[0,a_n)} u(W_{T_D}(t))\, \rho_{T_D}(dt),$$

which yields (141). □

Lemma 4.7.1 implies that $T_D < L_D$, $\mathbb{N}_x^D$ a.s. Since we know that $\rho_{T_D}(\{H_{T_D}\}) = 0$, $\mathbb{N}_x^D$ a.s., an application of the strong Markov property at time T_D shows that $m_D < H_{T_D}$, $\mathbb{N}_x^D$ a.s. We define W_0^D as the stopped path which is the restriction of W_{T_D} to $[0, m_D]$. Then we define the excursions of W above level m_D in a way analogous to subsection 4.2.2. If

$$R_D = \sup\{s \leqslant T_D : H_s = m_D\}\ , \quad S_D = \inf\{s \geqslant L_D : H_s = m_D\},$$

we let (a_j, b_j), $j \in J$ be the connected components of the open set $(R_D, S_D) \cap \{s \geqslant 0 : H_s > m_D\}$. For each $j \in J$, we can then define the process $W^{(j)} \in C(\mathbb{R}_+, \mathcal{W})$ by setting

$$\begin{aligned} &W_s^{(j)}(r) = W_{a_j+s}(m_D + r),\ \zeta_{W_s^{(j)}} = H_{a_j+s} - m_D && \text{if } 0 < s < b_j - a_j \\ &W_s^{(j)} = \widehat{W}_0^D && \text{if } s = 0 \text{ or } s \geqslant b_j - a_j. \end{aligned}$$

By a simple continuity argument, the set $\{j \in J : T_D(W^{(j)}) < \infty\}$ is finite a.s., and we set

$$N_D = \mathrm{Card}\{j \in J : T_D(W^{(j)}) < \infty\}.$$

We write $W^{D,1}, W^{D,2}, \ldots, W^{D,N_D}$ for the excursions $W^{(j)}$ such that $T_D(W^{(j)}) < \infty$, listed in chronological order.

We are now ready to state our main result.

Theorem 4.7.2. — *For every $r \geqslant 0$, set $\theta(r) = \psi'(r) - \widetilde{\psi}(r)$. Then the law of W_0^D is characterized by the following formula, valid for any nonnegative measurable function F on $\mathcal{W}^*$:*

$$\begin{aligned} (143)\quad &\mathbb{N}_x(1_{\{T_D<\infty\}} F(W_0^D)) \\ &= \int_0^\infty db\, \Pi_x\Big[1_{\{b<\tau\}} u(\xi_b)\, \theta(u(\xi_b))\, \exp\Big(-\int_0^b \psi'(u(\xi_r))dr\Big) F(\xi_r, 0 \leqslant r \leqslant b)\Big]. \end{aligned}$$

The conditional distribution of N_D knowing W_0^D is given by:

$$(144)\qquad \mathbb{N}_x^D[r^{N_D} \mid W_0^D] = r\, \frac{\psi'(U) - \gamma_\psi(U, (1-r)U)}{\psi'(U) - \gamma_\psi(U, 0)}\ , \qquad 0 \leqslant r \leqslant 1,$$

where $U = u(\widehat{W}_0^D)$. Finally, conditionally on the pair (W_0^D, N_D), the processes $W^{D,1}, W^{D,2}, \ldots, W^{D,N_D}$ are independent and distributed according to $\mathbb{N}^D_{\widehat{W}_0^D}$.

Proof. — Our first objective is to compute the conditional distribution of m_D knowing W_{T_D}. To this end, we will apply the strong Markov property of the Lévy snake at time T_D. We have for every $b > 0$

$$\mathbb{N}_x^D[m_D > b \mid \rho_{T_D}, W_{T_D}] = \mathbb{P}^*_{\rho_{T_D}, W_{T_D}}[\inf_{0 \leqslant s \leqslant L_D} H_s > b].$$

By Lemma 4.2.4, the latter expression is equal to the probability that in a Poisson point measure with intensity

$$\rho_{T_D}(dh)\, \mathbb{N}_{W_{T_D}(h)}(d\rho dW)$$

there is no atom (h_i, ρ^i, W^i) such that $h_i \leqslant b$ and $T_D(W^i) < \infty$. We conclude that

$$\begin{aligned} \mathbb{N}_x^D[m_D > b \mid \rho_{T_D}, W_{T_D}] &= \exp - \int_{[0,b]} \rho_{T_D}(dh)\, \mathbb{N}_{W_{T_D}(h)}(T_D < \infty) \\ &= \exp - \int_{[0,b]} \rho_{T_D}(dh)\, u(W_{T_D}(h)). \end{aligned} \tag{145}$$

Recall that the conditional law of ρ_{T_D} knowing W_{T_D} is given in Theorem 4.6.2. Using this conditional distribution we see that

$$\mathbb{N}_x^D[m_D > b \mid W_{T_D}] = \exp\Big(-\beta \int_0^b da\, u(W_{T_D}(a))\Big)\, E\Big[\exp - \sum_{r_i \leqslant b} (v_i - \ell_i) u(W_{T_D}(r_i))\Big],$$

where $\sum \delta_{(r_i, v_i, \ell_i)}$ is a Poisson point measure with intensity (depending on W_{T_D}) given by (133). By exponential formulas for Poisson measures, we have

$$\begin{aligned} &E\Big[\exp - \sum_{r_i \leqslant b} (v_i - \ell_i) u(W_{T_D}(r_i))\Big] \\ &\qquad = \exp - \int_0^b dr \int \pi(dv) \int d\ell\, e^{-\ell u(W_{T_D}(r))}(1 - e^{-(v-\ell)u(W_{T_D}(r))}). \end{aligned}$$

By substituting this in the previous displayed formula, and using (130), we get

$$\mathbb{N}_x^D[m_D > b \mid W_{T_D}] = \exp\Big(-\int_0^b dr\, (\psi'(u(W_{T_D}(r))) - \widetilde{\psi}(u(W_{T_D}(r))))\Big). \tag{146}$$

Hence, if $\theta(r) = \psi'(r) - \widetilde{\psi}(r)$ as in the statement of the theorem, the conditional law of m_D knowing W_{T_D} has density

$$1_{[0, H_{T_D})}(b)\, \theta(u(W_{T_D}(r)))\, \exp\Big(-\int_0^b \theta(u(W_{T_D}(r)))\, dr\Big).$$

It follows that

$$\begin{aligned}
&\mathbb{N}_x\Big(1_{\{T_D<\infty\}}\, F(W_0^D)\Big)\\
&\quad = \mathbb{N}_x\Big(1_{\{T_D<\infty\}}\, F(W_{T_D}(t), 0\leqslant t\leqslant m_D)\Big)\\
&\quad = \mathbb{N}_x\Big(1_{\{T_D<\infty\}} \int_0^{H_{T_D}} db\, \theta(u(W_{T_D}(b)))\, \exp\Big(-\int_0^b \theta(u(W_{T_D}(r)))\, dr\Big)\\
&\qquad\qquad \times\ F(W_{T_D}(t), 0\leqslant t\leqslant b)\Big)\\
&\quad = \int_0^\infty db\, \mathbb{N}_x\Big(1_{\{T_D<\infty\}} 1_{\{b<H_{T_D}\}} \theta(u(W_{T_D}(b)))\, \exp\Big(-\int_0^b \theta(u(W_{T_D}(r)))\, dr\Big)\\
&\qquad\qquad \times\ F(W_{T_D}(t), 0\leqslant t\leqslant b)\Big)\\
&\quad = \int_0^\infty db\, \Pi_x\Big[1_{\{b<\tau\}} u(\xi_b)\theta(u(\xi_b))\exp(-\int_0^b \psi'(u(\xi_r))\, dr)\, F(\xi_r, 0\leqslant r\leqslant b)\Big],
\end{aligned}$$

using (132) in the last equality. This gives the first assertion of the theorem.

We now turn to the distribution of N_D. We use again the strong Markov property at time T_D and Lemma 4.2.4 to analyse the conditional distribution of the pair (m_D, N_D) knowing (ρ_{T_D}, W_{T_D}). Conditional on (ρ_{T_D}, W_{T_D}), let $\sum \delta_{(h_i,\rho^i,W^i)}$ be a Poisson point measure with intensity

$$\rho_{T_D}(dh)\, \mathbb{N}_{W_{T_D}(h)}(d\rho dW).$$

Set

$$\begin{aligned}
m &= \inf\{h_i : T_D(W^i) < \infty\},\\
M &= \mathrm{Card}\{i : h_i = m \text{ and } T_D(W^i) < \infty\}.
\end{aligned}$$

Then Lemma 4.2.4 and the strong Markov property show that the pairs $(m, 1+M)$ and (m_D, N_D) have the same distribution conditional on (ρ_{T_D}, W_{T_D}). Recall that the conditional distribution of m_D (or of m) is given by (145).

Now note that:

– If $\rho_{T_D}(\{m\}) = 0$, then $M = 1$ because the Poisson measure $\sum \delta_{(h_i,\rho^i,W^i)}$ cannot have two atoms at a level h such that $\rho_{T_D}(\{h\}) = 0$.

– Let $b \geqslant 0$ be such that $\rho_{T_D}(\{b\}) > 0$. The event $\{m = b\}$ occurs with probability

$$\exp\Big(-\int_{[0,b)} \rho_{T_D}(dh)\, u(W_{T_D}(h))\Big)\Big(1 - e^{-\rho_{T_D}(\{b\})u(W_{T_D}(b))}\Big).$$

Conditionally on this event, M is distributed as a Poisson variable with parameter $c = \rho_{T_D}(\{b\})u(W_{T_D}(b))$ and conditioned to be (strictly) positive, whose generating function is

$$\frac{e^{-c(1-r)} - e^{-c}}{1 - e^{-c}}.$$

Since the continuous part of the law of m has density

$$\beta\, u(W_{T_D}(b))\, \exp\Big(- \int_{[0,b)} \rho_{T_D}(dh)\, u(W_{T_D}(h))\Big)$$

we get by combining the previous two cases that

$$\begin{aligned}(147)\quad &\mathbb{N}_x^D[f(m_D) r^{N_D} \mid \rho_{T_D}, W_{T_D}] \\ &= \beta r^2 \int_0^{H_{T_D}} db\, f(b)\, u(W_{T_D}(b))\, \exp\Big(- \int_{[0,b)} \rho_{T_D}(dh)\, u(W_{T_D}(h))\Big) \\ &\qquad + r \sum_{\rho_{T_D}(\{b\})>0} f(b) \exp\Big(- \int_{[0,b)} \rho_{T_D}(dh)\, u(W_{T_D}(h))\Big) \\ &\qquad \times \Big(e^{-\rho_{T_D}(\{b\}) u(W_{T_D}(b))(1-r)} - e^{-\rho_{T_D}(\{b\}) u(W_{T_D}(b))}\Big).\end{aligned}$$

We now need to integrate the right side of (147) with respect to the conditional law of ρ_{T_D} knowing W_{T_D}. We get

$$\mathbb{N}_x^D[f(m_D) r^{N_D} \mid W_{T_D}] = A_1 + A_2$$

where

(148)

$$\begin{aligned}A_1 &= \beta r^2 \mathbb{N}_x^D\Big[\int_0^{H_{T_D}} db\, f(b)\, u(W_{T_D}(b))\, \exp\Big(- \int_{[0,b)} \rho_{T_D}(dh)\, u(W_{T_D}(h))\Big) \,\Big|\, W_{T_D}\Big] \\ &= \beta r^2 \int_0^{H_{T_D}} db\, f(b)\, u(W_{T_D}(b))\, \exp\Big(- \int_{[0,b)} \theta(u(W_{T_D}(h)))\, dh\Big),\end{aligned}$$

by the calculation used in the proof of (146). We then compute A_2. To this end, let $\mathcal{N}(dbdvd\ell)$ be (conditionally on W_{T_D}) a Poisson point measure in $\mathbb{R}_+^3$ with intensity

$$1_{[0,H_{T_D}]}(b) 1_{[0,v]}(\ell) e^{-\ell u(W_{T_D}(b))}\, db\pi(dv)d\ell.$$

From Theorem 4.6.2, we get

$$\begin{aligned}A_2 = r\, \mathbb{N}_x^D\Big[\int \mathcal{N}(dbdvd\ell)\, f(b)\, e^{-\beta \int_0^b da\, u(W_{T_D}(a)) - \int_{\{a<b\}} \mathcal{N}(dadv'd\ell')(v'-\ell') u(W_{T_D}(a))} \\ \times (e^{-(v-\ell)u(W_{T_D}(b))(1-r)} - e^{-(v-\ell)u(W_{T_D}(b))})\Big) \,\Big|\, W_{T_D}\Big].\end{aligned}$$

From Lemma 3.1.5 and (once again) the calculation used in proving (146), we arrive at

$$\begin{aligned}A_2 = r \int_0^{H_{T_D}} db\, f(b)\, \exp\Big(- \int_{[0,b)} \theta(u(W_{T_D}(a)))\, da\Big) \\ \times \int \pi(dv) \int_0^v d\ell\, e^{-\ell u(W_{T_D}(b))} \Big(e^{-(v-\ell)(1-r)u(W_{T_D}(b))} - e^{-(v-\ell)u(W_{T_D}(b))}\Big).\end{aligned}$$

From (130), we have

$$\int \pi(dv) \int_0^v d\ell\, e^{-\ell u(W_{T_D}(b))} \Big(e^{-(v-\ell)(1-r)u(W_{T_D}(b))} - e^{-(v-\ell)u(W_{T_D}(b))} \Big)$$
$$= \psi'(u(W_{T_D}(b))) - \gamma_\psi(u(W_{T_D}(b)), (1-r)u(W_{T_D}(b))) - \beta r u(W_{T_D}(b))).$$

By substituting this identity in the previous formula for A_2, and then adding the formula for A_1, we arrive at:

$$\mathbb{N}_x^D[f(m_D) r^{N_D} \mid W_{T_D}]$$
$$= r \int_0^{H_{T_D}} db\, f(b) \exp\Big(- \int_0^b da\, \theta(u(W_{T_D}(a)))\Big)$$
$$\times \Big(\psi'(u(W_{T_D}(b))) - \gamma_\psi(u(W_{T_D}(b)), (1-r)u(W_{T_D}(b)))\Big)$$
$$= \mathbb{N}_x^D[f(m_D)\, r\, \frac{\psi'(u(W_{T_D}(m_D))) - \gamma_\psi(u(W_{T_D}(m_D)), (1-r)u(W_{T_D}(m_D)))}{\psi'(u(W_{T_D}(m_D))) - \gamma_\psi(u(W_{T_D}(m_D)), 0)} \,\Big|\, W_{T_D}\Big].$$

In the last equality we used the conditional distribution of m_D knowing W_{T_D}, and the fact that $\theta(u) = \psi'(u) - \widetilde{\psi}(u) = \psi'(u) - \gamma_\psi(u, 0)$.

Finally, if $U = u(W_{T_D}(m_D)) = u(\widehat{W}_0^D)$, we have obtained

$$\mathbb{N}_x^D[r^{N_D} \mid W_0^D] = r\, \frac{\psi'(U) - \gamma_\psi(U, (1-r)U)}{\psi'(U) - \gamma_\psi(U, 0)},$$

which is formula (144) of the theorem.

It remains to obtain the last assertion of the theorem. Here again, we will rely on Lemma 4.2.4 and the strong Markov property at time T_D. We need to restate the result of Lemma 4.2.4 in a slightly different form. Let $(\mu, \mathrm{w}) \in \Theta_x$ with $\mu(\{H(\mu)\}) = 0$ and $\mathrm{w}(t) \in D$ for every $t < H(\mu)$. Under $\mathbb{P}^*_{\mu,\mathrm{w}}$, we write $Y_t = \langle \rho_t, 1\rangle$, $K_t = \inf_{r \leqslant t} Y_r$ and $I_t = K_t - \langle \mu, 1\rangle$. If (α_i, β_i), $i \in I$ are the excursion intervals of $Y - K$ away from 0, we introduce the "excursions" (ρ_i, W^i), $i \in I$ as defined before the statement of Lemma 4.2.4. The starting height of excursion (ρ_i, W^i) is $h_i = H_{\alpha_i} = H(k_{-I_{\alpha_i}}\mu)$. The proof of Lemma 4.2.4 shows that the point measure

$$\sum_{i \in I} \delta_{(-I_{\alpha_i}, \rho^i, W^i)}$$

is Poisson with intensity $1_{[0,\langle \mu, 1\rangle]}(u) du\, \mathbb{N}_{\mathrm{w}(H(k_u\mu))}(d\rho\, dW)$ (this is slightly more precise than the statement of Lemma 4.2.4).

We then write $i_1, i_2, \ldots$ for the indices $i \in I$ such that $T_D(W^i) < \infty$, ranked in such a way that $I_{\alpha_{i_1}} < I_{\alpha_{i_2}} < \cdots$. Our assumption (139) guarantees that this ordering is possible, and we have clearly $h_{i_1} \leqslant h_{i_2} \leqslant \cdots$. By well-known properties of Poisson measures, the processes $W^{i_1}, W^{i_2}, \ldots$ are independent conditionally on the sequence $h_{i_1}, h_{i_2}, \ldots$, and the conditional distribution of W^{i_ℓ} is $\mathbb{N}^D_{\mathrm{w}(h_{i_\ell})}$.

If we apply the previous considerations to the shifted process $(\rho_{T_D+s}, W_{T_D+s}; s \geqslant 0)$, taking $\mu = \rho_{T_D}$ and $\mathrm{w} = W_{T_D}$ and relying on the strong Markov property at T_D, we

can easily identify

$$\begin{aligned} m_D &= h_{i_1} \\ N_D &= 1 + \sup\{k \geqslant 1 : h_{i_k} = h_{i_1}\} \\ W^{D,N_D} &= W^{i_1},\ W^{D,N_D-1} = W^{i_2}, \dots,\ W^{D,2} = W^{i_{N_D-1}}. \end{aligned}$$

By a preceding observation, we know that conditionally on (m_D, N_D), the processes $W^{i_1}, \dots, W^{i_{N_D-1}}$ are independent and distributed according to $\mathbb{N}^D_{w(m_D)}$.

Combining this with the strong Markov property at time T_D, we see that, conditionally on (N_D, W_0^D), the processes $W^{D,2}, \dots, W^{D,N_D}$ are independent and distributed according to $\mathbb{N}^D_{\widehat{W}_0^D}$ (recall that $\widehat{W}_0^D = W_{T_D}(m_D)$). An argument similar to the end of the proof of Theorem 3.2.1 (relying on independence properties of Poisson measures) also shows that, conditionally on (N_D, W_0^D), the vector $(W^{D,2}, \dots, W^{D,N_D})$ is independent of $W^{D,1}$. Furthermore, denote by $\check{W}^{D,\ell}$ the time-reversed processes

$$\check{W}^{D,\ell}_s = W^{D,\ell}_{(\sigma(W^{D,\ell})-s)+}.$$

The time-reversal property already used in the proof of Theorem 4.6.2 implies that the vectors $(W^{D,1}, \dots, W^{D,N_D})$ and $(\check{W}^{D,N_D}, \dots, \check{W}^{D,1})$ have the same conditional distribution given (N_D, W_0^D). Hence, the conditional distribution of $\check{W}^{D,1}$, or equivalently that of $W^{D,1}$, is also equal to $\mathbb{N}^D_{\widehat{W}_0^D}$. This completes the proof of Theorem 4.7.2. □

Remarks

(i) By considering the special case where the spatial motion ξ_t is deterministic, $\xi_t = t$, and $E = \mathbb{R}_+$, $x = 0$ and $D = [0, T)$ for some fixed $T > 0$, we obtain an alternative proof of formulas derived in Theorem 2.7.1. In particular, formula (86) is a special case of (144). Similarly, (85) can be seen as a special case of (146).

(ii) In the stable case $\psi(u) = cu^\alpha$, the variable N_D is independent of W_0^D, and its law is given by

$$\mathbb{N}^D_x[r^{N_D}] = \frac{(1-r)^\alpha - 1 + \alpha r}{\alpha - 1}.$$

Of course when $\alpha = 2$, we have $N_D = 2$.

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NOTATION INDEX

X Lévy process (Sect. 1.1.1)
P probability measure under which X starts from 0
ψ Laplace functional of X (Sect. 1.1.1)
$\psi(u) = \alpha u + \beta u^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-ru} - 1 + ru)$
$\widetilde{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda}$
$\varphi(\lambda) = \frac{\lambda}{\psi^{-1}(\lambda)}$
$S_t = \sup_{s \leqslant t} X_s$
$I_t = \inf_{s \leqslant t} X_s$
$I_t^s = \inf_{s \leqslant r \leqslant t} X_r$
$T_x = \inf\{t \geqslant 0 : X_t = -x\}$

N excursion measure of $X - I$ (Sect. 1.1.2)
N^* excursion measure of $X - S$ (Sect. 1.1.2)
σ duration of the excursion under N or under N^*
L_t local time of $X - S$ at level 0
$L^{-1}(t) = \inf\{s \geqslant 0 : L_s > t\}$
$X_s^{(T)} = X_{T+s} - X_T$
$\widehat{X}_s^{(t)}$ process time-reversed at t: $\widehat{X}_s^{(t)} = X_t - X_{(t-s)-}$
H_t height process (Sect. 1.2)
$H(\mu) = \sup(\mathrm{supp}(\mu))$

ρ_t exploration process (Sect. 1.2)
$M_f(\mathbb{R}_+)$ set of all finite measures on $\mathbb{R}_+$
$k_a\mu$ "killing operator" on measures (Sect. 1.2, Sect. 3.1)
$[\mu, \nu]$ concatenation of the measures μ and ν (Sect. 1.2)
ρ^μ exploration process started at $\mu \in M_f(\mathbb{R}_+)$
$\mathbf{P}^\mu$ law of ρ^μ
$\mathbf{M}$ invariant measure of ρ (Sect. 1.2)
L_t^a local times of the height process

$u_t(\lambda) = N(1 - \exp(-\lambda L_\sigma^t))$
$v(t) = N\Big(\sup_{s \geqslant 0} H_s > t\Big)$
$\gamma_\psi(a,b) = \dfrac{\psi(a) - \psi(b)}{a - b}$
η_t dual of the exploration process (Sect. 3.1)
$\mathbb{M}$ invariant measure of (ρ, η) (Sect. 3.1)
$\mathbf{T}$ set of all finite rooted ordered trees (Sect. 0.1)
$k_v(\mathcal{T})$ number of children of the vertex v in the tree $\mathcal{T}$
h_v mark (lifetime) of v
$\theta(e, t_1, \ldots, t_p)$ tree associated with the function e and the times $t_1, \ldots, t_p$
ξ Markov process in E (Sect. 4.1)
Π_x law of ξ started at x
$\mathcal{W}$ set of all E-valued killed paths
$\zeta = \zeta_{\mathrm{w}}$ lifetime of the killed path w
$\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{\mathrm{w}} -)$
$m_h(s, s') = \inf_{s \leqslant r \leqslant s'} h(r)$
$\overline{W}_s = (\rho_s, W_s)$ Lévy snake (Sect. 4.1)
$\mathbb{P}_{\mu,\mathrm{w}}$ law of the Lévy snake started at (μ, w)
$\mathbf{Q_s}$ transition kernels of the Lévy snake
$\mathbb{N}_x$ excursion measure for the Lévy snake
$\mathbb{P}^*_{\mu,\mathrm{w}}$ law of the Lévy snake started at (μ, w) and stopped at $\inf\{s \geqslant 0 : \rho_s = 0\}$
$\langle \mathcal{Z}_a, f \rangle = \displaystyle\int_0^\sigma dL_s^a f(\widehat{W}_s)$
$\tau(\mathrm{w}) = \inf\{t \geqslant 0 : \mathrm{w}(t) \notin D\}$
L_s^D exit local time from D (Sect. 4.3)
$\mathcal{Z}^D$ exit measure from D (Sect. 4.3)
$\mathcal{W}^*$ set of all E-valued stopped paths
$T_D = \inf\{s \geqslant 0 : \tau(W_s^*) < \infty\}$

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